Chapter 1 Matrices and System of Equations

Section 1.1 Systems of Linear Equations

Definition (Linear equation) A *linear equation* is an equation of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,$$

where $a_1, \cdots, a_n$ and b are all known in advance.

$x_1, \cdots, x_n$ are called the *variables* in the above case.

We sometimes write x, y, z, w, etc. for variables instead of $x_1, \cdots, x_n$.

Example $4x_1 - 5x_2 + 2 = x_1$ and $x_2 = 2(\sqrt{6} - x_1) + x_3$ are linear equations.

$4x_1 - 5x_2 + 2 = x_1$	and	$x_2 = 2(\sqrt{6} - x_1) + x_3$
$\downarrow$		$\downarrow$
rearranged		rearranged
$\downarrow$		$\downarrow$
$3x_1 - 5x_2 = -2$		$2x_1 + x_2 - x_3 = 2\sqrt{6}$

Definition (Linear System) A *linear system of m equations in n unknowns* is a system of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = b_2 \\ \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = b_m \end{cases}$$

Example $\begin{cases} x_1 & +2x_2 & & +x_4 & = 7 \\ x_1 & +x_2 & +x_3 & -x_4 & = 3 \end{cases}$

Write a linear system using matrix multiplication

The linear system $\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = b_2 \\ \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = b_m \end{cases}$ can be expressed as

$$A\mathbf{x} = \mathbf{b}$$

where A is the **Coefficient Matrix**,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Definition (Solution) A *solution* to the system of m equations in n unknowns $x_1, x_2, \cdots, x_n$ is a list of numbers

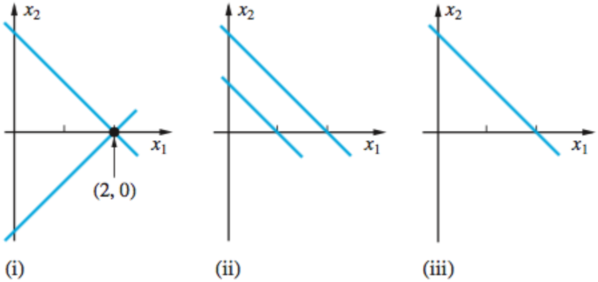
$$(x_1, x_2, \cdots, x_n) = (s_1, s_2, \cdots, s_n)$$

that satisfy *all* m equations.

Consistency of a system of equations

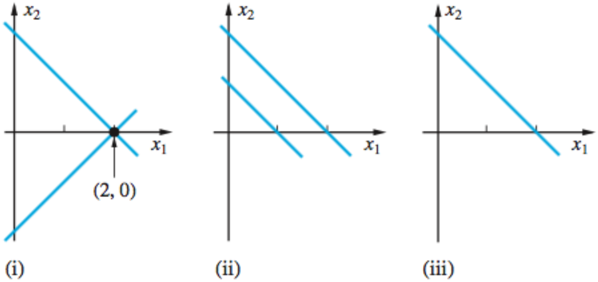
If a system has at least one solution, we say that it is consistent.
If a system has no solution, we say that the system is inconsistent.

$$\text{(i)}\ \begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 2 \end{aligned} \qquad \text{(ii)}\ \begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_2 &= 1 \end{aligned} \qquad \text{(iii)}\ \begin{aligned} x_1 + x_2 &= 2 \\ -x_1 - x_2 &= -2 \end{aligned}$$



(i) $x_1 + x_2 = 2$ (ii) $x_1 + x_2 = 2$ (iii) $x_1 + x_2 = 2$

 $x_1 - x_2 = 2$ $\quad$ $x_1 + x_2 = 1$ $\quad$ $-x_1 - x_2 = -2$



Basic Facts: A system of linear equations has either

(i) **unique solution** (consistent) or

(ii) **no solution** (inconsistent) or

(iii) **infinitely many solutions** (consistent) .

Definition (Equivalent Systems) Two systems of equations involving the same variables are said to be *equivalent* if they have the same set of solution.

Three elementary operations that give equivalent systems

I. The order in which any two equations are written may be interchanged.
Notation: $R_i \leftrightarrow R_j$.

Example Interchange two equations $(R_1 \leftrightarrow R_2)$.

The systems $\begin{cases} 2x_1 + x_2 = 8 \\ x_1 + x_2 = 3 \end{cases}$ and $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + x_2 = 8 \end{cases}$ are equivalent.

