Chapter Ten

Arbitrage Pricing Theory and Multifactor Models of Risk and Return

Multifactor Models: A Preview

- Recall stock variability may be decomposed into the following sources:
  - Market (i.e., systemic) risk
    - Largely due to macroeconomic events
  - Firm-specific (i.e., idiosyncratic) effects
- Risk premiums may depend on correlations with extra-market risk factors
  - E.g., inflation, interest rates, volatility, etc.

Factor Models of Security Returns

- **Single-factor model** of excess returns

$$R_i = E(R_i) + B_i F + e_i$$

$E(R_i) =$ expected excess return on stock i

$B_i =$ sensitivity of firm i

$F =$ deviation of the common factor from its expected value

$e_i =$ nonsystematic components of returns

Factor Models of Security Returns

- Extra market sources of risk may arise from several sources
  - E.g., uncertainty about interest rates or inflation.

- **Multifactor models** posit that returns respond to several systematic risk factors, as well as firm-specific influences
  - Useful in risk management applications

Factor Models of Security Returns

$$R_i = E(R_i) + \beta_{iGDP}GDP + \beta_{iIR}IR + e_i$$

R_i = excess return on security i

B_{GDP} = sensitivity of returns to GDP

B_{IR} = sensitivity of returns to interest rates

e_i = nonsystematic components of returns

Arbitrage Pricing Theory

- **Arbitrage pricing theory (APT)** was developed by Stephen Ross
  - Predicts a SML linking expected returns to risk, but the path it takes to the SML is quite different
  - APT relies on three key propositions:
    1. Security returns can be described by a factor model
    2. There are sufficient securities to diversify away idiosyncratic risk
    3. Well-functioning security markets do not allow for the persistence of arbitrage opportunities

Arbitrage, Risk Arbitrage, and Equilibrium

- *Arbitrage* is the exploitation of security mispricing in such a way that risk-free profits can be earned
  - Most basic principle of capital market theory is that well-functioning security markets rule out arbitrage opportunities

Arbitrage, Risk Arbitrage, and Equilibrium

- Arbitrage opportunity exists when an investor can earn riskless profits without making a net investment
  - E.g., shares of a stock sell for different prices on two different exchanges
- **Law of One Price**
  - Enforced by arbitrageurs; If they observe a violation, they will engage in *arbitrage activity*
  - This bids up (down) the price where it is low (high) until the arbitrage opportunity is eliminated

Diversification in a Single-Factor Security Market

- The excess return, R_p, on an n-stock portfolio with weights w_i, $\sum w_i = 1$, is

$$R_P = E(R_P) + \beta_P F + e_P$$

where $\beta_P = \sum w_i \beta_i$, $E(R_P) = \sum w_i E(R_i)$, and $e_P = \sum w_i e_i$

- The variance of the portfolio can be separated into its systematic and nonsystematic sources

$$\sigma_P^2 = \beta_P^2 \sigma_F^2 + \sigma^2(e_P) = \beta_P^2 \sigma_F^2 + \sum w_i^2 \sigma^2(e_i)$$

  - If the portfolio were equally weighted, $w_i = 1/\mathrm{n}$, then

$$\sigma^2(e_P) = \frac{1}{n} \sum \frac{\sigma^2(e_i)}{n} = \frac{1}{n} \bar{\sigma}^2(e_i)$$

  - Firm-specific risk becomes increasingly irrelevant as the portfolio becomes more diversified.

Well-Diversified Portfolios

- In a single-factor market, if a portfolio is well diversified, firm-specific risk becomes negligible, so that only systematic risk remains
  - One effect of diversification is that, when n is large, nonsystematic variance approaches zero
- A **well-diversified portfolio** is one with each weight small enough that for practical purposes the nonsystematic variance is negligible
  - For a well-diversified portfolio, $R_p = E(R_p) + \beta_P F$

