Solution to PT Assignment 13

1. Find the moment generating function of X

(a) Binomial(n, p)

(b) Poisson(λ)

(c) Negative Binomial (r, p)

(d) Uniform (a, b)

(e) Gamma (α, λ)

(f) Normal (μ, σ^2)

Solution

a Binomial distribution with parameters n and p

If X is a binomial random variable with parameters n and p, then

$$\begin{aligned} M(t) &= E[e^{tX}] \\ &= \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k} \\ &= (pe^t + 1 - p)^n \end{aligned}$$

where the last equality follows from the binomial theorem.

b **Poisson distribution with mean λ**

If X is a Poisson random variable with parameter λ, then

$$\begin{aligned} M(t) &= E[e^{tX}] \\ &= \sum_{n=0}^{\infty} \frac{e^{tn} e^{-\lambda} \lambda^n}{n!} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} \\ &= e^{-\lambda} e^{\lambda e^t} \\ &= \exp\{\lambda(e^t - 1)\} \end{aligned}$$

(c) $M_X(t) = E\left[e^{tX}\right] = \sum_{k=r}^{\infty} \binom{k-1}{r-1} p^r (1-p)^{k-r} e^{tk} = p^r e^{tr} \sum_{k=r}^{\infty} \binom{k-1}{r-1} \left((1-p)e^t\right)^{k-r} = \left(\frac{pe^t}{1-(1-p)e^t}\right)^r$

d **Normal distribution**

We first compute the moment generating function of a unit normal random variable with parameters 0 and 1. Letting Z be such a random variable, we have

$$\begin{aligned} M_Z(t) &= E[e^{tZ}] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x^2 - 2tx)}{2}\right\} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x - t)^2}{2} + \frac{t^2}{2}\right\} dx \\ &= e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2}\, dx \\ &= e^{t^2/2} \end{aligned}$$

Hence, the moment generating function of the unit normal random variable Z is given by $M_Z(t) = e^{t^2/2}$. To obtain the moment generating function of an arbitrary normal random variable, we recall (see Section 5.4) that $X = \mu + \sigma Z$ will have a normal distribution with parameters μ and σ^2 whenever Z is a unit normal random variable. Hence, the moment generating function of such a random variable is given by

$$
\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= E[e^{t(\mu+\sigma Z)}] \\
&= E[e^{t\mu} e^{t\sigma Z}] \\
&= e^{t\mu} E[e^{t\sigma Z}] \\
&= e^{t\mu} M_Z(t\sigma) \\
&= e^{t\mu} e^{(t\sigma)^2/2} \\
&= \exp\left\{\frac{\sigma^2 t^2}{2} + \mu t\right\}
\end{aligned}
$$

(e) $$M_X(t) = E\left[e^{tX}\right] = \int_a^b \frac{1}{b-a} e^{tx}\,dx = \begin{cases} \left.\dfrac{e^{tx}}{t(b-a)}\right|_a^b = \dfrac{e^{bt}-e^{at}}{t(b-a)} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

(f) $$
\begin{aligned}
M_X(t) &= E\left[e^{tX}\right] \\
&= \int_0^\infty \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} e^{tx}\,dx \\
&= \lambda^\alpha \int_0^\infty \frac{e^{-(\lambda-t)x} x^{\alpha-1}}{\Gamma(\alpha)}\,dx \\
&= \left(\frac{\lambda}{\lambda-t}\right)^\alpha \int_0^\infty \frac{(\lambda-t)e^{-(\lambda-t)x}\left((\lambda-t)x\right)^{\alpha-1}}{\Gamma(\alpha)}\,dx \\
&= \left(\frac{\lambda}{\lambda-t}\right)^\alpha \quad \text{for} \quad \lambda > t
\end{aligned}
$$

2. You are given the joint moment generating function of 2 random variables, X and Y:

$$M_{X,Y}(s,t) = \frac{1}{1-2s-3t+6st}$$

for $s < \frac{1}{2}$ and $t < \frac{1}{3}$. Find $P\left(\min(X,Y) > 0.95\right)$.

