MATH2033 Mathematical Statistics
Assignment 8 Suggested Solutions

1. Since H_1 is one-sided, H_0 is rejected when $\bar{y} \geq 30 + z_\alpha \cdot \frac{9}{\sqrt{16}}$. Also, $1-\beta$ = power = $P\left(\bar{Y} \geq 30 + z_\alpha \cdot \frac{9}{\sqrt{16}} \,\middle|\, \mu = 34\right) = 0.85$. Therefore, $1 - \beta = P\left(Z \geq \frac{30+z_\alpha \cdot 9/\sqrt{16} - 34}{9/\sqrt{16}}\right) = 0.85$. But $P(Z \geq -1.04) = 0.85$, so $\frac{30+z_\alpha \cdot 9/\sqrt{16}-34}{9/\sqrt{16}} = -1.04$, implying that $z_\alpha = 0.74$. Therefore, $\alpha = 0.23$.

2. (a) Not a good one. Question canceled.

   (b) The level of significance is

$$\alpha = P\left[X_1 \leq 3 \,\middle|\, p_0 = 0.5\right] = \sum_{x=0}^{3} \binom{10}{x} 0.5^x (1-0.5)^{10-x} = 0.172$$

   The power of the test is

$$1 - \beta = 1 - P\left[X_1 > 3 \,\middle|\, p_0 = 0.25\right] = \sum_{x=0}^{3} \binom{10}{x} 0.25^x (1-0.25)^{10-x} = 0.776$$

3. To test $H_0 : \mu = 60$ against $H_1 : \mu \neq 60$ at $\alpha = 0.05$.
   The test statistic is the sample mean of a random sample of size $n = 16$

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

   following a normal distribution with mean μ and variance σ^2/n. Thus, the rejection region can be determined from

$$0.05 = \alpha = P\left(\left|\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}\right| > z_{0.025} \,\middle|\, \mu = 60\right) \quad \rightarrow \quad z_{0.025} = 1.96$$

   and the rejection region is

$$C = \left\{\left|\overline{X} - 60\right| > 1.96\right\}$$

   The power function $\gamma(\mu_1)$ is

$$\gamma(\mu_1) = 1 - \beta = P\left(\overline{X} \in C \,\middle|\, \mu_1 \neq 60\right) = P\left(\overline{X} > 61.96 \text{ or } \overline{X} < 58.04 \,\middle|\, \mu_1\right)$$

   that is,

$$\gamma(\mu_1) = P(Z > 61.96 - \mu_1) + P(Z < 58.04 - \mu_1)$$

   For $\mu_1 = 56, 58$ and 60, we have

$$\gamma(56) = P(Z > 5.96) + P(Z < 2.04) = 0.9793$$

$$\gamma(58) = P(Z > 3.96) + P(Z < 0.04) = 0.5160$$

$$\gamma(60) = P(Z > 1.96) + P(Z < -1.96) = 0.05$$

4. To test $H_0 : \theta = 1$ against $H_1 : \theta < 1$ at the significance level α.
The test statistic is Y, having the distribution $f_Y(y) = (1+\theta)y^\theta$, $0 \le y \le 1$.
Reject H_0 if $Y \in C$, where

$$C = \{y \le 1/2\}$$

Then the power of the test is

$$1 - \beta = P\left(Y \in C \;\middle|\; H_1\right) = P\left(Y \le 1/2 \;\middle|\; \theta < 1\right) = \int_0^{1/2} (1+\theta)y^\theta \ \mathrm{d}y = \left(\frac{1}{2}\right)^{1+\theta}, \quad \theta < 1.$$

5. (a) To test $H_0 : \theta = \theta_0 = 0$ against $H_1 : \theta = \theta_1 = 1$ at the significance level $\alpha = 0.05$.
By Neyman-Pearson lemma, we consider the likelihood ratio under the two hypotheses. Let $\mathbf{x} \in \mathbb{R}^n$ be an observation, the likelihood function is

$$L(\theta; \mathbf{x}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left[-\sum_{i=1}^n (x_i - \theta)^2/2\right]$$

then

$$\frac{L(\theta_0; \mathbf{x})}{L(\theta_1; \mathbf{x})} = \frac{(1/\sqrt{2\pi})^n \exp\left[-\sum_1^n x_i^2/2\right]}{(1/\sqrt{2\pi})^n \exp\left[-\sum_1^n (x_i - 1)^2/2\right]}$$
$$= \exp\left(-\sum_1^n x_i + \frac{n}{2}\right).$$

If $k > 0$, the set of all points $(x_1, x_2, \ldots, x_n)$ such that

$$\exp\left(-\sum_1^n x_i + \frac{n}{2}\right) \le k$$

is a best critical region. Equivalently,

$$\sum_{i=1}^n x_i \ge \frac{n}{2} - \log k = c$$

The best critical region is the set

$$C = \left\{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^n x_i \ge c\right\}$$

where the value of c can be determined so that the size of the critical region is α. The event $\sum_i X_i \ge c$ is equivalent to the event $\bar{X} \ge c/n = c_1$, for example, so the test may be based upon the statistic $\bar{X}$.
If $n = 25$ and $\alpha = 0.05$, then we find $c_1 = 1.645/\sqrt{25} = 0.329$. The best critical region is $C = \{\sum_{i=1}^n x_i \ge 1.645\}$ or

$$C = \{\bar{x} \ge 0.329\}$$

(b) The power of this best test of H_0 against H_1 is

$$P(\bar{X} \ge 0.329 | \theta = 1) = \int_{0.329}^\infty \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{25}}} \exp\left[-\frac{(\bar{x}-1)^2}{2\left(\frac{1}{25}\right)}\right] d\bar{x} = \int_{-3.355}^\infty \frac{1}{\sqrt{2\pi}} e^{-w^2/2} \ \mathrm{d}w = 0.9996,$$

when H_1 is true.