2023-24 First Semester

MATH2023 Ordinary and Partial Differential Equations (1002)

Assignment 5 Suggested Solutions

1. (a) The functions $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{-4t}$. The particular solution is given by $Y_P(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, in which

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t}$$

$$u_1(t) = -\int \frac{te^{-2t}(t^{-2}e^{-2t})}{W(t)}\,\mathrm{d}t = -\int t^{-1}\,\mathrm{d}t = -\ln t$$

$$u_2(t) = \int \frac{e^{-2t}(t^{-2}e^{-2t})}{W(t)}\,\mathrm{d}t = \int t^{-2}\,\mathrm{d}t = -1/t$$

Hence a particular solution is $Y_P(t) = -e^{-2t}\ln t - e^{-2t}$. Since the second term is a solution of the homogeneous equation, the general solution is given by

$$y(t) = c_1e^{-2t} + c_2te^{-2t} - e^{-2t}\ln t.$$

(b) The solution of the homogeneous equation is $y_c(t) = c_1\cos 3t + c_2\sin 3t$. Denote $y_1(t) = \cos 3t$ and $y_2(t) = \sin 3t$, with $W(y_1, y_2) = 3$. A particular solution is assumed as $Y_P(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, in which

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} = 3\cos^2 3t + 3\sin^2 3t = 3.$$

$$u_1(t) = -\int \frac{\sin 3t(9\sec^2 3t)}{W(t)}\,\mathrm{d}t = -\int \frac{3\sin(3t)}{\cos^2(3t)}\,\mathrm{d}t = -\frac{1}{\cos(3t)}$$

$$\begin{aligned} u_2(t) &= \int \frac{\cos 3t(9\sec^2 3t)}{W(t)}\,\mathrm{d}t = \int \frac{3\cos(3t)}{\cos^2(3t)}\,\mathrm{d}t = 3\int \frac{\cos(3t)}{1-\sin^2(3t)}\,\mathrm{d}t \\ &= \frac{3}{2}\int \cos(3t)\left[\frac{1}{1-\sin(3t)} + \frac{1}{1+\sin(3t)}\right]\,\mathrm{d}t \\ &= \ln|\sec 3t + \tan 3t| \end{aligned}$$

The general solution to the nonhomogeneous problem is

$$y(t) = c_1\cos 3t + c_2\sin 3t + (\sin 3t)\ln|\sec 3t + \tan 3t| - 1.$$

2. (a) $t^2y'' - t(t+2)y' + (t+2)y = 2t^3, \quad t > 0; \quad y_1(t) = t, \quad y_2(t) = te^t$

The corresponding homogeneous equation:

$$t^2y'' - t(t+2)y' + (t+2)y = 0 \qquad \text{(H)}$$

For $y_1(x) = t$, $y_1'(x) = 1$, $y_1''(x) = 0$, substitution gives us

$$\rightarrow \quad 0 - t(t+2) + (t+2)t = 0$$

For $y_2(x) = te^t$, $y_2'(x) = (t+1)e^t$, $y_2''(x) = (t+2)e^t$,

$$\rightarrow \quad t^2(t+2)e^t - t(t+2)(t+1)e^t + (t+2)te^t = 0$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & te^t \\ 1 & (t+1)e^t \end{vmatrix} = t(t+1)e^t - te^t = t^2e^t \neq 0.$$

$\rightarrow$ Both y_1 and y_2 satisfy (H) and form a fundamental set of solutions to (H).

