Linear Algebra I

Section 1.1 Matrices

For any two positive integers m and n, an $m \times n$ matrix A is a rectangular array of mn numbers:

$$
A = \begin{array}{c}
\begin{array}{ccccc} \phantom{a_{11}} & & \textit{j-th column} & & \phantom{a_{1n}} \end{array} \\
\begin{bmatrix}
a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\
\vdots & & \vdots & & \vdots \\
a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & & \vdots & & \vdots \\
a_{m1} & \cdots & a_{mj} & \cdots & a_{mn}
\end{bmatrix}
\end{array}
\begin{array}{l} \\ \\ \\ \textit{i-th row} \\ \\ \\ \end{array}
$$

in m rows and n columns. We will sometimes shorten this to $A = (a_{ij})_{m \times n}$. If $m = n$, A is called a square matrix. The entry a_{ij} is located in the i-th row and the j-th column.

方阵

Two matrices A and B are equal if they have the same size (number of rows and columns) and corresponding entries are equal: $a_{ij} = b_{ij}$ for all i and j.

$A = B \begin{cases} ① \text{ 维度同} \\ ② \text{ 元素同} \end{cases}$

If $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ are both $m \times n$ matrices, then the **sum** $A + B$ is the $m \times n$ matrix whose (i, j) entry is $a_{ij} + b_{ij}$ for each ordered pair (i, j).

Definition 1.1.1 (Scalar Multiplication)

If A is an $m \times n$ matrix and α is a scalar, then αA is the $m \times n$ matrix whose (i, j) entry is αa_{ij}.

If all entries are all 0, then such matrix is called zero matrix and denoted by O. For instance, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a 2×3 zero matrix. 实例

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For any two positive integers m and n, an $m \times n$ matrix A is a rectangular array of mn numbers:

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 & \begin{array}{ccccc} & & j\text{-th column} & & \end{array} \\
A = & \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \begin{array}{c} \\ \\ i\text{-th row} \\ \\ \\ \end{array}
\end{array}
$$

in m rows and n columns. We will sometimes shorten this to $A = (a_{ij})_{m \times n}$. If $m = n$, A is called a square matrix. The entry a_{ij} is located in the i-th row and the j-th column.

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If O represents the zero matrix, with the same size as A, whose, then

$$\boxed{A + O = O + A = A.}$$

It acts as an additive identity on the set of all $m \times n$ matrices. Furthermore, each $m \times n$ matrix A has an additive inverse. Indeed,

$$A + (-1)A = O = (-1)A + A.$$

It is customary to denote the additive inverse by $-A$. Thus,

习惯

$$-A = (-1)A.$$

If A and B are both $m \times n$ matrices, we define $A - B$ to be $A + (-1)B$. Then it turns out that $A - B$ is formed by subtracting the corresponding entry of B from each entry of A.

Example 1.1.2

对应相减

$$3\begin{pmatrix} 2 & -1 & 7 \\ 3 & 0 & 4 \end{pmatrix} - 5\begin{pmatrix} 3 & -4 & 2 \\ 1 & 2 & -5 \end{pmatrix} = \begin{pmatrix} 3(2)-5(3) & 3(-1)-5(-4) & 3(7)-5(2) \\ 3(3)-5(1) & 3(0)-5(2) & 3(4)-5(-5) \end{pmatrix} = \begin{pmatrix} -9 & 17 & 11 \\ 4 & -10 & 37 \end{pmatrix}$$

Definition 1.1.3

If $A = (a_{ij})_{m\times k}$ is an $m \times k$ matrix and $B = (b_{ij})_{k\times n}$ is an $k \times n$ matrix, then the product $AB = C = (c_{ij})_{m\times n}$ is the $m \times n$ matrix whose entries are defined by

$$c_{ij} = \sum_{r=1}^{k} a_{ir} b_{rj}.$$

If number of column of A is not number of row of B, then the matrix multiplication of AB is not valid.

$$A_{m\times n} \cdot b_{n\times k} = C_{m\times k}.$$

Example 1.1.4

$$AB = \overset{A}{\begin{pmatrix} -3 & -1 & -1 & -5 & 1 \\ -3 & -3 & -4 & -5 & 3 \\ -1 & -5 & 3 & -1 & -3 \\ 3 & 2 & -1 & -4 & -4 \\ -5 & 3 & -2 & -1 & -1 \end{pmatrix}} \overset{B}{\begin{pmatrix} 0 & 5 & 3 & -3 & 0 \\ 5 & 5 & 2 & 0 & -1 \\ 3 & 0 & -4 & -1 & -4 \\ 4 & 0 & -3 & 2 & 4 \\ 4 & -2 & 0 & -1 & 3 \end{pmatrix}}$$

$$= \begin{pmatrix} -24 & -22 & \boxed{8} & -1 & -12 \\ -35 & -36 & 16 & 0 & 8 \\ -32 & -24 & -22 & 1 & -20 \\ -25 & 33 & 29 & -12 & -26 \\ 1 & -8 & 2 & 16 & -2 \end{pmatrix}$$

$$\begin{aligned} &(-3)(3) + (-1)(2) + (-1)(-4) + (-5)(-3) + (1)(0) \\ = &\ -9 + -2 + 4 + 15 + 0 \\ = &\ \boxed{8} \end{aligned}$$

Remark 1.1.5

一般 $AB \neq BA$.

$AB \neq BA$ even the matrix multiplication is valid. Consider $A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}$.

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 11 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The example in Remark 1.1.5 also shows us the following:

If a product AB is the zero matrix, you cannot conclude in general that either $A = O$ or $B = O$. Furthermore, if $AB = AC$ for a nonzero matrix A, then it is not true in general that $B = C$. If we consider $C = \begin{pmatrix} 5 & 6 \\ 0 & 0 \end{pmatrix}$,

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = AC.$$

However $B \neq C$.

$AB = AC \not\Rightarrow B = C$ (不能推出来)

Definition 1.1.6

The **transpose** of an $m \times n$ matrix $A = (a_{ij})_{m\times n}$ is the $n \times m$ matrix, denoted by $A^T = (b_{ij})_{n\times m}$, such that 转置.

$$b_{ij} = a_{ji}$$

for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. That is, rows of A^T are formed from the corresponding columns of A.

A^T 的行对应 A 的列

Example 1.1.7

$$A = \begin{pmatrix} -1 & -5 \\ 3 & 2 \\ -5 & 3 \end{pmatrix} \qquad A^T = \begin{pmatrix} -1 & 3 & -5 \\ -5 & 2 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & -3 & 0 \\ 2 & 0 & -1 \end{pmatrix} \qquad B^T = \begin{pmatrix} 3 & 2 \\ -3 & 0 \\ 0 & -1 \end{pmatrix}$$

Theorem 1.1.8

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

(i) $\left(A^T\right)^T = A$

(ii) $\left(A+B\right)^T = A^T + B^T$

(iii) For any scalar r, $\left(rA\right)^T = rA^T$

(iv) $\left(AB\right)^T = B^T A^T$

$(ABC)^T = C^T \cdot B^T \cdot A^T$

Example 1.1.9

$$AB = \begin{pmatrix} -1 & -5 \\ 3 & 2 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 3 & -3 & 0 \\ 2 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -13 & 3 & 5 \\ 13 & -9 & -2 \\ -9 & 15 & -3 \end{pmatrix}$$

$$B^T A^T = \begin{pmatrix} 3 & 2 \\ -3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 3 & -5 \\ -5 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -13 & 13 & -9 \\ 3 & -9 & 15 \\ 5 & -2 & -3 \end{pmatrix} = (AB)^T$$

$A = \begin{bmatrix} x & y & z \\ x & y & z \\ x & y & z \end{bmatrix}$

Definition 1.1.10

An $n \times n$ matrix A is said to be **symmetric** if $A^T = A$.

对称

反证法.

$B = AA^T$ proof $B^T = B$

$B^T = (AA^T)^T = A^T A$

Example 1.1.11

$a_{ij} = a_{ji}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} \text{ is symmetric but } \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix} \text{ is not.}$$

$\neq$

对角线.

off diagonal's

An $n \times n$ matrix A is **diagonal** if $a_{ij} = 0$ whenever $i \neq j$. The matrices

square

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

are all diagonal.

Definition 1.1.12

单位矩阵

The $n \times n$ **identity matrix** is the matrix $I_n = (\delta_{ij})_{n \times n}$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3x3

3 by 3

$I_4 = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}$

Just as the number 1 acts as an identity for the multiplication of real numbers, identity matrix I_n acts as an identity for matrix multiplication; that is,

$$I_n A = AI_n = A$$

for any $n \times n$ matrix A.

$I_n \cdot A = A \cdot I_n = A$

上三角 下三角

An $n \times n$ matrix A is said to be **upper triangular** if $a_{ij} = 0$ for $i > j$ and **lower triangular** if $a_{ij} = 0$ for $i < j$. For instance,

$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix}$ is an upper triangular matrix and $\begin{pmatrix} 1 & 0 & 0 \\ 6 & 0 & 0 \\ 1 & 4 & 3 \end{pmatrix}$ is a lower triangular matrix.

Theorem 1.1.13

Let A be an $m \times n$ matrix, and let α and β be scalars. For any matrices B and C having sizes for which the indicated sums and products are defined, we have

(i) $A + B = B + A$

(ii) $(A+B)+C = A+(B+C)$

(iii) $(AB)C = A(BC)$ 空间顺序不变（时间顺序可以）

(iv) $A(B+C) = AB + AC$

(v) $(A+B)C = AC + BC$ $A+(-A)=0$

(vi) $\alpha(\beta A) = (\alpha\beta)A$ $A+0=A$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(vii) $\alpha(AB) = (\alpha A)B = A(\alpha B)$

(viii) $(\alpha+\beta)A = \alpha A + \beta A$

(ix) $\alpha(A+B) = \alpha A + \alpha B$

(x) $I_m A = A I_n = A$

对角矩阵的k次幂

= 对角上的k次幂

Example 1.1.14

If

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

verify that $A(BC) = (AB)C$.

Solution

$$(AB)C=\left(\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}2&1\\-3&2\end{pmatrix}\right)\cdot\begin{pmatrix}1&0\\2&1\end{pmatrix}=\begin{pmatrix}-4&5\\-6&11\end{pmatrix}\begin{pmatrix}1&0\\2&1\end{pmatrix}=\begin{pmatrix}6&5\\16&11\end{pmatrix}$$

$$A(BC)=\begin{pmatrix}1&2\\3&4\end{pmatrix}\cdot\left(\begin{pmatrix}2&1\\-3&2\end{pmatrix}\begin{pmatrix}1&0\\2&1\end{pmatrix}\right)=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}4&1\\1&2\end{pmatrix}=\begin{pmatrix}6&5\\16&11\end{pmatrix}=(AB)C$$

Since $(AB)C = A(BC)$, we may simply omit the parentheses and write ABC.

A to the power k

If A is an $n\times n$ matrix and k a positive integer, then A^k denotes the product of k copies of A:

$(I_n)^k = I_n$

高次幂

$$A^k=\underbrace{A\cdots A}_{k\text{ times}}.$$

Suppose matrices A and B are partitioned into blocks as (分成块)

$$A=\begin{pmatrix}A_{11}&A_{12}&\cdots&A_{1k}\\A_{21}&A_{22}&\cdots&A_{2k}\\\vdots&\vdots&\ddots&\vdots\\A_{m1}&A_{m2}&\cdots&A_{mk}\end{pmatrix},\qquad B=\begin{pmatrix}B_{11}&B_{12}&\cdots&B_{1n}\\B_{21}&B_{22}&\cdots&B_{2n}\\\vdots&\vdots&\ddots&\vdots\\B_{k1}&B_{k2}&\cdots&B_{kn}\end{pmatrix}$$

where A_{ij} and B_{ij} are matrices.

Assume that the partitioning has been done in such a way that each product $A_{il}B_{lj}$ can be formed. Define

$$C_{ij}=\begin{pmatrix}A_{i1}&A_{i2}&\cdots&A_{ik}\end{pmatrix}\begin{pmatrix}B_{1j}\\B_{2j}\\\vdots\\B_{kj}\end{pmatrix}=\sum_{l=1}^{k}A_{il}B_{lj}.$$

Then $C = AB$ can be partitioned into blocks as follows:

$$D^k=\begin{bmatrix}d_{11}^k&&&\\&d_{22}^k&&\\&&d_{33}^k\cdots&\\&&&d_{nn}^k\end{bmatrix}_{n\times n}$$

$$C=\begin{pmatrix}C_{11}&C_{12}&\cdots&C_{1n}\\C_{21}&C_{22}&\cdots&C_{2n}\\\vdots&\vdots&\ddots&\vdots\\C_{m1}&C_{m2}&\cdots&C_{mn}\end{pmatrix}.$$

Example 1.1.15

An example of a 2×3 block partitioned matrix is

$$A=\left(\begin{array}{cc|cc|c} -3 & -1 & -1 & -5 & 1 \\ -3 & -3 & -4 & -5 & 3 \\ \hline -1 & -5 & 3 & -1 & -3 \\ 3 & 2 & -1 & -4 & -4 \\ -5 & 3 & -2 & -1 & -1 \end{array}\right)=\left(\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \end{array}\right)$$

where

$$A_{11}=\begin{pmatrix} -3 & -1 \\ -3 & -3 \end{pmatrix}, A_{12}=\begin{pmatrix} -1 & -5 \\ -4 & -5 \end{pmatrix}, A_{13}=\begin{pmatrix} 1 \\ 3 \end{pmatrix}, A_{21}=\begin{pmatrix} -1 & -5 \\ 3 & 2 \\ -5 & 3 \end{pmatrix}, A_{22}=\begin{pmatrix} 3 & -1 \\ -1 & -4 \\ -2 & -1 \end{pmatrix}, A_{23}=\begin{pmatrix} -3 \\ -4 \\ -1 \end{pmatrix}.$$

Consider a 3×2 block partitioned matrix

$$B=\left(\begin{array}{cc|ccc} 0 & 5 & 3 & -3 & 0 \\ 5 & 5 & 2 & 0 & -1 \\ \hline 3 & 0 & -4 & -1 & -4 \\ 4 & 0 & -3 & 2 & 4 \\ \hline 4 & -2 & 0 & -1 & 3 \end{array}\right)=\left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline B_{31} & B_{32} \end{array}\right)$$

where $B_{11}=\begin{pmatrix} 0 & 5 \\ 5 & 5 \end{pmatrix}, B_{12}=\begin{pmatrix} 3 & -3 & 0 \\ 2 & 0 & -1 \end{pmatrix}, B_{21}=\begin{pmatrix} 3 & 0 \\ 4 & 0 \end{pmatrix}, B_{22}=\begin{pmatrix} -4 & -1 & -4 \\ -3 & 2 & 4 \end{pmatrix}, B_{31}=\begin{pmatrix} 4 & -2 \end{pmatrix}$ and $B_{32}=\begin{pmatrix} 0 & -1 & 3 \end{pmatrix}$.

Then

$$AB=\left(\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \end{array}\right)\left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline B_{31} & B_{32} \end{array}\right)=\left(\begin{array}{c|c} A_{11}B_{11}+A_{12}B_{21}+A_{13}B_{31} & A_{11}B_{12}+A_{12}B_{22}+A_{13}B_{32} \\ \hline A_{21}B_{11}+A_{22}B_{21}+A_{23}B_{31} & A_{21}B_{12}+A_{22}B_{22}+A_{23}B_{32} \end{array}\right)$$

2×3 $3\times 2 \to 2\times 2$

$$\begin{aligned} A_{11}B_{11}+A_{12}B_{21}+A_{13}B_{31} &= \begin{pmatrix} -3 & -1 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} 0 & 5 \\ 5 & 5 \end{pmatrix}+\begin{pmatrix} -1 & -5 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 4 & 0 \end{pmatrix}+\begin{pmatrix} 1 \\ 3 \end{pmatrix}\begin{pmatrix} 4 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -5 & -20 \\ -15 & -30 \end{pmatrix}+\begin{pmatrix} -23 & 0 \\ -32 & 0 \end{pmatrix}+\begin{pmatrix} 4 & -2 \\ 12 & -6 \end{pmatrix}=\boxed{\begin{pmatrix} -24 & -22 \\ -35 & -36 \end{pmatrix}} \quad C_{11} \end{aligned}$$

$$A_{11}B_{12}+A_{12}B_{22}+A_{13}B_{32}=\begin{pmatrix}-3&-1\\-3&-3\end{pmatrix}\begin{pmatrix}3&-3&0\\2&0&-1\end{pmatrix}+\begin{pmatrix}-1&-5\\-4&-5\end{pmatrix}\begin{pmatrix}-4&-1&-4\\-3&2&4\end{pmatrix}+\begin{pmatrix}1\\3\end{pmatrix}\begin{pmatrix}0&-1&3\end{pmatrix}$$

$$=\begin{pmatrix}-11&9&1\\-15&9&3\end{pmatrix}+\begin{pmatrix}19&-9&-16\\31&-6&-4\end{pmatrix}+\begin{pmatrix}0&-1&3\\0&-3&9\end{pmatrix}=\boxed{\begin{pmatrix}8&-1&-12\\16&0&8\end{pmatrix}}$$ C_{12}

$$A_{21}B_{11}+A_{22}B_{21}+A_{23}B_{31}=\begin{pmatrix}-1&-5\\3&2\\-5&3\end{pmatrix}\begin{pmatrix}0&5\\5&5\end{pmatrix}+\begin{pmatrix}3&-1\\-1&-4\\-2&-1\end{pmatrix}\begin{pmatrix}3&0\\4&0\end{pmatrix}+\begin{pmatrix}-3\\-4\\-1\end{pmatrix}\begin{pmatrix}4&-2\end{pmatrix}$$

$$=\begin{pmatrix}-25&-30\\10&25\\15&-10\end{pmatrix}+\begin{pmatrix}5&0\\-19&0\\-10&0\end{pmatrix}+\begin{pmatrix}-12&6\\-16&8\\-4&2\end{pmatrix}=\boxed{\begin{pmatrix}-32&-24\\-25&33\\1&-8\end{pmatrix}}$$ C_{21}

$$A_{21}B_{12}+A_{22}B_{22}+A_{23}B_{32}=\begin{pmatrix}-1&-5\\3&2\\-5&3\end{pmatrix}\begin{pmatrix}3&-3&0\\2&0&-1\end{pmatrix}+\begin{pmatrix}3&-1\\-1&-4\\-2&-1\end{pmatrix}\begin{pmatrix}-4&-1&-4\\-3&2&4\end{pmatrix}+\begin{pmatrix}-3\\-4\\-1\end{pmatrix}\begin{pmatrix}0&-1&3\end{pmatrix}$$

$$=\begin{pmatrix}-13&3&5\\13&-9&-2\\-9&15&-3\end{pmatrix}+\begin{pmatrix}-9&-5&-16\\16&-7&-12\\11&0&4\end{pmatrix}+\begin{pmatrix}0&3&-9\\0&4&-12\\0&1&-3\end{pmatrix}=\boxed{\begin{pmatrix}-22&1&-20\\29&-12&-26\\2&16&-2\end{pmatrix}}$$ C_{22}

Hence $AB=\left(\begin{array}{cc|ccc}-24&-22&8&-1&-12\\-35&-36&16&0&8\\\hline -32&-24&-22&1&-20\\-25&33&29&-12&-26\\1&-8&2&16&-2\end{array}\right)$, which is the same as in Example 1.1.4.

Example 1.1.16

Suppose A and C are $m\times m$ matrices, B and D are $n\times n$ matrices. Then

$$\left(\begin{array}{c|c}A&O_{m\times n}\\\hline O_{n\times m}&B\end{array}\right)\left(\begin{array}{c|c}C&O_{m\times n}\\\hline O_{n\times m}&D\end{array}\right)=\left(\begin{array}{c|c}AC&O_{m\times n}\\\hline O_{n\times m}&BD\end{array}\right).$$

In particular, for any positive integer k, we have

$$\left(\begin{array}{cc|ccc}\lambda&1&0&0&0\\0&\lambda&0&0&0\\\hline 0&0&\lambda&1&0\\0&0&0&\lambda&1\\0&0&0&0&\lambda\end{array}\right)^k=\left(\begin{array}{c|c}\begin{pmatrix}\lambda&1\\0&\lambda\end{pmatrix}^k&\begin{matrix}0&0&0\\0&0&0\end{matrix}\\\hline \begin{matrix}0&0\\0&0\\0&0\end{matrix}&\begin{pmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix}^k\end{array}\right).$$

分好块可以简化运算过程.

Section 1.2 Solving system of linear equations

Given the system of linear equations

$$(*):\begin{cases} \quad\;\; 2y - 8z = 8 \\ x \;\, - 2y + \;z \;= 0 \\ -4x + 5y + 9z = -9 \end{cases}$$

with the coefficients of each variable aligned in columns, the matrix

$$\begin{pmatrix} 0 & 2 & -8 \\ 1 & -2 & 1 \\ -4 & 5 & 9 \end{pmatrix}$$

is called the coefficient matrix of the system $(*)$, and

$$\left(\begin{array}{ccc|c} 0 & 2 & -8 & 8 \\ 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \end{array}\right)$$

参数矩阵

is called the augmented matrix of the system.

基本战略

The basic strategy is to replace one system with an equivalent system (i.e., one with the same solution set) that is easier to solve.

Strategy

We proceed from equation to equation, from top to bottom.

(i) Rearrange the equations if the position of leading variables do not appear in the ascending order.

(ii) Suppose we get to the i-th equation, with leading variable x_j and leading (nonzero) coefficient c, so that the equation will be of the form $cx_j + \cdots = b$.

(iii) Divide the i-th equation by c bring it into the form $x_j + \cdots = b/c$.

(iv) Eliminate x_j from all the other equations, above and below the i-th equation, by subtracting suitable multiples of the i-th equation.

(v) If an equation zero = nonzero appears in this process, then the system has no solutions.

(vi) Proceed to the next equation.

Repeat such process until the last equation. Solve each equation for its leading variable.

Example 1.2.1

$$\begin{cases} 2y-8z=8 \\ x-2y+z=0 \\ -4x+5y+9z=-9 \end{cases} \qquad \left(\begin{array}{ccc|c} 0 & 2 & -8 & 8 \\ 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \end{array}\right)$$

$$\xrightarrow{\text{1st eq}\leftrightarrow\text{2nd eq}} \begin{cases} x-2y+z=0 \\ 2y-8z=8 \\ -4x+5y+9z=-9 \end{cases} \qquad \xrightarrow{R_1\leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right)$$

$$\xrightarrow{\text{3rd eq}+4\times(\text{1st eq})\to\text{3rd eq}} \begin{cases} x-2y+z=0 \\ 2y-8z=8 \\ -3y+13z=-9 \end{cases} \qquad \xrightarrow{R_3+4R_1\to R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right)$$

$$\xrightarrow{\text{2nd eq}\div 2\to\text{2nd eq}} \begin{cases} x-2y+z=0 \\ y-4z=4 \\ -3y+13z=-9 \end{cases} \qquad \xrightarrow{R_2\div 2\to R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right)$$

$$\xrightarrow{\text{3rd eq}+3\times(\text{2nd eq})\to\text{3rd eq}} \begin{cases} x-2y+z=0 \\ y-4z=4 \\ z=3 \end{cases} \qquad \xrightarrow{R_3+3R_2\to R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Using back substitution, we obtain $z=3,\ y=4+4z=16,\ x=2y-z=29.$

You may also keep elimination until the simplest form. 最簡形式.

$$\begin{cases} x-2y+z=0 \\ y-4z=4 \\ z=3 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{\text{2nd eq}+4\times(\text{3rd eq})\to\text{2nd eq}} \begin{cases} x-2y+z=0 \\ y=16 \\ z=3 \end{cases} \qquad \xrightarrow{R_2+4R_3\to R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{\text{1st eq}-\text{3rd eq}\to\text{1st eq}} \begin{cases} x-2y=-3 \\ y=16 \\ z=3 \end{cases} \qquad \xrightarrow{R_1-R_3\to R_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{\text{1st eq}+2\times(\text{2nd})\text{ eq}\to\text{1st eq}}\begin{cases} x \qquad = 29 \\ \quad y \quad = 16 \\ \qquad z = 3 \end{cases} \qquad \xrightarrow{R_1+2R_2\to R_1}\left(\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

We still obtain $x = 29$, $y = 16$, $z = 3$.

初等行变化

Definition 1.2.2 (**Elementary Row Operations**)

(I) Interchange two rows. 更换两行

(II) Multiply a row by a nonzero real number. 给其中一行乘一个常数

(III) Replace a row by its sum with a multiple of another row.

Remark 1.2.3 R_3作为变化组不可以乘数再去加

$R_3 + 4R_1 \to R_3$ is a type III row operation, but $R_3 + 4R_1 \to R_1$ is not.

Elementary row operations are equivalent to the left multiplications of an augmented matrix A by elementary matrices E.

左乘一些单位矩阵

An elementary matrix of type I is a matrix obtained by interchanging two rows of I_n. For example, the matrix

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

⇒换①②两行

is an elementary matrix of type I since it was obtained by interchanging the first two rows of I_3.

$$EA = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \end{pmatrix}$$

Left multiplication of A by E is equivalent to the elementary row operation (type I) of interchanging the first and second row.

An elementary matrix of type II is a matrix obtained by multiplying a row of I_n by a nonzero constant. For example, for any nonzero α,

$$E=\begin{pmatrix}1&0&0\\0&\boxed{\alpha}&0\\0&0&1\end{pmatrix}$$

行变化② 乘一个数.

is an elementary matrix of type II.

$$EA=\begin{pmatrix}1&0&0\\0&\alpha&0\\0&0&1\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\a_{31}&a_{32}&\cdots&a_{3n}\end{pmatrix}=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\\alpha a_{21}&\alpha a_{22}&\cdots&\alpha a_{2n}\\a_{31}&a_{32}&\cdots&a_{3n}\end{pmatrix}.$$

Left multiplication of A by E is equivalent to the elementary row operation (type II) of multiplying the second row by α.

An elementary matrix of type III is a matrix obtained from I_n by adding a multiple of one row to another row. For example, for any β,

$$E=\begin{pmatrix}1&0&\boxed{\beta}\\0&1&0\\0&0&1\end{pmatrix}$$

行变化③

is an elementary matrix of type III since it was obtained from I_n by adding a multiple β of third row to the first row.

$$EA=\begin{pmatrix}1&0&\beta\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\a_{31}&a_{32}&\cdots&a_{3n}\end{pmatrix}=\begin{pmatrix}a_{11}+\beta a_{31}&a_{12}+\beta a_{32}&\cdots&a_{1n}+\beta a_{3n}\\a_{21}&a_{22}&\cdots&a_{2n}\\a_{31}&a_{32}&\cdots&a_{3n}\end{pmatrix}.$$

Left multiplication of A by E is equivalent to the elementary row operation (type III) of adding β times the third row to the first row.

Remark 1.2.4

Right multiplication of A by elementary matrices performs elementary column operations.

右乘矩阵则是初等列变化

Example 1.2.5

$$\left(\begin{array}{ccc|c} 0 & 2 & -8 & 8 \\ 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 0 & 2 & -8 & 8 \\ 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \end{array}\right)$$

$$\xrightarrow{R_3 + 4R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right)$$

$$\xrightarrow{R_2 \div 2 \to R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right)$$

$$\xrightarrow{R_3 + 3R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right)$$

$$\xrightarrow{R_2 + 4R_3 \to R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_1 - R_3 \to R_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_1 + 2R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Definition 1.2.6

Let A and B be two $m \times n$ matrices. B is **row equivalent** to a matrix A if there exists a finite sequence $E_1, E_2, \cdots, E_k$ of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_1 A.$$

In other words, B is row equivalent to A if B can be obtained from A by a finite number of row operations.

如果 $B = E_k \cdots E_2 E_1 \cdot A$

则 $B \Rightarrow A$ 行等价

Since

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$= \underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R_1+2R_2\to R_1} \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R_1-R_3\to R_1} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}}_{R_2+4R_3\to R_2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}}_{R_3+3R_2\to R_3} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R_2\div 2\to R_2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}}_{R_3+4R_1\to R_3} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R_1\leftrightarrow R_2} \left(\begin{array}{ccc|c} 0 & 2 & -8 & 8 \\ 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \end{array}\right)$$

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right)$ is row equivalent to $\left(\begin{array}{ccc|c} 0 & 2 & -8 & 8 \\ 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \end{array}\right)$. Thus,

$$\begin{cases} \quad 2y - 8z = 8 \\ x - 2y + z = 0 \\ -4x + 5y + 9z = -9 \end{cases} \text{ and } \begin{cases} x = 29 \\ y = 16 \\ z = 3 \end{cases} \text{ have the same solution.}$$

Definition 1.2.7

行梯形式.

A matrix is said to be in **row echelon form** if

(i) the first nonzero entry in each nonzero row is 1,

(ii) row k does not consist entirely of zeros, the number of leading zero entries in row $k + 1$ is greater than the number of leading zero entries in row k,

(iii) there are rows whose entries are all zero, they are below the rows having nonzero entries.

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Definition 1.2.8

The process of using row operations I, II, and III to transform a linear system into one whose augmented matrix is in row echelon form is called **Gaussian elimination**.

Definition 1.2.9

A matrix is said to be in **reduced row echelon form** if

(i) the matrix is in row echelon form,

(ii) the first nonzero entry in each row is the only nonzero entry in its column.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

The reduced row echelon form of A is denoted by $\mathrm{rref}(A)$.

A system of linear equations may have a unique solution, infinitely many solutions, or no solution.

If a linear system has no solution, we say that the system is **inconsistent**. If the system has at least one solution, we say that it is **consistent**.

Does the system has solutions? If a solution exists, is it the only one; that is, is the solution unique?

If the row echelon form of the augmented matrix contains a row of the form

$$\left(0 \quad \cdots \quad 0 \mid k\right)$$

for nonzero k, the system is inconsistent. Otherwise, the system will be consistent. If the system is consistent and the nonzero rows of the row echelon form of the matrix form a strictly triangular system

$$\left(\begin{array}{cccc|c} 1 & * & \cdots & * & * \\ 0 & 1 & \ddots & \vdots & * \\ \vdots & \ddots & \ddots & * & \vdots \\ 0 & \cdots & 0 & 1 & * \end{array}\right),$$

the system will have a unique solution. Otherwise, the last row of the row echelon form must be 0.

$$\left(\begin{array}{cccc|c} * & * & \cdots & * & * \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & * & * & * \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right),$$

the system has infinitely many solutions.

Example 1.2.10

Consider the system of linear equations

$$(*):\begin{cases} x + \lambda y + z = \lambda \\ 3x - y + (\lambda+2)z = 7 \\ x - y + z = 3 \end{cases} \quad \text{where } \lambda \in \boldsymbol{R}.$$

Solve $(*)$ for

(a) $\lambda \neq \pm 1$,

(b) $\lambda = -1$,

(c) $\lambda = 1$.

Solution

The augmented matrix of $(*)$ is

$$\left(\begin{array}{ccc|c} 1 & \lambda & 1 & \lambda \\ 3 & -1 & \lambda+2 & 7 \\ 1 & -1 & 1 & 3 \end{array}\right) \xrightarrow[R_3 - R_1 \to R_3]{R_2 - 3R_3 \to R_2} \left(\begin{array}{ccc|c} 1 & \lambda & 1 & \lambda \\ 0 & 2 & \lambda-1 & -2 \\ 0 & -1-\lambda & 0 & 3-\lambda \end{array}\right)$$

$$\xrightarrow{R_2 \div 2 \to R_2} \left(\begin{array}{ccc|c} 1 & \lambda & 1 & \lambda \\ 0 & 1 & \frac{\lambda-1}{2} & -1 \\ 0 & -1-\lambda & 0 & 3-\lambda \end{array}\right)$$

$$\xrightarrow{R_3 + (\lambda+1)R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & \lambda & 1 & \lambda \\ 0 & 1 & \frac{\lambda-1}{2} & -1 \\ 0 & 0 & \frac{(\lambda-1)(\lambda+1)}{2} & -2\lambda+2 \end{array}\right)$$

(a) If $\lambda \neq \pm 1$, $\boxed{z=-\frac{4}{\lambda+1}}$, $y=-1-\frac{\lambda-1}{2}z=-1+2\frac{\lambda-1}{\lambda+1}=\boxed{\frac{\lambda-3}{\lambda+1}, = y}$

$x=\lambda-\lambda y-z=\lambda-\lambda\frac{\lambda-3}{\lambda+1}+\frac{4}{\lambda+1}=\frac{\lambda^2+\lambda-\lambda^2+3\lambda+4}{\lambda+1}=4$, exactly 1 solution.

(b) If $\lambda=-1$, $\left(\begin{array}{ccc|c} 1 & \lambda & 1 & \lambda \\ 0 & 1 & \frac{\lambda-1}{2} & -1 \\ 0 & 0 & \frac{(\lambda-1)(\lambda+1)}{2} & -2\lambda+2 \end{array}\right)=\left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 4 \end{array}\right)$. ⇒ 无解

$0z=4$! No solution. (*) is inconsistent.

(c) If $\lambda=1$, $\left(\begin{array}{ccc|c} 1 & \lambda & 1 & \lambda \\ 0 & 1 & \frac{\lambda-1}{2} & -1 \\ 0 & 0 & \frac{(\lambda-1)(\lambda+1)}{2} & -2\lambda+2 \end{array}\right)=\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$. ⇒ 无限解

$y=-1, x=1-y-z=2-z$.

The system has infinite many solutions. Solution set is $\{(2-t,-1,t)\in \boldsymbol{R}^3 : t\in \boldsymbol{R}\}$.

The variables corresponding to the first nonzero elements in each row of the reduced row echelon form are called **leading variables**. The remaining variables are called **free variables**.

Thus, in Example 1.2.10(c), x and y are the leading variables and z is the free variable.

"z is the free variable" means that you are free to choose any value for z. Once you choose, the solution of the system is determined by the choice of z.

Theorem 1.2.11

If a system of linear equation is consistent, then it has either

(i) infinitely many solutions (if there is at least one free variable), or

(ii) exactly one solution (if all variable are leading).

Section 2.1 Inverse of a Matrix

Consider the systems of linear equations of m equations and n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$A \cdot \vec{x} = b$

$\vec{x} = A^{-1}b$

It can be expressed as $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Such an equation is called a matrix equation

矩阵方程.

平均来说,

Roughly speaking, 1 equation eliminates 1 variable. If number of equations is greater than number of unknowns ($m > n$), the system has no solution in most cases. On the other hand, if number of equations is less than number of unknowns ($m < n$), the system has infinitely many solutions in most cases. If number of equations equal number of unknowns ($m = n$), the system has 1 solution in most cases.

方程数 > 未知数 一般无解

方程数 < 未知数 一般无限解

方程数 = 未知数 → 唯一解

Question 2.1.1

Under what condition does the system of linear equations with $m = n$ give us exactly 1 solution?

(Theorem 2.1.9 and Theorem 2.1.26)

Consider an equation $ax = b$.

Case (i): If $a \neq 0$, then the equation has exactly 1 solution, which is $x = a^{-1}b$.

Case (ii): If $a = 0$ and $b = 0$, then any value x is a solution (infinitely many solutions).

Case (iii): If $a = 0$ and $b \neq 0$, then $0 = b \neq 0$! No solution.

Question 2.1.2

If A is an $n \times n$ nonzero matrix, is it possible to find the inverse of A? If inverse of A exists, the solution must be unique. We need the definition of inverse of a square matrix.

Definition 2.1.3

An $n \times n$ matrix A is said to be **nonsingular** or **invertible** if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. The matrix B is said to be a **multiplicative inverse** of A and denoted by A^{-1}.

Remark 2.1.4 B、C 均为 A的逆

If B and C are both multiplicative inverses of A, then

$$B = BI_n = BAC = I_nC = C.$$

Thus, a matrix can have at most one multiplicative inverse.

Example 2.1.5

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For any nonzero α,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \boxed{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \boxed{\alpha^{-1}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For any β,

$$\begin{pmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Theorem 2.1.6 初等矩阵的逆

If E is an elementary matrix, then E is nonsingular and E^{-1} is an elementary matrix of the same type.

Theorem 2.1.7 上标运算

(a) If A is invertible, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A.$$

(b) If A and B are invertible, then so is AB, and

$$(AB)^{-1} = B^{-1}A^{-1} \qquad (ABC)^{-1} = C^{-1} \cdot B^{-1} \cdot A^{-1}$$

(c) If A is invertible, then so is A^T and

$$(A^T)^{-1} = (A^{-1})^T. \quad \text{可以交换.}$$

(d) If A is invertible, then so is αA for nonzero α and

$$\boxed{(\alpha A)^{-1} = \alpha^{-1}A^{-1}.} \ \triangle$$

$$(\alpha \cdot A)^{-1} = \alpha^{-1} \cdot A^{-1}$$

Example 2.1.8

The answer of Question 2.1.2 is NO. We can find a nonzero square matrix such that its inverse does not exist.

Consider

$$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

The matrix equation is

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}^{-1}$ does not exist. Suppose $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}^{-1}$ exists and we can find a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(1, 1) entry and (1, 2) entry on the left give us $a + 2b = 1$ and $a + 2b = 0$! Impossible. A nonzero square matrix does not guarantee the existence of inverse.

非零的方阵不能保证有逆矩阵

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right) \xrightarrow{R_2 - 2R_1 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

Indeed, this system of linear equations have infinitely many solutions $\{(1-t,t): t \in \boldsymbol{R}\}$, but not one solution. It is because the reduced row echelon form of $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ is not I_2.

Theorem 2.1.9

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, that is $\operatorname{rref}(A) = I_n$.

To find the inverse of an $n \times n$ matrix A, form the $n \times 2n$ matrix $(A \mid I_n)$ and compute $\operatorname{rref}(A \mid I_n)$.

If $\operatorname{rref}(A \mid I_n)$ is of the form $(I_n \mid C)$, then $\operatorname{rref}(A) = I_n$ and A is invertible. There exist a sequence of elementary matrices $E_1, E_2, \cdots, E_k$ such that $E_k E_{k-1} \cdots E_1 A = I_n$.

$$\begin{aligned} (A \mid I_n) &\to E_1(A \mid I_n) = (E_1 A \mid E_1) \\ &\to E_2(E_1 A \mid E_1) = (E_2 E_1 A \mid E_2 E_1) \\ &\vdots \\ &\to E_k(E_{k-1} \cdots E_1 A \mid E_{k-1} \cdots E_1) = (E_k E_{k-1} \cdots E_1 A \mid E_k E_{k-1} \cdots E_1) = (I_n \mid \underbrace{E_k E_{k-1} \cdots E_1}_{A^{-1}}) \end{aligned}$$

$A^{-1} = E_k E_{k-1} \cdots E_1$.

