Section 2.8 Implicit Differentiation（隐函数求导） and Inverse Trigonometric Functions

- So far the functions have all been given by equations of the form *y=f(x)*. A function in this form is said to be in explicit form. For example, the functions

$$y = x^2 + 3x + 1 \quad y = \frac{x^3 + 1}{2x - 3} \quad \text{and} \quad y = \sqrt{1 - x^2}$$

are all functions in explicit form（显函数表达）.

- Sometimes practical problems will lead to equations in which the function *y* is not written explicitly in terms of the independent variable *x*. For example, the equations such as

$$x^2 y^3 - 6 = 5y^3 + xy \quad \text{and} \quad x^2 y + 2y^3 = 3x + 2y$$

are said to be in implicit form（隐函数表达）.

Example Find *dy/dx* if $x^2 y + y^2 = x^3$

Solution:

We are going to differentiate both sides of the given equation with respect to *x*. Firstly, we temporarily replace *y* by *f(x)* and rewrite the equation as $x^2 f(x) + (f(x))^2 = x^3$. Secondly, we differentiate both sides of this equation term by term with respect to *x*:

$$\frac{d}{dx}[x^2 f(x) + (f(x))^2] = \frac{d}{dx}[x^3]$$

$$\underbrace{\left[x^2 \frac{df}{dx} + f(x)\frac{d}{dx}(x^2)\right]}_{\frac{d}{dx}[x^2 f(x)]} + \underbrace{2f(x)\frac{df}{dx}}_{\frac{d}{dx}[(f(x))^2]} = \underbrace{3x^2}_{\frac{d}{dx}(x^3)}$$

To be continued

Thus, we have

$$x^2\frac{df}{dx} + f(x)(2x) + 2f(x)\frac{df}{dx} = 3x^2 \qquad \text{gather all } \frac{df}{dx} \text{ terms}$$

$$x^2\frac{df}{dx} + 2f(x)\frac{df}{dx} = 3x^2 - 2xf(x) \quad \text{on one side of the equation}$$

$$[x^2 + 2f(x)]\frac{df}{dx} = 3x^2 - 2xf(x) \quad \text{combine terms}$$

$$\frac{df}{dx} = \frac{3x^2 - 2xf(x)}{x^2 + 2f(x)} \quad \text{solve for } \frac{df}{dx}$$

Finally, replace *f(x)* by *y* to get

$$\frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 + 2y}$$

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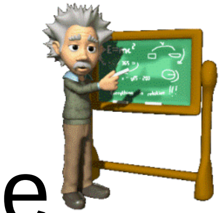
Implicit Differentiation: Suppose an equation defines y implicitly as a differentiable function of x. To find df/dx

1. Differentiate both sides of equation with respect to x. remember that y is really a function of x and use the chain rule when differentiating terms containing y.
2. Solve the differentiated equation algebraically for dy/dx.

Exercise

Find dy/dx by implicit differentiation where $x^3 + y^3 = xy$



Computing the slope of a tangent line by implicit differentiation

Example 27

Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point *(3,4)*. What is the slope at the point *(3,-4)?*

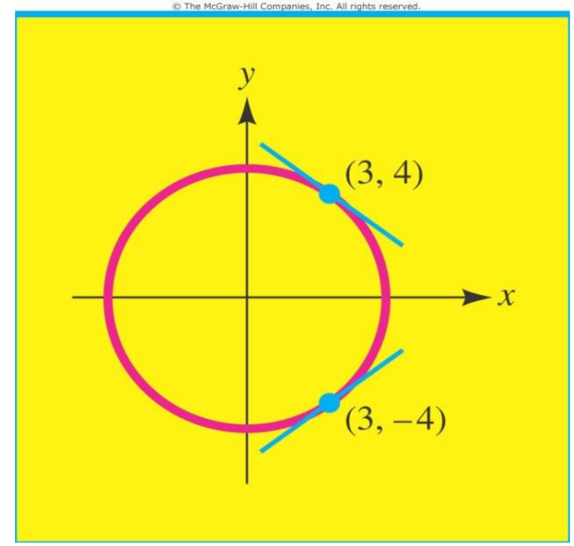
Solution:

Differentiating both sides of the equation with respect to *x*, we have

$$2x + 2y\frac{dy}{dx} = 0 \Leftrightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Thus, the slope at *(3,4)* is $\left.\frac{dy}{dx}\right|_{(3,4)} = -\frac{3}{4}$

The slope at *(3,-4)* is $\left.\frac{dy}{dx}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$



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Derivatives of the Inverse Trigonometric Functions （反三角函数求导）

The inverse sine (or arcsine) function is defined by restricting the domain of the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Specifically, we have

$$y = \sin^{-1} x \text{ if and only if } \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

Differentiating the equation *siny*=*x* implicitly, we have

$$\frac{d}{dx} \sin y = \frac{d}{dx} x$$

and so, $\cos y \dfrac{dy}{dx} = 1.$

Solving this for *dy/dx*, we find (for *cosy≠0*) that

$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

Notice that for $-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ \cos y \ge 0$ and hence,

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

This leaves us with $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}},$

for -1<x<1. That is,

$$\boxed{\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad \text{for } -1 < x < 1.}$$

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$$\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}},\quad \text{for } -1<x<1$$

$$\frac{d}{dx}\cos^{-1}x=\frac{-1}{\sqrt{1-x^2}},\quad \text{for } -1<x<1$$

$$\frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\cot^{-1}x=\frac{-1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x=\frac{1}{|x|\sqrt{x^2-1}},\quad \text{for } |x|>1$$

$$\frac{d}{dx}\csc^{-1}x=\frac{-1}{|x|\sqrt{x^2-1}},\quad \text{for } |x|>1$$

Section 2.9 The Mean Value Theorem (中值定理)

THEOREM 9.1 (Rolle's Theorem)

Suppose that f is continuous on the interval $[a, b]$, differentiable on the interval (a, b) and $f(a) = f(b)$. Then there is a number $c \in (a, b)$ such that $f'(c) = 0$.

