PT Assignment 7

1. A random variable X has the density function

$$f(x) = \begin{cases} cx, & \text{if } 0 < x < 4, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine c.

   (b) Compute $P(1 \le X \le 2)$.

   (c) Determine EX and $\mathrm{Var}(X)$.

2. After your complaint about their service, a representative of an insurance company promised to call you "between 7 and 9 this evening." Assume that this means that the time T of the call is uniformly distributed in the specified interval.

   (a) Compute the probability that the call arrives between 8:00 and 8:20.

   (b) At 8:30, the call still hasn't arrived. What is the probability that it arrives in the next 10 minutes?

   (c) Assume that you know in advance that the call will last exactly 1 hour. From 9 to 9:30, there is a game show on TV that you wanted to watch. Let M be the amount of time of the show that you miss because of the call. Compute the expected value of M.

3. An insurance company offers snow insurance, which pays nothing if the daily snow fall is below 2 inches, and for higher level of snow fall it pays an increasing amount, changing linearly from \$0 for 2 inches of snow to a maximum of \$2000, with paid rate \$250/inch of snow above 2 inches. The amount of snow falling in a given day during the policy term follows an exponential distribution with mean 0.5. Find the expected value of the amount of the claim under this snow insurance policy.

4. Find the variance of the following distributions:

   (a) Uniform: X is uniform on (a, b), i.e., $f_X(x) = \begin{cases} \frac{1}{b-a} & x \in (a, b) \\ 0 & \text{otherwise} \end{cases}$

   (b) Gamma: X has Gamma distribution with parameter (α, β), i.e.,

$$f_X(x) = \begin{cases} \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

   where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$. Show that $E\left[X^k\right] = \frac{\Gamma(\alpha+k)}{\beta^k \Gamma(\alpha)}$ $(k \ge 0)$. Hence find $\mathrm{Var}[X]$.

(c) Normal: X has Normal distribution $N(\mu, \sigma^2)$, i.e.,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty.$$

Show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2\sigma^2}} dxdy = 2\pi\sigma^2$. (Use polar coordinates) Hence show that $\int_{-\infty}^{\infty} f_X(x)dx = 1$. Furthermore, Show that

$$E\left[\left(\frac{X-\mu}{\sigma}\right)^{2n+1}\right] = 0 \text{ and } E\left[\left(\frac{X-\mu}{\sigma}\right)^{2n}\right] = \frac{(2n)!}{2^n n!} = (2n-1)(2n-3)\cdots 3\cdot 1.$$

Hence find $E[X]$ and $\text{Var}[X]$.

5. A random variable X has the density function

$$f(x) = \begin{cases} c(x+\sqrt{x}) & x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine c. (b) Compute $E(1/X)$. (c) Determine the probability density function of $Y = X^2$.

6. The density function of a random variable X is given by

$$f(x) = \begin{cases} a+bx, & 0 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

We also know that $E(X) = 7/6$. (a) Compute a and b. (b) Compute $\text{Var}(X)$.

7. Let X be a continuous random variable such that its probability density function is

$$f(x) = \begin{cases} c\left(1 - \frac{1}{2\sqrt{x}}\right), & 1 \le x \le 16, \\ 0, & \text{otherwise.} \end{cases}$$

Find the distribution function of X.

8. X is a random variable with probability density function

$$f_X(x) = \begin{cases} e^{-2x}, & x \ge 0 \\ 2e^{4x}, & x < 0. \end{cases}$$

Let $T = X^2$. Determine the probability density function for T.