6–12 Find parametric equations and symmetric equations for the line.

6. The line through the points $(-5, 2, 5)$ and $(1, 6, -2)$

7. The line through the origin and the point $(8, -1, 3)$

8. The line through the points $(0.4, -0.2, 1.1)$ and $(1.3, 0.8, -2.3)$

9. The line through the points $(12, 9, -13)$ and $(-7, 9, 11)$

10. The line through $(2, 1, 0)$ and perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$

11. The line through $(-6, 2, 3)$ and parallel to the line $\frac{1}{2}x = \frac{1}{3}y = z + 1$

12. The line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$

6. The vector $\mathbf{v} = \langle 1-(-5), 6-2, -2-5 \rangle = \langle 6, 4, -7 \rangle$ is parallel to the line. Letting $P_0 = (-5, 2, 5)$, parametric equations are $x = -5 + 6t$, $y = 2 + 4t$, $z = 5 - 7t$ and symmetric equations are $\dfrac{x+5}{6} = \dfrac{y-2}{4} = \dfrac{z-5}{-7}$.

7. The vector $\mathbf{v} = \langle 8-0, -1-0, 3-0 \rangle = \langle 8, -1, 3 \rangle$ is parallel to the line. Letting $P_0 = (0, 0, 0)$, parametric equations are $x = 8t$, $y = -t$, $z = 3t$ and symmetric equations are $\dfrac{x}{8} = \dfrac{y}{-1} = \dfrac{z}{3}$ or $\dfrac{x}{8} = -y = \dfrac{z}{3}$.

8. The vector $\mathbf{v} = \langle 1.3-0.4, 0.8-(-0.2), -2.3-1.1 \rangle = \langle 0.9, 1, -3.4 \rangle$ is parallel to the line. Letting $P_0 = (0.4, -0.2, 1.1)$, parametric equations are $x = 0.4 + 0.9t$, $y = -0.2 + t$, $z = 1.1 - 3.4t$ and symmetric equations are $\dfrac{x-0.4}{0.9} = \dfrac{y+0.2}{1} = \dfrac{z-1.1}{-3.4}$ or $\dfrac{x-0.4}{0.9} = y + 0.2 = \dfrac{z-1.1}{-3.4}$.

9. The vector $\mathbf{v} = \langle -7-12, 9-9, 11-(-13) \rangle = \langle -19, 0, 24 \rangle$ is parallel to the line. Letting $P_0 = (12, 9, -13)$, parametric equations are $x = 12 - 19t$, $y = 9$, $z = -13 + 24t$ and symmetric equations are $\dfrac{x-12}{-19} = \dfrac{z+13}{24}$, $y = 9$. Notice here that the direction number $b = 0$, so rather than writing $\dfrac{y-9}{0}$ in the symmetric equation, we must write $y = 9$ separately.

10. $\mathbf{v} = (\mathbf{i}+\mathbf{j}) \times (\mathbf{j}+\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ is the direction of the line perpendicular to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{j}+\mathbf{k}$.

With $P_0 = (2, 1, 0)$, parametric equations are $x = 2 + t$, $y = 1 - t$, $z = t$ and symmetric equations are $x - 2 = \dfrac{y-1}{-1} = z$ or $x - 2 = 1 - y = z$.

11. The given line $\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z+1}{1}$ has direction $\mathbf{v} = \langle 2, 3, 1 \rangle$. Taking $(-6, 2, 3)$ as P_0, parametric equations are $x = -6 + 2t$, $y = 2 + 3t$, $z = 3 + t$ and symmetric equations are $\dfrac{x+6}{2} = \dfrac{y-2}{3} = z - 3$.

12. Setting $z = 0$ we see that $(1, 0, 0)$ satisfies the equations of both planes, so they do in fact have a line of intersection. The line is perpendicular to the normal vectors of both planes, so a direction vector for the line is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 2, 3 \rangle \times \langle 1, -1, 1 \rangle = \langle 5, 2, -3 \rangle$. Taking the point $(1, 0, 0)$ as P_0, parametric equations are $x = 1 + 5t$, $y = 2t$, $z = -3t$, and symmetric equations are $\dfrac{x-1}{5} = \dfrac{y}{2} = \dfrac{z}{-3}$.

