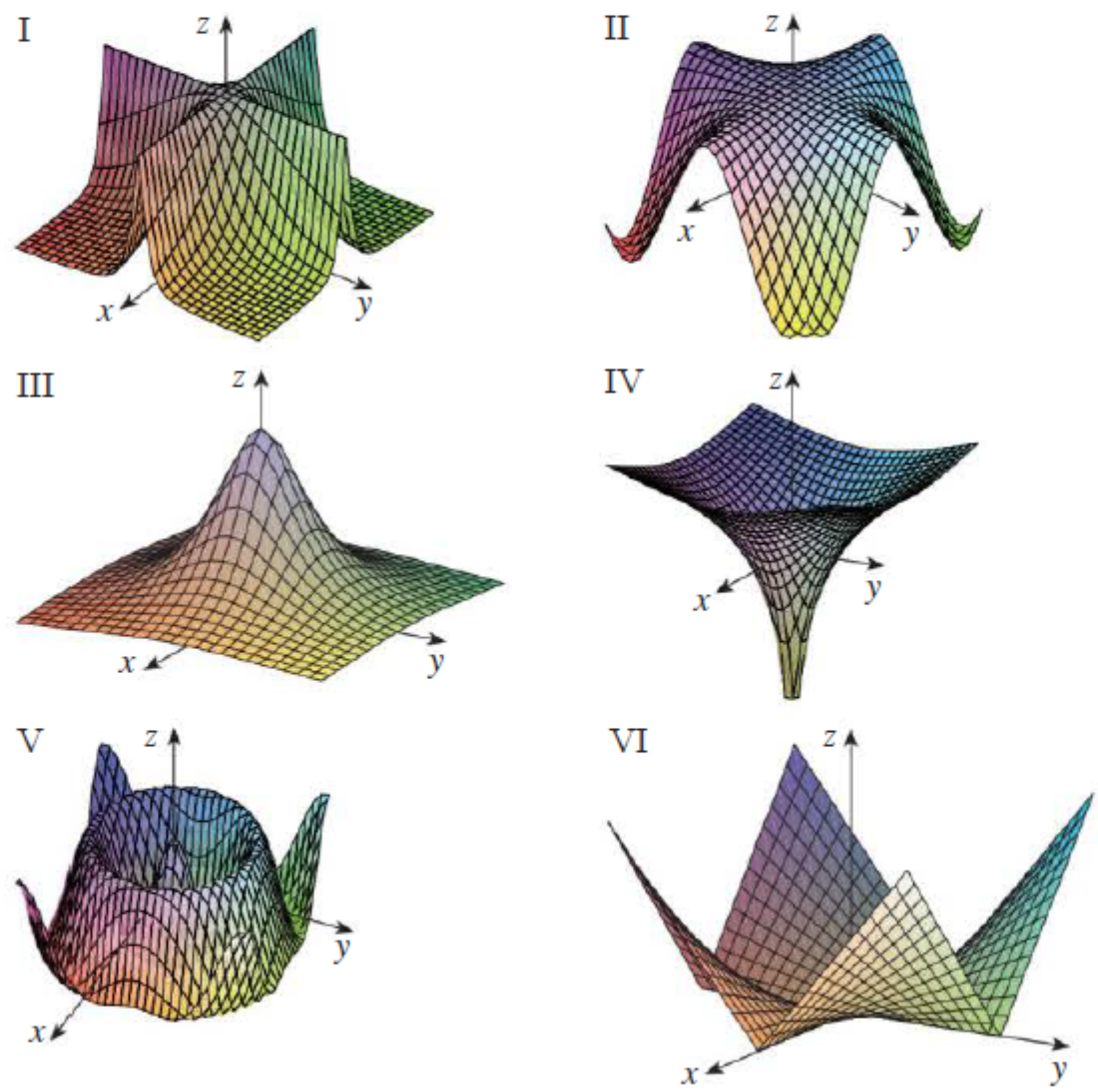
2021-22 First Semester
MATH1083 Calculus II (1002&1003)

