PT

Solution to Assignment 8

1.

(a) $a = 0.3, b = 0.05, c = 0.2, d = 0.1, e = 0.55$ and $f = 0.55$

(b) $P(X = Y) = 0.1 + 0.25 = 0.35$

$$P(X < Y) = a + b + d = 0.45$$

(c) $p_X(x) = \begin{cases} 0.45 & x = 0 \\ 0.55 & x = 1 \\ 0 & \text{otherwise} \end{cases}$ $\quad p_Y(y) = \begin{cases} 0.3 & y = 0 \\ 0.55 & y = 1 \\ 0.15 & y = 2 \\ 0 & \text{otherwise} \end{cases}$

2.

(a) To find $P(X = 0, Y \leq 1)$, we can write

$$P(X = 0, Y \leq 1) = P_{XY}(0, 0) + P_{XY}(0, 1) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}.$$

(b) Note that from the table,

$$R_X = \{0, 1\} \quad \text{and} \quad R_Y = \{0, 1, 2\}.$$

To find the marginal PMFs, for example, to find $P_X(0)$, we can write

$$\begin{aligned} P_X(0) &= P_{XY}(0, 0) + P_{XY}(0, 1) + P_{XY}(0, 2) \\ &= \frac{1}{6} + \frac{1}{4} + \frac{1}{8} \\ &= \frac{13}{24}. \end{aligned}$$

Similarly, we obtain

$$P_X(x) = \begin{cases} \frac{13}{24} & x = 0 \\ \frac{11}{24} & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \begin{cases} \frac{7}{24} & y = 0 \\ \frac{5}{12} & y = 1 \\ \frac{7}{24} & y = 2 \\ 0 & \text{otherwise} \end{cases}$$

(c) Using the formula for conditional probability, we have

$$\begin{aligned} P(Y = 1 \mid X = 0) &= \frac{P(X = 0, Y = 1)}{P(X = 0)} \\ &= \frac{P_{XY}(0,1)}{P_X(0)} \\ &= \frac{\frac{1}{4}}{\frac{13}{24}} = \frac{6}{13}. \end{aligned}$$

(d) X and Y are not independent, because as we just found out

$$P(Y = 1 \mid X = 0) = \frac{6}{13} \neq P(Y = 1) = \frac{5}{12}.$$

Caution: If we want to show that X and Y are independent, we need to check that $P(X = x_i, Y = y_j) = P(X = x_i)\, P(Y = y_j)$, for all $x_i \in R_X$ and all $y_j \in R_Y$. Thus, even if in the above calculation we had found $P(Y = 1 \mid X = 0) = P(Y = 1)$, we would not yet have been able to conclude that X and Y are independent. For that, we would need to check the independence condition for all $x_i \in R_X$ and all $y_j \in R_Y$.

3. For (a),

$$\int_{-1}^{1} dx \int_{x^2}^{1} cx^2 y dy = 1$$

$$c \cdot \frac{4}{21} = 1$$

and so

$$c = \frac{21}{4}$$

For (b), let S be the region between the graphs $y = x^2$ and $y = x$, for $x \in (0,1)$. Then,

$$\begin{aligned} P(X \geq Y) &= P((X,Y) \in S) \\ &= \int_0^1 dx \int_{x^2}^{x} \frac{21}{4} \cdot x^2 y dy \\ &= \frac{3}{20} \end{aligned}$$

Both probabilities in (c) and (d) are 0 because a two-dimensional integral over a line is 0 .

4.

(a) Assume that $x \in [0,1]$. As

$$f_X(x) = \int_0^x 3x dy = 3x^2$$

we have

$$f_Y(y \mid X = x) = \frac{f(x,y)}{f_X(x)} = \frac{3x}{3x^2} = \frac{1}{x},$$

for $0 \le y \le x$. In other words, Y is uniform on $[0, x]$.

(b) As the answer in (a) depends on x, the two random variables are not independent.

5.