26. The plane through the origin and perpendicular to the line

$$x = 1 - 8t \qquad y = -1 - 7t \qquad z = 4 + 2t$$

27. The plane through the point $(1, 3, -1)$ and perpendicular to the line

$$\frac{x + 3}{4} = -y = \frac{z - 1}{5}$$

28. The plane through the point $(9, -4, -5)$ and parallel to the plane $z = 2x - 3y$

29. The plane through the point $(2.1, 1.7, -0.9)$ and parallel to the plane $2x - y + 3z = 1$

30. The plane that contains the line $x = 1 + t$, $y = 2 - t$, $z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$

26. Since the line is perpendicular to the plane, its direction vector, $\langle -8, -7, 2\rangle$, is a normal vector to the plane. $(0, 0, 0)$ is a point on the plane. Setting $a = -8$, $b = -7$, $c = 2$ and $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ in Equation 7 gives $-8(x - 0) - 7(y - 0) + 2(z - 0) = 0$, or $-8x - 7y + 2z = 0$, as an equation of the plane.

27. Since the line is perpendicular to the plane, its direction vector, $\langle 4, -1, 5\rangle$, is a normal vector to the plane. $(1, 3, -1)$ is a point on the plane. Setting $a = 4$, $b = -1$, $c = 5$ and $x_0 = 1$, $y_0 = 3$, $z_0 = -1$ in Equation 7 gives $4(x - 1) - 1(y - 3) + 5(z + 1) = 0$, or $4x - y + 5z = -4$, as an equation of the plane.

28. Since the two planes are parallel, they will have the same normal vectors. The plane is $z = 2x - 3y \;\Leftrightarrow\; 2x - 3y - z = 0$, so we can take $\mathbf{n} = \langle 2, -3, -1\rangle$, and an equation of the plane is $2(x - 9) - 3(y + 4) - 1(z + 5) = 0$, or $2x - 3y - z = 35$.

29. Since the two planes are parallel, they will have the same normal vectors. The plane is $2x - y + 3z = 1$, so we can take $\mathbf{n} = \langle 2, -1, 3\rangle$, and an equation of the plane is $2(x - 2.1) - 1(y - 1.7) + 3(z + 0.9) = 0$, or $2x - y + 3z = -0.2$, or $10x - 5y + 15z = -1$.

30. First, a normal vector for the plane $5x + 2y + z = 1$ is $\mathbf{n} = \langle 5, 2, 1\rangle$. A direction vector for the line is $\mathbf{v} = \langle 1, -1, -3\rangle$, and since $\mathbf{n} \cdot \mathbf{v} = 0$, we know the line is perpendicular to $\mathbf{n}$ and hence parallel to the plane. Thus, there is a parallel plane which contains the line. By putting $t = 0$, we know that the point $(1, 2, 4)$ is on the line and hence the new plane. We can use the same normal vector $\mathbf{n} = \langle 5, 2, 1\rangle$, so an equation of the plane is $5(x - 1) + 2(y - 2) + 1(z - 4) = 0$ or $5x + 2y + z = 13$.

36. The plane that passes through the point $(6, -1, 3)$ and contains the line with symmetric equations $x/3 = y + 4 = z/2$

37. The plane that passes through the point $(3, 1, 4)$ and contains the line of intersection of the planes $x + 2y + 3z = 1$ and $2x - y + z = -3$

38. The plane that passes through the points $(0, -2, 5)$ and $(-1, 3, 1)$ and is perpendicular to the plane $2z = 5x + 4y$

39. The plane that passes through the point $(1, 5, 1)$ and is perpendicular to the planes $2x + y - 2z = 2$ and $x + 3z = 4$

40. The plane that passes through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and is perpendicular to the plane $x + y - 2z = 1$

36. Since the line $\frac{x}{3} = \frac{y+4}{1} = \frac{z}{2}$ lies in the plane, its direction vector $\mathbf{a} = \langle 3, 1, 2 \rangle$ is parallel to the plane. The point $(0, -4, 0)$ is on the line (put $t = 0$ in the corresponding parametric equations), and we can verify that the given point $(6, -1, 3)$ in the plane is not on the line. The vector connecting these two points, $\mathbf{b} = \langle 6, 3, 3 \rangle$, is therefore parallel to the plane, but not parallel to $\mathbf{a}$. Then $\mathbf{a} \times \mathbf{b} = \langle 3 - 6, 12 - 9, 9 - 6 \rangle = \langle -3, 3, 3 \rangle$ is a normal vector to the plane, and an equation of the plane is $-3(x - 0) + 3[y - (-4)] + 3(z - 0) = 0$ or $-3x + 3y + 3z = -12$ or $x - y - z = 4$.

