Caculus II Math 1038 (1002&1003)

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Week 7: Ch14 Partial differentiation

1. Limit

   To check whether a limit
   $$\lim_{(x,y)\to(a,b)} f(x,y) = L$$
   exist or not, if yes, find the limit, if not, prove the limit does not exsit (find two paths).

2. **Continuity**

   Definition: f is continuous on D if f is continuous at every point (a,b) in D.
   $$\lim_{(x,y)\to(a,b)} f(x,y) = L = f(a,b)$$
   $f(x,y)$ is **defined** at (a,b) and the limit **exsits** and **equals** to $f(a,b)$.

   (a) Any rational function **is continuous** on its domain.

   (b) To prove a function $f(x,y)$ is **continuous at a given point** (a,b)

      i. IF $(a,b) \in D$ (the point is in the domain) and $f(x,y)$ is a polynomial/rational/trig function and other simple functions, THEN $f(x,y)$ is continous at (a,b), and the value equals to $f(a,b)$ (no jump!)

      ii. If $(a,b) \notin D$ (not in the domain), then we have to check

         A. whether the limit L **exists**

         B. whether the limit **equal to** $f(a,b)$

   (c) To prove a function $f(x,y)$ is **continuous on a given region** D, then it is equivalent to prove f is continuous at every the point $(x,y) \in D$.

3. Partial derivative

   (a) **Definition**: partial derivative with respect to x at (a,b),
   $$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$
   $$f_x(a,b) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

   (b) to find partial derivatives $f_x(x,y)$

      i. **use definition as a limit.**

      ii. treat the other variable y as **a constant** and differentiate it as a function of single variable: important rules: **Product, Quotient and Chain Rule.**

   (c) notations:
   $$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = D_x f$$
   if $z = f(x,y)$, we can also use
   $$\frac{\partial z}{\partial x}$$

   (d) interpretation of $f_x(a,b)$: slope of the tangent lines at point $(a,b,f(a,b))$ to the traces C in the plane $y = b$.

(e) **Clairaut's Theorem**: $f(x,y)$ is defined on a disk D that contains (a,b). If f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

4. **Differentiability**

   (a) Increment/difference of z: Δz

   $$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

   (b) Definition: The function is **differentiable at** (a,b) if $f_x(a,b)$ and $f_y(a,b)$ exist and

   $$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

   where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

   (c) If $f_x(a,b)$ and $f_y(a,b)$ exist and are **continuous** at (a,b), then f is differentiable, and if a function f is differentiable at (a,b), then it is continuous at (a,b).

   (d) **Important relationships**: i $\to$ ii $\to$ iii $\to$ iv

      i. $f_x(a,b)$ and $f_y(a,b)$ are **continuous**
      ii. f **is differentiable at** (a,b)
      iii. f is continuous at (a,b)
      iv. $\lim_{(x,y)\to(a,b)} f(x,y) = L$ limit exist.

      iv $\not\to$ iii $\not\to$ ii $\not\to$ i.

   (e) To prove f is differentiable

      i. Use definition: difficulty
      ii. To show $f_x(a,b)$ and $f_y(a,b)$ exist AND are **continuous** at (a,b), .

   (f) To prove f is NOT differentiable:

      i. To show f is not continuous, or
      ii. by definition: ϵ_1 and $\epsilon_2 \not\to 0$.

5. **Tangent plane** through a point $P(x_0, y_0, z_0)$

   (a) two tangent directions: $\overrightarrow{u}_x = \langle 1, 0, f_x(x_0,y_0)\rangle$ and $\overrightarrow{u}_y = \langle 0, 1, f_y(x_0,y_0)\rangle$

   (b) normal vector of tangent plane $\overrightarrow{n} = \overrightarrow{u}_x \times \overrightarrow{u}_y = \langle f_x, f_y, -1\rangle$

   (c) **equation of a tangent plane** to the surface $z = f(x,y)$

   $$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

   (d) **equation of a normal line**

   $$(x,y,z) = (x_0,y_0,z_0) + t\,\langle f_x, f_y, -1\rangle$$

6. Linear approximations

   for $f(x,y)$

   $$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

7. Differentials

   total differential

   $$dz = f_x(a,b)dx + f_y(a,b)dy$$

   compare with the difference/increment $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$, we have $dz \approx \Delta z$.

