2023-24 First Semester
MATH2043 Ordinary Differential Equations (1002)

Assignment 10 Suggested Solutions

1. (a) Partition the time interval into equal pieces with step $h = 0.1$ and denote $t_i = t_0 + i * h$. Let w_n be an approximation of $y(t_n)$ by the Euler's method

$$\begin{aligned} w_1 &= y_0 + (3y_0 - 4t_0)h = 1.3 \\ w_2 &= w_1 + (3w_1 - 4t_1)h = 1.65 \end{aligned}$$

(b) Let $\hat{w}_n$ be an approximation of $y(t_n)$ by the backward Euler's method

$$\begin{aligned} \hat{w}_1 &= y_0 + (3\hat{w}_1 - 4t_1)h \quad \to \quad \hat{w}_1 = (y_0 - 4h * t_1)/(1-3h) \approx 1.3714 \\ \hat{w}_2 &= \hat{w}_1 + (3\hat{w}_2 - 4t_2)h \quad \to \quad \hat{w}_2 = (\hat{w}_1 - 4h * t_2)/(1-3h) \approx 1.8449 \end{aligned}$$

For this IVP, the exact solution is

$$y(t) = \frac{5}{9}e^{3t} + \frac{12t+4}{9},$$

where $y(0.1) = 1.3277$ and $y(0.2) = 1.7234$.

2. (a) Let w_n be an approximation of $y(t_n)$ by the Euler's method

$$\begin{aligned} w_1 &= y_0 + (t_0^2 + y_0^2)h = 1.1 \\ w_2 &= w_1 + (t_1^2 + w_1^2)h = 1.222 \end{aligned}$$

(b) Let $\hat{w}_n$ be an approximation of $y(t_n)$ by the backward Euler's method

$$\begin{aligned} \hat{w}_1 &= y_0 + (t_1^2 + \hat{w}_1^2)h \quad \to \quad h\hat{w}_1^2 - \hat{w}_1 + (y_0 + t_1^2 h) = 0, \hat{w}_1 \approx 1.1283 \\ \hat{w}_2 &= \hat{w}_1 + (t_2^2 + \hat{w}_2^2)h \quad \to \quad h\hat{w}_2^2 - \hat{w}_2 + (\hat{w}_1 + t_2^2 h) = 0, \hat{w}_2 \approx 1.3018 \end{aligned}$$

For this IVP, a solution in elementary functions does not exist.

3.

