Section 4.6 Integration By Substitution (换元积分法)

How to do the following integral? $\int (3x+5)^7 dx$

Answer

Let $u = 3x + 5$ so that $du = 3\,dx$ or $dx = \frac{1}{3}\,du$

$$\int (3x+5)^7\,dx = \int u^7 \left(\frac{1}{3}\,du\right)$$

$$= \frac{1}{3}\left(\frac{1}{8}u^8\right) + C = \frac{1}{24}u^8 + C$$

$$= \frac{1}{24}(3x+5)^8 + C$$

Substitution Rule

$$\int f(x)\,dx = \int g(u(x))\,u'(x)\,dx$$

Think of $u=u(x)$ as a change of variable whose differential is

$$du = u'(x)\,dx$$

substitute du for u′(x) dx

Then

$$\int f(x)\,dx = \int g(u(x))\,u'(x)\,dx = \int g(u)\,du$$

$$= G(u) + C = G(u(x)) + C$$

G is an antiderivative of g

substitute u(x) for u

INTEGRATION BY SUBSTITUTION

Integration by substitution consists of the following general steps, as illustrated in example 6.2.

- **Choose a new variable *u*:** a common choice is the innermost expression or "inside" term of a composition of functions. (In example 6.2, note that $x^3 + 5$ is the inside term of $(x^3 + 5)^{100}$.)
- **Compute** $du = \frac{du}{dx}\, dx$.
- **Replace all terms** in the original integrand with expressions involving u and du.
- **Evaluate** the resulting (u) integral. If you still can't evaluate the integral, you may need to try a different choice of u.
- **Replace each occurrence of *u*** in the antiderivative with the corresponding expression in x.

Example

Find $\int 3(8y-1)\mathbf{e}^{4y^2-y}\,dy$

Solution:

You make the substitution $u = 4y^2 - y$ with

$$du = (8y-1)\,dy$$

to obtain

$$\int 3(8y-1)\mathbf{e}^{4y^2-y}\,dy = 3\int \mathbf{e}^{u}\,du$$

$$= 3\mathbf{e}^{u} + c$$

$$= 3\mathbf{e}^{4y^2-y} + c$$

Example

Find $\displaystyle\int \frac{x}{x-1}\,dx.$

Solution:

You make the substitution $u = x - 1$ with $du = dx$

to obtain

$$\int \frac{x}{x-1}\,dx = \int \frac{u+1}{u}\,du$$

$$= \int 1 + \frac{1}{u}\,du$$

$$= u + \ln|u| + C$$

$$= x - 1 + \ln|x-1| + C$$

Example

Find $\int \frac{3x+6}{\sqrt{2x^2+8x+3}}\,dx.$

Solution:

This time, our guidelines suggest substituting for the quantity inside the radical in the denominator; that is,

$$u = 2x^2 + 8x + 3 \qquad du = (4x + 8)\,dx$$

At first glance, it may seem that this substitution fails, since $du = (4x + 8)\,dx$ appears quite different from the term $(3x + 6)\,dx$ in the integral. However, note that

$$(3x + 6)\,dx = 3(x + 2)\,dx = \frac{3}{4}(4)[(x + 2)\,dx]$$

$$= \frac{3}{4}[(4x + 8)\,dx] = \frac{3}{4}\,du$$

to be continued

Substituting, we find that

$$\int \frac{3x+6}{\sqrt{2x^2+8x+3}}\,dx = \int \frac{1}{\sqrt{2x^2+8x+3}}[(3x+6)\,dx]$$

$$= \int \frac{1}{\sqrt{u}}\left(\frac{3}{4}\,du\right) = \frac{3}{4}\int u^{-1/2}\,du$$

$$= \frac{3}{4}\left(\frac{u^{1/2}}{1/2}\right) + C = \frac{3}{2}\sqrt{u} + C$$

$$= \frac{3}{2}\sqrt{2x^2+8x+3} + C$$

substitute
$u = 2x^2 + 8x + 3$

Example

Find $\int \frac{(\ln x)^2}{x}\, dx.$

Solution:

Because $\frac{d}{dx}(\ln x) = \frac{1}{x}$

$$\frac{(\ln x)^2}{x} = (\ln x)^2\left(\frac{1}{x}\right)$$

is a product in which one factor $\frac{1}{x}$ is the derivative of an expression $\ln x$ that appears in the other factor. This suggests that you let $u = \ln x$ with $du = \frac{1}{x}\,dx$. Substituting $u = \ln x$ and $du = \frac{1}{x}\,dx$, you get

$$\int \frac{(\ln x)^2}{x}\,dx = \int (\ln x)^2\left(\frac{1}{x}\,dx\right)$$

$$= \int u^2\,du = \frac{1}{3}u^3 + C$$

$$= \frac{1}{3}(\ln x)^3 + C$$

substitute ln x for u

Example Find $\displaystyle\int \frac{1}{1+e^{-x}}\,dx$

Solution:

You may try to substitute $w = 1 + e^{-x}$. However, this is a dead end because $dw = -e^{-x}\,dx$ but there is no e^{-x} term in the numerator of the integrand. Instead, note that

$$\frac{1}{1+e^{-x}} = \frac{1}{1+\dfrac{1}{e^x}} = \frac{1}{\dfrac{e^x+1}{e^x}}$$

$$= \frac{e^x}{e^x+1}$$

Now, if you substitute $u = e^x + 1$ with $du = e^x\,dx$ into the given integral, you get

$$\int \frac{1}{1+e^{-x}}\,dx = \int \frac{e^x}{e^x+1}\,dx = \int \frac{1}{e^x+1}(e^x\,dx)$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln|e^x+1| + C \qquad \textit{substitute } e^x + 1 \textit{ for } u$$

EXAMPLE 6.3 Using Substitution: A Power Function Inside a Cosine

Evaluate $\int x \cos x^2 \, dx$.

EXAMPLE 6.4 Using Substitution: A Trigonometric Function Inside a Power

Evaluate $\int (3 \sin x + 4)^5 \cos x \, dx$.

EXAMPLE 6.5 Using Substitution: A Root Function Inside a Sine

Evaluate $\displaystyle\int \frac{\sin \sqrt{x}}{\sqrt{x}} \, dx$.

EXAMPLE 6.6 Substitution: Where the Numerator Is the Derivative of the Denominator

Evaluate $\displaystyle\int \frac{x^2}{x^3 + 5} \, dx$.

THEOREM 6.1

For any continuous function, f

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c,$$

provided $f(x) \neq 0$.

EXAMPLE 6.7 An Antiderivative for the Tangent Function

Evaluate $\int \tan x\,dx$.

EXAMPLE 6.8 A Substitution for an Inverse Tangent

Evaluate $\displaystyle\int \frac{(\tan^{-1} x)^2}{1+x^2}\,dx$.

Substituting in a definite integral 定积分中的换元法

Change of Variables ("u-Substitution")

$$\boxed{\int \underbrace{f(u(x))}_{f(u)}\, \underbrace{u'(x)\,dx}_{du} = \int f(u)\,du}$$

Definite Integrals

$$\boxed{\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du}$$

Example

Option 1.

$$\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx = \frac{1}{3}\int_{u(0)}^{u(2)} \frac{1}{\sqrt{u}}\,du$$

$$u = x^3 + 1$$

$$du = 3x^2\,dx$$

$$x^2\,dx = \tfrac{1}{3}du$$

$$= \frac{1}{3}\int_1^9 \frac{1}{\sqrt{u}}\,du$$

$$= \frac{2}{3}\sqrt{u}\,\Big|_1^9 = \frac{2}{3}(3-1) = \frac{4}{3}$$

Option 2.

$$\int \frac{x^2}{\sqrt{x^3+1}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{u}}du = \frac{2}{3}\sqrt{u} = \frac{2}{3}\sqrt{x^3+1} \quad +C$$

$$\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx = \frac{2}{3}\sqrt{x^3+1}\,\Big|_0^2 = \frac{4}{3}$$

Substituting in a definite integral

When using a substitution $u = g(x)$ to evaluate a definite integral $\int_a^b f(x)\,dx$, you can proceed in either of these two ways:

1. Use the substitution to find an antiderivative $F(x)$ for $f(x)$ and then evaluate the definite integral using the fundamental theorem of calculus.

2. Use the substitution to express the integrand and dx in terms of u and du and to replace the original limits of integration, a and b, with transformed limits $c = g(a)$ and $d = g(b)$. The original integral can then be evaluated by applying the fundamental theorem of calculus to the transformed definite integral.

Example Evaluate $\int_0^1 8x(x^2+1)^3\,dx$.

The integrand is a product in which one factor $8x$ is a constant multiple of the derivative of an expression x^2+1 that appears in the other factor. This suggests that you let $u = x^2 + 1$. Then $du = 2x\,dx$, and so

$$\int 8x(x^2+1)^3\,dx = \int 4u^3\,du = u^4$$

The limits of integration, 0 and 1, refer to the variable x and not to u. You can, therefore, proceed in one of two ways. Either you can rewrite the antiderivative in terms of x, or you can find the values of u that correspond to $x = 0$ and $x = 1$.

If you choose the first alternative, you find that

$$\int 8x(x^2+1)^3\,dx = u^4 = (x^2+1)^4$$

and so $\displaystyle\int_0^1 8x(x^2+1)^3\,dx = (x^2+1)^4\Big|_0^1 = 16 - 1 = 15$

to be continued

If you choose the second alternative, use the fact that $u = x^2 + 1$ to conclude that $u = 1$ when $x = 0$ and $u = 2$ when $x = 1$. Hence

$$\int_0^1 8x(x^2+1)^3 = \int_1^2 4u^3 du = u^4|_1^2 = 15$$

Example

Evaluate the following definite integral.

$$\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt$$

Solution:

Solution 1 : In this case the substitution is,

$$u = 1-4t^3 \qquad du = -12t^2 dt \qquad \Rightarrow \qquad t^2 dt = -\frac{1}{12}du$$

$$\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt = -\frac{1}{9}\left(1-4t^3\right)^{\frac{3}{2}}\bigg|_{-2}^{0}$$

$$= -\frac{1}{9} - \left(-\frac{1}{9}(33)^{\frac{3}{2}}\right)$$

$$= \frac{1}{9}\left(33\sqrt{33}-1\right)$$

to be continued

Solution 2 :

$$u = 1-4t^3 \qquad du = -12t^2\,dt \qquad \Rightarrow \qquad t^2\,dt = -\frac{1}{12}\,du$$

$$t = -2 \qquad \Rightarrow \qquad u = 1-4(-2)^3 = 33$$

$$t = 0 \qquad \Rightarrow \qquad u = 1-4(0)^3 = 1$$

$$\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt = -\frac{1}{6}\int_{33}^{1} u^{\frac{1}{2}}\,du$$

$$= -\frac{1}{9}u^{\frac{3}{2}}\Big|_{33}^{1}$$

$$= -\frac{1}{9}-\left(-\frac{1}{9}(33)^{\frac{3}{2}}\right)$$

$$= \frac{1}{9}\left(33\sqrt{33}-1\right)$$

EXAMPLE 6.10 Using Substitution in a Definite Integral

Evaluate $\int_1^2 x^3\sqrt{x^4+5}\,dx$.

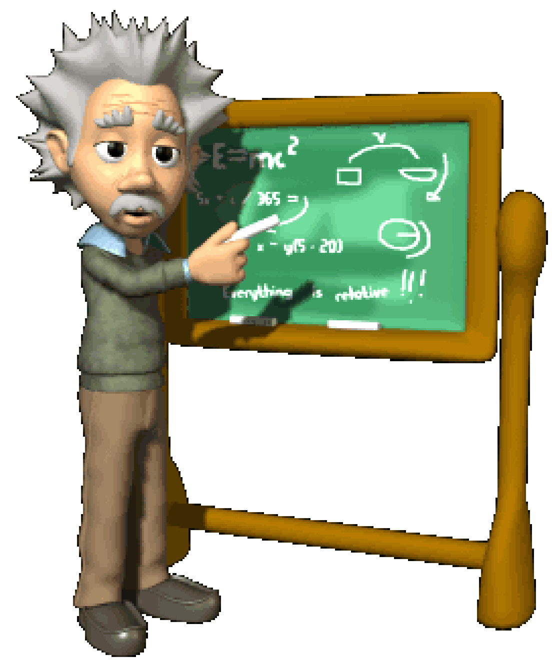
EXAMPLE 6.11 Substitution in a Definite Integral Involving an Exponential

Compute $\int_0^{15} te^{-t^2/2}dt$.

Chapter 5 Applications of the Definite Integral (定积分的应用)

In this Chapter, we will encounter some important concepts.

- Area Between Curves
- Volume: Slicing, Disks, and Washers
- Volumes by Cylindrical Shells
- Arc Length and Surface Area
- Probability



A Procedure for using Definite Integration in Applications

To use definite integration to "accumulate" a quantity Q over an interval $a \le x \le b$, proceed as follows:

Step 1. Divide the interval $a \le x \le b$ into n equal subintervals, each of length $\Delta x = \dfrac{b-a}{n}$. Choose a number x_j from the jth subinterval, for $j = 1, 2, \ldots, n$.

Step 2. Approximate small parts of the quantity Q by products of the form $f(x_j)\,\Delta x$, where $f(x)$ is an appropriate function that is continuous on $a \le x \le b$.