If $\operatorname{rref}(A \mid I_n)$ is of another form (that is, its left half fails to be I_n), then A is not invertible. Note that the left half of $\operatorname{rref}(A \mid I_n)$ is rref(A).

Example 2.1.10

Let $A = \begin{pmatrix} 0 & 2 & -8 \\ 1 & -2 & 1 \\ -4 & 5 & 9 \end{pmatrix}$. Find A^{-1} using row operation.

Solution

$$\left(\begin{array}{ccc|ccc} 0 & 2 & -8 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ -4 & 5 & 9 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 2 & -8 & 1 & 0 & 0 \\ -4 & 5 & 9 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 0 & 2 & -8 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ -4 & 5 & 9 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 + 4R_1 \to R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 2 & -8 & 1 & 0 & 0 \\ 0 & -3 & 13 & 0 & 4 & 1 \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 2 & -8 & 1 & 0 & 0 \\ -4 & 5 & 9 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \div 2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 1/2 & 0 & 0 \\ 0 & -3 & 13 & 0 & 4 & 1 \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 2 & -8 & 1 & 0 & 0 \\ 0 & -3 & 13 & 0 & 4 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 + 3R_2 \to R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & 4 & 1 \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 1/2 & 0 & 0 \\ 0 & -3 & 13 & 0 & 4 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 + 4R_3 \to R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 13/2 & 16 & 4 \\ 0 & 0 & 1 & 3/2 & 4 & 1 \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & 4 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 - R_3 \to R_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -3/2 & -3 & -1 \\ 0 & 1 & 0 & 13/2 & 16 & 4 \\ 0 & 0 & 1 & 3/2 & 4 & 1 \end{array}\right) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 13/2 & 16 & 4 \\ 0 & 0 & 1 & 3/2 & 4 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 + 2R_2 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 23/2 & 29 & 7 \\ 0 & 1 & 0 & 13/2 & 16 & 4 \\ 0 & 0 & 1 & 3/2 & 4 & 1 \end{array}\right) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -3/2 & -3 & -1 \\ 0 & 1 & 0 & 13/2 & 16 & 4 \\ 0 & 0 & 1 & 3/2 & 4 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 23/2 & 29 & 7 \\ 13/2 & 16 & 4 \\ 3/2 & 4 & 1 \end{pmatrix}.$$

Theorem 2.1.11 (*LU* decomposition)

Let A be an $n \times n$ matrix. Suppose A is row equivalent to an upper triangular matrix U using only elementary matrix of type III that add a multiple of one row to another row below it. In this case, there exist lower triangular elementary matrices $E_1, \cdots, E_k$ with 1's on the diagonal such that

$$E_k \cdots E_1 A = U.$$

Then $A = LU$ where $L = (E_k \cdots E_1)^{-1} = E_1^{-1} \cdots E_k^{-1}$ is a lower triangular matrix with 1's on the diagonal.

顺序变了

Example 2.1.12

$$\underbrace{\begin{pmatrix} 2 & 3 & 1 \\ 4 & 1 & 4 \\ 3 & 4 & 6 \end{pmatrix}}_{A} \xrightarrow{R_2 - 2R_1 \to R_2} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -5 & 2 \\ 3 & 4 & 6 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1} \underbrace{\begin{pmatrix} 2 & 3 & 1 \\ 4 & 1 & 4 \\ 3 & 4 & 6 \end{pmatrix}}_{A}$$

$$\xrightarrow{R_3 - 1.5R_1 \to R_3} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -5 & 2 \\ 0 & -0.5 & 4.5 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1.5 & 0 & 1 \end{pmatrix}}_{E_2} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -5 & 2 \\ 3 & 4 & 6 \end{pmatrix}$$

$$\xrightarrow{R_3 - 0.1R_2 \to R_3} \underbrace{\begin{pmatrix} 2 & 3 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 4.3 \end{pmatrix}}_{U} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.1 & 1 \end{pmatrix}}_{E_3} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -5 & 2 \\ 0 & -\frac{1}{2} & \frac{9}{2} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.1 & 1 \end{pmatrix}}_{E_3} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1.5 & 0 & 1 \end{pmatrix}}_{E_2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1} \underbrace{\begin{pmatrix} 2 & 3 & 1 \\ 4 & 1 & 4 \\ 3 & 4 & 6 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 2 & 3 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 4.3 \end{pmatrix}}_{U}$$

$$\underbrace{\begin{pmatrix} 2 & 3 & 1 \\ 4 & 1 & 4 \\ 3 & 4 & 6 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0.1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 3 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 4.3 \end{pmatrix}}_{U}$$

The (2, 1) entry of L comes from $R_2 - 2R_1 \to R_2$. The (3, 1) entry of L comes from $R_3 - 1.5R_1 \to R_3$. The (3, 2) entry of L comes from $R_3 - 0.1R_2 \to R_3$.

When $A = LU$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ can be written as $L(U\boldsymbol{x}) = \boldsymbol{b}$. Writing $\boldsymbol{y}$ for $U\boldsymbol{x}$, we can find $\boldsymbol{x}$ by solving the pair of equations

$$\boxed{L\boldsymbol{y} = \boldsymbol{b} \quad \text{and} \quad U\boldsymbol{x} = \boldsymbol{y}.}$$

简化运算

First solve $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$, and then solve $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$ by forward and backward substitution. Each equation is easy to solve because L and U are triangular.

Example 2.1.13

Solve the system the following system of linear equations by LU decomposition.

$$\begin{cases} 2x_1 + 3x_2 + x_3 = -4 \\ 4x_1 + x_2 + 4x_3 = 9 \\ 3x_1 + 4x_2 + 6x_3 = 0 \end{cases}$$

Solution

We have the matrix equation

$$\begin{pmatrix} 2 & 3 & 1 \\ 4 & 1 & 4 \\ 3 & 4 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \\ 0 \end{pmatrix}$$

where $\begin{pmatrix} 2 & 3 & 1 \\ 4 & 1 & 4 \\ 3 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0.1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 4.3 \end{pmatrix}$ as in Example 2.1.12.

Consider $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0.1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \\ 0 \end{pmatrix}$. We get

$$y_1 = -4, \quad y_2 = 9 - 2y_1 = 17, \quad y_3 = 0 - 1.5y_1 - 0.1y_2 = 4.3.$$

Consider $\begin{pmatrix} 2 & 3 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 4.3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 17 \\ 4.3 \end{pmatrix}$. We get

$$x_3 = 1, \quad x_2 = -\tfrac{1}{5}(17 - 2x_3) = -3, \quad x_1 = \tfrac{1}{2}(-4 - 3x_2 - x_3) = 2.$$

Definition 2.1.14

Let $A = (a_{ij})$ be an $n \times n$ matrix, and let M_{ij} denote the $(n-1)\times(n-1)$ matrix obtained from A by deleting the row and column containing a_{ij}. The determinant of M_{ij} is called the **minor** of a_{ij}. We define the **cofactor** $A_{i\,j}$ of a_{ij} by

$$A_{i\,j} = (-1)^{i+j}\det(M_{ij}).$$

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}, \quad M_{ij}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

Definition 2.1.15

The **determinant** of an $n\times n$ matrix A, denoted $\det(A)$ or $|A|$, is a scalar associated with the matrix A that is defined inductively as

$$\det(A)=\begin{cases} a_{11} & \text{if } n=1 \\ a_{11}A_{11}+a_{12}A_{12}+\cdots+a_{1n}A_{1n} & \text{if } n>1 \end{cases}$$

where

$$A_{1j}=(-1)^{1+j}\det(M_{1j}) \qquad j=1,\cdots,n$$

are cofactors associated with the entries in the first row of A.

The appropriate sign of cofactors of 1×1, 2×2, 3×3 matrices are $(+)$, $\begin{pmatrix} + & - \\ - & + \end{pmatrix}$, $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.

Example 2.1.16

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}-a_{12}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}+a_{13}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$=a_{11}\left(a_{22}a_{33}-a_{23}a_{32}\right)-a_{12}\left(a_{21}a_{33}-a_{23}a_{31}\right)+a_{13}\left(a_{21}a_{32}-a_{22}a_{31}\right)$$

$$=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}$$

Remark 2.1.17

Shortcut for calculating determinant of 2×2 and 3×3 matrices.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\left|\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right| \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Theorem 2.1.18

If A is an $n\times n$ matrix with $n\geq 2$, then $\det(A)$ can be expressed as a cofactor expansion using any row or column of A; that is,

$$\begin{aligned} \det(A) &= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \\ &= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \end{aligned}$$

for $i = 1,\cdots,n$ and $j = 1,\cdots,n$. In particular, $\det(A) = \det(A^T)$.

Example 2.1.19

Use cofactor expansion to compute $\det(A)$ for

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 9 & 1 & 3 & 0 \\ 9 & 2 & 2 & 0 \\ 5 & 0 & 0 & 3 \end{pmatrix}.$$

Solution

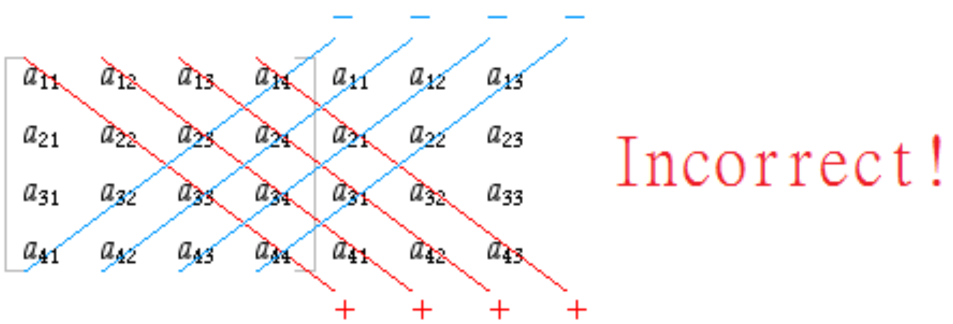
Looking for rows or columns with as many zeros as possible, we choose the second column:

$$
\begin{aligned}
\det(A) &= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} + a_{42}A_{42} \\
&= 1\cdot\det\begin{pmatrix} 1 & \cancel{0} & 1 & 2 \\ \cancel{9} & \cancel{1} & \cancel{3} & \cancel{0} \\ 9 & \cancel{2} & 2 & 0 \\ 5 & \cancel{0} & 0 & 3 \end{pmatrix} - 2\cdot\det\begin{pmatrix} 1 & \cancel{0} & 1 & 2 \\ 9 & \cancel{1} & 3 & 0 \\ \cancel{9} & \cancel{2} & \cancel{2} & \cancel{0} \\ 5 & \cancel{0} & 0 & 3 \end{pmatrix} \\
&= \det\begin{pmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{pmatrix} - 2\det\begin{pmatrix} 1 & 1 & 2 \\ 9 & 3 & 0 \\ 5 & 0 & 3 \end{pmatrix} \\
&= \underbrace{2\det\begin{pmatrix} 9 & 2 \\ 5 & 0 \end{pmatrix}}_{\det\begin{pmatrix} \cancel{1} & \cancel{1} & \cancel{2} \\ 9 & 2 & \cancel{0} \\ 5 & 0 & \cancel{3} \end{pmatrix}} + \underbrace{3\det\begin{pmatrix} 1 & 1 \\ 9 & 2 \end{pmatrix}}_{\det\begin{pmatrix} 1 & 1 & \cancel{2} \\ 9 & 2 & \cancel{0} \\ \cancel{5} & \cancel{0} & \cancel{3} \end{pmatrix}} - 2\left[\underbrace{2\det\begin{pmatrix} 9 & 3 \\ 5 & 0 \end{pmatrix}}_{\det\begin{pmatrix} \cancel{1} & \cancel{1} & \cancel{2} \\ 9 & 3 & \cancel{0} \\ 5 & 0 & \cancel{3} \end{pmatrix}} + \underbrace{3\det\begin{pmatrix} 1 & 1 \\ 9 & 3 \end{pmatrix}}_{\det\begin{pmatrix} 1 & 1 & \cancel{2} \\ 9 & 3 & \cancel{0} \\ \cancel{5} & \cancel{0} & \cancel{3} \end{pmatrix}}\right] \\
&= -20 - 21 - 2(-30 - 18) = 55
\end{aligned}
$$

If we expand along the last row, we also get the same answer.

$$
\begin{aligned}
\det(A) &= a_{41}A_{41} + a_{42}A_{42} + a_{43}A_{43} + a_{44}A_{44} \\
&= -5\det\begin{pmatrix} 0 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 2 & 0 \end{pmatrix} + 3\det\begin{pmatrix} 1 & 0 & 1 \\ 9 & 1 & 3 \\ 9 & 2 & 2 \end{pmatrix} \\
&= \underbrace{-10\det\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}}_{\det\begin{pmatrix} \cancel{0} & \cancel{1} & \cancel{2} \\ 1 & 3 & \cancel{0} \\ 2 & 2 & \cancel{0} \end{pmatrix}} + 3\left[\underbrace{\det\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}}_{\det\begin{pmatrix} \cancel{1} & \cancel{0} & \cancel{1} \\ \cancel{9} & 1 & 3 \\ \cancel{9} & 2 & 2 \end{pmatrix}} + \underbrace{\det\begin{pmatrix} 9 & 1 \\ 9 & 2 \end{pmatrix}}_{\det\begin{pmatrix} \cancel{1} & \cancel{0} & \cancel{1} \\ 9 & 1 & \cancel{3} \\ 9 & 2 & \cancel{2} \end{pmatrix}}\right] \\
&= 40 + 3(-4 + 9) = 55
\end{aligned}
$$

This shortcut in Remark 2.1.17 is not valid for any $n\times n$ matrix for $n\geq 4$.



If we use this shortcut in Example 2.1.19, we get

$$\det\begin{pmatrix} 1 & 0 & 1 & 2 \\ 9 & 1 & 3 & 0 \\ 9 & 2 & 2 & 0 \\ 5 & 0 & 0 & 3 \end{pmatrix} = (1)(1)(2)(3) - (2)(3)(2)(5) - (1)(1)(9)(3) = -81$$

which is incorrect.

Example 2.1.20 上三角/下三角行列式为对角元素相乘

Determinant of an upper and lower triangular matrices are product of diagonal entries, that is

$$\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} = a_{11}a_{22}\cdots a_{nn}, \quad \det\begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ b_{m1} & b_{2m} & \cdots & b_{mm} \end{pmatrix} = b_{11}b_{22}\cdots b_{mm}.$$

Remark 2.1.21

$\det(A+B) \neq \det(A) + \det(B)$ in general. Consider $A = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = -I_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

$\det A = \det(B) = 1$

$\det(A+B) = \det\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \neq 2 = \det(A) + \det(B)$

Theorem 2.1.22

If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det(EA) = \det(E)\det(A)$$

where

$$\det(E) = \begin{cases} -1 & \text{if } E \text{ is of type I} \\ \alpha \neq 0 & \text{if } E \text{ is of type II} \\ 1 & \text{if } E \text{ is of type III} \end{cases}$$

Similar results hold for elementary column operations, that is,

$$\det(AE) = \det(E)\det(A)$$

Theorem 2.1.23

Let $A = (a_{ij})_{n\times n}$. Then

(i) If A has a row or column consisting entirely of zeros, then $\det(A) = 0$. For example,

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$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

(ii) If A has two identical rows or two identical columns, then $\det(A) = 0$. For example,

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$$\det\begin{pmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 0.$$

(iii) A determinant can be expressed as the sum of two determinants by expressing every element in any row (or column) as the sum of two terms. For example,

$$\det\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \det\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

(iv) If we interchange any two rows (or columns) of $\det(A)$, then the result is the negative of $\det(A)$. For example,

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$$\det\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = -\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

(v) If all the elements of one row (or column) of $\det(A)$ have a common factor α, then we can take this common factor out of the determinant. For example,

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \alpha\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Hence, $\det(\alpha A) = \alpha^n \det(A)$ for any α.

(vi) The value of the determinant remains unchanged if the elements of a row (or column) are changed by adding to them any multiple of the corresponding elements in any other row (or column). For example,

$$\det\begin{pmatrix} a_{11}+\beta a_{31} & a_{12}+\beta a_{32} & a_{13}+\beta a_{33} \\ a_{21} & a_{22} & a_{2n} \\ a_{31} & a_{32} & a_{3n} \end{pmatrix} = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Example 2.1.24

Given 1445, 2091, 6188 are multiples of 17 and 126, 4508, 518 are multiples of 7.

Show that $\begin{vmatrix} 1 & 44 & 5 \\ 2 & 9 & 1 \\ 6 & 18 & 8 \end{vmatrix}$ is divisible by $2\times2\times7\times17$ without expansion.

Solution

$$\begin{vmatrix} 1 & 44 & 5 \\ 2 & 9 & 1 \\ 6 & 18 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 44 & 1445 \\ 2 & 9 & 2091 \\ 6 & 18 & 6188 \end{vmatrix} \quad [C_3 + 10C_2 + 1000C_1 \to C_3]$$

$$= 17\begin{vmatrix} 1 & 44 & 85 \\ 2 & 9 & 123 \\ 6 & 18 & 364 \end{vmatrix}$$

$\begin{vmatrix} 1 & 44 & 5 \\ 2 & 9 & 1 \\ 6 & 18 & 8 \end{vmatrix}$ is divisible by 17.

$$\begin{vmatrix} 1 & 44 & 5 \\ 2 & 9 & 1 \\ 6 & 18 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 44 & 5 \\ 2 & 9 & 1 \\ 126 & 4508 & 518 \end{vmatrix} \quad [R_3 + 10R_2 + 100R_1 \to R_3]$$

$$= 7\begin{vmatrix} 1 & 44 & 85 \\ 2 & 9 & 123 \\ 18 & 644 & 74 \end{vmatrix}$$

$\begin{vmatrix} 1 & 44 & 5 \\ 2 & 9 & 1 \\ 6 & 18 & 8 \end{vmatrix}$ is divisible by 7.

$$\begin{vmatrix} 1 & 44 & 5 \\ 2 & 9 & 1 \\ 6 & 18 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 44 & 5 \\ 2 & 9 & 1 \\ 12 & 124 & 20 \end{vmatrix} \quad [R_3 + 2R_2 + 2R_1 \to R_3]$$

$$= 2 \times 2 \begin{vmatrix} 1 & 44 & 5 \\ 2 & 9 & 1 \\ 3 & 31 & 5 \end{vmatrix}$$

$\begin{vmatrix} 1 & 44 & 5 \\ 2 & 9 & 1 \\ 6 & 18 & 8 \end{vmatrix}$ is divisible by 2×2.

Since $2 \times 2, 7, 17$ are relatively prime, $\begin{vmatrix} 1 & 44 & 5 \\ 2 & 9 & 1 \\ 6 & 18 & 8 \end{vmatrix}$ is divisible by $2 \times 2 \times 7 \times 17$.

Indeed, $\begin{vmatrix} 1 & 44 & 5 \\ 2 & 9 & 1 \\ 6 & 18 & 8 \end{vmatrix} = -2 \times 2 \times 7 \times 17$.

Theorem 2.1.25

An $n \times n$ matrix A is singular if and only if $\det(A) = 0$.

Proof

Let U = rref(A). Then there exist a sequence of elementary matrices $E_1, E_2, \cdots, E_k$ such that $E_k E_{k-1} \cdots E_1 A = U$. By Theorem 2.1.22, we have

$$\det(E_k)\det(E_{k-1})\cdots\det(E_1)\det(A) = \det(E_k E_{k-1} \cdots E_1 A) = \det(U).$$

Since $\det(E_i) \neq 0$ for $i = 1, \cdots, k$, $\det(A) = 0$ if and only if $\det(U) = 0$.

By Theorem 2.1.9, if A is nonsingular, then $U = I_n$. $\det(U) = 1$ implies $\det(A) \neq 0$.

By Theorem 2.1.9, if A is singular, the last row of U must be 0, that is,

$$U = \begin{pmatrix} * & \cdots & * & * \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & * & * \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then $\det(U) = 0$. Hence $\det(A) = 0$.

Theorem 2.1.26

The system of linear equation $A\boldsymbol{x} = \boldsymbol{b}$ of n linear equations in n unknowns has exactly 1 solution if and only if A is nonsingular. Equivalently, $\det(A) \neq 0$. Such solution is $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

Theorem 2.1.27

Let A and B be two $n \times n$ matrices. Then $\det(AB) = \det(A)\det(B)$. If A is invertible ($\det(A) \neq 0$), then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Theorem 2.1.28

If A is an $n \times n$ matrix where n is odd and $A = -A^T$, then $\det A = 0$.

Solution

$\det(\alpha A) = \alpha^n$

$$\begin{aligned} \det A &= \det\left(-A^T\right) \\ &= \det\left((-I_n) \cdot A^T\right) \\ &= \det(-I_n) \det A^T \\ &= (-1)^n \det A \\ &= -\det A \end{aligned}$$

Therefore, $\det A = 0$.

Clearly, $\begin{vmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix} = 0.$

Example 2.1.29

Factorize

(a) $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ 全部加在一起

(b) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$. Hence factorize $\begin{vmatrix} 1 & 1 & 1 \\ \cos a & \cos b & \cos c \\ \cos 2a & \cos 2b & \cos 2c \end{vmatrix}$.

(c) $\begin{vmatrix} a+b & b+c & c+a \\ a^2+b^2 & b^2+c^2 & c^2+a^2 \\ a^3+b^3 & b^3+c^3 & c^3+a^3 \end{vmatrix}$

Solution

(a) $$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c & a & b \\ b & c & a \end{vmatrix} \quad [R_1+R_2+R_3 \to R_1]$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ c & a-c & b-c \\ b & c-b & a-b \end{vmatrix} \quad [C_2-C_1 \to C_2, C_3-C_1 \to C_3]$$

$$= (a+b+c)\begin{vmatrix} a-c & b-c \\ c-b & a-b \end{vmatrix}$$

$$= (a+b+c)\big((a-c)(a-b)-(b-c)(c-b)\big)$$

$$= (a+b+c)\big((a^2-ac-ab+bc)-(-b^2+2bc-c^2)\big)$$

$$= (a+b+c)\big(a^2+b^2+c^2-ac-ab-bc\big)$$

(b) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \quad [C_2-C_1 \to C_2, C_3-C_1 \to C_3]$$

$$= \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix} \quad [\text{expand along the 1st column}]$$

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$$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} \quad [\text{take out the common factor in each column}]$$

$$= (b-a)(c-a)(c-b)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ \cos a & \cos b & \cos c \\ \cos 2a & \cos 2b & \cos 2c \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ \cos a & \cos b & \cos c \\ 2\cos^2 a - 1 & 2\cos^2 b - 1 & 2\cos^2 c - 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ \cos a & \cos b & \cos c \\ 2\cos^2 a & 2\cos^2 b & 2\cos^2 c \end{vmatrix} \quad [R_3 + R_1 \to R_3]$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ \cos a & \cos b & \cos c \\ \cos^2 a & \cos^2 b & \cos^2 c \end{vmatrix}$$

$$= 2(\cos c - \cos b)(\cos b - \cos a)(\cos c - \cos a)$$

(c) Clearly, $\begin{pmatrix} a+b & b+c & c+a \\ a^2+b^2 & b^2+c^2 & c^2+a^2 \\ a^3+b^3 & b^3+c^3 & c^3+a^3 \end{pmatrix} = \begin{pmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

取用矩阵

$$\begin{vmatrix} a+b & b+c & c+a \\ a^2+b^2 & b^2+c^2 & c^2+a^2 \\ a^3+b^3 & b^3+c^3 & c^3+a^3 \end{vmatrix} = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 2abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= 2abc(b-a)(c-a)(c-b)$$

Theorem 2.1.30

Let $\left(\begin{array}{c|c} A_{k\times k} & B_{k\times l} \\ \hline C_{l\times k} & D_{l\times l} \end{array}\right)$ be a $(k+l)\times(k+l)$ block matrix where $A_{k\times k}, B_{k\times l}, C_{l\times k}$ and $D_{l\times l}$ are $k\times k$, $k\times l$, $l\times k$ and $l\times l$ matrices respectively. If $C_{l\times k} = \mathbf{0}_{l\times k}$, then

$$\det\left(\begin{array}{c|c} A_{k\times k} & B_{k\times l} \\ \hline \mathbf{0}_{l\times k} & D_{l\times l} \end{array}\right) = \det A_{k\times k} \det D_{l\times l}.$$

Similarly, if $B_{k\times l} = \mathbf{0}_{k\times l}$, then

$$\det\left(\begin{array}{c|c} A_{k\times k} & \mathbf{0}_{k\times l} \\ \hline C_{l\times k} & D_{l\times l} \end{array}\right) = \det A_{k\times k} \det D_{l\times l}.$$

Proof

$$\begin{aligned}
\det\left(\begin{array}{c|c} A_{k\times k} & B_{k\times l} \\ \hline \mathbf{0}_{l\times k} & D_{l\times l} \end{array}\right) &= \det\left(\left(\begin{array}{c|c} I_{k\times k} & \mathbf{0}_{k\times l} \\ \hline \mathbf{0}_{l\times k} & D_{l\times l} \end{array}\right)\left(\begin{array}{c|c} A_{k\times k} & B_{k\times l} \\ \hline \mathbf{0}_{l\times k} & I_{l\times l} \end{array}\right)\right) \\
&= \det\left(\begin{array}{c|c} I_{k\times k} & \mathbf{0}_{k\times l} \\ \hline \mathbf{0}_{l\times k} & D_{l\times l} \end{array}\right)\det\left(\begin{array}{c|c} A_{k\times k} & B_{k\times l} \\ \hline \mathbf{0}_{l\times k} & I_{l\times l} \end{array}\right) \\
&= \det A_{k\times k} \det D_{l\times l}.
\end{aligned}$$

The proof of $\det\left(\begin{array}{c|c} A_{k\times k} & \mathbf{0}_{k\times l} \\ \hline C_{l\times k} & D_{l\times l} \end{array}\right) = \det A_{k\times k} \det D_{l\times l}$ is left as an exercise.

Example 2.1.31

Find $\det\left(\begin{array}{cc|cc} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ \hline 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{array}\right)$.

Solution

It is not easy to find the determinant directly. However, by Theorem 2.1.30,

$$\det\left(\begin{array}{cc|cc} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ \hline 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{array}\right) = \det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\cdot\det\begin{pmatrix} 9 & 10 \\ 11 & 12 \end{pmatrix} = (1\times 4 - 2\times 3)(9\times 12 - 10\times 11) = 4.$$

In general, $\det\left(\begin{array}{c|c} A_{k\times k} & B_{k\times k} \\ \hline C_{k\times k} & D_{k\times k} \end{array}\right)$, $\det\left(A_{k\times k}D_{k\times k} - B_{k\times k}C_{k\times k}\right)$ and $\det A_{k\times k} \det D_{k\times k} - \det B_{k\times k} \det C_{k\times k}$ are not equal to each other.

Example 2.1.32

Consider $\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$.

$$\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = -1.$$

$$\det\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = -2$$

$$\det\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\det\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} - \det\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\det\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$$

Theorem 2.1.33

Let $\left(\begin{array}{c|c} A_{k\times k} & B_{k\times k} \\ \hline C_{k\times k} & D_{k\times k} \end{array}\right)$ be a $2k\times 2k$ block matrix where $A_{k\times k}, B_{k\times k}, C_{k\times k}$ and $D_{k\times k}$ are $k\times k$ matrices. Suppose $C_{k\times k}D_{k\times k} = D_{k\times k}C_{k\times k}$ and $\det D_{k\times k} \neq 0$. Then

$$\boxed{\det\left(\begin{array}{c|c} A_{k\times k} & B_{k\times k} \\ \hline C_{k\times k} & D_{k\times k} \end{array}\right) = \det\left(A_{k\times k}D_{k\times k} - B_{k\times k}C_{k\times k}\right).}$$

Proof

$$\left(\begin{array}{c|c} A_{k\times k} & B_{k\times k} \\ \hline C_{k\times k} & D_{k\times k} \end{array}\right)\left(\begin{array}{c|c} D_{k\times k} & \mathbf{0}_{k\times k} \\ \hline -C_{k\times k} & I_k \end{array}\right) = \left(\begin{array}{c|c} A_{k\times k}D_{k\times k} - B_{k\times k}C_{k\times k} & B_{k\times k} \\ \hline C_{k\times k}D_{k\times k} - D_{k\times k}C_{k\times k} & D_{k\times k} \end{array}\right) = \left(\begin{array}{c|c} A_{k\times k}D_{k\times k} - B_{k\times k}C_{k\times k} & B_{k\times k} \\ \hline \mathbf{0}_{k\times k} & D_{k\times k} \end{array}\right)$$

$$\det\left(\left(\begin{array}{c|c} A_{k\times k} & B_{k\times k} \\ \hline C_{k\times k} & D_{k\times k} \end{array}\right)\left(\begin{array}{c|c} D_{k\times k} & \mathbf{0}_{k\times k} \\ \hline -C_{k\times k} & I_k \end{array}\right)\right) = \det\left(\begin{array}{c|c} A_{k\times k}D_{k\times k} - B_{k\times k}C_{k\times k} & B_{k\times k} \\ \hline \mathbf{0}_{k\times k} & D_{k\times k} \end{array}\right)$$

$$\det\left(\begin{array}{c|c} A_{k\times k} & B_{k\times k} \\ \hline C_{k\times k} & D_{k\times k} \end{array}\right)\det D_{k\times k} = \det\left(A_{k\times k}D_{k\times k} - B_{k\times k}C_{k\times k}\right)\det D_{k\times k}$$

$$\det\left(\begin{array}{c|c} A_{k\times k} & B_{k\times k} \\ \hline C_{k\times k} & D_{k\times k} \end{array}\right) = \det\left(A_{k\times k}D_{k\times k} - B_{k\times k}C_{k\times k}\right)$$

Definition 2.1.34

Let A be an $n \times n$ matrix. We define a new matrix called the **adjoint** of A by

$$\operatorname{adj}(A) = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}.$$

Thus, to form the adjoint, we must replace each term by its cofactor and then transpose the resulting matrix.

Theorem 2.1.35

Let A be an $n \times n$ matrix. If A_{ij} denotes the cofactor of a_{ij}, then

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

That is,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \det(A) \end{pmatrix}$$

$\boxed{\text{or } A(\operatorname{adj}(A)) = (\operatorname{adj}(A))A = \det(A)I_n.}$

Theorem 2.1.36

Let A be an invertible $n \times n$ matrix, that is, $\det(A) \neq 0$. Then

$$\boxed{A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A).}$$

Example 2.1.37

Find the inverse of $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $\det(A)\neq 0$.

Solution

Clearly, $\det(A)=ad-bc$.

$A_{11}=d$, $A_{12}=-c$, $A_{21}=-b$, $A_{22}=a$

$\operatorname{adj}(A)=\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$$A^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example 2.1.38

Let $A=\begin{pmatrix} 0 & 2 & -8 \\ 1 & -2 & 1 \\ -4 & 5 & 9 \end{pmatrix}$. Find A^{-1} using adjoint of A. Hence solve $\underbrace{\begin{pmatrix} 0 & 2 & -8 \\ 1 & -2 & 1 \\ -4 & 5 & 9 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{x}}=\underbrace{\begin{pmatrix} 8 \\ 0 \\ -9 \end{pmatrix}}_{\mathbf{b}}$.

Solution

Clearly, $\det\begin{pmatrix} 0 & 2 & -8 \\ 1 & -2 & 1 \\ -4 & 5 & 9 \end{pmatrix}=-40-8-(-64+18)=-2\neq 0$. Then A is invertible.

The 9 cofactors are

$$A_{11}=\begin{vmatrix} -2 & 1 \\ 5 & 9 \end{vmatrix}=-23,\quad A_{12}=-\begin{vmatrix} 1 & 1 \\ -4 & 9 \end{vmatrix}=-13,\quad A_{13}=\begin{vmatrix} 1 & -2 \\ -4 & 5 \end{vmatrix}=-3,$$

$$A_{21}=-\begin{vmatrix} 2 & -8 \\ 5 & 9 \end{vmatrix}=-58,\quad A_{22}=\begin{vmatrix} 0 & -8 \\ -4 & 9 \end{vmatrix}=-32,\quad A_{23}=-\begin{vmatrix} 0 & 2 \\ -4 & 5 \end{vmatrix}=-8,$$

$$A_{31}=\begin{vmatrix} 2 & -8 \\ -2 & 1 \end{vmatrix}=-14,\quad A_{32}=-\begin{vmatrix} 0 & -8 \\ 1 & 1 \end{vmatrix}=-8,\quad A_{33}=\begin{vmatrix} 0 & 2 \\ 1 & -2 \end{vmatrix}=-2.$$

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{-2}\begin{pmatrix} -23 & -58 & -14 \\ -13 & -32 & -8 \\ -3 & -8 & -2 \end{pmatrix} = \begin{pmatrix} 23/2 & 29 & 7 \\ 13/2 & 16 & 4 \\ 3/2 & 4 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 2 & -8 \\ 1 & -2 & 1 \\ -4 & 5 & 9 \end{pmatrix}^{-1}\begin{pmatrix} 8 \\ 0 \\ -9 \end{pmatrix} = \begin{pmatrix} 23/2 & 29 & 7 \\ 13/2 & 16 & 4 \\ 3/2 & 4 & 1 \end{pmatrix}\begin{pmatrix} 8 \\ 0 \\ -9 \end{pmatrix} = \begin{pmatrix} 29 \\ 16 \\ 3 \end{pmatrix}.$$

Theorem 2.1.39 (Cramer's Rule)

Let A be an $n \times n$ nonsingular matrix, and let $\boldsymbol{b} \in \boldsymbol{R}^n$. Let A_i be the matrix obtained by replacing the i-th column of A by $\boldsymbol{b}$. If $\boldsymbol{x}$ is the unique solution of $A\boldsymbol{x} = \boldsymbol{b}$, then

$$x_i = \frac{\det(A_i)}{\det(A)} \quad \text{for } i = 1, \cdots, n.$$

Example 2.1.40

$$\underbrace{\begin{pmatrix} 0 & 2 & -8 \\ 1 & -2 & 1 \\ -4 & 5 & 9 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 8 \\ 0 \\ -9 \end{pmatrix}}_{b}.$$

Consider Example 2.1.38. $\det(A) = -2$.

$$\det(A_1) = \det\begin{pmatrix} 8 & 2 & -8 \\ 0 & -2 & 1 \\ -9 & 5 & 9 \end{pmatrix} = -144 - 18 - (-144 + 40) = -58$$

$$\det(A_2) = \det\begin{pmatrix} 0 & 8 & -8 \\ 1 & 0 & 1 \\ -4 & -9 & 9 \end{pmatrix} = 72 - 32 - 72 = -32$$

$$\det(A_3) = \det\begin{pmatrix} 0 & 2 & 8 \\ 1 & -2 & 0 \\ -4 & 5 & -9 \end{pmatrix} = 40 - (64 - 18) = -6$$

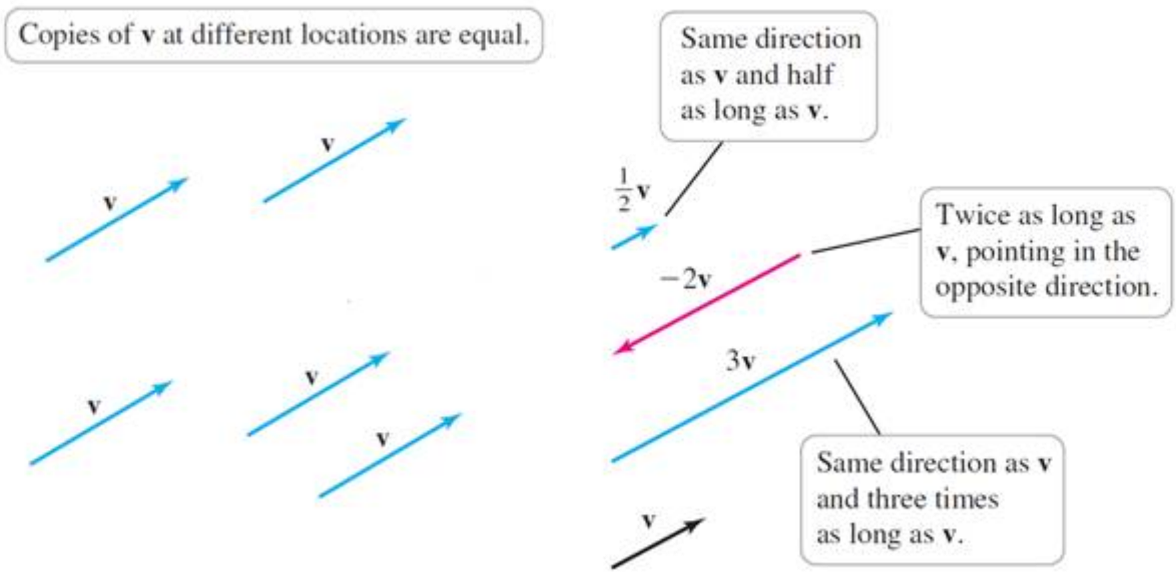
We obtain $x = \dfrac{\det(A_1)}{\det(A)} = 29$, $y = \dfrac{\det(A_2)}{\det(A)} = 16$, $z = \dfrac{\det(A_3)}{\det(A)} = 3$.

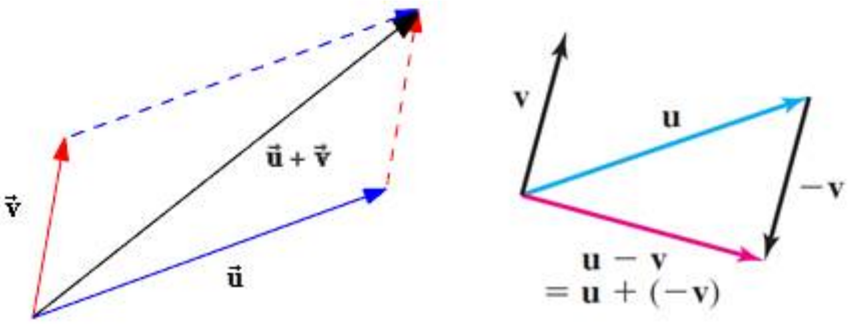
Section 2.2 Dot Products and Cross Products

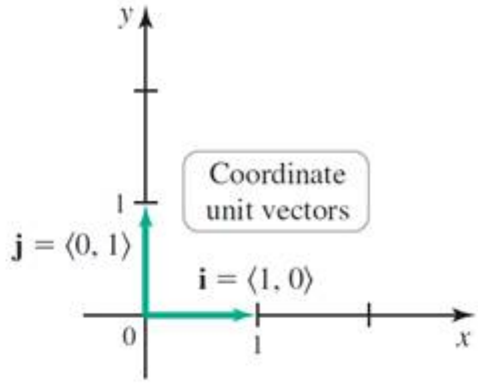
Vectors are quantities that have both **length** (or **magnitude**) and **direction**. Two vectors are **equal** if they have the same magnitude and direction. Quantities having magnitude but no direction are called **scalars**. One exception is the **zero vector**, denote **0**: It has length 0 and no direction.

