Solution to Midterm of ASP

1. (12 points)

   (a) (4 points) A class of sets $\mathcal{A} \subset 2^{\Omega}$ is called a σ-field if it fulfills the following three conditions:

      i. $\emptyset \in \mathcal{A}$;

      ii. if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$;

      iii. and if $A_1,\ A_2, \cdots \in \mathcal{A}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

   (b) (4 points) $\{\emptyset, \Omega\}$ and $\{\emptyset, A, A^c, \Omega\}$

   (c) (4 points) $\{\emptyset, A, \Omega\}$

2. (10 points) The transition matrix of the chain is

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

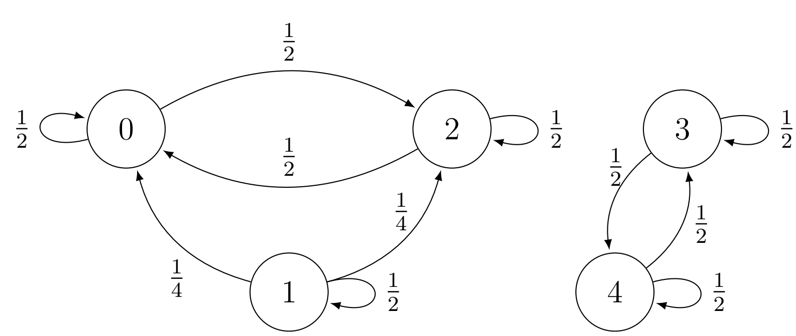
   Then

$$\boldsymbol{P^3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{93}{125} & 0 & \frac{24}{125} & 0 & \frac{8}{125} \\ \frac{9}{125} & \frac{36}{125} & 0 & \frac{24}{125} & \frac{4}{125} \\ \frac{27}{125} & 0 & \frac{36}{125} & 0 & \frac{62}{125} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.744 & 0 & 0.192 & 0 & 0.064 \\ 0.36 & 0.288 & 0 & 0.192 & 0.16 \\ 0.216 & 0 & 0.288 & 0 & 0.0496 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

   So $P^3_{1,4} = \frac{8}{125} = 0.064$ and $P^3_{1,0} = \frac{93}{125} = 0.744$

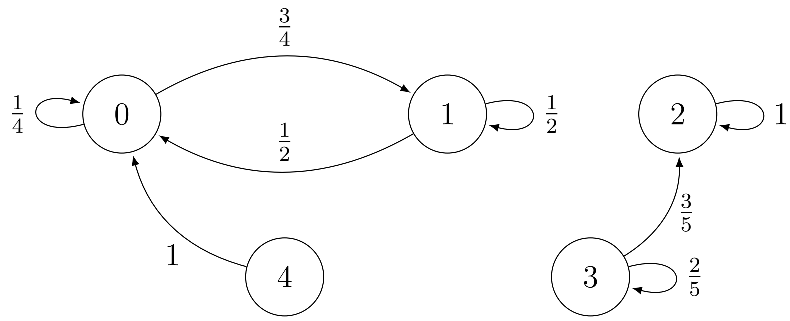
3. (16 points)

   (a) (8 points)



Recurrent classes: $\{0,2\}$, {3,4}. Transient class:$\{1\}$.

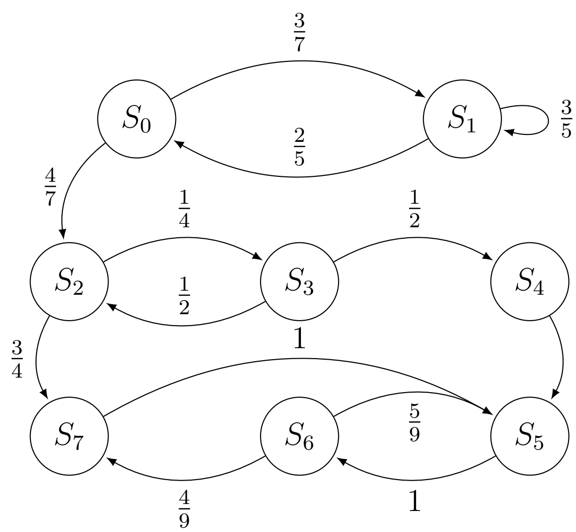
(b) (8 points)



Recurrent classes: $\{0,1\},\{2\}$. Transient classes: $\{3\},\{4\}$.