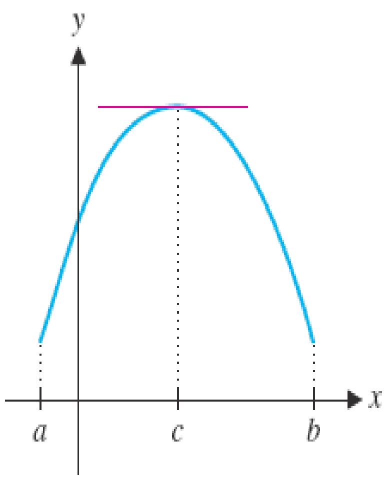

FIGURE 2.45a
Graph initially rising

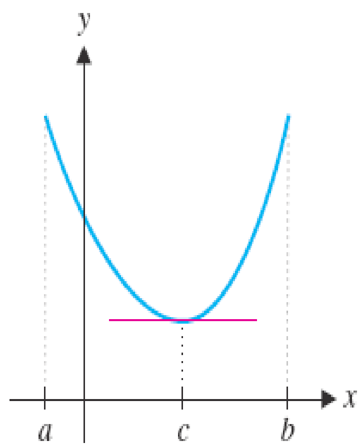

FIGURE 2.45b
Graph initially falling

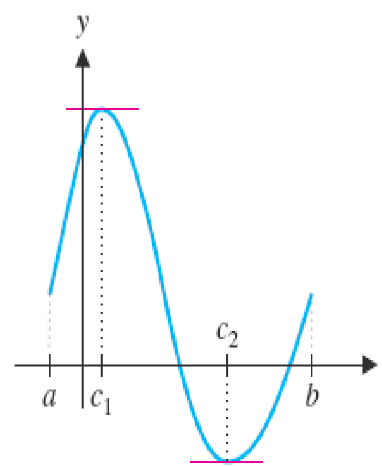

FIGURE 2.45c
Graph with two horizontal tangents

EXAMPLE 9.1 An Illustration of Rolle's Theorem

Find a value of c satisfying the conclusion of Rolle's Theorem for

$$f(x) = x^3 - 3x^2 + 2x + 2$$

on the interval [0, 1].

REMARK 9.1

We want to emphasize that example 9.1 is merely an *illustration* of Rolle's Theorem. Finding the number(s) c satisfying the conclusion of Rolle's Theorem is *not* the point of our discussion. Rather, Rolle's Theorem is of interest to us primarily because we use it to prove one of the fundamental results of elementary calculus, the Mean Value Theorem.

Rolle's Theorem can be helpful to determine how many zeros a given function has.

THEOREM 9.2

If f is continuous on the interval $[a, b]$, differentiable on the interval (a, b) and $f(x) = 0$ has two solutions in $[a, b]$, then $f'(x) = 0$ has at least one solution in (a, b).

THEOREM 9.3

For any integer $n > 0$, if f is continuous on the interval $[a, b]$ and differentiable on the interval (a, b) and $f(x) = 0$ has n solutions in $[a, b]$, then $f'(x) = 0$ has at least $(n - 1)$ solutions in (a, b).

EXAMPLE 9.2 Determining the Number of Zeros of a Function

Prove that $x^3 + 4x + 1 = 0$ has exactly one solution.

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We now generalize Rolle’s Theorem to one of the most significant results of elementary calculus.

THEOREM 9.4 (Mean Value Theorem)

Suppose that f is continuous on the interval $[a, b]$ and differentiable on the interval (a, b). Then there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{9.2}$$

EXAMPLE 9.3 An Illustration of the Mean Value Theorem

Find a value of c satisfying the conclusion of the Mean Value Theorem for

$$f(x) = x^3 - x^2 - x + 1$$

on the interval [0, 2].

Remark: The existence of *c* is not point of the theorem. In fact, these *c*'s usually remain unknown. The significance of the Mean Value Theorem is that is relates a difference of function values to the difference of the corresponding *x*-values, that is,

$$f(b) - f(a) = f'(c)(b - a).$$

Recall that for any constant *c*, $dc/dx=0$.

A question that you probably haven't thought to ask is: Are there any other functions whose derivative is zero?

THEOREM 9.5

Suppose that $f'(x) = 0$ for all x in some open interval I. Then, $f(x)$ is constant on I.

COROLLARY 9.1

Suppose that $g'(x) = f'(x)$ for all x in some open interval I. Then, for some constant c,

$$g(x) = f(x) + c, \text{ for all } x \in I.$$

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EXAMPLE 9.4 Finding Every Function with a Given Derivative

Find all functions that have a derivative equal to $3x^2 + 1$.

