2022-23 First Semester
MATH1053 Linear Algebra I

Assignment 7

Due Date: 19/Dec/2022 (Monday), on or before 15:00, iSpace.

- Write down your **CHN name** and **student ID**. Write neatly on **A4-sized** paper (*staple if necessary*) and **show your steps**.
- Hand in your homework in **one pdf file** on iSpace.
- **Late submissions or answers without steps won't be graded.**

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1. Let A be an $m \times n$ matrix with $m > n$. Let $\mathbf{b} \in \mathbb{R}^m$ and suppose that $N(A) = \{\mathbf{0}\}$.
   (a) What can you conclude about the column vectors of A? Are they linearly independent? Do they span $\mathbb{R}^m$? Explain.
   (b) How many solutions will the system $A\mathbf{x} = \mathbf{b}$ have if $\mathbf{b}$ is not in the column space of A? How many solutions will there be if $\mathbf{b}$ is in the column space of A? Explain.

2. Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and let $A = \mathbf{x}\mathbf{y}^T$.
   (a) Show that $\{\mathbf{x}\}$ is a basis for the column space of A and that $\{\mathbf{y}^T\}$ is a basis for the row space of A.
   (b) What is the dimension of $N(A)$?

3. Let A be a matrix. Show that $\operatorname{rank}(A) = \operatorname{rank}(AA^T)$.

4. Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times r}$, and $C = AB$. Show that
   (a) The column space of C is a subspace of the column space of A.
   (b) The row space of C is a subspace of the row space of B.
   (c) $\operatorname{rank}(C) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

   (*Hint for (a): Because* $\mathrm{Col}(C)$ *is already a subspace of* $\mathbb{R}^m$*. Here we only need to show that* $\mathrm{Col}(C)$ *is a subset of* $\mathrm{Col}(A)$*. Namely,* $\forall \mathbf{x} \in \mathbb{R}^r$, $C\mathbf{x} \in \mathrm{Col}(C)$*. Explain if* $C\mathbf{x} \in \mathrm{Col}(A)$.)

5. Let A and B be $n \times n$ matrices.
   (a) Show that $AB = O$ if and only if the column space of B is a subspace of the null space of A.
   (b) Show that if $AB = O$, then $\operatorname{rank}(A) + \operatorname{rank}(B) \le n$.

   (*Hint for (a): Partition the matrix B by columns, then consider $AB = O$.*)