2023-24 First Semester

MATH2023 Ordinary and Partial Differential Equations (1002)

Assignment 2

Due Date: **8/Oct/2023(Sun), on or before 16:00, T3-401-R3.**

- Write down your **CHN name** and **student ID**. Write neatly on **A4-sized** paper (*staple if necessary*) and **show your steps**.
- **Late submissions or answers without details will not be graded.**

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1. Determine the values for constants a, b and c so that the differential equation is exact, then solve the equation
$$y' = \frac{by - ax}{bx - cy}.$$

2. Solve the given differential equation
   (a) $y + (2xy - e^{-2y})y' = 0.$
   (b) $(x+2)\sin(y) + x\cos(y)y' = 0$

3. Find the solution of the following initial value problem and determine the interval in which the solution is valid.
   (a) $(2x - y) - (4y + x)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0, \quad y(1) = 3.$
   (b) $(4 - t^2)y' + 2ty = 3t^2, \quad y(0) = 4.$
   (c) $y' = -4t/y, \quad y(0) = y_0.$

4. (*Optional!*)
   (a) Show that if $(N_x - M_y)/(xM - yN) = R$, where R depends on the quantity xy only, then the differential equation
$$M + Ny' = 0$$
   has an integrating factor of the form $\mu(xy)$. Find a general formula for $\mu(xy)$.
   (b) Use the method suggested in part (a) to find an integrating factor and solve the given equation.
$$\left(3x + \frac{6}{y}\right) + \left(\frac{x^2}{y} + 3\frac{y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

5. A retired person has a sum $S(t)$ invested so as to draw interest at an annual rate r compounded continuously. Withdrawals for living expenses are made at a rate of k dollars/year, $k \geq 0$; assume that the withdrawals are made continuously.

   (a) Write down the differential equation that governs this process.

   (b) If the initial value of the investment is S_0, determine $S(t)$ at any time.

   (c) Assuming that S_0 and r are fixed, determine the withdrawal rate k_0 at which $S(t)$ will remain constant.

   (d) Is the equilibrium solution you find in part (c) asymptotically stable or unstable? Briefly state why.

   (e) If k exceeds the value k_0 found in part (b), then $S(t)$ will decrease and ultimately become zero. Find the time T at which $S(t) = 0$.

   (f) Suppose that a person retiring with capital S_0 wishes to withdraw funds at an annual rate k for not more than T years. Determine the maximum possible rate of withdrawal.

   (g) How large an initial investment is required to permit an annual withdrawal of \$2,000 for 20 years, assuming an interest rate of 8%?

6. Assume that a population is governed by a differential equation.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(aP - b)$$

   where a and b are positive constants.

   (a) Find all the equilibrium solutions.

   (b) Sketch the graph of $f(y)$ versus y and classify each critical point you found in part (a) as asymptotically stable or unstable.

   (c) Discuss what happens to the population P as time t increases.

7. Consider the equation $\mathrm{d}y/\mathrm{d}t = f(y)$ and suppose that y_1 is a critical point, that is, $f(y_1) = 0$. Show that the constant equilibrium solution $\phi(t) = y_1$ is asymptotically stable if $f'(y_1) < 0$ and unstable if $f'(y_1) > 0$.