EXAMPLE 9.5 Proving an Inequality for $\sin x$

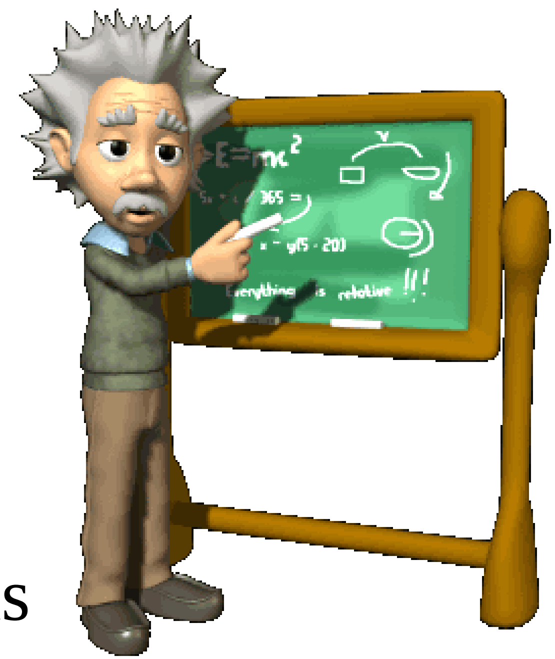
Prove that

$$|\sin a| \leq |a| \text{ for all } a.$$

Chapter 3 Applications of Differentiation

In this Chapter, we will know some important concepts.

- Newton’s Method (牛顿法)
- L’Hopital’s Rule (洛必达法则)
- Optimization (最优化)
- Increasing and Decreasing Functions
- Concavity and the Second Derivative Test (凹凸性和二阶导检验)



Section 3.1 Linear Approximations and Newton's Method

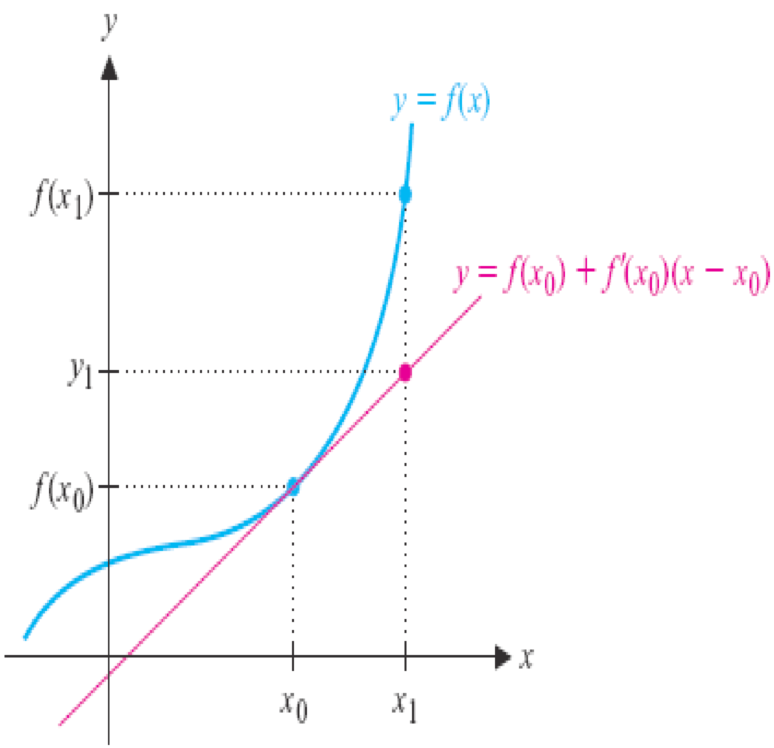


FIGURE 3.1
Linear approximation of $f(x_1)$

Linear Approximations (线性近似)

Suppose we wanted to find an approximation for $f(x_1)$, where $f(x_1)$ is unknown. For some x_0 "close" to x_1 , we know the value $f(x_0)$.

DEFINITION 1.1

The **linear** (or **tangent line**) **approximation** of $f(x)$ at $x = x_0$ is the function $L(x) = f(x_0) + f'(x_0)(x - x_0)$.

We define $\triangle x$ and $\triangle y$ by

$$\Delta x = x_1 - x_0 \quad \text{and} \quad \Delta y = f(x_1) - f(x_0).$$

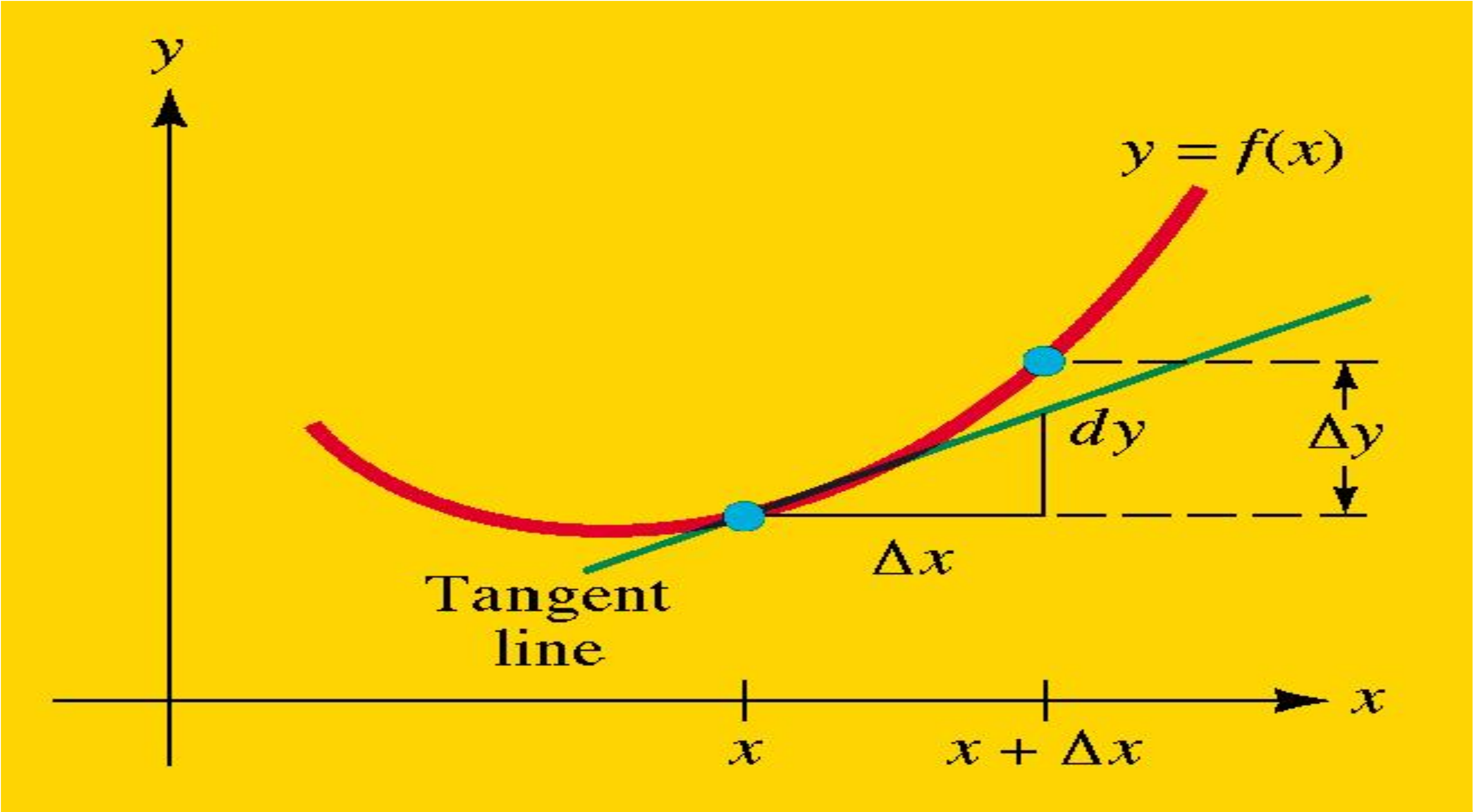
Using this notation, we have the approximation