(a) We have

$$\begin{aligned} P(\min(X,Y) > i) &= P(\{X > i\} \cap \{Y > i\}) \\ &= P(X > i)P(Y > i) \qquad \text{[independance]} \\ &= \left(\frac{1}{2^{i+1}} + \frac{1}{2^{i+2}} + \cdots\right)^2 \\ &= \left(\frac{1}{2^i}\right)^2. \end{aligned}$$

Thus $P(\min(X,Y) \le i) = 1 - \frac{1}{4^i}$.

(b) It holds

$$\begin{aligned} P(X = Y) &= P(\cup_{i=1}^{\infty} \{X = Y = i\}) \\ &= \sum_{i=1}^{\infty} P(X = i, Y = i) \\ &= \sum_{i=1}^{\infty} P(X = i)P(Y = i) \\ &= \sum_{i=1}^{\infty} \frac{1}{4^i} = \frac{1}{4} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{1}{3} \end{aligned}$$

(c) By symmetry,

$$P(X > Y) = P(X < Y).$$

So

$$\begin{aligned} &P(X > Y) + P(X > Y) + P(X = Y) = 1 \\ &\therefore P(X < Y) = \frac{1 - 1/3}{2} = 1/3. \end{aligned}$$

(d) By law of total probability,

$$
\begin{aligned}
P(X \text{ divides } Y) &= \sum_{i=1}^{\infty} P(X=i)P(X \text{ divides } Y|X=i) \\
&= \sum_{i=1}^{\infty} P(X=i) \sum_{k=1}^{\infty} P(Y=ki|X=i) \\
&= \sum_{i=1}^{\infty} P(X=i) \sum_{k=1}^{\infty} P(Y=ki) \quad \text{[independence]} \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i} \sum_{k=1}^{\infty} \left(\frac{1}{2^i}\right)^k \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{1}{2^i} \cdot \frac{1}{1-\frac{1}{2^i}} \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i} \left(\frac{1}{2^i-1}\right)
\end{aligned}
$$

(e) By law of total probability,

$$
\begin{aligned}
P(X \geq kY) &= \sum_{i=1}^{\infty} P(Y=i)P(X \geq kY|Y=i) \\
&= \sum_{i=1}^{\infty} P(Y=i)P(X \geq ki|Y=i) \\
&= \sum_{i=1}^{\infty} P(Y=i)P(X \geq ki) \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i} \left(\frac{1}{2^{ki}} + \frac{1}{2^{ki+1}} + \cdots\right) \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{2}{2^{ki}} = \frac{2}{2^{k+1}} \cdot \frac{1}{1-\frac{1}{2^{k+1}}} = \frac{2}{2^{k+1}-1}
\end{aligned}
$$

6. First note that

$$P(X=i) = (1-\lambda)^{i-1} \cdot \lambda$$

and

$$
\begin{aligned}
P(X \geqslant i) =& \lambda \cdot \left[(1-\lambda)^{i-1} + (1-\lambda)^i + \cdots\right] \\
=& \lambda \cdot \frac{(1-\lambda)^{i-1}}{\lambda} \\
=& (1-\lambda)^{i-1}.
\end{aligned}
$$

As in the previous question, we have

$$\begin{aligned}
P(Z \geqslant i) =& P[\min(X,Y) \geqslant i] \\
=& P(X \geqslant i, Y \geqslant i) \\
=& P(X \geqslant i) \cdot P(Y \geqslant i) \\
=& (1-\lambda)^{i-1} \cdot (1-\mu)^{i-1}
\end{aligned}$$

So

$$\begin{aligned}
P(Z = i) &= P(Z \geqslant i) - P(Z \geqslant i+1) \\
&= (1-\lambda)^{i-1} \cdot (1-\mu)^{i-1} - (1-\lambda)^{i} \cdot (1-\mu)^{i} \\
&= (1-\lambda)^{i-1} \cdot (1-\mu)^{i-1} \cdot [1-(1-\lambda) \cdot (1-\mu)] \\
&= [(1-\lambda) \cdot (1-\mu)]^{i-1} \cdot [1-(1-\lambda) \cdot (1-\mu)],
\end{aligned}$$

which indicates that Z is geometric with parameter $1-(1-\lambda) \cdot (1-\mu)$.