MATH 2023: Ordinary and Partial Differential Equations

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Contents

1 Introduction

1.1 Classification of Differential Equations

Several examples will be given to illustrate some basic concepts of differential equations. First of all, algebraic equations

$$2x+1=0 \to x=-\frac{1}{2}. \tag{1}$$

Then equations involve derivatives

$$\begin{aligned} &(a) \quad \frac{\mathrm{d}y}{\mathrm{d}x}=x, \\ &(b) \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \end{aligned} \tag{2}$$

Equations involving derivatives are called **differential equations**. If the derivatives are only ordinary such as $(2a)$, then these equations are called ordinary differential equations (ODEs). If the equations involve partial derivatives such as $(2b)$, then they are called partial differential equations (PDEs).

Furthermore, you may face several ODEs containing several unknowns, for example

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}x}=2v+u, \\ \frac{\mathrm{d}v}{\mathrm{d}x}=2u. \end{cases} \tag{3}$$

Equation (3) has two equations and contains two unknowns u,v, and it is called a system of ODEs.

Usually, for one ODE, it is classified by its order. For instance, for the following ODE,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}+x=0, \tag{4}$$

the highest order of derivative is 2, thus it is called second order ODE. Generally, an ODE may take the form

$$F(x,u(x),u'(x),\cdots,u^{(n)}(x))=0. \tag{5}$$

In (5), $F(\cdots)$ is a given function and the unknown is $u(x)$. The highest order of derivatives is n, therefore we say that the **order** of this ODE is n. The solution of (5) is the function $\phi(x)$ such that $u=\phi(x)$ satisfies (5), i.e.

$$F(x,\phi(x),\phi'(x),\cdots,\phi^{(n)}(x))\equiv 0. \tag{6}$$

For example, for the ODE $\frac{\mathrm{d}y}{\mathrm{d}x}=x$, the solution is $y=\frac{1}{2}x^2$.

The main task in this course: Find the solution for the given ODE. Accompanied by the task are several questions:

(a) Given an ODE, whether there exists a solution or not?

(b) If there is a solution, whether it is unique?

(c) If there is a solution, how to determine it?

1.2 Linear and Nonlinear Equation

The general form of ODE is

$$F(x,y,y',y'',\cdots,y^{(n)})=0. \tag{1}$$

If $F(\cdots)$ is a linear function of y, y', $\cdots$, $y^{(n)}$, i.e., (1) takes the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = q(x), \tag{2}$$

then, we say that (1) is a linear ODE. Otherwise, (1) is said to be a nonlinear ODE. For example, $\frac{\mathrm{d}y}{\mathrm{d}x} = x$ is linear while $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \sin\theta = 0$ is nonlinear.

1.3 Geometrical Aspect

Consider

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y), \tag{1}$$

where $f(x,y)$ is given. Suppose that the solution is given as $y = \phi(x)$. Geometrically, it represents a curve in the $x-y$ plane. $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the slope of the that solution curve. However, from (1), it can be seen that the slope is given by $f(x,y)$, which is known. Thus, even if we do not know the solution curve, we still know its slope.

1.4 Motivation

Suppose we know the value of the quantity now and we wish to predict its value in the future. This quantity can be, for example, the temperature of coffee in a cup, the number of people infected with a virus, the concentration of carbon dioxide in the atmosphere. To do the prediction, we must know how quickly these quantities are changing. Mathematically, the rate of change of one quantity is the derivative. And practically the rate of change of a quantity will depend on the quantity itself. Therefore, we may model the problem as $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t,y)$. This equation contains derivative, so it is a differential equation. Through this equation and the initial condition, we may know the behavior of this quantity at any time.

As one example, suppose that $N(t)$ is the number of bacteria growing on a plate of nutrients. At the start of the experiment, suppose that there are 1000 bacteria, so $N(0) = 1000$. The rate of change of N will be proportional to N itself: if there are twice as many bacteria, then N will grow twice as rapidly. So we have:

$$\mathrm{d}N/\mathrm{d}t = \sigma N. \tag{2}$$

where σ is a constant, and $\mathrm{d}N/\mathrm{d}t$ is the derivative (rate of change) of N with respect to time. We would have to do further experiments to find out the value of σ. We can easily verify that

$$N(t) = 1000e^{\sigma t} \tag{3}$$

is a solution of this differential equation with the given initial condition. To do this, first calculate $N(0)$ and verify that it is the same as the number given:

$$N(0) = 1000e^{0} = 1000. \tag{4}$$

Next, calculate $\mathrm{d}N/\mathrm{d}t$ and verify that it satisfies the differential equation:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 1000\sigma e^{\sigma t} = \sigma 1000 e^{\sigma t} = \sigma N \mathrm{d}t \tag{5}$$

as required.

As another example, Suppose a 25 year-old plans to set aside a fixed amount every year of his/her working life, invests at a real return of 6%, and retires at age 65. How much must he/she invest each year to have $8,000,000$ at retirement? Let $S(t)$ be the value of the investment at time t, and let r be the annual interest rate compounded after every time interval Δt. We can also include deposits and let k be the annual deposit amount, and suppose that an installment is deposited after every time interval Δt. The value of the investment at the time $t+\Delta t$ is then given by

$$S(t+\Delta t) = S(t) + (r\Delta t)S(t) + k\Delta t \tag{6}$$

where at the end of the time interval Δt, $r\Delta t S(t)$ is the amount of interest credited and $k\Delta t$ is the amount of money deposited $(k > 0)$.

Rearranging (6), we have

$$\frac{S(t+\Delta t) - S(t)}{\Delta t} = rS(t) + k. \tag{7}$$

The equation for continuous compounding of interest and continuous deposits is obtained by taking the limit $\Delta t \to 0$. The resulting differential equation is

$$\frac{\mathrm{d}S}{\mathrm{d}t} = rS + k \tag{8}$$

which can be solved with the initial condition $S(0) = S_0$, where S_0 is the initial capital.

The solution is

$$S = S_0 e^{rt} + \frac{k}{r} e^{rt}(1 - e^{-rt}). \tag{9}$$

Suppose $S_0 = 0$, then from (9), we have

$$k = \frac{rS(t)}{e^{rt} - 1}. \tag{10}$$

Then according to the problem, we have $r = 0.06$ and $t = 40$ and $S(t) = 8,000,000$, therefore,

$$k = \frac{0.06 \times 8,000,000}{e^{0.06\times 40} - 1} = \$47,889 \text{ year}^{-1}. \tag{11}$$

To save approximately eight million at retirement, the worker would need to save about 50,000 per year over his/her working life. Note that the amount saved over the worker's life is approximately $40 \times 50,000 = 2,000,000$, while the amount earned on the investment (at the assumed 6% real return) is approximately $8,000,000 - 2,000,000 = 6,000,000$. The amount earned from the investment is about $3\times$the amount saved, even with the modest real return of 6%. Sound investment planning is well worth the effort.

2 First Order Ordinary Differential Equations

We shall study the first order differential equations and they are in the form as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y). \tag{1}$$

2.1 Linear Equations

For a linear first order ODE, $f(x,y)$ need be a linear function of y, say

$$f(x,y) = -p(x)y + q(x). \tag{2}$$

Then, one has

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x). \tag{3}$$

Equation (3) is the first order linear ODE.

Consider a particular example:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2}y = \frac{3}{2}. \tag{4}$$

We want to find the solution to (4). One method is like this:

$$\begin{aligned}
&\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y-3}{2} \to \frac{1}{y-3}\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2} \to \frac{\mathrm{d}}{\mathrm{d}x}\ln|y-3| = -\frac{1}{2},\\
&\to \ln|y-3| = -\frac{1}{2}x + C_1 \to |y-3| = e^{C_1}e^{-\frac{1}{2}x}\\
&\to y-3 = \underline{\underline{\pm e^{C_1}}}\, e^{-\frac{1}{2}x} \to y = 3 + Ce^{-\frac{1}{2}x},
\end{aligned} \tag{5}$$

where C_1 is arbitrary constant and C is arbitrary constant not equal to zero. The other method to derive the solution to (4) is as follows. Multiplying $e^{\frac{1}{2}x}$ on both sides of (4), one has:

$$\begin{aligned}
&\underline{\underline{e^{\frac{1}{2}x}y' + \frac{1}{2}e^{\frac{1}{2}x}y}} = \frac{3}{2}e^{\frac{1}{2}x} \to \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\frac{1}{2}x}y\right) = \frac{3}{2}e^{\frac{1}{2}x}\\
&\to e^{\frac{1}{2}x}y = 3e^{\frac{1}{2}x} + C \to y = 3 + Ce^{-\frac{1}{2}x},
\end{aligned} \tag{6}$$

where C is an arbitrary constant and $e^{\frac{1}{2}x}$ is called the **integrating factor**. Through the integrating factor, the resulting equation is readily integrable. For a general linear first order ODE, it can not be solved by the direct method and one may consider using the integrating factor. The difficulty lies in **how to find the integration factor for general case (3)?**

Consider the general case

$$y' + p(x)y = q(x). \tag{7}$$

We multiply it by the integrating factor $u(x)$ (yet to be determined) and have

$$u(x)y' + u(x)p(x)y = u(x)q(x). \tag{8}$$

We want $u(x)$ to be such a function that

$$\begin{aligned}
&u(x)y' + u(x)p(x)y = \frac{\mathrm{d}}{\mathrm{d}x}(u(x)y) = u(x)y' + \frac{\mathrm{d}u(x)}{\mathrm{d}x}y,\\
&\to \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} = p(x) \to \frac{\mathrm{d}}{\mathrm{d}x}\ln|u(x)| = p(x) \to \ln|u(x)| = \int p(x)\mathrm{d}x + k,
\end{aligned} \tag{9}$$

where k is a constant. The simplest choice is to set $k = 0$. Then

$$|u(x)| = e^{\int p(x)\mathrm{d}x}. \tag{10}$$

Taking $u(x)$ to be non-negative, we have

$$u(x) = e^{\int p(x)\mathrm{d}x}. \tag{11}$$

Equation (11) gives one form of integrating factor.

With the above integrating factor, (8) becomes

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}(u(x)y) = u(x)q(x) \to u(x)y = \int u(x)q(x)\mathrm{d}x + C, \\ &\to y = \frac{\int u(x)q(x)\mathrm{d}x + C}{u(x)}, \end{aligned} \tag{12}$$

with $u(x)$ given by (11). Equation (12) involves an arbitrary constant C and includes every solution to (7) and it is called the **general solution**. Geometrically, (12) represents a family of curves, called integral curves. On the other hand, if we impose that

$$y(x_0) = y_0, \tag{13}$$

which is called an initial condition. Then under (13), one has

$$y_0 = \frac{\int u(x)q(x)\mathrm{d}x\Big|_{x=x_0} + C}{u(x_0)}, \tag{14}$$

which determines C. **ODE (7) together with initial condition (13) is called an initial value problem. The solution (12) with C determined by (14) is called the particular solution.**

Example 1 Find the general solution of the following ODE:

$$y' + \frac{y}{x} = 3\cos(2x), \quad x > 0. \tag{15}$$

Solution: Since $p(x) = \frac{1}{x}$, the integrating factor is

$$u(x) = \exp\left(\int p(x)\mathrm{d}x\right) = \exp\left(\int \frac{1}{x}\mathrm{d}x\right) = \exp(\ln x) = x. \tag{16}$$

Then multiplying the integrating factor $u(x)$ on both sides , one has

$$xy' + y = 3x\cos(2x) \to \frac{\mathrm{d}}{\mathrm{d}x}(xy) = 3x\cos(2x) \to xy = \int (3x\cos(2x))\mathrm{d}x + C = \frac{3}{2}x\sin(2x) + \frac{3}{4}\cos(2x) + C. \tag{17}$$

Therefore, the general solution is

$$y = \frac{\frac{3}{2}x\sin(2x) + \frac{3}{4}\cos(2x) + C}{x}. \tag{18}$$

Example 2 Find the solution of the following initial value problem:

$$x^3y' + 4x^2y = e^{-x}, \quad y(-1) = 0. \tag{19}$$

Solution: The equation is first transformed to

$$y' + \frac{4}{x}y = \frac{e^{-x}}{x^3}. \tag{20}$$

Then one can see $p(x) = \frac{4}{x}$ and the integrating factor is

$$u(x) = \exp(\int p(x)\mathrm{d}x) = \exp(\int \frac{4}{x}\mathrm{d}x) = \exp(4\ln x) = x^4. \tag{21}$$

Multiplying (19) with the integrating factor, one has

$$x^4y' + 4x^3y = xe^{-x} \to \frac{\mathrm{d}}{\mathrm{d}x}(x^4y) = xe^{-x} \to x^4y = -xe^{-x} - e^{-x} + C. \tag{22}$$

Therefore, the general solution is

$$y = \frac{-xe^{-x} - e^{-x} + C}{x^4}. \tag{23}$$

Applying the initial condition $y(-1) = 0$, one has $C = 0$. Therefore, the solution to the initial value problem (19) is

$$y = \frac{-xe^{-x} - e^{-x}}{x^4}. \tag{24}$$

Figure 1 shows the general solution and the particular solution.

Figure 1: The general solution and particular solution to Example 2.

2.2 Further Discussion of Linear Equations (For reading only)

Consider the following initial value problem

$$y' + p(x)y = q(x), \quad y(x_0) = y_0. \tag{1}$$

There are several questions need to be answered:

(1) Does there always exist a solution? (existence)

(2) Is there only one solution? (Uniqueness)

(3) Is the solution valid for all x or for x in certain interval?

Theorem If $p(x)$ and $q(x)$ are continuous in the interval $\alpha \le x \le \beta$ which contains x_0. Then the initial value problem (1) has a unique solution which is valid in the interval (α, β). (Theorem 2.4.1)

Proof From the previous section, we know that the form of the general solution is given by

$$y = \frac{\int u(x)q(x)\mathrm{d}x + C}{u(x)}, \quad u(x) = e^{\int p(x)\mathrm{d}x}. \tag{2}$$

Thus, to show that the solution exists and is valid in $\alpha \le x \le \beta$ is equivalent to show (2) satisfies the ODE in $\alpha \le x \le \beta$. We know that

$$p(x) \text{ is continuous } \to \int p(x)\mathrm{d}x \text{ is differentiable } \to u(x) \text{ is differentiable,}$$

$$q(x), u(x) \text{ are continuous } \to \int u(x)q(x)\mathrm{d}x \text{ are differentiable.}$$

Thus, (2) is differentiable. Substituting it into the ODE, after some calculation, it is indeed satisfied. Then we may conclude the solution exists and is valid in (α,β). To show that the solution is unique, we only need to show that under $y(x_0)=y_0$. C is uniquely determined.

$$y(x_0)=y_0 \ \rightarrow \ y_0=\frac{\int u(x)q(x)\mathrm{d}x\Big|_{x=x_0}+C}{u(x_0)} \ \rightarrow \ C=y_0\,u(x_0)-\underline{\underline{\int u(x)q(x)\mathrm{d}x\Big|_{x=x_0}}},$$

$$\rightarrow \ C \text{ has unique value.}$$

Example 1 Find the solution of

$$y'+y=\frac{1}{1+x^2}, \quad y(0)=0, \tag{3}$$

and state the interval in which the solution is valid.

Solution We know

$$p(x)=1, \quad q(x)=\frac{1}{1+x^2}. \tag{4}$$

One may observe $p(x)$, $q(x)$ are continuous for $-\infty \le x \le \infty$. According to the theorem, there is a unique solution which is valid in $-\infty \le x \le \infty$. The integrating factor is

$$u(x)=e^{\int p(x)\mathrm{d}x}=e^{\int \mathrm{d}x}=e^x. \tag{5}$$

Multiplying $u(x)$ on both sides of the ODE, one has

$$\begin{aligned} &e^x y'+e^x y=\frac{e^x}{1+x^2}, \ \rightarrow \ \frac{\mathrm{d}}{\mathrm{d}x}(e^x y)=\frac{e^x}{1+x^2}, \\ &\rightarrow e^x y=\int \frac{e^x}{1+x^2}\mathrm{d}x+C, \ \rightarrow \ y=e^{-x}\left[\int \frac{e^x}{1+x^2}\mathrm{d}x+C\right]. \end{aligned} \tag{6}$$

Equation (6) is the general solution. Then using the initial condition, one has

$$\begin{aligned} &y(0)=0 \ \rightarrow \ 0=e^0\left[\int \frac{e^x}{1+x^2}\mathrm{d}x+C\right]\Big|_{x=0,} \\ &\rightarrow C=-\int \frac{e^x}{1+x^2}\mathrm{d}x\Big|_{x=0.} \end{aligned} \tag{7}$$

2.3 Separable Equations

Consider the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y). \tag{1}$$

We write

$$f(x,y) = -\frac{M(x,y)}{N(x,y)}. \tag{2}$$

Then, we have

$$M(x,y) + N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{3}$$

In this section, we consider the special case that $M(x,y) = M(x), \quad N(x,y) = N(y)$, i.e., $f(x,y) = \frac{-M(x)}{N(y)}$. In this case, the ODE becomes

$$M(x) + N(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \tag{4}$$

or it can be written as

$$M(x)\mathrm{d}x = -N(y)\mathrm{d}y. \tag{5}$$

Equations (4), (5) are separable ODEs. And further integration of (5) gives

$$\int M(x)\mathrm{d}x = -\int N(y)\mathrm{d}y + C. \tag{6}$$

Example 1 Find the solution of the following initial value problem

$$y' = \frac{2x}{y + x^2 y}, \quad y(0) = -2, \tag{7}$$

and determine the interval in which it is defined.

Solution

$$\begin{aligned} &\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{(1+x^2)y} \;\rightarrow\; y\mathrm{d}y = \frac{2x}{1+x^2}\mathrm{d}x, \\ &\rightarrow\; \frac{1}{2}y^2 = \ln(1+x^2) + C \;\rightarrow\; y^2 = 2\ln(1+x^2) + C, \\ &\rightarrow\; y = \pm\sqrt{2\ln(1+x^2) + C}. \end{aligned} \tag{8}$$

Note, this ODE has two general solution, i.e., the solution is not unique. Further, to determine the integration constant, we use $y(0) = -2$ and have

$$-2 = \pm\sqrt{2\ln(1+0) + C}. \tag{9}$$

We must take "$-$" on the right hand side for the present problem and this gives $C = 4$. Thus, the solution is

$$y = -\sqrt{2\ln(1+x^2) + 4}. \tag{10}$$

Note, for the ODE, we require $y \neq 0$. For (10), y is less than zero and can never be zero for all x. Thus, it is valid in $-\infty \leq x \leq \infty$.

Example 2 Find the solution of the following initial value problem

$$y' = \frac{y\cos x}{1 + 2y^2}, \quad y(0) = 1. \tag{11}$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{\frac{1+2y^2}{y}} \ \to \ \frac{1+2y^2}{y}\mathrm{d}y = \cos x \mathrm{d}x \ \to \ \int \frac{1+2y^2}{y}\mathrm{d}y = \int \cos x \mathrm{d}x \ \to \ \ln|y| + y^2 = \sin x + C. \tag{12}$$

By the initial condition, one has $C = 1$ and the solution to the initial value problem is

$$\ln|y| + y^2 = \sin x + 1. \tag{13}$$

2.4 Difference Between Linear and Nonlinear Equations

Consider

$$y' = f(x,y), \quad y(x_0) = y_0. \tag{1}$$

There are three major questions: existence, uniqueness, valid interval of the solution.

Theorem Suppose that $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous in some rectangle $\alpha \le x \le \beta$, $\gamma \le y \le \delta$, which contains (x_0, y_0). Then (1) has a unique solution which is valid in some interval $x_0 - h \le x \le x_0 + h$ within $\alpha \le x \le \beta$.

Remark: Notice the difference of conditions between linear and nonlinear ODEs. Here, the exact values h is not stated in the theorem. It depends on the differential equation as well as the initial condition.

Example 1 Solve the initial value problem.

$$y' = y^2, \quad y(0) = 1, \tag{2}$$

and determine the interval in which the solution exists.

Solution: $f(x,y) = y^2$ and $\frac{\partial f}{\partial y} = 2y$. Thus, according to the theorem, there is a unique solution which is valid in some interval containing $x = 0$.

$$\begin{aligned} &\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 \ \to \ \frac{1}{y^2}\mathrm{d}y = \mathrm{d}x, \\ &-\frac{1}{y} = x + C \ \to \ y = -\frac{1}{x+C}. \end{aligned} \tag{3}$$

Then using the initial value condition $y(0) = 1$, one has

$$1 = -\frac{1}{C} \ \to \ C = -1. \tag{4}$$

Thus, the solution is

$$y = \frac{1}{1-x}. \tag{5}$$

We require that $x \ne 1$, i. e., $-\infty < x < 1$ is the interval in which the solution is valid. If we take $y(0) = -1$, then one has $C = 1$ and the solution to the initial value problem is

$$y = -\frac{1}{1+x}, \tag{6}$$

and the valid interval is then $(-1, \infty)$. This example shows that the valid intervals depend on the initial conditions.

Remark Equation (3) is not the general solution since it does not include the solution $y = 0$, although it contains an arbitrary constant. Only for linear ODE, we use the terminology “general solution”. You can also see other differences, such as the solution to the linear solution is explicit while the solution to the non-linear equation may be implicit.

2.5 Applications of modeling with first order ODE(For reading only)

First we review some results in mechanics.

1. Newton's second law:

This law gives $F = ma$, where F is the external force, m is the mass and a is the acceleration. The basic units are s for time, m for length and kg for mass. The unit for the force is N and for the acceleration is m/s^2.

2. Gravity of earth

Gravity is given by the formula

$$\text{Gravity} = -\frac{mgR^2}{(R+x)^2} = -\frac{mg}{\left(1+\frac{x}{R}\right)^2}, \tag{7}$$

where $g = 9.8m/s^2$. For small x, we have Taylor expansion of $(1+\frac{x}{R})^{-2}$ so that

$$\text{Gravity} \approx -mg(1-2\frac{x}{R}+\cdots) \approx -mg. \tag{8}$$

3. Other external force For example, the air resistance can be regarded as being proportional to the velocity, say $-kv$ where k is a known constant.

Example 1 A body of constant mass m is projected vertically upward from the surface of the earth with an initial velocity v_0. The gravitational acceleration of the earth is assumed to be constant. During the motion, the body is subjected to an air resistance which is proportional to the magnitude of the velocity, say, $k|v|$. Find

a) The time at which the maximum height is reached;

b) The maximum height attained by the body.

Solution: From Newton's second law, one has

$$\begin{aligned} &F = -kv - mg = ma = m\frac{\mathrm{d}v}{\mathrm{d}t}, \\ &v(0) = v_0. \end{aligned} \tag{9}$$

We want to solve the above initial value problem. The differential equation is separable equation and it is rewritten as

$$-\mathrm{d}t = \frac{\mathrm{d}v}{\frac{k}{m}v+g}. \tag{10}$$

Integrating on both sides gives

$$-t + C_1 = \frac{m}{k}\ln|\frac{k}{m}v+g| \ \to\ |\frac{k}{m}v+g| = e^{\frac{k}{m}C_1}e^{-\frac{k}{m}t} \ \to\ \frac{k}{m}v+g = \pm e^{\frac{k}{m}C_1}e^{-\frac{k}{m}t} = Ce^{-\frac{k}{m}t}. \tag{11}$$

So the general solution is

$$v = \frac{m}{k}\left(Ce^{-\frac{k}{m}t} - g\right). \tag{12}$$

Using the initial condition, one has

$$v_0 = \frac{m}{k}(C - g) \;\rightarrow\; C = \frac{k}{m}v_0 + g. \tag{13}$$

Therefore, the solution to the initial value problem is

$$v = \frac{m}{k}\left[\left(\frac{k}{m}v_0 + g\right)e^{-\frac{k}{m}t} - g\right] = \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t} - \frac{mg}{k}. \tag{14}$$

Further, one has the system governing the displacement as

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}t} = v = \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t} - \frac{mg}{k}, \\ &x(0) = 0. \end{aligned} \tag{15}$$

From (15), one has

$$x = -\frac{m}{k}\left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t} - \frac{mg}{k}t + D. \tag{16}$$

The initial condition gives

$$D = \frac{m}{k}\left(v_0 + \frac{mg}{k}\right). \tag{17}$$

Thus, the solution to the displacement is

$$x = \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)\left(1 - e^{-\frac{k}{m}t}\right) - \frac{mg}{k}t. \tag{18}$$

When the velocity is zero, the body reaches the maximum height. In (14), setting $v = 0$, one has

$$0 = \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t} - \frac{mg}{k}, \tag{19}$$

which leads to

$$t = \frac{m}{k}\ln\left(\frac{v_0 k}{mg} + 1\right). \tag{20}$$

This is the time when the body reaches the maximum height. And the maximum height is obtained by taking this time into (18), one has

$$x = \frac{mv_0}{k} - \frac{m^2 g}{k^2}\ln\left(\frac{v_0 k}{mg} + 1\right). \tag{21}$$

Example 2 A body of constant mass m is projected vertically upward from the surface of the earth with an initial velocity v_0. Neglecting the air resistance but taking into account the variation of the earth gravitational field with altitude. Find

a) The expression for the velocity during the motion;

b) The maximum height attained by the body;

c) The smallest initial velocity for which the body will not return to earth.

Solution: From Newton's second law, one has

$$\text{Gravity} = -\frac{mgR^2}{(R+x)^2} = m\frac{\mathrm{d}v}{\mathrm{d}t}. \tag{22}$$

There are two unknown functions $x(t)$ and $v(t)$ in the above differential equation. And there exists a relation between these two unknown functions

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}x} = v\frac{\mathrm{d}v}{\mathrm{d}x}. \tag{23}$$

Taking (23) into (22), one has

$$-\frac{gR^2}{(R+x)^2} = v\frac{\mathrm{d}v}{\mathrm{d}x} \ \rightarrow \ -\frac{gR^2\mathrm{d}x}{(R+x)^2} = v\mathrm{d}v \ \rightarrow \ \frac{gR^2}{R+x} + C = \frac{v^2}{2}. \tag{24}$$

Initially, one has

$$x(0) = 0, \quad v(0) = v_0. \tag{25}$$

This implies at $x = 0$, $v = v_0$ which further gives

$$\frac{gR^2}{R+0} + C = \frac{1}{2}v_0^2 \ \rightarrow \ C = \frac{1}{2}v_0^2 - gR. \tag{26}$$

From (24), one has

$$\frac{1}{2}v^2 = \frac{gR^2}{R+x} + \frac{1}{2}v_0^2 - gR \ \rightarrow \ v = \pm\sqrt{\frac{2gR^2}{R+x} + v_0^2 - 2gR}. \tag{27}$$

At the maximum height, denoted by ξ, the velocity is zero:

$$0 = \pm\sqrt{\frac{2gR^2}{R+\xi} + v_0^2 - 2gR} \ \rightarrow \ \xi = \frac{v_0^2 R}{2gR - v_0^2}. \tag{28}$$

Not returning to the earth implies $\xi \rightarrow \infty$. In (28), letting $\xi \rightarrow \infty$, we must have

$$2gR - v_0^2 = 0 \ \rightarrow \ v_0 = \sqrt{2gR}, \tag{29}$$

which is the smallest initial velocity. For $g = 9.8m/s^2, R \approx 6286224.5m$, $v_0 = 11100m/s$ or $v_0 = 11.1km/s$.

2.6 Exact Equations and Integrating Factors

Consider

$$M(x,y)+N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}=0. \tag{1}$$

Exact Equations: If we can find a function $\Psi(x,y)$ such that

$$M(x,y)=\frac{\partial\Psi}{\partial x},\quad N(x,y)=\frac{\partial\Psi}{\partial y}. \tag{2}$$

Then, the ODE becomes

$$\frac{\partial\Psi}{\partial x}+\frac{\partial\Psi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}=0\ \rightarrow\ \frac{\mathrm{d}\Psi(x,y)}{\mathrm{d}x}=0\ \rightarrow\ \Psi(x,y)=C. \tag{3}$$

In this case, (1) is called an exact ODE.

Question: Under which condition this $\Psi(x,y)$ exists?

Theorem Suppose that $M,N,\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous in rectangle $R:\alpha<x<\beta,\nu<y<\delta$. Then (1) is an exact ODE in R (i. e., there exists a $\Psi(x,y)$ such that (2) is satisfied) if and only if

$$\frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}\quad \text{in } R \tag{4}$$

Proof (i) Necessary Condition

Suppose that (1) is an exact ODE, i. e., (2) is satisfied, we have

$$M(x,y)=\frac{\partial\Psi}{\partial x},\quad N(x,y)=\frac{\partial\Psi}{\partial y}\ \rightarrow\ \frac{\partial M}{\partial y}=\frac{\partial^2\Psi}{\partial y\partial x},\qquad \frac{\partial N}{\partial x}=\frac{\partial^2\Psi}{\partial y\partial x},\ \rightarrow\ \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}. \tag{5}$$

So (4) is satisfied.

(ii) Sufficient Condition

Suppose that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\ \text{in } R. \tag{6}$$

Next, we consider a function $\Psi(x,y)$ defined by

$$\Psi(x,y)=\int M(x,y)\mathrm{d}x+\int\left[N(x,y)-\int\frac{\partial M(x,y)}{\partial y}\mathrm{d}x\right]\mathrm{d}y. \tag{7}$$

Then we have

$$\begin{aligned}\frac{\partial\Psi}{\partial x}&=M(x,y)+\int\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]\mathrm{d}y=M(x,y),\\ \frac{\partial\Psi}{\partial y}&=\int\frac{\partial M}{\partial y}\mathrm{d}x+N(x,y)-\int\frac{\partial M}{\partial y}\mathrm{d}x=N(x,y).\end{aligned} \tag{8}$$

Thus, for such a $\Psi(x,y)$, (2) is satisfied, i. e., (1) is an exact ODE.

Example 1 Find the solution of the ODE

$$(y\cos x+2xe^y)+(\sin x+x^2e^y-1)y'=0 \tag{9}$$

Solution:

$$\begin{aligned}
&M(x,y) = y\cos x + 2xe^y, \quad N(x,y) = \sin x + x^2e^y - 1,\\
&\frac{\partial M}{\partial y} = \cos x + 2xe^y, \quad \frac{\partial N}{\partial x} = \cos x + 2xe^y, \;\rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.
\end{aligned} \tag{10}$$

Therefore, the ODE is exact. Thus there exists a $\Psi(x,y)$ such that

$$\frac{\partial \Psi(x,y)}{\partial x} = M(x,y) = y\cos x + 2xe^y, \quad \frac{\partial \Psi(x,y)}{\partial y} = N(x,y) = \sin x + x^2e^y - 1. \tag{11}$$

Integrating $(11)_1$, we obtain

$$\Psi(x,y) = y\sin x + x^2e^y + h(y). \tag{12}$$

Further, from $(11)_2$, one has

$$\begin{aligned}
&\frac{\Psi(x,y)}{\partial y} = \sin x + x^2e^y + \frac{\mathrm{d}h(y)}{\mathrm{d}y} = N(x,y) = \sin x + x^2e^y - 1\\
&\rightarrow \frac{\mathrm{d}h(y)}{\mathrm{d}y} = -1 \;\rightarrow\; h(y) = -y + C_1\,.
\end{aligned} \tag{13}$$

Setting $C_1 = 0$, we have

$$\Psi(x,y) = y\sin x + x^2e^y + h(y) = y\sin x + x^2e^y - y. \tag{14}$$

Hence the solution to (9) is

$$y\sin x + x^2e^y - y = C. \tag{15}$$

Example 2 Find the solution of the ODE

$$(3xy + y^2) + (x^2 + xy)y' = 0. \tag{16}$$

Solution:

$$\frac{\partial M}{\partial y} = 3x + 2y, \quad \frac{\partial N}{\partial x} = 2x + y, \tag{17}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the given equation is not exact. Let's see whether there exists a function $\Psi(x,y)$ which satisfies

$$\frac{\partial \Psi}{\partial x} = M(x,y) = 3xy + y^2, \quad \frac{\partial \Psi}{\partial y} = N(x,y) = x^2 + xy. \tag{18}$$

Integrating the first equation above, one has

$$\Psi(x,y) = \frac{3}{2}x^2y + xy^2 + h(y). \tag{19}$$

Then from $(18)_2$, one has

$$\frac{\partial \Psi}{\partial y} = \frac{3}{2}x^2 + 2xy + \frac{\mathrm{d}h(y)}{\mathrm{d}y} = N(x,y) = x^2 + xy, \tag{20}$$

which implies

$$\frac{\mathrm{d}h(y)}{\mathrm{d}y} = -\frac{1}{2}x^2 - xy. \tag{21}$$

Since the right hand side depends on x as well, contradiction appears. So there is no $\Psi(x,y)$ satisfies (18).