4. (16 points)

(a) (4 points)



(b) (12 points) $\{S_0, S_1\}$: transient, $d=1$. $\{S_2, S_3\}$: transient, $d=2$. $\{S_4\}$: transient, $d=\infty$. $\{S_5, S_5, S_7\}$: recurrent, $d=1$.

5. (18 points)

(a) (6 points) We have the transition matrix

$$P=\begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

If $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ be a stationary distribution, then

$$\pi P = \pi. \tag{2}$$

From the structure of the above matrix, one can see that any vector $(\pi_1, 0, \pi_3, 0)$ satisfies (2), provided that

$$\begin{cases} \frac{2}{3}\pi_1 + \frac{1}{4}\pi_3 = \pi_1, \\ \frac{1}{3}\pi_1 + \frac{3}{4}\pi_3 = \pi_3. \end{cases}$$

Both equations imply

$$\frac{1}{4}\pi_3 = \frac{1}{3}\pi_1, \qquad \pi_3 = \frac{4}{3}\pi_1.$$

To have $(\pi_1, 0, \pi_3, 0)$ as a distribution, one needs

$$\pi_1 + \pi_3 = 1,$$

then

$$\pi_1 + \frac{4}{3}\pi_1 = 1, \qquad \pi_1 = \frac{3}{7}, \quad \pi_3 = \frac{4}{7}.$$

Next, two other rows, the second and the forth, corresponds to two stationary states $(0, 1, 0, 0)$ and $(0, 0, 0, 1)$, those both satisfy (2).

Then any linear combination

$$a\Big(\frac{3}{7}, 0, \frac{4}{7}, 0\Big) + b(0, 1, 0, 0) + c(0, 0, 0, 1)$$

also satisfies (2), for arbitrary $a, b, c \geq 0$. To have it as a stochastic vector, we replace a by $7a$ (since $a \geq 0$ is arbitrary, it does not change anything), and we get then the vector

$$(3a, b, 4a, c).$$

Its sum of coordinates is $7a + b + c$, hence, any stationary distribution of the Markov chain has the form

$$\pi = \frac{1}{7a + b + c}(3a, b, 4a, c),$$

where $a, b, c \geq 0$, excluding the case $a = b = c = 0$.

(b) (12 points)

i. (6 points) We have the transition matrix

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3}$$

By definition, a distribution $\pi = (\pi_1, \pi_2, \pi_3)$ is a stationary distribution, if

$$\pi P = \pi$$

can be rewritten as follows:

$$\begin{cases} \frac{1}{4}\pi_1 + \pi_3 = \pi_1, \\ \frac{1}{2}\pi_1 = \pi_2, \\ \frac{1}{4}\pi_1 + \pi_2 = \pi_3. \end{cases}$$

Then

$$\pi_3 = \frac{3}{4}\pi_1, \quad \pi_2 = \frac{1}{2}\pi_1$$

(and the third equality holds then). Since π is a distribution:

$$\pi_1 + \frac{1}{2}\pi_1 + \frac{3}{4}\pi_1 = 1, \quad \frac{9}{4}\pi_1 = 1,$$

hence,

$$\pi_1 = \frac{4}{9}, \quad \pi_2 = \frac{2}{9}, \quad \pi_3 = \frac{1}{3}.$$

The solution is unique, thus it is the unique stationary distribution for the Markov chain.

ii. (6 points) Consider expected times needed to reach state 2 starting from different states:

$$t_i = \mathbb{E}(\exists n \geq 0 : X_n = 2 \mid X_0 = i), \quad i = 1, 2, 3.$$

We have then $t_2 = 0$, and

$$t_i = 1 + \sum_{k=1}^{3} t_k p_{i,k}, \qquad i = 1, 3,$$

i.e.