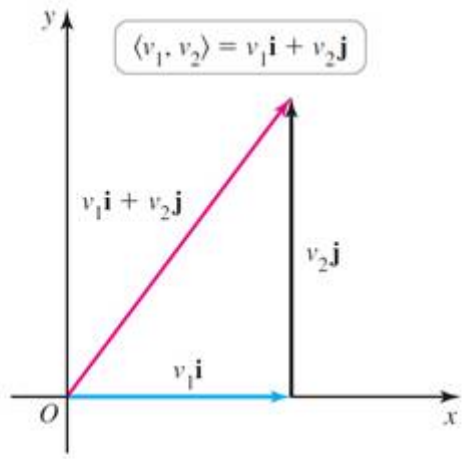
Definition 2.2.1

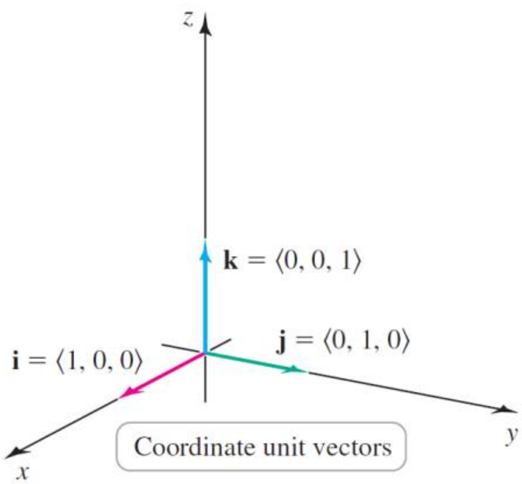
Given a scalar c and a vector $\boldsymbol{v}$, the **scalar multiple** of $c\boldsymbol{v}$ is a vector whose magnitude is $|c|$ multiplied by the magnitude of $\boldsymbol{v}$. If $c > 0$, then $c\boldsymbol{v}$ has the same direction as $\boldsymbol{v}$. If $c < 0$, then $c\boldsymbol{v}$ and $\boldsymbol{v}$ point in opposite directions. Two vectors are **parallel** if they are scalar multiples of each other.

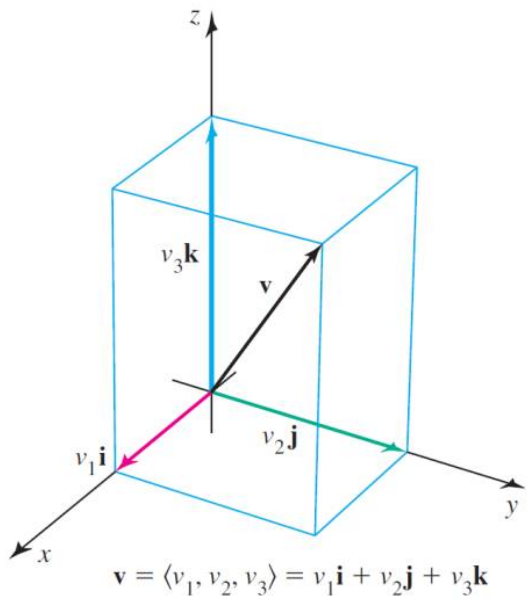












Definition 2.2.2

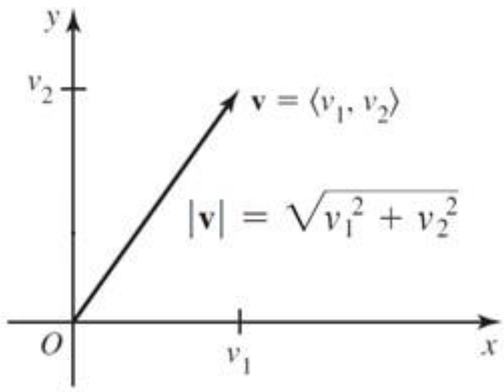
Let c be a scalar, $\boldsymbol{u}=(u_1,u_2)$ and $\boldsymbol{v}=(v_1,v_2)$ be two vectors in $\boldsymbol{R}^2$. Then

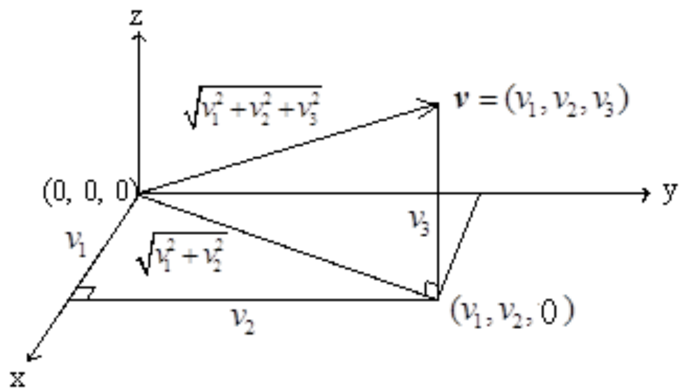
$$\boldsymbol{u}+\boldsymbol{v}=(u_1+v_1,u_2+v_2),\quad \boldsymbol{u}-\boldsymbol{v}=(u_1-v_1,u_2-v_2),\quad c\boldsymbol{u}=(cu_1,cu_2).$$

Definition 2.2.3

Let c be a scalar, $\boldsymbol{u}=(u_1,u_2,u_3)$ and $\boldsymbol{v}=(v_1,v_2,v_3)$ be two vectors in $\boldsymbol{R}^3$. Then

$$\boldsymbol{u}+\boldsymbol{v}=(u_1+v_1,u_2+v_2,u_3+v_3),\quad \boldsymbol{u}-\boldsymbol{v}=(u_1-v_1,u_2-v_2,u_3-v_3),\quad c\boldsymbol{u}=(cu_1,cu_2,cu_3).$$



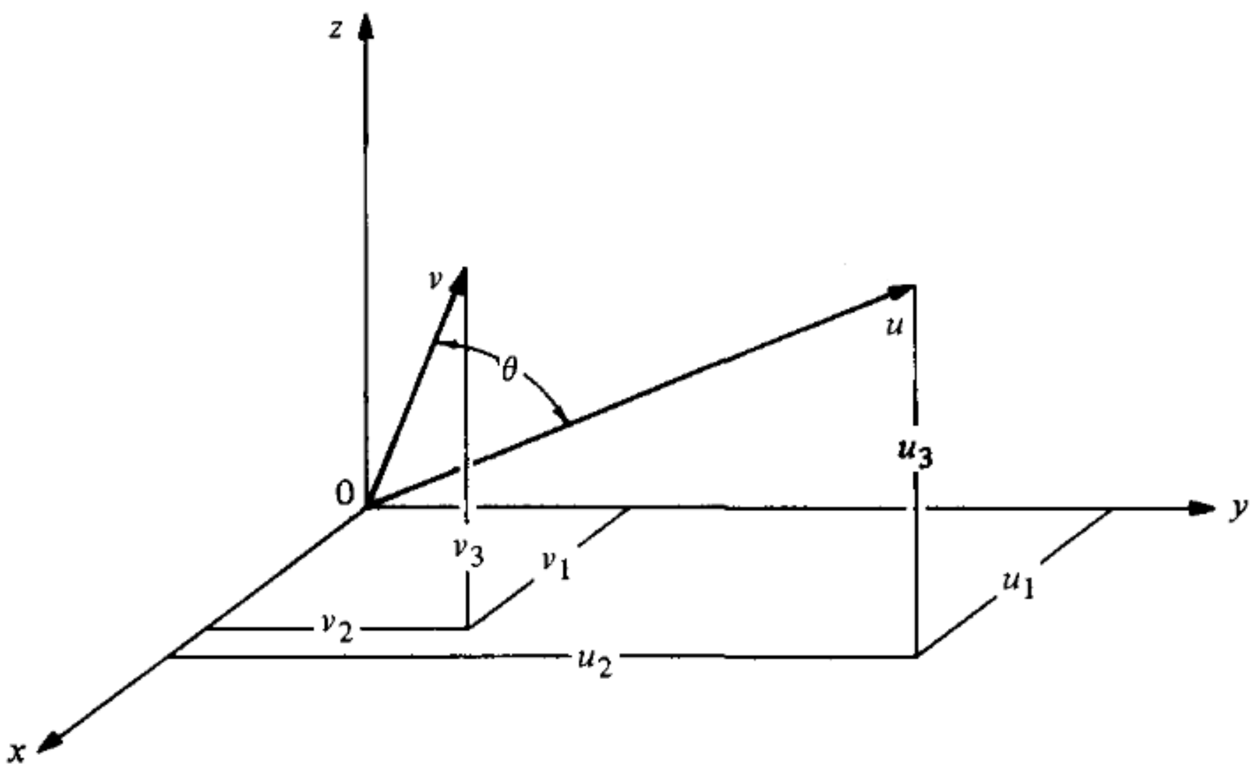


Definition 2.2.4 (Dot Product) 点乘

Given two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$, their **dot product** is defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\theta,$$

where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ with $0 \le \theta \le \pi$. If $\boldsymbol{u} = \mathbf{0}$ or $\boldsymbol{v} = \mathbf{0}$, then $\boldsymbol{u} \cdot \boldsymbol{v} = 0$, and θ is undefined.

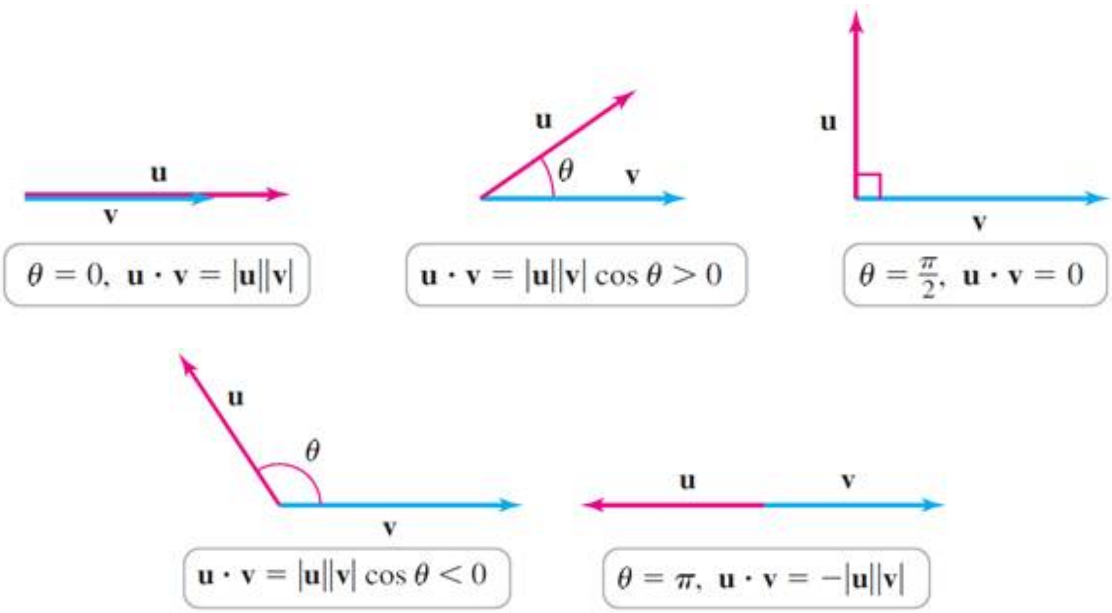


Remark 2.2.5

平行 (i) $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\boldsymbol{u} \cdot \boldsymbol{v} = \pm|\boldsymbol{u}||\boldsymbol{v}|$ respectively.

垂直 (ii) $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular ($\theta = \pi / 2$) if and only if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

(iii) $\boldsymbol{u} \cdot \boldsymbol{u} = u_1^2 + u_2^2 = |\boldsymbol{u}|^2$ if $\boldsymbol{u} \in \boldsymbol{R}^2$ and $\boldsymbol{u} \cdot \boldsymbol{u} = u_1^2 + u_2^2 + u_3^2 = |\boldsymbol{u}|^2$ if $\boldsymbol{u} \in \boldsymbol{R}^3$.



Theorem 2.2.6 坐标运算

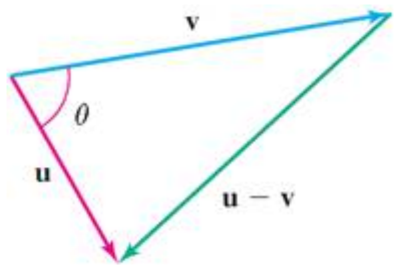
Let $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$. Then $\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$.

Proof

Suppose θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. The vector $\boldsymbol{u} - \boldsymbol{v}$ forms the third side of a triangle. By the Law of Cosines,

余弦定理：

$$|\boldsymbol{u} - \boldsymbol{v}|^2 = |\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - 2|\boldsymbol{u}||\boldsymbol{v}|\cos\theta.$$



Since $|\boldsymbol{u}|^2 = u_1^2 + u_2^2 + u_3^2$, $|\boldsymbol{v}|^2 = v_1^2 + v_2^2 + v_3^2$, and $|\boldsymbol{u}-\boldsymbol{v}|^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2$,

推导：

$$\begin{aligned}
\boldsymbol{u}\cdot\boldsymbol{v} &= |\boldsymbol{u}||\boldsymbol{v}|\cos\theta \\
&= \frac{1}{2}\left(|\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - |\boldsymbol{u}-\boldsymbol{v}|^2\right) \\
&= \frac{1}{2}\left(u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2 - (u_3 - v_3)^2\right) \\
&= \frac{1}{2}\left(u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - u_1^2 + 2u_1v_1 - v_1^2 - u_2^2 + 2u_2v_2 - v_2^2 - u_3^2 + 2u_3v_3 - v_3^2\right) \\
&= u_1v_1 + u_2v_2 + u_3v_3
\end{aligned}$$

Remark 2.2.7 $R^2 \to R^3$

(i) Let $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ be two vectors in $\boldsymbol{R}^2$. Extend $\boldsymbol{u}$ to $\tilde{\boldsymbol{u}} = (u_1, u_2, 0)$ and $\boldsymbol{v}$ to $\tilde{\boldsymbol{v}} = (v_1, v_2, 0)$ in $\boldsymbol{R}^3$. Then $\boldsymbol{u}\cdot\boldsymbol{v} = \tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{v}} = u_1v_1 + u_2v_2$.

(ii) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are both nonzero, then

$$-1 \le \cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\boldsymbol{u}||\boldsymbol{v}|} = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}|\cdot|\boldsymbol{v}|} \le 1.$$

$\cos\theta = \dfrac{u\cdot v}{|u|\cdot|v|}$

Theorem 2.2.8

Suppose $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors and let c be a scalar.

(i) $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u}$ 乘法交换律

(ii) $c(\boldsymbol{u}\cdot\boldsymbol{v}) = (c\boldsymbol{u})\cdot\boldsymbol{v} = \boldsymbol{u}\cdot(c\boldsymbol{v})$

(iii) $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{u}\cdot\boldsymbol{w}$

(iv) $(\boldsymbol{u}+\boldsymbol{v})\cdot\boldsymbol{w} = \boldsymbol{u}\cdot\boldsymbol{w} + \boldsymbol{v}\cdot\boldsymbol{w}$

Definition 2.2.9 (Cross Product) 叉乘

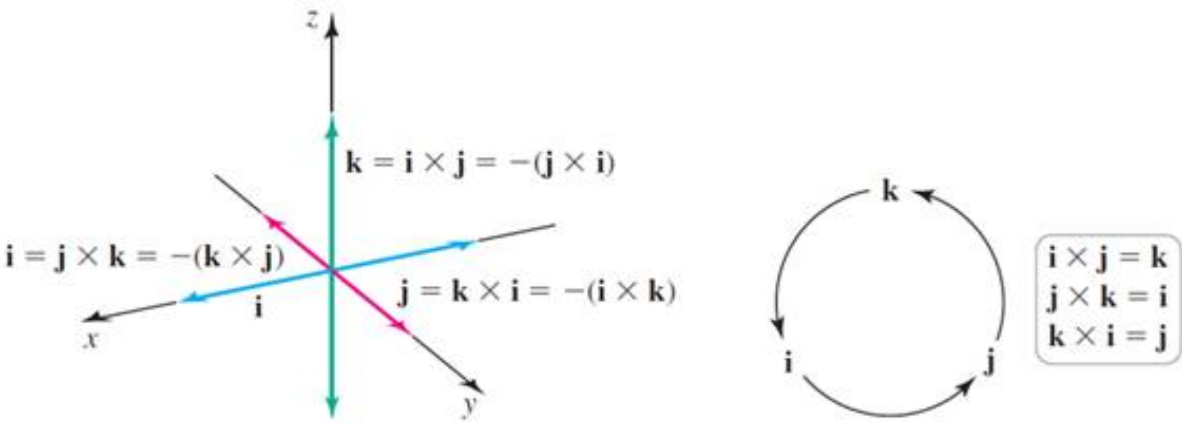
Let $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ be unit vectors in $\mathbb{R}^3$ on positive x, y and z-axis respectively. Let $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$ and $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$. The **cross product** $\boldsymbol{u}\times\boldsymbol{v}$ of $\boldsymbol{u}$ and $\boldsymbol{v}$ is the vector

叉乘：乘出来的是一个向量

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$$
$$=(u_2v_3-u_3v_2)\boldsymbol{i}+(u_3v_1-u_1v_3)\boldsymbol{j}+(u_1v_2-u_2v_1)\boldsymbol{k}$$
$$=(u_2v_3-u_3v_2,\,u_3v_1-u_1v_3,\,u_1v_2-u_2v_1)$$

Theorem 2.2.10

$$\boldsymbol{i}\times\boldsymbol{j}=-(\boldsymbol{j}\times\boldsymbol{i})=\boldsymbol{k},\quad \boldsymbol{j}\times\boldsymbol{k}=-(\boldsymbol{k}\times\boldsymbol{j})=\boldsymbol{i}$$
$$\boldsymbol{k}\times\boldsymbol{i}=-(\boldsymbol{i}\times\boldsymbol{k})=\boldsymbol{j},\quad \boldsymbol{i}\times\boldsymbol{i}=\boldsymbol{j}\times\boldsymbol{j}=\boldsymbol{k}\times\boldsymbol{k}=\boldsymbol{0}$$



Theorem 2.2.11

Let $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ be nonzero vectors in $\mathbb{R}^3$, and let a and b be scalars.

(i) $\boldsymbol{u}\times\boldsymbol{v}=-(\boldsymbol{v}\times\boldsymbol{u})$

(ii) $(a\boldsymbol{u})\times(b\boldsymbol{v})=ab(\boldsymbol{u}\times\boldsymbol{v})$

(iii) $\boldsymbol{u}\times(\boldsymbol{v}+\boldsymbol{w})=(\boldsymbol{u}\times\boldsymbol{v})+(\boldsymbol{u}\times\boldsymbol{w})$ 乘法结合律、

(iv) $(\boldsymbol{u}+\boldsymbol{v})\times\boldsymbol{w}=(\boldsymbol{u}\times\boldsymbol{w})+(\boldsymbol{v}\times\boldsymbol{w})$

用间顺序也不可以交

Q? Is $(\boldsymbol{u}\times\boldsymbol{v})\times\boldsymbol{w}=\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w})$ for every vector $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$?

A. No. $(\boldsymbol{i}\times\boldsymbol{i})\times\boldsymbol{j}=\boldsymbol{0}$, $\quad\boldsymbol{i}\times(\boldsymbol{i}\times\boldsymbol{j})=\boldsymbol{i}\times\boldsymbol{k}=-\boldsymbol{j}$.

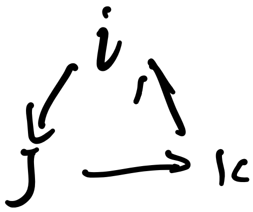
Theorem 2.2.12

Let $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$, $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$ and $\boldsymbol{w} = w_1\boldsymbol{i} + w_2\boldsymbol{j} + w_3\boldsymbol{k}$. Then

$$\begin{aligned}(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{w} &= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)\cdot(w_1, w_2, w_3)\\ &= (u_2v_3 - u_3v_2)w_1 + (u_3v_1 - u_1v_3,)w_2 + (u_1v_2 - u_2v_1)w_3\\ &= \begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}\end{aligned}$$

Clearly, $\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) = (\boldsymbol{v}\times\boldsymbol{w})\cdot\boldsymbol{u} = \begin{vmatrix} v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\\ u_1 & u_2 & u_3\end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix} = (\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{w}$

Suppose $\boldsymbol{u}\times\boldsymbol{v} \neq \boldsymbol{0}$, i.e, $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel. Clearly, $(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{u} = \begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ u_1 & u_2 & u_3\end{vmatrix} = 0$ and $(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{v} = \begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ v_1 & v_2 & v_3\end{vmatrix} = 0$. $\boldsymbol{u}\times\boldsymbol{v}$ is perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$.



Furthermore, we also have

$$(\boldsymbol{u}\times\boldsymbol{v})\cdot(\boldsymbol{u}\times\boldsymbol{v}) = |\boldsymbol{u}\times\boldsymbol{v}|^2 > 0.$$

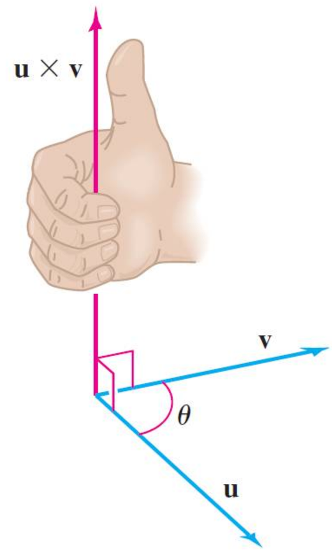
On the other hand, $\boldsymbol{i}\cdot(\boldsymbol{j}\times\boldsymbol{k}) = \boldsymbol{i}\cdot\boldsymbol{i} = 1 > 0$. $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}\times\boldsymbol{v}$ has the same orientation as $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

The direction of $\boldsymbol{u}\times\boldsymbol{v}$ is given by the right hand rule:

When you put the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ tail to tail and let the fingers of you right hand curl from $\boldsymbol{u}$ to $\boldsymbol{v}$, the direction of $\boldsymbol{u}\times\boldsymbol{v}$ is the direction of your thumb, perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$ (see the below figure). However switching the order of $\boldsymbol{u}$ and $\boldsymbol{v}$ gives the opposite direction but in the same magnitude, that means

$$\boldsymbol{v}\times\boldsymbol{u} = -(\boldsymbol{u}\times\boldsymbol{v}).$$

When $\boldsymbol{u}\times\boldsymbol{v} = \boldsymbol{0}$, the direction of $\boldsymbol{u}\times\boldsymbol{v}$ is undefined.



Theorem 2.2.13

对比 $|u \cdot v| = |u| \cdot |v| \cdot \cos\theta$

Given two nonzero vectors $\boldsymbol{u} = (u_1, u_2, u_3)$, and $\boldsymbol{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$, the magnitude of $\boldsymbol{u} \times \boldsymbol{v}$ is

$$|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}|\sin\theta,$$

where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ with $0 \le \theta \le \pi$.

Solution

$$\begin{aligned}
|\boldsymbol{u} \times \boldsymbol{v}|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\
&= (u_2v_3)^2 - 2(u_2v_3)(u_3v_2) + (u_3v_2)^2 \\
&+ (u_3v_1)^2 - 2(u_3v_1)(u_1v_3) + (u_1v_3)^2 \\
&+ (u_1v_2)^2 - 2(u_1v_2)(u_2v_1) + (u_2v_1)^2 \\
&= (u_1v_1)^2 + (u_1v_2)^2 + (u_1v_3)^2 + (u_2v_1)^2 + (u_2v_2)^2 + (u_2v_3)^2 + (u_3v_1)^2 + (u_3v_2)^2 + (u_3v_3)^2 \\
&- \left[(u_1v_1)^2 + (u_2v_2)^2 + (u_3v_3)^2 + 2(u_1v_1)(u_2v_2) + 2(u_2v_2)(u_3v_3) + 2(u_1v_1)(u_3v_3)\right] \\
&= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\
&= |\boldsymbol{u}|^2|\boldsymbol{v}|^2 - (\boldsymbol{u} \cdot \boldsymbol{v})^2 \\
&= |\boldsymbol{u}|^2|\boldsymbol{v}|^2 - |\boldsymbol{u}|^2|\boldsymbol{v}|^2\cos^2\theta \\
&= |\boldsymbol{u}|^2|\boldsymbol{v}|^2\sin^2\theta
\end{aligned}$$

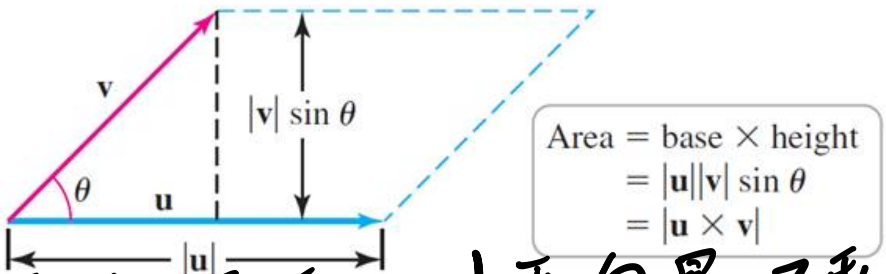
$|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}|\sin\theta$ where $0 \le \theta \le \pi$.

Corollary 2.2.14

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two nonzero vectors in $\mathbb{R}^3$, If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two sides of a parallelogram, then the area of the parallelogram is

$$|\boldsymbol{u}\times\boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}|\sin\theta,$$

where $0 \le \theta \le \pi$ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.



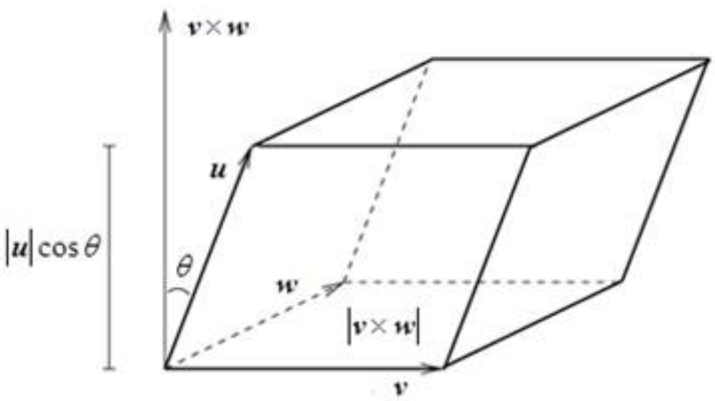
平行四边形的面积=|两向量叉乘绝对值|

Theorem 2.2.15

六面体

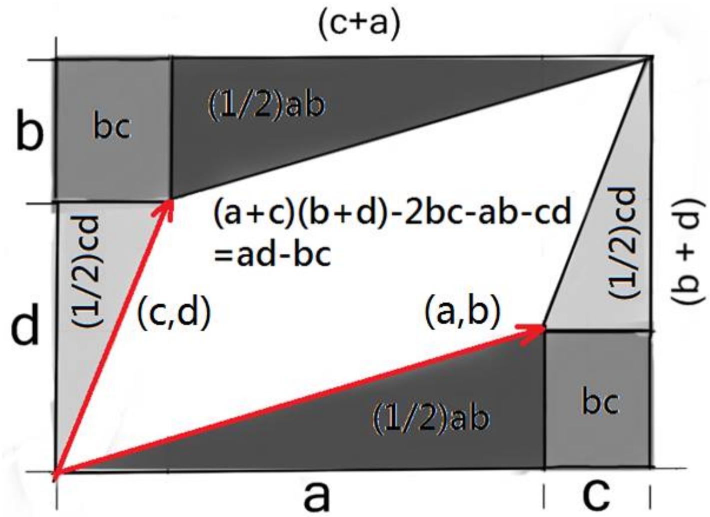
The volume of the parallelepiped generated by the vectors generated by $\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{v} = (v_1, v_2, v_3)$ and $\boldsymbol{w} = (w_1, w_2, w_3)$ is the base area times its height

$$|\boldsymbol{v}\times\boldsymbol{w}||\boldsymbol{u}|\cos\theta = \boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$



Theorem 2.2.16

The area of the parallelogram generated by the vectors (a, b) and (c, d) is $\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|$.



直线　　　　　　　　线性方程

Every straight line L in the plane is defined by a linear equation

$$① \quad a_1x + a_2y = c.$$

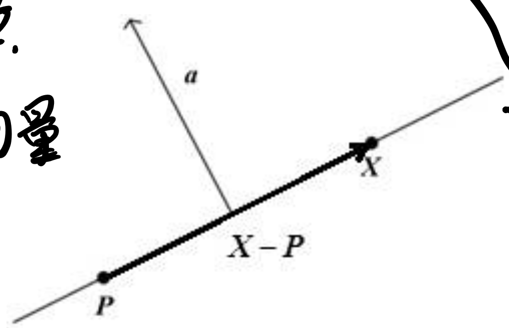
Let $\boldsymbol{a} = (a_1, a_2)$, $\boldsymbol{X} = (x, y)$ and $\boldsymbol{P} = (p, q)$ be two points on L. Then $\boldsymbol{a} \cdot \boldsymbol{X} = \boldsymbol{a} \cdot \boldsymbol{P} = c$. Hence

$$② \quad \boldsymbol{a} \cdot (\boldsymbol{X} - \boldsymbol{P}) = 0.$$

That means $\boldsymbol{a}$ and $\boldsymbol{X} - \boldsymbol{P}$ are perpendicular. $\boldsymbol{a}$ is a normal vector of the straight line L.

a是直线法向量.

x-P 是直线方向向量



$(a_1, a_2)(x-p, y-q)$

$a_1x - a_1p + a_2y - a_2q = 0$

$\Rightarrow a_1x + a_2y = a_1p + a_2q$

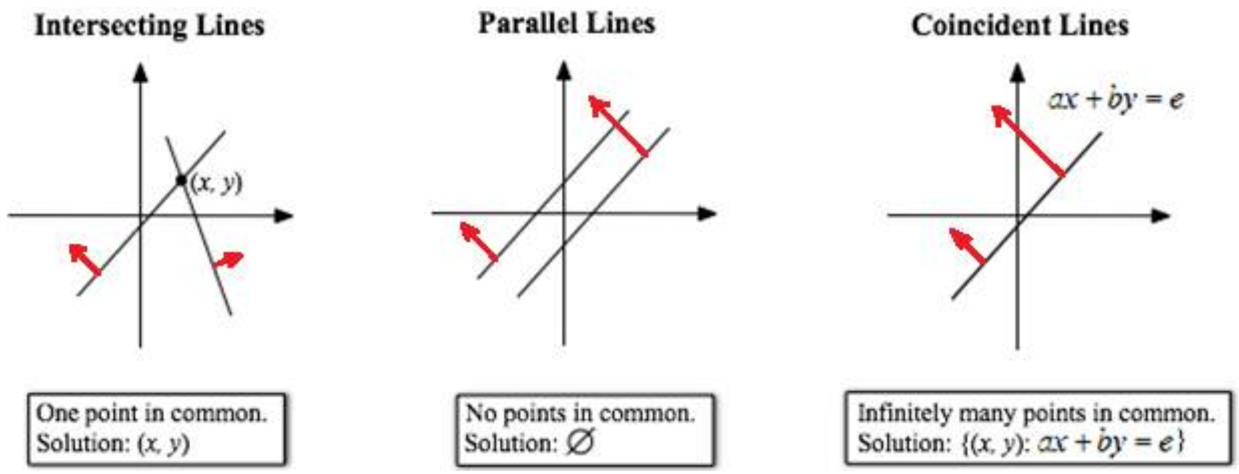
Changing c will move the straight line in parallel.

Consider the system of linear equations

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

There are 2 straight lines in $\boldsymbol{R}^2$ with normal vectors (a, b) and (c, d).

直线的法向量 (a,b)　(c,d)



The system of linear equations has exactly one solution if and only if the area of the parallelogram generated by the normal vectors of two straight lines is nonzero. Equivalently,

由两个法向量组成的

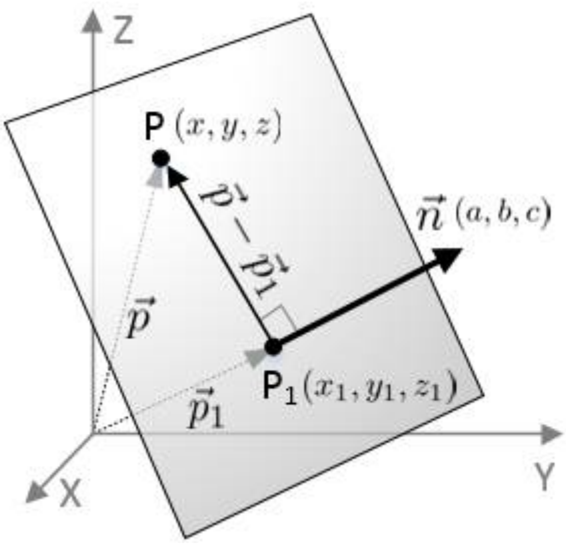
$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.$$

平四面积不为0 $\Rightarrow$ 有交点(有且一个)

If $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$, whether the system has infinitely solutions depends on e and f.

$\begin{cases} \text{要么平行} \\ \text{要么重合} \end{cases}$

A plane in $\mathbb{R}^3$ can be represented by a linear equation in variables x, y, z: $ax + by + cz = d$.



Choose two distinct points $P = (x, y, z)$ and $P_1 = (x_1, y_1, z_1)$ on the plane.

$$(a,b,c)\cdot(x - x_1, y - y_1, z - z_1) = ax + by + cz - (ax_1 + by_1 + cz_1) = 0$$

法向量 平面内一向量

$\vec{n} \cdot \vec{PX} = \vec{n} \cdot \vec{OX} - \vec{n} \cdot \vec{OP} = 0$

$ax+by+cz=d$

Any vector in the plane is perpendicular to vector (a, b, c). (a, b, c) is the normal vector of the plane.

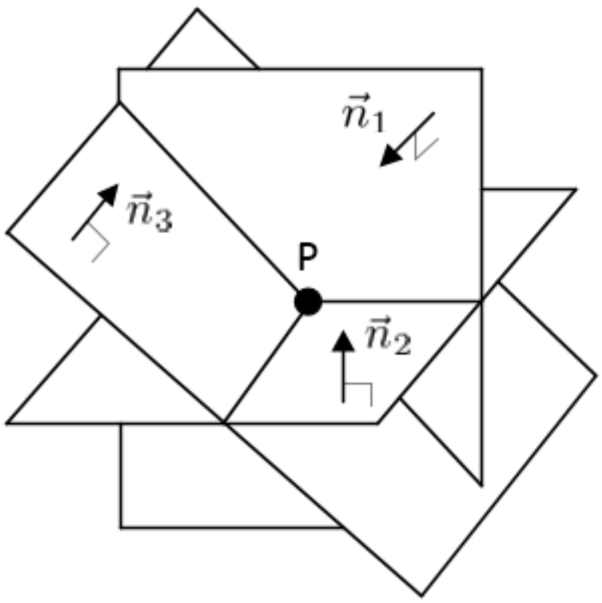
垂直

Changing d will move the plane in parallel.

Consider the system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases} \Rightarrow \text{三个平面.}$$

There are 3 planes in $\boldsymbol{R}^3$ with normal vectors $\vec{\boldsymbol{n}}_1 = (a_{11}, a_{12}, a_{13})$, $\vec{\boldsymbol{n}}_2 = (a_{21}, a_{22}, a_{23})$ and $\vec{\boldsymbol{n}}_3 = (a_{31}, a_{32}, a_{33})$.



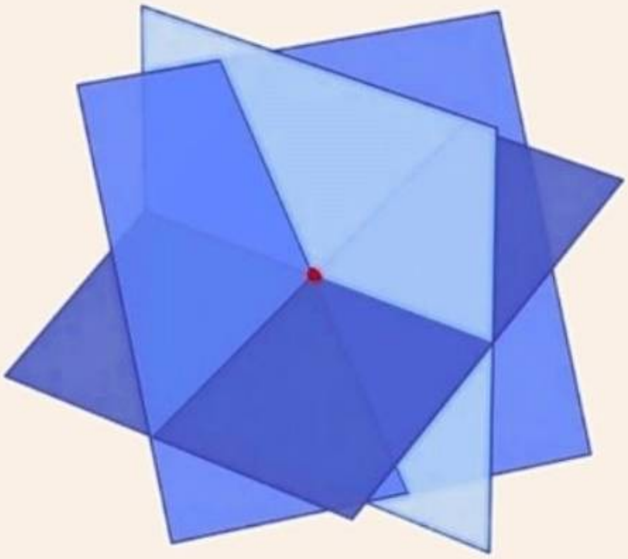
The system of linear equations has exactly one solution if and only if the volume of the parallelepiped generated by the normal vectors of three planes is nonzero. Equivalently,

三个平面有一交点

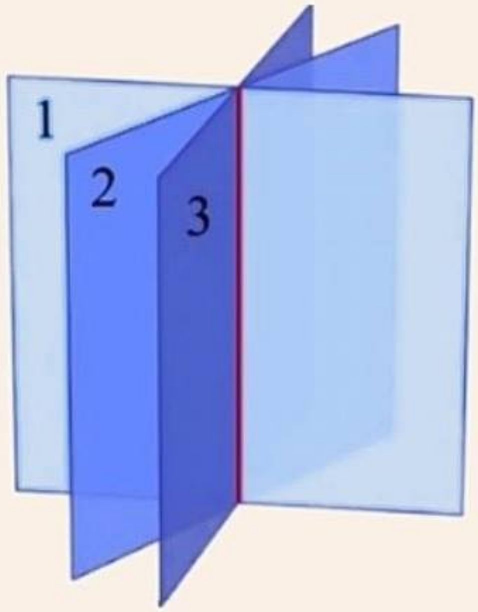
$$\Leftrightarrow \det|\vec{n}_1, \vec{n}_2, \vec{n}_3| \neq 0 \qquad \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \neq 0.$$

If $\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 0$, whether the system has infinitely solutions depends on b_1, b_2, b_3.

