Chap5-Week8

2023/03/28 16:00

Chap5.5

I. Orthogonal/Orthonormal Sets Sec 5.5

II. Orthogonal projections for inner product spaces

(III. Application: Fourier Series) optional Sec 5.4

IV. Orthogonal Matrix

V. Gram-Schmidt Process Sec 5.6

△ Inner Product and Inner Product Space (Section 5.4) optional

Def 5.4.1 An inner product on a vector space V is a function $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$

with the following properties: $\forall \vec{u}, \vec{v}, \vec{w} \in V$, $\alpha \in \mathbb{R}$

① $\langle \vec{u}, \vec{u} \rangle \geq 0$ with equality holding iff $\vec{u} = \vec{0}$. (Nonnegative)

② $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ (Symmetric)

③ $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

④ $\langle \alpha \vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle$

$\} \langle \alpha \vec{u} + \beta \vec{v}, \vec{w} \rangle = \alpha \langle \vec{u}, \vec{w} \rangle + \beta \langle \vec{v}, \vec{w} \rangle$ (Bilinear)

A vector space with an inner product is called an inner product space.

E.g. ($\mathbb{R}^n$) The dot product in $\mathbb{R}^n$ defined as

$$\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y} = x_1 y_1 + \cdots + x_n y_n \quad , \quad \forall \vec{x}, \vec{y} \in \mathbb{R}^n$$

Satisfying all properties as an inner product.

E.g. ($\mathbb{R}^n$) The weighted dot product in $\mathbb{R}^n$ defined by

$$\langle \vec{x}, \vec{y} \rangle = w_1 x_1 y_1 + w_2 x_2 y_2 + \cdots + w_n x_n y_n \quad , \quad \forall \vec{x}, \vec{y} \in \mathbb{R}^n$$

where $\{w_1, w_2, \cdots, w_n\}$ are positive weights.

Hence, the weighted dot product is also an inner product on $\mathbb{R}^n$.

E.g. $C[a,b]$ Define an inner product on $C[a,b]$ as

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx$$

nonnegativity / Symmetry / Bilinearity (as exercises)

If $w(x) > 0$ for all $x \in [a,b]$ and $w(x)$ is positive continuous on $[a,b]$, then

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x) \, dx$$

is also an inner product on $C[a,b]$.

△ Norm and Normed Space

Def 5.4.2 A norm on a vector space V is a function $\| \cdot \| : V \to \mathbb{R}$

with the following properties: $\forall \vec{u}, \vec{v} \in V$, $\alpha \in \mathbb{R}$

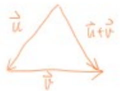
① $\| \vec{u} \| \geq 0$ with equality holding iff $\vec{u} = \vec{0}$.

② $\| \alpha \vec{u} \| = |\alpha| \cdot \| \vec{u} \|$

① $\|\vec{u}\| \geq 0$ with equality holding iff. $\vec{u} = \vec{0}$.

② $\|\alpha \vec{u}\| = |\alpha| \cdot \|\vec{u}\|$

③ $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ → triangle inequality



A vector space with a norm is called a normed space.
The distance between $\vec{u}, \vec{v} \in V$, $\|\vec{u} - \vec{v}\|$.

E.g. $\mathbb{R}^n$ Define the norm as

$$\|\vec{x}\| = \sqrt{<\vec{x}, \vec{x}>} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

→ the norm induced from inner product

is a norm on $\mathbb{R}^n$.

E.g. $\mathbb{R}^n$ Let $\vec{x} = (x_1, x_2, \cdots, x_n)^T$, define

$$\|\vec{x}\|_p = \begin{cases} \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, & p \geq 1 \\ \max_{1 \leq i \leq n} |x_i|, & p = \infty \end{cases}$$

→ other norms on $\mathbb{R}^n$

verify for ①~③ properties :

E.g. Let $\vec{x} = (1, 2, -1)'$, then

$\|\vec{x}\|_1 = (|1| + |2| + |-1|)^1 = 4$

$\sqrt{<\vec{x}, \vec{x}>} = \|\vec{x}\|_2 = (1^2 + 2^2 + (-1)^2)^{\frac{1}{2}} = \sqrt{1+4+1} = \sqrt{6}$

$\|\vec{x}\|_\infty = \max\{|1|, |2|, |-1|\} = 2$

△ Orthogonal Sets and Orthonormal Sets (in general inner product spaces)

Def 5.5.4 (Orthogonal Set) A set of nonzero vectors in an inner product space

$$\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\}$$

is called an orthogonal set if $<\vec{v}_i, \vec{v}_j> = 0$ for every $i \neq j$.

