Ordinary Differential Equations

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Chapter 3: Second Order Linear Equations

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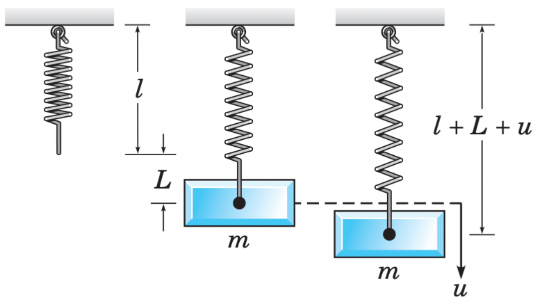
Motivation: spring-mass system



Newton's Law: $ma = f$

$a = u''$

$f = mg - k(L+u) - \gamma u' + F$

k: spring constant

γ: damping coefficient

$$mu'' = mg - k(L+u) - \gamma u' + F$$

$$mu'' + \gamma u' + ku = mg - kL + F$$

But $mg = kL$, so

$$mu'' + \gamma u' + ku = F$$

1 Homogeneous Equations with Constant Coefficients

$$ay'' + by' + cy = 0,$$

initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

....

Example 1.1. Solve the IVP

$$y'' - y = 0, \quad y(0) = 2, \quad y'(0) = -1.$$

By investigation, we know $y = ce^t$ satisfies the equation for any constant c. However, it doesn't satisfy the initial conditions. More investigation shows $y = ce^{-t}$ is also a solution for any constant c. It turns out

$$y = c_1 e^t + c_2 e^{-t}$$

is a solution for any constants c_1, c_2. Now, the initial conditions require

$$c_1 + c_2 = 2, \quad c_1 - c_2 = -1.$$

This is a system of linear equations. The matrix form is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

The matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is nonsingular ($\det A = -2 \neq 0$). So the system has a unique solution. The solution is

$$c_1 = \frac{1}{2}, \quad c_2 = -\frac{3}{2}.$$

So the IVP has a solution

$$y = \frac{1}{2} e^t - \frac{3}{2} e^{-t}.$$

....

Example 1.2. Solve

$$y'' + 5y' + 6y = 0.$$

Answer: We assume the ansatz of the solution: $y = e^{rt}$ for some constant r. Then

$$\begin{aligned} y'' + 5y' + 6y &= r^2 e^{rt} + 5r e^{rt} + 6 e^{rt} \\ &= (r^2 + 5r + 6) e^{rt} \\ &= 0 \\ \Rightarrow \quad r^2 + 5r + 6 &= 0 \end{aligned}$$

The equation

$$r^2 + 5r + 6 = 0$$

is called the **characteristic equation** for the ODE. The solutions are

$$r_1 = -3, \quad r_2 = -2.$$

So have two solutions of the ODE:

$$y_1 = e^{-3t}, \quad y_2 = e^{-2t}.$$

So we can let the solution be

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{-3t} + c_2 e^{-2t}.$$

Note that

$$y \to 0 \quad \text{as} \quad t \to \infty$$

Example 1.3. Solve the IVP

$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}.$$

Answer: The characteristic equation is

$$4r^2 - 8r + 3 = (2r - 3)(2r - 1) = 0 \quad \Rightarrow \quad r_1 = \frac{3}{2}, \quad r_2 = \frac{1}{2}.$$

So we have solutions

$$y = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t}.$$

Plugging initial conditions

$$\begin{aligned} c_1 + c_2 &= 2 \\ \frac{3}{2} c_1 + \frac{1}{2} c_2 &= \frac{1}{2} \end{aligned}$$

We can find unique solutions

$$c_1 = -\frac{1}{2}, \quad c_2 = \frac{5}{2}.$$

So a solution of the IVP is

$$y = -\frac{1}{2} e^{\frac{3}{2}t} + \frac{5}{2} e^{\frac{1}{2}t}.$$

Note that

$$y \to -\infty \quad \text{as} \quad t \to \infty$$

2 Theory of 2nd Order Linear Equations

Consider the general 2nd order linear equation

$$L[y] := y''(t) + p(t)\,y' + q(t)\,y = g(t).$$

Note that L is a linear operator.

Existence and Uniqueness Theorem *Consider the IVP*

$$y''(t) + p(t)\,y' + q(t)\,y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

If p, q, g are continuous on an interval I containing t_0, then there exists a unique solution to this IVP on I.

Example 2.1. Find the longest interval in which the solution of the initial value problem

$$(t^2 - 3t)\,y'' + ty' - (t+3)\,y = 0, \quad y(1) = 2, \quad y'(1) = 1$$

is certain to exist.

Answer: Assuming $t \neq 0, t \neq 3$, rewrite the equation in the standard form

$$y'' + \frac{1}{t-3}y' - \frac{t+3}{t^2 - 3t}y = 0.$$

So p, q, g are continuous in $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$. Since $1 \in (0, 3)$. By the E&U theorem, there exists a unique solution to the IVP on $(0, 3)$.

Principle of Superposition *Consider the **homogeneous** linear equation*

$$L[y] = 0.$$

If y_1 and y_2 are both solutions, then $c_1 y_1 + c_2 y_2$ is also a solution for any constants c_1 and c_2.

Proof.

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= (c_1y_1 + c_2y_2)'' + p(t)\,(c_1y_1 + c_2y_2)' + q(t)\,(c_1y_1 + c_2y_2) \\ &= c_1\,(y_1'' + p(t)\,y_1' + q(t)\,y_1) + c_2\,(y_2'' + p(t)\,y_2' + q(t)\,y_2) \\ &= c_1L[y_1] + c_2L[y_2] = 0 + 0 = 0. \end{aligned}$$

So $c_1 y_2 + c_2 y_2$ is also a solution. □

Let's consider the homogeneous equation with constant coefficients

$$ay'' + by' + cy = 0.$$

Suppose there are two different roots r_1, r_2 of the characteristic polynomial $ar^2 + br + c$. Then we have two solutions

$$y_1 = e^{r_1 t}, \quad y_2 = e^{r_2 t}.$$

By the principle of superposition,

$$y = c_1 y_1 + c_2 y_2$$

is a solution for any constants c_1, c_2.

The next question: can we always find c_1, c_2 such that a given initial conditions are satisfied?

Pluggin the initial conditions, we obtain a linear system for c_1, c_2:

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0' \end{aligned}$$

or, in matrix form

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}.$$

In order to have a solution for arbitrary values of y_0, y_0', we need the matrix to be nonsingular, i.e.

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)\, y_2'(t_0) - y_1'(t_0)\, y_2(t_0) \neq 0.$$

Definition 2.2 *Suppose y_1, y_2 are two solutions of the ODE $L[y] = 0$. Then the Wronskian of them is*

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

Theorem 2.3 *Let y_1, y_2 are solutions of the equation $L[y] = 0$. Then one can find constants c_1 and c_2 such that $c_1 y_1 + c_2 y_2$ solves the IVP*

$$L[y] = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

regardless of the values y_0 and y_0' if and only if $W(y_1, y_2)(t_0) \neq 0$.

Next we show all solutions of $L[y] = 0$ can are actually in the form $c_1 y_1 + c_2 y_2$ if and only if the Wronskian is nonzero.

Theorem 2.4 *Let y_1, y_2 are solutions of the equation $L[y] = 0$ on some interval I. Then every solution of $L[y] = 0$ on I can be written as $c_1 y_1 + c_2 y_2$ if and only if $W(y_1, y_2)(t) \neq 0$ for some $t \in I$.*

Proof. Suppose $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$. Let $\phi(t)$ to be a solution of $L[y] = 0$. Let $y_0 = \phi(t_0)$ and $y_0' = \phi'(t_0)$. Now consider the IVP

$$L[y] = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{2.1}$$

Clearly ϕ is a solution of the IVP (2.1). On the other hand, we can find c_1 and c_2 such that $c_1 y_1 + c_2 y_2$ is a solution of the IVP (2.1) for some c_1, c_2 since $W(y_1, y_2)(t_0) \neq 0$. By the uniqueness part of the E&U theorem, we have $\phi = c_1 y_1 + c_2 y_2$.

Next, suppose $W(y_1, y_2)(t) = 0$ for any $t \in I$. Then $W(y_1, y_2)(t_0) = 0$ for some $t_0 \in I$. So there exists some numbers y_0, y_0' such that

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix} \tag{2.2}$$

has no solution. Let $\phi(t)$ to be the solution of the IVP (2.1). Suppose $\phi = c_1 y_1 + c_2 y_2$ for some c_1, c_2, then c_1, c_2 must satisfy the linear system (2.2). A contradition! $\square$

If $W(y_1, y_2)(t) \neq 0$ for some t, we call the solutions $\{y_1, y_2\}$ a **fundamental set of solutions**.

Example 2.5. If $r_1 \neq r_2$ are real numbers, and $y_1 = e^{r_1 t}, y_2 = e^{r_2 t}$ are solutions of some ODE. Then

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)t} \neq 0$$

for any t. So $\{y_1, y_2\}$ form a fundamental set of solutions.

Example 2.6. Show that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions of

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0.$$

Answer:

$$W(y_1, y_2)(t) = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2} t^{-1/2} & -t^{-2} \end{vmatrix} = -\frac{3}{2} t^{-3/2} \neq 0$$

for any $t \neq 0$. So $\{y_1, y_2\}$ form a fundamental set of solutions for $t \neq 0$.

Theorem 2.7 *Let y_1 to be the solution of the IVP*

$$L[y] = 0, \quad y(t_0) = 1, \quad y'(t_0) = 0.$$

Let y_2 to be the solution of the IVP

$$L[y] = 0, \quad y(t_0) = 0, \quad y'(t_0) = 1.$$

Then the Wronskian of y_1, y_2 is $W(t) = 1$. So $\{y_1, y_2\}$ form a fundamental set of solutions.

This theorem says fundamental set of solutions always exist. In fact, we can find infinitely many fundamental set of solutions.

Example 2.8. Find the fundamental set of solutions y_1 and y_2 specified by Theorem 2.7 for the differential equation

$$y'' - y = 0.$$

Answer: Using the characteristic equations, we find two solutions

$$y_1 = e^t, \quad y_2 = e^{-t}.$$

We can check they form a fundamental set of solutions. Let's find another fundamental set of solutions defined as in Theorem 2.7 for $t_0 = 0$. Let

$$y_3 = \frac{e^t + e^{-t}}{2} = \cosh t, \quad y_4 = \frac{e^t - e^{-t}}{2} = \sinh t.$$

Then $W(y_3, y_4) = 1$. So the general solution can be written as

$$c_1 y_1 + c_2 y_2 \text{ or } c_3 y_3 + c_4 y_4.$$

Theorem 2.9 **(Abel)** *Let y_1, y_2 are solutions of the equation*

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0.$$

Then the Wronskian

$$W(y_1, y_2)(t) = c e^{-\int p(t)\,dt}$$

for some constant c, which may depend on y_1, y_2 but otherwise independent of p, q.

Proof. We have

$$\begin{aligned} y_1'' + p(t)\,y_1' + q(t)\,y_1 &= 0, \\ y_2'' + p(t)\,y_2' + q(t)\,y_2 &= 0. \end{aligned}$$

Then

$$\begin{aligned} y_2\,[y_1'' + p(t)\,y_1' + q(t)\,y_1] &= 0, \\ y_1\,[y_2'' + p(t)\,y_2' + q(t)\,y_2] &= 0. \end{aligned}$$

Subtracting two equations, we obtain

$$y_1\,y_2'' - y_1''\,y_2 + p(t)\,(y_1\,y_2' - y_1'\,y_2) = 0.$$

Note that

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1\,y_2' - y_2\,y_1'.$$

$$W'(t) = y_1'\,y_2' + y_1\,y_2'' - y_2'\,y_1' - y_2\,y_1'' = y_1\,y_2'' - y_2\,y_1''.$$

So we obtain

$$W'(t) + p(t)\,W(t) = 0.$$

Solving this first order linear ODE, we obtain

$$W(t) = c e^{-\int p(t)\,dt}.$$

□

Remark 2.10. From the Abel's theorem, we now know the Wronskian of two solutions of an ODE is either identically zero or nowhere zero.

Theorem 2.11 *Suppose p, q are real-valued functions. Let $y(t) = u(t) + i v(t)$ be a complex-valued solution of*

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0,$$

where u, v are real-valued functions. Then u, v are also solutions of $L[y] = 0$.

Proof. We have

$$\begin{aligned} L[y] &= (u + iv)'' + p(t)\,(u + iv)' + q(t)\,(u + iv) \\ &= (u'' + iv'') + p(t)\,(u' + iv') + q(t)\,(u + iv) \\ &= (u'' + p(t)\,u' + q(t)\,u) + i\,(v'' + p(t)\,v' + q(t)\,v) \\ &= 0. \end{aligned}$$

So

$$u'' + p(t)u' + q(t)u = 0, \quad v'' + p(t)v' + q(t)v = 0.$$

That is, u, v are both solutions of $L[y] = 0$.

$\square$

3 Complex roots of the characteristic equation

Let's go back to equations with constant coefficients

$$ay'' + by' + cy = 0.$$

The characteristic equation is

$$ar^2 + br + c = 0.$$

The solution is given by the quadratic formula

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1. $b^2 > 4ac$. Then r_1, r_2 are both real and $r_1 \neq r_2$.
2. $b^2 = 4ac$. Then r_1, r_2 are both real and $r_1 = r_2$.
3. $b^2 < 4ac$. Then r_1, r_2 are both complex, and $r_2 = \bar{r}_1$.

Now consider case (3). Let $r_{1,2} = \lambda \pm i\mu$. So then we have two complex-valued solutions

$$y_1 = e^{r_1 t} = e^{(\lambda + i\mu)t}, \quad y_2 = e^{r_2 t} = e^{(\lambda - i\mu)t}.$$

What does it to raise a number to a complex exponent?

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Then we define

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\ &= \cos x + i \sin x \end{aligned}$$

Euler's formula

$$e^{ix} = \cos x + i \sin x.$$

Then we define

$$e^{\lambda + i\mu} = e^{\lambda} e^{i\mu} = e^{\lambda}(\cos x + i \sin x) = e^{\lambda} \cos x + i e^{\lambda} \sin x$$

One can verify the exponential function defined this way has the usual algebraic properties of exponential functions with real exponent.
The two complex solutions are

$$y_1 = e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos \mu t + i \sin \mu t), \quad y_2 = e^{(\lambda - i\mu)t} = e^{\lambda t}(\cos \mu t - i \sin \mu t).$$

One can verify y_1, y_2 form a fundamental set of solutions.

By the previous theorem, we know

$$y_3 = e^{\lambda t} \cos \mu t, \quad y_4 = e^{\lambda t} \sin \mu t.$$

are real-valued solutions. One can verify y_3, y_4 also form a fundamental set of solutions.

Example 3.1. Find the general solution (real-valued) of the differential equation

$$y'' + y' + 9.25 y = 0,$$

Also find the solution that satisfies the initial conditions

$$y(0) = 2, \quad y'(0) = 8,$$

and draw its graph.

Answer: The roots of the characteristic polynomial is

$$r_{1,2} = \frac{-1 \pm 6i}{2} = -\frac{1}{2} \pm 3i.$$

The real-valued solutions are

$$y_1 = e^{-\frac{t}{2}} \cos 3t, \quad y_2 = e^{-\frac{t}{2}} \sin 3t.$$

The general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{-\frac{t}{2}} \cos 3t + c_2 e^{-\frac{t}{2}} \sin 3t,$$

and

$$y' = c_1 e^{-\frac{t}{2}} \left(-\frac{1}{2} \cos 3t - 3 \sin 3t \right) + c_2 e^{-\frac{t}{2}} \left(-\frac{1}{2} \sin 3t + 3 \cos 3t \right).$$

Plugging the initial conditions,

$$\begin{aligned} c_1 &= 2, \\ -\frac{1}{2} c_1 + 3 c_2 &= 8. \end{aligned}$$

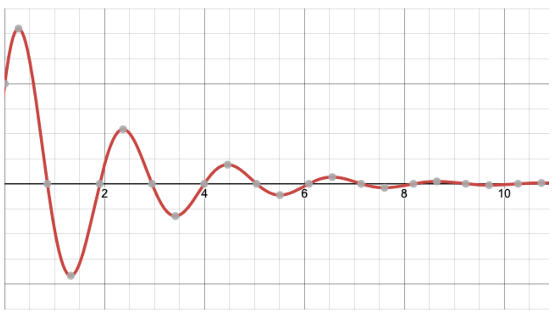
Solving the linear systm,

$$c_1 = 2, \quad c_2 = 3.$$

So the solution of the IVP is

$$y = 2e^{-\frac{t}{2}} \cos 3t + 3e^{-\frac{t}{2}} \sin 3t = e^{-\frac{t}{2}} (2 \cos 3t + 3 \sin 3t).$$

The graph is a damped oscillation.



Example 3.2. Find the general solution of

$$y'' + 9y = 0.$$

Answer:

$$r_{1,2} = \pm 3i$$

The general solution is

$$y = c_1 \cos 3t + c_2 \sin 3t.$$

The graph is an undamped oscillation.

4 Repeated Roots; Reduction of order

4.1 Repeated roots

Suppose the charateristic equation have one repeated root $r = -\dfrac{b}{2a}$. Then we have a solution

$$\boxed{y_1 = e^{rt}.}$$

Then $y_2 = c y_1 = c e^{rt}$ is also a solution for any constant c, but $\{y_1, y_2\}$ is not a fundamental set of solutions. To find another independent solution, let

$$y_2 = v(t)\, y_1.$$

Plugging y_2 into the equation,

$$\begin{aligned}
a(vy_1)'' + b(vy_1)' + c(vy_1) &= a(v'' y_1 + 2v' y_1' + v y_1'') + b(v' y_1 + v y_1') + c v y_1 \\
&= v(a y_1'' + b y_1' + c y_1) + a v'' y_1 + 2 a v' y_1' + b v' y_1 \\
&= a y_1 v'' + (2 a y_1' + b y_1) v' \\
&= a e^{rt} v'' + (2 a r e^{rt} + b e^{rt}) v' \\
&= e^{rt}(a v'' + (2ar + b) v') = 0 \\
\Rightarrow \quad & a v'' + (2ar + b) v' = a v'' = 0 \\
\Rightarrow \quad & v'' = 0 \quad \Rightarrow \quad v = c_1 t + c_2.
\end{aligned}$$

Then

$$y_2 = (c_1 t + c_2) e^{rt} = c_1 t e^{rt} + c_2 e^{rt}.$$

Choose

$$\boxed{y_2 = t e^{rt}.}$$

Then one can verify y_1, y_2 form a fundamental set of solutions (check $W(y_1, y_2) \neq 0$).

Example 4.1.

$$y'' + 4y' + 4y = 0.$$

Answer: The characteristic equation is $r^2 + 4r + 4 = 0$. The (repeated) root is $r = -2$. So

$$y_1 = e^{-2t}, \quad y_2 = t e^{-2t}.$$

The general solution is

$$y = c_1 e^{-2t} + c_2 t e^{-2t}.$$

We have $y(t) \to 0$ as $t \to \infty$.

4.2 Reduction of order

The idea to find y_2 can be generalized to a general second order linear equation. If y_1 is a solution of

$$y'' + p(t)\, y' + q(t)\, y = 0.$$

Let $y_2 = v(t)\, y_1$ be another solution. Then plugging y_2 into the equation we can obtain an second order linear ODE for $v(t)$:

$$y_1 v'' + (y_1' + p(t)\, y_1)\, v' = 0.$$

Let $w = v'$, then we obtain a first order ODE for w

$$y_1 w' + (y_1' + p(t)\, y_1)\, w = 0.$$

Solve w, then let $v = \displaystyle\int w$.

Example 4.2. Given the variable coefficient equation and solution y_1,

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0; \quad y_1(t) = t^{-1},$$

use reduction of order method to find a second solution.

Answer: Let $y_2 = v y_1$. Then

$$\begin{aligned} 2t^2 y_2'' + 3t y_2' - y_2 &= 2t^2 (v'' y_1 + 2v' y_1' + v y_1'') + 3t (v' y_1 + v y_1') - v y_1 \\ &= 2t^2 (t^{-1} v'' - 2t^{-2} v' + 2t^{-3} v) + 3t (t^{-1} v' - t^{-2} v) - t^{-1} v \\ &= 2t v'' - v' = 0. \end{aligned}$$

Let $w = v'$,

$$2tw' - w = 0 \quad \Rightarrow \quad \frac{dw}{w} = \frac{dt}{2t} \quad \Rightarrow \quad \ln w = \frac{1}{2} \ln t \quad \Rightarrow \quad w = c\sqrt{t} \quad \Rightarrow \quad v = c\frac{2}{3} t^{\frac{3}{2}}.$$

So

$$y_2 = c\frac{2}{3} t^{\frac{3}{2}} t^{-1} = c\frac{2}{3}\sqrt{t}.$$

Choose

$$y_2 = \sqrt{t}.$$

Exercise 4.1. Check y_2 satisfies the equation and $W(y_1, y_2) \neq 0$.

5 Method of Undetermined Coefficients

Consider the nonhomogeneous equation

$$L[y] = y'' + p(t)\, y' + q(t)\, y = g(t).$$

Let y_1, y_2 be solutions. Then

$$L[y_1 - y_2] = L[y_1] - L[y_2] = g(t) - g(t) = 0.$$

So $y_1 - y_2$ is a solution of the homogeneous equation $L[y] = 0$.

Theorem 5.1 *The general solution of the nonhomogeneous equation $L[y]=g$ is*

$$y=c_1y_1+c_2y_2+Y,$$

where c_1,c_2 are arbitrary constant, y_1,y_2 form a fundamental set of solutions for the homogeneous equation $L[y]=0$, and Y is a particular solution of the nonhomogeneous equation $L[y]=g$.

Proof. Let y be any solution of $L[y]=g$. Then $y-Y$ is a solution of $L[y]=0$. Then

$$y-Y=c_1y_1+c_2y_2 \quad \Rightarrow \quad y=c_1y_1+c_2y_2+Y.$$

for some constants c_1,c_2. □

How to find a particular solution?

Example 5.2. Find a particular solution of

$$y''-3y'-4y=3e^{2t}.$$

Answer: Suppose the solution is of the form (ansatz) $y=Ae^{2t}$, where A is an undetermined coefficient. To find A, just plug the ansatz into the equation.

$$\begin{aligned} 4Ae^{2t}-6Ae^{2t}-4Ae^{2t} &= -6Ae^{2t}=3e^{2t} \\ &\Rightarrow \quad A=-\frac{1}{2}. \end{aligned}$$

So

$$Y=-\frac{1}{2}e^{2t}$$

is a particular solution.

Example 5.3. Find a particular solution of

$$y''-3y'-4y=2\sin t.$$

Answer: Suppose the solution is of the form

$$y=A\sin t+B\cos t.$$

Then

$$(-A\sin t-B\cos t)-3(A\cos t-B\sin t)-4(A\sin t+B\cos t)=2\sin t.$$

Comparing the coefficients on the LHS and RHS,

$$\begin{cases} -5A+3B=2 \\ -5B-3A=0 \end{cases} \Rightarrow \begin{cases} -25A+15B=10 \\ -15B-9A=0 \end{cases} \Rightarrow \begin{cases} A=-\dfrac{5}{17} \\ B=\dfrac{3}{17} \end{cases}.$$

So

$$Y=-\frac{5}{17}\sin t+\frac{3}{17}\cos t.$$

Remark 5.4. The method also works if the RHS is a cosine function.

Example 5.5. Find a particular solution of

$$y'' - 3y' - 4y = -8e^t \cos 2t.$$

Answer: Suppose the solution is of the form

$$y = e^t(A\sin 2t + B\cos 2t).$$

Then

$$\begin{aligned} y' &= e^t(A\sin 2t + B\cos 2t) + e^t(2A\cos 2t - 2B\sin 2t) \\ &= e^t[(A-2B)\sin 2t + (2A+B)\cos 2t] \\ y'' &= e^t[(A-2B)\sin 2t + (2A+B)\cos 2t] + e^t[2(A-2B)\cos 2t - 2(2A+B)\sin 2t] \\ &= e^t[(-3A-4B)\sin 2t + (4A-3B)\cos 2t]. \end{aligned}$$

$$y'' - 3y' - 4y = e^t[(-3A-4B-3A+6B-4A)\sin 2t + (4A-3B-6A-3B-4B)\cos 2t] = -8e^t\cos 2t$$

$$\Rightarrow \begin{cases} -10A + 2B = 0 \\ -2A - 10B = -8 \end{cases} \Rightarrow \begin{cases} -5A + B = 0 \\ -A - 5B = -4 \end{cases} \Rightarrow \begin{cases} A = \dfrac{2}{13}, \\ B = \dfrac{10}{13}. \end{cases}$$

So

$$Y = e^t\left(\frac{2}{13}\sin 2t + \frac{10}{13}\cos 2t\right) = \frac{2}{13}e^t(\sin 2t + 5\cos 2t).$$

is a particular solution.

Example 5.6. Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t\cos 2t.$$

Answer: A particular solution is

$$Y = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{2}{13}e^t(\sin 2t + 5\cos 2t).$$

Example 5.7. Find a particular solution of

$$y'' - 3y' - 4y = 2e^{-t}.$$

Answer: Try the ansatz $y = Ae^{-t}$. Then

$$y'' - 3y' - 4y = (A + 3A - 4A)e^{-t} = 0.$$

Not work! Try another one

$$y = Ate^{-t}.$$

Then

$$y' = A(1-t)e^{-t}, \quad y'' = A(-2+t)e^{-t}$$

$$y'' - 3y' - 4y = Ae^{-t}(-2 + t - 3(1-t) - 4t) = -5Ae^{-t} = 2e^{-t} \quad \Rightarrow \quad A = -\frac{2}{5}.$$

So

$$y = -\frac{2}{5} t e^{-t}$$

is a particular solution.

Question 1. Why Ate^{-t} works?

Exercise 5.1. Derive the solution ansatz $y = At^2 e^{\alpha t}$ if α is a repeated root of the characteristic polynomial.

Example 5.8. Find a particular solution of

$$y'' - 4y' + 4y = e^{2t}.$$

Answer: Try the ansatz $y = Ae^{2t}$, not work. Try $y = Ate^{2t}$, not work. Try

$$y = At^2 e^{2t}.$$

$$y' = 2A(t + t^2)e^{2t}, \quad y'' = 2A(1 + 4t + 2t^2)$$

$$y'' - 4y' + 4y = Ae^{2t}[2(1 + 4t + 2t^2) - 8(t + t^2) + 4t^2] = 2Ae^{2t} = e^{2t}.$$

So $A = 1/2$ and

$$y = \frac{1}{2} t^2 e^{2t}$$

is a particular solution.

Example 5.9. Find a particular solution of

$$y'' - 4y' + 3y = t^2 + t + 1.$$

Answer: Consider the ansatz

$$y = At^2 + Bt + C.$$

Then

$$y' = 2At + B, \quad y'' = 2A.$$

$$\begin{aligned} y'' - 4y' + 3y &= 2A - 4(2At + B) + 3(At^2 + Bt + C) \\ &= 3At^2 + (3B - 8A)t + (2A - 4B + 3C) \end{aligned}$$

$$\Rightarrow \begin{cases} 3A &= 1 \\ 3B - 8A &= 1 \\ 2A - 4B + 3C &= 1 \end{cases} \Rightarrow \begin{cases} A = \dfrac{1}{3} \\ B = \dfrac{11}{9} \\ C = \dfrac{1}{3}\left(1 - \dfrac{2}{3} + \dfrac{44}{9}\right) = \dfrac{47}{27} \end{cases}$$

TABLE 3.5.1 The Particular Solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t} \begin{cases} \sin \beta t \\ \cos \beta t \end{cases}$	$t^s[(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t} \cos \beta t$ $+ (B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t} \sin \beta t]$

Notes. Here s is the smallest nonnegative integer ($s = 0, 1,$ or 2) that will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation, respectively.

