# Chapter 3 Vector Spaces

## Section 3.6 Row Space and Column Space

Definition (Row space) If $A$ is an $m \times n$ matrix, then the subspace of $\mathbf{R}^{1xn}$ spanned by the row vectors of $A$ is called the **row space** of $A$.

Definition (Column space) If $A$ is an $m \times n$ matrix, then the subspace of $\mathbf{R}^m$ spanned by the column vectors of $A$ is called the **column space** of $A$.

Example Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
The row space of $A$

$$\{a(1,1,0) + b(0,0,1) | a, b \in \mathbf{R}\} = \{(a,a,b) | a, b \in \mathbf{R}\}.$$

The column space of $A$ is

$$\left\{ a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid a, b, c \in \mathbf{R} \right\} = \mathbf{R}^2.$$

Theorem Two row equivalent matrices have the same row space.

Proof If $B$ is row equivalent to $A$, then $B$ can be formed from $A$ by a finite sequence of row operations. Thus, the row vectors of $B$ must be linear combinations of the row vectors of $A$. Consequently, the row space of $B$ must be a subspace of the row space of $A$. Since $A$ is row equivalent to $B$, by the same reasoning, the row space of $A$ is a subspace of the row space of $B$.

Example Find a basis of the row space of $A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{pmatrix}$.

Solution $A \longrightarrow \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$.

$\{(1, -2, 3), (0, 1, 5)\}$ is a basis for the row space of $A$.

Definition (Rank) The **rank of a matrix** $A$, denoted by $\operatorname{rank}(A)$, is the dimension of the row space of $A$.

Example In the previous example, $\operatorname{rank}(A) = 2$.

Definition in Sc3.2 (Null space) Let $A$ be an $m \times n$ matrix. Let $N(A)$ denote the set of all solutions of the homogenous system $A\mathbf{x} = \mathbf{0}$. That is,

$$N(A) = \{\mathbf{x} \in \mathbf{R}^n | A\mathbf{x} = \mathbf{0}\}$$

where $N(A)$ is a subspace of $\mathbf{R}^n$, and is called the **null space** of $A$.

Definition (Nullity) The dimension of the null space of a matrix is called the **nullity** of the matrix.

**Example** Let $A = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{pmatrix}$. Find the nullity of $A$.

**Solution** The reduced row echelon form of $A$ is $\begin{pmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

$$\begin{aligned} N(A) &= \{\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^T : A\mathbf{x} = \mathbf{0}\} \\ &= \left\{ (x_1, x_2, x_3, x_4, x_5)^T : \begin{pmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} -3\alpha - 7\beta \\ -\alpha - 3\beta \\ \alpha \\ \beta \\ 0 \end{pmatrix} \right\} = \left\{ \alpha \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \end{aligned}$$

$\left\{ \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis of $N(A)$. The nullity of $A$ is 2.

Theorem (The Rank-Nullity Theorem) If $A$ is an $m \times n$ matrix, then the rank of $A$ plus the nullity of $A$ equals $n$.

Proof Let $U$ be the reduced row echelon form of $A$. The system $A\mathbf{x} = \mathbf{0}$ is equivalent to the system $U\mathbf{x} = \mathbf{0}$. If $\text{rank}(A) = r$, then $U$ will have $r$ nonzero rows, and consequently the system $U\mathbf{x} = \mathbf{0}$ will involve $r$ lead variables and $n - r$ free variables. The dimension of $N(A)$ will equal the number of free variables, i.e; $\text{rank}(A) + \dim N(A) = n$.

Theorem (Consistency Theorem for Linear Systems) A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of $A$.

Proof Let $\mathbf{a}_i$ be the $i$th column of $A$, and let $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$. Note that $A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \quad + x_n\mathbf{a}_n$. Thus the system $A\mathbf{x} = \mathbf{b}$ is consistent, if and only if there exists $\mathbf{x}' = (x_1', x_2', \cdots, x_n')^T$ such that $\mathbf{b} = x_1'\mathbf{a}_1 + x_2'\mathbf{a}_2 + \quad + x_n'\mathbf{a}_n$, if and only if $\mathbf{b}$ is in the column space of $A$.

