MS Week12

2024年5月10日 星期五 08:08

decision \ fact	H_0 is true	H_1 true
Reject H_0		✓
Fail to reject H_0	✓	

Prob. of Type I error : $P(\text{Reject } H_0 \mid H_0 \text{ is true}) = \alpha$ — The significance level

decision Rule — $\theta = \theta_0$ in H_0.

Prob. of Type II error : $P(\text{Fail to reject } H_0 \mid H_1 \text{ is true}) = \beta$

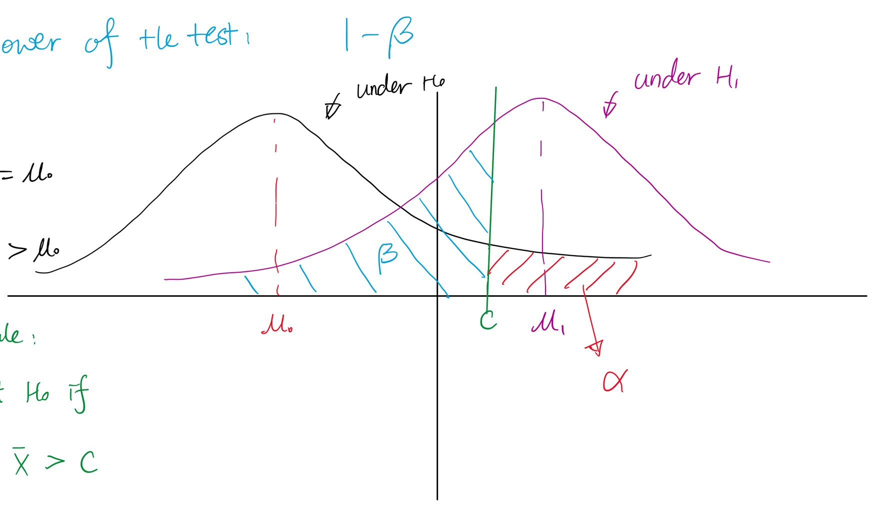
The Power of the test : $1-\beta$

E.g.

$H_0: \mu = \mu_0$

v.s.

$H_1: \mu > \mu_0$



Decision Rule :

Reject H_0 if

$\bar{X} > C$

△ Best rejection region of level α.

def 4.4.1 $H_0: \theta = \theta_0$ v.s. $H_1: \theta = \theta_1$ simple hypotheses

"Best" : $P(C \mid H_1) \geq P(D \mid H_1)$

"largest power"

C is chosen by comparing $L(\theta_0)$ with $L(\theta_1)$

The points in the "best" critical region are selected by finding the smallest value of $\dfrac{L(\vec{x} \mid \theta_0)}{L(\vec{x} \mid \theta_1)}$

$$C = \left\{ \vec{x} \;\middle|\; \frac{L(\vec{x} \mid \theta_0)}{L(\vec{x} \mid \theta_1)} \text{ has as small values as possible} \right\}$$

△ Neymann – Pearson Lemma

(i) $\dfrac{L(\vec{x} \mid \theta_0)}{L(\vec{x} \mid \theta_1)} \leq k$? for $\vec{x} \in C$

(ii) $P(\vec{X} \in C \mid \theta_0) = \alpha$

↳ The critical value C^* for Test statistic

△ Some Extension :

$H_0: \theta = \theta_0$ v.s. $H_1: \theta > \theta_0$

simple — composite

modify the method as .

$$\frac{L(\vec{x} \mid \theta_0)}{L(\vec{x} \mid \theta_1)} < k \quad \xleftrightarrow{\text{transform}} \quad \text{The critical value for } T(\vec{X})$$

test statistic (known distribution)

where $\theta_1 > \theta_0$ is a value in the alternative hypothesis.

$$P(\vec{X} \in C \mid H_0) = \alpha$$

Remark : In this case, the critical region C is called the "uniformly most powerful" critical region of level α.

Remark : Not every test has the uniformly most powerful critical region of level α.

For instance

$H_0: \theta = \theta_0$ v.s. $H_1: \theta \neq \theta_0$

Section 1 : Some examples

① Tesing on μ with known σ^2 (normal population)

Test-statistic : $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

or $Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$ — Z-test

one-sided test or two-sided test depends on H_1.

$H_1: \mu > \mu_0$,	$Z > Z_\alpha$	$P(Z > z)$
$H_1: \mu < \mu_0$,	$Z < -Z_\alpha$	$P(Z < z)$ $< \alpha$
$H_1: \mu \neq \mu_0$,	$\lvert Z \rvert > Z_{\alpha/2}$	$P(\lvert Z \rvert > z)$
	critical values method.	p-values $< \alpha$

② Testing μ with unknown σ^2 (normal population)

Test statistic : $\bar{X} \sim N(\mu, \sigma^2)$ ✗.

$$\frac{Z}{\sqrt{U/df}} = T = \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1) \quad \text{— t-test}$$

$$T = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2} / (n-1)}} = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{S}$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2, \quad \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

③ Testing μ with known σ^2

Use clt to approximate the distribution of $\bar{X} \,\dot{\sim}\, N(\mu, \frac{\sigma^2}{n})$

④ Testing σ^2 (normal population)

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2, \quad \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) \quad \text{— } \chi^2\text{-test}$$

Section 2 : Binomial Data Binom(n, p)

$H_0: p = p_0$ v.s. H_1 : $p > p_0$

$p < p_0$

$p \neq p_0$.

A random sample of $X_1, \cdots, X_n$ from a Bernoulli distribution with p.

↳ A large sample test ⟶ $\bar{X} = \frac{1}{n}\sum X_i \,\dot{\sim}\, N(p, \frac{p(1-p)}{n})$

$0 < np_0 - 3\sqrt{np_0(1-p_0)} < np_0 + 3\sqrt{np_0(1-p_0)} < n$

or $Z = \dfrac{\bar{X} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \,\dot{\sim}\, N(0,1)$ — Z-test

↳ A small sample test ⟶ $X = \sum_{i=1}^{n} X_i \sim \text{binom}(n, p)$

Section 3 : α, β, $1-\beta$.

Type I :

$\alpha = P(\text{Reject } H_0 \mid H_0)$

Type II :

$\beta = P(\text{Fail to reject } H_0 \mid H_1)$ — as a function of θ

power :

$1-\beta = P(\text{Reject } H_0 \mid H_1)$ — as a function of θ.