MS: Solution to Test 2

1. (a) By computing the first moment, we yield $\theta = 3/2 - E(X)/2$, since

$$E(X) = 1 \cdot P(X = 1) + 3 \cdot P(X = 3) = 1 \cdot \theta + 3 \cdot (1 - \theta) = 3 - 2\theta$$

Replacing $E(X)$ by the sample mean $\bar{X}$, we have the MOM estimator of θ as

$$\hat{\theta}_{MOM} = \frac{3}{2} - \frac{1}{2}\bar{X}$$

where

$$\bar{X} = \frac{1+3+1+1}{4} = \frac{3}{2}.$$

Then the MOM estimate of θ is

$$\hat{\theta}_{MOM} = \frac{3}{2} - \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}.$$

(b) The likelihood function is

$$\begin{aligned}\mathcal{L}(\theta) &= f(x_1, \ldots, x_n \mid \theta) = f(x_1 \mid \theta) \cdots f(x_n \mid \theta) \\ &= P(X = x_1 \mid \theta) \cdots P(X = x_n \mid \theta)\end{aligned}$$

Substituting $n = 4$ and the realizations of X yields

$$\mathcal{L}(\theta) = P(X = 3 \mid \theta) \cdot P(X = 1 \mid \theta)^3 = (1 - \theta) \cdot \theta^3$$

(c) To find the MLE of θ, it's easier to work with the natural logarithm of $\mathcal{L}(\theta)$,

$$l(\theta) = \ln(\mathcal{L}(\theta)) = 3\ln(\theta) + \ln(1 - \theta),$$

$$l'(\theta) = \frac{3}{\theta} - \frac{1}{1-\theta} = \frac{3 - 4 \cdot \theta}{\theta \cdot (1 - \theta)}.$$

So

$$l'(\theta) = \frac{3 - 4 \cdot \theta}{\theta \cdot (1 - \theta)} = 0 \Longleftrightarrow 3 - 4 \cdot \theta = 0 \Longleftrightarrow \quad \hat{\theta}_{MLE} = \frac{3}{4}.$$

2. (a) Let n be the sample size, and let $X_1, \ldots, X_n \sim N(0, \theta)$ be independent identically distributed random variables with the same density function. Remember that the density function of $X \sim N(0, \theta)$ is

$$f(x) = \frac{1}{\sqrt{\theta} \cdot \sqrt{2\pi}} \cdot e^{-\frac{x^2}{2\theta}}, \quad x \in \mathbb{R}.$$

To find the MLE of θ, we first define the likelihood function:

$$\mathcal{L}(\theta) = f(x_1, \ldots, x_n \mid \theta) = f(x_1 \mid \theta) \cdots f(x_n \mid \theta).$$

Substituting the definition of the density function of X yields

$$\begin{aligned}\mathcal{L}(\theta) &= \frac{1}{\sqrt{\theta} \cdot \sqrt{2\pi}} \cdot e^{-\frac{x_1^2}{2\theta}} \cdots \frac{1}{\sqrt{\theta} \cdot \sqrt{2\pi}} \cdot e^{-\frac{x_n^2}{2\theta}} \\ &= \left(\frac{1}{\sqrt{\theta} \cdot \sqrt{2\pi}}\right)^n \cdot e^{-\frac{x_1^2 + \cdots + x_n^2}{2\theta}}\end{aligned}$$

It's easier to work with the natural logarithm of the given expression, so we define

$$l(\theta) = \ln(\mathcal{L}(\theta)) = n \cdot \ln\left(\frac{1}{\sqrt{\theta} \cdot \sqrt{2\pi}}\right) - \frac{1}{2\theta} \cdot \sum_{i=1}^{n} x_i^2,$$

and we need to find its global maximum on the interval $(0, +\infty)$ (where θ can take on values).

The derivative of l (with respect to θ) is

$$l'(\theta) = -\frac{n}{2\theta} + \frac{1}{2\theta^2} \cdot \sum_{i=1}^{n} x_i^2.$$

So

$$l'(\theta) = 0 \iff -\frac{n}{2\theta} + \frac{1}{2\theta^2} \cdot \sum_{i=1}^{n} x_i^2 = 0 \iff \sum_{i=1}^{n} x_i^2 = n \cdot \theta$$

$$\iff \theta = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i^2.$$

Therefore, the MLE for θ is

$$\hat{\theta} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i^2.$$

(b) The log-density function is

$$\log f(X|\theta) = -\log(\sqrt{2\pi\theta}) - \frac{X^2}{2\theta}$$

Then the score function is

$$\frac{\partial}{\partial\theta} \log f(X|\theta) = -\frac{1}{2\theta} + \frac{X^2}{2\theta^2} \quad \text{and} \quad \frac{\partial^2}{\partial\theta^2} \log f(X|\theta) = \frac{1}{2\theta^2} - \frac{X^2}{\theta^3}$$

$$I(\theta) = -E\left(\frac{\partial^2}{\partial\theta^2} \log f(X|\theta)\right) = -\frac{1}{2\theta^2} + \frac{E(X^2)}{\theta^3} = -\frac{1}{2\theta^2} + \frac{\mathrm{Var}(X) + E(X)^2}{\theta^3} = \frac{1}{2\theta^2}$$

The asymptotic variance of the mle $\hat{\theta}_{MLE}$ is $1/nI(\theta)$, that is, $2\theta^2/n$.