Step 3. Add the individual approximating products to estimate the total quantity Q by the Riemann sum

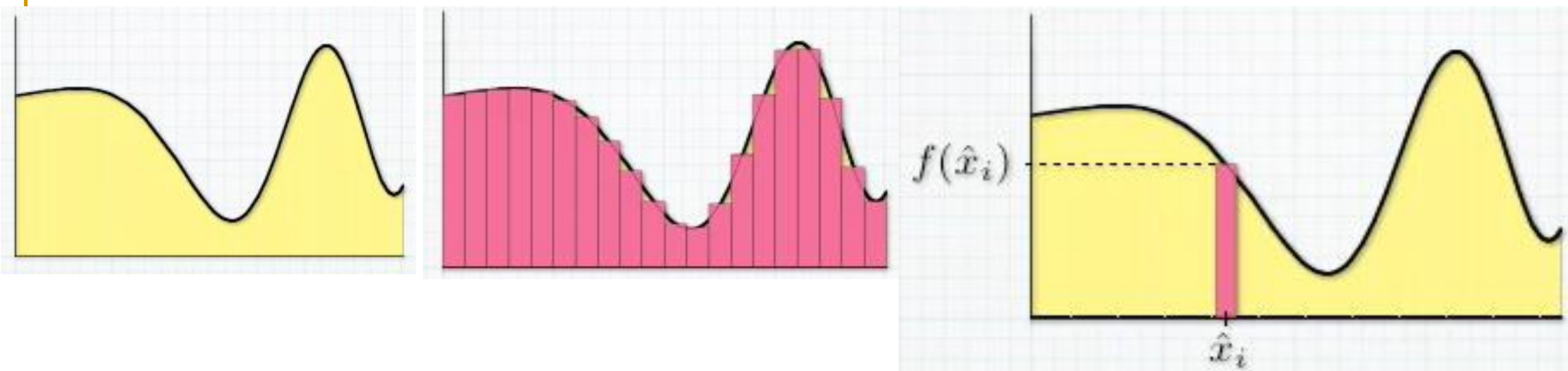
$$[f(x_1) + f(x_2) + \cdots + f(x_n)]\,\Delta x$$

Step 4. Make the approximation in step 3 exact by taking the limit of the Riemann sum as $n \to +\infty$ to express Q as a definite integral; that is,

$$Q = \lim_{n \to +\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\,\Delta x = \int_a^b f(x)\,dx$$

Then use the fundamental theorem of calculus to compute $\int_a^b f(x)\,dx$ and thus to obtain the required quantity Q.

Area of a Region Defined by $0 \leq y \leq f(x)$ and $a \leq x \leq b$



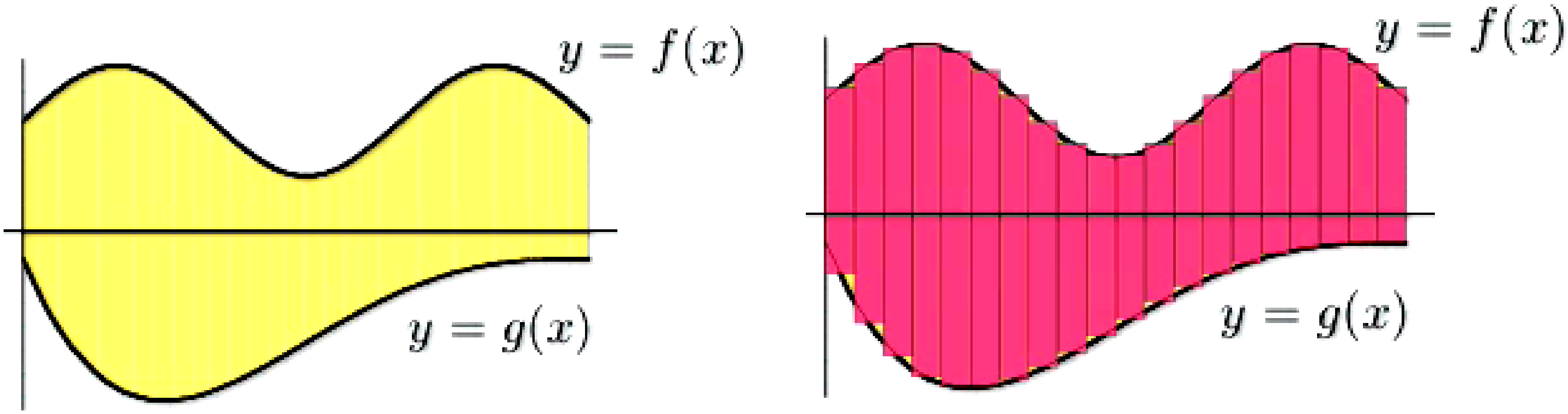
Area of a typical rectangle $\Delta A_i = f(\hat{x}_i)\Delta x$

Riemann sum $A \approx \sum \Delta A_i = \sum f(\hat{x}_i)\Delta x$

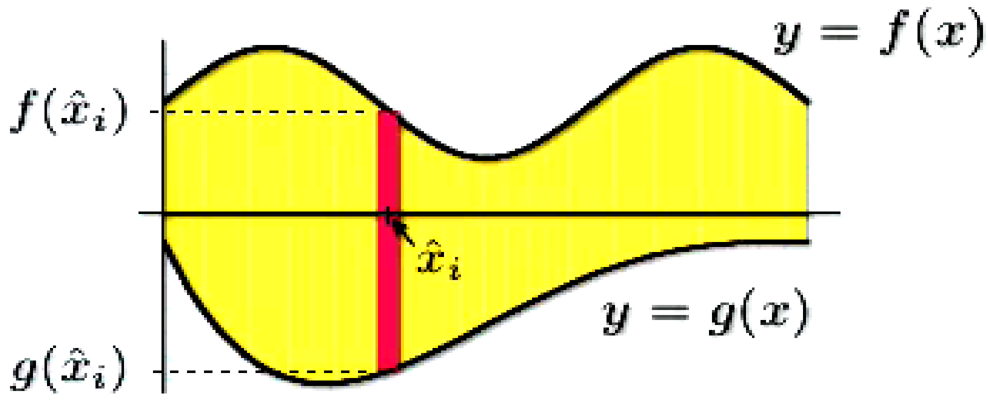
Area of a typical thin *slice* $dA = f(x)\,dx$

Definite integral $A = \int_a^b f(x)\,dx$

Area of a Region Defined by $g(x) \le y \le f(x)$ and $a \le x \le b$



Area of a Region Defined by $g(x) \le y \le f(x)$ and $a \le x \le b$



Area of a typical rectangle $\Delta A_i = \big(f(\hat{x}_i) - g(\hat{x}_i)\big)\Delta x$

Riemann sum $A \approx \sum \Delta A_i = \sum \big(f(\hat{x}_i) - g(\hat{x}_i)\big)\Delta x$

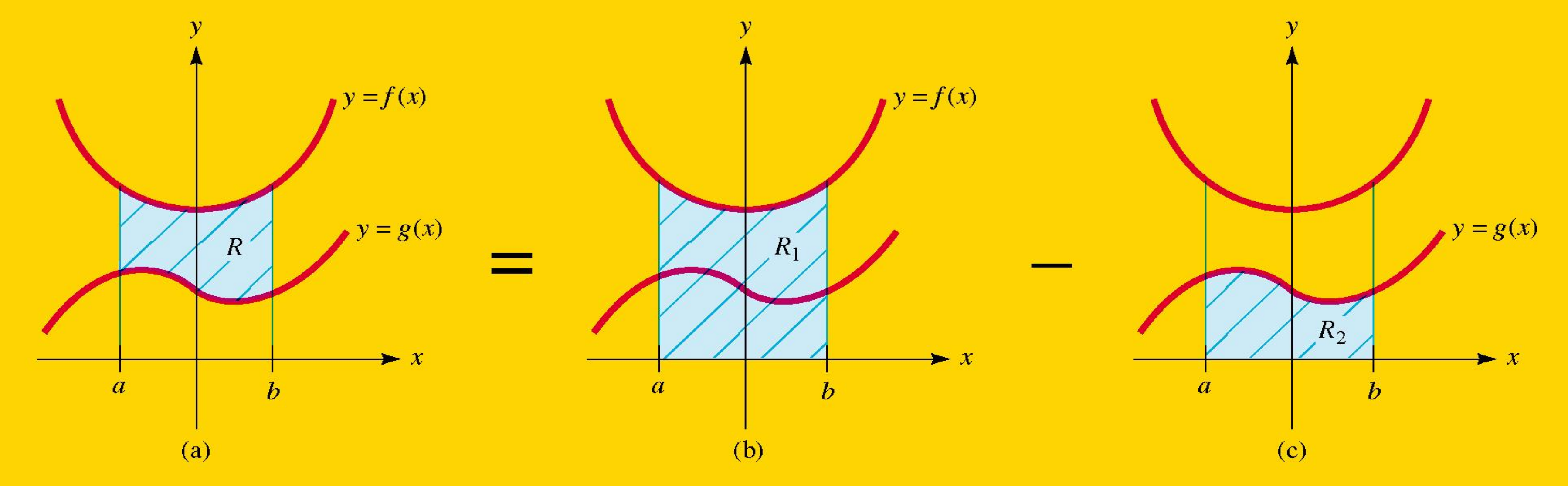
Area of a typical thin *slice* $dA = \big(f(x) - g(x)\big)\, dx$

Definite integral $A = \int_{x=a}^{x=b} dA \;=\; \int_a^b \big(f(x) - g(x)\big)\, dx$

Top curve

Bottom curve

Section 5.1 Area Between Curves



$$\text{area of } R = [\text{area under } y = f(x)] - [\text{area under } y = g(x)]$$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$

The Area Between Two Curve If $f(x)$ and $g(x)$ are continuous with $f(x) \geq g(x)$ on the interval $a \leq x \leq b$, then the area A between the curves $y=f(x)$ and $y=g(x)$ over the interval is given by

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Example

Find the area of the region R enclosed by the curves $y = x^3$ and $y = x^2$

Solution:

To find the points where the curves intersect, solve the equations simultaneously as follows:

$$x^3 = x^2$$

$$x^3 - x^2 = 0$$

$$x^2(x-1) = 0$$

$$x = 0,1$$

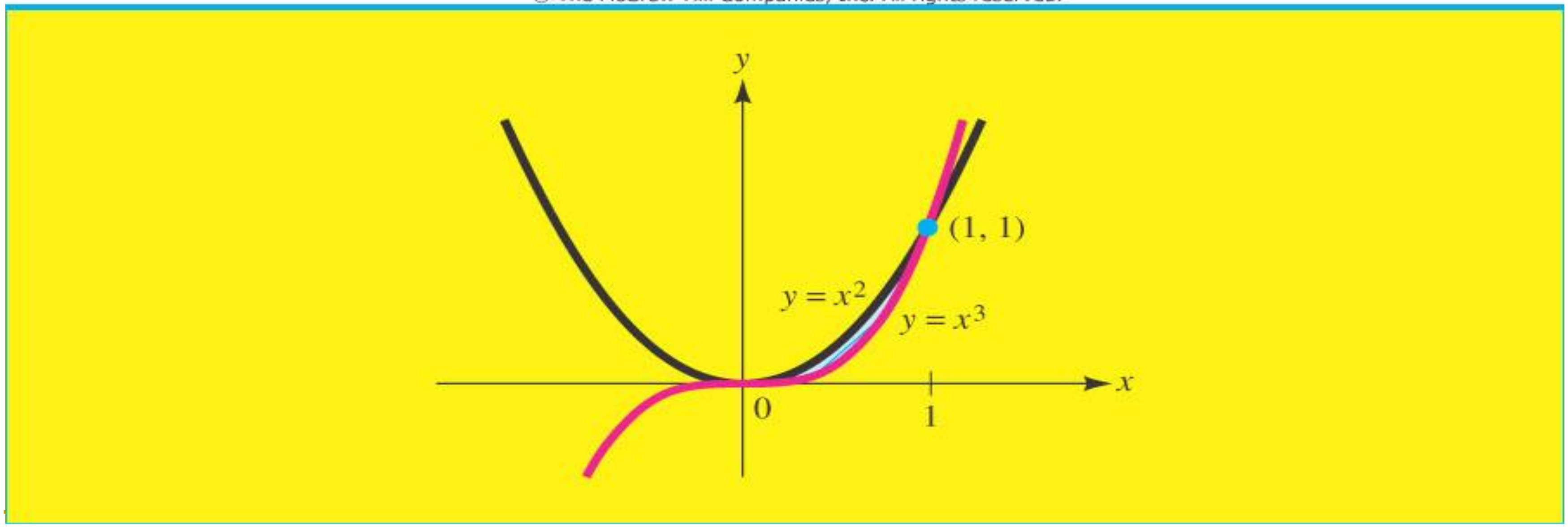
The corresponding points *(0,0)* and *(1,1)* are the only points of intersection.

to be continued

The region R enclosed by the two curves is bounded above by $y = x^2$ and below by $y = x^3$, over the interval $0 \le x \le 1$ (See the Figure). The area of this region is given by the integral

$$A = \int_0^1 (x^2 - x^3)dx = \left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Bigg|_0^1$$

$$= \left[\frac{1}{3}(1)^3 - \frac{1}{4}(1)^4\right] - \left[\frac{1}{3}(0)^3 - \frac{1}{4}(0)^4\right] = \frac{1}{12}$$



Example Find the area of the region enclosed by the line $y = 4x$ and the curve $y = x^3 + 3x^2$.