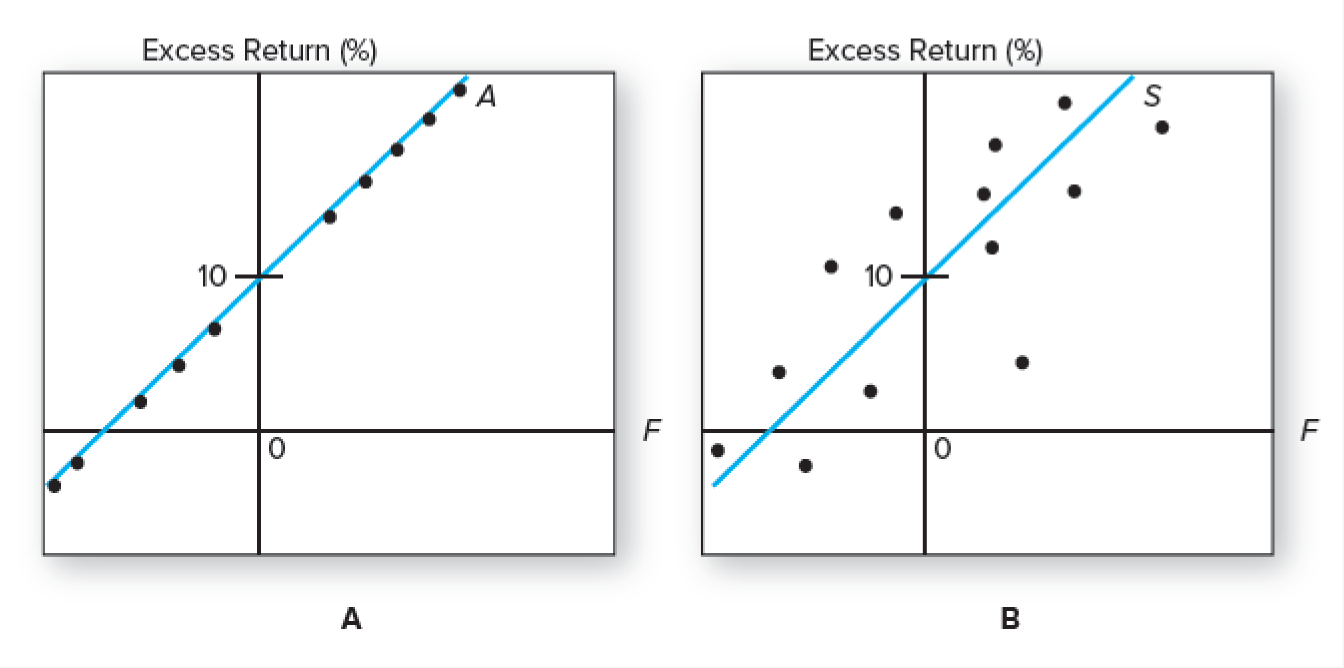
Excess Returns as a Function of the Systematic Factor

Panel A: a well-diversified portfolio A with $E(R_A) = 10\%$, $\beta_A = 1$.

- The well-diversified portfolio's return is determined completely by the systematic factor.

Panel B: a single stock S with $\beta_S = 1$.

- The undiversified stock is subject to nonsystematic risk, as seen in the scatter of points around the line.