Assignment 8

Due Date: 11:30am 10/Mar/2021(Mon). [Please pay attention to the deadline]

- Write down your **Chinese name** and **student number**. Write neatly on **A4-sized** paper and **show your steps**.
- **Late submissions or answers without details will not be graded.**

1. Match the function with its graph.



(a)
$$f(x,y) = \frac{1}{1+x^2+y^2}$$
$x \to 0, y \to 0$ $f(x,y) \to 1$ III
$x = 0$ $y \to \infty$ $f(x,y) = 0$

(b)
$$f(x,y) = \frac{1}{1+x^2y^2}$$
$x \to 0, y \to 0$ $f(x,y) \to 1$ I
$x = 0$ $y \to \infty$ $f(x,y) = 1$

(c)
$$\ln(x^2+y^2)$$ $x\to 0, y\to 0$ $f(x,y)\to -\infty$ IV

(d)
$$\cos\sqrt{x^2+y^2}$$ $x\to 0$ $y\to 0$ $f(x,y)\to 1$ V

$x=0$ $f(x,y)=\cos y$

$y=0$ $f(x,y)=\cos x$

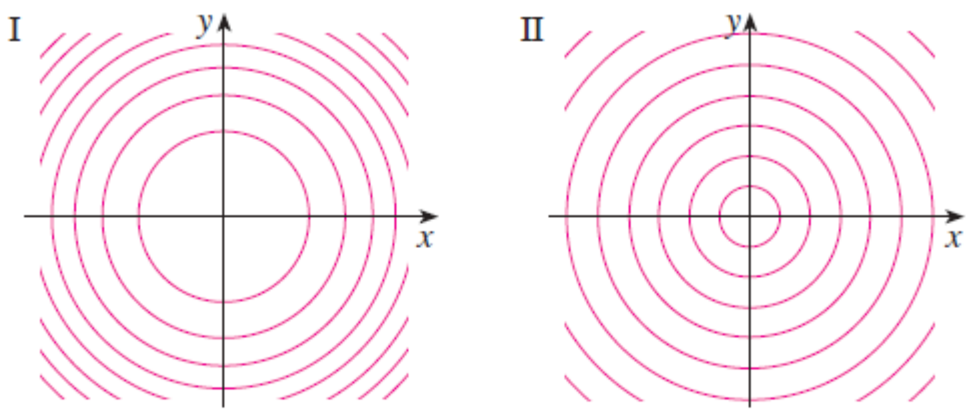
(e)
$f(x,y)\geqslant 0$ VI $$f(x,y)=|xy|$$

(f)
$$\cos(xy)$$ $x=0$ $y\to\infty$ $f(x,y)=1$ II

$y=0$ $x\to\infty$ $f(x,y)=1$

Solution: I (b),II (f), III (a), IV (c), V(d), VI (e)

2. Two contour maps are shown. One is for a function f whose graph is a cone. The other is for a function g whose graph is a paraboloid. Which one is which and why?



Solution: I, paraboloid, II, cone. For a paraboloid, the slope is increasing (steeper), so the gaps between the level curves are getting denser. While for the slope of a cone is a constant, so the gaps between the level curves are equidistant.

3. Discuss the continuity of the function

$$f(x,y)=\begin{cases}\frac{\sin xy}{xy} & xy\neq 0\\ 1 & xy=0\end{cases}$$

We can let $z=xy$

$$\lim_{(x,y)\to(0,0)}\frac{\sin xy}{xy}=\lim_{z\to 0}\frac{\sin z}{z}=1$$

so $f(x,y)$ is continuous on $\mathbb{R}^2$.

