Chapter 3: Estimation of Parameters

Mathematical Statistics

UIC

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Overview

Statistical Inference

Statistical inference, or "learning", is the process of using data to infer the distribution that generated the data.

Basic Problem

We observe $X_1, \ldots, X_n \sim \pi$. We want to infer (or estimate, or learn) π or some features of π such as its mean.

Definition 3.1.1

A statistical model is a set of distributions or a set of densities (or PMFs) $\mathcal{F}$.

1. A parametric model is a set $\mathcal{F}$ that can be parameterized by a finite number of parameters.
2. A nonparametric model is a set $\mathcal{F}$ that cannot be parameterized by a finite set of parameters.

Example 3.1.2

1. If assume the data come from a normal distribution, then the model is

$$\mathcal{F} = \left\{ \pi\left(x|\mu,\sigma^2\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad \mu, \sigma^2 \in \mathbb{R} \right\},$$

   which is a two-parameter model. In $\pi\left(x|\mu,\sigma^2\right)$, x is a possible value of the random variable, whereas μ and σ^2 are parameters.

2. A nonparametric model:

$$\mathcal{F}_{\text{all}} = \{ \text{ all PDFs } \}$$

We will focus on parametric models. In general, a parametric model takes the form

$$\mathcal{F} = \{\pi(x|\theta), \quad \theta \in \Theta\}$$

where θ is an unknown parameter and Θ is the parameter space.

Remark: θ can be a vector, for instance, $\theta = \left(\mu, \sigma^2\right)$

Statistical Inference

Given a parametric model, $\mathcal{F} = \{\pi(x|\theta), \quad \theta \in \Theta\}$, the problem of inference is then to estimate (to learn) the parameter θ from the data.

Almost all problems in statistical inference can be identified as being one of three types: **point estimates**, **confidence intervals**, and **hypothesis testing**.

Three types of statistical inferences:

- **Point Estimation** refers to providing a single "best guess."

  Suppose $X_1, \ldots, X_n \sim \pi(x|\theta)$, where $\pi(x|\theta) \in \mathcal{F}$. A point estimator $\hat{\theta}_n$ of a parameter θ is some function of $X_1, \ldots, X_n$:

  $$\hat{\theta}_n = f(X_1, \ldots, X_n)$$

  Remember: θ is fixed but unknown, $\hat{\theta}_n$ is random since it depends on $X_1, \ldots, X_n$. We say that $\hat{\theta}_n$ is **unbiased** if

  $$\mathbb{E}\left[\hat{\theta}_n\right] = \theta$$

Cont'd

- A $100(1-\alpha)\%$ **Confidence Interval** for a parameter θ is a random interval $I_n = (a, b)$ where $a = a(X_1, \ldots, X_n)$ and $b = b(X_1, \ldots, X_n)$ such that

  $$\mathbb{P}(\theta \in I_n) = 1 - \alpha$$

  In words: (a, b) traps θ with probability $1 - \alpha$.
  $(1 - \alpha)$ is called coverage of the confidence interval. In practice, $\alpha = 0.05$ is often used.

- In **Hypothesis Testing**, we start with some default theory, called a null hypothesis, and then ask if the data provide sufficient evidence to reject the theory. Otherwise, we fail to reject the null hypothesis.

Example 3.1.3

$X_1, \ldots, X_n \sim \text{Bernoulli}(p) : n$ independent coin flips. To test if the coin is fair, we test the null hypothesis $H_0 : p = 1/2$ against the alternative hypothesis $H_1 : p \neq 1/2$
It seems reasonable to reject H_0 if

$$\left| \frac{1}{n} \sum_{i=1}^{n} X_i - \frac{1}{2} \right| \quad \text{is large}$$

Summary

- A parametric model is a set $\mathcal{F}$ that can be parameterized by a finite number of parameters.
  - General parametric model:

    $$\mathcal{F} = \{\pi(x|\theta), \quad \theta \in \Theta\}$$

- A nonparametric model is a set $\mathcal{F}$ that cannot be parameterized by a finite set of parameters.
- Almost all problems in statistical inference can be identified as being one of three types:
  - Point Estimates
  - Confidence Intervals
  - Hypothesis Testing

Overview

Method of Moments: Problem Formulation

Suppose that
$$X_1, \ldots, X_n \sim \pi(x|\theta)$$
where $\theta \in \Theta$, and we want to estimate θ based on the data $X_1, \ldots, X_n$. The first method for constructing parametric estimators that we will study is called the method of moments.

- The estimators produced by this method are not optimal, but that are often easy to compute.
- They are also useful as starting values for other methods that require iterative numerical routines.

