FINM3123 Introduction to Econometrics

Chapter 2
The Simple Regression Model

The Simple Regression Model

- **<u>Definition of the simple linear regression model</u>**

„Explains variable y in terms of variable x"

Intercept

Slope parameter

$$y = \beta_0 + \beta_1 x + u$$

Dependent variable, explained variable, response variable,…

Independent variable, explanatory variable, regressor,…

Error term, disturbance, unobservables,…

The Simple Regression Model

- **Interpretation of the simple linear regression model**

Studies how y varies with changes in x :

$$\frac{\partial y}{\partial x} = \beta_1 \qquad \text{as long as} \qquad \frac{\partial u}{\partial x} = 0$$

By how much does the dependent variable change if the independent variable is increased by one unit?

Interpretation only correct if all other things remain equal when the indepen-dent variable is increased by one unit

- **The simple linear regression model is rarely applicable in prac-tice but its discussion is useful for pedagogical reasons**

The Simple Regression Model

- **Example: Soybean yield and fertilizer**

$$yield = \beta_0 + \beta_1 fertilizer + u$$

Rainfall,
land quality,
presence of parasites, …

Measures the effect of fertilizer on yield, holding all other factors fixed

- **Example: A simple wage equation**

$$wage = \beta_0 + \beta_1 educ + u$$

Labor force experience,
tenure with current employer,
work ethic, intelligence …

Measures the change in hourly wage given another year of education, holding all other factors fixed

The Simple Regression Model

- **When is there a causal interpretation?**

- **Zero mean assumption**

$$E(u) = 0$$

- **Conditional mean independence assumption**

$$E(u|x) = E(u)$$

The explanatory variable must not contain information about the mean of the unobserved factors

- **Example: wage equation**

$$wage = \beta_0 + \beta_1 educ + u$$

e.g. intelligence …

The conditional mean independence assumption is unlikely to hold because individuals with more education will also be more intelligent on average.

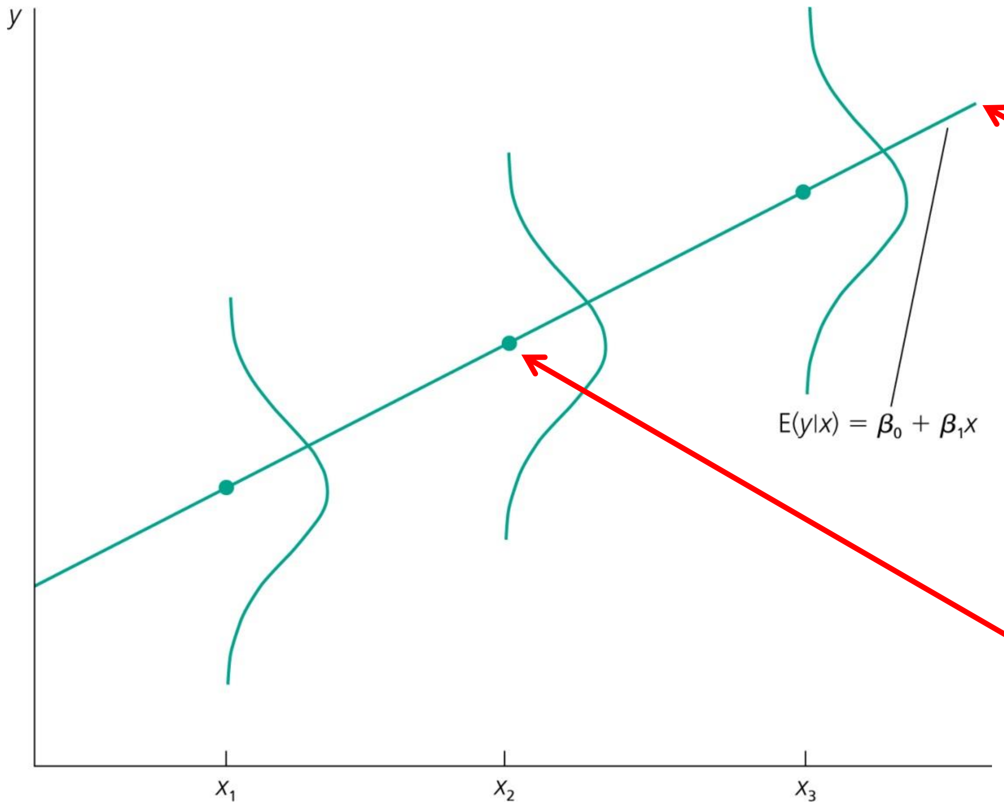
The Simple Regression Model

- **Population regression function (PRF)**
  - The zero conditional mean assumption implies that

$$
\begin{aligned}
E(y|x) &= E(\beta_0 + \beta_1 x + u|x) \\
&= \beta_0 + \beta_1 x + E(u|x) \\
&= \beta_0 + \beta_1 x
\end{aligned}
$$

  - This means that the average value of the dependent variable can be expressed as a linear function of the explanatory variable

The Simple Regression Model



Population regression function

For individuals with $x = x_2$, the average value of y is $\beta_0 + \beta_1 x$

The Simple Regression Model

- **In order to estimate the regression model one needs data**
- **A random sample of n observations**