8. Chain rule

   (a) Recall for $y = f(x)$ and $x = g(t)$

   $$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

   x is a intermediate variable and t is the sole independent variable.

(b) $z = f(x(t), y(t))$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

with x and y are intermediate variables and t is the sole independent variable.

(c) $z = f(x(s,t), y(s,t))$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial x} \cdot \frac{dy}{dt}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial z}{\partial x} \cdot \frac{dy}{ds}$$

(d) change of coordinate $(x,y) \to (r,\theta)$, with $x = r\cos\theta$ and $y = r\sin\theta$

(e) **Implicit differentiation** $F(x,y) = 0$, where $y = f(x)$ but f is not in a explicit form, and we need to find dy/dx

Differentiate both side

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

then

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

(f) Implicit Differentiation Theorem for $F(x,y,z) = 0$ where $z = f(x,y)$ is not explicit, to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ we can use the formula

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

9. Directional derivatives $D_{\overrightarrow{u}} f = \nabla f \cdot \overrightarrow{u}$

(a) Recall $z = f(x,y)$, $f_x = D_x f$ represents the rates of change of z in the x-direction, in the directions of the unit vector $\overrightarrow{i} = \langle 1, 0\rangle$, similarly, $f_y = D_y f$ represents the rates of change of z in the y-direction, in the directions of the unit vector $\overrightarrow{j} = \langle 0, 1\rangle$.

along direction $\overrightarrow{u} = \overrightarrow{i} = \langle 1, 0\rangle$ parallel to x-axis where y is fixed to be a constant,

$$f_x = D_x = \langle f_x, f_y\rangle \cdot \langle 1, 0\rangle = f_x$$

similarly, along direction $\overrightarrow{u} = \overrightarrow{j} = \langle 0, 1\rangle$ parallel to y-axis where x is fixed to be a constant,

$$f_y = D_y = \langle f_x, f_y\rangle \cdot \langle 0, 1\rangle = f_y$$

(b) Definition of **directional derivative** of f at (x_0, y_0) **in the direction** of a **unit vector** $\overrightarrow{u} = \langle a, b\rangle$

$$D_{\overrightarrow{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

or we can represent $\overrightarrow{u} = \langle \cos\theta, \sin\theta\rangle$.

(c) Theorem

$$D_{\overrightarrow{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b = \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a, b\rangle = \nabla f \cdot \overrightarrow{u}$$

or

$$D_{\overrightarrow{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta = \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle \cos\theta, \sin\theta\rangle$$

prove by **Chain Rule** via defining a new function $g(h) = f(x + ha, y + hb)$ then $g(h) = f(x,y)$ with $x = x_0 + ha$ and $y = y_0 + hb$, so $\frac{dx}{dh} = a$ and $\frac{dy}{dh} = b$.

$$g'(h) = \frac{dg}{dh} = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b$$

10. Gradient vector ∇f

(a) Recall $f(x)$, the slope/gradient is $f'(x)$

(b) The gradient of a function $f(x,y)$, grad$f$, ∇f (read "del f"), which is a **vector** function

$$\nabla f = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\overrightarrow{i} + \frac{\partial f}{\partial y}\overrightarrow{j}$$

$$\nabla f = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle = \frac{\partial f}{\partial x}\overrightarrow{i} + \frac{\partial f}{\partial y}\overrightarrow{j} + \frac{\partial f}{\partial z}\overrightarrow{k}$$

(c) **directional derivative**

$$D_{\overrightarrow{u}} f(x,y) = \nabla f \cdot \overrightarrow{u}$$

a projection of the gradient vector onto $\overrightarrow{u}$, which gives the

i. **rate of change** of a function z in a given direction $\overrightarrow{u}$.

