PT

Solution to Assignment 7

1. For (a), we use the fact that density integrates to 1 , so we have $\int_0^4 cxdx = 1$ and $c = \frac{1}{8}$. For (b), we compute

$$\int_1^2 \frac{x}{8}dx = \frac{3}{16}.$$

Finally, for (c) we get

$$EX = \int_0^4 \frac{x^2}{8}dx = \frac{8}{3}$$

and

$$E\left(X^2\right) = \int_0^4 \frac{x^3}{8}dx = 8.$$

So, $\mathrm{Var}(X) = 8 - \frac{64}{9} = \frac{8}{9}$.

2.

(a) $\frac{1}{6}$.

(b) Let T be the time of the call, from 7pm, in minutes; T is uniform on $[0, 120]$. Thus,

$$P(T \leq 100 \mid T \geq 90) = \frac{1}{3}.$$

(c) We have

$$M = \begin{cases} 0 & \text{if } 0 \leq T \leq 60 \\ T - 60 & \text{if } 60 \leq T \leq 90 \\ 30 & \text{if } 90 \leq T \leq 120. \end{cases}$$

Then,

$$EM = \frac{1}{120}\int_{60}^{90}(t-60)dx + \frac{1}{120}\int_{90}^{120} 30dx = 11.25.$$

3. Let us write X for an exponential distribution with mean 0.5, and Y for the random claim amount under the policy offered by the insurance company. Note that \$2000 represents 8 intervals of length \$250, and 8 intervals corresponds to 8 inches. As there is no payment for the first two inches of daily snowfall, this implies that the policy limit of \$2000 is reached when daily snowfall reaches 10 inches. Therefore

$$Y = \begin{cases} 0 & \text{if } X \leq 2, \\ 250(X-2) & \text{if } 2 < X \leq 10, \\ 2000 & \text{if } X > 10. \end{cases}$$

Note also that

$$f_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ 2e^{-2x} & \text{for } x \geq 0 \end{cases}$$

Based on this

$$\begin{aligned} E(Y) &= \int_2^{10} 250(x-2) \cdot 2e^{-2x} dx + \int_{10}^{+\infty} 2000 \cdot 2e^{-2x} dx \\ &= 500 \int_2^{10} xe^{-2x} dx - 1000 \int_2^{10} e^{-2x} dx + 4000 \int_{10}^{+\infty} e^{-2x} dx. \end{aligned}$$

We perform integration by parts in the first integral and obtain

$$\begin{aligned} 500 \int_2^{10} xe^{-2x} dx &= -250 \int_2^{10} xd\left(e^{-2x}\right) \\ &= \left(-250xe^{-2x}\right)\Big|_{x=2}^{x=10} + 250 \int_2^{10} e^{-2x} dx \\ &= \left(-2500e^{-20} + 500e^{-4}\right) + \left(-\ 125e^{-2x}\Big|_{x=2}^{x=10}\right) \\ &= \left(-2500e^{-20} + 500e^{-4}\right) + \left(-125e^{-20} + 125e^{-4}\right) \approx 11.4473. \end{aligned}$$

Furthermore,

$$\begin{aligned} -1000 \int_2^{10} e^{-2x} dx + 4000 \int_{10}^{+\infty} e^{-2x} dx &= -1000 \cdot \left(-\frac{1}{2}e^{-2x}\Big|_{x=2}^{x=10}\right) + 4000 \cdot \left(-\frac{1}{2}e^{-2x}\Big|_{x=10}^{x+\infty}\right) \\ &= -1000 \cdot \left(-\frac{1}{2}e^{-20} + \frac{1}{2}e^{-4}\right) + 4000 \cdot \left(0 + \frac{1}{2}e^{-20}\right) \\ &= -9.1578. \end{aligned}$$

Therefore,

$$E(Y) \approx 11.4473 - 9.1578 = 2.2895.$$

4.