Solution:

To find where the line and curve intersect, solve the equations simultaneously as follows:

$$x^3 + 3x^2 = 4x$$

$$x^3 + 3x^2 - 4x = 0 \qquad \textit{subtract } 4x \textit{ from both sides}$$

$$x(x^2 + 3x - 4) = 0 \qquad \textit{factor out } x$$

$$x(x - 1)(x + 4) = 0 \qquad \textit{factor } x^2 + 3x - 4$$

$$x = 0,\ 1,\ -4 \qquad uv = 0 \textit{ if and only if } u = 0 \textit{ or } v = 0$$

The corresponding points of intersection are (0, 0), (1, 4), and (−4, −16). The curve and the line are sketched in Figure 5.12.

Over the interval $-4 \le x \le 0$, the curve is above the line so $x^3 + 3x^2 \ge 4x$, and the region enclosed by the curve and line has area

$$A_1 = \int_{-4}^{0} [(x^3 + 3x^2) - 4x]\,dx = \frac{1}{4}x^4 + x^3 - 2x^2 \Big|_{-4}^{0}$$

$$= \left[\frac{1}{4}(0)^4 + (0)^3 - 2(0)^2\right] - \left[\frac{1}{4}(-4)^4 + (-4)^3 - 2(-4)^2\right] = 32$$

to be continued

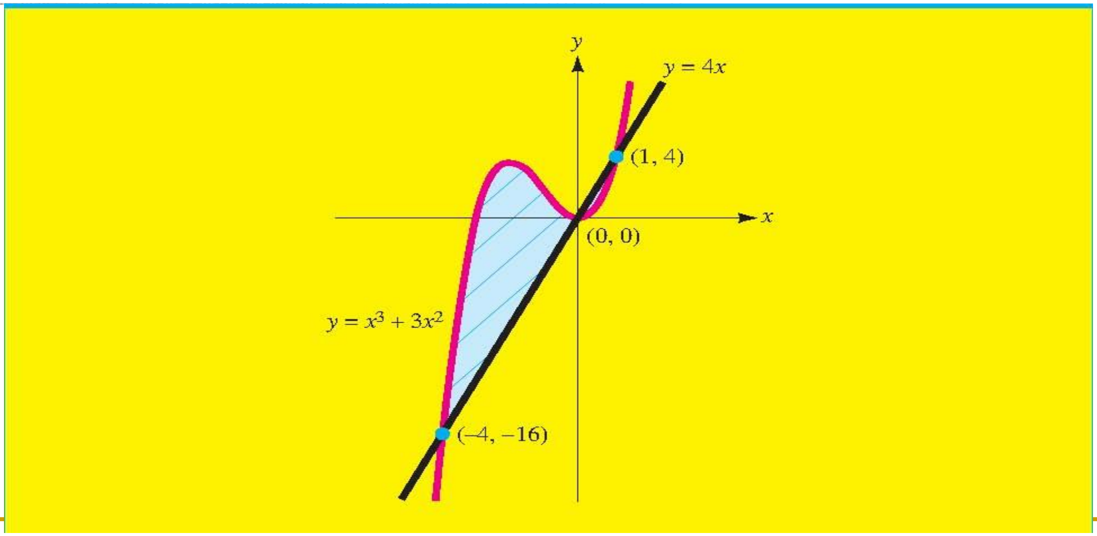
Over the interval $0 \le x \le 1$, the line is above the curve and the enclosed region has area

$$A_2 = \int_0^1 [4x - (x^3 + 3x^2)]\,dx = 2x^2 - \frac{1}{4}x^4 - x^3 \Big|_0^1$$

$$= \left[2(1)^2 - \frac{1}{4}(1)^4 - (1)^3\right] - \left[2(0)^2 - \frac{1}{4}(0)^4 - (0)^3\right] = \frac{3}{4}$$

Therefore, the total area enclosed by the line and the curve is given by the sum

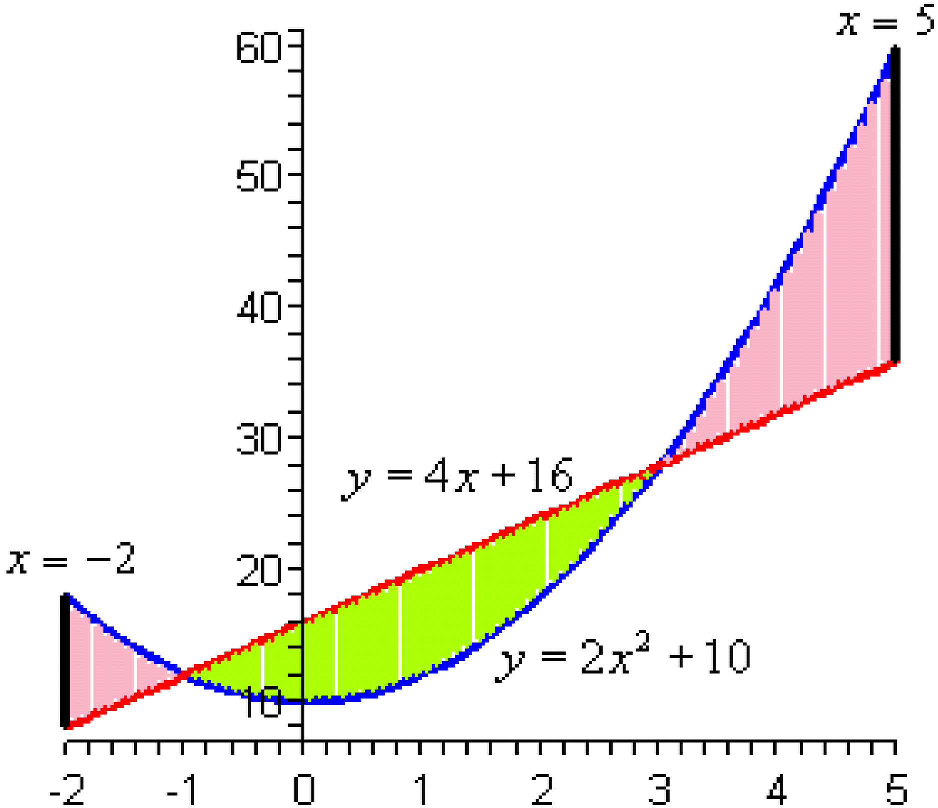
$$A = A_1 + A_2 = 32 + \frac{3}{4} = 32.75$$



Example

Determine the area of the region bounded by $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$ and $x = 5$

Solution:



$$2x^2 + 10 = 4x + 16$$

$$2x^2 - 4x - 6 = 0$$

$$x = -1, \quad x = 3$$

to be continued

Here is the area.

$$A = \int_{-2}^{-1} 2x^2 + 10 - (4x+16)\,dx + \int_{-1}^{3} 4x + 16 - \left(2x^2 + 10\right) dx$$

$$+ \int_{3}^{5} 2x^2 + 10 - (4x+16)\,dx$$

$$= \left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\Bigg|_{-2}^{-1} + \left(-\frac{2}{3}x^3 + 2x^2 + 6x\right)\Bigg|_{-1}^{3} + \left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\Bigg|_{3}^{5}$$

$$= \frac{14}{3} + \frac{64}{3} + \frac{64}{3}$$

$$= \frac{142}{3}$$

Example

Find the area of the region bounded by the graph of $y=x^2$ and $y=x+2$.

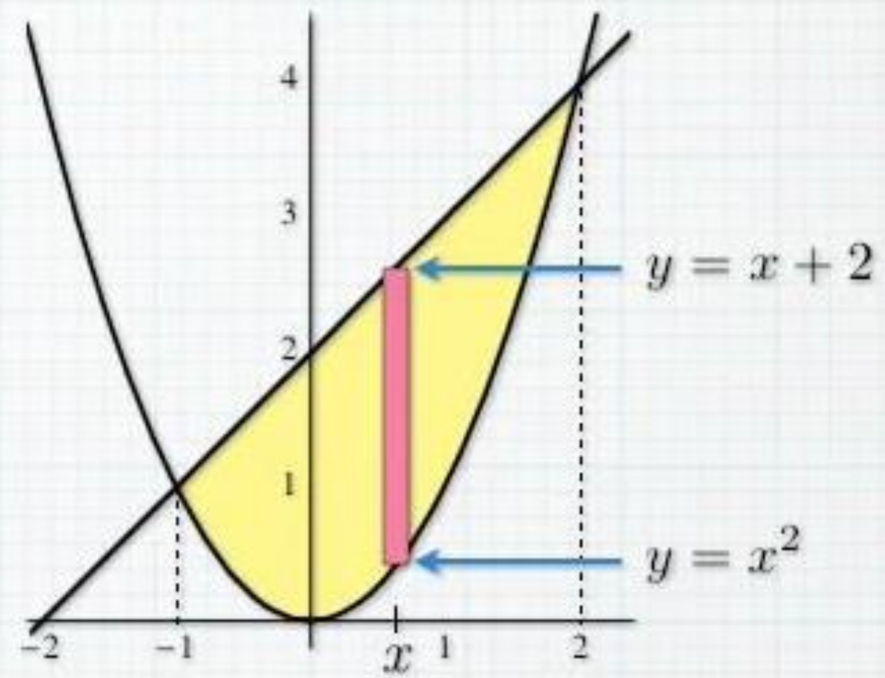
Solution:

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0$$

$$x = -1,\ 2$$



$$dA = (x + 2 - x^2)\,dx$$

$$A = \int_{-1}^{2} (x + 2 - x^2)\,dx$$

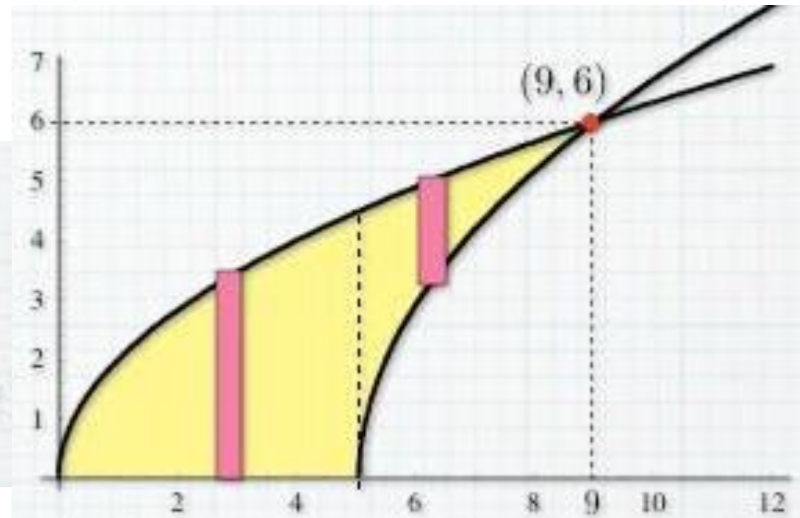
$$= \left(\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3\right)\Bigg|_{-1}^{2} = \frac{9}{2}$$

Example

Find the area of the region bounded by the graph of $y = 2\sqrt{x}$, $y=0$, and $y = 3\sqrt{x-5}$.

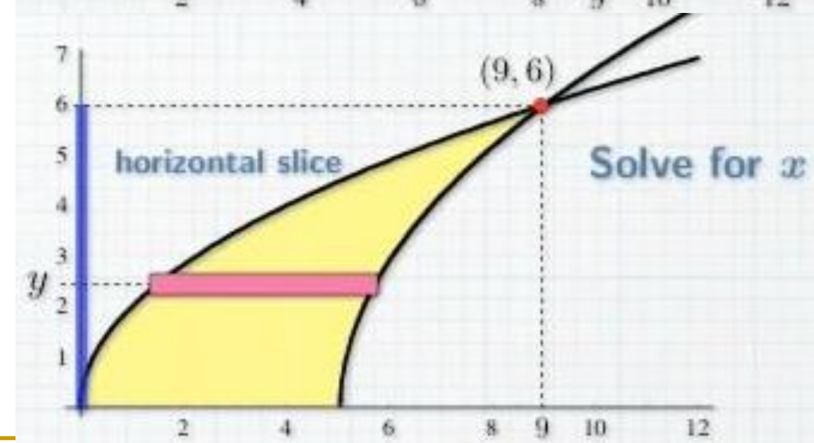
Solution:

$$A = \int_0^5 2\sqrt{x}\,dx + \int_5^9 \left(2\sqrt{x} - 3\sqrt{x-5}\right) dx$$



$$A = \frac{5}{36}\int_0^6 \left(36 - y^2\right) dy$$

$$= \frac{5}{36}(144) = 20$$



EXAMPLE 1.1 Finding the Area between Two Curves

Find the area bounded by the graphs of $y = 3 - x$ and $y = x^2 - 9$ (see Figure 5.4).

EXAMPLE 1.2 Finding the Area between Two Curves That Cross

Find the area bounded by the graphs of $y = x^2$ and $y = 2 - x^2$ for $0 \le x \le 2$.

EXAMPLE 1.4 The Area of a Region Determined by Three Curves

Find the area bounded by the graphs of $y = x^2$, $y = 2 - x$ and $y = 0$.