Solution

Recall that variables X and Y are independent if, and only if, their joint moment generating function is a product of individual MGF's of X and Y, where defined. Here we have

$$M_{X,Y}(s,t) = \frac{1}{1-2s-3t+6st} = \frac{1}{1-2s} \cdot \frac{1}{1-3t} = M_X(s) \cdot M_Y(t).$$

Therefore, X and Y are independent. Now also recall that the MGF of exponential distribution with mean μ and hazard rate $\lambda = \frac{1}{\mu}$ is $\frac{1}{1-\mu t} = \frac{\lambda}{\lambda - t}$. This tells us that X is exponential with mean 2, and Y is exponential with mean 3. Based on this, we conclude that

$$\Pr\left(\min(X,Y) > 0.95\right) = \Pr\left(\{X > 0.95\} \cap \{Y > 0.95\}\right) =$$

$$= \Pr(X > 0.95) \cdot \Pr(Y > 0.95) = e^{-\frac{0.95}{2}} \cdot e^{-\frac{0.95}{3}} \approx 0.4531.$$

3. Let X_1 and X_2 be random variables with joint moment generating function

$$M_{X_1,X_2}(t_1,t_2) = 0.3 + 0.1e^{t_1} + 0.2e^{t_2} + 0.4e^{t_1+t_2}.$$

What is $E[2X_1 - X_2]$?

Solution

$$E(2X_1 - X_2) = 2E(X_1) - E(X_2) = 2\left.\frac{\partial M_{X_1,X_2}(t_1,t_2)}{\partial t_1}\right|_{t_1=t_2=0} - \left.\frac{\partial M_{X_1,X_2}(t_1,t_2)}{\partial t_2}\right|_{t_1=t_2=0} =$$

$$= 2\left.\left(0.1e^{t_1} + 0.4e^{t_1+t_2}\right)\right|_{t_1=t_2=0} - \left.\left(0.2e^{t_2} + 0.4e^{t_1+t_2}\right)\right|_{t_1=t_2=0} = 2 \cdot 0.5 - 0.6 = 0.4.$$

4. Let X and Y be identically distributed independent random variables such that the moment generating function of $X + Y$ is

$$M(t) = 0.09e^{-2t} + 0.24e^{-t} + 0.34 + 0.24e^{t} + 0.09e^{2t}, -\infty < t < \infty.$$

Calculate $P[X \leq 0]$.

Solution

4. Because X and Y are independent and identically distributed, the moment generating function of $X + Y$ equals $K^2(t)$, where $K(t)$ is the moment generating function common to X and Y. Thus, $K(t) = 0.3e^{-t} + 0.4 + 0.3e^t$. This is the moment generating function of a discrete random variable that assumes the values −1, 0, and 1 with respective probabilities 0.3, 0.4, and 0.3. The value we seek is thus 0.3 + 0.4 = 0.7.

5. Let X_1 , X_2, $\cdots$, X_n be i.i.d. $Exponential(\lambda)$ random variables with $\lambda = 1$. Let

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

How large n should be such that

$$P\left(0.9 \leq \overline{X} \leq 1.1\right) \geq 0.95?$$

Solution

Let $Y = X_1 + X_2 + \cdots + X_n$, so $\overline{X} = \frac{Y}{n}$. Since $X_i \sim Exponential(1)$, we have

$$E(X_i) = \frac{1}{\lambda} = 1, \qquad \mathrm{Var}(X_i) = \frac{1}{\lambda^2} = 1.$$

Therefore,

$$E(Y) = nEX_i = n, \qquad \mathrm{Var}(Y) = n\mathrm{Var}(X_i) = n,$$

$$
\begin{aligned}
P(0.9 \le \overline{X} \le 1.1) &= P\left(0.9 \le \frac{Y}{n} \le 1.1\right) \\
&= P(0.9n \le Y \le 1.1n) \\
&= P\left(\frac{0.9n - n}{\sqrt{n}} \le \frac{Y-n}{\sqrt{n}} \le \frac{1.1n-n}{\sqrt{n}}\right) \\
&= P\left(-0.1\sqrt{n} \le \frac{Y-n}{\sqrt{n}} \le 0.1\sqrt{n}\right).
\end{aligned}
$$