Integrating factor It is sometimes possible to convert a differential equation not exact into an exact equation by multiplying the equation with a suitable integrating factor. Consider

$$M(x,y) + N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{22}$$

Suppose that it is not exact. Now, we multiplying it by an integrating factor $u(x,y)$ which gives

$$uM + uN\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{23}$$

The purpose is trying to make this new equation to be exact, i. e.,

$$\frac{\partial(uM)}{\partial y} = \frac{\partial(uN)}{\partial x} \rightarrow M\frac{\partial u}{\partial y} - N\frac{\partial u}{\partial x} + u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 0. \tag{24}$$

But (24) is not so easy to be solved, except in special cases.

Case 1 $u(x,y)$ is a function of x only

We want to find the necessary condition that $u(x,y)$ is a function of x only. From (24), we have

$$-N\frac{\mathrm{d}u}{\mathrm{d}x} + \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)u = 0 \rightarrow \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}. \tag{25}$$

The left hand side of the equation is a function of x only, thus it is necessary that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \tag{26}$$

is also a function of x only.

Case 2 $u(x,y)$ is a function of y only

Similarly, the necessary condition that $u(x,y)$ is a function of y only is established and it requires that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} \tag{27}$$

is a function of y only and in this case $u(y)$ satisfies

$$\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}y} = -\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} \tag{28}$$

Example 3 Find the solution of the ODE

$$(3xy + y^2) + (x^2 + xy)y' = 0. \tag{29}$$

Solution:

Case One: Let's first determine whether it has an integrating factor that depends on x only.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{1}{x}. \tag{30}$$

Thus there is an integrating factor u that is a function of x only and it satisfies the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u}{x}. \tag{31}$$

Hence

$$u(x) = x. \tag{32}$$

Multiplying this integrating factor on (29), we obtain

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0. \tag{33}$$

Check (33), we find that

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial(3x^2y+xy^2)}{\partial y} = 3x^2+2xy, \quad \frac{\partial N(x,y)}{\partial x} = \frac{\partial(x^3+x^2y)}{\partial x} = 3x^2+2xy. \tag{34}$$

Thus, the new equation is exact and it's solution is given implicitly by

$$x^3y+\frac{1}{2}x^2y^2 = C. \tag{35}$$

There is no integrating factor which is only a function of y. But you may also verify that a second integrating factor is

$$u(x,y) = \frac{1}{xy(2x+y)}, \tag{36}$$

and the same solution is obtained.

3 Second Order Linear Equations

3.1 Homogeneous equations with constant coefficient

The **general form of second order ODE** is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}). \tag{1}$$

In particular, we consider

$$f(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}) = -p(x)\frac{\mathrm{d}y}{\mathrm{d}x} - q(x)y + g(x), \tag{2}$$

i. e.,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = g(x). \tag{3}$$

This is the **general form of second order linear ODE**. Initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1. \tag{4}$$

Equations (3) together with (4) are called an **initial value problem**.

If $g(x) = 0$, (3) is called **homogeneous**, otherwise, it is **non-homogeneous**. In this section, we consider

$$p(x) = b, \quad q(x) = c, \tag{5}$$

where b, c are constants, i. e.,

$$y'' + by' + cy = 0. \tag{6}$$

First, we consider a particular example.

$$y'' - y = 0. \tag{7}$$

Inspect: $y'' = y$ and note that $((e)^x)' = e^x$, $(e^x)'' = e^x$. Thus, $y_1(x) = e^x$ is a solution. Also, $(e^{-x})'' = e^{-x}$. Thus, $y_2(x) = e^{-x}$ is another solution. Further, $C_1 e^x$ and $C_2 e^{-x}$ (C_1 and C_2 are any constants) are still two solutions. Also,

$$(C_1 e^x + C_2 e^{-x})'' = C_1 e^x + C_2 e^{-x}. \tag{8}$$

Thus, $C_1 e^x + C_2 e^{-x}$ is also the solution, representing double infinite family of solutions, see Figure 2.

Suppose that we further impose

$$y(0) = 2, \quad y'(0) = -1. \tag{9}$$

$$\begin{aligned} &y(0) = 2: \quad 2 = C_1 + C_2, \\ &y'(0) = -1: \quad -1 = C_1 - C_2, \quad \to C_1 = \frac{1}{2}, \quad C_2 = \frac{3}{2}. \end{aligned} \tag{10}$$

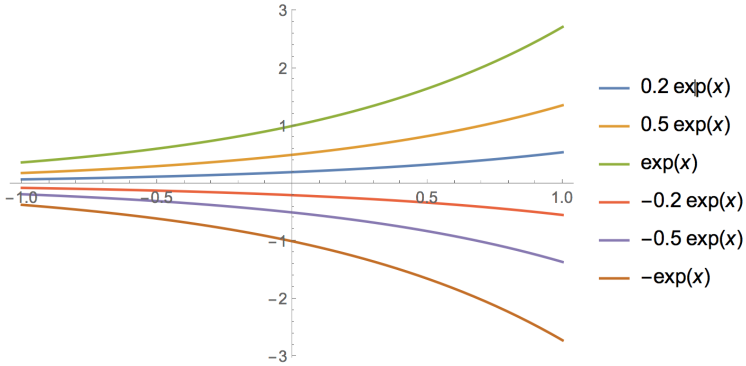
Thus,

$$y = \frac{1}{2}e^x + \frac{3}{2}e^{-x}. \tag{11}$$

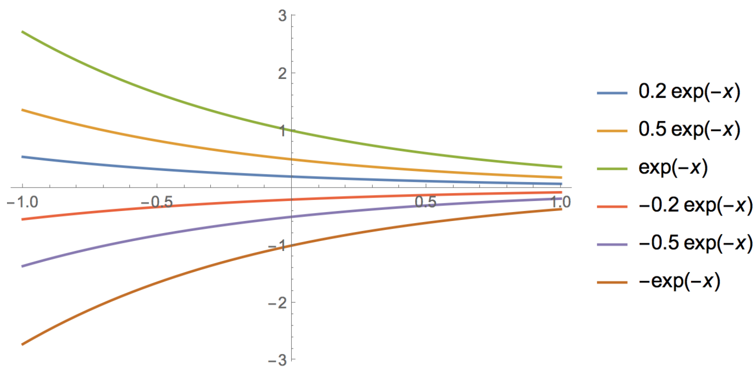
This is the solution to the initial value problem of (7) and (9).

Consider

$$y'' + by' + cy = 0. \tag{12}$$



(a) The family solutions $y = C_1 e^x$.



(b) The family solutions $y = C_2 e^{-x}$.

Figure 2: The two families of solutions to (7).

Seek a solution of the form

$$y = e^{rx}, \tag{13}$$

where r needs to be determined. Substituting (13) into (12):

$$r^2 e^{rx} + bre^{rx} + ce^{rx} = 0 \ \rightarrow \ r^2 + br + c = 0. \tag{14}$$

This algebraic equation for r is called the **characteristic equation** of (12).

For the solution of the characteristic equation, there are three cases:

(Case a) Two distinct real roots: r_1 and r_2.

(Case b) Two repeated roots: $r_1 = r_2$.

(Case c) Two complex conjugate roots: r_1, r_2 where $r_2 = \overline{r_1}$.

In this section, we only consider (Case a). Thus, we have two solutions:

$$y_1(x) = e^{r_1 x}, \quad y_2(x) = e^{r_2 x}. \tag{15}$$

It can be verified that

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}, \tag{16}$$

satisfies the (12) and thus is also a solution. Here, C_1, C_2 are arbitrary constants. It turns out that (16) contains every possible solutions, i.e., it is the **general solution** (this will be justified later on).

Next, suppose that we impose the initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1. \tag{17}$$

Then we have

$$\begin{aligned} &y(x_0) = y_0: \ y_0 = C_1 e^{r_1 x_0} + C_2 e^{r_2 x_0}; \\ &y'(x_0) = y_1: \ y_1 = C_1 r_1 e^{r_1 x_0} + C_2 r_2 e^{r_2 x_0}. \\ &\to C_1 = \frac{y_1 - y_0 r_2}{r_1 - r_2} e^{-r_1 x_0}, \quad C_2 = \frac{y_0 r_1 - y_1}{r_1 - r_2} e^{-r_2 x_0}. \end{aligned} \tag{18}$$

Since C_1 and C_2 are uniquely determined, we have the unique solution to the initial value problem (12) and (17), which is

$$y = \frac{y_1 - y_0 r_2}{r_1 - r_2} e^{-r_1 x_0} e^{r_1 x} + \frac{y_0 r_1 - y_1}{r_1 - r_2} e^{-r_2 x_0} e^{r_2 x}. \tag{19}$$

Example 1: Find the general solution of the following ODE:

$$2y'' - 3y' + y = 0. \tag{20}$$

Solution: Seek a solution of the form

$$y = e^{rx}. \tag{21}$$

Substituting it into the ODE,

$$2r^2 e^{rx} - 3re^{rx} + e^{rx} = 0 \ \to \ 2r^2 - 3r + 1 = 0 \ \to \ (2r-1)(r-1) = 0 \ \to \ r = \frac{1}{2} \text{ or } 1. \tag{22}$$

Thus, there are two solutions:

$$y_1 = e^{\frac{1}{2}x}, \quad y_2 = e^x. \tag{23}$$

The general solution is

$$y = C_1 e^{\frac{1}{2}x} + C_2 e^x, \tag{24}$$

where C_1 and C_2 are arbitrary constants.

3.2 Fundamental Solutions of Linear Homogeneous Equations

Consider

$$y'' + p(x)y' + q(x)y = g(x), \quad y(x_0) = y_0, \ y'(x_0) = y_1. \tag{1}$$

Theorem If $p(x)$, $q(x)$ and $g(x)$ are continuous in an open interval I: $\alpha < x < \beta$, which contains x_0, then the initial value problem has a unique solution which is valid in I.

Example 1: Determine the largest intervals in which the following initial value problem has a unique solution.

$$x(x-4)y'' + 3xy' + 4y = 2, \quad y(3) = 0, \quad y'(3) = 1. \tag{2}$$

Solution: The equation can be arranged as

$$y'' + \frac{3x}{x(x-4)} y' + \frac{4}{x(x-4)} y = \frac{2}{x(x-4)}. \tag{3}$$

$p(x)$, $q(x)$ and $g(x)$ are continuous for $x \neq 0$ and $x \neq 4$, i.e., in $-\infty < x < 0$ or $0 < x < 4$ or $4 < x < \infty$. The interval $0 < x < 4$ contains $x = 3$. Thus, it is the largest interval in which there is a unique solution.

Theorem If y_1 and y_2 are two solutions to the differential equations $y'' + p(x)y' + q(x)y = 0$, then the linear combination $C_1y_1 + C_2y_2$ is also a solution for any values of C_1 and C_2.
Proof

$$\begin{aligned} &y_1 \text{ is a solution}: \quad y_1'' + p(x)y_1' + q(x)y_1 = 0, \\ &y_2 \text{ is a solution}: \quad y_2'' + p(x)y_2' + q(x)y_2 = 0. \end{aligned} \tag{4}$$

Let $y = C_1y_1 + C_2y_2$, we have

$$\begin{aligned} LHS &= [C_1y_1 + C_2y_2]'' + p(x)[C_1y_1 + C_2y_2]' + q(x)[C_1y_1 + C_2y_2] \\ &= C_1[y_1'' + p(x)y_1' + q(x)y_1] + C_2[y_2'' + p(x)y_2' + q(x)y_2] \\ &= 0 = RHS. \end{aligned} \tag{5}$$

Question: Whether C_1 and C_2 can be uniquely chosen to satisfy any initial conditions, say $y(x_0) = a_0$, $y'(x_0) = a_1$?

Answer:

$$\begin{aligned} &y(x_0) = a_0: \quad a_0 = C_1y_1(x_0) + C_2y_2(x_0), \\ &y'(x_0) = a_1: \quad a_1 = C_1y_1'(x_0) + C_2y_2'(x_0), \\ &\to C_1 = \frac{\begin{vmatrix} a_0 & y_2(x_0) \\ a_1 & y_2'(x_0) \end{vmatrix}}{W_0}, \quad C_2 = \frac{\begin{vmatrix} y_1(x_0) & a_0 \\ y_1'(x_0) & a_1 \end{vmatrix}}{W_0}, \quad (\text{Cramer's rule}) \end{aligned} \tag{6}$$

where

$$W_0 = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}. \tag{7}$$

Terminology:

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \tag{8}$$

is called Wronskian for $y_1(x)$ and $y_2(x)$.

One can observe that $W_0 = W|_{x=x_0}$. Thus we conclude that if $W_0 = W|_{x=x_0} \neq 0$, then C_1 and C_2 can be uniquely chosen to satisfy the initial conditions

$$y(x_0) = a_0, \quad y'(x_0) = a_1. \tag{9}$$

Theorem Suppose that y_1 and y_2 are two solutions of

$$y'' + p(x)y' + q(x)y = 0. \tag{10}$$

If there is a point x_0 such that

$$W(y_1, y_2)|_{x=x_0} \neq 0 \tag{11}$$

then

$$y = C_1 y_1(x) + C_2 y_2(x) \tag{12}$$

is the general solution. If (11) is satisfied, we call y_1 and y_2 form **a fundamental set of solutions**.

Example 1 Suppose that the characteristic equation of

$$y'' + by' + cy = 0 \tag{13}$$

has two distinct real roots, r_1, r_2. Show that $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$ are a fundamental set of solutions.

Proof: We already know that y_1 and y_2 are two solutions.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix} = r_2 e^{(r_1+r_2)x} - r_1 e^{(r_1+r_2)x} = (r_2 - r_1) e^{(r_1+r_2)x} \neq 0. \tag{14}$$

The above inequality holds for any x. Thus y_1 and y_2 are a fundamental set of solutions.

Question: What is the general solution?

The general solution: $y = C_1 y_1 + C_2 y_2 = C_1 e^{r_1 x} + C_2 e^{r_2 x}$.

Proof: Suppose that $y = \phi(x)$ is any solution of (13), we need to show that C_1 and C_2 can be found such that

$$\phi(x) = C_1 y_1(x) + C_2 y_2(x). \tag{15}$$

Denote $\phi(x_0) = a_0$, $\phi'(x_0) = a_1$. Then, the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = a_0, \quad y'(x_0) = a_1 \tag{16}$$

has a unique solution according to the theorem on the uniqueness and existence. Then $y = \phi(x)$ should be this unique solution (it satisfies both the ODE and initial conditions). On the other hand, we know that for the function

$$\bar{\phi}(x) = C_1 y_1(x) + C_2 y_2(x) \tag{17}$$

to satisfy (16), we only have to choose

$$C_1 = \frac{\begin{vmatrix} a_0 & y_2(x_0) \\ a_1 & y_2'(x_0) \end{vmatrix}}{W_0}, \quad C_2 = \frac{\begin{vmatrix} y_1(x_0) & a_0 \\ y_1'(x_0) & a_1 \end{vmatrix}}{W_0}, \tag{18}$$

where

$$W_0 = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}. \tag{19}$$

Since $y_1(x)$ and $y_2(x)$ are a fundamental set of solutions as shown by (14), W_0 in (19) is nonzero. So C_1 and C_2 are uniquely determined. Therefore $\bar{\phi}(x)$ is also a solution to the initial value problem (16). We know both $\bar{\phi}(x)$

and $\phi(x)$ are solutions to the initial value problem (16) while this initial value problem has a unique solution. Therefore,

$$\phi(x) = \bar{\phi}(x) = C_1 y_1(x) + C_2 y_2(x). \tag{20}$$

So $y(x) = C_1 y_1(x) + C_2 y_2(x)$ is the general solution.

3.3 Linear Independence and Wronskian

Definition f and g are said to be linear dependent on interval I if there exist two constants k_1 and k_2, not both zero, such that

$$k_1 f(x) + k_2 g(x) = 0 \quad \text{for all } x \text{ in } I. \tag{1}$$

Otherwise, if k_1 and k_2 have to be zero to make the above equation hold, f and g are said to be linear independent.

Theorem Suppose that f and g are differentiable on an open interval I. If $W(f,g) \neq 0$ for a point x_0 in I, then f and g are linear independent. Equivalently, this theorem states that if f, g are linear dependent, then $W(f,g) = 0$ for every point x in I.

Proof We start from

$$k_1 f(x) + k_2 g(x) = 0 \quad \text{in } I. \tag{2}$$

We need to show that k_1 and k_2 must be zero.
Differentiation gives

$$k_1 f'(x) + k_2 g'(x) = 0. \tag{3}$$

Consider at $x = x_0$:

$$\begin{aligned} k_1 f(x_0) + k_2 g(x_0) &= 0, \\ k_1 f'(x_0) + k_2 g'(x_0) &= 0. \end{aligned} \tag{4}$$

The coefficient matrix determinant:

$$\begin{vmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{vmatrix} = W(f,g)|_{x=x_0} \neq 0. \tag{5}$$

This implies k_1 and k_2 must be zero.

Theorem Consider

$$y'' + p(x)y' + q(x)y = 0, \tag{6}$$

where $p(x)$ and $q(x)$ are supposed to be continuous in an open interval I. If y_1 and y_2 are two solutions, then

$$W(y_1, y_2) = C \cdot \exp\left[-\int p(x)\mathrm{d}x\right], \tag{7}$$

where C is a constant dependent on y_1 and y_2.

Proof We have

$$\begin{aligned} y_1'' + p(x)y_1' + q(x)y_1 &= 0, \\ y_2'' + p(x)y_2' + q(x)y_2 &= 0. \end{aligned} \tag{8}$$

Besides, for the Wronskian, one has

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1', \\ \frac{\mathrm{d}W(y_1, y_2)}{\mathrm{d}x} &= y_1 y_2'' + y_1' y_2' - y_1' y_2' - y_2 y_1'' = y_1 y_2'' - y_2 y_1''. \end{aligned} \tag{9}$$

From $(8)_2 \cdot y_1 - (8)_1 \cdot y_2$, we have

$$y_1 y_2'' - y_2 y_1'' + p(x) y_1 y_2' - p(x) y_2 y_1' = 0 \;\to\; \frac{\mathrm{d}W}{\mathrm{d}x} + p(x)W = 0 \;\to\; \frac{\mathrm{d}W}{W} = -p(x)\mathrm{d}x \tag{10}$$

which yields

$$W = C\exp\left[-\int p(x)\mathrm{d}x\right]. \tag{11}$$

Notes: (i) If $W = 0$ at one point x_0, then $C = 0$. Therefore, $W = 0$ in the whole interval.

(ii) If $W \neq 0$ at one point x_0, then $C \neq 0$. Thus $W \neq 0$ for all x.

Example 1 Find the Wronskian of two solutions of the following ODE without solving the ODE

$$x^2 y'' - x(x+2)y' + (x+2)y = 0. \tag{12}$$

Solution

$$y'' - \frac{x+2}{x}y' + \frac{x+2}{x^2}y = 0, \quad \text{for } x \neq 0. \tag{13}$$

The Wronskian is

$$W = C\exp\left[-\int p(x)\mathrm{d}x\right] = C\exp\left[\int \frac{x+2}{x}\mathrm{d}x\right] = C\exp[x + 2\ln|x|] = Ce^x \cdot e^{\ln x^2} = Cx^2 e^x. \tag{14}$$

Theorem Suppose that y_1 and y_2 are solutions of

$$y'' + p(x)y' + q(x)y = 0, \tag{15}$$

where $p(x), q(x)$ are continuous in an open interval I. Then y_1, y_2 are linear dependent if and only if $W(y_1, y_2)$ is zero for all x in I.

Proof The necessary condition is stated already by the previous theorem. To prove the sufficient condition, consider y_1 and y_2 are non-zero functions. If either one of them is zero function, then they are of course linear dependent. We know

$$0 = W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x), \tag{16}$$

which implies

$$\frac{y_1'}{y_1} = \frac{y_2'}{y_2}. \tag{17}$$

Integrating the above ODE, one has

$$\ln|y_1| = \ln|y_2| + C \;\to\; y_1 = Cy_2 \;\to\; y_1 - Cy_2 = 0, \tag{18}$$

where the value of C depend on y_1 and y_2. Anyway, it is nonzero and we can say y_1 and y_2 are linear dependent.

Summary

Following five statements are equivalent, regarding $y'' + p(x)y' + q(x) = 0$ with two solutions y_1 and y_2:

1. $W(y_1, y_2)|_{x=x_0} \neq 0$;

2. $W(y_1, y_2) \neq 0$ for all x;

3. y_1 and y_2 are linear independent.

4. y_1 and y_2 are a fundamental set of solutions;

5. $C_1 y_1 + C_2 y_2$ is the general solution;

3.4 Complex roots of the characteristic equations

Consider

$$y'' + by' + cy = 0. \tag{1}$$

Seek a solution of the form:

$$y = e^{rx}. \tag{2}$$

The characteristic equation is

$$r^2 + br + c = 0. \tag{3}$$

We consider the case it has two complex conjugate roots, say

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu, \quad \mu \neq 0. \tag{4}$$

There are two solutions

$$\begin{aligned} y_1 &= e^{(\lambda+i\mu)x} = e^{\lambda x}e^{i\mu x} = e^{\lambda x}(\cos\mu x + i\sin\mu x) \\ y_2 &= e^{(\lambda-i\mu)x} = e^{\lambda x}e^{-i\mu x} = e^{\lambda x}(\cos\mu x - i\sin\mu x). \end{aligned} \tag{5}$$

Any linear combination of y_1 and y_2 are still solutions, therefore,

$$\begin{aligned} y_{11} &= \frac{1}{2}y_1 + \frac{1}{2}y_2 = e^{\lambda x}\cos\mu x, \\ y_{12} &= \frac{1}{2i}(y_1 - y_2) = e^{\lambda x}\sin\mu x, \end{aligned} \tag{6}$$

are also two solutions to (1). Consider the Wronskian of y_{11} and y_{12}, we have

$$\begin{aligned} W(y_{11}, y_{12}) &= \begin{vmatrix} y_{11} & y_{12} \\ y_{11}' & y_{12}' \end{vmatrix} \\ &= \begin{vmatrix} e^{\lambda x}\cos\mu x & e^{\lambda x}\sin\mu x \\ \lambda e^{\lambda x}\cos\mu x - \mu e^{\lambda x}\sin\mu x & \lambda e^{\lambda x}\sin\mu x + \mu e^{\lambda x}\cos\mu x \end{vmatrix} \\ &= \mu e^{2\lambda x} \neq 0. \end{aligned} \tag{7}$$

Thus, y_{11} and y_{12} are a fundamental set of solutions and the general solution is

$$y = C_1 e^{\lambda x}\cos\mu x + C_2 e^{\lambda x}\sin\mu x. \tag{8}$$

Example 1 Find the general solution of the following ODE

$$y'' + 2y' + 2y = 0. \tag{9}$$

Solution Seek a solution of the form $y = e^{rx}$ and substitute it into the ODE, one has

$$r^2 + 2r + 2 = 0. \tag{10}$$

The complex conjugate roots are

$$r_1 = \frac{-2 + \sqrt{4 - 4\cdot 2}}{2} = -1 + i, \quad r_2 = -1 - i. \tag{11}$$

The general solution is

$$y = C_1 e^{\lambda x}\cos\mu x + C_2 e^{\lambda x}\sin\mu x = C_1 e^{-x}\cos x + C_2 e^{-x}\sin x. \tag{12}$$

Example 2 Find the solution of the following initial-value problem

$$y'' - 2y' + 5y = 0, \quad y\left(\frac{\pi}{2}\right) = 0, \quad y'\left(\frac{\pi}{2}\right) = 2. \tag{13}$$

Solution The characteristic equation is

$$r_1 = \frac{2 + \sqrt{4 - 4 \cdot 5}}{2} = 1 + 2i, \quad r_2 = 1 - 2i, \quad \lambda = 1, \quad \mu = 2. \tag{14}$$

The general solution is

$$y = C_1 e^{\lambda x} \cos \mu x + C_2 e^{\lambda x} \sin \mu x = C_1 e^x \cos(2x) + C_2 e^x \sin(2x). \tag{15}$$

Taking the initial conditions into the general solution, one has

$$\begin{aligned} &y\left(\frac{\pi}{2}\right) = 0 \;\rightarrow\; 0 = C_1 e^{\frac{\pi}{2}}(-1) \;\rightarrow\; C_1 = 0 \;\rightarrow\; y = C_2 e^x \sin(2x), \\ &y'\left(\frac{\pi}{2}\right) = 2 \;\rightarrow\; 2 = 2C_2 e^{\frac{\pi}{2}}(-1) \;\rightarrow\; C_2 = -e^{-\frac{\pi}{2}}. \end{aligned} \tag{16}$$

Therefore, the solution to the initial value problem is

$$y = -e^{-\frac{\pi}{2}} e^x \sin(2x). \tag{17}$$

3.5 Repeated Roots: Reduction of Order

Consider

$$y'' + by' + cy = 0. \tag{1}$$

The characteristic equation:

$$r^2 + br + c = 0. \tag{2}$$

We consider the case $b^2 - 4c = 0$, i.e., there are two repeated roots: $r_1 = r_2 = -\frac{b}{2}$. One solution is $y_1 = e^{-\frac{b}{2}x}$.

How to find another linear independent solution? (Reduction of order)

We know that $C_1 y_1$ satisfies the ODE. We replace C_1 by a function of x, say $v(x)$. Then, we try to determine $v(x)$ to make $y_2 = v(x)y_1$ be a solution which is also independent of y_1. Substituting $y_2 = v(x)y_1$ into the ODE, we have

$$\begin{aligned} &v''y_1 + 2v'y_1' + vy_1'' + b(v'y_1 + vy_1') + cvy_1 = 0 \\ \rightarrow\; &v(y_1'' + by_1' + cy_1) + v''y_1 + 2v'y_1' + bv'y_1 = 0 \\ \rightarrow\; &0 + v''y_1 + v'(2y_1' + by_1) = 0 \\ \rightarrow\; &v''e^{-\frac{b}{2}x} + v' \cdot 0 = 0 \\ \rightarrow\; &v'' = 0 \\ \rightarrow\; &v(x) = d_1 x + d_2, \end{aligned} \tag{3}$$

where d_1, d_2 are arbitrary constants.

We only need one form of $v(x)$ as long as $v(x)y_1$ is a solution linearly independent of y_1. Setting $d_1 = 1$ and $d_2 = 0$, we have

$$y_2 = xy_1 = xe^{-\frac{b}{2}x}. \tag{4}$$

Further,

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^{-\frac{b}{2}x} & xe^{-\frac{b}{2}x} \\ -\frac{b}{2}e^{-\frac{b}{2}x} & e^{-\frac{b}{2}x}-\frac{bx}{2}e^{-\frac{b}{2}x} \end{vmatrix}=e^{-bx}\neq 0. \tag{5}$$

This means y_1 and y_2 are linear independent. So the general solution is

$$y=C_1y_1+C_2y_2=C_1e^{-\frac{b}{2}x}+C_2xe^{-\frac{b}{2}x}. \tag{6}$$

Example 1 Find the general solution of the following ODE

$$4y''+12y'+9y=0. \tag{7}$$

Solution The characteristic equation is

$$4r^2+12r+9=0 \;\rightarrow\; (2r+3)^2=0 \;\rightarrow\; r_1=-\frac{3}{2}=r_2. \tag{8}$$

Thus one solution is

$$y_1=e^{-\frac{3}{2}x}. \tag{9}$$

The second solution is

$$y_2=xy_1=xe^{-\frac{3}{2}x}. \tag{10}$$

The general solution is

$$y=C_1e^{-\frac{3}{2}x}+C_2xe^{-\frac{3}{2}x}. \tag{11}$$

Example 2 Find the solution of the following initial value problem

$$y''-6y'+9y=0, \quad y(0)=0, \quad y'(0)=2. \tag{12}$$

Solution The characteristic equation is

$$r^2-6r+9=0 \;\rightarrow\; (r-3)^2=0 \;\rightarrow\; r_1=r_2=3. \tag{13}$$

One solution is

$$y_1=e^{3x}. \tag{14}$$

The second solution is

$$y_2=xy_1=xe^{3x}. \tag{15}$$

The general solution is

$$y=C_1e^{3x}+C_2xe^{3x}. \tag{16}$$

The initial conditions yield

$$\begin{aligned} &y(0)=0: \quad 0=C_1 \;\rightarrow\; y=C_2xe^{3x} \\ &y'(0)=2 \;\rightarrow\; 2=C_2. \end{aligned} \tag{17}$$

The solution is

$$y = 2xe^{3x}. \tag{18}$$

Reduction of Order (For reading only)

Suppose that $y_1(x)$ is a solution of

$$y'' + p(x)y' + q(x)y = 0. \tag{19}$$

To find another solution, we let $y_2 = v(x)y_1$. Substituting it into the ODE,

$$\begin{aligned} v''y_1 + 2v'y_1' + vy_1'' + p(x)(v'y_1 + vy_1') + q(x)vy_1 = 0 \\ v[y_1'' + p(x)y_1' + q(x)y_1] + v''y_1 + v'[2y_1' + p(x)y_1] = 0 \\ 0 + v''y_1 + v'[2y_1' + p(x)y_1] = 0. \end{aligned} \tag{20}$$

Setting $u = \frac{\mathrm{d}v}{\mathrm{d}x}$, one has

$$y_1\frac{\mathrm{d}u}{\mathrm{d}x} + [2y_1' + p(x)y_1]u = 0. \tag{21}$$

This is a first order ODE for u, you can solve it by using the integrating factor.

Example 1 Suppose that $y_1(x) = \sin x^2$ is a solution of

$$xy'' - y' + 4x^3y = 0, \quad x > 0. \tag{22}$$

Find the general solution.

