2022-23 First Semester
MATH1063 Linear Algebra II (1003)

Assignment 3 Suggested Solutions

1. Transition matrix V corresponding to a change of basis from$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ to $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$

$$V = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

then V^{-1} is a transition matrix corresponding to a change of basis from $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ to $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$

$$V^{-1} = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 1 & 3 \end{pmatrix}$$

thus, the matrix B representing L with respect to $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is

$$B = V^{-1}AV = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & -2 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 0 & -4 \\ 6 & 1 & 4 \\ 8 & 0 & 7 \end{pmatrix}.$$

2. (a) $\|k\mathbf{v}\|^2 = (k\mathbf{v})^T(k\mathbf{v}) = k^2(\mathbf{v}^T\mathbf{v}) = k^2\|\mathbf{v}\|^2$.
Take square roots of both sides; note that $\sqrt{k^2} = |k|$. Thus $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$.

(b) $\|\mathbf{u}\| = \underbrace{\left\| \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\|}_{\text{Based on (a)}} = 1$, as claimed.

3.

$$\mathbf{z}^T\mathbf{p} = \mathbf{x}^T\mathbf{p} - \mathbf{p}^T\mathbf{p} = \frac{(\mathbf{x}^T\mathbf{y})^2}{\mathbf{y}^T\mathbf{y}} - \left(\frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\right)^2 \mathbf{y}^T\mathbf{y} = 0.$$

4. No. For example, let $\mathbf{x}_1 = \mathbf{e}_1$, $\mathbf{x}_2 = \mathbf{e}_2$, $\mathbf{x}_3 = 2\mathbf{e}_1$, then $\mathbf{x}_1 \perp \mathbf{x}_2$, $\mathbf{x}_2 \perp \mathbf{x}_3$, but $\mathbf{x}_1$ is not orthogonal to $\mathbf{x}_3$.

5. By the plane equation, we know that a normal vector to the plane is $\mathbf{n} = (6, 2, 3)^T$ and the point $Q(1, 3, -2)$ lies in the plane. Then the distance from the point $P(2, 1, -2)$ to the plane is the absolute value of the scalar projection of $\overrightarrow{PQ} = (-1, 2, 0)^T$ onto $\mathbf{n}$.

$$d = \left| \frac{\overrightarrow{PQ} \cdot \mathbf{n}}{\|\mathbf{n}\|} \right| = \left| \frac{-2}{\sqrt{49}} \right| = \frac{2}{7}.$$

6. Let $\mathbf{v} = (1,2)^T$. To find the distance, we need to find the scalar projection of $\mathbf{v}$ onto a normal vector to the line $4x - 3y = 0$, say $\mathbf{n} = (-4,3)^T$:

$$\alpha = \frac{\mathbf{v}^T\mathbf{n}}{\|\mathbf{n}\|} = \frac{(1,2)(-4,3)^T}{5} = 0.4$$

7. Pick two points $(0,2)$ and $(2,6)$ on the line $y = 2x+2$.

   Let $\mathbf{v} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ be the vector from $(0,2)$ to the point $(5,2)$, and $\mathbf{w} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ be a vector along the line $y = 2x+2$, then the vector projection of $\mathbf{v}$ onto $\mathbf{w}$ is

$$\vec{p} = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}^T\mathbf{w}}\mathbf{w} = \frac{(5,0)(2,4)^T}{(2,4)(2,4)^T}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

   Thus, the closest point to $(5,2)$ on the line $y = 2x+2$ should be the point $(0,2)+\vec{p} = (1,4)$.

8. $Y^\perp$ is not empty since $\mathbf{0} \in Y^\perp$. If $\mathbf{x} \in Y^\perp$ and $\alpha \in \mathbb{R}$, then for all $\mathbf{y} \in Y$,

$$(\alpha\mathbf{x})^T\mathbf{y} = \alpha(\mathbf{x})^T\mathbf{y} = \alpha \cdot 0 = 0.$$

   Therefore, $\alpha\mathbf{x} \in Y^\perp$. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are in $Y^\perp$, then

$$(\mathbf{x}_1 + \mathbf{x}_2)^T\mathbf{y} = \mathbf{x}_1^T\mathbf{y} + \mathbf{x}_2^T\mathbf{y} = 0 + 0 = 0,$$

   for each $\mathbf{y} \in Y$. Thus, $\mathbf{x}_1 + \mathbf{x}_2 \in Y^\perp$. Hence, $Y^\perp$ is a subspace of $\mathbb{R}^n$.

9. (a)

$$\mathrm{Col}(A) = \mathrm{span}\left\{(1,2)', (3,4)'\right\}, \quad \mathrm{N}(A) = \mathrm{span}\left\{(2,-1,1)'\right\}$$
$$\mathrm{Col}(A^T) = \mathrm{span}\left\{(1,0,-2)', (0,1,1)'\right\}, \quad \mathrm{N}(A^T) = \mathrm{span}\left\{\mathbf{0}\right\}$$

   Notice that $\mathrm{Col}(A^T) \oplus \mathrm{N}(A) = \mathbb{R}^3$ and $\mathrm{Col}(A) \oplus \mathrm{N}(A^T) = \mathbb{R}^2$.

   (b)

$$\mathrm{Col}(A) = \mathrm{span}\left\{(1,2,1)', (3,4,4)'\right\}, \quad \mathrm{N}(A) = \mathrm{span}\left\{(2,-1,1)'\right\}$$
$$\mathrm{Col}(A^T) = \mathrm{span}\left\{(1,0,-2)', (0,1,1)'\right\}, \quad \mathrm{N}(A^T) = \mathrm{span}\left\{(-4,1,2)'\right\}$$

   Notice that $\mathrm{Col}(A^T) \oplus \mathrm{N}(A) = \mathbb{R}^3$ and $\mathrm{Col}(A) \oplus \mathrm{N}(A^T) = \mathbb{R}^3$.

10. Since V is the solution space of the linear system $A\mathbf{x} = \mathbf{0}$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + 2x_2 + 5x_3 + 4x_4 = 0 \end{cases} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 4 \end{pmatrix}\mathbf{x} = \mathbf{0} \qquad \text{i.e. } V = \mathrm{N}(A).$$

   Then $V^\perp = [\mathrm{N}(A)]^\perp = \mathrm{Col}(A^T) = \mathrm{Row}(A)$. Since $(1,1,1,1)$ and $(1,2,5,4)$ are linearly independent, they form a basis for $Row(A)$.