(x_1, y_1) First observation

(x_2, y_2) Second observation

(x_3, y_3) Third observation

$\vdots$

(x_n, y_n) n-th observation

$$\{(x_i, y_i) : \; i = 1, \ldots n\}$$

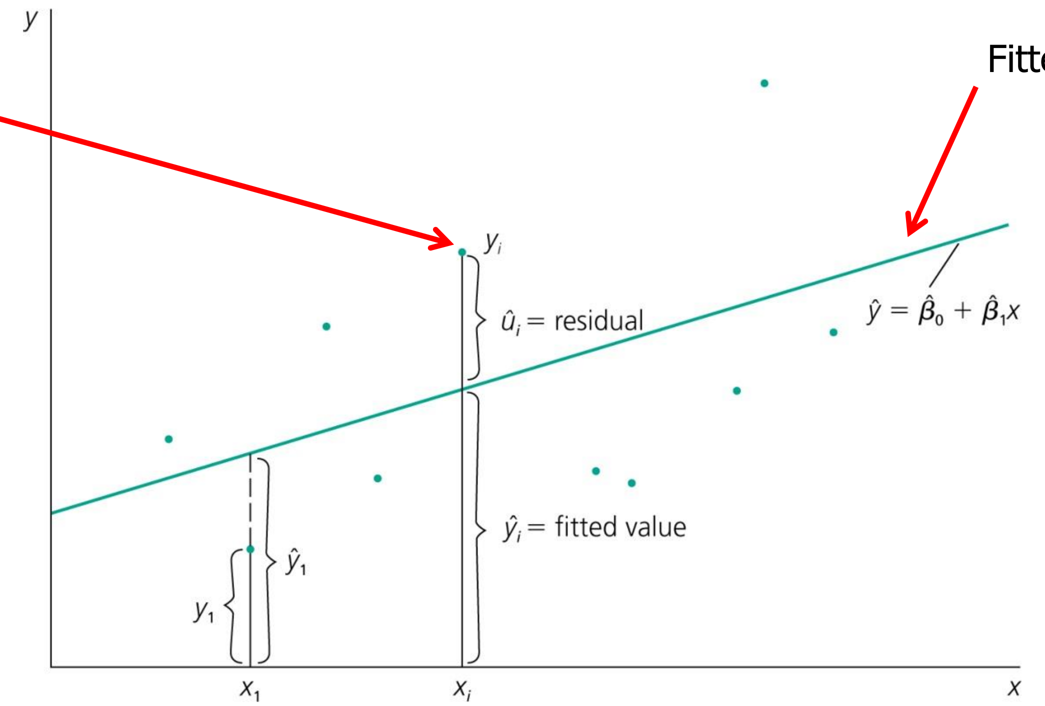
Value of the <u>explanatory variable</u> of the i-th observation

Value of the <u>dependent</u> variable of the i-th observation

The Simple Regression Model

- **Fit as good as possible a regression line through the data points:**

For example, the i-th data point (x_i, y_i)



Fitted regression line

The Simple Regression Model

- **<u>What does „as good as possible“ mean?</u>**
- **Regression residuals**

$$\widehat{u}_i = y_i - \widehat{y}_i = y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i$$

- **Minimize sum of squared regression residuals**

$$\min \sum_{i=1}^{n} \widehat{u}_i^2 \quad \rightarrow \quad \widehat{\beta}_0, \widehat{\beta}_1$$

- **<u>Ordinary Least Squares (OLS) estimates</u>**

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}, \quad \widehat{\beta}_0 = \bar{y} - \widehat{\beta}_1 \bar{x}$$