Solution Seek a solution of the form

$$y_2 = v(x)y_1 = v(x)\sin x^2. \tag{23}$$

Substituting it into the ODE, one has

$$\begin{aligned} & x[v''y_1 + 2v'y_1' + vy_1''] - (v'y_1 + vy_1') + 4x^3vy_1 = 0 \\ \to\ & v[xy_1'' - y_1' + 4x^3y_1] + xy_1v'' + v'[2xy_1' - y_1] = 0 \\ \to\ & 0 + xy_1v'' + v'[2xy_1' - y_1] = 0. \end{aligned} \tag{24}$$

Let $u = \frac{\mathrm{d}v}{\mathrm{d}x}$, we have

$$\begin{aligned} & x\sin x^2\frac{\mathrm{d}u}{\mathrm{d}x} + [4x^2\cos x^2 - \sin x^2]u = 0 \\ \to\ & \frac{\mathrm{d}u}{u} = -\frac{4x^2\cos x^2 - \sin x^2}{x\sin x^2}\mathrm{d}x = -\frac{4x\cos x^2}{\sin x^2}\mathrm{d}x + \frac{1}{x}\mathrm{d}x \\ & \ln|u| = -\int\frac{2\mathrm{d}\sin x^2}{\sin x^2} + \ln|x| + C_1 = -2\ln|\sin x^2| + \ln|x| + C_1 \\ \to\ & u = C_2\frac{x}{(\sin x^2)^2}. \end{aligned} \tag{25}$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = C_2\frac{x}{(\sin x^2)^2} \ \to\ v = C_2\int\frac{x}{(\sin x^2)^2}\mathrm{d}x + C_3. \tag{26}$$

Letting $t = x^2$, one has $\mathrm{d}t = 2x\mathrm{d}x$ and the above solution to v becomes

$$v = C_2 \int \frac{\frac{1}{2}}{(\sin t)^2}\mathrm{d}t + C_3 = -\frac{C_2}{2}\cot t + C_3 = -\frac{C_2}{2}\cot x^2 + C_3. \tag{27}$$

Choose $C_2 = -2, C_3 = 0$, we have the second solution to be

$$y_2 = vy_1 = \cot x^2 \sin x^2 = \cos x^2. \tag{28}$$

Check

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin x^2 & \cos x^2 \\ 2x\cos x^2 & -2x\sin x^2 \end{vmatrix} = -2x \neq 0. \tag{29}$$

Thus, the general solution is

$$y = C_1 \sin x^2 + C_2 \cos x^2. \tag{30}$$

3.6 Non-homogeneous Equations and Method of Undetermined Coefficients

Consider

$$y'' + p(x)y' + q(x)y = g(x), \tag{1}$$

and corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

Theorem If $Y_1(x)$ and $Y_2(x)$ are two solutions of (1), then $Y_2(x) - Y_1(x)$ is a solution of (2).

Proof: Substituting Y_1 and Y_2 into (1), one has

$$\begin{aligned} Y_1'' + p(x)Y_1' + q(x)Y_1 = g(x), \\ Y_2'' + p(x)Y_2' + q(x)Y_2 = g(x). \end{aligned} \tag{3}$$

Then $(3)_2 - (3)_1$ gives:

$$(Y_2 - Y_1)'' + p(x)(Y_2 - Y_1)' + q(x)(Y_2 - Y_1) = 0, \tag{4}$$

which implies $Y_2 - Y_1$ satisfies (2).

Theorem If $Y(x)$ is a particular solution of (1) and $y_1(x)$ and $y_2(x)$ are a set of fundamental solutions to (2), then the general solution to (1) is given by

$$y = C_1 y_1(x) + C_2 y_2(x) + Y(x), \quad C_1, C_2 \text{ are arbitrary constants.} \tag{5}$$

Proof: We need to show that any solution of (1) is included in (5). Consider any solution $Y_1(x)$, we know $Y_1(x) - Y(x)$ is a solution of (2). As $C_1y_1 + C_2y_2$ is the general solution of (2), there exist constants c_1 and c_2 such that

$$Y_1(x) - Y(x) = c_1y_1 + c_2y_2 \to Y_1(x) = c_1y_1 + c_2y_2 + Y(x), \tag{6}$$

which indicates $Y_1(x)$ is included in (5).

Method of Undetermined Coefficients

This method is used to find the particular solution. The procedure is as follows:

(i) Assume a form of solution with some coefficients;

(ii) Determine those coefficients.

Usually, this method is for the cases that the non-homogeneous term is exponential-, sine-, cosine-functions or polynomials and further the homogeneous ODE is the one with constant coefficients.

Example 1 Find the general solution of the ODE

$$y'' - 2y' - 3y = 3e^{2x}. \tag{7}$$

Solution: The general solution of the homogeneous equation:

$$y'' - 2y' - 3y = 0 \to r^2 - 2r - 3 = 0 \to r_1 = -1, r_2 = 3 \to y = C_1e^{-x} + C_2e^{3x}. \tag{8}$$

Assume a solution of the form

$$Y = Ae^{2x}, \quad A \text{ is to be determined.} \tag{9}$$

Substituting the solution into the ODE, one has

$$4Ae^{2x} - 2 \cdot 2Ae^{2x} - 3Ae^{2x} = 3e^{2x} \to -3A = 3 \to A = -1. \tag{10}$$

Thus, one particular solution is

$$Y = -e^{2x} \tag{11}$$

And the general solution is

$$y = C_1 e^{-x} + C_2 e^{3x} - e^{2x}. \tag{12}$$

Example 2 Find a particular solution of the ODE

$$y'' - 2y' - 3y = 2\sin x. \tag{13}$$

Solution: Assume a solution of the form

$$Y = A\sin x. \tag{14}$$

Substituting it into the ODE, one has

$$-A\sin x - 2A\cos x - 3A\sin x = 2\sin x \to -4A\sin x - 2A\cos x = 2\sin x \to \text{can not be satisfied.} \tag{15}$$

Instead, we assume

$$Y = A\sin x + B\cos x. \tag{16}$$

Substituting it into the ODE, one has

$$\begin{aligned}
&-A\sin x - B\cos x - 2(A\cos x - B\sin x) - 3A\sin x - 3B\cos x = 2\sin x \\
&\to (-4A + 2B)\sin x + (-4B - 2A)\cos x = 2\sin x \\
&\to -4A + 2B = 2, \quad -4B - 2A = 0 \\
&\to A = -\frac{2}{5}, \quad B = \frac{1}{5}.
\end{aligned} \tag{17}$$

Thus, the particular solution is

$$Y = -\frac{2}{5}\sin x + \frac{1}{5}\cos x. \tag{18}$$

Example 3 Find a particular solution of the ODE

$$y'' - 2y' - 3y = 2x^2. \tag{19}$$

Solution: Assume a solution of the form

$$Y = Ax^2 + Bx + C. \tag{20}$$

Substituting it into the ODE, one has

$$
\begin{aligned}
&2A-2(2Ax+B)-3(Ax^2+Bx+C)=2x^2\\
&\rightarrow -3Ax^2+(-4A-3B)x+2A-2B-3C=2x^2\\
&\rightarrow -3A=2,\quad -4A-3B=0,\quad 2A-2B-3C=0\\
&\rightarrow A=-\frac{2}{3},\quad B=\frac{8}{9},\quad C=-\frac{28}{27}.
\end{aligned}
\tag{21}
$$

Then the particular solution is

$$
Y=-\frac{2}{3}x^2+\frac{8}{9}x-\frac{28}{27}. \tag{22}
$$

Example 4 Find a particular solution to the ODE

$$
y''-2y'-3y=2e^x\sin x. \tag{23}
$$

Solution: Assume the solution in the form

$$
Y=e^x(A\sin x+B\cos x). \tag{24}
$$

Substituting it into the ODE, one has

$$
\begin{aligned}
&2e^x(A\cos x-B\sin x)-2e^x\left[(A-B)\sin x+(A+B)\cos x\right]-3e^x(A\sin x+B\cos x)=2e^x\sin x\\
&\rightarrow -5e^xA\sin x-5e^xB\cos x=2e^x\sin x\\
&\rightarrow -5A=2,\quad -5B=0\\
&\rightarrow A=-\frac{2}{5},\quad B=0.
\end{aligned}
\tag{25}
$$

Thus, the particular solution is

$$
Y=-\frac{2}{5}e^x\sin x. \tag{26}
$$

Example 5 Find a particular solution of the ODE

$$
y''-2y'-3y=3e^{2x}+2\sin x+2x^2+2e^x\sin x. \tag{27}
$$

Solution: Considering the ODE in this form

$$
y''+p(x)y'+q(x)y=g_1(x)+g_2(x). \tag{28}
$$

If y_1 is a solution of

$$
y''+p(x)y'+q(x)y=g_1(x), \tag{29}
$$

and y_2 is a solution of

$$
y''+p(x)y'+q(x)y=g_2(x), \tag{30}
$$

then y_1+y_2 is a solution of (28). For the current problem, we know

$$\begin{aligned}
&y''-2y'-3y=3e^{2x} \quad \text{solution is } y_1=-e^{2x},\\
&y''-2y'-3y=2\sin x \quad \text{solution is } y_2=-\frac{2}{5}\sin x+\frac{1}{5}\cos x,\\
&y''-2y'-3y=2x^2 \quad \text{solution is } y_3=-\frac{2}{3}x^2+\frac{8}{9}x-\frac{28}{27},\\
&y''-2y'-3y=2e^x\sin x \quad \text{solution is } y_4=-\frac{2}{5}e^x\sin x.
\end{aligned} \tag{31}$$

Thus, the solution for our problem is (note the superposition)

$$y=y_1+y_2+y_3+y_4=-e^{2x}-\frac{2}{5}\sin x+\frac{1}{5}\cos x-\frac{2}{3}x^2+\frac{8}{9}x-\frac{28}{27}-\frac{2}{5}e^x\sin x. \tag{32}$$

Example 6 Find a particular solution of

$$y''-y'-2y=e^{2x}. \tag{33}$$

Solution: Assume the solution in the form

$$Y=Ae^{2x}. \tag{34}$$

Substituting it into the ODE, one has

$$4Ae^{2x}-2Ae^{2x}-2Ae^{2x}=e^{2x}\to(4A-2A-2A)e^{2x}=e^{2x} \tag{35}$$

The equation can never be satisfied which means the assumed form is wrong! The reason is that e^{2x} is a solution of the corresponding homogeneous equation. Assume that

$$Y=Axe^{2x}. \tag{36}$$

Substituting it into the ODE, one has

$$\begin{aligned}
&4Ae^{2x}+4Axe^{2x}-A(e^{2x}+2xe^{2x})-2Axe^{2x}=e^{2x}\\
&\to(4A-2A-2A)xe^{2x}+(4A-A)e^{2x}=e^{2x}\\
&\to 3A=1\to A=\frac{1}{3}.
\end{aligned} \tag{37}$$

Thus the particular solution is

$$Y=\frac{1}{3}xe^{2x}. \tag{38}$$

Proof of the Method of Undetermined Coefficients

(i) Consider

$$y''+by'+cy=P_n(x)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n. \tag{39}$$

Assume that

$$Y=A_0x^n+A_1x^{n-1}+\cdots+A_{n-1}x+A_n. \tag{40}$$

Substituting it into the ODE, one has

$$\begin{aligned}
&n(n-1)A_0x^{n-2}+(n-1)(n-2)A_1x^{n-3}+\cdots+2A_{n-2}\\
&+b(nA_0x^{n-1}+(n-1)A_1x^{n-2}+\cdots+A_{n-1})\\
&+c(A_0x^n+A_1x^{n-1}\cdots+A_{n-1}x+A_n)\\
&=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n.
\end{aligned} \tag{41}$$

After arrangement of the two sides, we have

$$\begin{cases}
cA_0=a_0\\
cA_1+nbA_0=a_1\\
\cdots\\
cA_{n-1}+2bA_{n-2}+6A_{n-3}=a_{n-1}\\
cA_n+bA_{n-1}+2A_{n-2}=a_n
\end{cases} \tag{42}$$

If $c\neq 0$, $A_0,A_1,\cdots,A_n$ can be determined and this method works.

If $c=0$ but $b\neq 0$, then the left hand side of (41) is a polynomial of degree $n-1$, while the right hand side is always a polynomial of degree n. Thus (41) can never be satisfied. To make the left hand side of (41) to have degree n, we have to assume that

$$Y=x(A_0x^n+A_1x^{n-1}+\cdots+A_{n-1}x+A_n). \tag{43}$$

If both $b=0$ and $c=0$, then the left hand side of (41) has degree $n-2$. Thus, we have to assume

$$Y=x^2(A_0x^n+A_1x^{n-1}+\cdots+A_{n-1}x+A_n). \tag{44}$$

(ii) Consider

$$y''+by'+cy=e^{\alpha x}P_n(x). \tag{45}$$

Introduce

$$y=u(x)e^{\alpha x}. \tag{46}$$

Substituting it into the ODE, one has

$$\begin{aligned}
&\alpha^2e^{\alpha x}u+2\alpha u'e^{\alpha x}+u''e^{\alpha x}+b(u'+\alpha u)e^{\alpha x}+cue^{\alpha x}=e^{\alpha x}P_n(x)\\
&\to u''+(2\alpha+b)u'+(\alpha^2+b\alpha+c)u=P_n(x).
\end{aligned} \tag{47}$$

It is the form of (39) in part (i).

If $\alpha^2+b\alpha+c\neq 0$, we can assume that the solution u have the form (40). In other words, we can assume the particular solution to (45) is

$$Y=e^{\alpha x}(A_0x^n+a_1x^{n-1}+\cdots+A_{n-1}x+A_n). \tag{48}$$

If $\alpha^2+b\alpha+c=0$ and $2\alpha+b\neq 0$, (i.e., α is a root (not repeated root) of the characteristic equation $r^2+br+c=0$), then u has the form (43). In other words, we can assume

$$Y=xe^{\alpha x}(A_0x^n+a_1x^{n-1}+\cdots+A_{n-1}x+A_n). \tag{49}$$

If both $\alpha^2 + b\alpha + c = 0$ and $2\alpha + b = 0$ which implies

$$
\begin{aligned}
&\alpha = -\frac{b}{2}, \quad \left(-\frac{b}{2}\right)^2 + b\left(-\frac{b}{2}\right) + c = 0, \\
&\to -\frac{b^2}{4} + c = 0, \\
&\to b^2 - 4c = 0.
\end{aligned}
\tag{50}
$$

Therefore, $\alpha = -\frac{b}{2}$ is a repeated root of $r^2 + br + c = 0$. Then u should have the form (44). In other words,

$$
Y = x^2 e^{\alpha x}(A_0 x^n + a_1 x^{n-1} + \cdots + A_{n-1}x + A_n). \tag{51}
$$

(iii) Consider

$$
y'' + by' + cy = e^{\alpha x}\cos\beta x\, P_n(x) \quad \text{(discussions are similar for the case } e^{\alpha x}\sin\beta x\, P_n(x)\text{)}. \tag{52}
$$

We note that

$$
e^{i\beta x} = \cos\beta x + i\sin\beta x, \quad e^{-i\beta x} = \cos\beta x - i\sin\beta x \to \cos\beta x = \frac{e^{i\beta x} + e^{-i\beta x}}{2}. \tag{53}
$$

Then

$$
\begin{aligned}
&y'' + by' + cy = \frac{1}{2}e^{(\alpha+i\beta)x}P_n(x) + \frac{1}{2}e^{(\alpha-i\beta)x}P_n(x) \quad \text{(can be decomposed into two parts)} \\
&\to y'' + by' + cy = \frac{1}{2}e^{(\alpha+i\beta)x}P_n(x) \\
&\quad y'' + by' + cy = \frac{1}{2}e^{(\alpha-i\beta)x}P_n(x)
\end{aligned}
\tag{54}
$$

$(54)_2$ and $(54)_3$ degenerate to case (ii).

The discussions are as follows.

If $\alpha + i\beta$ is not a root of $r^2 + br + c = 0$, then correspondingly, $\alpha - i\beta$ is not a root to this equation neither. In this case, for $(54)_2$, the particular solution is given as

$$
y_1 = e^{(\alpha+i\beta)x}(A_0 x^n + A_1 x^{n-1} + \cdots + A_{n-1}x + A_n), \tag{55}
$$

and for $(54)_3$, the particular solution is given as

$$
y_2 = e^{(\alpha-i\beta)x}(B_0 x^n + B_1 x^{n-1} + \cdots + B_{n-1}x + B_n), \tag{56}
$$

Then the particular solution to (52) is given as

$$
\begin{aligned}
y =& y_1 + y_2 \\
=& \left[A_0 e^{(\alpha+i\beta)x} + B_0 e^{(\alpha-i\beta)x}\right]x^n + \left[A_1 e^{(\alpha+i\beta)x} + B_1 e^{(\alpha-i\beta)x}\right]x^{n-1} + \cdots + \left[A_n e^{(\alpha+i\beta)x} + B_n e^{(\alpha-i\beta)x}\right].
\end{aligned}
\tag{57}
$$

We may write

$$
\begin{aligned}
A_k e^{(\alpha+i\beta)x} + B_k e^{(\alpha-i\beta)x} &= e^{\alpha x}\left[(A_k + B_k)\cos\beta x + i(A_k - B_k)\sin\beta x\right] \\
&= e^{\alpha x}\left[C_k\cos\beta x + D_k\sin\beta x\right],
\end{aligned}
\tag{58}
$$

where $C_k = A_k + B_k$ and $D_k = i(A_k - B_k)$.

Then the particular solution is rewritten as

$$\begin{aligned} y &= e^{\alpha x}\left[C_0 \cos\beta x + D_0 \sin\beta x\right] x^n + e^{\alpha x}\left[C_1 \cos\beta x + D_1 \sin\beta x\right] x^{n-1} + \cdots + e^{\alpha x}\left[C_n \cos\beta x + D_n \sin\beta x\right] \\ &= e^{\alpha x}\cos\beta x\left[C_0 x^n + C_1 x^{n-1} + \cdots + C_n\right] + e^{\alpha x}\sin\beta x\left[D_0 x^n + D_1 x^{n-1} + \cdots + D_n\right]. \end{aligned} \tag{59}$$

If $\alpha + i\beta$ is a root to $r^2 + br + c = 0$, then so is $\alpha - i\beta$. In this case, we can get the particular solution is the form

$$y = e^{\alpha x}(\cos\beta x)\, x\left[C_0 x^n + C_1 x^{n-1} + \cdots + C_n\right] + e^{\alpha x}(\sin\beta x)\, x\left[D_0 x^n + D_1 x^{n-1} + \cdots + D_n\right] \tag{60}$$

3.7 Variation of Parameters

Consider

$$y'' + p(x)y' + q(x)y = g(x), \tag{1}$$

and

$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

Suppose that y_1 and y_2 are a fundamental set of solutions to (2), i.e.,

$$y = C_1 y_1 + C_2 y_2 \tag{3}$$

is the general solution of (2).

Based on the information, how can we find a particular solution to (1)?

Idea: Replacing C_1 and C_2 by two functions $u_1(x)$ and $u_2(x)$. Then determine $u_1(x)$ and $u_2(x)$ to make

$$Y = u_1(x)y_1 + u_2(x)y_2 \tag{4}$$

be a particular solution to (1).

Determine u_1 and u_2

We intend to substitute (4) into (1) to find u_1 and u_2, and we do this step by step. First of all,

$$Y' = u_1 y_1' + u_2 y_2' + \underline{u_1' y_1 + u_2' y_2}. \tag{5}$$

We always choose that

$$u_1' y_1 + u_2' y_2 = 0. \tag{6}$$

Then

$$Y'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''. \tag{7}$$

Substituting (4), (5) and (7) into (1), one has

$$\begin{aligned} &u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' + p(x)(u_1 y_1' + u_2 y_2') + q(x)(u_1 y_1 + u_2 y_2) = g(x) \\ \to\ &u_1 \underline{\left[y_1'' + p(x)y_1' + q(x)y_1\right]} + u_2 \underline{\left[y_2'' + p(x)y_2' + q(x)y_2\right]} + u_1' y_1' + u_2' y_2' = g(x), \end{aligned} \tag{8}$$

which implies

$$u_1' y_1' + u_2' y_2' = g(x). \tag{9}$$

We can regard (6) and (9) as two linear algebraic equations for u_1' and u_2' and they are solved as

$$\begin{aligned} &u_1' = -\frac{y_2 g(x)}{W(y_1,y_2)}, \quad u_2' = \frac{y_1 g(x)}{W(y_1,y_2)}, \\ &\to u_1 = \int -\frac{y_2 g(x)}{W(y_1,y_2)}\mathrm{d}x + k_1, \quad u_2 = \int \frac{y_1 g(x)}{W(y_1,y_2)}\mathrm{d}x + k_2 \\ &\to u_1 = \int -\frac{y_2 g(x)}{W(y_1,y_2)}\mathrm{d}x, \quad u_2 = \int \frac{y_1 g(x)}{W(y_1,y_2)}\mathrm{d}x, \end{aligned} \tag{10}$$

where k_1, k_2 are two arbitrary constants and they are set to be 0.

Example 1 Find the general solution to the following ODE

$$y'' + y = \tan x, \quad 0 < x < \frac{\pi}{2}. \tag{11}$$

Solution The general solution to the corresponding homogeneous equation

$$y'' + y = 0 \tag{12}$$

is

$$y = C_1 \cos x + C_2 \sin x. \tag{13}$$

Let

$$Y = u_1(x) \cos x + u_2(x) \sin x. \tag{14}$$

Then

$$Y' = -u_1 \sin x + u_2 \cos x + \underline{u_1' \cos x + u_2' \sin x}. \tag{15}$$

Choose that

$$u_1' \cos x + u_2' \sin x = 0. \tag{16}$$

Further, one has

$$Y'' = -u_1' \sin x - u_1 \cos x + u_2' \cos x - u_2 \sin x. \tag{17}$$

Substituting (14) and (17) into (11),

$$-u_1' \sin x - u_1 \cos x + u_2' \cos x - u_2 \sin x + u_1 \cos x + u_2 \sin x = \tan x, \tag{18}$$

which implies

$$-u_1' \sin x + u_2' \cos x = \tan x. \tag{19}$$

Solve u_1' and u_2' from (16) and (19),

$$u_1' = \frac{(\cos x)^2 - 1}{\cos x}, \quad u_2' = \sin x. \tag{20}$$

Then one solution to u_1 and u_2 are respectively

$$u_1 = \sin x + \ln \sqrt{\frac{1 - \sin x}{1 + \sin x}}, \quad u_2 = -\cos x. \tag{21}$$

Thus,

$$Y = \left(\sin x + \ln \sqrt{\frac{1 - \sin x}{1 + \sin x}} \right) \cos x - \cos x \sin x = \cos x \cdot \ln \sqrt{\frac{1 - \sin x}{1 + \sin x}}. \tag{22}$$

Then the general solution is

$$y = C_1 \cos x + C_2 \sin x + \cos x \cdot \ln \sqrt{\frac{1 - \sin x}{1 + \sin x}}. \tag{23}$$

Example 2 Find the general solution of the ODE

$$x^2 y'' - 2y = 3x^2 - 1, \quad x > 0. \tag{24}$$

Solution The corresponding homogeneous equation is

$$x^2y'' - 2y = 0. \tag{25}$$

Observation: coefficients are $x^2, 2$, and we may seek a solution of the form

$$y = Ax^2 + Bx + C. \tag{26}$$

Taking the solution into (25),

$$x^2 2A - 2Ax^2 - 2Bx - 2C = 0 \to B = C = 0. \tag{27}$$

Thus $y_1 = Ax^2$ is a solution and we may set $A = 1$. To find another solution, we use the method of reduction of order and let

$$y_2 = v(x)y_1 = v(x)x^2. \tag{28}$$

Substituting it into (25),

$$x^2(v''x^2 + 4xv' + 2v) - 2vx^2 = 0. \tag{29}$$

Setting $u = v'$, (29) becomes

$$\begin{aligned} x^4\frac{\mathrm{d}u}{\mathrm{d}x} + 4x^3u = 0 \to \frac{\mathrm{d}u}{u} = -\frac{4}{x}\mathrm{d}x \\ \ln|u| = -4\ln|x| + k_1 \to u = \frac{k_1}{x^4} = \frac{-3}{x^4}, \end{aligned} \tag{30}$$

where the arbitrary constant k_1 is taken to be -3. Then

$$v' = u = \frac{-3}{x^4} \to v = \frac{1}{x^3} + k_2 = \frac{1}{x^3}, \tag{31}$$

where the arbitrary constant k_2 is set to be 0. Thus

$$y_2 = v(x)y_1 = \frac{1}{x}. \tag{32}$$

The general solution for the homogeneous equation is

$$y = C_1x^2 + C_2\frac{1}{x}. \tag{33}$$

Then we intend to find a particular solution. Let

$$Y = u_1x^2 + u_2\frac{1}{x}. \tag{34}$$

Then

$$Y' = 2xu_1 - x^{-2}u_2 + \underline{u_1'x^2 + u_2'x^{-1}}. \tag{35}$$

Choose

$$u_1'x^2 + u_2'x^{-1} = 0. \tag{36}$$

Then

$$Y'' = 2u_1 + 2x^{-3}u_2 + 2xu_1' - x^{-2}u_2'. \tag{37}$$

Substituting (34) and (37) into (24),

$$x^2(2u_1 + 2x^{-3}u_2 + 2xu_1' - x^{-2}u_2') - 2(u_1x^2 + u_2x^{-1}) = 3x^2 - 1, \tag{38}$$

which gives

$$2x^3u_1' - u_2' = 3x^2 - 1. \tag{39}$$

From (36) and (39), one has

$$u_1' = \frac{1}{x} - \frac{1}{3x^3}, \quad u_2' = \frac{1}{3} - x^2. \tag{40}$$

Then the solutions to u_1 and u_2 are

$$u_1 = \ln|x| + \frac{1}{6x^2}, \quad u_2 = \frac{x}{3} - \frac{x^3}{3}. \tag{41}$$

Thus, the particular solution is

$$Y = x^2 \ln|x| + \frac{1}{6} + \frac{1}{3} - \frac{x^2}{3} = x^2 \ln|x| - \frac{x^2}{3} + \frac{1}{2}. \tag{42}$$

The general solution is

$$y = C_1x^2 + C_2x^{-1} + x^2 \ln|x| - \frac{x^2}{3} + \frac{1}{2} = C_1x^2 + C_2x^{-1} + x^2 \ln|x| + \frac{1}{2}. \tag{43}$$

4 Higher Order Linear Equations

4.1 General Theory of the n-th Order Linear Equations

All the results for the second order linear equations can be naturally extended to the n-th order linear equations. Consider

$$\frac{d^n y}{dx^n}+P_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+P_{n-1}(x)\frac{dy}{dx}+P_n(x)y=g(x). \tag{1}$$

Introduce

$$L[y]=\frac{d^n y}{dx^n}+P_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+P_{n-1}(x)\frac{dy}{dx}+P_n(x)y. \tag{2}$$

Then (1) can be rewritten as

$$L[y]=g(x). \tag{3}$$

Initial conditions are proposed as

$$y(x_0)=b_0,\quad y'(x_0)=b_1,\cdots,y^{(n-1)}(x_0)=b_{n-1}. \tag{4}$$

Theorem If $P_1(x),P_2(x),\cdots,P_n(x),g(x)$ are continuous on an open interval I, then the initial value problem (3) and (4) has a unique solution which is valid in I.

The Homogeneous Equation

$$L[y]=0. \tag{5}$$

Suppose that $y_1,y_2,\cdots,y_n$ are n solutions. Then

$$y=C_1y_1+C_2y_2+\cdots+C_ny_n \tag{6}$$

is also a solution.

Is (6) the general solution? To answer this, we need to see whether $C_1,C_2,\cdots,C_n$ can be uniquely determined under any initial conditions.

$$\begin{aligned}
&y(x_0)=b_0\to C_1y_1(x_0)+C_2y_2(x_0)+\cdots+C_ny_n(x_0)=b_0,\\
&y'(x_0)=b_1\to C_1y_1'(x_0)+C_2y_2'(x_0)+\cdots+C_ny_n'(x_0)=b_1,\\
&\cdots\\
&y^{(n-1)}(x_0)=b_{n-1}\to C_1y_1^{(n-1)}(x_0)+C_2y_2^{(n-1)}(x_0)+\cdots+C_ny_n^{(n-1)}(x_0)=b_{n-1}.
\end{aligned} \tag{7}$$

These are the n equations for $C_1,C_2,\cdots,C_n$. These equations have unique solution to $C_1,C_2,\cdots,C_n$ if and only if

$$W_0=\begin{vmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0)\\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0)\\ & & \cdots & \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0)\end{vmatrix}\neq 0. \tag{8}$$

We define the Wronskian for $y_1, y_2, \cdots, y_n$ as

$$W(y_1, \cdots, y_n) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ & & \cdots & \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}. \tag{9}$$

Then

$$W_0 = W|_{x=x_0}. \tag{10}$$

Theorem Suppose that $P_1(x), P_2(x), \cdots, P_n(x)$ are continuous on an open interval I. If $W(y_1, y_2, \cdots, y_n)$ is not equal to zero at a point x_0 in I (it can be shown that this implies $W(y_1, y_2, \cdots, y_n) \neq 0$ on the whole interval). Then (6) is the general solution and $y_1, y_2, \cdots, y_n$ are then called a fundamental set of solutions.

Consider $f_1(x), f_2(x), \cdots, f_n(x)$. If

$$k_1 f_1 + k_2 f_2 + \cdots k_n f_n = 0, \tag{11}$$

implies $k_1 = k_2 = \cdots = k_n = 0$, then $f_1, f_2, \cdots, f_n$ are linear independent. Otherwise, they are linear dependent.

Necessary and Sufficient Conditions A set of solutions to $L[y] = 0$, $\{y_1, y_2, \cdots, y_n\}$, are linear independent if and only if $W(y_1, y_2, \cdots, y_n) \neq 0$ at a point.

The Nonhomogeneous Equations

$$L[y] = g(x). \tag{12}$$

If Y_1 and Y_2 are two solutions of (12). Then $Y_1 - Y_2$ is a solution of

$$L[y] = 0. \tag{13}$$

If $y_1, y_2, \cdots, y_n$ are a fundamental set of solutions of (13), and $Y(x)$ is a particular solution of (12), then

$$y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n + Y(x) \tag{14}$$

is the general solution to (12).

4.2 Homogeneous Equations with Constant Coefficients

Consider

$$L[y] = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0. \tag{1}$$

We seek a solution of the form

$$y = e^{rx}. \tag{2}$$

Substituting it into the ODE,

$$\begin{aligned} & r^n e^{rx} + a_1 r^{n-1} e^{rx} + \cdots + a_{n-1} r e^{rx} + a_n e^{rx} = 0 \\ & \to r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0. \end{aligned} \tag{3}$$

This equation is called the characteristic equation and it has n roots, say $r_1, r_2, \cdots, r_n$. Then we have solutions

$$e^{r_1 x}, \quad e^{r_2 x}, \cdots, e^{r_n x}. \tag{4}$$

Real and Unique Roots We need to show that

$$W(e^{r_1 x}, e^{r_2 x}, \cdots, e^{r_n x}) \neq 0, \tag{5}$$

or equivalently, $e^{r_1 x}, e^{r_2 x}, \cdots, e^{r_n x}$ are linear independent.