Section 5.2 Volume: Slicing, Disks and Washers

Volumes by Slicing

- Cylinder (圆柱体): *Any* solid whose cross sections perpendicular to some axis running through the solid are all the same.

- If either the cross-sectional area or width of a solid is not constant, we will need to modify our approach.

- ✓ We start by partitioning the interval *[a,b]* on the *x*-axis into *n* subintervals, each of width $\triangle x=(b-a)/n$. As usual, we denote $x_0 = a,\ x_1 = a + \Delta x$ and so on, so that $$x_i = a + i\Delta x, \quad \text{for } i = 0, 1, 2, \ldots, n.$$

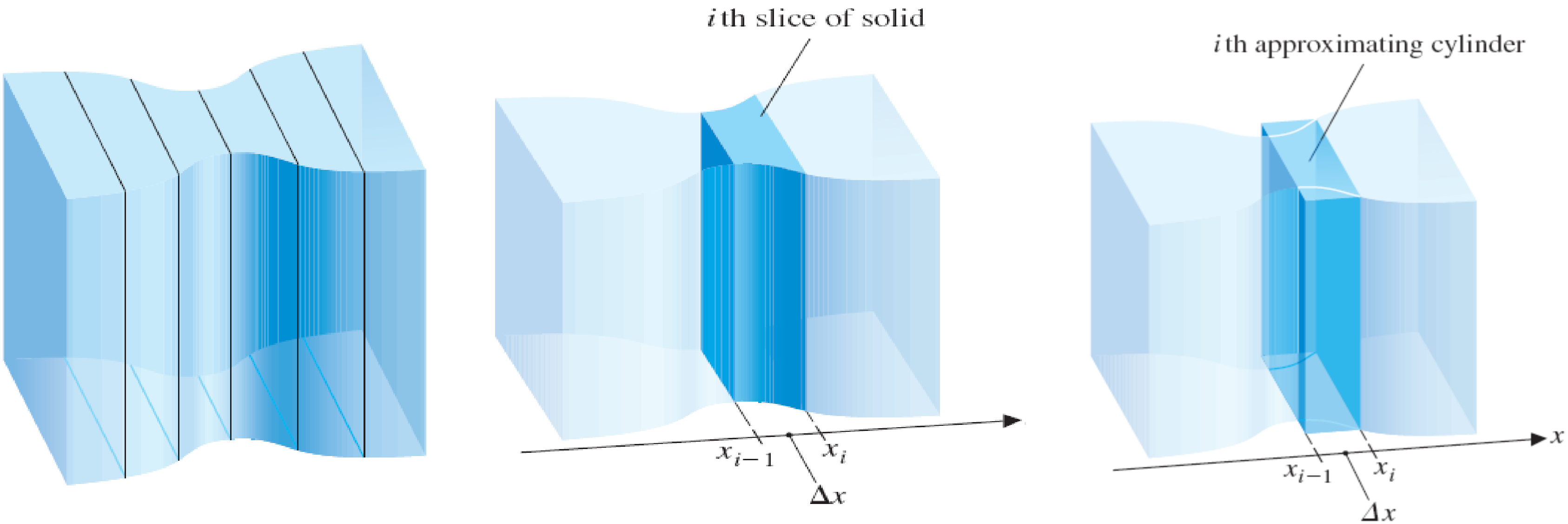


FIGURE 5.14a Sliced solid

FIGURE 5.14b *i*th slice of solid

FIGURE 5.14c *i*th approximating cylinder

The Volume of the *ith* slice is given as following.

$$V_i \approx \underbrace{A(c_i)}_{\text{cross-sectional area}} \underbrace{\Delta x}_{\text{width}},$$

We get the exact volume by computing

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(c_i)\,\Delta x = \int_a^b A(x)\,dx.$$

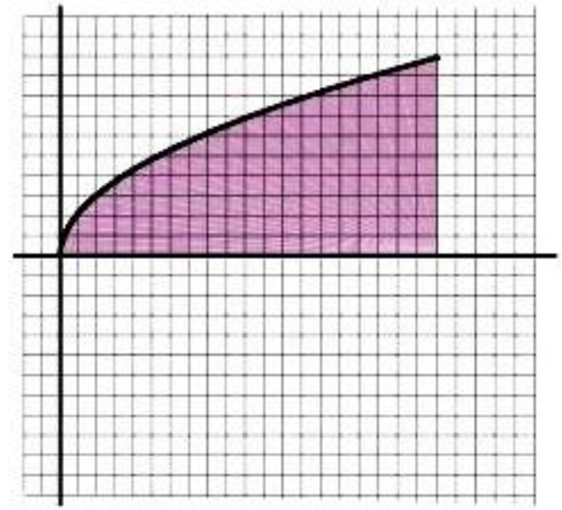
REMARK 2.1

We use the same process followed here to derive many important formulas. In each case, we divide an object into n smaller pieces, approximate the quantity of interest for each of the small pieces, sum the approximations and then take a limit, ultimately recognizing that we have derived a definite integral. For this reason, it is essential that you understand the concept behind formula (2.1). Memorization will not do this for you. However, if you understand how the various pieces of this puzzle fit together, then the rest of this chapter should fall into place for you nicely.

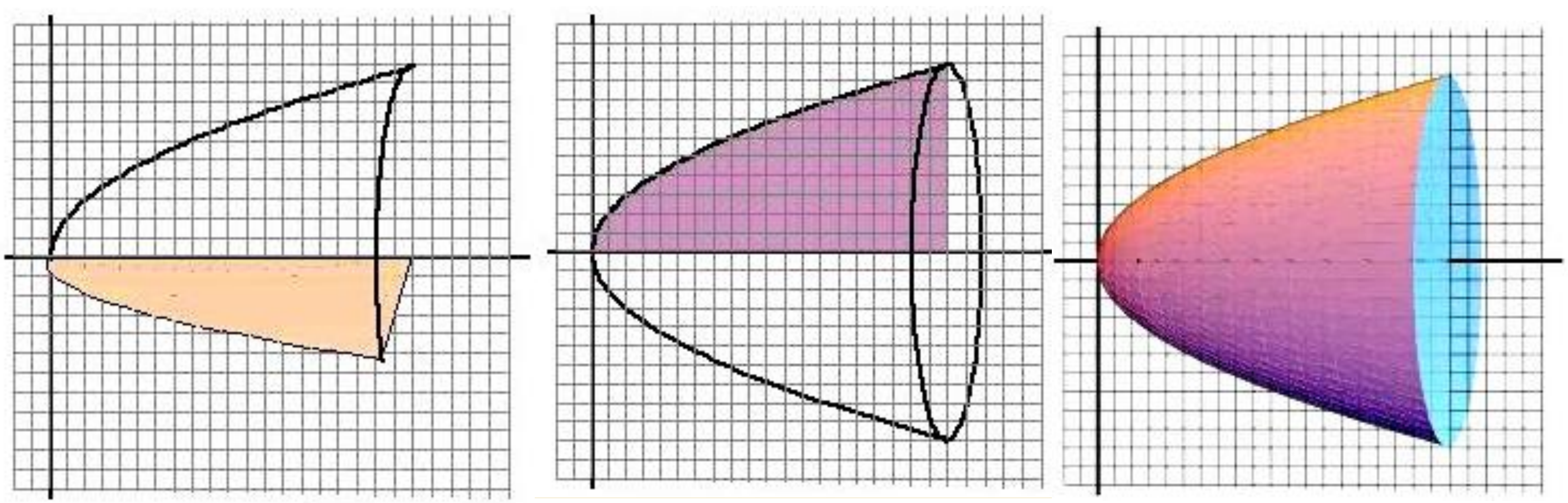
The Method of Disks: The volume of a Solid of Revolution (旋转体的体积)

Consider, for example, the planar region in which

$$0 \le y \le \sqrt{x}, \;\; 0 \le x \le 4.$$

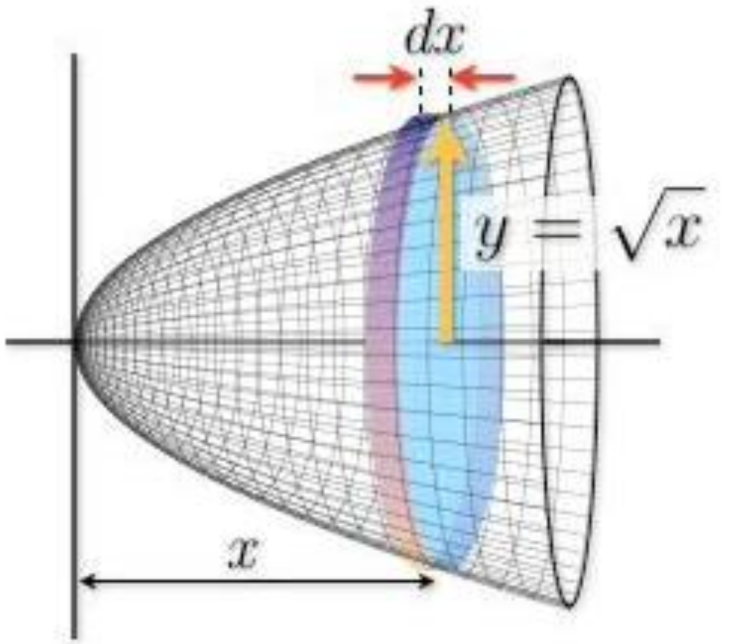


By revolving the region about the x-axis we obtain a **solid of revolution**.



Cross-sections perpendicular to the axis of rotation are circular

So a typical slice may be viewed as a thin disk



disk radius $\quad r = y = \sqrt{x}$

Solids with cross-sectional area $A(x)$

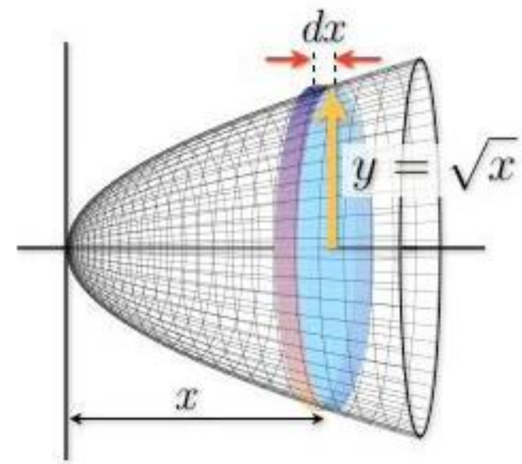
$$A = \pi r^2 = \pi [f(x)]^2$$

slice volume $= A(x_i)\Delta x$ where $A(x) =$ cross-sectional area perpendicular to the x-axis

Divide the interval a≤x≤b into n equal subintervals of length △x

disk volume $\quad dV = \pi r^2 dx = \pi (\sqrt{x})^2 dx = \pi x \, dx$

The total volume V can be approximated by



$$V \approx \sum A(x_i)\Delta x$$

$$\approx \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x$$

The approximation improves as n increase without bound ($\triangle x$ approach *0*) and

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x$$

$$= \pi \int_a^b [f(x)]^2 dx = \int_a^b A(x)dx$$

Volume is the integral of cross-sectional area.

total volume $\quad V = \int_0^4 \pi\, x\, dx = \frac{\pi}{2} x^2 \Big|_0^4 = \frac{\pi}{2}(4^2 - 0^2) = 8\pi$