Method of Moments

Recall that the k^{th} moment of a probability distribution $\pi(x|\theta)$ is
$$\mu_k(\theta) = \mathbb{E}_\theta\left[X^k\right]$$
where $\mathbb{E}_\theta$ denotes expectation with respect to $\pi(x|\theta)$, i.e.
$$\mathbb{E}_\theta[f(X)] = \int f(x)\pi(x|\theta)\,\mathrm{d}x$$

If $X_1, \ldots, X_n$ are i.i.d from $\pi(x|\theta)$, then the k^{th} sample moment is defined as
$$\hat{\mu}_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$$

We can view $\hat{\mu}_k$ as an estimate of μ_k. Suppose that the parameter θ has k components:
$$\theta = (\theta_1, \ldots, \theta_k)$$

Method of Moments

Definition 3.2.1 (Method of Moments Estimator)

The **method of moments estimator** $\hat{\theta}$ is defined to be the value of θ such that
$$\begin{cases} \mu_1(\theta) = \hat{\mu}_1 \\ \mu_2(\theta) = \hat{\mu}_2 \\ \quad \cdots\cdots \\ \mu_k(\theta) = \hat{\mu}_k \end{cases} \tag{1}$$

- System (1) is a system of k equations with k unknowns: $\theta_1, \ldots, \theta_k$
- The solutions of this system $\hat{\theta}$ is the method of moments estimate of the parameter θ.

Example 3.2.2 (Bernoulli)

Let $X_1, \ldots, X_n \sim \mathrm{Bernoulli}(p)$. Find the method of moments estimate of the parameter p.

Example 3.2.3 (Normal)

Let $X_1, \dots, X_n \sim \mathcal{N}\left(\mu, \sigma^2\right)$. Find the method of moments estimates of μ and σ^2.

Consistency of the MoM estimator

Question: How good is the estimator $\hat{\theta}$ obtained by the method of moments?

Definition 3.2.4 (Consistency)

Let $\hat{\theta}_n$ be an estimate of a parameter θ based on a sample of size n. Then $\hat{\theta}_n$ is said to be consistent if

$$\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta$$

That is, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\hat{\theta}_n - \theta\right| \geq \varepsilon\right) \to 0 \quad \text{as} \quad n \to \infty$$

Theorem 3.2.5

The method of moments estimate is consistent.

Summary

- If $X_1, \dots, X_n \sim \pi(x|\theta)$, then the method of moments estimate $\hat{\theta}$ of $\theta = (\theta_1, \dots, \theta_k)$ is the solution of

$$\begin{cases} \mu_1(\theta) = \hat{\mu}_1 \\ \mu_2(\theta) = \hat{\mu}_2 \\ \quad\vdots \\ \mu_k(\theta) = \hat{\mu}_k \end{cases}$$

  where

  - $\mu_k(\theta)$ is the k^{th} moment
  $$\mu_k(\theta) = \mathbb{E}_\theta\left[X^k\right]$$
  - $\hat{\mu}_k$ is the k^{th} sample moment
  $$\hat{\mu}_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$$

- The method of moments estimate $\hat{\theta}$ is a consistent estimate of θ.

Overview

The most common method for estimating parameters in a parametric model is the method of maximum likelihood.

Suppose $X_1, \dots, X_n$ are i.i.d. from $\pi(x|\theta)$.

Definition 3.3.1 (Likelihood Function)

The **likelihood function** is defined by

$$\boxed{\mathcal{L}(\theta) = \prod_{i=1}^{n} \pi\left(X_i|\theta\right)}$$

Important Remarks:

- The likelihood function is just the joint pdf/pmf of the data, except that we treat it as a function of the parameter θ.
- Thus, $\mathcal{L} : \Theta \to [0, \infty)$
- The likelihood function is not a density function: it is not true that $\mathcal{L}$ integrates to one, i.e $\int_\Theta \mathcal{L}(\theta)\, \mathrm{d}\theta \neq 1$.

Definition 3.3.2 (The Maximum Likelihood Estimate)

The **maximum likelihood estimate (MLE)** of θ, denoted $\hat{\theta}_{\mathrm{MLE}}$, is the value of θ that maximizes the likelihood $\mathcal{L}(\theta)$

$$\hat{\theta}_{\textsf{MLE}} = \arg\max_{\theta \in \Theta} \mathcal{L}(\theta)$$

$\hat{\theta}_{\textsf{MLE}}$ makes the observed data $X_1, \dots, X_n$ "most probable" or "most likely"

Important Remark:
Rather than maximizing the likelihood itself, it is often easier to maximize its natural logarithm (which is equivalent since the log is a monotonic function). The log-likelihood is

$$\boxed{l(\theta) = \log \mathcal{L}(\theta) = \sum_{i=1}^{n} \log \pi\left(X_i|\theta\right)}$$

Example 3.3.3 (Bernoulli)

$X_1, \dots, X_n \sim \mathrm{Bernoulli}(p)$. Find the MLE of p.

