Chap5-Week5

2023/03/07 16:00

Chapter 5 Orthogonality

i. Inner product, length, distance, orthogonality, direct sum...

ii. Orthogonality

a. Orthogonal vectors

b. Orthogonal subspaces

c. Orthogonal sets

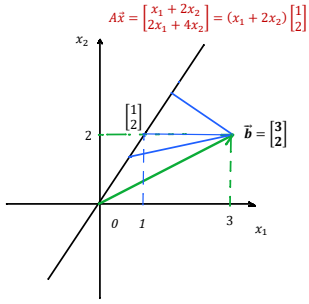
E.g.

$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{\vec{b}}{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}$$

$$A\vec{x} = \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix} = (x_1 + 2x_2)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

What can you say about the solution ?

No solution.



↳ Find an $\hat{x}$ s.t. $A\hat{x}$ is the "closest" to $\vec{b}$.

Q1. We need to define "distance"

Q2. How to find the "closest" $A\hat{x}$

Def 5.1.1 (Dot Product) or (Inner Product)

$\forall\ \vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \vec{x}^T \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \langle \vec{y}, \vec{x} \rangle$$

Def 5.1.2 (Euclidean Length)

The length of a vector $\vec{x} \in \mathbb{R}^n$ is

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{\vec{x}^T \vec{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Def 5.1.3 (Unit Vector)

A vector with length 1, that is, $\|\vec{u}\| = 1$.

Remark: Can you scale an arbitrary vector $\vec{x}$ to a unit vector in the same direction of $\vec{x}$?

Normalization: $\hat{x} = \frac{1}{\|\vec{x}\|}\vec{x}$, then $\|\hat{x}\| = 1$

E.g. 5.1.5

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

Then $\vec{x} + \vec{y} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\vec{x} - \vec{y} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

$$\vec{x} \cdot \vec{y} = (1\ 1\ 2)\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 1 \qquad = \vec{y}^T \vec{x}$$

$$\vec{x}\cdot\vec{y} = (1\ 1\ 2)\begin{pmatrix}2\\-1\\0\end{pmatrix} = 1 \qquad = \vec{y}^T\vec{x}$$

$$\vec{x}\cdot\vec{x} = 6 \qquad , \quad \vec{y}^T\vec{y} = (2\ -1\ 0)\begin{pmatrix}2\\-1\\0\end{pmatrix} = 5$$

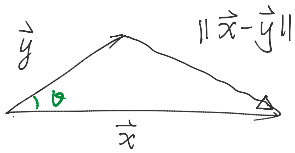
$$\|\vec{x}\| = \sqrt{\vec{x}^T\vec{x}} = \sqrt{6} \quad , \quad \|\vec{x}+\vec{y}\| = \sqrt{(3\ 0\ 2)\begin{pmatrix}3\\0\\2\end{pmatrix}} = \sqrt{13}$$

$$\|\vec{y}\| = \sqrt{5} \qquad \|\vec{x}-\vec{y}\| = \sqrt{(-1)^2+2^2+2^2} = 3$$

$$\hat{x} = \frac{1}{\|\vec{x}\|}\vec{x} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\2\end{pmatrix} \quad , \quad \hat{y} = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\-1\\0\end{pmatrix}$$

check: $\|\hat{x}\| = \sqrt{(\frac{1}{\sqrt{6}})^2+(\frac{1}{\sqrt{6}})^2+(\frac{2}{\sqrt{6}})^2} = \sqrt{\frac{1}{6}+\frac{1}{6}+\frac{4}{6}} = 1$. a unit vector.

<u>Def 5.1.4</u> (Distance between vectors in $\mathbb{R}^n$)



$\forall\ \vec{x}, \vec{y} \in \mathbb{R}^n$, the distance between $\vec{x}$ and $\vec{y}$ is

$$\|\vec{x}-\vec{y}\| = \sqrt{(\vec{x}-\vec{y})^T(\vec{x}-\vec{y})} = \sqrt{(\vec{x}^T-\vec{y}^T)(\vec{x}-\vec{y})}$$

$$= \sqrt{\vec{x}^T\vec{x}-\vec{x}^T\vec{y}-\vec{y}^T\vec{x}+\vec{y}^T\vec{y}} = \sqrt{\|\vec{x}\|^2-2\vec{x}\cdot\vec{y}+\|\vec{y}\|^2}$$

