# Solution to Assignment 7

1. Let $N(t) = N_1(t) + N_2(t)$.

(i) $N(0) = N_1(0) + N_2(0) = 0 + 0 = 0$.

(ii) For each $n \in \mathbb{N}$ and each $0 \le t_0 < t_1 < \cdots < t_n$, $N(t_1) - N(t_0)$, $N(t_2) - N(t_1)$, $\cdots$, $N(t_n) - N(t_{n-1})$ are independent since $N(t_i) - N(t_{i-1}) = [N_1(t_i) - N_1(t_{i-1})] + [N_2(t_i) - N_2(t_{i-1})]$ $(i \ge 1)$, are independent to each other with different $i$.

(iii) For all $s, t \ge 0$,

$$
\begin{aligned}
P[N(t+s) - N(s) = k] &= P[N_1(t+s) - N_1(s) + N_2(t+s) - N_2(s) = k] \\
&= \sum_{i=0}^{k} P[N_1(t+s) - N_1(s) = i, N_2(t+s) - N_2(s) = k-i] \\
&= \sum_{i=0}^{k} P[N_2(t+s) - N_2(s) = k-i \mid N_1(t+s) - N_1(s) = i]\, P[N_1(t+s) - N_1(s) = i] \\
&= \sum_{i=0}^{k} P[N_2(t+s) - N_2(s) = k-i]\, P[N_1(t+s) - N_1(s) = i] \\
&= \sum_{i=0}^{k} \frac{(\lambda_2 t)^{(k-i)}}{(k-i)!} e^{-\lambda_2 t} \cdot \frac{(\lambda_1 t)^i}{i!} e^{-\lambda_1 t} \\
&= e^{-(\lambda_1+\lambda_2)t} \frac{1}{k!} \sum_{i=0}^{k} \frac{k!}{(k-i)!\, i!} (\lambda_2 t)^{(k-i)} (\lambda_1 t)^i \\
&= e^{-(\lambda_1+\lambda_2)t} \frac{1}{k!} (\lambda_2 t + \lambda_1 t)^k \\
&= e^{-(\lambda_1+\lambda_2)t} \frac{1}{k!} [(\lambda_1+\lambda_2)t]^k
\end{aligned}
$$

(i), (ii) and (iii) indicate that $\{N(t) = N_1(t) + N_2(t), t \ge 0\}$ is a Poisson process with rate $(\lambda_1+\lambda_2)$.

2. Denote $N(t)$ as the number of tickets the policewoman writes in $[0,t]$. $N(t) \sim Poi(6t)$. Denote $Y_i$ as the cost of the $i^{th}$ ticket. The distribution of $Y_i$ is

| $Y_i$ | 100 | 400 |
|---|---|---|
| $P$ | $2/3$ | $1/3$ |

. Denote $S(t) = \begin{cases} 0, & N(t) = 0; \\ \sum_{i=1}^{N(t)} Y_i, & N(t) > 0. \end{cases}$

(a) According to theorem 12 in the slide,

$$\mathbb{E}[S(1)] = \lambda \mathbb{E}[Y_i] = 6 \times (100 \times 2/3 + 400 \times 1/3) = 1200$$

$$\sqrt{D[S(1)]} = \sqrt{\lambda \mathbb{E}[Y_i^2]} = \sqrt{6 \times (100^2 \times 2/3 + 400^2 \times 1/3)} = 600$$

(b) Let $N_1(t)$ and $N_2(t)$ be the number of ticket for speeding and DWI up to time $t$ respectively. Then $\{N_1(t), t \ge 0\}$ is a Poisson process with rate $\lambda_1 = 6 \times 2/3 = 4$ and $\{N_2(t), t \ge 0\}$ is a Poisson process with rate

$\lambda_2 = 6 \times 1/3 = 2$. Also $\{N_1(t), t \geq 0\}$ is independent to $\{N_2(t), t \geq 0\}$. So

$$P[N_1(3) - N_1(2) = 5, N_2(3) - N_2(2) = 1] = P[N_1(3) - N_1(2) = 5]\, P[N_2(3) - N_2(2) = 1]$$

$$= e^{-4} \cdot \frac{4^4}{5!} \cdot e^{-2} \cdot \frac{2^1}{1!} = e^{-6}\, \frac{4^4}{15}$$

(c) $P(A) = P[N(1.5) - N(1) = 0] = \frac{3^0}{0!} e^{-3} = e^{-3} \approx 0.04979$

$$P(A \mid N=5) = \frac{P(A, N=5)}{P(N=5)} = \frac{P[N(1.5) - N(1) = 0,\ N(2) - N(1) = 5]}{P(N(2) - N(1) = 5]}$$

$$= \frac{P[N(1.5) - N(1) = 0,\ N(2) - N(1.5) = 5]}{P[N(2) - N(1) = 5]} = \frac{e^{-3} \cdot \frac{3^5}{5!} e^{-3}}{e^{-6} \cdot 6^5/5!} = (1/2)^5 = 0.03125 < P(A)$$

3. Denote $N(t)$ as the number of road accidents in time $[0, t]$. Then $N(t) \sim Poi(0.1t)$. Denote $Y_i$ as the number of casualties in the $i^{th}$ accident. $Y_i \sim Bin(4, 0.05)$. Denote

$$S(t) = \begin{cases} 0, & N(t) = 0; \\ \sum_{i=1}^{N(t)} Y_i, & N(t) > 0. \end{cases}$$

$$\mathbb{E}[S(30)] = 30\lambda\, \mathbb{E}[Y_i] = 30 \times 0.1 \times (4 \times 0.05) = 0.6$$

$$\mathbb{D}[S(30)] = 30\lambda\, \mathbb{E}[Y_i^2] = 30 \times 0.1 \times [4 \times 0.05 \times 0.95 + (4 \times 0.05)^2] = 0.69$$