Integration by Parts

Given 2 differentiable functions u and v, the Product Rule states that

$$(uv)' = uv' + u'v.$$

By integrating both sides, we have

$$uv = \int (uv)' \, dx = \int uv'dx + \int u'vdx = \int udv + \int vdu.$$

Rearranging this expression, integration by parts for indefinite integral is

$$\int udv = uv - \int vdu.$$

Similarly, integration by parts for definite integral is

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx.$$

Remark:

We distinguish 3 cases:

Case 1: If $f(x) = x^a e^{bx}$, $x^a \sin bx$ or $x^a \cos bx$ then $u = x^a$ and the rest is dv.

Case 2: If $f(x) = x^a \ln x$, $x^a \sin^{-1} bx$, $x^a \cos^{-1} bx$, $x^a \tan^{-1} bx$ or $x^a \cot^{-1} bx$ then $x^a dx = dv$ and the rest is u.

Case 3: If $f(x) = e^{ax} \sin bx$ or $e^{ax} \cos bx$ then $u = e^{ax}$ or $dv = e^{ax} dx$.

Example

Evaluate the following integrals.

(a) $\int \ln x dx.$

(b) $\int x \ln x dx.$

(c) $\int x\cos 4x dx.$

(d) $\int_0^{1/2} \cos^{-1} x dx.$

Solution

(a) $$\begin{aligned}\int \ln x dx &= x\ln x - \int x d\ln x\\ &= x\ln x - \int x\cdot\frac{1}{x}dx\\ &= x\ln x - \int 1dx\\ &= x\ln x - x + C\end{aligned}$$

(b) $$\begin{aligned}\int x\ln x dx &= \frac{1}{2}\int \ln x dx^2\\ &= \frac{x^2}{2}\ln x - \frac{1}{2}\int x^2 d\ln x\\ &= \frac{x^2}{2}\ln x - \frac{1}{2}\int x^2\cdot\frac{1}{x}dx\\ &= \frac{x^2}{2}\ln x - \frac{1}{2}\int x dx\\ &= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C\end{aligned}$$

(c) $$\begin{aligned}\int x\cos 4x dx &= \frac{1}{4}\int x d\sin 4x\\ &= \frac{x\sin 4x}{4} - \frac{1}{4}\int \sin 4x dx\\ &= \frac{x\sin 4x}{4} + \frac{1}{16}\cos 4x + C\end{aligned}$$

(d) $$\begin{aligned}\int_0^{1/2} \cos^{-1} x dx &= x\cos^{-1} x\Big|_0^{1/2} - \int_0^{1/2} x d\cos^{-1} x\\ &= \frac{\pi}{6} - \int_0^{1/2} x\left(-\frac{1}{\sqrt{1-x^2}}\right)dx\\ &= \frac{\pi}{6} - \int_0^{1/2} \frac{\left(1-x^2\right)^{-\frac{1}{2}}}{2} d(1-x^2)\\ &= \frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{1-x^2}}{\frac{1}{2}}\Bigg|_0^{1/2}\\ &= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1\end{aligned}$$

Example

Find $\int xe^x dx$.

Solution

Bad Method:

$$\begin{aligned}\int xe^x dx &= \int e^x d\left(\frac{x^2}{2}\right)\\ &= e^x\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2} de^x\\ &= e^x\left(\frac{x^2}{2}\right) - \frac{1}{2}\int x^2 e^x dx.\end{aligned}$$

Try to decrease the power of x but not increase the power of x.

Good Method:

$$\begin{aligned}\int xe^x dx &= \int xde^x\\ &= xe^x - \int e^x dx\\ &= xe^x - e^x + C.\end{aligned}$$

Example

Find $\int e^x \cos x dx$.

Solution

Bad Method

$$\begin{aligned}\int e^x \cos x dx &= \int e^x d\sin x\\ &= e^x \sin x - \int \sin x de^x\\ &= e^x \sin x - \int e^x \sin x dx\\ &= e^x \sin x - \int \sin x de^x\end{aligned}$$

$$= e^x \sin x - \left(e^x \sin x - \int e^x d \sin x \right)$$
$$= \int e^x d \sin x$$
$$= \int e^x \cos x dx \qquad \text{Do nothing.}$$

Method 1:

$$\int e^x \cos x dx = \int e^x d \sin x$$
$$= e^x \sin x - \int \sin x de^x$$
$$= e^x \sin x - \int e^x \sin x dx$$
$$= e^x \sin x + \int e^x d \cos x$$
$$= e^x \sin x + \left(e^x \cos x - \int \cos x de^x \right)$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x dx$$
$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x$$
$$\int e^x \cos x dx = \frac{e^x \sin x + e^x \cos x}{2} + C$$

Method 2:

$$\int e^x \cos x dx = \int \cos x de^x$$
$$= e^x \cos x - \int e^x d \cos x$$
$$= e^x \cos x + \int e^x \sin x dx$$
$$= e^x \cos x + \int \sin x de^x$$
$$= e^x \cos x + e^x \sin x - \int e^x d \sin x$$
$$= e^x \cos x + e^x \sin x - \int e^x \cos x dx$$
$$2\int e^x \cos x dx = e^x \cos x + e^x \sin x$$
$$\int e^x \cos x dx = \frac{e^x \cos x + e^x \sin x}{2} + C$$

Trigonometric Integrals

Reduction Formulas

Assume n is a positive integer.

1. $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

2. $\displaystyle\int \cos^n x\,dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

3. $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx,\ n \neq 1$

4. $\displaystyle\int \sec^n x\,dx = \frac{\sec^{n-2} x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx,\ n \neq 1$

Proof:

$$
\begin{aligned}
\int \sin^n xdx &= \int \sin^{n-1} x\sin xdx = \int -\sin^{n-1} xd\cos x\\
&= -\sin^{n-1} x\cos x - \int -\cos xd\sin^{n-1} x\\
&= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 xdx\\
&= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\left(1-\sin^2 x\right)dx\\
&= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} xdx - (n-1)\int \sin^n xdx
\end{aligned}
$$

$$n\int \sin^n xdx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} xdx$$

$$\int \sin^n xdx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} xdx.$$

Substituting $u = \dfrac{\pi}{2} - x$ into the reduction formula of $\int \sin^n x\,dx$ will gives us the reduction formula of $\int \cos^n x\,dx$ (D.I.Y.)