Theorem

1. The linear system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbf{R}^n$ if and only if the column vectors of $A$ span $\mathbf{R}^n$.
2. The system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b} \in \mathbf{R}^n$ if and only if the column vectors of $A$ are linearly independent.
3. The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbf{R}^n$ if and only if the column vectors of $A$ form a basis of $\mathbf{R}^n$.

Proof of (1) By the theorem of the previous slide, the system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of $A$. It follows that $A\mathbf{x} = \mathbf{b}$ will be consistent for every $\mathbf{b} \in \mathbf{R}^n$ if and only if the column vectors of $A$ span $\mathbf{R}^n$ .

Proof of (2) If $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b}$, then, in particular, the system $A\mathbf{x} = \mathbf{0}$ can have only the trivial solution, and hence the column vectors of $A$ must be linearly independent.
Conversely, if the column vectors of $A$ are linearly independent, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Now, if $\mathbf{x}_1$ and $\mathbf{x}_2$ were both solutions of $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_2$ would be a solution of $A\mathbf{x} = \mathbf{0}$,

$$A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

It follows that $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$, and hence $\mathbf{x}_1$ must equal $\mathbf{x}_2$.

Proof of (3) It follows from (1), (2) and the definition of a basis.

Theorem For any matrix $A$, the dimension of the row space of $A$ equals the dimension of the column space of $A$.

Proof If $A$ is an $m \times n$ matrix of rank $r$, the row echelon form $U$ of $A$ will have $r$ leading 1's. The columns of $U$ corresponding to the lead 1's will be linearly independent.

Let $U_L$ denote the matrix obtained from $U$ by deleting all the columns corresponding to the free variables. Delete the same columns from $A$ and denote the new matrix by $A_L$. The matrices $A_L$ and $U_L$ are row equivalent. Thus, if $\mathbf{x}$ is a solution of $A_L\mathbf{x} = \mathbf{0}$, then x must also be a solution of $U_L\mathbf{x} = \mathbf{0}$. Since the columns of $U_L$ are linearly independent, $\mathbf{x}$ must equal $\mathbf{0}$. From the previous theorem, the columns of $A_L$ are linearly independent.

Proof (continuity) Since $A_L$ has $r$ columns, the dimension of the column space of $A$ is at least $r$. We have proved that, for any matrix, the dimension of the column space is greater than or equal to the dimension of the row space. Applying this result to the matrix $A^T$ , we see that

$$\begin{aligned} \dim(\text{row space of } A) &= \dim(\text{column space of } A^T) \\ &\geq \dim(\text{row space of } A^T) \\ &= \dim(\text{column space of } A) \end{aligned}$$

Thus, for any matrix $A$, the dimension of the row space must equal the dimension of the column space.

**Example** Let $A = \begin{pmatrix} 2 & -4 & 3 & 0 & 1 & 6 \\ 1 & -2 & -2 & 14 & -4 & 15 \\ 1 & -2 & 1 & 2 & 1 & -1 \\ -2 & 4 & 0 & -12 & 1 & -7 \end{pmatrix}$. Find a basis of column space of a matrix $A$.

**Solution** $A \longrightarrow \begin{pmatrix} 1 & -2 & 0 & 6 & 0 & 1 \\ 0 & 0 & 1 & -4 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\{(2, 1, 1, -2)^T, (3, -2, 1, 0)^T, (1, -4, 1, 1)^T\}$ is a basis of the column space of $A$.

## Theorem (Equivalent Conditions for Nonsingularity)

Let $A$ be an $n \times n$ matrix. The following are equivalent:

- $A$ is nonsingular.
- $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$.
- $A$ is row equivalent to $I$. ($I$ is the reduced row echelon form of $A$. $A$ can be written as a product of elementary matrices.)
- The system $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b} \in \mathbf{R}^m$, (which is $\mathbf{x} = A^{-1}\mathbf{b}$).
- $\det A \neq 0$.
- Columns of $A$ form a basis of $\mathbf{R}^n$.
- The rank of $A$ is $n$.
- Rows of $A$ form a basis of $\mathbf{R}^{1 \times n}$.
- The nullity of $A$ is 0.