4. Find the indicated partial derivative

(a)
$$R(s,t)=te^{s/t}; \qquad R_t(0,1)$$

(b)
$$f(x,y,z)=x^{yz}, \qquad f_z(e,1,0)$$

Solution (a) $R_t(s,t)=e^{s/t}-\frac{s}{t}e^{s/t}$ and $R_t(0,1)=1$

(b) $f_z=x^{yz}y\ln x$ and $f_z(e,1,0)=e^0 1\ln e=1$

5. Find all the second partial derivatives

(a) $f(x,y)=\ln(ax+by)$

Solution:

$$f_x = \frac{a}{ax+by}, \qquad f_y = \frac{b}{ax+by}$$

$$f_{xx} = \frac{-a^2}{(ax+by)^2}, \qquad f_{xy} = \frac{-ab}{(ax+by)^2} \qquad f_{yy} = \frac{-b^2}{(ax+by)^2}$$

6. Verify that the conclusion of Clairaut's Theorem holds, that is $u_{xy} = u_{yx}$, $u(x,y) = \cos(x^2y)$

   Solution: $u_x = -2xy\sin\left(x^2y\right)$ and $u_y = -x^2\sin\left(x^2y\right)$, then

   $$u_{xy} = -2x\sin\left(x^2y\right) - 2x^3y\cos\left(x^2y\right)$$

   and

   $$u_{yx} = -2x\sin\left(x^2y\right) - 2x^3y\cos\left(x^2y\right)$$

   both u_{xy} and u_{yx} are continuous and $u_{xy} = u_{yx}$.

7. Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$ for

   $$x^2 + 2y^2 + 3z^2 = 1$$

   Solution: Treat y as a constant and z as a function of x, we differentiate implicitly with respect to x

   $$2x + 6z\frac{\partial z}{\partial x} = 0$$

   so $\frac{\partial z}{\partial x} = -\frac{x}{3z}$. differentiate implicitly with respect to y

   $$4y + 6z\frac{\partial z}{\partial y} = 0$$

   $\frac{\partial z}{\partial x} = -\frac{2y}{3z}$.

8. Assume the third partial derivatives of f are continuous, $f_{xyz} = f_{yxz}$

$f_y = 2xyz^3$ $f_{yx} = 2yz^3$ $f_{yxz} = 6yz^2$

8. If $f(x,y,z) = xy^2z^3 + \arcsin(x\sqrt{z})$, find f_{xyz} in the easist order.

9. If

   $$u = e^{a_1x_1 + a_2x_2 + a_3x_3}$$

   where $a_1^2 + a_2^2 + a_3^2 = 1$, show that

   $$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = u$$

   Solution: for $i = 1,2,3$

   $$\frac{\partial u}{\partial x_i} = a_i e^{a_1x_1 + a_2x_2 + a_3x_3}$$

   and

   $$\frac{\partial^2 u}{\partial x_i^2} = a_i^2 e^{a_1x_1 + a_2x_2 + a_3x_3}$$

   so

   $$\sum_{i=1}^{3}\frac{\partial^2 u}{\partial x_i^2} = e^{a_1x_1 + a_2x_2 + a_3x_3}\sum_{i=1}^{3} a_i^2 = e^{a_1x_1 + a_2x_2 + a_3x_3} = u$$

10. Find an equation of the tangent plane to the given surface at the specified point

    (a) $z = (x+2)^2 - 2(y-1)^2 - 5$, $(2,3,3)$

    (b) $z = \frac{x}{y^2}$, $(-4,2,-1)$

Solution: (a) $f_x = 2(x+2)$ and $f_x(2,3) = 8$, $f_y = -4(y-1)$ and $f_y(2,3) = -8$, the tangent plane

$$z - 3 = 8(x-2) - 8(y-3)$$

so

$$z = 8x - 8y + 11$$

(b) $f_x = \frac{1}{y^2}$ and $f_x(-4,2) = 1/4$, $f_y = -\frac{2x}{y^3}$ and $f_y(-4,2) = 1$, so the tangent plane

$$z + 1 = \frac{1}{4}(x+4) + (y-2)$$

理解看懂即可

11. Prove that if f is a function of two variables that is differentiable at (a,b), then f is continuous at (a,b)

Proof: If $f(x,y)$ is differentiable at (a,b) which means f_x and f_y both exist, and