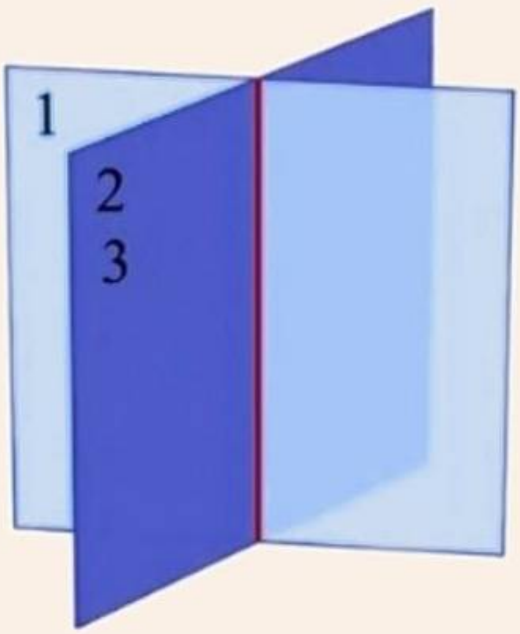
$$\det(\vec{n}_1, \vec{n}_2, \vec{n}_3) = 0 \Rightarrow \begin{cases} \text{无解} \\ \text{有无限解} \end{cases}$$



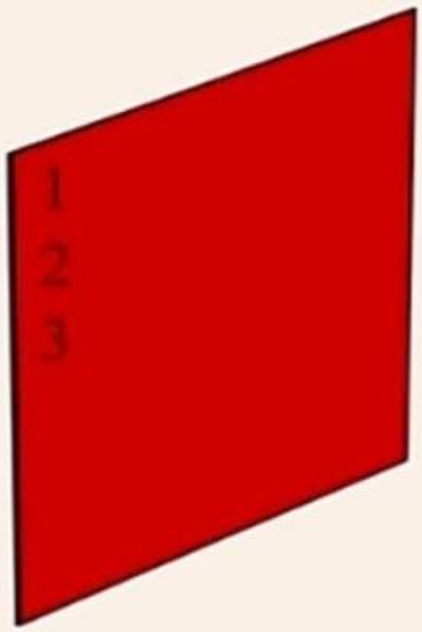
One solution
(Intersection is a point)



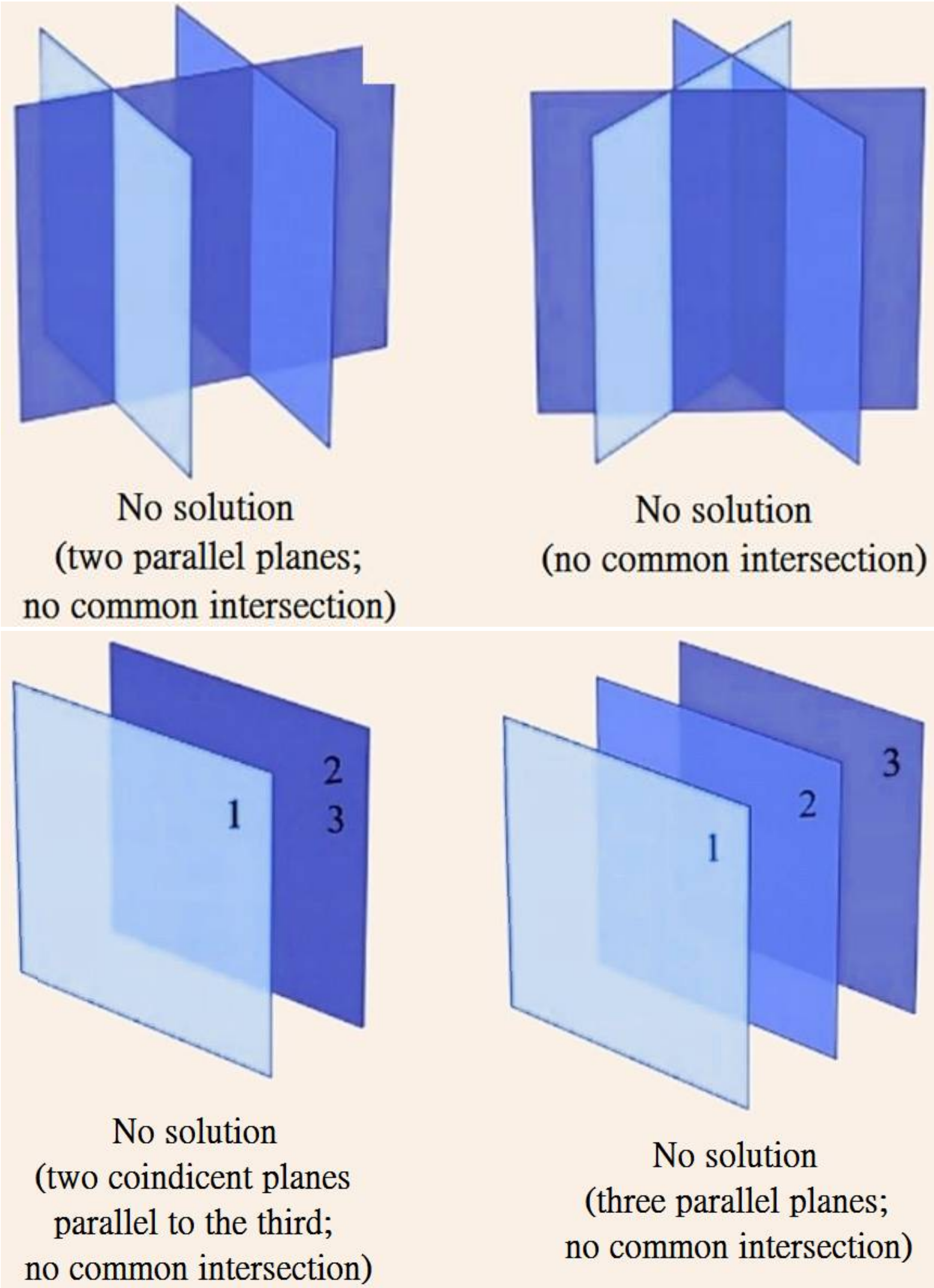
Infinitely many solutions
(Intersection is a line)



Infinitely many solutions
(two coincident planes;
intersection is a line)



Infinitely many solutions
(planes are all coincident;
intersection is a plane)



No solution
(two parallel planes;
no common intersection)

No solution
(no common intersection)

No solution
(two coindicent planes
parallel to the third;
no common intersection)

No solution
(three parallel planes;
no common intersection)

两向量构成的平行四边形面积=0

↑

二维：两向量平行 $\det(u,v)=0$

Let $\boldsymbol{u}$, $\boldsymbol{v}$ be two nonzero vectors in $\boldsymbol{R}^2$. $\boldsymbol{u}//\boldsymbol{v}$ (that is $\boldsymbol{u} = k\boldsymbol{v}$ for some $k \neq 0$) if and only if $\det(\boldsymbol{u} \mid \boldsymbol{v}) = 0$ (no area in Theorem 2.2.16). Let $\boldsymbol{u}$, $\boldsymbol{v}$ be two nonzero vectors in $\boldsymbol{R}^3$. $\boldsymbol{u}//\boldsymbol{v}$ (that is $\boldsymbol{u} = k\boldsymbol{v}$ for some $k \neq 0$) if and only if $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{0}$ (no area in Corollary 2.2.14).

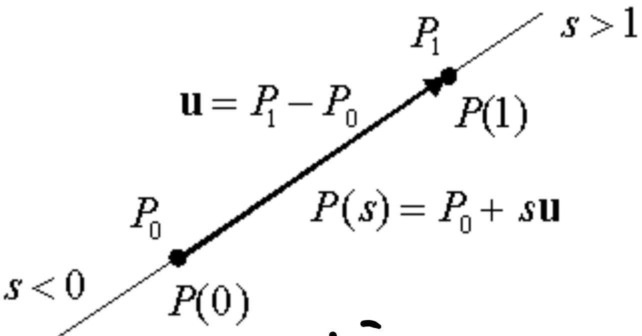
三维：两向量平行 $|u\times v|=0 \Rightarrow u\times v=0$

构成的平行四边形面积为0

In any dimension, the parametric equation of a line defined by two points P_0 and P_1 can be represented as:

$$P(s) = P_0 + s(P_1 - P_0) = P_0 + s\boldsymbol{u},$$

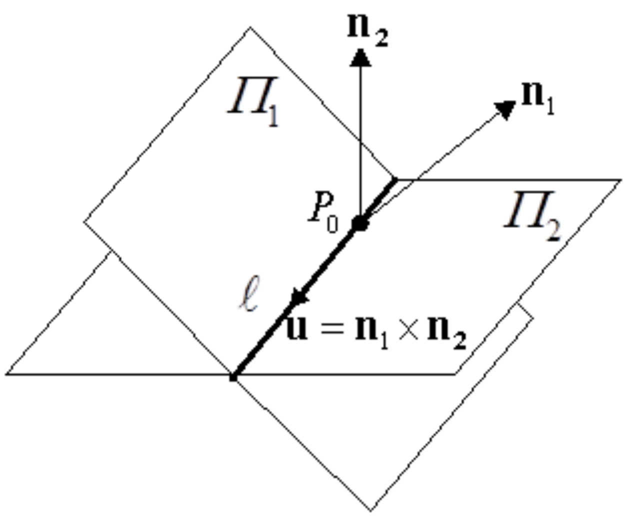
where the parameter s is a real number and $\boldsymbol{u} = P_1 - P_0$ is a direction vector. $P(s) = sP_1 + (1-s)P_0$ is a point on the finite segment P_0P_1 when $0 \le s \le 1$. Further, if $s < 0$ then $P(s)$ is outside the segment on the P_0 side, and if $s > 1$ then $P(s)$ is outside the segment on the P_1 side. Particularly in $\boldsymbol{R}^3$, the equation of $P(s)$ can be expressed as $(P(s) - P_0) \times \boldsymbol{u} = \boldsymbol{0}$ since $P(s) - P_0 // \boldsymbol{u}$.



(x, y, z)

交线

A straight line ℓ in $\boldsymbol{R}^3$ can be expressed as intersection of 2 non-parallel planes Π_1 and Π_2.



$\vec{PX}\cdot\vec{n}=0$

$\Rightarrow (\vec{OX}-\vec{OP})\vec{n}=0$

$\vec{OX}\cdot\vec{n}=\vec{OP}\cdot\vec{n}$

$r\cdot\vec{n}=\underbrace{\vec{OP}\cdot\vec{n}}_{c}$

u(方向向量) = $\vec{n_1} \times \vec{n_2}$

Let $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be normal vectors of Π_1 and Π_2 respectively. $\boldsymbol{u} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$ is a direction vector for the intersection line ℓ since $\boldsymbol{u}$ is perpendicular to both $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$.

$\vec{n_1} \times \vec{n_2} = u$ u分别垂直于 $\vec{n_1}$, $\vec{n_2}$

Example 2.2.17

Let $\boldsymbol{r} = (x, y, z)$. The planes Π_1 and Π_2 have equations

$$\boldsymbol{r} \cdot (2,1,7) = 10 \quad \text{and} \quad \boldsymbol{r} \cdot (3,1,-4) = 7.$$

Express $\boldsymbol{r}$ in the form of $P(s) = P_0 + s\boldsymbol{u}$ and $(P(s) - P_0) \times \boldsymbol{u} = \boldsymbol{0}$.

Solution

We need to solve the system of linear equations

$$\begin{cases} 2x + y + 7z = 10 \\ 3x + y - 4z = 7 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 7 & 10 \\ 3 & 1 & -4 & 7 \end{array}\right) \xrightarrow{R_2 - R_1 \to R_2} \left(\begin{array}{ccc|c} 2 & 1 & 7 & 10 \\ 1 & 0 & -11 & -3 \end{array}\right) \xrightarrow{R_1 - 2R_2 \to R_1} \left(\begin{array}{ccc|c} 0 & 1 & 29 & 16 \\ 1 & 0 & -11 & -3 \end{array}\right)$$

Then $\dfrac{x+3}{11} = \dfrac{y-16}{-29} = z$, $\quad (x, y, z) = (11s - 3, -29s + 16, s) = (-3, 16, 0) + s(11, -29, 1)$ and

$((x, y, z) - (-3, 16, 0)) \times (11, -29, 1) = (0, 0, 0)$.

$(2,1,7) \times (3,1,-4) = (-11, 29, -1)$ is parallel to the direction vector $(11, -29, 1)$.

Example 2.2.18

两个方向向量

Find the equation of a plane defined by $(x, y, z) = (2,1,1) + s(3,1,2) + t(4,-1,1)$.

Solution

$u_1 \times u_2 = \vec{n}$

The normal vector of the plane is $(3,1,2) \times (4,-1,1) = (3,5,-7)$

The equation of the plane is $3(x-2) + 5(y-1) - 7(z-1) = 0$ or $3x + 5y - 7z = 4$.

$\vec{n_1} = (3, 5, -7)$

$P = (2, 1, 1)$

Example 2.2.19

Po.

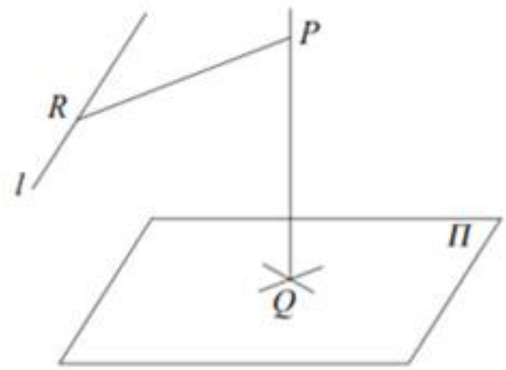
Let $Q = (7, 4, 6)$, the plane Π defined by $(x, y, z)\cdot(2,1,3) = 36$, and the line l defined by $(x, y, z) = (20, -8, 1) + \mu(-7, 5, 3)$. 方向向量. 法向量.

P(5)

(a) Show that Q lies in Π.

(b) Show that l is parallel to Π.

(c) The diagram shows the point P, which lies on the normal to Π that passes through Q. The point R is the point on l which is closest to P, and $PQ = PR$. Determine the coordinate of P.



Solution

(a) Since $(7,4,6)\cdot(2,1,3) = 36$, Q lies in Π.

(b) Since $(-7,5,3)\cdot(2,1,3) = 0$, l is perpendicular to normal of Π. l is parallel to Π.

(c) $P = Q + \lambda \boldsymbol{n} = (7,4,6) + \lambda(2,1,3)$ for some λ where $\boldsymbol{n}$ is normal of Π. $\left|\overrightarrow{PQ}\right| = |\lambda|\sqrt{14}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (20,-8,1) + \mu(-7,5,3) - \big((7,4,6) + \lambda(2,1,3)\big) = (13,-12,-5) + \mu(-7,5,3) - \lambda(2,1,3)$

Since $\overrightarrow{PR}$ is perpendicular to l, $\overrightarrow{PR}\cdot(-7,5,3) = 0$. We have $\mu = 2$.

Hence $\overrightarrow{PR} = (-1,-2,1) - \lambda(2,1,3)$.

Since $\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{PR}\right|$, we get

$$(1+2\lambda)^2 + (2+\lambda)^2 + (1-3\lambda)^2 = 14\lambda^2$$
$$1 + 4\lambda + 4\lambda^2 + 4 + 4\lambda + \lambda^2 + 1 - 6\lambda + 9\lambda^2 = 14\lambda^2$$
$$\lambda = -3$$

$P = (7,4,6) - 3(2,1,3) = (1,1,-3)$.

歪斜

△ Example 2.2.20 (distance between 2 skew lines)

Show that 2 lines $P(s)=(-3,-8,7)+(0,-1,1)s$ and $Q(t)=(6,3,0)+(-4,-3,0)t$ never meet. Find the distance between these 2 skew lines.

Solution

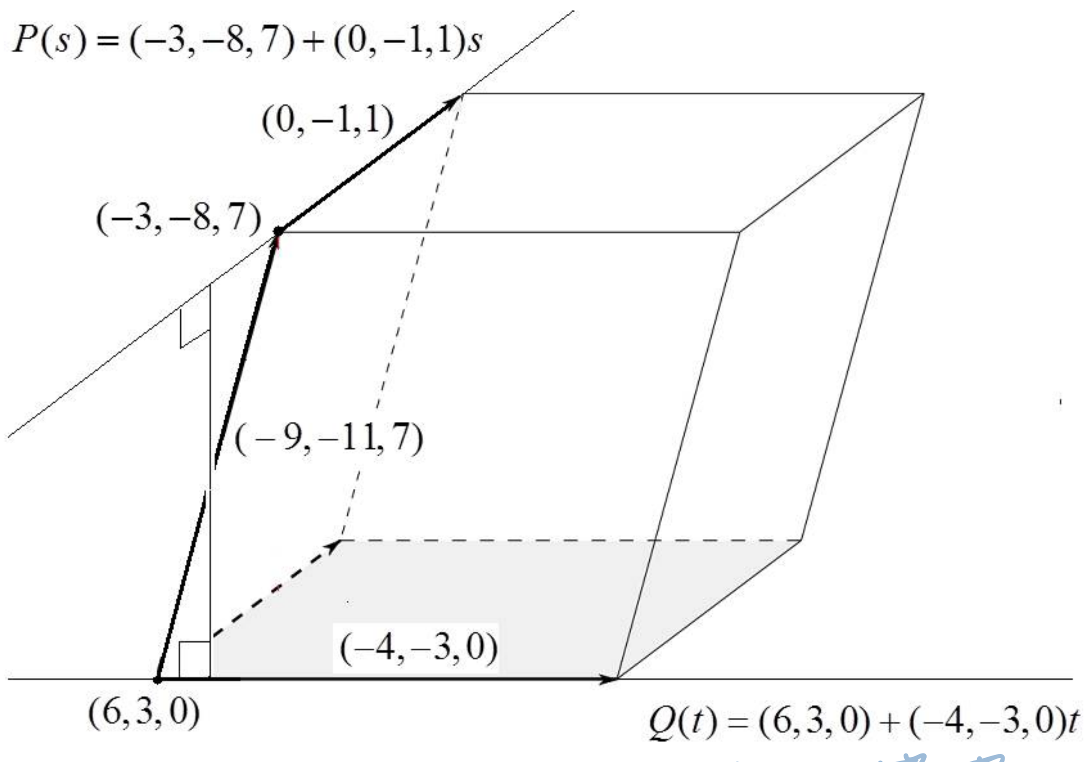
? Suppose $(-3,-8,7)+(0,-1,1)s=(6,3,0)+(-4,-3,0)t$ for some s and t. That means

先要验证不相交

$$(-9,-11,7)+(0,-1,1)s-(-4,-3,0)t=(0,0,0) \quad \text{or} \quad \begin{pmatrix} -4 & 0 & -9 \\ -3 & -1 & -11 \\ 0 & 1 & 7 \end{pmatrix}\begin{pmatrix} -t \\ s \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

? However, $\det\begin{pmatrix} -4 & 0 & -9 \\ -3 & -1 & -11 \\ 0 & 1 & 7 \end{pmatrix}=11\neq 0$. Hence $\begin{pmatrix} -t \\ s \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ impossible.

Distance between these 2 skew lines is height of parallelepiped.



⇒ 类似 以前

$$d=\frac{\text{volume of parallelepiped}}{\text{base area of parallelogram}}=\frac{\left|\det\begin{pmatrix} -4 & -3 & 0 \\ 0 & -1 & 1 \\ -9 & -11 & 7 \end{pmatrix}\right|}{|(-4,-3,0)\times(0,-1,1)|}=\frac{11}{|(-3,4,4)|}=\frac{11}{\sqrt{41}}$$

向量 子空间

Section 3.1 Vector Spaces and Subspaces

Definition 3.1.1

集合

A vector space V over $\boldsymbol{R}$ consists of a set V on which two operations (called **addition** + and scalar **multiplication** $\cdot$, respectively) are defined so that for each pair of elements $\boldsymbol{u},\boldsymbol{v}\in V$, $\boldsymbol{u}+\boldsymbol{v}\in V$, and for each $\alpha\in\boldsymbol{R}$ and each element $\boldsymbol{v}\in V$, $\alpha\cdot\boldsymbol{v}\in V$, such that the following conditions hold:

验证封闭性

(1) $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$ for all $\boldsymbol{u},\boldsymbol{v}\in V$.

(2) $\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})=(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}$ for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V$.

(3) there exists an elements $\boldsymbol{e}\in V$ such that $\boldsymbol{u}+\boldsymbol{e}=\boldsymbol{u}$ for all $\boldsymbol{u}\in V$.

(4) for each element $\boldsymbol{u}\in V$ there exists an elements $\boldsymbol{v}\in V$ such that $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{e}$.

(5) for each element $\boldsymbol{u}\in V$, $1\cdot\boldsymbol{u}=\boldsymbol{u}$.

(6) for each pair of scalar $a,b\in\boldsymbol{R}$ and each $\boldsymbol{u}\in V$, $(ab)\cdot\boldsymbol{u}=a\cdot(b\boldsymbol{u})$.

(7) for each scalar $a\in\boldsymbol{R}$ and each pair of elements $\boldsymbol{u},\boldsymbol{v}\in V$, $a\cdot(\boldsymbol{u}+\boldsymbol{v})=a\cdot\boldsymbol{u}+a\cdot\boldsymbol{v}$.

(8) for each pair of scalar $a,b\in\boldsymbol{R}$ and each $\boldsymbol{u}\in V$, $(a+b)\cdot\boldsymbol{u}=a\cdot\boldsymbol{u}+b\cdot\boldsymbol{u}$.

Remark 3.1.2

The vector $\boldsymbol{e}$ in (3) is unique, called zero vector. The vector $\boldsymbol{v}$ in (4) is unique, called the additive inverse of $\boldsymbol{u}$. However the zero vector $\boldsymbol{e}$ is not necessary 0 in all positions. (Example 3.1.4)

Example 3.1.3

Let $\boldsymbol{R}^n=\{(a_1,\cdots,a_n):a_1,\cdots,a_n\in\boldsymbol{R}\}$. Define the standard addition and scalar multiplication on $\boldsymbol{R}^n$ by

$$(a_1,\cdots,a_n)+(b_1,\cdots,b_n)=(a_1+b_1,\cdots,a_n+b_n)$$

and

$$\alpha(a_1,\cdots,a_n)=(\alpha a_1,\cdots,\alpha a_n)$$

for any $(a_1,\cdots,a_n),(b_1,\cdots,b_n)\in\boldsymbol{R}^n$ and $\alpha\in\boldsymbol{R}$. In this case, the zero vector is $(0,\cdots,0)$.

It is easily verified that all the vector space axioms hold. Thus, $\boldsymbol{R}^n$, with the standard addition and scalar multiplication, is a vector space.

Example 3.1.4

Let $V=\{(a,1): a\in \boldsymbol{R}\}\subset \boldsymbol{R}^2$. Define a new addition $+_{\text{new}}$ and scalar multiplication $\cdot_{\text{new}}$ of V by

$$(a,1)+_{\text{new}}(b,1)=(a+b,1)$$

and

$$\alpha\cdot_{\text{new}}(a,1)=(\alpha a,1)$$

for any $(a,1),(b,1)\in V$ and $\alpha\in \boldsymbol{R}$. In this case, the zero vector is (0, 1), but not (0, 0).

说明零向量不一定是 (0,0)

$\boldsymbol{R}^n$ is not the only vector space over $\boldsymbol{R}$. Besides $\boldsymbol{R}^n$, there are a lot of vector spaces over $\boldsymbol{R}$.

Example 3.1.5

多项式.

Let P_n denote the set of all polynomials of degree less than n, that is

$$P_n=\{a_0+a_1x+\cdots+a_{n-1}x^{n-1}: a_0,a_1,\cdots,a_{n-1}\in \boldsymbol{R}\}.$$

Define the standard addition and scalar multiplication of polynomials by

$$p(x)+q(x)=(a_0+b_0)+(a_1+b_1)x+\cdots+(a_{n-1}+b_{n-1})x^{n-1}$$

and

$$\alpha p(x)=\alpha a_0+\alpha a_1x+\cdots+\alpha a_{n-1}x^{n-1}$$

for any $p(x)=a_0+a_1x+\cdots+a_{n-1}x^{n-1}, q(x)=b_0+b_1x+\cdots+b_{n-1}x^{n-1}\in P_n$ and $\alpha\in \boldsymbol{R}$.

In this case, the zero vector is the zero polynomial 0.

It is easily verified that all the vector space axioms hold. Thus, P_n, with the standard addition and scalar multiplication of polynomials, is a vector space.

Example 3.1.6

Let V denote the set of all polynomials of degree equal to $n \geq 1$, that is

$$V = \{a_0 + a_1 x + \cdots + a_n x^n : a_0, a_1, \cdots, a_n \in \boldsymbol{R}, a_n \neq 0\}.$$

只是 a_n 不为0

Define the standard addition and scalar multiplication of polynomials by

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

and

$$\alpha\, p(x) = \alpha a_0 + \alpha a_1 x + \cdots + \alpha a_n x^n$$

for any $p(x) = a_0 + a_1 x + \cdots + a_n x^n,\ q(x) = b_0 + b_1 x + \cdots + b_n x^n \in V$ and $\alpha \in \boldsymbol{R}$.

However, V is not a vector space because $\underbrace{(1+x^n)}_{\deg n} + \underbrace{(-x^n)}_{\deg n} = 1 \notin V$.

Example 3.1.7

Let $M_{m\times n}$ denote the set of all $m \times n$ matrix with real entries, that is

定义

$$M_{m\times n} = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} : a_{ij} \in \boldsymbol{R} \right\}.$$

矩阵的向量空间

Define the standard addition and scalar multiplication of matrices by

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{121} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

and

$$\alpha \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix}$$

for any $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \in M_{m\times n}$ and $\alpha \in \boldsymbol{R}$.

In this case, the zero vector is the zero matrix $\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m\times n}$.

It is easily verified that all the vector space axioms hold. Thus, $M_{m\times n}$, with the standard addition and scalar multiplication of matrices, is a vector space.

行列式一定是 n×n 矩阵

Example 3.1.8

Let $C(I)$ be the set of all continuous functions on I. Define the standard addition and scalar multiplication of functions by

$$(f+g)(x) = f(x) + g(x)$$

and

$$(\alpha f)(x) = \alpha f(x).$$

for all $x \in I$. In this case, the zero vector is the zero function $f(x)=0$ for all $x \in I$.

It is easily verified that all the vector space axioms hold. Thus, $C(I)$, with the standard addition and scalar multiplication of functions, is a vector space.

Example 3.1.9

Define the standard addition on $\boldsymbol{R}^2$ by

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

and a new scalar multiplication $\cdot_{\text{new}}$ on $\boldsymbol{R}^2$ by

$$\alpha \cdot_{\text{new}} (a_1, a_2) = (a_1, 0).$$

Is $\boldsymbol{R}^2$ a vector space over $\boldsymbol{R}$ with these operations? Justify your answer.

一般性思路：①从陌生的运算法则出发（新定义）

②寻找不满足的性质.

乘法 ⇒ 找乘法. (5~8)

⇒ 找加法 (1~4)

Solution

It is not a vector because condition (5) fails: $1 \cdot_{new} (1,1) = (1,0) \neq (1,1)$.

Definition 3.1.10

子集 子空间

A subset W of a vector space V is said to be a **subspace** of V if

(i) $\boldsymbol{e} \in W$,

(ii) $\alpha \boldsymbol{v} \in W$ for any vector $\boldsymbol{v} \in W$ and scalar $\alpha \in \boldsymbol{R}$,

(iii) $\boldsymbol{u} + \boldsymbol{v} \in W$ for any vectors $\boldsymbol{u}, \boldsymbol{v} \in W$.

陈金铭

Condition (i), (ii) and (iii) guarantee that a subspace W of V is itself a vector space, under the vector space operations already defined in V. Fortunately, it is not necessary to verify all of the vector space properties to prove that W is a vector space. Conditions 1, 2, 5, 6, 7, 8 hold for all vectors in V, they also hold for the vectors in W. Condition 3 is condition (i). Substitute $\alpha = -1$ in condition (ii), we have condition (4).

Example 3.1.11

Let $W = \{\text{all polynomial } p(x)\}$. W is a subset of $C(-\infty, \infty)$. Clearly, zero polynomial $0 \in W$. W satisfies condition (ii) and (iii). Thus, W, with the standard addition and scalar multiplication of polynomials, is a subspace of $C(-\infty, \infty)$.

Example 3.1.12

Let $W = \{f : f'' - 3f' + 2f = 0\}$. W is a subset of $C(-\infty, \infty)$. Clearly, zero function $0 \in W$.

For any $f, g \in W$ and $\alpha \in \boldsymbol{R}$, we have $f'' - 3f' + 2f = 0$ and $g'' - 3g' + 2g = 0$.

$$(f+g)'' - 3(f+g)' + 2(f+g) = f'' + g'' - 3f' - 3g' + 2f + 2g = 0$$
$$(\alpha f)'' - 3(\alpha f)' + 2(\alpha f) = \alpha(f'' - 3f' + 2f) = 0$$

Thus, W, with the standard addition and scalar multiplication of functions, is a subspace of $C(-\infty, \infty)$.

Section 3.2 Span and linear independence

由… 张成

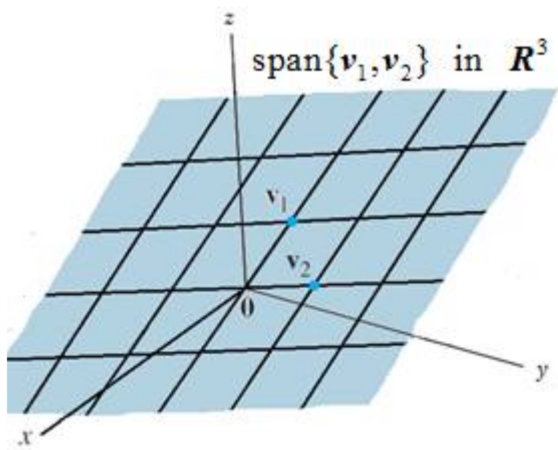
Definition 3.2.1

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ be vectors in a vector space V. A sum of the form $\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \cdots + \alpha_n\boldsymbol{v}_n$, where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are scalars, is called a **linear combination** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$. The set of all linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$

线性组合

$$\{\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \cdots + \alpha_n\boldsymbol{v}_n : \alpha_1, \alpha_2, \cdots, \alpha_n \in \boldsymbol{R}\}$$

is called the **span** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ and denoted by $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$.



① 有e

② 满足封闭性

Theorem 3.2.2

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ be vectors in a vector space V. Then $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ is a subspace of V.

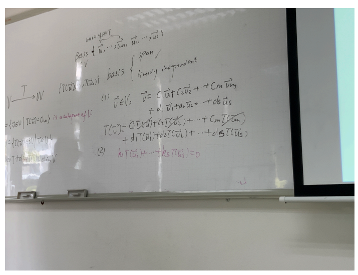
Example 3.2.3

Show that $\operatorname{span}\left\{(2,-3,5)^T, (8,-12,20)^T, (1,0,-2)^T, (0,2,-1)^T, (7,2,0)^T\right\} = \boldsymbol{R}^3$.

Solution

For any $(x,y,z)^T \in \boldsymbol{R}^3$, if we can find a, b, c, d, e such that

$$a\begin{pmatrix}2\\-3\\5\end{pmatrix} + b\begin{pmatrix}8\\-12\\20\end{pmatrix} + c\begin{pmatrix}1\\0\\-2\end{pmatrix} + d\begin{pmatrix}0\\2\\-1\end{pmatrix} + e\begin{pmatrix}7\\2\\0\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \qquad (*)$$



then $\operatorname{span}\left\{(2,-3,5)^T,(8,-12,20)^T,(1,0,-2)^T,(0,2,-1)^T,(7,2,0)^T\right\}=\boldsymbol{R}^3$.

If not, then $\operatorname{span}\left\{(2,-3,5)^T,(8,-12,20)^T,(1,0,-2)^T,(0,2,-1)^T,(7,2,0)^T\right\}\neq\boldsymbol{R}^3$.

Equation (*) becomes

$$\begin{pmatrix} 2 & 8 & 1 & 0 & 7 \\ -3 & -12 & 0 & 2 & 2 \\ 5 & 20 & -2 & -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\left(\begin{array}{ccccc|c} 2 & 8 & 1 & 0 & 7 & x \\ -3 & -12 & 0 & 2 & 2 & y \\ 5 & 20 & -2 & -1 & 0 & z \end{array}\right)\xrightarrow{R_1\div 2\to R_1}\left(\begin{array}{ccccc|c} 1 & 4 & 1/2 & 0 & 7/2 & x/2 \\ -3 & -12 & 0 & 2 & 2 & y \\ 5 & 20 & -2 & -1 & 0 & z \end{array}\right)$$

$$\xrightarrow{R_2+3R_1\to R_2,R_3-5R_1\to R_3}\left(\begin{array}{ccccc|c} 1 & 4 & 1/2 & 0 & 7/2 & x/2 \\ 0 & 0 & 3/2 & 2 & 25/2 & y+3x/2 \\ 0 & 0 & -9/2 & -1 & -35/2 & z-5x/2 \end{array}\right)$$

$$\xrightarrow{R_2\times 2/3\to R_2}\left(\begin{array}{ccccc|c} 1 & 4 & 1/2 & 0 & 7/2 & x/2 \\ 0 & 0 & 1 & 4/3 & 25/3 & 2y/3+x \\ 0 & 0 & -9/2 & -1 & -35/2 & z-5x/2 \end{array}\right)$$

$$\xrightarrow{R_3+(9/2)R_2\to R_3}\left(\begin{array}{ccccc|c} 1 & 4 & 1/2 & 0 & 7/2 & x/2 \\ 0 & 0 & 1 & 4/3 & 25/3 & 2y/3+x \\ 0 & 0 & 0 & 5 & 20 & z+3y+2x \end{array}\right)$$

$$\xrightarrow{R_3\div 5\to R_3}\left(\begin{array}{ccccc|c} 1 & 4 & 1/2 & 0 & 7/2 & x/2 \\ 0 & 0 & 1 & 4/3 & 25/3 & 2y/3+x \\ 0 & 0 & 0 & 1 & 4 & z/5+3y/5+2x/5 \end{array}\right)$$

$$\xrightarrow{R_2-(4/3)R_3\to R_2}\left(\begin{array}{ccccc|c} 1 & 4 & 1/2 & 0 & 7/2 & x/2 \\ 0 & 0 & 1 & 0 & 3 & 7x/15-2y/15-4z/15 \\ 0 & 0 & 0 & 1 & 4 & z/5+3y/5+2x/5 \end{array}\right)$$

$$\xrightarrow{R_1-(1/2)R_2\to R_1}\left(\begin{array}{ccccc|c} 1 & 4 & 0 & 0 & 2 & 4x/15+y/15+2z/15 \\ 0 & 0 & 1 & 0 & 3 & 7x/15-2y/15-4z/15 \\ 0 & 0 & 0 & 1 & 4 & 2x/5+3y/5+z/5 \end{array}\right)$$

Choose $a=4x/15+y/15+2z/15$, $b=0$, $c=7x/15-2y/15-4z/15$, $d=2x/5+3y/5+z/5$, $e=0$,
then $\operatorname{span}\left\{(2,-3,5)^T,(8,-12,20)^T,(1,0,-2)^T,(0,2,-1)^T,(7,2,0)^T\right\}=\boldsymbol{R}^3$.

乘法封闭

Example 3.2.4

Let $V=\{(a_1,a_2,a_3,a_4)^T\in\boldsymbol{R}^4 : a_1+a_2+a_3+a_4=0\}$. Clearly, $(0,0,0,0)\in V$.

加法封闭 $\underbrace{a_1+a_2+a_3+a_4=0}_{(a_1,a_2,a_3,a_4)^T\in V},\underbrace{b_1+b_2+b_3+b_4=0}_{(b_1,b_2,b_3,b_4)^T\in V}\Rightarrow\underbrace{a_1+b_1+a_2+b_2+a_3+b_3+a_4+b_4=0}_{(a_1,a_2,a_3,a_4)^T+(b_1,b_2,b_3,b_4)^T\in V}$

乘法封闭 $\underbrace{a_1+a_2+a_3+a_4=0}_{(a_1,a_2,a_3,a_4)^T\in V}\Rightarrow\underbrace{\alpha a_1+\alpha a_2+\alpha a_3+\alpha a_4=0}_{\alpha(a_1,a_2,a_3,a_4)^T\in V}$

for any $(a_1,a_2,a_3,a_4)^T,(b_1,b_2,b_3,b_4)^T\in V$ and $\alpha\in\boldsymbol{R}$. V is a subspace of $\boldsymbol{R}^4$.

For any $(a_1,a_2,a_3,a_4)^T\in V$,

$$\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\\-a_1-a_2-a_3\end{pmatrix}=a_1\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}+a_2\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}+a_3\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$$

Hence $\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},\begin{pmatrix}0\\1\\0\\-1\end{pmatrix},\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$ spans V, that is, $\operatorname{span}\left\{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},\begin{pmatrix}0\\1\\0\\-1\end{pmatrix},\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}\right\}=V$.

Definition 3.2.5

The vectors $\boldsymbol{v}_1,\boldsymbol{v}_2,\cdots,\boldsymbol{v}_n$ in a vector space V are said to be **linearly independent** if

$$c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2+\cdots+c_n\boldsymbol{v}_n=\boldsymbol{0}$$

implies that all the scalars $c_1,c_2,\cdots,c_n$ must equal 0.

Definition 3.2.6

The vectors $\boldsymbol{v}_1,\boldsymbol{v}_2,\cdots,\boldsymbol{v}_n$ in a vector space V are said to be **linearly dependent** if there exist scalars $c_1,c_2,\cdots,c_n$, not all zero, such that

$$c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2+\cdots+c_n\boldsymbol{v}_n=\boldsymbol{0}$$

Example 3.2.7

Show that $(1,1,0,0)^T,(0,1,1,0)^T,(0,0,1,1)^T,(1,0,0,1)^T$ are linearly dependent in $\boldsymbol{R}^4$.

Solution

Suppose

$$a\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\1\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\1\\1\end{pmatrix}+d\begin{pmatrix}1\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \quad \text{that is,} \begin{pmatrix}1&0&0&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

If $a=b=c=d=0$ is the only solution, then $(1,1,0,0)^T,(0,1,1,0)^T,(0,0,1,1)^T,(1,0,0,1)^T$ are linearly independent. If not, they are linearly dependent.