$$
\begin{aligned}
\int \tan^n x\,dx &= \int \tan^{n-2} x\tan^2 x\,dx\\
&= \int \tan^{n-2} x\left(\sec^2 x - 1\right)dx\\
&= \int \tan^{n-2} x\sec^2 xdx - \int \tan^{n-2} xdx\\
&= \int \tan^{n-2} xd\tan x - \int \tan^{n-2} xdx\\
&= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} xdx.
\end{aligned}
$$

$$\begin{aligned}
\int \sec^n x\,dx &= \int \sec^2 x\,\sec^{n-2} x\,dx \\
&= \tan x\,\sec^{n-2} x - \int (\tan x)\,.\,(n-2)\sec^{n-3} x\,(\sec x\,\tan x)\,dx \\
&= \tan x\,\sec^{n-2} x - (n-2)\int \sec^{n-2} x\,(\sec^2 x - 1)\,dx \\
&= \tan x\,\sec^{n-2} x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx\,. \\
\therefore (n-1)\int \sec^n x\,dx &= \tan x\,\sec^{n-2} x + (n-2)\int \sec^{n-2} x\,dx \\
\therefore \int \sec^n x\,dx &= \frac{1}{n-1}\tan x\,\sec^{n-2} x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx\,.
\end{aligned}$$

Strategies for solving $\int \sin^m x \cos^n x dx$

Case 1 $\quad m = 2k+1$ odd, n real

$$\begin{aligned}
\int \sin^{2k+1} x \cos^n x dx &= \int \sin^{2k} x \cos^n x \sin x dx && (\text{split off } \sin x) \\
&= \int \left(1-\cos^2 x\right)^k \cos^n x \sin x dx && \begin{pmatrix}\text{rewrite the resulting even power} \\ \text{of } \sin x \text{ in terms of } \cos x\end{pmatrix} \\
&= -\int \left(1-\cos^2 x\right)^k \cos^n x d\cos x && (\text{substitute } u = \cos x)
\end{aligned}$$

Case 2 $\quad m$ real, $n = 2l+1$ odd

$$\begin{aligned}
\int \sin^m x \cos^{2l+1} x dx &= \int \sin^m x \cos^{2l} x \cos x dx && (\text{split off } \cos x) \\
&= \int \sin^m x \left(1-\sin^2 x\right)^l \cos x dx && \begin{pmatrix}\text{rewrite the resulting even power} \\ \text{of } \cos x \text{ in terms of } \sin x\end{pmatrix} \\
&= \int \sin^m x \left(1-\sin^2 x\right)^l d\sin x && (\text{substitute } u = \sin x)
\end{aligned}$$

Case 3 $\quad$ both $m = 2k$ and $n = 2l$ nonnegative even integers

Use $\sin^2 x = \dfrac{1-\cos 2x}{2}$ and $\cos^2 x = \dfrac{1+\cos 2x}{2}$ to transform $\sin^{2k} x \cos^{2l} x$ into a polynomial in $\cos 2x$; i.e.,

$$\int \sin^{2k} x \cos^{2l} x dx = \frac{1}{2^{k+l}} \int (1-\cos 2x)^k (1+\cos 2x)^l dx$$

and apply the preceding strategies once again to powers of $\cos 2x$ greater than 1.

Example

Evaluate $\int \sin^3 x dx.$

Solution

$$\begin{aligned}\int \sin^3 x dx &= \int \sin^2 x \cdot \sin x dx \\ &= \int (1-\cos^2 x)\cdot \sin x dx \\ &= -\int \left(1-u^2\right) du \qquad u=\cos x;\ du=-\sin x dx \\ &= -u+\frac{u^3}{3}+C \\ &= -\cos x+\frac{\cos^3 x}{3}+C.\end{aligned}$$

By Reduction Formula,

$$\begin{aligned}\int \sin^3 x dx &= -\frac{\sin^2 x\cos x}{3}+\frac{2}{3}\int \sin x dx \\ &= -\frac{\sin^2 x\cos x}{3}-\frac{2}{3}\cos x+C \\ &= -\frac{\left(1-\cos^2 x\right)\cos x}{3}-\frac{2}{3}\cos x+C \\ &= -\frac{\cos x}{3}+\frac{\cos^3 x}{3}-\frac{2}{3}\cos x+C \\ &= -\cos x+\frac{\cos^3 x}{3}+C.\end{aligned}$$

Example

Evaluate $\int \sin^2 x\cos^3 x dx.$

Solution

$$\begin{aligned}\int \sin^2 x\cos^3 x dx &= \int \sin^2 x\cos^2 x\cdot \cos x dx \\ &= \int \sin^2 x\left(1-\sin^2 x\right)\cdot \cos x dx \\ &= \int \sin^2 x\left(1-\sin^2 x\right) d\sin x \\ &= \int \sin^2 x-\sin^4 x d\sin x \\ &= \frac{\sin^3 x}{3}-\frac{\sin^5 x}{5}+C\end{aligned}$$

Example

Evaluate $\int \sin^2 x \cos^2 x dx.$

Solution

$$\begin{aligned}
\int \sin^2 x \cos^2 x dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx \\
&= \frac{1}{4}\int 1-\cos^2 2x dx \\
&= \frac{1}{4}\int 1-\frac{1+\cos 4x}{2} dx \\
&= \frac{1}{8}\int 1-\cos 4x dx \\
&= \frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)+C
\end{aligned}$$

THEOREM 7.1 Integrals of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

$$\int \tan x\, dx = -\ln|\cos x| + C = \ln|\sec x| + C \qquad \int \cot x\, dx = \ln|\sin x| + C$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C \qquad \int \csc x\, dx = -\ln|\csc x + \cot x| + C$$

Solution

$$\begin{aligned}
\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\
&= -\int \frac{1}{\cos x} d\cos x \\
&= -\ln|\cos x| + C \\
&= \ln|\sec x| + C
\end{aligned}$$

$$\begin{aligned}
\int \sec x dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\
&= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} \\
&= \ln|\sec x + \tan x| + C
\end{aligned}$$

Substituting $u = \frac{\pi}{2} - x$ gives us the strategy for the integral $\int \cot x\, dx$ and $\int \csc x\, dx$. (D.I.Y.)

Strategies for solving $\int \tan^m x \sec^n x dx$

Case 1 $\quad n = 2l$ even

$$\int \tan^m x \sec^{2l} x dx = \int \tan^m x \sec^{2l-2} x \sec^2 x dx \qquad \text{(split off } \sec^2 x\text{)}$$

$$= \int \tan^m x \left(\tan^2 x + 1\right)^{l-1} \sec^2 x dx \qquad \left(\begin{array}{c}\text{rewrite the resulting even power} \\ \text{of } \sec x \text{ in terms of } \tan x\end{array}\right)$$

$$= \int \tan^m x \left(\tan^2 x + 1\right)^{l-1} d \tan x \qquad \text{(substitute } u = \tan x\text{)}$$

Case 2 $\quad m = 2k + 1$ odd

$$\int \tan^{2k+1} x \sec^n x dx = \int \tan^{2k} x \sec^{n-1} x \sec x \tan x dx \qquad \text{(split off } \sec x \tan x\text{)}$$

$$= \int \left(\sec^2 x - 1\right)^k \sec^{n-1} x \sec x \tan x dx \qquad \left(\begin{array}{c}\text{rewrite the resulting even power} \\ \text{of } \tan x \text{ in terms of } \sec x\end{array}\right)$$

$$= \int \left(\sec^2 x - 1\right)^k \sec^{n-1} x d \sec x \qquad \text{(substitute } u = \sec x\text{)}$$

Case 3 $\quad m = 2k$ even and $n = 2l + 1$ odd

Rewrite the even power of $\tan x$ in terms of $\sec x$ to produce a polynomial in $\sec x$; i.e.,

$$\int \tan^{2k} x \sec^{2l+1} x dx = \int \left(\sec^2 x - 1\right)^k \sec^{2l+1} x dx$$

and apply reduction formula 4 to each term.