The Simple Regression Model

- **CEO Salary and return on equity**

$$salary = \beta_0 + \beta_1 roe + u$$

Salary in thousands of dollars

Return on equity of the CEO's firm

- **Fitted regression**

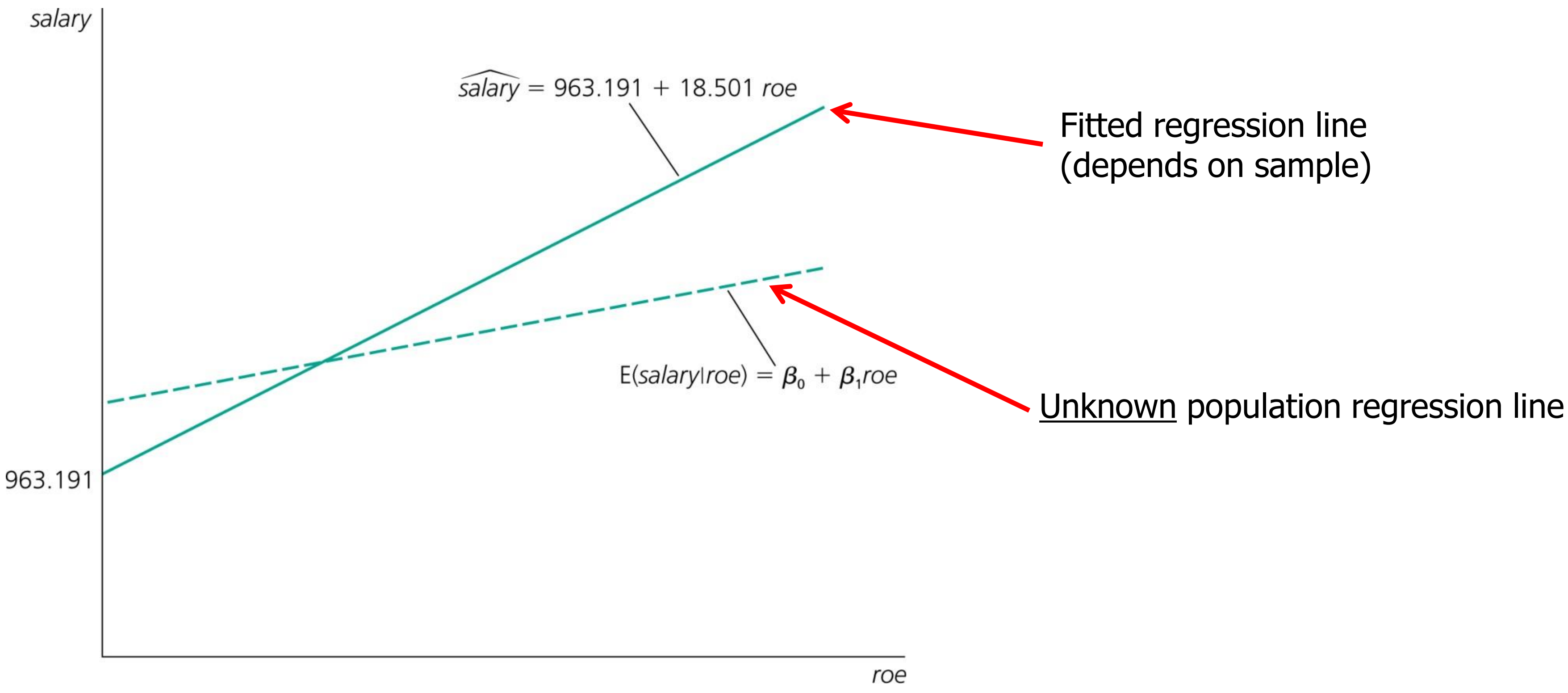
$$\widehat{salary} = 963.191 + 18.501\ roe$$

Intercept

If the return on equity increases by 1 percent,
then salary is predicted to change by 18,501 $

- **Causal interpretation?**

The Simple Regression Model



The Simple Regression Model

- **Wage and education**

$$wage = \beta_0 + \beta_1 educ + u$$

Hourly wage in dollars

Years of education

- **Fitted regression**

$$\widehat{wage} = -0.90 + 0.54\ educ$$

Intercept

In the sample, one more year of education was associated with an increase in hourly wage by 0.54 $

The Simple Regression Model

- **Voting outcomes and campaign expenditures (two parties)**

$$voteA = \beta_0 + \beta_1 shareA + u$$

Percentage of vote for candidate A

Percentage of campaign expenditures candidate A

- **Fitted regression**

$$\widehat{voteA} = 26.81 + 0.464\ shareA$$

Intercept

If candidate A's share of spending increases by one percentage point, he or she receives 0.464 percentage points more of the total vote

The Simple Regression Model

- <u>**Properties of OLS on any sample of data**</u>
- **Fitted values and residuals**

$$\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_i \qquad \widehat{u}_i = y_i - \widehat{y}_i$$

Fitted or predicted values

Deviations from regression line (= residuals)

- **Algebraic properties of OLS regression**

$$\sum_{i=1}^{n} \widehat{u}_i = 0 \qquad \sum_{i=1}^{n} x_i \widehat{u}_i = 0 \qquad \bar{y} = \widehat{\beta}_0 + \widehat{\beta}_1 \bar{x}$$

Deviations from regression line sum up to zero

Correlation between deviations and regressors is zero

Sample averages of y and x lie on regression line

The Simple Regression Model

x_i

y_i

$\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_i$

$\widehat{u}_i = y_i - \widehat{y}_i$

TABLE 2.2 Fitted Values and Residuals for the First 15 CEOs

obsno	roe	salary	salaryhat	uhat
1	14.1	1095	1224.058	−129.0581
2	10.9	1001	1164.854	−163.8542
3	23.5	1122	1397.969	−275.9692
4	5.9	578	1072.348	−494.3484
5	13.8	1368	1218.508	149.4923
6	20.0	1145	1333.215	−188.2151
7	16.4	1078	1266.611	−188.6108
8	16.3	1094	1264.761	−170.7606
9	10.5	1237	1157.454	79.54626
10	26.3	833	1449.773	−616.7726
11	25.9	567	1442.372	−875.3721
12	26.8	933	1459.023	−526.0231
13	14.8	1339	1237.009	101.9911
14	22.3	937	1375.768	−438.7678
15	56.3	2011	2004.808	6.191895

For example, CEO number 12's salary was 526,023 $ lower than predicted using the the information on his firm's return on equity

The Simple Regression Model

- **Goodness-of-Fit**

„How well does the explanatory variable explain the dependent variable?"