$$\boxed{f(x_1) \approx y_1 = f(x_0) + f'(x_0)\Delta x.}$$

So we have

$$\Delta y = f(x_1) - f(x_0) \approx f'(x_0)\,\Delta x = dy$$

Differentials(微分): The differential of x is $dx=\triangle x$, and if $y=f(x)$ is a differentiable function of x, then $dy=f'(x)dx$ is the differential of y.



EXAMPLE 1.1 Finding a Linear Approximation

Find the linear approximation to $f(x) = \cos x$ at $x_0 = \pi/3$ and use it to approximate $\cos(1)$.

EXAMPLE 1.2 Linear Approximation of sin x

Find the linear approximation of $f(x) = \sin x$, for x close to 0.

EXAMPLE 1.3 Linear Approximation to Some Cube Roots

Use a linear approximation to approximate $\sqrt[3]{8.02}$, $\sqrt[3]{8.07}$, $\sqrt[3]{8.15}$ and $\sqrt[3]{25.2}$.

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Taylor Polynomials Approximation

To approximate a function f by a quadratic function P near a number a, it is best to write P in the form

$$P(x) = A + B(x - a) + C(x - a)^2$$

Show that the quadratic function that satisfies conditions (i), (ii), and (iii) is

$$P(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2$$

Instead of being satisfied with a linear or quadratic approximation to $f(x)$ near $x = a$, let's try to find better approximations with higher-degree polynomials. We look for an nth-degree polynomial

$$T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots + c_n(x - a)^n$$

such that T_n and its first n derivatives have the same values at $x = a$ as f and its first n derivatives. By differentiating repeatedly and setting $x = a$, show that these conditions are satisfied if $c_0 = f(a)$, $c_1 = f'(a)$, $c_2 = \frac{1}{2}f''(a)$, and in general

$$c_k = \frac{f^{(k)}(a)}{k!}$$

where $k! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot k$. The resulting polynomial

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

is called the **nth-degree Taylor polynomial of f centered at a.**

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Newton's Method (牛顿法) (Newton-Raphson method)

HISTORICAL NOTES

Sir Isaac Newton (1642–1727) An English mathematician and scientist known as the co-inventor of calculus. In a 2-year period from 1665 to 1667, Newton made major discoveries in several areas of calculus, as well as optics and the law of gravitation. Newton's mathematical results were not published in a timely fashion. Instead, techniques such as Newton's method were quietly introduced as useful tools in his scientific papers. Newton's *Mathematical Principles of Natural Philosophy* is widely regarded as one of the greatest achievements of the human mind.

We now return to the question of finding zeros of a function.

Here, we explore a method that is usually much more efficient than bisection method.

The question is

how are we to find them (roots of equation or zeros of the function f) ?

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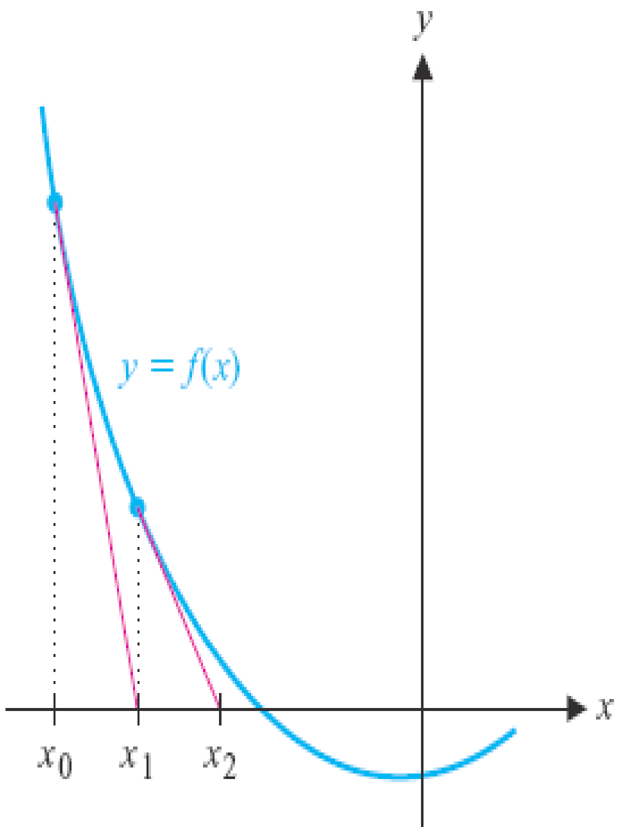