6 Variation of Parameters

The method of undetermined coefficients has some limitations.

- only equations with constant coefficients
- $g(t)$ must be in one of those special forms

Now consider the more general nonhomogeneous equation

$$y'' + p(t)\,y' + q(t)\,y = g(t).$$

Suppose $y = c_1 y_1 + c_2 y_2$ is a general solution of the homogeneous equation

$$y'' + p(t)\,y' + q(t)\,y = 0.$$

Let

$$Y = u_1 y_1 + u_2 y_2,$$

where u_1, u_2 are functions to be determined. Then

$$Y' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'.$$

Let's pose the condition

$$u_1' y_1 + u_2' y_2 = 0. \tag{6.1}$$

Then

$$Y' = u_1 y_1' + u_2 y_2' \quad \text{and} \quad Y'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''.$$

So

$$\begin{aligned} Y'' + p(t) Y' + q(t) Y &= u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' + p(t)\,(u_1 y_1' + u_2 y_2') + q(t)\,(u_1 y_1 + u_2 y_2) \\ &= u_1\,[y_1'' + p(t)\,y_1' + q(t)\,y_1] + u_2\,[y_2'' + p(t)\,y_2' + q(t)\,y_2] + u_1' y_1' + u_2' y_2' \\ &= u_1' y_1' + u_2' y_2'. \end{aligned}$$

So

$$u_1' y_1' + u_2' y_2' = g(t). \tag{6.2}$$

So from (6.1) and (6.2) we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix}=\begin{pmatrix} 0 \\ g \end{pmatrix}.$$

Note this system has a unique solution because $W(y_1,y_2)\neq 0$. The solution is (given by Cramer's rule):

$$u_1'=\frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}=\frac{-y_2}{W(y_1,y_2)}g, \quad u_2'=\frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}=\frac{y_1}{W(y_1,y_2)}g.$$

Integrating in t, we obtain

$$u_1=\int\frac{-y_2}{W(y_1,y_2)}g\,dt, \quad u_2=\int\frac{y_1}{W(y_1,y_2)}g\,dt.$$

This is called the method of **variation of parameters**.

Example 6.1. Find the general solution of

$$y''+4y=3\csc t.$$

Answer: We have $y_1=\sin 2t$, $y_2=\cos 2t$,

$$W(y_1,y_2)=\begin{vmatrix} \sin 2t & \cos 2t \\ 2\cos 2t & -2\sin 2t \end{vmatrix}=-4.$$

So

$$u_1=\int\frac{-y_2}{W(y_1,y_2)}g\,dt=\int-\frac{\cos 2t}{-4}3\csc t=\frac{3}{4}\int\frac{\cos 2t}{\sin t}dt=\frac{3}{4}\int\frac{1+2\cos^2 t}{\sin t}dt$$
$$=\frac{3}{4}\left[\int\csc t\,dt+2\int\frac{1-\sin^2 t}{\sin t}dt\right]=\frac{3}{4}\left[3\int\csc t\,dt+2\int\sin t\,dt\right]=\frac{3}{4}[3\ln|\csc t-\cot t|-2\cos t]$$

Similarly we can find u_2 (exercise). So the general solution is

$$y=c_1\sin 2t+c_2\cos 2t+u_1\sin 2t+u_2\cos 2t$$

$$\int\csc t\,dt=\int\frac{1}{\sin t}dt=\int\frac{\sin t}{\sin^2 t}dt=\int\frac{\sin t}{1-\cos^2 t}dt=\int\frac{1}{2}\left[\frac{\sin t}{1+\cos t}+\frac{\sin t}{1-\cos t}\right]dt$$
$$=\frac{1}{2}\left[\int\frac{-1}{1+\cos t}d(1+\cos t)+\int\frac{1}{1-\cos t}d(1-\cos t)\right]=\frac{1}{2}[-\ln(1+\cos t)+\ln(1-\cos t)]$$
$$=\frac{1}{2}\ln\frac{1-\cos t}{1+\cos t}=\frac{1}{2}\ln\frac{(1-\cos t)^2}{1-\cos^2 t}=\frac{1}{2}\ln\frac{(1-\cos t)^2}{\sin^2 t}=\ln\sqrt{\frac{(1-\cos t)^2}{\sin^2 t}}=\ln\left|\frac{1-\cos t}{\sin t}\right|$$
$$=\ln|\csc t-\cot t|$$

7 Free Vibrations

Consider the equation for the spring-mass system

$$mu''+\gamma u'+ku=0.$$

7.1 Undamped free vibrations

Let $\gamma = 0$, i.e. there is no damping force. Then the equation reduces to

$$mu'' + ku = 0.$$

The roots of the characteristic equation are

$$r_{1,2} = \pm\omega_0 i, \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

So the general solution is

$$u = A\cos\omega_0 t + B\sin\omega_0 t = R\cos(\omega_0 t - \delta) = R(\cos\omega_0 t\cos\delta + \sin\omega_0 t\sin\delta).$$

So

$$A = R\cos\delta, \quad B = R\sin\delta.$$

So

$$R = \sqrt{A^2 + B^2}, \quad \cos\delta = \frac{A}{R}, \quad \sin\delta = \frac{B}{R} \quad \Rightarrow \quad \delta =$$

Here R is the **amplitude**, ω_0 is the **angular frequency** (natural frequency of the system), δ is the **phase**, and $T = \dfrac{2\pi}{w_0}$ is the **period**.

7.2 Damped free vibrations

Now consider the case when $\gamma > 0$ (damped vibration). Then the roots of the characteristic equation is

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = -\frac{\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m}.$$

1. If $\gamma^2 > 4mk$ (**overdamped**), then $r_1 \neq r_2$ are real and both negative. The general solution is

   $$u = Ae^{r_1 t} + Be^{r_2 t}.$$

   The solution is nonoscillatory and $u(t) \to 0$ as $t \to \infty$.
2. If $\gamma^2 = 4mk$ (**critically damped)**, then we have repeated root $r = -\dfrac{\gamma}{2m}$. So the general solution is

   $$u = Ae^{rt} + Bte^{rt}.$$

   The solution is nonoscillatory and $u(t) \to 0$ as $t \to \infty$.
3. If $\gamma^2 < 4mk$, then the roots are

   $$r_{1,2} = \lambda \pm i\mu, \quad \lambda = -\frac{\gamma}{2m}, \quad \mu = \frac{\sqrt{4mk - \gamma^2}}{2m}.$$

   The general solution is

   $$u = e^{\lambda t}(A\cos\mu t + B\sin\mu t) = Re^{\lambda t}\cos(\mu t - \delta).$$

   It's a **damped oscillation**, and $u(t) \to 0$ as $t \to \infty$.

   $u(t)$ is nonperiodic, but we call $T = \dfrac{2\pi}{\mu}$ the **quasi period**. In fact

   $$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} < \omega_0.$$

7.3 Electric circuits (skip)

8 Forced Vibrations (optional)

8.1 Forced vibrations with damping

$$mu'' + \gamma u' + ku = F$$

We consider periodic forces $F = F_0 \cos \omega t$. The general solution is

$$u = [c_1 u_1(t) + c_2 u_2(t)] + [A \cos \omega t + B \sin \omega t] = u_c(t) + U(t).$$

Note that $u_c(t) \to 0$ as $t \to \infty$, but $U(t)$ is periodic. So we call $u_c(t)$ the **transient solution** and $U(t)$ the **steady-state solution**.

Example 8.1. Consider the IVP

$$u'' + u' + \frac{5}{4}u = 3\cos t, \quad u(0) = 2, \quad u'(0) = 3.$$

Then

$$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = -\frac{1}{2} \pm i,$$

So

$$u_c(t) = e^{-\frac{t}{2}}(c_1 \cos t + c_2 \sin t).$$

Let $U = A\cos t + B \sin t$. Then

$$U'' + U' + \frac{5}{4}U = -A\cos t - B\sin t - A\sin t + B\cos t + \frac{5}{4}(A\cos t + B\sin t)$$
$$= \left(-A + B + \frac{5}{4}A\right)\cos t + \left(-B - A + \frac{5}{4}B\right)\sin t = \left(\frac{1}{4}A + B\right)\cos t + \left(\frac{1}{4}B - A\right)\sin t$$

So

$$\frac{1}{4}A + B = 3, \quad \frac{1}{4}B - A = 0 \quad \Rightarrow \quad \begin{cases} A = \frac{12}{17}, \\ B = \frac{48}{17}. \end{cases}$$

So the steady-state solution is

$$U(t) = \frac{12}{17}\cos t + \frac{48}{17}\sin t.$$

So the general solution is

$$u(t) = e^{-\frac{t}{2}}(c_1 \cos t + c_2 \sin t) + \frac{12}{17}\cos t + \frac{48}{17}\sin t.$$

Plugging initial conditions, we obtain $c_1 = \frac{22}{17}, c_2 = \frac{14}{17}$. So the solution of the IVP is

$$u(t) = \frac{2}{17}\left[e^{-\frac{t}{2}}(11\cos t + 7\sin t) + 6\cos t + 24\sin t\right].$$

Resonance. Steady-state solution $U = A\cos\omega t + B\sin\omega t$

$$U' = \omega(-A\sin\omega t + B\cos\omega t), \quad U'' = \omega^2(-A\cos\omega t - B\sin\omega t)$$

$$\begin{aligned}
\Rightarrow \quad & mU'' + \gamma U' + kU \\
= \quad & m\omega^2(-A\cos\omega t - B\sin\omega t) + \gamma\omega(-A\sin\omega t + B\cos\omega t) + k(A\cos\omega t + B\sin\omega t) \\
= \quad & (-m\omega^2 A + \gamma\omega B + kA)\cos\omega t + (-Bm\omega^2 - A\gamma\omega + kB)\sin\omega t \\
= \quad & [(k - m\omega^2)A + \gamma\omega B]\cos\omega t + [-\gamma\omega A + (k - m\omega^2)B]\sin\omega t \\
= \quad & F_0\cos\omega t
\end{aligned}$$

$$\begin{cases} (k - m\omega^2)A + \gamma\omega B = F_0 \\ -\gamma\omega A + (k - m\omega^2)B = 0 \end{cases} \Rightarrow \begin{pmatrix} k - m\omega^2 & \gamma\omega \\ -\gamma\omega & k - m\omega^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} F_0 \\ 0 \end{pmatrix}$$

$$\begin{aligned}
A &= \frac{k - m\omega^2}{(k - m\omega^2)^2 + \gamma^2\omega^2}F_0 = \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}F_0 \\
B &= \frac{\gamma\omega}{(k - m\omega^2)^2 + \gamma^2\omega^2}F_0 = \frac{\gamma\omega}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}F_0
\end{aligned}$$

$$A\cos\omega t + B\sin\omega t = R\cos(\omega t - \delta) \quad \Rightarrow \quad R = \frac{F_0}{\Delta}, \quad \Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

Nondimensionalize (无量纲化)

$$R = \frac{F_0}{m\omega_0^2\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{\gamma^2\omega^2}{m^2\omega_0^4}}} = \frac{F_0}{k\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{\gamma^2}{mk}\frac{\omega^2}{\omega_0^2}}}$$

$$\Rightarrow \frac{R}{(F_0/k)} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0^2}}}, \quad \Gamma = \frac{\gamma^2}{mk}$$

$$\gamma u' = F \Rightarrow \gamma = \frac{F}{u'}: \frac{\text{N}}{\text{m}\cdot\text{s}^{-1}} = \frac{\text{N}\cdot\text{s}}{\text{m}} \Rightarrow \Gamma = \frac{\gamma^2}{mk}: \frac{\text{N}^2\cdot\text{s}^2}{\text{m}^2\cdot\text{kg}\cdot\text{N}\cdot\text{m}^{-1}} = \frac{\text{N}\cdot\text{s}^2}{\text{m}\cdot\text{kg}} = \frac{\text{N}}{\text{m}\cdot\text{s}^{-2}\cdot\text{kg}} = 1$$

Clearly $\frac{R}{(F_0/k)}$ and $\frac{\omega^2}{\omega_0^2}$ are also dimensionless. Rewrite the equation as

$$y = \frac{1}{\sqrt{(1-x)^2 + \Gamma x}}, \quad y = \frac{R}{(F_0/k)}, \quad x = \frac{\omega^2}{\omega_0^2}$$

$$y' = -\frac{1}{2}[(1-x)^2 + \Gamma x]^{-\frac{3}{2}}[\Gamma - 2 + 2x]$$

If $0 < \Gamma < 2$, then $y' > 0$ for $x \in \left[0, 1 - \frac{\Gamma}{2}\right)$, $y' < 0$ for $x \in \left(1 - \frac{\Gamma}{2}, \infty\right)$ and $y' = 0$ for $x = 1 - \frac{\Gamma}{2}$.

So $y_{\max}$ is obtained at $x_{\max} = 1 - \frac{\Gamma}{2}$:

$$y_{\max} = \frac{1}{\sqrt{(1-x)^2 + \Gamma x}} = \frac{1}{\sqrt{\Gamma - \Gamma^2/4}} \to \infty \quad \text{as} \quad \Gamma \to 0.$$

Hence for lightly damped system (Γ is small), the amplitude of the steady-state solution when ω is near ω_0 can be very large for small external force. This phenomenon is known as **resonance**.

Asymptote

```
% -width 0.5par
import graph;
size(8cm, 0);
xaxis("$x$", RightTicks, Arrow);
yaxis("$y$", LeftTicks, Arrow);
real G1 = 1;
real G2 = 0.2;
real G3 = 0.1;
real f1(real x) {return ((1-x)**2+G1*x)**(-0.5);}
real f2(real x) {return ((1-x)**2+G2*x)**(-0.5);}
real f3(real x) {return ((1-x)**2+G3*x)**(-0.5);}
draw(graph(f1, 0, 3, Hermite), blue+linewidth(1pt));
draw(graph(f2, 0, 3, Hermite), purple+linewidth(1pt));
draw(graph(f3, 0, 3, Hermite), red+linewidth(1pt));
label("$\displaystyle y=\frac{1}{\sqrt{(1 - x)^2 + \Gamma x}}$", (1.3, 3.5),
align=E);
label("$\Gamma=1$", (0.8, 1.2), align=E, blue);
label("$\Gamma=0.2$", (0.6, 1.7), align=E, purple);
label("$\Gamma=0.1$", (1.2, 2.6), align=E, red);
```

8.2 Forced vibrations without damping

Consider

$$mu'' + ku = F_0 \cos\omega t.$$

The general solution of the homogeneous part is

$$u_c(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t, \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

8.2.1 $\omega \neq \omega_0$

The general solution is

$$u = u_c(t) + U(t), \quad U(t) = A\cos\omega t + B\sin\omega t.$$

$$U' = \omega(-A\sin\omega t + B\cos\omega t), \quad U'' = \omega^2(-A\cos\omega t - B\sin\omega t)$$

$$\begin{aligned} mU'' + kU &= m\omega^2(-A\cos\omega t - B\sin\omega t) + k(A\cos\omega t + B\sin\omega t) \\ &= (-Am\omega^2 + kA)\cos\omega t + (-Bm\omega^2 + kB)\sin\omega t \\ &= A(k - m\omega^2)\cos\omega t + B(k - m\omega^2)\sin\omega t \\ &= F_0\cos\omega t \end{aligned}$$

$$\Rightarrow \quad A = \frac{F_0}{k - m\omega^2}, \quad B = 0$$

The general solution is

$$u = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{F_0}{k - m\omega^2}\cos\omega t.$$

Suppose the initial condition is $u(0)=u'(0)=0$, then

$$c_1+\frac{F_0}{k-m\omega^2}=0, \quad c_2\omega_0=0 \quad \Rightarrow \quad c_1=-\frac{F_0}{k-m\omega^2}, \quad c_2=0.$$

$$u=\frac{F_0}{k-m\omega^2}(\cos\omega t-\cos\omega_0 t)=\frac{-2F_0}{k-m\omega^2}\sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right)$$

If ω is close to ω_0, then we have a **beat**. Also used in **amplitude modulation**.

Asymptote

```
% -width 0.6par
import graph;
size(10cm, 0);
real f(real x) {return cos(10*x)-cos(11*x);}
draw(graph(f, -10, 10, Hermite), black+linewidth(1pt));
label("$y=\cos(10 x)-\cos(11 x)$", (0, 3));
```

8.2.2 $\omega=\omega_0$

A general solution is

$$U=c_1\cos\omega t+c_2\sin\omega t+t(A\cos\omega t+B\sin\omega t).$$

9 Higher Order Linear Equations

9.1 General theory

An n-th order linear ODE is in the form

$$L[y]=y^{(n)}+p_{n-1}(t)\,y^{(n-1)}+\cdots+p_1(t)\,y'+p_0(t)\,y=g(t)$$

An initial value problem is the equation $L[y]=g$ together with initial conditions

$$y(t_0)=y_0, \quad y'(t_0)=y_0', \quad \ldots \quad y^{(n-1)}(t_0)=y_0^{(n-1)}.$$

Definition 9.1 *The **Wronskian** of n solutions $y_1,\ldots,y_n$ of $L[y]=0$ is*

$$W[y_1,\ldots,y_n]=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

Theorem 9.2 *Let $y_1,\ldots,y_n$ be solutions of $L[y]=0$. Then $y_1,\ldots,y_n$ form a fundamental set of solutions if and only if they are linearly independent.*

Proof. Suppose $y_1,\ldots,y_n$ form a fundamental set of solutions, that is, $W[y_1,\ldots,y_n]\neq 0$. Let $c_1,\ldots,c_n$ be constants such that

$$c_1y_1+\cdots+c_ny_n=0.$$

Differentiate the above equation in t,

$$c_1y_1'+\cdots+c_ny_n'=0.$$

Repeat differentiating, we obtain

$$\begin{aligned} c_1 y_1'' + \cdots + c_n y_n'' &= 0 \\ &\vdots \\ c_1 y_1^{(n-1)} + \cdots + c_n y_n^{(n-1)} &= 0 \end{aligned}$$

Hence we have a linear system

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Hence $y_1, \ldots, y_n$ are linearly independent.
Now assume $y_1, \ldots, y_n$ do not form a fundamental set of solutions, i.e. $W[y_1, \ldots, y_n](t_0) = 0$ for some t_0. Then there exists constants $c_1, \ldots, c_n$, not all zero, such that

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let

$$Y = c_1 y_1 + \cdots + c_n y_n.$$

Then consider the IVP

$$L[y] = 0, \quad y(t_0) = Y(t_0) = 0, \quad y'(t_0) = Y'(t_0) = 0, \quad \cdots \quad y^{(n-1)}(t_0) = Y^{(n-1)}(t_0) = 0$$

Clearly Y is the solution of the IVP. On the other hand, 0 is also a solution of the IVP. By the E&U theorem, we have $Y = 0$. Thus $y_1, \ldots, y_n$ are linearly independent. $\square$

9.2 Homogeneous constant coefficients

Example 9.3.

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$

Answer: The characteristic equation is

$$r^4 + r^3 - 7r^2 - r + 6 = 0.$$

To find the roots, we factorize it:

$$\begin{aligned} (r^4 + r^3) - (7r^2 + r - 6) &= r^3(r+1) - (r+1)(7r-6) \\ &= (r+1)(r^3 - 7r + 6) = (r+1)(r^3 - r - 6r + 6) \\ &= (r+1)[r(r^2-1) - 6(r-1)] \\ &= (r+1)(r-1)(r^2 + r - 6) \\ &= (r+1)(r-1)(r-2)(r+3) \end{aligned}$$

So the roots are$f$

$$r = -3, -1, 1, 2$$

So there are four solutions

$$y = e^{-3t}, e^{-t}, e^{t}, e^{2t}$$

and they form a fundamental set of solutions. So the general solution is

$$y = c_1 e^{-3t} + c_2 e^{-t} + c_3 e^{t} + c_4 e^{2t}.$$

Note 9.4. Another method to find the roots. All possible rational roots are

$$\frac{\pm 1, \pm 2, \pm 3, \pm 6}{1}.$$

Then verify directly it they are roots. Once one or more roots are found, we can use **polynomial division** to reduce the order and help find other roots.

Example 9.5.

$$y^{(4)} - y = 0$$

Answer:

$$r^4 - 1 = (r^2 + 1)(r^2 - 1) \quad \Rightarrow \quad r = \pm i, \pm 1 \quad \Rightarrow \quad y = \cos t, \sin t, e^{-t}, e^{t}$$

Example 9.6.

$$y^{(4)} + 2y'' + y = 0$$

Answer:

$$r^4 + 2r^2 + 1 = (r^2 + 1)^2 = 0 \quad \Rightarrow \quad r = \pm i, \pm i \quad \Rightarrow \quad y = \cos t, \sin t, t\cos t, t\sin t$$

(We say the root $r = \pm i$ has multiplicity 2).

Example 9.7.

$$y^{(4)} + y = 0.$$

Answer:

$$\begin{aligned}
&r^4 + 1 = 0 \quad \Rightarrow \quad r^4 = -1 = e^{i(\pi + 2n\pi)} \\
\Rightarrow \quad & r = \exp\left(i\frac{(2n+1)}{4}\pi\right) = e^{\frac{\pi}{4}i}, e^{\frac{3\pi}{4}i}, e^{\frac{5\pi}{4}i}, e^{\frac{7\pi}{4}i} \\
\Rightarrow \quad & r = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \quad \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \\
\Rightarrow \quad & r = \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i, \quad -\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i \\
\Rightarrow \quad & y = e^{\frac{\sqrt{2}}{2}t}\cos\frac{\sqrt{2}}{2}t, \quad e^{\frac{\sqrt{2}}{2}t}\sin\frac{\sqrt{2}}{2}t, \quad e^{-\frac{\sqrt{2}}{2}t}\cos\frac{\sqrt{2}}{2}t, \quad e^{-\frac{\sqrt{2}}{2}t}\sin\frac{\sqrt{2}}{2}t
\end{aligned}$$

9.3 The method of undetermined coefficients

Example 9.8. Find the general solution of

$$y''' - 3y'' + 3y' - y = 4e^t$$

Answer:

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0 \quad \Rightarrow \quad r = 1, 1, 1 \quad \Rightarrow \quad y_1 = e^t, te^t, t^2e^t.$$

Let

$$Y = At^3e^t.$$

Then

$$Y' = A(3t^2 + t^3)e^t, \quad Y'' = A(6t + 6t^2 + t^3)e^t, \quad Y''' = A(6 + 18t + 9t^2 + t^3)e^t.$$

$$\Rightarrow \quad [(6 + 18t + 9t^2 + t^3) - 3(6t + 6t^2 + t^3) + 3(3t^2 + t^3) - t^3]Ae^t = 4e^t$$

$$\Rightarrow \quad 6A = 4 \quad \Rightarrow \quad A = \frac{2}{3}.$$

So the general solution is

$$y = c_1e^t + c_2te^t + c_3t^2e^t + \frac{2}{3}t^3e^t.$$

9.4 The method of variation of parameters

Suppose $y_1, \ldots, y_n$ form a fundamental set of solutions for $L[y] = 0$. Consider the nonhomogeneous equation

$$L[y] = g(t).$$

Let

$$Y = u_1y_1 + \cdots + u_ny_n$$

Then differentiate Y and make some assumption as in the case of 2nd order equations. We obtain

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g \end{pmatrix}$$

Then

$$u_m' = \frac{W_m}{W}g \quad \Rightarrow \quad u_m = \int \frac{W_m}{W}g$$

where W is the Wronskian, and W_m is the determinant of the above matrix with the m-th column replaced by the vector $(0, \ldots, 0, 1)^T$.

Example 9.9. Find the general solution to

$$y''' - y'' - y' + y = g(t).$$

Answer:

$$r^3 - r^2 - r + 1 = r^2(r-1) - (r-1) = (r-1)^2(r+1) \quad \Rightarrow \quad r = 1, 1, -1$$

$$y_1 = e^{-t}, \quad y_2 = e^t, \quad y_3 = te^t.$$

$$\begin{aligned}
W &= \begin{vmatrix} e^{-t} & e^t & te^t \\ -e^{-t} & e^t & (t+1)e^t \\ e^{-t} & e^t & (t+2)e^t \end{vmatrix} = e^t \begin{vmatrix} 1 & 1 & t \\ -1 & 1 & t+1 \\ 1 & 1 & t+2 \end{vmatrix} \\
&= e^t \begin{vmatrix} 1 & 0 & t \\ -1 & 2 & t+1 \\ 1 & 0 & t+2 \end{vmatrix} = 2e^t \begin{vmatrix} 1 & t \\ 1 & t+2 \end{vmatrix} = 4e^t, \\
W_1 &= \begin{vmatrix} 0 & e^t & te^t \\ 0 & e^t & (t+1)e^t \\ 1 & e^t & (t+2)e^t \end{vmatrix} = e^{2t} \begin{vmatrix} 0 & 1 & t \\ 0 & 1 & (t+1) \\ 1 & 1 & (t+2) \end{vmatrix} = e^{2t}, \\
W_2 &= \begin{vmatrix} e^{-t} & 0 & te^t \\ -e^{-t} & 0 & (t+1)e^t \\ e^{-t} & 1 & (t+2)e^t \end{vmatrix} = \begin{vmatrix} 1 & 0 & t \\ -1 & 0 & (t+1) \\ 1 & 1 & (t+2) \end{vmatrix} = -(2t+1), \\
W_3 &= \begin{vmatrix} e^{-t} & e^t & 0 \\ -e^{-t} & e^t & 0 \\ e^{-t} & e^t & 1 \end{vmatrix} = 2. \\
u_1 &= \int \frac{W_1}{W} g = \int \frac{1}{4} e^t g(t)\,dt, \\
u_2 &= \int \frac{W_2}{W} g = \int -\frac{2t+1}{4e^t} g(t)\,dt, \\
u_3 &= \int \frac{W_3}{W} g = \int \frac{1}{2e^t} g(t)\,dt.
\end{aligned}$$

So the general solution is

$$y = (c_1 e^{-t} + c_2 e^t + c_3 te^t) + e^{-t} \int \frac{1}{4} e^t g(t)\,dt - e^t \int \frac{2t+1}{4e^t} g(t)\,dt + te^t \int \frac{1}{2e^t} g(t)\,dt.$$

Ordinary Differential Equations

BY YULIANG WANG

Chapter 3: Second Order Linear Equations

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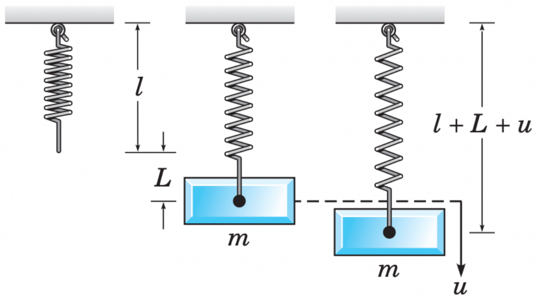
Motivation: spring-mass system



Newton's Law: $ma = f$

$a = u''$

$f = mg - k(L+u) - \gamma u' + F$

k: spring constant

γ: damping coefficient

$$mu'' = mg - k(L+u) - \gamma u' + F$$

$$mu'' + \gamma u' + ku = mg - kL + F$$

But $mg = kL$, so

$$mu'' + \gamma u' + ku = F$$

1 Homogeneous Equations with Constant Coefficients

$$ay'' + by' + cy = 0,$$

initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

Example 1.1. Solve the IVP

$$y'' - y = 0, \quad y(0) = 2, \quad y'(0) = -1.$$

By investigation, we know $y = ce^t$ satisfies the equation for any constant c. However, it doesn't satisfy the initial conditions. More investigation shows $y = ce^{-t}$ is also a solution for any constant c. It turns out

$$y = c_1 e^t + c_2 e^{-t}$$

is a solution for any constants c_1, c_2. Now, the initial conditions require

$$c_1 + c_2 = 2, \quad c_1 - c_2 = -1.$$

This is a system of linear equations. The matrix form is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

The matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is nonsingular ($\det A = -2 \neq 0$). So the system has a unique solution. The solution is

$$c_1 = \frac{1}{2}, \quad c_2 = -\frac{3}{2}.$$

So the IVP has a solution

$$y = \frac{1}{2} e^t - \frac{3}{2} e^{-t}.$$

Example 1.2. Solve

$$y'' + 5y' + 6y = 0.$$

Answer: We assume the ansatz of the solution: $y = e^{rt}$ for some constant r. Then

$$\begin{aligned} y'' + 5y' + 6y &= r^2 e^{rt} + 5re^{rt} + 6e^{rt} \\ &= (r^2 + 5r + 6)e^{rt} \\ &= 0 \\ \Rightarrow \quad r^2 + 5r + 6 &= 0 \end{aligned}$$

The equation

$$r^2 + 5r + 6 = 0$$

is called the **characteristic equation** for the ODE. The solutions are

$$r_1 = -3, \quad r_2 = -2.$$

So have two solutions of the ODE:

$$y_1 = e^{-3t}, \quad y_2 = e^{-2t}.$$

So we can let the solution be

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{-3t} + c_2 e^{-2t}.$$

Note that

$$y \to 0 \quad \text{as} \quad t \to \infty$$

Example 1.3. Solve the IVP

$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}.$$

Answer: The characteristic equation is

$$4r^2 - 8r + 3 = (2r - 3)(2r - 1) = 0 \quad \Rightarrow \quad r_1 = \frac{3}{2}, \quad r_2 = \frac{1}{2}.$$

So we have solutions

$$y = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t}.$$

Plugging initial conditions

$$\begin{aligned} c_1 + c_2 &= 2 \\ \frac{3}{2}c_1 + \frac{1}{2}c_2 &= \frac{1}{2} \end{aligned}$$

We can find unique solutions

$$c_1 = -\frac{1}{2}, \quad c_2 = \frac{5}{2}.$$

So a solution of the IVP is

$$y = -\frac{1}{2}e^{\frac{3}{2}t} + \frac{5}{2}e^{\frac{1}{2}t}.$$

Note that

$$y \to -\infty \quad \text{as} \quad t \to \infty$$

2 Theory of 2nd Order Linear Equations

Consider the general 2nd order linear equation

$$\boxed{L[y] := y''(t) + p(t)\,y' + q(t)\,y} = g(t).$$

Note that L is a linear operator.