- **Measures of Variation**

$$SST = \sum_{i=1}^{n} (y_i - \bar{y})^2 \qquad SSE = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 \qquad SSR = \sum_{i=1}^{n} \hat{u}_i^2$$

<u>Total sum of squares</u>, represents total variation in dependent variable

<u>Explained sum of squares</u>, represents variation explained by regression

<u>Residual sum of squares</u>, represents variation <u>not</u> explained by regression

The Simple Regression Model

- **Decomposition of total variation**

$$SST = SSE + SSR$$

Total variation

Explained part

Unexplained part

- **Goodness-of-fit measure (R-squared)**

$$R^2 = \frac{SSE}{SST} = 1 - \frac{SSR}{SST}$$

R-squared measures the fraction of the total variation that is explained by the regression

The Simple Regression Model

- **CEO Salary and return on equity**

$$\widehat{salary} = 963.191 + 18.501\ roe$$

$$n = 209, \quad R^2 = 0.0132$$

The regression explains only 1.3 %
of the total variation in salaries

- **Voting outcomes and campaign expenditures**

$$\widehat{voteA} = 26.81 + 0.464\ shareA$$

$$n = 173, \quad R^2 = 0.856$$

The regression explains 85.6% of the total
variation in election outcomes

- **<u>Caution:</u> A high R-squared does not necessarily mean that the regression has a causal interpretation!**

The Simple Regression Model

- **<u>Incorporating nonlinearities: Semi-logarithmic form</u>**

- **Regression of log wages on years of eduction**

$$\log(wage) = \beta_0 + \beta_1 educ + u$$

Natural logarithm of wage

- **This changes the interpretation of the regression coefficient:**

$$\beta_1 = \frac{\partial \log(wage)}{\partial educ} = \frac{1}{wage} \cdot \frac{\partial wage}{\partial educ} = \frac{\frac{\partial wage}{wage}}{\partial educ}$$

Percentage change of wage

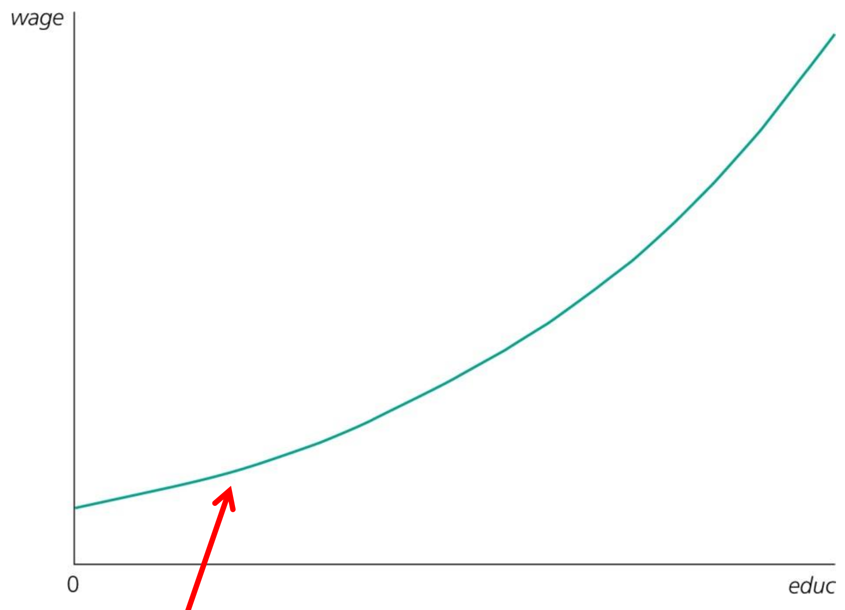
... if years of education are increased by one year

The Simple Regression Model

- **Fitted regression**

$$\widehat{\log}(wage) = 0.584 + 0.083\ educ$$

The wage increases by 8.3 % for every additional year of education (= return to education)



For example:

$$\frac{\frac{\partial wage}{wage}}{\partial educ} = \frac{\frac{+0.83\$}{10\$}}{+1 \text{ year}} = 0.083 = +8.3\%$$

Growth rate of wage is 8.3 % per year of education

The Simple Regression Model

- **<u>Incorporating nonlinearities: Log-logarithmic form</u>**

- **CEO salary and firm sales**

$$\log(salary) = \beta_0 + \beta_1 \log(sales) + u$$

Natural logarithm of CEO salary

Natural logarithm of his/her firm‘s sales

- **This changes the interpretation of the regression coefficient:**

$$\beta_1 = \frac{\partial \log(salary)}{\partial \log(sales)} = \frac{\frac{\partial salary}{salary}}{\frac{\partial sales}{sales}}$$

<u>Percentage</u> change of salary

… if sales increase <u>by 1 %</u>

<u>Logarithmic changes are always percentage changes</u>

The Simple Regression Model

- **CEO salary and firm sales: fitted regression**

$$\widehat{\log}(salary) = 4.822 + 0.257 \log(sales)$$

+ 1 % sales ! + 0.257 % salary

- **For example:**

$$\frac{\frac{\partial salary}{salary}}{\frac{\partial sales}{sales}} = \frac{\frac{+2{,}570\$}{1{,}000{,}000\$}}{\frac{+10{,}000{,}000\$}{1{,}000{,}000{,}000\$}} = \frac{+0.257\%\ \text{salary}}{+1\%\ \text{sales}} = 0.257$$

- **The log-log form postulates a <u>constant elasticity</u> model, whereas the semi-log form assumes a <u>semi-elasticity</u> model**