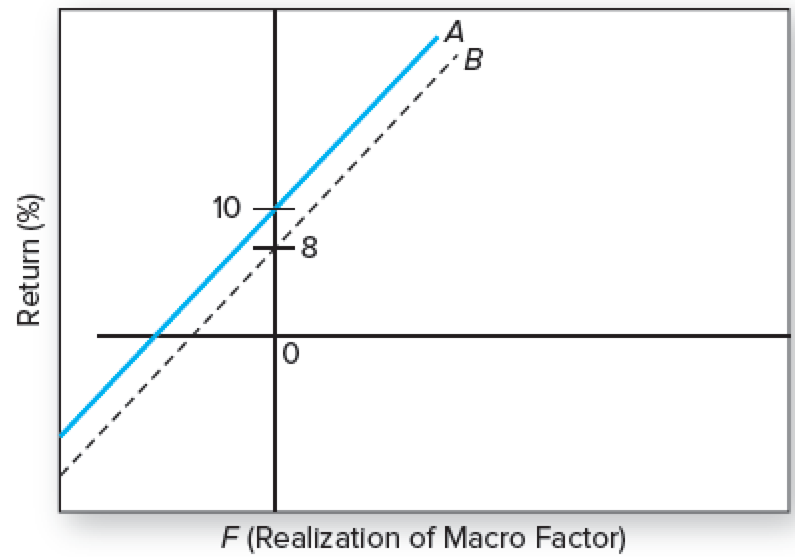
The SML of the APT

- First, we show that all well-diversified portfolios with the same beta must have the same expected return.
- Suppose we have two well-diversified portfolios, A and B, both with betas of 1, but with differing expected returns: $E[r_A] = 10\%$, $E[r_B] = 8\%$.
- No matter what the systematic factor turns out to be, portfolio A outperforms portfolio B, leading to an arbitrage opportunity.

Returns as a Function of the Systematic Factor: An Arbitrage Opportunity

If you sell short \$ 1 million of B and buy \$1 million of A, a zero-net-investment strategy, you would have a riskless payoff of \$20,000, as follows:

From long position in A:
$(.10 + 1.0 * F) * \$1\ million$
From short position in B:
$-(.08 + 1.0 * F) * \$1\ million$
The net proceeds is:
$.02 * \$1\ million = \$20{,}000$



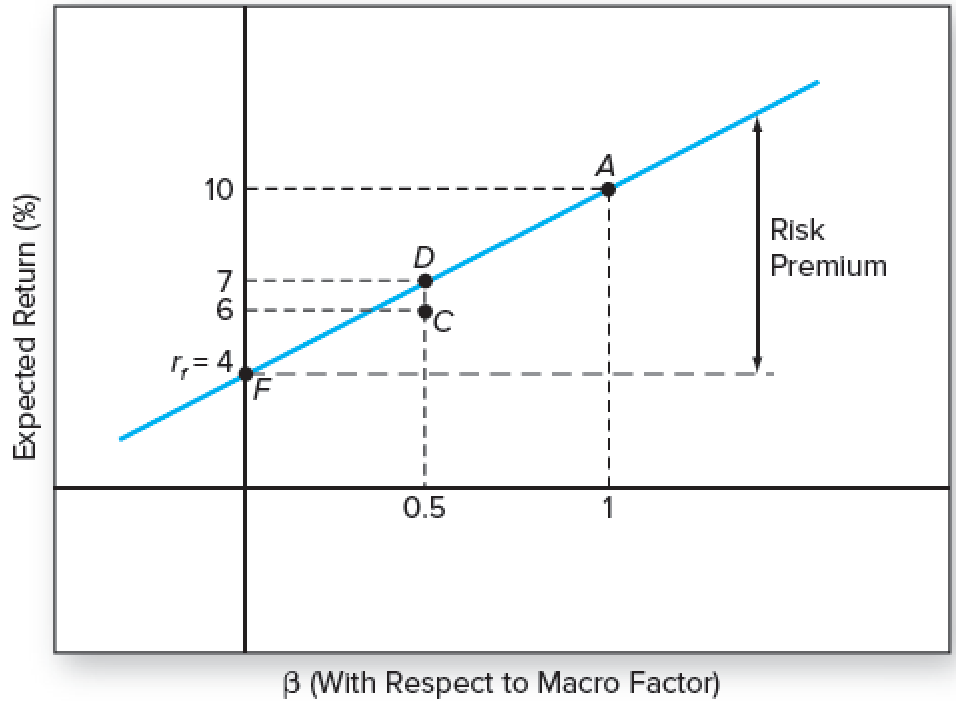
The SML of the APT

- Second, we show that risk premiums of portfolios with different betas must be proportional to beta.
- Suppose that risk-free rate is 4% and that a well-diversified portfolio, C, with a beta of .5, has an expected return of 6%
- Consider a new portfolio D (half in A and half in risk-free).

An Arbitrage Opportunity

Portfolio D's beta is 0.5, with expected return 7%.

To preclude arbitrage opportunities, the expected return on all well-diversified portfolios must lie on the straight line from the risk-free asset.