ii. gradient of function of three variables

(d) **maximizing** $D_{\overrightarrow{u}}$

$$D_{\overrightarrow{u}} f(x,y) = \nabla f \cdot \overrightarrow{u} = |\nabla f| \cdot |\overrightarrow{u}| \cos\theta$$

it attains its maxima when $\theta = 0$ which means $\overrightarrow{u}$ is in the same direction as the gradient of f. Since $|\overrightarrow{u}| = 1$ and $\cos\theta = 1$

$$\max D_{\overrightarrow{u}} f(x,y) = |\nabla f|$$

direction of steepest ascent. When $\theta = \pi$, $\cos\theta = -1$, $\overrightarrow{u}$ is the direction of greatest descent.

$$\min D_{\overrightarrow{u}} f(x,y) = -|\nabla f|$$

(e) **To find the maximum rate of change**,

i. find the gradient vector $\nabla f = \langle f_x, f_y\rangle$

ii. compute its length $|\nabla f|$ and get the maximum rate and $-|\nabla f|$ is the mininum rate ,

iii. the direction of the maximum and minimum rate of change is the normalized ∇f:

$$\overrightarrow{u} = \frac{\nabla f}{|\nabla f|}, \qquad \text{and} \qquad \overrightarrow{u} = -\frac{\nabla f}{|\nabla f|}$$

(f) other conclusions

i. $D_{\overrightarrow{u}} = 0$, when $\theta = \pi/2$ $\cos\theta = 0$, $\overrightarrow{u}$ is perpendicular to ∇f (tangent to the level curves).

ii. The paths of steepest ascent/descent is a curve that remains **perpendicular to each level curves** through which it passes.

(g) Theorem: tangent to the level curve of f at (a,b) is orthogonal to the gradient $\nabla f(a,b)$ given $\nabla f(a,b) = 0$. at level curves $f(x,y) = k$, so $f_x + f_y y'(x) = 0$ and $y' = -f_x/f_y$, so tangent direction

$$\overrightarrow{t} = \langle 1, y'\rangle = \langle 1, -f_x/f_y\rangle$$

gradient vector:

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle$$

therefore

$$\overrightarrow{t} \cdot \nabla f(a,b) = 0$$

(h) Equation of the tangent line for $f(x,y) = z$

$$\nabla f(a,b) \cdot \langle x-a, y-b\rangle = 0$$

or

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) = 0$$

(i) Theorem: the gradient of function $f(x,y,z)$ is normal to the tangent plane to the **level surface** $f(x,y,z) = k$ at the point (a,b,c). The equation of the tangent plane

$$f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0$$

11. Extreme values
    (a) local maximum/minimum values
    (b) absolute maximum/minimum values
    (c) saddle point is a critical point which is a local max in one direction and local min in another direction
    (d) critical (stationary) point (a, b)
    $$f_x(a, b) = f_y(a, b)$$
    (e) Discriminant: $D = f_{xx}f_{yy} - f_{xy}^2$
    (f) Second Derivatives Test for critical points
        i. $f_{xx} > 0$, $D > 0$, local min
        ii. $f_{xx} < 0$, $D > 0$, local max
        iii. $D < 0$, saddle point
    (g) Extreme value theorem: if $f(x, y)$ is continuous on the closed and bounded region $R \subset \mathbb{R}^2$, then f has absolute max and min on R.
    (h) To find absolute max and min:
        i. find all the critical points
        ii. find all the boundary points
        iii. compare the values at these points to get the greatest and least values to the absolute max and mean.
12. Optimization problems with two independent variables
    (a) objective function $f(x, y)$ to maximize or to minimize
    (b) constaints $g(x, y) = 0$
    (c) Lagrange multiplier λ
    $$\nabla f(a, b) = \lambda \nabla g(a, b)$$
    or for three variables
    $$\nabla f(a, b, c) = \lambda \nabla g(a, b, c)$$
    (d) double constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$
    $$\nabla f(a, b, c) = \lambda \nabla g(a, b, c) + \mu \nabla h(a, b, c)$$
    (e) Our goal is to find the point (a, b, c) where extreme value locates and then compute $f(a, b, c)$.