(a) We have

$$
\begin{aligned}
E\left(X^k\right) &= \int_{-\infty}^{+\infty} x^k f_X(x)dx = \frac{1}{b-a}\int_a^b x^k dx \\
&= \frac{1}{b-a}\cdot \left.\frac{x^{k+1}}{k+1}\right|_a^b = \frac{1}{b-a}\cdot\frac{b^{k+1}-a^{k+1}}{k+1} \\
E\left(X\right) &= \frac{1}{b-a}\cdot\frac{b^2-a^2}{2} = \frac{1}{2}(a+b) \\
E\left(X^2\right) &= \frac{1}{b-a}\cdot\frac{b^3-a^3}{2+1} = \frac{a^2+b^2+ab}{3} \\
\sigma_x^2 &= E\left(X^2\right) - \left[E\left(X\right)\right]^2 \\
&= \frac{1}{12}(b-a)^2
\end{aligned}
$$

(b) We have

$$
\begin{aligned}
&E\left(X^k\right) = \int_0^{+\infty} x^k \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx \\
&= \int_0^{+\infty} x^{k-1}\cdot\frac{\alpha\beta^{\alpha+1}}{\beta\cdot\Gamma(\alpha+1)} x^{(\alpha+1)-1} e^{-\beta x} dx \\
&= \frac{\alpha}{\beta}\int_0^{+\infty} x^{k-1}\frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} x^{(\alpha+1)-1} e^{-\beta x} dx. \\
&= \frac{\alpha(\alpha+1)}{\beta^2}\int_0^{\infty} x^{k-2}\frac{\beta^{\alpha+2}}{\Gamma(\alpha+2)} x^{(\alpha+2)-1} e^{-\beta x} dx \\
&\ldots \\
&= \frac{(\alpha+k-1)!}{\beta^k\cdot(\alpha-1)!}\int_0^{+\infty} \frac{\beta^{\alpha+k}}{\Gamma(\alpha+k)} x^{(\alpha+k)-1} e^{-\beta x} dx \\
&= \frac{\Gamma(\alpha+k)}{\beta\Gamma(\alpha)} \\
&k=2 \Rightarrow E\left(X^2\right) = \frac{\Gamma(\alpha+2)}{\beta\cdot\Gamma(\alpha)} = \frac{\alpha\cdot(\alpha+1)}{\beta^2} \\
&k=1 \Rightarrow E(X) = \alpha/\beta \\
&\therefore \operatorname{Var}(X) = E\left(X^2\right) - E^2(X) = \frac{\alpha(\alpha+1)}{\beta^2} - \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}
\end{aligned}
$$

(c) Let $x = r\cos\theta, \quad y = r\sin\theta$. Then

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{x^2+y^2}{2\sigma^2}}dxdy \\
&= \int_{-\pi}^{\pi}\int_{0}^{+\infty} e^{-\frac{r^2}{2\sigma^2}} \cdot rdrd\theta \\
&= \int_{-\pi}^{\pi} d\theta \int_{0}^{+\infty} -\sigma^2 de^{-\frac{r^2}{2\sigma^2}} \\
&= 2\pi \cdot \left(-\sigma^2 e^{-\frac{r^2}{2\sigma^2}}\right)\Bigg|_{0}^{+\infty} \\
&= 2\pi\sigma^2
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{x^2+y^2}{2\sigma^2}}dxdy = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2}}dx \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2\sigma^2}}dy \\
&= \left[\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2}}dx\right]^2 = 2\pi\sigma^2 \\
&\therefore \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2}}dx = \sqrt{2\pi}\sigma \\
&\therefore \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}dx = \frac{1}{\sqrt{2\pi}\sigma} \cdot \sqrt{2\pi}\sigma = 1.
\end{aligned}$$

