PT

Solution to Assignment 3

1. Here $A = \{\text{ both H}\}, B = \{\text{ at least one H}\}$, and

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\text{ both H})}{P(\text{ at least one H})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

2. If $A \subset B$, then

$$P(A \mid B) = \frac{P(A)}{P(B)}, P(A \mid B^c) = 0, \quad P(B \mid A) = 1, \quad P(B \mid A^c) = \frac{P(BA^c)}{P(A^c)}.$$

3. For (a),

$$P\left(\text{ first toss H} \mid \text{ exactly 7H's}\right) = \frac{\binom{9}{6} \cdot \frac{1}{2^{10}}}{\binom{10}{7} \cdot \frac{1}{2^{10}}} = \frac{7}{10}.$$

Why is this not surprising? Conditioned on 7 Heads, they are equally likely to occur on any given 7 tosses. If you choose 7 tosses out of 10 at random, the first toss is included in your choice with probability $\frac{7}{10}$. For (b), the answer is, after canceling $\frac{1}{2^{10}}$,

$$\frac{\binom{9}{6} + \binom{9}{7} + \binom{9}{8} + \binom{9}{9}}{\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}} = \frac{65}{88} \approx 0.7386$$

Clearly, the answer should be a little larger than before, because this condition is more advantageous for Heads.

4. We apply the multiplication rule.

(a) Let $E_1 = \{\text{the first ball is black}\}$, $E_2 = \{\text{the second ball is black}\}$, $E_3 = \{\text{the third ball is white}\}$, $E_4 = \{\text{the fourth ball is white}\}$. It holds

$$\begin{aligned}
&P(\text{the first 2 balls selected are black and the next 2 are white})\\
&= P(E_1 \cap E_2 \cap E_3 \cap E_4)\\
&= P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2)P(E_4|E_1 \cap E_2 \cap E_3)\\
&= \frac{7}{12} \cdot \frac{9}{14} \cdot \frac{5}{16} \cdot \frac{7}{18} = \frac{35}{768}.
\end{aligned}$$

(b) Similarly as in (a), we have

$$\begin{aligned}
&P(\text{tof the first 3 balls selected, exactly 2 are black})\\
&= P(\text{the first 2 balls selected are black and the next is white})\\
&\quad + P(\text{the first and third balls selected are black and the second is white})\\
&\quad + P(\text{the second and third balls selected are black and the first is white})\\
&= \frac{7}{12} \cdot \frac{9}{14} \cdot \frac{5}{16} + \frac{7}{12} \cdot \frac{5}{14} \cdot \frac{9}{16} + \frac{5}{12} \cdot \frac{7}{14} \cdot \frac{9}{16}\\
&= \frac{45}{128}.
\end{aligned}$$

5. For (a) we use induction.

(a) If $n = 2$, then

$$\begin{aligned}
P\left(E_1^c \cap E_2^c\right) &= P((E_1 \cup E_2)^c)\\
&= 1 - P(E_1 \cup E_2)\\
&= 1 - P(E_1) - P(E_2) + P(E_1 \cap E_2)\\
&= 1 - P(E_1) - P(E_2) + P(E_1)P(E_2)\\
&= (1 - P(E_1))(1 - P(E_2))\\
&= P\left(E_1^c\right) P\left(E_1^c\right).
\end{aligned}$$

So the assertion is true for $n = 2$. Suppose the assertion is true for $n \leq k-1$. We show that it's also true for $n = k$. Consider $E_1, E_2, \ldots, E_k$ that are independent events. Let $i_1, i_2, \ldots, i_m \in \{1, \ldots, k\}$ be different indices. If $m \leq k-1$, then, by induction hypothesis,

$$P(\bigcap_{i=1}^{m} E_i^c) = \prod_{i=1}^{m} P(E_i^c).$$

Suppose $m = k$. It's easy to check that $E_1 \cap E_2, E_3, \ldots, E_k$ are independent. So

$$P((E_1 \cap E_2)^c \cap E_3^c \cap \cdots \cap E_k^c) = P\left[(E_1 \cap E_2)^c\right] \cdot \prod_{i=3}^{n} P(E_i^c).$$

On the other hand,

$$
\begin{aligned}
P((E_1\cap E_2)^c\cap E_3^c\cap\cdots\cap E_k^c) &= P((E_1^c\cup E_2^c)\cap E_3^c\cap\cdots\cap E_k^c)\\
&= P((E_1^c\cap E_3^c\cap\cdots\cap E_k^c)\cup(E_2^c\cap E_3^c\cap\cdots\cap E_k^c))\\
&= P((E_1^c\cap E_3^c\cap\cdots\cap E_k^c)+P((E_1^c\cap E_3^c\cap\cdots\cap E_k^c)\\
&\quad -P\left(E_1^c\cap E_2^c\cap\cdots\cap E_k^c\right)\\
&= P(E_1^c)\cdot\prod_{i=3}^{n}P(E_i^c)+P(E_2^c)\cdot\prod_{i=3}^{n}P(E_i^c)\\
&\quad -P\left(E_1^c\cap E_2^c\cap\cdots\cap E_k^c\right).
\end{aligned}
$$

From the above two identities, we obtain

$$
\begin{aligned}
P\left(E_1^c\cap E_2^c\cap\cdots\cap E_k^c\right) &= P(E_1^c)\cdot\prod_{i=3}^{n}P(E_i^c)+P(E_2^c)\cdot\prod_{i=3}^{n}P(E_i^c)\\
&\quad -P\left[(E_1\cap E_2)^c\right]\cdot\prod_{i=3}^{n}P(E_i^c)\\
&= \left[P(E_1^c)+P(E_2^c)-P(E_1^c\cup E_2^c)\right]\cdot\prod_{i=3}^{n}P(E_i^c)\\
&= P(E_1^c\cap E_2^c)\cdot\prod_{i=3}^{n}P(E_i^c)\\
&= \prod_{i=1}^{n}P(E_i^c).
\end{aligned}
$$

Summarizing the above two cases, we see that $E_1^c, E_2^c, \ldots, E_m^c$ are independent events. So the assertion is true for $n=k$ and thus, by induction, true for any n.