Proof Consider

$$k_1 e^{r_1 x} + k_2 e^{r_2 x} + \cdots + k_n e^{r_n x} = 0. \tag{6}$$

We multiply $e^{-r_1 x}$ on both sides of the equation, we have

$$k_1 + k_2 e^{(r_2 - r_1)x} + k_3 e^{(r_3 - r_1)x} + \cdots + k_n e^{(r_n - r_1)x} = 0. \tag{7}$$

Differentiation gives

$$k_2 (r_2 - r_1) e^{(r_2 - r_1)x} + k_3 (r_3 - r_1) e^{(r_3 - r_1)x} + \cdots + k_n (r_n - r_1) e^{(r_n - r_1)x} = 0. \tag{8}$$

Multiplying $e^{-(r_2 - r_1)x}$ on both sides,

$$k_2 (r_2 - r_1) + k_3 (r_3 - r_1) e^{(r_3 - r_2)x} + \cdots + k_n (r_n - r_1) e^{(r_n - r_2)x} = 0. \tag{9}$$

Differentiation gives

$$k_3 (r_3 - r_1)(r_3 - r_2) e^{(r_3 - r_2)x} + \cdots + k_n (r_n - r_1)(r_n - r_2) e^{(r_n - r_2)x} = 0. \tag{10}$$

We do this continuously and finally have

$$k_n (r_n - r_1)(r_n - r_2) \cdots (r_n - r_{n-1}) e^{(r_n - r_{n-1})x} = 0. \tag{11}$$

The last equation gives $k_n = 0$. Then (6) becomes

$$k_1 e^{r_1 x} + k_2 e^{r_2 x} + \cdots + k_n e^{r_n x} = 0. \tag{12}$$

By the same procedure, it can be shown that

$$k_{n-1} = 0, \quad k_{n-2} = 0, \quad k_1 = 0. \tag{13}$$

This is to say $e^{r_1 x}, e^{r_2 x}, \cdots, e^{r_n x}$ are linear independent. The general solution is

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots C_n e^{r_n x}. \tag{14}$$

Complex Roots The characteristic equation is

$$r^n + a_1 r^{n-1} + a_{n-1} r + a_n = 0. \tag{15}$$

It has total n roots. Suppose that there is a pair of complex conjugate roots. $\lambda + i\mu$ and $\lambda - i\mu$. Then we have solutions

$$e^{(\lambda+i\mu)x}, e^{(\lambda-i\mu)x}, \tag{16}$$

which can be replaced by

$$e^{\lambda x}\cos\mu x, \quad e^{\lambda x}\sin\mu x. \tag{17}$$

Repeated Roots

(a) A real root, say r_1 is repeated s times, then

$$e^{r_1 x}, \quad xe^{r_1 x}, \cdots, x^{s-1}e^{r_1 x} \tag{18}$$

are the s solutions.

(b) A complex root, say $\lambda + i\mu$ is repeated s times (it implies that $\lambda - i\mu$ is also a s time repeated root), then

$$e^{\lambda x}\cos\mu x, \quad e^{\lambda x}\sin\mu x, \quad xe^{\lambda x}\cos\mu x, \quad xe^{\lambda x}\sin\mu x, \cdots, x^{s-1}e^{\lambda x}\cos\mu x, \quad x^{s-1}e^{\lambda x}\sin\mu x \tag{19}$$

are the $2s$ solutions.

Example 1 Find the general solution of the following ODE

$$y''' - 3y' - 2y = 0. \tag{20}$$

Solution: The characteristic equation is

$$r^3 - 3r - 2 = 0. \tag{21}$$

The root to this equation is obtained as

$$\begin{aligned} 0 = r^3 - 3r - 2 = r^3 - r - (2r+2) &= r(r-1)(r+1) - 2(r+1) \\ &= (r+1)(r^2 - r - 2) = (r+1)^2(r-2) \\ &\rightarrow r = 2 \text{ or } -1 \text{ (repeated)}. \end{aligned} \tag{22}$$

So e^{2x}, e^{-x}, xe^{-x} are a fundamental set of solutions and the general solution is

$$y = C_1 e^{2x} + C_2 e^{-x} + C_3 xe^{-x}. \tag{23}$$

Example 2 Find the solution of the following initial-value problem

$$y^{(4)} - y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 0. \tag{24}$$

Solution: The characteristic equation is

$$r^4 - 1 = 0 \to (r^2+1)(r^2-1) = 0 \to (r-1)(r+1)(r-i)(r+i) = 0. \tag{25}$$

Thus,

$$r_1 = 1, \quad r_2 = -1, \quad r_3 = i, \quad r_4 = -i. \tag{26}$$

There are four solutions:

$$e^x, \quad e^{-x}, \quad e^{ix}, \quad e^{-ix}. \tag{27}$$

The last two can be replaced by $e^{\lambda x}\cos\mu x = \cos x$ and $e^{\lambda x}\sin\mu x = \sin x$. Therefore, the general solution is

$$y = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x. \tag{28}$$

From the initial conditions, we have

$$\begin{aligned}
&y(0) = 1 \to C_1 + C_2 + C_3 = 1, \\
&y'(0) = 0 \to C_1 - C_2 + C_4 = 0, \\
&y''(0) = -1 \to C_1 + C_2 - C_3 = -1, \\
&y'''(0) = 0 \to C_1 - C_2 - C_4 = 0.
\end{aligned} \tag{29}$$

This gives

$$C_1 = C_2 = C_4 = 0, \quad C_3 = 1. \tag{30}$$

Thus, the solution is

$$y = \cos x. \tag{31}$$

4.3 The Method of Undetermined Coefficients

Consider

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = g(x). \tag{1}$$

The corresponding homogeneous equation is

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = 0. \tag{2}$$

The characteristic equation is then

$$a_0r^n + a_1r^{n-1} + \cdots + a_n = 0. \tag{3}$$

We discuss the particular solution under three cases.

(**Case 1**) $g(x) = P_m(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m$.
(1 a) $a_n \neq 0$: Form of the particular solution:

$$Y = A_0x^m + A_1x^{m-1} + \cdots + A_m. \tag{4}$$

(1 b) $a_n = a_{n-1} = \cdots = a_{n-s} = 0$ (there are $s+1$ lower order coefficients equal to zero) which implies $x, x^2, \cdots, x^s$ are solutions to the homogeneous equation. Then the form of the particular solution is

$$Y = x^{s+1}(A_0x^m + A_1x^{m-1} + \cdots + A_m). \tag{5}$$

(**Case 2**) $g(x) = e^{\alpha x}P_m(x)$.
(2 a) α is not a root of the characteristic equation, then the form for the particular solution is

$$Y = e^{\alpha x}(A_0x^m + A_1x^{m-1} + \cdots + A_m). \tag{6}$$

(2 b) α is a s-time repeated root, then the form for the particular solution is

$$Y = x^se^{\alpha x}(A_0x^m + A_1x^{m-1} + \cdots + A_m). \tag{7}$$

(**Case 3**) $g(x) = e^{\alpha x}\cos\beta x P_m(x)$.
(3 a) $\alpha + i\beta$ is not a root of the characteristic equation, then the particular solution is

$$Y = e^{\alpha x}\cos\beta x(A_0x^m + A_1x^{m-1} + \cdots + A_m) + e^{\alpha x}\sin\beta x(B_0x^m + B_1x^{m-1} + \cdots + B_m). \tag{8}$$

(3 b) $\alpha + i\beta$ is a s-time repeated root, then the particular solution is

$$Y = x^se^{\alpha x}\cos\beta x(A_0x^m + A_1x^{m-1} + \cdots + A_m) + x^s\sin\beta x(B_0x^m + B_1x^{m-1} + \cdots + B_m). \tag{9}$$

4.4 The Method of Variation of Parameters

Consider

$$L[y] = y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = g(x), \tag{1}$$

and

$$L[y] = 0. \tag{2}$$

Suppose that we have found a fundamental set of solutions for (2), say $y_1, y_2, \cdots, y_n$. Then the general solution is

$$y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n. \tag{3}$$

To find a particular solution, we suppose that it has the following form:

$$Y = u_1(x) y_1 + u_2(x) y_2 + \cdots + u_n(x) y_n, \tag{4}$$

where $u_1, u_2, \cdots, u_n$ are n unknown functions (we can impose $n-1$ equations on $u_1, u_2, \cdots, u_n$ ourself). Then

$$Y' = u_1 y_1' + u_2 y_2' + \cdots u_n y_n' + \underline{u_1' y_1 + u_2' y_2 + \cdots u_n' y_n}. \tag{5}$$

Impose that

$$u_1' y_1 + u_2' y_2 + \cdots u_n' y_n = 0. \tag{6}$$

Then

$$Y'' = u_1 y_1'' + u_2 y_2'' + \cdots + u_n y_n'' + \underline{u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n'}. \tag{7}$$

Impose

$$u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' = 0. \tag{8}$$

We repeat this process and finally have

$$Y^{(n-1)} = u_1 y_1^{(n-1)} + u_2 y_2^{(n-1)} + \cdots + u_n y_n^{(n-1)} + \underline{u_1' y_1^{(n-2)} + u_2' y_2^{(n-2)} + \cdots + u_n' y_n^{(n-2)}}. \tag{9}$$

Impose that

$$u_1' y_1^{(n-2)} + u_2' y_2^{(n-2)} + \cdots + u_n' y_n^{(n-2)} = 0. \tag{10}$$

Substituting (4), (5), (7), (9) into the ODE,

$$\begin{aligned} & u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} \\ & \qquad + u_1 y_1^{(n)} + u_2 y_2^{(n)} + \cdots + u_n y_n^{(n)} \\ & + P_1(x) \left[u_1 y_1^{(n-1)} + u_2 y_2^{(n-1)} + \cdots + u_n y_n^{(n-1)} \right] \\ & + P_2(x) \left[u_1 y_1^{(n-2)} + u_2 y_2^{(n-2)} + \cdots + u_n y_n^{(n-2)} \right] \\ & + \cdots \\ & + P_n(x) \left[u_1 y_1 + u_2 y_2 + \cdots + u_n y_n \right] = g(x), \end{aligned} \tag{11}$$

which is further arranged as

$$\begin{aligned} & u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} \\ & + u_1 \left[y_1^{(n)} + P_1 y_1^{(n-1)} + \cdots + P_n y_1 \right] \\ & + u_2 \left[y_2^{(n)} + P_1 y_2^{(n-1)} + \cdots + P_n y_2 \right] \\ & \cdots \\ & + u_n \left[y_n^{(n)} + P_1 y_n^{(n-1)} + \cdots + P_n y_n \right] = g(x). \end{aligned} \tag{12}$$

The expressions inside the brackets are zero, so finally we have

$$u_1'y_1^{(n-1)} + u_2'y_2^{(n-1)} + \cdots + u_n'y_n^{(n-1)} = g(x). \tag{13}$$

Up till now, we have obtained n linear algebraic equations for $u_1', u_2', \cdots, u_n'$, see (6), (8), (10), (13). They can be solved by using Cramer's rule:

$$u_1' = \frac{g(x)W_1}{W(y_1, y_2, \cdots, y_n)}, \cdots, u_m' = \frac{g(x)W_m}{W(y_1, y_2, \cdots, y_n)}, \cdots, u_n' = \frac{g(x)W_n}{W(y_1, y_2, \cdots, y_n)}, \tag{14}$$

Note $W(y_1, y_2, \cdots, y_n)$ is the Wronskian as follows

$$W(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & \cdots & y_m & \cdots & y_n \\ y_1' & \cdots & y_m' & \cdots & y_n' \\ \vdots & & \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_m^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}. \tag{15}$$

And W_m is obtained by replacing the m-th column of $W(y_1, y_2, \cdots, y_n)$ with $(0, 0, \cdots, 1)$:

$$W_m = \begin{vmatrix} y_1 & \cdots & 0 & \cdots & y_n \\ y_1' & \cdots & 0 & \cdots & y_n' \\ \vdots & & \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & 1 & \cdots & y_n^{(n-1)} \end{vmatrix}, \quad m = 1, 2, \cdots, n. \tag{16}$$

Taking an integration to (14), one has

$$u_m = \int \frac{g(x)W_m}{W(y_1, y_2, \cdots, y_n)} \mathrm{d}x + D_m, \quad m = 1, 2, \cdots, n, \tag{17}$$

where D_m is the arbitrary constant and we take it to be zero. Therefore, the particular solution is

$$Y = u_1y_1 + u_2y_2 + \cdots + u_ny_n = \sum_{m=1}^{n} u_my_m = \sum_{m=1}^{n} y_m \int \frac{g(x)W_m}{W(y_1, y_2, \cdots, y_n)} \mathrm{d}x. \tag{18}$$

Example 1 Find the general solution of the following ODE

$$y''' - y' = x. \tag{19}$$

Solution: Consider

$$y''' - y' = 0. \tag{20}$$

The characteristic equation is

$$r^3 - r = 0 \to r(r+1)(r-1) = 0 \to r_1 = 0,\ r_2 = 1,\ r_3 = -1. \tag{21}$$

The general solution to (20) is

$$y = C_1 + C_2e^x + C_3e^{-x}. \tag{22}$$

To find a particular solution to (19), we use the formula we have derived

$$Y = \sum_{m=1}^{3} y_m \int \frac{g(x)W_m}{W(y_1, y_2, y_3)} \mathrm{d}x. \tag{23}$$

$$W(y_1, y_2, y_3) = \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = \begin{vmatrix} e^x & -e^{-x} \\ e^x & e^{-x} \end{vmatrix} = 2. \tag{24}$$

$$W_1 = \begin{vmatrix} 0 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 1 & e^x & e^{-x} \end{vmatrix} = -2, \quad W_2 = \begin{vmatrix} 1 & 0 & e^{-x} \\ 0 & 0 & -e^{-x} \\ 0 & 1 & e^{-x} \end{vmatrix} = e^{-x}, \quad W_3 = \begin{vmatrix} 1 & e^x & 0 \\ 0 & e^x & 0 \\ 0 & e^x & 1 \end{vmatrix} = e^x. \tag{25}$$

Thus,

$$Y = 1 \cdot \int \frac{x(-2)}{2} \mathrm{d}x + e^x \int \frac{xe^{-x}}{2} \mathrm{d}x + e^{-x} \int \frac{xe^x}{2} \mathrm{d}x = -\frac{1}{2}x^2 - 1. \tag{26}$$

The general solution is

$$y = C_1 + C_2 e^x + C_3 e^{-x} - \frac{x^2}{2} - 1 = C_1 + C_2 e^x + C_3 e^{-x} - \frac{x^2}{2}, \tag{27}$$

where C_1, C_2, C_3 are updated arbitrary constants.

5 Series Solutions of Second Order Linear Equations

The representations are somehow different from the textbook.

5.0 Brief Review on Power Series

Recall from calculus that a power series in $x - x_0$ is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

Such a series is also said to be a **power series centered at** x_0. For example, the power series $\sum_{n=0}^{\infty}(x+1)^n$ is centered at $x_0 = -1$.

The following list summarizes some important facts about power series.

- **Convergence** A power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ **converges** at a point x if

  $$\lim_{m\to\infty} \sum_{n=0}^{m} a_n(x-x_0)^n$$

  exists for that x.

  The series certainly converges for $x = x_0$; it may converge for all x, or it may converge for some x and not for others.

- **Absolute Convergence** A power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ **converges absolutely** at a point x if

  $$\sum_{n=0}^{\infty} |a_n(x-x_0)^n| = \sum_{n=0}^{\infty} |a_n||x-x_0|^n$$

  converges.

  If the series converges absolutely, then the series also converges; however, the converse is not necessarily true.

- **Ratio Test** Convergence of a power series can often be determined by the ratio test. If $a_n \neq 0$, and if for a fixed value of x

  $$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L|x-x_0|.$$

  The power series converges absolutely at that value of x if $|x-x_0| < 1/L$, and diverges if $|x-x_0| > 1/L$. If $|x-x_0| = 1/L$, the test is inconclusive.

- The **radius of convergence** (about x_0): a nonnegative ρ, such that

  $$\sum_{n=0}^{\infty} a_n(x-x_0)^n \begin{cases} \text{converges absolutely} & \text{for } |x-x_0| < \rho, \\ \text{diverges} & \text{for } |x-x_0| > \rho. \end{cases}$$

  For a series that converges only at x_0, $\rho = 0$;

  for a series that converges for all x, ρ is infinite.

  If $\rho > 0$, then the interval $|x-x_0| < \rho$ is called **the interval of convergence**.

- **A Power Series Defines a Function**

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

whose domain is the interval of convergence of the series.

  - Within the interval of convergence, $f'(x)$ and $\int f(x)\mathrm{d}x$ can be found by term-wise differentiation and integration. For example,

$$f'(x)=\sum_{n=0}^{\infty}na_n(x-x_0)^{n-1}=\sum_{n=1}^{\infty}na_n(x-x_0)^{n-1},$$

$$f''(x)=\sum_{n=0}^{\infty}n(n-1)a_n(x-x_0)^{n-2}=\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2}$$

  - The differentiated (integrated) series has *the same radius of convergence* as the original series.
  - However, *convergence at an endpoint* may be either lost by differentiation or gained through integration.

These results are important and will be used shortly.

- **Useful Expansions** Some power series expansions about 0 that you should be familiar with.

Except for the last one, the expansions are valid for all z (i.e. the radius of convergence is $\rho=\infty$).

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

$$\cos(z)=\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k}}{(2k)!},\quad \sin(z)=\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!}$$

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n\quad(|z|<1)$$

- **Identity Property**

  - If $\sum_{n=0}^{\infty}c_n(x-x_0)^n=0$ for all x in the interval of convergence, then

$$c_n=0\quad\text{for all } n.$$

  - If $\sum_{n=0}^{\infty}a_n(x-x_0)^n=\sum_{n=0}^{\infty}b_n(x-x_0)^n$ for all x in the interval of convergence, then

$$a_n=b_n\quad\text{for all } n.$$

  Namely, if two power series are **equal**, then they must have the same coefficients.

- **Arithmetic of Power Series**

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n,\qquad g(x)=\sum_{n=0}^{\infty}b_n(x-x_0)^n$$

whose domain is the interval of convergence of the series.

–
$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)\ (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

–
$$f(x)g(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right] \left[\sum_{n=0}^{\infty} b_n (x - x_0)^n\right] = \sum_{n=0}^{\infty} d_n (x - x_0)^n,$$

where $d_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$.

– If $g(x_0) \neq 0$, the series for $f(x)/g(x)$ can also be found accordingly.

- **Analytic at a Point** A function f is **analytic at a point** x_0 if it can be represented by a power series in $x - x_0$ with a positive or infinite radius of convergence (i.e. $\rho > 0$).

  In calculus, functions like e^x, $\cos(x)$, $\sin(x)$ can be represented by Taylor series with $|x| < \infty$.

  Taylor series centered at 0 are called **Maclaurin series**. Show that e^x, $\cos(x)$, $\sin(x)$, are analytic at $x = 0$.

- **Shifting the Summation Index**

$$\sum_{n=1}^{\infty} n a_n\ x^{n-1} \quad \xrightarrow[\text{then } k=0,1,2,\cdots]{\text{let } \ k=n-1} \quad \sum_{k=0}^{\infty} (k+1) a_{k+1}\ x^k$$

$$\sum_{n=2}^{\infty} n(n-1) a_n\ (x - x_0)^{n-2} \quad \xrightarrow[\text{then } n=k+2 \text{ and } k=0,1,2,\cdots]{\text{let } \ k=n-2} \quad \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2}\ (x - x_0)^k$$

5.1 Introduction

Consider

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{1}$$

This is variable coefficient second order homogeneous equation. A powerful method to solve it is using power series.

Example Find the general solution of

$$y' - y = 0. \tag{2}$$

Solution $y = Ce^x$ is the general solution.

Idea: Many functions can be expanded as a power series. Thus, we first assume that the solution has a power series expansion:

$$y = \sum_{n=0}^{\infty} a_n x^n. \tag{3}$$

If a_n can be determined, then we have found a solution.

Substituting (3) into the ODE, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0, \\
&\to \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k - \sum_{n=0}^{\infty} a_n x^n = 0, \\
&\to \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0, \\
&\to \sum_{n=0}^{\infty} [(n+1) a_{n+1} - a_n] x^n = 0.
\end{aligned} \tag{4}$$

If a power series is equal to zero, then all the coefficients must vanish. So we have

$$(n+1) a_{n+1} = a_n \to a_{n+1} = \frac{a_n}{n+1}. \tag{5}$$

Equation (5) is called the **recurrence relation**. From this relation, we have

$$\begin{aligned}
&n = 0 : a_1 = a_0, \\
&n = 1 : a_2 = \frac{a_1}{2} = \frac{a_0}{1 \cdot 2} = \frac{a_0}{2!}, \\
&n = 2 : a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \\
&\cdots \\
&a_n = \frac{a_0}{n!},
\end{aligned} \tag{6}$$

where a_0 is an arbitrary constant. Thus, the solution is

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x. \tag{7}$$

Classification of Types of Points:

Consider

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{8}$$

A point x_0 such that $P(x_0) \neq 0$ is called an **ordinary point**. If $P(x_0) = 0$, then x_0 is a **singular point**. These are the definitions suitable when $P(x), Q(x)$ and $R(x)$ are polynomials and have no common factors.

More generally, the ordinary and singular points are defined as:

If at $x = x_0$, both $Q(x)/P(x)$ and $R(x)/P(x)$ are analytic, then $x = x_0$ is called an **ordinary point**. Otherwise, $x = x_0$ is a **singular point**.

Example 1

Is $x = 0$ an ordinary point or singular point of

$$xy'' + (\sin x)\, y' + x^3 y = 0? \tag{9}$$

Solution:

$$\begin{aligned} \frac{Q(x)}{P(x)} &= \frac{\sin x}{x} = \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right] \\ &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots, \quad |x| < \infty, \\ \frac{R(x)}{P(x)} &= x^2. \end{aligned} \tag{10}$$

Since both $Q(x)/P(x)$ and $R(x)/P(x)$ are analytic at $x = 0$, $x = 0$ is an ordinary point.

Example 2

For

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \tag{11}$$

where α is a constant. Is $x = 1$ an ordinary point or singular point?

Solution:

$$\frac{Q(x)}{P(x)} = \frac{-2x}{1 - x^2}, \tag{12}$$

which is not continuous at $x = 1$. The Taylor series does not exist at this point, thus $x = 1$ is a singular point.

Regular Singular Point and Irregular Singular Point:

Suppose that $x = x_0$ is a singular point. If both $(x - x_0)Q(x)/P(x)$ and $(x - x_0)^2 R(x)/P(x)$ are analytic, then $x = x_0$ is called a **regular singular point**. Otherwise, $x = x_0$ is called an **irregular singular point**.

Example 3

For

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \tag{13}$$

is $x = 1$ a regular singular point or irregular singular point?

Solution:

$$(x - 1)\frac{Q(x)}{P(x)} = (x - 1)\frac{-2x}{1 - x^2} = -2\frac{x}{1 + x}, \tag{14}$$

which are analytic at $x = 1$ because the rational function has no singularities at $x = 1$. Besides,

$$(x-1)^2 \frac{R(x)}{P(x)} = (x-1)^2 \frac{\alpha(\alpha+1)}{1-x^2} = (1-x)\frac{\alpha(\alpha+1)}{1+x}, \tag{15}$$

which is also analytic at $x = 1$. In conclusion, $x = 1$ is a regular singular point.

Note: In this course, we will not consider the case of irregular singular points.

5.2 Series Solutions Near an Ordinary Point

Theorem If $x = x_0$ is an ordinary point of

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{1}$$

then the general solution of this equation is given by

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1(x) + a_1 y_2(x), \tag{2}$$

where a_0 and a_1 are arbitrary constants and $y_1(x)$ and $y_2(x)$ are two linear independent solutions. Further, the radius of convergence for each of series solutions y_1 and y_2 is at least as large as the minimum of the radius of convergence of the series for $Q(x)/P(x)$ and $R(x)/P(x)$.

Example 1

Determine a lower bound for the radius of convergence of series solutions about $x = 4$ for

$$(x^2 - 2x - 3)y'' + xy' + 4y = 0. \tag{3}$$

Solution:

$$\frac{Q(x)}{P(x)} = \frac{x}{x^2 - 2x - 3} = \frac{x}{(x-3)(x+1)} = \frac{1}{4}\left[\frac{3}{x-3} + \frac{1}{x+1}\right]. \tag{4}$$

$$\frac{3}{x-3} = \frac{3}{1 + x - 4} = \frac{3}{1+t} = 3[1 - t + t^2 - t^3 + \cdots] = 3[1 - (x-4) + (x-4)^2 - (x-4)^3 + \cdots], \tag{5}$$

where $t = x - 4$ and the series is convergent for $|t| < 1$ or $|x - 4| < 1$.

$$\begin{aligned} \frac{1}{x+1} = \frac{1}{5 + (x-4)} = \frac{1}{5}\frac{1}{1 + \frac{x-4}{5}} &= \frac{1}{5}[1 - t + t^2 - t^3 + \cdots] \\ &= \frac{1}{5}\left[1 - \frac{x-4}{5} + \left(\frac{x-4}{5}\right)^2 - \left(\frac{x-4}{5}\right)^3 + \cdots\right], \end{aligned} \tag{6}$$

where $t = (x-4)/5$ and the series is convergent for $\left|\frac{x-4}{5}\right| < 1$ or $|x - 4| < 5$. Therefore, $\frac{Q(x)}{P(x)}$ is convergent for $|x - 4| < 1$ and the radius of convergence is 1. Similarly,

$$\frac{R(x)}{P(x)} = \frac{4}{(x-3)(x+1)} = \frac{1}{x-3} - \frac{1}{x+1}, \tag{7}$$

which is convergent for $|x - 4| < 1$ and the radius of convergence is 1. According to the theorem, a lower bound for the radius of convergence for the series solution is 1.

Example 2

Use the method of power series to find two linear independent series solutions about $x = 2$ (first four non-zero terms only) for

$$2y'' + (x+1)y' + 3y = 0. \tag{8}$$

Solution: $x = 2$ is an ordinary point. According to the theorem, we have the following general solution:

$$y = \sum_{n=0}^{\infty} a_n (x-2)^n = a_0 y_1(x) + a_1 y_2(x). \tag{9}$$

Substituting it into the ODE,

$$\begin{aligned}
&2\sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} + (x-2+3)\sum_{n=1}^{\infty} na_n(x-2)^{n-1} + 3\sum_{n=0}^{\infty} a_n(x-2)^n = 0,\\
&\to 2\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}(x-2)^k + \sum_{n=0}^{\infty} na_n(x-2)^n + 3\sum_{n=1}^{\infty} na_n(x-2)^{n-1} + 3\sum_{n=0}^{\infty} a_n(x-2)^n = 0,\\
&\to 2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-2)^n + + \sum_{n=0}^{\infty} na_n(x-2)^n + 3\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-2)^n + 3\sum_{n=0}^{\infty} a_n(x-2)^n = 0,\\
&\to \sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2} + 3(n+1)a_{n+1} + (n+3)a_n\right](x-2)^n = 0,
\end{aligned} \tag{10}$$

which finally gives

$$2(n+2)(n+1)a_{n+2} + 3(n+1)a_{n+1} + (n+3)a_n = 0. \tag{11}$$

According to the theorem,

$$y_1(x) = \sum_{n=0}^{\infty} a_n(x-2)^n, \tag{12}$$

with $a_1 = 0, a_0 = 1$. In (11), let $a_1 = 0, a_0 = 1$. Then

$$\begin{aligned}
n = 0 &: 2\cdot 2\cdot 1\cdot a_2 + 3\cdot 1\cdot 0 + 3\cdot 1 = 0 \to a_2 = -\frac{3}{4},\\
n = 1 &: 2\cdot 3\cdot 2\cdot a_3 + 3\cdot 2\cdot a_2 + 4\cdot 0 = 0 \to a_3 = -\frac{1}{2}a_2 = \frac{3}{8},\\
n = 2 &: 2\cdot 4\cdot 3\cdot a_4 + 3\cdot 3\cdot \frac{3}{8} + 5\cdot\left(-\frac{3}{4}\right) = 0 \to a_4 = \frac{1}{64}.
\end{aligned} \tag{13}$$

Thus,

$$y_1(x) = 1 - \frac{3}{4}(x-2)^2 + \frac{3}{8}(x-2)^3 + \frac{1}{64}(x-2)^4 + \cdots \tag{14}$$

According to the theorem,

$$y_2(x) = \sum_{n=0}^{\infty} a_n(x-2)^n, \tag{15}$$

with $a_0 = 0, a_1 = 1$. In (11), let $a_0 = 0, a_1 = 1$. Then

$$\begin{aligned}
n = 0 &: 2\cdot 2\cdot 1\cdot a_2 + 3\cdot 1\cdot a_1 = 0 \to a_2 = -\frac{3}{4},\\
n = 1 &: 2\cdot 3\cdot 2\cdot a_3 + 3\cdot 2\cdot\left(-\frac{3}{4}\right) + 4\cdot 1 = 0 \to a_3 = \frac{1}{24},\\
n = 2 &: 2\cdot 4\cdot 3\cdot a_4 + 3\cdot 3\cdot \frac{1}{24} + 5\cdot\left(-\frac{3}{4}\right) = 0 \to a_4 = \frac{9}{64}.
\end{aligned} \tag{16}$$

Thus,

$$y_2(x) = (x-2) - \frac{3}{4}(x-2)^2 + \frac{1}{24}(x-2)^3 + \frac{9}{64}(x-4)^4 + \cdots. \tag{17}$$

Example 3 (The Legendre Equation)

Find two linear independent solutions for

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \quad (\alpha > -1) \tag{18}$$

in terms of power series about $x=0$ for $|x|<1$.

Solution: $x=0$ is an ordinary point. According to the theorem, the general solution is given by

$$y=\sum_{n=0}^{\infty} a_n x^n = a_0 y_1(x) + a_1 y_2(x). \tag{19}$$

Substituting the solution into the ODE:

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\to [2a_1+(\alpha^2+\alpha)a_0] + [6a_2 - 2a_1 + (\alpha^2+\alpha)a_1]x + \sum_{n=2}^{\infty}\Big[(n+2)(n+1)a_{n+2} + \big[-n(n-1)-2n+\alpha(\alpha+1)\big]a_n\Big]x^n = 0, \tag{20}$$

which gives the recurrence relation as

$$2a_1+(\alpha^2+\alpha)a_0=0, \qquad 6a_2+(\alpha^2+\alpha-2)a_1=0$$

and

$$a_{n+2}=\frac{-n(n+1)+\alpha(\alpha+1)}{-(n+2)(n+1)}a_n = -\frac{(\alpha-n)(\alpha+n+1)}{(n+2)(n+1)}a_n, \qquad n=2,3,\cdots \tag{21}$$

From (19):

$$y_1(x)=\sum_{n=0}^{\infty} a_n x^n, \quad \text{with } a_0=1, a_1=0. \tag{22}$$

In (21), let $a_0=1, a_1=0$. Then

$$\begin{aligned}
n=0:&\quad a_2 = -\frac{\alpha(\alpha+1)}{2!},\\
n=1:&\quad a_3 = 0,\\
n=2:&\quad a_4 = -\frac{(\alpha-2)(\alpha+3)}{4\cdot 3}a_2 = \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!},\\
n=3:&\quad a_5 = 0,\\
n=4:&\quad a_6 = -\frac{(\alpha-4)(\alpha+5)}{6\cdot 5}a_4 = -\frac{\alpha(\alpha-2)(\alpha-4)(\alpha+1)(\alpha+3)(\alpha+5)}{6!}, \quad (k=3)\\
&\quad \cdots\\
&\quad a_{2k+1}=0,\\
&\quad a_{2k} = (-1)^k\frac{\alpha(\alpha-2)(\alpha-4)\cdots(\alpha-2(k-1))(\alpha+1)(\alpha+3)(\alpha+5)\cdots(\alpha+2k-1)}{(2k)!}, \quad k\ge 1.
\end{aligned} \tag{23}$$

Thus,

$$\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{k=1}^{\infty} a_{2k}x^{2k} + \sum_{k=0}^{\infty} a_{2k+1}x^{2k+1}\\
&= 1+\sum_{k=1}^{\infty}(-1)^k\frac{\alpha(\alpha-2)\cdots(\alpha-2k+2)(\alpha+1)(\alpha+3)\cdots(\alpha+2k-1)}{(2k)!}x^{2k}.
\end{aligned} \tag{24}$$

From (19),

$$y_2(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{with } a_0 = 0, a_1 = 1. \tag{25}$$

In (21), let $a_0 = 0, a_1 = 1$. Then

$$\begin{aligned}
n = 0: \quad & a_2 = 0, \\
n = 1: \quad & a_3 = -\frac{(\alpha-1)(\alpha+2)}{3!}, \quad (k=1) \\
n = 2: \quad & a_4 = 0, \\
n = 3: \quad & a_5 = -\frac{(\alpha-3)(\alpha+4)}{5\cdot 4} a_3 = \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}, \quad (k=2) \\
n = 4: \quad & a_6 = 0, \\
n = 5: \quad & a_7 = -\frac{(\alpha-5)(\alpha+6)}{7\cdot 6} = -\frac{(\alpha-1)(\alpha-3)(\alpha-5)(\alpha+2)(\alpha+4)(\alpha+6)}{7!}, \quad (k=3) \\
& \cdots \\
& a_{2k} = 0, \\
& a_{2k+1} = (-1)^k \frac{(\alpha-1)(\alpha-3)(\alpha-5)\cdots(\alpha-2k+1)(\alpha+2)(\alpha+4)\cdots(\alpha+2k)}{(2k+1)!}, \quad k \geq 1.
\end{aligned} \tag{26}$$

Thus,

$$\begin{aligned}
y_2 = \sum_{n=0}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} a_n x^n &= x + \sum_{k=1}^{\infty} a_{2k} x^{2k} + \sum_{k=1}^{\infty} a_{2k+1} x^{2k+1} \\
&= x + \sum_{k=1}^{\infty} (-1)^k \frac{(\alpha-1)(\alpha-3)\cdots(\alpha-2k+1)(\alpha+2)(\alpha+4)\cdots(\alpha+2k)}{(2k+1)!} x^{2k+1}.
\end{aligned} \tag{27}$$

Note: Consider the case that α is a positive integer.