FIGURE 3.8
Newton's method

To find approximate solutions to *f(x)=0.*

a. We first make an initial guess, denoted x_0*, of the location of a solution.*

b. We get $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}.$

c. Using x_1 *as our new guess, we should produce a further improved approximation.*

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

d. In this way, we generate a sequence of successive approximations determined by

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{for } n = 0, 1, 2, 3, \ldots .}$$

The approximate solution will be generated until convergence.

Remark: Newton’s method does not always work. Newton’s method is sensitive to the initial guess.

EXAMPLE 1.5 Using Newton's Method to Approximate a Zero

Find a zero of $f(x) = x^5 - x + 1$.

EXAMPLE 1.6 Using Newton's Method to Approximate a Cube Root

Use Newton's method to approximate $\sqrt[3]{7}$.

EXAMPLE 1.7 The Effect of a Bad Guess on Newton's Method

Use Newton's method to find an approximate zero of $f(x) = x^3 - 3x^2 + x - 1$.

Section 3.2 Indeterminate Forms and L'Hopital's Rule （不定式与洛必达法则）

We reconsider the problem of computing limits as follows

$$\lim_{x\to a}\frac{f(x)}{g(x)},$$

where $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or where $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty\ (\text{or} -\infty).$

We call this form as *indeterminate forms.*

L’Hopital’s Rule (洛必达法则)

THEOREM 2.1 (L’Hôpital’s Rule)

Suppose that f and g are differentiable on the interval (a, b), except possibly at some fixed point $c \in (a, b)$ and that $g'(x) \neq 0$ on (a, b), except possibly at c. Suppose further that $\lim_{x \to c} \frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and that $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$ (or $\pm\infty$). Then,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

REMARK 2.1

The conclusion of Theorem 2.1 also holds if $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$ is replaced with any of the limits $\lim\limits_{x\to c^+} \dfrac{f(x)}{g(x)}$, $\lim\limits_{x\to c^-} \dfrac{f(x)}{g(x)}$, $\lim\limits_{x\to \infty} \dfrac{f(x)}{g(x)}$ or $\lim\limits_{x\to -\infty} \dfrac{f(x)}{g(x)}$. (In each case, we must make appropriate adjustments to the hypotheses.)

EXAMPLE 2.1 The Indeterminate Form $\frac{0}{0}$

Evaluate $\lim_{x\to 0} \dfrac{1-\cos x}{\sin x}$.

EXAMPLE 2.3 A Limit Requiring Two Applications of L'Hôpital's Rule

Evaluate $\lim_{x\to\infty} \dfrac{x^2}{e^x}$.

EXAMPLE 2.4 An Erroneous Use of L'Hôpital's Rule

Find the mistake in the string of equalities

$$\lim_{x\to 0} \frac{x^2}{e^x-1} = \lim_{x\to 0} \frac{2x}{e^x} = \lim_{x\to 0} \frac{2}{e^x} = \frac{2}{1} = 2.$$

EXAMPLE 2.9 The Indeterminate Form 1^∞

Evaluate $\lim_{x \to 1^+} x^{\frac{1}{x-1}}$.

EXAMPLE 2.10 The Indeterminate Form 0^0

Evaluate $\lim_{x \to 0^+} (\sin x)^x$.

EXAMPLE 2.11 The Indeterminate Form ∞^0

Evaluate $\lim_{x \to \infty} (x+1)^{2/x}$.

Section 3.3 Maximum and Minimum Values(最大和最小值)

DEFINITION 3.1

For a function f defined on a set S of real numbers and a number $c \in S$,

(i) $f(c)$ is the **absolute maximum** of f on S if $f(c) \geq f(x)$ for all $x \in S$ and

(ii) $f(c)$ is the **absolute minimum** of f on S if $f(c) \leq f(x)$ for all $x \in S$.

An absolute maximum or an absolute minimum is referred to as an **absolute extremum.** If a function has more than one extremum, we refer to these as **extrema** (the plural form of extremum).

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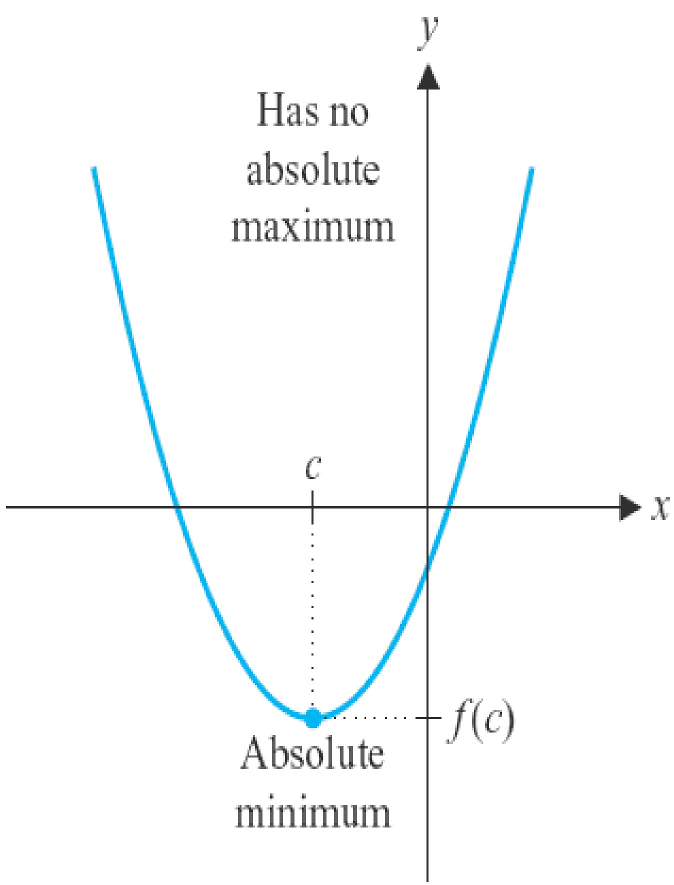

