2022-23 First Semester
MATH1063 Linear Algebra II (1003)

Assignment 4

Due Date: **24/Mar/2023 (Friday), 09:00 in tutorial class.**

- Write down your **CHN name** and **student number**. Write neatly on **A4-sized** paper (*staple if necessary*) and **show your steps**.
- **Late submissions or answers without steps won't be graded.**

---

1. Let $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$ be the inner product on $\mathbb{R}^n$. Prove that for any $n \times n$ matrix A,

   (a) $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^T \mathbf{y} \rangle$;

   (b) $\langle A^T A\mathbf{x}, \mathbf{x} \rangle = \|A\mathbf{x}\|^2$.

2. Show that $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is orthogonal to $N(A^T)$.

3. Find the projection of the vector $\mathbf{v} = (2, 7, 10)^T$ onto the subspace $S = \text{Span}\{(1, 0, 1)^T, (0, 1, 1)^T\}$.

4. Find the least-square solution $\hat{\mathbf{x}}$ of each of the following $A\mathbf{x} = \mathbf{b}$ systems:

   (a) $\begin{cases} x_1 & +x_2 & = 3 \\ 2x_1 & -3x_2 & = 1 \\ 0x_1 & +0x_2 & = 2 \end{cases}$. (b) $\begin{cases} x_1 & +x_2 & +x_3 & = 4 \\ -x_1 & +x_2 & +x_3 & = 0 \\ & -x_2 & +x_3 & = 1 \\ x_1 & & +x_3 & = 2 \end{cases}$.

5. Find all least squares solutions $\hat{\mathbf{x}}$ for the following $A\mathbf{x} = \mathbf{b}$: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

   (Remark: The least-square solution $\hat{\mathbf{x}}$ is not unique now. Comparing to Thm 5.3.5, what changes? Is $A\hat{\mathbf{x}}$ unique?)

6. Let $\mathbf{x}$ and $\mathbf{y}$ be linearly independent vectors in $\mathbb{R}^n$ and let $S = \text{Span}(\mathbf{x}, \mathbf{y})$. We can use $\mathbf{x}$ and $\mathbf{y}$ to define a matrix A by setting

   $$A = \mathbf{x}\mathbf{y}^T + \mathbf{y}\mathbf{x}^T$$

   (a) Show that A is symmetric.

   (b) Show that $N(A) = S^{\perp}$.

   (c) Show that the rank of A must be 2.