Linear Algebra II, 2023 spring
Midterm Exam Suggested Answers

1. (a) For any $\alpha, \beta \in \mathbb{R}$ and any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$,

$$L(\alpha\mathbf{x}+\beta\mathbf{y}) = \begin{bmatrix}2(\alpha x_2 + \beta y_2), -(\alpha x_1 + \beta y_1)\end{bmatrix} = \alpha\begin{bmatrix}2x_2, -x_1\end{bmatrix} + \beta\begin{bmatrix}2y_2, -y_1\end{bmatrix} = \alpha L(\mathbf{x}) + \beta L(\mathbf{y}).$$

Thus, L is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.

(b) Since

$$L(\mathbf{u}_1) = \begin{bmatrix}0\\-1\end{bmatrix} = -\frac{2}{3}\mathbf{b}_1 + \frac{1}{3}\mathbf{b}_2, \qquad L(\mathbf{u}_2) = \begin{bmatrix}4\\-1\end{bmatrix} = \frac{2}{3}\mathbf{b}_1 + \frac{5}{3}\mathbf{b}_2$$

$$L(\mathbf{u}_3) = \begin{bmatrix}2\\1\end{bmatrix} = \frac{4}{3}\mathbf{b}_1 + \frac{1}{3}\mathbf{b}_2$$

Then

$$[L]_E^F = \frac{1}{3}\begin{bmatrix}-2 & 2 & 4\\ 1 & 5 & 1\end{bmatrix}.$$

2. (a) The rank of A is 2, then

$$\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix}2\\1\end{bmatrix},$$

where

$$A^TA = \begin{bmatrix}6 & -3\\ -3 & 6\end{bmatrix}, \quad (A^TA)^{-1} = \frac{1}{9}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}, \quad A^T\mathbf{b} = \begin{bmatrix}9\\0\end{bmatrix}.$$

(b) Since $\mathrm{Col}(A)^\perp = \mathrm{N}(A^T)$, then

$$\mathrm{rref}(A^T) = \mathrm{rref}\begin{bmatrix}1 & 2 & -1 & 0\\ 0 & -1 & 1 & 2\end{bmatrix} = \begin{bmatrix}1 & 0 & 1 & 4\\ 0 & 1 & -1 & -2\end{bmatrix} \quad \rightarrow \quad \mathrm{N}(A^T) = \mathrm{span}\left\{\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}-4\\2\\0\\1\end{bmatrix}\right\}$$

(c) Using the Gram-Schmidt Process, let $\mathbf{v}_1 = \begin{bmatrix}-1 & 1 & 1 & 0\end{bmatrix}^T$, then

$$\mathbf{v}_2 = \begin{bmatrix}-4\\2\\0\\1\end{bmatrix} - \frac{\begin{pmatrix}-4 & 2 & 0 & 1\end{pmatrix}\begin{pmatrix}-1\\1\\1\\0\end{pmatrix}}{\begin{pmatrix}-1 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}-1\\1\\1\\0\end{pmatrix}}\begin{bmatrix}-1\\1\\1\\0\end{bmatrix} = \begin{bmatrix}-4\\2\\0\\1\end{bmatrix} - \frac{6}{3}\begin{bmatrix}-1\\1\\1\\0\end{bmatrix} = \begin{bmatrix}-2\\0\\-2\\1\end{bmatrix}.$$

Normalizing $\mathbf{v}_1$ and $\mathbf{v}_2$, we have an orthonormal basis for $\mathrm{N}(A^T)$ as $\left\{\frac{1}{3}\begin{pmatrix}-1\\1\\1\\0\end{pmatrix}, \frac{1}{3}\begin{pmatrix}-4\\2\\0\\1\end{pmatrix}\right\}$.

3. The eigenvalues of A are $\lambda_{1,2} = 1$ and $\lambda_3 = 3$ since

$$\det(A - \lambda I_3) = (2-\lambda)(1-\lambda)(2-\lambda) + (-1)\begin{vmatrix} 0 & 1-\lambda \\ -1 & 0 \end{vmatrix} = (1-\lambda)[(2-\lambda)^2 - 1] = (1-\lambda)^2(3-\lambda) = 0.$$

For $\lambda = 1$, $(A - I_3)\mathbf{v}_1 = \mathbf{0}$, i.e. $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \mathbf{v}_1 = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$,

α, β are not all zero.

For $\lambda = 3$, $(A - 3I_3)\mathbf{v}_3 = \mathbf{0}$, i.e. $\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \mathbf{v}_3 = \gamma \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\gamma \neq 0$.

4. Let θ be the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$, and α be the angle between $\mathbf{x}$ and $\mathbf{y}$, then

$$\cos\theta = \frac{\langle Q\mathbf{x}, Q\mathbf{y}\rangle}{\|Q\mathbf{x}\|\|Q\mathbf{y}\|} = \frac{\mathbf{x}^T Q^T Q\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \cos\alpha,$$

since Q is orthogonal and $Q^TQ = I$, then $\|Q\mathbf{z}\| = \sqrt{\mathbf{z}^T Q^T Q\mathbf{z}} = \sqrt{\mathbf{z}^T\mathbf{z}} = \|\mathbf{z}\|$, $\forall \mathbf{z} \in \mathbb{R}^n$.

5. (a) $P^T = (A(A^TA)^{-1}A^T)^T = (A^T)^T(A^TA)^{-T}A^T = A(A^TA)^{-1}A^T = P$.

(b) For every $\mathbf{b} \in \mathrm{Col}(A)$, $\mathbf{b} = A\mathbf{y}$ for some $\mathbf{y} \in \mathbb{R}^n$,

$$P\mathbf{b} = A(A^TA)^{-1}A^T\mathbf{b} = A(A^TA)^{-1}A^TA\mathbf{y} = A\mathbf{y} = \mathbf{b}.$$

Notice that P is the linear transformation of projecting a vector onto the column space of A, so any vector $\mathbf{b}$ in $\mathrm{Col}(A)$, it definitely will be projected to itself.

(c) For any $\mathbf{x} \in \mathrm{N}(A^T)$,

$$P\mathbf{x} = A(A^TA)^{-1}A^T\mathbf{x} = A(A^TA)^{-1}\left[A^T\mathbf{x}\right] = A(A^TA)^{-1}\mathbf{0} = \mathbf{0}, \quad \rightarrow \quad \mathbf{x} \in \mathrm{N}(P).$$

For any $\mathbf{y} \in \mathrm{N}(P)$,

$$A^T\mathbf{y} = A^TA(A^TA)^{-1}A^T\mathbf{y} = A^TP\mathbf{y} = \mathbf{0}, \quad \rightarrow \quad \mathbf{y} \in \mathrm{N}(A^T).$$

Thus, $\mathrm{N}(A^T) = \mathrm{N}(P)$.

(d) Based on the results in part (a) and (c),

$$\mathrm{Col}(A) = \mathrm{N}(A^T)^{\perp} = \mathrm{N}(P)^{\perp} = \mathrm{Col}(P^T) = \mathrm{Col}(P)$$

6. Assume that there exists a nonsingular matrix S such that $A = SBS^{-1}$, then

$$\det(A - \lambda I) = \det(SBS^{-1} - \lambda I) = \det(SBS^{-1} - \lambda SIS^{-1})$$
$$= \det[S(B - \lambda I)S^{-1}] = \det(S)\det(B - \lambda I)\det(S^{-1}) = \det(B - \lambda I)$$

Thus, A and B have the same eigenvalues since they have the same characteristic polynomials.