Chapter 3 Vector Spaces

Section 3.6 Row Space and Column Space (Extra)

Theorem Let A be an $m \times n$ matrix.

1. $\text{rank}(A) + \dim N(A) = n$.
2. $\text{rank}(A) = \text{rank}(A^T)$ [The dimension of column space is the rank].
3. $0 \leq \text{rank}(A) \leq min(n, m)$.
4. $\text{rank}(A) = 0$ if and only if $A = O$.

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We are going to show

Theorem Let A be an $m \times n$ matrix.

5. $\text{rank}(A) = \text{rank}(A^T A) = \text{rank}(AA^T)$.
6. If P is $m \times m$, Q is $n \times n$, and both P and Q are nonsingular, then $\text{rank}(PAQ) = \text{rank}(A)$.
7. For any $m \times n$ matrix B, $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.
8. For any $n \times r$ matrix B, $\text{rank}(AB) \leq min\{\text{rank}(A), \text{rank}(B)\}$.

Theorem (5) Let A be an $m \times n$ matrix. $\operatorname{rank}(A) = \operatorname{rank}(A^T A) = \operatorname{rank}(AA^T)$.

Example

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Then $A^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A^T A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $AA^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$\operatorname{rank}(A) = 2$, $\operatorname{rank}(A^T A) = 2$, $\operatorname{rank}(AA^T) = 2$.

Theorem (6) Let A be an $m \times n$ matrix. If P is $m \times m$, Q is $n \times n$, and both P and Q are nonsingular, then $\operatorname{rank}(PAQ) = \operatorname{rank}(A)$.

Example Let $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Note that P, Q are nonsingular because $\det(P) = -1 = \det(Q)$. But R is singular because $\det(R) = 0$.

Then $PAI = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$, $IAQ = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$, $IAR = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}$.

$\operatorname{rank}(A) = 2$, $\operatorname{rank}(PAI) = \operatorname{rank}(A) = 2$, $\operatorname{rank}(IAQ) = \operatorname{rank}(A) = 2$, $\operatorname{rank}(IAR) = 1 \neq \operatorname{rank}(A)$.

Theorem (5) Let A be a matrix. $\text{rank}(A) = \text{rank}(A^TA) = \text{rank}(AA^T)$.

Proof of $\text{rank}(A) = \text{rank}(A^TA)$

Let $\mathbf{x} \in N(A)$. Then $A\mathbf{x} = \mathbf{0}$ and $(A^TA)\mathbf{x} = A^T(A\mathbf{x}) = A^T\mathbf{0} = \mathbf{0}$. So $\mathbf{x} \in N(A^TA)$. Hence $N(A)$ is a subspace in $N(A^TA)$.
Let $\mathbf{y} \in N(A^TA)$ and $\mathbf{z} = A\mathbf{y}$. Then $(A^TA)\mathbf{y} = \mathbf{0}$ and

$$\mathbf{z}^T\mathbf{z} = (A\mathbf{y})^T(A\mathbf{y}) = (\mathbf{y}^TA^T)A\mathbf{y} = \mathbf{y}^T(A^TA\mathbf{y}) = \mathbf{y}^T\mathbf{0} = \mathbf{0}.$$

Note the $\mathbf{y}^T\mathbf{0}$ is the 1×1 matrix $\mathbf{0}$. Since $\mathbf{z}$ is a column vector such that $\mathbf{z}^T\mathbf{z} = \mathbf{0}$, we have $A\mathbf{y} = \mathbf{z} = \mathbf{0}$ (why?). So $\mathbf{y} \in N(A)$, and $N(A^TA)$ is a subspace of $N(A)$.

So $N(A^TA) = N(A)$. By the Rank-Nullity Theorem,
$\text{rank}(A^TA) = n - \dim(N(A^TA)) = n - \dim(N(A)) = \text{rank}(A)$.

Theorem (Half of 6) Let A be an $m \times n$ matrix, P be $m \times m$ nonsingular matrix. Then $\text{rank}(PA) = \text{rank}(A)$.

Proof

Let $\mathbf{x} \in N(A)$. Then $PA\mathbf{x} = P\mathbf{0} = \mathbf{0}$ and hence $\mathbf{x} \in N(PA)$. Thus $N(A)$ is a subspace of $N(PA)$.