$$\begin{cases} t_1 = 1 + \frac{1}{4}t_1 + \frac{1}{2}\underbrace{t_2}_{=0} + \frac{1}{4}t_3, \\ t_3 = 1 + 1 \cdot t_1 + 0. \end{cases}$$

Therefore, one can find the needed t_1:

$$\frac{3}{4}t_1 = 1 + \frac{1}{4}t_3 = 1 + \frac{1}{4}(1 + t_1), \quad \frac{1}{2}t_1 = \frac{5}{4},$$

hence,

$$t_1 = \frac{5}{2}.$$

6. (15 points)

(a) $P(N(2) = 5) = \frac{6^5}{5!}e^{-6} = \frac{324}{5}e^{-6} \approx 0.1606.$

(b) $P(N(1) = 1 \mid N(3) = 4) = \frac{P(N(1)=1, N(3)=4)}{P(N(3)=4)} = \frac{P(N(1)=1)P(N(3)-N(1)=3)}{P(N(3)=4)} = \frac{\frac{3^1}{1!}e^{-3}\frac{6^1}{3!}e^{-6}}{\frac{9^4}{4!}e^{-9}} = \frac{32}{81} \approx 0.3951.$

(c) $E(S_{12}) = E(T_1 + T_2 + \cdots + T_{12}) = 12E(T_1) = \frac{12}{3} = 4.$

(d) $E(S_{12} \mid N(2) = 5) = E(S_5 + T_6 + \cdots + T_{12} \mid S_5 \le 2, S_6 > 2) = E(2 + T_1' + T_2' + \cdots + T_7') = 2 + 7E(T_1') = 2 + \frac{7}{3} = \frac{13}{3} \approx 4.3333.$

(e) $E(N(5) \mid N(2) = 5) = E(N(5) - N(2) + N(2) \mid N(2) = 5) = E(N(5) - N(2)) + 5 = 9 + 5 = 14.$

7. (13 points)

(a) (5 points) Let $N_1(t)$ and $N_2(t)$ be the number of vegetarians and meat eaters up to time t respectively. Then $\{N_1(t), t \ge 0\}$ is a Poisson process with rate $\lambda_1 = 15 * \frac{4}{5} = 12$ and $\{N_2(t), t \ge 0\}$ is a Poisson process with rate $\lambda_1 = 15 * \frac{1}{5} = 3$. Also $\{N_1(t), t \ge 0\}$ is independent to $\{N_2(t), t \ge 0\}$. So $P(N_1(\frac{1}{3}) = 3, N_2(\frac{1}{3}) = 2) = P(N_1(\frac{1}{3}) = 3)P(N_2(\frac{1}{3}) = 2) = \frac{4^3}{3!}e^{-4}\frac{1^2}{2!}e^{-1} = \frac{16}{3}e^{-5} \approx 0.0359.$

(b) (8 points) Let Z_{1j} and Z_{2j} be the amount of money that the j^{th} vegetarian and the j^{th} meat eater, respectively, spend. Z_{1j} is of mean 7 and standard deviation 3. Z_{2j} is of mean 15 and standard deviation 8. Let $Y_i(t) = \sum_{j=0}^{N_i(t)} Z_{ij}$ with $Y_i(t) = 0$ when $N_i(t) = 0,\ i = 1, 2$. Then $S(t) = Y_1(t) + Y_2(t)$ is the amount of money that people spend up to time t. Hence

$$
\begin{aligned}
E(S_t) &= E(Y_1(t)) + E(Y_2(t)) = \lambda_1 t E(Z_{1i}) + \lambda_2 t E(Z_{2i}) = 84t + 45t = 129t \\
D(S_t) &= D(Y_1(t) + Y_2(t)) = D(Y_1(t)) + D(Y_2(t)) = \lambda_1 t E(Z_{1i}^2) + \lambda_2 t E(Z_{2i}^2) \\
&= 12t(3^2 + 7^2) + 3t(8^2 + 15^2) = 1563t
\end{aligned}
$$

The mean and standard deviation of the amount of money spent during the four hours the market is open is $E(S(4)) = 516$ and $\sqrt{D(S(4))} = \sqrt{6252} \approx 79.0696.$