You may use row operation (D.I.Y.). By observation, 求det也可以

$$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\1\\1\\0\end{pmatrix}+\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}1\\1\\0\\0\end{pmatrix}-\begin{pmatrix}0\\1\\1\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\\1\end{pmatrix}-\begin{pmatrix}1\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$(1,1,0,0)^T,(0,1,1,0)^T,(0,0,1,1)^T,(1,0,0,1)^T$ are linearly dependent in $\boldsymbol{R}^4$.

Example 3.2.8

Show that $(x-1)(x-2),(x-1),1$ are linearly independent in P_3.

Solution

Suppose

$$a(x-1)(x-2)+b(x-1)+c=0.$$

At $x=1$, $c=0$. Then we have $a(x-1)(x-2)+b(x-1)=0$.

At $x=2$, $b=0$. Then we have $a(x-1)(x-2)=0$.

At $x=3$, $a=0$. 赋值法求出特殊情况的a、b、c值

Therefore, $(x-1)(x-2),(x-1),1$ are linearly independent in P_3.

$$\begin{cases} a\vec{v_1} + b\vec{v_2} = 0 \\ a, b \text{ must be a constant} \end{cases}$$

Example 3.2.9

? Consider e^x and e^{2x}. Then $\underset{c_1}{e^x}(\underset{v_1}{e^x}) + \underset{c_2}{(-1)}(\underset{v_2}{e^{2x}}) = 0$. Are they linearly dependent in $C(-\infty,\infty)$?

No. It is because $c_1 = e^x$ is not a constant. Indeed, they are linearly independent. Consider

$$c_1e^x + c_2e^{2x} = 0$$

W行列式. 对函数求多次导, 形成行列式(n×n)

Taking derivative, we have $c_1e^x + 2c_2e^{2x} = 0$. Then

原则n个元素求n次导 形成 n×n 形式.

$$\begin{pmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

求 $\det A \begin{cases} =0 & \text{线性相关} \\ \neq 0 & \text{线性无关} \end{cases}$

Since $\det\begin{pmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{pmatrix} = 2e^{2x}e^x - e^{2x}e^x = e^{3x} \neq 0$, $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

e^x and e^{2x} are linearly independent.

Definition 3.2.10

The **Wronskian** of n differentiable functions $f_1,\cdots,f_n$ is defined by

朗斯基行列式.

$$W(f_1,\cdots,f_n)(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

△: 一定是 n×n 形式.

Theorem 3.2.11

If $W(f_1,\cdots,f_n)(x) \neq 0$ then $f_1,\cdots,f_n$ are linearly independent.

Example 3.2.12

Since $\begin{vmatrix} (x-1)(x-2) & x-1 & 1 \\ 2x-3 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = -2 \neq 0$, $(x-1)(x-2), (x-1), 1$ are linearly independent in P_3.

Section 3.3 Basis and Dimension

Definition 3.3.1

The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ form a **basis** for a vector space *V* if

(i) $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ are linearly independent.

(ii) $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ span *V*.

A basis is an "efficient" spanning set that contains no unnecessary vectors. In fact, a basis can be constructed from a spanning set by discarding unneeded vectors.

去除无用的向量

Theorem 3.3.2

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ be a spanning set for a vector space *V*, that is, $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\} = V$. If one of the vectors in *S*, say $\boldsymbol{v}_k$, is a combination of the remaining vectors in *S*, then the set formed from *S* by removing $\boldsymbol{v}_k$ still span *V*, that is, V_k能被剩下的线性表示时

$$\text{span}\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_{k+1}, \cdots, \boldsymbol{v}_n\} = V.$$

If the original spanning set *S* is linear independent, then it is already a basis for *V*. Otherwise, one of the vectors in *S* depends on the others and can be deleted. We can repeat this process until the spanning set is linearly independent and hence is a basis for *V*.

Example 3.3.3

Let $S = \left\{(2,-3,5)^T, \cancel{(8,-12,20)^T}, (1,0,-2)^T, (0,2,-1)^T, \cancel{(7,2,0)^T}\right\}$. By Example 3.2.3, $\text{span}(S) = \boldsymbol{R}^3$.

Since $(8,-12,20)^T = 4(2,-3,5)^T$, by Theorem 3.3.2

$$\text{span}\left\{(2,-3,5)^T, (1,0,-2)^T, (0,2,-1)^T, (7,2,0)^T\right\} = \boldsymbol{R}^3.$$

Since $(7,2,0)^T = 2(2,-3,5)^T + 3(1,0,-2)^T + 4(0,2,-1)^T$, by Theorem 3.3.2

$$\operatorname{span}\left\{(2,-3,5)^T,(1,0,-2)^T,(0,2,-1)^T\right\} = \boldsymbol{R}^3.$$

Suppose $\alpha(2,-3,5)^T + \beta(1,0,-2)^T + \gamma(0,2,-1)^T = (0,0,0)^T$, that is

$$\begin{pmatrix} 2 & 1 & 0 \\ -3 & 0 & 2 \\ 5 & -2 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \alpha\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \gamma\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since $\det\begin{pmatrix} 2 & 1 & 0 \\ -3 & 0 & 2 \\ 5 & -2 & -1 \end{pmatrix} = 15 \neq 0, \alpha = \beta = \gamma = 0.$ $\left\{(2,-3,5)^T,(1,0,-2)^T,(0,2,-1)^T\right\}$ are linearly independent. We conclude that $\left\{(2,-3,5)^T,(1,0,-2)^T,(0,2,-1)^T\right\}$ is a basis for $\boldsymbol{R}^3$.

如何找到 basis

A basis is also a linearly independent set that is as large as possible. Let S be a linearly independent subset of a vector space V. If $\operatorname{span}(S) \underset{\neq}{\subset} V$, then choose a vector $\boldsymbol{v} \notin \operatorname{span}(S)$. Clearly,

$$\operatorname{span}(S) \underset{\neq}{\subset} \operatorname{span}(S \cup \boldsymbol{v}).$$

Since $\boldsymbol{v}$ is not a linear combination of any vector in S, and $S \cup \boldsymbol{v}$ is linearly independent. If $\operatorname{span}(S \cup \boldsymbol{v}) = V$, then $S \cup \boldsymbol{v}$ is a basis for V. If not, repeat the process, enlarge S by one more vector until the spanning set spans V.

Example 3.3.4

子集

Let $S = \left\{(2,-3,5)^T\right\}$ be subset of $\boldsymbol{R}^3$. Clearly, $\operatorname{span}(S) \neq \boldsymbol{R}^3$. Choose a vector, say $(1,0,-2)^T \notin \operatorname{span}(S)$. Then $(2,-3,5)^T$ and $(1,0,-2)^T$ are linearly independent and

$$\operatorname{span}\left\{(2,-3,5)^T,(1,0,-2)^T\right\} \underset{\neq}{\subset} \boldsymbol{R}^3.$$

Choose a vector, say $(0,2,-1)^T \notin \operatorname{span}\left\{(2,-3,5)^T,(1,0,-2)^T\right\}$. Then $(2,-3,5)^T,(1,0,-2)^T$ and $(0,2,-1)^T$ are linearly independent (Example 3.3.3). For any $(x,y,z)^T \in \boldsymbol{R}^3$, there exists $(\alpha,\beta,\gamma)^T \in \boldsymbol{R}^3$ such that

13

$$\alpha\begin{pmatrix}2\\-3\\5\end{pmatrix}+\beta\begin{pmatrix}1\\0\\-2\end{pmatrix}+\gamma\begin{pmatrix}0\\2\\-1\end{pmatrix}=\underbrace{\begin{pmatrix}2&1&0\\-3&0&2\\5&-2&-1\end{pmatrix}}_{\det\neq 0}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}.$$

That is, $\operatorname{span}\{(2,-3,5)^T,(1,0,-2)^T,(0,2,-1)^T\}=\boldsymbol{R}^3$. Then $\{(2,-3,5)^T,(1,0,-2)^T,(0,2,-1)^T\}$ is a basis for $\boldsymbol{R}^3$.

Theorem 3.3.5

有限 包含

Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors.

Definition 3.3.6

有限维 维度

A vector space is called **finite-dimensional** if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the **dimension** of V and is denoted by $\dim(V)$. A vector space that is not finite-dimensional is called **infinite-dimensional**.

唯一的 无限维.

Example 3.3.7

为了方便 0维.

For convenience, we define the basis of the zero-dimensional vector space is ϕ. The vector space $\{\mathbf{0}\}$ has dimension zero.

Example 3.3.8

维度

Clearly, the set $\{(1,0,\cdots 0),(0,1,\cdots,0),\cdots,(0,\cdots,0,1)\}$ is basis for $\boldsymbol{R}^n$. Then $\dim(\boldsymbol{R}^n)=n$.

Example 3.3.9

Consider $M_{2\times3}$, the set of all 2×3 matrix with real entries.

$$\left\{\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&0\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}\right\}$$

is a basis for $M_{2\times3}$. Hence $\dim(M_{2\times3})=2\times3=6$. In general, $\dim(M_{m\times n})=mn$.

Example 3.3.10

Clearly, the set $\{1, x, \cdots, x^{n-1}\}$ is basis for P_n. Then $\dim(P_n) = n$.

Example 3.3.11

Let $V = \{\text{all polynomial } p(x)\}$. The set $\{1, x, x^2, \cdots\}$ is basis for V. V is an infinite dimensional vector space.

Example 3.3.12

Find the basis and dimension for $V = \{p(x) \in P_n : p(1) = 0\}$.

Solution

Clearly, the set $\{x-1, x(x-1), \cdots, x^{n-2}(x-1)\}$ is basis for V. Then $\dim(V) = n-1$.

Theorem 3.3.13 n维向量空间

n个元素

Let V be an n-dimensional vector space, $n \geq 1$. Any linearly independent set of exactly n elements in V is a basis for V. Any set of exactly n vectors that spans V is a basis for V.

Corollary 3.3.14

如果V是n维向量空间

If V is a vector space of dimension $n > 0$, then

(i) any set of fewer than n vectors cannot span V. 向量数少于n不可能是span

(ii) any set of fewer than n linearly independent vectors can be extended to form a basis for V. 可以扩展成basis

(iii) any collection of m vectors in V, where $m > n$, is linearly dependent.

有(n+1)个 n维向量 ⇒ 一定线性相关

Example 3.3.15

Show that $\{(x-1)(x-2), (x-1)(x-3), (x-2)(x-3)\}$ forms a basis for P_3.

Solution

Suppose

$$a(x-1)(x-2)+b(x-1)(x-3)+c(x-2)(x-3)=0.$$

At $x=3$, $a=0$. At $x=2$, $b=0$. At $x=1$, $c=0$.

Hence $(x-1)(x-2),(x-1)(x-3),(x-2)(x-3)$ are linearly independent. By Theorem 3.3.13 and $\dim(P_3)=3$, $\{(x-1)(x-2),(x-1)(x-3),(x-2)(x-3)\}$ forms a basis for P_3.

Theorem 3.3.16 ⇒加封闭性. ⇒有限维

Let W be a subspace of a finite-dimensional vector space V. Then W is finite-dimensional and $\dim(W)\le\dim(V)$. Moreover, if $\dim(W)=\dim(V)$, then $V=W$.

W是V的子空间 W的维数 ≤ V的维数

Example 3.3.17

The subspaces of $\boldsymbol{R}^3$ can be classified (分类) by dimension.

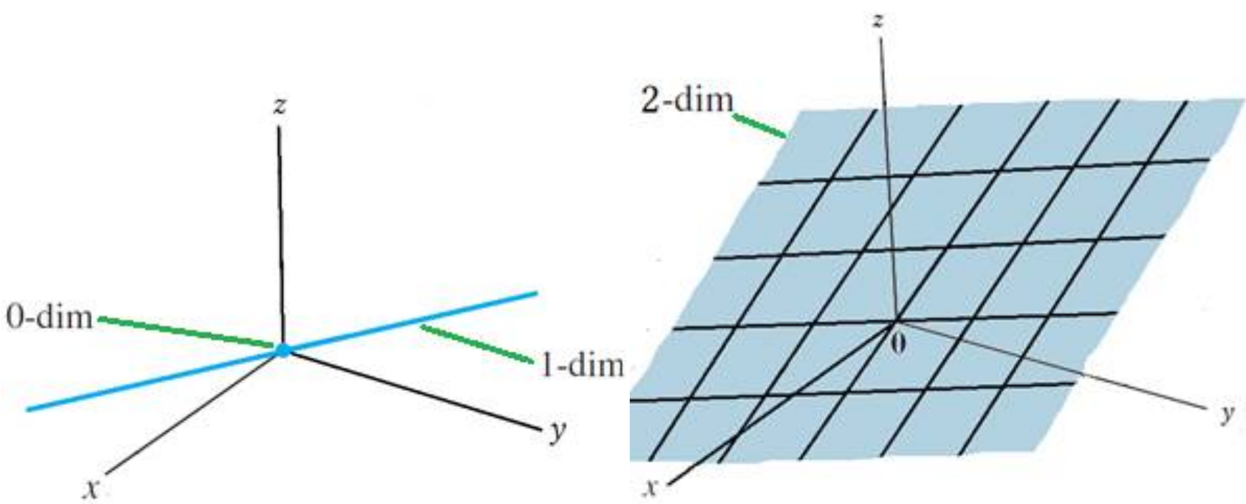
3维子空间:
0维 ⇒ 是一个点
1维 ⇒ 是一条直线
2维 ⇒ 是一个平面
3维 ⇒ 是一个空间

0-dimensional subspaces: Only the zero subspace.

1-dimensional subspaces: Any subspace spanned by a single nonzero vector. Such subspaces are lines through the origin.

2-dimensional subspaces: Any subspace spanned by two linearly independent vectors. Such subspaces are planes through the origin.

3-dimensional subspaces: Only $\boldsymbol{R}^3$ itself. Any three linearly independent vectors in $\boldsymbol{R}^3$ span all of $\boldsymbol{R}^3$.



"不同基底下坐标不同"

Section 3.4 Change of basis

更换基底、

Theorem 3.4.1

Let $\beta=\{\boldsymbol{v}_1,\boldsymbol{v}_2,\cdots,\boldsymbol{v}_n\}$ be a basis for a vector space V. Then for each $\boldsymbol{v}\in V$, there exists a unique set of scalar $c_1,c_2,\cdots,c_n\in\boldsymbol{R}$ such that

$$\boldsymbol{v}=c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2+\cdots+c_n\boldsymbol{v}_n.$$

Let $\beta=\{\boldsymbol{v}_1,\boldsymbol{v}_2,\cdots,\boldsymbol{v}_n\}$ be a basis for a vector space V. If we order the basis vector in β, it is called an ordered basis for V. Let us order the basis elements so that $\boldsymbol{v}_i$ is the i-th basis vector.

Definition 3.4.2

Suppose $\beta=\{\boldsymbol{v}_1,\boldsymbol{v}_2,\cdots,\boldsymbol{v}_n\}$ be an ordered basis for a vector space V and $\boldsymbol{v}\in V$. By Theorem 3.4.1, there exists a unique set of scalar $c_1,c_2,\cdots,c_n\in\boldsymbol{R}$ such that

唯一

$$\boldsymbol{v}=c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2+\cdots+c_n\boldsymbol{v}_n.$$

Then

$$\begin{pmatrix} c_1\\ c_2\\ \vdots\\ c_n\end{pmatrix}\in\boldsymbol{R}^n$$

坐标向量

is called the **coordinate vector** of $\boldsymbol{v}$ with respect to the ordered basis β and is denoted $[\boldsymbol{v}]_\beta$.

如果更换 V1 => V2 位置

If we switch the order of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, we have different ordered basis $\gamma=\{\boldsymbol{v}_2,\boldsymbol{v}_1,\cdots,\boldsymbol{v}_n\}$. Then $\boldsymbol{v}=c_2\boldsymbol{v}_2+c_1\boldsymbol{v}_1+\cdots+c_n\boldsymbol{v}_n$. The entry of the coordinate vector is switched in the same order, i.e.,

那么坐标也会跟着变，

$$[\boldsymbol{v}]_\gamma=\begin{pmatrix} c_2\\ c_1\\ \vdots\\ c_n\end{pmatrix}.$$

坐标与 order basis 有关

Example 3.4.3

Consider two bases $\beta = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ and $\gamma = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ for $\boldsymbol{R}^2$, where $\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \boldsymbol{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{b}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Suppose $\boldsymbol{x} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$. $\boldsymbol{x}$ can be expressed as

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1\boldsymbol{e}_1 + 6\boldsymbol{e}_2.$$

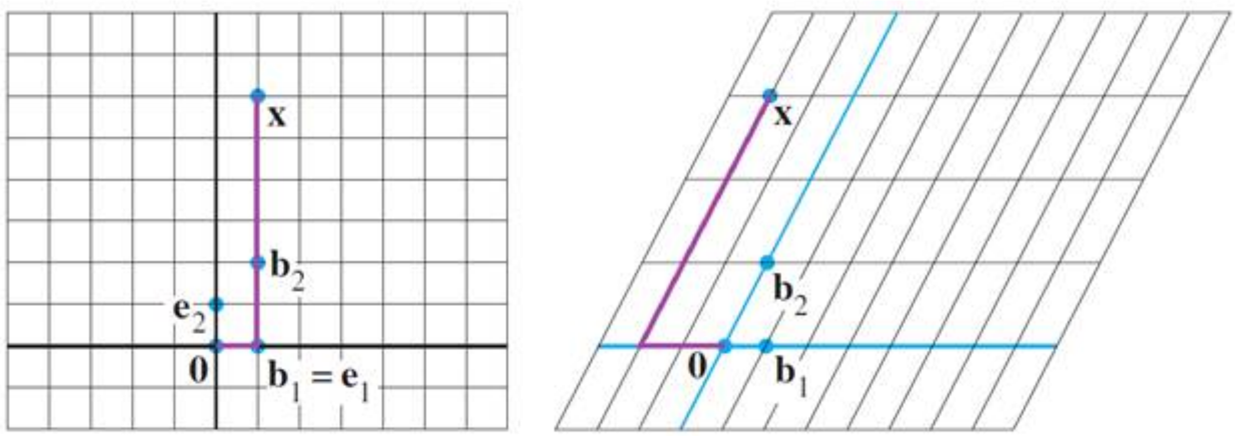
Hence $[\boldsymbol{x}]_\beta = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$. On the other hand, $\boldsymbol{x}$ can be expressed as

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -2\boldsymbol{b}_1 + 3\boldsymbol{b}_2.$$

Hence $[\boldsymbol{x}]_\gamma = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

不同基底 ⇒ 不同的坐标

Different basis gives the same vector different coordinate vector.



Theorem 3.4.4

If β is a basis of a vector space V, then

(i) $[\boldsymbol{x}+\boldsymbol{y}]_\beta = [\boldsymbol{x}]_\beta + [\boldsymbol{y}]_\beta$, for all vectors $\boldsymbol{x}, \boldsymbol{y} \in V$, and

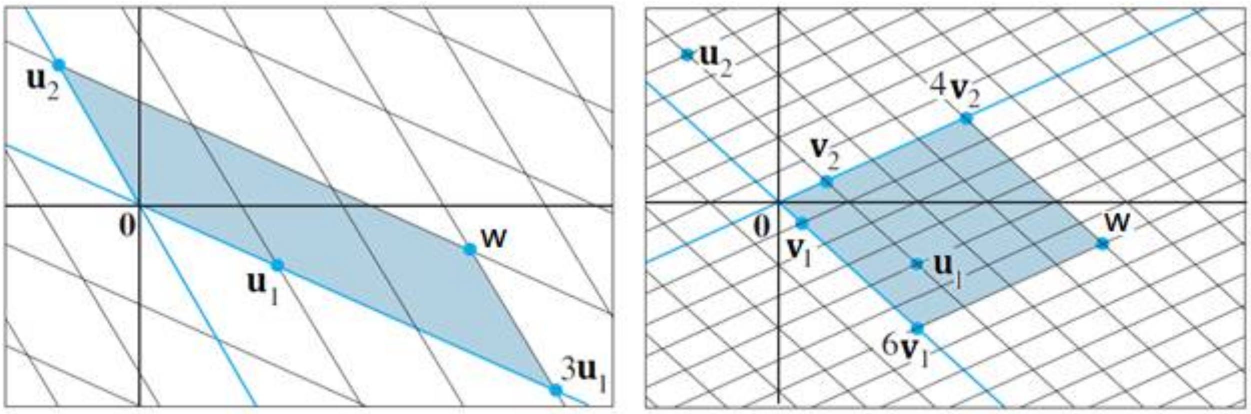
(ii) $[k\boldsymbol{x}]_\beta = k[\boldsymbol{x}]_\beta$, for all vectors $\boldsymbol{x} \in V$ and all scalars $k \in \boldsymbol{R}$.

$[x+y]_\beta = [x]_\beta + [y]_\beta$

Example 3.4.5

Consider two bases $\beta = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $\gamma = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ for a vector space V. Let $\boldsymbol{w} \in V$ such that

$$\boldsymbol{w} = 3\boldsymbol{u}_1 + \boldsymbol{u}_2 \text{ and } \boldsymbol{w} = 6\boldsymbol{v}_1 + 4\boldsymbol{v}_2.$$



What is the relation between the two coordinate vectors $[\boldsymbol{w}]_\beta$ and $[\boldsymbol{w}]_\gamma$?

Clearly, $[\boldsymbol{w}]_\beta = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $[\boldsymbol{w}]_\gamma = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$.

Let $\boldsymbol{u}_1 = a\boldsymbol{v}_1 + b\boldsymbol{v}_2$ and $\boldsymbol{u}_2 = c\boldsymbol{v}_1 + d\boldsymbol{v}_2$. Then

$$\boldsymbol{w} = 3\boldsymbol{u}_1 + \boldsymbol{u}_2 = 3(a\boldsymbol{v}_1 + b\boldsymbol{v}_2) + (c\boldsymbol{v}_1 + d\boldsymbol{v}_2) = (3a+c)\boldsymbol{v}_1 + (3b+d)\boldsymbol{v}_2.$$

We have

$$\begin{pmatrix} 6 \\ 4 \end{pmatrix} = [\boldsymbol{w}]_\gamma = \begin{pmatrix} 3a+c \\ 3b+d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \left([\boldsymbol{u}_1]_\gamma \;\vdots\; [\boldsymbol{u}_2]_\gamma \right)[\boldsymbol{w}]_\beta.$$

Theorem 3.4.6

Let $\beta = \{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_n\}$ and $\gamma = \{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$ be two bases of a vector space V. Let $\boldsymbol{w} \in V$. Then

$$[\boldsymbol{w}]_\gamma = \left([\boldsymbol{u}_1]_\gamma \;\vdots\; \cdots \;\vdots\; [\boldsymbol{u}_n]_\gamma \right)[\boldsymbol{w}]_\beta.$$

The $n \times n$ matrix $\left([\boldsymbol{u}_1]_\gamma \;\vdots\; \cdots \;\vdots\; [\boldsymbol{u}_n]_\gamma \right)$ is called the transition matrix from β to γ, denoted by $[I]_\beta^\gamma$.

过渡矩阵

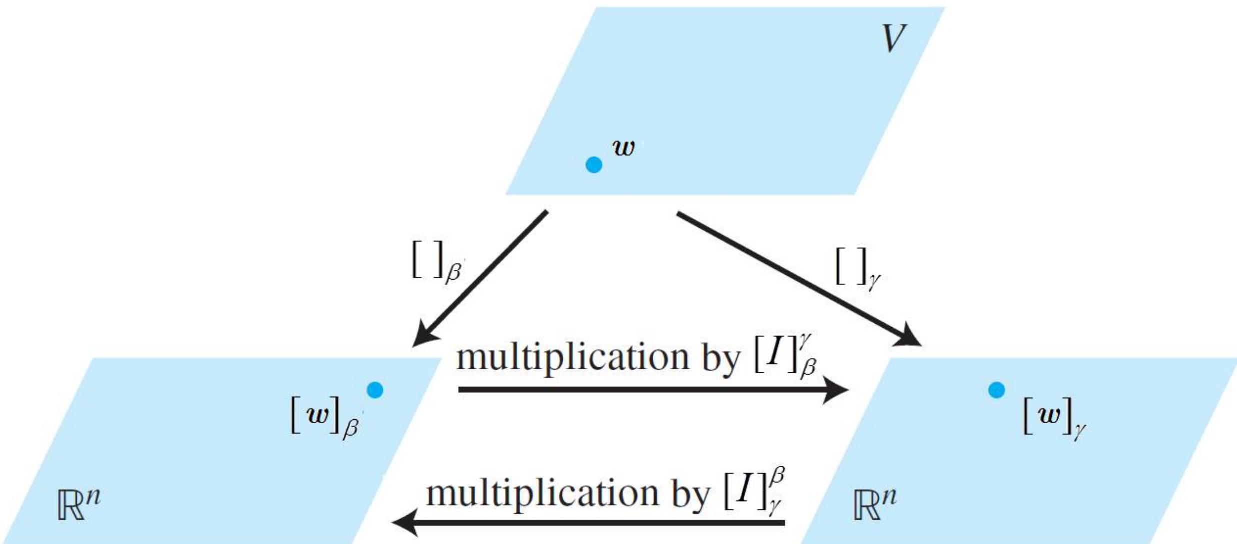
从β系到γ系

$[I]^{\gamma}_{\beta}$ $\gamma \to$ 目标 $\beta \to$ 初始

Theorem 3.4.7

$[\beta \to \gamma$ 的过渡矩阵$]^{-1} = \gamma \to \beta$ 的过渡矩阵

Let β and γ be two bases of a vector space V. Then $[I]_\beta^\gamma$ is invertible and $\left([I]_\beta^\gamma\right)^{-1} = [I]_\gamma^\beta$.



Example 3.4.8

Let $\beta = \{(x-1)(x-2), (x-1)(x-3), (x-2)(x-3)\}$ and $\gamma = \{(x-1)(x-2), (x-1), 1\}$ be two bases of P_3. Find $[I]_\beta^\gamma$, $[I]_\gamma^\beta$, $\left[x^2\right]_\beta$ and $\left[x^2\right]_\gamma$. Verify Theorem 3.4.6 and 3.4.7.

证实

Solution

Clearly,

$$[I]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{pmatrix}, \quad [I]_\gamma^\beta = \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & -1 & -1 \\ 0 & 0 & 1/2 \end{pmatrix}, \quad \left[x^2\right]_\beta = \begin{pmatrix} 9/2 \\ -4 \\ 1/2 \end{pmatrix}, \quad \left[x^2\right]_\gamma = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$$[I]_\beta^\gamma \left[x^2\right]_\beta = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 9/2 \\ -4 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \left[x^2\right]_\gamma, \quad [I]_\gamma^\beta \left[x^2\right]_\gamma = \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & -1 & -1 \\ 0 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 9/2 \\ -4 \\ 1/2 \end{pmatrix} = \left[x^2\right]_\beta$$

[证明过程]

$$\left([I]_\beta^\gamma\right)^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{pmatrix}^{-1} = [I]_\gamma^\beta = \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & -1 & -1 \\ 0 & 0 & 1/2 \end{pmatrix}$$

胡闲性

$A \to \mathrm{Row}(A) \to$ basis
最重要
$A \to \mathrm{Col}(A)$ basis

Section 3.5 Row Spaces and Column Spaces

Definition 3.5.1

全部 线性组合 行向量 行空间

Let A be an $m \times n$ matrix. The set of all linear combinations of the row vectors is called the **row space** of A and is denoted by Row(A). Each row has n entries, so Row(A) is a subspace of $\boldsymbol{R}^n$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}. \quad \mathrm{Row}(A) = \mathrm{span}\,\{(a_{11}, a_{12} \cdots, a_{1n}), (a_{21}, a_{22} \cdots, a_{2n}), \cdots, (a_{m1}, a_{m2} \cdots, a_{mn})\}$$

列空间

The set of all linear combinations of the column vectors is called the **column space** of A and is denoted by Col (A). Each column has m entries, so Col (A) is a subspace of $\boldsymbol{R}^m$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}. \quad \mathrm{Col}(A) = \mathrm{span}\left\{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}\right\}$$

Theorem 3.5.2

basis Row(A) ⟺ basis Row(U)

行等价

If two matrices A and B are row equivalent, then their row spaces are the same. If U is in row echelon form, the nonzero rows of U form a basis for the row space of A as well as for that of U.

梯型形式.

梯形 A 初等行变化 U

? We can use the row echelon form U of A to find a basis for the column space of A. We need only determine the columns of U that correspond to the leading 1's. These same columns of A will be linearly independent and form a basis for the column space of A.

Remark 3.5.3

The row echelon form U tells us only which columns of A to use to form a basis. We cannot use the column vectors from U, since, in general, U and A have different column spaces.

Basis of Col(U) ≠ Basis of col(A)

$\beta_1 = \alpha_1 \cdot e_{11} + \alpha_2 \cdots e_{12} \cdots + e_{1n} \cdot \alpha_n \in \mathrm{Row}A$

$\boxed{E_k \cdots E_2 \cdot E_1} \cdot A = U$ $\quad \boxed{\gamma \in \mathrm{Row}(U)} \Rightarrow \gamma = k_1 \cdot \beta_1 + \cdots k_n \cdot \beta_n$

β 可以用 Row A 表示

$$\begin{bmatrix} e_{11} & \cdots & e_{1n} \\ \vdots & & \vdots \\ e_{n1} & \cdots & e_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

$\therefore \gamma \in \text{Row}(A)$

$\text{Row}(L) \subseteq \text{Row}(A)$

Example 3.5.4

$A = (E_k \cdots E_2 E_1)^{-1} \cdot L \Rightarrow \text{Row}(A) \subseteq \text{Row}(L)$

Let

$$A = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{pmatrix} \xrightarrow{R_2+R_1\to R_2,\, R_4-R_1\to R_4} \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 3 & 4 \\ 0 & 4 & 4 & 12 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3-R_2\to R_3,\, R_4-4R_2\to R_4} \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 3 & 4 \\ 0 & 4 & 4 & 12 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3-R_2\to R_3,\, R_4-4R_2\to R_4} \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3\div 4\to R_3} \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow{R_4-3R_3\to R_4} \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$(1,-2,1,1,2), (0,1,1,3,0), (0,0,0,0,1)$, form a basis of Row($A$).

任意两个basis可以线性表示

$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 4 \\ 5 \end{pmatrix}$ form a basis of Col(A) but $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ do not.

如果这个是basis那么最后一行一定是0

Number of basis vectors in Row(A) equal number of basis vectors in dim Col(A)

1. $\dim \text{Col} A \geq \dim \text{Row} A$ (先证明这一条)

2. $\dim \text{Row} A = \dim(\text{Col} A^T) \geq \dim \text{Row}(A^T) = \dim(\text{Col} A)$

Theorem 3.5.5

If A is an $m \times n$ matrix, then the dimension of the row space of A equals the dimension of the column space of A.

Definition 3.5.6

rank (A) = dim Row(A).

The **rank** of a matrix A, denoted rank(A), is the dimension of the row space of A.

秩

Definition 3.5.7

Let A be an $m \times n$ matrix. Let $N(A)$ denote the set of all solutions of system of linear equations $A\boldsymbol{x} = \boldsymbol{0}$, that is,

$$N(A) = \{x \in \boldsymbol{R}^n : A\boldsymbol{x} = \boldsymbol{0}\}$$

Theorem 3.5.8

思维的转化：证明 N(A) 是 vector space

Let A be an $m \times n$ matrix. $N(A)$ is a subspace of $\boldsymbol{R}^n$.

⇓ 高级转化

证明 N(A) 是 subspace

Proof

How to prove a set is a vector space?

Clearly, $\boldsymbol{0} \in N(A)$. If $\boldsymbol{x} \in N(A)$ and $\alpha \in \boldsymbol{R}$, then

乘法封闭性

$$A(\alpha \boldsymbol{x}) = \alpha A\boldsymbol{x} = \boldsymbol{0}$$

and hence $\alpha \boldsymbol{x} \in N(A)$. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are elements of $N(A)$, then

加法封闭性

$$A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y} = \boldsymbol{0} + \boldsymbol{0} = \boldsymbol{0}.$$

Therefore, $\boldsymbol{x} + \boldsymbol{y} \in N(A)$. It then follows that $N(A)$ is a subspace of $\boldsymbol{R}^n$.

Definition 3.5.9

Let A be an $m \times n$ matrix. $N(A)$ is called the **null space** of A. The dimension of the null space of a matrix is called the **nullity** of the matrix.

dim N(A) = nullity

Theorem 3.5.10 (Rank Nullity Theorem)

If A is an $m\times n$ matrix, then the rank of A + the nullity of $A = n$.

Outline of the proof

If rank(A) = r, number of leading variables equal r. Thus, number of free variables equal $n-r$, which is nullity of A.

自由变量

Example 3.5.11

Consider Example 3.5.4, the reduced row echelon form of A is

每一列就它一个

$$\begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1-2R_3\to R_1} \begin{pmatrix} 1 & -2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1+2R_2\to R_2} \begin{pmatrix} \boxed{1} & 0 & 3 & 7 & 0 \\ 0 & \boxed{1} & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

leading 1's

$$N(A) = \left\{\boldsymbol{x} = (x_1,x_2,x_3,x_4,x_5)^T : A\boldsymbol{x} = \boldsymbol{0}\right\}$$

$$= \left\{(x_1,x_2,x_3,x_4,x_5)^T : \begin{pmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

$$= \left\{(x_1,x_2,x_3,x_4,x_5)^T : x_1+3x_3+7x_4=0, x_2+x_3+3x_4=0, x_5=0\right\}$$

$$= \left\{\begin{pmatrix} -3x_3-7x_4 \\ -x_3-3x_4 \\ x_3 \\ x_4 \\ 0 \end{pmatrix}\right\}$$

$$= \left\{x_3\begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

rank(A) = 3 (3 leading variables x_1,x_2,x_5), dim $N(A)$ = 2 (2 free variables x_3,x_4), 3 + 2 = n = 5.

附: 求 $A\vec{x}=\vec{b}$ 的 span

① 求 $A\vec{x}=0$ 的通解

结合在一起就是 $A\vec{x}=\vec{b}$ 通解

② 求 $A\vec{x}=b$ 的眉解

Theorem 3.5.12

Let $A=\left(\boldsymbol{a}_1 \mid \boldsymbol{a}_2 \mid \cdots \mid \boldsymbol{a}_n\right)$ and $\boldsymbol{x}=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. A linear system $A\boldsymbol{x}=\boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ is in the column space of A, i.e.,

$$\boxed{\boldsymbol{b}=x_1\boldsymbol{a}_1+x_2\boldsymbol{a}_2+\cdots+x_n\boldsymbol{a}_n}$$

Consider $A=\begin{pmatrix} 0 & 2 & -8 \\ 1 & -2 & 1 \\ -4 & 5 & 9 \end{pmatrix}$ in Example 1.2.1. We have $rref(A)=\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ in Example 1.2.5, an invertible A in Example 2.1.10 and $\det A=-2\neq 0$ in Example 2.1.38.

Furthermore, $\begin{pmatrix} 8 \\ 0 \\ -9 \end{pmatrix}=29\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}+16\begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}+3\begin{pmatrix} -8 \\ 1 \\ 9 \end{pmatrix}\in \mathrm{Col}(A)$ in Example 2.1.38.

Summary 3.5.13

等价

For an $n\times n$ matrix A, the following statements are equivalent:

(i) A is invertible.

(ii) $\det(A)\neq 0$.

(ii) The linear system $A\boldsymbol{x}=\boldsymbol{b}$ has a unique solution $\boldsymbol{x}$, for all $\boldsymbol{b}$ in $\boldsymbol{R}^n$.

(iii) $rref(A)=I_n$.

(iv) $\mathrm{rank}(A)=n$.

(v) $\mathrm{Col}(A)=\boldsymbol{R}^n$.

(vi) $N(A)=\{\boldsymbol{0}\}$.

(vii) The column vectors of A form a basis of $\boldsymbol{R}^n$.

(viii) The column vectors of A span $\boldsymbol{R}^n$.

(ix) The column vectors of A are linearly independent.

A → Rank A → det A, nullity A.

A → Row A.

A → col A

A → N(A) $A\vec{x}=0$

$A \to \to U \Rightarrow$ (核心)

Section 4.1 Linear Transformations

Definition 4.1.1

映射、

A mapping $T: V \to W$ from a vector space V into a vector space W is said to be a **linear transformation** over $\boldsymbol{R}$ if

$$T(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2) = \alpha T(\boldsymbol{v}_1) + \beta T(\boldsymbol{v}_2) \tag{1}$$

for all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and for all scalars $\alpha, \beta \in \boldsymbol{R}$.

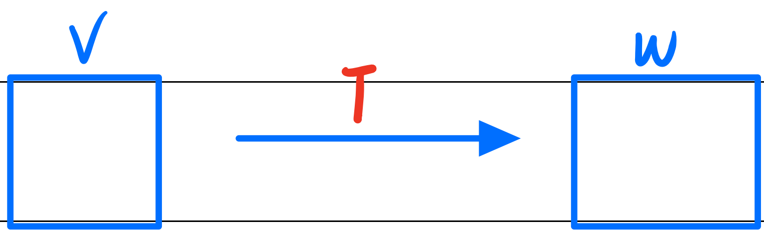
$\alpha = \beta = 1$

Remark 4.1.2

Condition (1) is equivalent to

$$\begin{cases} T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2) \\ T(\alpha \boldsymbol{v}_1) = \alpha T(\boldsymbol{v}_1) \end{cases}$$

for all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and for all scalars $\alpha \in \boldsymbol{R}$.



证明 linear transformation 两种方法二选一

$$\begin{cases} T(\alpha v_1 + \beta v_2) = \alpha T(\vec{v}_1) + \beta T(\vec{v}_2) \\ \begin{cases} T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) \\ T(\alpha v_1) = \alpha T(\vec{v}_1) \end{cases} \end{cases}$$

Remark 4.1.3

If $T: V \to W$ is a linear transformation from V to W, then

$$T(\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n) = \alpha_1 T(\boldsymbol{v}_1) + \alpha_2 T(\boldsymbol{v}_2) + \cdots + \alpha_n T(\boldsymbol{v}_n).$$

for all $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n \in V$ and for all scalars $\alpha_1, \cdots, \alpha_n \in \boldsymbol{R}$.

Remark 4.1.4

If $T: V \to W$ is a linear transformation from V to W, then $T(\mathbf{0}_V) = \mathbf{0}_W$.