The Simple Regression Model

TABLE 2.3 Summary of Functional Forms Involving Logarithms

Model	Dependent Variable	Independent Variable	Interpretation of β_1
Level-level	y	x	$\Delta y = \beta_1 \Delta x$
Level-log	y	$\log(x)$	$\Delta y = (\beta_1/100)\%\Delta x$
Log-level	$\log(y)$	x	$\%\Delta y = (100\beta_1)\Delta x$
Log-log	$\log(y)$	$\log(x)$	$\%\Delta y = \beta_1 \%\Delta x$

The Simple Regression Model

- **<u>Expected values and variances of the OLS estimators</u>**
- **The estimated regression coefficients are random variables because they are calculated from a random sample**

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}, \qquad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

Data is random and depends on particular sample that has been drawn

- **The question is what the estimators will estimate on average and how large their variability in repeated samples is**

$$E(\hat{\beta}_0) = ?, \; E(\hat{\beta}_1) = ? \qquad Var(\hat{\beta}_0) = ?, \; Var(\hat{\beta}_1) = ?$$

The Simple Regression Model

- <u>**Standard assumptions for the linear regression model**</u>
- **Assumption SLR.1 (Linear in parameters)**

$$y = \beta_0 + \beta_1 x + u$$

In the population, the relationship between y and x is linear

- **Assumption SLR.2 (Random sampling)**

$$\{(x_i, y_i) : \ i = 1, \ldots n\}$$

The data is a random sample drawn from the population

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

Each data point therefore follows the population equation

The Simple Regression Model

- **Discussion of random sampling: Wage and education**
  - The population consists, for example, of all workers of country A
  - In the population, a linear relationship between wages (or log wages) and years of education holds
  - Draw completely randomly a worker from the population
  - The wage and the years of education of the worker drawn are random because one does not know beforehand which worker is drawn
  - Throw back worker into population and repeat random draw n times
  - The wages and years of education of the sampled workers are used to estimate the linear relationship between wages and education

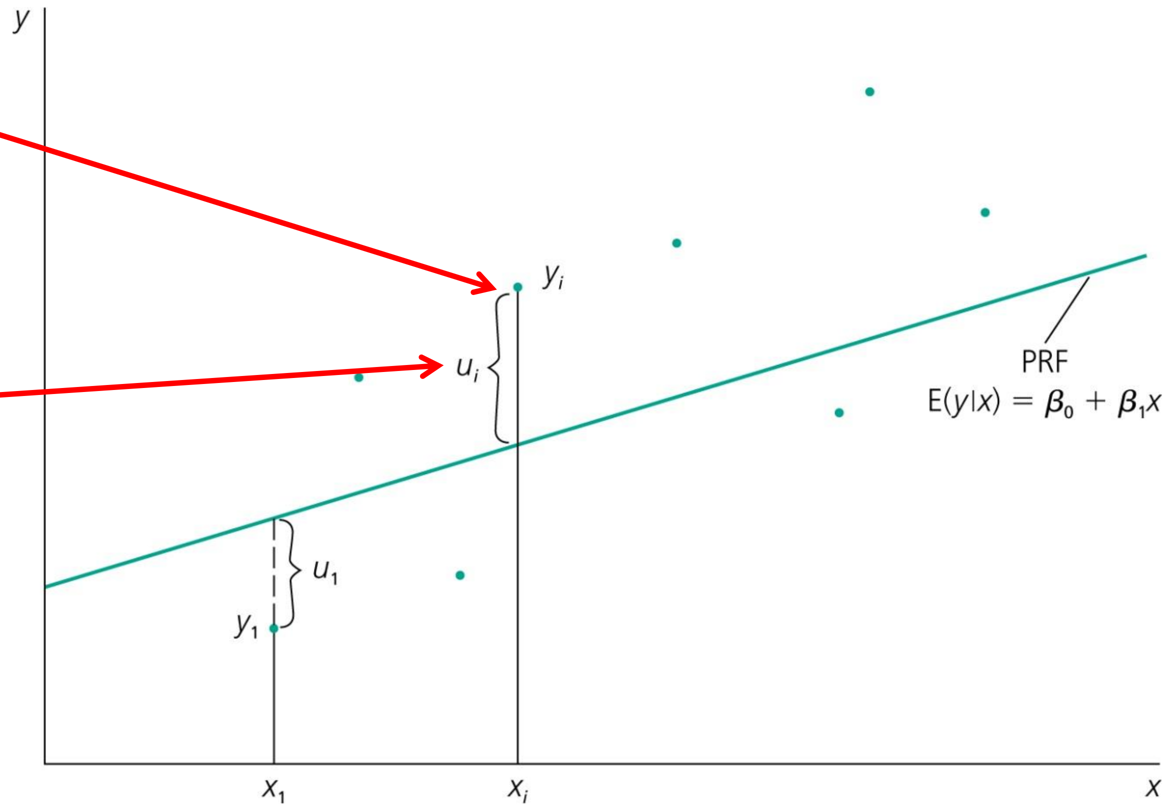
The Simple Regression Model

The values drawn for the i-th worker

(x_i, y_i)