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \epsilon_1(x-a) + \epsilon_2(y-b)$$

with $h,k \to 0$, we have $\epsilon_1, \epsilon_2 \to 0$. So we can prove using $\epsilon - \delta$ definition

$$\begin{aligned} f(x,y) - f(a,b) &= f_x(a,b)(x-a) + f_y(a,b)(y-b) + \epsilon_1(x-a) + \epsilon_2(y-b) \\ |f(x,y) - f(a,b)| &\le M\left(|x-a| + |y-b|\right) + \epsilon_1 |x-a| + \epsilon_2 |y-b| \end{aligned}$$

where $M = \max\{|f_x(a,b)|, |f_y(a,b)|\}$ that is
$\forall \epsilon > 0$, $\exists \delta < \frac{\epsilon}{2M+2}$, when $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ which implies $|x-a| < \delta$ and $|y-b| < \delta$

$$|f(x,y) - f(a,b)| \le M \cdot 2\delta + \epsilon_1\delta + \epsilon_2\delta = \delta(2M + \epsilon_1 + \epsilon_2) < \epsilon$$

so $f(x,y)$ is continuous at (a,b).

12. Find the linearization $L(x,y)$ of the function $f(x,y) = y + \sin(x/y)$ at the point $(0,3)$.

Solution: $f_x = \frac{1}{y}\cos\frac{x}{y}$ and $f_x(0,3) = \frac{1}{3}$, $f_y = 1 - \frac{x}{y^2}\cos\frac{x}{y}$ and $f(0,3) = 1$, so the linear function

$$\begin{aligned} L(x,y) &= f(0,3) + \frac{1}{3}x + (y-3) \\ &= \frac{1}{3}x + y \end{aligned}$$

13. Find the linear approximation of the function $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ at point $(3,2,6)$ and use it to approximate the number $\sqrt{3.02^2 + 1.97^2 + 5.99^2}$

Solution:

$$f_x = \frac{x}{\sqrt{x^2+y^2+z^2}}, \qquad f_y = \frac{y}{\sqrt{x^2+y^2+z^2}}, \qquad f_z = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

and $f_x(3,2,6) = \frac{3}{7}$, $f_y(3,2,6) = \frac{2}{7}$ and $f_z(3,2,6) = \frac{6}{7}$, so the linear function

$$\begin{aligned} L(x,y,z) &= f(3,2,6) + f_x(3,2,6)(x-3) + f_y(3,2,6)(y-2) + f_z(3,2,6)(z-6) \\ &= 7 + \frac{3}{7}(x-3) + \frac{2}{7}(y-2) + \frac{6}{7}(z-6) \end{aligned}$$

so

$$\begin{aligned} \sqrt{3.02^2 + 1.97^2 + 5.99^2} &\approx L(3.02, 1.97, 5.99) \\ &= 7 + \frac{3}{7}0.02 - \frac{2}{7}0.03 - \frac{6}{7}0.01 \\ &= 7 - \frac{6}{7}0.01 \end{aligned}$$

14. Use Chain Rule to find dz/dt

$$z = \sqrt{1+xy}, \qquad x = \tan t, \quad y = \arctan t$$

Solution:

$$\frac{dz}{dx} = \frac{y}{2\sqrt{1+xy}} = \frac{\arctan t}{2\sqrt{1+\tan t \arctan t}}, \qquad \frac{dz}{dy} = \frac{\tan t}{2\sqrt{1+\tan t \arctan t}}$$

and

$$\frac{dx}{dt} = \frac{1}{\cos^2 t} \qquad \frac{dy}{dt} = \frac{1}{1+t^2}$$

$$\begin{aligned}\frac{dz}{dt} &= \frac{dz}{dx}\cdot\frac{dx}{dt} + \frac{dz}{dy}\cdot\frac{dy}{dt}\\ &= \frac{\arctan t}{2\sqrt{1+\tan t \arctan t}}\cdot\frac{1}{\cos^2 t} + \frac{\tan t}{2\sqrt{1+\tan t \arctan t}}\cdot\frac{1}{1+t^2}\end{aligned}$$

15. Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$

$$z = \frac{\sin\theta}{r} \qquad r = st, \quad \theta = s^2+t^2$$

Solution:

$$\frac{dz}{dr} = -\frac{\sin\theta}{r^2} = -\frac{\sin\left(s^2+t^2\right)}{s^2t^2}, \qquad \frac{dz}{d\theta} = \frac{\cos\theta}{r} = \frac{\cos\left(s^2+t^2\right)}{st}$$

$$\frac{dr}{ds} = t, \quad \frac{dr}{dt} = s, \quad \frac{d\theta}{ds} = 2s, \quad \frac{d\theta}{dt} = 2t,$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{dz}{dr}\cdot\frac{dr}{dt} + \frac{dz}{d\theta}\cdot\frac{d\theta}{dt}\\ &= -\frac{\sin\left(s^2+t^2\right)}{s^2t^2}\cdot s + \frac{\cos\left(s^2+t^2\right)}{st}\cdot 2t\\ &= \frac{2st^2\cos\left(s^2+t^2\right) - s\sin\left(s^2+t^2\right)}{s^2t^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{dz}{dr}\cdot\frac{dr}{ds} + \frac{dz}{d\theta}\cdot\frac{d\theta}{ds}\\ &= -\frac{\sin\left(s^2+t^2\right)}{s^2t^2}\cdot t + \frac{\cos\left(s^2+t^2\right)}{st}\cdot 2s\\ &= \frac{2s^2t\cos\left(s^2+t^2\right) - t\sin\left(s^2+t^2\right)}{s^2t^2}\end{aligned}$$

16. If $z = f(x,y)$ where $x = r\cos\theta$, $y = r\sin\theta$

(a) Find $\partial z/\partial r$ and $\partial z/\partial\theta$

(b) Show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial\theta}\right)^2$$

17. Find the directional derivative of $f = \sqrt{2x+3y}$ at the given point $(3,1)$ in the direction indicated by the angle $\theta = -\frac{\pi}{6}$.

Solution: first compute the gradient vector

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{1}{\sqrt{2x+3y}}\vec{i} + \frac{3}{2\sqrt{2x+3y}}\vec{j}$$

$$\nabla f(3,1) = \frac{1}{3}\vec{i} + \frac{1}{2}\vec{j}$$

the direction $\overrightarrow{u} = \cos\theta \overrightarrow{i} + \sin\theta \overrightarrow{j} = \frac{\sqrt{3}}{2}\overrightarrow{i} + \frac{1}{2}\overrightarrow{j}$, so

$$D_{\overrightarrow{u}} f(x,y) = \nabla f(3,1) \cdot \overrightarrow{u} = \left(\frac{1}{3}\overrightarrow{i} + \frac{1}{2}\overrightarrow{j}\right) \cdot \left(\frac{\sqrt{3}}{2}\overrightarrow{i} + \frac{1}{2}\overrightarrow{j}\right) = \frac{2\sqrt{3}+3}{12}$$

18. For the function $f(x,y) = x^2 e^y$

   (a) Find the gradient of f.

   (b) Evaluate the gradient at point $P(3,0)$

   (c) Find the rate of change of f at P in the direction of the vector $\overrightarrow{u} = \frac{1}{5}(3\overrightarrow{i} - 4\overrightarrow{j})$
   Solution: (a)

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = 2xe^y \overrightarrow{i} + x^2 e^y \overrightarrow{j}$$

   (b)

$$\nabla f(3,0) = 6\overrightarrow{i} + 9\overrightarrow{j}$$

   (c) the rate of change in the direction

$$|D_{\overrightarrow{u}} f(x,y)| = |\nabla f(3,0) \cdot \overrightarrow{u}| = \left|\left(6\overrightarrow{i} + 9\overrightarrow{j}\right) \cdot \frac{1}{5}(3\overrightarrow{i} - 4\overrightarrow{j})\right| = \frac{18}{5}$$