Remark 4.1.5

If $V = W$, the linear transformation T is called the linear operator on V.

Example 4.1.6

Let $T: V \to W$ be zero mapping defined by $T(\mathbf{v}) = \mathbf{0}_W$ for all $\mathbf{v} \in V$. Then T is linear.

$T(\vec{\alpha}) = T(0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2) = 0 \cdot T(\vec{v}_1) + 0 \cdot T(\vec{v}_2) = \vec{0}_W$

Example 4.1.7

Let $I: V \to V$ be identity mapping defined by $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$. Then I is linear.

$I(\vec{v}_1) = I(\alpha \vec{v}_1 + 0 \cdot \vec{v}_2) = 1 \cdot I(\vec{v}_1) + 0 \cdot I(\vec{v}_2) = \vec{v}_1$

Example 4.1.8

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be mapping defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix}$ for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$. Then T is linear.

$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \vec{v} = A \cdot \vec{v}$

$T(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2) = A(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2) = \alpha_1 A\vec{v}_1 + \alpha_2 A\vec{v}_2$

$= \alpha_1 T(\vec{v}_1) + \alpha_2 T(\vec{v}_2)$

Example 4.1.9

前提：小方阵

The **trace** of an n-by-n square matrix $A=(a_{ij})_{n\times n}$ is defined to be $\mathrm{Tr}(A)=a_{11}+a_{22}+\cdots+a_{nn}$. Let $T: M_{n\times n}\to \boldsymbol{R}$ be mapping defined by $T(A)=Tr(A)$ for all $A\in M_{n\times n}$. Then T is linear.

$T: M_{n\times n} \longrightarrow R$

① $A \to \det(A)$

② $A \longrightarrow tr(A)$

$T(\alpha_1 A_{n\times n} + \alpha_2 B_{n\times n})$
$= \alpha_1 a_{11} + \alpha_2 b_{11} + \cdots \alpha_1 a_{nn} + \alpha_2 b_{nn}$
$= \alpha_1 Tr(A) + \alpha_2 Tr(B)$

$L(\alpha_1 A_{nn} + \alpha_2 B_{nn}) = \det(\alpha_1 A_{nn} + \alpha_2 B_{nn})$

$\triangle \det(A+B) \neq \det A + \det B$

$\neq \alpha_1 \det A_{nn} + \alpha_2 \det B_{nn}$

Example 4.1.10

Let $P_n=\{a_0+a_1x+\cdots+a_{n-1}x^{n-1}\mid a_0,\cdots,a_{n-1}\in\boldsymbol{R}\}$ where $n\ge 1$, be set of polynomials having degree $\le n-1$. It is an n-dimensional vector space. Let $T:P_3\to P_4$ be mapping defined by $T(p(x))=(1+x)p(x)$ for all $p(x)\in P_3$, i.e., $T(a_0+a_1x+a_2x^2)=(1+x)(a_0+a_1x+a_2x^2)$ $=a_0+(a_0+a_1)x+(a_1+a_2)x^2+a_2x^3$. Then T is linear.

$T[\alpha_1 P_1(x) + \alpha_2 P_2(x)] = (1+x)[\alpha_1 P_1(x) + \alpha_2 P_2(x)]$

$= (1+x)\alpha_1 P_1(x) + (1+x)\alpha_2 P_2(x) = \alpha_1 T[P_1(x)] + \alpha_2 T[P_2(x)]$

Example 4.1.11

Let $T:P_3\to P_2$ be differentiation mapping defined by $T(p(x))=p'(x)$ for all $p(x)\in P_3$, i.e., $T(a_0+a_1x+a_2x^2)=a_1+2a_2x$. Then T is linear.

$P_3 \to P_3/P_2$ 都可以

$T(\alpha_1 h_1(x) + \alpha_2 h_2(x)) = [\alpha_1 h_1(x) + \alpha_2 h_2(x)]' = \alpha_1 h_1'(x) + \alpha_2 h_2'(x) = \alpha_1 T(h_1(x)) + \alpha_2 T(h_2(x))$

$P_3 \longrightarrow P_2,$

$p(x) \longrightarrow p'(x)$

引入概念：{ injective, surjective

① linear transformation $V \longrightarrow W$

② $\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = 0_W\}$ is a subspace of V

Injective $\Leftrightarrow \ker(T) = \{\vec{0}_V\}$

Surjective $\Leftrightarrow \dim T(\vec{V}) = W$

③ Range $T(\vec{V}) \Leftrightarrow \{ T(\vec{v}) \mid \vec{v} \in V, T(\vec{v}) \in W \}$ is a subspace of W

Example 4.1.12

Let $T: P_3 \to P_4$ be integration mapping defined by $T(p(x)) = \int_1^t p(x)dx$ for all $p(x) \in P_3$, i.e.,
$T(a_0 + a_1x + a_2x^2) = \int_1^t a_0 + a_1x + a_2x^2 dx = \left[a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} \right]_1^t = -\left(a_0 + \frac{a_1}{2} + \frac{a_2}{3} \right) + a_0t + a_1\frac{t^2}{2} + a_2\frac{t^3}{3}$.
Then T is linear.

Example 4.1.13

Let $T: P_3 \to P_3$ be mapping defined by $T(p(x)) = p(x+1)$ for all $p(x) \in P_3$, i.e.,
$T(a_0 + a_1x + a_2x^2) = a_0 + a_1(x+1) + a_2(x+1)^2 = (a_0 + a_1 + a_2) + (a_1 + 2a_2)x + a_2x^2$. Then T is linear.

Example 4.1.14

Let $T: P_3 \to \boldsymbol{R}^3$ be mapping defined by $T(p(x)) = (p(1), p(2), p(3))$ for all $p(x) \in P_3$, i.e.,
$T(a_0 + a_1x + a_2x^2) = (a_0 + a_1 + a_2, a_0 + 2a_1 + 2^2a_2, a_0 + 3a_1 + 3^2a_2)$. Then T is linear.

Example 4.1.15

$\dim A_{n\times m} = n\times m$

Let $M_{2\times 3}=\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}\end{pmatrix}\middle| a_{11},a_{12},a_{13},a_{21},a_{22},a_{23}\in \boldsymbol{R}\right\}$ consists of 2×3 matrices. It is a 6-dimensional vector space. Let $T: M_{2\times 3}\to M_{2\times 3}$ be mapping defined by $T(A)=\begin{pmatrix} a & b \\ c & d\end{pmatrix}A$. for all $A\in M_{2\times 3}$. Then T is linear.

$$M_{2\times 3} \longrightarrow M_{2\times 3}$$
$$A \longrightarrow \begin{bmatrix} a & b \\ c & d\end{bmatrix}A.$$

$$T(\alpha_1 A+\alpha_2 B)=\begin{bmatrix} a & b \\ c & d\end{bmatrix}[\alpha_1 A+\alpha_2 B]$$
$$=\alpha_1\begin{bmatrix} a & b \\ c & d\end{bmatrix}A+\alpha_2\begin{bmatrix} a & b \\ c & d\end{bmatrix}B$$
$$=\alpha_1 T(A)+\alpha_2 T(B)$$

Example 4.1.16 (Rotation) 旋转

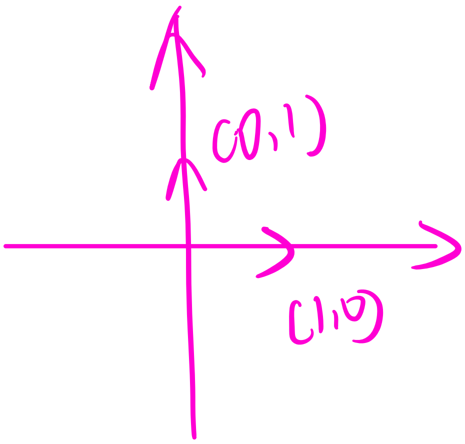
$\begin{bmatrix} a & b \\ 0 & 0\end{bmatrix} \Rightarrow T(a)=T(b)$

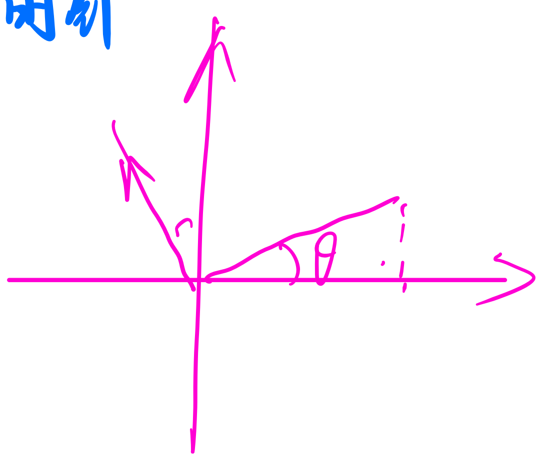
正: abcd 全为 0 ⇒ T(a) 总为 T(b)，

逆否: T(a) 存在≠T(b) ⇒ abcd 不全为0，

Let $T:\boldsymbol{R}^2\to \boldsymbol{R}^2$ be rotation mapping defined by counterclockwise rotating vector $\begin{pmatrix} x \\ y\end{pmatrix}$ by angle θ. Then T is linear.

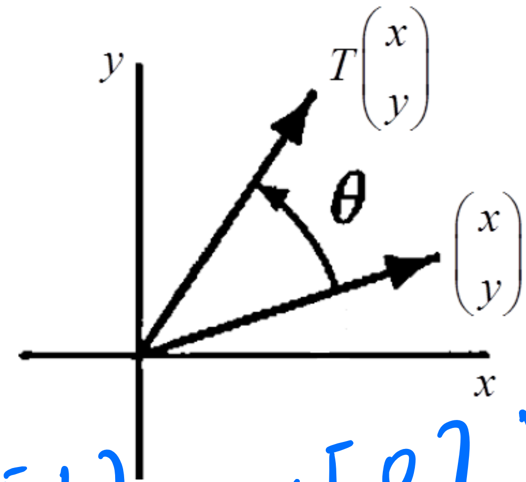
逆用析





理解成基底变化后整体的线性变换

① 就是基底旋转 θ 再去计算 $T\begin{pmatrix} x \\ y \end{pmatrix}$



$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = T\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$= xT\begin{pmatrix} 1 \\ 0 \end{pmatrix} + yT\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

→ 利用三角变换 $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Example 4.1.17

Recall complex number $C = \{z = x + yi \mid x, y \in \boldsymbol{R}\}$ is a vector space over $\boldsymbol{R}$. Let $T : \boldsymbol{C} \to \boldsymbol{C}$ be mapping defined by $T(z) = (a+bi)z$ for all $z \in \boldsymbol{C}$, i.e., $T(x+yi) = (a+bi)(x+yi)$ $= (ax - by) + (ay + bx)i$. Then T is linear.

Example 4.1.18 (Translation)

Let $T : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be mapping defined by $T(x, y) = (x+1, y+2)$ for all $(x, y) \in \boldsymbol{R}^2$. Is T linear?

$R^2 \to R^2$.

$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+1 \\ y+2 \end{pmatrix}$

平移不是线性变换
旋转是线性变化

Example 4.1.19

Is it possible to find a linear transformation $T: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ such that $T(1,2)=(3,4)$ and $T(2,4)=(5,6)$?

$T\begin{pmatrix}2\\4\end{pmatrix} = T\left(2\begin{pmatrix}1\\2\end{pmatrix}\right) = 2T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}6\\8\end{pmatrix} \neq \begin{pmatrix}5\\6\end{pmatrix}$

不是线性变换.

Example 4.1.20

basis

先通过线性组合表示出来移后的东西再通过线性变换求出.

Let $T: span\{(-1,0,1),(0,1,0)\} \to \boldsymbol{R}^2$ be linear transformation defined by $T(-1,0,1)=(1,3)$ and $T(0,1,0)=(2,4)$. What is $T(2,3,-2)$?

$T\begin{pmatrix}2\\3\\-2\end{pmatrix} = T\left[-2\begin{bmatrix}-1\\0\\1\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix}\right] = -2T\begin{bmatrix}-1\\0\\1\end{bmatrix} + 3T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix}$

Definition 4.1.21

Proof:

① ker T is a subspace of V

$\begin{cases}\vec{0}_V \in \ker(T)\\ \vec{v}_1, \vec{v}_2 \in \ker(T) \quad \vec{v}_1+\vec{v}_2 \in \ker(T)\\ \vec{v} \in \ker(T) \quad \alpha\vec{v} \in \ker T\end{cases}$

Let $T: V \to W$ be a linear transformation. The **kernel** of T, denoted ker(T), is defined by

$$\ker(T) = \{\boldsymbol{v} \in V \mid T(\boldsymbol{v}) = \boldsymbol{0}_W\}.$$

一个空间通过T运算映射出0结果的空间

Remark 4.1.22

Let $T: V \to W$ be a linear transformation. Then T is injective if and only if $\ker(T) = \{\boldsymbol{0}_V\}$.

单射

T is injective $\begin{cases}\forall \vec{v}_1, \vec{v}_2 \in V\\ T(\vec{v}_1) \neq T(\vec{v}_2)\end{cases} \Leftrightarrow \ker(T)$

① $\forall \vec{v} \in \ker(T)$, Assume $\vec{v} \neq \vec{0}$ $\Rightarrow T(\vec{v}) \neq T(\vec{0}) = 0_W$ 矛盾

② $\ker(T) = \{\vec{0}\}$ $\forall \vec{v}_1 \neq \vec{v}_2 \in V \Rightarrow \vec{v}_1 - \vec{v}_2 \neq \vec{0}$

$T(\vec{v}_1 - \vec{v}_2) \neq 0_W \Rightarrow T(\vec{v}_1) - T(\vec{v}_2) \neq 0_W \Rightarrow T(\vec{v}_1) \neq T(\vec{v}_2)$

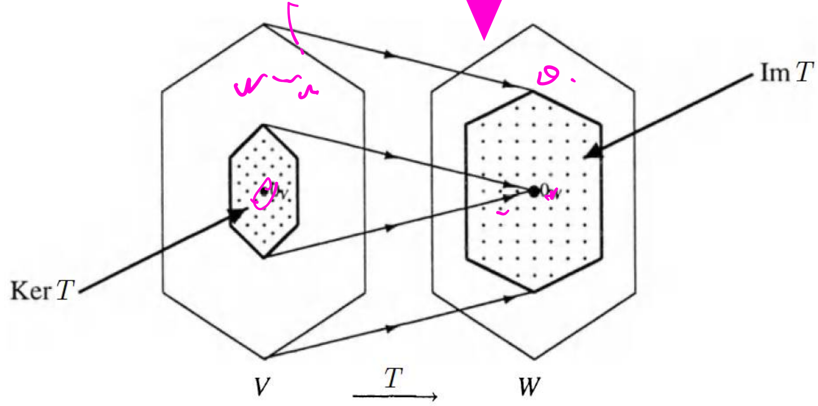
Definition 4.1.23

Let $T: V \to W$ be linear. The **image or range** of T, denoted $T(V)$, is defined by

$$T(V) = \{T(\boldsymbol{v}) \in W \mid \boldsymbol{v} \in V\}.$$

Theorem 4.1.24

Let $T: V \to W$ be a linear transformation. Then ker(T) and $T(V)$ are subspaces of V and W respectively.



Example 4.1.25

Let $T:V \to W$ be zero mapping defined by $T(\boldsymbol{v}) = \boldsymbol{0}_W$ for all $\boldsymbol{v} \in V$. Then $\ker(T) = V$ and $T(V) = \{\boldsymbol{0}_W\}$.

Example 4.1.26

Let $T:V \to V$ be identity mapping defined by $T(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in V$. Then $\ker(T) = \{\boldsymbol{0}_V\}$ and $T(V) = V$.

A → nullity
A → Row(A)

Theorem 4.1.27 (Dimension Theorem)

Let $T:V \to W$ be a linear transformation. If $\dim V$ is finite, then

$$\dim V = \dim \ker(T) + \dim T(V).$$

类比 Rank(A) = dim RowA + nullity

In particular, if $V = \boldsymbol{R}^n$ and $W = \boldsymbol{R}^m$, this is exactly rank-nullity theorem.

Outline of proof

Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m\}$ be the basis of $\ker(T)$. Extend this basis to form a basis of V.

① linear transformation: $V \to W$
$\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = 0_W\}$ is a subspace of V
Injective ⟺ $\ker(T) = \{\vec{0}\}$
surjective ⟺ $\dim(T(V)) = W$
② Range $T(V) = \{T(\vec{v}) \in W \mid \vec{v} \in V\}$ is a subspace of W

$$\underbrace{\{\overbrace{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m}^{\text{basis of } \ker(T)}, \boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}}_{\text{basis of } V}$$

① span
② linear independent.

Then $\{T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \cdots, T(\boldsymbol{u}_k)\}$ is a basis of $T(V)$.

步骤: ① 找到 ker(T) 的 basis
② Extend 成 V 的 basis
③ 找到 Range T(V) 的 basis.

Theorem 4.1.28 小前提:

Let V and W be vector spaces of equal finite dimension, and let $T: V \to W$ be linear. Then the following are equivalent.

(a) T is one-to-one ($\ker(T) = \{\mathbf{0}_V\}$). injective

(b) T is onto ($\dim T(V) = \dim W$). surjective

Example 4.1.29

Find ker(T) and $T(\vec{v})$

$T(\vec{v}) = A \cdot \vec{v}$

Let $T: \boldsymbol{R}^3 \to \boldsymbol{R}^3$ be linear transformation defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+3z \\ 4x+5y+6z \\ 7x+8y+9z \end{pmatrix}$ for all $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \boldsymbol{R}^3$.

Consider ker(T). By elementary row operation, $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \ker(T)$. We have $x+2y+3z = y+2z = 0$. Then $y = -2z$ and $x = -2y-3z$ $= -2(-2z)-3z = z$.

$\ker T = \mathrm{Nul}(A)$

$$\ker(T) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\} = \left\{ \begin{pmatrix} z \\ -2z \\ z \end{pmatrix} \right\} = span\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

$T(\vec{v}) = \mathrm{Col}(A)$

Consider $T(\boldsymbol{R}^3)$.

leading 1's

Method 1: By elementary column operation, $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 4 & -3 & -6 \\ 7 & -6 & -12 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 0 \end{pmatrix}$. Then

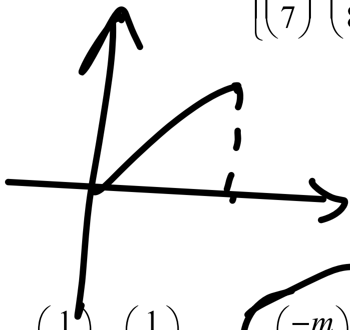
$$T(\boldsymbol{R}^3) = span\left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \right\}.$$

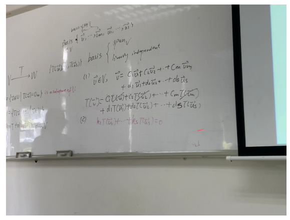
列变化不是行变化.

Method 2 思路:
① 找到 ker T 的 basis.
② 扩展成 总 的 basis (补 $\vec{v_1}, \vec{v_2} \cdots \vec{v_n}$)
③ $T(\vec{v_1})\ T(\vec{v_2}) \cdots T(\vec{v_n}) \Rightarrow$ Range

Method 2: $\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}$ is a basis of $\ker(T)$. Extend it to form a basis of $\boldsymbol{R}^3$, $\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$. Then

$\left\{T\begin{pmatrix}1\\0\\0\end{pmatrix},T\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}=\left\{\begin{pmatrix}1\\4\\7\end{pmatrix},\begin{pmatrix}2\\5\\8\end{pmatrix}\right\}$ form a basis of $T(V)$. Hence $T(V)=span\left\{\begin{pmatrix}1\\4\\7\end{pmatrix},\begin{pmatrix}2\\5\\8\end{pmatrix}\right\}$.





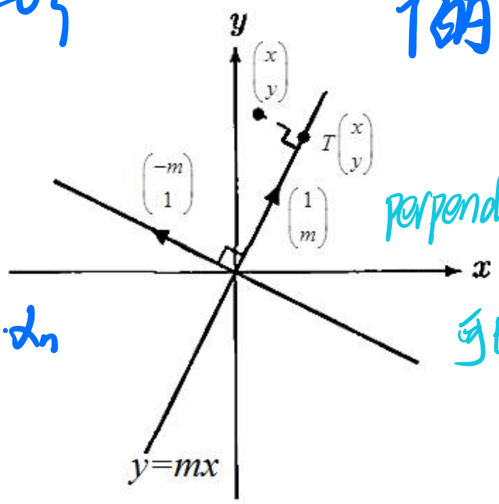
$\dim \boldsymbol{R}^3=3=1+2=\dim\ker(T)+\dim T(\boldsymbol{R}^3)$.

Example 4.1.30 (Projection) 投影

Let $T:\boldsymbol{R}^2\to\boldsymbol{R}^2$ be linear transformation defined by $T\begin{pmatrix}1\\m\end{pmatrix}=\begin{pmatrix}1\\m\end{pmatrix}$ and $T\begin{pmatrix}-m\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.

$\ker T = \{\vec{x} \in \vec{V} \mid A\vec{x} = 0\}$
(Nul(A))
$T(\vec{V}) = \{A\vec{x} \mid \vec{x} \in \vec{V}\}$
$A = (\alpha_1, \cdots\cdots \alpha_n)$
$= x_1\alpha_1 + x_2\alpha_2 + \cdots\cdots x_n\alpha_n$

T的几何视角: 求长度.



perpendicular.

可以类比成力的分解

Method 1: Clearly, $\ker(T)=span\left\{\begin{pmatrix}-m\\1\end{pmatrix}\right\}$ and

两个不共线平行的向量可以表示一切向量.

$T(\boldsymbol{R}^2)=\left\{T\left(\alpha\begin{pmatrix}1\\m\end{pmatrix}+\beta\begin{pmatrix}-m\\1\end{pmatrix}\right)\right\}=\left\{\alpha T\begin{pmatrix}1\\m\end{pmatrix}+\beta T\begin{pmatrix}-m\\1\end{pmatrix}\right\}=\left\{\alpha\begin{pmatrix}1\\m\end{pmatrix}\right\}=span\left\{\begin{pmatrix}1\\m\end{pmatrix}\right\}$. ⇒ 利用定义

Method 2: ① 由 V 的 basis ② ker (T) 的 basis

③ 倒推 T(V) 的 basis ⟺ how to find ker(T) and image.

$\dim \boldsymbol{R}^2=2=1+1=\dim\ker(T)+\dim T(\boldsymbol{R}^2)$.

R^2: $\begin{pmatrix}1\\m\end{pmatrix}, \begin{pmatrix}-m\\1\end{pmatrix}$

ker T image.

ker T: $\begin{pmatrix}-m\\1\end{pmatrix}$ ⇒ $u = \begin{pmatrix}1\\m\end{pmatrix}$
image: $T(\vec{u})$

Example 4.1.31

Let $T: P_3 \to P_2$ be differentiation mapping defined by $T(p(x)) = p'(x)$ for all $p(x) \in P_3$, i.e., $T(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x$.

$\ker T = \{1\}$ is right too

Method 1: $T(a_0 + a_1 x + a_2 x^2) = 0$ if and only if $a_1 = a_2 = 0$. $\ker(T) = \{a_0\}$, constant function. Clearly, $T(P_3) = \{a_1 + 2a_2 x\} = span\{1, x\} = P_2$.

Method 2: $P_3 = span\{1, x, x^2\}$ $\quad \ker(T) = span\{1\}$

$\Rightarrow P_2: span\{T(x), T(x^2)\} = \{1, 2x\} \Leftrightarrow \{1, x\}$

$\dim P_3 = 3 = 1 + 2 = \dim \ker(T) + \dim T(P_3)$.

Example 4.1.32

Let $T: P_3 \to \boldsymbol{R}^3$ be mapping defined by $T(p(x)) = (p(1), p(2), p(3))$ for all $p(x) \in P_3$, i.e., $T(a_0 + a_1 x + a_2 x^2) = (a_0 + a_1 + a_2, a_0 + 2a_1 + 2^2 a_2, a_0 + 3a_1 + 3^2 a_2)$.

Consider $\ker(T) = \{p(x) = a_0 + a_1 x + a_2 x^2 \in P_3 \mid (p(1), p(2), p(3)) = (0,0,0)\}$.

Method 1: We need to solve

三点可确定一条抛物线.

$$\begin{cases} a_0 + a_1 + a_2 = 0 \\ a_0 + 2a_1 + 2^2 a_2 = 0 \\ a_0 + 3a_1 + 3^2 a_2 = 0 \end{cases}$$

变成抛物线求x轴交点.
但抛物线不可能存在三个点皆知
⇒ 无解

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \end{pmatrix}$ is called Vandermonde matrix. Its determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \end{vmatrix} = (3-1)(3-2)(2-1) \neq 0$. $= 2$

$T = \left\{ \begin{pmatrix} a_0 + a_1 + a_2 \\ a_0 + 2a_1 + 4a_2 \\ a_0 + 3a_1 + 9a_2 \end{pmatrix} \middle| a_0, a_1, a_2 \in R \right\} = \left\{ a_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + a_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix} \right\}$

$\Rightarrow T(\vec{V}) = span\left\{ \boxed{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}} \right\} \to$ ① 线性无关 {定义, det ② basis.

Hence $a_0 = a_1 = a_2 = 0$, i.e., $\ker(T) = \{0\}$, zero polynomial.

$\ker(T) = \{0\} \Rightarrow$ 无基底 $\Rightarrow \dim \ker(T) = 0$

$\dim \ker(T) + \dim T(\vec{V}) = \dim V$

$\Downarrow$

$\dim T(\vec{V}) = \dim V$

$T(\vec{V}) \subseteq V$ 子集 但空间维数相同 则二者相同.

Method 2: $\ker(T) = \{p(x) = a_0 + a_1 x + a_2 x^2 \in P_3 \mid (p(1), p(2), p(3)) = (0,0,0)\}$.

That means the quadratic equation $p(x) = a_0 + a_1 x + a_2 x^2 = 0$ has 3 roots, 1, 2 and 3. Such polynomial $p(x)$ must be a zero polynomial. (By factor theorem, $p(x)$ must contain factor $(x-1)$ and $(x-2)$. Hence $p(x) = a_2(x-1)(x-2)$. Since $p(3) = 0$, $a_2(3-1)(3-2) = 0$. Then $a_2 = 0$, i.e., $p(x)$ is a zero polynomial and $\ker(T) = \{0\}$.)

Remark 4.1.33 ???

升纲 ⇒ Let $T : P_n \to \mathbf{R}^3$ be mapping defined by $T(p(x)) = (p(1), p(2), p(3))$ for all $p(x) \in P_n$, where $n \geq 4$. Then $\ker(T) = \{(x-1)(x-2)(x-3)q(x) \mid q(x) \in P_{n-3}\}$. Hence $\dim \ker(T) = n-3$.

Example 4.1.34

Find $a_0 + a_1 x + a_2 x^2$ so that the graph of $y = a_0 + a_1 x + a_2 x^2$ passes through $(1,-1),(2,-3),(3,-3)$.

Solution 抛物线三点即可确定

Method 1: We need to solve

$$\begin{cases} a_0 + a_1 + a_2 = -1 \\ a_0 + 2a_1 + 2^2 a_2 = -3 \\ a_0 + 3a_1 + 3^2 a_2 = -3 \end{cases} \qquad \text{or} \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ -3 \end{pmatrix}$$

Finding $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \end{pmatrix}^{-1}$ is not easy or you may use Cramer's rule. (Do it yourself.)

Method 2: The question is:

Find $p(x) = a_0 + a_1 x + a_2 x^2 \in P_3$ so that $T(p(x)) = (p(1), p(2), p(3)) = (-1,-3,-3)$.

Since $\ker(T) = \{0\}$ and $\dim T(P_3) = \dim(P_3) = 3$, by Theorem 4.1.28, $T(P_3) = \mathbf{R}^3$. Let $p_1(x), p_2(x), p_3(x) \in P_3$ such that $T(p_1(x)) = (1,0,0)$, $T(p_2(x)) = (0,1,0)$, $T(p_3(x)) = (0,0,1)$. Then $p_1(x) = c_1(x-2)(x-3)$. Since $p_1(1) = 1, c_1(1-2)(1-3) = 1$. Then $c_1 = \dfrac{1}{(1-2)(1-3)}$ and $p_1(x) = \dfrac{(x-2)(x-3)}{(1-2)(1-3)}$. Similarly, $p_2(x) = \dfrac{(x-1)(x-3)}{(2-1)(2-3)}$ and $p_3(x) = \dfrac{(x-1)(x-2)}{(3-1)(3-2)}$.

$$\begin{aligned}(-1,-3,-3) &= -1(1,0,0)+(-3)(0,1,0)+(-3)(0,0,1)\\ &= -1T(p_1(x))+(-3)T(p_2(x))+(-3)T(p_3(x))\\ &= T\left((-1)\frac{(x-2)(x-3)}{(1-2)(1-3)}+(-3)\frac{(x-1)(x-3)}{(2-1)(2-3)}+(-3)\frac{(x-1)(x-2)}{(3-1)(3-2)}\right).\end{aligned}$$

The required polynomial is

$$(-1)\frac{(x-2)(x-3)}{(1-2)(1-3)}+(-3)\frac{(x-1)(x-3)}{(2-1)(2-3)}+(-3)\frac{(x-1)(x-2)}{(3-1)(3-2)}=x^2-5x+3.$$

Remark 4.1.35

In general, the degree 2 polynomial passes through $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ is

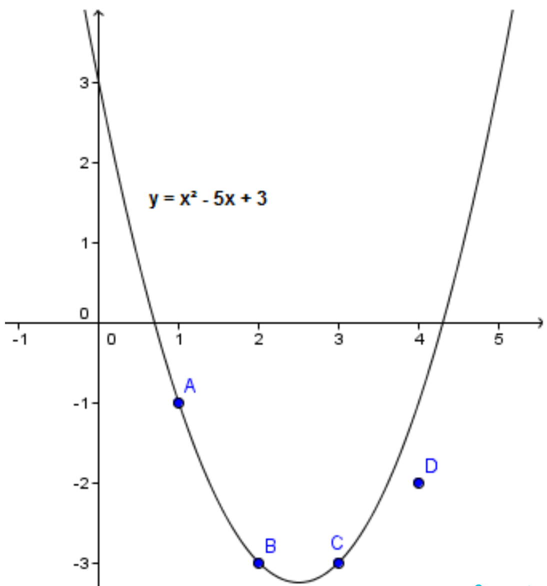
$$y_1\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}+y_2\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}+y_3\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}.$$

Example 4.1.36

Is it possible to find $a_0+a_1x+a_2x^2$ so that the graph of $y=a_0+a_1x+a_2x^2$ passes through $(1,-1),(2,-3),(3,-3),(4,-2)$?

Solution

No. x^2-5x+3 is the only polynomial so that the graph of $y=a_0+a_1x+a_2x^2$ passes through $(1,-1),(2,-3),(3,-3)$. When $x=4$, $y=4^2-5(4)+3=-1\neq-2$. It is impossible.



Δ.抛物线可由三点确定

Section 4.2 Matrix representations of Linear Transformations

Let $\beta=\{v_1,v_2,\cdots,v_n\}$ and $\gamma=\{w_1,w_2,\cdots,w_m\}$ be ordered bases of finite dimensional vector spaces V and W respectively. Let $T:V\to W$ be linear. Then

$V = \text{span}\{\vec{v_1}, \vec{v_2} \cdots \vec{v_n}\}$

$W = \text{span}\{\vec{w_1}, \vec{w_2} \cdots \vec{w_n}\}$

$$\begin{cases} T(\boldsymbol{v}_1)=a_{11}\boldsymbol{w}_1+a_{21}\boldsymbol{w}_2+\cdots+a_{m1}\boldsymbol{w}_m \\ T(\boldsymbol{v}_2)=a_{12}\boldsymbol{w}_1+a_{22}\boldsymbol{w}_2+\cdots+a_{m2}\boldsymbol{w}_m \\ \vdots \\ T(\boldsymbol{v}_n)=a_{1n}\boldsymbol{w}_1+a_{2n}\boldsymbol{w}_2+\cdots+a_{mn}\boldsymbol{w}_m \end{cases}$$

$T(\vec{v}) = T(c_1\vec{v_1} + c_2\vec{v_2} + \cdots c_n\vec{v_n})$
$= c_1 T(\vec{v_1}) + c_2 T(\vec{v_2}) + \cdots c_n T(\vec{v_n})$
$= c_1 T(a_{11} w_1 + \cdots a_{m1} w_m) + \cdots c_n T(a_{1n} w_1 + \cdots a_{mn} w_m)$
$= (c_1 a_{11} \cdots c_n a_{1n}) w_1 + \cdots (c_1 a_{m1} + \cdots c_n a_{mn}) \vec{w_m}$

Definition 4.2.1

We call the $m\times n$ matrix

$$A=\left(\begin{array}{c|c|c|c} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right)$$

the **matrix representation of T in the ordered bases β and γ** and write $A=[T]_\beta^\gamma$. If $V=W$ and $\beta=\gamma$, then we write $A=[T]_\beta$.

The following theorem is special case of Definition 4.2.1 by taking $V=\boldsymbol{R}^n, W=\boldsymbol{R}^m$, and β and γ the standard bases of $\boldsymbol{R}^n$ and $\boldsymbol{R}^m$ respectively.

Theorem 4.2.2

If $T:\boldsymbol{R}^n\to\boldsymbol{R}^m$ is linear, then there is an $m\times n$ matrix A such that

$$T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}=\underbrace{\left(\begin{array}{c|c|c|c} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right)}_{A}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Indeed, $A=[T]_\beta^\gamma$ and $\ker(T)=N(A)$.

Example 4.2.3

Let $T:\boldsymbol{R}^2\to\boldsymbol{R}^2$ be linear transformation defined by $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x-2y\\-2x+y\end{pmatrix}$ for all $\begin{pmatrix}x\\y\end{pmatrix}\in\boldsymbol{R}^2$. Let $\beta=\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}3\\4\end{pmatrix}\right\}$, $\beta'=\left\{\begin{pmatrix}3\\4\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right\}$, $\gamma=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ be ordered bases of $\boldsymbol{R}^2$. We have

$$T\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}-3\\0\end{pmatrix}=-3\begin{pmatrix}1\\0\end{pmatrix}+0\begin{pmatrix}0\\1\end{pmatrix}\quad\text{and}\quad T\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}-5\\-2\end{pmatrix}=-5\begin{pmatrix}1\\0\end{pmatrix}-2\begin{pmatrix}0\\1\end{pmatrix}.$$

Then $[T]_\beta^\gamma=\left(\begin{array}{c:c}-3&-5\\0&-2\end{array}\right)$ and $[T]_{\beta'}^\gamma=\left(\begin{array}{c:c}-5&-3\\-2&0\end{array}\right)$.

Switching order of basis changes the matrix representation of T.

Example 4.2.4

$$T\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Let $T:\boldsymbol{R}^2\to\boldsymbol{R}^2$ be linear transformation defined by $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+2y\\3x+4y\end{pmatrix}$ for all $\begin{pmatrix}x\\y\end{pmatrix}\in\boldsymbol{R}^2$. Let $\beta=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ be the standard basis of $\boldsymbol{R}^2$. We have

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}=1\begin{pmatrix}1\\0\end{pmatrix}+3\begin{pmatrix}0\\1\end{pmatrix}\quad\text{and}\quad T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix}=2\begin{pmatrix}1\\0\end{pmatrix}+4\begin{pmatrix}0\\1\end{pmatrix}.$$

Then $A=[T]_\beta=$ $\begin{bmatrix}1&2\\3&4\end{bmatrix}$ and $\boxed{T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+2y\\3x+4y\end{pmatrix}=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}}$.

Example 4.2.5

Let $T:\boldsymbol{R}^2\to\boldsymbol{R}^2$ be linear transformation defined by $T\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$ and $T\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}$. Find $T\begin{pmatrix}x\\y\end{pmatrix}$?

$\begin{pmatrix}x\\y\end{pmatrix}=k_1\begin{pmatrix}2\\1\end{pmatrix}+k_2\begin{pmatrix}3\\2\end{pmatrix}$

$T\begin{pmatrix}x\\y\end{pmatrix}=k_1T\begin{pmatrix}2\\1\end{pmatrix}+k_2T\begin{pmatrix}3\\2\end{pmatrix}$
$=k_1\begin{pmatrix}1\\2\end{pmatrix}+k_2\begin{pmatrix}3\\4\end{pmatrix}$

$T\begin{pmatrix}2\\1\end{pmatrix}=T\left(2\begin{pmatrix}1\\0\end{pmatrix}+1\begin{pmatrix}0\\1\end{pmatrix}\right)=2T\begin{pmatrix}1\\0\end{pmatrix}+T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$

$T\begin{pmatrix}3\\2\end{pmatrix}=T\left(3\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix}\right)=3T\begin{pmatrix}1\\0\end{pmatrix}+2T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}$

$T\begin{pmatrix}x\\y\end{pmatrix}=xT\begin{pmatrix}1\\0\end{pmatrix}+yT\begin{pmatrix}0\\1\end{pmatrix}$

$\Rightarrow T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}k_1\\k_2\end{pmatrix}$

$T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}k_3\\k_4\end{pmatrix}$

$\mathbb{R}^n\xrightarrow{T}\mathbb{R}^m$

$T(\vec{x})=A_{m\times n}\vec{x}$ $\quad\ker T=\mathrm{Nul}(A)$

$T(\vec{v})=\{A\vec{x}\mid\vec{x}\in\mathbb{R}^n\}$ $\quad T(V)=\mathrm{col}(A)$

$=\left\{(\vec{a}_1,\vec{a}_2\cdots\vec{a}_n)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\right\}$

① 先线性表示

②

$\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$

$\begin{cases}2a_{11}+a_{12}=1\\2a_{21}+a_{22}=2.\end{cases}$

$\begin{cases}3a_{11}+2a_{12}=3\\3a_{21}+2a_{22}=4\end{cases}$

Solution

Since $\det\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = 1 \neq 0$, $\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\}$ is linearly independent. Hence $\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\}$ is a basis of $\boldsymbol{R}^2$. Consider

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 2 \\ 1 \end{pmatrix} + b\begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c\begin{pmatrix} 2 \\ 1 \end{pmatrix} + d\begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

That is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}.$$

By Cramer's rule,

$$a = \frac{\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}} = 2 \quad b = \frac{\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}} = -1 \quad c = \frac{\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}} = -3 \quad d = \frac{\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}} = 2.$$

Then

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -3\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

$$\begin{aligned} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= T\left(2\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) \\ &= 2T\begin{pmatrix} 2 \\ 1 \end{pmatrix} - T\begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ &= -1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= T\left[-3\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right] \\ &= -3T\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2T\begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= -3\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

$T\begin{pmatrix}0\\1\end{pmatrix} = T\left(-3\begin{pmatrix}2\\1\end{pmatrix}+2\begin{pmatrix}3\\2\end{pmatrix}\right) = -3T\begin{pmatrix}2\\1\end{pmatrix}+2T\begin{pmatrix}3\\2\end{pmatrix}$

$= -3\begin{pmatrix}1\\2\end{pmatrix}+2\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}$

$= 3\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix}$

Let $\beta=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ be the standard basis of $\boldsymbol{R}^2$. Then $A=[T]_\beta=\begin{pmatrix}-1&3\\0&2\end{pmatrix}$. Hence $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1&3\\0&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$.