The implied deviation from the population relationship for the i-th worker:

$$u_i = y_i - \beta_0 - \beta_1 x_i$$



The Simple Regression Model

- **<u>Assumptions for the linear regression model (cont.)</u>**

- **Assumption SLR.3 (Sample variation in explanatory variable)**

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 > 0$$

The values of the explanatory variables are not all the same (otherwise it would be impossible to study how different values of the explanatory variable lead to different values of the dependent variable)

- **Assumption SLR.4 (Zero conditional mean)**

$$E(u_i|x_i) = 0$$

The value of the explanatory variable must contain no information about the mean of the unobserved factors

The Simple Regression Model

- **<u>Theorem 2.1 (Unbiasedness of OLS)</u>**

$$SLR.1 - SLR.4 \quad \Rightarrow \quad E(\widehat{\beta}_0) = \beta_0,\ E(\widehat{\beta}_1) = \beta_1$$

- **Interpretation of unbiasedness**
  - The estimated coefficients may be smaller or larger, depending on the sample that is the result of a random draw
  - However, on average, they will be equal to the values that characterize the true relationship between y and x in the population
  - „On average“ means if sampling was repeated, i.e. if drawing the random sample and doing the estimation was repeated many times
  - In a given sample, estimates may differ considerably from true values

The Simple Regression Model

- **<u>Variances of the OLS estimators</u>**
  - Depending on the sample, the estimates will be nearer or farther away from the true population values
  - How far can we expect our estimates to be away from the true population values on average (= sampling variability)?
  - Sampling variability is measured by the estimator‘s variances

$$Var(\widehat{\beta}_0),\ \ Var(\widehat{\beta}_1)$$

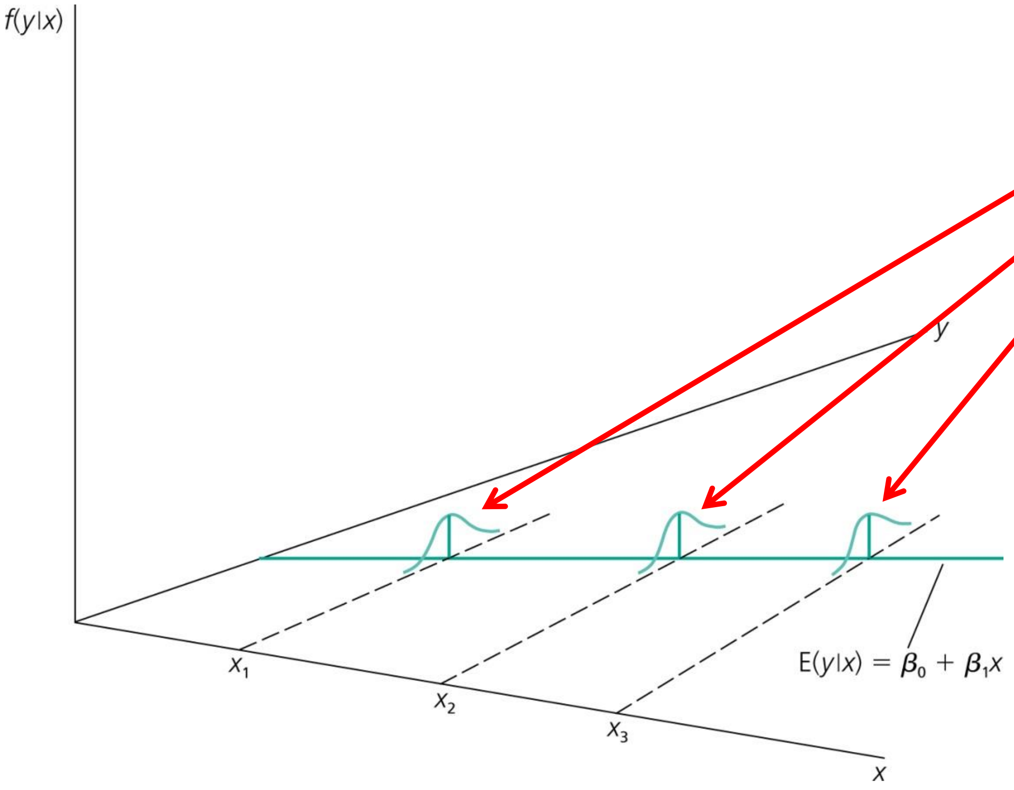
- **Assumption SLR.5 (Homoskedasticity)**

$$Var(u_i|x_i) = \sigma^2$$

The value of the explanatory variable must contain no information about the <u>variability</u> of the unobserved factors

