2021-22 First Semester
MATH1053 Linear Algebra II (1002)

Assignment 6 Suggested Solutions

1. (a) The eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 1$ since

$$\det(A-\lambda I) = \begin{vmatrix} -3-\lambda & -4 \\ -1 & -\lambda \end{vmatrix} = (3+\lambda)\lambda - 4 = (\lambda+4)(\lambda-1) = 0.$$

For $\lambda_1 = -4$, $\operatorname{almu}(-4) = 1$,

$$[A-\lambda_1 I_2 |\, \mathbf{0}] = \left[\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \rightarrow \mathbf{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 1$, $\operatorname{almu}(1) = 1$,

$$[A-\lambda_2 I_2 |\, \mathbf{0}] = \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(b) The eigenvalues are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$.
For $\lambda_1 = \lambda_2 = 1$, $\operatorname{almu}(1) = 2$,

$$[B-\lambda_1 I_3 |\, \mathbf{0}] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} x_1, x_3 \in \mathbb{R} \\ x_2 = -x_3 \end{cases} \rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

For $\lambda_3 = 2$, $\operatorname{almu}(2) = 1$,

$$[B-\lambda_3 I_3 |\, \mathbf{0}] = \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \rightarrow \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

(c) By observation, $\operatorname{rank}(C) = 1, \dim N(C) = 3 - 1 = 2$. The null space $N(C) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$, where the vectors are eigenvectors corresponding to $\lambda = 0$.

Let λ_3 be the last eigenvalue, then $\sum \lambda_i = 0 + 0 + \lambda_3 = \operatorname{Tr}(C) = 3k$.

Consider $N(C - 3kI_3)$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of C corresponding to $\lambda_3 = 3k$.

Thus, $\operatorname{almu}(0) = \operatorname{gemu}(0) = 2$ and $\operatorname{almu}(3k) = \operatorname{gemu}(3k) = 1$.

2. (a) All diagonalizable, since each matrix has enough linearly independent eigenvectors.

(b) Using the results in HW5-Q4, we have

$$A = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{5} \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}.$$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}.$$

$$C = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3k \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}.$$

<u>Also note that the answers are not unique. Use MATLAB to help you verify.</u>

(c) Using the results in part(b), we have

$$\begin{aligned}
A^3 = PD^3P^{-1} &= \frac{1}{5} \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} (-4)^3 & 0 \\ 0 & 1^3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}} \\
&= \frac{1}{5} \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -4^3 & -4^3 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} -51 & -52 \\ -13 & -12 \end{bmatrix}, \\
A^n = PD^nP^{-1} &= \frac{1}{5} \begin{bmatrix} -4^{n+1}+1 & -4^{n+1}-4 \\ -4^n-1 & -4^n+4 \end{bmatrix}, \\
B^n = PD^nP^{-1} &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 1^n \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2^n & 2^n \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2^n-1 & 2^n-1 \\ 0 & 2^n & 2^n-1 \\ 0 & 0 & 1 \end{bmatrix}, \\
B^3 = PD^3P^{-1} &= \begin{bmatrix} 1 & 2^3-1 & 2^3-1 \\ 0 & 2^3 & 2^3-1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 7 & 7 \\ 0 & 8 & 7 \\ 0 & 0 & 1 \end{bmatrix}, \\
C^n = PD^nP^{-1} &= \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (-3k)^n \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0 & 0 & (-3k)^n \\ 0 & 0 & (-3k)^n \\ 0 & 0 & (-3k)^n \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} = (-3k)^{n-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.
\end{aligned}$$

3. 0 is an eigenvalue of A $\Leftrightarrow$ $N(A)$ has nonzero vectors $\Leftrightarrow$ A is singular.

4. Since $\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I)$, then A and A^T have the same characteristic polynomials and must have the same eigenvalues.

   The eigenspaces, however, are not necessarily the same. For instance,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

   both have eigenvalues $\lambda_1 = \lambda_2 = 1$. The eigenspace of A corresponding to $\lambda = 1$ is spanned by $\{(1,0)^T\}$ while the eigenspace of A^T is spanned by $\{(0,1)^T\}$.

5. **An $n \times n$ matrix is diagonalizable $\Leftrightarrow$ it has n linearly independent eigenvectors.**

   (a) The eigenvalues of A are $\lambda_1 = \lambda_2 = 1$, with eigenspace $E_1 = N(A - I_2) = N\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$.
   If $a = 0$, then $\dim(E_1) = 2 =$almu(1) and A is diagonalizable.

   (b) The eigenvalues of B are 1 and b. If $b \neq 1$, B is diagonalizable for any value of a.
   If $b = 1$, then B is diagonalizable when $a = 0$ by part (a).