The SML of the APT

- The last step is to transition from a factor model to a market index model.
- Because all well-diversified portfolios are perfectly correlated with the macro factor.
- Therefore, if a market index portfolio is well diversified, its return will perfectly reflect the value of the macro factor.
- Therefore, we can write the excess return on a well-diversified portfolio P as:

$$R_p = \alpha_p + \beta_p R_M$$

The SML of the APT

- All well-diversified portfolio with the same beta must have the same expected return
- Generally, for any well-diversified *P*, the expected excess return must be:

$$E(R_P) = \beta_P E(R_M)$$

- Risk premium on portfolio *P* is the product of its beta and the risk premium of the market index
  - SML of the CAPM must also apply to well-diversified portfolios

APT and CAPM

APT

- Built on the foundation of well-diversified portfolios
  - Cannot rule out a violation of the expected return-beta relationship for any particular asset
- Does not assume investors are mean-variance optimizers
- Uses an observable market index

CAPM

- Model is based on an inherently unobservable “market” portfolio
- Provides unequivocal statement on the expected return-beta relationship for all securities

A Multifactor APT
(1 of 2)

- To this point, we have examined the APT in a one-factor world
  - However, there are several sources of systematic risk and exposure to these factors will affect a stock’s appropriate expected return
  - APT can be generalized to accommodate these multiple sources of risk
  - Assume a two-factor model:

$$R_i = E(R_i) + \beta_{i1}F_1 + \beta_{i2}F_2 + e_i$$

A Multifactor APT
(2 of 2)

- Benchmark portfolios in the APT are **factor portfolios,** which are well-diversified portfolios constructed to have β=1 for one of the factors and β= 0 for any other factors
  - Factor portfolios track a particular source of macroeconomic risk, but are uncorrelated with other sources of risk
    - Referred to as a “tracking portfolio”

Multifactor APT

- The multifactor SML predicts that the contribution of each source of risk to the security's total risk premium equals the factor beta times the risk premium of the factor portfolio tracking that source of risk.

$$E(r_P) = r_f + \beta_{P1}[E(r_1) - r_f] + \beta_{P2}[E(r_2) - r_f]$$

Example: Multifactor SML

- Suppose that for the two factor portfolios, $E(r_1) = 10\%$, $E(r_2) = 12\%$ and r_f =4%. Now consider a well-diversified portfolio *P*, with beta on the first factor portfolio, $\beta_{P1} = .5$, and beta on the second factor portfolio, $\beta_{P2} = .75$.

4%	Risk-free rate	$r_f = 4\%$
+3%	Risk premium for exposure to factor 1	$\beta_{P1} \times [E(r_1) - r_f] = .5 \times 6\%$
+6%	Risk premium for exposure to factor 2	$\beta_{P2} \times [E(r_2) - r_f] = .75 \times 8\%$
= 13%	Total expected return	$r_f + \beta_{P1}[E(r_1) - r_f] + \beta_{P2}[E(r_2) - r_f]$

Summary

- APT relies on three key propositions
  1. Security returns can be described by a factor model
  2. There are sufficient securities to diversify away idiosyncratic risk
  3. Well-functioning security markets do not allow for the persistence of arbitrage opportunities
- Single factor model $R_i = E(R_i) + \beta_i F + e_i$
  - Variance can be separated into systematic and firm-specific risks

$$\sigma_P^2 = \beta_P^2 \sigma_F^2 + \sigma^2(e_P) = \beta_P^2 \sigma_F^2 + \sum w_i^2 \sigma^2(e_i)$$

Summary

- Multifactor model

$$R_i = E(R_i) + \beta_{i1}F_1 + \beta_{i2}F_2 + e_i$$

- Multifactor SML

$$E(r_P) = r_f + \beta_{P1}[E(r_1) - r_f] + \beta_{P2}[E(r_2) - r_f]$$

- Well-diversified portfolios
  - Firm-specific risk becomes negligible, so that only factor risk remains.
  - A well-diversified portfolio is one with each weight small enough that for practical purposes the nonsystematic variance is negligible

Summary

- Consider a single factor APT. Portfolio A has a beta of 2.0 and an expected return of 22%. Portfolio B has a beta of 1.5 and an expected return of 17%. The risk-free rate of return is 4%. If you wanted to take advantage of an arbitrage opportunity, you should take a short position in portfolio ___________ and a long position in portfolio ________.

    A) A; A
    B) A; B
    C) B; A
    D) B; B
    E) A; the riskless asset