E.g. $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \right\}$ is an orthogonal set.

Since $(1, 1, 1) \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 2 + 1 - 3 = 0$.

E.g. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$? an orthogonal set ?

Yes, it is an orthogonal set since
$<\vec{e}_1, \vec{e}_2> = 0$, $<\vec{e}_1, \vec{e}_4> = 0$, $<\vec{e}_2, \vec{e}_4> = 0$

Thm 5.5.5 If $\{\vec{v}_1, \cdots, \vec{v}_m\}$ is an orthogonal set, then it is linear independent.

outline of proof: Assume that $\vec{v}_k$ in the set is linearly dependent on the others.

$$\vec{v}_k = c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1} + c_{k+1}\vec{v}_{k+1} + \cdots + c_m\vec{v}_m \quad \text{for some } c_i \in \mathbb{R}$$

Then for $j = 1, \cdots, m$, $j \neq k$

$$0 = \langle \vec{v}_k, \vec{v}_1 \rangle = c_1\langle \vec{v}_1, \vec{v}_1\rangle + \cdots + c_{k-1}\langle \vec{v}_{k-1}, \vec{v}_1\rangle + c_{k+1}\langle \vec{v}_{k+1}, \vec{v}_1\rangle + \cdots + c_m\langle \vec{v}_m, \vec{v}_1\rangle$$

$$= c_1\langle \vec{v}_1, \vec{v}_1\rangle + 0 \quad \rightarrow \quad c_1 = 0$$

for $j \neq k$, $j = 2, 3, \cdots, m$:
$$0 = \langle \vec{v}_k, \vec{v}_j\rangle = c_j\langle \vec{v}_j, \vec{v}_j\rangle + 0 \quad \rightarrow \quad c_j = 0, \quad \text{for } j = 2, \cdots, m,\ j \neq k$$

Since $\vec{v}_k \neq \vec{0}$, then we have a contradiction as $\vec{v}_k = 0\cdot\vec{v}_1 + \cdots + 0\cdot\vec{v}_m = \vec{0}$.

E.g. $\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\-3\end{pmatrix}, \begin{pmatrix}4\\-5\\1\end{pmatrix} \right\}$ as $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$\langle \vec{v}_1, \vec{v}_2\rangle = (1\ 1\ 1)\begin{pmatrix}2\\1\\-3\end{pmatrix} = 0$, $\quad \langle \vec{v}_2, \vec{v}_3\rangle = (2\ 1\ -3)\begin{pmatrix}4\\-5\\1\end{pmatrix} = 0$

$\langle \vec{v}_1, \vec{v}_3\rangle = (1\ 1\ 1)\begin{pmatrix}4\\-5\\1\end{pmatrix} = 0$, $\quad$ this is an orthogonal set.

Question: Can $\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\-3\end{pmatrix}, \begin{pmatrix}4\\-5\\1\end{pmatrix} \right\}$ form a basis for $\mathbb{R}^3$?

Def 5.5.6 (Orthogonal Basis) A basis $\{\vec{v}_1, \cdots, \vec{v}_m\}$ is an orthogonal basis if it is an orthogonal set.

Def 5.5.7 (Orthonormal Set) A set $\{\vec{u}_1, \cdots, \vec{u}_m\}$ of nonzero vectors in an inner product space is said to be orthonormal if

$$\langle \vec{u}_i, \vec{u}_j\rangle = 0 \quad \text{for every } i \neq j$$

and $\quad \|\vec{u}_i\| = 1 \quad$ for every i.

$$\underset{\{\vec{v}_1, \cdots, \vec{v}_k\}}{\text{Orthogonal Set}} \xrightarrow[\text{each } \vec{v}_i]{\text{normalize}} \underset{\left\{\frac{\vec{v}_1}{\|\vec{v}_1\|}, \cdots, \frac{\vec{v}_k}{\|\vec{v}_k\|}\right\}}{\text{Orthonormal Set}} \ ?$$

