2021-22 First Semester
MATH1083 Calculus II (1003)

Assignment 7

Due Date: 11:30am 4/Mar/2021(Tue).

- Write down your **Chinese name** and **student number**. Write neatly on **A4-sized** paper and **show your steps**.
- **Late submissions or answers without details will not be graded.**

1. Find the limit

   (a)
$$\lim_{(x,y)\to(3,2)} \frac{x^2y+xy^2}{x^2-y^2}$$

   (b)
$$\lim_{(x,y)\to(\pi,\pi/2)} \frac{\cos y-\sin 2y}{\cos x\cos y}$$

   Solution: (a) The point $(3,2)$ is in the domain of this rational function, so we can use **direct subsitution** method and get $\lim = 6$;
   (b) We can first simplify the function by dividing $\cos y$ for both numerator and denominator and then use direct substitution

$$\begin{aligned} L &= \lim_{(x,y)\to(\pi,\pi/2)} \frac{\cos y-2\sin y\cos y}{\cos x\cos y} \\ &= \lim_{(x,y)\to(\pi,\pi/2)} \frac{1-2\sin y}{\cos x} \end{aligned}$$

   since $\cos\pi=-1$ and $\sin\pi/2=0$, then $L=-1$.

2. Show that the limit does not exist

   (a)
$$\lim_{(x,y)\to(0,0)} \frac{xy^2\cos y}{x^2+y^4}$$

   (b)
$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2}$$

   - Solution: (a) first approaching from x-axis, we have $y=0$ on the paths for all the point, thus

$$f(x,y)\to 0 \qquad \text{as} \qquad (x,y)\to(0,0) \qquad \text{along the x-axis}$$

     Second path: $x=y^2$, then

$$f(x,y)=f(y^2,y)=\lim_{(x,y)\to(0,0)} \frac{y^4\cos y}{2y^4}=\lim_{(x,y)\to(0,0)} \frac{\cos y}{2}=\frac{1}{2}$$

     Since we obatained different limits along different paths, the given limit does not exist.
     (b) First let's simplify the function, as $(x,y)\to(1,0)$ so, $(x-1,y)\to(0,0)$

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2}=\lim_{(x,y)\to(1,0)} \frac{(x-1)\,y}{(x-1)^2+y^2}=\lim_{(u,y)\to(0,0)} \frac{uy}{u^2+y^2}$$

     where $u=x-1$.
     Path 1: If $y=0$ (approaching along x-axis), then $f(x,0)=0$.
     Path 2:If $y=x-1$ (or you can think the path as $y=u$), then

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)\,y}{(x-1)^2+y^2}=\lim_{(u,y)\to(0,0)} \frac{uy}{u^2+y^2}=\lim_{(x,y)\to(1,0)} \frac{y^2}{2y^2}=\frac{1}{2}$$

3. Use the Squeeze Theorem to find the limit

(a)

$$\lim_{(x,y)\to(0,0)} xy\sin\frac{1}{x^2+y^2}$$

(b)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

Solution: (a)

$$-|xy| \le \left|xy\sin\frac{1}{x^2+y^2}\right| \le |xy| \qquad \text{since} \qquad -1\le \sin\frac{1}{x^2+y^2} \le 1$$

$|xy| \to 0$ as $x\to 0$ and $y\to 0$, so $\lim_{(x,y)\to(0,0)} |xy| = 0$. so the limit is 0. because do not forget the absolute sign.

(b) since

$$-1 \le \frac{x}{\sqrt{x^2+y^2}} \le 1$$

$$-|y| \le \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le |y|$$

both sides approaches to 0, so the limit is 0.

4. Determine the set of points at which the given function is continuous.

(a)

$$f(x,y,z) = \frac{x^3}{y} + \sin z$$

(b)

$$f(x,y) = \begin{cases} \frac{\sin\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}} & \text{if} x^2+y^2\neq 1 \\ 1 & \text{if } x^2+y^2=1 \end{cases}$$

(c)

$$f(x,y,z) = \sqrt{y-x^2}\ln z$$

(a) This function is a sum of rational function and a trignometric function, so it is continuous on its domain $D = \{(x,y,z)\in\mathbb{R}^3 | y\neq 0\}$

(b) This function is a composite function with domain $x^2+y^2\neq 1$. At point $(0,0)$,

$$\lim_{x^2+y^2\to 1} \frac{\sin\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}} = \lim_{r\to 1}\frac{\sin\sqrt{1-r^2}}{\sqrt{1-r^2}} = 1$$

This can be proved using polar coordinate as well. so it is also continuous at $(0,0)$, then the function is continuous on $D = \{(x,y) | x^2+y^2\le 1\}$,

(c) $D = \{(x,y,z)\in\mathbb{R}^3 | y\ge x^2 \quad \text{and} \quad z\ge 0\}$ This function is continuous on its domain.

5. Use polar coordinates to find the limit

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$$

Solution: let $x = r\cos\theta$ and $y = r\sin\theta$, then $x^2+y^2 = r^2$ and $r\to 0^+$ when $(x,y)\to(0,0)$, applying Taylor series for e^{-r^2}

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2} = \lim_{r\to 0^+}\frac{e^{-r^2}-1}{r^2} \overset{L'}{\to} \lim_{r\to 0^+}\frac{-2re^{-r^2}}{2r} = \lim_{r\to 0^+} -e^{-r^2} = -1$$

$$= \lim_{r\to 0^+}\frac{(1-r^2+\frac{r^4}{2}-\cdots)-1}{r^2}$$

$$= -1$$

6. Find the first partial derivatives of the functions

(a)

$$f(x,y) = \frac{x}{(x+y)^2}$$

(b)

$$R(p,q) = \tan^{-1}\left(pq^2\right)$$

(c)

$$z = x\sin(xy)$$

Solution: (a) Using quotient rule

$$\frac{\partial f}{\partial x} = \frac{(x+y)^2 - 2\,(x+y)\,x}{(x+y)^4} = \frac{y-x}{(x+y)^3}$$

$$\frac{\partial f}{\partial y} = \frac{-\,2x}{(x+y)^4}$$

(b) Using Chain rule, (recall $d(\tan^{-1}(x)) = \frac{1}{1+x^2}dx$)

$$\frac{\partial R}{\partial p} = \frac{q^2}{1+p^2q^4}$$

$$\frac{\partial R}{\partial q} = \frac{2pq}{1+p^2q^4}$$

(c) Using product rule,

$$\frac{\partial z}{\partial x} = \sin(xy) + xy\cos(xy)$$

Using Chain rule

$$\frac{\partial z}{\partial y} = x^2\cos(xy)$$