MATH1063: Linear Algebra II (1003), Quiz 1, Mar/29/2023

Chinese Name ______________________ Student ID ______________________

1. (20 points) Let the mapping defined by $L : \mathbb{R}^3 \to \mathbb{R}^4$, $L((x,y,z)') = (2x, x+y, y+z, x+z)'$.

(a). Find the matrix representing L with respect to the ordered bases

$$\alpha = \{(2,0,1)', (0,2,1)', (1,2,1)'\}, \quad \beta = \{(1,0,0,1)', (0,1,0,1)', (1,0,1,0)', (1,1,0,0)'\}.$$

(b). Find the bases for the kernel and the image of L, respectively.

(c). Let $\mathbf{v} = (2,2,2)'$. Find $L(\mathbf{v})$ using the matrix you find in part (a).

10'

(a) $L\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}4\\2\\1\\3\end{pmatrix}$, $L\begin{pmatrix}0\\2\\1\end{pmatrix} = \begin{pmatrix}0\\2\\3\\1\end{pmatrix}$, $L\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}2\\3\\3\\2\end{pmatrix}$ 3'

$$\begin{pmatrix}4\\2\\1\\3\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\\1\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\\1\end{pmatrix} + 1\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + 1\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$ 2'

$$\begin{bmatrix}1&0&1&1\\0&1&0&1\\0&0&1&0\\1&1&0&0\end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix}1&-1&-1&1\\-1&1&1&1\\0&0&2&0\\1&1&-1&-1\end{bmatrix}$$

$$\begin{pmatrix}0\\2\\3\\1\end{pmatrix} = -2\begin{pmatrix}1\\0\\0\\1\end{pmatrix} + 3\begin{pmatrix}0\\1\\0\\1\end{pmatrix} + 3\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + (-1)\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$ 2'

$$\begin{pmatrix}2\\3\\3\\2\end{pmatrix}_\beta = \frac{1}{2}\begin{bmatrix}1&-1&-1&1\\-1&1&1&1\\0&0&2&0\\1&1&-1&-1\end{bmatrix}\begin{bmatrix}2\\3\\3\\2\end{bmatrix} = \begin{bmatrix}-1\\3\\3\\0\end{bmatrix}$$ 2'

$$[L]_\alpha^\beta = \begin{bmatrix}2&-2&-1\\1&3&3\\1&3&3\\1&-1&0\end{bmatrix}$$ 1'

6'

(b) Let $L\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2x\\x+y\\y+z\\x+z\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \to \begin{bmatrix}2&0&0\\1&1&0\\0&1&1\\1&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

~~Rank(~~ $\text{rref}\begin{pmatrix}2&0&0\\1&1&0\\0&1&1\\1&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{pmatrix}$, $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$. 3'

$\ker(L) = \{\vec{0}\}.$

Let $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$L(\mathbb{R}^3) = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ 3'

(c) $\vec{v} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, $[\vec{v}]_\alpha = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

4'

$[L(\vec{v})]_\beta = [L]_\alpha^\beta (\vec{v})_\alpha = \begin{bmatrix} 2 & -2 & -1 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cancel{8}\ 0 \\ 4 \\ 4 \\ 0 \end{bmatrix}$

$L(\vec{v}) = \cancel{4}\,0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cancel{0} = \begin{pmatrix} \cancel{8}\ 4 \\ 4 \\ 4 \\ \cancel{8}\ 4 \end{pmatrix}$

$L(\vec{v}) = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 4 \end{pmatrix}.$

MATH1063: Linear Algebra II (1003), Quiz 2, Mar/29/2023

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This quiz contains 2 questions and full mark is 25.

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1. (10 points) Find the orthogonal complement $W^\perp$ of W and give a basis for $W^\perp$.

$$W = \{(x, y)' : 3x + 2y = 0\}$$

$$W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : (3,2)\begin{pmatrix} x \\ y \end{pmatrix} = 0 \right\}.$$

$$\begin{cases} x = -\frac{2}{3}y \\ y \in \mathbb{R} \end{cases}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}\alpha, \quad \alpha \in \mathbb{R}$$

Hence $W = \left\{ \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}\alpha, \alpha \in \mathbb{R} \right\}.$

Let $W^\perp = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 ; (a, b)\begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}\alpha = 0 \right\}.$

$$\to -\frac{2}{3}a\alpha + b\alpha = 0$$

$$a = \frac{3}{2}b, \quad b \in \mathbb{R}.$$

$$W^\perp = \left\{ \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}\beta, \beta \in \mathbb{R} \right\}.$$

A basis for $W^\perp$ is $\left\{ \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \right\}.$

2. (15 points)

(a) Find the orthogonal projection of $\mathbf{x} = (2, 4, 3)'$ onto $\mathbf{y} = (1, 0, -1)'$.

(b) Find the orthogonal projection of any vectors $\mathbf{x} = (x_1, x_2, x_3)' \in \mathbb{R}^3$ onto $\mathbf{y} = (1, 0, -1)'$ and find its matrix representation with respect to the standard basis.

5' (a) $\mathrm{proj}_{\vec{y}} \vec{x} = \dfrac{\vec{x}^T \vec{y}}{\vec{y}^T \vec{y}} \vec{y} = \dfrac{-1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$

10' (b) $L(\vec{x}) = \mathrm{proj}_{\vec{y}} \vec{x} = \dfrac{\vec{x}^T \vec{y}}{\vec{y}^T \vec{y}} \vec{y} = \dfrac{x_1 - x_3}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ 5'

$L(\vec{e}_1) = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad L(\vec{e}_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad L(\vec{e}_3) = \frac{-1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ 3'

$L_A = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$ 2'