2023-24 First Semester

MATH2023 Ordinary and Partial Differential Equations (1002)

Assignment 7 Suggested Solutions

1. **Solution:**

   (a) $\sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \rho = \infty$

   (b) $1-2(x+1)+(x+1)^2, \quad \rho = \infty$

   (c) $\sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n, \quad \rho = 1$

2. **Solution:**

   (a) $x = 0$ is an **ordinary point** since the coefficients are all polynomials and $P(0) = 1 \neq 0$. Assume the solution has the **form**

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 y_1(x) + a_1 y_2(x)$$

Substitution gives:

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - (n+1)a_n]x^n &= 0
\end{aligned}$$

By equating the coefficients on both sides, the **recurrence relation** is given as

$$\begin{aligned}
(n+2)(n+1)a_{n+2} - (n+1)a_n &= 0 \\
a_{n+2} &= \frac{a_n}{(n+2)}, \quad n = 0,1,2,\cdots
\end{aligned}$$

To find y_1**,** we take $a_0 = 1$ and $a_1 = 0$

$$a_0 = 1,\ a_2 = \frac{1}{2},\ a_4 = \frac{1}{8},\ a_6 = \frac{1}{2\cdot 4\cdot 6},\ \cdots,\ a_{2n} = \frac{1}{2^n(n)!}$$

$$a_1 = a_3 = a_5 = \cdots = a_{2n+1} = 0$$

Thus,

$$y_1(x) = 1 + \frac{x^2}{2} + \frac{x^4}{2\cdot 4} + \frac{x^6}{2\cdot 4\cdot 6} + ... = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} \tag{1}$$

To find y_2, we take $a_0 = 0$ and $a_1 = 1$

$$a_1 = 1,\ a_3 = \frac{2}{2\cdot 3},\ a_5 = \frac{2\cdot 4}{2\cdot 3\cdot 4\cdot 5},\ a_7 = \frac{1}{3\cdot 5\cdot 7},\ \cdots,\ a_{2n+1} = \frac{2^n n!}{(2n+1)!}$$

$$a_2 = a_4 = a_6 = \cdots = a_{2n} = 0$$

Another solution is

$$y_2(x) = x + \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} + \frac{x^7}{3 \cdot 5 \cdot 7} + \ldots = \sum_{n=0}^{\infty} \frac{2^n n! x^{2n+1}}{(2n+1)!} \tag{2}$$

The power series solution about $x = 0$ to this differential equation is

$$y(x) \; = \; a_0 \sum_{n=0}^{\infty} \frac{1}{2^n n!} \, x^{2n} \; + \; a_1 \sum_{n=0}^{\infty} \frac{2^n n!}{(2n+1)!} \, x^{2n+1}, \qquad a_0, a_1 \in \mathbb{R}.$$

(b) $x = 1$ is an **ordinary point** since $P(1) = 1 \neq 0$. Assume the solution has the form:

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0 y_1(x) + a_1 y_2(x)$$

Substitution gives

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - x \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$
$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n - (x-1+1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$
$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=1}^{\infty} n a_n (x-1)^n - \sum_{n=0}^{\infty} (n+1)a_{n+1}(x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Recurrence relation:

$$(n+2)a_{n+2} - a_{n+1} - a_n = 0$$

Let $a_0 = 1$, $a_1 = 0$:

$$\begin{aligned} n = 0: &\quad 2a_2 - 0 - 1 = 0 \to \quad a_2 = \frac{1}{2} \\ n = 1: &\quad 3a_3 - \frac{1}{2} - 0 = 0 \to \quad a_3 = \frac{1}{6} \\ n = 2: &\quad 4a_4 - \frac{1}{6} - \frac{1}{2} = 0 \to \quad a_4 = \frac{1}{6} \end{aligned}$$

Thus, one solution is

$$y_1(x) = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{6} + \ldots \tag{3}$$

and let $a_0 = 0$, $a_1 = 1$, another solution is

$$y_2(x) = (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{2} + \frac{(x-1)^4}{4} + \ldots \tag{4}$$

(c) $x = 1$ is an ordinary point since $P(1) = 4 \neq 0$. Assume the solution has the form

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0 y_1(x) + a_1 y_2(x)$$

Then

$$\left[1-\frac{1}{2}(x-1)^2\right]y''-3(x-1)y'-3y=0$$

$$\left[1-\frac{1}{2}(x-1)^2\right]\sum_{n=0}^{\infty}n(n-1)a_n(x-1)^{n-2}-3(x-1)\sum_{n=0}^{\infty}na_n(x-1)^{n-1}-3\sum_{n=0}^{\infty}a_n(x-1)^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-\frac{1}{2}n(n-1)a_n-3na_n-3a_n\right]x^n=0$$

Recurrence relation:

$$(n+2)(n+1)a_{n+2}-\frac{1}{2}(n+2)(n+3)a_n=0$$

we can derive the formulas as

$$a_{2n}=\frac{2n+1}{2^n}a_0, \quad \text{and} \quad a_{2n+1}=\frac{n+1}{2^n}a_1, \quad n\geq 0$$

