$|\vec{v}|$ ① $\mathbb{R}^2$: $\vec{v}=(v_1, v_2)$ $|\vec{v}|=\sqrt{v_1^2+v_2^2}$

② $\mathbb{R}^3$: $\vec{v}=(v_1, v_2, v_3)$ $|\vec{v}|=\sqrt{v_1^2+v_2^2+v_3^2}$

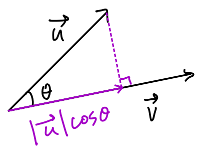
unit vector $\dfrac{\vec{v}}{|\vec{v}|}$

dot plot $\vec{u}\cdot\vec{v}$ ① $\mathbb{R}^2$ $\vec{u}:(u_1, u_2)$ $\vec{v}=(v_1, v_2)$ $\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2$

$\mathbb{R}^3$ $\vec{u}=(u_1, u_2, u_3)$ $\vec{v}=(v_1, v_2, v_3)$ $\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2+u_3v_3$

② $\vec{v}\cdot\vec{v}=|\vec{v}|^2$ $\theta=0$ $\cos\theta=1$

③ $\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cos\theta$



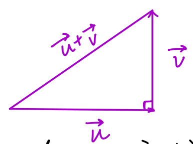
④ $|\vec{u}\cdot\vec{v}|\leq|\vec{u}||\vec{v}|$

1) $|\vec{u}\cdot\vec{v}|=|\vec{u}||\vec{v}| \longleftrightarrow$ $\vec{u}$ and $\vec{v}$ are parallel.

2) $\vec{u}\cdot\vec{v}=0 \xleftrightarrow{(|\vec{u}|\neq0,\ |\vec{v}|\neq0)}$ $\vec{u}$ and $\vec{v}$ are perpendicular / orthogonal

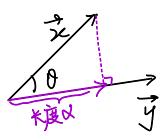
$\downarrow$

$|\vec{u}+\vec{v}|^2=|\vec{u}|^2+|\vec{v}|^2$



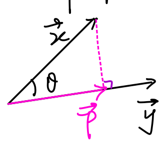
$|\vec{u}|^2+|\vec{v}|^2=|\vec{u}+\vec{v}|^2$ if and only if $\vec{u}\cdot\vec{v}=0$

projection ① scalar projection onto y $\alpha=|\vec{x}|\cos\theta=\dfrac{\vec{x}\cdot\vec{y}}{|\vec{y}|}$



② vector projection onto y $p=\alpha\dfrac{\vec{y}}{|\vec{y}|}=\dfrac{\vec{x}\cdot\vec{y}}{|\vec{y}|^2}\vec{y}$

在方向上的单位向量.



cross product $\vec{u}\times\vec{v}$ ① $\vec{u}=(u_1, u_2, u_3)$ or $u_1\vec{i}+u_2\vec{j}+u_3k$ $\vec{v}=(v_1, v_2, v_3)$ or $v_1\vec{i}+v_2\vec{j}+v_3k$

$$\vec{u}\times\vec{v}=\begin{vmatrix} \vec{i} & \vec{j} & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}=\vec{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}-\vec{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}+k\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$=(u_2v_3-u_3v_2)\vec{i}-(u_1v_3-u_3v_1)\vec{j}+(u_1v_2-u_2v_1)k$$

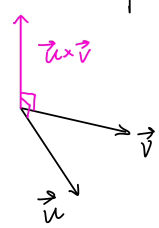
② $\vec{i}\ \vec{j}\ k$ 1) $\vec{i}\times\vec{j}=k$ $\vec{j}\times k=\vec{i}$ $k\times\vec{i}=\vec{j}$

2) $\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=k\times k=0$

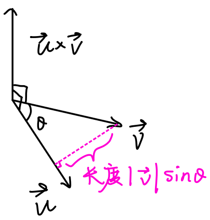
③ $\vec{u}\times\vec{v}=-(\vec{v}\times\vec{u})$

④ $(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (\vec{v}\times\vec{w})\cdot\vec{u} = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$

⑤ $\vec{u}\times\vec{v}$ is perpendicular to $\vec{u}$ and $\vec{v}$.



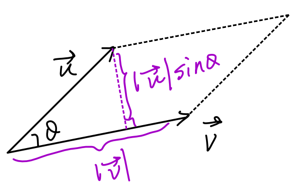
⑥ $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$



1) $\vec{u}\times\vec{v}=0 \longleftrightarrow \vec{u}$ and $\vec{v}$ are parallel

☆ 2) **area of parallelgram**

R^2: Area $= |\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$

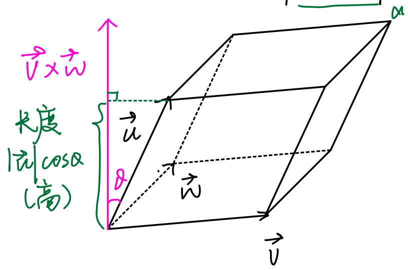


R^3: Area $= |\vec{u}\times\vec{v}|$

☆ 3) **volume of parallelepiped**

$\vec{u}\cdot(\vec{v}\times\vec{w}) = |\vec{u}|\underline{|\vec{v}\times\vec{w}|}\cos\theta$

area of parallelgram.



