Chapter 1 Matrices and System of Equations

Section 1.4 Matrix Algebra

Theorem (Addition) For any $m \times n$ matrices A, B, and C, the following statements are true.

$$\begin{aligned} A + B &= B + A \\ (A + B) + C &= A + (B + C) \end{aligned}$$

Definition (Zero matrix) An $m \times n$ matrix whose entries are all 0 is called a *zero matrix*, denoted by $O_{m,n}$, $O_{m\times n}$ or simply O.
If $\mathbf{a}$ is a column vector with all entries to be zero, it is called a zero column vector, denoted by $\mathbf{0}$.

The zero matrix is the identity element for matrix addition. For example,

$$\begin{pmatrix} 3 & 1 & 2 \\ 5 & 0 & 9 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 5 & 0 & 9 \end{pmatrix}$$

Definition (Additive inverse) If $A = (a_{ij})$ is an $m \times n$ matrix, then the *additive inverse* of A is $(-1)A$.

Example The additive inverse of $\begin{pmatrix} 3 & 1 \\ 2 & 5 \\ 0 & 9 \end{pmatrix}$ is $\begin{pmatrix} -3 & -1 \\ -2 & -5 \\ -0 & -9 \end{pmatrix}$, because

$$\begin{pmatrix} 3 & 1 \\ 2 & 5 \\ 0 & 9 \end{pmatrix} + \begin{pmatrix} -3 & -1 \\ -2 & -5 \\ -0 & -9 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Notation Denote the additive inverse of A by $-A$. If B is a matrix, denote $B + (-A)$ by $B - A$.

Here are some properties for any $m \times n$ matrices A, B and C.

	Addition
Closure	$A + B$ is an $m \times n$ matrix
Associativity	$A + (B + C) = (A + B) + C$
Commutativity	$A + B = B + A$
Existence of an identity element	There is O such that $A + O = A$.
Existence of inverse elements	$A + (-A) = O$

Theorem (Scalar Multiplication) Each of the following statements are true for any scalars α and β and for any matrices A, B for which *the indicated operations are defined*.

$$\begin{aligned}
(\alpha\beta)A &= \alpha(\beta A)\\
\alpha(A+B) &= \alpha A+\alpha B\\
(\alpha+\beta)A &= \alpha A+\beta A\\
\alpha(AB) &= (\alpha A)B = A(\alpha B)
\end{aligned}$$

Theorem (Multiplication and distributive law) Each of the following statements are true for any matrices A, B, and C for which *the indicated operations are defined*.

$$\begin{aligned} AB &\neq BA, \quad \text{in general} \\ (AB)C &= A(BC) \\ A(B+C) &= AB + AC \\ (A+B)C &= AC + BC \end{aligned}$$

Example $(A+B)(A-B) = A(A-B) + B(A-B) = AA - AB + BA - BB = A^2 - AB + BA - B^2$, which might not be $A^2 - B^2$.

Definition (Power) For a square matrix A, A^k is the product of k A's, i.e. $A^k = AAAA \cdots A$ (k copies).

Definition (Identity (Unit) matrix) An $n \times n$ matrix $A = (a_{ij})$ is called the *identity matrix* of order n if

$$a_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

and denoted by $A = I_n$, or simply $A = I$.

Example $I_1 = (1)$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, etc.

Theorem For any $m \times n$ matrix A, $AI_n = A$ and $I_m A = A$.

Here are some properties for any $n \times n$ matrices A, B and C

	Addition	Multiplication
Closure	$A + B$ is an $n \times n$ matrix	AB is an $n \times n$ matrix
Associativity	$A+(B+C)=(A+B)+C$	$A(BC) = (AB)C$
Commutativity	$A + B = B + A$	No!
Existence of an identity element	$A + O = A$	$AI = A$ $IA = A$
Existence of inverse elements	$A + (-A) = O$	?
Distributivity	$A(B + C) = AB + AC$ $(A + B)C = AC + BC$	

Definition (Nonsingular matrix / Invertible matrix) A square matrix A of order n is said to be *nonsingular/invertible* if there exists a matrix B such that $AB = BA = I_n$. The matrix B is said to be a *multiplicative inverse* of A, and denote $B = A^{-1}$.

Uniqueness of inverse If matrices B and C are (multiplicative) inverses of a matrix A, then $B = C$.

Proof If B and C are both multiplicative inverses of A, then

$$B = BI = B(AC) = (BA)C = IC = C.$$

Thus, a matrix can have at most one multiplicative inverse.

Example Are the matrices $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ multiplicative inverse of each other?

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Solution Since $AB = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $BA = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, A and B are multiplicative inverse of each other.

It is not necessary that a square matrix has its inverse.

Example Show that $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ has no inverse.

Solution Suppose A has an inverse $B = (b_{ij})$. Then

$$I = BA = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b_{11} & 0 \\ b_{21} & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is contradictory to have $I \neq I$. Hence the assumption that A has an inverse is false.

Definition (Singular) A square matrix is said to be *singular* if it does not have a multiplicative inverse.

Theorem If A and B are nonsingular matrices, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$$

Hence, AB is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.

Exercise Prove by the induction that, if $A_1, \cdots, A_n$ are all nonsingular with the same size, then the product $A_1A_2\cdots A_n$ is nonsingular and

$$(A_1A_2\cdots A_n)^{-1} = A_n^{-1}\cdots A_2^{-1}A_1^{-1}.$$

(This n is not the size of the matrix.)

Theorem If A is a nonsingular matrix and $r \neq 0$ is a scalar, then rA and A^T are nonsingular. Furthermore,

- $(rA)^{-1} = \frac{1}{r}A^{-1}$;
- $(A^T)^{-1} = (A^{-1})^T$.

Proof Exercise.