(i) For $\alpha = 2l$,

$$y_1 = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{\alpha(\alpha-2)\cdots(\alpha-2k+2)(\alpha+1)(\alpha+3)\cdots(\alpha+2k-1)}{(2k)!} x^{2k}. \tag{28}$$

Consider $k = l+1$, then $\alpha - 2k + 2 = 2l - 2l = 0$. For $k \geq l+1$, all the coefficients will be zero. Therefore,

$$y_1 = 1 + \sum_{k=1}^{l} (-1)^k \frac{\alpha(\alpha-2)\cdots(\alpha-2k+2)(\alpha+1)(\alpha+3)\cdots(\alpha+2k-1)}{(2k)!} x^{2k}, \tag{29}$$

which is a polynomial of degree $2l$.

(ii) For $\alpha = 2l+1$,

$$y_2 = x + \sum_{k=1}^{\infty} (-1)^k \frac{(\alpha-1)(\alpha-3)\cdots(\alpha-2k+1)(\alpha+2)(\alpha+4)\cdots(\alpha+2k)}{(2k+1)!} x^{2k+1}. \tag{30}$$

Consider $k = l+1$, $\alpha - 2k + 1 = 0$. For $k \geq l+1$, all the coefficients will be zero. Therefore,

$$y_2 = x + \sum_{k=1}^{l} (-1)^k \frac{(\alpha-1)(\alpha-3)\cdots(\alpha-2k+1)(\alpha+2)(\alpha+4)\cdots(\alpha+2k)}{(2k+1)!} x^{2k+1}, \tag{31}$$

which is a polynomial of degree $2l+1$.

5.3 Euler's Equation

The Euler's equation is

$$L[y] = x^2 y'' + \alpha x y' + \beta y = 0, \quad \alpha, \beta \text{ are constants.} \tag{1}$$

In the following, we assume that $x > 0$. If $x < 0$, by a change of variables $t = -x(> 0)$, we obtain the same equation.

Since $Q(x)/P(x) = 1/x$ has a singularity at $x = 0$, $x = 0$ is a singular point. On the other hand, consider

$$\begin{aligned} x\frac{Q(x)}{P(x)} &= x\frac{\alpha x}{x^2} = \alpha, \\ x^2\frac{R(x)}{P(x)} &= x^2\frac{\beta}{x^2} = \beta. \end{aligned} \tag{2}$$

Since both $x\frac{Q(x)}{P(x)}$ and $x^2\frac{R(x)}{P(x)}$ are analytic at $x = 0$, $x = 0$ is a regular singular point.

To convert (1) into a constant coefficient equation, we introduce a change of variables

$$x = e^z \to z = \ln x. \tag{3}$$

Then,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}z}\cdot\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}z}, \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}z}\right) = -\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right) = -\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{1}{x^2}\frac{\mathrm{d}^2y}{\mathrm{d}z^2}. \end{aligned} \tag{4}$$

Substituting above equations into the ODE, one has

$$-\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{\mathrm{d}^2y}{\mathrm{d}z^2} + \alpha\frac{\mathrm{d}y}{\mathrm{d}z} + \beta y = 0 \to \frac{\mathrm{d}^2y}{\mathrm{d}z^2} + (\alpha - 1)\frac{\mathrm{d}y}{\mathrm{d}z} + \beta y = 0. \tag{5}$$

The characteristic equation of (5) is

$$r^2 + (\alpha - 1)r + \beta = 0. \tag{6}$$

Real and distinct roots r_1, r_2:

$$y = C_1 e^{r_1 z} + C_2 e^{r_2 z} = C_1 e^{r_1 \ln x} + C_2 e^{r_2 \ln x} = C_1 x^{r_1} + C_2 x^{r_2}. \tag{7}$$

Equal roots r_1 and r_2:

$$y = C_1 e^{r_1 z} + C_2 z e^{r_1 z} = C_1 x^{r_1} + C_2 \ln x \cdot x^{r_1}. \tag{8}$$

Complex roots: $(r_{1,2} = \lambda \pm i\mu)$

$$y = C_1 e^{\lambda z}\cos(\mu z) + C_2 e^{\lambda z}\sin(\mu z) = C_1 e^{\lambda \ln x}\cos(\mu \ln x) + C_2 e^{\lambda \ln x}\sin(\mu \ln x). \tag{9}$$

Comments: For the Euler's equation $x^2 y'' + \alpha x y' + \beta y = 0$,

(i) $x = 0$ is a regular singular point;

(ii) According to (1) and (7), one solution is $y_1 = x^{r_1}$ (r_1 is not necessarily positive integers).

(iii) This suggests that for a regular singular point, we may seek a solution of the form:

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \tag{10}$$

where r is to be determined. More generally, one may set

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}. \tag{11}$$

5.4 Series Solution near a Regular Singular Point

Consider

$$P(x)y'' + Q(x)y' + R(x)y = 0, \quad x > 0. \tag{1}$$

Suppose that $x = 0$ is a regular singular point. (Note, if the point is $x = x_0$, let $t = x - x_0$, then the corresponding point is changed to $t = 0$.) Then $xQ(x)/P(x)$ and $x^2R(x)/P(x)$ are analytic at $x = 0$. Suppose that

$$p(x) = x\frac{Q(x)}{P(x)} = \sum_{n=0}^{\infty} p_n x^n, \quad q(x) = x^2\frac{R(x)}{P(x)} = \sum_{n=0}^{\infty} q_n x^n, \quad 0 < x < \rho, \tag{2}$$

where

$$p_0 = \lim_{x\to 0} x\frac{Q(x)}{P(x)}, \quad q_0 = \lim_{x\to 0} x^2\frac{R(x)}{P(x)}. \tag{3}$$

Seek a solution in the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = a_0 x^r + a_1 x^{1+r} + a_2 x^{2+r} + \cdots, \tag{4}$$

where $a_0 \neq 0$.

Write (1) as

$$x^2 y'' + x\left(x\frac{Q(x)}{P(x)}\right)y' + \left(x^2\frac{R(x)}{P(x)}\right)y = 0. \tag{5}$$

Substituting (2) and (4) into the (5):

$$\begin{aligned}
&x^2\left[r(r-1)a_0x^{r-2} + r(r+1)a_1x^{r-1} + \cdots\right] + x(p_0 + p_1x + \cdots)\left[ra_0x^{r-1} + (r+1)a_1x^r + \cdots\right]\\
&\quad + (q_0 + q_1x + \cdots)\left[a_0x^r + a_1x^{1+r} + \cdots\right] = 0\\
&\to r(r-1)a_0x^r + (r+1)ra_1x^{r+1} + \cdots\\
&\qquad + rp_0a_0x^r + [rp_1a_0 + (r+1)p_0a_1]x^{r+1} + \cdots\\
&\qquad + q_0a_0x^r + (q_1a_0 + q_0a_1)x^{r+1} + \cdots = 0
\end{aligned} \tag{6}$$

Dividing both sides by the factor x^r:

$$[r^2 + r(p_0 - 1) + q_0]a_0 + [(r+1)ra_1 + rp_1a_0 + (r+1)p_0a_1 + q_1a_0 + q_0a_1]x + \cdots = 0, \tag{7}$$

which further gives

$$[r^2 + r(p_0 - 1) + q_0]a_0 = 0, \tag{8}$$

and

$$(r+1)ra_1 + rp_1a_0 + (r+1)p_0a_1 + q_1a_0 + q_0a_1 = 0. \tag{9}$$

From (8), since $a_0 \neq 0$, we have

$$r^2 + r(p_0 - 1) + q_0 = 0. \tag{10}$$

Equation (10) is the **indicial equation** and the r is called the **exponents at the singularity**.

Therefore, suppose (10) has two real roots r_1 and r_2 ($r_1 \geq r_2$). Then, there exists a solution of the form

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n \quad \text{with } a_0 = 1, \quad \text{for } 0 < x < \rho. \tag{11}$$

For the second linear independent solution, there are three cases:

Case 1 $\quad r_1 - r_2 \notin \mathbb{Z}$,

$$y_2 = x^{r_2} \sum_{n=0}^{\infty} b_n x^n \quad \text{with } b_0 = 1 \quad \text{for } 0 < x < \rho. \tag{12}$$

Case 2 $\quad r_1 - r_2 = 0$,

$$y_2 = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} c_n x^n, \quad \text{for } 0 < x < \rho. \tag{13}$$

Case 3 $\quad r_1 - r_2 \in \mathbb{Z}$,

$$y_2(x) = a y_1(x) \ln x + x^{r_2} \sum_{n=0}^{\infty} d_n x^n \quad \text{with } d_0 = 1 \quad \text{for } 0 < x < \rho. \tag{14}$$

In (11) to (14), $a_n, b_n, c_n, d_n, (n \geq 1)$ and a can be determined by substituting the solution into (1).

Example 1 (The Bessel Equation)

Find the two linear independent solutions of the Bessel equation of order $1/2$:

$$x^2 y'' + x y' + [x^2 - \underline{(1/2)}^2] y = 0. \tag{15}$$

Note $1/2$ is the order of the Bessel equation.

Solution:

$$p(x) = x\frac{Q(x)}{P(x)} = x\frac{x}{x^2} = 1, \quad q(x) = x^2\frac{R(x)}{P(x)} = x^2\frac{x^2 - (1/2)^2}{x^2} = -\frac{1}{4} + x^2. \tag{16}$$

Since both $xQ(x)/P(x)$ and $x^2R(x)/P(x)$ are analytic at $x = 0$, $x = 0$ is a regular singular point.

$$p_0 = \lim_{x\to 0} x\frac{Q(x)}{P(x)} = 1, \quad q_0 = \lim_{x\to 0} x^2\frac{R(x)}{P(x)} = -\frac{1}{4}. \tag{17}$$

The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = 0 \to r^2 - r + r - \frac{1}{4} = 0 \to r^2 - \frac{1}{4} = 0 \to r_1 = \frac{1}{2}, \quad r_2 = -\frac{1}{2}. \tag{18}$$

$r_1 - r_2 = 1$ corresponds to Case 3 of the theorem. According to the theorem, the first solution is given by

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1/2} \quad \text{with } a_0 = 1. \tag{19}$$

Substituting (19) into (15):

$$\begin{aligned} &x^2 \sum_{n=0}^{\infty} (n+1/2)(n-1/2) a_n x^{n-3/2} + x \sum_{n=0}^{\infty} (n+1/2) a_n x^{n-1/2} + (x^2 - 1/4) \sum_{n=0}^{\infty} a_n x^{n+1/2} = 0, \\ &\to \sum_{n=0}^{\infty} (n^2 - 1/4) a_n x^{n+1/2} + \sum_{n=0}^{\infty} (n+1/2) a_n x^{n+1/2} + \sum_{n=0}^{\infty} a_n x^{n+5/2} - \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{n+1/2} = 0. \end{aligned} \tag{20}$$

Dividing both sides by the factor $x^{1/2}$:

$$\begin{aligned}
&\to \sum_{n=0}^{\infty}(n^2-1/4)a_n x^n + \sum_{n=0}^{\infty}(n+1/2)a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} - \frac{1}{4}\sum_{n=0}^{\infty} a_n x^n = 0\\
&\to \sum_{n=0}^{\infty}[n^2-\frac{1}{4}+n+\frac{1}{2}-\frac{1}{4}]a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0\\
&\to \sum_{n=0}^{\infty} n(n+1)a_n x^n + \sum_{k=2}^{\infty} a_{k-2}x^k = 0\\
&\to 0+1\times 2\times a_1 x + \sum_{n=2}^{\infty} n(n+1)a_n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0\\
&\to 2a_1 x + \sum_{n=2}^{\infty}[n(n+1)a_n + a_{n-2}]x^n = 0\\
&\to a_1 = 0, \quad n(n+1)a_n + a_{n-2} = 0,\\
&\to a_1 = 0, \quad a_n = -\frac{a_{n-2}}{n(n+1)}, \quad n \geq 2.
\end{aligned} \tag{21}$$

Based on this recurrence relation, we have

$$\begin{aligned}
&n=2: \quad a_2 = -\frac{a_0}{3\cdot 2} = -\frac{1}{3!},\\
&n=3: \quad a_3 = 0,\\
&n=4: \quad a_4 = -\frac{a_2}{5\cdot 4} = +\frac{1}{5!},\\
&n=5: \quad a_5 = 0,\\
&n=6: \quad a_6 = -\frac{a_4}{7\cdot 6} = -\frac{1}{7!},\\
&\cdots\\
&a_{2k+1} = 0,\\
&a_{2k} = (-1)^k \frac{1}{(2k+1)!}.
\end{aligned} \tag{22}$$

Thus,

$$\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+1/2} = \sum_{k=0}^{\infty} a_{2k}x^{2k+1/2} + \sum_{k=0}^{\infty} a_{2k+1}x^{2k+1+1/2}\\
&= \sum_{n=0}^{\infty}(-1)^k \frac{x^{2k+1/2}}{(2k+1)!} = x^{-1/2}\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\\
&= x^{-1/2}\sin x.
\end{aligned} \tag{23}$$

$$\left(\frac{2}{\pi}\right)^{1/2} y_1 = \left(\frac{2}{\pi x}\right)^{1/2}\sin x \xlongequal{\text{denoted as}} J_{1/2}(x), \tag{24}$$

is also a solution, which is called Bessel function of order $1/2$.

According to the theorem, the second solution is given by

$$y_2 = a y_1 \ln x + x^{r_2}\sum_{n=0}^{\infty} d_n x^n = a y_1 \ln x + \sum_{n=0}^{\infty} d_n x^{n-1/2} \quad \text{with } d_0 = 1. \tag{25}$$

Substituting this solution into the ODE, one has

$$a\ln x[\underline{x^2 y_1'' + x y_1' + (x^2-1/4)y_1}] + 2axy_1' + \sum_{n=0}^{\infty}(n^2-n)d_n x^{n-1/2} + \sum_{n=0}^{\infty} d_n x^{n+3/2} = 0, \tag{26}$$

where the term with underline is zero. We also have

$$\begin{aligned}2axy_1' &= 2ax\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1/2}}{(2n+1)!}\right)' \\ &= 2ax\cdot\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1/2)x^{2n-1/2}}{(2n+1)!} \\ &= 2a\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1/2)x^{2n+1/2}}{(2n+1)!}.\end{aligned} \tag{27}$$

Therefore,

$$2a\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1/2)x^{2n+1/2}}{(2n+1)!}+\sum_{n=0}^{\infty}(n^2-n)d_nx^{n-1/2}+\sum_{k=2}^{\infty}d_{k-2}x^{k-1/2}=0. \tag{28}$$

Multiplying both sides by the factor $x^{1/2}$,

$$\begin{aligned}&2a\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n+1}}{(2n+1)!}+\sum_{n=0}^{\infty}(n^2-n)d_nx^n+\sum_{k=2}^{\infty}d_{k-2}x^k=0 \\ &\to 2a\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n+1}}{(2n+1)!}+\sum_{n=2}^{\infty}[n(n-1)d_n+d_{n-2}]x^n=0.\end{aligned} \tag{29}$$

Consider the first few terms:

$$\begin{aligned}&2a\frac{1}{1!}x+2a(-1)\frac{1}{2!}x^3+\cdots+[(2\cdot 1d_2+d_0)x^2+(3\cdot 2d_3+d_1)x^3+\cdots]=0 \\ &\to 2ax+(2d_2+d_0)x^2+[-a+(6d_3+d_1)]x^3+\cdots=0,\end{aligned} \tag{30}$$

which gives

$$a=0,\quad 2d_2+d_0=0,\cdots. \tag{31}$$

Then $(29)_2$ becomes

$$\sum_{n=0}^{\infty}[n(n-1)d_n+d_{n-2}]x^n=0, \tag{32}$$

which gives the recurrence relation

$$\begin{aligned}&d_n=-\frac{d_{n-2}}{n(n-1)},\quad n\geq 2 \\ &d_{2k}=(-1)^k\frac{1}{(2k)!},\quad d_{2k+1}=(-1)^k\frac{d_1}{(2k+1)!},\end{aligned} \tag{33}$$

where d_1 is arbitrary. We set $d_1=0$ so that $d_{2k+1}=0$. Then

$$\begin{aligned}y_2 &= ay_1\ln x+\sum_{n=0}^{\infty}d_nx^{n-1/2} \\ &= 0+x^{-1/2}\left[\sum_{k=0}^{\infty}d_{2k}x^{2k}+\sum_{k=0}^{\infty}d_{2k+1}x^{2k+1}\right] \\ &= x^{-1/2}\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}=x^{-1/2}\cos x.\end{aligned} \tag{34}$$

$\left(\frac{2}{\pi}\right)^{1/2}y_1=\left(\frac{2}{\pi x}\right)^{1/2}\cos x \xlongequal{\text{denoted as}} J_{-1/2}(x)$ is also a solution, which is called the Bessel function of order $-1/2$. Thus, the general solution is

$$y=C_1J_{1/2}(x)+C_2J_{-1/2}(x). \tag{35}$$

Note

(i) For a regular singular point, the corresponding power series solutions are called Frobenius solutions.

(ii) The procedures to obtain series solution:

(1) Identify the point concerned is an ordinary point or regular singular point.

(2) Ordinary point: The independent solutions are given by

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1 + a_1 y_2.$$

For y_1, setting $a_0 = 1, a_1 = 0$.

For y_2, setting $a_0 = 0, a_1 = 1$.

(3) Regular singular point at x_0:

A change of variables: $t = x - x_0$, which changes the regular singular point to $t = 0$.

(3 a) Find p_0 and q_0 and solve the indicial equations:

$$r^2 + r(p_0 - 1) + q_0 = 0$$

to obtain two roots r_1 and r_2 (set $r_1 \geq r_2$).

(3 b) Construct the first solution as

$$y = t^{r_1} \sum_{n=0}^{\infty} a_n t^n$$

where $a_0 = 1$.

(3 c) To construct the second solution: there are three cases.

6 System of First Order Linear Equations

The lecture will cover sections $7.1, 7.4-7.9$ and please read sections $7.2, 7.3$ yourself.

6.1 Introduction & Basic Theory

We have learned one equation involves only one unknown, for example:

$$y' = f(x,y), \quad y'' = f(x,y,y'). \tag{1}$$

A system of equations, involving several (or many) unknowns.

A higher order ODE can be converted into a system of first order equations. For example, for (1), we introduce

$$x_1 = y, \quad x_2 = y'. \tag{2}$$

Then we may convert the second equation in (1) as

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= f(x,x_1,x_2). \end{aligned} \tag{3}$$

A system of two first order equations for two unknowns x_1 and x_2.

Consider

$$y^{(n)} = F(t,y,y',\cdots,y^{(n-1)}), \tag{4}$$

where t is the variable and y is the unknown. Introduce

$$x_1 = y, \quad x_2 = y', x_3 = y'', \cdots x_{n-1} = y^{(n-2)}, x_n = y^{(n-1)}. \tag{5}$$

Then

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_3, \\ &\cdots, \\ x_{n-1}' &= x_n, \\ x_n' &= F(t,x_1,x_2,\cdots,x_n). \end{aligned} \tag{6}$$

Generally, a system of n first-order equations may take the following forms:

$$\begin{aligned} x_1' &= F_1(t,x_1,\cdots,x_n), \\ x_2' &= F_2(t,x_1,\cdots,x_n), \\ &\cdots, \\ x_n' &= F_n(t,x_1,\cdots,x_n). \end{aligned} \tag{7}$$

We shall only consider the case that $F_1, F_2, \cdots, F_n$ are all linear functions:

$$
\begin{aligned}
& x_1' = F_1(t, x_1, \cdots, x_n) = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t), \\
& x_2' = F_2(t, x_1, \cdots, x_n) = p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + g_2(t), \\
& \cdots, \\
& x_n' = F_n(t, x_1, \cdots, x_n) = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + g_n(t).
\end{aligned}
\tag{8}
$$

The system (8) is said to be linear. If further $g_1(t) = g_2(t) = \cdots = g_n(t) = 0$, (8) is said to be homogeneous.

Propose Initial conditions as

$$
x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \cdots, x_n(t_0) = x_n^0. \tag{9}
$$

The system of (8) and (9) is called an initial value problem.

Theorem For initial-value problem, (8) and (9) , if $p_{11}(t), p_{12}(t), \cdots, p_{nn}(t)$ and $g_1(t), g_2(t), \cdots g_n(t)$ are all continuous on $I : \alpha < t < \beta$, then there exists a unique solution which is valid on I.

Introduce matrix notations:

$$
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{P}(t) = \begin{pmatrix} p_{11}(t), & p_{12}(t), & \cdots, & p_{1n}(t) \\ p_{21}(t), & p_{22}(t), & \cdots, & p_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1}(t), & p_{n2}(t), & \cdots, & p_{nn}(t) \end{pmatrix}, \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \cdots \\ g_n(t) \end{pmatrix}. \tag{10}
$$

Then (8) becomes

$$
\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t). \tag{11}
$$

If a vector

$$
\mathbf{x} = \vec{\phi}(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \\ \vdots \\ \phi_n(t) \end{pmatrix} \tag{12}
$$

satisfies (11), it is called a solution. We denote the solutions as

$$
\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \\ \cdots \\ x_n^{(1)}(t) \end{pmatrix}, \cdots, \mathbf{x}^{(n)}(t) = \begin{pmatrix} x_1^{(n)}(t) \\ x_2^{(n)}(t) \\ \cdots \\ x_n^{(n)}(t) \end{pmatrix}. \tag{13}
$$

Consider the homogeneous equation

$$
\mathbf{x}' = \mathbf{P}(t)\mathbf{x}. \tag{14}
$$

Theorem If $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t), \cdots$, $\mathbf{x}^{(k)}(t)$ are solutions, then $C_1\mathbf{x}^{(1)}(t) + C_2\mathbf{x}^{(2)}(t) + \cdots + C_k\mathbf{x}^{(k)}(t)$ is still a solution, where $C_1, C_2, \cdots, C_k$ are called arbitrary constants.

Let $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \cdots, \mathbf{x}^{(n)}(t)$ to be n solutions of (14). Then the necessary and sufficient conditions for $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}$ to be linear independent is for the matrix

$$\mathbf{X} = \begin{pmatrix} x_1^{(1)}(t), & x_1^{(2)}(t), & \cdots, & x_1^{(n)}(t) \\ x_2^{(1)}(t), & x_2^{(2)}(t), & \cdots, & x_2^{(n)}(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_n^{(1)}(t), & x_n^{(2)}(t), & \cdots, & x_n^{(n)}(t) \end{pmatrix}, \tag{15}$$

we have

$$\det \mathbf{X} \neq 0. \tag{16}$$

We denote

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}] = \det \mathbf{X}, \tag{17}$$

and call it to be the Wronskian.

Theorem Suppose that $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}$ are solutions of (14). If $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)})] \neq 0$, (i.e., $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}$ are linear independent), the general solution of (14) is given by

$$\mathbf{x} = C_1\mathbf{x}^{(1)} + C_2\mathbf{x}^{(2)} + \cdots + C_n\mathbf{x}^{(n)}. \tag{18}$$

Theorem If (18) is the general solution of the homogeneous equation and $\mathbf{x}^{(p)}$ is a particular solution of the inhomogeneous equation (11), the general solution for (11) is given by

$$\mathbf{x} = C_1\mathbf{x}^{(1)} + C_2\mathbf{x}^{(2)} + \cdots C_n\mathbf{x}^{(n)} + \mathbf{x}^{(p)} \tag{19}$$

6.2 Homogeneous System with Constant Coefficients

Consider

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

where $\mathbf{A}$ is a constant coefficient $n \times n$ matrix. Seek a solution of the form

$$\mathbf{x} = \vec{\xi} e^{rt}, \tag{2}$$

where $\vec{\xi}$ and r are to be determined. Substituting the solution into (1):

$$\vec{\xi} r e^{rt} = \mathbf{A}\vec{\xi} e^{rt} \to \mathbf{A}\vec{\xi} = r\vec{\xi} \to (\mathbf{A} - r\mathbf{I})\vec{\xi} = \mathbf{0}, \tag{3}$$

where $\mathbf{I}$ is an identity matrix. This is a problem of finding the eigenvalues and eigenvectors for the matrix $\mathbf{A}$.

In the case that $\mathbf{A}$ is real and symmetric, we have the following results:

(a) The eigenvalues $r_1, r_2, \cdots, r_n$ are real (some of them may be the same).

(b) Corresponding to $r_1, r_2, \cdots, r_n$, there are n linear independent and orthogonal eigenvectors $\vec{\xi}^{(1)}, \vec{\xi}^{(2)}, \cdots, \vec{\xi}^{(n)}$ $((\vec{\xi}^{(i)}, \vec{\xi}^{(j)}) = 0$, for $i \neq j)$.

Thus, in this case, we have n linear independent solutions:

$$\mathbf{x}^{(1)} = \vec{\xi}^{(1)} e^{r_1 t}, \quad \mathbf{x}^{(2)} = \vec{\xi}^{(2)} e^{r_2 t}, \cdots \mathbf{x}^{(n)} = \vec{\xi}^{(n)} e^{r_n t}. \tag{4}$$

The general solution is

$$\mathbf{x} = C_1 \vec{\xi}^{(1)} e^{r_1 t} + C_2 \vec{\xi}^{(2)} e^{r_2 t} + \cdots \vec{\xi}^{(n)} e^{r_n t}. \tag{5}$$

Example 1 Find the general solution of the following equations

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x}. \tag{6}$$

Solution Consider

$$(\mathbf{A} - r\mathbf{I})\vec{\xi} = \mathbf{0}. \tag{7}$$

For non-zero solutions of $\vec{\xi}$, we require that

$$|\mathbf{A} - r\mathbf{I}| = 0 \to \begin{vmatrix} -2-r & 1 \\ 1 & -2-r \end{vmatrix} = 0 \to r^2 + 4r + 3 = 0 \to r_1 = -1, r_2 = -3. \tag{8}$$

For $r_1 = -1$, from (7):

$$\begin{pmatrix} -2-r_1 & 1 \\ 1 & -2-r_1 \end{pmatrix} \vec{\xi}^{(1)} = \mathbf{0} \to \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \mathbf{0} \to \xi_1^{(1)} = \xi_2^{(1)}. \tag{9}$$

Choose that $\xi_1^{(1)} = \xi_2^{(1)} = 1$, then

$$\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{10}$$

and

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}. \tag{11}$$

For $r_2 = -3$, from (7), we have

$$\begin{pmatrix} -2-r_2 & 1 \\ 1 & -2-r_2 \end{pmatrix} \vec{\xi}^{(2)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} = 0 \to \xi_1^{(2)} = -\xi_2^{(2)}. \tag{12}$$

Choose $\xi_1^{(2)} = 1$, we have

$$\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{13}$$

and

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}. \tag{14}$$

The general solution is

$$\mathbf{x} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}, \tag{15}$$

i.e.,

$$x_1 = C_1 e^{-t} + C_2 e^{-3t}, \quad x_2 = C_1 e^{-t} - C_2 e^{-3t}. \tag{16}$$

Generally, for (1), if $\mathbf{A}$ is arbitrary $n \times n$ constant coefficient matrix. Then there are three possibilities:

(1) All eigenvalues are real and distinct.

(2) Some eigenvalues occur in complex conjugate pairs.

(3) Some eigenvalues are repeated.

For case (1), there are n linear independent eigenvectors. Case (2) and (3) will be considered in the next two sections.

Example 2 Solve the following initial value problem:

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}. \tag{17}$$

Solution Consider

$$(\mathbf{A} - r\mathbf{I})\vec{\xi} = \mathbf{0}. \tag{18}$$

We require that

$$|\mathbf{A} - r\mathbf{I}| = 0 \to \begin{vmatrix} 1-r & 1 & 2 \\ 0 & 2-r & 2 \\ -1 & 1 & 3-r \end{vmatrix} = 0 \to (1-r)(2-r)(3-r) = 0 \to r_1 = 1, r_2 = 2, r_3 = 3. \tag{19}$$

For $r_1 = 1$, from (18),

$$\begin{pmatrix} 1-r_1 & 1 & 2 \\ 0 & 2-r_1 & 2 \\ -1 & 1 & 3-r_1 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \\ \xi_3^{(1)} \end{pmatrix} = \mathbf{0} \to \begin{array}{c} \xi_2^{(1)} + 2\xi_3^{(1)} = 0 \\ \xi_2^{(1)} + 2\xi_3^{(1)} = 0 \\ -\xi_1^{(1)} + \xi_2^{(1)} + 2\xi_3^{(1)} = 0 \end{array} \to \xi_1^{(1)} = 0, \xi_2^{(1)} = -2\xi_3^{(1)}. \tag{20}$$

Choose $\xi_3^{(1)} = 1$, we have

$$\vec{\xi}^{(1)} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}. \tag{21}$$

For $r_2 = 2$, we find

$$\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \tag{22}$$

and for $r_3 = 3$, we find

$$\vec{\xi}^{(3)} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}. \tag{23}$$

Thus, the general solution is

$$\mathbf{x} = C_1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + C_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} e^{3t}. \tag{24}$$

From the initial condition, we have

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \rightarrow \begin{array}{c} C_2 + 2C_3 = 2 \\ -2C_1 + C_2 + 2C_3 = 0 \\ C_1 + C_3 = 1 \end{array} \rightarrow C_1 = 1, C_2 = 2, C_3 = 0. \tag{25}$$

Thus, the solution for the initial value problem is

$$\mathbf{x} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^t + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}. \tag{26}$$

6.3 Complex Eigenvalues

Consider

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x}, \quad \mathbf{A} \text{ is real.} \tag{1}$$

The corresponding eigenvalue problem is

$$(\mathbf{A} - r\mathbf{I})\vec{\xi} = \mathbf{0}. \tag{2}$$

Suppose there is a complex eigenvalue $r_1 = \lambda + i\mu$ and a corresponding eigenvector $\vec{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ where $\mathbf{a}, \mathbf{b}$ are real. Then

$$(\mathbf{A} - r_1\mathbf{I})\vec{\xi}^{(1)} = \mathbf{0}. \tag{3}$$

Taking complex conjugate gives

$$(\mathbf{A} - \bar{r}_1\mathbf{I})\bar{\vec{\xi}}^{(1)} = \mathbf{0}. \tag{4}$$

This shows that

$$r_2 = \bar{r}_1 = \lambda - i\mu \tag{5}$$

is also an eigenvalue and

$$\vec{\xi}^{(2)} = \bar{\vec{\xi}}^{(1)} = \mathbf{a} - i\mathbf{b} \tag{6}$$

is also an eigenvector.