FIGURE 3.25a

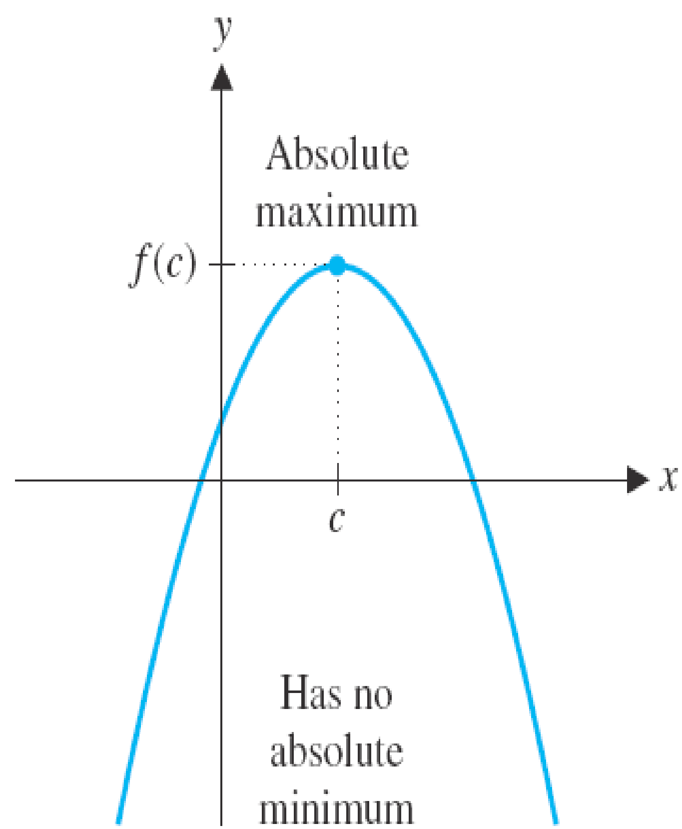

FIGURE 3.25b

EXAMPLE 3.1 Absolute Maximum and Minimum Values

(a) Locate any absolute extrema of $f(x) = x^2 - 9$ on the interval $(-\infty, \infty)$. (b) Locate any absolute extrema of $f(x) = x^2 - 9$ on the interval $(-3, 3)$. (c) Locate any absolute extrema of $f(x) = x^2 - 9$ on the interval $[-3, 3]$.

EXAMPLE 3.2 A Function with No Absolute Maximum or Minimum

Locate any absolute extrema of $f(x) = 1/x$, on the interval $[-3, 3]$.

THEOREM 3.1 (Extreme Value Theorem) （极值定理）

A continuous function f defined on a *closed, bounded* interval $[a, b]$ attains both an absolute maximum and an absolute minimum on that interval.

Our objective is to determine how to locate the absolute extrema of a given function. Before we do this, we need to consider an additional type of extremum.

DEFINITION 3.2 (局部极值)

(i) $f(c)$ is a **local maximum** of f if $f(c) \geq f(x)$ for all x in some *open* interval containing c.

(ii) $f(c)$ is a **local minimum** of f if $f(c) \leq f(x)$ for all x in some *open* interval containing c.

In either case, we call $f(c)$ a **local extremum** of f.