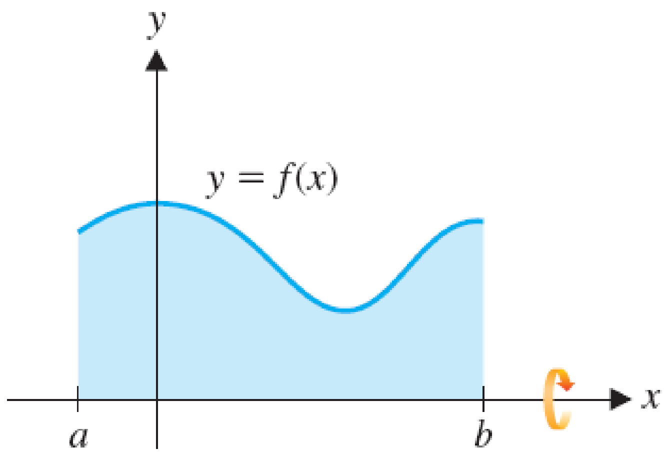


FIGURE 5.16a
$y = f(x) \geq 0$

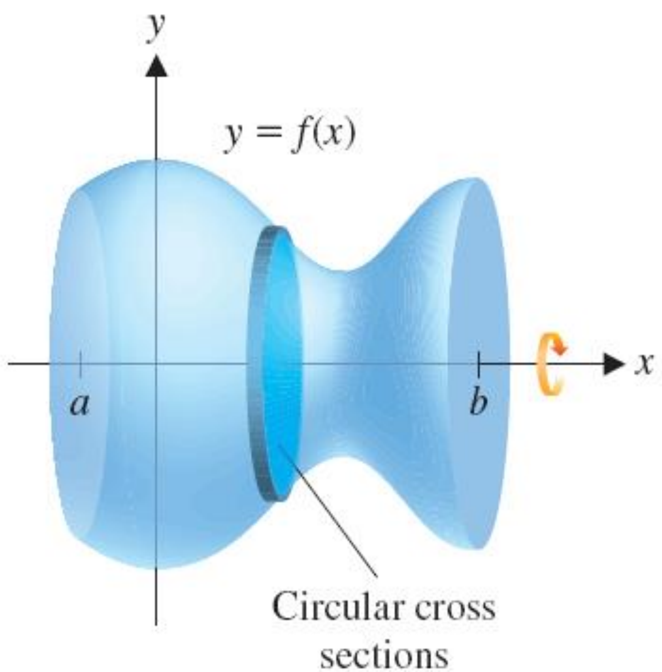


FIGURE 5.16b
Solid of revolution

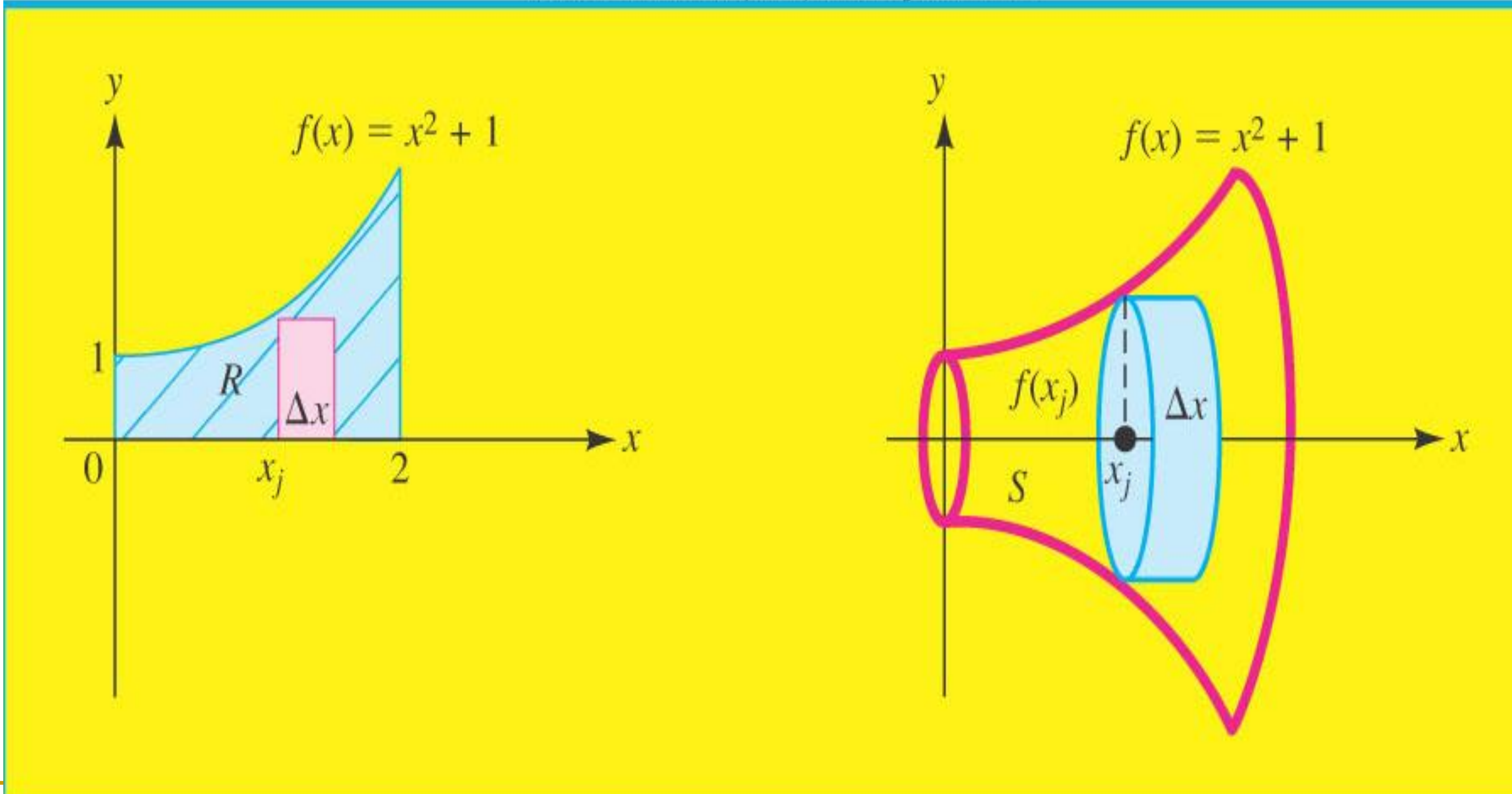
Volume Formula

Suppose $f(x)$ is continuous and $f(x) \geq 0$ and $a \leq x \leq b$ and let R be the region under the curve $y=f(x)$ between $x=a$ and $x=b$. Then the solid S formed by revolving R about the x axis has volume

$$\text{Volume of S} = \pi \int_a^b [f(x)]^2 \, dx$$

Example

Find the volume of the solid S formed by revolving the region under the curve $y = x^2 + 1$ from $x=0$ to $x=2$ about the x axis.



Solution:

The region, the solid of revolution, and the jth disk are shown in the Figure. The radius of the jth disk is $f(x_j) = x_j^2 + 1$. Hence,

$$\text{Volume of } j\text{th disk} = \pi[f(x_j)]^2 \Delta x = \pi(x_j^2 + 1)^2 \Delta x$$

and

$$\begin{aligned}
\text{Volume of } S &= \lim_{n\to\infty} \sum_{j=1}^{n} \pi(x_j^2 + 1)^2 \Delta x \\
&= \pi \int_0^2 (x^2 + 1)^2 dx \\
&= \pi \int_0^2 (x^4 + 2x^2 + 1) dx \\
&= \pi \left(\frac{1}{5} x^5 + \frac{2}{3} x^3 + x \right) \Bigg|_0^2 = \frac{206}{15} \pi \approx 43.14
\end{aligned}$$

EXAMPLE 2.4 Using the Method of Disks to Compute Volume

Revolve the region under the curve $y = \sqrt{x}$ on the interval $[0, 4]$ about the x-axis and find the volume of the resulting solid of revolution.

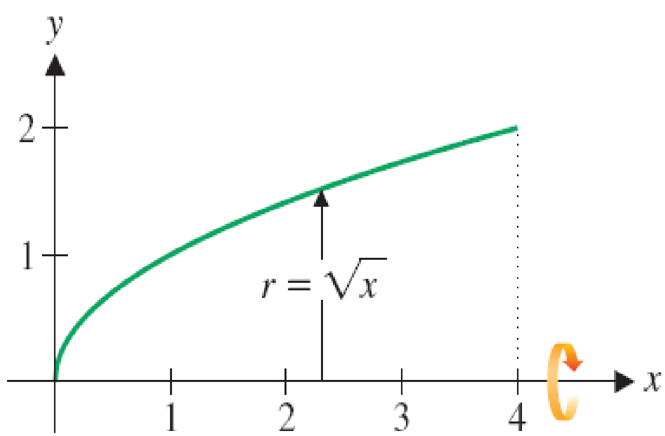


FIGURE 5.17a
$y = \sqrt{x}$

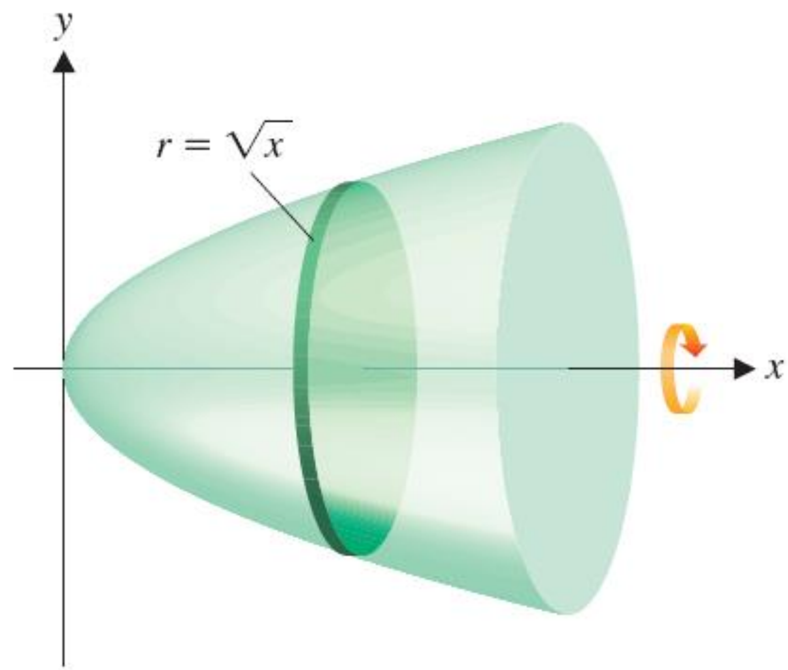


FIGURE 5.17b
Solid of revolution

Suppose that $g(y) \geq 0$ and g is continuous on the interval $[c,d]$. Then revolving the region bounded by the curve $x=g(y)$ and the y-axis, for $c \leq y \leq d$, about the y-axis generates a solid. The volume of the solid is then given by

$$\boxed{V = \int_c^d \underbrace{\pi [g(y)]^2}_{\text{cross-sectional area} = \pi r^2} dy.}$$

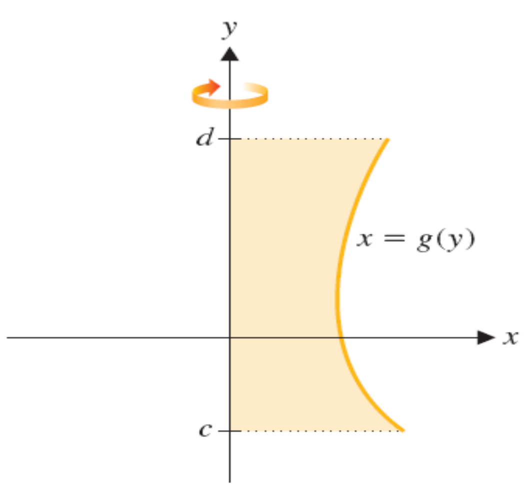


FIGURE 5.18a
Revolve about the y-axis

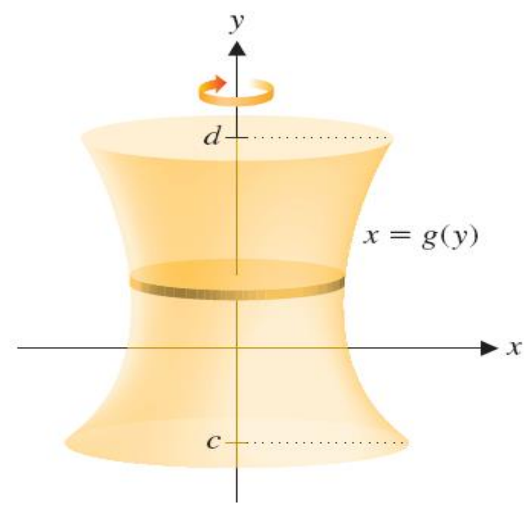


FIGURE 5.18b
Solid of revolution

REMARK 2.2

When using the method of disks, the variable of integration depends solely on the axis about which you revolve the two-dimensional region: revolving about the x-axis requires integration with respect to x, while revolving about the y-axis requires integration with respect to y. This is easily determined by looking at a sketch of the solid. Don't make the mistake of simply looking for what you can plug in where. This is a recipe for disaster, for the rest of this chapter will require you to make similar choices, each based on distinctive requirements of the problem at hand.

EXAMPLE 2.5 Using the Method of Disks with y as the Independent Variable

Find the volume of the solid resulting from revolving the region bounded by the curves $y = 4 - x^2$ and $y = 1$ from $x = 0$ to $x = \sqrt{3}$ about the y-axis.

The Method of Washers

One complication that occurs in computing volumes is that the solid may have a cavity or “hole” in it. Another occurs when a region is revolved about a line other than the *x*-axis or the *y*- axis.

EXAMPLE 2.6 Computing Volumes of Solids with and without Cavities

Let R be the region bounded by the graphs of $y = \frac{1}{4}x^2$, $x = 0$ and $y = 1$. Compute the volume of the solid formed by revolving R about (a) the y-axis, (b) the x-axis and (c) the line $y = 2$.