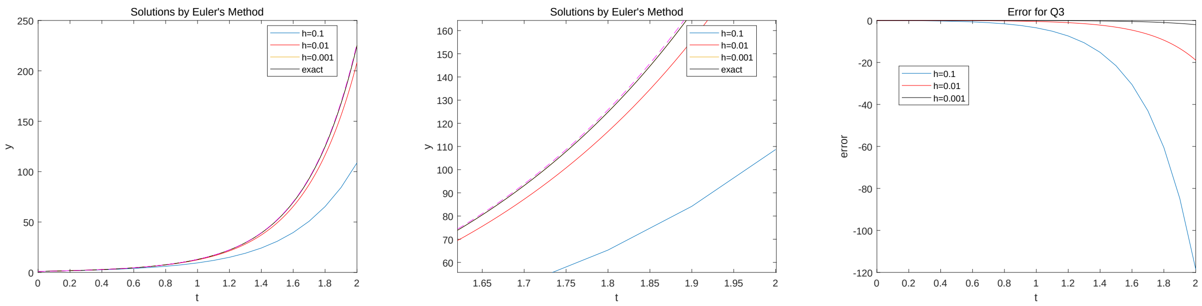

Figure 1: Left to right: Q3, Q3-zoom in, and Q3-error

$h=0.1$	$h=0.01$	$h=0.001$
0	0	0
-0.0276993375422239	-0.00330134901771162	-0.000336715671941157
-0.0733993335502827	-0.00889309206727473	-0.000908833151440636
-0.145890617309416	-0.0179670222040480	-0.00183978155726416
-0.257787179298082	-0.0322661837430394	-0.00331051718346398
-0.427088372410035	-0.0543239175727615	-0.00558465971286193
-0.679354702451636	-0.0878024224262246	-0.00904417225107235
-1.05073234031536	-0.137971111358837	-0.0142398851151722
-1.59214953924533	-0.212381013450777	-0.0219628811044235
-2.37512908437380	-0.321813675154502	-0.0333451532972138
-3.49982652127093	-0.481613856850784	-0.0500012776299990
-5.10613054460438	-0.713558118332443	-0.0742274748754923
-7.38897184510943	-1.04847072670922	-0.109280872759729
-10.6194102479471	-1.52988052821797	-0.159770713848395
-15.1736484355179	-2.21912622713886	-0.232205648981235
-21.5729101591283	-3.20247493796082	-0.335758436106040
-30.5381981113428	-4.60103649156085	-0.483333161818521
-43.0654156144935	-6.58455660048573	-0.693053044826343
-60.5283384734610	-9.39058702865911	-0.990332452127049
-84.8196510719529	-13.3511035741220	-1.41075976431381
-118.543975413293	-18.9294323761912	-2.00410478611155

4.

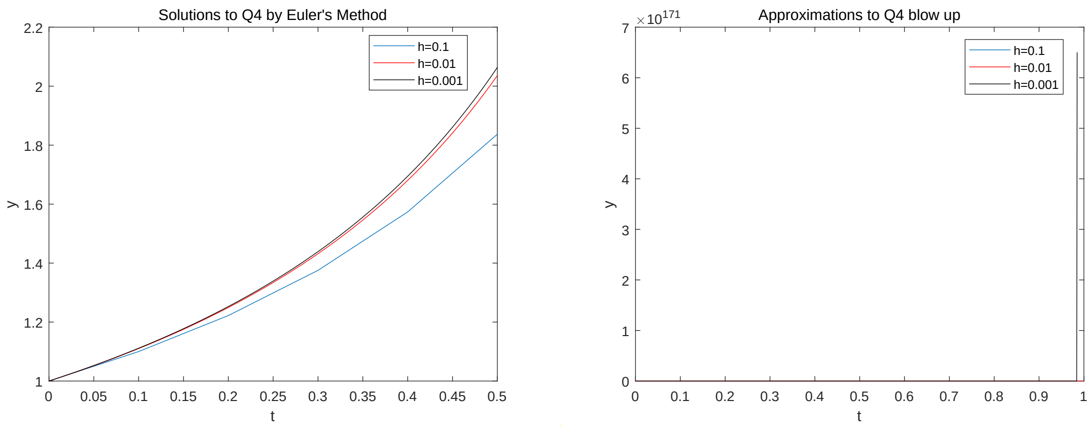

Figure 2: Left to right: Q4 with T=0.5, and Q4 with T=1

5. (a) $y(t) = (y_0 - t_0)e^{t-t_0} + t$.

   (b) By the Euler's method,

$$\begin{aligned} \hat{y}_k &= \hat{y}_{k-1} + (1 - t_{k-1} + \hat{y}_{k-1})h \\ &= (1+h)\hat{y}_{k-1} + (1 - t_{k-1})h, \qquad k = 1, 2, \cdots \end{aligned}$$

   where $\hat{y}_0 = y_0$.

(c) From part (b), we have

$$
\begin{aligned}
\hat{y}_k &= (1+h)(\hat{y}_{k-1} - t_{k-1}) + t_k \\
&= (1+h)\left[((1+h)(\hat{y}_{k-2} - t_{k-2}) + t_{k-1}) - t_{k-1}\right] + t_k \\
&= (1+h)^2(\hat{y}_{k-2} - t_{k-2}) + t_k \\
&= (1+h)^k(y_0 - t_0) + t_k
\end{aligned}
$$

Since when $k = 1$, $\hat{y}_1 = (1+h)(y_0 - t_0) + t_1$.

(d)

$$
\begin{aligned}
\lim_{n\to\infty} \hat{y}_n &= \lim_{n\to\infty} \left(1 + \frac{t-t_0}{n}\right)^n (y_0 - t_0) + t_n \\
&= \lim_{n\to\infty} \left(1 + \frac{t-t_0}{n}\right)^{\frac{n}{t-t_0}(t-t_0)} (y_0 - t_0) + t \\
&= e^{t-t_0}(y_0 - t_0) + t
\end{aligned}
$$