On the other hand, if $\mathbf{y} \in N(PA)$, then $P(Ay) = PA\mathbf{y} = \mathbf{0}$ and hence $A\mathbf{y} \in N(P)$. But $N(P) = \{\mathbf{0}\}$ since P is nonsingular. Therefore $A\mathbf{y} = \mathbf{0}$ and hence $\mathbf{y} \in N(A)$. Thus $N(PA)$ is a subspace of $N(A)$.

Now, $N(PA) = N(A)$. It follows from the Rank-Nullity Theorem that $\text{rank}(A) = n - dimN(A) = n - dimN(PA) = \text{rank}(PA)$.

Theorem (7) Let A be an $m \times n$ matrix. Let B be an $n \times r$ matrix. Then $\text{rank}(AB) \leq min\{\text{rank}(A), \text{rank}(B)\}$.

Example Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$. $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$
$\text{rank}(A) = 2$, $\text{rank}(B) = 2$, $\text{rank}(AB) = 1$.

Example Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$
$\text{rank}(A) = 1$, $\text{rank}(B) = 2$, $\text{rank}(AB) = 1$.

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$\text{rank}(A) = 2$, $\text{rank}(B) = 2$, $\text{rank}(AB) = 2$.

Theorem (8) Let A, B be $m \times n$ matrices. Then $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

Example Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. $A + B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\text{rank}(A) = 2$, $\text{rank}(B) = 2$, $\text{rank}(A + B) = 2$.

Example Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $A + B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\text{rank}(A) = 2$, $\text{rank}(B) = 1$, $\text{rank}(A + B) = 3$.

Theorem (7) Let A be an $m \times n$ matrix. Let B be an $n \times r$ matrix. Then $\text{rank}(AB) \leq min\{\text{rank}(A), \text{rank}(B)\}$.

Proof

Let $Col(A)$, $Col(AB)$ be the column space of A and AB respectively.

Let $\mathbf{d} \in Col(AB)$. By Theorem $(*)$, $\mathbf{d} = AB\mathbf{x}$ for some $\mathbf{x} \in \mathbf{R}^r$. Let $\mathbf{y} = B\mathbf{x}$. Since $\mathbf{d} = A\mathbf{y}$, it follows that $\mathbf{d}$ is in $Col(A)$ by Theorem $(*)$. Hence $Col(AB)$ is a subspace of $Col(A)$. Hence
$\text{rank}(AB) = \dim(Col(AB)) \leq \dim(Col(A)) = \text{rank}(A)$.

$$
\begin{aligned}
\text{rank}(AB) &= \text{rank}((AB)^T) \quad \text{dim. of col. sp.=rank} \\
&= \text{rank}(B^T A^T) \\
&\leq \text{rank}(B^T) \quad \text{we have just shown } \text{rank}(CD) \leq \text{rank}(C) \\
&= \text{rank}(B) \quad \text{dim. of col. sp.=rank}
\end{aligned}
$$

Theorem (81) Let A, B be $m \times n$ matrices. Then $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.

Proof Let $Col(A) + Col(B) = \{\mathbf{x} + \mathbf{y} | \mathbf{x} \in Col(A), \mathbf{y} \in Col(B)\}$.
Let $\mathbf{d} \in Col(A+B)$. Then $(A+B)\mathbf{x} = \mathbf{d}$ for some $\mathbf{x}$. Then $\mathbf{d} = A\mathbf{x} + B\mathbf{x}$. Since $A\mathbf{x}$ is in $Col(A)$ and $B\mathbf{x}$ is in $Col(B)$, we have $\mathbf{d} \in Col(A) + Col(B)$. So $Col(A+B)$ is a subspace of $Col(A) + Col(B)$.

Hence,

$$
\begin{aligned}
& \text{rank}(A+B) \\
=& \dim(Col(A+B)) \\
\leq& \dim(Col(A) + Col(B)) \\
=& \dim(Col(A)) + \dim(Col(B)) - \dim(Col(A) \cap Col(B))
\end{aligned}
$$

by the dimension theorem at the end of the slides

$$
\begin{aligned}
\leq& \dim(Col(A)) + \dim(Col(B)) \\
=& \text{rank}(A) + \text{rank}(B)
\end{aligned}
$$

Theorem (8) Let A, B be $m \times n$ matrices. Then $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.

Proof 2 Let $Q = \left(\begin{array}{c|c} A & O \\ \hline O & B \end{array}\right)$. Then $\operatorname{rank}(Q) = \operatorname{rank}(A) + \operatorname{rank}(B)$. (You can see this by converting Q into row-echelon form).

Since Q and $Q' = \left(\begin{array}{c|c} A & O \\ \hline A & B \end{array}\right)$ are row equivalent, $\operatorname{rank}(Q) = \operatorname{rank}(Q')$.

