Chapter 2 Determinants

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Section 2.1 The Determinant of a Matrix

Definition (Minor and Cofactor) Let $A = (a_{ij})$ be an $n \times n$ matrix, and let M_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the row and column containing a_{ij}. The determinant of M_{ij} is called the **minor** of a_{ij}. We define the **cofactor** A of a_{ij} by

$$A_{ij} = (-1)^{i+j} \det(M_{ij}).$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}, \quad M_{ij} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

Definition (Determinant) The **determinant** of an $n \times n$ matrix A, denoted $\det(A)$ or $|A|$, is a scalar associated with the matrix A that is defined inductively as

$$\det(A) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & \text{if } n > 1 \end{cases},$$

where

$$A_{1j} = (-1)^{1+j} \det(M_{1j}) \qquad j = 1, 2, \cdots, n$$

are cofactors associated with the entries in the first row of A.

Example Find $\det(A)$ and $\det(B)$ for $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{pmatrix}$.

Solution

$$\det(A) = 1 \times 4 - 2 \times 3 = -2$$

$$\begin{aligned} \det(B) &= b_{11}\det(M_{11}) - b_{12}\det(M_{12}) + b_{13}\det(M_{13}) \\ &= 2\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} - 5\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 4\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} \\ &= 2(6-8) - 5(18-10) + 4(12-5) \\ &= -16 \end{aligned}$$

Exercise*

Find the determinant of $C = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix}$.

Answer: -24.

Remark

If A is a 2×2 matrix, then

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Theorem If A is an $n \times n$ matrix with $n \geq 2$, then $\det(A)$ can be expressed as a cofactor expansion **using any row or column** of A, i.e.

$$\begin{aligned} \det(A) &= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \\ &= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \end{aligned},$$

where $i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, n$.

$\rightarrow$ Reduce computational cost by using the row or column that contains the most zeros.

Example Compute $\det(C)$ for $C = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix}$.

Example Let $B = \begin{pmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{pmatrix}$.

To find $\det(B)$ by expanding along the second column, we have

$$\det(B) = -5\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = -5(18-10) + (12-20) - 4(4-12) = -16.$$

To find $\det(B)$ by expanding along the third row, we have

$$\det(B) = 5\begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} + 6\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = 5(10-4) - 4(4-12) + 6(2-15) = -16.$$

Example

$$\begin{aligned}
\begin{vmatrix} 2 & 3 & -1 & 3 & 3 \\ 0 & -1 & 5 & 2 & -1 \\ 0 & 0 & 3 & 9 & 2 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{vmatrix} &= 2(-1)^{1+1} \begin{vmatrix} -1 & 5 & 2 & -1 \\ 0 & 3 & 9 & 2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{vmatrix} \\
&= 2(-1)(-1)^{1+1} \begin{vmatrix} 3 & 9 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 5 \end{vmatrix} \\
&= 2(-1)(3)(-1)^{1+1} \begin{vmatrix} -1 & 3 \\ 0 & 5 \end{vmatrix} \\
&= 2(-1)(3)(-1)(-1)^{1+1} \begin{vmatrix} 5 \end{vmatrix} \\
&= 2(-1)(3)(-1)(5) = 30
\end{aligned}$$

Theorem If A is an $n \times n$ triangular (diagonal) matrix, then the determinant of A equals the product of the diagonal elements of A, i.e.

$$\det \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = a_{11}a_{22}\cdots a_{nn}, \quad \det \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix} = \prod_{k=1}^{n} b_{kk}$$

Proof Exercise.

Theorem If A is a square matrix of order n, then $\det(A) = \det(A^T)$.

Proof The proof is by induction on n. Clearly, the result holds if $n = 1$, since 1×1 matrix must be a symmetric matrix.
Assume that the result holds for all $k \times k$ matrices.
Suppose that A is a $(k+1) \times (k+1)$ matrix. Expanding $\det(A)$ along the first row of A, we get

$$\det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + \cdots \pm a_{1,k+1}\det(M_{1,k+1}).$$

Since the M_{ij}'s are all $k \times k$ matrices, it follows from the induction hypothesis that

$$\det(A) = a_{11}\det(M_{11}^T) - a_{12}\det(M_{12}^T) + \cdots \pm a_{1,k+1}\det(M_{1,k+1}^T).$$

The right-hand side of the above equation is just the expansion by minors of $\det(A^T)$ using the first column of A^T. Therefore, $\det(A^T) = \det(A)$.

Theorem Let A be an $n \times n$ matrix,

(i) If A has a row or column consisting entirely of zeros, then $\det(A) = 0$.

(ii) If A has two identical rows or two identical columns, then $\det(A) = 0$.

Proof Exercise.

Exercise*

Compute $\det(A)$, $\det(B)$ and $\det(A+B)$ for $A = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Remark

$\det(A+B) \neq \det(A) + \det(B)$ in general!!