Remark: Substituting $u = \dfrac{\pi}{2} - x$ gives us the strategy for the integral $\int \cot^m x \csc^n x\, dx$. (D.I.Y.)

Example

Evaluate $\int \tan^2 x \sec x dx$.

Solution

$$\begin{aligned}\int \tan^2 x \sec x dx &= \int (\sec^2 x - 1) \sec x dx \\ &= \int \sec^3 x dx - \int \sec x dx \\ &= \frac{\sec x \tan x}{2} + \frac{1}{2}\int \sec x dx - \int \sec x dx \\ &= \frac{\sec x \tan x}{2} - \frac{1}{2} \ln\left|\sec x + \tan x\right| + C\end{aligned}$$

Example

Evaluate $\int \tan x \sec^2 x \, dx.$

Solution

$$\text{Method 1: } \int \tan x \sec^2 x \, dx = \int u du \qquad u = \tan x;\ du = \sec^2 x \, dx$$

$$= \frac{u^2}{2} + C_1$$

$$= \frac{\tan^2 x}{2} + C_1$$

$$\text{Method 2: } \int \tan x \sec^2 x \, dx = \int v dv \qquad v = \sec x;\ dv = \sec x \tan x \, dx$$

$$= \frac{v^2}{2} + C_2$$

$$= \frac{\sec^2 x}{2} + C_2 = \frac{\tan^2 x}{2} + \frac{1}{2} + C_2$$

To evaluate the integrals $\int \sin mx \cos nx \, dx$, $\int \sin mx \sin nx \, dx$, or $\int \cos mx \cos nx \, dx$, use the corresponding identities

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

Example

Evaluate $\int \sin 4x \cos 5x \, dx.$

Solution

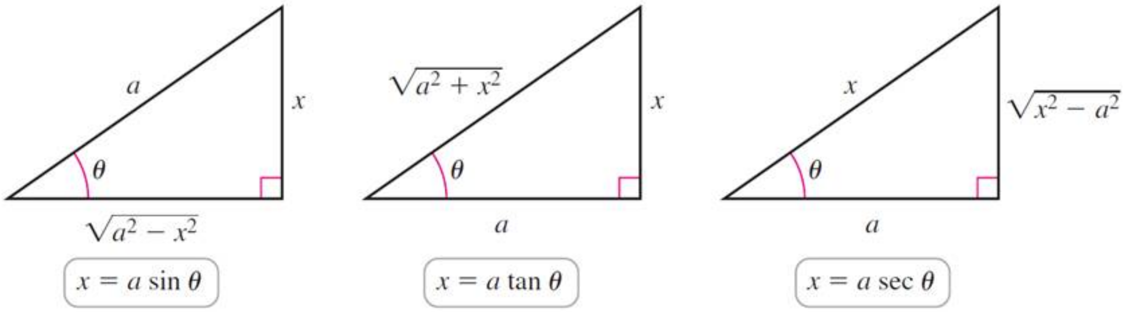
$$\int \sin 4x \cos 5x \, dx = \int \frac{1}{2}\left[\sin(-x) + \sin 9x\right] dx$$

$$= \frac{1}{2}\int \left(-\sin x + \sin 9x\right) dx$$

$$= \frac{1}{2}\left(\cos x - \frac{1}{9}\cos 9x\right) + C.$$

Trigonometric Substitutions

The Integral Contains . . .	Corresponding Substitution	Useful Identity
$a^2 - x^2$	$x = a\sin\theta, -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$, for $\lvert x\rvert \le a$	$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta, -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$	$a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta, \begin{cases} 0 \le \theta < \dfrac{\pi}{2}, \text{ for } x \ge a \\ \dfrac{\pi}{2} < \theta \le \pi, \text{ for } x \le -a \end{cases}$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$



Example

Verify that the area of a circle of radius a is πa^2.

Solution

One-fourth of the area of a circle of radius a is

$$\int_0^a \sqrt{a^2 - x^2}\,dx = \int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta}\, a\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} a^2\cos^2\theta\, d\theta$$

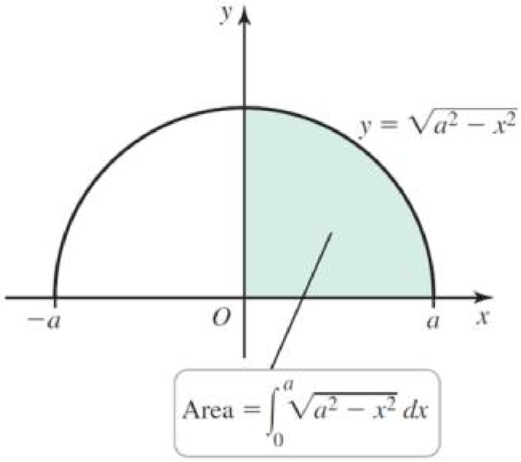
$$= a^2\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_0^{\pi/2}$$

$$= \frac{\pi a^2}{4}.$$

$x = a\sin\theta,\ dx = a\cos\theta\, d\theta,$

$x = 0,\ \theta = 0,\ x = a,\ \theta = \pi/2.$



The area of a circle of radius a is πa^2.

Example

Evaluate

(a) $\displaystyle\int \frac{dx}{(16-x^2)^{3/2}}$

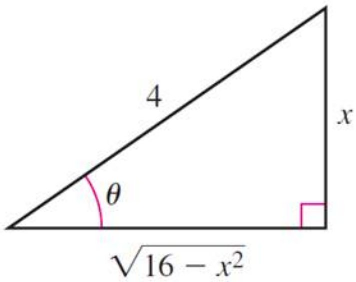
(b) $\displaystyle\int \frac{dx}{(1+x^2)^2}.$

(c) $\displaystyle\int \frac{1}{\sqrt{x^2-6x}}\,dx.$

Solution

(a) Let $x = 4\sin\theta, \quad dx = 4\cos\theta d\theta.$

$$\begin{aligned}\int \frac{dx}{(16-x^2)^{3/2}} &= \int \frac{4\cos\theta}{(16-16\sin^2\theta)^{3/2}}d\theta \\ &= \int \frac{4\cos\theta}{(16\cos^2\theta)^{3/2}}d\theta \\ &= \int \frac{4\cos\theta}{64\cos^3\theta}d\theta \\ &= \frac{1}{16}\int \frac{1}{\cos^2\theta}d\theta \\ &= \frac{1}{16}\int \sec^2\theta d\theta \\ &= \frac{\tan\theta}{16}+C \\ &= \frac{x}{16\sqrt{16-x^2}}+C\end{aligned}$$