E.g. Turn $\left\{ \underset{\vec{v}_1}{\begin{pmatrix}1\\1\\1\end{pmatrix}}, \underset{\vec{v}_2}{\begin{pmatrix}2\\1\\-3\end{pmatrix}} \right\}$ and $\left\{ \underset{\vec{v}_1}{\begin{pmatrix}1\\1\\1\end{pmatrix}}, \underset{\vec{v}_2}{\begin{pmatrix}2\\1\\-3\end{pmatrix}}, \underset{\vec{v}_3}{\begin{pmatrix}4\\-5\\1\end{pmatrix}} \right\}$ into orthonormal sets.

Recall normalizing a vector: $\vec{v}$ is nonzero, let $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$, then $\|\vec{u}\| = 1$.

Let $\vec{u}_1 = \vec{v}_1 / \|\vec{v}_1\| = \frac{1}{\sqrt{1+1+1}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\quad \left(\|\vec{u}_1\| = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = \left|\frac{1}{\sqrt{3}}\right|\sqrt{1+1+1} = 1. \right)$

$\vec{u}_2 = \vec{v}_2 / \|\vec{v}_2\| = \frac{1}{\sqrt{2^2+1^2+(-3)^2}}\begin{pmatrix}2\\1\\-3\end{pmatrix} = \frac{1}{\sqrt{14}}\begin{pmatrix}2\\1\\-3\end{pmatrix}$

$\vec{u}_3 = \vec{v}_3 / \|\vec{v}_3\| = \frac{1}{\sqrt{42}}\begin{pmatrix}4\\-5\\1\end{pmatrix}$

Then $\left\{ \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{14}}\begin{pmatrix}2\\1\\-3\end{pmatrix} \right\}$ is an orthormal set, $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is also an orthonormal set. $\mathbb{R}^3$

Def. 5.5.8 (Orthonormal Basis) A basis $\{\vec{u}_1, \cdots \vec{u}_m\}$ is called an orthonormal basis if it is an orthonormal set.

E.g. $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is an orthonormal basis of $\mathbb{R}^3$. $\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$ is an orthonormal basis of $\mathbb{R}^4$.

E.g. $\left\{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{14}}\begin{pmatrix}2\\1\\-3\end{pmatrix}, \frac{1}{\sqrt{42}}\begin{pmatrix}4\\-5\\1\end{pmatrix}\right\}$ is an orthonormal basis of $\mathbb{R}^3$.

Thm 5.5.9 Let $\{\vec{u}_1, \cdots, \vec{u}_m\}$ be an orthonormal basis of an inner product space V.

Then $\forall \vec{x} \in V$, $\vec{x} \in V$

$$\vec{x} = \langle \vec{x}, \vec{u}_1 \rangle \vec{u}_1 + \langle \vec{x}, \vec{u}_2 \rangle \vec{u}_2 + \cdots + \langle \vec{x}, \vec{u}_m \rangle \vec{u}_m .$$

outline of proof: $\forall \vec{x} \in V$, $\vec{x} = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_m\vec{u}_m$ for some $a_i \in \mathbb{R}$.

To find a_i: $\langle \vec{x}, \vec{u}_i \rangle = a_1 \langle \vec{u}_1, \vec{u}_i \rangle + a_2 \langle \vec{u}_2, \vec{u}_i \rangle + \cdots + a_m \langle \vec{u}_m, \vec{u}_i \rangle$

$\langle \vec{x}, \vec{u}_i \rangle = a_i \langle \vec{u}_i, \vec{u}_i \rangle = a_i$, $i = 1, 2, \cdots, m$

Since $\{\vec{u}_1, \cdots \vec{u}_m\}$ is an orthonormal basis of V.