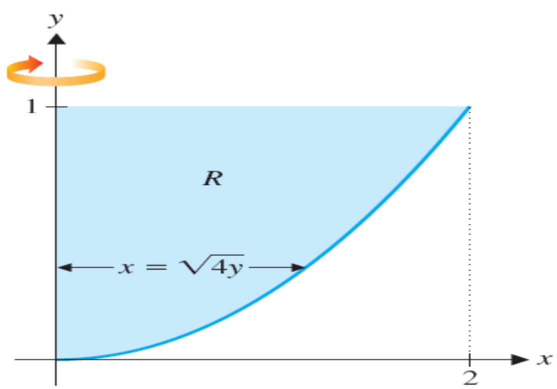


FIGURE 5.20a
$x = \sqrt{4y}$

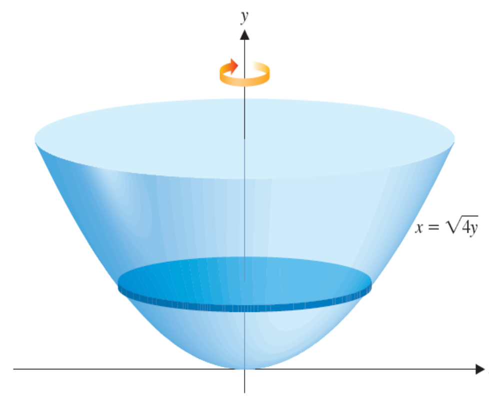


FIGURE 5.20b
Solid of revolution

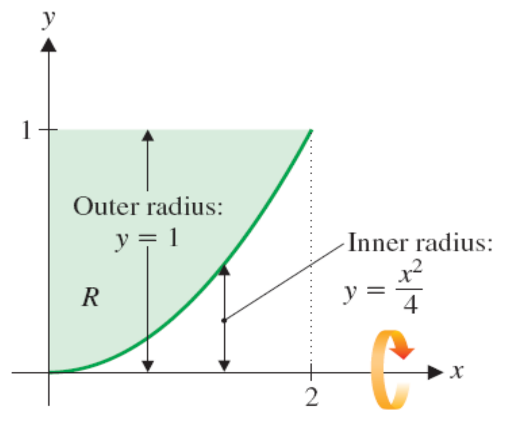


FIGURE 5.21a

$$y = \frac{1}{4}x^2$$

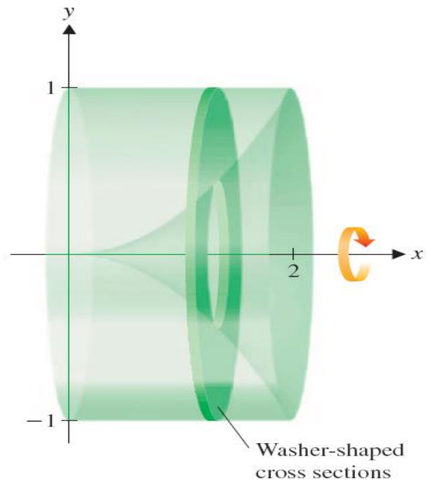


FIGURE 5.21b
Solid with cavity

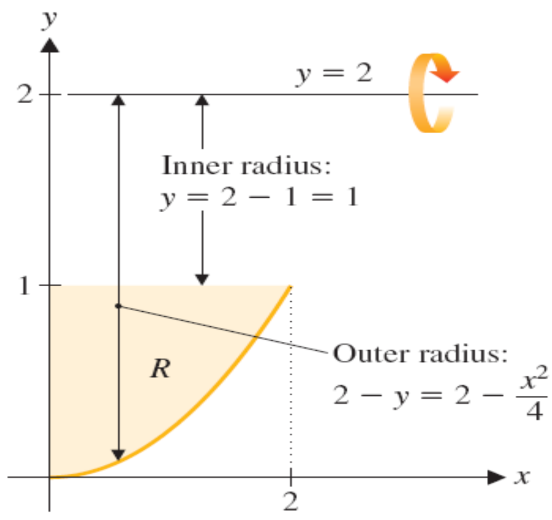


FIGURE 5.22a
Revolve about $y = 2$

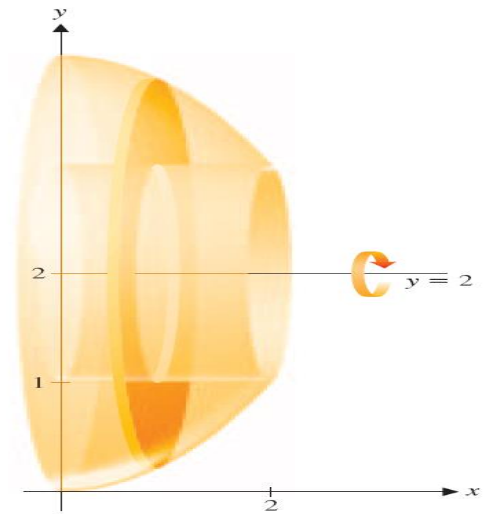


FIGURE 5.22b
Solid of revolution

EXAMPLE 2.7 Revolving a Region about Different Lines

Let R be the region bounded by $y = 4 - x^2$ and $y = 0$. Find the volume of the solids obtained by revolving R about each of the following: (a) the y-axis, (b) the line $y = -3$, (c) the line $y = 7$ and (d) the line $x = 3$.

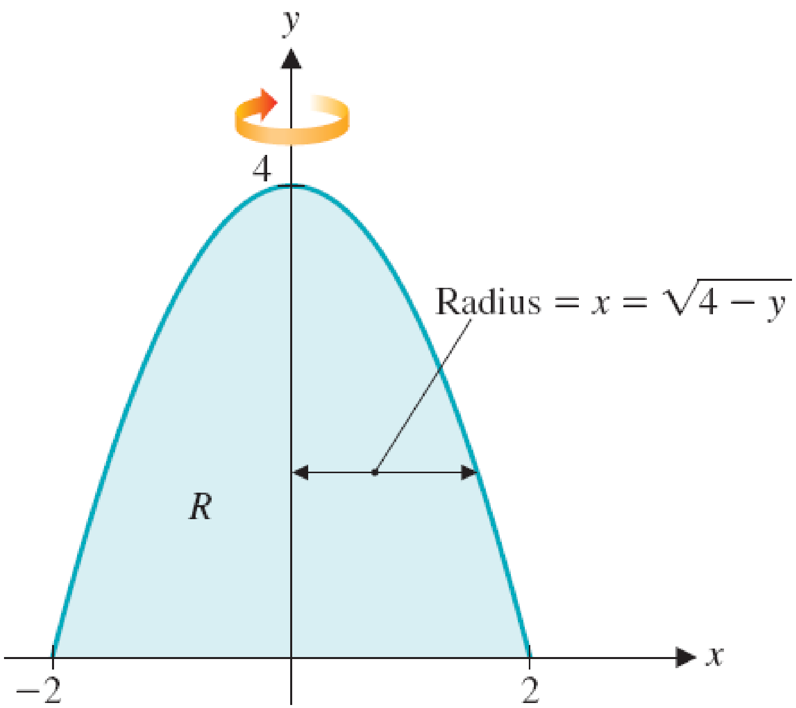


FIGURE 5.23a
Revolve about y-axis

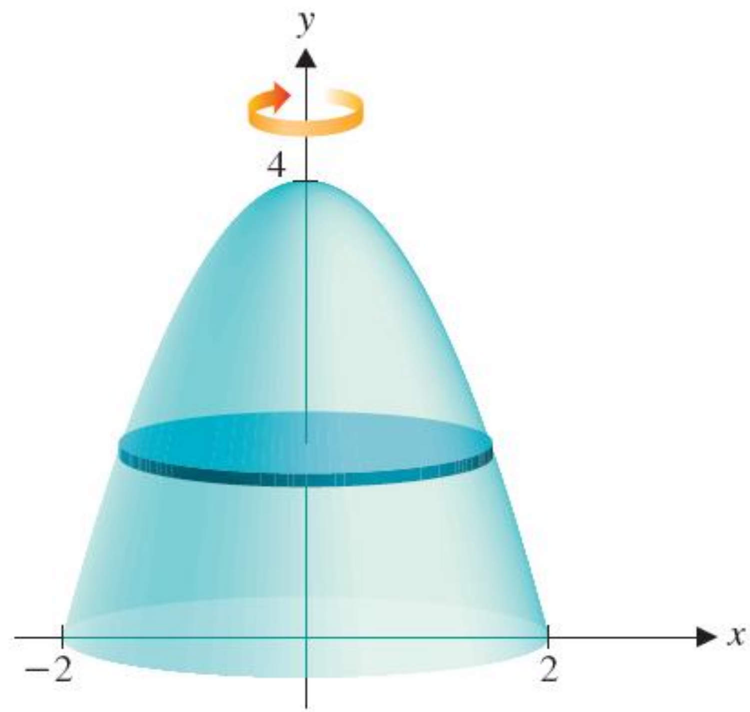
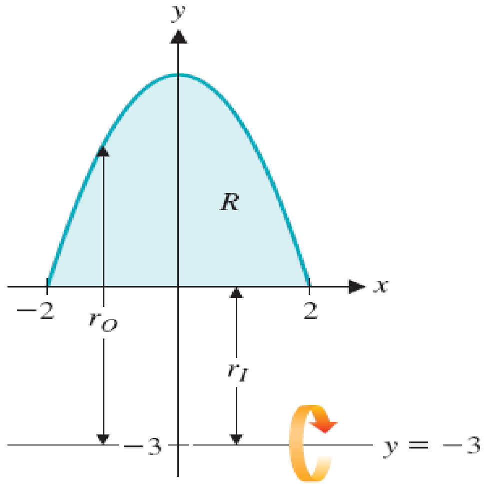


FIGURE 5.23b
Solid of revolution



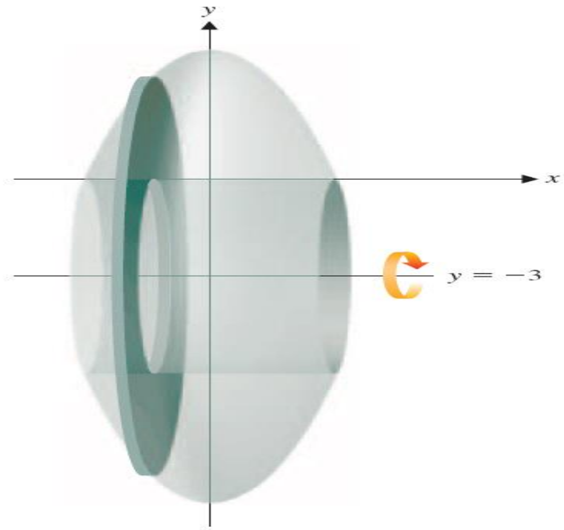


FIGURE 5.24b
Solid of revolution

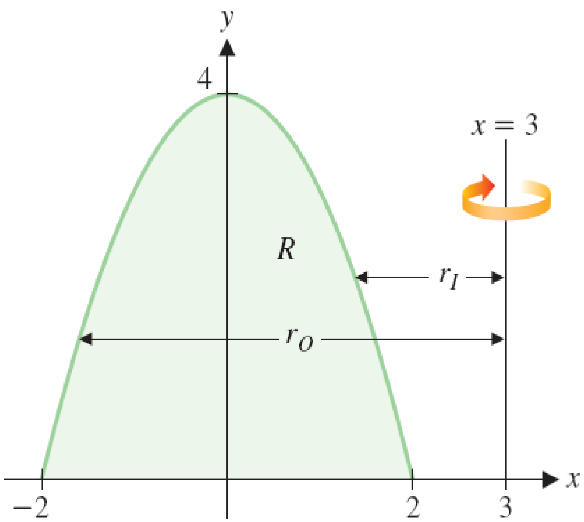


FIGURE 5.26a
Revolve about $x = 3$

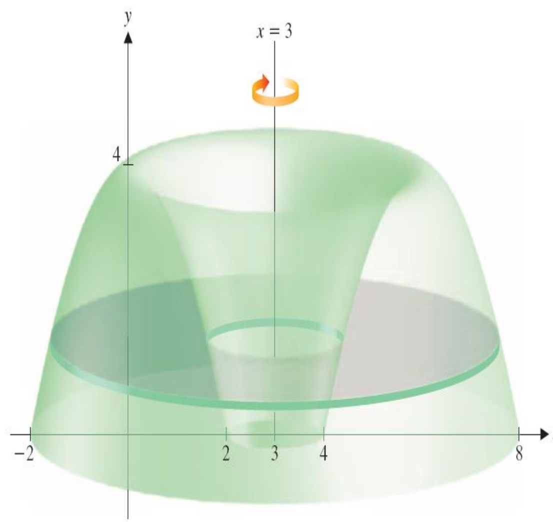


FIGURE 5.26b
Solid of revolution

Section 5.3 Volumes by Cylindrical Shells

In this section, we present an alternative to the method of washers discussed in section 5.2.

We consider the general case for a region revolved about y-axis. Let R denote the region bounded by the graph of $y=f(x)$ and the x-axis on the interval $[a,b]$, where $0<a<b$ and $f(x) \geq 0$ on $[a,b]$.