Thus the general solution is

$$y\ =\ a_0\sum_{n=0}^{\infty}\frac{2n+1}{2^n}(x-1)^{2n}\ +\ a_1\sum_{n=0}^{\infty}\frac{n+1}{2^n}(x-1)^{2n+1}, \qquad a_0,a_1\in\mathbb{R}.$$

(d) $x=0$ is an **ordinary point**, since $p(x)=0$ and $q(x)=x/e^x$ are both analytic at $x=0$ due to the fact that the Taylor series of xe^{-x} about $x_0=0$ is

$$f(x)=xe^{-x}=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n-1)!}x^n, \quad \text{with} \quad \rho=\infty$$

where $f^{(n)}(x)=(-1)^{n+1}(n-x)e^{-x}$.

Assume the solution has the form:

$$y(x)=\ \sum_{n=0}^{\infty}a_n\,x^n\ =a_0y_1(x)+a_1y_2(x)$$

Substitution gives

$$\left[\sum_{n=0}^{\infty}\frac{x^n}{n!}\right]\left[\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n\right]-x\sum_{n=0}^{\infty}a_nx^n=0$$

The coefficient of x^n in the *first* product of two series is

$$c_n=2a_2\frac{1}{n!}+6a_3\frac{1}{(n-1)!}+12a_4\frac{1}{(n-2)!}+\cdots(n+1)na_{n+1}+(n+2)(n+1)a_{n+2}$$

The coefficient of x^n in the *second* series is a_{n-1}. Combining the series

$$2a_2+(2a_2+6a_3+a_0)x+(a_2+6a_3+12a_4+a_1)x^2+(a_2+6a_3+12a_4+20a_5+a_2)x^3+\cdots=0$$

and equating the coefficients to zero, we obtain

$$\begin{aligned} 2a_2 &= 0 \\ 2a_2 + 6a_3 + a_0 &= 0 \\ a_2 + 6a_3 + 12a_4 + a_1 &= 0 \\ a_2 + 6a_3 + 12a_4 + 20a_5 + a_2 &= 0 \\ &\vdots \end{aligned}$$

The general solution is

$$y(x) = a_0\left[1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{40} + \frac{x^6}{90} + \cdots\right] + a_1\left[x - \frac{x^4}{12} + \frac{x^5}{20} - \frac{x^6}{60} + \cdots\right]$$

3. This can be done in one of two ways. One possibility is simply to compute the power series for $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$, and then to determine the radii of convergence by using one of the convergence tests for infinite series.

   There is an easier way when P, Q, and R are polynomials. It is shown in the theory of functions of a complex variable that the ratio of two polynomials, say Q/P, has a convergent power series expansion about a point $x = x_0$ if $P(x_0) \neq 0$. Further, if we assume that any factors common to Q and P have been canceled, then the radius of convergence of the power series for Q/P about x_0 is precisely the distance from x_0 to the nearest zero of P in the complex plane.

   (a) The zero of $P(x) = x$ is 0. Then the shortest distance from $x_0 = 1$ to 0 in the complex plane is $\rho = 1$.

   (b) The zero of $P(x) = x^2 - 2x - 3$ are -1 and 3. Then the shortest distance from $x_0 = 4$ to the zeros in the complex plane is $\rho = |4 - 3| = 1$.

   (c) The zero of $P(x) = x^2 - 2x + 3$ are $1 \pm \sqrt{2}i$. Then the shortest distance from x_0 to $1 \pm \sqrt{2}$ in the complex plane is $\rho = \sqrt{11}$, $\rho = \sqrt{3}$ and $\rho = 3\sqrt{3}$, respectively.

4. **Solution:** Denote $t = \ln x$ and $u(t) = y(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}t}\frac{1}{x} = u'\frac{1}{x},$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\cdot\frac{1}{x}\right) = \frac{\mathrm{d}^2u}{\mathrm{d}t^2}\frac{\mathrm{d}t}{\mathrm{d}x}\cdot\frac{1}{x} + \frac{\mathrm{d}u}{\mathrm{d}t}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right) = (u'' - u')\frac{1}{x^2}.$$

   After substitution, the original DE becomes a linear DE with constant coefficients,

$$3u'' + (12-3)u' + 9u = 0, \quad \to \quad r = \frac{-3 \pm \sqrt{3}i}{2}$$

$$u(t) = c_1 e^{-1.5t}\cos(\sqrt{3}t/2) + c_2 e^{-1.5t}\sin(\sqrt{3}t/2).$$

   Substitute $t = \ln x$ back and we yield the general solution for y is

$$y(x) = c_1 x^{-3/2}\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_2 x^{-3/2}\sin\left(\frac{\sqrt{3}}{2}\ln x\right), \quad c_1, c_2 \in \mathbb{R}.$$