II. Both sides of an equation may be multiplied by the same nonzero real number c. Notation: $cR_i \to R_i$.

Example Multiply the second equation by 3 $(3R_2 \to R_2)$.

The systems $\begin{cases} 2x_1 + x_2 = 8 \\ x_1 + x_2 = 3 \end{cases}$ and $\begin{cases} 2x_1 + x_2 = 8 \\ 3x_1 + 3x_2 = 9 \end{cases}$ are equivalent.

III. A multiple of the i^{th} equation is added to j^{th} equation. Notation: $cR_i + R_j \to R_j$.

Example $-1/2$ times the first equation, added to the second equation $-1/2R_1 + R_2 \to R_2$.

The systems $\begin{cases} 2x_1 + x_2 = 8 \\ x_1 + x_2 = 3 \end{cases}$ and $\begin{cases} 2x_1 + x_2 = 8 \\ 1/2x_2 = -1 \end{cases}$ are equivalent.

Definition (Strict triangular form) A system is said to be in *strict triangular form* if, in the kth equation, the coefficients of the first $k-1$ variables are all zero and the coefficient of x_k is nonzero $(k = 1, \cdots, n)$.

Example The system of equations

$$\begin{cases} 3x_1 + & 2x_2 - & x_3 = & 1 \\ 0x_1 + & 1x_2 + & 0x_3 = & 3 \\ 0x_1 + & 0x_2 + & 2x_3 = & 4 \end{cases}$$

is of strict triangular form.

Example Solve $\begin{cases} x_1 + 2x_2 + x_3 & = 3 \\ 3x_1 - x_2 - 3x_3 & = -1 \\ 2x_1 + 3x_2 + x_3 & = 4 \end{cases}$

Outline of solution

Stage 1 Reduce the system to a strict/upper triangular form

Perform elementary operations to give an equivalent system

$$\begin{cases} x_1 + 2x_2 & +x_3 & = 3 \\ -7x_2 & -6x_3 & = -10 \\ & (-1/7)x_3 & = (-4/7) \end{cases}$$

Stage 2 Backward substitution method

In the last equation of that system, x_1 and x_2 are eliminated. Then we solve for x_3 and then substitute the value of x_3 to find x_2, and then x_1.

Stage 1 Perform elementary operations to change

$$\begin{cases} x_1 + 2x_2 + x_3 & = 3 \\ 3x_1 - x_2 - 3x_3 & = -1 \\ 2x_1 + 3x_2 + x_3 & = 4 \end{cases} \quad \text{to} \quad \begin{cases} x_1 + 2x_2 & +x_3 & = 3 \\ -7x_2 & -6x_3 & = -10 \\ & (-1/7)x_3 & = (-4/7) \end{cases}$$

Detail: Eliminate x_1 in Row 2 by replacing Row 2 with "-3(Row 1)+Row 2"

$$\begin{cases} x_1 & +2x_2 & +x_3 & = 3 \\ 0x_1 & -7x_2 & -6x_3 & = -10 \\ 2x_1 & +3x_2 & +x_3 & = 4 \end{cases}$$

Eliminate x_1 in Row 3 by replacing Row 3 with "-2(Row 1)+Row 3"

$$\begin{cases} x_1 & +2x_2 & +x_3 & = 3 \\ & -7x_2 & -6x_3 & = -10 \\ 0x_1 & -x_2 & -x_3 & = -2 \end{cases}$$

Eliminate x_2 in Row 3 by replacing Row 3 with "$\frac{-1}{7}$(Row 2)+Row 3"

$$\begin{cases} x_1 & +2x_2 & +x_3 & = 3 \\ & -7x_2 & -6x_3 & = -10 \\ & 0x_2 & +(-1/7)x_3 & = (-4/7) \end{cases}$$

Stage 2 Solve x_1, x_2, x_3 from $\begin{cases} x_1 + 2x_2 \quad + x_3 = 3 \\ \quad -7x_2 \quad -6x_3 = -10 \\ \quad\quad -\frac{1}{7}x_3 = -\frac{4}{7} \end{cases}$ by back substitution.