Since $Q'^T = \left(\begin{array}{c|c} A^T & A^T \\ \hline O & B^T \end{array}\right)$ and $Q'' = \left(\begin{array}{c|c} A^T & A^T \\ \hline A^T & A^T + B^T \end{array}\right)$ are row equivalent, $\operatorname{rank}(Q'^T) = \operatorname{rank}(Q'')$.

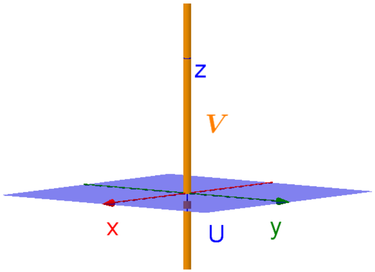
Since dimension of the column space of a matrix equals the rank, $\operatorname{rank}(Q) = \operatorname{rank}(Q') = \operatorname{rank}(Q'^T) = \operatorname{rank}(Q'') = \operatorname{rank}(Q''^T)$.

Note that Q''^T has a submatrix $A + B$. Since the rank of a submatrix cannot exceed the rank of the whole matrix (why?), $\operatorname{rank}(A+B) \leq \operatorname{rank}(Q) = \operatorname{rank}(A) + \operatorname{rank}(B)$.

Dimension Theorem Let U, V be subspaces of a vector space W. Let $U+V=\{\mathbf{u}+\mathbf{v}|\mathbf{u}\in U, \mathbf{v}\in V\}$. Then $\dim(U)+\dim(V)=\dim(U+V)+\dim(U\cap V)$.

Example

$U=\{(x,y,0)^T|x,y\in\mathbf{R}\}$
$V=\{(0,0,z)^T|z\in\mathbf{R}\}$
$U\cap V=\{\mathbf{0}\}$,
$U+V=\mathbf{R}^3$
$\dim U=2$
$\dim V=1$
$\dim(U\cap V)=0$
$\dim(U+V)=3$



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$V=\{(0,0,z)^T|z\in\mathbf{R}\}$
$U\cap V=\{\mathbf{0}\}$, $\quad U+V=\{(0,y,z)^T|y,z\in\mathbf{R}\}$
$\dim U=1$, $\dim V=1$
$\dim(U\cap V)=0$, $\dim(U+V)=2$

Dimension Theorem Let U, V be subspaces of a vector space W. Let $U+V=\{\mathbf{u}+\mathbf{v}|\mathbf{u}\in U,\mathbf{v}\in V\}$. Then
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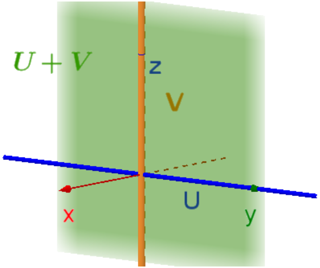

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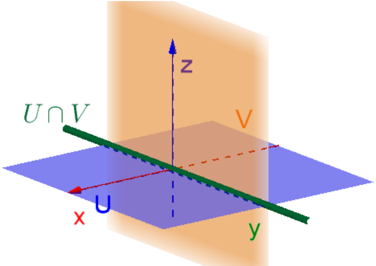

Example
$U=\{(x,y,0)^T|x,y\in\mathbf{R}\}$
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$U\cap V=\{(0,y,0)^T|y,0\in\mathbf{R}\}$, $\quad U+V=\mathbf{R}^3$
$\dim U=2$, $\dim V=2$
$\dim(U\cap V)=1$, $\dim(U+V)=3$

Dimension Theorem Let U, V be subspaces of a vector space W. Let $U + V = \{\mathbf{u} + \mathbf{v} | \mathbf{u} \in U, \mathbf{v} \in V\}$. Then $\dim(U) + \dim(V) = \dim(U + V) + \dim(U \cap V)$.

Idea of a proof Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_k\}$ be a basis for $U \cap V$. Then we can extend $\{\mathbf{x}_1, \cdots, \mathbf{x}_k\}$ to a basis $\{\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{y}_1, \cdots, \mathbf{y}_\ell\}$ of U and $\{\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{z}_1, \cdots, \mathbf{z}_m\}$ of V.