- Answer:

$$\hat{p}_{\textsf{MLE}} = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}_n$$

- In this example, $\hat{p}_{\mathrm{MLE}} = \hat{p}_{\mathrm{MoM}}$

Example 3.3.4 (Normal)

$X_1, \dots, X_n \sim \mathcal{N}\left(\mu, \sigma^2\right)$. Find the MLEs of μ and σ^2.

- Answer:

$$\hat{\mu}_{\mathrm{MLE}} = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}_n \qquad \hat{\sigma}^2_{\mathrm{MLE}} = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2$$

- Again, in this example, MLEs are the same as the MoM estimates.

Properties of MLE

Under certain conditions on the model

$$\mathcal{F} = \{\pi(x|\theta), \quad \theta \in \Theta\}$$

(under some smoothness conditions of π), the MLE $\hat{\theta}_{\text{MLE}}$ possesses many attractive properties that make it an appealing choice of estimate.

Main properties of the MLE:

- MLE is consistent:
$$\hat{\theta}_{\text{MLE}} \xrightarrow{\mathbb{P}} \theta_0$$
where θ_0 denotes the true value of θ.
- MLE is equivariant: if $\hat{\theta}_{\text{MLE}}$ is the MLE of $\theta \Rightarrow f\left(\hat{\theta}_{\text{MLE}}\right)$ is the MLE of $f(\theta)$.
- MLE is asymptotically optimal: the MLE has the smallest variance for large sample sizes n.

Properties of MLE

Main properties of the MLE (cont'd):

- MLE is asymptotically normal:
$$\hat{\theta}_{\text{MLE}} \to \mathcal{N}\left(\theta_0, \frac{1}{nI(\theta_0)}\right)$$
where
$$I(\theta) \overset{\text{def}}{=} \mathbb{E}_\theta\left[\left(\frac{\partial}{\partial\theta}\log\pi(X|\theta)\right)^2\right] = \int\left(\frac{\partial}{\partial\theta}\log\pi(x|\theta)\right)^2\pi(x|\theta)\,\mathrm{d}x$$
    - $I(\theta)$ is called Fisher Information.
- MLE is asymptotically unbiased:
$$\lim_{n\to\infty}\mathbb{E}\left[\hat{\theta}_{\text{MLE}}\right] = \theta_0$$

Example: when MoM and MLE produce different estimates

Example 3.3.5 (Uniform)

Let $X_1, \ldots, X_n \sim U(0, \theta)$. Find the MoM estimate and MLE of θ.

- Answer:
$$\hat{\theta}_{\text{MoM}} = 2\overline{X}_n \quad \hat{\theta}_{\text{MLE}} = X_{(n)}$$
- In this example, the MLE and MoM estimate are different.

Summary

- The Likelihood Function:
$$\mathcal{L}(\theta) = \prod_{i=1}^{n}\pi(X_i|\theta) \quad X_1, \ldots, X_n \sim \pi(x|\theta)$$
- The Maximum Likelihood Estimate:
$$\hat{\theta}_{\text{MLE}} = \arg\max_{\theta\in\Theta}\mathcal{L}(\theta) = \arg\max_{\theta\in\Theta}\log\mathcal{L}(\theta)$$
- MLE is consistent, equivariant, asymptotically optimal, asymptotically normal, and asymptotically unbiased.
- Examples: Bernoulli (p), $N(\mu, \sigma^2)$, and $U(0, \theta)$.

Overview

Confidence Interval

Recall the definition of a confidence interval (see also Definition 2.4.12 and Theorem 2.4.13):

Definition 3.4.1 (Confidence Interval)

A $100(1-\alpha)\%$ **confidence interval** for a parameter θ is a *random* interval calculated from the sample,

$$X_1, \dots, X_n \sim \pi(x|\theta)$$

which contains θ with probability $1-\alpha$.