$\sin\theta = \dfrac{x}{4}$

$\tan\theta = \dfrac{x}{\sqrt{16-x^2}}$

(b) Let $x = \tan\theta, \quad dx = \sec^2\theta d\theta.$

$$\begin{aligned}\int \frac{dx}{(1+x^2)^2} &= \int \frac{\sec^2\theta}{(1+\tan^2\theta)^2}d\theta \\ &= \int \frac{\sec^2\theta}{\sec^4\theta}d\theta \\ &= \int \frac{1}{\sec^2\theta}d\theta\end{aligned}$$

$$=\int \cos^2\theta d\theta$$

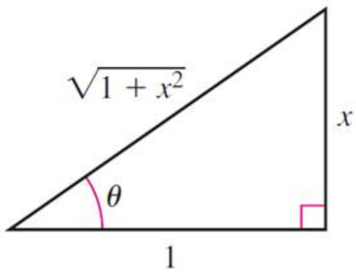
$$=\int \frac{1+\cos 2\theta}{2} d\theta$$

$$=\frac{1}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]+C$$

$$=\frac{1}{2}\tan^{-1}x+\frac{1}{4}2\sin\theta\cos\theta+C$$

$$=\frac{1}{2}\tan^{-1}x+\frac{1}{2}\cdot\frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}}+C$$

$$=\frac{1}{2}\tan^{-1}x+\frac{x}{2(1+x^2)}+C$$



$$\sin\theta=\frac{x}{\sqrt{1+x^2}}$$

$$\cos\theta=\frac{1}{\sqrt{1+x^2}}$$

(c) Clearly, $\sqrt{x^2-6x}=\sqrt{(x-3)^2-3^2}$.

Let $x-3=3\sec\theta$, $d\theta=3\sec\theta\tan\theta d\theta$

$$\int\frac{1}{\sqrt{x^2-6x}}dx=\int\frac{dx}{\sqrt{(x-3)^2-3^2}}$$

$$=\int\frac{3\sec\theta\tan\theta}{\sqrt{9\sec^2\theta-9}}d\theta$$

$$=\int\frac{3\sec\theta\tan\theta}{\sqrt{9\tan^2\theta}}d\theta$$

$$=\int\frac{3\sec\theta\tan\theta}{3\tan\theta}d\theta$$

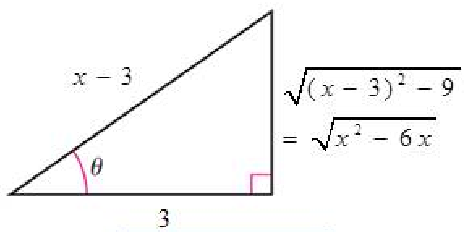
$$=\int\sec\theta d\theta$$

$$=\ln|\sec\theta+\tan\theta|+C$$

$$=\ln\left|\frac{\sqrt{x^2-6x}}{3}+\frac{x-3}{3}\right|+C$$

$$=\ln\left|\sqrt{x^2-6x}+x-3\right|-\ln 3+C$$

$$=\ln\left|\sqrt{x^2-6x}+x-3\right|+C_1$$



$$\tan\theta=\frac{\sqrt{x^2-6x}}{3}$$

$$\sec\theta=\frac{x-3}{3}$$

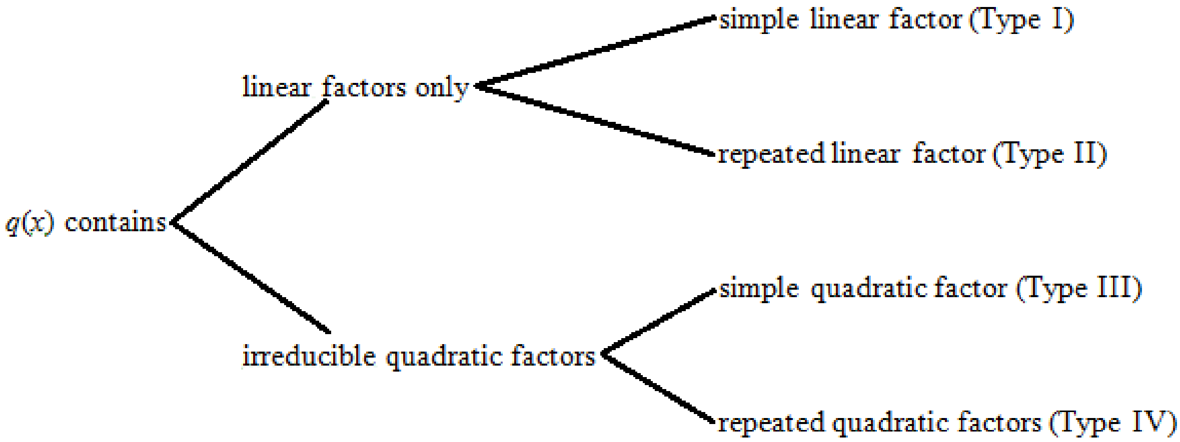
Partial Fractions

Rational function	*method of partial fractions*	**Partial fraction decomposition**
$\dfrac{3x}{x^2+2x-8}$	$\longrightarrow$	$\dfrac{1}{x-2}+\dfrac{2}{x+4}$
Difficult to integrate		**Easy to integrate**
$\displaystyle\int \frac{3x}{x^2+2x-8}\,dx$		$\displaystyle\int \left(\frac{1}{x-2}+\frac{2}{x+4}\right)dx$

Suppose $f(x)=p(x)/q(x)$, where p and q are polynomials with no common factors. If $\deg p \ge \deg q$, by long division, $p(x)$ can be written as $p(x)=q(x)d(x)+r(x)$ where d and r are polynomials with $\deg r < \deg q$. Hence

$$f(x)=\frac{p(x)}{q(x)}=d(x)+\frac{r(x)}{q(x)}.$$

The proper rational function $\dfrac{r(x)}{q(x)}$ (if $\deg p \ge \deg q$) or $\dfrac{p(x)}{q(x)}$ (if $\deg p < \deg q$) is categorized into the following partial fractions.



Type I

PROCEDURE Partial Fractions with Simple Linear Factors

Suppose $f(x)=p(x)/q(x)$, where p and q are polynomials with no common factors and with the degree of p less than the degree of q. Assume that q is the product of simple linear factors. The partial fraction decomposition is obtained as follows.

Step 1. **Factor the denominator q** in the form $(x - r_1)(x - r_2)\cdots(x - r_n)$, where $r_1, \ldots, r_n$ are real numbers.

Step 2. **Partial fraction decomposition** Form the partial fraction decomposition by writing

$$\frac{p(x)}{q(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

Step 3. **Clear denominators** Multiply both sides of the equation in Step 2 by $q(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$, which produces conditions for $A_1, \ldots, A_n$.

Step 4. **Solve for coefficients** Solve for the undetermined coefficients $A_1, \ldots, A_n$ in Step 3 by substitution or comparing coefficients.