Standard form of (N):

$$y'' - \frac{(t+2)}{t}y' + \frac{(t+2)}{t^2}y = 2t, \qquad t > 0$$

Assume $Y_P = u_1(x)y_1 + u_2(x)y_2$, then by the method of variation of parameter,

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(x) \end{bmatrix}, \quad \text{i.e.} \quad \begin{bmatrix} t & te^t \\ 1 & e^t(1+t) \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2t \end{bmatrix}$$

Use Cramer's Rule to solve for u_1' and u_2' and direct integration gives u_1 and u_2:

$$\begin{cases} u_1' = \dfrac{\begin{vmatrix} 0 & te^t \\ 2t & e^t(1+t) \end{vmatrix}}{W} = -\dfrac{2t^2e^t}{t^2e^t} = -2 \\ u_2' = \dfrac{\begin{vmatrix} t & 0 \\ 1 & 2t \end{vmatrix}}{W} = \dfrac{2t^2}{t^2e^t} = 2e^{-t} \end{cases} \rightarrow \quad \begin{cases} u_1 = -2t \\ u_2 = -2e^{-t} \end{cases}$$

$$Y_P(t) = u_1y_1 + u_2y_2 = -2t^2 - 2t$$

Notice that $y = -2t$ is a solution to (H), we may choose $Y = -2t^2$, as a solution to (N).

(b) $x^2y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0; \quad y_1(x) = x^2, \quad y_2(x) = x^2 \ln x$

It is trivial to verify that both y_1 and y_2 satisfy equation (H).

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix} = x^3 \neq 0.$$

The nonzero Wronskian of y_1 and y_2 suggests linear independency between y_1 and y_2. Therefore, $\{y_1, y_2\}$ form a fundamental set of solutions to (H).

Standard form of (N):

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = \ln x, \qquad x > 0$$

Assume $Y_P = u_1(x)y_1 + u_2(x)y_2$, then by the variation of parameter,

$$\begin{bmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \ln x \end{bmatrix}$$

$$u_1' = \begin{vmatrix} 0 & x^2 \ln x \\ \ln x & 2x\ln x + x \end{vmatrix} \Big/ W = -\frac{\ln^2 x}{x} \quad \rightarrow \quad u_1 = -\frac{(\ln x)^3}{3},$$

$$u_2' = \begin{vmatrix} x^2 & 0 \\ 2x & \ln x \end{vmatrix} \Big/ W = \frac{\ln x}{x} \quad \rightarrow \quad u_2 = \frac{(\ln x)^2}{2}$$

A particular solution to (N) is

$$Y_P(t) = u_1 y_1 + u_2 y_2 = \frac{1}{6} x^2 (\ln x)^3.$$

3. (a) Consider

$$\begin{aligned} (D - r_1)(D - r_2)y &= (D - r_1)(y' - r_2 y) = D(y') - r_1 y' - D(r_2 y) + r_1 r_2 y \\ &= y'' - (r_1 + r_2)y' + r_1 r_2 y \end{aligned}$$

Since $r_1 + r_2 = -b$ and $r_1 r_2 = c$, then

$$(D - r_1)(D - r_2)y = y'' + by' + cy$$

(b) The solution to the original DE can be found by solving a system of coupled first order DEs. Taking

$$u = (D - r_2)y, \tag{1}$$

the equation becomes

$$(D - r_1)(D - r_2)y = g(t) \qquad \rightarrow \qquad (D - r_1)u(t) = g(t). \tag{2}$$

Both equations are linear and of first order. Eq (2) is an equation in u while Eq (1) is one in y. Solve Eq (2) for u first,

$$u(t) = e^{r_1 t} \int e^{-r_1 \tau} g(\tau)\, \mathrm{d}\tau + c_1 e^{r_1 t}.$$

From above, we substitute $u(t)$ back into Eq (1) and solve for $y(t)$

$$y(t) = e^{r_2 t} \int e^{-r_2 \tau} u(\tau)\, \mathrm{d}\tau + c_2 e^{r_2 t}.$$

Note that the solution $y(t)$ contains two arbitrary constants.