Detail From Row (3),

$$x_3 = 4.$$

Substitute $x_3 = 4$ into Row (2),

$$7x_2 = -10 + 6(4)$$

$$\text{and so} \quad x_2 = -2.$$

Substitute $x_3 = 4$ and $x_2 = -2$ into Row (1),

$$x_1 = 3 - 4 - 2(-2) = 3.$$

So, the solution is $(x_1, x_2, x_3) = (4, -2, 3)$.

Matrix Representation of a Linear System

$$\left\{\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m \end{aligned}\right. \quad \rightarrow \quad A\mathbf{x} = \mathbf{b}$$

Define $[A|\mathbf{b}]$ as the $m \times (n+1)$ augmented matrix

$$[A|\mathbf{b}] = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right)$$

Example The system of linear equations

$$\begin{aligned} 2x_1 + 4x_2 - 3x_3 + 5x_4 + x_5 &= 9 \\ 3x_1 + x_2 + x_4 - 3x_5 &= 0 \\ -2x_1 + 7x_2 - 5x_3 + 2x_4 + 2x_5 &= -3 \end{aligned}$$

has matrix representation $A\mathbf{x} = \mathbf{b}$ with the augmented matrix $[A|\mathbf{b}]$ as

$$\left(\begin{array}{ccccc|c} 2 & 4 & -3 & 5 & 1 & 9 \\ 3 & 1 & 0 & 1 & -3 & 0 \\ -2 & 7 & -5 & 2 & 2 & -3 \end{array}\right).$$

Elementary Row Operations

(I) (*Interchange*) Interchange two rows.

$$R_i \leftrightarrow R_j$$

(II) (*Scaling*) Multiply a row by a nonzero constant.

$$cR_i \rightarrow R_i, \quad c \neq 0$$

(III) (*Replacement*) Replace a row by its sum with a multiple of another row.

$$R_i + cR_j \rightarrow R_i$$

Elementary row operations Corresponding to each type of operations on equations, we can perform the following operations on the augmented matrix.

I. Interchange two rows.

Example

$$\left(\begin{array}{cc|c} 2 & 1 & 8 \\ 1 & 1 & 3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 8 \end{array}\right)$$

II. Multiply a row by a nonzero real number.

Example

$$\left(\begin{array}{cc|c} 2 & 1 & 8 \\ 1 & 1 & 3 \end{array}\right) \xrightarrow{3R_2 \to R_2} \left(\begin{array}{cc|c} 2 & 1 & 8 \\ 3 & 3 & 9 \end{array}\right)$$

III. Replace a row by its sum with a multiple of another row.

Example

$$\left(\begin{array}{cc|c} 2 & 1 & 8 \\ 1 & 1 & 3 \end{array}\right) \xrightarrow{-1/2R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 2 & 1 & 8 \\ 0 & 1/2 & -1 \end{array}\right)$$

Example Solve $\left\{\begin{array}{rcrcrcrcr} 2x & - & 3y & - & z & + & 2w & = & -2 \\ x & + & 0y & + & 3z & + & 1w & = & 6 \\ 2x & - & 3y & - & z & + & 3w & = & -3 \\ 0x & + & y & + & z & - & 2w & = & 4 \end{array}\right.$

Solution

Then we apply back substitution method.
By Row 4, $w = -1$.
Substitute $w = -1$ into Row 3, $z = 2$.
Substitute $w = -1$ and $z = 2$ into Row 2, $y = 0$.
Substitute $w = -1$, $z = 2$ and $y = 0$ into Row 1, $x = 1$.
The solution is $(1, 0, 2, -1)$.

Exercise Solve $\begin{cases} x_1 + 2x_2 + 2x_3 &= 4 \\ x_1 + 3x_2 + 3x_3 &= 5 \\ 2x_1 + 6x_2 + 5x_3 &= 6 \end{cases}$.

Solution

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{array}\right) \xrightarrow{-1R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 2 & 6 & 5 & 6 \end{array}\right) \xrightarrow{-2R_1+R_3\to R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 \end{array}\right) \xrightarrow{-2R_2+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -4 \end{array}\right)$$

Solve x_3 from Row 3, $x_3 = 4$.
Substitute $x_3 = 4$ back into Row 2, $x_2 = -3$.
Substitute $x_3 = 4$ and $x_2 = -3$ into Row 1, $x_1 = 2$.
So $(x_1, x_2, x_3) = (2, -3, 4)$ is the solution.