Example

Evaluate $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}dx.$

Solution

Since the degree of the numerator is greater than the degree of the denominator, we first perform the long division.

$$\begin{array}{r} 2x \\ x^2 - 2x - 3 \,\overline{\smash{)}\, 2x^3 - 4x^2 - x - 3} \\ \underline{2x^3 - 4x^2 - 6x } \\ 5x - 3 \end{array}$$

We have $\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = \frac{2x(x^2 - 2x - 3) + (5x - 3)}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}.$

Since $x^2 - 2x - 3 = (x+1)(x-3),$ therefore we find constants A and B such that

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x+1} + \frac{B}{x-3} = \frac{A(x-3) + B(x+1)}{(x+1)(x-3)}.$$

Therefore $A(x-3) + B(x+1) = 5x - 3.$ (1)

If we put $x=3$ in (1), we get $4B=12.\quad B=3.$

If we put $x=-1$ in (1), we get $-4A=-8.\quad A=2.$

So, we have $\dfrac{2x^3-4x^2-x-3}{x^2-2x-3}=2x+\dfrac{5x-3}{x^2-2x-3}=2x+\dfrac{2}{x+1}+\dfrac{3}{x-3}.$

$$\begin{aligned}\int\frac{2x^3-4x^2-x-3}{x^2-2x-3}dx&=\int 2x+\frac{2}{x+1}+\frac{3}{x-3}dx\\&=\int 2xdx+\int\frac{2}{x+1}dx+\int\frac{3}{x-3}dx\\&=x^2+2\ln|x+1|+3\ln|x-3|+C\end{aligned}$$

Example

Evaluate $\displaystyle\int\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx.$

Solution

Clearly, this is a Type I integral. Therefore we find constants A, B and C such that

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x+3}=\frac{A(x+1)(x+3)+B(x-1)(x+3)+C(x-1)(x+1)}{(x-1)(x+1)(x+3)}.$$

Therefore $x^2+4x+1=A(x+1)(x+3)+B(x-1)(x+3)+C(x-1)(x+1).$ (2)

If we put $x=1$ in (2), we get $1+4+1=8A,\quad A=\dfrac{3}{4}.$

If we put $x=-1$ in (2), we get $1-4+1=-4B,\quad B=\dfrac{1}{2}.$

If we put $x=-3$ in (2), we get $9-12+1=8C,\quad C=-\dfrac{1}{4}.$

So, we have $\dfrac{x^2+4x+1}{(x-1)(x+1)(x+3)}=\dfrac{3}{4}\cdot\dfrac{1}{x-1}+\dfrac{1}{2}\cdot\dfrac{1}{x+1}-\dfrac{1}{4}\cdot\dfrac{1}{x+3}$

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx = \frac{3}{4}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x+3}$$
$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + C$$

Type II

PROCEDURE Partial Fractions for Repeated Linear Factors

We can write

$$q(x) = (x-r_1)^{m_1} \cdots (x-r_k)^{m_k}$$

where some $m_i > 1$. For example, $(x+1)^2, x^3(x-2)(x+4)^2, \cdots$ etc. Suppose the repeated linear factor $(x-r)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $(x-r)$ up to and including the m-th power; that is, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m},$$

where $A_1, \cdots, A_m$ are constants to be determined by substitution or comparing coefficients.

Example

Evaluate $\int \frac{6x+7}{(x+2)^2}dx.$

Solution

Clearly, this is a Type II integral. Therefore we find constants A and B such that

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2}.$$

Therefore $6x+7 = A(x+2)+B.$ (3)

From (3), we have $6x+7 = Ax + (2A+B).$ Hence

$$\begin{cases} A = 6 \\ 2A + B = 7 \end{cases} \Rightarrow \quad A = 6 \text{ and } B = -5.$$

So, we have $\dfrac{6x+7}{(x+2)^2} = \dfrac{6}{x+2} - \dfrac{5}{(x+2)^2}$.

$$\begin{aligned} \int \frac{6x+7}{(x+2)^2}dx &= 6\int \frac{dx}{x+2} - 5\int \frac{dx}{(x+2)^2} \\ &= 6\ln|x+2| + \frac{5}{x+2} + C \end{aligned}$$

Example

Evaluate $\displaystyle\int \frac{2x+4}{x^3 - 2x^2}dx$.

Solution

Since $x^3 - 2x^2 = x^2(x-2)$, this is a Type II integral. Therefore we find constants A, B and C such that

$$\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{Ax(x-2) + B(x-2) + Cx^2}{x^2(x-2)}.$$

Therefore $2x + 4 = Ax(x-2) + B(x-2) + Cx^2$. (4)

From (4), we have $2x + 4 = (A+C)x^2 + (-2A+B)x - 2B$. Hence

$$\begin{cases} A + C = 0 \\ -2A + B = 2 \\ -2B = 4 \end{cases} \Rightarrow \quad A = B = -2 \text{ and } C = 2.$$

So, we have $\dfrac{2x+4}{x^2(x-2)} = \dfrac{-2}{x} + \dfrac{-2}{x^2} + \dfrac{2}{x-2}$.

$$\begin{aligned} \int \frac{2x+4}{x^2(x-2)}dx &= -2\int \frac{dx}{x} - 2\int \frac{dx}{x^2} + 2\int \frac{dx}{x-2} \\ &= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C \end{aligned}$$

Type III

PROCEDURE **Partial Fractions with Simple Irreducible Quadratic Factors**

For example, $q(x)$ can be written as

$$x^2+1, x^2\left(x^2+4\right)\left(x^2+x+1\right), \cdots \text{etc.}$$

Suppose a simple irreducible factor ax^2+bx+c appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition contains a term of the form

$$\frac{Ax+B}{ax^2+bx+c},$$

where A and B are constants to be determined by substitution or comparing coefficients.

Example

Evaluate $\int \frac{2x^2+2x+1}{x^3+x^2+x}dx.$

Solution

Since $x^3+x^2+x=x(x^2+x+1)$, this is a Type III integral. Therefore we find constants A, B and C such that

$$\frac{2x^2+2x+1}{x^3+x^2+x}=\frac{A}{x}+\frac{Bx+C}{x^2+x+1}=\frac{A(x^2+x+1)+(Bx+C)x}{x(x^2+x+1)}.$$

Therefore $2x^2+2x+1=A(x^2+x+1)+(Bx+C)x.$ (5)

From (5), we have $2x^2+2x+1=(A+B)x^2+(A+C)x+A$. Hence

$$\begin{cases} A+B=2 \\ A+C=2 \\ A=1 \end{cases} \Rightarrow \quad A=B=C=1.$$

So, we have $\frac{2x^2+2x+1}{x^3+x^2+x}=\frac{1}{x}+\frac{x+1}{x^2+x+1}.$

$$
\begin{aligned}
\int \frac{2x^2+2x+1}{x^3+x^2+x}dx &= \int \frac{dx}{x} + \int \frac{x+1}{x^2+x+1}dx \\
&= \ln|x| + \int \frac{x+1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx \\
&= \ln|x| + \int \frac{x+\frac{1}{2}+\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx \\
&= \ln|x| + \int \frac{x+\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx + \int \frac{\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx \\
&= \ln|x| + \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}d\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right) + \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx \\
&= \ln|x| + \frac{1}{2}\ln\left|\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right| + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + C \\
&= \ln|x| + \frac{1}{2}\ln\left|x^2+x+1\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C
\end{aligned}
$$

