FINM3123 Introduction to Econometrics

Chapter 5

Multiple Regression Analysis: OLS Asymptotics

Multiple Regression Analysis: OLS Asymptotics

So far we focused on properties of OLS that hold for any sample

- **Properties of OLS that hold for any sample/sample size**
  - Expected values/unbiasedness under MLR.1 – MLR.4
  - Variance formulas under MLR.1 – MLR.5
  - Gauss-Markov Theorem under MLR.1 – MLR.5
  - Exact sampling distributions/tests under MLR.1 – MLR.6

- **Properties of OLS that hold for large samples**
  - Consistency under MLR.1 – MLR.4
  - Asymptotic normality/tests under MLR.1 – MLR.5

Without assuming normality of the error term!

Probability reminder: convergence of random variables

Let X be a random variable with cumulative distribution function (CDF) F, and let $X_1, X_2, \ldots$ be a sequence of real-valued random variables with CDF $F_1, F_2, \ldots$

- **convergence in distribution/law**, a.k.a. **weak convergence**:

  $\lim_{n\to\infty} F_n(x) = F(x)$ for every $x \in \mathbb{R}$. We denote $X_n \overset{\mathcal{L}}{\to} X$ or $X_n \overset{D}{\to} X$

- **convergence in probability**

  $\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$ for every $\varepsilon > 0$. We denote $X_n \overset{P}{\to} X$ or $\operatorname*{plim}_{n\to\infty} X_n = X$

- **almost sure convergence**, a.k.a. **strong convergence**

  $\mathbb{P}\left(\lim_{n\to\infty} X_n = X\right) = 1$. We denote $X_n \overset{a.s.}{\longrightarrow} X$

- **L^p convergence**, a.k.a. **convergence in the p^{th} mean**

  $\lim_{n\to\infty} \mathbb{E}(|X_n - X|^p) = 0$. We denote $X_n \overset{L^p}{\to} X$

- For $q > p \geq 1$ $\left(X_n \overset{L^q}{\to} X\right) \Rightarrow \left(X_n \overset{L^p}{\to} X\right) \Rightarrow \left(X_n \overset{P}{\to} X\right) \Rightarrow \left(X_n \overset{D}{\to} X\right)$; and $\left(X_n \overset{a.s.}{\longrightarrow} X\right) \Rightarrow \left(X_n \overset{P}{\to} X\right)$

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Consistency

An estimator θ_n is **consistent** for a population parameter θ if

$\mathbb{P}(|\boldsymbol{\theta_n} - \boldsymbol{\theta}| < \boldsymbol{\varepsilon}) \rightarrow \mathbf{1}$ **for arbitrary** $\boldsymbol{\varepsilon} > \mathbf{0}$ **and** $\boldsymbol{n} \rightarrow \infty$**.**

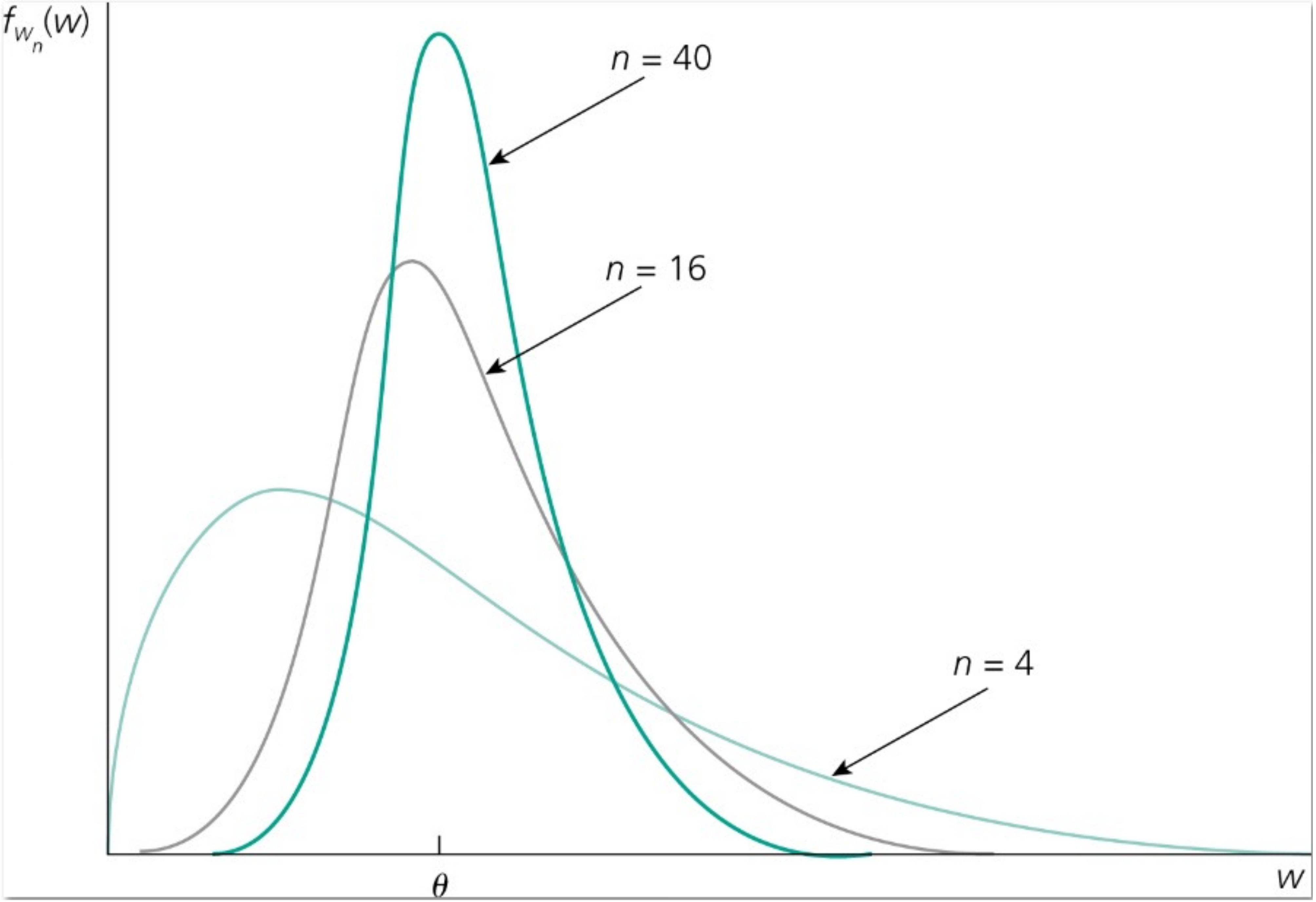
Alternative notation: $\mathbf{plim}\ \boldsymbol{\theta_n} = \boldsymbol{\theta}$