Existence and Uniqueness Theorem *Consider the IVP*

$$y''(t) + p(t)\,y' + q(t)\,y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

If p, q, g are continuous on an interval I containing t_0, then there exists a unique solution to this IVP on I.

Example 2.1. Find the longest interval in which the solution of the initial value problem

$$(t^2 - 3t)\,y'' + ty' - (t+3)\,y = 0, \quad y(1) = 2, \quad y'(1) = 1$$

is certain to exist.

Answer: Assuming $t \neq 0, t \neq 3$, rewrite the equation in the standard form

$$y'' + \frac{1}{t-3}y' - \frac{t+3}{t^2 - 3t}y = 0.$$

So p, q, g are continuous in $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$. Since $1 \in (0, 3)$. By the E&U theorem, there exists a unique solution to the IVP on $(0, 3)$.

Principle of Superposition *Consider the **homogeneous** linear equation*

$$L[y] = 0.$$

If y_1 and y_2 are both solutions, then $c_1 y_1 + c_2 y_2$ is also a solution for any constants c_1 and c_2.

Proof.

$$\begin{aligned}
L[c_1y_1 + c_2y_2] &= (c_1y_1 + c_2y_2)'' + p(t)\,(c_1y_1 + c_2y_2)' + q(t)\,(c_1y_1 + c_2y_2) \\
&= c_1\,(y_1'' + p(t)\,y_1' + q(t)\,y_1) + c_2\,(y_2'' + p(t)\,y_2' + q(t)\,y_2) \\
&= c_1\,L[y_1] + c_2\,L[y_2] = 0 + 0 = 0.
\end{aligned}$$

So $c_1\,y_2 + c_2\,y_2$ is also a solution. □

Let's consider the homogeneous equation with constant coefficients

$$ay'' + by' + cy = 0.$$

Suppose there are two different roots r_1, r_2 of the characteristic polynomial $ar^2 + br + c$. Then we have two solutions

$$y_1 = e^{r_1 t}, \quad y_2 = e^{r_2 t}.$$

By the principle of superposition,

$$y = c_1 y_1 + c_2 y_2$$

is a solution for any constants c_1, c_2.

The next question: can we always find c_1, c_2 such that a given initial conditions are satisfied?

Pluggin the initial conditions, we obtain a linear system for c_1, c_2:

$$\begin{array}{rcl} c_1 y_1(t_0) + c_2 y_2(t_0) & = & y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) & = & y_0' \end{array}$$

or, in matrix form

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}.$$

In order to have a solution for arbitrary values of y_0, y_0', we need the matrix to be nonsingular, i.e.

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)\, y_2'(t_0) - y_1'(t_0)\, y_2(t_0) \neq 0.$$

Definition 2.2 *Suppose y_1, y_2 are two solutions of the ODE $L[y] = 0$. Then the Wronskian of them is*

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

Theorem 2.3 *Let y_1, y_2 are solutions of the equation $L[y] = 0$. Then one can find constants c_1 and c_2 such that $c_1 y_1 + c_2 y_2$ solves the IVP*

$$L[y] = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

regardless of the values y_0 and y_0' if and only if $W(y_1, y_2)(t_0) \neq 0$.

Next we show all solutions of $L[y] = 0$ can are actually in the form $c_1 y_1 + c_2 y_2$ if and only if the Wronskian is nonzero.

Theorem 2.4 *Let y_1, y_2 are solutions of the equation $L[y] = 0$ on some interval I. Then every solution of $L[y] = 0$ on I can be written as $c_1 y_1 + c_2 y_2$ if and only if $W(y_1, y_2)(t) \neq 0$ for some $t \in I$.*

Proof. Suppose $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$. Let $\phi(t)$ to be a solution of $L[y] = 0$. Let $y_0 = \phi(t_0)$ and $y_0' = \phi'(t_0)$. Now consider the IVP

$$L[y] = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{2.1}$$

Clearly ϕ is a solution of the IVP (2.1). On the other hand, we can find c_1 and c_2 such that $c_1 y_1 + c_2 y_2$ is a solution of the IVP (2.1) for some c_1, c_2 since $W(y_1, y_2)(t_0) \neq 0$. By the uniqueness part of the E&U theorem, we have $\phi = c_1 y_1 + c_2 y_2$.

Next, suppose $W(y_1, y_2)(t) = 0$ for any $t \in I$. Then $W(y_1, y_2)(t_0) = 0$ for some $t_0 \in I$. So there exists some numbers y_0, y_0' such that

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix} \tag{2.2}$$

has no solution. Let $\phi(t)$ to be the solution of the IVP (2.1). Suppose $\phi = c_1 y_1 + c_2 y_2$ for some c_1, c_2, then c_1, c_2 must satisfy the linear system (2.2). A contradition! $\square$

If $W(y_1, y_2)(t) \neq 0$ for some t, we call the solutions $\{y_1, y_2\}$ a **fundamental set of solutions**.

Example 2.5. If $r_1 \neq r_2$ are real numbers, and $y_1 = e^{r_1 t}, y_2 = e^{r_2 t}$ are solutions of some ODE. Then

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)t} \neq 0$$

for any t. So $\{y_1, y_2\}$ form a fundamental set of solutions.

Example 2.6. Show that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions of

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0.$$

Answer:

$$W(y_1, y_2)(t) = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2} t^{-1/2} & -t^{-2} \end{vmatrix} = -\frac{3}{2} t^{-3/2} \neq 0$$

for any $t \neq 0$. So $\{y_1, y_2\}$ form a fundamental set of solutions for $t \neq 0$.

Theorem 2.7 *Let y_1 to be the solution of the IVP*

$$L[y] = 0, \quad y(t_0) = 1, \quad y'(t_0) = 0.$$

Let y_2 to be the solution of the IVP

$$L[y] = 0, \quad y(t_0) = 0, \quad y'(t_0) = 1.$$

Then the Wronskian of y_1, y_2 is $W(t) = 1$. So $\{y_1, y_2\}$ form a fundamental set of solutions.

This theorem says fundamental set of solutions always exist. In fact, we can find infinitely many fundamental set of solutions.

Example 2.8. Find the fundamental set of solutions y_1 and y_2 specified by Theorem 2.7 for the differential equation

$$y'' - y = 0.$$

Answer: Using the characteristic equations, we find two solutions

$$y_1 = e^t, \quad y_2 = e^{-t}.$$

We can check they form a fundamental set of solutions. Let's find another fundamental set of solutions defined as in Theorem 2.7 for $t_0 = 0$. Let

$$y_3 = \frac{e^t + e^{-t}}{2} = \cosh t, \quad y_4 = \frac{e^t - e^{-t}}{2} = \sinh t.$$

Then $W(y_3, y_4) = 1$. So the general solution can be written as

$$c_1 y_1 + c_2 y_2 \text{ or } c_3 y_3 + c_4 y_4.$$

Theorem 2.9 **(Abel)** *Let y_1, y_2 are solutions of the equation*

$$L[y] = y'' + p(t) y' + q(t) y = 0.$$

Then the Wronskian

$$W(y_1, y_2)(t) = c e^{-\int p(t) dt}$$

for some constant c, which may depend on y_1, y_2 but otherwise independent of p, q.

Proof. We have

$$\begin{aligned} y_1'' + p(t) y_1' + q(t) y_1 &= 0, \\ y_2'' + p(t) y_2' + q(t) y_2 &= 0. \end{aligned}$$

Then

$$\begin{aligned} y_2 [y_1'' + p(t) y_1' + q(t) y_1] &= 0, \\ y_1 [y_2'' + p(t) y_2' + q(t) y_2] &= 0. \end{aligned}$$

Subtracting two equations, we obtain

$$y_1 y_2'' - y_1'' y_2 + p(t) (y_1 y_2' - y_1' y_2) = 0.$$

Note that

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

$$W'(t) = y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'' = y_1 y_2'' - y_2 y_1''.$$

So we obtain

$$W'(t) + p(t) W(t) = 0.$$

Solving this first order linear ODE, we obtain

$$W(t) = ce^{-\int p(t)dt}.$$

□

Remark 2.10. From the Abel's theorem, we now know the Wronskian of two solutions of an ODE is either identically zero or nowhere zero.

Theorem 2.11 *Suppose p, q are real-valued functions. Let $y(t) = u(t) + iv(t)$ be a complex-valued solution of*

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0,$$

where u, v are real-valued functions. Then u, v are also solutions of $L[y] = 0$.

Proof. We have

$$\begin{aligned} L[y] &= (u+iv)'' + p(t)\,(u+iv)' + q(t)\,(u+iv) \\ &= (u''+iv'') + p(t)\,(u'+iv') + q(t)\,(u+iv) \\ &= (u''+p(t)\,u'+q(t)\,u) + i\,(v''+p(t)\,v'+q(t)\,v) \\ &= 0. \end{aligned}$$

So

$$u'' + p(t)\,u' + q(t)\,u = 0, \quad v'' + p(t)\,v' + q(t)\,v = 0.$$

That is, u, v are both solutions of $L[y] = 0$.

□

3 Complex roots of the characteristic equation

Let's go back to equations with constant coefficients

$$ay'' + by' + cy = 0.$$

The characteristic equation is

$$ar^2 + br + c = 0.$$

The solution is given by the quadratic formula

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1. $b^2 > 4ac$. Then r_1, r_2 are both real and $r_1 \neq r_2$.
2. $b^2 = 4ac$. Then r_1, r_2 are both real and $r_1 = r_2$.
3. $b^2 < 4ac$. Then r_1, r_2 are both complex, and $r_2 = \bar{r}_1$.

Now consider case (3). Let $r_{1,2} = \lambda \pm i\mu$. So then we have two complex-valued solutions

$$y_1 = e^{r_1 t} = e^{(\lambda + i\mu)t}, \quad y_2 = e^{r_2 t} = e^{(\lambda - i\mu)t}.$$

What does it to raise a number to a complex exponent?

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Then we define

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\ &= \cos x + i \sin x \end{aligned}$$

Euler's formula

$$e^{ix} = \cos x + i \sin x.$$

Then we define

$$e^{\lambda + i\mu} = e^{\lambda} e^{i\mu} = e^{\lambda}(\cos x + i \sin x) = e^{\lambda} \cos x + i e^{\lambda} \sin x$$

One can verify the exponential function defined this way has the usual algebraic properties of exponential functions with real exponent.
The two complex solutions are

$$y_1 = e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos \mu t + i \sin \mu t), \quad y_2 = e^{(\lambda - i\mu)t} = e^{\lambda t}(\cos \mu t - i \sin \mu t).$$

One can verify y_1, y_2 form a fundamental set of solutions.

By the previous theorem, we know

$$y_3 = e^{\lambda t} \cos \mu t, \quad y_4 = e^{\lambda t} \sin \mu t.$$

are real-valued solutions. One can verify y_3, y_4 also form a fundamental set of solutions.

Example 3.1. Find the general solution (real-valued) of the differential equation

$$y'' + y' + 9.25 y = 0,$$

Also find the solution that satisfies the initial conditions

$$y(0) = 2, \quad y'(0) = 8,$$

and draw its graph.

Answer: The roots of the characteristic polynomial is

$$r_{1,2} = \frac{-1 \pm 6i}{2} = -\frac{1}{2} \pm 3i.$$

The real-valued solutions are

$$y_1 = e^{-\frac{t}{2}} \cos 3t, \quad y_2 = e^{-\frac{t}{2}} \sin 3t.$$

The general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{-\frac{t}{2}} \cos 3t + c_2 e^{-\frac{t}{2}} \sin 3t,$$

and

$$y' = c_1 e^{-\frac{t}{2}}\left(-\frac{1}{2}\cos 3t - 3\sin 3t\right) + c_2 e^{-\frac{t}{2}}\left(-\frac{1}{2}\sin 3t + 3\cos 3t\right).$$

Plugging the initial conditions,

$$\begin{aligned} c_1 &= 2, \\ -\frac{1}{2}c_1 + 3c_2 &= 8. \end{aligned}$$

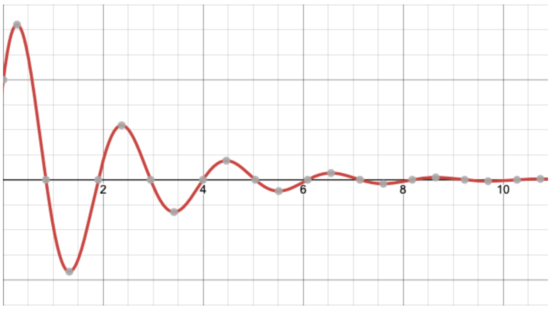
Solving the linear systm,

$$c_1 = 2, \quad c_2 = 3.$$

So the solution of the IVP is

$$y = 2e^{-\frac{t}{2}}\cos 3t + 3e^{-\frac{t}{2}}\sin 3t = e^{-\frac{t}{2}}(2\cos 3t + 3\sin 3t).$$

The graph is a damped oscillation.



Example 3.2. Find the general solution of

$$y'' + 9y = 0.$$

Answer:

$$r_{1,2} = \pm 3i$$

The general solution is

$$y = c_1\cos 3t + c_2\sin 3t.$$

The graph is an undamped oscillation.

4 Repeated Roots; Reduction of order

4.1 Repeated roots

Suppose the charateristic equation have one repeated root $r = -\dfrac{b}{2a}$. Then we have a solution

$$y_1 = e^{rt}.$$

Then $y_2 = cy_1 = ce^{rt}$ is also a solution for any constant c, but $\{y_1, y_2\}$ is not a fundamental set of solutions. To find another independent solution, let

$$y_2 = v(t)\, y_1.$$

Plugging y_2 into the equation,

$$\begin{aligned} a(vy_1)''+b(vy_1)'+c(vy_1) &= a(v''y_1+2v'y_1'+vy_1'')+b(v'y_1+vy_1')+cvy_1 \\ &= v(ay_1''+by_1'+cy_1)+av''y_1+2av'y_1'+bv'y_1 \\ &= ay_1v''+(2ay_1'+by_1)v' \\ &= ae^{rt}v''+(2are^{rt}+be^{rt})v' \\ &= e^{rt}(av''+(2ar+b)v')=0 \\ \Rightarrow \quad & av''+(2ar+b)v'=av''=0 \\ \Rightarrow \quad & v''=0 \quad \Rightarrow \quad v=c_1t+c_2. \end{aligned}$$

Then

$$y_2=(c_1t+c_2)e^{rt}=c_1te^{rt}+c_2e^{rt}.$$

Choose

$$y_2=te^{rt}.$$

Then one can verify y_1, y_2 form a fundamental set of solutions (check $W(y_1,y_2)\neq 0$).

Example 4.1.

$$y''+4y'+4y=0.$$

Answer: The characteristic equation is $r^2+4r+4=0$. The (repeated) root is $r=-2$. So

$$y_1=e^{-2t}, \quad y_2=te^{-2t}.$$

The general solution is

$$y=c_1e^{-2t}+c_2te^{-2t}.$$

We have $y(t)\to 0$ as $t\to\infty$.

4.2 Reduction of order

The idea to find y_2 can be generalized to a general second order linear equation. If y_1 is a solution of

$$y''+p(t)y'+q(t)y=0.$$

Let $y_2=v(t)y_1$ be another solution. Then plugging y_2 into the equation we can obtain an second order linear ODE for $v(t)$:

$$y_1v''+(y_1'+p(t)y_1)v'=0.$$

Let $w=v'$, then we obtain a first order ODE for w

$$y_1w'+(y_1'+p(t)y_1)w=0.$$

Solve w, then let $v=\displaystyle\int w$.

Example 4.2. Given the variable coefficient equation and solution y_1,

$$2t^2y''+3ty'-y=0, \quad t>0; \quad y_1(t)=t^{-1},$$

use reduction of order method to find a second solution.

Answer: Let $y_2 = vy_1$. Then

$$\begin{aligned} 2t^2 y_2'' + 3ty_2' - y_2 &= 2t^2(v''y_1 + 2v'y_1' + vy_1'') + 3t(v'y_1 + vy_1') - vy_1 \\ &= 2t^2(t^{-1}v'' - 2t^{-2}v' + 2t^{-3}v) + 3t(t^{-1}v' - t^{-2}v) - t^{-1}v \\ &= 2tv'' - v' = 0. \end{aligned}$$

Let $w = v'$,

$$2tw' - w = 0 \quad \Rightarrow \quad \frac{dw}{w} = \frac{dt}{2t} \quad \Rightarrow \quad \ln w = \frac{1}{2}\ln t \quad \Rightarrow \quad w = c\sqrt{t} \quad \Rightarrow \quad v = c\frac{2}{3}t^{\frac{3}{2}}.$$

So

$$y_2 = c\frac{2}{3}t^{\frac{3}{2}}t^{-1} = c\frac{2}{3}\sqrt{t}.$$

Choose

$$y_2 = \sqrt{t}.$$

Exercise 4.1. Check y_2 satisfies the equation and $W(y_1, y_2) \neq 0$.

5 Method of Undetermined Coefficients

Consider the nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t).$$

Let y_1, y_2 be solutions. Then

$$L[y_1 - y_2] = L[y_1] - L[y_2] = g(t) - g(t) = 0.$$

So $y_1 - y_2$ is a solution of the homogeneous equation $L[y] = 0$.

Theorem 5.1 *The general solution of the nonhomogeneous equation $L[y] = g$ is*

$$y = c_1 y_1 + c_2 y_2 + Y,$$

where c_1, c_2 are arbitrary constant, y_1, y_2 form a fundamental set of solutions for the homogeneous equation $L[y] = 0$, and Y is a particular solution of the nonhomogeneous equation $L[y] = g$.

Proof. Let y be any solution of $L[y] = g$. Then $y - Y$ is a solution of $L[y] = 0$. Then

$$y - Y = c_1 y_1 + c_2 y_2 \quad \Rightarrow \quad y = c_1 y_1 + c_2 y_2 + Y.$$

for some constants c_1, c_2. □

How to find a particular solution?

Example 5.2. Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t}.$$

Answer: Suppose the solution is of the form (ansatz) $y = Ae^{2t}$, where A is an undetermined coefficient. To find A, just plug the ansatz into the equation.

$$\begin{aligned} 4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} &= -6Ae^{2t} = 3e^{2t} \\ &\Rightarrow \quad A = -\frac{1}{2}. \end{aligned}$$

So

$$Y = -\frac{1}{2}e^{2t}$$

is a particular solution.

Example 5.3. Find a particular solution of

$$y'' - 3y' - 4y = 2\sin t.$$

Answer: Suppose the solution is of the form

$$y = A\sin t + B\cos t.$$

Then

$$(-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) = 2\sin t.$$

Comparing the coefficients on the LHS and RHS,

$$\begin{cases} -5A + 3B = 2 \\ -5B - 3A = 0 \end{cases} \Rightarrow \begin{cases} -25A + 15B = 10 \\ -15B - 9A = 0 \end{cases} \Rightarrow \begin{cases} A = -\dfrac{5}{17} \\ B = \dfrac{3}{17} \end{cases}.$$

So

$$Y = -\frac{5}{17}\sin t + \frac{3}{17}\cos t.$$

Remark 5.4. The method also works if the RHS is a cosine function.

Example 5.5. Find a particular solution of

$$y'' - 3y' - 4y = -8e^t\cos 2t.$$

Answer: Suppose the solution is of the form

$$y = e^t(A\sin 2t + B\cos 2t).$$

Then

$$\begin{aligned} y' &= e^t(A\sin 2t + B\cos 2t) + e^t(2A\cos 2t - 2B\sin 2t) \\ &= e^t[(A - 2B)\sin 2t + (2A + B)\cos 2t] \\ y'' &= e^t[(A - 2B)\sin 2t + (2A + B)\cos 2t] + e^t[2(A - 2B)\cos 2t - 2(2A + B)\sin 2t] \\ &= e^t[(-3A - 4B)\sin 2t + (4A - 3B)\cos 2t]. \end{aligned}$$

$$y'' - 3y' - 4y = e^t[(-3A - 4B - 3A + 6B - 4A)\sin 2t + (4A - 3B - 6A - 3B - 4B)\cos 2t] = -8e^t\cos 2t$$

$$\Rightarrow \begin{cases} -10A + 2B = 0 \\ -2A - 10B = -8 \end{cases} \Rightarrow \begin{cases} -5A + B = 0 \\ -A - 5B = -4 \end{cases} \Rightarrow \begin{cases} A = \dfrac{2}{13}, \\ B = \dfrac{10}{13}. \end{cases}$$

So

$$Y = e^t\left(\frac{2}{13}\sin 2t + \frac{10}{13}\cos 2t\right) = \frac{2}{13}e^t(\sin 2t + 5\cos 2t).$$

is a particular solution.

Example 5.6. Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t.$$

Answer: A particular solution is

$$Y = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{2}{13}e^t(\sin 2t + 5\cos 2t).$$

Example 5.7. Find a particular solution of

$$y'' - 3y' - 4y = 2e^{-t}.$$

Answer: Try the ansatz $y = Ae^{-t}$. Then

$$y'' - 3y' - 4y = (A + 3A - 4A)e^{-t} = 0.$$

Not work! Try another one

$$y = Ate^{-t}.$$

Then

$$y' = A(1-t)e^{-t}, \quad y'' = A(-2+t)e^{-t}$$

$$y'' - 3y' - 4y = Ae^{-t}(-2 + t - 3(1-t) - 4t) = -5Ae^{-t} = 2e^{-t} \quad \Rightarrow \quad A = -\frac{2}{5}.$$

So

$$y = -\frac{2}{5}te^{-t}$$

is a particular solution.

Question 2. Why Ate^{-t} works?

Answer. Consider the general case:

$$ay'' + by' + cy = de^{\alpha t}.$$

Suppose α is a root (not repeated) of the characteristic equation $ar^2 + br + c = 0$. Let $y = v(t)e^{\alpha t}$.

Then

$$\begin{aligned} y' &= (v' + \alpha v)e^{\alpha t}, \\ y'' &= (v'' + 2\alpha v' + \alpha^2 v)e^{\alpha t}. \end{aligned}$$

Plugging into the equation

$$\begin{aligned} ay'' + by' + cy &= [a(v'' + 2\alpha v' + \alpha^2 v) + b(v' + \alpha v) + cv]e^{\alpha t} \\ &= [av'' + (2a\alpha + b)v' + (a\alpha^2 + b\alpha + c)v]e^{\alpha t} \\ &= [av'' + (2a\alpha + b)v']e^{\alpha t} = de^{\alpha t} \\ \Rightarrow \quad av'' + (2a\alpha + b)v' &= d. \end{aligned}$$

Let $w = v'$, then

$$aw' + (2a\alpha + b)w = d \quad \Rightarrow \quad w = \frac{d}{2a\alpha + b} := A \quad \Rightarrow \quad v = At + B.$$

So

$$y = (At + B)e^{\alpha t} = Ate^{\alpha t}$$

by choosing $B = 0$.

Exercise 5.1. Derive the solution ansatz $y = At^2 e^{\alpha t}$ if α is a repeated root of the characteristic polynomial.

Example 5.8. Find a particular solution of

$$y'' - 4y' + 4y = e^{2t}.$$

Answer: Try the ansatz $y = Ae^{2t}$, not work. Try $y = Ate^{2t}$, not work. Try

$$y = At^2 e^{2t}.$$

$$y' = 2A(t + t^2)e^{2t}, \quad y'' = 2A(1 + 4t + 2t^2)$$

$$y'' - 4y' + 4y = Ae^{2t}[2(1 + 4t + 2t^2) - 8(t + t^2) + 4t^2] = 2Ae^{2t} = e^{2t}.$$

So $A = 1/2$ and

$$y = \frac{1}{2}t^2 e^{2t}$$

is a particular solution.

Example 5.9. Find a particular solution of

$$y'' - 4y' + 3y = t^2 + t + 1.$$

Answer: Consider the ansatz

$$y = At^2 + Bt + C.$$

Then

$$y' = 2At + B, \quad y'' = 2A.$$

$$\begin{aligned} y'' - 4y' + 3y &= 2A - 4(2At + B) + 3(At^2 + Bt + C) \\ &= 3At^2 + (3B - 8A)t + (2A - 4B + 3C) \end{aligned}$$

$$\Rightarrow \quad \begin{cases} 3A &= 1 \\ 3B - 8A &= 1 \\ 2A - 4B + 3C &= 1 \end{cases} \quad \Rightarrow \quad \begin{cases} A = \dfrac{1}{3} \\ B = \dfrac{11}{9} \\ C = \dfrac{1}{3}\left(1 - \dfrac{2}{3} + \dfrac{44}{9}\right) = \dfrac{47}{27} \end{cases}$$

TABLE 3.5.1 The Particular Solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin\beta t\\ \cos\beta t\end{cases}$	$t^s[(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\cos\beta t$ $+ (B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\sin\beta t]$

Notes. Here s is the smallest nonnegative integer ($s = 0, 1,$ or 2) that will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation, respectively.

6 Variation of Parameters

The method of undetermined coefficients has some limitations.