37. Normal vectors for the given planes are $\mathbf{n}_1 = \langle 1, 2, 3 \rangle$ and $\mathbf{n}_2 = \langle 2, -1, 1 \rangle$. A direction vector, then, for the line of intersection is $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2 + 3, 6 - 1, -1 - 4 \rangle = \langle 5, 5, -5 \rangle$, and $\mathbf{a}$ is parallel to the desired plane. Another vector parallel to the plane is the vector connecting any point on the line of intersection to the given point $(3, 1, 4)$ in the plane. Setting $z = 0$, the equations of the planes reduce to $x + 2y = 1$ and $2x - y = -3$ with simultaneous solution $x = -1$ and $y = 1$. So a point on the line is $(-1, 1, 0)$ and another vector parallel to the plane is $\mathbf{b} = \langle 3 - (-1), 1 - 1, 4 - 0 \rangle = \langle 4, 0, 4 \rangle$. Then a normal vector to the plane is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 20 - 0, -20 - 20, 0 - 20 \rangle = \langle 20, -40, -20 \rangle$. Equivalently, we can take $\langle 1, -2, -1 \rangle$ as a normal vector, and an equation of the plane is $1(x - 3) - 2(y - 1) - 1(z - 4) = 0$ or $x - 2y - z = -3$.

38. The points $(0, -2, 5)$ and $(-1, 3, 1)$ lie in the desired plane, so the vector $\mathbf{v}_1 = \langle -1, 5, -4 \rangle$ connecting them is parallel to the plane. The desired plane is perpendicular to the plane $2z = 5x + 4y$ or $5x + 4y - 2z = 0$ and for perpendicular planes, a normal vector for one plane is parallel to the other plane, so $\mathbf{v}_2 = \langle 5, 4, -2 \rangle$ is also parallel to the desired plane.
A normal vector to the desired plane is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle -10 + 16, -20 - 2, -4 - 25 \rangle = \langle 6, -22, -29 \rangle$.
Taking $(x_0, y_0, z_0) = (0, -2, 5)$, the equation we are looking for is $6(x - 0) - 22(y + 2) - 29(z - 5) = 0$ or $6x - 22y - 29z = -101$.

39. If a plane is perpendicular to two other planes, its normal vector is perpendicular to the normal vectors of the other two planes. Thus $\langle 2, 1, -2 \rangle \times \langle 1, 0, 3 \rangle = \langle 3 - 0, -2 - 6, 0 - 1 \rangle = \langle 3, -8, -1 \rangle$ is a normal vector to the desired plane. The point $(1, 5, 1)$ lies on the plane, so an equation is $3(x - 1) - 8(y - 5) - (z - 1) = 0$ or $3x - 8y - z = -38$.

40. $\mathbf{n}_1 = \langle 1, 0, -1 \rangle$ and $\mathbf{n}_2 = \langle 0, 1, 2 \rangle$. Setting $z = 0$, it is easy to see that $(1, 3, 0)$ is a point on the line of intersection of $x - z = 1$ and $y + 2z = 3$. The direction of this line is $\mathbf{v}_1 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, -2, 1 \rangle$. A second vector parallel to the desired plane is $\mathbf{v}_2 = \langle 1, 1, -2 \rangle$, since it is perpendicular to $x + y - 2z = 1$. Therefore, a normal of the plane in question is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 4 - 1, 1 + 2, 1 + 2 \rangle = \langle 3, 3, 3 \rangle$, or we can use $\langle 1, 1, 1 \rangle$. Taking $(x_0, y_0, z_0) = (1, 3, 0)$, the equation we are looking for is $(x - 1) + (y - 3) + z = 0 \quad \Leftrightarrow \quad x + y + z = 4$.

69–70 Use the formula in Exercise 12.4.45 to find the distance from the point to the given line.

69. $(4, 1, -2)$; $\quad x = 1 + t, \quad y = 3 - 2t, \quad z = 4 - 3t$

70. $(0, 1, 3)$; $\quad x = 2t, \quad y = 6 - 2t, \quad z = 3 + t$

71–72 Find the distance from the point to the given plane.

71. $(1, -2, 4)$, $\quad 3x + 2y + 6z = 5$

72. $(-6, 3, 5)$, $\quad x - 2y - 4z = 8$

73–74 Find the distance between the given parallel planes.