The Simple Regression Model

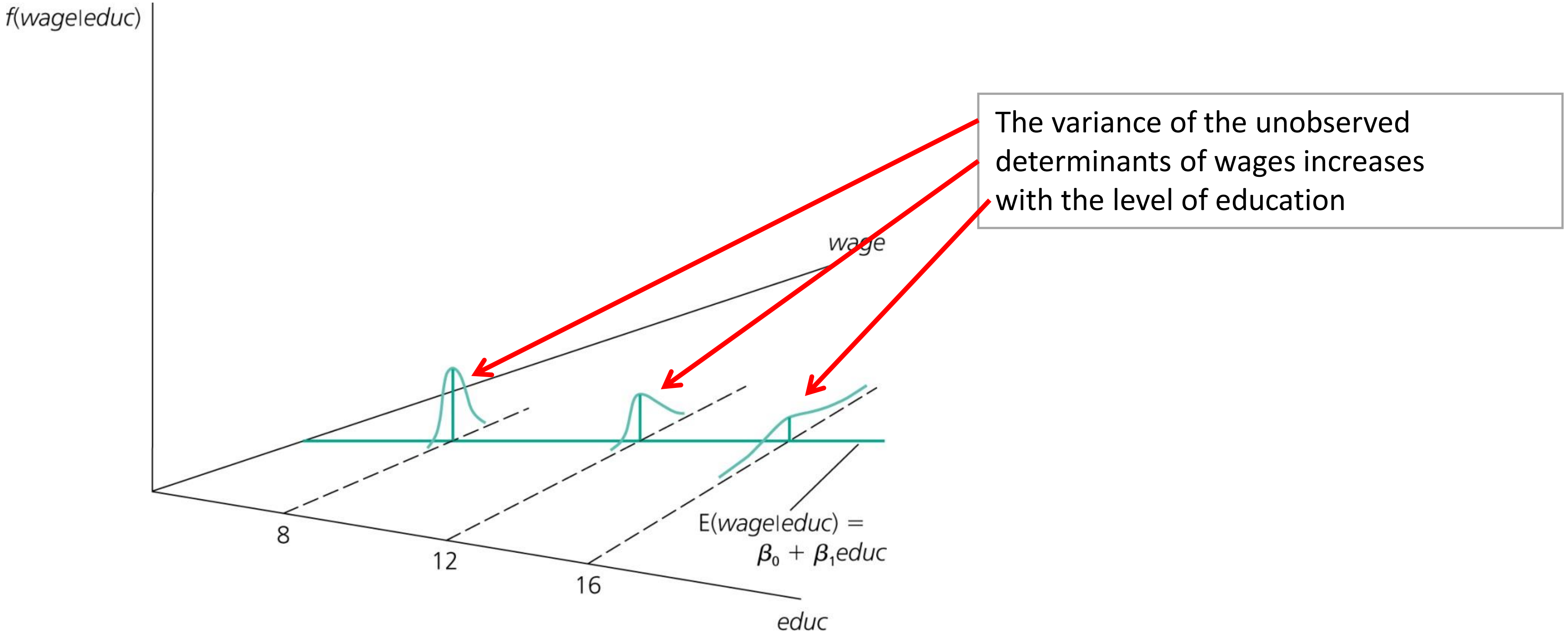
- **Graphical illustration of homoskedasticity**



The variability of the unobserved influences does not depend on the value of the explanatory variable

The Simple Regression Model

- **An example for heteroskedasticity: Wage and education**



The Simple Regression Model

- **Theorem 2.2 (Variances of OLS estimators)**

  Under assumptions SLR.1 – SLR.5:

$$Var(\widehat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\sigma^2}{SST_x}$$

$$Var(\widehat{\beta}_0) = \frac{\sigma^2 n^{-1} \sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\sigma^2 n^{-1} \sum_{i=1}^{n} x_i^2}{SST_x}$$

- **Conclusion:**
  - The sampling variability of the estimated regression coefficients will be the higher the larger the variability of the unobserved factors, and the lower, the higher the variation in the explanatory variable

The Simple Regression Model

- **Estimating the error variance**

$$Var(u_i|x_i) = \sigma^2 = Var(u_i)$$

The variance of u does not depend on x, i.e. is equal to the unconditional variance

$$\tilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(\hat{u}_i - \bar{\hat{u}}_i)^2 = \frac{1}{n}\sum_{i=1}^{n}\hat{u}_i^2$$

One could estimate the variance of the errors by calculating the variance of the residuals in the sample; unfortunately this estimate would be biased

$$\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n}\hat{u}_i^2$$

An unbiased estimate of the error variance can be obtained by substracting the number of estimated regression coefficients from the number of observations

The Simple Regression Model

- **<u>Theorem 2.3 (Unbiasedness of the error variance)</u>**

$$SLR.1 - SLR.5 \quad \Rightarrow \quad E(\hat{\sigma}^2) = \sigma^2$$

- **Calculation of standard errors for regression coefficients**

$$se(\hat{\beta}_1) = \sqrt{\widehat{Var}(\hat{\beta}_1)} = \sqrt{\hat{\sigma}^2 / SST_x}$$

$$se(\hat{\beta}_0) = \sqrt{\widehat{Var}(\hat{\beta}_0)} = \sqrt{\hat{\sigma}^2 n^{-1} \sum_{i=1}^{n} x_i^2 / SST_x}$$

Plug in $\hat{\sigma}^2$ for
the unknown σ^2

The estimated standard deviations of the regression coefficients are called „standard errors“.
<u>They measure how precisely the regression coefficients are estimated.</u>