- only equations with constant coefficients
- $g(t)$ must be in one of those special forms

Now consider the more general nonhomogeneous equation

$$y'' + p(t)\,y' + q(t)\,y = g(t).$$

Suppose $y = c_1y_1 + c_2y_2$ is a general solution of the homogeneous equation

$$y'' + p(t)\,y' + q(t)\,y = 0.$$

Let

$$Y = u_1y_1 + u_2y_2,$$

where u_1, u_2 are functions to be determined. Then

$$Y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'.$$

Let's pose the condition

$$u_1'y_1 + u_2'y_2 = 0. \tag{6.1}$$

Then

$$Y' = u_1y_1' + u_2y_2' \quad \text{and} \quad Y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''.$$

So

$$\begin{aligned} Y'' + p(t)Y' + q(t)Y &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' + p(t)(u_1y_1' + u_2y_2') + q(t)(u_1y_1 + u_2y_2) \\ &= u_1[y_1'' + p(t)\,y_1' + q(t)\,y_1] + u_2[y_2'' + p(t)\,y_2' + q(t)\,y_2] + u_1'y_1' + u_2'y_2' \\ &= u_1'y_1' + u_2'y_2'. \end{aligned}$$

So

$$u_1'y_1' + u_2'y_2' = g(t). \tag{6.2}$$

So from (6.1) and (6.2) we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

Note this system has a unique solution because $W(y_1, y_2) \neq 0$. The solution is (given by Cramer's rule):

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2}{W(y_1, y_2)} g, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1}{W(y_1, y_2)} g.$$

Integrating in t, we obtain

$$u_1 = \int \frac{-y_2}{W(y_1, y_2)} g\, dt, \quad u_2 = \int \frac{y_1}{W(y_1, y_2)} g\, dt.$$

This is called the method of **variation of parameters**.

Example 6.1. Find the general solution of

$$y'' + 4y = 3\csc t.$$

Answer: We have $y_1 = \sin 2t$, $y_2 = \cos 2t$,

$$W(y_1, y_2) = \begin{vmatrix} \sin 2t & \cos 2t \\ 2\cos 2t & -2\sin 2t \end{vmatrix} = -4.$$

So

$$u_1 = \int \frac{-y_2}{W(y_1, y_2)} g\, dt = \int -\frac{\cos 2t}{-4} 3\csc t = \frac{3}{4}\int \frac{\cos 2t}{\sin t} dt = \frac{3}{4}\int \frac{1 + 2\cos^2 t}{\sin t} dt$$

$$= \frac{3}{4}\left[\int \csc t\, dt + 2\int \frac{1 - \sin^2 t}{\sin t} dt\right] = \frac{3}{4}\left[3\int \csc t\, dt + 2\int \sin t\, dt\right] = \frac{3}{4}[3\ln|\csc t - \cot t| - 2\cos t]$$

Similarly we can find u_2 (exercise). So the general solution is

$$y = c_1 \sin 2t + c_2 \cos 2t + u_1 \sin 2t + u_2 \cos 2t$$

$$\int \csc t\, dt = \int \frac{1}{\sin t} dt = \int \frac{\sin t}{\sin^2 t} dt = \int \frac{\sin t}{1 - \cos^2 t} dt = \int \frac{1}{2}\left[\frac{\sin t}{1 + \cos t} + \frac{\sin t}{1 - \cos t}\right] dt$$

$$= \frac{1}{2}\left[\int \frac{-1}{1 + \cos t} d(1 + \cos t) + \int \frac{1}{1 - \cos t} d(1 - \cos t)\right] = \frac{1}{2}[-\ln(1 + \cos t) + \ln(1 - \cos t)]$$

$$= \frac{1}{2}\ln\frac{1 - \cos t}{1 + \cos t} = \frac{1}{2}\ln\frac{(1 - \cos t)^2}{1 - \cos^2 t} = \frac{1}{2}\ln\frac{(1 - \cos t)^2}{\sin^2 t} = \ln\sqrt{\frac{(1 - \cos t)^2}{\sin^2 t}} = \ln\left|\frac{1 - \cos t}{\sin t}\right|$$

$$= \ln|\csc t - \cot t|$$

7 Free Vibrations

Consider the equation for the spring-mass system

$$mu'' + \gamma u' + ku = 0.$$

7.1 Undamped free vibrations

Let $\gamma = 0$, i.e. there is no damping force. Then the equation reduces to

$$mu'' + ku = 0.$$

The roots of the characteristic equation are

$$r_{1,2} = \pm\omega_0 i, \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

So the general solution is

$$u = A\cos\omega_0 t + B\sin\omega_0 t = R\cos(\omega_0 t - \delta) = R(\cos\omega_0 t\cos\delta + \sin\omega_0 t\sin\delta).$$

So

$$A = R\cos\delta, \quad B = R\sin\delta.$$

So

$$R = \sqrt{A^2 + B^2}, \quad \cos\delta = \frac{A}{R}, \quad \sin\delta = \frac{B}{R} \quad \Rightarrow \quad \delta =$$

Here R is the **amplitude**, ω_0 is the **angular frequency** (natural frequency of the system), δ is the **phase**, and $T = \dfrac{2\pi}{w_0}$ is the **period**.

7.2 Damped free vibrations

Now consider the case when $\gamma > 0$ (damped vibration). Then the roots of the characteristic equation is

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = -\frac{\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m}.$$

1. If $\gamma^2 > 4mk$ (**overdamped**), then $r_1 \neq r_2$ are real and both negative. The general solution is

$$u = Ae^{r_1 t} + Be^{r_2 t}.$$

   The solution is nonoscillatory and $u(t) \to 0$ as $t \to \infty$.

2. If $\gamma^2 = 4mk$ (**critically damped)**, then we have repeated root $r = -\dfrac{\gamma}{2m}$. So the general solution is

$$u = Ae^{rt} + Bte^{rt}.$$

   The solution is nonoscillatory and $u(t) \to 0$ as $t \to \infty$.

3. If $\gamma^2 < 4mk$, then the roots are

$$r_{1,2} = \lambda \pm i\mu, \quad \lambda = -\frac{\gamma}{2m}, \quad \mu = \frac{\sqrt{4mk - \gamma^2}}{2m}.$$

   The general solution is

$$u = e^{\lambda t}(A\cos\mu t + B\sin\mu t) = Re^{\lambda t}\cos(\mu t - \delta).$$

   It's a **damped oscillation**, and $u(t) \to 0$ as $t \to \infty$.

   $u(t)$ is nonperiodic, but we call $T = \dfrac{2\pi}{\mu}$ the **quasi period**. In fact

$$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} < \omega_0.$$

7.3 Electric circuits (skip)

8 Forced Vibrations (optional)

8.1 Forced vibrations with damping

$$mu'' + \gamma u' + ku = F$$

We consider periodic forces $F = F_0 \cos \omega t$. The general solution is

$$u = [c_1 u_1(t) + c_2 u_2(t)] + [A \cos \omega t + B \sin \omega t] = u_c(t) + U(t).$$

Note that $u_c(t) \to 0$ as $t \to \infty$, but $U(t)$ is periodic. So we call $u_c(t)$ the **transient solution** and $U(t)$ the **steady-state solution**.

Example 8.1. Consider the IVP

$$u'' + u' + \frac{5}{4}u = 3\cos t, \quad u(0) = 2, \quad u'(0) = 3.$$

Then

$$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = -\frac{1}{2} \pm i,$$

So

$$u_c(t) = e^{-\frac{t}{2}}(c_1 \cos t + c_2 \sin t).$$

Let $U = A\cos t + B\sin t$. Then

$$U'' + U' + \frac{5}{4}U = -A\cos t - B\sin t - A\sin t + B\cos t + \frac{5}{4}(A\cos t + B\sin t)$$
$$= \left(-A + B + \frac{5}{4}A\right)\cos t + \left(-B - A + \frac{5}{4}B\right)\sin t = \left(\frac{1}{4}A + B\right)\cos t + \left(\frac{1}{4}B - A\right)\sin t$$

So

$$\frac{1}{4}A + B = 3, \quad \frac{1}{4}B - A = 0 \quad \Rightarrow \quad \begin{cases} A = \dfrac{12}{17}, \\ B = \dfrac{48}{17}. \end{cases}$$

So the steady-state solution is

$$U(t) = \frac{12}{17}\cos t + \frac{48}{17}\sin t.$$

So the general solution is

$$u(t) = e^{-\frac{t}{2}}(c_1 \cos t + c_2 \sin t) + \frac{12}{17}\cos t + \frac{48}{17}\sin t.$$

Plugging initial conditions, we obtain $c_1 = \frac{22}{17}$, $c_2 = \frac{14}{17}$. So the solution of the IVP is

$$u(t) = \frac{2}{17}\left[e^{-\frac{t}{2}}(11\cos t + 7\sin t) + 6\cos t + 24\sin t\right].$$

Resonance. Steady-state solution $U = A\cos\omega t + B\sin\omega t$

$$U' = \omega(-A\sin\omega t + B\cos\omega t), \quad U'' = \omega^2(-A\cos\omega t - B\sin\omega t)$$

$$\begin{aligned}
\Rightarrow\ & mU'' + \gamma U' + kU \\
= \ & m\omega^2(-A\cos\omega t - B\sin\omega t) + \gamma\omega(-A\sin\omega t + B\cos\omega t) + k(A\cos\omega t + B\sin\omega t) \\
= \ & (-m\omega^2 A + \gamma\omega B + kA)\cos\omega t + (-Bm\omega^2 - A\gamma\omega + kB)\sin\omega t \\
= \ & [(k - m\omega^2)A + \gamma\omega B]\cos\omega t + [-\gamma\omega A + (k - m\omega^2)B]\sin\omega t \\
= \ & F_0\cos\omega t
\end{aligned}$$

$$\left\{\begin{array}{rcl} (k - m\omega^2)A + \gamma\omega B & = & F_0 \\ -\gamma\omega A + (k - m\omega^2)B & = & 0 \end{array}\right. \quad \Rightarrow \quad \begin{pmatrix} k - m\omega^2 & \gamma\omega \\ -\gamma\omega & k - m\omega^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} F_0 \\ 0 \end{pmatrix}$$

$$\begin{aligned}
A &= \frac{k - m\omega^2}{(k - m\omega^2)^2 + \gamma^2\omega^2}F_0 = \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}F_0 \\
B &= \frac{\gamma\omega}{(k - m\omega^2)^2 + \gamma^2\omega^2}F_0 = \frac{\gamma\omega}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}F_0
\end{aligned}$$

$$A\cos\omega t + B\sin\omega t = R\cos(\omega t - \delta) \quad \Rightarrow \quad R = \frac{F_0}{\Delta}, \quad \Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

Nondimensionalize (无量纲化)

$$R = \frac{F_0}{m\omega_0^2\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{\gamma^2\omega^2}{m^2\omega_0^4}}} = \frac{F_0}{k\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{\gamma^2}{mk}\frac{\omega^2}{\omega_0^2}}}$$

$$\Rightarrow \frac{R}{(F_0/k)} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0^2}}}, \quad \boxed{\Gamma = \frac{\gamma^2}{mk}}$$

$$\gamma u' = F \Rightarrow \gamma = \frac{F}{u'}: \frac{\text{N}}{\text{m}\cdot\text{s}^{-1}} = \frac{\text{N}\cdot\text{s}}{\text{m}} \Rightarrow \Gamma = \frac{\gamma^2}{mk}: \frac{\text{N}^2\cdot\text{s}^2}{\text{m}^2\cdot\text{kg}\cdot\text{N}\cdot\text{m}^{-1}} = \frac{\text{N}\cdot\text{s}^2}{\text{m}\cdot\text{kg}} = \frac{\text{N}}{\text{m}\cdot\text{s}^{-2}\cdot\text{kg}} = 1$$

Clearly $\frac{R}{(F_0/k)}$ and $\frac{\omega^2}{\omega_0^2}$ are also dimensionless. Rewrite the equation as

$$y = \frac{1}{\sqrt{(1-x)^2 + \Gamma x}}, \quad y = \frac{R}{(F_0/k)}, \quad x = \frac{\omega^2}{\omega_0^2}$$

$$y' = -\frac{1}{2}[(1-x)^2 + \Gamma x]^{-\frac{3}{2}}[\Gamma - 2 + 2x]$$

If $0 < \Gamma < 2$, then $y' > 0$ for $x \in \left[0, 1 - \frac{\Gamma}{2}\right)$, $y' < 0$ for $x \in \left(1 - \frac{\Gamma}{2}, \infty\right)$ and $y' = 0$ for $x = 1 - \frac{\Gamma}{2}$.

So $y_{\max}$ is obtained at $x_{\max} = 1 - \frac{\Gamma}{2}$:

$$y_{\max} = \frac{1}{\sqrt{(1-x)^2 + \Gamma x}} = \frac{1}{\sqrt{\Gamma - \Gamma^2/4}} \to \infty \quad \text{as} \quad \Gamma \to 0.$$

Hence for lightly damped system (Γ is small), the amplitude of the steady-state solution when ω is near ω_0 can be very large for small external force. This phenomenon is known as **resonance**.

Asymptote

```
% -width 0.5par
import graph;
size(8cm, 0);
xaxis("$x$", RightTicks, Arrow);
yaxis("$y$", LeftTicks, Arrow);
real G1 = 1;
real G2 = 0.2;
real G3 = 0.1;
real f1(real x) {return ((1-x)**2+G1*x)**(-0.5);}
real f2(real x) {return ((1-x)**2+G2*x)**(-0.5);}
real f3(real x) {return ((1-x)**2+G3*x)**(-0.5);}
draw(graph(f1, 0, 3, Hermite), blue+linewidth(1pt));
draw(graph(f2, 0, 3, Hermite), purple+linewidth(1pt));
draw(graph(f3, 0, 3, Hermite), red+linewidth(1pt));
label("$\displaystyle y=\frac{1}{\sqrt{(1 - x)^2 + \Gamma x}}$", (1.3, 3.5),
align=E);
label("$\Gamma=1$", (0.8, 1.2), align=E, blue);
label("$\Gamma=0.2$", (0.6, 1.7), align=E, purple);
label("$\Gamma=0.1$", (1.2, 2.6), align=E, red);
```

8.2 Forced vibrations without damping

Consider

$$mu''+ku=F_0\cos\omega t.$$

The general solution of the homogeneous part is

$$u_c(t)=c_1\cos\omega_0 t+c_2\sin\omega_0 t,\quad \omega_0=\sqrt{\frac{k}{m}}.$$

8.2.1 $\omega\neq\omega_0$

The general solution is

$$u=u_c(t)+U(t),\quad U(t)=A\cos\omega t+B\sin\omega t.$$

$$U'=\omega(-A\sin\omega t+B\cos\omega t),\quad U''=\omega^2(-A\cos\omega t-B\sin\omega t)$$

$$\begin{aligned}
mU''+kU &= m\omega^2(-A\cos\omega t-B\sin\omega t)+k(A\cos\omega t+B\sin\omega t)\\
&= (-Am\omega^2+kA)\cos\omega t+(-Bm\omega^2+kB)\sin\omega t\\
&= A(k-m\omega^2)\cos\omega t+B(k-m\omega^2)\sin\omega t\\
&= F_0\cos\omega t
\end{aligned}$$

$$\Rightarrow\quad A=\frac{F_0}{k-m\omega^2},\quad B=0$$

The general solution is

$$u=c_1\cos\omega_0 t+c_2\sin\omega_0 t+\frac{F_0}{k-m\omega^2}\cos\omega t.$$

Suppose the initial condition is $u(0)=u'(0)=0$, then

$$c_1+\frac{F_0}{k-m\omega^2}=0,\quad c_2\omega_0=0\quad\Rightarrow\quad c_1=-\frac{F_0}{k-m\omega^2},\quad c_2=0.$$

$$u=\frac{F_0}{k-m\omega^2}(\cos\omega t-\cos\omega_0 t)=\frac{-2F_0}{k-m\omega^2}\sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right)$$

If ω is close to ω_0, then we have a **beat**. Also used in **amplitude modulation**.

Asymptote
```
% -width 0.6par
import graph;
size(10cm, 0);
real f(real x) {return cos(10*x)-cos(11*x);}
draw(graph(f, -10, 10, Hermite), black+linewidth(1pt));
label("$y=\cos(10 x)-\cos(11 x)$", (0, 3));
```

8.2.2 $\omega = \omega_0$

A general solution is

$$U = c_1 \cos\omega t + c_2 \sin\omega t + t(A\cos\omega t + B\sin\omega t).$$

9 Higher Order Linear Equations

9.1 General theory

An n-th order linear ODE is in the form

$$L[y] = y^{(n)} + p_{n-1}(t)\,y^{(n-1)} + \cdots + p_1(t)\,y' + p_0(t)\,y = g(t)$$

An initial value problem is the equation $L[y] = g$ together with initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots \quad y^{(n-1)}(t_0) = y_0^{(n-1)}.$$

Definition 9.1 *The **Wronskian** of n solutions $y_1, \ldots, y_n$ of $L[y] = 0$ is*

$$W[y_1, \ldots, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

Theorem 9.2 *Let $y_1, \ldots, y_n$ be solutions of $L[y] = 0$. Then $y_1, \ldots, y_n$ form a fundamental set of solutions if and only if they are linearly independent.*

Proof. Suppose $y_1, \ldots, y_n$ form a fundamental set of solutions, that is, $W[y_1, \ldots, y_n] \neq 0$. Let $c_1, \ldots, c_n$ be constants such that

$$c_1 y_1 + \cdots + c_n y_n = 0.$$

Differentiate the above equation in t,

$$c_1 y_1' + \cdots + c_n y_n' = 0.$$

Repeat differentiating, we obtain

$$\begin{aligned} c_1 y_1'' + \cdots + c_n y_n'' &= 0 \\ &\vdots \\ c_1 y_1^{(n-1)} + \cdots + c_n y_n^{(n-1)} &= 0 \end{aligned}$$

Hence we have a linear system

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Hence $y_1, \ldots, y_n$ are linearly independent.

Now assume $y_1, \ldots, y_n$ do not form a fundamental set of solutions, i.e. $W[y_1, \ldots, y_n](t_0) = 0$ for some t_0. Then there exists constants $c_1, \ldots, c_n$, not all zero, such that

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let

$$Y = c_1 y_1 + \cdots + c_n y_n.$$

Then consider the IVP

$$L[y] = 0, \quad y(t_0) = Y(t_0) = 0, \quad y'(t_0) = Y'(t_0) = 0, \quad \cdots \quad y^{(n-1)}(t_0) = Y^{(n-1)}(t_0) = 0$$

Clearly Y is the solution of the IVP. On the other hand, 0 is also a solution of the IVP. By the E&U theorem, we have $Y = 0$. Thus $y_1, \ldots, y_n$ are linearly independent. ☐

9.2 Homogeneous constant coefficients

Example 9.3.

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$

Answer: The characteristic equation is

$$r^4 + r^3 - 7r^2 - r + 6 = 0.$$

To find the roots, we factorize it:

$$\begin{aligned} (r^4 + r^3) - (7r^2 + r - 6) &= r^3(r+1) - (r+1)(7r-6) \\ &= (r+1)(r^3 - 7r + 6) = (r+1)(r^3 - r - 6r + 6) \\ &= (r+1)[r(r^2-1) - 6(r-1)] \\ &= (r+1)(r-1)(r^2 + r - 6) \\ &= (r+1)(r-1)(r-2)(r+3) \end{aligned}$$

So the roots are*f*

$$r = -3, -1, 1, 2$$

So there are four solutions

$$y = e^{-3t}, e^{-t}, e^t, e^{2t}$$

and they form a fundamental set of solutions. So the general solution is

$$y = c_1 e^{-3t} + c_2 e^{-t} + c_3 e^t + c_4 e^{2t}.$$

Note 9.4. Another method to find the roots. All possible rational roots are

$$\frac{\pm 1, \pm 2, \pm 3, \pm 6}{1}.$$

Then verify directly it they are roots. Once one or more roots are found, we can use **polynomial division** to reduce the order and help find other roots.

Example 9.5.

$$y^{(4)} - y = 0$$

Answer:

$$r^4 - 1 = (r^2+1)(r^2-1) \quad \Rightarrow \quad r = \pm i, \pm 1 \quad \Rightarrow \quad y = \cos t, \sin t, e^{-t}, e^t$$

Example 9.6.

$$y^{(4)}+2y''+y=0$$

Answer:

$$r^4+2r^2+1=(r^2+1)^2=0 \quad \Rightarrow \quad r=\pm i,\pm i \quad \Rightarrow \quad y=\cos t,\sin t,t\cos t,t\sin t$$

(We say the root $r=\pm i$ has multiplicity 2).

Example 9.7.

$$y^{(4)}+y=0.$$

Answer:

$$\begin{aligned}
r^4+1=0 \quad &\Rightarrow \quad r^4=-1=e^{i(\pi+2n\pi)}\\
\Rightarrow \quad r=\exp\left(i\frac{(2n+1)}{4}\pi\right)&=e^{\frac{\pi}{4}i},e^{\frac{3\pi}{4}i},e^{\frac{5\pi}{4}i},e^{\frac{7\pi}{4}i}\\
\Rightarrow \quad r=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i,\quad -\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i,&\quad -\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i,\quad \frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\\
\Rightarrow \quad r=\frac{\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}i,&\quad -\frac{\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}i\\
\Rightarrow \quad y=e^{\frac{\sqrt{2}}{2}t}\cos\frac{\sqrt{2}}{2}t,\quad e^{\frac{\sqrt{2}}{2}t}\sin\frac{\sqrt{2}}{2}t,&\quad e^{-\frac{\sqrt{2}}{2}t}\cos\frac{\sqrt{2}}{2}t,\quad e^{-\frac{\sqrt{2}}{2}t}\sin\frac{\sqrt{2}}{2}t
\end{aligned}$$

9.3 The method of undetermined coefficients

Example 9.8. Find the general solution of

$$y'''-3y''+3y'-y=4e^t$$

Answer:

$$r^3-3r^2+3r-1=(r-1)^3=0 \quad \Rightarrow \quad r=1,1,1 \quad \Rightarrow \quad y_1=e^t,te^t,t^2e^t.$$

Let

$$Y=At^3e^t.$$

Then

$$\begin{aligned}
Y'=A(3t^2+t^3)e^t,\quad Y''=A(6t+6t^2+t^3)e^t,\quad Y'''=A(6+18t+9t^2+t^3)e^t.\\
\Rightarrow \quad [(6+18t+9t^2+t^3)-3(6t+6t^2+t^3)+3(3t^2+t^3)-t^3]Ae^t=4e^t\\
\Rightarrow \quad 6A=4 \quad \Rightarrow \quad A=\frac{2}{3}.
\end{aligned}$$

So the general solution is

$$y=c_1e^t+c_2te^t+c_3t^2e^t+\frac{2}{3}t^3e^t.$$

9.4 The method of variation of parameters

Suppose $y_1,\ldots,y_n$ form a fundamental set of solutions for $L[y]=0$. Consider the nonhomogeneous equation

$$L[y]=g(t).$$

Let

$$Y=u_1y_1+\cdots+u_ny_n$$

Then differentiate Y and make some assumption as in the case of 2nd order equations. We obtain

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g \end{pmatrix}$$

Then

$$u_m' = \frac{W_m}{W} g \quad \Rightarrow \quad u_m = \int \frac{W_m}{W} g$$

where W is the Wronskian, and W_m is the determinant of the above matrix with the m-th column replaced by the vector $(0, \ldots, 0, 1)^T$.

Example 9.9. Find the general solution to

$$y''' - y'' - y' + y = g(t).$$

Answer:

$$r^3 - r^2 - r + 1 = r^2(r-1) - (r-1) = (r-1)^2(r+1) \quad \Rightarrow \quad r = 1, 1, -1$$

$$y_1 = e^{-t}, \quad y_2 = e^t, \quad y_3 = te^t.$$

$$\begin{aligned}
W &= \begin{vmatrix} e^{-t} & e^t & te^t \\ -e^{-t} & e^t & (t+1)e^t \\ e^{-t} & e^t & (t+2)e^t \end{vmatrix} = e^t \begin{vmatrix} 1 & 1 & t \\ -1 & 1 & t+1 \\ 1 & 1 & t+2 \end{vmatrix} \\
&= e^t \begin{vmatrix} 1 & 0 & t \\ -1 & 2 & t+1 \\ 1 & 0 & t+2 \end{vmatrix} = 2e^t \begin{vmatrix} 1 & t \\ 1 & t+2 \end{vmatrix} = 4e^t, \\
W_1 &= \begin{vmatrix} 0 & e^t & te^t \\ 0 & e^t & (t+1)e^t \\ 1 & e^t & (t+2)e^t \end{vmatrix} = e^{2t} \begin{vmatrix} 0 & 1 & t \\ 0 & 1 & (t+1) \\ 1 & 1 & (t+2) \end{vmatrix} = e^{2t}, \\
W_2 &= \begin{vmatrix} e^{-t} & 0 & te^t \\ -e^{-t} & 0 & (t+1)e^t \\ e^{-t} & 1 & (t+2)e^t \end{vmatrix} = \begin{vmatrix} 1 & 0 & t \\ -1 & 0 & (t+1) \\ 1 & 1 & (t+2) \end{vmatrix} = -(2t+1), \\
W_3 &= \begin{vmatrix} e^{-t} & e^t & 0 \\ -e^{-t} & e^t & 0 \\ e^{-t} & e^t & 1 \end{vmatrix} = 2. \\
u_1 &= \int \frac{W_1}{W} g = \int \frac{1}{4} e^t g(t)\, dt, \\
u_2 &= \int \frac{W_2}{W} g = \int -\frac{2t+1}{4e^t} g(t)\, dt, \\
u_3 &= \int \frac{W_3}{W} g = \int \frac{1}{2e^t} g(t)\, dt.
\end{aligned}$$

So the general solution is

$$y = (c_1 e^{-t} + c_2 e^t + c_3 t e^t) + e^{-t} \int \frac{1}{4} e^t g(t)\,dt - e^t \int \frac{2t+1}{4e^t} g(t)\,dt + t e^t \int \frac{1}{2e^t} g(t)\,dt.$$

Ordinary Differential Equations

BY YULIANG WANG

Chapter 3: Second Order Linear Equations

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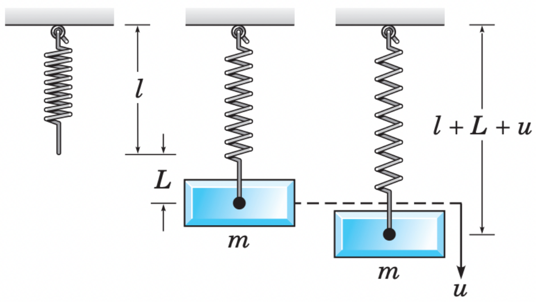
Motivation: spring-mass system



Newton's Law: $ma = f$

$a = u''$

$f = mg - k(L + u) - \gamma u' + F$

k: spring constant

γ: damping coefficient

$$mu'' = mg - k(L + u) - \gamma u' + F$$

$$mu'' + \gamma u' + ku = mg - kL + F$$

But $mg = kL$, so

$$mu'' + \gamma u' + ku = F$$

1 Homogeneous Equations with Constant Coefficients

$$ay'' + by' + cy = 0,$$

initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

Example 1.1. Solve the IVP

$$y'' - y = 0, \quad y(0) = 2, \quad y'(0) = -1.$$

By investigation, we know $y = ce^t$ satisfies the equation for any constant c. However, it doesn't satisfy the initial conditions. More investigation shows $y = ce^{-t}$ is also a solution for any constant c. It turns out

$$y = c_1 e^t + c_2 e^{-t}$$

is a solution for any constants c_1, c_2. Now, the initial conditions require

$$c_1 + c_2 = 2, \quad c_1 - c_2 = -1.$$

This is a system of linear equations. The matrix form is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

The matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is nonsingular ($\det A = -2 \neq 0$). So the system has a unique solution. The solution is

$$c_1 = \frac{1}{2}, \quad c_2 = -\frac{3}{2}.$$

So the IVP has a solution

$$y = \frac{1}{2} e^t - \frac{3}{2} e^{-t}.$$

Example 1.2. Solve

$$y''+5y'+6y=0.$$

Answer: We assume the ansatz of the solution: $y=e^{rt}$ for some constant r. Then

$$\begin{aligned} y''+5y'+6y &= r^2e^{rt}+5re^{rt}+6e^{rt} \\ &= (r^2+5r+6)e^{rt} \\ &= 0 \\ \Rightarrow \quad r^2+5r+6 &= 0 \end{aligned}$$

The equation

$$r^2+5r+6=0$$

is called the **characteristic equation** for the ODE. The solutions are

$$r_1=-3, \quad r_2=-2.$$

So have two solutions of the ODE:

$$y_1=e^{-3t}, \quad y_2=e^{-2t}.$$

So we can let the solution be

$$y=c_1y_1+c_2y_2=c_1e^{-3t}+c_2e^{-2t}.$$

Note that

$$y\to 0 \quad \text{as} \quad t\to\infty$$

Example 1.3. Solve the IVP

$$4y''-8y'+3y=0, \quad y(0)=2, \quad y'(0)=\frac{1}{2}.$$

Answer: The characteristic equation is

$$4r^2-8r+3=(2r-3)(2r-1)=0 \quad \Rightarrow \quad r_1=\frac{3}{2}, \quad r_2=\frac{1}{2}.$$

So we have solutions

$$y=c_1e^{\frac{3}{2}t}+c_2e^{\frac{1}{2}t}.$$

Plugging initial conditions

$$\begin{aligned} c_1+c_2 &= 2 \\ \frac{3}{2}c_1+\frac{1}{2}c_2 &= \frac{1}{2} \end{aligned}$$

We can find unique solutions

$$c_1=-\frac{1}{2}, \quad c_2=\frac{5}{2}.$$

So a solution of the IVP is

$$y=-\frac{1}{2}e^{\frac{3}{2}t}+\frac{5}{2}e^{\frac{1}{2}t}.$$

Note that

$$y \to -\infty \quad \text{as} \quad t \to \infty$$

2 Theory of 2nd Order Linear Equations

Consider the general 2nd order linear equation

$$\boxed{L[y] := y''(t) + p(t)\,y' + q(t)\,y} = g(t).$$

Note that L is a linear operator.