E.g. Express $\begin{pmatrix}1\\2\\3\end{pmatrix}$ as a linear combination of $\left\{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{14}}\begin{pmatrix}2\\1\\-3\end{pmatrix}, \frac{1}{\sqrt{42}}\begin{pmatrix}4\\-5\\1\end{pmatrix}\right\}$.

$$\begin{pmatrix}1\\2\\3\end{pmatrix} = a_1 \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} + a_2 \frac{1}{\sqrt{14}}\begin{pmatrix}2\\1\\-3\end{pmatrix} + a_3 \frac{1}{\sqrt{42}}\begin{pmatrix}4\\-5\\1\end{pmatrix}$$

Where $a_1 = (1\ 2\ 3)\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} = 2\sqrt{3}$, $a_2 = -\frac{5}{\sqrt{14}}$, $a_3 = -\frac{3}{\sqrt{42}}$

Verify $\begin{pmatrix}1\\2\\3\end{pmatrix} \overset{?}{=} 2\sqrt{3} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} + a_2\vec{u}_2 + a_3\vec{u}_3$

E.g. Express $\begin{pmatrix}1\\2\\3\end{pmatrix}$ as .. .. .. of $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ — orthonormal basis

$$\begin{pmatrix}1\\2\\3\end{pmatrix} = \langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \vec{e}_1 \rangle \vec{e}_1 + \langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \vec{e}_2 \rangle \vec{e}_2 + \langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \vec{e}_3 \rangle \vec{e}_3 = \vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3$$

Thm 5.5.10 (Parseval's Formula) If $\{\vec{u}_1, \cdots, \vec{u}_m\}$ is an orthonormal basis of an inner product space V and for any $\vec{x} \in V$, if $\vec{x} = \sum_{i=1}^{m} c_i \vec{u}_i$, then

$$\|\vec{x}\|^2 = \sum_{i=1}^{m} c_i^2 .$$

proof: $\|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle = \langle \sum_{i=1}^{m} c_i\vec{u}_i, \sum_{i=1}^{m} c_i\vec{u}_i \rangle = \underbrace{\cdots}_{\text{leave as exercise}} = \sum_{i=1}^{m} c_i^2$

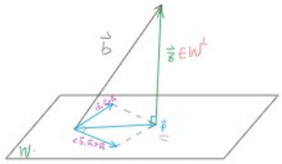
Thm 5.5.11 (Orthogonal Projection onto a Subspace) given an orthonormal basis & key

$W \oplus W^\perp = V$

Let W be a subspace of an inner product space V and $\{\vec{u}_1, \cdots, \vec{u}_k\}$ be an orthonormal basis of W.

Then for any $\vec{b} \in V$, the projection of $\vec{b}$ onto W is

$$\text{proj}_W \vec{b} = \vec{p} = \langle \vec{b}, \vec{u}_1 \rangle \vec{u}_1 + \langle \vec{b}, \vec{u}_2 \rangle \vec{u}_2 + \cdots + \langle \vec{b}, \vec{u}_k \rangle \vec{u}_k \leftarrow$$



Outline of proof: $\vec{b} = \vec{p} + \vec{z}$, where $\vec{p} \in W$ and $\vec{z} \in W^\perp$

By taking $\langle \vec{b}, \vec{u}_i \rangle =$, $i = 1, \cdots, k$

$=$

$=$

$=$

$\langle \underset{\in W^\perp}{\vec{z}}, \underset{\in W}{\vec{u}_1} \rangle = 0$

$\langle \underset{\in W^\perp}{\vec{z}}, \underset{\in W}{\vec{u}_i} \rangle = 0$, $i = 1, \cdots, k$

Def 5.5.12 (Orthogonal Matrix)

An $n \times n$ matrix is called an orthogonal matrix if its columns form an orthonormal basis for $\mathbb{R}^n$.

$\|\vec{a}_i\| = 1$, $\langle \vec{a}_i, \vec{a}_j \rangle = 0$, for $i \neq j$

E.g. Rotation Matrix $Q = \begin{array}{c} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ \begin{matrix} \vec{q}_1 & \quad \vec{q}_2 \end{matrix} \end{array}$ is orthogonal.

verification: $\vec{q}_1 \perp \vec{q}_2$: $(\cos\theta \ \ \sin\theta) \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = -\cos\theta \sin\theta + \sin\theta \cos\theta = 0$

$\|\vec{q}_1\| = \sqrt{\vec{q}_1^{\,T} \vec{q}_1} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$\|\vec{q}_2\| = \sqrt{\vec{q}_2^{\,T} \vec{q}_2} = \sqrt{\sin^2\theta + \cos^2\theta} = 1$

So Q is an orthogonal matrix.