There are three methods for constructing confidence intervals using MLEs $\hat{\theta}_{\text{MLE}}$:

- Exact Method
- Approximate Method
- Bootstrap Method

Exact Method. Example: Normal distribution $\mathcal{N}(\mu, \sigma^2)$

Let $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$, then the MLEs for μ and σ^2 are (Example 3.3.4):

$$\hat{\mu}_{\mathrm{MLE}} = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}_n \qquad \hat{\sigma}^2_{\mathrm{MLE}} = \frac{1}{n}\sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2$$

- A confidence interval for μ is based on the following fact (Theorem 1.7.29):

$$\frac{\sqrt{n}\left(\overline{X}_n - \mu\right)}{S_n} \sim t_{n-1}$$

where S_n^2 is the sample variance $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2 = \frac{n}{n-1}\hat{\sigma}^2_{\text{MLE}}$

Result

A $100(1-\alpha)\%$ confidence interval for μ is

$$\hat{\mu}_{\mathrm{MLE}} \pm \frac{1}{\sqrt{n-1}}\hat{\sigma}_{\mathrm{MLE}} t_{n-1}(\alpha/2)$$

where $t_{n-1}(\alpha)$ is the point beyond which the t-distribution with $(n-1)$ degrees of freedom has probability α.

Exact Method. Example: Normal distribution $\mathcal{N}(\mu, \sigma^2)$

- A confidence interval for σ^2 is based on the following fact (Theorem 1.7.29):

$$\frac{(n-1)\, S_n^2}{\sigma^2} \sim \chi^2_{n-1}$$

Result

A $100(1-\alpha)\%$ confidence interval for σ^2 is

$$\left(\frac{n\hat{\sigma}^2_{\mathrm{MLE}}}{\chi^2_{n-1}\left(\frac{\alpha}{2}\right)}, \frac{n\hat{\sigma}^2_{\mathrm{MLE}}}{\chi^2_{n-1}\left(1-\frac{\alpha}{2}\right)}\right)$$

where $\chi^2_{n-1}(\alpha)$ is the point beyond which the χ^2-distribution with $(n-1)$ degrees of freedom has probability α.

Remark:
The main drawback of the exact method is that in practice the sampling distributions like t_{n-1} and χ^2_{n-1} in our example are unknown.

Approximate Method

One of the most important properties of MLE is that it is asymptotically normal:

$$\hat{\theta}_{\text{MLE}} \rightarrow \mathcal{N}\left(\theta_0, \frac{1}{nI(\theta_0)}\right), \quad \text{as } n \rightarrow \infty$$

where $l(\theta_0)$ is Fisher information

$$I(\theta) = \mathbb{E}_\theta\left[\left(\frac{\partial}{\partial\theta}\log\pi(X|\theta)\right)^2\right]$$

Since the true value θ_0 is unknown, we will use $I\left(\hat{\theta}_{\text{MLE}}\right)$ instead of $I(\theta_0)$:

Result

An **approximate** $100(1-\alpha)\%$ confidence interval for θ_0 is

$$\hat{\theta}_{\text{MLE}} \pm \frac{z_{\alpha/2}}{\sqrt{nI\left(\hat{\theta}_{\text{MLE}}\right)}}$$

where z_α is the point beyond which the standard normal distribution has probability α.

Approximate Method. Example: Bernoulli (p)

Example 3.4.2 (Bernoulli (p))

Let $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$. Find an approximate confidence interval for p

- Answer:

$$\overline{X}_n \pm z_{\alpha/2}\sqrt{\frac{\overline{X}_n\left(1-\overline{X}_n\right)}{n}}$$

Bootstrap Method

Suppose $\hat{\theta}$ is an estimate of a parameter θ, the true unknown value of which is θ_0. $\hat{\theta}$ can be any estimate, not necessarily MLE,

$$X_1, \ldots, X_n \sim \pi(x|\theta) \quad \hat{\theta} = \hat{\theta}(X_1, \ldots, X_n)$$

Define a new random variable

$$\Delta = \hat{\theta} - \theta_0$$

- Step 1: Assume (for the moment) that the distribution of Δ is known. Let (as before) q_α be the number such that $\mathbb{P}(\Delta > q_\alpha) = \alpha$. Then

$$\mathbb{P}\left(q_{1-\frac{\alpha}{2}} \leq \hat{\theta} - \theta_0 \leq q_{\frac{\alpha}{2}}\right) = 1 - \alpha$$

And therefore a $100(1-\alpha)\%$ confidence interval for θ_0 is

$$\left(\hat{\theta} - q_{\frac{\alpha}{2}}, \hat{\theta} - q_{1-\frac{\alpha}{2}}\right)$$

The problem is that the distribution of Δ is unknown and, therefore, q_α are unknown.

