Caculus II Math 1038 (1002&1003)

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Midterm exam: 13 Apr Thu 6:30-8:00pm Venue: T2-101

Week 7: Ch14 Partial differentiation

1. Plot functions of two variables $z = f(x, y)$
   (a) Online tools: WolframAlpha Plotting and Graphics https://www.wolframalpha.com/
   (b) Python: Matplotlib package
   (c) Matlab plot
2. Limit and continuity
   To check whether a limit
   $$\lim_{(x,y)\to(a,b)} f(x, y)$$
   exist or not, if yes, find the limit, if not, prove the limit does not exsit. You need to follow these step:
   (a) Check domain D:
      i. if $(a, b) \in D$ and $f(x, y)$ are polynomials, **rational**, trignometrics, and common functions, then the limit exists, you can use **direct substitute** and the limit:
      $$L = f(a, b)$$
      e.g. since $(1, 2) \in D = \{(x, y)|x^2 + y^2 \neq 0\}$
      $$\lim_{(x,y)\to(1,2)} \frac{xy}{x^2 + y^2} = \frac{2}{5}$$
   (b) If $(a, b) \notin D$, then we have two situations, not exist or exist.
      i. If you think it may NOT EXIST, find two different paths $y = 0$, $x = 0$, $y = x$ or $y = x^2$, etc. try to obtain two different limits. If you found two paths which leads to two different limits, then you proved the limit does NOT exist. However if you failed to find such two paths, then you can try to prove the limit exists. [**If you find two or more paths which give you the same limit, it does NOT mean anything**, we cannot draw any conclusion about existence of the limit.]
      ii. If you think the limit EXIST, you can try **Squeezed Theorem** or other techniques such as change of coordinates (to polar coordinate) to find the limit. An example using polar coordinate and Taylor's series:
      $$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2} &= \lim_{r\to0} \frac{e^{-r^2} - 1}{r^2} \\ &= \lim_{r\to0} \frac{(1 - r^2 + \frac{r^4}{2} - ...) - 1}{r^2} \\ &= -1\end{aligned}$$
   (c) If you cannot find two paths gives different limit nor can you find the limit, you may do something wrong and have to try again.
   (d) Remembering all the examples can give you more experience about the initial guess.
3. Continuity
   Definition: f is continuous on D if f is continuous at every point (a, b) in D.
   $$\lim_{(x,y)\to(a,b)} f(x, y) = L = f(a, b)$$

(a) $f(x,y)$ is not continuous at where it is not defined, e.g.

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

is NOT continuous (is discontinuous) at $(0,0)$, because it is not defined (is undefined).

(b) $f(x,y)$ is not continuous at where the limit does not exist, e.g.

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if} \quad x^2 + y^2 \neq 0 \\ 0 & \text{if} \quad x^2 + y^2 = 0 \end{cases}$$

Although $f(x,y)$ is defined at $(0,0)$, the limit does not exist. You can choose two paths: $x = 0$ $(L = -1)$ and $y = 0$ $(L = 1)$

(c) Any rational function **is continuous** on its domain.

(d) Composite function of continuous functions is also a continuous function.

4. Partial derivative

(a) Definition: partial derivative with respect to x at (a,b),

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

(b) to find partial derivatives $f_x(x,y)$

i. use definition

ii. treat the other variable y as a constant and differentiate it as a function of single variable.

(c) other notations:

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = D_x f$$

if $z = f(x,y)$, we can also use

$$\frac{\partial z}{\partial x}$$

(d) Interpretation of $f_x(a,b)$: slope of the tangent lines at point $(a,b,f(a,b))$ to the traces C in the plane $y = b$.

(e) Higher derivatives

$$f_{xx}(x,y) = f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2}{\partial x^2} f(x,y) = D_{xx} f$$

(f) **Clairaut's Theorem**: $f(x,y)$ is defined on a disk D that contains (a,b). If f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

(g) Partial differential equation

i. Laplace's equation

ii. wave equation

5. **Differentiability**

(a) increment of z: Δz

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a,b)$$

(b) Definition: The function is **differentiable at** (a,b) if $f_x(a,b)$ and $f_y(a,b)$ exist and

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

(c) If $f_x(a,b)$ and $f_y(a,b)$ exist and are **continuous** at (a,b), then f is differentiable, and if a function f is differentiable at (a,b), then it is continuous at (a,b).