4. (a) Since $r^2 + 4 = 0$, then $r_{1,2} = \pm 2i$ and

$$(D + 2i)(D - 2i)y(t) = 3e^t + t^2$$

Denote $k = 2i$ and take $u(t) = (D - k)y(t)$, we get

$$u' + ku = 3e^t + t^2, \qquad y' - ky = u(t).$$

The general solution for $u(t)$:

$$\begin{aligned}
[e^{kt}u(t)]' &= e^{kt}[3e^t + t^2] = 3e^{(k+1)t} + t^2 e^{kt} \\
e^{kt}u(t) &= \frac{3}{k+1}e^{(k+1)t} + \frac{1}{k}t^2 e^{kt} - \frac{2}{k}\int te^{kt}\ \mathrm{d}t + c_1 \\
&= \frac{3}{k+1}e^{(k+1)t} + \frac{1}{k}t^2 e^{kt} - \frac{2}{k^2}te^{kt} + \frac{2}{k^3}e^{kt} + c_1
\end{aligned}$$

We obtain

$$u(t) = \frac{3}{k+1}e^t + \frac{1}{k}t^2 - \frac{2}{k^2}t + \frac{2}{k^3} + c_1 e^{-kt}.$$

From above, solve another equation for $y(t)$:

$$\begin{aligned}
[e^{-kt}y(t)]' &= \frac{3}{k+1}e^{(1-k)t} + \frac{1}{k}t^2 e^{-kt} - \frac{2}{k^2}te^{-kt} + \frac{2}{k^3}e^{-kt} + c_1 e^{-2kt} \\
e^{-kt}y(t) &= \frac{3}{(1+k)(1-k)}e^{(1-k)t} - \frac{1}{k^2}t^2 e^{-kt} - \frac{2}{k^4}e^{-kt} - \frac{c_1}{2k}e^{-2kt} + c_2
\end{aligned}$$

The general solution to the original DE is

$$y(t) = \frac{3}{5}e^t + \frac{1}{4}t^2 - \frac{1}{8} + c_1\sin(2t) + c_2\cos(2t).$$

(b) By taking $u = (D+2)y$, we get

$$(D+2)u = t^{-2}e^{-2t}, \qquad (D+2)y = u$$

Solve $u' + 2u = t^{-2}e^{-2t}$:

$$\begin{aligned}
u(t) &= e^{-2t}\left[\int e^{2t}(t^{-2}e^{-2t})dt + C_1\right] \\
&= e^{-2t}\left[-\frac{1}{t} + C_1\right]
\end{aligned}$$

Solve $y' + 2y = u(t)$:

$$\begin{aligned}
y(t) &= e^{-2t}\left[\int e^{2t}(-t^{-1}e^{-2t} + C_1 e^{-2t})dt + C_2\right] \\
&= e^{-2t}\left[-\ln t + C_1 t + C_2\right]
\end{aligned}$$

Thus the general solution to the original DE is given by

$$y(t) = -e^{-2t}\ln t + C_1 te^{-2t} + C_2 e^{-2t}.$$

Remarks: Compare with HW4-Q5(a) and HW5-Q1(a). Can you solve them by using reduction of order?

5. The characteristic equation of the governing equation is $\frac{3}{2}r^2 + k = 0$ with $r_{1,2} = \pm i\sqrt{\frac{2k}{3}}$. The general solution is

$$u(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t), \qquad \text{where } \omega = \sqrt{\frac{2k}{3}}.$$

Use the initial conditions to determine the values of $c_{1,2}$:

$$c_1 = 2, \quad c_2 = v/\omega$$

Solution to this IVP:

$= \sqrt{2^2 + (\frac{v}{\omega})^2} \cos(\omega t - \mu)$

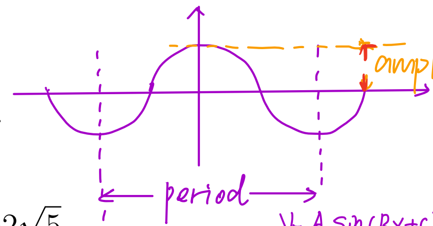
$$u(t) = 2\cos(\omega t) + v/\omega \sin(\omega t) = R\cos(\omega t - \mu), \qquad \text{where } R = \sqrt{4 + \frac{v^2}{\omega^2}}, \quad \cos\mu = 2/R$$

