2023-24 First Semester

MATH2023 Ordinary and Partial Differential Equations (1002)

Assignment 1 Suggested Solution

1. (a) Second Order, linear, nonhomogeneous.
   (b) Third Order, non-linear, homogeneous.
   (c) First Order, non-linear, homogeneous.
   (d) Second Order, non-linear, nonhomogeneous.
   (e) Third Order, linear, nonhomogeneous.

2. We have

$$y'(t) = -(\sin t)\ln\cos t + t\cos t, \quad y'' = -(\cos t)\ln\cos t + \frac{1}{\cos t} - t\sin t$$

Substituting $y(t) = (\cos t)\ln\cos t + t\sin t$ into the equation, then

$$-(\cos t)\ln\cos t + \frac{1}{\cos t} - t\sin t + (\cos t)\ln\cos t + t\sin t = \frac{1}{\cos t} = \sec t$$

Thus, the function satisfies the equation and it is a solution.

3. (a) $y' + \frac{1}{t}y = 3\cos 2t,\ t > 0$ is in **standard form**, hence the **integrating factor** is

$$u(t) = e^{\int \frac{1}{t}\ \mathrm{d}t} = e^{\ln|t|} = t.$$

Multiply $u(t)$ on both sides of the standard form:

$$ty' + y = 3t\cos 2t$$

$$\text{i.e.}\quad \frac{\mathrm{d}}{\mathrm{d}t}(ty) = 3t\cos 2t$$

Integrating on both sides with respect to t,

$$ty = 3\int t\cos 2t\mathrm{d}t = 3\left(\frac{1}{2}t\sin 2t - \int \frac{1}{2}\sin 2t\mathrm{d}t\right) = \frac{3}{2}\left(t\sin 2t + \frac{1}{2}\cos 2t\right) + C.$$

Divide $u(t) = t$ on both sides to obtain the **general solution**:

$$y(t) = \frac{3}{2}\sin 2t + \frac{3}{4t}\cos 2t + \frac{C}{t}, t > 0.$$

(b) **Standard form:**

$$y' + \frac{4}{t}y = \frac{e^{-t}}{t^3}, \qquad t \neq 0$$

Integrating factor:

$$\mu(t) = e^{\int \frac{4}{t}\mathrm{d}t} = t^4$$

Multiplying $\mu(t)$ to the standard form: $(t^4 y)' = te^{-t}$

$$\begin{aligned} t^4 y &= \int te^{-t}\mathrm{d}t + C \\ t^4 y &= -(t+1)e^{-t} + C \end{aligned}$$

General solution:

$$y(t) = -t^{-4}\left[(t+1)e^{-t} + C\right], \quad C \in \mathbb{R}, t \neq 0$$

4. (a) $y' - y = 2te^{2t}, \quad y(0) = -1$
Integrating factor: $u(x) = e^{\int -1\mathrm{d}t} = e^{-t}$,

$$\begin{aligned} e^{-t}y' - e^{-t}y &= 2te^t \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t}y\right) &= 2te^t \\ e^{-t}y &= 2\int te^t\mathrm{d}t = 2(te^t - \int e^t\mathrm{d}t) \\ &= 2e^t(t-1) + C, \quad C \in \mathbb{R}. \end{aligned}$$

General solution: $y(t) = 2e^{2t}(t-1) + Ce^t, \quad C \in \mathbb{R}$.
Plug in the **initial condition**:

$$y(0) = 2(-1) + C = -1, \quad \to \quad C = 1.$$

Hence the solution to this IVP is

$$y = 2e^{2t}(t-1) + e^t.$$

(b) **Standard form:** $y' + \frac{2}{t}y = \frac{\sin t}{t}$
Integrating factor: $u(x) = \exp(\int \frac{2}{t}\mathrm{d}t) = t^2$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(t^2 y\right) &= t\sin t \\ t^2 y &= t\cos t + \sin t + C \end{aligned}$$

General solution: $y(t) = t^{-1}\cos t + t^{-2}\sin t + Ct^{-2}, \quad C \in \mathbb{R}$.
Plug in the **initial condition**:

$$y(\pi/2) = \frac{4}{\pi^2} + C\frac{4}{\pi^2} = 1, \quad \to \quad C = \frac{\pi^2}{4} - 1.$$

Solution to this IVP is

$$y = t^{-1}\cos t + t^{-2}\sin t + (\frac{\pi^2}{4} - 1)t^{-2}.$$

(c) This is a **separable equation**, since

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1-2x}{y}, \quad y \neq 0 \\ \int y\ \mathrm{d}y &= \int (1-2x)\ \mathrm{d}x \\ \frac{1}{2}y^2 &= x - x^2 + C\end{aligned}$$

The solution is

$$y(x) = \pm\sqrt{2x-2x^2+C}, \quad y \neq 0.$$

Plug in the **initial condition** and we have

$$y(1) = -2, \quad \rightarrow \quad y(1) = -\sqrt{2-2+C} = -2, \quad \rightarrow \quad C = 4.$$

$$y \neq 0 \quad \rightarrow \quad 2x - 2x^2 + 4 \neq 0 \quad \rightarrow \quad x \neq -1, 2.$$

Due to the negative initial value, we discard the positive solution.
Solution to the IVP:

$$y(x) = -\sqrt{2x-2x^2+4}, \quad -1 < x < 2.$$

(d) This is a **separable equation**, since

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3x^2}{3y^2-4} \\ \int 3y^2 - 4\ \mathrm{d}y &= \int 3x^2\ \mathrm{d}x + C \\ y^3 - 4y &= x^3 + C\end{aligned}$$

The initial condition $y(0) = 1$ yields $C = -3$.