The estimate converges in probability to the true population value

Interpretation:

- Consistency means that the probability that the estimate is arbitrarily close to the true population value can be made arbitrarily high by increasing the sample size

Consistency is a minimum requirement for sensible estimators


$f_{W_n}(w)$
$n = 40$
$n = 16$
$n = 4$
θ
w

Multiple Regression Analysis: OLS Asymptotics

- **Theorem 5.1 (Consistency of OLS)**

$$MLR.1 - MLR.4 \quad \Rightarrow \quad plim\ \widehat{\beta}_j = \beta_j, \quad j = 0, 1, \ldots, k$$

- **Special case of simple regression model**

$$plim\ \widehat{\beta}_1 = \beta_1 + \boxed{Cov(x_1, u)}/Var(x_1)$$

One can see that the slope estimate is consistent if the explanatory variable is exogenous, i.e. uncorrelated with the error term.

- **Assumption MLR.4'**

$$E(u) = 0$$

$$Cov(x_j, u) = 0$$

All explanatory variables must be uncorrelated with the error term. This assumption is <u>weaker</u> than the zero conditional mean assumption MLR.4.

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- **For consistency of OLS, only the weaker MLR.4‘ is needed**
- **Asymptotic analog of omitted variable bias**

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + v$$

True model

$$y = \beta_0 + \beta_1 x_1 + [\beta_2 x_2 + v] = \beta_0 + \beta_1 x_1 + u$$

Misspecified model

$$\Rightarrow \quad plim\ \tilde{\beta}_1 = \beta_1 + Cov(x_1, u)/Var(x_1)$$

Bias

$$= \beta_1 + \beta_2 Cov(x_1, x_2)/Var(x_1) = \beta_1 + \boxed{\beta_2 \delta_1}$$

There is no omitted variable bias if the omitted variable is irrelevant or uncorrelated with the included variable

Multiple Regression Analysis: OLS Asymptotics

- **Asymptotic normality and large sample inference**
  - In practice, the normality assumption MLR.6 is often questionable
  - If MLR.6 does not hold, the results of t- or F-tests may be wrong
  - Fortunately, F- and t-tests still work if the sample size is large enough
  - Also, OLS estimates are normal in large samples even without MLR.6

- **Theorem 5.2 (Asymptotic normality of OLS)**

Under assumptions MLR.1 – MLR.5:

$$\frac{(\widehat{\beta}_j - \beta_j)}{se(\widehat{\beta}_j)} \overset{a}{\sim} N(0,1)$$

In large samples, the standardized estimates are normally distributed

and $plim\ \widehat{\sigma}^2 = \sigma^2$

Multiple Regression Analysis: OLS Asymptotics

Practical consequences

- In large samples, the t-distribution is close to the N(0,1) distribution
- As a consequence, t-tests are valid in large samples without MLR.6
- The same is true for confidence intervals and F-tests
- <u>Important</u>: MLR.1 – MLR.5 are still necessary, especially homoskedasticity

Asymptotic analysis of the OLS sampling errors

$$\widehat{Var}(\hat{\beta}_j) = \frac{\hat{\sigma}^2}{SST_j(1 - R_j^2)}$$

Converges to σ^2

Converges to a fixed number

Converges to $n \times \mathrm{Var}(x_j)$

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Asymptotic analysis of the OLS sampling errors (cont.)

$\widehat{Var}(\widehat{\beta}_j)$ shrinks with the rate $\frac{1}{n}$

$se(\widehat{\beta}_j)$ shrinks with the rate $\frac{1}{\sqrt{n}}$

<u>This is why large samples are better</u>

Example: Standard errors in a birth weight equation

$$n = 1,388 \;\Rightarrow se(\widehat{\beta}_{cigs}) = .00086$$

$$n = 694 \;\Rightarrow se(\widehat{\beta}_{cigs}) = .0013$$

$$\frac{.00086}{.0013} \approx \sqrt{\frac{694}{1,388}}$$

Use only the first half of observations

Summary

Consistency

- Definition:
  $\mathbb{P}(|\theta_n - \theta| < \varepsilon) \to 1$ for arbitrary $\varepsilon > 0$ and $n \to \infty$.
- Theorem 5.1 Consistency of OLS
- Inconsistency (or asymptotic bias) for simple regression case.
  $\text{plim}\, \hat{\beta}_1 = \beta_1 + \text{Cov}(x_1, u)/\text{Var}(x_1)$
- Asymptotic analog omitted variable bias
  $\text{plim}\, \tilde{\beta}_1 = \beta_1 + \beta_2 \delta_1$

Asymptotic normality

- Theorem 5.2 Asymptotic normality of OLS
- In large samples, both standard normal and t tests are valid