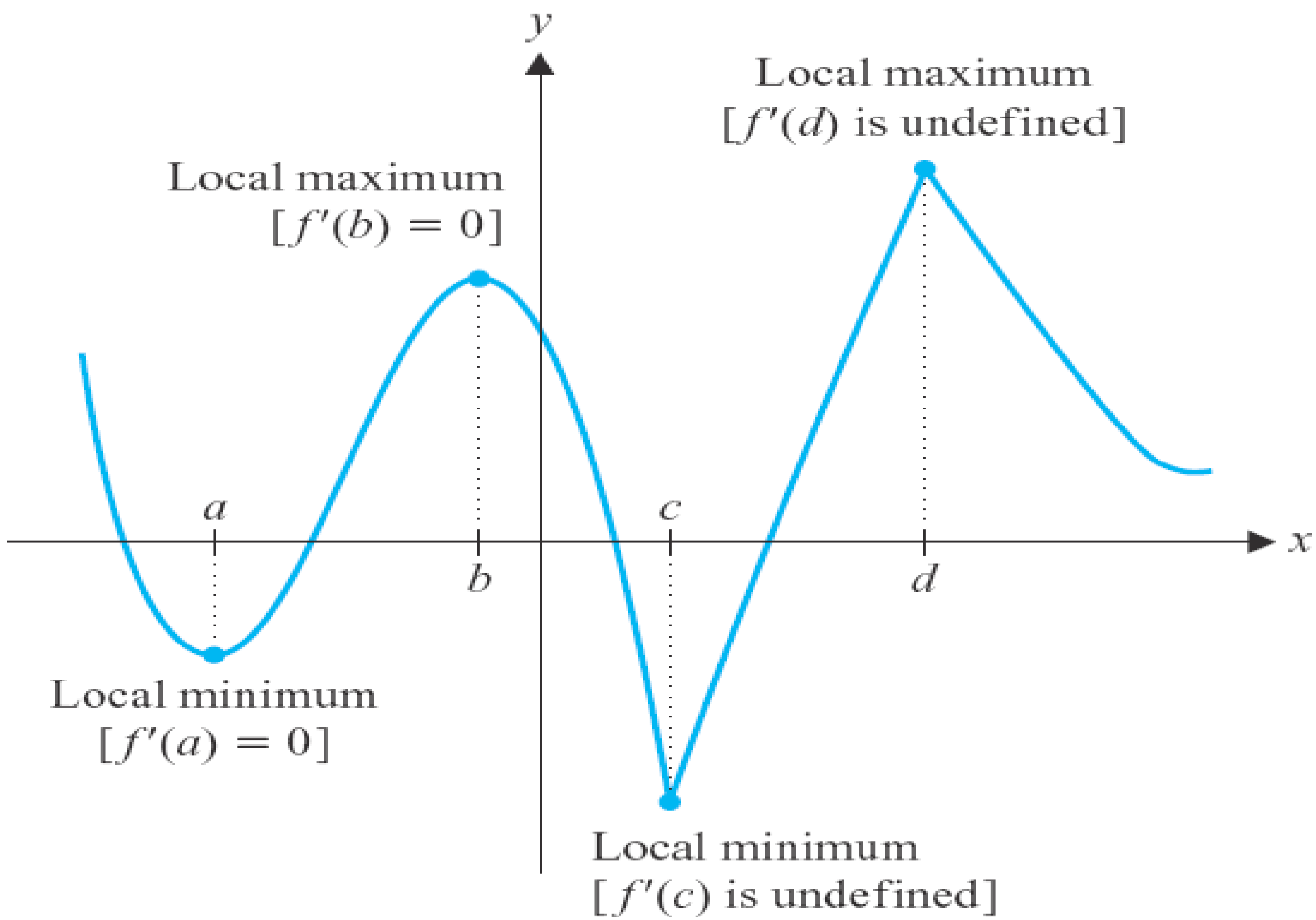

FIGURE 3.30
Local extrema

EXAMPLE 3.4 A Function with a Zero Derivative at a Local Maximum

Locate any local extrema for $f(x) = 9 - x^2$ and describe the behavior of the derivative at the local extremum.

EXAMPLE 3.5 A Function with an Undefined Derivative at a Local Minimum

Locate any local extrema for $f(x) = |x|$ and describe the behavior of the derivative at the local extremum.

DEFINITION 3.3 （关键点）

A number c in the domain of a function f is called a **critical number** of f if $f'(c) = 0$ or $f'(c)$ is undefined.

Relative (local) extrema can only occur at critical numbers!

Not all critical points correspond to relative（local）extrema!

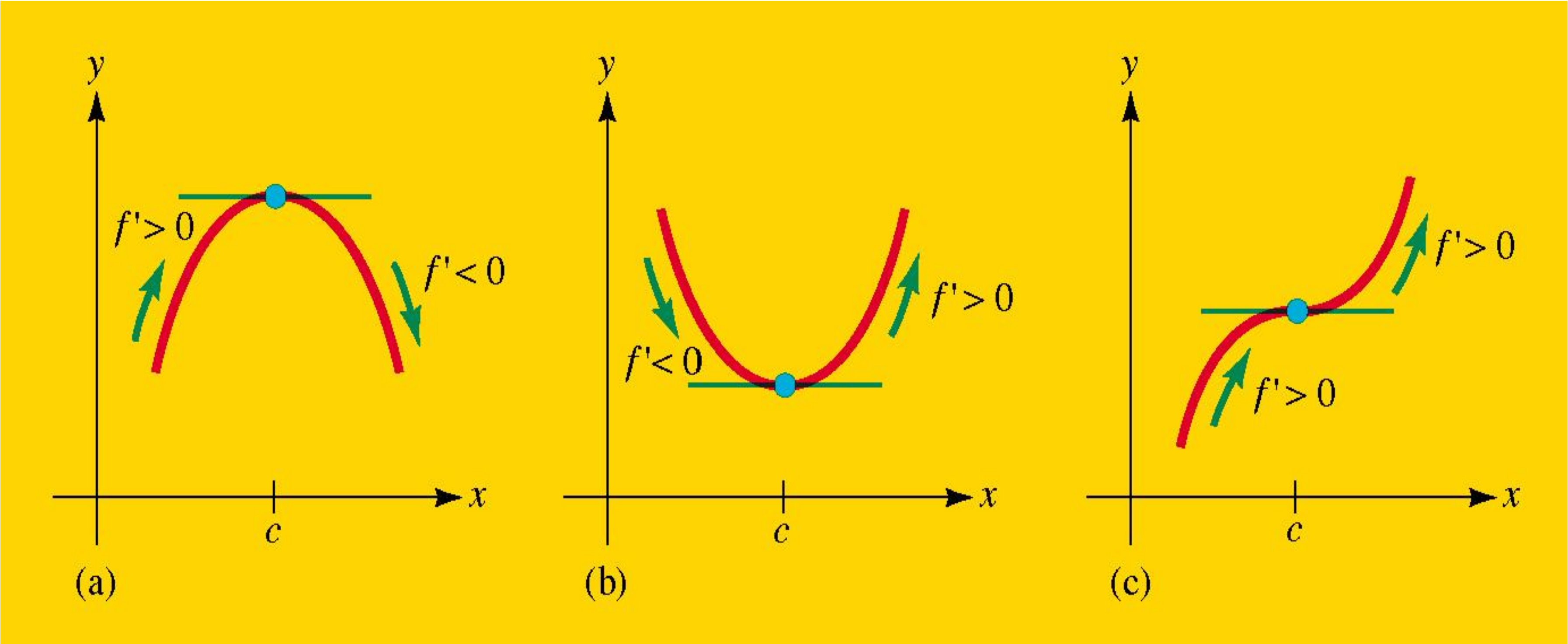

Figure. Three critical points where $f'(x) = 0$: (a) relative maximum, (b) relative minimum (c) not a relative extremum.