Since the period is π and the amplitude is 3, so

$\frac{2\pi}{\omega}$

$$\frac{2\pi}{\omega} = \pi \quad \rightarrow \quad \omega = 2, k = 6.$$

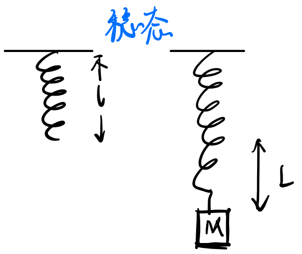
$$R = \sqrt{4 + \frac{v^2}{\omega^2}} = 3 \quad \rightarrow \quad v = \pm 2\sqrt{5}$$



$y = A\sin(Bx + C) + D$

amplitude A

period $\frac{2\pi}{B}$.



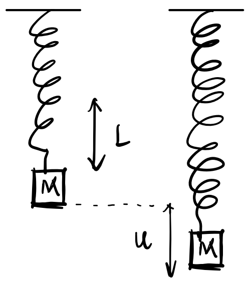
弹簧原长度为 l

栓上重为 mg 的物体 M 后，稳态下弹簧伸长 L ⇒ 长度变为 $l(l+L)$

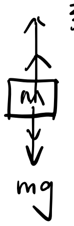
物体因此受向上的弹力 kL

稳态：$mg = kL$ $\qquad k = \frac{mg}{L}$

t时刻 物体M在外力作用下移动 $u(t)$ ： $u'(t) = v(t)$ $\quad u''(t) = a(t)$



受力分析：



弹力 $= -k(L+u)$

阻力 $= -\gamma v = F_d$

外力使弹簧向下移动 $= F(t)$

$$ma(t) = mu''(t) = mg + F(t) - k(L+u) - \gamma v$$

$$mu''(t) + \gamma u'(t) + ku = \underbrace{mg}_{} + F(t) \underbrace{- kL}_{} = 0$$

$$= F(t)$$

$u(0) =$ 初位置 u_0

$u'(0) =$ 初速度 V_0.

$F(t) = 10 \sin \frac{t}{2}$

$m = 5$ $L = 10\,cm$

6. (Forced Vibration) A mass of 5kg stretches a spring 10cm. The mass is acted on by an external force of $10\sin(t/2)$ N(newtons) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4cm/sec. $F_d = 2$

$V = 4\,cm/s$ $u_0 = 0$ $V_0 = 3\,cm/s.$

(a) If the mass is set in motion from its equilibrium position with an initial velocity of 3cm/sec, formulate the initial value problem describing the motion of the mass.

(b) Find the solution of the initial value problem.

(c) Identify the transient and steady-state parts of the solution.

(d) (*Optional!*) If the given external force is replaced by a force $2\cos(\omega t)$ of frequency ω, find the value of ω for which the amplitude of the forced response is maximum.

homogeneous solution. $\lim_{t \to \infty} u(t) = \frac{1}{153281}\left(-160\cos\frac{t}{2} + 3128\sin\frac{t}{2}\right)$

6. The gravity acceleration $g = 9.8m/sec^2$, then the spring constant $k = mg/L = 5 \times 9.8/0.1 = 490(N/m)$ and the damping constant $\gamma = F_d/v = 50(N \cdot sec/m)$.