Assume $X \sim N\left(\mu, \sigma^2\right)$. Let $Y = \frac{X-u}{\sigma}$. Then $Y \backsim N(0,1)$. Note that

$$\begin{aligned}
&E\left(\frac{X-\mu}{\sigma}\right)^{2n+1} = E\left(Y^{2n+1}\right) = \int_{-\infty}^{+\infty} y^{2n+1}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy = 0 \\
&E\left(\frac{X-\mu}{\sigma}\right)^{2n} = E\left(Y^{2n}\right) = \int_{-\infty}^{+\infty} y^{2n}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\
&= -\int_{-\infty}^{+\infty} y^{2n-1} \cdot \frac{1}{\sqrt{2\pi}}de^{-\frac{y^2}{2}} = -\frac{y^{2n-1}}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\Bigg|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \cdot (2n-1)y^{2n-2}dy \\
&= (2n-1)\int_{-\infty}^{+\infty} \frac{y^{2n-2}}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy = (2n-1)(2n-3)\int_{-\infty}^{+\infty} \frac{y^{2n-4}}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy = \cdots \\
&= (2n-1)!!
\end{aligned}$$

5.

(a) As

$$1 = c\int_0^1 (x+\sqrt{x})dx = c\left(\frac{1}{2}+\frac{2}{3}\right) = \frac{7}{6}c$$

it follows that $c = \frac{6}{7}$.

(b) We have

$$\frac{6}{7}\int_0^1 \frac{1}{x}(x+\sqrt{x})dx = \frac{18}{7}$$

(c) It holds

$$\begin{aligned} F_r(y) &= P(Y \le y) \\ &= P(X \le \sqrt{y}) \\ &= \frac{6}{7}\int_0^{\sqrt{y}} (x+\sqrt{x})dx \end{aligned}$$

and so

$$f_Y(y) = \begin{cases} \frac{3}{7}\left(1+y^{-\frac{1}{4}}\right) & \text{if } y \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

6.

(a) From $\int_0^2 f(x)dx = 1$ we get $2a+2b=1$ and from $\int_0^2 xf(x)dx = \frac{7}{6}$ we get $2a+\frac{8}{3}b = \frac{7}{6}$. The two equations give $a=b=\frac{1}{4}$.

(b) $E\left(X^2\right) = \int_0^2 x^2 f(x)dx = \frac{5}{3}$ and so $\operatorname{Var}(X) = \frac{5}{3} - \left(\frac{7}{6}\right)^2 = \frac{11}{36}$.

7.

(a) Since

$$1 = F_X(\infty) = \int_1^{16} c\left(1 - \frac{1}{2\sqrt{x}}\right)dx = c[x-\sqrt{x}]_1^{16} = 12c,$$

we get

$$c = \frac{1}{12}.$$

(b) So

$$F_X(x) = \begin{cases} 1 & x > 16 \\ \int_1^x \frac{1}{12}\left(1-\frac{1}{2\sqrt{t}}\right)dt = \frac{1}{12}[t-\sqrt{t}]_1^x = \frac{1}{12}(x-\sqrt{x}) & 1 \le x \le 16 \\ 0 & x < 1 \end{cases}$$

8. We have

$F(t) = P(T \le t) = P\left(X^2 \le t\right) = P(-\sqrt{t} \le X \le \sqrt{t})$

$= \int_{-\sqrt{t}}^{\sqrt{t}} f(x)dx = \int_{-\sqrt{t}}^{0} 2e^{4x}dx + \int_0^{\sqrt{t}} e^{-2x} = 0.5e^{4x}\big|_{-\sqrt{t}}^{0} - 0.5e^{-2x}\big|_0^{\sqrt{t}} = 0.5 - 0.5e^{-4\sqrt{t}} - 0.5e^{-2\sqrt{t}} + 0.5$

$= 1 - 0.5e^{-4\sqrt{t}} - 0.5e^{-2\sqrt{t}}$,

so

$f(t) = F'(t) = -0.5e^{-4\sqrt{t}}[-4(0.5)/\sqrt{t}] - 0.5e^{-2\sqrt{t}}[-2(0.5)/\sqrt{t}] = e^{-4\sqrt{t}}/\sqrt{t} + 0.5e^{-2\sqrt{t}}/\sqrt{t}$

$= \dfrac{e^{-2\sqrt{t}}}{2\sqrt{t}} + \dfrac{e^{-4\sqrt{t}}}{\sqrt{t}}.$