73. $2x - 3y + z = 4, \quad 4x - 6y + 2z = 3$

74. $6z = 4y - 2x, \quad 9z = 1 - 3x + 6y$

69. Let $Q = (1, 3, 4)$ and $R = (2, 1, 1)$, points on the line corresponding to $t = 0$ and $t = 1$. Let $P = (4, 1, -2)$. Then $\mathbf{a} = \overrightarrow{QR} = \langle 1, -2, -3 \rangle$, $\mathbf{b} = \overrightarrow{QP} = \langle 3, -2, -6 \rangle$. The distance is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 1, -2, -3 \rangle \times \langle 3, -2, -6 \rangle|}{|\langle 1, -2, -3 \rangle|} = \frac{|\langle 6, -3, 4 \rangle|}{|\langle 1, -2, -3 \rangle|} = \frac{\sqrt{6^2 + (-3)^2 + 4^2}}{\sqrt{1^2 + (-2)^2 + (-3)^2}} = \frac{\sqrt{61}}{\sqrt{14}} = \sqrt{\frac{61}{14}}.$$

70. Let $Q = (0, 6, 3)$ and $R = (2, 4, 4)$, points on the line corresponding to $t = 0$ and $t = 1$. Let $P = (0, 1, 3)$. Then $\mathbf{a} = \overrightarrow{QR} = \langle 2, -2, 1 \rangle$ and $\mathbf{b} = \overrightarrow{QP} = \langle 0, -5, 0 \rangle$. The distance is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 2, -2, 1 \rangle \times \langle 0, -5, 0 \rangle|}{|\langle 2, -2, 1 \rangle|} = \frac{|\langle 5, 0, -10 \rangle|}{|\langle 2, -2, 1 \rangle|} = \frac{\sqrt{5^2 + 0^2 + (-10)^2}}{\sqrt{2^2 + (-2)^2 + 1^2}} = \frac{\sqrt{125}}{\sqrt{9}} = \frac{5\sqrt{5}}{3}.$$

71. By Equation 9, the distance is $D = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \dfrac{|3(1) + 2(-2) + 6(4) - 5|}{\sqrt{3^2 + 2^2 + 6^2}} = \dfrac{|18|}{\sqrt{49}} = \dfrac{18}{7}$.

72. By Equation 9, the distance is $D = \dfrac{|1(-6) - 2(3) - 4(5) - 8|}{\sqrt{1^2 + (-2)^2 + (-4)^2}} = \dfrac{|-40|}{\sqrt{21}} = \dfrac{40}{\sqrt{21}}$.

73. Put $y = z = 0$ in the equation of the first plane to get the point $(2, 0, 0)$ on the plane. Because the planes are parallel, the distance D between them is the distance from $(2, 0, 0)$ to the second plane. By Equation 9,

$$D = \frac{|4(2) - 6(0) + 2(0) - 3|}{\sqrt{4^2 + (-6)^2 + (2)^2}} = \frac{5}{\sqrt{56}} = \frac{5}{2\sqrt{14}} \text{ or } \frac{5\sqrt{14}}{28}.$$

74. Put $x = y = 0$ in the equation of the first plane to get the point $(0, 0, 0)$ on the plane. Because the planes are parallel the distance D between them is the distance from $(0, 0, 0)$ to the second plane $3x - 6y + 9z - 1 = 0$. By Equation 9,

$$D = \frac{|3(0) - 6(0) + 9(0) - 1|}{\sqrt{3^2 + (-6)^2 + 9^2}} = \frac{1}{\sqrt{126}} = \frac{1}{3\sqrt{14}}.$$

75. Distance between Parallel Planes Show that the distance between the parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

76. Find equations of the planes that are parallel to the plane $x + 2y - 2z = 1$ and two units away from it.

77. Show that the lines with symmetric equations $x = y = z$ and $x + 1 = y/2 = z/3$ are skew, and find the distance between these lines.

78. Find the distance between the skew lines with parametric equations $x = 1 + t$, $y = 1 + 6t$, $z = 2t$, and $x = 1 + 2s$, $y = 5 + 15s$, $z = -2 + 6s$.