Bootstrap Method

- Step 2: Assume that the distribution of Δ is not known, but θ_0 is known. Then we can approximate the distribution of Δ as follows:

$$\begin{gathered} X_1^{(1)}, \ldots, X_n^{(1)} \sim \pi(x|\theta_0) \rightsquigarrow \hat{\theta}^{(1)} - \theta_0 = \Delta^{(1)} \\ X_1^{(2)}, \ldots, X_n^{(2)} \sim \pi(x|\theta_0) \rightsquigarrow \hat{\theta}^{(2)} - \theta_0 = \Delta^{(2)} \\ \vdots \\ X_1^{(B)}, \ldots, X_n^{(B)} \sim \pi(x|\theta_0) \rightsquigarrow \hat{\theta}^{(B)} - \theta_0 = \Delta^{(B)} \end{gathered}$$

From these realizations $\Delta^{(1)}, \ldots, \Delta^{(B)}$ of Δ we can approximate the distribution of Δ by its empirical distribution, and, therefore, we can approximate q_α. The problem is that θ_0 is not known!

Bootstrap Method

- Step 3: **Bootstrap strategy**: Use $\hat{\theta}$ instead of θ_0.

$$
\begin{aligned}
X_1^{(1)},\dots,X_n^{(1)} &\sim \pi\left(x|\hat{\theta}\right) \rightsquigarrow \hat{\theta}^{(1)} - \hat{\theta} \approx \Delta^{(1)} \\
X_1^{(2)},\dots,X_n^{(2)} &\sim \pi\left(x|\hat{\theta}\right) \rightsquigarrow \hat{\theta}^{(2)} - \hat{\theta} \approx \Delta^{(2)} \\
&\vdots \\
X_1^{(B)},\dots,X_n^{(B)} &\sim \pi\left(x|\hat{\theta}\right) \rightsquigarrow \hat{\theta}^{(B)} - \hat{\theta} \approx \Delta^{(B)}
\end{aligned}
$$

Distribution of Δ is approximated from realizations $\Delta^{(1)},\dots,\Delta^{(B)}$.

Remark: $\hat{\theta}^{(i)}$ is the estimate of θ that is obtained from $X_1^{(i)},\dots,X_n^{(i)}$ by the same method (for example, MLE) as $\hat{\theta}$ was obtained from $X_1,\dots,X_n$.

Summary

- Three methods for constructing confidence intervals using MLEs:
- Exact method provides exact confidence intervals, but it's hard to use in practice
  - Example: $X_1,\dots,X_n \sim \mathcal{N}\left(\mu,\sigma^2\right)$

$$
\mu : \quad \hat{\mu}_{\mathrm{MLE}} \pm \frac{1}{\sqrt{n-1}}\hat{\sigma}^2_{\mathrm{MLE}} t_{n-1}(\alpha/2)
$$

$$
\sigma^2 : \quad \left(\frac{n\hat{\sigma}^2_{\mathrm{MLE}}}{\chi^2_{n-1}\left(\frac{\alpha}{2}\right)}, \frac{n\hat{\sigma}^2_{\mathrm{MLE}}}{\chi^2_{n-1}\left(1-\frac{\alpha}{2}\right)}\right)
$$

- Approximate method provides an approximate confidence interval for θ_0, which is constructed using asymptotic properties of MLE:

$$
\hat{\theta}_{\text{MLE}} \pm \frac{z_{\alpha/2}}{\sqrt{nI\left(\hat{\theta}_{\text{MLE}}\right)}}
$$

- Bootstrap method provides an approximate confidence interval. Bootstrap is the most popular method in practice since it is easy to implement.

Overview

Example: Gaussian Model

Suppose that:

- $X_1,\dots,X_n \sim \mathcal{N}\left(\mu,\sigma^2\right)$, true values: $\mu = 1$ and $\sigma = 2$
- Exact Confidence Intervals:

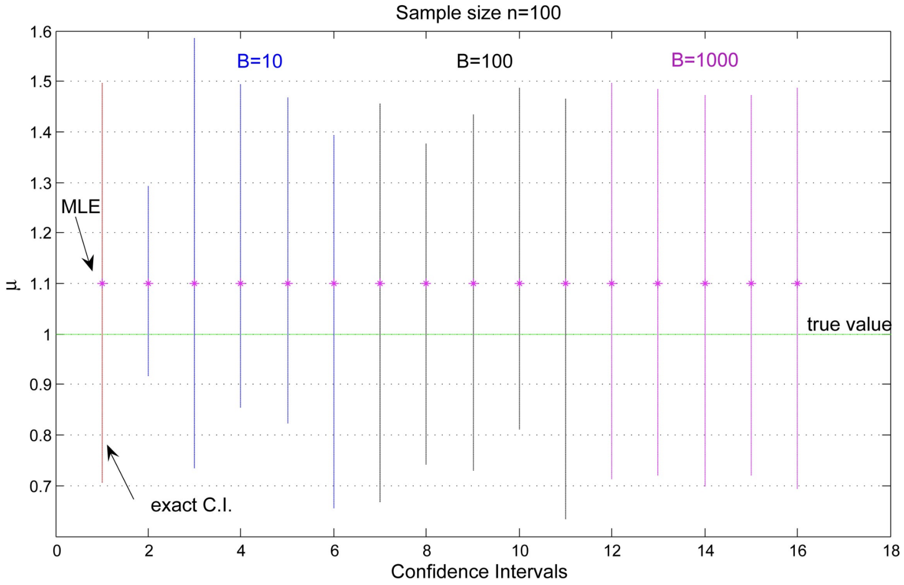
$$
\mu : \quad \hat{\mu}_{\mathrm{MLE}} \pm \frac{1}{\sqrt{n-1}}\hat{\sigma}_{\mathrm{MLE}} t_{n-1}(\alpha/2) \qquad \sigma^2 : \quad \left(\frac{n\hat{\sigma}^2_{\mathrm{MLE}}}{\chi^2_{n-1}\left(\frac{\alpha}{2}\right)}, \frac{n\hat{\sigma}^2_{\mathrm{MLE}}}{\chi^2_{n-1}\left(1-\frac{\alpha}{2}\right)}\right)
$$

```
%---- Data:
mu0=1;                     % true mean
sigma0=2;                  % true sigma
n=100;                     % sample size;
X=mu0+sigma0*randn(1,n); % data
%---- MLEs:
mu_mle=mean(X);
sigma_mle=std(X,1);
%---- Level of Confidence:
alpha=0.05;                % 100(1-alpha) CI
%---- Exact Confidence Intervals:
CImu_exact=[mu_mle-sigma_mle*tinv(1-alpha/2,n-1)/sqrt(n-1),
mu_mle+sigma_mle*tinv(1-alpha/2,n-1)/sqrt(n-1)];
CIsigma_exact=[sqrt(n*sigma_mle^2/chi2inv(1-alpha/2,n-1)),
sqrt(n*sigma_mle^2/chi2inv(alpha/2,n-1))];
%[phat,pci] = mle(X);
```

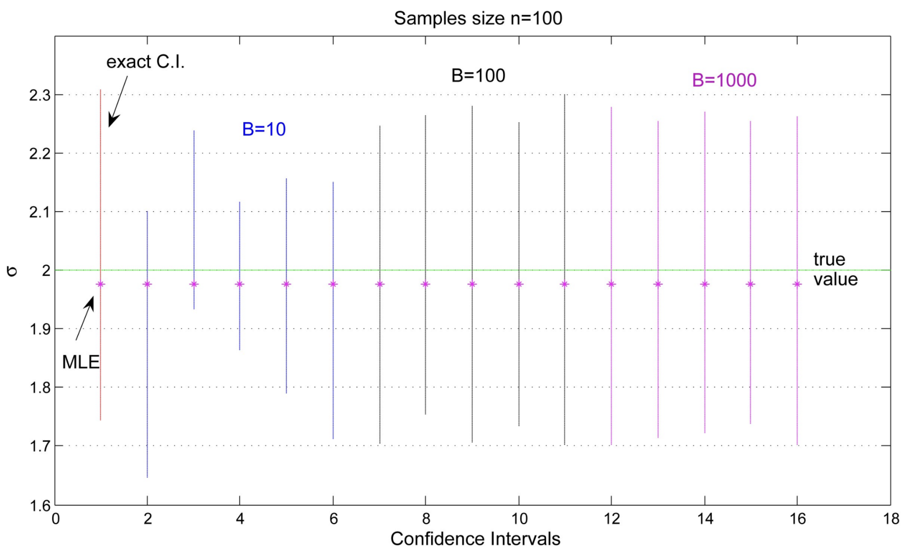
Bootstrap

```
%---- Bootstrap Confidence Intervals:
B=10;     % number of the bootstrap samples
for i=1:B
    Z(i,:)=mu_mle+sigma_mle*randn(1,n); % "bootstrap data"
    mu_b(i)=mean(Z(i,:));               % MLE from b-data
    sigma_b(i)=std(Z(i,:),1);           % MLE from b-data
    Delta_mu(i)=mu_b(i)-mu_mle;
    Delta_sigma(i)=sigma_b(i)-sigma_mle;
end
CImu_bootstrap=[mu_mle-quantile(Delta_mu,1-alpha/2),
    mu_mle-quantile(Delta_mu,alpha/2)];
CIsigma_bootstrap=[sigma_mle-quantile(Delta_sigma,1-alpha/2),
    sigma_mle-quantile(Delta_sigma,alpha/2)];
```