One solution is

$$\vec{y}^{(1)}(t) = \vec{\xi}^{(1)}e^{r_1 t} = (\mathbf{a} + i\mathbf{b})e^{(\lambda+i\mu)t} = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) + ie^{\lambda t}(\mathbf{b}\cos\mu t + \mathbf{a}\sin\mu t), \tag{7}$$

where $e^{(\lambda+i\mu)t} = e^{\lambda t}(\cos\mu t + i\sin\mu t)$. The other solution is

$$\vec{y}^{(2)}(t) = \vec{\xi}^{(2)}e^{r_2 t} = (\mathbf{a} - i\mathbf{b})e^{(\lambda-i\mu)t} = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) - ie^{\lambda t}(\mathbf{b}\cos\mu t + \mathbf{a}\sin\mu t). \tag{8}$$

Linear combination of $\vec{y}^{(1)}$ and $\vec{y}^{(2)}$ is still a solution. Thus

$$\mathbf{x}^{(1)} = \frac{1}{2}(\vec{y}^{(1)} + \vec{y}^{(2)}) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t), \quad \mathbf{x}^{(2)} = \frac{1}{2i}(\vec{y}^{(1)} - \vec{y}^{(2)}) = e^{\lambda t}(\mathbf{b}\cos\mu t + \mathbf{a}\sin\mu t) \tag{9}$$

are two real solutions. Note, they are also linear independent.

Example 1 Find the general solution of the following system

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}\mathbf{x}. \tag{10}$$

Solution Consider

$$(\mathbf{A} - r\mathbf{I})\vec{\xi} = \mathbf{0}. \tag{11}$$

We require

$$|\mathbf{A} - r\mathbf{I}| = 0 \to \begin{vmatrix} 1-r & 0 & 0 \\ 2 & 1-r & -2 \\ 3 & 2 & 1-r \end{vmatrix} = 0 \to (1-r)(r^2 - 2r + 2) = 0 \to r_1 = 1-2i, r_2 = 1+2i, r_3 = 1. \tag{12}$$

For $r_1 = 1-2i$, from (11),

$$\begin{pmatrix} 2i & 0 & 0 \\ 2 & 2i & -2 \\ 3 & 2 & 2i \end{pmatrix}\begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \\ \xi_3^{(1)} \end{pmatrix} = \mathbf{0} \to \begin{array}{c} 2i\xi_1^{(1)} = 0 \\ 2\xi_1^{(1)} + 2i\xi_2^{(1)} - 2\xi_3^{(1)} = 0 \\ 3\xi_1^{(1)} + 2\xi_2^{(1)} + 2i\xi_3^{(1)} = 0 \end{array} \to \xi_1^{(1)} = 0, \quad \xi_3^{(1)} = i\xi_2^{(1)}. \tag{13}$$

Choose $\xi_2^{(1)} = 1$, the eigenvector is

$$\xi^{(1)} = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{a} + i\mathbf{b}. \tag{14}$$

For $r_2 = 1+2i$, $\bar{\vec{\xi}}^{(1)}$ is the eigenvector. Two linear independent solutions are

$$\begin{aligned} \mathbf{x}^{(1)} &= e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) = e^t\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\sin 2t\right], \\ \mathbf{x}^{(2)} &= e^{\lambda t}(\mathbf{b}\cos\mu t + \mathbf{a}\sin\mu t) = e^t\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\cos 2t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\sin 2t\right]. \end{aligned} \tag{15}$$

For $r_3 = 1$, the eigenvector is

$$\vec{\xi}^{(3)} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \tag{16}$$

and the third solution is

$$\mathbf{x}^{(3)} = \vec{\xi}^{(3)} e^{r_3 t} = e^t\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}. \tag{17}$$

Thus, the general solution is

$$\mathbf{x} = C_1 e^t\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\sin 2t\right] + C_2 e^t\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\cos 2t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\sin 2t\right] + C_3 e^t\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \tag{18}$$

6.4 Repeated Eigenvalues

Consider

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{A} \text{ is real.} \tag{1}$$

The corresponding eigenvalue problem is

$$(\mathbf{A} - r\mathbf{I})\vec{\xi} = \mathbf{0}. \tag{2}$$

Suppose that there is a $k-$ repeated eigenvalues $r = \rho$, i.e. ρ is the $k-$ fold root of

$$|\mathbf{A} - r\mathbf{I}| = 0. \tag{3}$$

For the corresponding eigenvectors, there are two possibilities:

(i) There are $k-$ linear independent eigenvectors $\vec{\xi}^{(1)}, \vec{\xi}^{(2)}, \cdots \vec{\xi}^{(k)}$. In this case, we have $k-$ linear independent solutions

$$\mathbf{x}^{(1)} = \vec{\xi}^{(1)} e^{\rho t}, \quad \mathbf{x}^{(2)} = \vec{\xi}^{(2)} e^{\rho t}, \cdots \mathbf{x}^{(k)} = \vec{\xi}^{(k)} e^{\rho t}. \tag{4}$$

(ii) There are less than k linear independent eigenvectors. In this case, we have to find other solutions.

Example 1 Find the general solution of the following equation

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x} = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \mathbf{x}. \tag{5}$$

Solution Consider

$$(\mathbf{A} - r\mathbf{I})\vec{\xi} = \mathbf{0}. \tag{6}$$

We require

$$|\mathbf{A} - r\mathbf{I}| = 0 \to \begin{vmatrix} 4-r & -2 \\ 8 & -4-r \end{vmatrix} = 0 \to r^2 = 0 \to r_1 = r_2 = 0. \tag{7}$$

For $r_1 = r_2 = 0$, from (6),

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \mathbf{0} \to \begin{array}{c} 4\xi_1^{(1)} - 2\xi_2^{(1)} = 0 \\ 8\xi_1^{(1)} - 4\xi_2^{(1)} = 0 \end{array} \to \xi_2^{(1)} = 2\xi_1^{(1)}. \tag{8}$$

Choose $\xi_1^{(1)} = 1$, the eigenvector is

$$\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{9}$$

and one solution is

$$\mathbf{x}^{(1)} = \vec{\xi}^{(1)} e^{r_1 t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{10}$$

Seek another solution of the form

$$\mathbf{x}^{(2)} = t\vec{\xi} e^{r_1 t} = t\vec{\xi}, \tag{11}$$

where $\vec{\xi}$ is to be determined. Substituting (11) into (5), we have

$$\vec{\xi} = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} t\vec{\xi}. \tag{12}$$

The left hand side is irrelevant with t while the right hand side is and this can not be satisfied except $\vec{\xi} = \mathbf{0}$.

Try a solution of the form

$$\mathbf{x}^{(2)} = (t\vec{\xi} + \vec{\eta}) e^{r_1 t} = t\vec{\xi} + \vec{\eta}, \tag{13}$$

where $\vec{\xi}, \vec{\eta}$ are to be determined. Substituting it into (5):

$$\vec{\xi} = \mathbf{A}(t\vec{\xi} + \vec{\eta}) = t\mathbf{A}\,\vec{\xi} + \mathbf{A}\,\vec{\eta}. \tag{14}$$

Compare both sides:

$$\mathbf{A}\,\vec{\xi} = \mathbf{0}, \quad \mathbf{A}\,\vec{\eta} = \vec{\xi}. \tag{15}$$

$(15)_1$ is the same as (8), thus

$$\vec{\xi} = \vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{16}$$

From $(15)_2$,

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \to \eta_2 = 2\eta_1 - 1/2. \tag{17}$$

Choose $\eta_1 = 0$, we have

$$\vec{\eta} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}. \tag{18}$$

Then, the second solution is

$$\mathbf{x}^{(2)} = t\vec{\xi} + \vec{\eta} = t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}. \tag{19}$$

It can be shown that $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] \neq 0$, thus the general solution is

$$\mathbf{x} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 \left[t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} \right] \tag{20}$$

Remark Consider the situation where $r = \rho$ is a triple-repeated eigenvalue of (3):

Case i. There is only one eigenvector $\vec{\xi}^{(1)}$ and one solution is

$$\mathbf{x}^{(1)} = \vec{\xi}^{(1)} e^{\rho t}. \tag{21}$$

The second and third solutions have the form of

$$\mathbf{x}^{(2)} = (t\vec{\xi}^{(2)} + \vec{\eta}^{(2)}) e^{\rho t}, \quad \mathbf{x}^{(3)} = (t^2 \vec{\xi}^{(3)} + t\vec{\eta}^{(3)} + \vec{\zeta}^{(3)}) e^{\rho t}, \tag{22}$$

where $\vec{\xi}^{(2)}, \vec{\eta}^{(2)}, \vec{\xi}^{(3)}, \vec{\eta}^{(3)}, \vec{\zeta}^{(3)}$ can be determined from (1).

Case ii There are two linear independent eigenvectors $\vec{\xi}^{(1)}$ and $\vec{\xi}^{(2)}$. Two solutions are

$$\mathbf{x}^{(1)} = \vec{\xi}^{(1)} e^{\rho t}, \quad \mathbf{x}^{(2)} = \vec{\xi}^{(2)} e^{\rho t}. \tag{23}$$

The third solution has the form

$$\mathbf{x}^{(3)} = (t\vec{\xi}^{(3)} + \vec{\eta}^{(3)}) e^{\rho t} \tag{24}$$

and by taking it into the (1), we have

$$(\mathbf{A} - \rho\mathbf{I})\vec{\xi}^{(3)} = 0, \quad (\mathbf{A} - \rho\mathbf{I})\vec{\eta}^{(3)} = \vec{\xi}^{(3)}, \tag{25}$$

where $\vec{\xi}^{(3)}$ should be one eigenvector corresponding to $r = \rho$. Most generally, it should be

$$\vec{\xi}^{(3)} = k_1 \vec{\xi}^{(1)} + k_2 \vec{\xi}^{(2)} \tag{26}$$

where we should choose such values for k_1 and k_2 that $(25)_2$ has a solution for $\vec{\eta}^{(3)}$.

6.5 Fundamental Matrices

Consider

$$\mathbf{x}' = \mathbf{P}(t)\,\mathbf{x}, \quad \alpha \le t \le \beta, \tag{1}$$

where $\mathbf{P}(t)$ is a variable-coefficient $n \times n$ matrix. Suppose that $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \cdots, \mathbf{x}^{(n)}(t)$ are n linear independent solutions. (We say that they form a fundamental set solutions). Introduce

$$\mathbf{\Psi}(t) = (\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \cdots, \mathbf{x}^{(n)}(t)) = \begin{pmatrix} \mathbf{x}_1^{(1)}(t) & \mathbf{x}_1^{(2)}(t) & \cdots & \mathbf{x}_1^{(n)}(t) \\ \mathbf{x}_2^{(1)}(t) & \mathbf{x}_2^{(2)}(t) & \cdots & \mathbf{x}_2^{(n)}(t) \\ \vdots & \vdots & & \vdots \\ \mathbf{x}_n^{(1)}(t) & \mathbf{x}_n^{(2)}(t) & \cdots & \mathbf{x}_n^{(n)}(t) \end{pmatrix} \tag{2}$$

is called the fundamental matrix for the system (1). The general solution of (1) is

$$\mathbf{x} = C_1\mathbf{x}^{(1)} + C_2\mathbf{x}^{(2)} + \cdots + C_n\mathbf{x}^{(n)} = \mathbf{\Psi}(t)\,\mathbf{c}, \tag{3}$$

where

$$\mathbf{c} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}. \tag{4}$$

Suppose that the initial condition is

$$\mathbf{x}(t_0) = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{pmatrix}, \quad \alpha \le t \le \beta. \tag{5}$$

In (1), let $t = t_0$, we have

$$\mathbf{x}(t_0) = \mathbf{\Psi}(t_0)\,\mathbf{c}. \tag{6}$$

We note that

$$\det \mathbf{\Psi}(t) = \det[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}] = W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}] \ne 0, \quad \alpha \le t \le \beta. \tag{7}$$

Thus, $\mathbf{\Psi}(t)$ has an inverse $\mathbf{\Psi}^{-1}(t)$. Therefore, from (6),

$$\mathbf{c} = \mathbf{\Psi}^{-1}(t_0)\,\mathbf{x}(t_0). \tag{8}$$

The solution for the initial value problem is

$$\mathbf{x} = \mathbf{\Psi}(t)\,\mathbf{\Psi}^{-1}(t_0)\,\mathbf{x}(t_0). \tag{9}$$

Diagonalization

Consider

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x}. \tag{10}$$

If **A** is diagonal, i.e.

$$\mathbf{x}' = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \mathbf{x}, \tag{11}$$

then

$$\begin{aligned} x_1' &= a_{11} x_1, \\ x_2' &= a_{22} x_2, \\ &\vdots \\ x_n' &= a_{nn} x_n, \end{aligned} \tag{12}$$

and these are decoupled equations for $x_1, x_2, \cdots, x_n$. The solution is

$$\begin{aligned} x_1 &= C_1 e^{a_{11} t}, \\ x_2 &= C_2 e^{a_{22} t}, \\ &\vdots \\ x_n &= C_n e^{a_{nn} t}. \end{aligned} \tag{13}$$

For an arbitrary **A**, we shall try to use transforms to obtain a diagonal matrix. We first need to find a linear independent eigenvectors $\vec{\xi}^{(1)}, \vec{\xi}^{(2)}, \cdots, \vec{\xi}^{(n)}$ (corresponding to $r_1, r_2, \cdots, r_n$). Denote

$$\mathbf{T} = (\vec{\xi}^{(1)}, \vec{\xi}^{(2)}, \cdots, \vec{\xi}^{(n)}) = \begin{pmatrix} \xi_1^{(1)} & \xi_1^{(2)} & \cdots & \xi_1^{(n)} \\ \xi_2^{(1)} & \xi_2^{(2)} & \cdots & \xi_2^{(n)} \\ \vdots & \vdots & & \vdots \\ \xi_n^{(1)} & \xi_n^{(2)} & \cdots & \xi_n^{(n)} \end{pmatrix}. \tag{14}$$

Introduce a change of variables:

$$\mathbf{x} = \mathbf{T} \vec{y} \tag{15}$$

where $\vec{y}$ are new variables. Then $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$ becomes

$$\mathbf{T} \vec{y}' = \mathbf{A}\,\mathbf{T} \vec{y}. \tag{16}$$

Multiplying the inverse of **T** on both sides:

$$\vec{y}' = \mathbf{T}^{-1} \mathbf{A}\,\mathbf{T} \vec{y}. \tag{17}$$

According to what we have learnt in linear algebra,

$$\mathbf{T}^{-1} \mathbf{A} \mathbf{T} = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{pmatrix}. \tag{18}$$

Thus,

$$\vec{y} = \begin{pmatrix} C_1 e^{r_1 t} \\ C_2 e^{r_2 t} \\ \vdots \\ C_n e^{r_n t} \end{pmatrix}, \tag{19}$$

and

$$\mathbf{x} = \mathbf{T}\vec{y} = \mathbf{T} \begin{pmatrix} C_1 e^{r_1 t} \\ C_2 e^{r_2 t} \\ \vdots \\ C_n e^{r_n t} \end{pmatrix}. \tag{20}$$

6.6 Non-homogeneous linear systems

Consider

$$\mathbf{x}' = \mathbf{P}(t)\,\mathbf{x} + \mathbf{g}(t), \quad \alpha \le t \le \beta, \tag{1}$$

where $\mathbf{P}(t)$ and $\mathbf{g}(t)$ are assumed to be continuous on $\alpha \le t \le \beta$.

Suppose that $\mathbf{x}_1$ is a particular solution to (1) and the general solution for

$$\mathbf{x}' = \mathbf{P}(t)\,\mathbf{x}, \quad \alpha \le t \le \beta, \tag{2}$$

is

$$C_1\mathbf{x}^{(1)} + C_2\mathbf{x}^{(2)} + \cdots + C_n\mathbf{x}^{(n)}. \tag{3}$$

Then

$$\mathbf{x} = C_1\mathbf{x}^{(1)} + C_2\mathbf{x}^{(2)} + \cdots + C_n\mathbf{x}^{(n)} + \mathbf{x}_1 \tag{4}$$

is the general solution to (1).

Variation of Parameters

The general solution to (2) may be written as

$$\mathbf{x} = \mathbf{\Psi}(t)\,\mathbf{c}, \tag{5}$$

where $\mathbf{\Psi}(t)$ is the fundamental matrix. To find a particular solution, we let

$$\mathbf{x}_1 = \mathbf{\Psi}(t)\,\mathbf{u}(t). \tag{6}$$

We want to determine $\mathbf{u}(t)$ such that (6) is a solution.

Substitute (6) into (1):

$$\mathbf{\Psi}'(t)\,\mathbf{u}(t) + \mathbf{\Psi}(t)\,\mathbf{u}'(t) = \mathbf{P}(t)\mathbf{\Psi}(t)\mathbf{u}(t) + \mathbf{g}(t), \tag{7}$$

where

$$\mathbf{\Psi}'(t) = \left[\mathbf{x}'^{(1)}, \mathbf{x}'^{(2)}, \cdots, \mathbf{x}'^{(n)}\right] = \left[\mathbf{P}(t)\,\mathbf{x}^{(1)}, \mathbf{P}(t)\,\mathbf{x}^{(2)}, \cdots, \mathbf{P}(t)\,\mathbf{x}^{(n)}\right] = \mathbf{P}(t)\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}\right] = \mathbf{P}(t)\,\mathbf{\Psi}(t). \tag{8}$$

Based on (7) and (8)

$$\mathbf{\Psi}(t)\,\mathbf{u}'(t)=\mathbf{g}(t). \tag{9}$$

Multiplying $\mathbf{\Psi}^{-1}(t)$, we have

$$\mathbf{u}'(t)=\mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t)\to\mathbf{u}(t)=\int\mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t)\mathrm{d}t+k_1, \tag{10}$$

where we may choose $k_1=0$. The general solution for (1) is then

$$\mathbf{x}=\mathbf{\Psi}(t)\,\mathbf{c}+\mathbf{\Psi}(t)\int\mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t)\mathrm{d}t. \tag{11}$$

Example 1 Find the general solution of

$$\mathbf{x}'=\mathbf{A}\,\mathbf{x}+\mathbf{g}(t)=\begin{pmatrix}2&-5\\1&-2\end{pmatrix}\mathbf{x}+\begin{pmatrix}-5\cos t\\ \sin t\end{pmatrix}. \tag{12}$$

Solution Consider

$$(\mathbf{A}-r\mathbf{I})\vec{\xi}=\mathbf{0}. \tag{13}$$

We have

$$|\mathbf{A}-r\mathbf{I}|=0\to\begin{vmatrix}2-r&-5\\1&-2-r\end{vmatrix}=0\to r_1=i,\quad r_2=-i.\quad(\lambda=0,\mu=1) \tag{14}$$

For $r_1=i$,

$$\begin{pmatrix}2-i&-5\\1&-2-i\end{pmatrix}\begin{pmatrix}\xi_1^{(1)}\\ \xi_2^{(1)}\end{pmatrix}=\mathbf{0}, \tag{15}$$

which gives

$$\begin{aligned}&(2-i)\xi_1^{(1)}-5\xi_2^{(1)}=0\\&\xi_1^{(1)}-(2+i)\xi_2^{(1)}=0.\end{aligned} \tag{16}$$

Note $(16)_1\times(2+i)$ leads to $\xi_1^{(1)}-(2+i)\xi_2^{(1)}=0$ which is exactly $(16)_2$. From $(16)_2$, we let $\xi_1^{(1)}=5$, then $\xi_2^{(1)}=2-i$. So

$$\vec{\xi}^{(1)}=\begin{pmatrix}5\\2-i\end{pmatrix}=\begin{pmatrix}5\\2\end{pmatrix}+i\begin{pmatrix}0\\-1\end{pmatrix}=\mathbf{a}+i\mathbf{b}. \tag{17}$$

The two solutions are

$$\begin{aligned}\mathbf{x}^{(1)}&=e^{\lambda t}(\mathbf{a}\cos\mu t-\mathbf{b}\sin\mu t)=\begin{pmatrix}5\\2\end{pmatrix}\cos t-\begin{pmatrix}0\\-1\end{pmatrix}\sin t=\begin{pmatrix}5\cos t\\2\cos t+\sin t\end{pmatrix},\\ \mathbf{x}^{(2)}&=e^{\lambda t}(\mathbf{b}\cos\mu t+\mathbf{a}\sin\mu t)=\begin{pmatrix}0\\-1\end{pmatrix}\cos+\begin{pmatrix}5\\2\end{pmatrix}\sin t=\begin{pmatrix}5\sin t\\-\cos t+2\sin t\end{pmatrix}.\end{aligned} \tag{18}$$

The fundamental matrix is

$$\mathbf{\Psi}(t)=\begin{pmatrix}5\cos t&5\sin t\\2\cos t+\sin t&-\cos t+2\sin t\end{pmatrix}. \tag{19}$$

For a particular solution, we let

$$\mathbf{x}^{(p)}=\mathbf{\Psi}(t)\,\mathbf{u}(t). \tag{20}$$

Substituting it into (12), we have

$$\mathbf{\Psi}(t)\,\mathbf{u}'(t) = \mathbf{g}(t) \to \begin{pmatrix} 5\cos t & 5\sin t \\ 2\cos t + \sin t & -\cos t + 2\sin t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} -5\cos t \\ \sin t \end{pmatrix} \tag{21}$$

which further gives

$$u_1' = -\cos 2t + \sin 2t, \quad u_2' = -2\cos t(\cos t + \sin t). \tag{22}$$

Integration gives one set of solutions to u_1, u_2 as

$$u_1 = -\frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t, \quad u_2 = -t + \cos^2 t - \frac{1}{2}\sin 2t. \tag{23}$$

The general solution is

$$\begin{aligned} \mathbf{x} &= \mathbf{\Psi}(t)\,\mathbf{c} + \mathbf{\Psi}(t)\,\mathbf{u}(t) \\ &= \left(-\frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t + C_1\right) \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + \left(-t + \cos^2 t - \frac{1}{2}\sin 2t + C_2\right) \begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix}. \end{aligned} \tag{24}$$

Diagonalization Consider

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x} + \mathbf{g}(t), \quad \alpha \le t \le \beta, \tag{25}$$

where $\mathbf{A}$ is a constant coefficient matrix. Introduce $\mathbf{x} = \mathbf{T}\,\vec{y}$, where $\mathbf{T} = (\vec{\xi}^{(1)}, \vec{\xi}^{(2)}, \cdots, \vec{\xi}^{(n)})$ and $\vec{\xi}^{(1)}, \vec{\xi}^{(2)}, \cdots, \vec{\xi}^{(n)}$ are eigenvectors corresponding to eigenvalues $r_1, r_2, \cdots, r_n$. Then

$$\mathbf{T}\,\vec{y}' = \mathbf{A}\,\mathbf{T}\,\vec{y} + \mathbf{g}(t) \to \vec{y}' = \mathbf{T}^{-1}\mathbf{A}\,\mathbf{T}\,\vec{y} + \mathbf{T}^{-1}\mathbf{g}(t) = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & r_n \end{pmatrix} \vec{y} + \begin{pmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_n(t) \end{pmatrix}. \tag{26}$$

In component form

$$\begin{aligned} y_1' &= r_1 y_1 + h_1(t) \\ y_2' &= r_2 y_2 + h_2(t) \\ &\vdots \\ y_n' &= r_n y_n + h_n(t) \end{aligned} \tag{27}$$

which are decoupled equations. Using the method of integrating factor, we have solutions

$$\begin{aligned} y_1 &= C_1 e^{r_1 t} + e^{r_1 t}\textstyle\int e^{-r_1 t} h_1(t)\mathrm{d}t, \\ y_2 &= C_2 e^{r_2 t} + e^{r_2 t}\textstyle\int e^{-r_2 t} h_2(t)\mathrm{d}t, \\ &\vdots \\ y_n &= C_n e^{r_n t} + e^{r_n t}\textstyle\int e^{-r_n t} h_n(t)\mathrm{d}t. \end{aligned} \tag{28}$$

The solution is then

$$\mathbf{x} = \mathbf{T}\,\vec{y}. \tag{29}$$

Example 2 Find the general solution of

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x} + \mathbf{g}(t) = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}. \tag{30}$$

Solution Two eigenvalues for $\mathbf{A}$ are $r_1 = -3, r_2 = 2$. Two corresponding eigenvectors are

$$\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \quad \vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{31}$$

Thus,

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}. \tag{32}$$

Let $\mathbf{x} = \mathbf{T}\,\vec{y}$, then the equation becomes

$$\vec{y}' = \mathbf{T}^{-1}\,\mathbf{A}\,\mathbf{T}\,\vec{y} + \mathbf{T}^{-1}\,\mathbf{g}(t), \tag{33}$$

where

$$\mathbf{T}^{-1} = \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}, \quad \mathbf{T}^{-1}\,\mathbf{g}(t) = \frac{1}{5} \begin{pmatrix} e^{-2t} + 2e^{t} \\ 4e^{-2t} - 2e^{t} \end{pmatrix}. \tag{34}$$

In component form, we have

$$\begin{aligned} y_1' &= -3y_1 + \tfrac{1}{5}(e^{-2t} + 2e^{t}), \\ y_2' &= 2y_2 + \tfrac{1}{5}(4e^{-2t} - 2e^{t}), \end{aligned} \tag{35}$$

which lead to solutions

$$\begin{aligned} y_1 &= C_1 e^{-3t} + \tfrac{1}{5}(e^{-2t} + e^{t}/2) \\ y_2 &= C_2 e^{2t} + \tfrac{1}{5}(2e^{t} - e^{-2t}). \end{aligned} \tag{36}$$

Then

$$\mathbf{x} = \mathbf{T}\,\vec{y} = C_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{t}. \tag{37}$$

7 Partial Differential Equations

This chapter treats one important method for solving partial differential equations, a method known as separation of variables. Its essential feature is the replacement of the partial differential equation by a set of ordinary differential equations, which must be solved subject to given initial or boundary conditions.

The first section of this chapter deals with some basic properties of boundary value problems for ordinary differential equations. The desired solution of the partial differential equation is then expressed as a sum, usually an infinite series, formed from solutions of the ordinary differential equations.

In many cases we ultimately need to deal with a series of sines and/or cosines, so part of the chapter is devoted to a discussion of such series, which are known as Fourier series.

With the necessary mathematical background in place, we then illustrate the use of separation of variables on a variety of problems arising from heat conduction, wave propagation, and potential theory.

7.1 Two-Point Boundary Value Problems

Consider

$$y'' + p(x)y' + q(x)y = g(x), \tag{1}$$

over the interval $x \in [x_1, x_2]$ with the **boundary conditions**(B.C.'s)

$$y(x_1) = y_1, \quad y(x_2) = y_2 \tag{2}$$

which is called a **two-point boundary value problem**(BVP).

- If $g(x) = y_1 = y_2 = 0$, problem (1) and (2) are **homogeneous**.
  - If $y_1 = y_2 = 0$, then they are homogeneous B.C's.
- Otherwise, the problem is **non-homogeneous**.

Except for boundary conditions in equation (2), we also have the other kinds of B.C.'s, such as $y'(x_1) = 0$, $y(-L) = y(L)$ etc.

Solution strategy:

1. Find the general solution; 2. Use B.C.s to determine c_1, c_2.

Example 1 Consider

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 0.$$

Solution By characteristic equation $r^2 + 1 = 0$, $r_{1,2} = \pm i$, we get the general solution

$$y(x) = c_1 \cos x + c_2 \sin x.$$

Plug the general solution y into B.C.'s

$$c_1 = 1, \quad c_2 = 0.$$

The solution to this DE is $y(x) = \cos x$.

In the following examples, you will see that if we change the BC's, the solution to the differential equation will also change.

Example 2

$$y'' + y = 0, \quad y(0) = 0, \quad y(\pi) = 2.$$

$$y(x) = c_1 \cos x + c_2 \sin x.$$

B.C.'s yield

$$c_1 = 0, \quad -c_1 = 2,$$

Contradiction! There is *no solution*.

Example 3 If assume different boundary conditions

$$y(0) = 0, \quad y(\pi) = 0.$$

Then $c_1 = 0$, c_2 can be arbitrary, so $y(x) = c_2 \sin x$ is a solution for any c_2. There are *infinitely many solutions*.

Example 4 For a non-homogeneous BVP, consider

$$y'' + 4y = \cos x, \quad y'(0) = 0, \quad y'(\pi) = 0. \tag{3}$$

The general solution of the corresponding homogeneous DEs to (3) is

$$Y_H(x) = c_1 \cos 2x + c_2 \sin 2x.$$

Assume a particular solution to (3) has the form

$$Y_P = A \cos x + B \sin x.$$

$$(-A \cos x - B \sin x) + 4(A \cos x + B \sin x) = \cos x, \quad \to \quad A = 1/3, \ \ B = 0.$$

Then the general solution to (3) is

$$y = Y_H + Y_P = c_1 \cos 2x + c_2 \sin 2x + 1/3 \cos x.$$

Set in B.C.'s:

$$y'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x - 1/3 \sin x$$

$$y'(0) = 0 \ \Rightarrow \ c_2 = 0, \quad y'(\pi) = 0 \ \Rightarrow \ c_2 = 0.$$

Then the solution is

$$y(x) = c_1 \cos 2x + 1/3 \cos x, \quad c \in \mathbb{R}.$$

Important Observations: For two-point Boundary Value Problems, the existence and uniqueness are automatic! This is very different from the initial value problem we studied in previous chapters.

For a linear BVP, three possibilities exist: (i) No solutions; (ii) Infinitely many solutions; (iii) Unique solution.

7.2 Eigenvalue Problems

Consider the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0, \tag{4}$$

We are interested in

- non-trivial solutions ($y \equiv 0$ is not considered since it is a trivial solution),
- and possible values of λ that lead to non-trivial solutions.

Note that it is important to have *homogeneous* BCs for eigenvalue problems! And the eigenvalue problem is a two-point BVP. Depending on the BC, it might or might not have nontrivial solutions.

Definition For some λ_n, if we are able to find a nontrivial solution $y_n(x)$, then, such λ_n is called an **eigenvalue**, and y_n is the corresponding **eigenfunction**.

This type of problems is an important building block in series solutions of partial differential equations.

Example 1(Dirichlet BC) We now attempt to solve the problem in (4).

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0,$$

The general solution depends on the roots, i.e., on the sign of λ, which has 3 possibilities:

1. If $\lambda < 0$, we write $\lambda = -k^2$, where $k = \sqrt{|\lambda|} > 0$,
   Then the characteristic equation yields $r^2 = k^2 \rightarrow r = \pm k$ and the general solution is

   $$y(x) = c_1 e^{kx} + c_2 e^{-kx}.$$

   By boundary conditions, we must have

   $$c_1 + c_2 = 0, \;\; c_1 e^{kL} + c_2 e^{-kL} = 0,$$

   which gives the solution $c_1 = c_2 = 0$. Then $y(x) = 0$, which is a trivial solution. Discard it.

2. If $\lambda = 0$,

   $$y'' = 0 \quad \rightarrow \quad y(x) = Ax + B, \;\; A, B \in \mathbb{R}.$$

   With homogeneous boundary conditions, $y(x) = 0$, therefore trivial. We also discard it.