(d) **Important relationships**: i $\to$ ii $\to$ iii $\to$ iv

i. $f_x(a,b)$ and $f_y(a,b)$ are **continuous**

ii. f **is differentiable at** (a,b)

iii. f is continuous at (a,b)

iv. $\lim_{(x,y)\to(a,b)} f(x,y) = L$ limit exist.

iv $\not\to$ iii $\not\to$ ii $\not\to$ i.

(e) To prove f is differentiable

i. Use definition: difficulty

ii. To show $f_x(a,b)$ and $f_y(a,b)$ exist AND are **continuous** at (a,b), .

(f) To prove f is NOT differentiable:

i. To show f is not continuous, or

ii. by definition: ϵ_1 and $\epsilon_2 \not\to 0$.

6. Tangent plane through a point $P(x_0, y_0, z_0)$

(a) Recall for function of single variable $y = f(x)$

i. tangent direction: $\overrightarrow{u} = \langle 1, f'(x_0)\rangle$, normal direction $\overrightarrow{n} = \langle f'(x_0), -1\rangle$ since $\overrightarrow{u} \cdot \overrightarrow{n} = 0$

ii. equation of tangent line: $y - y_0 = f'(x_0)(x - x_0)$

(b) two tangent directions: $\overrightarrow{u}_x = \langle 1, 0, f_x(x_0, y_0)\rangle$ and $\overrightarrow{u}_y = \langle 0, 1, f_y(x_0, y_0)\rangle$

(c) normal vector of tangent plane $\overrightarrow{n} = \overrightarrow{u}_x \times \overrightarrow{u}_y = \langle f_x, f_y, -1\rangle$

(d) **equation of a tangent plane** to the surface $z = f(x,y)$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

(e) vector equation of a normal line

$$(x, y, z) = (x_0, y_0, z_0) + t\,\langle f_x, f_y, -1\rangle$$

7. Linear approximations

(a) Recall for $f(x)$, linear approximation (first degree Taylor polynomial):

$$L(x) = f(a) + f'(a)(x - a)$$

(b) for $f(x,y)$

$$L(x,y) = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b)$$

8. Differentials

(a) Recall for $f(x)$:

i. differential dx is the independent variable

ii. differential $dy = f'(x)dx$: change of height of the tangent line

iii. difference or increment of x is $\Delta x = dx$

iv. difference or increment of y is $\Delta y = f(x + \Delta x) - f(x)$

v. $\Delta y \approx dy = f'(x)dx = f'(x)\Delta x$

(b) total differential

$$dz = f_x(a,b)dx + f_y(a,b)dy$$

compare with the difference/increment $\Delta z = f(a + \Delta x, b + \Delta y) - f(a,b)$, we have $dz \approx \Delta z$.

9. Chain rule

(a) Recall for $y = f(x)$ and $x = g(t)$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

x is a intermediate variable and t is the sole independent variable.

(b) $z = f(x(t), y(t))$

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

with x and y are intermediate variables and t is the sole independent variable.

(c) $z = f(x(s,t), y(s,t))$

$$\frac{\partial z}{\partial t} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

$$\frac{\partial z}{\partial s} = \frac{dz}{dx} \cdot \frac{dx}{ds} + \frac{dz}{dy} \cdot \frac{dy}{ds}$$

(d) change of coordinate $(x, y) \to (r, \theta)$, with $x = r\cos\theta$ and $y = r\sin\theta$

(e) **Implicit differentiation** $F(x, y) = 0$, find dy/dx
Differentiate both side

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

then

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

10. Directional derivatives

(a) Recall $z = f(x, y)$, $f_x = D_x f$ represents the rates of change of z in the x-direction, in the directions of the unit vector $\overrightarrow{i} = \langle 1, 0\rangle$, similarly, $f_y = D_y f$ represents the rates of change of z in the y-direction, in the directions of the unit vector $\overrightarrow{j} = \langle 0, 1\rangle$

(b) Definition of **directional derivative** of f at (x_0, y_0) **in the direction** of a **unit vector** $\overrightarrow{u} = \langle a, b\rangle$

$$D_{\overrightarrow{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

or we can represent $\overrightarrow{u} = \langle \cos\theta, \sin\theta\rangle$.