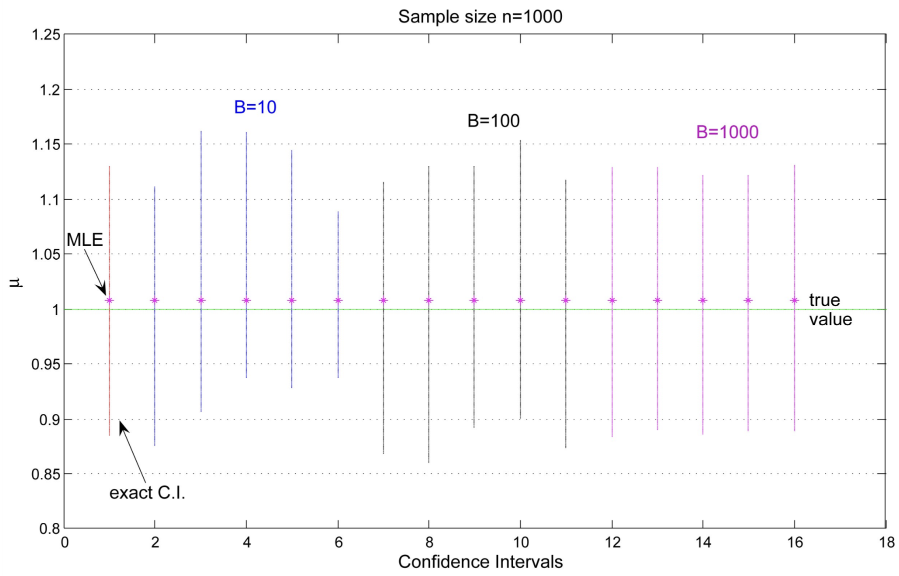

Confidence Intervals for μ when $n = 100$



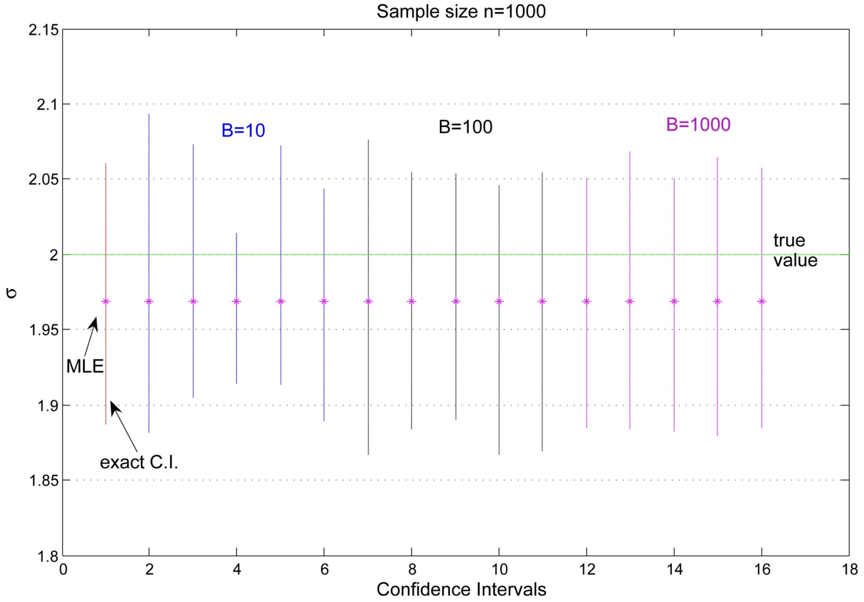
Confidence Intervals for σ when $n = 100$



Confidence Intervals for μ when $n = 1000$



Confidence Intervals for σ when $n = 1000$



Overview

Measure of Efficiency: Mean Squared Error

In most estimation problems, there are many possible estimates $\hat{\theta}$ of θ. For example, the MoM estimate $\hat{\theta}_{\text{MoM}}$ or the MLE estimate $\hat{\theta}_{\text{MLE}}$.

Question: How would we choose which estimate to use?

Qualitatively, it is reasonable to choose that estimate whose distribution is most highly concentrated about the true parameter value θ_0. To make this idea work, we need to define a quantitative measure of such concentration.