3. If $\lambda > 0$, we write $\lambda = k^2$, where $k = \sqrt{|\lambda|} > 0$, then $r^2 = -k^2 \rightarrow r = \pm ki$, and

   $$y(x) = c_1 \cos kx + c_2 \sin kx.$$

   By setting in boundary conditions:

   $$\begin{cases} y(0) = c_1 \;= 0 \\ y(L) = c_2 \sin kL = 0 \end{cases}$$

So it must be $c_2 \neq 0$ and $\sin kL = 0$ leads to

$$kL = n\pi, \quad \Rightarrow \quad k = \frac{n\pi}{L}, \;\; n = 1, 2, 3, \cdots$$

We have found a family (infinite size) of eigenvalues and eigenfunctions! Using n as the index, they are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \cdots$$

In the next example, we will change the boundary conditions to another type, and we usually call it the Neumann type boundary conditions.

Example 2(Neumann BC) Consider

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0,$$

To solve this problem, we still consider the same 3 cases of λ: Denote $k = \sqrt{|\lambda|}$,

1. If $\lambda = -k^2 < 0$, then

   $$y(x) = c_1 e^{kx} + c_2 e^{-kx}, \quad y'(x) = kc_1 e^{kx} - kc_2 e^{-kx}$$

   To determine c_1, c_2, use the boundary conditions:

   $$kc_1 - kc_2 = 0, \quad kc_1 e^{kL} - kc_2 e^{-kL} = 0, \quad \Rightarrow \quad c_1 = c_2 = 0$$

   which gives only trivial solution (Discard).

2. If $\underline{\lambda = 0}$

   $$y(x) = Ax + B, \quad \rightarrow \quad y'(x) = A$$

   By boundary conditions, we have $A = 0,\ B \in \mathbb{R}$. So we found an eigenpair:

   $$\lambda_0 = 0, \qquad y_0(x) = 1.$$

3. If $\lambda = k^2 > 0$, then $r^2 = -k^2 \ \rightarrow \ r = \pm ki$,

   $$y(x) = c_1 \cos kx + c_2 \sin kx, \quad y'(x) = -kc_1 \sin kx + kc_2 \cos kx.$$

   We now check the boundary conditions:

   $$\begin{cases} y'(0) = kc_2 = 0 \\ y'(L) = -kc_1 \sin kL = 0 \end{cases} \Rightarrow \begin{cases} c_2 = 0 \\ c_1 \sin kL = 0 \end{cases}$$

   If $c_1 = 0$, then $y(x) \equiv 0$ which is trivial. So $c_1 \neq 0$ and we must have $\sin kL = 0$:

   $$kL = n\pi, \quad \Rightarrow \quad k = \frac{n\pi}{L}, \;\; n = 1, 2, 3, \cdots$$

   For each k, we get a pair of eigenvalue and eigenfunction

   $$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \cos\frac{n\pi x}{L}, \quad n = 1, 2, 3, \cdots$$

One could combine the results in 2 and 3, and get

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \cos\frac{n\pi x}{L}, \quad n = 0, 1, 2, 3, \cdots$$

Note that these are also a part of the trig set used in Fourier series! We will go to that part in the section 7.3.

Except for Dirichlet type and Neumann type of boundary conditions, an eigenvalue problem can also adopt boundary conditions of mixed type, or sometimes even more complicated one.

Example 3(Mixed BC) Find all positive eigenvalues and their corresponding eigenfunctions of the problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0.$$

Solution If $\lambda > 0$, we write $\lambda = k^2$ where $k > 0$, and the general solution is

$$y(x) = c_1 \cos kx + c_2 \sin kx, \quad y'(x) = -kc_1 \sin kx + kc_2 \cos kx.$$

We now check the boundary conditions.

$$\begin{cases} y(0) = c_1 = 0 \\ y'(L) = kc_2 \cos kL = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 \cos kL = 0 \end{cases}$$

which implies

$$kL = (n - \frac{1}{2})\pi, \quad \Rightarrow \quad k = \frac{\pi(n-\frac{1}{2})}{L}, \;\; n = 1, 2, 3, \cdots$$

We get the eigenvalues λ_n and the corresponding eigenfunction y_n as

$$\lambda_n = \left(\frac{\pi(n-\frac{1}{2})}{L}\right)^2, \; y_n = \sin\frac{\pi(n-\frac{1}{2})x}{L}, \; n = 1, 2, 3, \cdots$$

For many other different boundary conditions, please refer to the textbook.

Observation.

- Different types of boundary conditions would give very different eigenvalues and eigenfunctions.
- In these examples, the eigenfunctions are sine and cosine functions, in the same form as the trig set we use in Fourier series. Recall that the trig set is a mutually orthogonal set. So, for each of these eigenvalue problems, the set of eigenfunctions are mutually orthogonal. In fact, this is a more general property for eigenfunctions. One can define proper inner product such that eigenfunctions for the same eigenvalue problem would always form a mutually orthogonal set.

7.3 Fourier Series

The solutions to many important problems involving partial differential equations can be expressed as an infinite sum of sines and/or cosines. In this and the following two sections we explain in detail how this can be done. These trigonometric series are called **Fourier series**; they are somewhat analogous to Taylor series in that both types of series provide a means of expressing quite complicated functions in terms of certain familiar elementary functions.

Objective: Representing **periodic** functions as a series of sine and cosine functions.

Definition A function $f(x)$ is **periodic** with period T, if

$$f(x+T) = f(x), \quad \forall x$$

Note: If T is a period, so are $2T, 3T, \cdots$.

The smallest period T is called the **fundamental period**.

Properties: If $f(x)$ and $g(x)$ are both periodic with period T, so will any linear combination $af(x)+bg(x)$ for $a,b \in \mathbb{R}$; Also, the product $f(x)g(x)$ is periodic with the same period T.

Known examples of periodic functions: trig functions.

With period 2π:

$$\sin x, \sin 2x, \sin 3x, \cdots, \cos x, \cos 2x, \cos 3x, \cdots$$

With period $2L$: $(L > 0)$

$$\sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \sin\frac{3\pi x}{L}, \cdots, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \cos\frac{3\pi x}{L}, \cdots$$

We have the **trig set**:

$$\left\{1, \sin\frac{m\pi x}{L}, \cos\frac{m\pi x}{L}\right\}, \qquad m = 1, 2, \cdots$$

Definition Let $f(x)$ be periodic with period $2L$. **Fourier series** for $f(x)$ is:

$$\hat{f}(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right) \tag{5}$$

Here the constants a_0, a_m, b_m are called: **Fourier coefficients**.

How to compute the Fourier coefficients? Use the orthogonality of the trig set!

Definition Given two functions $u(x)$, $v(x) \in C[a,b]$, define the **inner product** on $C[a,b]$ as

$$< u, v > := \int_a^b u(x)v(x)\, \mathrm{d}x.$$

In our case, $a = -L$, $b = L$ (period $2L$), i.e.,

$$< u, v > := \int_{-L}^{L} u(x)v(x)\, \mathrm{d}x.$$

Definition The functions $u(x)$ and $v(x)$ are **orthogonal** if $< u, v >= 0$.

Claim 1: The trig set is *mutually orthogonal*, i.e, any two distinct functions in the set are orthogonal to each other. This means

$$
\begin{aligned}
< 1, \sin\frac{m\pi x}{L} >= 0, \qquad < 1, \cos\frac{m\pi x}{L} >= 0, \qquad \forall m \\
< \sin\frac{m\pi x}{L}, \sin\frac{n\pi x}{L} >= 0, \quad < \cos\frac{m\pi x}{L}, \cos\frac{n\pi x}{L} >= 0, \qquad \forall m \neq n \\
< \sin\frac{m\pi x}{L}, \cos\frac{n\pi x}{L} >= 0, \qquad \forall m, n
\end{aligned}
$$

Proof. Check the inner product by direct integration:

$$\int_{-L}^{L} \sin\frac{m\pi x}{L} \mathrm{d}x = 0, \quad \int_{-L}^{L} \cos\frac{m\pi x}{L} \mathrm{d}x = 0, \quad \forall m$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L} \mathrm{d}x = \frac{1}{2}\int_{-L}^{L} \left[\cos\frac{(m-n)\pi x}{L} - \cos\frac{(m+n)\pi x}{L}\right] \mathrm{d}x = 0, \ \forall m \neq n.$$

All other identities can be proven in a similar way.

Claim 2: The inner product of any two equal functions in the trig set equals some constants, that is,

$$< 1, 1 >= 2L,$$

$$< \sin\frac{n\pi x}{L}, \sin\frac{n\pi x}{L} >= L, \qquad < \cos\frac{n\pi x}{L}, \cos\frac{n\pi x}{L} >= L, \quad \forall n$$

Proof. Direct integration gives

$$\int_{-L}^{L} \sin^2\frac{n\pi x}{L} \mathrm{d}x = \frac{1}{2}\int_{-L}^{L} \left[1 - \cos\frac{2n\pi x}{L}\right] \mathrm{d}x = L.$$

We omit the proof for the other identities.

Back to Fourier series.

For $f(x)$ periodic with period $2L$, Fourier series for $f(x)$ is defined as:

$$\hat{f}(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right), \quad (*)$$

To determine a_i and b_i, we use the **orthogonality** of the trig set!

Steps: Multiply $(*)$ by $\cos\frac{n\pi x}{L}$ and integrate over $[-L, L]$.

$$
\begin{aligned}
\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L} = \quad & \int_{-L}^{L} \frac{a_0}{2}\cos\frac{n\pi x}{L} + \sum_{m=1}^{\infty}\int_{-L}^{L} a_m \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L} \mathrm{d}x \\
& + \sum_{m=1}^{\infty}\int_{-L}^{L} b_m \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L} \mathrm{d}x.
\end{aligned}
$$

All the terms are 0 except one:

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L} = \int_{-L}^{L} a_n \cos\frac{n\pi x}{L}\cos\frac{n\pi x}{L} \mathrm{d}x = a_n L$$

This gives us the formula to compute a_n:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L} \mathrm{d}x, \qquad n = 1, 2, 3, \cdots$$

Similarly, multiplying $(*)$ by $\sin\frac{n\pi x}{L}$ and integrating over $[-L,L]$, we derive the formula for b_n:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\mathrm{d}x, \qquad n = 1,2,3,\cdots$$

In a similar way, we get

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\mathrm{d}x.$$

Note that $a_0/2$ is the average of $f(x)$ over a period. The formula for a_0 fit in the one for a_n with $n = 0$. Hence we summarize the formulae in a more compacted way. These formulae for computing the Fourier coefficients are called **Euler-Fourier formula**. If the period is 2π, i.e., $2L = 2\pi$, we get simpler looking formulas

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\mathrm{d}x, \qquad n = 0,1,2,3,\cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\mathrm{d}x, \qquad n = 1,2,3,\cdots.$$

Example 1 Find the Fourier series for a periodic function $f(x)$ with period 2π

$$f(x) = \begin{cases} -1, & \text{if } -\pi < x < 0 \\ 1, & \text{if } 0 < x < \pi \end{cases}, \qquad f(x+2\pi) = f(x).$$

Solution: We use the Euler-Fourier formulas with $L = \pi$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\left[\int_{-\pi}^{0} -1\mathrm{d}x + \int_{0}^{\pi} 1\mathrm{d}x\right].$$

(Or note that $f(x)$ is an odd function. While integrating over a period, one get 0.)

For $n \geq 1$, we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0} -\cos nx\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi}\cos nx\mathrm{d}x \\ &= \frac{1}{\pi}(-\frac{1}{n})\sin nx|_{x=-\pi}^{0} + \frac{1}{\pi}\frac{1}{n}\sin nx|_{0}^{\pi} = 0. \end{aligned}$$

(Or, $f(x)$ is odd, and $\cos nx$ is even. Then, the product $f(x)\cos nx$ is odd. The integral over an entire period is 0.)

Finally, we compute b_n as

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0} -\sin nx\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi}\sin nx\mathrm{d}x \\ &= -\frac{1}{\pi n}(-\cos nx)|_{x=-\pi}^{0} + \frac{1}{\pi n}(-\cos nx)|_{0}^{\pi} \\ &= \frac{2}{n\pi}(1-\cos n\pi). \end{aligned}$$

(Or, $\sin nx$ is odd, so $f(x)\sin nx$ is even. The integrals on $[-\pi,0]$ and $[0,\pi]$ are the same. So one needs to do only one integral, and multiply the result by 2.)

Then

$$b_n = \frac{2}{n\pi}(1-(-1)^n) = \begin{cases} \frac{4}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Since all $a_n's$ are 0 and b_n is nonzero only for odd n, we will only have sine functions. We can now write out the Fourier series with the first few terms.

$$\hat{f}(x) = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots\right].$$

Partial sum of a series: The sum of the first few terms, while truncating the rest.

We can write $y_n(x)$ to be the sum of the first n term in the Fourier series. For our example, we have

$$\begin{aligned}
y_1(x) &= \frac{4}{\pi}\sin x \\
y_2(x) &= \frac{4}{\pi}[\sin x + \frac{1}{3}\sin 3x] \\
y_3(x) &= \frac{4}{\pi}[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x] \\
&\cdots
\end{aligned}$$

Then, the limit $\lim_{n\to\infty} y_n(x)$ (if it converges) gives the whole Fourier series.

Figure 3: Partial Sums on Example 1

Example 2 Find the Fourier series for a periodic function $f(x)$ with period 4

$$f(x) = \begin{cases} 0, & \text{if } -2 < x < -1 \\ K, & \text{if } -1 < x < 1 \\ 0, & \text{if } 1 < x < 2 \end{cases}, \qquad f(x+4) = f(x).$$

<u>Solution:</u> We use the Euler-Fourier formulas with $L = 4/2 = 2$:

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,\mathrm{d}x = \frac{1}{2}\cdot 2\cdot\int_0^1 K\,\mathrm{d}x = K.$$

(or, $f(x)$ is even. Integrating f over a period, one get twice of the integration over half of a period.)

For $n \geq 1$, we have

$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\mathrm{d}x \\
&= \int_0^2 f(x)\cos\frac{n\pi x}{2}\mathrm{d}x \\
&= \frac{2K}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^1 = \frac{2K}{n\pi}\sin\frac{n\pi}{2}.
\end{aligned}
\qquad \text{i.e. } a_n = \begin{cases} 0, & n \text{ even} \\ \frac{2K}{n\pi}, & n = 1, 5, 9, \cdots \\ -\frac{2K}{n\pi}, & n = 3, 7, 11, \cdots. \end{cases}$$

(Both $f(x)$ and $\cos\frac{n\pi x}{2}$ are even. Then, the product $f(x)\cos\frac{n\pi x}{2}$ is even.)

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\mathrm{d}x = 0.$$

(The product $f(x)\sin\frac{n\pi x}{2}$ is odd.)

The Fourier series is

$$f(x) = \frac{K}{2} + \frac{2K}{\pi}\left[\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \frac{1}{7}\cos\frac{7\pi x}{2} + \cdots\right].$$

Figure 4: Partial Sums on Example 2

Next example is a dummy one, but might be useful.

Example 3 Find the Fourier coefficients for $f(x)$, with period 2π, given as

$$f(x) = 2 + 4\sin x - 0.5\cos 4x - 99\sin 100x.$$

Since $f(x)$ here is already in terms of sine and cosine functions, there is no need to compute the Fourier coefficients. We just need to figure out where each term would fit, by comparing it with a Fourier series. Only the following Fourier coefficients are nonzero:

$$\frac{a_0}{2} = 2, \quad b_1 = 4, \quad a_4 = -0.5, \quad b_{100} = -99.$$

Observation.

- If $f(x)$ is an odd function, then there are no cosine functions in the Fourier series.
- If $f(x)$ is an even function, then there are no sine functions in the Fourier series.

In the next section, we will verify this observation as a general rule.

7.4 The Fourier Convergence Theorem

Given a function $f(x)$ defined on $[-L, L]$, one can always take integrations to write out a Fourier series $\hat{f}(x)$. The problem is if we can use $\hat{f}(x)$ to approximate $f(x)$ on $[-L, L]$? Or simply, does $\hat{f}(x)$ converges to $f(x)$ on $[-L, L]$? Is it possible for a series in (5) to converge at a point x in the interval $(-L, L)$, yet not be equal to $f(x)$? The answer is Yes.

To guarantee convergence of a Fourier series to the function from which its coefficients were computed it is essential to place additional conditions on the function. From a practical point of view, such conditions should be broad enough to cover all situations of interest, yet simple enough to be easily checked for particular functions.

Before stating a convergence theorem for Fourier series, we define a term that appears in the theorem.

Definition A function f is said to be **piecewise continuous** on an interval $a \leq x \leq b$ if the interval can be partitioned by a finite number of points $a = x_0 < x_1 < \cdots < x_n = b$ so that

1. f is continuous on each open subinterval $x \in (x_{i-1}, x_i)$.
2. f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

Theorem (Sufficient Condition for Convergence.) Suppose that f and f' are piecewise continuous on the interval $[-L, L]$. Then the Fourier series of f

- converges to $f(x)$ at all points x where f is continuous;

- converges to the mid value of the left and right limit, at a point x where f is discontinuous, i.e.,

$$\frac{f(x+)+f(x-)}{2}$$

This is confirmed by the previous examples.

7.5 Even and Odd Functions

The effort that is expended in evaluation of the definite integrals that define the coefficients the a_0, a_n, and b_n in the expansion of a function f in a Fourier series is reduced significantly when f is either an even or an odd function. Recall that a function f is said to be

even if $f(-x) = f(x)$ and **odd** if $f(-x) = -f(x)$.

The following theorem lists some properties of even and odd functions.

Properties of Even/Odd Functions

1. The product(quotient) of two even(odd) functions is even.
2. The product of an even function and an odd function is odd.
3. The sum(difference) of two even functions is even.
4. The sum(difference) of two odd functions is odd.
5. If f is even, then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\mathrm{d}x$.
6. If f is odd, then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$.

We have already observed that, if $f(x)$ is an even function, then its Fourier series will NOT have sine functions. If $f(x)$ is an odd function, then its Fourier series will NOT have cosine functions.

The formulas for the Fourier coefficients could be simplified, as we have already observed.

- If $f(x)$ is an even, periodic function with $2L$, it has a **Fourier cosine series**

$$\hat{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L},$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x, \qquad n = 0,1,2,3,\cdots$$

- If $f(x)$ is an odd, periodic function with $2L$, it has a **Fourier sine series**

$$\hat{f}(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x, \qquad n = 1,2,3,\cdots$$

Half-Range Expansions If a function $f(x)$ is only defined on an interval $[0,L]$, we can extend/expand the domain into the whole real line by periodic expansion. There are two ways of doing this:

- Extend $f(x)$ onto the interval $[-L,L]$ such that f is an even function, i.e., $f(-x) = f(x)$, then extend it into a periodic function with $2L$;
- Extend $f(x)$ onto the interval $[-L,L]$ such that f is an odd function, i.e., $f(-x) = -f(x)$, then extend it into a periodic function with $2L$.

These are called **even/odd periodic extensions** of f, or **half-range expansions**.

Example 1 Let $f(x) = x$ for $x \in [0,L]$. Sketch 3 periods of the even/odd extension of f, and then compute the corresponding Fourier sine or cosine series.

- For the odd periodic extension, we have Fourier sine series:

$$\hat{f}_{\text{odd}}(x) = \frac{2L}{\pi}\left[\sin\frac{\pi x}{L} - \frac{1}{2}\sin\frac{2\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} - \frac{1}{4}\sin\frac{4\pi x}{L} + \cdots\right].$$

- For the even periodic extension, we have Fourier cosine series:

$$\hat{f}_{\text{even}}(x) = \frac{L}{2} - \frac{4L}{\pi^2}\left[\cos\frac{\pi x}{L} + \frac{1}{9}\cos\frac{3\pi x}{L} + \frac{1}{5^2}\cos\frac{5\pi x}{L} + \frac{1}{7^2}\cos\frac{7\pi x}{L} + \cdots\right].$$

Choice of the half range expansion with concerns on convergence. We note that the Fourier cosine series, i.e, the even expansion seems to have smaller error for the same number of terms in the partial sum. This is because the even extension is a continuous function, while the odd extension is a piecewise continuous function with discontinuity points. All sine and cosine functions are smooth. Using smooth functions to represent discontinuous function would give larger error.

From the convergence Theorem, we know that, at a discontinuous point, the Fourier series converges to the mid value of the left and right limits. This implies an error that is equal to half of the size of the jump at this point. This error will not become smaller by taking more terms in the partial sum.

In practice, when one has the choice, it would always be recommended to choose the expansion that does NOT has discontinuities, if possible. So even expansions should be preferred for accuracy.

7.6 Introduction to Partial Differential Equations

Some basic concepts

- PDE: A differential equation that has **unknown functions** and **their partial derivatives**.
  - $u_t = c^2 u_{xx}$, 1D heat eqn
  - $u_{tt} = a^2 u_{xx}$, 1D wave eqn
  - $\left(\dfrac{\partial^2 u}{\partial x^2}\right)^2 + \dfrac{\partial^2 u}{\partial x \partial y} = x^2 + y^2$.
  - $u_t = c^2(u_{xx} + u_{yy})$, 2D heat eqn
  - $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, 2D Laplace eqn
  - $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x,y)$, 2D Poisson eqn

- Order: The highest order of derivatives.

- Linearity: The terms with u and its derivatives are in a **linear** form. Otherwise, **nonlinear**.

- Homogeneity: If all the terms contains u and its derivatives, then it is **homogeneous**, otherwise **non-homogeneous**.

Theorem (Superposition Principle)
If $u_1, u_2, \cdots, u_k$ are solutions of a homogeneous linear partial differential equation,

$$\mathscr{L}(u) = 0 \qquad \text{(H)}$$

then the linear combination

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k,$$

where c_i, $i = 1, 2, \cdots, k$ are constants, is also a solution.
The principle of superposition also valid in non-homogeneous linear PDEs:

$$\mathscr{L}(u) = f \qquad \text{(N)}$$

- If u_H solves (H) and u_P solves (N), then u solves (N), where

$$u = u_H + u_P.$$

- If u_1 solves $\mathscr{L}(u) = f_1$ and u_2 solves $\mathscr{L}(u) = f_2$, then

$$u = c_1 u_1 + c_2 u_2$$

  solves

$$\mathscr{L}(u) = c_1 f_1 + c_2 f_2.$$

7.7 1D Heat Equation; Solutions by Separation of Variable and Fourier Series

Consider a heat conduction problem for a straight bar of uniform cross section and homogeneous material. Let the x-axis be chosen to lie along the axis of the bar, and let $x = 0$ and $x = L$ denote the ends of the bar. Suppose further that the sides of the bar are perfectly insulated so that no heat passes through them and the ends of the bar are held at fixed temperature 0. We also assume that the cross-sectional dimensions are so small that the temperature u can be considered as constant on any given cross section. Then u is a function only of the axial coordinate x and the time t.

The variation of temperature in the bar is governed by a partial differential equation. The equation is called the **heat conduction equation**, and has the form

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \tag{6}$$

where α^2 is a constant known as the thermal diffusivity. In addition, we assume that the initial temperature distribution in the bar is given; thus

$$u(x,0) = f(x), \quad 0 \le x \le L, \tag{7}$$

where f is a given function. Finally, we assume that the ends of the bar are held at fixed temperature zero:

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0. \tag{8}$$

The fundamental problem of heat conduction is to find $u(x,t)$ that satisfies the **boundary-value problem** (BVP), that is the partial differential equation (6) together with the initial condition (7) and boundary conditions (8).

Solution to the IBVP by Separation of Variable

Step 1: Separating variables. To find $u(x,t)$, we start by making a basic assumption about the form of the solutions: $u(x,t)$ is a product of two functions, one depending only on x and the other depending only on t; thus

$$u(x,t) = X(x)T(t). \tag{9}$$

Substituting from Eq. (9) for u in the differential equation (6) yields

$$XT' = \alpha^2 X''T, \quad \to \quad \frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}, \tag{10}$$

in which the variables are separated; that is, the left side depends only on x and the right side only on t. The only way to make it happen is the radio equals a constant. If we call this separation constant $-\lambda$, then

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda, \tag{11}$$

We end up with 2 ODEs,

$$X'' + \lambda X = 0 \tag{12}$$

$$T' + \alpha^2 \lambda T = 0 \tag{13}$$

The assumption (9) has led to the replacement of the partial differential equation (6) by two ODEs (34) and (13). Each of these equations can be readily solved for any value of λ. The product of two solutions of Eq. (34) and (13), respectively, provides a solution of the partial differential equation (6).

However, we are interested only in those solutions of Eq. (6) that also satisfy the boundary conditions (8).

Step 2: Solve for $X(x)$. Substituting $u(x,t) = XT$ in the boundary condition at $x = 0$, we obtain

$$u(0,t) = X(0)T(t) = 0. \tag{14}$$

If above equation is satisfied by choosing $T(t)$ to be zero for all t, then $u(x,t)$ is zero for all x and t, and we have already rejected this possibility. Therefore it must be satisfied by requiring that

$$X(0) = 0. \tag{15}$$

Similarly, the boundary condition at $x = L$ requires that

$$X(L) = 0. \tag{16}$$

We have the following *eigenvalue problem* for $X(x)$

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0,$$

which is an example we had earlier. The nontrivial solutions should be eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1,2,3,\cdots \tag{17}$$

Step 3: Solve for $T(t)$. For a given n, we get a solution $T_n(t)$, which solves

$$\begin{aligned} T'(t) + \alpha^2\lambda_n T &= T'(t) + \alpha^2\tfrac{n^2\pi^2}{L^2}T = 0 \\ T_n(t) = C_n \cdot \exp&\left[-\left(\tfrac{n\pi\alpha}{L}\right)^2 t\right], \end{aligned} \tag{18}$$

where C_n are arbitrary constants, $n = 1,2,3,\cdots$.

Step 4: Formal solution. Multiplying Eqs. (17) and (18) together, we have

$$u_n = X_n(x)T_n(t),$$

satisfy the PDE (6) and the BCs (8) for each $n = 1,2,\cdots$. The functions u_n are sometimes called the **fundamental solutions** to the heat conduction problem (6),(7) and (8). Then the formal solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2\alpha^2 t/L^2}\sin(n\pi x/L). \tag{19}$$

It remains only to satisfy the initial condition (7).

Step 5: Determine C_n by initial condition. To satisfy the initial condition, set $t = 0$, we get

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} = f(x), \qquad 0 \le x \le L.$$

In other words, we need to choose the coefficients C_n so that the series of sine functions converges to the initial temperature distribution $f(x)$ for $0 \le x \le L$. The series is just the Fourier sine series for f and its coefficients are given by

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\mathrm{d}x, \qquad n = 1,2,3,\cdots \tag{20}$$

Summary: The formal solution to the following initial and boundary value problem

$$\begin{aligned} u_t = \alpha^2 u_{xx}, &\qquad 0 < x < L, \quad t > 0 \\ u(x,0) = f(x), &\qquad 0 \le x \le L \\ u(0,t) = 0, \quad u(L,t) = 0, &\qquad t > 0. \end{aligned}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin\frac{n\pi x}{L},$$

where

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\mathrm{d}x, \qquad n = 1,2,3,\cdots.$$

Discussions on solutions:

- Harmonic oscillation in x, exponential decay in t:
- Speed of decay depending on $\lambda_n = n\pi\alpha/L$. Faster decay for larger n, meaning the high frequency components are vanished quickly. After a while, what remain in the solution are the terms with small n.
- The **asymptotic** or **steady state solution** of the problem can be obtained as $\lim_{t\to\infty} u(x,t)$, which is 0 here, $\forall x$.

Example 1 Let $\alpha = 1$ and $L = 1$. If $f(x) = 10\sin\pi x$, then we have $C_1 = 10$ and all other $C_n = 0$, the solution is

$$u(x,t) = 10e^{-\pi^2 t}\sin\pi x.$$

If now let $f(x) = 10\sin 3\pi x$, then $C_3 = 10$ and all other $C_n = 0$, and the solution is

$$u(x,t) = 10e^{-9\pi^2 t}\sin 3\pi x.$$

If the initial temperature is

$$f(x) = 10\sin\pi x + 10\sin 3\pi x,$$

the solution would be

$$u(x,t) = 10e^{-\pi^2 t}\sin\pi x + 10e^{-9\pi^2 t}\sin 3\pi x.$$

Example 2 Let $\alpha = 1$. If $f(x) = x$ on $[0,L]$, then the solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{n^2\pi^2 t}{L^2}} \sin\frac{n\pi x}{L}, \qquad n = 1,2,3,\cdots$$

where

$$C_n = \frac{2L}{n\pi}(-1)^{n+1}.$$

based on the result of the Fourier sine expansion in the last example from Section 7.5.

7.8 Other Heat Conduction Problems

Neumann Boundary Condition. We now consider a set of new BCs: (insulated)

$$u_x(0,t) = 0, \quad u_x(L,t) = 0.$$

This means that both ends are insulated, and no heat flows through. Following the same setting, we get the eigenvalue problem for $X(x)$ as

$$X'' + \lambda X = 0, \quad X'(0) = X'(L) = 0,$$

From Example 2 in Section 7.2, we have only nonnegative eigenvalues λ_n:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = \cos\frac{n\pi x}{L}, \quad n = 0, 1, 2, 3, \cdots$$

The solution for $T(t)$ remains the same

$$T_n(t) = C_n \cdot \exp\left[-\left(\frac{n\pi\alpha}{L}\right)^2 t\right], \quad n = 0, 1, 2, 3, \cdots$$

which leads to the formal solution

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} X_n(x)T_n(t) = C_0 + \sum_{n=1}^{\infty} C_n e^{-\frac{n^2\pi^2\alpha^2}{L^2}t} \cos\frac{n\pi x}{L},$$

Finally, by fitting in the initial condition, C_n can be determined as the Fourier cosine coefficient for the even half-range expansion of $f(x)$, i.e.,

$$C_0 = \frac{1}{2}\frac{2}{L}\int_0^L f(x)\,\mathrm{d}x = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x, \qquad C_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x, \quad n = 1, 2, 3, \cdots.$$

Discussions on solutions:

- Harmonic oscillation in x, exponential decay in t.
- exponential decay in t, except the term C_0. Decay faster for larger n.
- As $t \to \infty$, $u(x,t) \to C_0$, which is the average of $f(x)$ (initial temperature). This is reasonable because the bar is insulated.

Steady State: As $t \to \infty$, solution does not change in time anymore, as it reaches a steady state, say $U(x)$. Then

$$U_t = 0 \quad \to \quad U_{xx} = U_t = 0 \quad \to \quad U(x) = Ax + B,$$

where A, B are determined by boundary conditions.

Example 1 (1) If $u(0,t) = a$, $u(L,t) = b$, then $U(0) = a$, $U(L) = b$, we get

$$U(x) = a + \frac{b-a}{L}x.$$

(2) If $u(0,t) = a$, $u_x(L,t) = 0$, then $U(0) = a$, $U'(L) = 0$, we get $U(x) = a$.

(3) If $u(0,t) = a$, $u_x(L,t) = b$, then $U(0) = a$, $U'(L) = b$, we get $U(x) = bx + a$.