(c) Theorem

$$D_{\overrightarrow{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = \langle f_x(x, y), f_y(x, y)\rangle \cdot \langle a, b\rangle$$

prove by **Chain Rule** via defining a new function $g(h) = f(x + ha, y + hb)$ then $g(h) = f(x, y)$ with $x = x_0 + ha$ and $y = y_0 + hb$, so $\frac{dx}{dh} = a$ and $\frac{dy}{dh} = b$.

$$g'(h) = \frac{dg}{dh} = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b$$

11. Gradient vector

$$D_{\overrightarrow{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = \langle f_x(x, y), f_y(x, y)\rangle \cdot \langle a, b\rangle = \langle f_x, f_y\rangle \cdot \overrightarrow{u} = \nabla f \cdot \overrightarrow{u}$$

$$D_{\overrightarrow{u}} f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c = \nabla f \cdot \overrightarrow{u}$$

(a) Recall $f(x)$, the slope/gradient is $f'(x)$

(b) The gradient of a function $f(x, y)$, grad$f$, ∇f (read "del f"), which is a vector function

$$\nabla f = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\overrightarrow{i} + \frac{\partial f}{\partial y}\overrightarrow{j}$$

(c) directional derivative

$$D_{\overrightarrow{u}} f(x, y) = \nabla f \cdot \overrightarrow{u}$$

a projection of the gradient vector onto $\overrightarrow{u}$

(d) gradient of function of three variables

$$\nabla f = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \frac{\partial f}{\partial x}\overrightarrow{i} + \frac{\partial f}{\partial y}\overrightarrow{j} + \frac{\partial f}{\partial z}\overrightarrow{k}$$

$$D_{\overrightarrow{u}} f(x,y) = \nabla f \cdot \overrightarrow{u}$$

where $\overrightarrow{u} = \langle a, b, c\rangle$

(e) maximizing $D_{\overrightarrow{u}}$

$$D_{\overrightarrow{u}} f(x,y) = \nabla f \cdot \overrightarrow{u} = |\nabla f| \cdot |\overrightarrow{u}| \cos\theta$$

it attains its maxima when $\theta = 0$ which means $\overrightarrow{u}$ is in the same direction as the gradient of f. Since $|\overrightarrow{u}| = 1$ and $\cos\theta = 1$

$$\max D_{\overrightarrow{u}} f(x,y) = |\nabla f|$$

direction of steepest aescent.

(f) **To find the maximum rate of change**, just need to find ∇f and then compute its length $|\nabla f|$ and the direction is the normalized ∇f,

$$\overrightarrow{u} = \frac{\nabla f}{|\nabla f|}$$

(g) other important conclusions

i. When $\theta = \pi$, $\cos\theta = -1$

$$\min D_{\overrightarrow{u}} f(x,y) = -|\nabla f|$$

direction of steepest descent

ii. $D_{\overrightarrow{u}} = 0$, when $\theta = \pi/2$ $\cos\theta = 0$, $\overrightarrow{u}$ is perpendicular to∇f (tangent to the level curves).

iii. The paths of steepest ascent/descent is a curve that remains **perpendicular to each level curves** through which it passes.

(h) Theorem: tangent to the level curve of f at (a,b) is orthogonal to the gradient $\nabla f(a,b)$ given $\nabla f(a,b) = 0$. at level curves $f(x,y) = k$, so $f_x + f_y y'(x) = 0$ and $y' = -f_x/f_y$, so

tangent direction

$$\overrightarrow{t} = \langle 1, y'\rangle = \langle 1, -f_x/f_y\rangle$$

or times f_y and get

$$\overrightarrow{t} = \langle -f_y(a,b), f_x(a,b)\rangle$$

gradient direction:

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle$$

therefore

$$\overrightarrow{t} \cdot \nabla f(a,b) = 0$$

(i) Equation of the tangent line for $f(x,y) = z$

$$\nabla f(a,b) \cdot \langle x-a, y-b\rangle = 0$$

or

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) = 0$$

(j) Theorem: the gradient of function $f(x,y,z)$ is normal to the tangent plane to the level surface $f(x,y,z) = k$ at the point (a,b,c). The equation of the tangent plane

$$f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0$$