75. The distance between two parallel planes is the same as the distance between a point on one of the planes and the other plane. Let $P_0 = (x_0, y_0, z_0)$ be a point on the plane given by $ax + by + cz + d_1 = 0$. Then $ax_0 + by_0 + cz_0 + d_1 = 0$ and the distance between P_0 and the plane given by $ax + by + cz + d_2 = 0$ is, from Equation 9,

$$D = \frac{|ax_0 + by_0 + cz_0 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|-d_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

76. The planes must have parallel normal vectors, so if $ax + by + cz + d = 0$ is such a plane, then for some $t \neq 0$, $\langle a, b, c \rangle = t\langle 1, 2, -2 \rangle = \langle t, 2t, -2t \rangle$. So this plane is given by the equation $x + 2y - 2z + k = 0$, where $k = d/t$. By Exercise 75, the distance between the planes is $2 = \dfrac{|1 - k|}{\sqrt{1^2 + 2^2 + (-2)^2}} \Leftrightarrow 6 = |1 - k| \Leftrightarrow k = 7$ or -5. So the desired planes have equations $x + 2y - 2z = 7$ and $x + 2y - 2z = -5$.

77. L_1: $x = y = z \;\Rightarrow\; x = y$ **(1)**. L_2: $x + 1 = y/2 = z/3 \;\Rightarrow\; x + 1 = y/2$ **(2)**. The solution of **(1)** and **(2)** is $x = y = -2$. However, when $x = -2$, $x = z \;\Rightarrow\; z = -2$, but $x + 1 = z/3 \;\Rightarrow\; z = -3$, a contradiction. Hence the lines do not intersect. For L_1, $\mathbf{v}_1 = \langle 1, 1, 1\rangle$, and for L_2, $\mathbf{v}_2 = \langle 1, 2, 3\rangle$, so the lines are not parallel. Thus the lines are skew lines. If two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\langle 1, 1, 1\rangle$ and $\langle 1, 2, 3\rangle$, the direction vectors of the two lines. So set $\mathbf{n} = \langle 1, 1, 1\rangle \times \langle 1, 2, 3\rangle = \langle 3 - 2, -3 + 1, 2 - 1\rangle = \langle 1, -2, 1\rangle$. From above, we know that $(-2, -2, -2)$ and $(-2, -2, -3)$ are points of L_1 and L_2, respectively. So in the notation of Equation 8, $1(-2) - 2(-2) + 1(-2) + d_1 = 0 \;\Rightarrow\; d_1 = 0$ and $1(-2) - 2(-2) + 1(-3) + d_2 = 0 \;\Rightarrow\; d_2 = 1$.

By Exercise 75, the distance between these two skew lines is $D = \dfrac{|0 - 1|}{\sqrt{1 + 4 + 1}} = \dfrac{1}{\sqrt{6}}$.

Alternate solution (without reference to planes): A vector which is perpendicular to both of the lines is $\mathbf{n} = \langle 1, 1, 1\rangle \times \langle 1, 2, 3\rangle = \langle 1, -2, 1\rangle$. Pick any point on each of the lines, say $(-2, -2, -2)$ and $(-2, -2, -3)$, and form the vector $\mathbf{b} = \langle 0, 0, 1\rangle$ connecting the two points. The distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is, $D = \dfrac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \dfrac{|1 \cdot 0 - 2 \cdot 0 + 1 \cdot 1|}{\sqrt{1 + 4 + 1}} = \dfrac{1}{\sqrt{6}}$.

78. First notice that if two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\mathbf{v}_1 = \langle 1, 6, 2\rangle$ and $\mathbf{v}_2 = \langle 2, 15, 6\rangle$, the direction vectors of the two lines, respectively. Thus, set $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 36 - 30, 4 - 6, 15 - 12\rangle = \langle 6, -2, 3\rangle$. Setting $t = 0$ and $s = 0$ gives the points $(1, 1, 0)$ and $(1, 5, -2)$. So in the notation of Equation 8, $6 - 2 + 0 + d_1 = 0 \;\Rightarrow\; d_1 = -4$ and $6 - 10 - 6 + d_2 = 0 \;\Rightarrow\; d_2 = 10$.

Then by Exercise 75, the distance between the two skew lines is given by $D = \dfrac{|-4 - 10|}{\sqrt{36 + 4 + 9}} = \dfrac{14}{7} = 2$.

Alternate solution (without reference to planes): We already know that the direction vectors of the two lines are $\mathbf{v}_1 = \langle 1, 6, 2\rangle$ and $\mathbf{v}_2 = \langle 2, 15, 6\rangle$. Then $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 6, -2, 3\rangle$ is perpendicular to both lines. Pick any point on each of the lines, say $(1, 1, 0)$ and $(1, 5, -2)$, and form the vector $\mathbf{b} = \langle 0, 4, -2\rangle$ connecting the two points. Then the distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is,

$$D = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{1}{\sqrt{36 + 4 + 9}}\,|0 - 8 - 6| = \frac{14}{7} = 2.$$