PT

Solution to Assignment 4

1. If $a > 0, x \in \mathbb{R}$, then $\{\omega : aX(\omega) \le x\} = \{\omega : X(\omega) \le x/a\} \in \mathcal{A}$ since X is a random variable. If $a < 0$,

$$\{\omega : aX(\omega) \le x\} = \{\omega : X(\omega) \ge x/a\} = \left\{ \bigcup_{n \ge 1} \left\{ \omega : X(\omega) \le \frac{x}{a} - \frac{1}{n} \right\} \right\}^c$$

which lies in $\mathcal{A}$ since it is the complement of a countable union of members of $\mathcal{A}$. If $a = 0$,

$$\{\omega : aX(\omega) \le x\} = \begin{cases} \varnothing & \text{if } x < 0 \\ \Omega & \text{if } x \ge 0 \end{cases}$$

in either case, the event lies in $\mathcal{A}$.

2. Set $Y = aX + b$. We have that

$$\mathbb{P}(Y \le y) = \begin{cases} \mathbb{P}(X \le (y-b)/a) = F((y-b)/a) & \text{if } a > 0 \\ \mathbb{P}(X \ge (y-b)/a) = 1 - \lim_{x \uparrow (y-b)/a} F(x) & \text{if } a < 0 \end{cases}$$

Finally, if $a = 0$, then $Y = b$, so that $\mathbb{P}(Y \le y)$ equals 0 if $b > y$ and 1 if $b \le y$.

3. Assume that any specified sequence of heads and tails with length n has probability 2^{-n}. There are exactly $\begin{pmatrix} n \\ k \end{pmatrix}$ such sequences with k heads. If heads occurs with probability p then, assuming the independence of outcomes, the probability of any given sequence of k heads and $n-k$ tails is $p^k(1-p)^{n-k}$. The answer is therefore $\begin{pmatrix} n \\ k \end{pmatrix} p^k(1-p)^{n-k}$.

4. By the independence of the tosses,

$$\mathbb{P}(X > m) = \mathbb{P}(\text{ first } m \text{ tosses are tails }) = (1-p)^m$$

Hence

$$\mathbb{P}(X \le x) = \begin{cases} 1 - (1-p)^{\lfloor x \rfloor} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Remember that $\lfloor x \rfloor$ denotes the integer part of x.

5. (a) $\lim_{x\to-\infty} \lambda F(x) + (1-\lambda)G(x) = 0,\ \lim_{x\to\infty} \lambda F(x) + (1-\lambda)G(x) = 1.$
   For $x < y$,

$$\begin{aligned}&\lambda F(y) + (1-\lambda)G(y) - \lambda F(x) - (1-\lambda)G(x)\\&= \lambda(F(y) - F(y)) + (1-\lambda)(G(y) - G(x))\\&\geq 0\end{aligned}$$

   and

$$\begin{aligned}&\lim_{h\to0^+} \lambda F(x+h) + (1-\lambda)G(x+h) - \lambda F(x) - (1-\lambda)G(x)\\=&\lim_{h\to0^+} \lambda(F(x+h) - F(x)) + (1-\lambda)(G(x+h) - G(x)) = 0\end{aligned}$$

   Thus, $\lambda F(x) + (1-\lambda)G(x)$ is a distribution function.

   (b) $\lim_{x\to-\infty} F(x)G(x) = 0,\ \lim_{x\to\infty} F(x)G(x) = 1.$
   For $x < y$,

$$F(y)G(y) - F(x)G(x) \geq F(y)G(x) - F(x)G(x) = (F(y) - F(x))G(x) \geq 0,$$

   and

$$\lim_{h\to0^+} F(x+h)G(x+h) - F(x)G(x) = 0.$$

   Hence, $F(x)G(x)$ is a distribution.

6. The function $g(F(x))$ is a distribution function whenever g is continuous and non-decreasing on $[0,1]$, with $g(0) = 0, g(1) = 1$. In fact:

   - $\lim_{x\to-\infty} g(F(x)) = g(0) = 0,\ \lim_{x\to\infty} g(F(x)) = g(1) = 1.$
   - For $x < y$, we have $F(x) \leq F(y)$ and thus

$$\begin{aligned}&g(F(y)) - g(F(x))\\&\geq g(F(x)) - g(F(x))\\&= 0\end{aligned}$$

   - It holds

$$\begin{aligned}&\lim_{h\to0^+} g(F(x+h)) - g(F(x))\\=&g(F(x)) - g(F(x)) = 0\end{aligned}$$

   Thus $g(F(x))$ is a distribution function.