☆ **equation** ① Vector equation $P = P_0 + su$

through point P_0 direction vector

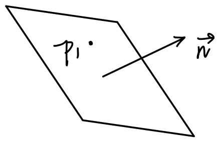
② parameter equation $P_0 = (x_0, y_0, z_0)$ $u = (a, b, c)$

$(x, y, z) = (x_0+sa,\ y_0+sb,\ z_0+sc)$

③ equation of the plane

法一: normal vector $\vec{n}=(a,b,c)$ $P_1=(x_1,y_1,z_1)$

△随便取个平面上的点

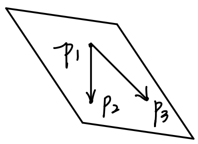


$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

or

$ax+by+cz+d=0$ $d=-ax_1-by_1-cz_1$

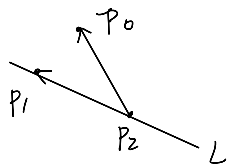
法二: 找平面上三点 p_1, p_2, p_3



$(x,y,z)=P_1+s\,\overrightarrow{P_1P_2}+t\,\overrightarrow{P_1P_3}$

两平面夹角为法向量(normal vector)夹角

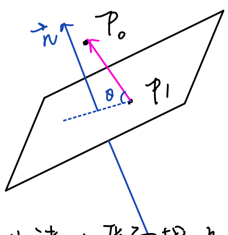
distance ①点到线问题 $P_0(a_1,b_1,c_1)$ Line $ax+by+cz+d=0$



在线上找到两点 P_1, P_2

$$d=\frac{|\overrightarrow{P_2P_0}\times\overrightarrow{P_2P_1}|}{|\overrightarrow{P_2P_1}|}$$

② 点到面距离 $P_0=(x_1,y_1,z_1)$ plane $ax+by+cz+d=0$



1) 法一: 平面找一点 P_1

$D=|\overrightarrow{P_0P_1}|\sin\theta$

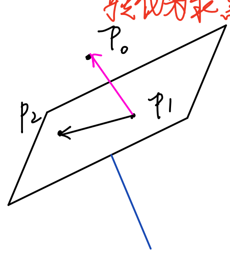
$=\dfrac{|\overrightarrow{P_0P_1}\times\vec{n}|}{|\vec{n}|}$

2) 法二: $D=\dfrac{|ax_1+by_1+cz_1+d|}{|\vec{n}|}$

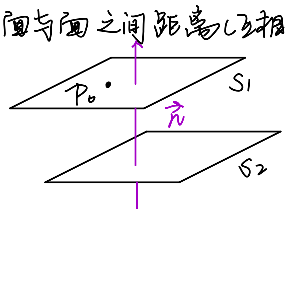
3) 法三: 在平面上找两点 P_1, P_2

⇑

转化为求点到线距离



③ 面与面之间距离(互相平行)



plane $D_1: a_1x+b_1y+c_1z+d_1=0$ $(a_1,b_1,c_1)\parallel(a_2,b_2,c_2)$

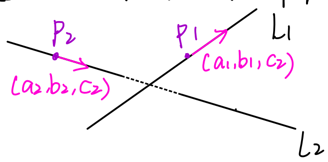
$D_2: a_2x+b_2y+c_2z+d_2=0$

平面S_1 找一点 P_0, 计算 P_0 到平面S_2距离

⇑

转化为求点到面距离

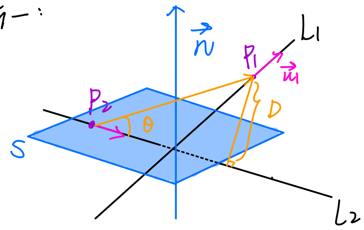
④ 线与线之间距离（非平行，不相交 ⇒ skew lines）



line 1: $\overset{P_1}{(x_1,y_1,z_1)}+\overset{\vec{u_1}}{(a_1,b_1,c_1)}t$

line 2: $\overset{P_2}{(x_2,y_2,z_2)}+\overset{\vec{u_2}}{(a_2,b_2,c_2)}t$

1) 法一:



根据 $\vec{u_1}$ 与 $\vec{u_2}$ 找到法向量 $\vec{n}=(a,b,c)$

创立经过点 P_2 且法向量为 $\vec{n}$ 的平面 S

$$a(x-x_2)+b(y-y_2)+c(z-z_2)=0$$

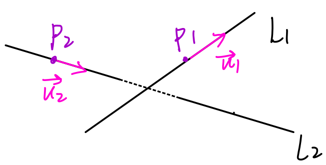
之后求点 P_1 到平面 S 的距离.

↑

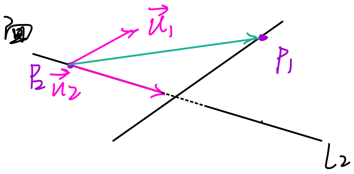
转化为求点到面距离

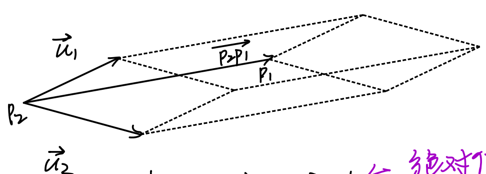
$$D=|\overrightarrow{P_1P_2}|\sin\theta=\frac{|a(x_1-x_2)+b(y_1-y_2)+c(z_1-z_2)|}{|\vec{n}|}$$

2) 法二:



平移 $\vec{u_1}$, $\vec{u_2}$ 到一个平面





扩展成平行六面体（parallelpiped）

求距离转化为求平行六面体的高.

$$D=\frac{|(\vec{u_2}\times\vec{u_1})\cdot\overrightarrow{P_2P_1}|}{|\vec{u_2}\times\vec{u_1}|}$$

← 绝对值

→ 体积

→ 底面积