Type IV

PROCEDURE Partial Fractions for Repeated Irreducible Quadratic Factors

For example, $q(x)$ can be written as

$$\left(x^2+1\right)^2,\ x^2(x-1)\left(x^2+4\right)^3\left(x^2+x+4\right)^4,\cdots \text{etc.}$$

Suppose the repeated irreducible factor $\left(ax^2+bx+c\right)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $\left(ax^2+bx+c\right)$ up to and including the m-th power; that is, the partial fraction decomposition contain the sum

$$\frac{A_1x+B_1}{\left(ax^2+bx+c\right)} + \frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2} + \frac{A_3x+B_3}{\left(ax^2+bx+c\right)^3} + \cdots + \frac{A_mx+B_m}{\left(ax^2+bx+c\right)^m},$$

where $A_1, B_1, \cdots, A_m, B_m$ are constants to be determined by substitution or comparing coefficients.

Example

Evaluate $\int \frac{dx}{x^5 + 2x^3 + x}$.

Solution

Since $x^5 + 2x^3 + x = x(x^2 + 1)^2$, this is a Type IV integral. Therefore we find constants $A, B, C, D,$ and E such that

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} = \frac{A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x}{x(x^2+1)^2}.$$

Therefore $1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x.$ (6)

From (6), we have $1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A.$ Hence

$$\begin{cases} A+B=0 \\ C=0 \\ 2A+B+D=0 \\ C+E=0 \\ A=1 \end{cases} \Rightarrow \quad A=1, B=-1, C=0, D=-1 \text{ and } E=0.$$

So, we have $\frac{1}{x^5+2x^3+x} = \frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{\left(x^2+1\right)^2}.$

$$\begin{aligned} \int \frac{dx}{x^5+2x^3+x} &= \int \frac{dx}{x} - \int \frac{x}{x^2+1}dx - \int \frac{x}{\left(x^2+1\right)^2}dx \\ &= \ln|x| - \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} - \frac{1}{2}\int \frac{d(x^2+1)}{\left(x^2+1\right)^2} \\ &= \ln|x| - \frac{1}{2}\ln\left|x^2+1\right| + \frac{1}{2(x^2+1)} + C \end{aligned}$$

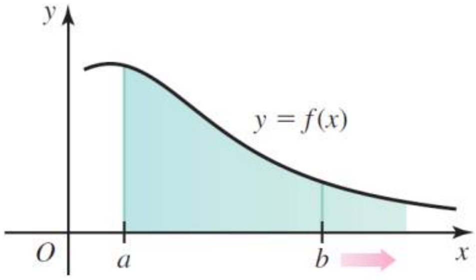
Improper Integrals

DEFINITIONS Improper Integrals over Infinite Intervals

1. If f is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx,$$

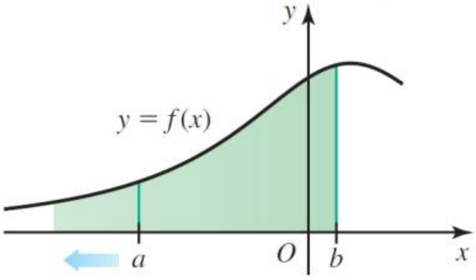
provided the limit exists.



2. If f is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx,$$

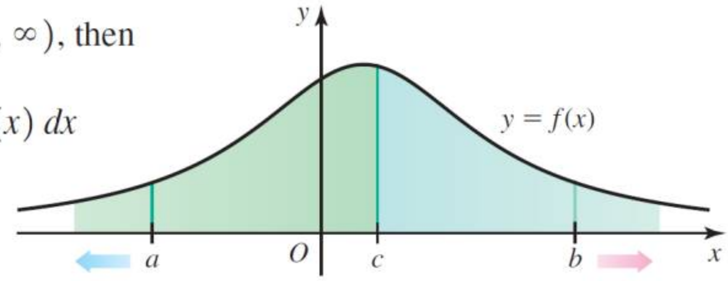
provided the limit exists.



3. If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx,$$

provided both limits exist, where c is any real number.



In each case, if the limit exists, the improper integral is said to **converge**; if it does not exist, the improper integral is said to **diverge**.

Remark: $\int_{-\infty}^{\infty} f(x)dx \neq \lim_{a\to\infty}\int_{-a}^{a} f(x)dx$ even $\lim_{a\to\infty}\int_{-a}^{a} f(x)dx$ exists.

Consider $f(x)=x$. Since x is an odd function, $\int_{-a}^{a} f(x)dx = 0$. Hence $\lim_{a\to\infty}\int_{-a}^{a} f(x)dx = 0$. But $\int_0^b f(x)dx = \frac{b^2}{2} \to \infty$ as $b\to\infty$, that means $\int_{-\infty}^{\infty} f(x)dx$ diverges.

Example

Evaluate the following integrals

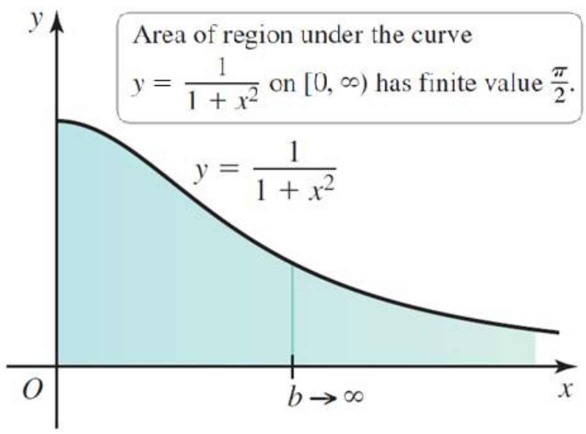
a) $\int_0^{\infty} xe^{-x}dx$

b) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Solution

a) $$\begin{aligned}\int_0^{\infty} xe^{-x}dx &= \lim_{b\to\infty}\int_0^b xe^{-x}dx \\ &= \lim_{b\to\infty}\int_0^b -xde^{-x} \\ &= \lim_{b\to\infty}\left(\left(-xe^{-x}\right)\Big|_0^b + \int_0^b e^{-x}dx\right) \\ &= \lim_{b\to\infty}\left(-\frac{b}{e^b} + \left(-e^{-x}\right)\Big|_0^b\right) \\ &= \lim_{b\to\infty}\left(-\frac{b}{e^b} + 1 - e^{-b}\right) \qquad \because \lim_{b\to\infty}\frac{b}{e^b} = \lim_{b\to\infty}\frac{1}{e^b} = 0 \quad \text{by L'Hospital's Rule} \\ &= 1.\end{aligned}$$

b) $$\begin{aligned}\int_0^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} \\ &= \lim_{b\to\infty}\left(\tan^{-1}x\right)\Big|_0^b \\ &= \lim_{b\to\infty}\left(\tan^{-1}b - \tan^{-1}0\right) \\ &= \frac{\pi}{2} - 0 = \frac{\pi}{2}.\end{aligned}$$



Since $\frac{1}{1+x^2}$ is an even function,

$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \int_0^{\infty} \frac{dx}{1+x^2}$. Hence $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\times\frac{\pi}{2} = \pi$.