Example 4.2.5

Define $T:span\{e^x\sin x,e^x\cos x\}\to span\{e^x\sin x,e^x\cos x\}$ by $T(f(x))=f(x+\theta)$. Let $\beta=\{e^x\sin x,e^x\cos x\}$ be the basis.

We have $T(e^x\sin x)=e^{x+\theta}\sin(x+\theta)=$ $= e^\theta\cos\theta\cdot e^x\sin x+e^\theta\sin\theta\cdot e^x\cos x$

and $T(e^x\cos x)=e^{x+\theta}\cos(x+\theta)=$ $= -e^\theta\sin\theta\cdot e^x\sin x+e^\theta\cos\theta\cdot e^x\cos x$.

Then $[T]_\beta=\begin{pmatrix}e^\theta\cos\theta&-e^\theta\sin\theta\\e^\theta\sin\theta&e^\theta\cos\theta\end{pmatrix}=e^\theta\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$.

△有顺序要求 {先 $e^x\sin x$ 再 $e^x\cos x$

Example 4.2.6 (Rotation) 旋转.

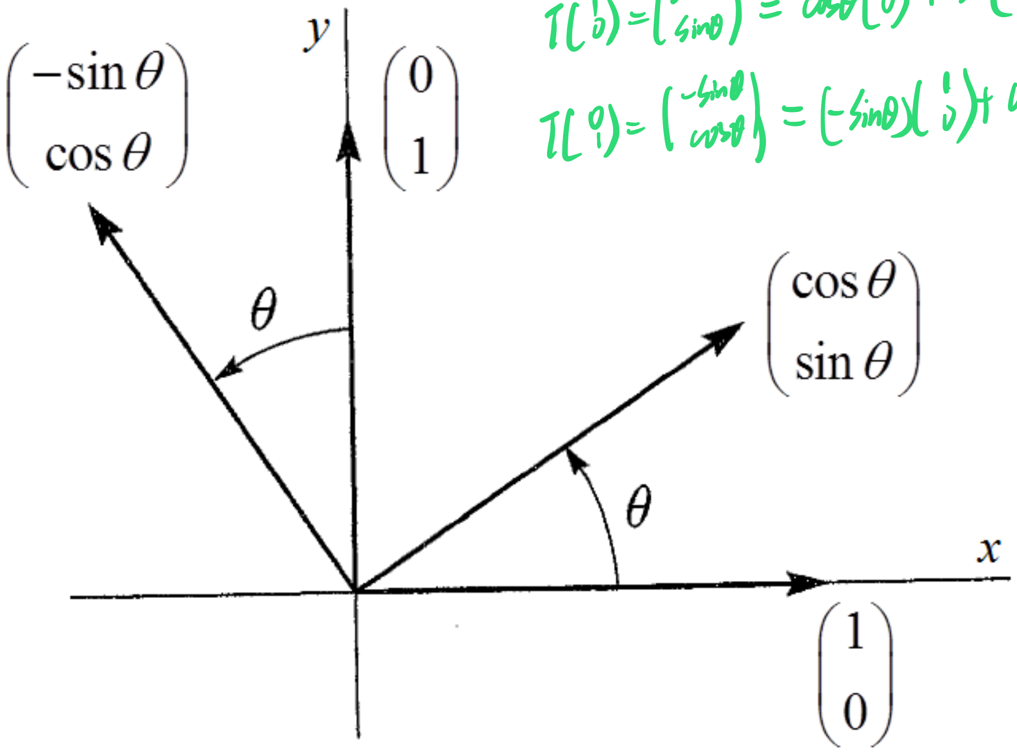
Let $T:\boldsymbol{R}^2\to\boldsymbol{R}^2$ be rotation mapping by counterclockwise rotating vector $\begin{pmatrix}x\\y\end{pmatrix}$ by angle θ. Let $\beta=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ be the standard basis of $\boldsymbol{R}^2$. We have

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}\text{ and }T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}.$$

先对 basis 进行变换

$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}=\cos\theta\begin{pmatrix}1\\0\end{pmatrix}+\sin\theta\begin{pmatrix}0\\1\end{pmatrix}$

$T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}=(-\sin\theta)\begin{pmatrix}1\\0\end{pmatrix}+\cos\theta\begin{pmatrix}0\\1\end{pmatrix}$



Then $A=[T]_\beta=\left(\begin{array}{c|c}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{array}\right)$ and $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$.

We call $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ the rotation matrix.

Example 4.2.7

Let $T:\boldsymbol{C}\to\boldsymbol{C}$ be mapping defined by $T(z)=(a+bi)z$ for all $z\in\boldsymbol{C}$, i.e., $T(x+yi)=(a+bi)(x+yi)=(ax-by)+(ay+bx)i$. Let $\beta=\{1,i\}$ be the standard basis of $\boldsymbol{C}$. We have

$$T(1)=a+bi \quad\text{and}\quad T(i)=(a+bi)i=-b+ai.$$

Then $A=[T]_\beta=\left(\begin{array}{c|c}a & -b\\ b & a\end{array}\right)$.

In particular, if $a+bi=\cos\theta+i\sin\theta$, then $A=[T]_\beta=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ and T is a linear transformation by counterclockwise rotating complex number $x+yi$ by angle θ.

逆时针旋转 θ 角

$[T]_\beta=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

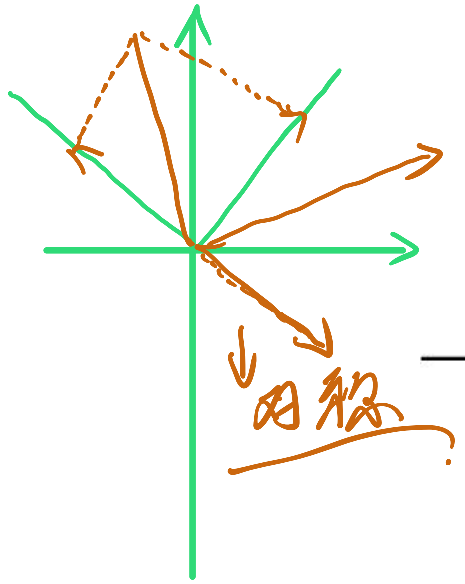
变换矩阵

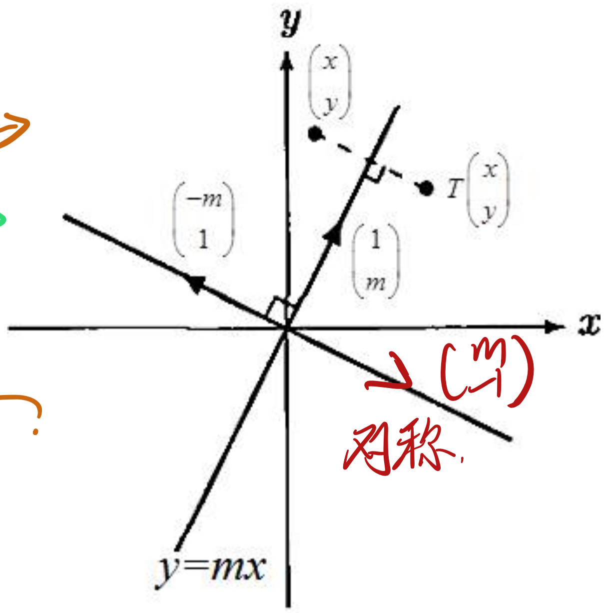
视角 ① 代数视角
② 几何视角

Example 4.2.8 (Reflection) 对称，

Let $T:\boldsymbol{R}^2 \to \boldsymbol{R}^2$ be linear transformation defined by $T\begin{pmatrix}1\\m\end{pmatrix}=\begin{pmatrix}1\\m\end{pmatrix}$ and $T\begin{pmatrix}-m\\1\end{pmatrix}=-\begin{pmatrix}-m\\1\end{pmatrix}=\begin{pmatrix}m\\-1\end{pmatrix}$.

We call T **reflection along** $y=mx$.





Method 1: Since $\det\begin{pmatrix}1 & -m\\ m & 1\end{pmatrix}=1+m^2\neq 0$, $\left\{\begin{pmatrix}1\\m\end{pmatrix},\begin{pmatrix}-m\\1\end{pmatrix}\right\}$ is linearly independent. Hence $\left\{\begin{pmatrix}1\\m\end{pmatrix},\begin{pmatrix}-m\\1\end{pmatrix}\right\}$ is a basis of $\boldsymbol{R}^2$. Consider 利用信息先找到 basis.

$$\begin{pmatrix}1\\0\end{pmatrix}=a\begin{pmatrix}1\\m\end{pmatrix}+b\begin{pmatrix}-m\\1\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\end{pmatrix}=c\begin{pmatrix}1\\m\end{pmatrix}+d\begin{pmatrix}-m\\1\end{pmatrix}.$$

That is 找标准基底与所给基底的关系.

$$\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1 & -m\\ m & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1 & -m\\ m & 1\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}.$$

By Cramer's rule,

$$a=\frac{\begin{vmatrix}1 & -m\\ 0 & 1\end{vmatrix}}{\begin{vmatrix}1 & -m\\ m & 1\end{vmatrix}}=\frac{1}{1+m^2}\quad b=\frac{\begin{vmatrix}1 & 1\\ m & 0\end{vmatrix}}{\begin{vmatrix}1 & -m\\ m & 1\end{vmatrix}}=\frac{-m}{1+m^2}\quad c=\frac{\begin{vmatrix}0 & -m\\ 1 & 1\end{vmatrix}}{\begin{vmatrix}1 & -m\\ m & 1\end{vmatrix}}=\frac{m}{1+m^2}\quad d=\frac{\begin{vmatrix}1 & 0\\ m & 1\end{vmatrix}}{\begin{vmatrix}1 & -m\\ m & 1\end{vmatrix}}=\frac{1}{1+m^2}.$$

Then

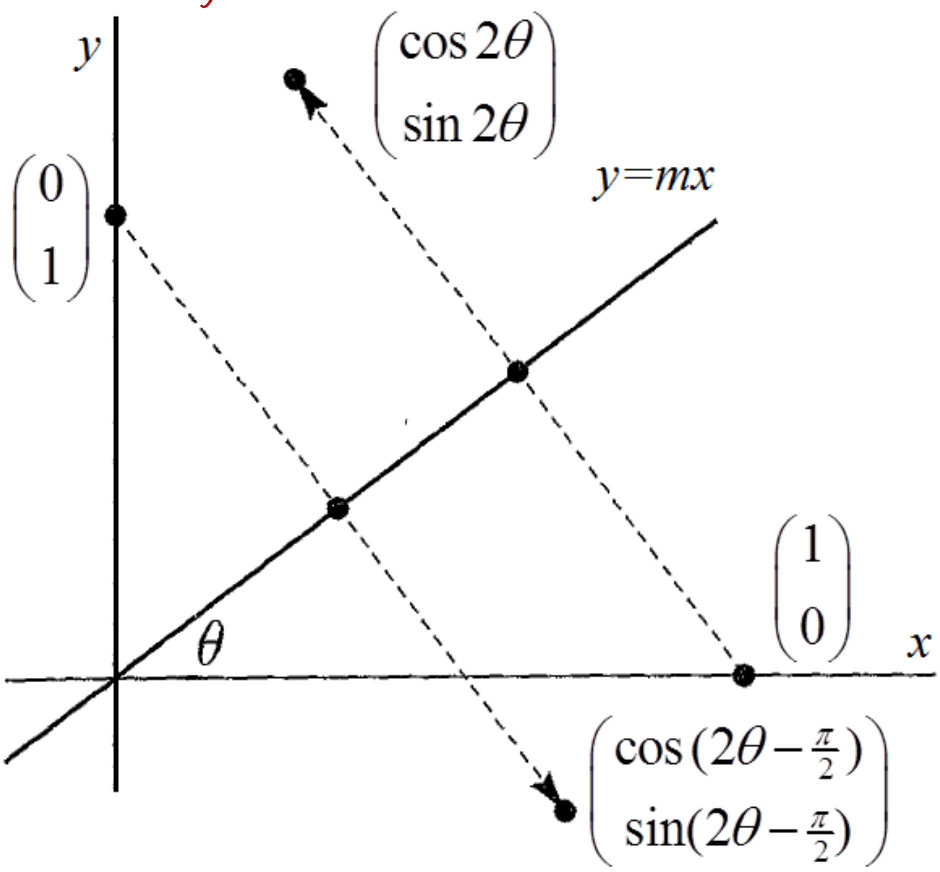
$$\begin{pmatrix}1\\0\end{pmatrix}=\frac{1}{1+m^2}\begin{pmatrix}1\\m\end{pmatrix}-\frac{m}{1+m^2}\begin{pmatrix}-m\\1\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}0\\1\end{pmatrix}=\frac{m}{1+m^2}\begin{pmatrix}1\\m\end{pmatrix}+\frac{1}{1+m^2}\begin{pmatrix}-m\\1\end{pmatrix}. \qquad (2)$$

$$\begin{aligned}
T\begin{pmatrix}1\\0\end{pmatrix}&=T\left(\frac{1}{1+m^2}\begin{pmatrix}1\\m\end{pmatrix}-\frac{m}{1+m^2}\begin{pmatrix}-m\\1\end{pmatrix}\right)\\
&=\frac{1}{1+m^2}T\begin{pmatrix}1\\m\end{pmatrix}-\frac{m}{1+m^2}T\begin{pmatrix}-m\\1\end{pmatrix}\\
&=\frac{1}{1+m^2}\begin{pmatrix}1\\m\end{pmatrix}-\frac{m}{1+m^2}\begin{pmatrix}m\\-1\end{pmatrix}\\
&=\begin{pmatrix}\frac{1-m^2}{1+m^2}\\ \frac{2m}{1+m^2}\end{pmatrix}\\
&=\frac{1-m^2}{1+m^2}\begin{pmatrix}1\\0\end{pmatrix}+\frac{2m}{1+m^2}\begin{pmatrix}0\\1\end{pmatrix}
\end{aligned}$$

$\cos 2\theta \begin{pmatrix}1\\0\end{pmatrix} + \sin 2\theta \begin{pmatrix}0\\1\end{pmatrix}$

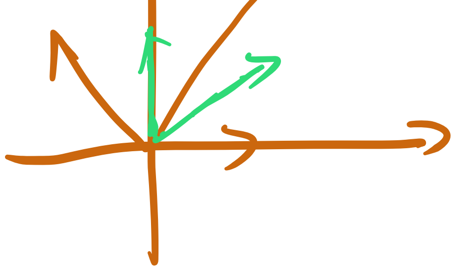
$$\begin{aligned}
T\begin{pmatrix}0\\1\end{pmatrix}&=T\left(\frac{m}{1+m^2}\begin{pmatrix}1\\m\end{pmatrix}+\frac{1}{1+m^2}\begin{pmatrix}-m\\1\end{pmatrix}\right)\\
&=\frac{m}{1+m^2}T\begin{pmatrix}1\\m\end{pmatrix}+\frac{1}{1+m^2}T\begin{pmatrix}-m\\1\end{pmatrix}\\
&=\frac{m}{1+m^2}\begin{pmatrix}1\\m\end{pmatrix}+\frac{1}{1+m^2}\begin{pmatrix}m\\-1\end{pmatrix}\\
&=\begin{pmatrix}\frac{2m}{1+m^2}\\ -\frac{1-m^2}{1+m^2}\end{pmatrix}\\
&=\frac{2m}{1+m^2}\begin{pmatrix}1\\0\end{pmatrix}-\frac{1-m^2}{1+m^2}\begin{pmatrix}0\\1\end{pmatrix}
\end{aligned}$$

$\sin 2\theta \begin{pmatrix}0\\1\end{pmatrix} - \cos 2\theta \begin{pmatrix}0\\1\end{pmatrix}$



一个V中稿 $\begin{pmatrix}0\\1\end{pmatrix}$.

一个 W中稿



Method 2: Since $m=\tan\theta$, we have $\sin\theta=\dfrac{m}{\sqrt{1+m^2}}$ and $\cos\theta=\dfrac{1}{\sqrt{1+m^2}}$.

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos 2\theta\\ \sin 2\theta\end{pmatrix}=\begin{pmatrix}\cos^2\theta-\sin^2\theta\\ 2\sin\theta\cos\theta\end{pmatrix}=\begin{pmatrix}\frac{1-m^2}{1+m^2}\\ \frac{2m}{1+m^2}\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\cos(2\theta-\frac{\pi}{2})\\ \sin(2\theta-\frac{\pi}{2})\end{pmatrix}=\begin{pmatrix}\sin 2\theta\\ -\cos 2\theta\end{pmatrix}=\begin{pmatrix}\frac{2m}{1+m^2}\\ -\frac{1-m^2}{1+m^2}\end{pmatrix}$$

Let $\beta=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ be the standard basis of $\boldsymbol{R}^2$. Then

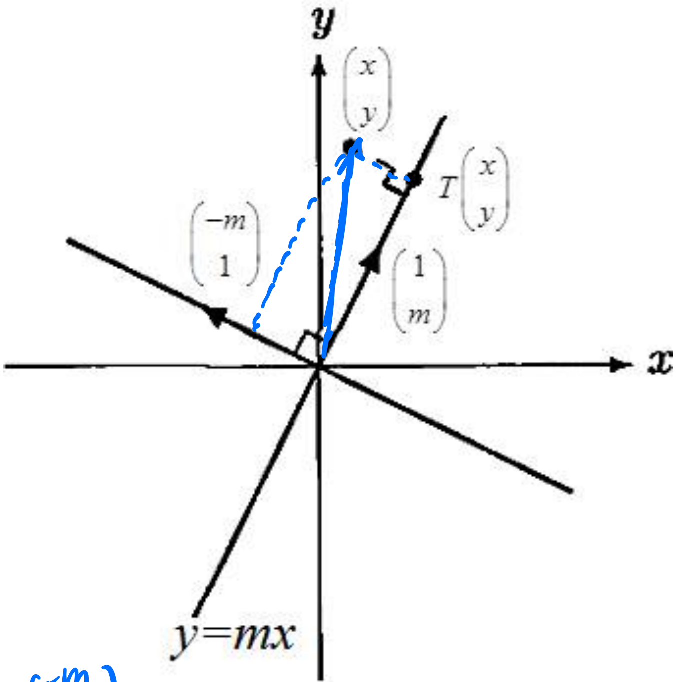
$$A=[T]_\beta=\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}=\begin{pmatrix}\dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2}\\ \dfrac{2m}{1+m^2} & -\dfrac{1-m^2}{1+m^2}\end{pmatrix}.\ \text{Hence}\ T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2}\\ \dfrac{2m}{1+m^2} & -\dfrac{1-m^2}{1+m^2}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}.$$

Example 4.2.9 (Projection) 映射

Let $T:\boldsymbol{R}^2\to\boldsymbol{R}^2$ be linear transformation defined by $T\begin{pmatrix}1\\m\end{pmatrix}=\begin{pmatrix}1\\m\end{pmatrix}$ and $T\begin{pmatrix}-m\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$. We call T **projection along** $y=mx$.

basis: $\left\{\begin{pmatrix}1\\m\end{pmatrix} \begin{pmatrix}-m\\1\end{pmatrix}\right.$

or $\left\{\begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix}\right.$



①左边操作.

$T(b)=k_1T\begin{pmatrix}1\\m\end{pmatrix}+k_2T\begin{pmatrix}-m\\1\end{pmatrix}.$

$=k_1\begin{pmatrix}1\\m\end{pmatrix}+k_2\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}k_1\\k_1m\end{pmatrix}$

$=k_1\begin{pmatrix}1\\0\end{pmatrix}+k_1m\begin{pmatrix}0\\1\end{pmatrix}$

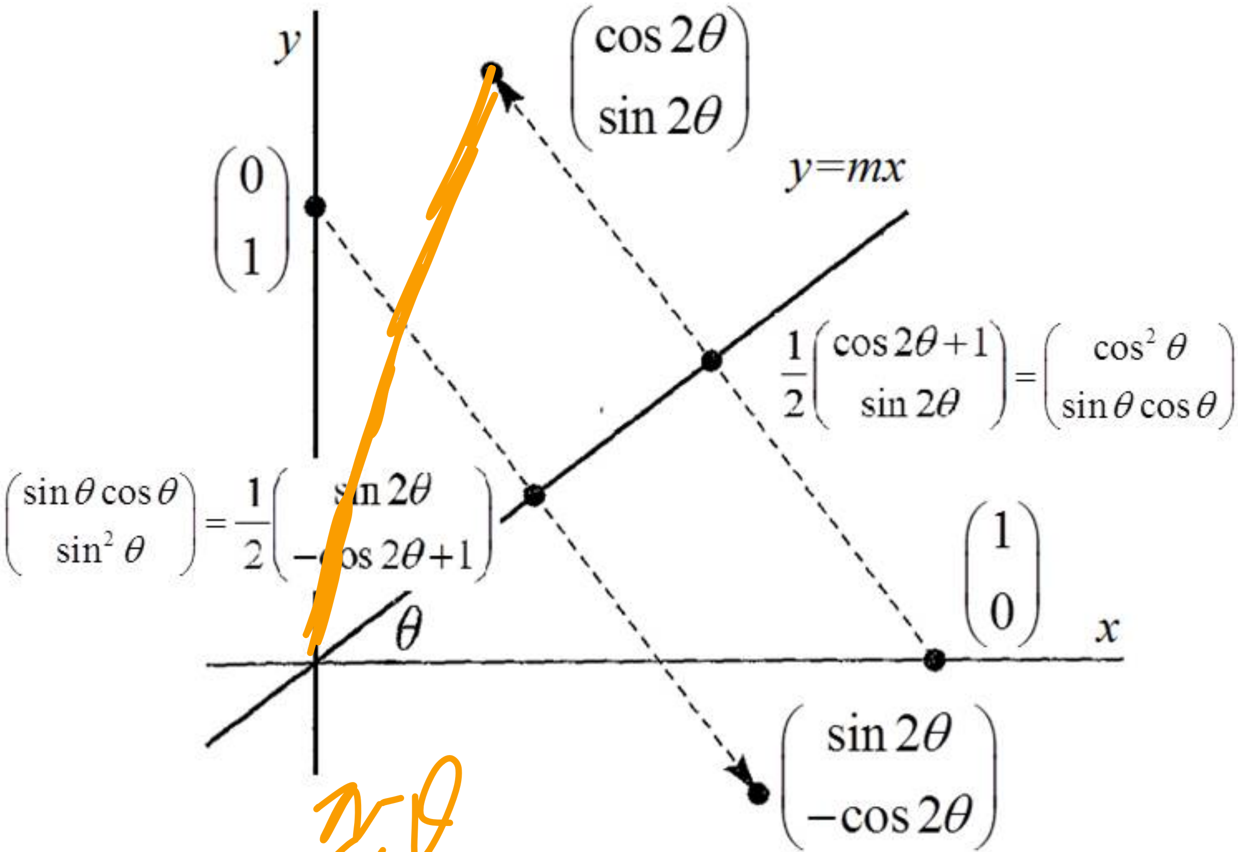
$\begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ cramer rule,

Method 1: Let $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard basis of $\boldsymbol{R}^2$. From (2),

$$
\begin{aligned}
T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= T\left(\frac{1}{1+m^2}\begin{pmatrix} 1 \\ m \end{pmatrix} - \frac{m}{1+m^2}\begin{pmatrix} -m \\ 1 \end{pmatrix} \right) \\
&= \frac{1}{1+m^2} T\begin{pmatrix} 1 \\ m \end{pmatrix} - \frac{m}{1+m^2} T\begin{pmatrix} -m \\ 1 \end{pmatrix} \\
&= \frac{1}{1+m^2}\begin{pmatrix} 1 \\ m \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{1+m^2} \\ \frac{m}{1+m^2} \end{pmatrix} \\
&= \frac{1}{1+m^2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{m}{1+m^2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= T\left(\frac{m}{1+m^2}\begin{pmatrix} 1 \\ m \end{pmatrix} + \frac{1}{1+m^2}\begin{pmatrix} -m \\ 1 \end{pmatrix} \right) \\
&= \frac{m}{1+m^2} T\begin{pmatrix} 1 \\ m \end{pmatrix} + \frac{1}{1+m^2} T\begin{pmatrix} -m \\ 1 \end{pmatrix} \\
&= \frac{m}{1+m^2}\begin{pmatrix} 1 \\ m \end{pmatrix} \\
&= \begin{pmatrix} \frac{m}{1+m^2} \\ \frac{m^2}{1+m^2} \end{pmatrix} \\
&= \frac{m}{1+m^2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{m^2}{1+m^2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

代数三角



Method 2: Since $m = \tan\theta$, we have $\sin\theta = \frac{m}{\sqrt{1+m^2}}$ and $\cos\theta = \frac{1}{\sqrt{1+m^2}}$.

$(\pi - 2\theta)$

$\cos^2\theta$

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos^2\theta\\ \sin\theta\cos\theta\end{pmatrix}=\begin{pmatrix}\frac{1}{1+m^2}\\ \frac{m}{1+m^2}\end{pmatrix} \text{ and } T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\sin\theta\cos\theta\\ \sin^2\theta\end{pmatrix}=\begin{pmatrix}\frac{m}{1+m^2}\\ \frac{m^2}{1+m^2}\end{pmatrix}$$

Then $A=[T]_\beta=\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{pmatrix}=\begin{pmatrix}\frac{1}{1+m^2} & \frac{m}{1+m^2}\\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2}\end{pmatrix}$. Hence

$$T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{1}{1+m^2} & \frac{m}{1+m^2}\\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}.$$

几何视角

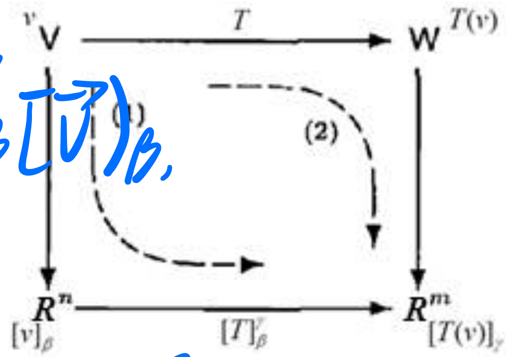
Theorem 4.2.10

Let V and W be finite dimensional vector spaces having ordered bases β and γ respectively, and let $T:V\to W$ be linear. Then, for each $\boldsymbol{v}\in V$, we have

通过 matrix 连接·

$$[T(\boldsymbol{v})]_\gamma=[T]_\beta^\gamma[\boldsymbol{v}]_\beta.$$

$[T(\vec{v})]_v = [T]^\gamma_\beta [\vec{v}]_\beta$,



$\vec{v} \longrightarrow T(\vec{v})$

$[\vec{v}]_\beta \longrightarrow T(\vec{v})_v$

坐标向量

坐标向量.

$[T(v)]_\gamma = [T]^\gamma_\beta [v]_\beta$

Example 4.2.11

Let $\beta=\{\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3\}$ be a basis of $\boldsymbol{R}^3$ where $\boldsymbol{x}_1=\begin{pmatrix}1\\1\\1\end{pmatrix},\boldsymbol{x}_2=\begin{pmatrix}1\\1\\0\end{pmatrix},\boldsymbol{x}_3=\begin{pmatrix}1\\0\\0\end{pmatrix}$. Define $T:\boldsymbol{R}^3\to\boldsymbol{R}^3$ the linear transformation by $T(c_1\boldsymbol{x}_1+c_2\boldsymbol{x}_2+c_3\boldsymbol{x}_3)=(c_1+c_2+c_3)\boldsymbol{x}_1+(2c_1+c_3)\boldsymbol{x}_2-(2c_2+c_3)\boldsymbol{x}_3$.

We have

$$T(\boldsymbol{x}_1)=\boldsymbol{x}_1+2\boldsymbol{x}_2+0\boldsymbol{x}_3,\quad T(\boldsymbol{x}_2)=\boldsymbol{x}_1+0\boldsymbol{x}_2-2\boldsymbol{x}_3,\quad T(\boldsymbol{x}_3)=\boldsymbol{x}_1+\boldsymbol{x}_2-\boldsymbol{x}_3$$

$[T(v)]_\beta = [T]_\beta \cdot [\vec{v}]_\beta$

$\begin{bmatrix}c_1+c_2+c_3\\ 2c_1+c_3\\ -2c_2-c_3\end{bmatrix} = [T]_\beta \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} \Rightarrow \begin{bmatrix}1&1&1\\2&0&1\\0&-2&-1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$

两大思路: ① T定义直接视角

② 坐标矩阵思路.

Then $[T]_\beta = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix}$.

$$\begin{pmatrix} 7 \\ 5 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \left[\begin{pmatrix} 7 \\ 5 \\ 2 \end{pmatrix}\right]_\beta = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

$$\left[T\begin{pmatrix} 7 \\ 5 \\ 2 \end{pmatrix}\right]_\beta = [T]_\beta \left[\begin{pmatrix} 7 \\ 5 \\ 2 \end{pmatrix}\right]_\beta = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ -8 \end{pmatrix}$$

$T\begin{pmatrix} 7 \\ 5 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 7 \\ 6 \\ -8 \end{pmatrix}$ but $T\begin{pmatrix} 7 \\ 5 \\ 2 \end{pmatrix} = 7\boldsymbol{x}_1 + 6\boldsymbol{x}_2 - 8\boldsymbol{x}_3$.

On the other hand, you may calculate $T\begin{pmatrix} 7 \\ 5 \\ 2 \end{pmatrix}$ directly.

$$\begin{aligned} T\begin{pmatrix} 7 \\ 5 \\ 2 \end{pmatrix} &= T\left(2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \\ &= 2T(\boldsymbol{x}_1) + 3T(\boldsymbol{x}_2) + 2T(\boldsymbol{x}_3) \\ &= 2(\boldsymbol{x}_1 + 2\boldsymbol{x}_2) + 3(\boldsymbol{x}_1 - 2\boldsymbol{x}_3) + 2(\boldsymbol{x}_1 + \boldsymbol{x}_2 - \boldsymbol{x}_3) \\ &= 7\boldsymbol{x}_1 + 6\boldsymbol{x}_2 - 8\boldsymbol{x}_3 \end{aligned}$$

$UT \neq TU$ 有顺序

$V \xrightarrow{T} W \xrightarrow{U} Z$

Theorem 4.2.12

Let V, W and Z be vector spaces. Let $T: V \to W$ and $U: W \to Z$ be linear transformations. Then $\ker(T) \subseteq \ker(UT)$. Equality holds if U is injective (by Remark 4.1.22, i.e., $\ker(U) = \{\mathbf{0}_W\}$).

$\ker T) \leq \ker(UT)$

① $\forall \vec{v} \in \ker T$ $T(\vec{v}) = \vec{0_W}$

$UT(\vec{v}) = U(\vec{0_W}) = 0_Z \in \ker UT$

Theorem 4.2.13

Let V, W and Z be finite dimensional vector spaces with ordered bases α, β and γ respectively. Let $T: V \to W$ and $U: W \to Z$ be linear transformations. Then,

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta.$$

$A \subseteq B$

$\forall \vec{v} \in A \Rightarrow \vec{v} \in B$

逻辑.

$\ker U = \ker T$

② $\forall \vec{v} \in \ker UT$ $UT(\vec{v}) = \vec{0_Z}$

Since $\ker U = \{\vec{0_W}\}$ we have

If U is injective

$\ker U = \{\vec{0}\}$　　$T(\vec{v}) = 0_W \Rightarrow \vec{v} \in \ker T$

Corollary 4.2.14

Let V be finite dimensional vector spaces with an ordered basis β, and let $T, U : V \to V$ be linear. Then $[UT]_\beta = [U]_\beta [T]_\beta$ and $\left[T^n\right]_\beta = \left([T]_\beta\right)^n$ for all positive integer n.

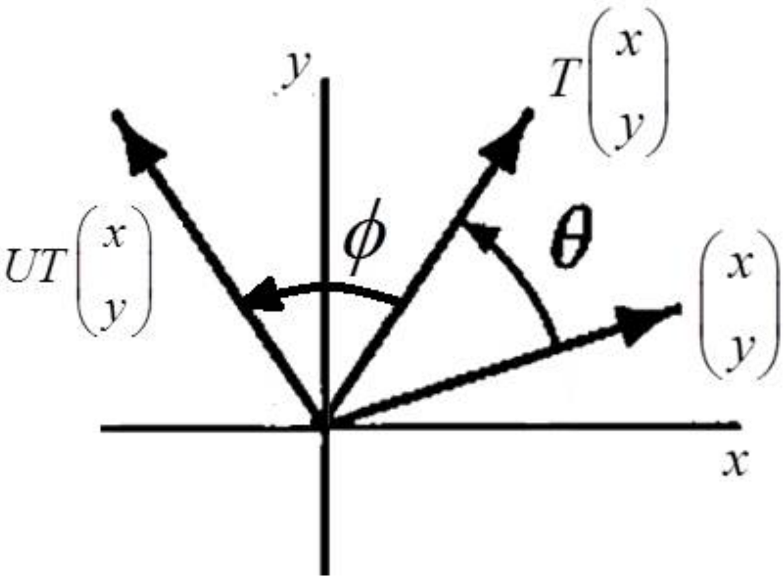
Example 4.2.15 (Rotation)

Let $T, U : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be rotation mapping defined by counterclockwise rotating vector $\begin{pmatrix} x \\ y \end{pmatrix}$ by angle θ and ϕ respectively. Let $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard basis of $\boldsymbol{R}^2$. We have

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad U\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Clearly, $UT : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is rotation mapping defined by counterclockwise rotating vector $\begin{pmatrix} x \\ y \end{pmatrix}$ by angle $\theta + \phi$.

$$\boldsymbol{R}^2 \xrightarrow[\text{rotate } \theta]{T} \boldsymbol{R}^2 \xrightarrow[\text{rotate } \phi]{U} \boldsymbol{R}^2$$



Then $UT\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

$$
\begin{aligned}
[UT]_\beta &= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\sin\theta\cos\phi - \cos\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{pmatrix} \\
&\boxed{= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} \\
&= [U]_\beta [T]_\beta .
\end{aligned}
$$

Example 4.2.16

Let $T: P_4 \to P_3$ be differentiation mapping at t defined by $T(p(x)) = p'(t) = \left.\dfrac{dp}{dx}\right|_{x=t}$ for all $p(x) \in P_4$, and $U: P_3 \to P_4$ be integration mapping defined by $U(q(t)) = \int_1^x q(t)dt$ for all $q(t) \in P_3$. Let $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \{1, t, t^2\}$ be the standard bases of P_4 and P_3 respectively. We have

$$T(1) = 0, \quad T(x) = 1, \quad T(x^2) = 2t, \quad T(x^3) = 3t^2$$

$$U(1) = \int_1^x 1dt = -1 + x, \quad U(t) = \int_1^x tdt = \left[\frac{t^2}{2}\right]_1^x = -\frac{1}{2} + \frac{1}{2}x^2, \quad U(t^2) = \int_1^x t^2 dt = \left[\frac{t^3}{3}\right]_1^x = -\frac{1}{3} + \frac{1}{3}x^3$$

Then $[T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ and $[U]_\gamma^\beta = \begin{bmatrix} -1 & -\frac{1}{2} & -\frac{1}{3} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$.

Consider $UT: P_4 \to P_4$.

$$P_4 \xrightarrow{T} P_3 \xrightarrow{U} P_4$$

$UT(1) = U(0) =$ ______________________,

$UT(x) = U(1) =$ ______________________,

$UT(x^2) = U(2t) = 2U(t) =$ ______________________,

$UT(x^3) = U(3t^2) = 3U(t^2) =$ ______________________

若要 $UT = TU \Rightarrow \begin{cases} \text{法则相同} \\ \text{空间相同} \end{cases}$

On the other hand, you may get the answers by Fundamental theorem of Calculus.

$$UT(1)=\int_1^x \frac{d}{dt}1dt=0,\quad UT(x)=\int_1^x \frac{d}{dt}tdt=\int_1^x 1dt=-1+x,\quad UT(x^2)=\int_1^x \frac{d}{dt}t^2dt=\int_1^x 1dt^2=-1+x^2$$

$$UT(x^2)=\int_1^x \frac{d}{dt}t^3dt=\int_1^x 1dt^3=-1+x^3$$

$[UT]_\beta =$

Consider $TU:P_3\to P_3$.

$$P_3 \xrightarrow{U} P_4 \xrightarrow{T} P_3$$

$$TU(1)=T(-1+x)=1,\quad TU(t)=T\left(-\frac{1}{2}+\frac{1}{2}x^2\right)=t,\quad TU(t^2)=T\left(-\frac{1}{3}+\frac{1}{3}x^3\right)=t^2$$

On the other hand, you may get the answers by Fundamental theorem of Calculus.

$$TU(1)=\frac{d}{dx}\int_1^x 1dt=1,\quad TU(t)=\frac{d}{dx}\int_1^x tdt=x\,|_t=t,\quad TU(t^2)=\frac{d}{dx}\int_1^x t^2dt=x^2\,|_t=t^2$$

Then $[TU]_\gamma=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

It is easy to see that $[T]_\beta^\gamma[U]_\gamma^\beta=\begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix}\begin{pmatrix}-1&-\frac{1}{2}&-\frac{1}{3}\\1&0&0\\0&\frac{1}{2}&0\\0&0&\frac{1}{3}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=[TU]_\gamma$ and

$$[U]_\gamma^\beta[T]_\beta^\gamma=\begin{pmatrix}-1&-\frac{1}{2}&-\frac{1}{3}\\1&0&0\\0&\frac{1}{2}&0\\0&0&\frac{1}{3}\end{pmatrix}\begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix}=\begin{pmatrix}0&-1&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=[UT]_\beta.$$

Clearly, $\ker(T)=\boldsymbol{R}$, $\ker(U)=0$. Also, $\ker(UT)=\boldsymbol{R}$. This verifies Theorem 4.2.12.

$$\begin{cases}\ker T \subseteq \ker UT\\ \ker U \subseteq \ker UT\end{cases}$$

Example 4.2.17

Let $T: P_3 \to P_3$ be differentiation mapping defined by $T(p(x)) = p'(x)$ for all $p(x) \in P_3$, i.e., $T(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$. Let $\beta = \{1, x, x^2\}$ be the standard basis of P_3. We have

$$T(1) = 0, \quad T(x) = 1, \quad T(x^2) = 2x$$

Then $[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

Hence $\left[T^3\right]_\beta = \left([T]_\beta\right)^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and

$$[T^3(a_0 + a_1x + a_2x^2)]_\beta = \left[T^3\right]_\beta [a_0 + a_1x + a_2x^2]_\beta = \left([T]_\beta\right)^3 \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

That means T^3 is a zero transformation and it sends all degree 2 polynomials to 0. This can be explained by Calculus. Derivative of a degree 2 polynomial 3 times must be 0, i.e.,

$$\frac{d^3}{dx^3}(a_0 + a_1x + a_2x^2) = 0.$$

Example 4.2.18 (Rotation)

Let $T: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be rotation mapping defined by counterclockwise rotating vector $\begin{pmatrix} x \\ y \end{pmatrix}$ by angle θ. Let $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard basis of $\boldsymbol{R}^2$. We have

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $U = T^n : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be rotation mapping defined by counterclockwise rotating vector $\begin{pmatrix} x \\ y \end{pmatrix}$ by angle $\underbrace{\theta + \theta + \cdots + \theta}_{n \text{ times}} = n\theta$ where n is a positive integer.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$

$$\underbrace{\boldsymbol{R}^2 \xrightarrow[\text{rotate }\theta]{T} \boldsymbol{R}^2 \xrightarrow[\text{rotate }\theta]{T} \cdots \xrightarrow[\text{rotate }\theta]{T} \boldsymbol{R}^2 \xrightarrow[\text{rotate }\theta]{T} \boldsymbol{R}^2}_{n \text{ times}} \quad = \quad \boldsymbol{R}^2 \xrightarrow[\text{rotate } n\theta]{U} \boldsymbol{R}^2$$

We have $U\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Then $\begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} = [U]_\beta = [T^n]_\beta = ([T]_\beta)^n = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^n$.