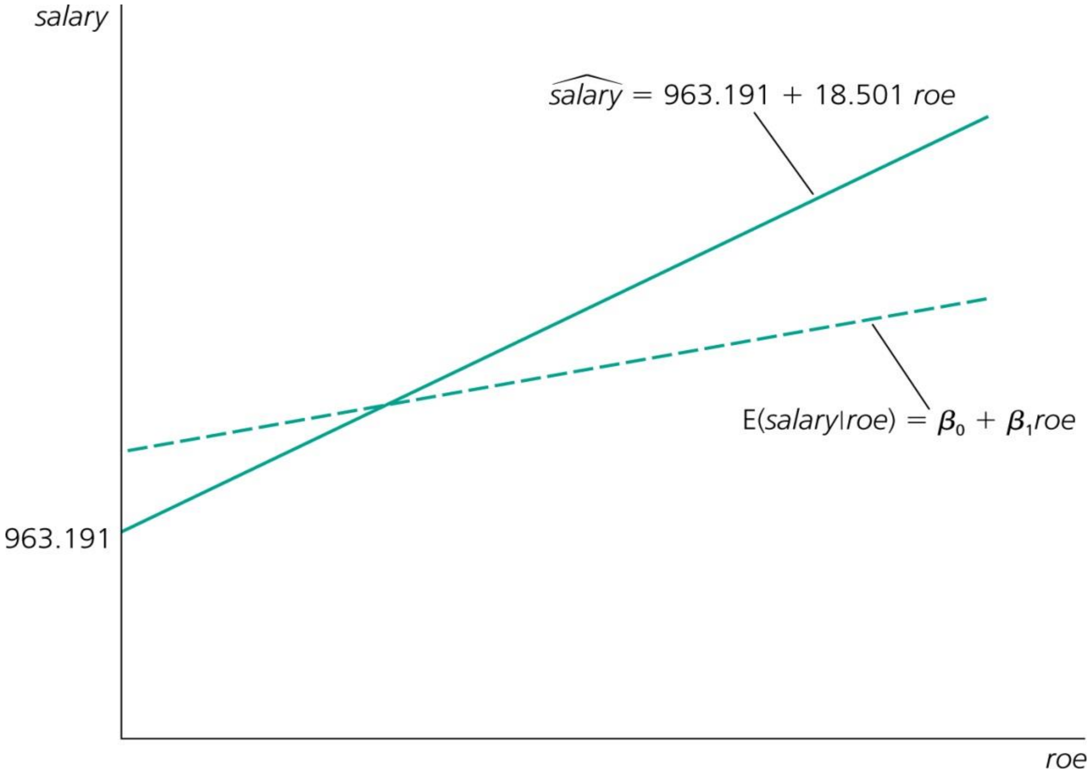
Summary

- Interpretation of slope coefficient
  - β_1 measures how y changes if x is increased by one unit, holding all other factors fixed.
- When is there a causal interpretation?
  - Conditional mean independence assumption

$$E(u|x) = E(u)$$

Summary

- Population regression function (PRF) and Sample regression function (SRF)



Summary

- OLS estimates

$$\widehat{u}_i = y_i - \widehat{y}_i = y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i$$

$$\min \sum_{i=1}^{n} \widehat{u}_i^2 \quad \rightarrow \quad \widehat{\beta}_0, \widehat{\beta}_1$$

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}, \qquad \widehat{\beta}_0 = \bar{y} - \widehat{\beta}_1 \bar{x}$$

- Algebraic properties

$$\sum_{i=1}^{n} \widehat{u}_i = 0 \qquad \sum_{i=1}^{n} x_i \widehat{u}_i = 0 \qquad \bar{y} = \widehat{\beta}_0 + \widehat{\beta}_1 \bar{x}$$

Summary

- Goodness-of-fit measure

$$R^2 = \frac{SSE}{SST} = 1 - \frac{SSR}{SST}$$

$$SST = \sum_{i=1}^{n} (y_i - \bar{y})^2 \qquad SSE = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 \qquad SSR = \sum_{i=1}^{n} \hat{u}_i^2$$

  - R-squared measures the fraction of the total variation that is explained by the regression
  - Between 0 and 1

Summary

- Incorporating nonlinearities

TABLE 2.3 Summary of Functional Forms Involving Logarithms

Model	Dependent Variable	Independent Variable	Interpretation of β_1
Level-level	y	x	$\Delta y = \beta_1 \Delta x$
Level-log	y	$\log(x)$	$\Delta y = (\beta_1/100)\%\Delta x$
Log-level	$\log(y)$	x	$\%\Delta y = (100\beta_1)\Delta x$
Log-log	$\log(y)$	$\log(x)$	$\%\Delta y = \beta_1 \%\Delta x$

Summary

- Standard assumptions
  - SLR.1: Linear in parameters
  - SLR.2: Random sampling
  - SLR.3: Sample variation in explanatory variable
  - SLR.4: Zero conditional mean
  - SLR.5: Homoskedasticity

Summary

- Three theorems
  - Unbiasedness of OLS

$$SLR.1 - SLR.4 \quad \Rightarrow \quad E(\hat{\beta}_0) = \beta_0,\ E(\hat{\beta}_1) = \beta_1$$

  - Variances of OLS estimators

$$SLR.1 - SLR.5 \Rightarrow \quad Var(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\sigma^2}{SST_x}$$

$$Var(\hat{\beta}_0) = \frac{\sigma^2 n^{-1} \sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\sigma^2 n^{-1} \sum_{i=1}^{n} x_i^2}{SST_x}$$

  - Unbiasedness of the error variance

$$SLR.1 - SLR.5 \quad \Rightarrow \quad E(\hat{\sigma}^2) = \sigma^2 \quad \text{where} \quad \hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n} \hat{u}_i^2$$