Solution to ASP Quiz 1

1. (32')

(a) (8') $Y_1 = e_1$, $Y_2 = Y_1 + e_2 = e_1 + e_2$, $Y_3 = Y_2 + e_3 = Y_1 + e_2 + e_3 = e_1 + e_2 + e_3$

$Y_t = Y_{t-1} + e_t = Y_{t-2} + e_{t-1} + e_t = \cdots = e_1 + e_2 + \cdots + e_t$. Then $\mathbb{E}(Y_t) = \mathbb{E}(e_1 + e_2 + \cdots + e_t) = \mathbb{E}(e_1) + \mathbb{E}(e_2) + \cdots + \mathbb{E}(e_t)$. Since $e_t \sim N(0,1)$ for all t, $\mathbb{E}(Y_t) = 0$.

(b) (12') Following (a), the autocovariance function is, for $k \geq 0$,

$$
\begin{aligned}
Cov(Y_t, Y_{t+k}) &= Cov(e_1 + e_2 + \cdots + e_t,\ e_1 + e_2 + \cdots + e_t + e_{t+1} + \cdots + e_{t+k}) \\
&= \sum_{i=1}^{t+k} \left(\sum_{j=1}^{t} Cov(e_j, e_i) \right) \\
&= \sum_{j=1}^{t} Cov(e_j, e_j) + \sum_{i \neq j} Cov(e_i, e_j) \\
&= t\, Var(e_1) + 0 \\
&= t
\end{aligned}
$$

Equivalently, we can write $Cov(Y_t, Y_s) = \min(s,t)$, $s,t \geq 0$.

(c) (12')

$$
\begin{bmatrix} Y_3 \\ Y_4 \\ Y_7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix} = A \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix}
$$

Since $e_1, e_2, \cdots, e_7$ are independent, identically distributed standard normal random variables, their joint distribution is easy to verify to be $N(\vec{0}, I_7)$. Hence (Y_3, Y_4, Y_7) is a linear transformation of normal vector $[e_1, e_2, \cdots, e_7]^T$ and then $(Y_3, Y_4, Y_7) \sim N(A\vec{0}, AI_7A^T)$, where

$$
A\vec{0} = \vec{0}, \quad AI_7A^T = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 4 & 4 \\ 3 & 4 & 7 \end{bmatrix}
$$

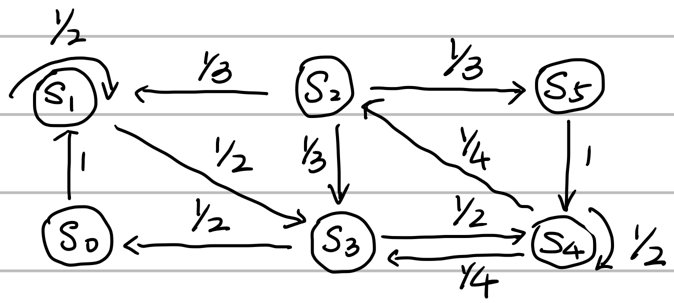
2. (14')

$$
P^3 = \begin{bmatrix} 1/4 & 3/4 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/4 & 3/4 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/4 & 3/4 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} \frac{83}{192} & \frac{109}{192} \\ \frac{109}{216} & \frac{107}{216} \end{bmatrix}
$$

Hence, $P(X_3 = 0 \mid X_0 = 1) = \frac{109}{216}$.

3. (36')

(a) (12')



(b) (12')

Solution 1:

Case 1: $S_2 \to S_5 \to S_4 \to$ leave S_4

$P_1 = \frac{1}{3} \times 1 \times \frac{1}{2} = \frac{1}{6}$

Case 2: $S_2 \to S_3 \to (S_0 \to \overset{k_i \text{ times}}{S_1} \to S_3)^{j \text{ times}} \to S_4 \to$ leave S_4

$P_2 = \frac{1}{3} \times \left[1 + \sum_{j=1}^{\infty} \sum_{k_j=0}^{\infty} \cdots \sum_{k_1=0}^{\infty} \left[\prod_{i=1}^{j} \left(\frac{1}{2} \times \frac{1}{2}^{k_i} \times \frac{1}{2}\right)\right]\right] \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{3} \times \left[1 + \sum_{j=1}^{\infty} \left[\frac{1}{4^j} \cdot \left[\sum_{k_j=0}^{\infty} \cdots \sum_{k_2=0}^{\infty} \left(\frac{1}{2}\right)^{\sum_{i=2}^{j} k_i}\right] \cdot \sum_{k_1=0}^{\infty} \left(\frac{1}{2}\right)^{k_1}\right]\right] \times \frac{1}{4}$

$= \frac{1}{3} \times \left[1 + \sum_{j=1}^{\infty} \frac{1}{4^j} \cdot 2^j\right] \times \frac{1}{4}$

$= \frac{1}{3} \times 2 \times \frac{1}{4}$

$= \frac{1}{6}$

Case 3: $S_2 \to \overset{i \text{ times}}{S_1} \to S_3 \to (S_0 \to \overset{k_i \text{ times}}{S_1} \to S_3)^{j \text{ times}} \to S_4 \to$ leave S_4

$P_3 = \frac{1}{3} \times \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i \times \frac{1}{2} \times 2 \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{6}$

Hence, P(process enter S_4 and leaves S_4 at the next step)

$= P_1 + P_2 + P_3$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$

$= \frac{1}{2}$

Solution 2:

$$\begin{cases} f_{2,4} = \frac{1}{3} f_{1,4} + \frac{1}{3} f_{3,4} + \frac{1}{3} f_{5,4} \\ f_{1,4} = \frac{1}{2} f_{1,4} + \frac{1}{2} f_{3,4} \qquad (*) \\ f_{3,4} = \frac{1}{2} + \frac{1}{2} f_{1,4} \\ f_{5,4} = 1 \end{cases}$$

Solve (*) and get $f_{1,4} = f_{2,4} = f_{3,4} = f_{5,4} = 1$. Actually, the chain is irreducible and has finite many states, which implies that all states are recurrent. In particular, $S_4 \to S_2$, and S_4 is recurrent. Theorem 1.14 in the slide of Chapter 2 Part I tells that $f_{2,4} = 1$. Hence starting from S_2,

P(process enter S_4 and leave S_4 at the next step)

$= f_{2,4} \times (P_{4,2} + P_{4,3}) = 1 \times \frac{1}{2} = \frac{1}{2}$

(c) (12')

Starting from S_2,

P(process enters S_3 for the 1^{st} time at 3^{rd} step)

$= P_{25} \times P_{54} \times P_{43} + P_{21} \times P_{11} \times P_{13}$

$= \frac{1}{6}$

4. (18')

The equivalent classes are $\{0\}$, $\{1\}$, $\{2,3\}$ and $\{4\}$. Classes $\{2,3\}$ and $\{4\}$ are recurrent, and classes $\{0\}$ and $\{1\}$ are transient.