(a) $u'' + 10u' + 98u = 2\sin(t/2), \quad u(0) = 0, \quad u'(0) = 0.03.$

(b) Solving the associated homogeneous equation for u_H:

$$r^2 + 10r + 98 = 0 \quad \rightarrow \quad r_{1,2} = \frac{-10 \pm \sqrt{100 - 4 * 98}}{2} = -5 \pm i\sqrt{73}$$

The general solution to the associated homogeneous problem is

$$u_H(t) = e^{-5t}\left[c_1 \cos(\mu t) + c_2 \sin(\mu t)\right], \quad \text{where } \mu = \sqrt{73}.$$

Assume that a particular solution to the non-homogeneous problem has the form

$$u_P(t) = A\cos(t/2) + B\sin(t/2).$$

Substituting u_P back into the DE,

$$-\frac{1}{4}\left[A\cos\frac{t}{2} + B\sin\frac{t}{2}\right] + \frac{10}{2}\left[-A\sin\frac{t}{2} + B\cos\frac{t}{2}\right] + 98\left[A\cos\frac{t}{2} + B\sin\frac{t}{2}\right] = 2\sin\frac{t}{2}$$

$$\left[-\frac{A}{4} + 5B + 98A\right]\cos\frac{t}{2} + \left[-\frac{B}{4} - 5A + 98B\right]\sin\frac{t}{2} = 2\sin\frac{t}{2}$$

$$u_P(t) = \frac{1}{153281}\left[-160\cos\frac{t}{2} + 3128\sin\frac{t}{2}\right].$$

Thus, the general solution of this problem is $u(t) = u_H(t) + u_P(t)$. By considering the initial conditions,

$$\begin{cases} c_1 - \frac{160}{153281} = 0 \\ -5c_1 + c_2\mu + \frac{1564}{153281} = 0.03 \end{cases} \quad \rightarrow \quad \begin{cases} c_1 = \frac{160}{153281} \\ c_2 = \frac{383443}{153281} \cdot \frac{1}{100\mu} \end{cases}$$

The solution is

$$u(t) = \frac{1}{153281}\left[160e^{-5t}\cos(\mu t) + \frac{383443}{100\mu}e^{-5t}\sin(\mu t) - 160\cos\frac{t}{2} + 3128\sin\frac{t}{2}\right], \quad \mu = \sqrt{73}.$$

(c) $u_H(t)$ is the transient part and $u_P(t)$ is the steady state part of the solution.

(d) Assume that the forced response $u_P(t) = A\cos(\omega t) + B\sin(\omega t)$ satisfies

$u'' + 10u' + 98u = 2\cos(\omega t)$

$$\underbrace{\left[-\omega^2 A + 10\omega B + 98A\right]}_{=2}\cos(\omega t) + \underbrace{\left[-\omega^2 B - 10\omega A + 98B\right]}_{=0}\sin(\omega t) = 2\cos(\omega t)$$

$\begin{cases} -\omega^2 A + 10\omega B + 98A = 2 \\ -\omega^2 B - 10\omega A + 98B = 0. \end{cases}$

$\begin{pmatrix} -\omega^2+98 & 10\omega \\ -10\omega & 98-\omega^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$\begin{pmatrix} A \\ B \end{pmatrix} = \dfrac{\begin{pmatrix} 98-\omega^2 & -10\omega \\ 10\omega & -\omega^2+98 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix}}{\begin{vmatrix} -\omega^2+98 & 10\omega \\ -10\omega & 98-\omega^2 \end{vmatrix}}$

$$\rightarrow \quad \begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} (98-\omega^2) & -10\omega \\ 10\omega & (98-\omega^2) \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} 2(98-\omega^2) \\ 20\omega \end{bmatrix}$$

where $\Delta = (98-\omega^2)^2 + 100\omega^2$. Since the amplitude of the forced response is

$$R = \sqrt{A^2 + B^2} = \frac{2}{\sqrt{\Delta}},$$

then R will reach its maximum when $\sqrt{\Delta}$ attains its minimum, namely,

$$\frac{\mathrm{d}\Delta}{\mathrm{d}\omega} = -2\omega \cdot 2(98-\omega^2) + 200\omega = \omega(4\omega^2 - 192) = 0$$

When $\omega = 4\sqrt{3}$, the amplitude of the forced response is maximum $\frac{2}{\sqrt{7300}}$.