Chapter 3 Vector Spaces

Section 3.5 Change of basis

Theorem in Sc3.3 Let $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ be a basis for a vector space V. Then for each $\mathbf{v} \in V$, there exists a *unique* set of scalar $c_1, c_2, \cdots, c_n \in \mathbf{R}$ such that

$$\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}.$$

Definition (Coordinate vector) Suppose $\beta = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ be an **ordered** basis for a vector space V and $\mathbf{v} \in V$. If

$$\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}$$

for some $c_1, c_2, \cdots, c_n \in \mathbf{R}$, then

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbf{R}^n$$

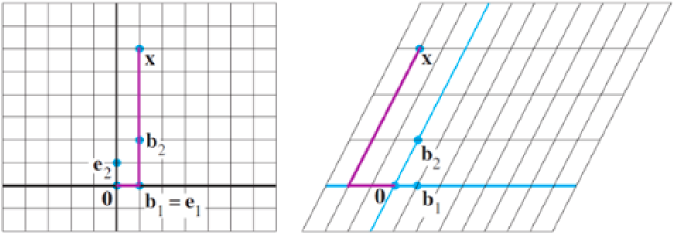
is called the **coordinate vector** of $\mathbf{v}$ with respect to β and is denoted $[\mathbf{v}]_\beta$. The c_i's are called the **coordinates** of $\mathbf{v}$ relative to β.

Example Consider two bases $\beta = \{\mathbf{u_1}, \mathbf{u_2}\}$ and $\gamma = \{\mathbf{e_1}, \mathbf{e_2}\}$ for $\mathbf{R}^2$, where $\mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{u_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Suppose $\mathbf{x} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$. Find $[\mathbf{x}]_\gamma$ and $[\mathbf{x}]_\beta$.

Solution Since $\mathbf{x} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{e_1} + 6\mathbf{e_2}$, we have $[\mathbf{x}]_\gamma = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$.

Since $\mathbf{x} = -2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -2\mathbf{u_1} + 3\mathbf{u_2}$, we have $[\mathbf{x}]_\beta = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

Different basis gives the same vector different coordinate vector.



Definition (Transition matrix) Let $\beta = \{\mathbf{u_1}, \cdots, \mathbf{u_n}\}$ and $\gamma = \{\mathbf{w_1}, \cdots, \mathbf{w_n}\}$ be two bases of a vector space V. The $n \times n$ matrix $\left([\mathbf{u_1}]_\gamma \,\middle|\, \cdots \,\middle|\, [\mathbf{u_n}]_\gamma \right)$ is called the **transition matrix** from β to γ.

In some books, the transition matrix from β to γ is denoted by $[I]_\beta^\gamma$ or $P_{\beta \to \gamma}$.

Example Consider two bases $\beta = \{\mathbf{u_1}, \mathbf{u_2}\}$ and $\gamma = \{\mathbf{e_1}, \mathbf{e_2}\}$ for $\mathbf{R}^2$, where $\mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{u_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then the transition matrix from β to γ is $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

Theorem Let $\beta = \{\mathbf{u_1}, \cdots, \mathbf{u_n}\}$ and $\gamma = \{\mathbf{w_1}, \cdots, \mathbf{w_n}\}$ be two bases of a vector space V. Let $[I]_\beta^\gamma$ be the transition matrix from β to γ. Then

1. $[\mathbf{v}]_\gamma = [I]_\beta^\gamma [\mathbf{v}]_\beta$ for any $\mathbf{v} \in V$.
2. $[I]_\beta^\gamma$ is nonsingular and $\left([I]_\beta^\gamma\right)^{-1}$ is the transition matrix from γ to β.

Proof of (1) Let a_{ij} such that

$$\begin{aligned}
\mathbf{u}_1 &= a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{n1}\mathbf{w}_n \\
\mathbf{u}_2 &= a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{n2}\mathbf{w}_n \\
&\cdots \\
\mathbf{u}_n &= a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{nn}\mathbf{w}_n
\end{aligned}$$

Let $\mathbf{v} \in V$. Suppose $[\mathbf{v}]_\beta = (c_1, c_2, \cdots, c_n)^T$. Then

$$\begin{aligned}
\mathbf{v} &= c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n \\
&= c_1(a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{n1}\mathbf{w}_n) + \\
&\quad c_2(a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{n2}\mathbf{w}_n) + \\
&\quad \cdots \\
&\quad c_n(a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{nn}\mathbf{w}_n)
\end{aligned}$$

$$
\begin{aligned}
=&(c_1a_{11} + c_2a_{12} + \cdots + c_na_{1n})\mathbf{w}_1 + \\
&(c_1a_{21} + c_2a_{22} + \cdots + c_na_{2n})\mathbf{w}_2 + \\
&\vdots \\
&(c_1a_{n1} + c_2a_{n2} + \cdots + c_na_{nn})\mathbf{w}_n
\end{aligned}
$$

Thus

$$
\begin{aligned}
[\mathbf{v}]_\gamma &= \begin{pmatrix} c_1a_{11} + c_2a_{12} + \cdots + c_na_{1n} \\ c_1a_{21} + c_2a_{22} + \cdots + c_na_{2n} \\ \vdots \\ c_1a_{n1} + c_2a_{n2} + \cdots + c_na_{nn} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = U[\mathbf{v}]_\beta
\end{aligned}
$$

Example Let $\beta = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $\gamma = \{\mathbf{e}_1, \mathbf{e}_2\}$ be two ordered bases of $\mathbf{R}^2$ where $\mathbf{u}_1 = (2,1)^T, \mathbf{u}_2 = (-1,3)^T$. If the coordinate vector of $\mathbf{v}$ with respect to γ is $[\mathbf{v}]_\gamma = (5,-1)^T$, find its coordinate vector $[\mathbf{v}]_\beta$?

Solution The transition matrix from β to γ is $U = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

The transition matrix from γ to β is $U^{-1} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$.

So, $[\mathbf{v}]_\beta = U^{-1}[\mathbf{v}]_\gamma = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Let β and γ be two non-standard bases of a vector space V. If W is the transition matrix from to β to the standard basis and U^{-1} is the transition matrix from the standard basis to γ, then the transition matrix from β to γ is $U^{-1}W$.

Example Let $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{(1,1,1)^T, (2,3,2)^T, (1,5,4)^T\}$ and $\gamma = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1,1,0)^T, (1,2,0)^T, (1,2,1)^T\}$.
(a) Which is the transition matrix from β to γ?
(b) Let $\mathbf{x} = 3\mathbf{u}_1 + 2\mathbf{u}_2 - 1\mathbf{u}_3$. Find the coordinates $[\mathbf{x}]_\gamma$ of $\mathbf{x}$ with respect to γ.

Solution (a) The transition matrix from β to γ is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 5 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -3 \\ -1 & -1 & 0 \\ 1 & 2 & 4 \end{pmatrix}$$

(b) The coordinates of $\mathbf{x}$ w.r.t. γ is

$$[\mathbf{x}]_\gamma = \begin{pmatrix} -1 & 1 & -3 \\ -1 & -1 & 0 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ -5 \\ 3 \end{pmatrix}.$$