Existence and Uniqueness Theorem *Consider the IVP*

$$y''(t) + p(t)\,y' + q(t)\,y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

If p, q, g are continuous on an interval I containing t_0, then there exists a unique solution to this IVP on I.

Example 2.1. Find the longest interval in which the solution of the initial value problem

$$(t^2 - 3t)\,y'' + ty' - (t+3)\,y = 0, \quad y(1) = 2, \quad y'(1) = 1$$

is certain to exist.

Answer: Assuming $t \neq 0, t \neq 3$, rewrite the equation in the standard form

$$y'' + \frac{1}{t-3}y' - \frac{t+3}{t^2 - 3t}y = 0.$$

So p, q, g are continuous in $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$. Since $1 \in (0, 3)$. By the E&U theorem, there exists a unique solution to the IVP on $(0, 3)$.

Principle of Superposition *Consider the **homogeneous** linear equation*

$$L[y] = 0.$$

If y_1 and y_2 are both solutions, then $c_1 y_1 + c_2 y_2$ is also a solution for any constants c_1 and c_2.

Proof.

$$\begin{aligned} L[c_1 y_1 + c_2 y_2] &= (c_1 y_1 + c_2 y_2)'' + p(t)\,(c_1 y_1 + c_2 y_2)' + q(t)\,(c_1 y_1 + c_2 y_2) \\ &= c_1\,(y_1'' + p(t)\,y_1' + q(t)\,y_1) + c_2\,(y_2'' + p(t)\,y_2' + q(t)\,y_2) \\ &= c_1 L[y_1] + c_2 L[y_2] = 0 + 0 = 0. \end{aligned}$$

So $c_1 y_2 + c_2 y_2$ is also a solution. $\square$

Let's consider the homogeneous equation with constant coefficients

$$ay'' + by' + cy = 0.$$

Suppose there are two different roots r_1, r_2 of the characteristic polynomial $ar^2 + br + c$. Then we have two solutions

$$y_1 = e^{r_1 t}, \quad y_2 = e^{r_2 t}.$$

By the principle of superposition,

$$y = c_1 y_1 + c_2 y_2$$

is a solution for any constants c_1, c_2.

The next question: can we always find c_1, c_2 such that a given initial conditions are satisfied?

Pluggin the initial conditions, we obtain a linear system for c_1, c_2:

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0' \end{aligned}$$

or, in matrix form

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}.$$

In order to have a solution for arbitrary values of y_0, y_0', we need the matrix to be nonsingular, i.e.

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)\, y_2'(t_0) - y_1'(t_0)\, y_2(t_0) \neq 0.$$

Definition 2.2 *Suppose y_1, y_2 are two solutions of the ODE $L[y] = 0$. Then the Wronskian of them is*

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

Theorem 2.3 *Let y_1, y_2 are solutions of the equation $L[y] = 0$. Then one can find constants c_1 and c_2 such that $c_1 y_1 + c_2 y_2$ solves the IVP*

$$L[y] = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

regardless of the values y_0 and y_0' if and only if $W(y_1, y_2)(t_0) \neq 0$.

Next we show all solutions of $L[y] = 0$ can are actually in the form $c_1 y_1 + c_2 y_2$ if and only if the Wronskian is nonzero.

Theorem 2.4 *Let y_1, y_2 are solutions of the equation $L[y] = 0$ on some interval I. Then every solution of $L[y] = 0$ on I can be written as $c_1 y_1 + c_2 y_2$ if and only if $W(y_1, y_2)(t) \neq 0$ for some $t \in I$.*

Proof. Suppose $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$. Let $\phi(t)$ to be a solution of $L[y] = 0$. Let $y_0 = \phi(t_0)$ and $y_0' = \phi'(t_0)$. Now consider the IVP

$$L[y] = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{2.1}$$

Clearly ϕ is a solution of the IVP (2.1). On the other hand, we can find c_1 and c_2 such that $c_1 y_1 + c_2 y_2$ is a solution of the IVP (2.1) for some c_1, c_2 since $W(y_1, y_2)(t_0) \neq 0$. By the uniqueness part of the E&U theorem, we have $\phi = c_1 y_1 + c_2 y_2$.

Next, suppose $W(y_1, y_2)(t) = 0$ for any $t \in I$. Then $W(y_1, y_2)(t_0) = 0$ for some $t_0 \in I$. So there exists some numbers y_0, y_0' such that

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix} \tag{2.2}$$

has no solution. Let $\phi(t)$ to be the solution of the IVP (2.1). Suppose $\phi = c_1 y_1 + c_2 y_2$ for some c_1, c_2, then c_1, c_2 must satisfy the linear system (2.2). A contradition! ☐

If $W(y_1, y_2)(t) \neq 0$ for some t, we call the solutions $\{y_1, y_2\}$ a **fundamental set of solutions**.

Example 2.5. If $r_1 \neq r_2$ are real numbers, and $y_1 = e^{r_1 t}, y_2 = e^{r_2 t}$ are solutions of some ODE. Then

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)t} \neq 0$$

for any t. So $\{y_1, y_2\}$ form a fundamental set of solutions.

Example 2.6. Show that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions of

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0.$$

Answer:

$$W(y_1, y_2)(t) = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2} t^{-1/2} & -t^{-2} \end{vmatrix} = -\frac{3}{2} t^{-3/2} \neq 0$$

for any $t \neq 0$. So $\{y_1, y_2\}$ form a fundamental set of solutions for $t \neq 0$.

Theorem 2.7 *Let y_1 to be the solution of the IVP*

$$L[y] = 0, \quad y(t_0) = 1, \quad y'(t_0) = 0.$$

Let y_2 to be the solution of the IVP

$$L[y] = 0, \quad y(t_0) = 0, \quad y'(t_0) = 1.$$

Then the Wronskian of y_1, y_2 is $W(t) = 1$. So $\{y_1, y_2\}$ form a fundamental set of solutions.

This theorem says fundamental set of solutions always exist. In fact, we can find infinitely many fundamental set of solutions.

Example 2.8. Find the fundamental set of solutions y_1 and y_2 specified by Theorem 2.7 for the differential equation

$$y'' - y = 0.$$

Answer: Using the characteristic equations, we find two solutions

$$y_1 = e^t, \quad y_2 = e^{-t}.$$

We can check they form a fundamental set of solutions. Let's find another fundamental set of solutions defined as in Theorem 2.7 for $t_0 = 0$. Let

$$y_3 = \frac{e^t + e^{-t}}{2} = \cosh t, \quad y_4 = \frac{e^t - e^{-t}}{2} = \sinh t.$$

Then $W(y_3, y_4) = 1$. So the general solution can be written as

$$c_1 y_1 + c_2 y_2 \text{ or } c_3 y_3 + c_4 y_4.$$

Theorem 2.9 **(Abel)** *Let y_1, y_2 are solutions of the equation*

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0.$$

Then the Wronskian

$$W(y_1, y_2)(t) = ce^{-\int p(t)dt}$$

for some constant c, which may depend on y_1, y_2 but otherwise independent of p, q.

Proof. We have

$$\begin{aligned} y_1'' + p(t)\,y_1' + q(t)\,y_1 &= 0, \\ y_2'' + p(t)\,y_2' + q(t)\,y_2 &= 0. \end{aligned}$$

Then

$$\begin{aligned} y_2\,[y_1'' + p(t)\,y_1' + q(t)\,y_1] &= 0, \\ y_1\,[y_2'' + p(t)\,y_2' + q(t)\,y_2] &= 0. \end{aligned}$$

Subtracting two equations, we obtain

$$y_1\,y_2'' - y_1''\,y_2 + p(t)\,(y_1\,y_2' - y_1'\,y_2) = 0.$$

Note that

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1\,y_2' - y_2\,y_1'.$$

$$W'(t) = y_1'\,y_2' + y_1\,y_2'' - y_2'\,y_1' - y_2\,y_1'' = y_1\,y_2'' - y_2\,y_1''.$$

So we obtain

$$W'(t) + p(t)\,W(t) = 0.$$

Solving this first order linear ODE, we obtain

$$W(t) = ce^{-\int p(t)dt}.$$

□

Remark 2.10. From the Abel's theorem, we now know the Wronskian of two solutions of an ODE is either identically zero or nowhere zero.

Theorem 2.11 *Suppose p, q are real-valued functions. Let $y(t) = u(t) + iv(t)$ be a complex-valued solution of*

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0,$$

where u, v are real-valued functions. Then u, v are also solutions of $L[y] = 0$.

Proof. We have

$$\begin{aligned} L[y] &= (u + iv)'' + p(t)\,(u + iv)' + q(t)\,(u + iv) \\ &= (u'' + iv'') + p(t)\,(u' + iv') + q(t)\,(u + iv) \\ &= (u'' + p(t)\,u' + q(t)\,u) + i\,(v'' + p(t)\,v' + q(t)\,v) \\ &= 0. \end{aligned}$$

So

$$u'' + p(t)u' + q(t)u = 0, \quad v'' + p(t)v' + q(t)v = 0.$$

That is, u, v are both solutions of $L[y] = 0$.

□

3 Complex roots of the characteristic equation

Let's go back to equations with constant coefficients

$$ay'' + by' + cy = 0.$$

The characteristic equation is

$$ar^2 + br + c = 0.$$

The solution is given by the quadratic formula

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1. $b^2 > 4ac$. Then r_1, r_2 are both real and $r_1 \neq r_2$.
2. $b^2 = 4ac$. Then r_1, r_2 are both real and $r_1 = r_2$.
3. $b^2 < 4ac$. Then r_1, r_2 are both complex, and $r_2 = \bar{r}_1$.

Now consider case (3). Let $r_{1,2} = \lambda \pm i\mu$. So then we have two complex-valued solutions

$$y_1 = e^{r_1 t} = e^{(\lambda + i\mu)t}, \quad y_2 = e^{r_2 t} = e^{(\lambda - i\mu)t}.$$

What does it to raise a number to a complex exponent?

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Then we define

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\ &= \cos x + i \sin x \end{aligned}$$

Euler's formula

$$e^{ix} = \cos x + i \sin x.$$

Then we define

$$e^{\lambda + i\mu} = e^{\lambda} e^{i\mu} = e^{\lambda}(\cos x + i \sin x) = e^{\lambda} \cos x + i e^{\lambda} \sin x$$

One can verify the exponential function defined this way has the usual algebraic properties of exponential functions with real exponent.
The two complex solutions are

$$y_1 = e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos \mu t + i \sin \mu t), \quad y_2 = e^{(\lambda - i\mu)t} = e^{\lambda t}(\cos \mu t - i \sin \mu t).$$

One can verify y_1, y_2 form a fundamental set of solutions.

By the previous theorem, we know

$$y_3 = e^{\lambda t} \cos \mu t, \quad y_4 = e^{\lambda t} \sin \mu t.$$

are real-valued solutions. One can verify y_3, y_4 also form a fundamental set of solutions.

Example 3.1. Find the general solution (real-valued) of the differential equation

$$y'' + y' + 9.25\, y = 0,$$

Also find the solution that satisfies the initial conditions

$$y(0) = 2, \quad y'(0) = 8,$$

and draw its graph.

Answer: The roots of the characteristic polynomial is

$$r_{1,2} = \frac{-1 \pm 6i}{2} = -\frac{1}{2} \pm 3i.$$

The real-valued solutions are

$$y_1 = e^{-\frac{t}{2}} \cos 3t, \quad y_2 = e^{-\frac{t}{2}} \sin 3t.$$

The general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{-\frac{t}{2}} \cos 3t + c_2 e^{-\frac{t}{2}} \sin 3t,$$

and

$$y' = c_1 e^{-\frac{t}{2}} \left(-\frac{1}{2} \cos 3t - 3 \sin 3t \right) + c_2 e^{-\frac{t}{2}} \left(-\frac{1}{2} \sin 3t + 3 \cos 3t \right).$$

Plugging the initial conditions,

$$\begin{aligned} c_1 &= 2, \\ -\frac{1}{2} c_1 + 3 c_2 &= 8. \end{aligned}$$

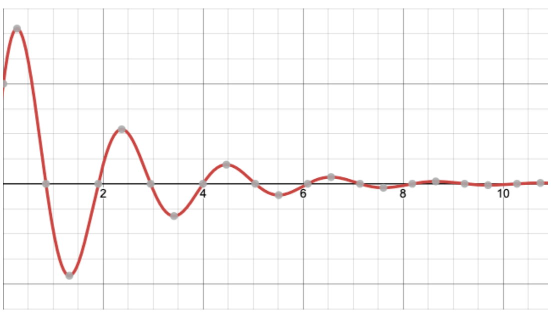
Solving the linear systm,

$$c_1 = 2, \quad c_2 = 3.$$

So the solution of the IVP is

$$y = 2e^{-\frac{t}{2}} \cos 3t + 3e^{-\frac{t}{2}} \sin 3t = e^{-\frac{t}{2}} (2 \cos 3t + 3 \sin 3t).$$

The graph is a damped oscillation.



Example 3.2. Find the general solution of

$$y''+9y=0.$$

Answer:

$$r_{1,2}=\pm 3i$$

The general solution is

$$y=c_1\cos 3t+c_2\sin 3t.$$

The graph is an undamped oscillation.

4 Repeated Roots; Reduction of order

4.1 Repeated roots

Suppose the charateristic equation have one repeated root $r=-\dfrac{b}{2a}$. Then we have a solution

$$y_1=e^{rt}.$$

Then $y_2=cy_1=ce^{rt}$ is also a solution for any constant c, but $\{y_1,y_2\}$ is not a fundamental set of solutions. To find another independent solution, let

$$y_2=v(t)\,y_1.$$

Plugging y_2 into the equation,

$$\begin{aligned}
a(vy_1)''+b(vy_1)'+c(vy_1) &= a(v''y_1+2v'y_1'+vy_1'')+b(v'y_1+vy_1')+cvy_1 \\
&= v(ay_1''+by_1'+cy_1)+av''y_1+2av'y_1'+bv'y_1 \\
&= ay_1v''+(2ay_1'+by_1)v' \\
&= ae^{rt}v''+(2are^{rt}+be^{rt})v' \\
&= e^{rt}(av''+(2ar+b)v')=0 \\
\Rightarrow \quad & av''+(2ar+b)v'=av''=0 \\
\Rightarrow \quad & v''=0 \quad \Rightarrow \quad v=c_1t+c_2.
\end{aligned}$$

Then

$$y_2=(c_1t+c_2)e^{rt}=c_1te^{rt}+c_2e^{rt}.$$

Choose

$$y_2=te^{rt}.$$

Then one can verify y_1,y_2 form a fundamental set of solutions (check $W(y_1,y_2)\neq 0$).

Example 4.1.

$$y''+4y'+4y=0.$$

Answer: The characteristic equation is $r^2+4r+4=0$. The (repeated) root is $r=-2$. So

$$y_1=e^{-2t},\quad y_2=te^{-2t}.$$

The general solution is

$$y=c_1e^{-2t}+c_2te^{-2t}.$$

We have $y(t) \to 0$ as $t \to \infty$.

4.2 Reduction of order

The idea to find y_2 can be generalized to a general second order linear equation. If y_1 is a solution of

$$y'' + p(t)\,y' + q(t)\,y = 0.$$

Let $y_2 = v(t)\,y_1$ be another solution. Then plugging y_2 into the equation we can obtain an second order linear ODE for $v(t)$:

$$y_1 v'' + (y_1' + p(t)\,y_1)\,v' = 0.$$

Let $w = v'$, then we obtain a first order ODE for w

$$y_1 w' + (y_1' + p(t)\,y_1)\,w = 0.$$

Solve w, then let $v = \displaystyle\int w$.

Example 4.2. Given the variable coefficient equation and solution y_1,

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0; \quad y_1(t) = t^{-1},$$

use reduction of order method to find a second solution.

Answer: Let $y_2 = v y_1$. Then

$$\begin{aligned} 2t^2 y_2'' + 3t y_2' - y_2 &= 2t^2(v'' y_1 + 2v' y_1' + v y_1'') + 3t(v' y_1 + v y_1') - v y_1 \\ &= 2t^2(t^{-1} v'' - 2t^{-2} v' + 2t^{-3} v) + 3t(t^{-1} v' - t^{-2} v) - t^{-1} v \\ &= 2t v'' - v' = 0. \end{aligned}$$

Let $w = v'$,

$$2tw' - w = 0 \quad \Rightarrow \quad \frac{dw}{w} = \frac{dt}{2t} \quad \Rightarrow \quad \ln w = \frac{1}{2}\ln t \quad \Rightarrow \quad w = c\sqrt{t} \quad \Rightarrow \quad v = c\frac{2}{3}t^{\frac{3}{2}}.$$

So

$$y_2 = c\frac{2}{3}t^{\frac{3}{2}}t^{-1} = c\frac{2}{3}\sqrt{t}.$$

Choose

$$y_2 = \sqrt{t}.$$

Exercise 4.1. Check y_2 satisfies the equation and $W(y_1, y_2) \neq 0$.

5 Method of Undetermined Coefficients

Consider the nonhomogeneous equation

$$L[y] = y'' + p(t)\,y' + q(t)\,y = g(t).$$

Let y_1, y_2 be solutions. Then

$$L[y_1 - y_2] = L[y_1] - L[y_2] = g(t) - g(t) = 0.$$

So $y_1 - y_2$ is a solution of the homogeneous equation $L[y] = 0$.

Theorem 5.1 *The general solution of the nonhomogeneous equation $L[y]=g$ is*

$$y=c_1y_1+c_2y_2+Y,$$

where c_1,c_2 are arbitrary constant, y_1,y_2 form a fundamental set of solutions for the homogeneous equation $L[y]=0$, and Y is a particular solution of the nonhomogeneous equation $L[y]=g$.

Proof. Let y be any solution of $L[y]=g$. Then $y-Y$ is a solution of $L[y]=0$. Then

$$y-Y=c_1y_1+c_2y_2 \quad\Rightarrow\quad y=c_1y_1+c_2y_2+Y.$$

for some constants c_1,c_2. $\square$

How to find a particular solution?

Example 5.2. Find a particular solution of

$$y''-3y'-4y=3e^{2t}.$$

Answer: Suppose the solution is of the form (ansatz) $y=Ae^{2t}$, where A is an undetermined coefficient. To find A, just plug the ansatz into the equation.

$$\begin{aligned}4Ae^{2t}-6Ae^{2t}-4Ae^{2t} &= -6Ae^{2t}=3e^{2t}\\ &\Rightarrow \quad A=-\frac{1}{2}.\end{aligned}$$

So

$$Y=-\frac{1}{2}e^{2t}$$

is a particular solution.

Example 5.3. Find a particular solution of

$$y''-3y'-4y=2\sin t.$$

Answer: Suppose the solution is of the form

$$y=A\sin t+B\cos t.$$

Then

$$(-A\sin t-B\cos t)-3(A\cos t-B\sin t)-4(A\sin t+B\cos t)=2\sin t.$$

Comparing the coefficients on the LHS and RHS,

$$\begin{cases}-5A+3B=2\\ -5B-3A=0\end{cases} \quad\Rightarrow\quad \begin{cases}-25A+15B=10\\ -15B-9A=0\end{cases} \quad\Rightarrow\quad \begin{cases}A=-\dfrac{5}{17}\\ B=\dfrac{3}{17}\end{cases}.$$

So

$$Y=-\frac{5}{17}\sin t+\frac{3}{17}\cos t.$$

Remark 5.4. The method also works if the RHS is a cosine function.

Example 5.5. Find a particular solution of

$$y'' - 3y' - 4y = -8e^t \cos 2t.$$

Answer: Suppose the solution is of the form

$$y = e^t (A \sin 2t + B \cos 2t).$$

Then

$$\begin{aligned}
y' &= e^t (A \sin 2t + B \cos 2t) + e^t (2A \cos 2t - 2B \sin 2t) \\
&= e^t [(A - 2B) \sin 2t + (2A + B) \cos 2t] \\
y'' &= e^t [(A - 2B) \sin 2t + (2A + B) \cos 2t] + e^t [2(A - 2B) \cos 2t - 2(2A + B) \sin 2t] \\
&= e^t [(-3A - 4B) \sin 2t + (4A - 3B) \cos 2t].
\end{aligned}$$

$$y'' - 3y' - 4y = e^t [(-3A - 4B - 3A + 6B - 4A) \sin 2t + (4A - 3B - 6A - 3B - 4B) \cos 2t] = -8e^t \cos 2t$$

$$\Rightarrow \begin{cases} -10A + 2B = 0 \\ -2A - 10B = -8 \end{cases} \Rightarrow \begin{cases} -5A + B = 0 \\ -A - 5B = -4 \end{cases} \Rightarrow \begin{cases} A = \dfrac{2}{13}, \\ B = \dfrac{10}{13}. \end{cases}$$

So

$$Y = e^t \left(\frac{2}{13} \sin 2t + \frac{10}{13} \cos 2t \right) = \frac{2}{13} e^t (\sin 2t + 5 \cos 2t).$$

is a particular solution.

Example 5.6. Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2 \sin t - 8e^t \cos 2t.$$

Answer: A particular solution is

$$Y = -\frac{1}{2} e^{2t} - \frac{5}{17} \sin t + \frac{3}{17} \cos t + \frac{2}{13} e^t (\sin 2t + 5 \cos 2t).$$

Example 5.7. Find a particular solution of

$$y'' - 3y' - 4y = 2e^{-t}.$$

Answer: Try the ansatz $y = Ae^{-t}$. Then

$$y'' - 3y' - 4y = (A + 3A - 4A) e^{-t} = 0.$$

Not work! Try another one

$$y = Ate^{-t}.$$

Then

$$y' = A(1 - t) e^{-t}, \quad y'' = A(-2 + t) e^{-t}$$

$$y'' - 3y' - 4y = Ae^{-t}(-2 + t - 3(1 - t) - 4t) = -5Ae^{-t} = 2e^{-t} \quad \Rightarrow \quad A = -\frac{2}{5}.$$

So

$$y = -\frac{2}{5} te^{-t}$$

is a particular solution.

Question 3. Why Ate^{-t} works?

Answer. Consider the general case:

$$ay'' + by' + cy = de^{\alpha t}.$$

Suppose α is a root (not repeated) of the characteristic equation $ar^2 + br + c = 0$. Let $y = v(t)e^{\alpha t}$.

Then

$$\begin{aligned} y' &= (v' + \alpha v)e^{\alpha t}, \\ y'' &= (v'' + 2\alpha v' + \alpha^2 v)e^{\alpha t}. \end{aligned}$$

Plugging into the equation

$$\begin{aligned} ay'' + by' + cy &= [a(v'' + 2\alpha v' + \alpha^2 v) + b(v' + \alpha v) + cv]e^{\alpha t} \\ &= [av'' + (2a\alpha + b)v' + (a\alpha^2 + b\alpha + c)v]e^{\alpha t} \\ &= [av'' + (2a\alpha + b)v']e^{\alpha t} = de^{\alpha t} \\ \Rightarrow \quad av'' + (2a\alpha + b)v' &= d. \end{aligned}$$

Let $w = v'$, then

$$aw' + (2a\alpha + b)w = d \quad \Rightarrow \quad w = \frac{d}{2a\alpha + b} := A \quad \Rightarrow \quad v = At + B.$$

So

$$y = (At + B)e^{\alpha t} = Ate^{\alpha t}$$

by choosing $B = 0$.

Exercise 5.1. Derive the solution ansatz $y = At^2 e^{\alpha t}$ if α is a repeated root of the characteristic polynomial.

Example 5.8. Find a particular solution of

$$y'' - 4y' + 4y = e^{2t}.$$

Answer: Try the ansatz $y = Ae^{2t}$, not work. Try $y = Ate^{2t}$, not work. Try

$$y = At^2 e^{2t}.$$

$$y' = 2A(t + t^2)e^{2t}, \quad y'' = 2A(1 + 4t + 2t^2)$$

$$y'' - 4y' + 4y = Ae^{2t}[2(1 + 4t + 2t^2) - 8(t + t^2) + 4t^2] = 2Ae^{2t} = e^{2t}.$$

So $A = 1/2$ and

$$y = \frac{1}{2}t^2 e^{2t}$$

is a particular solution.

Example 5.9. Find a particular solution of

$$y'' - 4y' + 3y = t^2 + t + 1.$$

Answer: Consider the ansatz

$$y = At^2 + Bt + C.$$

Then

$$y' = 2At + B, \quad y'' = 2A.$$

$$\begin{aligned} y'' - 4y' + 3y &= 2A - 4(2At + B) + 3(At^2 + Bt + C) \\ &= 3At^2 + (3B - 8A)t + (2A - 4B + 3C) \end{aligned}$$

$$\Rightarrow \begin{cases} 3A &= 1 \\ 3B - 8A &= 1 \\ 2A - 4B + 3C &= 1 \end{cases} \Rightarrow \begin{cases} A = \dfrac{1}{3} \\ B = \dfrac{11}{9} \\ C = \dfrac{1}{3}\left(1 - \dfrac{2}{3} + \dfrac{44}{9}\right) = \dfrac{47}{27} \end{cases}$$

TABLE 3.5.1 The Particular Solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$	$t^s(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin \beta t \\ \cos \beta t\end{cases}$	$t^s[(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)e^{\alpha t} \cos \beta t$ $+ (B_0 t^n + B_1 t^{n-1} + \cdots + B_n)e^{\alpha t} \sin \beta t]$

Notes. Here s is the smallest nonnegative integer ($s = 0, 1,$ or 2) that will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation, respectively.

6 Variation of Parameters

The method of undetermined coefficients has some limitations.

