2021-22 First Semester
MATH1083 Calculus II (1003)

Assignment 4

Due Date: 11:30am 14/Mar/2021(Tue).

- Write down your **Chinese name** and **student number**. Write neatly on **A4-sized** paper and **show your steps**.
- **Late submissions or answers without details will not be graded.**

1. [Taylor Remainder Theorem]
   (a) Find the first degree Taylor polynomial $p_1(x)$ of function $f(x) = x^{1/3}$ about $a = 27$ radius of convergence and interval of convergence of the power series.
   (b) Find the value of $\sqrt[3]{28}$ using $p_1(x)$
   (c) Estimate the error $R_1(x)$

   补充(a)

   $$\text{for } k \geq 1,\quad a_k = \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\cdots\left(\frac{1}{3}-k+1\right)27^{\frac{1}{3}-k}}{k!}(x-27)^k$$

   $$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\left(\frac{1}{3}-k\right)(x-27)}{(k+1)27}\right| = \lim_{k\to\infty}\left|\frac{x-27}{27}\right| < 1$$

   $|x-27| < 27 \quad R = 27$

   $0 < x < 54$ interval of convergence

   Solution:

   $$a_0 \quad f(x) = x^{1/3} \qquad f(27) = 3$$
   $$a_1 \quad f'(x) = \frac{1}{3}x^{-2/3} \qquad f'(27) = \frac{1}{27}$$
   $$a_2 \quad f''(x) = -\frac{2}{9}x^{-5/3} \qquad f''(27) = -\frac{2}{2457}$$

   a) The first-degree Taylor Polynomial

   $$x^{1/3} \approx p_1(x) = 3 + \frac{1}{27}(x-27)$$

   b) at point $x = 28$

   $$28^{1/3} \approx p_1(28) = 3 + \frac{1}{27}(28-27) = 3.037$$

   c)

   $$|R_n| \leq \frac{|f''(z)|}{2}(x-27)^2$$

   where $z \in (27, 28)$, $\max|f''(z)| = \frac{2}{9}(27)^{-5/3}$

   $$|R_n| \leq \frac{2}{9}(27)^{-5/3}$$

   泰勒展开可以换元条件是：在$x=a$处值相同

   $f(x)$在$x=a$处泰勒展开式为$T_n(x)$

   有$f(g(x))$，若$u=g(x)$，那么$f(u)$在$u=a$处泰勒展开式为$T_n(u)$

   $\Downarrow$

   $f(g(x))$在$g(x)=a$处泰勒展开处为$T_n(g(x))$.

2. Evaluate the limit using **Taylor series**

   $$\lim_{x\to 0}\frac{e^{x^2} - x^2 - 1}{x^4}$$

   Solution: Taylor series for e^{x^2} at $x = 0$

   $$e^{x^2} \approx 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{3!}x^6$$

   then

   $$\frac{e^{x^2} - x^2 - 1}{x^4} \approx \frac{1 + x^2 + \frac{1}{2}x^4 - x^2 - 1}{x^4} = \frac{1}{2}$$

   [Easier than using L'Hospital rule.]

   所以换元条件是在$x=a$处展开值相同.
   (也可以理解为 $f(x)$在x处求导代入$x=a$
   相当于 $f(g(x))$在$g(x)$处求导代入$g(x)=a$)

3. Expand $\frac{1}{\sqrt[4]{1+x}}$ as a power series and estimate $\frac{1}{\sqrt[4]{1.1}}$ to **three decimal places.**

Solution: $f(x)=(1+x)^{-\frac{1}{4}} = \sum_{k=0}^{\infty}\binom{-\frac{1}{4}}{k}x^k$

$$= 1+\left(-\frac{1}{4}\right)x + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!}x^2 + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!}x^3+\cdots$$

$$= 1-\frac{1}{4}x+\frac{5}{32}x^2-\frac{15}{4^3\times 2}x^3+\cdots$$

$$f(1.1)\approx 1-\frac{1}{4}\times 0.1+\frac{5}{32}\times 0.1^2-\frac{15}{4^3\times 2}0.1^3\approx 0.976$$