(a) In this case, $g(x) = x^r$ is continuous and non-decreasing on $[0,1]$, with $g(0) = 0, g(1) = 1$. Thus $g(F(x))$ is a distribution function.

(b) In this case, $g(x) = 1 - (1-x)^r$ is continuous and non-decreasing on $[0,1]$, with $g(0) = 0, g(1) = 1$. Thus $g(F(x))$ is a distribution function.

7.

(a) It holds

$$P(X=1) = P(\{w_1\}) = \frac{1}{3}$$
$$P(X=2) = P(\{w_2\}) = \frac{1}{3}$$
$$P(X=3) = P(\{w_3\}) = \frac{1}{3}.$$

and

$$P(Y=1) = P(\{w_3\}) = \frac{1}{3}$$
$$P(Y=2) = P(\{w_1\}) = \frac{1}{3}$$
$$P(Y=3) = P(\{w_2\}) = \frac{1}{3}.$$

and

$$P(Z=1) = P(\{w_2\}) = \frac{1}{3}$$
$$P(Z=2) = P(\{w_3\}) = \frac{1}{3}$$
$$P(Z=3) = P(\{w_1\}) = \frac{1}{3}.$$

Clearly $f_X(i) = f_Y(i) = f_Z(i) = \frac{1}{3}$ for $i = 1, 2, 3$.

(b)

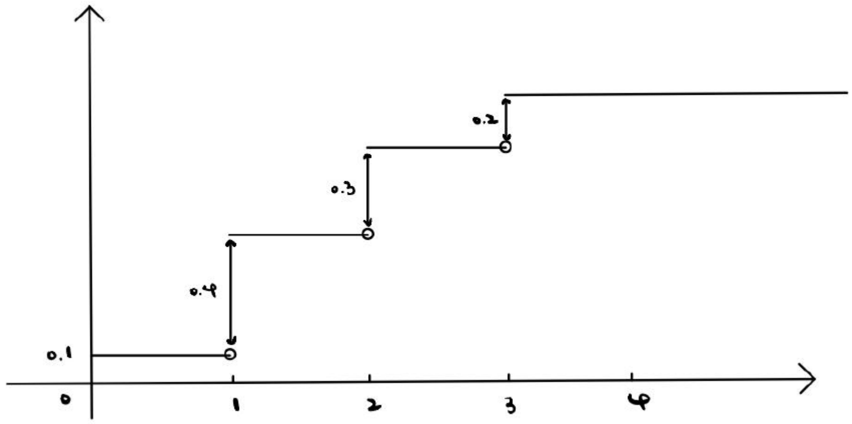
i. $(X+Y)(\omega_1) = 3, (X+Y)(\omega_2) = 5, (X+Y)(\omega_3) = 4$, and therefore $f_{X+Y}(i) = \frac{1}{3}$ for $i = 3, 4, 5$.

ii. $(Y+Z)(\omega_1) = 5, (Y+Z)(\omega_2) = 4, (Y+Z)(\omega_3) = 3$, and therefore $f_{Y+Z}(i) = \frac{1}{3}$ for $i = 3, 4, 5$.

iii. $(Z+X)(\omega_1) = 4, (Z+X)(\omega_2) = 3, (Z+X(\omega_3) = 5$, and therefore $f_{Z+X}(i) = \frac{1}{3}$ for $i = 3, 4, 5$.

8.

(a) Since $0.1 + c + 0.3 + 0.2 = 1$, we have $c = 0.4$.



(b)

(c) We have

$$
\begin{aligned}
P(X \le 2 \mid X \ge 1) &= \frac{P(1 \le X \le 2)}{P(X \ge 1)} \\
&= \frac{0.4 + 0.3}{0.4 + 0.3 + 0.2} \\
&= \frac{7}{9}.
\end{aligned}
$$