- only equations with constant coefficients
- $g(t)$ must be in one of those special forms

Now consider the more general nonhomogeneous equation

$$y'' + p(t)\,y' + q(t)\,y = g(t).$$

Suppose $y = c_1 y_1 + c_2 y_2$ is a general solution of the homogeneous equation

$$y'' + p(t)\,y' + q(t)\,y = 0.$$

Let

$$Y = u_1 y_1 + u_2 y_2,$$

where u_1, u_2 are functions to be determined. Then

$$Y' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'.$$

Let's pose the condition

$$u_1' y_1 + u_2' y_2 = 0. \tag{6.1}$$

Then

$$Y' = u_1 y_1' + u_2 y_2' \quad \text{and} \quad Y'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''.$$

So

$$\begin{aligned} Y'' + p(t)Y' + q(t)Y &= u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' + p(t)(u_1 y_1' + u_2 y_2') + q(t)(u_1 y_1 + u_2 y_2) \\ &= u_1[y_1'' + p(t) y_1' + q(t) y_1] + u_2[y_2'' + p(t) y_2' + q(t) y_2] + u_1' y_1' + u_2' y_2' \\ &= u_1' y_1' + u_2' y_2'. \end{aligned}$$

So

$$u_1' y_1' + u_2' y_2' = g(t). \tag{6.2}$$

So from (6.1) and (6.2) we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

Note this system has a unique solution because $W(y_1, y_2) \neq 0$. The solution is (given by Cramer's rule):

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2}{W(y_1, y_2)} g, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1}{W(y_1, y_2)} g.$$

Integrating in t, we obtain

$$u_1 = \int \frac{-y_2}{W(y_1, y_2)} g\,dt, \quad u_2 = \int \frac{y_1}{W(y_1, y_2)} g\,dt.$$

This is called the method of **variation of parameters**.

Example 6.1. Find the general solution of

$$y'' + 4y = 3\csc t.$$

Answer: We have $y_1 = \sin 2t$, $y_2 = \cos 2t$,

$$W(y_1, y_2) = \begin{vmatrix} \sin 2t & \cos 2t \\ 2\cos 2t & -2\sin 2t \end{vmatrix} = -4.$$

So

$$u_1 = \int \frac{-y_2}{W(y_1, y_2)} g\,dt = \int -\frac{\cos 2t}{-4} 3\csc t = \frac{3}{4} \int \frac{\cos 2t}{\sin t} dt = \frac{3}{4} \int \frac{1 + 2\cos^2 t}{\sin t} dt$$

$$= \frac{3}{4}\left[\int \csc t\,dt + 2\int \frac{1 - \sin^2 t}{\sin t} dt\right] = \frac{3}{4}\left[3\int \csc t\,dt + 2\int \sin t\,dt\right] = \frac{3}{4}[3\ln|\csc t - \cot t| - 2\cos t]$$

Similarly we can find u_2 (exercise). So the general solution is

$$y = c_1 \sin 2t + c_2 \cos 2t + u_1 \sin 2t + u_2 \cos 2t$$

$$\boxed{\begin{aligned}&\int \csc t\,dt = \int \frac{1}{\sin t}dt = \int \frac{\sin t}{\sin^2 t}dt = \int \frac{\sin t}{1-\cos^2 t}dt = \int \frac{1}{2}\left[\frac{\sin t}{1+\cos t} + \frac{\sin t}{1-\cos t}\right]dt \\ &= \frac{1}{2}\left[\int \frac{-1}{1+\cos t}d(1+\cos t) + \int \frac{1}{1-\cos t}d(1-\cos t)\right] = \frac{1}{2}[-\ln(1+\cos t) + \ln(1-\cos t)] \\ &= \frac{1}{2}\ln\frac{1-\cos t}{1+\cos t} = \frac{1}{2}\ln\frac{(1-\cos t)^2}{1-\cos^2 t} = \frac{1}{2}\ln\frac{(1-\cos t)^2}{\sin^2 t} = \ln\sqrt{\frac{(1-\cos t)^2}{\sin^2 t}} = \ln\left|\frac{1-\cos t}{\sin t}\right| \\ &= \ln|\csc t - \cot t|\end{aligned}}$$

7 Free Vibrations

Consider the equation for the spring-mass system

$$mu'' + \gamma u' + ku = 0.$$

7.1 Undamped free vibrations

Let $\gamma = 0$, i.e. there is no damping force. Then the equation reduces to

$$mu'' + ku = 0.$$

The roots of the characteristic equation are

$$r_{1,2} = \pm \omega_0 i, \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

So the general solution is

$$u = A\cos\omega_0 t + B\sin\omega_0 t = R\cos(\omega_0 t - \delta) = R(\cos\omega_0 t\cos\delta + \sin\omega_0 t\sin\delta).$$

So

$$A = R\cos\delta, \quad B = R\sin\delta.$$

So

$$R = \sqrt{A^2 + B^2}, \quad \cos\delta = \frac{A}{R}, \quad \sin\delta = \frac{B}{R} \quad \Rightarrow \quad \delta =$$

Here R is the **amplitude**, ω_0 is the **angular frequency** (natural frequency of the system), δ is the **phase**, and $T = \dfrac{2\pi}{w_0}$ is the **period**.

7.2 Damped free vibrations

Now consider the case when $\gamma > 0$ (damped vibration). Then the roots of the characteristic equation is

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = -\frac{\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m}.$$

1. If $\gamma^2 > 4mk$ (**overdamped**), then $r_1 \neq r_2$ are real and both negative. The general solution is

$$u = Ae^{r_1 t} + Be^{r_2 t}.$$

The solution is nonoscillatory and $u(t) \to 0$ as $t \to \infty$.

2. If $\gamma^2 = 4mk$ (**critically damped**), then we have repeated root $r = -\frac{\gamma}{2m}$. So the general solution is
$$u = Ae^{rt} + Bte^{rt}.$$
The solution is nonoscillatory and $u(t) \to 0$ as $t \to \infty$.

3. If $\gamma^2 < 4mk$, then the roots are
$$r_{1,2} = \lambda \pm i\mu, \quad \lambda = -\frac{\gamma}{2m}, \quad \mu = \frac{\sqrt{4mk - \gamma^2}}{2m}.$$
The general solution is
$$u = e^{\lambda t}(A\cos \mu t + B\sin \mu t) = Re^{\lambda t}\cos(\mu t - \delta).$$
It's a **damped oscillation**, and $u(t) \to 0$ as $t \to \infty$.

$u(t)$ is nonperiodic, but we call $T = \frac{2\pi}{\mu}$ the **quasi period**. In fact
$$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} < \omega_0.$$

7.3 Electric circuits (skip)

8 Forced Vibrations (optional)

8.1 Forced vibrations with damping

$$mu'' + \gamma u' + ku = F$$

We consider periodic forces $F = F_0 \cos \omega t$. The general solution is
$$u = [c_1 u_1(t) + c_2 u_2(t)] + [A\cos \omega t + B\sin \omega t] = u_c(t) + U(t).$$
Note that $u_c(t) \to 0$ as $t \to \infty$, but $U(t)$ is periodic. So we call $u_c(t)$ the **transient solution** and $U(t)$ the **steady-state solution**.

Example 8.1. Consider the IVP
$$u'' + u' + \frac{5}{4}u = 3\cos t, \quad u(0) = 2, \quad u'(0) = 3.$$
Then
$$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = -\frac{1}{2} \pm i,$$
So
$$u_c(t) = e^{-\frac{t}{2}}(c_1 \cos t + c_2 \sin t).$$
Let $U = A\cos t + B\sin t$. Then
$$U'' + U' + \frac{5}{4}U = -A\cos t - B\sin t - A\sin t + B\cos t + \frac{5}{4}(A\cos t + B\sin t)$$
$$= \left(-A + B + \frac{5}{4}A\right)\cos t + \left(-B - A + \frac{5}{4}B\right)\sin t = \left(\frac{1}{4}A + B\right)\cos t + \left(\frac{1}{4}B - A\right)\sin t$$

So

$$\frac{1}{4}A+B=3, \quad \frac{1}{4}B-A=0 \quad \Rightarrow \quad \begin{cases} A=\dfrac{12}{17}, \\ B=\dfrac{48}{17}. \end{cases}$$

So the steady-state solution is

$$U(t)=\frac{12}{17}\cos t+\frac{48}{17}\sin t.$$

So the general solution is

$$u(t)=e^{-\frac{t}{2}}(c_1\cos t+c_2\sin t)+\frac{12}{17}\cos t+\frac{48}{17}\sin t.$$

Plugging initial conditions, we obtain $c_1=\frac{22}{17}, c_2=\frac{14}{17}$. So the solution of the IVP is

$$u(t)=\frac{2}{17}\left[e^{-\frac{t}{2}}(11\cos t+7\sin t)+6\cos t+24\sin t\right].$$

Resonance. Steady-state solution $U=A\cos\omega t+B\sin\omega t$

$$U'=\omega(-A\sin\omega t+B\cos\omega t), \quad U''=\omega^2(-A\cos\omega t-B\sin\omega t)$$

$$\begin{aligned}
\Rightarrow\ & mU''+\gamma U'+kU \\
=\ & m\omega^2(-A\cos\omega t-B\sin\omega t)+\gamma\omega(-A\sin\omega t+B\cos\omega t)+k(A\cos\omega t+B\sin\omega t) \\
=\ & (-m\omega^2A+\gamma\omega B+kA)\cos\omega t+(-Bm\omega^2-A\gamma\omega+kB)\sin\omega t \\
=\ & [(k-m\omega^2)A+\gamma\omega B]\cos\omega t+[-\gamma\omega A+(k-m\omega^2)B]\sin\omega t \\
=\ & F_0\cos\omega t
\end{aligned}$$

$$\begin{cases} (k-m\omega^2)A+\gamma\omega B &= F_0 \\ -\gamma\omega A+(k-m\omega^2)B &= 0 \end{cases} \Rightarrow \begin{pmatrix} k-m\omega^2 & \gamma\omega \\ -\gamma\omega & k-m\omega^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=\begin{pmatrix} F_0 \\ 0 \end{pmatrix}$$

$$\begin{aligned}
A &= \frac{k-m\omega^2}{(k-m\omega^2)^2+\gamma^2\omega^2}F_0=\frac{m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}F_0 \\
B &= \frac{\gamma\omega}{(k-m\omega^2)^2+\gamma^2\omega^2}F_0=\frac{\gamma\omega}{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}F_0
\end{aligned}$$

$$A\cos\omega t+B\sin\omega t=R\cos(\omega t-\delta) \quad \Rightarrow \quad R=\frac{F_0}{\Delta}, \quad \Delta=\sqrt{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

Nondimensionalize (无量纲化)

$$R=\frac{F_0}{m\omega_0^2\sqrt{\left(1-\dfrac{\omega^2}{\omega_0^2}\right)^2+\dfrac{\gamma^2\omega^2}{m^2\omega_0^4}}}=\frac{F_0}{k\sqrt{\left(1-\dfrac{\omega^2}{\omega_0^2}\right)^2+\dfrac{\gamma^2}{mk}\dfrac{\omega^2}{\omega_0^2}}}$$

$$\Rightarrow \frac{R}{(F_0/k)}=\frac{1}{\sqrt{\left(1-\dfrac{\omega^2}{\omega_0^2}\right)^2+\Gamma\dfrac{\omega^2}{\omega_0^2}}}, \quad \boxed{\Gamma=\frac{\gamma^2}{mk}}$$

$$\gamma u'=F\Rightarrow\gamma=\frac{F}{u'}: \frac{\mathrm{N}}{\mathrm{m}\cdot\mathrm{s}^{-1}}=\frac{\mathrm{N}\cdot\mathrm{s}}{\mathrm{m}}\Rightarrow\Gamma=\frac{\gamma^2}{mk}: \frac{\mathrm{N}^2\cdot\mathrm{s}^2}{\mathrm{m}^2\cdot\mathrm{kg}\cdot\mathrm{N}\cdot\mathrm{m}^{-1}}=\frac{\mathrm{N}\cdot\mathrm{s}^2}{\mathrm{m}\cdot\mathrm{kg}}=\frac{\mathrm{N}}{\mathrm{m}\cdot\mathrm{s}^{-2}\cdot\mathrm{kg}}=1$$

Clearly $\dfrac{R}{(F_0/k)}$ and $\dfrac{\omega^2}{\omega_0^2}$ are also dimensionless. Rewrite the equation as

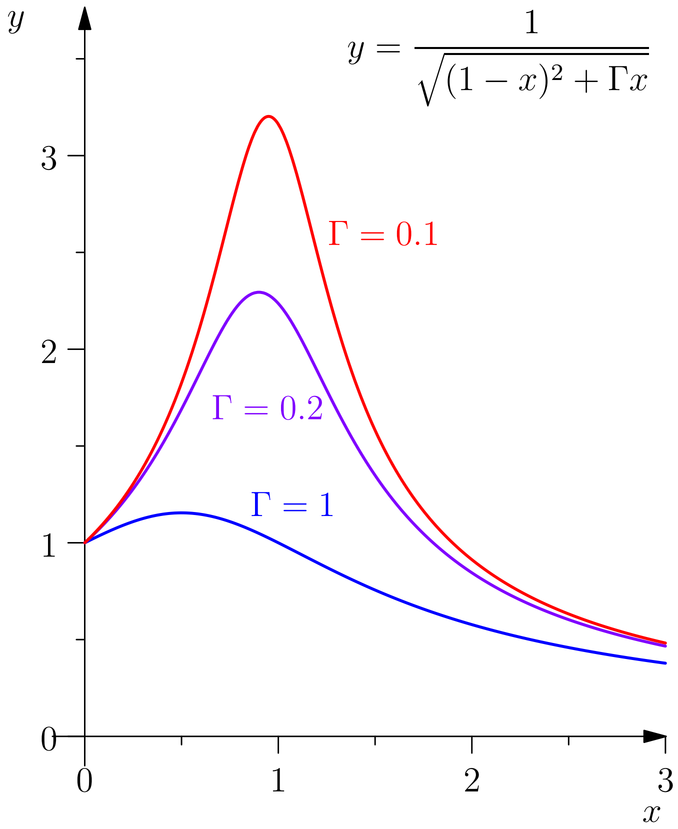
$$y=\frac{1}{\sqrt{(1-x)^2+\Gamma x}}, \quad y=\frac{R}{(F_0/k)}, \quad x=\frac{\omega^2}{\omega_0^2}$$

$$y'=-\frac{1}{2}[(1-x)^2+\Gamma x]^{-\frac{3}{2}}[\Gamma-2+2x]$$

If $0<\Gamma<2$, then $y'>0$ for $x\in\left[0,1-\dfrac{\Gamma}{2}\right)$, $y'<0$ for $x\in\left(1-\dfrac{\Gamma}{2},\infty\right)$ and $y'=0$ for $x=1-\dfrac{\Gamma}{2}$.
So $y_{\max}$ is obtained at $x_{\max}=1-\dfrac{\Gamma}{2}$:

$$y_{\max}=\frac{1}{\sqrt{(1-x)^2+\Gamma x}}=\frac{1}{\sqrt{\Gamma-\Gamma^2/4}}\to\infty \quad \text{as} \quad \Gamma\to 0.$$

Hence for lightly damped system (Γ is small), the amplitude of the steady-state solution when ω is near ω_0 can be very large for small external force. This phenomenon is known as **resonance**.



8.2 Forced vibrations without damping

Consider

$$mu''+ku=F_0\cos\omega t.$$

The general solution of the homogeneous part is

$$u_c(t)=c_1\cos\omega_0 t+c_2\sin\omega_0 t, \quad \omega_0=\sqrt{\frac{k}{m}}.$$

8.2.1 $\omega \neq \omega_0$

The general solution is

$$u = u_c(t) + U(t), \quad U(t) = A\cos\omega t + B\sin\omega t.$$

$$U' = \omega(-A\sin\omega t + B\cos\omega t), \quad U'' = \omega^2(-A\cos\omega t - B\sin\omega t)$$

$$\begin{aligned} mU'' + kU &= m\omega^2(-A\cos\omega t - B\sin\omega t) + k(A\cos\omega t + B\sin\omega t) \\ &= (-Am\omega^2 + kA)\cos\omega t + (-Bm\omega^2 + kB)\sin\omega t \\ &= A(k - m\omega^2)\cos\omega t + B(k - m\omega^2)\sin\omega t \\ &= F_0\cos\omega t \end{aligned}$$

$$\Rightarrow \quad A = \frac{F_0}{k - m\omega^2}, \quad B = 0$$

The general solution is

$$u = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{F_0}{k - m\omega^2}\cos\omega t.$$

Suppose the initial condition is $u(0) = u'(0) = 0$, then

$$c_1 + \frac{F_0}{k - m\omega^2} = 0, \quad c_2\omega_0 = 0 \quad \Rightarrow \quad c_1 = -\frac{F_0}{k - m\omega^2}, \quad c_2 = 0.$$

$$u = \frac{F_0}{k - m\omega^2}(\cos\omega t - \cos\omega_0 t) = \frac{-2F_0}{k - m\omega^2}\sin\left(\frac{\omega + \omega_0}{2}t\right)\sin\left(\frac{\omega - \omega_0}{2}t\right)$$

If ω is close to ω_0, then we have a **beat**. Also used in **amplitude modulation**.

Asymptote

```
% -width 0.6par
import graph;
size(10cm, 0);
real f(real x) {return cos(10*x)-cos(11*x);}
draw(graph(f, -10, 10, Hermite), black+linewidth(1pt));
label("$y=\cos(10 x)-\cos(11 x)$", (0, 3));
```

8.2.2 $\omega = \omega_0$

A general solution is

$$U = c_1\cos\omega t + c_2\sin\omega t + t(A\cos\omega t + B\sin\omega t).$$

9 Higher Order Linear Equations

9.1 General theory

An n-th order linear ODE is in the form

$$L[y] = y^{(n)} + p_{n-1}(t)\,y^{(n-1)} + \cdots + p_1(t)\,y' + p_0(t)\,y = g(t)$$

An initial value problem is the equation $L[y] = g$ together with initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots \quad y^{(n-1)}(t_0) = y_0^{(n-1)}.$$

Definition 9.1 *The **Wronskian** of n solutions $y_1, \ldots, y_n$ of $L[y] = 0$ is*

$$W[y_1, \ldots, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

Theorem 9.2 *Let $y_1, \dots, y_n$ be solutions of $L[y]=0$. Then $y_1, \dots, y_n$ form a fundamental set of solutions if and only if they are linearly independent.*

Proof. Suppose $y_1, \dots, y_n$ form a fundamental set of solutions, that is, $W[y_1, \dots, y_n] \neq 0$. Let $c_1, \dots, c_n$ be constants such that

$$c_1 y_1 + \cdots + c_n y_n = 0.$$

Differentiate the above equation in t,

$$c_1 y_1' + \cdots + c_n y_n' = 0.$$

Repeat differentiating, we obtain

$$\begin{array}{rcl} c_1 y_1'' + \cdots + c_n y_n'' & = & 0 \\ & \vdots & \\ c_1 y_1^{(n-1)} + \cdots + c_n y_n^{(n-1)} & = & 0 \end{array}$$

Hence we have a linear system

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Hence $y_1, \dots, y_n$ are linearly independent.
Now assume $y_1, \dots, y_n$ do not form a fundamental set of solutions, i.e. $W[y_1, \dots, y_n](t_0) = 0$ for some t_0. Then there exists constants $c_1, \dots, c_n$, not all zero, such that

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let

$$Y = c_1 y_1 + \cdots + c_n y_n.$$

Then consider the IVP

$$L[y] = 0, \quad y(t_0) = Y(t_0) = 0, \quad y'(t_0) = Y'(t_0) = 0, \quad \cdots \quad y^{(n-1)}(t_0) = Y^{(n-1)}(t_0) = 0$$

Clearly Y is the solution of the IVP. On the other hand, 0 is also a solution of the IVP. By the E&U theorem, we have $Y = 0$. Thus $y_1, \dots, y_n$ are linearly independent. $\square$

9.2 Homogeneous constant coefficients

Example 9.3.

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$

Answer: The characteristic equation is

$$r^4 + r^3 - 7r^2 - r + 6 = 0.$$

To find the roots, we factorize it:

$$\begin{aligned}(r^4 + r^3) - (7r^2 + r - 6) &= r^3(r+1) - (r+1)(7r-6) \\ &= (r+1)(r^3 - 7r + 6) = (r+1)(r^3 - r - 6r + 6) \\ &= (r+1)[r(r^2-1) - 6(r-1)] \\ &= (r+1)(r-1)(r^2 + r - 6) \\ &= (r+1)(r-1)(r-2)(r+3)\end{aligned}$$

So the roots are$f$

$$r = -3, -1, 1, 2$$

So there are four solutions

$$y = e^{-3t}, e^{-t}, e^{t}, e^{2t}$$

and they form a fundamental set of solutions. So the general solution is

$$y = c_1 e^{-3t} + c_2 e^{-t} + c_3 e^{t} + c_4 e^{2t}.$$

Note 9.4. Another method to find the roots. All possible rational roots are

$$\frac{\pm 1, \pm 2, \pm 3, \pm 6}{1}.$$

Then verify directly it they are roots. Once one or more roots are found, we can use **polynomial division** to reduce the order and help find other roots.

Example 9.5.

$$y^{(4)} - y = 0$$

Answer:

$$r^4 - 1 = (r^2+1)(r^2-1) \quad \Rightarrow \quad r = \pm i, \pm 1 \quad \Rightarrow \quad y = \cos t, \sin t, e^{-t}, e^{t}$$

Example 9.6.

$$y^{(4)} + 2y'' + y = 0$$

Answer:

$$r^4 + 2r^2 + 1 = (r^2+1)^2 = 0 \quad \Rightarrow \quad r = \pm i, \pm i \quad \Rightarrow \quad y = \cos t, \sin t, t\cos t, t\sin t$$

(We say the root $r = \pm i$ has multiplicity 2).

Example 9.7.

$$y^{(4)} + y = 0.$$

Answer:

$$r^4+1=0 \quad \Rightarrow \quad r^4=-1=e^{i(\pi+2n\pi)}$$
$$\Rightarrow \quad r=\exp\left(i\frac{(2n+1)}{4}\pi\right)=e^{\frac{\pi}{4}i},e^{\frac{3\pi}{4}i},e^{\frac{5\pi}{4}i},e^{\frac{7\pi}{4}i}$$
$$\Rightarrow \quad r=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i, \quad -\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i, \quad -\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i, \quad \frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$$
$$\Rightarrow \quad r=\frac{\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}i, \quad -\frac{\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}i$$
$$\Rightarrow \quad y=e^{\frac{\sqrt{2}}{2}t}\cos\frac{\sqrt{2}}{2}t, \quad e^{\frac{\sqrt{2}}{2}t}\sin\frac{\sqrt{2}}{2}t, \quad e^{-\frac{\sqrt{2}}{2}t}\cos\frac{\sqrt{2}}{2}t, \quad e^{-\frac{\sqrt{2}}{2}t}\sin\frac{\sqrt{2}}{2}t$$

9.3 The method of undetermined coefficients

Example 9.8. Find the general solution of

$$y'''-3y''+3y'-y=4e^t$$

Answer:

$$r^3-3r^2+3r-1=(r-1)^3=0 \quad \Rightarrow \quad r=1,1,1 \quad \Rightarrow \quad y_1=e^t,te^t,t^2e^t.$$

Let

$$Y=At^3e^t.$$

Then

$$Y'=A(3t^2+t^3)e^t, \quad Y''=A(6t+6t^2+t^3)e^t, \quad Y'''=A(6+18t+9t^2+t^3)e^t.$$
$$\Rightarrow \quad [(6+18t+9t^2+t^3)-3(6t+6t^2+t^3)+3(3t^2+t^3)-t^3]Ae^t=4e^t$$
$$\Rightarrow \quad 6A=4 \quad \Rightarrow \quad A=\frac{2}{3}.$$

So the general solution is

$$y=c_1e^t+c_2te^t+c_3t^2e^t+\frac{2}{3}t^3e^t.$$

9.4 The method of variation of parameters

Suppose $y_1,\ldots,y_n$ form a fundamental set of solutions for $L[y]=0$. Consider the nonhomogeneous equation

$$L[y]=g(t).$$

Let

$$Y=u_1y_1+\cdots+u_ny_n$$

Then differentiate Y and make some assumption as in the case of 2nd order equations. We obtain

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n)} & \cdots & y_n^{(n-1)} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ g \end{pmatrix}$$

Then

$$u_m' = \frac{W_m}{W} g \quad \Rightarrow \quad u_m = \int \frac{W_m}{W} g$$

where W is the Wronskian, and W_m is the determinant of the above matrix with the m-th column replaced by the vector $(0, \ldots, 0, 1)^T$.

Example 9.9. Find the general solution to

$$y''' - y'' - y' + y = g(t).$$

Answer:

$$r^3 - r^2 - r + 1 = r^2(r-1) - (r-1) = (r-1)^2(r+1) \quad \Rightarrow \quad r = 1, 1, -1$$

$$y_1 = e^{-t}, \quad y_2 = e^t, \quad y_3 = te^t.$$

$$\begin{aligned}
W &= \begin{vmatrix} e^{-t} & e^t & te^t \\ -e^{-t} & e^t & (t+1)e^t \\ e^{-t} & e^t & (t+2)e^t \end{vmatrix} = e^t \begin{vmatrix} 1 & 1 & t \\ -1 & 1 & t+1 \\ 1 & 1 & t+2 \end{vmatrix} \\
&= e^t \begin{vmatrix} 1 & 0 & t \\ -1 & 2 & t+1 \\ 1 & 0 & t+2 \end{vmatrix} = 2e^t \begin{vmatrix} 1 & t \\ 1 & t+2 \end{vmatrix} = 4e^t, \\
W_1 &= \begin{vmatrix} 0 & e^t & te^t \\ 0 & e^t & (t+1)e^t \\ 1 & e^t & (t+2)e^t \end{vmatrix} = e^{2t} \begin{vmatrix} 0 & 1 & t \\ 0 & 1 & (t+1) \\ 1 & 1 & (t+2) \end{vmatrix} = e^{2t}, \\
W_2 &= \begin{vmatrix} e^{-t} & 0 & te^t \\ -e^{-t} & 0 & (t+1)e^t \\ e^{-t} & 1 & (t+2)e^t \end{vmatrix} = \begin{vmatrix} 1 & 0 & t \\ -1 & 0 & (t+1) \\ 1 & 1 & (t+2) \end{vmatrix} = -(2t+1), \\
W_3 &= \begin{vmatrix} e^{-t} & e^t & 0 \\ -e^{-t} & e^t & 0 \\ e^{-t} & e^t & 1 \end{vmatrix} = 2. \\
u_1 &= \int \frac{W_1}{W} g = \int \frac{1}{4} e^t g(t)\, dt, \\
u_2 &= \int \frac{W_2}{W} g = \int -\frac{2t+1}{4e^t} g(t)\, dt, \\
u_3 &= \int \frac{W_3}{W} g = \int \frac{1}{2e^t} g(t)\, dt.
\end{aligned}$$

So the general solution is

$$y = (c_1 e^{-t} + c_2 e^t + c_3 te^t) + e^{-t} \int \frac{1}{4} e^t g(t)\, dt - e^t \int \frac{2t+1}{4e^t} g(t)\, dt + te^t \int \frac{1}{2e^t} g(t)\, dt.$$

Ordinary Differential Equations

BY YULIANG WANG

Chapter 3: Second Order Linear Equations

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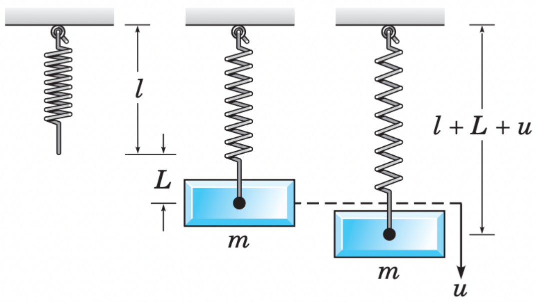
Motivation: spring-mass system



Newton's Law: $ma = f$

$a = u''$

$f = mg - k(L+u) - \gamma u' + F$

k: spring constant

γ: damping coefficient

$$mu'' = mg - k(L+u) - \gamma u' + F$$

$$mu'' + \gamma u' + ku = mg - kL + F$$

But $mg = kL$, so

$$mu'' + \gamma u' + ku = F$$

1 Homogeneous Equations with Constant Coefficients

$$ay'' + by' + cy = 0,$$

initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

Example 1.1. Solve the IVP

$$y'' - y = 0, \quad y(0) = 2, \quad y'(0) = -1.$$

By investigation, we know $y = ce^t$ satisfies the equation for any constant c. However, it doesn't satisfy the initial conditions. More investigation shows $y = ce^{-t}$ is also a solution for any constant c. It turns out

$$y = c_1 e^t + c_2 e^{-t}$$

is a solution for any constants c_1, c_2. Now, the initial conditions require

$$c_1 + c_2 = 2, \quad c_1 - c_2 = -1.$$

This is a system of linear equations. The matrix form is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

The matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is nonsingular ($\det A = -2 \neq 0$). So the system has a unique solution. The solution is

$$c_1 = \frac{1}{2}, \quad c_2 = -\frac{3}{2}.$$

So the IVP has a solution

$$y = \frac{1}{2} e^t - \frac{3}{2} e^{-t}.$$

Example 1.2. Solve

$$y'' + 5y' + 6y = 0.$$

Answer: We assume the ansatz of the solution: $y = e^{rt}$ for some constant r. Then

$$\begin{aligned} y'' + 5y' + 6y &= r^2 e^{rt} + 5r e^{rt} + 6 e^{rt} \\ &= (r^2 + 5r + 6) e^{rt} \\ &= 0 \\ \Rightarrow \quad r^2 + 5r + 6 &= 0 \end{aligned}$$

The equation

$$r^2 + 5r + 6 = 0$$

is called the **characteristic equation** for the ODE. The solutions are

$$r_1 = -3, \quad r_2 = -2.$$

So have two solutions of the ODE:

$$y_1 = e^{-3t}, \quad y_2 = e^{-2t}.$$

So we can let the solution be

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{-3t} + c_2 e^{-2t}.$$

Note that

$$y \to 0 \quad \text{as} \quad t \to \infty$$

....