Do some work (next two slides) to argue $\{\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{y}_1, \cdots, \mathbf{y}_\ell, \mathbf{z}_1, \cdots, \mathbf{z}_m\}$ is a basis of $U + V$. So

$$\begin{aligned}\dim(U) + \dim(V) &= (k + \ell) + (k + m) = (k + \ell + m) + k \\ &= \dim(U + V) + \dim(U \cap V).\end{aligned}$$

(Span) Let $\mathbf{w} \in U + V$. Then $\mathbf{w} = \mathbf{u} + \mathbf{v}$ for some $\mathbf{u} \in U$ and $\mathbf{v} \in V$.
Since $\mathbf{u} \in U$ and $\{\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{y}_1, \cdots, \mathbf{y}_\ell\}$ is a basis of U, we have
$\mathbf{u} = a_1\mathbf{x}_1 + \cdots + a_k\mathbf{x}_k + a_{k+1}\mathbf{y}_1 + \cdots + a_{k+\ell}\mathbf{y}_\ell$ for some a_i's.
Since $\mathbf{v} \in V$ and $\{\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{z}_1, \cdots, \mathbf{z}_m\}$ is a basis of V, we have
$\mathbf{v} = b_1\mathbf{x}_1 + \cdots + b_k\mathbf{x}_k + b_{k+1}\mathbf{z}_1 + \cdots + b_{k+m}\mathbf{z}_m$ for some b_i's.
So,

$$\begin{aligned}
\mathbf{w} =& \mathbf{u} + \mathbf{v} \\
=& a_1\mathbf{x}_1 + \cdots + a_k\mathbf{x}_k + a_{k+1}\mathbf{y}_1 + \cdots + a_{k+\ell}\mathbf{y}_\ell \\
& + b_1\mathbf{x}_1 + \cdots + b_k\mathbf{x}_k + b_{k+1}\mathbf{z}_1 + \cdots + b_{k+m}\mathbf{z}_m \\
=& (a_1 + b_1)\mathbf{x}_1 + \cdots + (a_k + b_k)\mathbf{x}_k + a_{k+1}\mathbf{y}_1 + \cdots + a_{k+\ell}\mathbf{y}_\ell \\
& + b_{k+1}\mathbf{z}_1 + \cdots + b_{k+m}\mathbf{z}_m \\
\in& \mathrm{Span}\{\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{y}_1, \cdots, \mathbf{y}_\ell, \mathbf{z}_1, \cdots, \mathbf{z}_m\}
\end{aligned}$$

(Linear independence) Let $a_1, \cdots, a_k, b_1, \cdots, b_\ell, c_1, \cdots, c_m$ such that

$$a_1\mathbf{x}_1 + \cdots + a_k\mathbf{x}_k + b_1\mathbf{y}_1 + \cdots + b_\ell\mathbf{y}_\ell + c_1\mathbf{z}_1 + \cdots + c_m\mathbf{z}_m = \mathbf{0}.$$

Then the vector

$$\begin{aligned} \mathbf{v} &= a_1\mathbf{x}_1 + \cdots + a_k\mathbf{x}_k + b_1\mathbf{y}_1 + \cdots + b_\ell\mathbf{y}_\ell && --(1) \\ &= -c_1\mathbf{z}_1 - \cdots - c_m\mathbf{z}_m && --(2) \end{aligned}$$

is in both U (by (1)) and in V (by (2)), and so $\mathbf{v} \in U \cap V$ and $\mathbf{v} = d_1\mathbf{x}_1 + \cdots + d_k\mathbf{x}_k$. So,

$$\begin{cases} -c_1\mathbf{z}_1 - \cdots - c_m\mathbf{z}_m = d_1\mathbf{x}_1 + \cdots + d_k\mathbf{x}_k & --(3) \\ a_1\mathbf{x}_1 + \cdots + a_k\mathbf{x}_k + b_1\mathbf{y}_1 + \cdots + b_\ell\mathbf{y}_\ell = d_1\mathbf{x}_1 + \cdots + d_k\mathbf{x}_k & --(4) \end{cases}$$

Since $\{\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{z}_1, \cdots, \mathbf{z}_m\}$ is a basis, it is a linearly independent set, and so (3) implies $d_1 = \cdots = d_k = -c_1 = \cdots = -c_m = 0$.
Since $\{\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{y}_1, \cdots, \mathbf{y}_\ell\}$ is a basis, it is a linearly independent set, and so (4) implies $d_1 - a_1 = \cdots = d_k - a_k = -b_1 = \cdots = -b_\ell = 0$.
Since $d_1 = \cdots = d_k = 0$ and $d_1 - a_1 = \cdots = d_k - a_k = 0$, we have $a_1 = \cdots = a_k = 0$.