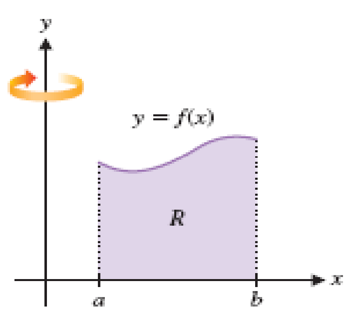


FIGURE 5.28a
Revolve about y-axis

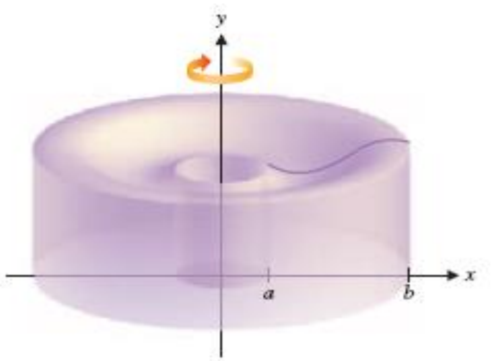


FIGURE 5.28b
Solid of revolution

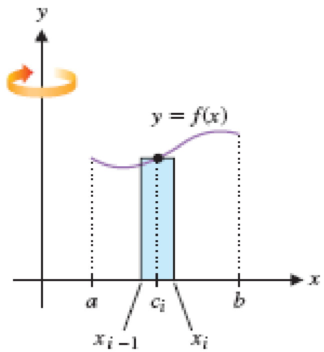


FIGURE 5.29a
*i*th rectangle

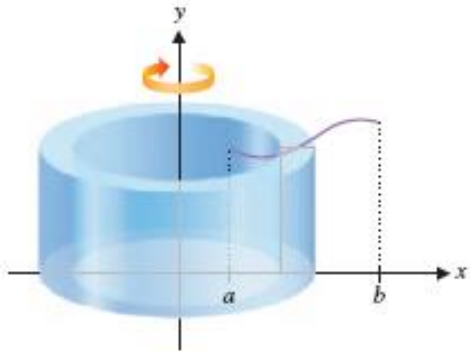


FIGURE 5.29b
Cylindrical shell

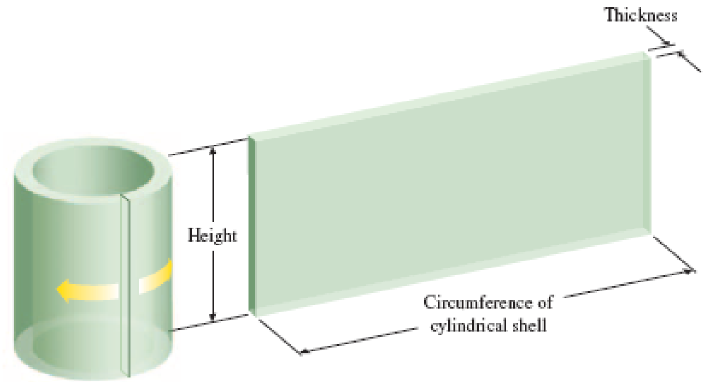


FIGURE 5.29c
Flattened cylindrical shell

We first partition the closed interval *[a,b]* into *n* subintervals of equal width $\triangle x=(b-a)/n$. On each subintervals $[x_{i-1}, x_i]$, pick a point c_i and construct the rectangle of height $f(c_i)$ as indicated in Figure 5.29a. Revolving this rectangle about the *y*-axis forms a thin cylindrical shell.

Notice that the length of such a thin sheet corresponds to the circumference of the cylindrical shell, which is $2\pi \cdot \text{radius} = 2\pi c_i$. So the volume V_i of the *i*th cylindrical shell is approximately

$$
\begin{aligned}
V_i &\approx \text{length} \times \text{width} \times \text{height} \\
&= (2\pi \times \text{radius}) \times \text{thickness} \times \text{height} \\
&= (2\pi c_i)\, \Delta x\, f(c_i).
\end{aligned}
$$

The total volume *V* of the solid can then be approximated by the sum of the volumes of the *n* cylindrical shells:

$$
V \approx \sum_{i=1}^{n} 2\pi \underbrace{c_i}_{\text{radius}} \underbrace{f(c_i)}_{\text{height}} \underbrace{\Delta x}_{\text{thickness}} .
$$

$$
\boxed{V = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi c_i\, f(c_i)\, \Delta x = \int_a^b 2\pi \underbrace{x}_{\text{radius}} \underbrace{f(x)}_{\text{height}} \underbrace{dx}_{\text{thickness}} .}
$$

EXAMPLE 3.1 Using the Method of Cylindrical Shells

Revolve the region bounded by the graphs of $y = x$ and $y = x^2$ in the first quadrant about the y-axis.

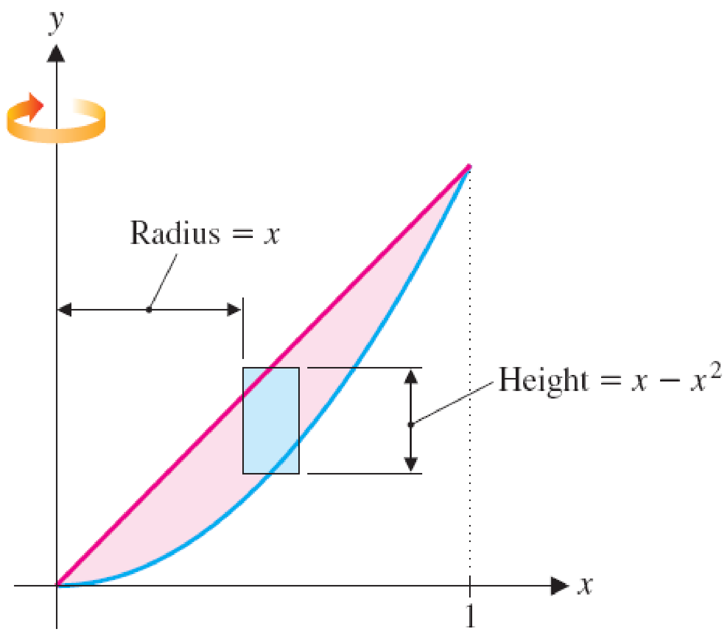


FIGURE 5.30a
Sample rectangle generating a cylindrical shell

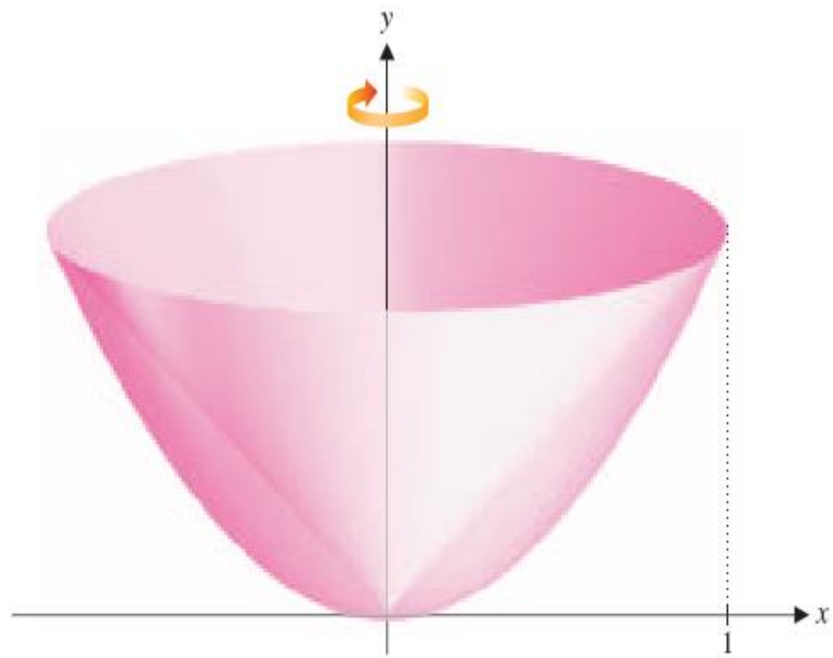


FIGURE 5.30b
Solid of revolution

EXAMPLE 3.2 A Volume Where Shells Are Simpler Than Washers

Find the volume of the solid formed by revolving the region bounded by the graph of $y = 4 - x^2$ and the x-axis about the line $x = 3$.

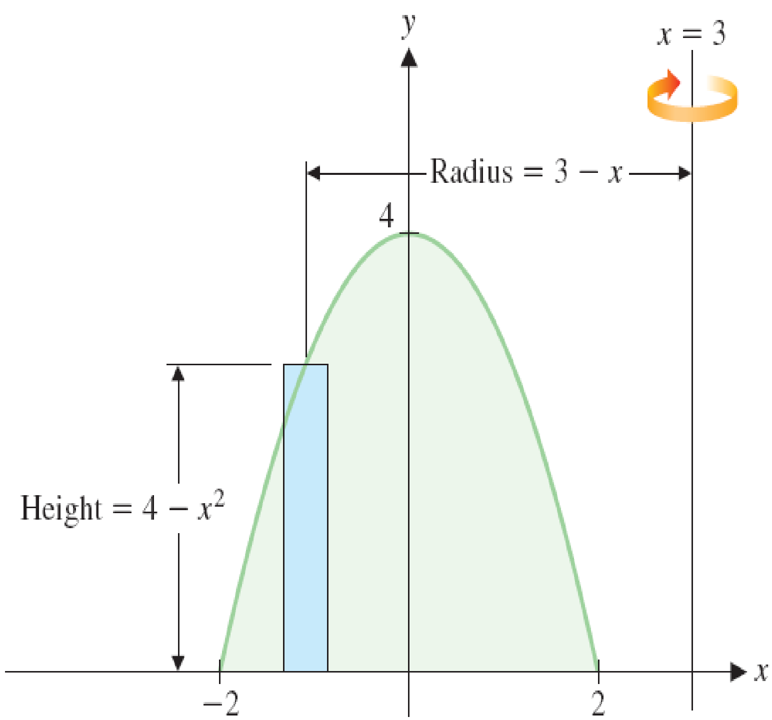


FIGURE 5.31a
Typical rectangle generating a cylindrical shell

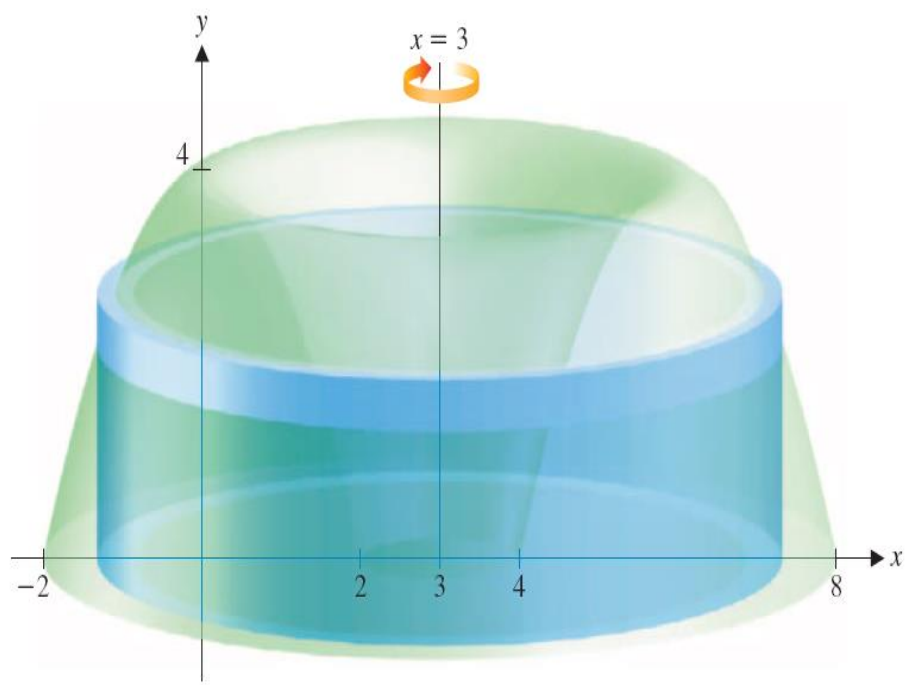


FIGURE 5.31b
Solid of revolution

EXAMPLE 3.3 Computing Volumes Using Shells and Washers

Let R be the region bounded by the graphs of $y = x$, $y = 2 - x$ and $y = 0$. Compute the volume of the solid formed by revolving R about the lines (a) $y = 2$, (b) $y = -1$ and (c) $x = 3$.

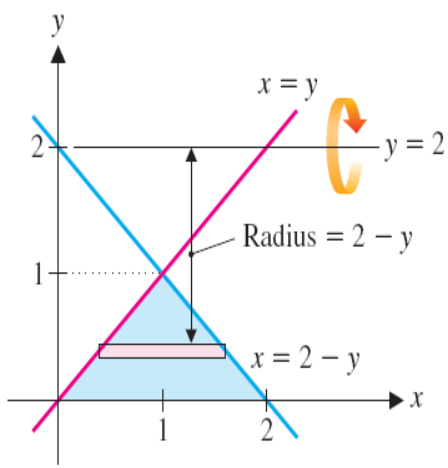


FIGURE 5.32b
Revolve about $y = 2$

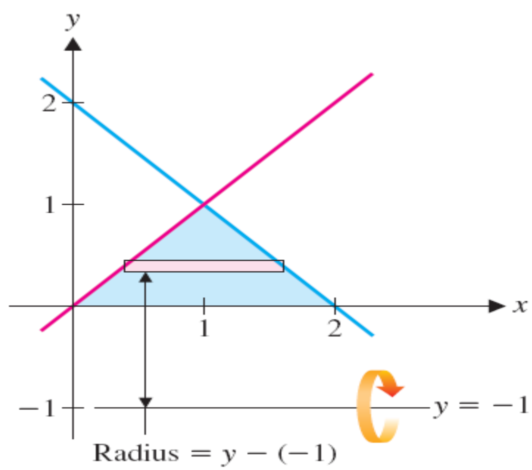


FIGURE 5.32c
Revolve about $y = -1$

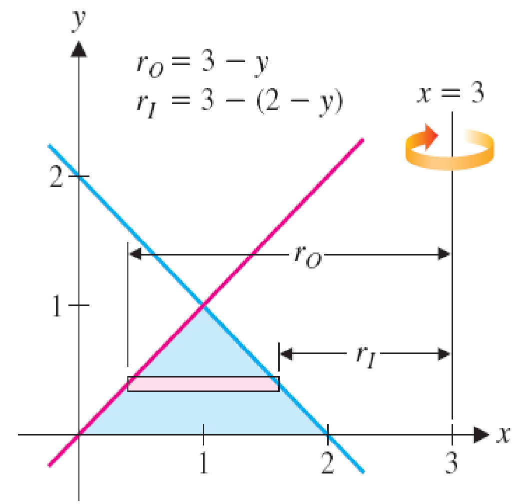


FIGURE 5.32d
Solid of revolution

VOLUME OF A SOLID OF REVOLUTION

- Sketch the region to be revolved.
- Determine the variable of integration (x if the region has a well-defined top and bottom, y if the region has well-defined left and right boundaries).
- Based on the axis of revolution and the variable of integration, determine the method (disks or washers for x-integration about a horizontal axis or y-integration about a vertical axis, shells for x-integration about a vertical axis or y-integration about a horizontal axis).
- Label your picture with the inner and outer radii for disks or washers; label the radius and height for cylindrical shells.
- Set up the integral(s) and evaluate.

