2022-23 First Semester
MATH1053 Linear Algebra II (1003)

Assignment 1 Suggested Solutions

1. (a). No, since $T(\mathbf{0}) = (1,0) \neq \mathbf{0}$. Or $T(\alpha x, \alpha y) = (1, \alpha y) \neq (\alpha, \alpha y) = \alpha T(x,y)$;
   (b). Yes. Since $L(\alpha p_1 + \beta p_2) = x(\alpha p_1 + \beta p_2) = \alpha x p_1 + \beta x p_2 = \alpha L(p_1) + \beta L(p_2)$.
   (c). Yes. Since $L(\alpha A + \beta B) = (\alpha A + \beta B)^T = \alpha A^T + \beta B^T = \alpha L(A) + \beta L(B)$.
   (d). No, since $L(5A) = 5A + I \neq 5(A+I) = 5L(A)$, for $A \in M_{n\times n}$;
   (e). No, since $L(6A) = \det(6A) = 6^2 \det(A) \neq 6L(A)$.

2. Notice that $\{1, 2x+x^2, x^2-1\}$ form a basis for P_3, and

$$\begin{aligned} T(2-6x+x^2) &= T[6-3(2x+x^2)+4(x^2-1)] \\ &= 6T(1) - 3T(2x+x^2) + 4T(x^2-1) = 6x - 3(1+x) + 4(1+x+x^2) \\ &= 4x^2+7x+1. \end{aligned}$$

3. (a) Consider $L(\mathbf{x}) = (2z, y+3x, 2x-z)^T = \mathbf{0}$ for some $\mathbf{x} = (x,y,z)^T \in \mathbb{R}^3$. Then

$$\begin{cases} 2z &= 0 \\ y+3x &= 0 \\ 2x-z &= 0 \end{cases} \quad \rightarrow \quad \begin{cases} x &= 0 \\ y &= 0 \\ z &= 0 \end{cases}$$

that is, $\ker(L) = \{(0,0,0)^T\}$.
For any $\mathbf{x} = (a,b,c)^T \in \mathbb{R}^3$, $L(\mathbf{x}) = (2c, b+3a, 2a-c)^T = a(0,3,2)^T + b(0,1,0)^T + c(2,0,-1)^T$. Then the image is

$$L(\mathbb{R}^3) = \operatorname{span}\left\{(0,3,2)^T, (0,1,0)^T, (2,0,-1)^T\right\}.$$

Check on the dimension: $\dim[\ker(L)] + \dim[L(\mathbb{R}^3)] = 0 + 3 = \dim(\mathbb{R}^3)$.

(b) Let $p(x) = a_0 + a_1 x + a_2 x^2$, consider $L(p(x)) = p(x) - p'(x) = 0, \ \forall x$, i.e.

$$p'(x) = p(x) \ \rightarrow \ a_1 + 2a_2 x = a_0 + a_1 x + a_2 x^2 \ \rightarrow \ a_1 = a_0, \ 2a_2 = a_1, \ a_2 = 0.$$

The kernel of L contains only the zero polynomial, $\ker(L) = \{0\}$.
On the other hand, for any $p(x) = b_0 + b_1 x + b_2 x^2 \in P_3$,

$$L(p(x)) = b_0 - b_1 + (b_1 - 2b_2)x + b_2 x^2 = b_0 + b_1(x-1) + b_2(x^2-2x)$$

for arbitrary $b_0, b_1, b_2 \in \mathbb{R}$. Then $L(P_3) = \operatorname{span}\{1, x-1, x^2-2x\}$.

(c) Consider $L(A) = A - A^T = O$, we have $A = A^T$, then A must be symmetric. Hence the kernel of L is the set of 3×3 symmetric matrices.

As for the range of L, for any 3×3 matrices $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, then

$$L(A) = A - A^T = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} - \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} 0 & b-d & c-g \\ d-b & 0 & f-h \\ g-c & h-f & 0 \end{pmatrix}.$$

The range of L is the set of 3×3 skew-symmetric matrices with zero diagonal entries. Check on the dimension: $\dim[\ker(L)] + \dim[L(\mathbb{R}^{3\times 3})] = 6 + 3 = \dim(\mathbb{R}^{3\times 3})$.

4. (a) The standard basis of $\mathbb{R}^3$ is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and that of $\mathbb{R}^2$ is $\{(1,0)^T, (0,1)^T\}$, then

$$A = [L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) The matrix representing L with respect to the standard basis is

$$A = [L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)] = \begin{bmatrix} 0 & 0 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

(c) The standard basis of P_3 is $E = \{1, x, x^2\}$. Since

$$L(1) = x \cdot 0 = 0, \quad L(x) = x \cdot 1 = x, \quad L(x^2) = x \cdot 2x = 2x^2,$$

$$\rightarrow \quad [L]_E = \big[[L(1)]_E, [L(x)]_E, [L(x^2)]_E\big] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

5. (a) The matrix representation of L with respect to the bases E and F is

$$[L]_E^F = [[L(\mathbf{u}_1)]_F, [L(\mathbf{u}_2)]_F, [L(\mathbf{u}_3)]_F] = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -4 & 1 \end{bmatrix},$$

since due to the calculation and observation, we have

$$L(\mathbf{u}_1) = (0,1)^T = \mathbf{b}_1 - 2\mathbf{b}_2, \quad L(\mathbf{u}_2) = (2,1)^T = 3\mathbf{b}_1 - 4\mathbf{b}_2, \text{ and } L(\mathbf{u}_3) = (1,-1)^T = \mathbf{b}_2.$$

(b) Based on the linear transformation L, we have

$$L(\mathbf{u}_1) = (0,2)^T, \quad L(\mathbf{u}_2) = (4,0)^T, \text{ and } L(\mathbf{u}_3) = (2,-2)^T.$$

To find the coordinate vectors $[L(\mathbf{u}_i)]_F$, let $B = [\mathbf{b}_1, \mathbf{b}_2]$, then $B^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$ and

$$[L(\mathbf{u}_1)]_F = B^{-1}L(\mathbf{u}_1) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}, \quad [L(\mathbf{u}_2)]_F = B^{-1}L(\mathbf{u}_2) = \begin{pmatrix} 4 \\ -4 \end{pmatrix},$$

$$[L(\mathbf{u}_3)]_F = B^{-1}L(\mathbf{u}_3) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Thus, the matrix representation of L with respect to the bases E and F is

$$A = [[L(\mathbf{u}_1)]_F, [L(\mathbf{u}_2)]_F, [L(\mathbf{u}_3)]_F] = \begin{bmatrix} 2 & 4 & 0 \\ -4 & -4 & 2 \end{bmatrix}.$$

6. Let $E = \{1,\ 1 + 2x,\ 4x^2 - 3\}$ and $F = \{1, x, x^2\}$. then based on the differential operator L, we have

$$L(1) = 0, \quad L(1 + 2x) = 2, \text{ and } L(4x^2 - 3) = 8x.$$

Thus, the matrix representation of L with respect to the bases E and F is

$$[L]_E^F = [[0]_F, [2]_F, [8x]_F] = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}.$$

7. MATLAB code:

(a). A=[3 -1 -2; 2 0 -2; 2 -1 -1]; rref(A)　　(b)-(c) similar