△ Hypothesis Testing

Step 1: State the null hypothesis H_0 & H_1

$H_0: \theta = \theta_0$ ($\theta \geq \theta_0$) v.s. $H_1: \theta < \theta_0$ — left-tail Test

$\theta \leq \theta_0$ — $H_1: \theta > \theta_0$ — Right-tail Test } One tail Test

$\theta = \theta_0$ — $H_1: \theta \neq \theta_0$ — Two-sided Test

Step 2: Determine the significance level α.

$\alpha = P(\text{Type I error}) = P(\text{Reject } H_0 \mid H_0 \text{ is true})$

e.g. α taken to be 0.1, 0.05, 0.01, ...

Step 3: Determine the $\underbrace{\text{test statistic}}_{\text{estimator of } \theta}$ and its sampling distribution

↑ closely related

E.g. Sometimes people prefer to standardize the test statistics.

$$\text{Test statistic} = \frac{\text{Observed value} - \text{Expected value}}{\text{Standard error}}$$

Step 4: Determine the Decision Rule (with rejection region(s) & critical value(s))

Reject H_0, if my observation is too extreme.

The rejection region and critical value w^* depend on α, two-tails test / one-tail test

$$C = \{ W > w^* \}$$

or $$C = \{ W < w^* \}$$

or $$C = \{ W > w_1^* \} \cup \{ W < w_2^* \}$$

Step 5: Collect sample data and compute the value of the test statistic

Step 6: To make decisions, we may use either { Critical value Method / p-value method.

p-value: the prob. of observing more extreme observations than the current sample.

Step 7: State "reject H_0" or "Fail to reject H_0" based on observation and interpret the result.

△ Testing on Population Mean μ

$H_0: \mu = \mu_0$ ($\mu \geq \mu_0$, $\leq$)

Assume (Normal population) $H_1: \mu \neq \mu_0$ ($\mu < \mu_0$, $>$)

↳ ① When population variance is known (σ^2 known)

Since $\bar{X} = \frac{1}{n}\sum X_i \overset{\text{exactly}}{\sim} N(\mu, \frac{\sigma^2}{n})$,

then $Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$ follows $N(0,1)$

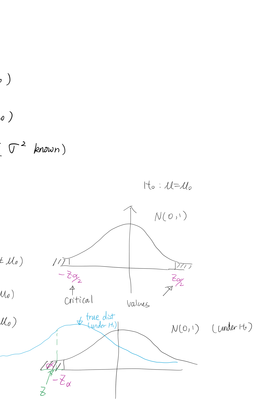
Reject H_0 if

① $Z > \boxed{Z_{\frac{\alpha}{2}}}$ or $Z < \boxed{-Z_{\frac{\alpha}{2}}}$ ($H_1: \mu \neq \mu_0$)

② $Z < \boxed{-Z_\alpha}$ ($H_1: \mu < \mu_0$)

③ $Z > \boxed{Z_\alpha}$ ($H_1: \mu > \mu_0$)

critical values



↳ ② When σ^2 unknown

we need to estimate the unknown σ^2 in

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

by $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$, the sample variance.

Under normality assumption about the population

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

As a result:

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \frac{\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)}{\sqrt{(n-1)S^2/\sigma^2(n-1)}} = \frac{Z}{\sqrt{U/\nu}} = T \sim t \text{ distribution with df} = \nu$$

where $U \sim \chi^2(\nu)$

Critical value method:

Reject H_0 if

① $t > t_{\frac{\alpha}{2}, n-1}$ or $t < -t_{\frac{\alpha}{2}, n-1}$ ($H_1: \mu \neq \mu_0$)

② $t > t_{\alpha, n-1}$ ($H_1: \mu > \mu_0$)

③ $t < -t_{\alpha, n-1}$ ($H_1: \mu < \mu_0$)

p-value method:

Reject H_0 if

① $P(T > |t|) + P(T < -|t|) < \alpha$ ($H_1: \mu \neq \mu_0$)

② $P(T > |t|) < \alpha$ ($H_1: \mu > \mu_0$)

the realization of test statistic

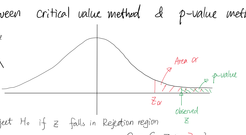
③ $P(T < -|t|) < \alpha$ ($H_1: \mu < \mu_0$)

△ Relationship between critical value method & p-value method

Take $N(0,1)$ as example

Z-test

one-sided



Reject H_0 if Z falls in Rejection region $C = \{ Z > Z_\alpha \}$

or p-value $< \alpha$

p-value $= P(Z > |z|) < \alpha$

Reject H_0 if we are observing an extremely rare event

Remarks (on Z-test): In practice, for a large sample, we want to test on $\mu = \mu_0$. even without normality assumption on population, we can use CLT

$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$ is approximately normally distributed.

△ Testing Binomial Data on $H_0: p = p_0$

$$\boxed{0 < np_0 - 3\sqrt{np_0(1-p_0)} < np_0 + 3\sqrt{np_0(1-p_0)} < n}$$

if true ⟷ "Large" sample

otherwise "Small" sample.

↳ For "Large" Sample Test: Z-test

Use CLT. $X = \sum_{i=1}^{n} Y_i$, $Y_i \sim \text{Bernoulli}(p)$

then $\bar{Y} = \frac{X}{n} = \frac{1}{n}\sum_{i=1}^{n} Y_i \overset{\text{approx}}{\sim} N(p, \frac{p(1-p)}{n})$

By CLT, an approximate Z-test statistic is

$$Z = \frac{\bar{Y} - p}{\sqrt{p(1-p)/n}} = \frac{X - np_0}{\sqrt{np_0(1-p_0)}} \overset{\text{approx}}{\sim} N(0,1)$$

Reject H_0 if

① $Z > Z_{\alpha/2}$ or $Z < -Z_{\alpha/2}$ ($H_1: p \neq p_0$)

② $Z > Z_\alpha$ ($H_1: p > p_0$)

③ $Z < -Z_\alpha$ ($H_1: p < p_0$)

↳ For "Small" Sample Test: Binomial Test

Unable to use normal approximation, use binomial distribution instead

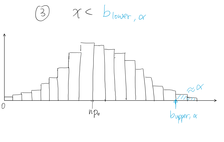
Test statistic:

$$X = \sum_{i=1}^{n} Y_i \overset{\text{exactly}}{\sim} \text{Binom}(n, p)$$

Reject H_0 if

① $x > b_{\text{upper}, \alpha/2}$ or $x < b_{\text{lower}, \alpha/2}$ ($H_1: p \neq p_0$)

where $P(X > b_{\text{upper}, \alpha/2}) \approx \alpha/2$ and $P(X < b_{\text{lower}, \alpha/2}) \approx \alpha/2$

② $x > b_{\text{upper}, \alpha}$ ($H_1: p > p_0$)

③ $x < b_{\text{lower}, \alpha}$ ($H_1: p < p_0$)