Chapter 2 Determinants

Section 2.3 Cramer's Rule

Definition (Adjoint) Let A be an $n \times n$ matrix. We define a new matrix called the **adjoint** of A by

$$\operatorname{adj}(A) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^T .$$

Thus, to form the adjoint, we replace each entry by its cofactor and then transpose the resulting matrix.

Theorem Let A be a nonsingular $n \times n$ matrix, that is, $\det(A) \neq 0$. Then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Proof By a lemma in Sc2.2,

$$A(\operatorname{adj}(A)) = \det(A) I_n.$$

If A is nonsingular, $\det(A) \neq 0$, and we may write

$$A \left(\frac{1}{\det(A)} \operatorname{adj}(A) \right) = I_n.$$

Thus

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A), \quad \text{when } \det(A) \neq 0.$$

Example Find the inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $\det(A) \neq 0$.

Solution

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example re-visit

Let $A = \begin{bmatrix} 0 & 2 & -8 \\ 1 & -2 & 1 \\ -4 & 5 & 9 \end{bmatrix}$. Find A^{-1} using adjoint of A. Hence solve

$$\begin{bmatrix} 0 & 2 & -8 \\ 1 & -2 & 1 \\ -4 & 5 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 8 \\ 0 \\ 9 \end{pmatrix}$$

Answer:

$$A^{-1} = \begin{bmatrix} 23/2 & 29 & 7 \\ 13/2 & 16 & 4 \\ 3/2 & 4 & 1 \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}b = \begin{bmatrix} 23/2 & 29 & 7 \\ 13/2 & 16 & 4 \\ 3/2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 29 \\ 16 \\ 3 \end{bmatrix}$$

Theorem (Cramer's Rule) Let A be an nonsingular $n \times n$ matrix, and let $\mathbf{b} \in \mathbf{R}^n$. Let A_i be the matrix obtained by replacing the i-th column of A by $\mathbf{b}$. If $\mathbf{x}$ is the unique solution of $A\mathbf{x} = \mathbf{b}$, then

$$x_i = \frac{\det(A_i)}{\det(A)}, \quad \text{for } i = 1, \cdots, n.$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{pmatrix} \xrightarrow[\text{by } \mathbf{b}]{\text{Replace } i\text{-th col.}} A_i = \begin{pmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{pmatrix}$$

Proof Since

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)}(\operatorname{adj}(A))\mathbf{b},$$

it follows that

$$x_i = \frac{b_1 A_{1i} + b_2 A_{2i} + \cdots + b_n A_{ni}}{\det(A)} = \frac{\det(A_i)}{\det(A)}.$$

Example Consider

$$\begin{pmatrix} 0 & 2 & -8 \\ 1 & -2 & 1 \\ -4 & 5 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ -9 \end{pmatrix}$$

Solution $\det(A) = -2$.

$$|A_1| = \begin{vmatrix} 8 & 2 & -8 \\ 0 & -2 & 1 \\ -9 & 5 & 9 \end{vmatrix} = -58, \quad |A_2| = \begin{vmatrix} 0 & 8 & -8 \\ 1 & 0 & 1 \\ -4 & -9 & 9 \end{vmatrix} = -32, \quad |A_3| = \begin{vmatrix} 0 & 2 & 8 \\ 1 & -2 & 0 \\ -4 & 5 & -9 \end{vmatrix} = -6$$

We obtain $x = \dfrac{|A_1|}{|A|} = 29, \ y = \dfrac{|A_2|}{|A|} = 16, \ z = \dfrac{|A_3|}{|A|} = 3.$

Exercise Use Cramer's rule to solve

$$\begin{array}{llll} x_1 & +2x_2 & +x_3 & = 5 \\ 2x_1 & +2x_2 & +x_3 & = 6 \\ x_1 & +2x_2 & +3x_3 & = 9 \end{array}$$

Answers: 1,1,2.