(b) By (a), we have

$$
P\left(\bigcup_{i=1}^{n}E_i\right)=1-P\left(\bigcap_{i=1}^{n}E_i^c\right)=1-\prod_{i=1}^{n}\left[1-P\left(E_i\right)\right].
$$

6. Let $F_i=\{$ number shown on the die is $i\}$, for $i=1,\ldots,6$. Clearly, $P\left(F_i\right)=\frac{1}{6}$. If A is the event that you get at least one Ace, Then:

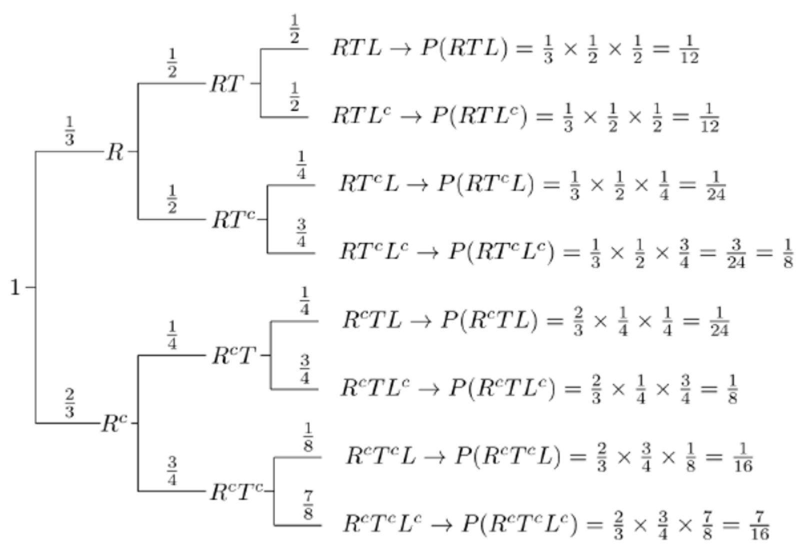
   (a) $P\left(A\mid F_1\right)=\frac{1}{13}$,

   (b) In general, for $i\geq 1, P\left(A\mid F_i\right)=1-\dfrac{\binom{48}{i}}{\binom{52}{i}}$.

Therefore, by the law of total probability,

$$P(A) = \frac{1}{6}\left(\frac{1}{13} + 1 - \frac{\binom{48}{2}}{\binom{52}{2}} + 1 - \frac{\binom{48}{3}}{\binom{52}{3}} + 1 - \frac{\binom{48}{4}}{\binom{52}{4}} + 1 - \frac{\binom{48}{5}}{\binom{52}{5}} + 1 - \frac{\binom{48}{6}}{\binom{52}{6}}\right).$$

7. We first have the following diagram:



(a) The probability that it's not raining and there is heavy traffic and I am not late can be found using the tree diagram which is in fact applying the chain rule:

$$\begin{aligned}
P(R^c \cap T \cap L^c) &= P(R^c)\,P(T \mid R^c)\,P(L^c \mid R^c \cap T) \\
&= \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{3}{4} \\
&= \frac{1}{8}.
\end{aligned}$$

(b) The probability that I am late can be found from the tree. All we need to do is sum the probabilities of the outcomes that correspond to me being late. In fact, we are using the law of total probability here.

$$\begin{aligned}
P(L) &= P(R,T,L) + P(R,T^c,L) + P(R^c,T,L) + P(R^c,T^c,L) \\
&= \frac{1}{12} + \frac{1}{24} + \frac{1}{24} + \frac{1}{16} \\
&= \frac{11}{48}
\end{aligned}$$

(c) We can find $P(R \mid L)$ using $P(R \mid L) = \frac{P(R \cap L)}{P(L)}$. We have already found $P(L) = \frac{11}{48}$ and we can find $P(R \cap L)$ similarly by adding the probabilities of the outcomes that belong to $R \cap L$. In particular,

$$\begin{aligned} P(R \cap L) &= P(R, T, L) + P\left(R, T^c, L\right) \\ &= \frac{1}{12} + \frac{1}{24} \\ &= \frac{1}{8}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} P(R \mid L) &= \frac{P(R \cap L)}{P(L)} \\ &= \frac{1}{8} \cdot \frac{48}{11} \\ &= \frac{6}{11}. \end{aligned}$$

8. Let H = heavy smoker, L = light smoker, N = non-smoker, D = death within five-year period. We are given that $P[D \mid L] = 2P[D \mid N]$ and $P[D \mid L] = \frac{1}{2}P[D \mid H]$ Therefore,

$$\begin{aligned} P[H \mid D] &= \frac{P[D \mid H]P[H]}{P[D \mid N]P[N] + P[D \mid L]P[L] + P[D \mid H]P[H]} \\ &= \frac{2P[D \mid L](0.2)}{\frac{1}{2}P[D \mid L](0.5) + P[D \mid L](0.3) + 2P[D \mid L](0.2)} = \frac{0.4}{0.25 + 0.3 + 0.4} = 0.42. \end{aligned}$$