E.g. Permutation Matrix is a matrix formed from the identity matrix by reordering its columns

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \text{ etc.}$$

Permutation matrices are orthogonal.

Thm 5.5.13 If Q is an $n \times n$ orthorgonal matrix, then

(i) $Q^T Q = I$

(ii) $Q^T = Q^{-1}$

(iii) $\langle Q\vec{x}, Q\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$ for any $\vec{x}, \vec{y} \in \mathbb{R}^n$

(iv) $\|Q\vec{x}\| = \|\vec{x}\|$, $\forall \vec{x} \in \mathbb{R}^n$.

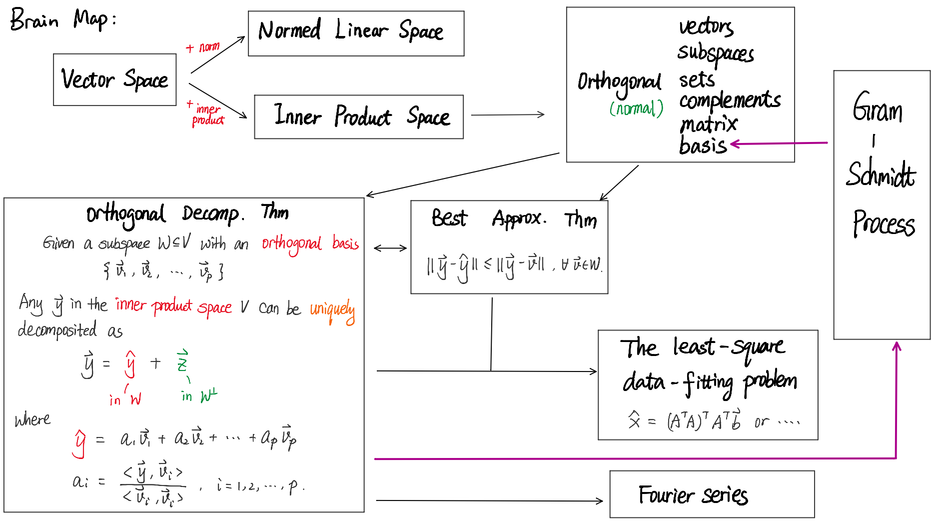
Outline of proof: (Hints)

$$Q = \begin{bmatrix} | & | & & | \\ \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_n \\ | & | & & | \end{bmatrix}_{n \times n}, \text{ then } Q^T = \begin{bmatrix} - & \vec{q}_1^{\,T} & - \\ - & \vec{q}_2^{\,T} & - \\ & \vdots & \\ - & \vec{q}_n^{\,T} & - \end{bmatrix}, \quad Q^T Q = \begin{bmatrix} - & \vec{q}_1^{\,T} & - \\ - & \vec{q}_i^{\,T} & - \\ & \vdots & \\ - & \vec{q}_n^{\,T} & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ & & & 0 \\ 0 & & & 1 \end{bmatrix}$$

$\langle Q\vec{x}, Q\vec{y} \rangle = (Q\vec{x})^T Q\vec{y} = \cdots$

$\|Q\vec{x}\| = \sqrt{\langle Q\vec{x}, Q\vec{x} \rangle} = \cdots$

Brain Map:


Vector Space
+ norm
Normed Linear Space
+ inner product
Inner Product Space
Orthogonal (normal)
vectors
subspaces
sets
complements
matrix
basis
Gram - Schmidt Process
Orthogonal Decomp. Thm
Given a subspace $W \subseteq V$ with an orthogonal basis $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_p\}$
Any $\vec{y}$ in the inner product space V can be uniquely decomposited as
$\vec{y} = \hat{y} + \vec{z}$
in W
in $W^\perp$
where
$\hat{y} = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_p\vec{u}_p$
$a_i = \frac{\langle \vec{y}, \vec{u}_i \rangle}{\langle \vec{u}_i, \vec{u}_i \rangle}$, $i = 1, 2, \cdots, p$.
Best Approx. Thm
$\|\vec{y} - \hat{y}\| \le \|\vec{y} - \vec{v}\|$, $\forall \vec{v} \in W$.
The least-square data-fitting problem
$\hat{x} = (A^TA)^{-1}A^T\vec{b}$ or ....
Fourier series