Example 1.3. Solve the IVP

$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}.$$

Answer: The characteristic equation is

$$4r^2 - 8r + 3 = (2r - 3)(2r - 1) = 0 \quad \Rightarrow \quad r_1 = \frac{3}{2}, \quad r_2 = \frac{1}{2}.$$

So we have solutions

$$y = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t}.$$

Plugging initial conditions

$$\begin{aligned} c_1 + c_2 &= 2 \\ \frac{3}{2} c_1 + \frac{1}{2} c_2 &= \frac{1}{2} \end{aligned}$$

We can find unique solutions

$$c_1 = -\frac{1}{2}, \quad c_2 = \frac{5}{2}.$$

So a solution of the IVP is

$$y = -\frac{1}{2} e^{\frac{3}{2}t} + \frac{5}{2} e^{\frac{1}{2}t}.$$

Note that

$$y \to -\infty \quad \text{as} \quad t \to \infty$$

2 Theory of 2nd Order Linear Equations

Consider the general 2nd order linear equation

$$\boxed{L[y] := y''(t) + p(t)\, y' + q(t)\, y} = g(t).$$

Note that L is a linear operator.

Existence and Uniqueness Theorem *Consider the IVP*

$$y''(t) + p(t)\, y' + q(t)\, y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

If p, q, g are continuous on an interval I containing t_0, then there exists a unique solution to this IVP on I.

Example 2.1. Find the longest interval in which the solution of the initial value problem

$$(t^2 - 3t)\, y'' + t y' - (t+3)\, y = 0, \quad y(1) = 2, \quad y'(1) = 1$$

is certain to exist.

Answer: Assuming $t \neq 0, t \neq 3$, rewrite the equation in the standard form

$$y'' + \frac{1}{t-3} y' - \frac{t+3}{t^2 - 3t} y = 0.$$

So p, q, g are continuous in $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$. Since $1 \in (0, 3)$. By the E&U theorem, there exists a unique solution to the IVP on $(0, 3)$.

Principle of Superposition *Consider the **homogeneous** linear equation*

$$L[y] = 0.$$

If y_1 and y_2 are both solutions, then $c_1 y_1 + c_2 y_2$ is also a solution for any constants c_1 and c_2.

Proof.

$$\begin{aligned} L[c_1 y_1 + c_2 y_2] &= (c_1 y_1 + c_2 y_2)'' + p(t)\, (c_1 y_1 + c_2 y_2)' + q(t)\, (c_1 y_1 + c_2 y_2) \\ &= c_1 (y_1'' + p(t)\, y_1' + q(t)\, y_1) + c_2 (y_2'' + p(t)\, y_2' + q(t)\, y_2) \\ &= c_1 L[y_1] + c_2 L[y_2] = 0 + 0 = 0. \end{aligned}$$

So $c_1 y_2 + c_2 y_2$ is also a solution. $\square$

Let's consider the homogeneous equation with constant coefficients

$$a y'' + b y' + c y = 0.$$

Suppose there are two different roots r_1, r_2 of the characteristic polynomial $a r^2 + b r + c$. Then we have two solutions

$$y_1 = e^{r_1 t}, \quad y_2 = e^{r_2 t}.$$

By the principle of superposition,

$$y = c_1 y_1 + c_2 y_2$$

is a solution for any constants c_1, c_2.

The next question: can we always find c_1, c_2 such that a given initial conditions are satisfied?

Pluggin the initial conditions, we obtain a linear system for c_1, c_2:

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0' \end{aligned}$$

or, in matrix form

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}.$$

In order to have a solution for arbitrary values of y_0, y_0', we need the matrix to be nonsingular, i.e.

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)\, y_2'(t_0) - y_1'(t_0)\, y_2(t_0) \neq 0.$$

Definition 2.2 *Suppose y_1, y_2 are two solutions of the ODE $L[y] = 0$. Then the Wronskian of them is*

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

Theorem 2.3 *Let y_1, y_2 are solutions of the equation $L[y] = 0$. Then one can find constants c_1 and c_2 such that $c_1 y_1 + c_2 y_2$ solves the IVP*

$$L[y] = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

regardless of the values y_0 and y_0' if and only if $W(y_1, y_2)(t_0) \neq 0$.

Next we show all solutions of $L[y] = 0$ can are actually in the form $c_1 y_1 + c_2 y_2$ if and only if the Wronskian is nonzero.

Theorem 2.4 *Let y_1, y_2 are solutions of the equation $L[y] = 0$ on some interval I. Then every solution of $L[y] = 0$ on I can be written as $c_1 y_1 + c_2 y_2$ if and only if $W(y_1, y_2)(t) \neq 0$ for some $t \in I$.*

Proof. Suppose $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$. Let $\phi(t)$ to be a solution of $L[y] = 0$. Let $y_0 = \phi(t_0)$ and $y_0' = \phi'(t_0)$. Now consider the IVP

$$L[y] = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{2.1}$$

Clearly ϕ is a solution of the IVP (2.1). On the other hand, we can find c_1 and c_2 such that $c_1 y_1 + c_2 y_2$ is a solution of the IVP (2.1) for some c_1, c_2 since $W(y_1, y_2)(t_0) \neq 0$. By the uniqueness part of the E&U theorem, we have $\phi = c_1 y_1 + c_2 y_2$.

Next, suppose $W(y_1,y_2)(t)=0$ for any $t\in I$. Then $W(y_1,y_2)(t_0)=0$ for some $t_0\in I$. So there exists some numbers y_0, y_0' such that

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} y_0 \\ y_0' \end{bmatrix} \tag{2.2}$$

has no solution. Let $\phi(t)$ to be the solution of the IVP (2.1). Suppose $\phi=c_1y_1+c_2y_2$ for some c_1,c_2, then c_1,c_2 must satisfy the linear system (2.2). A contradition! $\square$

If $W(y_1,y_2)(t)\neq 0$ for some t, we call the solutions $\{y_1,y_2\}$ a **fundamental set of solutions**.

Example 2.5. If $r_1\neq r_2$ are real numbers, and $y_1=e^{r_1t}, y_2=e^{r_2t}$ are solutions of some ODE. Then

$$W(y_1,y_2)(t)=\begin{vmatrix} e^{r_1t} & e^{r_2t} \\ r_1e^{r_1t} & r_2e^{r_2t} \end{vmatrix}=(r_2-r_1)e^{(r_1+r_2)t}\neq 0$$

for any t. So $\{y_1,y_2\}$ form a fundamental set of solutions.

Example 2.6. Show that $y_1(t)=t^{1/2}$ and $y_2(t)=t^{-1}$ form a fundamental set of solutions of

$$2t^2y''+3ty'-y=0,\quad t>0.$$

Answer:

$$W(y_1,y_2)(t)=\begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix}=-\frac{3}{2}t^{-3/2}\neq 0$$

for any $t\neq 0$. So $\{y_1,y_2\}$ form a fundamental set of solutions for $t\neq 0$.

Theorem 2.7 *Let y_1 to be the solution of the IVP*

$$L[y]=0,\quad y(t_0)=1,\quad y'(t_0)=0.$$

Let y_2 to be the solution of the IVP

$$L[y]=0,\quad y(t_0)=0,\quad y'(t_0)=1.$$

Then the Wronskian of y_1,y_2 is $W(t)=1$. So $\{y_1,y_2\}$ form a fundamental set of solutions.

This theorem says fundamental set of solutions always exist. In fact, we can find infinitely many fundamental set of solutions.

Example 2.8. Find the fundamental set of solutions y_1 and y_2 specified by Theorem 2.7 for the differential equation

$$y''-y=0.$$

Answer: Using the characteristic equations, we find two solutions

$$y_1=e^t,\quad y_2=e^{-t}.$$

We can check they form a fundamental set of solutions. Let's find another fundamental set of solutions defined as in Theorem 2.7 for $t_0=0$. Let

$$y_3=\frac{e^t+e^{-t}}{2}=\cosh t,\quad y_4=\frac{e^t-e^{-t}}{2}=\sinh t.$$

Then $W(y_3,y_4)=1$. So the general solution can be written as

$$c_1y_1+c_2y_2 \text{ or } c_3y_3+c_4y_4.$$

Theorem 2.9 **(Abel)** *Let y_1, y_2 are solutions of the equation*

$$L[y] = y'' + p(t)\, y' + q(t)\, y = 0.$$

Then the Wronskian

$$W(y_1, y_2)(t) = c e^{-\int p(t)\, dt}$$

for some constant c, which may depend on y_1, y_2 but otherwise independent of p, q.

Proof. We have

$$\begin{aligned} y_1'' + p(t)\, y_1' + q(t)\, y_1 &= 0, \\ y_2'' + p(t)\, y_2' + q(t)\, y_2 &= 0. \end{aligned}$$

Then

$$\begin{aligned} y_2\, [y_1'' + p(t)\, y_1' + q(t)\, y_1] &= 0, \\ y_1\, [y_2'' + p(t)\, y_2' + q(t)\, y_2] &= 0. \end{aligned}$$

Subtracting two equations, we obtain

$$y_1\, y_2'' - y_1''\, y_2 + p(t)\, (y_1\, y_2' - y_1'\, y_2) = 0.$$

Note that

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1\, y_2' - y_2\, y_1'.$$

$$W'(t) = y_1'\, y_2' + y_1\, y_2'' - y_2'\, y_1' - y_2\, y_1'' = y_1\, y_2'' - y_2\, y_1''.$$

So we obtain

$$W'(t) + p(t)\, W(t) = 0.$$

Solving this first order linear ODE, we obtain

$$W(t) = c e^{-\int p(t)\, dt}.$$

□

Remark 2.10. From the Abel's theorem, we now know the Wronskian of two solutions of an ODE is either identically zero or nowhere zero.

Theorem 2.11 *Suppose p, q are real-valued functions. Let $y(t) = u(t) + i v(t)$ be a complex-valued solution of*

$$L[y] = y'' + p(t)\, y' + q(t)\, y = 0,$$

where u, v are real-valued functions. Then u, v are also solutions of $L[y] = 0$.

Proof. We have

$$\begin{aligned} L[y] &= (u + i v)'' + p(t)\, (u + i v)' + q(t)\, (u + i v) \\ &= (u'' + i v'') + p(t)\, (u' + i v') + q(t)\, (u + i v) \\ &= (u'' + p(t)\, u' + q(t)\, u) + i\, (v'' + p(t)\, v' + q(t)\, v) \\ &= 0. \end{aligned}$$

So

$$u'' + p(t)u' + q(t)u = 0, \quad v'' + p(t)v' + q(t)v = 0.$$

That is, u, v are both solutions of $L[y] = 0$.

$\square$

3 Complex roots of the characteristic equation

Let's go back to equations with constant coefficients

$$ay'' + by' + cy = 0.$$

The characteristic equation is

$$ar^2 + br + c = 0.$$

The solution is given by the quadratic formula

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1. $b^2 > 4ac$. Then r_1, r_2 are both real and $r_1 \neq r_2$.
2. $b^2 = 4ac$. Then r_1, r_2 are both real and $r_1 = r_2$.
3. $b^2 < 4ac$. Then r_1, r_2 are both complex, and $r_2 = \bar{r}_1$.

Now consider case (3). Let $r_{1,2} = \lambda \pm i\mu$. So then we have two complex-valued solutions

$$y_1 = e^{r_1 t} = e^{(\lambda + i\mu)t}, \quad y_2 = e^{r_2 t} = e^{(\lambda - i\mu)t}.$$

What does it to raise a number to a complex exponent?

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Then we define

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\ &= \cos x + i \sin x \end{aligned}$$

Euler's formula

$$e^{ix} = \cos x + i \sin x.$$

Then we define

$$e^{\lambda + i\mu} = e^{\lambda} e^{i\mu} = e^{\lambda}(\cos x + i \sin x) = e^{\lambda} \cos x + i e^{\lambda} \sin x$$

One can verify the exponential function defined this way has the usual algebraic properties of exponential functions with real exponent.

The two complex solutions are

$$y_1 = e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos \mu t + i \sin \mu t), \quad y_2 = e^{(\lambda - i\mu)t} = e^{\lambda t}(\cos \mu t - i \sin \mu t).$$

One can verify y_1, y_2 form a fundamental set of solutions.

By the previous theorem, we know

$$y_3 = e^{\lambda t} \cos \mu t, \quad y_4 = e^{\lambda t} \sin \mu t.$$

are real-valued solutions. One can verify y_3, y_4 also form a fundamental set of solutions.

Example 3.1. Find the general solution (real-valued) of the differential equation

$$y'' + y' + 9.25 y = 0,$$

Also find the solution that satisfies the initial conditions

$$y(0) = 2, \quad y'(0) = 8,$$

and draw its graph.

Answer: The roots of the characteristic polynomial is

$$r_{1,2} = \frac{-1 \pm 6i}{2} = -\frac{1}{2} \pm 3i.$$

The real-valued solutions are

$$y_1 = e^{-\frac{t}{2}} \cos 3t, \quad y_2 = e^{-\frac{t}{2}} \sin 3t.$$

The general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{-\frac{t}{2}} \cos 3t + c_2 e^{-\frac{t}{2}} \sin 3t,$$

and

$$y' = c_1 e^{-\frac{t}{2}} \left(-\frac{1}{2} \cos 3t - 3 \sin 3t \right) + c_2 e^{-\frac{t}{2}} \left(-\frac{1}{2} \sin 3t + 3 \cos 3t \right).$$

Plugging the initial conditions,

$$\begin{aligned} c_1 &= 2, \\ -\frac{1}{2} c_1 + 3 c_2 &= 8. \end{aligned}$$

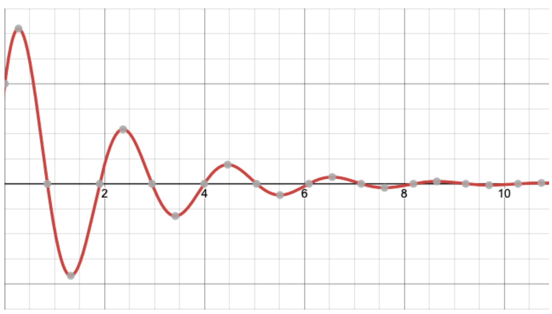
Solving the linear systm,

$$c_1 = 2, \quad c_2 = 3.$$

So the solution of the IVP is

$$y = 2e^{-\frac{t}{2}} \cos 3t + 3e^{-\frac{t}{2}} \sin 3t = e^{-\frac{t}{2}} (2 \cos 3t + 3 \sin 3t).$$

The graph is a damped oscillation.



Example 3.2. Find the general solution of

$$y'' + 9y = 0.$$

Answer:

$$r_{1,2} = \pm 3i$$

The general solution is

$$y = c_1 \cos 3t + c_2 \sin 3t.$$

The graph is an undamped oscillation.

4 Repeated Roots; Reduction of order

4.1 Repeated roots

Suppose the charateristic equation have one repeated root $r = -\dfrac{b}{2a}$. Then we have a solution

$$y_1 = e^{rt}.$$

Then $y_2 = cy_1 = ce^{rt}$ is also a solution for any constant c, but $\{y_1, y_2\}$ is not a fundamental set of solutions. To find another independent solution, let

$$y_2 = v(t)\, y_1.$$

Plugging y_2 into the equation,

$$\begin{aligned} a(vy_1)'' + b(vy_1)' + c(vy_1) &= a(v''y_1 + 2v'y_1' + vy_1'') + b(v'y_1 + vy_1') + cvy_1 \\ &= v(ay_1'' + by_1' + cy_1) + av''y_1 + 2av'y_1' + bv'y_1 \\ &= ay_1 v'' + (2ay_1' + by_1)v' \\ &= ae^{rt}v'' + (2are^{rt} + be^{rt})v' \\ &= e^{rt}(av'' + (2ar + b)v') = 0 \\ \Rightarrow \quad & av'' + (2ar + b)v' = av'' = 0 \\ \Rightarrow \quad & v'' = 0 \quad \Rightarrow \quad v = c_1 t + c_2. \end{aligned}$$

Then

$$y_2 = (c_1 t + c_2)e^{rt} = c_1 t e^{rt} + c_2 e^{rt}.$$

Choose

$$y_2 = te^{rt}.$$

Then one can verify y_1, y_2 form a fundamental set of solutions (check $W(y_1, y_2) \neq 0$).

Example 4.1.

$$y'' + 4y' + 4y = 0.$$

Answer: The characteristic equation is $r^2 + 4r + 4 = 0$. The (repeated) root is $r = -2$. So

$$y_1 = e^{-2t}, \quad y_2 = te^{-2t}.$$

The general solution is

$$y = c_1 e^{-2t} + c_2 t e^{-2t}.$$

We have $y(t) \to 0$ as $t \to \infty$.

4.2 Reduction of order

The idea to find y_2 can be generalized to a general second order linear equation. If y_1 is a solution of

$$y'' + p(t)\, y' + q(t)\, y = 0.$$

Let $y_2 = v(t)\, y_1$ be another solution. Then plugging y_2 into the equation we can obtain an second order linear ODE for $v(t)$:

$$y_1 v'' + (y_1' + p(t)\, y_1)\, v' = 0.$$

Let $w = v'$, then we obtain a first order ODE for w

$$y_1 w' + (y_1' + p(t)\, y_1)\, w = 0.$$

Solve w, then let $v = \displaystyle\int w$.

Example 4.2. Given the variable coefficient equation and solution y_1,

$$2t^2 y'' + 3t y' - y = 0, \quad t > 0; \quad y_1(t) = t^{-1},$$

use reduction of order method to find a second solution.

Answer: Let $y_2 = v y_1$. Then

$$\begin{aligned} 2t^2 y_2'' + 3t y_2' - y_2 &= 2t^2 (v'' y_1 + 2v' y_1' + v y_1'') + 3t (v' y_1 + v y_1') - v y_1 \\ &= 2t^2 (t^{-1} v'' - 2t^{-2} v' + 2t^{-3} v) + 3t (t^{-1} v' - t^{-2} v) - t^{-1} v \\ &= 2t v'' - v' = 0. \end{aligned}$$

Let $w = v'$,

$$2t w' - w = 0 \quad \Rightarrow \quad \frac{dw}{w} = \frac{dt}{2t} \quad \Rightarrow \quad \ln w = \frac{1}{2} \ln t \quad \Rightarrow \quad w = c\sqrt{t} \quad \Rightarrow \quad v = c\frac{2}{3} t^{\frac{3}{2}}.$$

So

$$y_2 = c\frac{2}{3} t^{\frac{3}{2}} t^{-1} = c\frac{2}{3}\sqrt{t}.$$

Choose

$$y_2 = \sqrt{t}.$$

Exercise 4.1. Check y_2 satisfies the equation and $W(y_1, y_2) \neq 0$.

5 Method of Undetermined Coefficients

Consider the nonhomogeneous equation

$$L[y] = y'' + p(t)\, y' + q(t)\, y = g(t).$$

Let y_1, y_2 be solutions. Then

$$L[y_1 - y_2] = L[y_1] - L[y_2] = g(t) - g(t) = 0.$$

So $y_1 - y_2$ is a solution of the homogeneous equation $L[y] = 0$.

Theorem 5.1 *The general solution of the nonhomogeneous equation $L[y]=g$ is*

$$y=c_1y_1+c_2y_2+Y,$$

where c_1,c_2 are arbitrary constant, y_1,y_2 form a fundamental set of solutions for the homogeneous equation $L[y]=0$, and Y is a particular solution of the nonhomogeneous equation $L[y]=g$.

Proof. Let y be any solution of $L[y]=g$. Then $y-Y$ is a solution of $L[y]=0$. Then

$$y-Y=c_1y_1+c_2y_2 \quad\Rightarrow\quad y=c_1y_1+c_2y_2+Y.$$

for some constants c_1,c_2. □

How to find a particular solution?

Example 5.2. Find a particular solution of

$$y''-3y'-4y=3e^{2t}.$$

Answer: Suppose the solution is of the form (ansatz) $y=Ae^{2t}$, where A is an undetermined coefficient. To find A, just plug the ansatz into the equation.

$$\begin{aligned} 4Ae^{2t}-6Ae^{2t}-4Ae^{2t} &= -6Ae^{2t}=3e^{2t} \\ \Rightarrow \quad A=-\frac{1}{2}. \end{aligned}$$

So

$$Y=-\frac{1}{2}e^{2t}$$

is a particular solution.

Example 5.3. Find a particular solution of

$$y''-3y'-4y=2\sin t.$$

Answer: Suppose the solution is of the form

$$y=A\sin t+B\cos t.$$

Then

$$(-A\sin t-B\cos t)-3(A\cos t-B\sin t)-4(A\sin t+B\cos t)=2\sin t.$$

Comparing the coefficients on the LHS and RHS,

$$\begin{cases} -5A+3B=2 \\ -5B-3A=0 \end{cases} \Rightarrow \begin{cases} -25A+15B=10 \\ -15B-9A=0 \end{cases} \Rightarrow \begin{cases} A=-\dfrac{5}{17} \\ B=\dfrac{3}{17} \end{cases}.$$

So

$$Y=-\frac{5}{17}\sin t+\frac{3}{17}\cos t.$$

Remark 5.4. The method also works if the RHS is a cosine function.

Example 5.5. Find a particular solution of

$$y'' - 3y' - 4y = -8e^t \cos 2t.$$

Answer: Suppose the solution is of the form

$$y = e^t(A\sin 2t + B\cos 2t).$$

Then

$$\begin{aligned} y' &= e^t(A\sin 2t + B\cos 2t) + e^t(2A\cos 2t - 2B\sin 2t) \\ &= e^t[(A-2B)\sin 2t + (2A+B)\cos 2t] \\ y'' &= e^t[(A-2B)\sin 2t + (2A+B)\cos 2t] + e^t[2(A-2B)\cos 2t - 2(2A+B)\sin 2t] \\ &= e^t[(-3A-4B)\sin 2t + (4A-3B)\cos 2t]. \end{aligned}$$

$$y'' - 3y' - 4y = e^t[(-3A-4B-3A+6B-4A)\sin 2t + (4A-3B-6A-3B-4B)\cos 2t] = -8e^t\cos 2t$$

$$\Rightarrow \begin{cases} -10A + 2B = 0 \\ -2A - 10B = -8 \end{cases} \quad \Rightarrow \begin{cases} -5A + B = 0 \\ -A - 5B = -4 \end{cases} \quad \Rightarrow \begin{cases} A = \dfrac{2}{13}, \\ B = \dfrac{10}{13}. \end{cases}$$

So

$$Y = e^t\left(\frac{2}{13}\sin 2t + \frac{10}{13}\cos 2t\right) = \frac{2}{13}e^t(\sin 2t + 5\cos 2t).$$

is a particular solution.

Example 5.6. Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t\cos 2t.$$

Answer: A particular solution is

$$Y = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{2}{13}e^t(\sin 2t + 5\cos 2t).$$

Example 5.7. Find a particular solution of

$$y'' - 3y' - 4y = 2e^{-t}.$$

Answer: Try the ansatz $y = Ae^{-t}$. Then

$$y'' - 3y' - 4y = (A + 3A - 4A)e^{-t} = 0.$$

Not work! Try another one

$$y = Ate^{-t}.$$

Then

$$y' = A(1-t)e^{-t}, \quad y'' = A(-2+t)e^{-t}$$

$$y'' - 3y' - 4y = Ae^{-t}(-2 + t - 3(1-t) - 4t) = -5Ae^{-t} = 2e^{-t} \quad \Rightarrow \quad A = -\frac{2}{5}.$$

So

$$y = -\frac{2}{5}te^{-t}$$

is a particular solution.

Question 4. Why Ate^{-t} works?

Answer. Consider the general case:

$$ay'' + by' + cy = de^{\alpha t}.$$

Suppose α is a root (not repeated) of the characteristic equation $ar^2 + br + c = 0$. Let $y = v(t)e^{\alpha t}$.

Then

$$\begin{aligned} y' &= (v' + \alpha v)e^{\alpha t}, \\ y'' &= (v'' + 2\alpha v' + \alpha^2 v)e^{\alpha t}. \end{aligned}$$

Plugging into the equation

$$\begin{aligned} ay'' + by' + cy &= [a(v'' + 2\alpha v' + \alpha^2 v) + b(v' + \alpha v) + cv]e^{\alpha t} \\ &= [av'' + (2a\alpha + b)v' + (a\alpha^2 + b\alpha + c)v]e^{\alpha t} \\ &= [av'' + (2a\alpha + b)v']e^{\alpha t} = de^{\alpha t} \\ \Rightarrow \quad av'' + (2a\alpha + b)v' &= d. \end{aligned}$$

Let $w = v'$, then

$$aw' + (2a\alpha + b)w = d \quad \Rightarrow \quad w = \frac{d}{2a\alpha + b} := A \quad \Rightarrow \quad v = At + B.$$

So

$$y = (At + B)e^{\alpha t} = Ate^{\alpha t}$$

by choosing $B = 0$.

Exercise 5.1. Derive the solution ansatz $y = At^2e^{\alpha t}$ if α is a repeated root of the characteristic polynomial.

Example 5.8. Find a particular solution of

$$y'' - 4y' + 4y = e^{2t}.$$

Answer: Try the ansatz $y = Ae^{2t}$, not work. Try $y = Ate^{2t}$, not work. Try

$$y = At^2e^{2t}.$$

$$y' = 2A(t + t^2)e^{2t}, \quad y'' = 2A(1 + 4t + 2t^2)$$

$$y'' - 4y' + 4y = Ae^{2t}[2(1 + 4t + 2t^2) - 8(t + t^2) + 4t^2] = 2Ae^{2t} = e^{2t}.$$

So $A = 1/2$ and

$$y = \frac{1}{2}t^2e^{2t}$$

is a particular solution.