Non-homogeneous boundary condition.

$$u_t = \alpha^2 u_{xx}, \quad u(0,t) = a, \;\; u(L,t) = b.$$

We know that the steady state is $U(x) = a + \frac{b-a}{L}x$. Now define a new variable

$$w(x,t) = u(x,t) - U(x).$$

Then

$$w_t = u_t, \quad w_x = u_x - U'(x), \quad w(L,t) = u(L,t) - U(L) = b - b = 0$$

which are homogeneous. Then, one can find the solution for w by the standard separation of variables and Fourier series. Once this is done, one can go back to u by

$$u(x,t) = w(x,t) + U(x).$$

Example 2 Find the solution to the heat equation $u_t = u_{xx}$ with the following BCs

$$u(0,t) = 2, \quad u(1,t) = 4,$$

and IC

$$u(x,0) = 2 + 2x - \sin\pi x - 3\sin 3\pi x.$$

Solution. Denote the steady state solution as $U(x)$, then it satisfies the following two-point boundary value problem

$$U'' = 0, \quad U(0) = 2, \quad U(1) = 4,$$

which gives the solution

$$U(x) = 2 + 2x.$$

Let $w(x,t)$ be the solution of the heat equation with homogeneous boundary condition. Then

$$w_t = w_{xx}, \quad w(0,t) = w(1,t) = 0, \quad w(x,0) = u(x,0) - U(x) = -\sin\pi x - 3\sin 3\pi x.$$

The formal solution for w is

$$w(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t}\sin(n\pi x), \qquad n = 1,2,3,\cdots$$

Here C_n are Fourier coefficients of the initial data $w(x,0)$. We find only two coefficients that are not 0, namely

$$C_1 = -1, \quad C_3 = -3.$$

i.e.

$$w(x,t) = -e^{-\pi^2 t}\sin(\pi x) - 3e^{-9\pi^2 t}\sin(3\pi x).$$

Finally combine $w(x,t)$ together with $U(x)$, we get the solution

$$u(x,t) = w(x,t) + U(x) = 2 + 2x - e^{-\pi^2 t}\sin(\pi x) - 3e^{-9\pi^2 t}\sin(3\pi x).$$

This separation of variables technique could be applied to a more general class of PDEs. It is not difficult to check whether an equation is separable. After separating the variables, one needs to fit in the boundary condition. Other than the heat conduct problem, we can use the separation of variable to solve a 1-D Wave equation or a 2D Laplace equation. You may find more details in the textbook.

8 Laplace transform

In this chapter, we will learn a new technique to solve differential equations (including both ODEs and PDEs): Laplace transform.

8.1 What are Laplace Transforms, and Why?

This is much easier to state than to motivate! We state the definition in two ways, first in words to explain it intuitively, then in symbols so that we can calculate transforms.

Definition 1

Given f, a function of time, with value $f(t)$ at time t, the Laplace transform of f is denoted F and it gives an average value of f taking over all positive values of t such that the value $F(s)$ represents an average of f taken over all possible time intervals of length s.

Definition 2

$$\mathscr{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt, \text{ for } s > 0. \tag{21}$$

A short table of commonly encountered Laplace Transforms is given in Section 8.6. Note that this definition involves integration of a product so it will involve frequent use of integration by parts.

This immediately raises the question of why to use such a procedure. In fact the reason is strongly motivated by real engineering problems. There, typically we encounter models for the dynamics of phenomena which depend on rates of change of functions, e.g. velocities and accelerations of particles or points on rigid bodies, which prompts the use of ordinary differential equations (ODEs). We can use ordinary calculus to solve ODEs, provided that the functions are nicely behaved—which means continuous and with continuous derivatives. Unfortunately, there is much interest in engineering dynamical problems involving functions that input step change or spike impulses to systems—playing pool is one example. Now, there is an easy way to smooth out discontinuities in functions of time: simply take an average value over all time. But an ordinary average will replace the function by a constant, so we use a kind of moving average which takes continuous averages over all possible intervals of t. This very neatly deals with the discontinuities by encoding them as a smooth function of interval length s.

The amazing thing about using Laplace Transforms is that we can convert a whole ODE initial value problem into a Laplace transformed version as functions of s, simplify the algebra, find the transformed solution $F(s)$, then undo the transform to get back to the required solution f as a function of t.

Interestingly, it turns out that the transform of a derivative of a function is a simple combination of the transform of the function and its initial value. So a calculus problem is converted into an algebraic problem involving polynomial functions, which is easier.

There is one further point of great importance: calculus operations of differentiation and integration are linear. So the Laplace Transform of a sum of functions is the sum of their Laplace Transforms and multiplication of a function by a constant can be done before or after taking its transform.

In this course we find some Laplace Transforms from first principles, i.e. from the definition (21), describe some theorems that help finding more transforms, then use Laplace Transforms to solve problems involving ODEs.

8.2 Finding Laplace Transforms

We have three methods to find $F(s)$ for a given $f(t)$;

From the definition: Here we use (21) directly: eg

$$\text{For } f(t)=1,\ \mathscr{L}[1]=\int_0^\infty e^{-st}\,dt\ =\left[-\frac{1}{s}e^{-st}\right]_0^\infty=\frac{1}{s}.$$

$$\text{For } f(t)=t,\ \mathscr{L}[t]=\int_0^\infty e^{-st}t\,dt\ =\left[-\frac{1}{s}e^{-st}t\right]_0^\infty+\int_0^\infty\frac{1}{s}e^{-st}=\frac{1}{s^2}.$$

$$\text{For } f(t)=y'(t),\ \mathscr{L}[f(t)]=\int_0^\infty e^{-st}\frac{dy}{dt}\,dt\ =\left[e^{-st}y\right]_0^\infty+\int_0^\infty se^{-st}y\,dt\ =-y(0)+sF(s),$$

where $F(s)=\mathscr{L}(y(t))$. More generally,

$$\mathscr{L}(y^{(n)})=s^nF(s)-s^{n-1}y(0)-\cdots-sy^{(n-1)}(0)-y^{(n-1)}(0).$$

For $f(t)=e^{at},\ a$ constant,

$$\mathscr{L}[e^{at}]=\int_0^\infty e^{-st}e^{at}\,dt\ =\int_0^\infty e^{-(s-a)t}\,dt\ =\left[-\frac{1}{s-a}e^{-(s-a)t}\right]_0^\infty=\frac{1}{s-a},\ \ s>a.$$

From a property: There are a number of powerful theorems about the properties of transforms and these theorems can be proved by the definition. In the following, some frequently used properties are listed.

- **Linearity**

$$\mathscr{L}[af(t)+bg(t)]=aF(s)+bG(s).$$

  Examples:

$$\mathscr{L}[3t+4]=3\frac{1}{s^2}+4\frac{1}{s}.$$

$$\mathscr{L}[\cos at+i\sin at]=\mathscr{L}[e^{iat}]\ \text{ by DeMoivre.}$$

$$\mathscr{L}[e^{iat}]=\frac{1}{s-ia}=\frac{s}{s^2+a^2}+\frac{ia}{s^2+a^2}.$$

  Hence, equating real and imaginary parts and using linearity

$$\mathscr{L}[\cos at]=\frac{s}{s^2+a^2}$$

$$\mathscr{L}[\sin at]=\frac{a}{s^2+a^2}.$$

- **Time Shifting**

$$\mathscr{L}[f(t-t_0)]=e^{-t_0s}F(s).$$

- **Shifting in s-Domain**

$$\mathscr{L}[e^{s_0t}f(t)]=F(s-s_0).$$

- **Time Scaling**

$$\mathscr{L}[f(at)] = \frac{1}{|a|}F(\frac{s}{a})$$

- **Convolution**

$$\mathscr{L}[f(t) * g(t)] = F(s)G(s),$$

where $f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau$.

Example:

$$\mathscr{L}^{-1}\left[\frac{f(s)}{s}\right].$$

$$\mathscr{L}^{-1}[f(s)] = f(t), \text{ and } \mathscr{L}^{-1}\left[\frac{1}{s}\right] = 1 = g(t),$$

so

$$\mathscr{L}^{-1}\left[\frac{f(s)}{s}\right] = \int_0^t f(\theta)\,d\theta.$$

- **Differentiation in Time Domain**

$$\mathscr{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0).$$

In general, we have

$$\mathscr{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

- **Differentiation in s-Domain**

$$\mathscr{L}[tf(t)] = -\frac{d}{ds}F(s).$$

This can be proven by differentiating the Laplace transform:

$$\frac{d}{ds}F(s) = \int_0^\infty f(t)\frac{d}{ds}e^{-st}dt = \int_0^\infty (-t)f(t)e^{-st}dt$$

Repeat this process we get

$$\mathscr{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s).$$

- **Integration in Time Domain**

$$\mathscr{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}.$$

This can be proven by realizing that

$$f(t) * u_0(t) = \int_0^t f(\tau)u_0(t-\tau)d\tau = \int_0^t f(\tau)d\tau$$

and therefore by convolution property we have

$$\mathscr{L}[f(t) * u_0(t)] = F(s)\frac{1}{s}.$$

Note $u_0(t) = 1$ and $\mathscr{L}[u_0(t)] = 1/s$.

From a list: Computer algebra packages like Mathematica, Matlab and Maple know Laplace Transforms of all the functions you are likely to encounter, so you have access to these online, and the packages have also an inversion routine to find a function f from a given F. There are books with long lists of transforms of known functions and compositions of functions; we give some in Section 8.6, which you should read through, eg some that are harder to calculate:

$$\mathscr{L}[t^n] = \frac{n!}{s^{n+1}},\ n = 0,1,2,\ldots,\ \ \mathscr{L}[t^{1/2}] = \frac{1}{2}\left(\frac{\pi}{s^3}\right)^{1/2},\ \ \mathscr{L}[t^{-1/2}] = \left(\frac{\pi}{s}\right)^{1/2}.$$

8.3 Finding inverse transforms using partial fractions

Given a function f, of t, we denote its Laplace Transform by $\mathscr{L}[f(t)] = F(s)$; the inverse process is written:

$$\mathscr{L}^{-1}[F(s)] = f(t).$$

Frequently, a Laplace transform $F(s)$ is expressible as a sum of several terms,

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s).$$

Suppose that $f_1(t) = \mathscr{L}^{-1}\{F_1(s)\}, \cdots, f_n(t) = \mathscr{L}^{-1}\{F_n(s)\}$. Then the function

$$f(t) = f_1(t) + f_2(t) + \cdots f_n(t)$$

has the Laplace transform $F(s)$. By the uniqueness property, no other continuous function f having the same transform. Thus

$$\mathscr{L}^{-1}[F(s)] = \mathscr{L}^{-1}[F_1(s)] + \mathscr{L}^{-1}[F_2(s)] + \cdots + \mathscr{L}^{-1}[F_n(s)],$$

that is, the inverse Laplace transform is also a linear operator.

A common situation is when $F(s)$ is a polynomial in s, or more generally, a ratio of polynomials; then we use partial fractions to simplify the expressions. Given an expression for a Laplace transform of the form N/D where numerator N and denominator D are both polynomials of s, possibly in the form of factors, and N may be constant; use partial fractions:

(i) if N has degree equal to or higher than D, divide N by D until the remainder is of lower degree than D

(ii) For every linear factor like $(as+b)$ in D, write a partial fraction of the form $A/(as+b)$

(iii) For every repeated factor like $(as+b)^2$ in D write two partial fractions of the form $A/(as+b)$ and $B/(as+b)^2$. Similarly for every repeated factor like $(as+b)^3$ in D write three partial fractions of the form $A/(as+b)$, $B/(as+b)^2$ and $C/(as+b)^3$; and so on.

(iv) For quadratic factor (as^2+bs+c) write a partial fraction $(As+B)/(as^2+bs+c)$.

For repeated quadratic factors write a series of partial fractions as in (iii), but with numerators of the form $(As+B)$ and successive powers of the quadratic factor as the denominators.

With a little more algebra you should in this way be able to write the original expression as a sum of simpler transforms, which are found in your table. You then add their inverse transforms together, to get the inverse of the original transform.

Examples: 1. For

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}, \tag{22}$$

using partial fractions, we can write $Y(s)$ in the form of

$$Y(s) = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4} = \frac{(as+b)(s^2+4)+(cs+d)(s^2+1)}{(s^2+1)(s^2+4)}. \tag{23}$$

By expanding the numerator on the right hand side of (23) and equating it to the numerator in (22), we find that

$$2s^3 + s^2 + 8s + 6 = (a+c)s^3 + (b+d)s^2 + (4a+c)s + (4b+d) \tag{24}$$

for all s. Then, comparing coefficients of like powers of s, we have

$$a + c = 2, \quad b + d = 1, \quad 4a + c = 8, \quad 4b + d = 6. \tag{25}$$

Consequently, $a = 2, c = 0, b = 5/3, d = -2/3$, from which it follows that

$$Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}. \tag{26}$$

Then from the table, the inverse Laplace transform of $Y(s)$ gives

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t. \tag{27}$$

2. For

$$Y(s) = \frac{s^2}{s^4-1}, \tag{28}$$

a partial fraction expansion of $Y(s)$ is

$$Y(s) = \frac{as+b}{s^2-1} + \frac{cs+d}{s^2+1}. \tag{29}$$

It follows that

$$(as+b)(s^2+1) + (cs+d)(s^2-1) = s^2 \tag{30}$$

for all s. By setting $s = 1$ and $s = -1$ respectively in (30), we have

$$2(a+b) = 1, \quad 2(-a+b) = 1, \tag{31}$$

and therefore $a = 0, b = 1/2$. If we set $s = 0$ in (30), then $b - d = 0$, so $d = 1/2$. Finally, equating the coefficients of the cubic terms on each side of (30), we find that $a + c = 0$, so $c = 0$. Thus

$$Y(s) = \frac{1/2}{s^2-1} + \frac{1/2}{s^2+1} \tag{32}$$

and

$$y(t) = \frac{\sinh t + \sin t}{2}. \tag{33}$$

8.4 Solving ODEs and ODE Systems

The application of Laplace Transform methods is particularly effective for linear ODEs, and for systems of such ODEs. To transform an ODE, we need the appropriate initial values of the function involved and initial values of its derivatives. We illustrate the methods with the following examples.

Examples:

1. For the ODE problem
$$2\frac{dy}{dt} - y = \sin t, \quad y(0) = 1. \tag{34}$$
   (a) obtain the transformed version as
$$2[sY(s) - 1] - Y(s) = \frac{1}{s^2+1}.$$
   (b) Rearrange to get
$$Y(s) = \frac{2s^2+3}{(2s-1)(s^2+1)} = \frac{A}{2s-1} + \frac{Bs+C}{s^2+1}$$
   (c) Show that $A = \frac{14}{5}$, $B = \frac{-2}{5}$, $C = \frac{-1}{5}$, and take the inverse transform to obtain the final solution to (34) as
$$y(t) = \frac{7}{5}e^{t/2} - \frac{2}{5}\cos t - \frac{1}{5}\sin t.$$

2. For the system of ODEs
$$\begin{aligned} \frac{dy}{dt} - \frac{dx}{dt} + y + 2x &= e^t \qquad (35)\\ \frac{dy}{dt} + \frac{dx}{dt} - x &= e^{2t} \qquad (36)\\ \text{Initial data}: \ x(0), y(0) &= 1, \qquad (37) \end{aligned}$$
   (a) transform to obtain
$$\begin{aligned} [sY(s) - y_0] - [sX(s) - x_0] + Y(s) + 2X(s) &= \frac{1}{s-1} \qquad (38)\\ sY(s) - y_0 + [sX(s) - x_0] - X(s) &= \frac{1}{s-2}. \qquad (39) \end{aligned}$$
   (b) Rearranging,
$$\begin{aligned} (s+1)Y - (s-2)X &= \frac{1}{s-1} + 1 - 1 = \frac{1}{s-1} \qquad (40)\\ sY + (s-1)X &= \frac{1}{s-2} + 1 + 1 = \frac{2s-3}{s-2}. \qquad (41) \end{aligned}$$
   (c) To eliminate $Y(s)$, multiply (40) by s and (41) by $(s+1)$ then subtract, and deduce as follows
$$\begin{aligned} [(s-1)(s+1) + s(s-2)]X(s) &= \frac{(2s-3)(s+1)}{s-2} - \frac{s}{s-1}, \qquad (42)\\ X(s) &= \frac{2s^3 - 4s^2 + 3}{(s-1)(s-2)(2s^2-2s-1)}. \qquad (43) \end{aligned}$$
   Then, by partial fractions,
$$X(s) = \frac{1}{s-1} + \frac{1}{s-2} - \frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2 - (\frac{\sqrt{3}}{2})^2} - \frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{(s-\frac{1}{2})^2 - (\frac{\sqrt{3}}{2})^2}. \tag{44}$$

(d) From the table of transforms, we can find $x(t)$ as

$$x(t) = e^t + e^{2t} - e^{t/2}\cosh\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{t/2}\sinh\left(\frac{\sqrt{3}}{2}t\right).$$

(e) You can find $y(t)$ by differentiating and substituting $\frac{dx}{dt}$ in either of the system equations. Quicker here is to subtract the second equation from the first to obtain

$$-2\frac{dx}{dt} + y + x + 2x = e^t - e^{2t}$$

so

$$y(t) = 2\frac{dx}{dt} - 3x + e^t - e^{2t}.$$

3. For the IVP with linear ODEs in variable coefficients,

$$ty'' - ty' + y = 2, \qquad y(0) = 2,\ \ y'(0) = -4. \tag{45}$$

We have $\mathscr{L}[y'] = sY(s) - y(0)$, then $\mathscr{L}[ty'] = -\frac{\mathrm{d}}{\mathrm{d}s}[sY(s) - y(0)] = -[Y(s) + sY'(s)]$.

Similarly, $\mathscr{L}[ty''] = -\frac{\mathrm{d}}{\mathrm{d}s}[s^2Y(s) - sy(0) - y'(0)] = -[2sY(s) + s^2Y'(s) - y(0)]$.

Thus, rewrite the differential equation in s-domain:

$$\begin{aligned}
-[2sY(s) + s^2Y'(s) - y(0)] + [Y(s) + sY'(s)] + Y(s) &= \frac{2}{s} \\
(-s^2 + s)Y'(s) + 2(1-s)Y(s) &= \frac{2-2s}{s} \\
Y'(s) + \frac{2}{x}Y(s) &= \frac{2}{s^2}. \\
Y(s) &= \frac{2}{s} + \frac{c}{s^2}, \quad c \in \mathbb{R}
\end{aligned}$$

The inverse transform gives

$$y(t) = 2 + ct.$$

Match with the second initial condition which we haven't used yet, the solution to this IVP is

$$y(t) = 2 - 4t.$$

8.5 Step input and Impulse problems

Laplace transform methods are particularly valuable in handling differential equations involving impulse and step functions. In the following, we first give the definitions for step function and Delta function and then give examples involving these two types of functions.

8.5.1 Step function and Delta function

The unit step function is defined by

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases} \tag{46}$$

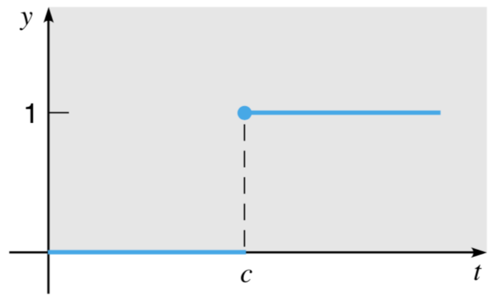


Figure 5: The unit step function $y = u_c(t)$.

The graph of $y = u_c(t)$ is given in Figure 5. The Laplace transform of $u_c(t)$ is easily determined

$$\mathscr{L}[u_c(t)] = \int_0^{\infty} e^{-st} u_c(t) \mathrm{d}t = \int_c^{\infty} e^{-st} \mathrm{d}t = \frac{e^{-cs}}{s}, \quad s > 0. \tag{47}$$

For the Delta function, we may understand it in the sense of limit. Define a sequence of functions

$$d_\tau(t) = \begin{cases} 1/(2\tau), & -\tau < t < \tau, \\ 0, & t \leq -\tau \text{ or } t \geq \tau, \end{cases} \tag{48}$$

where τ is a small positive constant (see the illustration of $d_\tau(t)$ in Figure 6). One can see that

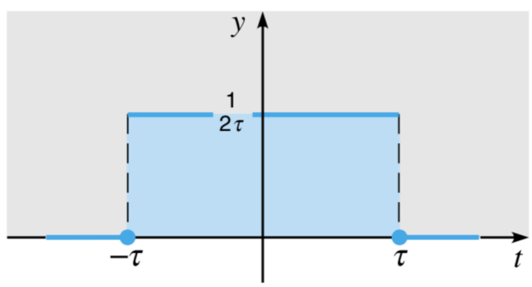


Figure 6: The function of $y = d_\tau(t)$.

$$\lim_{\tau \to 0} d_\tau(t) = 0, \quad t \neq 0. \tag{49}$$

And let

$$I(\tau) = \int_{-\infty}^{\infty} d_\tau(t) \mathrm{d}t, \tag{50}$$

then it is to calculate $I(\tau) = 1$ for each $\tau \neq 0$ and

$$\lim_{\tau \to 0} I(\tau) = 1. \tag{51}$$

Equations (49) and (51) can be used to define the unit impulse function δ which have the properties

$$\delta(t) = 0, \quad t \neq 0 \tag{52}$$

and

$$\int_{-\infty}^{\infty} \delta(t) \mathrm{d}t = 1. \tag{53}$$

8.5.2 Step Input problems

Here we consider the following initial value problem which involves a step input function—typical of many control-type problems:

$$\ddot{x} + 3\dot{x} + 2x = u_0(t), \;\; x(0) = \dot{x}(0) = 0. \tag{54}$$

Here we have a second order ODE representing a system that is at rest until time $t = 0$, when a unit step input $u_0(t)$ is applied; we seek the output $x(t)$.

Example

1. Transform (54) to get

$$(s^2 + 3s + 2)X(s) = \frac{1}{s},$$

$$X(s) = \frac{1}{s} \cdot \frac{1}{(s+2)(s+1)} = \frac{1}{s}\left(\frac{-1}{s+2} + \frac{1}{s+1}\right).$$

2. Apply the Integration property to obtain

$$\begin{aligned} x(t) &= \int_0^t \left(-e^{-2\theta} + e^{-\theta}\right) d\theta && (55)\\ &= \left[\frac{1}{2}e^{-2\theta} - e^{-\theta}\right]_0^t && (56)\\ &= \frac{1}{2}e^{-2t} - e^{-t} - \frac{1}{2} + 1 && (57)\\ &= \frac{1}{2}e^{-2t} - e^{-t} + \frac{1}{2}. && (58) \end{aligned}$$

3. Show that the solution $x(t)$ is asymptotic to $x = \frac{1}{2}$, why is this obvious as a steady state solution?

8.5.3 Impulse problem

The problem in the examples below represents the dynamics of a point, initially at rest, moving away from the origin along the y-axis under a constant acceleration of value 10 for $0 \le t < 1$ and an extra impulse acceleration of size 10 is applied at $t = 1$. This is like a simple rocket boost, but can you solve it any other way? We use the Dirac impulse function $\delta(t-a)$ which is nonzero at $t = a$, but zero elsewhere while having unit total area under it:

$$\delta(t-a) = 0 \text{ if } (t \neq a) \;\text{ and }\; \int_{-\infty}^{\infty} \delta(t-a)\, dt = 1. \tag{59}$$

Example

Consider the ODE initial value problem given by

$$y'' = 10 + 10\delta(t-1), \quad y(0) = y'(0) = 0. \tag{60}$$

1. Begin by sketching the graph of the acceleration, y'', to show the step increase.

2. Transforming according to the table, to get

$$s^2Y - sy(0) - y'(0) = \frac{10}{s} + 10e^{-s}$$

$$\text{so, rearranging} \quad Y(s) = \frac{10}{s^3} + \frac{10e^{-s}}{s^2} = 5\frac{2}{s^3} + 10e^{-s}\frac{1}{s^2}$$

3. From the table use the Delay property to deduce that

$$y(t) = 5t^2 + 10(t-1)\,u(t-1)$$

4. By interpreting the step function $u(t-1)$ up to and after $t=1$, show that the impulse at $t=1$ produces what you would expect: a discontinuity in **velocity** at $t=1$. Sketch the full solution:

$$y(t) = 5t^2 \text{ for } t \le 1 \quad (\text{so here } y'(t) = 10t)$$

$$y(t) = 5t^2 + 10(t-1) \text{ for } t > 1 \quad (\text{so here } y'(t) = 10t + 10).$$

8.6 Laplace transform for PDE (heat equation)

In this section, we show how to use the Laplace transform to solve the one-dimensional heat equation. There are three main steps in order to solve a PDE using the Laplace transform:
1. Begin by taking the Laplace transform with one of the two variables, usually t. This will give an ODE of the transform of the unknown function.
2. Solving the ODE, we shall obtain the transform of the unknown function.
3. By taking the inverse Laplace transform, we obtain the solution to the original problem.

Given a function $u(x,t)$ defined for all $t > 0$ and assumed to be bounded we can apply the Laplace transform in t considering x as a parameter.

$$\mathscr{L}[u(x,t)] = \int_0^\infty e^{-st} u(x,t)\mathrm{d}t = U(x,s). \tag{61}$$

In applications to PDEs we need the following:

$$\begin{aligned}\mathscr{L}[u_t(x,t)] &= \int_0^\infty e^{-st} u_t(x,t)\mathrm{d}t = e^{-st}u(x,t)\Big|_0^\infty + s\int_0^\infty e^{-st}u(x,t)\mathrm{d}t \\ &= sU(x,s) - u(x,0),\end{aligned} \tag{62}$$

so we have

$$\mathscr{L}[u_t(x,t)] = sU(x,s) - u(x,0). \tag{63}$$

In exactly the same way we obtain

$$\mathscr{L}[u_{tt}(x,t)] = s^2U(x,s) - su(x,0) - u_t(x,0). \tag{64}$$

We also need the corresponding transforms of the x derivatives:

$$\mathscr{L}[u_x(x,t)] = \int_0^\infty e^{-st} u_x(x,t)\mathrm{d}t = U_x(x,s), \tag{65}$$

and

$$\mathscr{L}[u_{xx}(x,t)] = \int_0^\infty e^{-st} u_{xx}(x,t)\mathrm{d}t = U_{xx}(x,s). \tag{66}$$

Examples

1. Solve the following system

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x > 0,\ t > 0, \\ u(x,0^+) = 0, \quad x > 0 \\ u(0,t) = \delta(t), \quad \lim\limits_{x\to\infty} u(x,t) = 0, \quad t > 0 \end{cases} \tag{67}$$

(a) We begin by taking the Laplace transform, with respect to t, of both sides

$$\mathscr{L}[u_t(x,t)] = s\mathscr{L}[u(x,t)] = \mathscr{L}[u_{xx}(x,t)]. \tag{68}$$

Let $L[u(x,t)] = U(x,s)$, then

$$sU = \frac{\mathrm{d}^2U}{\mathrm{d}x^2} \quad \rightarrow \quad \frac{\mathrm{d}^2U}{\mathrm{d}x^2} - sU = 0. \tag{69}$$

Notice that we have obtained an ODE for the unknown function U.

(b) The general solution of (69) is

$$U(x,s) = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x} \tag{70}$$

Apply the boundary conditions, with $\mathscr{L}[f(t)] = F(s)$, we obtain

$$U(0,s) = \mathscr{L}[u(0,t)] = \mathscr{L}(\delta(t)) = 1, \tag{71}$$

and

$$\lim_{x\to\infty} U(x,s) = \lim_{x\to\infty}\int_0^\infty e^{-st}u(x,t)\mathrm{d}t = \int_0^\infty e^{-st}\lim_{x\to\infty} u(x,t)\mathrm{d}t = 0. \tag{72}$$

The boundary condition

$$\lim_{x\to\infty} U(x,s) = 0 \quad \rightarrow \quad A(s) = 0, \tag{73}$$

as for every fixed $s > 0$, $e^{\sqrt{s}x}$ increases as $x \to \infty$. Hence

$$U(0,s) = B(s) = 1. \tag{74}$$

Therefore,

$$U(x,s) = e^{-\sqrt{s}x}. \tag{75}$$

(c) From the table of Laplace transform, we obtain the inverse Laplace transform as

$$\mathscr{L}^{-1}(e^{-\sqrt{s}x}) = \frac{x}{2\sqrt{\pi t^3}}e^{-\frac{x^2}{4t}}. \tag{76}$$

Hence

$$u(x,t) = \frac{x}{2\sqrt{\pi t^3}}e^{-\frac{x^2}{4t}}. \tag{77}$$

2. Solve the following system

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2,\ t > 0, \\ u(x,0) = 3\sin(2\pi x), \quad 0 < x < 2, \\ u(0,t) = 0, \quad u(2,t) = 0, \quad t > 0. \end{cases} \tag{78}$$

(a) Take the Laplace transform and apply the initial condition

$$\frac{\mathrm{d}^2U(x,s)}{\mathrm{d}x^2} = sU(x,s) - u(x,0) = sU(x,s) - 3\sin(2\pi x). \tag{79}$$

We write this equation as a non-homogeneous, second order linear constant coefficient equation

$$\frac{\mathrm{d}^2U(x,s)}{\mathrm{d}x^2} - sU(x,s) = -3\sin(2\pi x). \tag{80}$$

(b)The general solution of (80) can be obtained by applying the method we have learned in ODE and we get

$$U(x,s) = C_1 e^{\sqrt{s}x} + C_2 e^{-\sqrt{s}x} + \frac{3}{(s+4\pi^2)} \sin(2\pi x). \tag{81}$$

We note the Laplace transform of the boundary conditions give

$$u(0,t) = 0 \;\rightarrow\; U(0,s) = 0, \qquad u(2,t) = 0 \;\rightarrow\; U(2,s) = 0. \tag{82}$$

So we have

$$0 = U(0,s) = C_1 + C_2, \quad 0 = U(2,s) = C_1 e^{2\sqrt{s}} + C_2 e^{-2\sqrt{s}} \tag{83}$$

which gives $C_1 = 0, C_2 = 0$ and we have

$$U(x,s) = \frac{3}{(s+4\pi^2)} \sin(2\pi x). \tag{84}$$

(c) From the table, we find that

$$\mathscr{L}^{-1}[U(x,s)] = 3e^{-4\pi^2 t} \sin(2\pi x). \tag{85}$$

Thus

$$u(x,t) = 3e^{-4\pi^2 t} \sin(2\pi x). \tag{86}$$

Table for Laplace Transforms

Function $f(t)$	**Transform** $F(s) = \int_0^\infty e^{-st} f(t)\,dt$
1	$1/s$
$t^n, for\ n = 0, 1, 2, \ldots$	$n!/s^{n+1}$
$t^{1/2}$	$\frac{1}{2}(\pi/s^3)^{1/2}$
$t^{-1/2}$	$(\frac{\pi}{s})^{1/2}$
e^{at}	$1/(s-a)$
$\sin \omega t$	$\omega/(s^2+\omega^2)$
$cos \omega t$	$s/(s^2+\omega^2)$
$t \sin \omega t$	$2\omega s/(s^2+\omega^2)^2$
$t \cos \omega t$	$(s^2-\omega^2)/(s^2+\omega^2)^2$
$e^{at} t^n$	$n!/(s-a)^{n+1}$
$e^{at} \sin \omega t$	$\omega/\left((s-a)^2+\omega^2\right)$
$e^{at} \cos \omega t$	$(s-a)/\left((s-a)^2+\omega^2\right)$
$\sinh \omega t$	$\omega/(s^2-\omega^2)$
$\cosh \omega t$	$s/(s^2-\omega^2)$
$\dfrac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
Impulse (Dirac δ): $\delta(t-a)$ $(\neq 0$ at $t = a,$ else $= 0)$	$e^{-as}, \quad a \geq 0$
Step function: $u_a(t)$ $(= 0$ for $t < a$ and $= 1,\ t \geq a)$	e^{-as}/s
Delay of g : $u_a(t) g(t-a)$	$e^{-as} G(s)$
Shift of g: $e^{at} g(t)$	$G(s-a)$
Convolution: $f(t) * g(t) = \int_0^t f(t-\tau) g(\tau)\,d\tau$	$G(s)F(s)$
Integration: $1 * g(t) = \int_0^t g(\tau)\,d\tau$	$\dfrac{1}{s} G(s)$
Derivative:	
f'	$sF(s) - f(0)$
f''	$s^2 F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$