By the CLT $\frac{Y-n}{\sqrt{n}}$ is approximately $N(0,1)$, so

$$
\begin{aligned}
P(0.9 \le \overline{X} \le 1.1) &\approx \Phi\left(0.1\sqrt{n}\right) - \Phi\left(-0.1\sqrt{n}\right) \\
&= 2\Phi\left(0.1\sqrt{n}\right) - 1 \quad (\text{since} \quad \Phi(-x) = 1 - \Phi(x)).
\end{aligned}
$$

We need to have

$$
2\Phi\left(0.1\sqrt{n}\right) - 1 \ge 0.95, \qquad \text{so} \quad \Phi\left(0.1\sqrt{n}\right) \ge 0.975.
$$

Thus,

$$
\begin{aligned}
0.1\sqrt{n} &\ge \Phi^{-1}(0.975) = 1.96 \\
\sqrt{n} &\ge 19.6 \\
n &\ge 384.16
\end{aligned}
$$

Since n is an integer, we conclude $n \ge 385$.

6. Let X and Y be the number of hours that a randomly selected person watches movies and sporting events, respectively, during a three-month period. The following information is known about X and Y:

$$E[X] = 50,\ E[Y] = 20,\ Var[X] = 50,\ Var[Y] = 30,\ Cov(X, Y) = 10.$$

The totals of hours that different individuals watch movies and sporting events during the three months are mutually independent.

One hundred people are randomly selected and observed for these three months. Let T be the total number of hours that these one hundred people watch movies or sporting events during this three-month period.

Approximate the value of $P[T < 7100]$.

Solution

Observe that (where Z is total hours for a randomly selected person)
$E[Z] = E[X+Y] = E[X] + E[Y] = 50 + 20 = 70$
$Var[Z] = Var[X+Y] = Var[X] + Var[Y] + 2Cov[X,Y] = 50 + 30 + 20 = 100.$
It then follows from the Central Limit Theorem that T is approximately normal with mean 100(70) = 7000 and variance 100(100) = 10,000 and standard deviation 100. The probability of being less than 7100 is the probability that a standard normal variable is less than (7100 – 7000)/100 = 1. From the tables, this is 0.8413.

7. Let $X_1, \cdots, X_{100}$ and $Y_1, \cdots, Y_{100}$ be independent random samples from uniform distributions on the intervals $[-10\sqrt{3}, 10\sqrt{3}]$ and $[-30, 30]$, respectively. According to the Central Limit Theorem, what is the approximate value of $P\left(\bar{X} - \bar{Y} < 1\right)$?

Solution

The mean of each of the X's is 0, and the variance of each of them is $\frac{\left(20\sqrt{3}\right)^2}{12} = 100$. The mean of each of the Y's is 0 and the variance of each of them is $\frac{60^2}{12} = 5 \cdot 60 = 300$. Therefore, the mean of $\bar{X}$ is 0 and the variance of it is 1, while the mean of $\bar{Y}$ is 0 and its variance is 3. The random variable $\bar{Y} - \bar{X}$ can be expressed as a sum of IID random variables in the following way:

$$\bar{Y} - \bar{X} = \frac{Y_1 - X_1}{100} + \frac{Y_2 - X_2}{100} + \ldots + \frac{Y_{100} - X_{100}}{100}.$$

and by the Central Limit Theorem, $\bar{Y} - \bar{X}$ can be approximated by a normal random variable W with the same mean and variance as $\bar{Y} - \bar{X}$, i.e., $W \sim N(0,4)$. Hence, if we denote a standard normal random variable by Z, we have:

$$\Pr\left(\bar{Y} - \bar{X} < 1\right) = \Pr(W < 1) = \Pr\left(\frac{W-0}{2} < \frac{1}{2}\right) = \Pr\left(Z < \frac{1}{2}\right) = 0.6915.$$