2022-23 First Semester

MATH1063 Linear Algebra II (1003)

Assignment 2

Due Date: **10/Mar/2023 (Friday), 09:00 in tutorial class.**

- Write down your **CHN name** and **student number**. Write neatly on **A4-sized** paper (*staple if necessary*) and **show your steps**.
- **Late submissions or answers without steps won't be graded.**

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1. Let $L : P_3 \to P_3$ be the linear transformation defined by
$$L[p(x)] = p(2x-1).$$
   (a) Find the matrix representing L with respect to the standard basis $\alpha = \{1, x, x^2\}$.
   (b) Find the matrix representing L with respect to the basis $\beta = \{1, x-1, (x-1)^2\}$.
   (c) (Optional!) Find the matrix representing L^k with respect to the basis $\alpha = \{1, x, x^2\}$.

2. Let A and B be similar matrices and let λ be any scalar. Show that
   (a) $A - \lambda I$ and $B - \lambda I$ are similar.
   (b) $\det(A - \lambda I) = \det(B - \lambda I)$.

3. Consider an $n \times k$ matrix A and a $k \times m$ matrix B
   (a) What is the relationship between $\ker(AB)$ and $\ker(B)$? (Always equal? One is always contained in the other?)
   (b) What is the relationship between the image of A and image of AB?

4. (a) Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ be linearly independent vectors in $\mathbb{R}^n$. Let A be an $n \times n$ nonsingular matrix. Show that $A\mathbf{v}_1, A\mathbf{v}_2, \cdots, A\mathbf{v}_k$ are linearly independent.
   (b) Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ be linearly independent vectors in $\mathbb{R}^n$. Let A be an $m \times n$ matrix $(m > n)$, $\text{rank}(A) = n$. Show that $A\mathbf{v}_1, A\mathbf{v}_2, \cdots, A\mathbf{v}_k$ are linearly independent.
   (c) (*Optional!*) Let $L : V \to W$ be a linear transformation with $\dim V = n$ and $\dim W = m$ $(m \geq n)$. Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ be linearly independent vectors in V. Under what condition, $\{L(\mathbf{v}_1), L(\mathbf{v}_2), \cdots, L(\mathbf{v}_k)\}$ is linearly independent?
   Make a conjecture based on part (a) and (b), then prove your conjecture.

5. Consider the vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$,
$$\mathbf{v} = (1, 1, \cdots, 1)^T, \quad \mathbf{w} = (1, 0, \cdots, 0)^T.$$
   Express the angle θ between $\mathbf{v}$ and $\mathbf{w}$ in terms of n; then find the limit of θ as $n \to \infty$.

6. Find the vector projection of $\mathbf{y}$ onto $\mathbf{x}$
   (a). $\mathbf{x} = (4, 1)^T$, $\mathbf{y} = (3, 2)^T$. (b). $\mathbf{x} = (-2, 3, 1)^T$, $\mathbf{y} = (1, 2, 4)^T$.