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Recall : Matrix Representation of linear transformation

E.g. $L: P_3 \to P_3$

$$L(p(x)) = \frac{d}{dx}(p(x))$$

$E = \{1, x, x^2\}$, $F = \{1, 2x, 4x^2-2\}$

① Find the matrix representation of L from E to F.

$$[A]_E^F = \left[[L(1)]_F \ [L(x)]_F \ [L(x^2)]_F \right]$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since

$$L(1) = \frac{d}{dx}(1) = 0$$

$$L(x) = \frac{d}{dx}(x) = 1 = 1 \times 1 + 0 \times 2x + 0 \times (4x^2-2)$$

$$L(x^2) = 2x = 0 \times 1 + 1 \times 2x + 0 \times (4x^2-2)$$

② Find the matrix representation of L from F to E?

$$[A]_F^E = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$L(1) = \frac{d}{dx}(1) = 0$ $\qquad$ $L(4x^2-2) = 8x$

$L(2x) = \frac{d}{dx}(2x) = 2$

③ Find the matrix representation of L w.r.t. E.

$$[A]_E^E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$L(1) = 0$

$L(x) = 1$

$L(x^2) = 2x$

④ Find the matrix representation of L w.r.t. F

$$[A]_F^F = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$L(1) = 0$

$L(2x) = 2$

$L(4x^2-2) = 8x$

<u>Def</u> (Similar)

Two $n \times n$ matrices A and B are said to be similar if there exists a nonsingular matrix S s.t.

$$B = S^{-1} A S.$$

<u>Thm</u> If B is similar to A, is A similar to B?

$$B = S^{-1} A S \longrightarrow SBS^{-1} = SS^{-1}ASS^{-1}$$

$$A = SBS^{-1} = P^{-1}BP$$

with $P = S^{-1}$.

Prove or disprove:

If A is similar to B, B is similar to C. Is A similar to C?

<u>Thm</u> Let L be a linear operator on V, and E, F be two bases of V.

If A is the matrix representation of L w.r.t. E,

and B is 〃 〃 〃 〃 F.

Then

$$[B]_F = S^{-1}[A]_E S$$

where S is the transition matrix from F to E.

$$[B]_F^F = [I]_S^F [A]_S^S [I]_F^S$$

E.g. P_3. $E = \{1, x, x^2\}$, $F = \{1, 2x, 4x^2-2\}$

$$[I]_F^E = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad [I]_E^F = (\quad)^{-1} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\frac{1}{2}R_2 \to R_2]{\frac{1}{4}R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{4} \end{array}\right) \xrightarrow{2R_3 + R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{4} \end{array}\right)$$

$$[A]_E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad [B]_F = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[I]_E^F \underbrace{[A]_E^E [I]_F^E} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$[I]_E^F \underbrace{[A]_E^E \, [I]_F^E}_{[A]_F^E} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} = [B]_F$$

Thm If A and B are similar, then

① A^T and B^T are similar

② $\det(A) = \det(B)$

③ A is nonsingular iff. B is nonsingular

④ $\operatorname{rank} A = \operatorname{rank}(B)$