Example 4.2.19

Define $T : span\{e^x \sin x, e^x \cos x\} \to span\{e^x \sin x, e^x \cos x\}$ by $T(f(x)) = f'(x)$. Let $\beta = \{e^x \sin x, e^x \cos x\}$ be the basis. We have

$T(e^x \sin x) = (e^x \sin x)' = e^x \sin x + e^x \cos x$ and $T(e^x \cos x) = (e^x \cos x)' = -e^x \sin x + e^x \cos x$.

Then $[T]_\beta = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \sqrt{2}\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}$.

calculus 角度：一次一次求导

linear algebra: 对全局的把控.

非常巧妙的一步，利用三角减少运算次数并且利用结论

$$\begin{aligned}
[T^5]_\beta &= ([T]_\beta)^5 \\
&= \left(\sqrt{2}\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}\right)^5 \\
&= \sqrt{2}^5 \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}^5 \\
&= 4\sqrt{2}\begin{pmatrix} \cos\frac{5\pi}{4} & -\sin\frac{5\pi}{4} \\ \sin\frac{5\pi}{4} & \cos\frac{5\pi}{4} \end{pmatrix} \\
&= 4\sqrt{2}\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} -4 & 4 \\ -4 & -4 \end{pmatrix}.
\end{aligned}$$

great

$[(e^x \sin x)^5]_\beta = [T^5(\vec{v})]_\beta$

$= [T^5]_\beta [\vec{v}]_\beta$

$[T^5(e^x \sin x)]_\beta = [T^5]_\beta [e^x \sin x]_\beta = \begin{pmatrix} -4 & 4 \\ -4 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$,

i.e., $\frac{d^5}{dx^5}(e^x \sin x) = T^5(e^x \sin x) = -4e^x \sin x - 4e^x \cos x$

$$\left[T^5(e^x\cos x)\right]_\beta=\left[T^5\right]_\beta[e^x\cos x]_\beta=\begin{pmatrix}-4 & 4\\ -4 & -4\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}4\\-4\end{pmatrix},$$

i.e., $\dfrac{d^5}{dx^5}(e^x\cos x)=T^5(e^x\cos x)=4e^x\sin x-4e^x\cos x.$

These can be explained by Calculus. Recall

$(\sin x)'=\cos x\quad(\sin x)''=-\sin x\quad(\sin x)'''=-\cos x\quad(\sin x)^{(4)}=\sin x\quad(\sin x)^{(5)}=\cos x$ and
$(\cos x)'=-\sin x\quad(\cos x)''=-\cos x\quad(\cos x)'''=\sin x\quad(\cos x)^{(4)}=\cos x\quad(\cos x)^{(5)}=-\sin x$

and Leibniz's rule

$$(f\cdot g)^{(n)}=\sum_{k=0}^{n}\binom{n}{k}f^{(n-k)}g^{(k)}.$$

$$\begin{aligned}\frac{d^5(e^x\sin x)}{dx^5}&=\left(\frac{d^5e^x}{dx^5}\right)\sin x+5\left(\frac{d^4e^x}{dx^4}\right)\left(\frac{d\sin x}{dx}\right)+10\left(\frac{d^3e^x}{dx^3}\right)\left(\frac{d^2\sin x}{dx^2}\right)+10\left(\frac{d^2e^x}{dx^2}\right)\left(\frac{d^3\sin x}{dx^3}\right)\\&\quad+5\left(\frac{de^x}{dx}\right)\left(\frac{d^4\sin x}{dx^4}\right)+e^x\left(\frac{d^5\sin x}{dx^5}\right)\\&=e^x\sin x+5e^x\cos x-10e^x\sin x-10e^x\cos x+5e^x\sin x+e^x\cos x\\&=-4e^x\sin x-4e^x\cos x\end{aligned}$$

$$\begin{aligned}\frac{d^5(e^x\cos x)}{dx^5}&=\left(\frac{d^5e^x}{dx^5}\right)\cos x+5\left(\frac{d^4e^x}{dx^4}\right)\left(\frac{d\cos x}{dx}\right)+10\left(\frac{d^3e^x}{dx^3}\right)\left(\frac{d^2\cos x}{dx^2}\right)+10\left(\frac{d^2e^x}{dx^2}\right)\left(\frac{d^3\cos x}{dx^3}\right)\\&\quad+5\left(\frac{de^x}{dx}\right)\left(\frac{d^4\cos x}{dx^4}\right)+e^x\left(\frac{d^5\cos x}{dx^5}\right)\\&=e^x\cos x-5e^x\sin x-10e^x\cos x+10e^x\sin x+5e^x\cos x-e^x\sin x\\&=4e^x\sin x-4e^x\cos x\end{aligned}$$

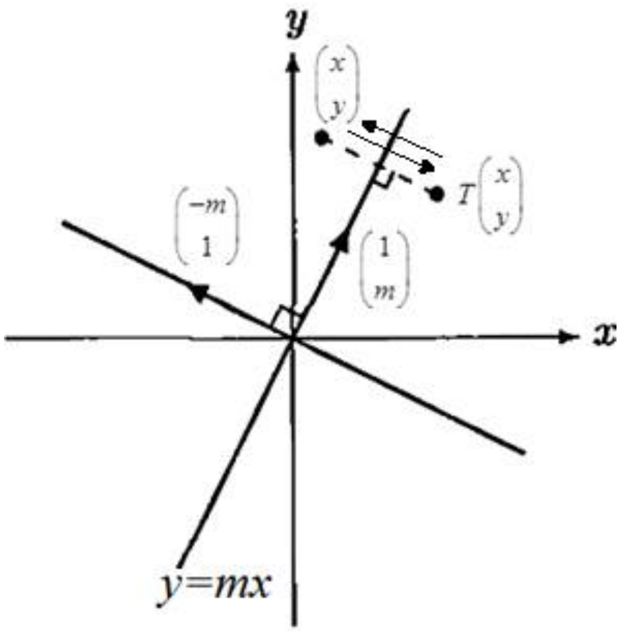
Example 4.2.20 (Reflection)

Let T be **reflection along** $y=mx$, i.e., $T:\boldsymbol{R}^2\to\boldsymbol{R}^2$ a linear transformation defined by $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2}\\ \dfrac{2m}{1+m^2} & -\dfrac{1-m^2}{1+m^2}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$. Let $\beta=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ be the standard basis of $\boldsymbol{R}^2$.

$$\left[T^2\right]_\beta=\left([T]_\beta\right)^2$$
$$=\begin{pmatrix}\frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2}\\ \frac{2m}{1+m^2} & -\frac{1-m^2}{1+m^2}\end{pmatrix}\begin{pmatrix}\frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2}\\ \frac{2m}{1+m^2} & -\frac{1-m^2}{1+m^2}\end{pmatrix}$$
$$=\begin{pmatrix}\frac{(1-m^2)^2+(2m)^2}{(1+m^2)^2} & \frac{2m(1-m^2)-(1-m^2)2m}{(1+m^2)^2}\\ \frac{2m(1-m^2)-(1-m^2)2m}{(1+m^2)^2} & \frac{(1-m^2)^2+(2m)^2}{(1+m^2)^2}\end{pmatrix}$$
$$=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

几何意义

Hence $T^2=I$ and $T=T^{-1}$. That means, reflection twice will go back to the original position.



Example 4.2.21 (Projection)

Let T be **projection along** $y=mx$, i.e., $T:\boldsymbol{R}^2\to\boldsymbol{R}^2$ a linear transformation defined by $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{1}{1+m^2} & \frac{m}{1+m^2}\\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$. Let $\beta=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ be the standard basis of $\boldsymbol{R}^2$.

$$
\begin{aligned}
\left[T^2\right]_\beta &= ([T]_\beta)^2 \\
&= \begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{pmatrix}\begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1+m^2}{(1+m^2)^2} & \frac{m+m^3}{(1+m^2)^2} \\ \frac{m+m^3}{(1+m^2)^2} & \frac{m^2+m^4}{(1+m^2)^2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{pmatrix} = [T]_\beta
\end{aligned}
$$

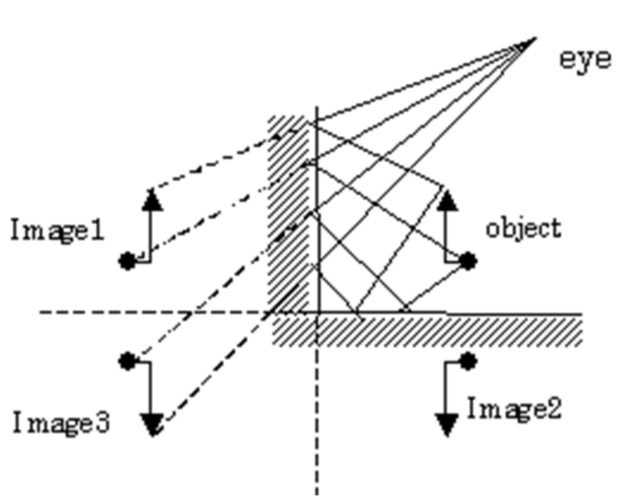
Hence $T^2 = T$. That means projection of a shadow is shadow itself.

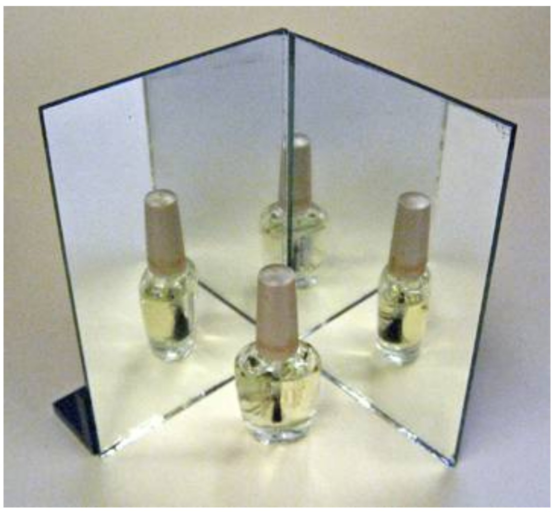
Example 4.2.22

Let $T : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be the reflection about x-axis and $U : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be the reflection about y-axis, i.e., $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$ and $U\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$.

Then $UT\begin{pmatrix} x \\ y \end{pmatrix} = U\begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$ and $TU\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} -x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$. Hence $UT = TU$.

reflect about x-axis then reflect about y-axis = reflect about y-axis then reflect about x-axis





T is invertible.
$\Updownarrow$
$[T]_\beta^\gamma$ is invertible
det / 梯形最簡式.

T is invertible

$\Updownarrow$

$[T]^{\gamma}_{\beta}$ is invertible $\Rightarrow$ det/梯形最简形式.

Theorem 4.2.23

Let V and W be finite dimensional vector spaces having ordered bases β and γ respectively, and let $T: V \to W$ be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Furthermore, if T is invertible, then $T^{-1}: W \to V$ is linear, $\dim V = \dim W$ and $\left[T^{-1}\right]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$.

Example 4.2.24

Define $T: span\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\} \to span\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ by $T(\boldsymbol{v}) = \begin{pmatrix}1 & 1 & 1 & 1\\1 & 0 & 0 & 0\\1 & 0 & 0 & 0\end{pmatrix}\boldsymbol{v}$. Let $\beta = \left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\}$ and $\gamma = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$. Show that T is invertible. Find $T^{-1}\begin{pmatrix}1\\0\\0\end{pmatrix}$, $T^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $T^{-1}\begin{pmatrix}a\\b\\b\end{pmatrix}$.

Solution

$[T]^{\gamma}_{\beta}$ 可逆 $\Rightarrow [T^{-1}]^{\gamma}_{\beta} = [T]^{\beta}_{\gamma}$

We have

$$T\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1 & 1 & 1 & 1\\1 & 0 & 0 & 0\\1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} = 0\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}1\\1\\1\end{pmatrix} \text{ and } T\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1 & 1 & 1 & 1\\1 & 0 & 0 & 0\\1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}4\\1\\1\end{pmatrix} = 3\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

Then $[T]_\beta^\gamma = \begin{pmatrix}0 & 3\\1 & 1\end{pmatrix}$.

Since $\det[T]_\beta^\gamma = \det\begin{pmatrix}0 & 3\\1 & 1\end{pmatrix} = -3 \neq 0$, T is invertible, i.e. $T^{-1}: span\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \to span\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\}$

$T^{-1}\begin{pmatrix}1\\0\\0\end{pmatrix} = c_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + c_2\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

exists.

$\Downarrow$ $\begin{pmatrix}1\\0\\0\end{pmatrix} = c_1 T\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + c_2 T\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

By Theorem 4.2.23, $\left[T^{-1}\right]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1} = \begin{pmatrix}0 & 3\\1 & 1\end{pmatrix}^{-1} = \frac{-1}{3}\begin{pmatrix}1 & -3\\-1 & 0\end{pmatrix} = \begin{pmatrix}-\frac{1}{3} & 1\\\frac{1}{3} & 0\end{pmatrix}$.

$$T^{-1}\begin{pmatrix}1\\0\\0\end{pmatrix} = c_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + c_2\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$\Downarrow$

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = c_1 T\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + c_2 T\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$$\begin{tikzcd} \overset{\beta}{V} \arrow[r, "T"] & \overset{\gamma}{W} \arrow[r, "\tilde{T}"] & \overset{\beta}{V} \\ \vec{v} \arrow[r] & T(\vec{v}) \arrow[r] & \vec{v} \end{tikzcd}$$

when T is invertible, we have exist $\tilde{T}$ such that $\tilde{T}\cdot T = I_n$, $[\tilde{T}T]_\beta = [\tilde{T}]^\beta_\gamma [T]^\gamma_\beta = [I_n] = [I]_\beta$

$T(\vec{v}_1) = \vec{v}_1$ ......
$T(\vec{v}_n) = \vec{v}_n$ $\Rightarrow [I]_\beta = I_n$

Then $T^{-1}\begin{pmatrix}1\\0\\0\end{pmatrix} = -\frac{1}{3}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{pmatrix}$ and $T^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$.

$$T^{-1}\begin{pmatrix}a\\b\\b\end{pmatrix} = T^{-1}\left(\begin{pmatrix}a-b\\0\\0\end{pmatrix} + \begin{pmatrix}b\\b\\b\end{pmatrix}\right) = (a-b)T^{-1}\begin{pmatrix}1\\0\\0\end{pmatrix} + bT^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\\frac{1}{3}(a-b)\\\frac{1}{3}(a-b)\\\frac{1}{3}(a-b)\end{pmatrix} + \begin{pmatrix}b\\0\\0\\0\end{pmatrix} = \begin{pmatrix}b\\\frac{1}{3}(a-b)\\\frac{1}{3}(a-b)\\\frac{1}{3}(a-b)\end{pmatrix}.$$

根据已有条件进行配凑.

Corollary 4.2.25

Let V be finite dimensional vector spaces with an ordered basis β, and let $T: V \to V$ be linear. Then T is invertible if and only if $[T]_\beta$ is invertible. Furthermore, $\left[T^{-1}\right]_\beta = \left([T]_\beta\right)^{-1}$.

△: 只是将W换成V的特殊情况罢了.

Example 4.2.26 (Rotation)

Let $T: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be rotation mapping defined by counterclockwise rotating vector $\begin{pmatrix}x\\y\end{pmatrix}$ by angle θ. Let $\beta = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ be the standard basis of $\boldsymbol{R}^2$. Then

$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}.$$

Let $U: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be rotation mapping defined by counterclockwise rotating vector $\begin{pmatrix}x\\y\end{pmatrix}$ by angle $-\theta$, i.e., clockwise rotating vector $\begin{pmatrix}x\\y\end{pmatrix}$ by angle θ. Then

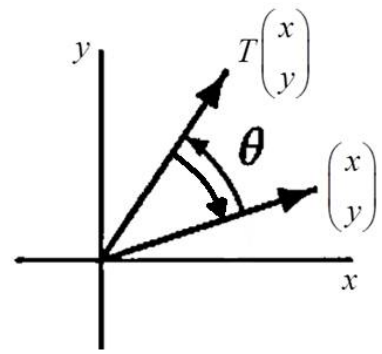
$$U\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}.$$

Given $[T]^\gamma_\beta$ is invertible, let $A = \left[[T]^\gamma_\beta\right]^{-1} = [\tilde{T}]^\gamma_\beta$

$\vec{v} \in W$ $\quad \vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots\cdots c_n\vec{u}_n$

$\tilde{T}(\vec{v}) = c_1\tilde{T}[\vec{u}_1] + \cdots\cdots + c_n\tilde{T}[\vec{u}_n]$

$T(\vec{v_1}) = \vec{v_1} + 0 + 0 + \cdots + 0$
$T(\vec{v_n}) = 0 + 0 + \cdots + \vec{v_n}$ } 通过 matrix 建立联系



Clearly, $U = T^{-1}$.

$$\mathbf{R}^2 \xrightarrow[\text{rotate } \theta]{T} \mathbf{R}^2 \xrightarrow[\text{rotate } -\theta]{U} \mathbf{R}^2 \quad \text{and} \quad \mathbf{R}^2 \xrightarrow[\text{rotate } -\theta]{U} \mathbf{R}^2 \xrightarrow[\text{rotate } \theta]{T} \mathbf{R}^2$$

$$\left[T^{-1}\right]_\beta = [U]_\beta = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} = \left([T]_\beta\right)^{-1}.$$

Example 4.2.27

$[T]_\beta \neq 0 \Leftrightarrow ad - bc \neq 0 \Rightarrow \det \neq 0$

Let $T: M_{2\times 2} \to M_{2\times 2}$ be linear transformation defined by $T(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} A$ for all $A \in M_{2\times 2}$. Let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be the standard basis of $M_{2\times 2}$. We have

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$$

$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}$$

$$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$$

$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$$

Then $[T]_\beta = \left(\begin{array}{cc|cc} a & b & 0 & 0 \\ c & d & 0 & 0 \\ \hline 0 & 0 & a & b \\ 0 & 0 & c & d \end{array}\right)$.

Solve $[T]^{-1}_\beta$:
1. $\to [T]_\beta^{-1}$
2. $\to T^{-1} \longrightarrow [T^{-1}]_\beta$

didn't understand.

Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible. Let $U : M_{2\times 2} \to M_{2\times 2}$ be linear transformation defined by $U(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} A = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} A$ for all $A \in M_{2\times 2}$.

Clearly, $U = T^{-1}$.

逻辑

$$\begin{array}{ccccc} M_{2\times 2} & \xrightarrow{T} & M_{2\times 2} & \xrightarrow{U} & M_{2\times 2} \\ A & & \begin{pmatrix} a & b \\ c & d \end{pmatrix} A & & \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} A = A \end{array}$$ and

$$\begin{array}{ccccc} M_{2\times 2} & \xrightarrow{U} & M_{2\times 2} & \xrightarrow{T} & M_{2\times 2} \\ A & & \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} A & & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} A = A \end{array}$$

$$\left[T^{-1}\right]_\beta = [U]_\beta = \left(\begin{array}{cc|cc} \frac{d}{ad-bc} & \frac{-b}{ad-bc} & 0 & 0 \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ \hline 0 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 0 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right) = \left(\begin{array}{c|c} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \end{array}\right) = \left(\begin{array}{cc|cc} a & b & 0 & 0 \\ c & d & 0 & 0 \\ \hline 0 & 0 & a & b \\ 0 & 0 & c & d \end{array}\right)^{-1} = \left([T]_\beta\right)^{-1}.$$

Example 4.2.28

Let $T : P_3 \to P_3$ be linear transformation defined by $T(p(x)) = p(x+1)$ for all $p(x) \in P_3$, i.e., $T(a_0 + a_1 x + a_2 x^2) = a_0 + a_1(x+1) + a_2(x+1)^2 = (a_0 + a_1 + a_2) + (a_1 + 2a_2)x + a_2 x^2$. Let $\beta = \{1, x, x^2\}$ be the standard basis of P_3. We have

$$T(1) = 1, \quad T(x) = 1 + x, \quad T(x^2) = (1+x)^2 = 1 + 2x + x^2.$$

Then $[T]_\beta = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

$\overset{\beta}{V} \xrightarrow{T} \overset{\gamma}{W} \xrightarrow{U} \overset{\alpha}{H}$.

1. $[T]_\beta^\gamma =$ 2. $[T^{-1}]_V^\beta = \left([T]_\beta^\gamma\right)^{-1}$

3. $[UT]_\beta^\alpha = [U]_\gamma^\alpha [T]_\beta^\gamma$.

Consider $T^2 : P_3 \to P_3$, $T(T(p(x))) = T(p(x+1)) = p((x+1)+1) = p(x+2)$

$$\begin{array}{ccccc} P_3 & \xrightarrow{T} & P_3 & \xrightarrow{T} & P_3 \\ x & & x+1 & & (x+1)+1 = x+2 \end{array}$$

$$\overset{\beta}{V} \longrightarrow \overset{\gamma}{W} \longrightarrow \overset{\alpha}{H}$$

1. $[T]_\beta^\gamma$ 2. $[T^{-1}]_\gamma^\beta = \left[[T]_\beta^\gamma\right]^{-1}$

3. $[UT]^{\alpha}_{\beta} = [U]^{\alpha}_{\gamma}[T]^{\gamma}_{\beta}$

We have $T^2(1)=1$, $\quad T^2(x)=T(1+x)=2+x$ and

$T^2(x^2)=T(1+2x+x^2)=T(1)+2T(x)+T(x^2)=1+2(2+x)+(1+2x+x^2)=4+4x+x^2$.

Then $\left[T^2\right]_\beta=\begin{pmatrix}1&2&4\\0&1&4\\0&0&1\end{pmatrix}$.

$$\left([T]_\beta\right)^2=\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}=\begin{pmatrix}1&2&4\\0&1&4\\0&0&1\end{pmatrix}=\left[T^2\right]_\beta.$$

You may find $\left([T]_\beta\right)^{-1}=\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}^{-1}$ directly by row operation:

$$\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&2&0&1&0\\0&0&1&0&0&1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&1&0&1&0&-1\\0&1&0&0&1&-2\\0&0&1&0&0&1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&0&0&1&-1&1\\0&1&0&0&1&-2\\0&0&1&0&0&1\end{array}\right).$$

Hence $\left([T]_\beta\right)^{-1}=\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}^{-1}=\begin{pmatrix}1&-1&1\\0&1&-2\\0&0&1\end{pmatrix}$.

$T(p(x)) = p(x+1)$

On the other hand, you may try to find the inverse of T. Define $U:P_3\to P_3$ by $U(p(x))=p(x-1)$ for all $p(x)\in P_3$. Then

$$U(1)=1,\quad U(x)=-1+x,\quad U(x)=(-1+x)^2=1-2x+x^2.$$

Clearly, $U=T^{-1}$.

$$\underset{x}{P_3}\xrightarrow{T}\underset{x+1}{P_3}\xrightarrow{U}\underset{(x-1)+1=x}{P_3}\quad\text{and}\quad\underset{x}{P_3}\xrightarrow{U}\underset{x-1}{P_3}\xrightarrow{T}\underset{(x+1)-1=x}{P_3}$$

Hence $[U]_\beta=\left[T^{-1}\right]_\beta=\begin{pmatrix}1&-1&1\\0&1&-2\\0&0&1\end{pmatrix}$. Therefore, $\left[T^{-1}\right]_\beta=\left([T]_\beta\right)^{-1}$.

$\vec{x} \Rightarrow A\vec{x}$ standard basis

$\downarrow$ $[T]_\gamma = A$.

Section 4.3 Similarity

Definition 4.3.1

now equal.

Let A and B be $n \times n$ matrices. B is said to be **similar** to A if there exists nonsingular matrix Q such that $B = Q^{-1}AQ$.

$\Rightarrow Q \cdot B \cdot Q^{-1} = A \Rightarrow \det A = \det B$

Theorem 4.3.2

Let $T : V \to V$ be a linear operator on a finite dimensional vector space V, and let β and γ be bases of V. Suppose that Q is the transition matrix from γ to β, i.e., $Q = [I]_\gamma^\beta$. Then

$$[T]_\gamma = Q^{-1}[T]_\beta Q.$$

$\overset{\beta}{V} \xrightarrow{I} \overset{\gamma}{V}$

$[\vec{v}]_\beta = [I]_\beta^\gamma [\vec{v}]_\gamma$

$[I]_\gamma$

$\overset{\gamma}{V} \xrightarrow{I} \overset{\beta}{V}$

$[I]_\beta$ $[\vec{v}]_\beta = [I]_\gamma^\beta [\vec{v}]_\gamma$

$$\begin{tikzcd}
\text{Basis } \gamma: V \arrow[r, "T", "{[T]_\gamma}"'] \arrow[d, "I"', "Q"] & V \arrow[d, "I"', "Q"] \\
\text{Basis } \beta: V \arrow[r, "T", "{[T]_\beta}"'] & V
\end{tikzcd}$$

Proof

$Q[T]_\gamma = \underbrace{[I]_\gamma^\beta [T]_\gamma = [I \circ T]_\gamma^\beta}_{\text{Theorem 4.2.13}} = [T]_\gamma^\beta = \underbrace{[T \circ I]_\gamma^\beta = [T]_\beta [I]_\gamma^\beta}_{\text{Theorem 4.2.13}} = [T]_\beta Q.$ Hence $[T]_\gamma = Q^{-1}[T]_\beta Q.$

$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$

$Q = [I]_\gamma^\beta$

$Q^{-1} = [I]_\beta^\gamma$ $\Rightarrow$ $\det[T]_\gamma = \det[T]_\beta$

Example 4.3.3 (Reflection)

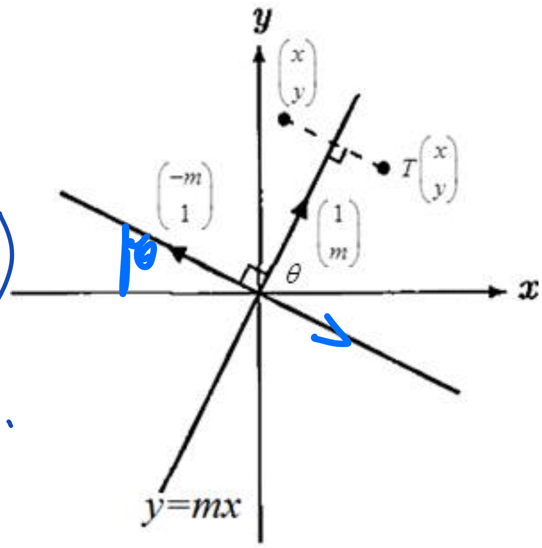
Let T be **reflection along** $y = mx$, i.e., $T : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ a linear transformation defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & -\frac{1-m^2}{1+m^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Let $\beta = \left\{ \begin{pmatrix} \frac{1}{\sqrt{1+m^2}} \\ \frac{m}{\sqrt{1+m^2}} \end{pmatrix}, \begin{pmatrix} -\frac{m}{\sqrt{1+m^2}} \\ \frac{1}{\sqrt{1+m^2}} \end{pmatrix} \right\}$ and $\gamma = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be bases of $\boldsymbol{R}^2$.

$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

$T\begin{pmatrix}1\\m\end{pmatrix} = \begin{matrix}1\\m\end{matrix}$ $\quad T\begin{pmatrix}-m\\1\end{pmatrix} = -\begin{pmatrix}-m\\1\end{pmatrix} = \begin{pmatrix}m\\-1\end{pmatrix}$

$\beta \to \gamma$

$\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} = \cos\theta\begin{pmatrix}1\\0\end{pmatrix} + \sin\begin{pmatrix}0\\1\end{pmatrix}.$



Since $m=\tan\theta$, we have $\sin\theta=\dfrac{m}{\sqrt{1+m^2}}$ and $\cos\theta=\dfrac{1}{\sqrt{1+m^2}}$. Then $\beta=\left\{\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix},\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\right\}$.

Clearly, $Q^{-1}=[I]_\beta^\gamma=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ and $Q=[I]_\gamma^\beta=\begin{pmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$.

Clearly, $[T]_\gamma=\begin{pmatrix}\dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2}\\ \dfrac{2m}{1+m^2} & -\dfrac{1-m^2}{1+m^2}\end{pmatrix}=\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$.

$$T\begin{pmatrix}\dfrac{1}{\sqrt{1+m^2}}\\ \dfrac{m}{\sqrt{1+m^2}}\end{pmatrix}=\frac{1}{\sqrt{1+m^2}}T\begin{pmatrix}1\\m\end{pmatrix}=\frac{1}{\sqrt{1+m^2}}\begin{pmatrix}1\\m\end{pmatrix}=\begin{pmatrix}\dfrac{1}{\sqrt{1+m^2}}\\ \dfrac{m}{\sqrt{1+m^2}}\end{pmatrix}\quad\text{and}$$

$$T\begin{pmatrix}-\dfrac{m}{\sqrt{1+m^2}}\\ \dfrac{1}{\sqrt{1+m^2}}\end{pmatrix}=\frac{1}{\sqrt{1+m^2}}T\begin{pmatrix}-m\\1\end{pmatrix}=\frac{-1}{\sqrt{1+m^2}}\begin{pmatrix}-m\\1\end{pmatrix}=-\begin{pmatrix}-\dfrac{m}{\sqrt{1+m^2}}\\ \dfrac{1}{\sqrt{1+m^2}}\end{pmatrix}.$$

Then $[T]_\beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

$$Q^{-1}[T]_\beta Q=\overbrace{\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}}^{\text{rotate }\theta\text{ counterclockwise}}\overbrace{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}^{\text{reflect along }x\text{-axis}}\overbrace{\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}}^{\text{rotate }\theta\text{ clockwise}}$$
$$=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}$$

$[b] = G_1\vec{v} + G_2A\vec{v}_2$

$\binom{0}{1} = c_3\vec{v} + c_4 A\vec{v}$

$B\binom{1}{0} = (A^2-3A+2I)\binom{1}{0}$
$= [A^2-3A+2I](c_1\vec{v} + c_2 A\vec{v})$
$= c_1(A^2-3A+2I)\cdot\vec{v} +$
$c_2(A^2-3A+2I)A\vec{v} = \vec{0}$

$$=\begin{pmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{pmatrix}=[T]_\gamma.$$

Example 4.3.4

联系线性变换.

Let A be a 2×2 matrix and $\boldsymbol{v}\in\boldsymbol{R}^2$ such that $\boldsymbol{v}$ and $A\boldsymbol{v}$ are linearly independent in $\boldsymbol{R}^2$. Suppose $A^2\boldsymbol{v}=3A\boldsymbol{v}-2\boldsymbol{v}$.

$A^2-3A+2I_2=0$

$(A-I_2)(A-2I_2)=0$

(a) Show that $A^2-3A+2I_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$.

$A\neq I_2$ $A\neq 2I_2$ $(BA=0 \not\Rightarrow B=0/A=0)$

(b) Find $\det(A+I_2)$.

(c) Find $(A+I_2)^{-1}(A\boldsymbol{v})$ in terms of $\boldsymbol{v}$ and $A\boldsymbol{v}$.

$\det(A-I_2)\cdot\det(A-2I)=0$

Solution

(a) Since $A^2\boldsymbol{v}-3A\boldsymbol{v}+2\boldsymbol{v}=\boldsymbol{0}$, $A^2(A\boldsymbol{v})-3A(A\boldsymbol{v})+2(A\boldsymbol{v})=\boldsymbol{0}$ and $\{A\boldsymbol{v},\boldsymbol{v}\}$ form a basis of $\boldsymbol{R}^2$, $(A^2-3A+2I_2)\boldsymbol{u}=\boldsymbol{0}$ for any $\boldsymbol{u}\in\boldsymbol{R}^2$. Then $A^2-3A+2I_2$ is a zero transformation, that is,

$R^2 \xrightarrow{T} R^2$
$(\vec{v}, A\vec{v})$
$\vec{x} \Rightarrow T(\vec{x})=(A^2-3A+2I)\cdot\vec{x}$
$T(\vec{v})=0$, $T(A\vec{v})=0$ $\} \Rightarrow$ zero mapping

$$A^2-3A+2I_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

(b) Define $T:\boldsymbol{R}^2\to\boldsymbol{R}^2$ a linear transformation by $T\begin{pmatrix}x\\y\end{pmatrix}=(A+I_2)\begin{pmatrix}x\\y\end{pmatrix}$. Since $\dim\boldsymbol{R}^2=2$ and $\boldsymbol{v}$ and $A\boldsymbol{v}$ are linearly independent, $\{A\boldsymbol{v},\boldsymbol{v}\}$ is a basis of $\boldsymbol{R}^2$. Let $\beta=\{A\boldsymbol{v},\boldsymbol{v}\}$ and $\gamma=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ be bases of $\boldsymbol{R}^2$. Clearly,

$R^2 \xrightarrow{T} R^2$

$\vec{v} \xrightarrow{T} (A+I_2)\vec{v}$

$$T(A\vec{v})=(A+I_2)(A\boldsymbol{v})=A^2\boldsymbol{v}+A\boldsymbol{v}=4A\boldsymbol{v}-2\boldsymbol{v}.$$
$$T(\vec{v})=(A+I_2)(\boldsymbol{v})=A\boldsymbol{v}+\boldsymbol{v}.$$

标准坐标系β下: $[T]_\beta = A+I_2$

Then $[A+I_2]_\beta=\begin{pmatrix}4&1\\-2&1\end{pmatrix}$

$$\det(A+I_2)=\det[T]_\gamma=\det\left(Q^{-1}[T]_\beta Q\right)=\det[T]_\beta=\det[A+I_2]_\beta=\det\begin{pmatrix}4&1\\-2&1\end{pmatrix}=6.$$

$T(\vec{x}) = (A+I_2)\vec{x}$

$T^{-1}(A\vec{v}) = w$

$T^{-1}(\vec{x}) = (A+I_2)^{-1}\vec{x}$

$\vec{x} = T[(A+I_2)(\vec{x})]$

$T(w) = A\vec{v}$.

(c) By Corollary 4.2.25, $\left[(A+I_2)^{-1}\right]_\beta = [A+I_2]_\beta^{-1} = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$. We have

$$\left[(A+I_2)^{-1}(A\boldsymbol{v})\right]_\beta = \left[(A+I_2)^{-1}\right]_\beta [A\boldsymbol{v}]_\beta = \frac{1}{6}\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Hence $(A+I_2)^{-1}(A\boldsymbol{v}) = \frac{1}{6}A\boldsymbol{v} + \frac{1}{3}\boldsymbol{v}$.

Remark 4.3.5

$$\begin{aligned}
&A^2\boldsymbol{v} = 3A\boldsymbol{v} - 2\boldsymbol{v} \\
&A^2\boldsymbol{v} - 3A\boldsymbol{v} + 2\boldsymbol{v} = \boldsymbol{0} \\
&(A^2 - 3A + 2I_2)\boldsymbol{v} = \boldsymbol{0} \\
&(A^2 - 3A + 2I_2)\boldsymbol{v}\boldsymbol{v}^{-1} = \boldsymbol{0}\boldsymbol{v}^{-1} \\
&A^2 - 3A + 2I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Such argument is not correct because there is no inverse for $\boldsymbol{v}$.

Even $A^2 - 3A + 2I_2 = (A - I_2)(A - 2I_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ together with $\boldsymbol{v}$ and $A\boldsymbol{v}$ linearly independent in $\boldsymbol{R}^2$, it does not mean $A = I_2$ or $A = 2I_2$. There are infinite many choices for A and $\boldsymbol{v}$ satisfying such condition but $A \neq I_2$ and $A \neq 2I_2$.

Consider $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Clearly, $\boldsymbol{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A\boldsymbol{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ are linearly independent in $\boldsymbol{R}^2$. Furthermore,

$$A^2\boldsymbol{v} = A \cdot A\boldsymbol{v} = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \end{pmatrix} = 3\begin{pmatrix} 3 \\ -2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3A\boldsymbol{v} - 2\boldsymbol{v}.$$

But $A \neq I_2$ and $A \neq 2I_2$.

In (c), keep in mind that

$$(A+I_2)^{-1}(A\boldsymbol{v}) \neq (A^{-1} + I_2^{-1})(A\boldsymbol{v}) = \boldsymbol{v} + A\boldsymbol{v}.$$

如果基底是 standard basis

那么 $T(\vec{v}) = A \cdot \vec{v}$

$$[\,]_y = A$$