Caculus II Math 1038 (1002&1003)

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Midterm exam: 13 Apr Thu 6-7:30pm

Week 7: finish Ch13 Vector-valued function and start Ch14 Partial differentiation

1. Chapter 10 Parametric Equations and Polar Coordinate

   We have not learnt this Chapter, because you have learnt most of them in high school already. This chapter is about parametric equations and polar coordiante in two-dimensional space, which is very important for you to learn the same things in three-dimensional space. So I recommend all of you to review Chapter 10, and do some exercise including these concepts:

   (a) parametric curve
   (b) eliminating the parameter: find the Cartesian equation in x and y
   (c) initial point, terminal point
   (d) Calculus with parametric curvers:
      i. tangent
      ii. area under the curve: change of variables
      iii. arc length
   (e) Polar coordinate (r, θ)
      i. reparametrization $x = r\cos\theta$ and $y = r\sin\theta$; $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$
      ii. Polar curves
      iii. arc length
      iv. area under the curve: change of variables
   (f) Conic sections: a set of points in a plane
      i. parabolas (equidistant from a fixed point F focus), directrix, vertex
      ii. ellipses: two fixed points F_1 and F_2, foci (p.l., focus. s.l.), major axis, minor axis
      iii. hyperbolas: asymptotes

2. Motion

   (a) Velocity vector

   $$\overrightarrow{v}(t) = \lim_{h\to 0} \frac{\overrightarrow{r}(t+h) - \overrightarrow{r}(t)}{h} = \overrightarrow{r}'(t)$$

   The velocity is also the tangent vector and points in the direction of the tangent line,

   (b) Speed: the magnitude of the velocity vector that is $|\overrightarrow{v}(t)|$.

   (c) Acceleration

   $$\overrightarrow{a}(t) = \lim_{h\to 0} \frac{\overrightarrow{v}(t+h) - \overrightarrow{v}(t)}{h} = \overrightarrow{v}'(t) = \overrightarrow{r}''(t)$$

   (d) Integrals

   $$\overrightarrow{v}(t) = \overrightarrow{v}(t_0) + \int_{t_0}^{t} \overrightarrow{a}(u)du, \qquad \overrightarrow{r}(t) = \overrightarrow{r}(t_0) + \int_{t_0}^{t} \overrightarrow{v}(u)du$$

3. Chapter 14 Partial Derivative

   This chapter is about the calculus of functions of two variabes

   $$z = f(x, y)$$

   with independent variables x, y and dependent variable: z.

   or more

   $$f(x, y, z, ...)$$

(a) Domain (x and y) $D \subset \mathbb{R}^2$ or $D \subset \mathbb{R}^3$: Domain D: **a set** of points at which the function is continous

(b) Range (z)

(c) Close set: contains all the boundary points, e.g. a close disk $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq r^2\}$

(d) Open set: contains none of the boundary points, e.g. an open disk $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < r^2\}$

(e) Points:

  i. Interior point

  ii. Boundary point

(f) Function $z = f(x, y)$,

(g) sketch the domain

(h) graph of z, a surface

(i) contour plot: level curves $f(x, y) = c$: tightness of the curves corresponds to the change of speed of the function.

(j) A **polynomials function** of two variables: a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers.

(k) A **rational function** is a ratio of two polynomials,e.g.

$$f_1(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \qquad f_2(x, y) = \frac{xy}{x^2 + y^2}, \qquad f_3(x, y) = \frac{2xy + 1}{x^2 + y^4},$$

4. Limit

(a) Paths of independent variables

$$\lim_{x \to 0} f(x)$$

Path of $x \to 0$ is along x-axis

$$\lim_{(x,y) \to (0,0)} f(x, y)$$

Inifinitely many paths of $(x, y) \to (0, 0)$.

(b) $\epsilon - \delta$ definition of limit

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

for all $\epsilon > 0$, there is a corresponding $\delta > 0$, such that
if $(x, y) \in D$ and $0 < |(x, y) - (a, b)| = \sqrt{(x - a)^2 + (y - b)^2} < \delta$, then

$$|f(x, y) - L| < \epsilon$$

e.g.

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

and

$$\lim_{(x,y) \to (0,0)} f(x, y) = 0$$

Remark: If the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) appoaches (a, b). If approaches to origin $(0, 0)$, the paths you can consider: x-axis, y-axis, a line e.g. $y = x$, a curve e.g. $y = x^2$ or $x = y^2$.

(c) Remember three important examples, for which limits do **NOT** exist

$$f_1(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \qquad f_2(x, y) = \frac{xy}{x^2 + y^2}, \qquad f_3(x, y) = \frac{xy^2}{x^2 + y^4},$$

To prove the limit does not exist, you can find two paths which results in **two different limits**.

(d) Find the limit **if it exists**.

  i. Direct substitution

   ii. Squeeze Theorem

   (e) Limit Laws for functions of two variables.

5. Continuity

   Definition: f is continuous on D if f is continuous at every point (a, b) in D.

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

   (a) All polynomials are continuous on $\mathbb{R}^2$

   (b) Any rational functions is continuous on its domain.

   (c) Composite function of continous functions is alos a continuous function.

6. Limit for functions of three or more variables

$$\lim_{\overrightarrow{x}\to\overrightarrow{a}} f(\overrightarrow{x}) = L$$

   for every $\epsilon > 0$ there is a corresponding number $\delta > 0$ s.t.
   if $x \in D$ and $0 < |\overrightarrow{x} - \overrightarrow{a}| < \delta$ then $|f(x) - L| < \epsilon$

7. Partial derivative

   (a) with respect to x at (a, b),

$$f_x(a, b)$$

   (b) other notations:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = D_x f$$

   if $z = f(x, y)$, we can also use

$$\frac{\partial z}{\partial x}$$

   (c) Higher derivatives

$$f_{xx}(x, y) = f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2}{\partial x^2} f(x, y) = D_{xx} f$$

   (d) Partial differential equation

   i. Laplace's equation

   ii. wave equation