Section 5.4 Arc Length and Surface Area (弧长和表面积)

Arc Length (弧长)

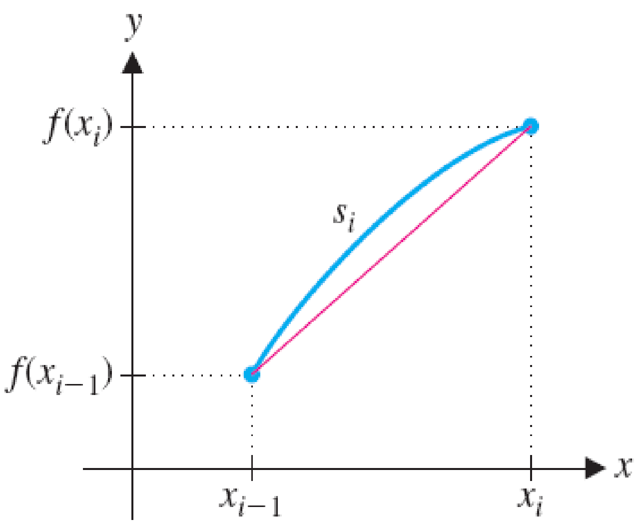


FIGURE 5.35
Straight-line approximation of arc length

We begin by partitioning the interval *[a,b]* into n equal pieces:

$$a = x_0 < x_1 < \cdots < x_n = b$$,

where $x_i - x_{i-1} = \Delta x = \frac{b-a}{n}$, for each *i=1,2,…,n*. From the usual distance formula, we have

$$s_i \approx d\{(x_{i-1}, f(x_{i-1})), (x_i, f(x_i))\}$$

$$= \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}.$$

Since *f* is continuous on the subinterval $[x_{i-1}, x_i]$ and differentiable on (x_{i-1}, x_i). By the mean value theorem, we then have

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}),$$

for some number $c_i \in (x_{i-1}, x_i)$. This gives us the approximation

$$\begin{aligned} s_i &\approx \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2} \\ &= \sqrt{(x_i - x_{i-1})^2 + [f'(c_i)(x_i - x_{i-1})]^2} \\ &= \sqrt{1 + [f'(c_i)]^2}\,\underbrace{(x_i - x_{i-1})}_{\Delta x} = \sqrt{1 + [f'(c_i)]^2}\,\Delta x. \end{aligned}$$

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\,\Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

EXAMPLE 4.1 Using the Arc Length Formula

Find the arc length of the portion of the curve $y = \sin x$ with $0 \leq x \leq \pi$. (We estimated this as 3.79 in our introductory example.)

EXAMPLE 4.2 Estimating an Arc Length

Find the arc length of the portion of the curve $y = x^2$ with $0 \leq x \leq 1$.

EXAMPLE 4.3 A Comparison of Arc Lengths of Power Functions

Find the arc length of the portion of the curve $y = x^4$ with $0 \leq x \leq 1$ and compare to the arc length of the portion of the curve $y = x^2$ on the same interval.

Surface Area (**表面积**)

We begin by partitioning the interval *[a,b]* into n equal pieces:

$$a = x_0 < x_1 < \cdots < x_n = b$$

where $x_i - x_{i-1} = \Delta x = \dfrac{b-a}{n}$, for each *i=1,2,…,n*.

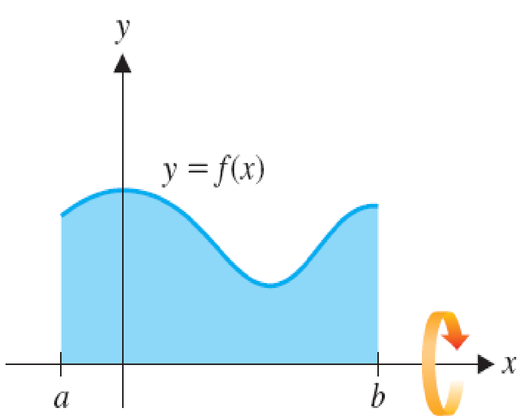


FIGURE 5.41a
Revolve about *x*-axis

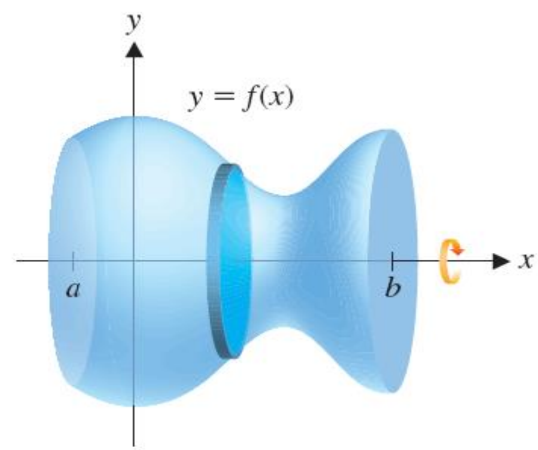


FIGURE 5.41b
Surface of revolution

The surface area of this frustum will give us an approximation to the actual surface area on the interval $[x_{i-1}, x_i]$. First, observe that the slant height of this frustum is

$$L_i = d\{(x_{i-1}, f(x_{i-1})), (x_i, f(x_i))\}$$

$$= \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2},$$

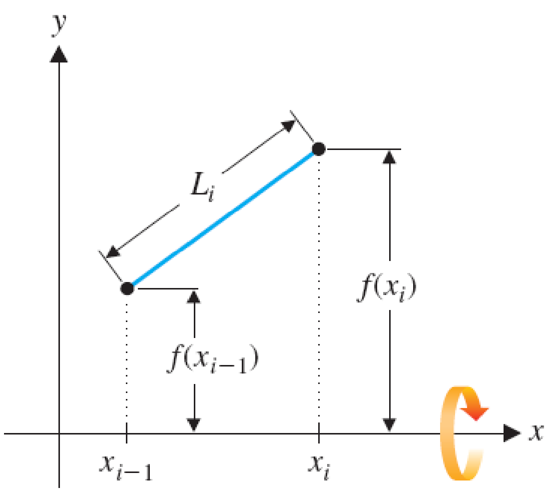


FIGURE 5.42
Revolve about x-axis

We can apply the mean value theorem, to obtain

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}),$$

For some number $c_i \in (x_{i-1}, x_i)$. This give us

$$L_i = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}$$

$$= \sqrt{1 + [f'(c_i)]^2}\,\underbrace{(x_i - x_{i-1})}_{\Delta x}.$$

The surface area S_i of that portion of the surface on the interval $[x_{i-1}, x_i]$ is approximately the surface area of the frustum of the cone,

$$\begin{aligned} S_i &\approx \pi[f(x_i) + f(x_{i-1})]\sqrt{1 + [f'(c_i)]^2}\,\Delta x \\ &\approx 2\pi f(c_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x, \end{aligned}$$

Since if $\triangle x$ is small, $f(x_i) + f(x_{i-1}) \approx 2f(c_i)$.

The total surface area S,

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(c_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x$$

$$= \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

SURFACE AREA OF A SOLID OF REVOLUTION

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx, \tag{4.3}$$

EXAMPLE 4.5 Using the Surface Area Formula

Find the surface area of the surface generated by revolving $y = x^4$, for $0 \le x \le 1$, about the x-axis.

Solution Using the surface area formula (4.3), we have

$$S = \int_0^1 2\pi x^4 \sqrt{1 + (4x^3)^2}\, dx = \int_0^1 2\pi x^4 \sqrt{1 + 16x^6}\, dx \approx 3.4365,$$

where we have used a numerical method to approximate the value of the integral.

Section 5.7 Probability (概率)

Relative Frequency Method(概率的频率解释)

Let *E* be an outcome of an experiment.

If the experiment is performed many times, *P(E)* is the relative frequency of *E*.

P(E) is the percentage of times *E* occurs in many repetitions of the experiment.

Use sampled or historical data to calculate probabilities.

Example

Suppose that of *1000* randomly selected consumers, *140* preferred brand *X*.

The probability of randomly picking a person who prefers brand *X* is

$$140/1000 = 0.14 \text{ or } 14\%.$$

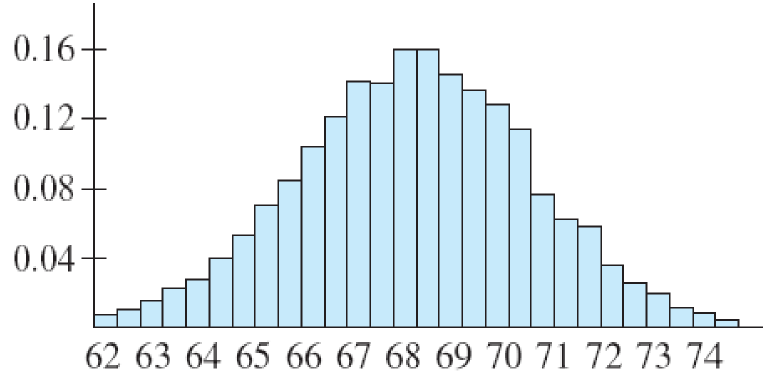


FIGURE 5.61
Histogram for heights

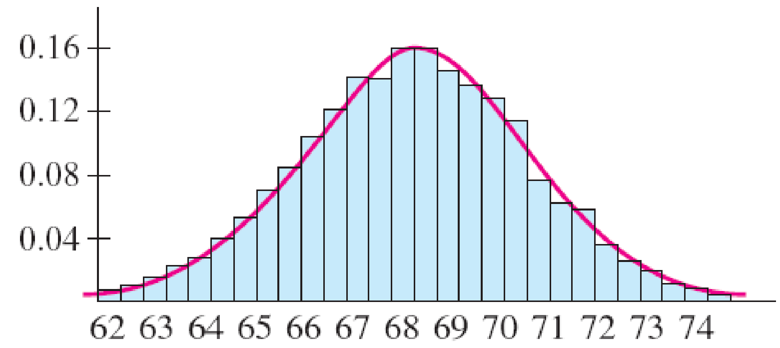


FIGURE 5.62
Probability density function and histogram for heights

We call this limiting function *f(x),* the **probability density function (pdf)** for heights.

DEFINITION 7.1

Suppose that X is a random variable that may assume any value x with $a \leq x \leq b$. A probability density function for X is a function $f(x)$ satisfying

(i) $f(x) \geq 0$ for $a \leq x \leq b$. Probability density functions are never negative.

and

(ii) $\int_a^b f(x)\,dx = 1.$ The total probability is 1.

The probability that the (observed) value of X falls between c and d is given by the area under the graph of the pdf on that interval. That is,

$$P(c \leq X \leq d) = \int_c^d f(x)\,dx.$$

Probability corresponds to area under the curve.

EXAMPLE 7.1 Verifying That a Function Is a pdf on an Interval

Show that $f(x) = 3x^2$ defines a pdf on the interval [0, 1] by verifying properties (i) and (ii) of Definition 7.1.

EXAMPLE 7.3 Computing Probability with an Exponential pdf

Suppose that the lifetime in years of a certain brand of lightbulb is exponentially distributed with pdf $f(x) = 4e^{-4x}$. Find the probability that a given lightbulb lasts 3 months or less.

EXAMPLE 7.4 Determining the Coefficient of a pdf

Suppose that the pdf for a random variable has the form $f(x) = ce^{-3x}$ for some constant c, with $0 \le x \le 1$. Find the value of c that makes this a pdf.

Given a *pdf*, it is possible to compute various statistics to summarize the properties of the random variable, The most common statistic is the ***mean***, the best-known measure of average value.

DEFINITION 7.2

The **mean** μ of a random variable with pdf $f(x)$ on the interval $[a, b]$ is given by

$$\mu = \int_a^b x f(x)\, dx. \tag{7.2}$$

An alternative measurement of average is the ***median***, the *x*-value that divides the probability in half. That is, half of all values of the random variable lie at or below the median and half lie at or above the median.

EXAMPLE 7.5 Finding the Mean Age and Median Age of a Group of Cells

Suppose that the age in days of a type of single-celled organism has pdf $f(x) = (\ln 2)e^{-kx}$, where $k = \frac{1}{2}\ln 2$. The domain is $0 \le x \le 2$. (The assumption here is that upon reaching an age of 2 days, each cell divides into two daughter cells.) Find (a) the mean age of the cells, (b) the proportion of cells that are younger than the mean and (c) the median age of the cells.