Question 1.

Consider a standard Brownian Motion W_u.

A useful formula:

$$E(W_u^{2k}) = (2k-1)!!u^k = \frac{(2k)!}{k!2^k}u^k, \quad \text{for } u > 0 \text{ and } k = 1, 2, 3, \cdots$$

Evaluate the following expectations.

(a) $E[(W_t^2 + 7)^2]$, for $t \geq 0$. (10 pts)

(b) $E(W_{12} - W_5 + 7)^3$, (10 pts)

(c) $E[(W_s)^4 (W_t)^2]$, for $t \geq s \geq 0$ (10 pts)

Question 2. (15 pts)

Consider the equation $dX_t = X_t(ae^{-at}dt + bdW_t)$ with the initial condition $X_0 = c$, where a, b and c are constants. It is known that the solution for X_t has the form $f(t)e^{\lambda W_t}$. Here λ is a constant and $f(t)$ is a function of t. Determine the constant λ and the function $f(t)$ in terms of t, a, b and c.

Question 3. (15 pts)

Given

$$dX_t = X_t(adt + bdW_t)$$

with the initial condition $X(t = 0) = 9$. Here a and b are constants.

Consider

$$Y_t = e^{\lambda t} X_t^n, \text{ where } \lambda \text{ and } n \text{ are constants.}$$

Which equations govern the process Y_t?

Question 4. (15 pts)

Solve the stochastic differential equation

$$dS_t = S_t(\lambda cos(t)dt + \sigma dW_t), \text{with } S_0 = x,$$

where λ and σ are constants. (Hint: apply Itô lemma to $ln(S_t)$).

Question 5. (10 pts)

Determine the solution for the equation

$$U_t = U_{xx} \quad -\infty < x < \infty$$

with the initial condition $X(t = 0) = 3 + 7x$.

Question 6. (15 pts)

Determine the solution for the equation

$$U_t = (3 + 7t)U_{xx} \quad -\infty < x < \infty$$

with the initial condition $X(t = 0) = 11x^2$.