Example 5.9. Find a particular solution of

$$y'' - 4y' + 3y = t^2 + t + 1.$$

Answer: Consider the ansatz

$$y = At^2 + Bt + C.$$

Then

$$y' = 2At + B, \quad y'' = 2A.$$

$$\begin{aligned} y'' - 4y' + 3y &= 2A - 4(2At + B) + 3(At^2 + Bt + C) \\ &= 3At^2 + (3B - 8A)t + (2A - 4B + 3C) \end{aligned}$$

$$\Rightarrow \begin{cases} 3A & = 1 \\ 3B - 8A & = 1 \\ 2A - 4B + 3C & = 1 \end{cases} \Rightarrow \begin{cases} A = \dfrac{1}{3} \\ B = \dfrac{11}{9} \\ C = \dfrac{1}{3}\left(1 - \dfrac{2}{3} + \dfrac{44}{9}\right) = \dfrac{47}{27} \end{cases}$$

TABLE 3.5.1 The Particular Solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$	$t^s(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin \beta t \\ \cos \beta t\end{cases}$	$t^s[(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)e^{\alpha t} \cos \beta t$ $+ (B_0 t^n + B_1 t^{n-1} + \cdots + B_n)e^{\alpha t} \sin \beta t]$

Notes. Here s is the smallest nonnegative integer ($s = 0, 1,$ or 2) that will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation, respectively.

6 Variation of Parameters

The method of undetermined coefficients has some limitations.

- only equations with constant coefficients
- $g(t)$ must be in one of those special forms

Now consider the more general nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t).$$

Suppose $y = c_1 y_1 + c_2 y_2$ is a general solution of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

Let

$$Y = u_1 y_1 + u_2 y_2,$$

where u_1, u_2 are functions to be determined. Then

$$Y' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'.$$

Let's pose the condition

$$u_1' y_1 + u_2' y_2 = 0. \tag{6.1}$$

Then

$$Y' = u_1 y_1' + u_2 y_2' \quad \text{and} \quad Y'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''.$$

So

$$\begin{aligned} Y'' + p(t) Y' + q(t) Y &= u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' + p(t)(u_1 y_1' + u_2 y_2') + q(t)(u_1 y_1 + u_2 y_2) \\ &= u_1 [y_1'' + p(t) y_1' + q(t) y_1] + u_2 [y_2'' + p(t) y_2' + q(t) y_2] + u_1' y_1' + u_2' y_2' \\ &= u_1' y_1' + u_2' y_2'. \end{aligned}$$

So

$$u_1' y_1' + u_2' y_2' = g(t). \tag{6.2}$$

So from (6.1) and (6.2) we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

Note this system has a unique solution because $W(y_1, y_2) \neq 0$. The solution is (given by Cramer's rule):

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2}{W(y_1, y_2)} g, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1}{W(y_1, y_2)} g.$$

Integrating in t, we obtain

$$u_1 = \int \frac{-y_2}{W(y_1, y_2)} g dt, \quad u_2 = \int \frac{y_1}{W(y_1, y_2)} g dt.$$

This is called the method of **variation of parameters**.

Example 6.1. Find the general solution of

$$y'' + 4y = 3 \csc t.$$

Answer: We have $y_1 = \sin 2t$, $y_2 = \cos 2t$,

$$W(y_1, y_2) = \begin{vmatrix} \sin 2t & \cos 2t \\ 2\cos 2t & -2\sin 2t \end{vmatrix} = -4.$$

So

$$u_1 = \int \frac{-y_2}{W(y_1, y_2)} g dt = \int -\frac{\cos 2t}{-4} 3 \csc t = \frac{3}{4} \int \frac{\cos 2t}{\sin t} dt = \frac{3}{4} \int \frac{1 + 2\cos^2 t}{\sin t} dt$$

$$= \frac{3}{4} \left[\int \csc t \, dt + 2 \int \frac{1 - \sin^2 t}{\sin t} dt \right] = \frac{3}{4} \left[3 \int \csc t \, dt + 2 \int \sin t \, dt \right] = \frac{3}{4} [3 \ln |\csc t - \cot t| - 2\cos t]$$

Similarly we can find u_2 (exercise). So the general solution is

$$y = c_1 \sin 2t + c_2 \cos 2t + u_1 \sin 2t + u_2 \cos 2t$$

$$\boxed{\begin{aligned}&\int \csc t\,dt=\int\frac{1}{\sin t}dt=\int\frac{\sin t}{\sin^2 t}dt=\int\frac{\sin t}{1-\cos^2 t}dt=\int\frac{1}{2}\left[\frac{\sin t}{1+\cos t}+\frac{\sin t}{1-\cos t}\right]dt\\ &=\frac{1}{2}\left[\int\frac{-1}{1+\cos t}d(1+\cos t)+\int\frac{1}{1-\cos t}d(1-\cos t)\right]=\frac{1}{2}[-\ln(1+\cos t)+\ln(1-\cos t)]\\ &=\frac{1}{2}\ln\frac{1-\cos t}{1+\cos t}=\frac{1}{2}\ln\frac{(1-\cos t)^2}{1-\cos^2 t}=\frac{1}{2}\ln\frac{(1-\cos t)^2}{\sin^2 t}=\ln\sqrt{\frac{(1-\cos t)^2}{\sin^2 t}}=\ln\left|\frac{1-\cos t}{\sin t}\right|\\ &=\ln|\csc t-\cot t|\end{aligned}}$$

7 Free Vibrations

Consider the equation for the spring-mass system

$$mu''+\gamma u'+ku=0.$$

7.1 Undamped free vibrations

Let $\gamma=0$, i.e. there is no damping force. Then the equation reduces to

$$mu''+ku=0.$$

The roots of the characteristic equation are

$$r_{1,2}=\pm\omega_0 i,\quad \omega_0=\sqrt{\frac{k}{m}}.$$

So the general solution is

$$u=A\cos\omega_0 t+B\sin\omega_0 t=R\cos(\omega_0 t-\delta)=R(\cos\omega_0 t\cos\delta+\sin\omega_0 t\sin\delta).$$

So

$$A=R\cos\delta,\quad B=R\sin\delta.$$

So

$$R=\sqrt{A^2+B^2},\quad \cos\delta=\frac{A}{R},\quad \sin\delta=\frac{B}{R}\quad\Rightarrow\quad\delta=$$

Here R is the **amplitude**, ω_0 is the **angular frequency** (natural frequency of the system), δ is the **phase**, and $T=\frac{2\pi}{w_0}$ is the **period**.

7.2 Damped free vibrations

Now consider the case when $\gamma>0$ (damped vibration). Then the roots of the characteristic equation is

$$r_{1,2}=\frac{-\gamma\pm\sqrt{\gamma^2-4mk}}{2m}=-\frac{\gamma}{2m}\pm\frac{\sqrt{\gamma^2-4mk}}{2m}.$$

1. If $\gamma^2>4mk$ (**overdamped**), then $r_1\neq r_2$ are real and both negative. The general solution is
$$u=Ae^{r_1 t}+Be^{r_2 t}.$$
The solution is nonoscillatory and $u(t)\to 0$ as $t\to\infty$.
2. If $\gamma^2=4mk$ (**critically damped**), then we have repeated root $r=-\frac{\gamma}{2m}$. So the general solution is
$$u=Ae^{rt}+Bte^{rt}.$$
The solution is nonoscillatory and $u(t)\to 0$ as $t\to\infty$.
3. If $\gamma^2<4mk$, then the roots are
$$r_{1,2}=\lambda\pm i\mu,\quad \lambda=-\frac{\gamma}{2m},\quad \mu=\frac{\sqrt{4mk-\gamma^2}}{2m}.$$

The general solution is

$$u = e^{\lambda t}(A\cos\mu t + B\sin\mu t) = Re^{\lambda t}\cos(\mu t - \delta).$$

It's a **damped oscillation**, and $u(t) \to 0$ as $t \to \infty$.

$u(t)$ is nonperiodic, but we call $T = \dfrac{2\pi}{\mu}$ the **quasi period**. In fact

$$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} < \omega_0.$$

7.3 Electric circuits (skip)

8 Forced Vibrations (optional)

8.1 Forced vibrations with damping

$$mu'' + \gamma u' + ku = F$$

We consider periodic forces $F = F_0\cos\omega t$. The general solution is

$$u = [c_1u_1(t) + c_2u_2(t)] + [A\cos\omega t + B\sin\omega t] = u_c(t) + U(t).$$

Note that $u_c(t) \to 0$ as $t \to \infty$, but $U(t)$ is periodic. So we call $u_c(t)$ the **transient solution** and $U(t)$ the **steady-state solution**.

Example 8.1. Consider the IVP

$$u'' + u' + \frac{5}{4}u = 3\cos t, \quad u(0) = 2, \quad u'(0) = 3.$$

Then

$$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = -\frac{1}{2} \pm i,$$

So

$$u_c(t) = e^{-\frac{t}{2}}(c_1\cos t + c_2\sin t).$$

Let $U = A\cos t + B\sin t$. Then

$$U'' + U' + \frac{5}{4}U = -A\cos t - B\sin t - A\sin t + B\cos t + \frac{5}{4}(A\cos t + B\sin t)$$

$$= \left(-A + B + \frac{5}{4}A\right)\cos t + \left(-B - A + \frac{5}{4}B\right)\sin t = \left(\frac{1}{4}A + B\right)\cos t + \left(\frac{1}{4}B - A\right)\sin t$$

So

$$\frac{1}{4}A + B = 3, \quad \frac{1}{4}B - A = 0 \quad \Rightarrow \quad \begin{cases} A = \dfrac{12}{17}, \\ B = \dfrac{48}{17}. \end{cases}$$

So the steady-state solution is

$$U(t) = \frac{12}{17}\cos t + \frac{48}{17}\sin t.$$

So the general solution is

$$u(t) = e^{-\frac{t}{2}}(c_1\cos t + c_2\sin t) + \frac{12}{17}\cos t + \frac{48}{17}\sin t.$$

Plugging initial conditions, we obtain $c_1=\dfrac{22}{17}, c_2=\dfrac{14}{17}$. So the solution of the IVP is

$$u(t)=\frac{2}{17}\left[e^{-\frac{t}{2}}(11\cos t+7\sin t)+6\cos t+24\sin t\right].$$

Resonance. Steady-state solution $U=A\cos\omega t+B\sin\omega t$

$$U'=\omega(-A\sin\omega t+B\cos\omega t),\quad U''=\omega^2(-A\cos\omega t-B\sin\omega t)$$

$$\begin{aligned}
\Rightarrow\ & mU''+\gamma U'+kU\\
=\ & m\omega^2(-A\cos\omega t-B\sin\omega t)+\gamma\omega(-A\sin\omega t+B\cos\omega t)+k(A\cos\omega t+B\sin\omega t)\\
=\ & (-m\omega^2A+\gamma\omega B+kA)\cos\omega t+(-Bm\omega^2-A\gamma\omega+kB)\sin\omega t\\
=\ & [(k-m\omega^2)A+\gamma\omega B]\cos\omega t+[-\gamma\omega A+(k-m\omega^2)B]\sin\omega t\\
=\ & F_0\cos\omega t
\end{aligned}$$

$$\begin{cases}(k-m\omega^2)A+\gamma\omega B &= F_0\\ -\gamma\omega A+(k-m\omega^2)B &= 0\end{cases}\quad\Rightarrow\quad\begin{pmatrix}k-m\omega^2 & \gamma\omega\\ -\gamma\omega & k-m\omega^2\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=\begin{pmatrix}F_0\\0\end{pmatrix}$$

$$\begin{aligned}
A &= \frac{k-m\omega^2}{(k-m\omega^2)^2+\gamma^2\omega^2}F_0=\frac{m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}F_0\\
B &= \frac{\gamma\omega}{(k-m\omega^2)^2+\gamma^2\omega^2}F_0=\frac{\gamma\omega}{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}F_0
\end{aligned}$$

$$A\cos\omega t+B\sin\omega t=R\cos(\omega t-\delta)\quad\Rightarrow\quad R=\frac{F_0}{\Delta},\quad \Delta=\sqrt{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

Nondimensionalize (无量纲化)

$$R=\frac{F_0}{m\omega_0^2\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\frac{\gamma^2\omega^2}{m^2\omega_0^4}}}=\frac{F_0}{k\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\frac{\gamma^2}{mk}\frac{\omega^2}{\omega_0^2}}}$$

$$\Rightarrow\frac{R}{(F_0/k)}=\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\Gamma\frac{\omega^2}{\omega_0^2}}},\quad \boxed{\Gamma=\frac{\gamma^2}{mk}}$$

$$\gamma u'=F\Rightarrow\gamma=\frac{F}{u'}:\frac{\text{N}}{\text{m}\cdot\text{s}^{-1}}=\frac{\text{N}\cdot\text{s}}{\text{m}}\Rightarrow\Gamma=\frac{\gamma^2}{mk}:\frac{\text{N}^2\cdot\text{s}^2}{\text{m}^2\cdot\text{kg}\cdot\text{N}\cdot\text{m}^{-1}}=\frac{\text{N}\cdot\text{s}^2}{\text{m}\cdot\text{kg}}=\frac{\text{N}}{\text{m}\cdot\text{s}^{-2}\cdot\text{kg}}=1$$

Clearly $\dfrac{R}{(F_0/k)}$ and $\dfrac{\omega^2}{\omega_0^2}$ are also dimensionless. Rewrite the equation as

$$y=\frac{1}{\sqrt{(1-x)^2+\Gamma x}},\quad y=\frac{R}{(F_0/k)},\quad x=\frac{\omega^2}{\omega_0^2}$$

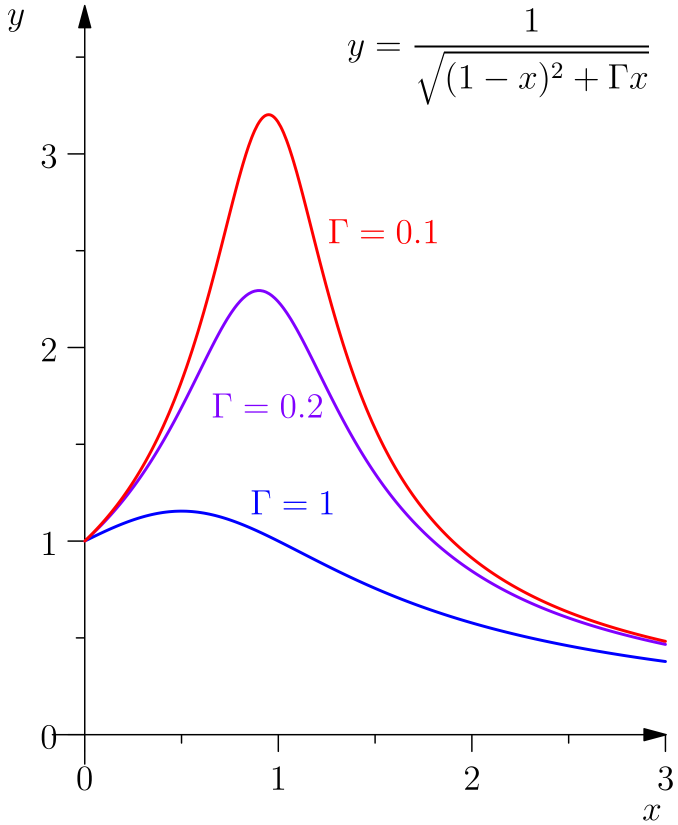
$$y'=-\frac{1}{2}[(1-x)^2+\Gamma x]^{-\frac{3}{2}}[\Gamma-2+2x]$$

If $0<\Gamma<2$, then $y'>0$ for $x\in\left[0,1-\dfrac{\Gamma}{2}\right)$, $y'<0$ for $x\in\left(1-\dfrac{\Gamma}{2},\infty\right)$ and $y'=0$ for $x=1-\dfrac{\Gamma}{2}$.

So $y_{\max}$ is obtained at $x_{\max}=1-\dfrac{\Gamma}{2}$:

$$y_{\max}=\frac{1}{\sqrt{(1-x)^2+\Gamma x}}=\frac{1}{\sqrt{\Gamma-\Gamma^2/4}}\to\infty\quad\text{as}\quad\Gamma\to 0.$$

Hence for lightly damped system (Γ is small), the amplitude of the steady-state solution when ω is near ω_0 can be very large for small external force. This phenomenon is known as **resonance**.



8.2 Forced vibrations without damping

Consider

$$mu'' + ku = F_0 \cos\omega t.$$

The general solution of the homogeneous part is

$$u_c(t) = c_1\cos\omega_0 t + c_2 \sin\omega_0 t, \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

8.2.1 $\omega \neq \omega_0$

The general solution is

$$u = u_c(t) + U(t), \quad U(t) = A\cos\omega t + B\sin\omega t.$$

$$U' = \omega(-A\sin\omega t + B\cos\omega t), \quad U'' = \omega^2(-A\cos\omega t - B\sin\omega t)$$

$$\begin{aligned} mU'' + kU &= m\omega^2(-A\cos\omega t - B\sin\omega t) + k(A\cos\omega t + B\sin\omega t) \\ &= (-Am\omega^2 + kA)\cos\omega t + (-Bm\omega^2 + kB)\sin\omega t \\ &= A(k - m\omega^2)\cos\omega t + B(k - m\omega^2)\sin\omega t \\ &= F_0\cos\omega t \end{aligned}$$

$$\Rightarrow \quad A = \frac{F_0}{k - m\omega^2}, \quad B = 0$$

The general solution is

$$u = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{F_0}{k - m\omega^2}\cos\omega t.$$

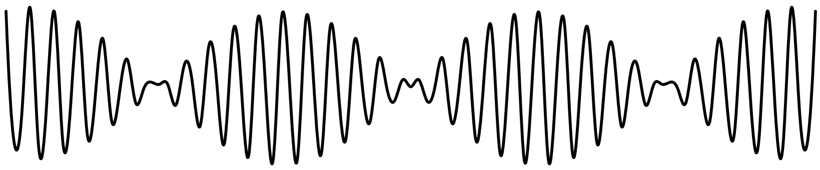
Suppose the initial condition is $u(0)=u'(0)=0$, then

$$c_1+\frac{F_0}{k-m\omega^2}=0, \quad c_2\omega_0=0 \quad \Rightarrow \quad c_1=-\frac{F_0}{k-m\omega^2}, \quad c_2=0.$$

$$u=\frac{F_0}{k-m\omega^2}(\cos\omega t-\cos\omega_0 t)=\frac{-2F_0}{k-m\omega^2}\sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right)$$

If ω is close to ω_0, then we have a **beat**. Also used in **amplitude modulation**.

$$y=\cos(10x)-\cos(11x)$$



8.2.2 $\omega=\omega_0$

A general solution is

$$U=c_1\cos\omega t+c_2\sin\omega t+t(A\cos\omega t+B\sin\omega t).$$

9 Higher Order Linear Equations

9.1 General theory

An n-th order linear ODE is in the form

$$L[y]=y^{(n)}+p_{n-1}(t)\,y^{(n-1)}+\cdots+p_1(t)\,y'+p_0(t)\,y=g(t)$$

An initial value problem is the equation $L[y]=g$ together with initial conditions

$$y(t_0)=y_0, \quad y'(t_0)=y_0', \quad \ldots \quad y^{(n-1)}(t_0)=y_0^{(n-1)}.$$

Definition 9.1 *The **Wronskian** of n solutions $y_1,\ldots,y_n$ of $L[y]=0$ is*

$$W[y_1,\ldots,y_n]=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

Theorem 9.2 *Let $y_1,\ldots,y_n$ be solutions of $L[y]=0$. Then $y_1,\ldots,y_n$ form a fundamental set of solutions if and only if they are linearly independent.*

Proof. Suppose $y_1,\ldots,y_n$ form a fundamental set of solutions, that is, $W[y_1,\ldots,y_n]\neq 0$. Let $c_1,\ldots,c_n$ be constants such that

$$c_1y_1+\cdots+c_ny_n=0.$$

Differentiate the above equation in t,

$$c_1y_1'+\cdots+c_ny_n'=0.$$

Repeat differentiating, we obtain

$$\begin{aligned} c_1y_1''+\cdots+c_ny_n'' &= 0 \\ &\vdots \\ c_1y_1^{(n-1)}+\cdots+c_ny_n^{(n-1)} &= 0 \end{aligned}$$

Hence we have a linear system

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Hence $y_1, \dots, y_n$ are linearly independent.
Now assume $y_1, \dots, y_n$ do not form a fundamental set of solutions, i.e. $W[y_1, \dots, y_n](t_0) = 0$ for some t_0. Then there exists constants $c_1, \dots, c_n$, not all zero, such that

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let

$$Y = c_1 y_1 + \cdots + c_n y_n.$$

Then consider the IVP

$$L[y] = 0, \quad y(t_0) = Y(t_0) = 0, \quad y'(t_0) = Y'(t_0) = 0, \quad \cdots \quad y^{(n-1)}(t_0) = Y^{(n-1)}(t_0) = 0$$

Clearly Y is the solution of the IVP. On the other hand, 0 is also a solution of the IVP. By the E&U theorem, we have $Y = 0$. Thus $y_1, \dots, y_n$ are linearly independent. $\square$

9.2 Homogeneous constant coefficients

Example 9.3.

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$

Answer: The characteristic equation is

$$r^4 + r^3 - 7r^2 - r + 6 = 0.$$

To find the roots, we factorize it:

$$\begin{aligned} (r^4 + r^3) - (7r^2 + r - 6) &= r^3(r+1) - (r+1)(7r-6) \\ &= (r+1)(r^3 - 7r + 6) = (r+1)(r^3 - r - 6r + 6) \\ &= (r+1)[r(r^2-1) - 6(r-1)] \\ &= (r+1)(r-1)(r^2 + r - 6) \\ &= (r+1)(r-1)(r-2)(r+3) \end{aligned}$$

So the roots are$f$

$$r = -3, -1, 1, 2$$

So there are four solutions

$$y = e^{-3t}, e^{-t}, e^{t}, e^{2t}$$

and they form a fundamental set of solutions. So the general solution is

$$y = c_1 e^{-3t} + c_2 e^{-t} + c_3 e^{t} + c_4 e^{2t}.$$

Note 9.4. Another method to find the roots. All possible rational roots are

$$\frac{\pm 1, \pm 2, \pm 3, \pm 6}{1}.$$

Then verify directly it they are roots. Once one or more roots are found, we can use **polynomial division** to reduce the order and help find other roots.

Example 9.5.

$$y^{(4)} - y = 0$$

Answer:

$$r^4 - 1 = (r^2 + 1)(r^2 - 1) \quad \Rightarrow \quad r = \pm i, \pm 1 \quad \Rightarrow \quad y = \cos t, \sin t, e^{-t}, e^t$$

Example 9.6.

$$y^{(4)} + 2y'' + y = 0$$

Answer:

$$r^4 + 2r^2 + 1 = (r^2 + 1)^2 = 0 \quad \Rightarrow \quad r = \pm i, \pm i \quad \Rightarrow \quad y = \cos t, \sin t, t\cos t, t\sin t$$

(We say the root $r = \pm i$ has multiplicity 2).

Example 9.7.

$$y^{(4)} + y = 0.$$

Answer:

$$r^4 + 1 = 0 \quad \Rightarrow \quad r^4 = -1 = e^{i(\pi + 2n\pi)}$$

$$\Rightarrow \quad r = \exp\left(i\frac{(2n+1)}{4}\pi\right) = e^{\frac{\pi}{4}i}, e^{\frac{3\pi}{4}i}, e^{\frac{5\pi}{4}i}, e^{\frac{7\pi}{4}i}$$

$$\Rightarrow \quad r = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \quad \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

$$\Rightarrow \quad r = \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i, \quad -\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i$$

$$\Rightarrow \quad y = e^{\frac{\sqrt{2}}{2}t}\cos\frac{\sqrt{2}}{2}t, \quad e^{\frac{\sqrt{2}}{2}t}\sin\frac{\sqrt{2}}{2}t, \quad e^{-\frac{\sqrt{2}}{2}t}\cos\frac{\sqrt{2}}{2}t, \quad e^{-\frac{\sqrt{2}}{2}t}\sin\frac{\sqrt{2}}{2}t$$

9.3 The method of undetermined coefficients

Example 9.8. Find the general solution of

$$y''' - 3y'' + 3y' - y = 4e^t$$

Answer:

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0 \quad \Rightarrow \quad r = 1, 1, 1 \quad \Rightarrow \quad y_1 = e^t, te^t, t^2e^t.$$

Let

$$Y = At^3e^t.$$

Then

$$Y' = A(3t^2 + t^3)e^t, \quad Y'' = A(6t + 6t^2 + t^3)e^t, \quad Y''' = A(6 + 18t + 9t^2 + t^3)e^t.$$

$$\Rightarrow \quad [(6 + 18t + 9t^2 + t^3) - 3(6t + 6t^2 + t^3) + 3(3t^2 + t^3) - t^3]Ae^t = 4e^t$$

$$\Rightarrow \quad 6A = 4 \quad \Rightarrow \quad A = \frac{2}{3}.$$

So the general solution is

$$y = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + \frac{2}{3} t^3 e^t.$$

9.4 The method of variation of parameters

Suppose $y_1, \dots, y_n$ form a fundamental set of solutions for $L[y] = 0$. Consider the nonhomogeneous equation

$$L[y] = g(t).$$

Let

$$Y = u_1 y_1 + \cdots + u_n y_n$$

Then differentiate Y and make some assumption as in the case of 2nd order equations. We obtain

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g \end{pmatrix}$$

Then

$$u_m' = \frac{W_m}{W} g \quad \Rightarrow \quad u_m = \int \frac{W_m}{W} g$$

where W is the Wronskian, and W_m is the determinant of the above matrix with the m-th column replaced by the vector $(0, \dots, 0, 1)^T$.

Example 9.9. Find the general solution to

$$y''' - y'' - y' + y = g(t).$$

Answer:

$$r^3 - r^2 - r + 1 = r^2(r-1) - (r-1) = (r-1)^2(r+1) \quad \Rightarrow \quad r = 1, 1, -1$$

$$y_1 = e^{-t}, \quad y_2 = e^t, \quad y_3 = te^t.$$

$$\begin{aligned} W &= \begin{vmatrix} e^{-t} & e^t & te^t \\ -e^{-t} & e^t & (t+1)e^t \\ e^{-t} & e^t & (t+2)e^t \end{vmatrix} = e^t \begin{vmatrix} 1 & 1 & t \\ -1 & 1 & t+1 \\ 1 & 1 & t+2 \end{vmatrix} \\ &= e^t \begin{vmatrix} 1 & 0 & t \\ -1 & 2 & t+1 \\ 1 & 0 & t+2 \end{vmatrix} = 2e^t \begin{vmatrix} 1 & t \\ 1 & t+2 \end{vmatrix} = 4e^t, \\ W_1 &= \begin{vmatrix} 0 & e^t & te^t \\ 0 & e^t & (t+1)e^t \\ 1 & e^t & (t+2)e^t \end{vmatrix} = e^{2t} \begin{vmatrix} 0 & 1 & t \\ 0 & 1 & (t+1) \\ 1 & 1 & (t+2) \end{vmatrix} = e^{2t}, \end{aligned}$$

$$
\begin{aligned}
W_2 &= \begin{vmatrix} e^{-t} & 0 & te^t \\ -e^{-t} & 0 & (t+1)e^t \\ e^{-t} & 1 & (t+2)e^t \end{vmatrix} = \begin{vmatrix} 1 & 0 & t \\ -1 & 0 & (t+1) \\ 1 & 1 & (t+2) \end{vmatrix} = -(2t+1), \\
W_3 &= \begin{vmatrix} e^{-t} & e^t & 0 \\ -e^{-t} & e^t & 0 \\ e^{-t} & e^t & 1 \end{vmatrix} = 2. \\
u_1 &= \int \frac{W_1}{W} g = \int \frac{1}{4} e^t g(t)\, dt, \\
u_2 &= \int \frac{W_2}{W} g = \int -\frac{2t+1}{4e^t} g(t)\, dt, \\
u_3 &= \int \frac{W_3}{W} g = \int \frac{1}{2e^t} g(t)\, dt.
\end{aligned}
$$

So the general solution is

$$
y = (c_1 e^{-t} + c_2 e^t + c_3 t e^t) + e^{-t} \int \frac{1}{4} e^t g(t)\, dt - e^t \int \frac{2t+1}{4e^t} g(t)\, dt + te^t \int \frac{1}{2e^t} g(t)\, dt.
$$