Not all critical points correspond to relative (local) extrema!

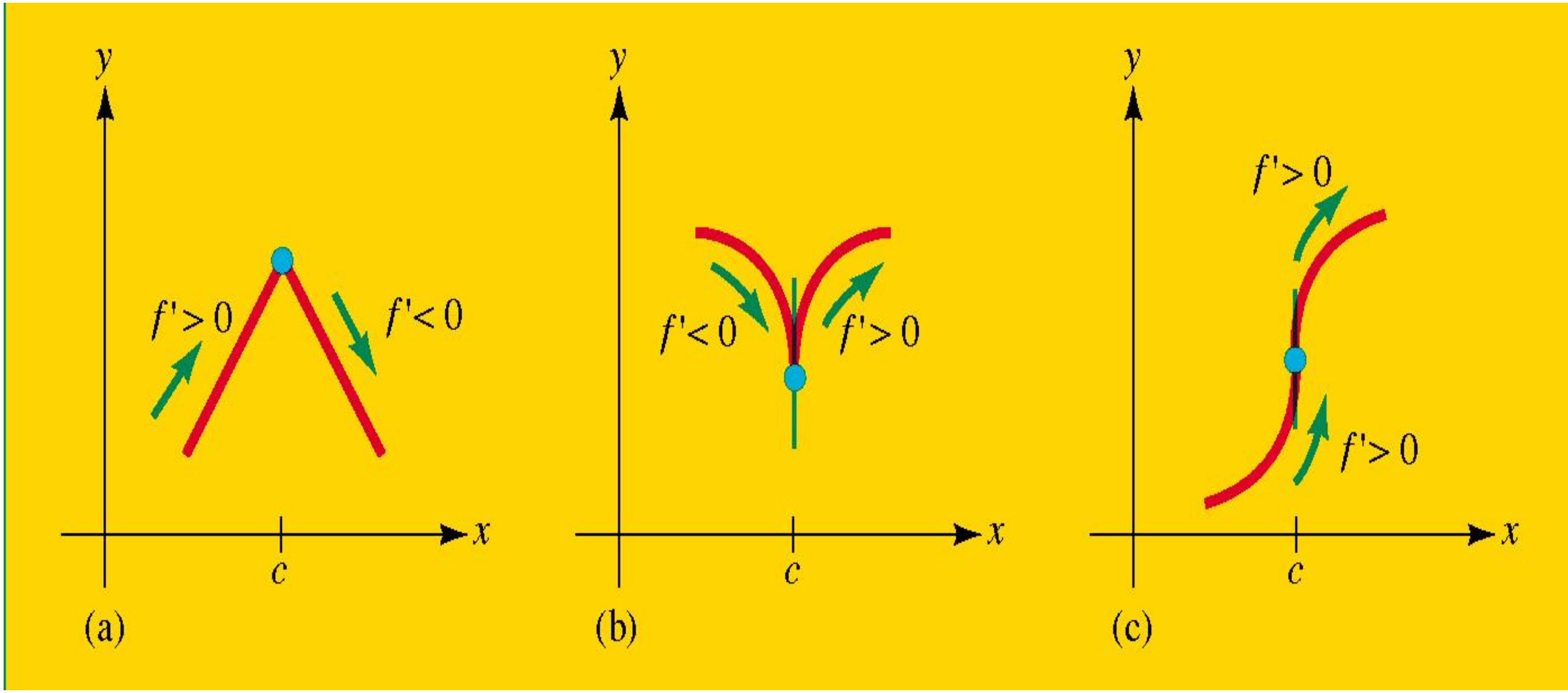

Figure Three critical points where $f'(x)$ is undefined:(a) relative maximum, (b) relative minimum (c) not a relative extremum.

THEOREM 3.2 (Fermat's Theorem) (费马定理)

Suppose that $f(c)$ is a local extremum (local maximum or local minimum). Then c must be a critical number of f.

EXAMPLE 3.6 Finding Local Extrema of a Polynomial

Find the critical numbers and local extrema of $f(x) = 2x^3 - 3x^2 - 12x + 5$.

EXAMPLE 3.7 An Extremum at a Point Where the Derivative Is Undefined

Find the critical numbers and local extrema of $f(x) = (3x + 1)^{2/3}$.

EXAMPLE 3.10 Finding Critical Numbers of a Rational Function

Find all the critical numbers of $f(x) = \dfrac{2x^2}{x+2}$.

THEOREM 3.3

Suppose that f is continuous on the closed interval $[a, b]$. Then, the absolute extrema of f must occur at an endpoint (a or b) or at a critical number.

REMARK 3.3

Theorem 3.3 gives us a simple procedure for finding the absolute extrema of a continuous function on a closed, bounded interval:

1. Find all critical numbers in the interval and compute function values at these points.
2. Compute function values at the endpoints.
3. The largest function value is the absolute maximum and the smallest function value is the absolute minimum.

EXAMPLE 3.11 Finding Absolute Extrema on a Closed Interval

Find the absolute extrema of $f(x) = 2x^3 - 3x^2 - 12x + 5$ on the interval $[-2, 4]$.