Example (p-test)

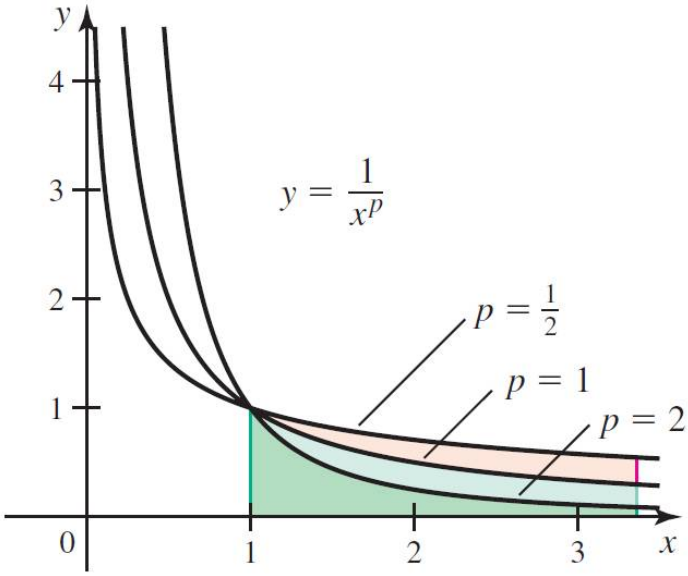
Consider the family of functions $f(x) = \frac{1}{x^p}$, where p is real number. For what values of p does

$\int_1^{\infty} f(x)dx$ converge?

Solution

Assuming $p \neq 1$.

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}dx$$
$$= \frac{1}{1-p}\lim_{b\to\infty}\left(x^{1-p}\right)\Big|_1^b$$
$$= \frac{1}{1-p}\lim_{b\to\infty}\left(b^{1-p}-1\right).$$



$$\boxed{\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1}, \text{ if } p > 1.}$$

Case 1: $p > 1$.

$$\int_1^\infty \frac{1}{x^p}dx = \frac{1}{1-p}\lim_{b\to\infty}\left(b^{1-p}-1\right) = \frac{1}{p-1}.$$

Case 2: $p < 1$.

$$\int_1^\infty \frac{1}{x^p}dx = \frac{1}{1-p}\lim_{b\to\infty}\left(b^{1-p}-1\right) = \infty.$$

Case 3: $p = 1$.

$$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}dx = \lim_{b\to\infty}(\ln x)\Big|_1^b = \lim_{b\to\infty}(\ln b) = \infty.$$

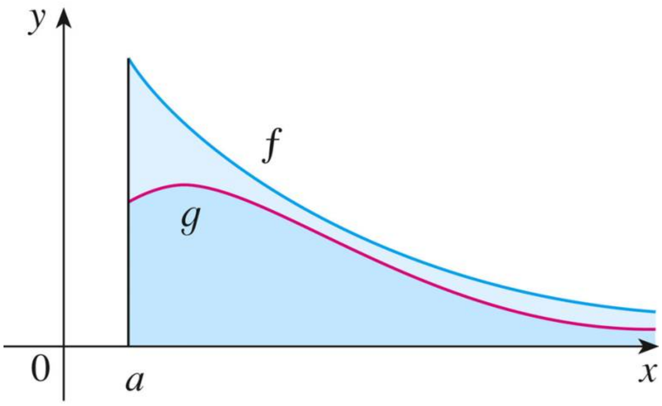
In conclusion, $\int_1^\infty \frac{1}{x^p}dx = \frac{1}{p-1}$ if $p > 1$, and it diverges if $p \leq 1$.

Comparison test

Assume f and g are continuous functions with $0 \leq g(x) \leq f(x)$ for $x \geq a$. Then $0 \leq \int_a^\infty g(x)dx \leq \int_a^\infty f(x)dx$. Furthermore

a) $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges.

b) $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges.



Example

Determine whether the following integral converges or not.

a) $\lim_{x\to\infty}\int_1^x \frac{dt}{\sqrt{1+t^3}}$

b) $\lim_{x\to\infty}\int_2^x \frac{dt}{\ln t}$

c) $\lim_{x\to\infty}\int_1^x \frac{\sin t}{t}dt$

Solution

a) Since $\sqrt{t^3} < \sqrt{1+t^3}$ for $t>1$, $\frac{1}{\sqrt{1+t^3}} < \frac{1}{t^{3/2}}$. By comparison test,

$$\int_1^x \frac{1}{\sqrt{1+t^3}}\,dt < \int_1^x \frac{1}{t^{3/2}}\,dt.$$

By p-test, $\int_1^\infty \frac{1}{t^{3/2}}dt$ converges. Hence $\int_1^\infty \frac{1}{\sqrt{1+t^3}}\,dt$ converges.

b) Since $\ln t \le t$ for $t\ge 1$, $\frac{1}{t} \le \frac{1}{\ln t}$. By comparison test,

$$\int_2^x \frac{1}{t}\,dt < \int_2^x \frac{1}{\ln t}\,dt.$$

By p-test, $\int_2^\infty \frac{1}{t}dt$ diverges. Hence $\int_2^\infty \frac{1}{\ln t}\,dt$ diverges.

c) $$\begin{aligned}\lim_{x\to\infty}\int_1^x \frac{\sin t}{t}dt &= \lim_{x\to\infty}\int_1^x \frac{-d\cos t}{t} \\ &= \lim_{x\to\infty}\left.\frac{-\cos t}{t}\right|_1^x + \lim_{x\to\infty}\int_1^x \cos t\; dt^{-1} \\ &= \lim_{x\to\infty}\frac{-\cos x}{x} + \cos 1 - \lim_{x\to\infty}\int_1^x \frac{\cos t}{t^2}\,dt \\ &= \cos 1 - \lim_{x\to\infty}\int_1^x \frac{\cos t}{t^2}\,dt\end{aligned}$$

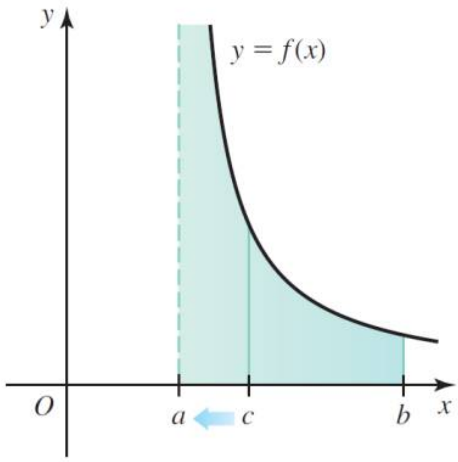
Since $0 \le \frac{1-\cos t}{t^2} \le \frac{2}{t^2}$ for $1\le t$, we have $\int_1^x \frac{1-\cos t}{t^2}\,dt \le \int_1^x \frac{2}{t^2}\,dt$ which is convergent. Hence $\int_1^\infty \frac{1-\cos t}{t^2}\,dt$ converges. Since $\int_1^x \frac{\cos t}{t^2}\,dt = \int_1^x \frac{1}{t^2}\,dt - \int_1^x \frac{1-\cos t}{t^2}\,dt$, $\int_1^\infty \frac{\cos t}{t^2}\,dt$ converges.
Hence $\int_1^\infty \frac{\sin t}{t}\,dt$ converges.