(c) Since $\sqrt{nI(\theta)}(\hat{\theta}_{MLE} - \theta)$ approximately follows $N(0, 1)$, namely,

$$\frac{\sqrt{n}(\hat{\theta}_{MLE} - \theta)}{\sqrt{2\theta^2}} \sim N(0, 1)$$

$$-z_{0.05} \le \frac{\sqrt{n}(\hat{\theta}_{MLE} - \theta)}{\sqrt{2\theta^2}} \le z_{0.05}$$

1. With $z_{0.05} = 1.645$, then

$$1 - 1.645 \cdot \sqrt{\frac{2}{n}} \le \frac{\hat{\theta}_{MLE}}{\theta} \le 1 + 1.645 \cdot \sqrt{\frac{2}{n}}$$

An approximate 90% interval estimator for θ:

$$\left[\hat{\theta}_{MLE}/\left(1 + 1.645 \cdot \sqrt{\frac{2}{n}}\right), \ \hat{\theta}_{MLE}/\left(1 - 1.645 \cdot \sqrt{\frac{2}{n}}\right)\right]$$

where n is large enough and $n \geq 6$.
2. Replace unknown $I(\theta)$ by $I(\hat{\theta}_{MLE})$, an approximate 90% interval estimator for θ:

$$\left[\hat{\theta}_{MLE} \cdot \left(1 - 1.645 \cdot \sqrt{\frac{2}{n}}\right),\ \hat{\theta}_{MLE} \cdot \left(1 + 1.645 \cdot \sqrt{\frac{2}{n}}\right)\right]$$

(d) The mle of θ is unbiased since

$$\hat{\theta}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \quad \text{with mean} \quad E(\hat{\theta}_{MLE}) = \frac{n\theta}{n} = \theta$$

The exact distribution of $\dfrac{n\hat{\theta}_{MLE}}{\theta}$ is $\chi^2(n)$. An exact 90% confidence interval of θ can be constructed from the statistics $n\hat{\theta}_{MLE}/\theta$:

$$\left[10\hat{\theta}_{MLE}/\chi^2_{0.05}(10),\ 10\hat{\theta}_{MLE}/\chi^2_{0.95}(10)\right]$$

where $\chi^2_{0.05}(10) = 18.307$ and $\chi^2_{0.95}(10) = 3.94$.

3. (a) Accept H_0, since

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = -1.739$$

and $|z| < 1.96 = z_{0.025}$.

(b) Reject H_0, since

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = -1.739$$

and $z < -1.645 = -z_{0.05}$.

4. (a) Suppose $X_1, \ldots, X_n$ are samples from $U(0, \theta)$. Then for $0 \leq x \leq \theta$,

$$F_{X_{\max}}(x) = P(X_{\max} \leq x) = P(X_1 \leq x, \cdots, X_n \leq x) = \prod_{i=1}^{n} P(X_i \leq x) = \left(\frac{x}{\theta}\right)^n$$

$$\Longrightarrow f_{X_{\max}}(x) = \frac{nx^{n-1}}{\theta^n}, \quad 0 \leq x \leq \theta.$$

We set

$$P(X_{\max} \leq 2.5 \mid H_0 \text{ is true}) = \alpha, \tag{1}$$

and the decision rule is "Reject H_0 if $X_{\max} \leq 2.5$".
The pdf of $X_{\max}$ given that H_0 is true is

$$f_{X_{\max}}(x \mid \theta = 3) = \frac{nx^{n-1}}{3^n}, \quad 0 \leq x \leq 3$$

$$\begin{aligned} & P(X_{\max} \leq 2.5 \mid H_0 \text{ is true }) = \alpha \\ \Rightarrow \quad & \int_0^{2.5} \frac{nx^{n-1}}{3^n} dx = \alpha \\ \Rightarrow \quad & \alpha = \left(\frac{2.5}{3}\right)^n \end{aligned}$$

(b) To make $\alpha \leq 0.05$, we have to choose n such that

$$\left(\frac{2.5}{3}\right)^n \leq 0.05.$$

We thus obtain

$$n \ln(2.5/3) \leq \ln(0.05) \quad \Rightarrow \quad n \geq 16.431.$$

So $n = 17$.