Definition 3.5.1 (Mean-squared Error)

The **mean squared error** of $\hat{\theta}$ as an estimate of θ_0 is

$$\boxed{\mathrm{MSE}(\hat{\theta}) = \mathbb{E}\left[\left(\hat{\theta} - \theta_0\right)^2\right]}$$

- The mean squared error can be also written as follows:

$$\mathrm{MSE}(\hat{\theta}) = \mathrm{Var}[\hat{\theta}] + \underbrace{\left(\mathbb{E}(\hat{\theta}) - \theta_0\right)^2}_{\text{squared bias}}$$

- If $\hat{\theta}$ is unbiased, then $\mathrm{MSE}(\hat{\theta}) = \mathrm{Var}[\hat{\theta}]$.

Cramer-Rao Inequality

- Given two unbiased estimates, $\hat{\theta}$ and $\tilde{\theta}$, the **efficiency** of $\hat{\theta}$ relative to $\tilde{\theta}$ is defined to be

$$\mathrm{eff}(\hat{\theta}, \tilde{\theta}) = \frac{\mathrm{Var}(\tilde{\theta})}{\mathrm{Var}(\hat{\theta})}$$

- $\hat{\theta}$ is more efficient than $\tilde{\theta}$ $\Leftrightarrow$ $\mathrm{eff}(\hat{\theta}, \tilde{\theta}) > 1$
- In general, the mean squared error is a measure of efficiency of an estimate:

the smaller $\mathrm{MSE}(\hat{\theta})$, the better the estimate $\hat{\theta}$

Theorem 3.5.2 (Cramer-Rao Inequality)

Let $X_1, \ldots, X_n$ be i.i.d. from $\pi(x|\theta)$. Let $\hat{\theta} = \hat{\theta}(X_1, \ldots, X_n)$ be any unbiased estimate of a parameter θ whose true values is θ_0. Then, under smoothness assumptions on $\pi(x|\theta)$,

$$\mathrm{MSE}(\hat{\theta}) = \mathrm{Var}[\hat{\theta}] \geq \frac{1}{nI(\theta_0)}$$

Cramer-Rao: $\boxed{\mathrm{MSE}(\hat{\theta}) = \mathrm{Var}[\hat{\theta}] \geq \frac{1}{nI(\theta_0)}}$

Important Remarks:

- $\hat{\theta}$ can't have arbitrary small MSE
- The Cramer-Rao inequality gives a lower bound on the variance of any unbiased estimate.

Definition 3.5.3 (Efficient)

An unbiased estimate whose variance achieves this lower bound is said to be **efficient**.

Recall that MLE is asymptotically Normal: $\hat{\theta}_{\text{MLE}} \to \mathcal{N}\left(\theta_0, \frac{1}{nI(\theta_0)}\right)$

- Therefore, MLE is asymptotically efficient
- However, for a finite sample size n, MLE may not be efficient
- MLEs are not the only asymptotically efficient estimates.

Example: Poisson Distribution

Recall: The Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events k occurring in a fixed interval of time if these events occur with a known average rate λ and independently of the time since the last event.

$$\mathbb{P}(X = k|\lambda) = \frac{\lambda^k}{k!} e^{-\lambda} \quad \mathbb{E}[X] = \lambda \quad \mathrm{Var}[X] = \lambda$$

Example 3.5.4 (Poisson)

Let $X_1, \ldots, X_n \sim \mathrm{Pois}(\lambda)$.

- Find the MLE of λ
- Show that $\hat{\lambda}_{\text{MLE}}$ is efficient.

- The theorem does not exclude the possibility that there is a biased estimator of λ that has a smaller MSE than $\hat{\lambda}_{\text{MLE}}$

Summary

- Mean squared error is a measure of efficiency of an estimate

$$\mathrm{MSE}(\hat{\theta}) = \mathbb{E}\left[\left(\hat{\theta} - \theta_0\right)^2\right]$$

- If $\hat{\theta}$ is unbiased, then

$$\mathrm{MSE}(\hat{\theta}) = \mathrm{Var}[\hat{\theta}]$$

- Cramer-Rao Inequality:

$$\mathrm{MSE}(\hat{\theta}) = \mathrm{Var}[\hat{\theta}] \geq \frac{1}{nI(\theta_0)}$$

- An unbiased estimate whose variance achieves this lower bound is said to be efficient
- Any MLE is asymptotically efficient (as $n \to \infty$)
- Example: if $X_1, \ldots, X_n \sim \mathrm{Poisson}(\lambda)$, then $\hat{\lambda}_{\mathrm{MLE}}$ is efficient