DEFINITIONS Improper Integrals with an Unbounded Integrand

1. Suppose f is continuous on $(a, b]$ with $\lim_{x \to a^+} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx,$$

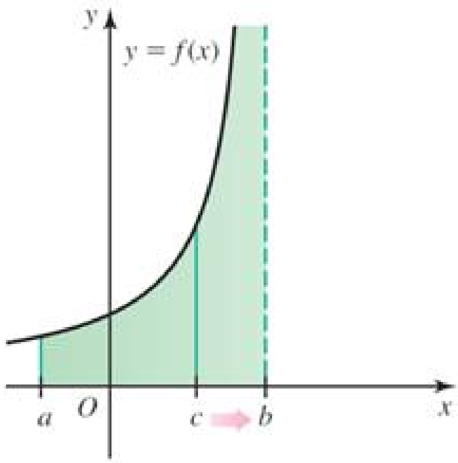
provided the limit exists.



2. Suppose f is continuous on $[a, b)$ with $\lim_{x \to b^-} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx,$$

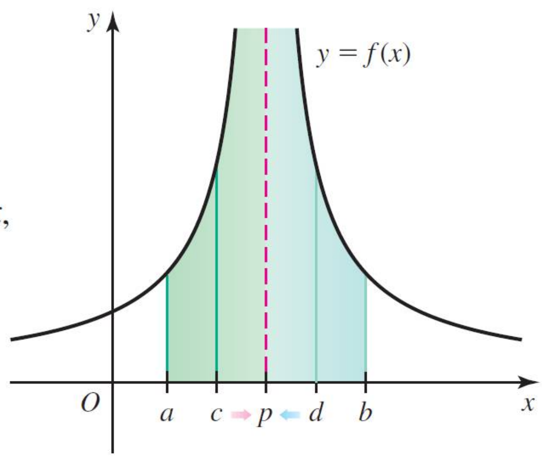
provided the limit exists.



3. Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then

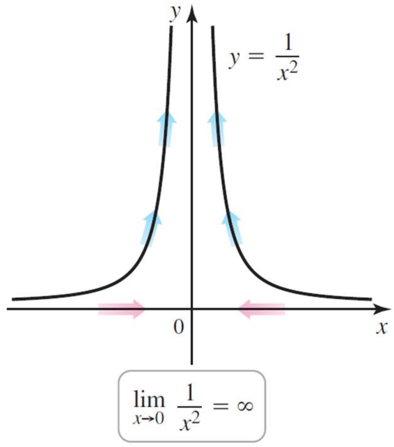
$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx,$$

provided the improper integrals on the right side exist.



In each case, if the limit exists, the improper integral is said to **converge**; if it does not exist, the improper integral is said to **diverge**.

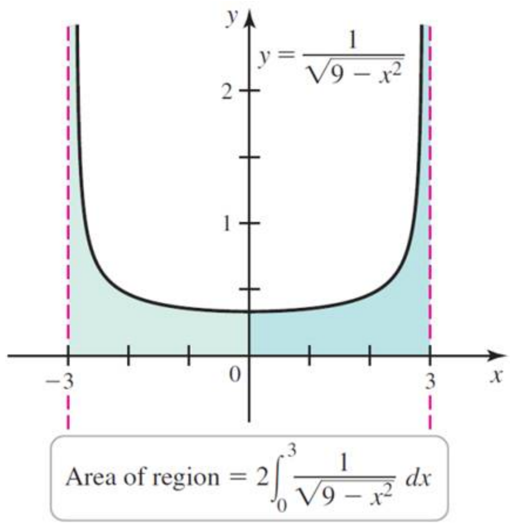
Question. Is $\int_{-1}^{1}\frac{1}{x^2}\,dx=\left.\frac{-1}{x}\right|_{-1}^{1}=-1-1=-2$ correct?



Answer. No. Actually, this is an improper integral. Indeed, $\int_{-1}^{1}\frac{1}{x^2}\,dx=\lim_{b\to 0^-}\int_{-1}^{b}\frac{1}{x^2}\,dx+\lim_{a\to 0^+}\int_{a}^{1}\frac{1}{x^2}\,dx.$ It diverges as $\lim_{a\to 0^+}\int_{a}^{1}\frac{1}{x^2}\,dx=\lim_{a\to 0^+}\left.\frac{-1}{x}\right|_{a}^{1}=\lim_{a\to 0^+}\left(-1+\frac{1}{a}\right)=\infty.$ When we calculate the definite integral $\int_a^b f(x)\,dx$, please check whether f is defined on $[a,b]$.

Example

Find the area of the region R between the graph of $f(x)=\frac{1}{\sqrt{9-x^2}}$ and the x-axis on the interval $(-3,3)$ (if it exists).



Solution

Clearly, $f(x)$ has vertical asymptotes at $x=\pm 3$. Since $f(x)$ is an even function, the area of R is

$$\begin{aligned}
\int_{-3}^{3}\frac{1}{\sqrt{9-x^2}}\,dx &= 2\int_0^3\frac{1}{\sqrt{9-x^2}}\,dx\\
&= 2\lim_{c\to 3^-}\int_0^c\frac{1}{\sqrt{9-x^2}}\,dx\\
&= 2\lim_{c\to 3^-}\int_{x=0}^{x=c}\frac{3\cos t}{\sqrt{9-9\sin^2 t}}\,dt \qquad \begin{array}{c} x=3\sin t\\ dx=3\cos t\,dt\end{array}\\
&= 2\lim_{c\to 3^-}\int_{x=0}^{x=c}1\,dt\\
&= 2\lim_{c\to 3^-}\left.t\right|_{x=0}^{x=c}\\
&= 2\lim_{c\to 3^-}\left.\sin^{-1}\left(\frac{x}{3}\right)\right|_{x=0}^{x=c}\\
&= 2\lim_{c\to 3^-}\sin^{-1}\left(\frac{c}{3}\right)-\sin^{-1}0=\pi
\end{aligned}$$