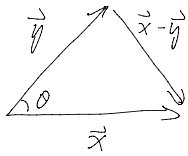
<u>Thm 5.1.6</u> (Angle) The angle between two nonzero vectors in $\mathbb{R}^n$

$$\cos\theta = \frac{\vec{x}^T\vec{y}}{\|\vec{x}\|\,\|\vec{y}\|} \quad , \quad 0 \le \theta \le \pi .$$

$\cos\theta = 0 \rightarrow \theta = \frac{\pi}{2}$

$\vec{x}^T\vec{y} = 0 \rightarrow$ $\vec{x}=\vec{0}$ or $\vec{y}=\vec{0}$ or $\vec{x}$ is perpendicular to $\vec{y}$. (orthogonal)

~~proof~~ : In the triangle formed by $\vec{x}$, $\vec{y}$, $\vec{x}-\vec{y}$, by the Law of Cosine.



$$\|\vec{x}-\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$$

$$\cancel{\|\vec{x}\|^2} - 2\vec{x}\cdot\vec{y} + \cancel{\|\vec{y}\|^2} = \cancel{\|\vec{x}\|^2} + \cancel{\|\vec{y}\|^2} - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$$

$$\vec{x}\cdot\vec{y} = \|\vec{x}\|\|\vec{y}\|\cos\theta$$

$$\rightarrow \quad \cos\theta = \frac{\vec{x}^T\vec{y}}{\|\vec{x}\|\cdot\|\vec{y}\|}$$

Remark:

If $\theta = \frac{\pi}{2}$, then $\|\vec{x}-\vec{y}\|^2 = \|\vec{x}\|^2+\|\vec{y}\|^2$, which is the Pythagorean Law.

<u>E.g.</u> $\vec{x} = (1\ 2\ 3)^T$, $\vec{y} = (-2\ 1\ 0)^T$

① Find out the length of $\vec{x}$ and $\vec{y}$.

② Distance between $\vec{x}$ and $\vec{y}$. Angle?

③ whether $\vec{x}$ and $\vec{y}$ are orthogonal?

↳ ① $\|\vec{x}\| = \sqrt{14}$

$\|\vec{y}\| = \sqrt{5}$

② $\|\vec{x}-\vec{y}\| = \sqrt{19}$, $\cos\theta = \frac{\vec{x}^T\vec{y}}{\|\vec{x}\|\|\vec{y}\|} = \frac{-2+2+0}{\sqrt{14}\cdot\sqrt{5}} = 0$

② $\|\vec{x}-\vec{y}\| = \sqrt{19}$, $\cos\theta = \dfrac{\vec{x}^T\vec{y}}{\|\vec{x}\|\,\|\vec{y}\|} = \dfrac{-2+2+0}{\sqrt{14}\cdot\sqrt{5}} = 0$

③ Yes, since $\cos\theta = 0$ and $\arccos 0 = \frac{\pi}{2}$.

<u>Thm 5.1.8</u> (Cauchy–Schwarz Inequality)

Let $\vec{x}$ and $\vec{y} \in \mathbb{R}^n$. Then

$$|\vec{x}^T\vec{y}| \le \|\vec{x}\|\cdot\|\vec{y}\|$$

with equality holding iff one of the vectors is $\vec{0}$, or one vector is a scalar multiple of another.

<u>proof</u>:

<u>Method 2</u> $\|\hat{x}-\hat{y}\|^2 = \left\|\frac{1}{\|\vec{x}\|}\vec{x} - \frac{1}{\|\vec{y}\|}\vec{y}\right\|^2 \ge 0$ (Assume $\vec{x}, \vec{y}$ both nonzero)

$$= \frac{\|\vec{x}\|^2}{\|\vec{x}\|^2} - 2\frac{\vec{x}^T}{\|\vec{x}\|}\frac{\vec{y}}{\|\vec{y}\|} + \frac{\|\vec{y}\|^2}{\|\vec{y}\|^2}$$

$$= 1 - 2\frac{\vec{x}^T\vec{y}}{\|\vec{x}\|\|\vec{y}\|} + 1 = 2 - 2\frac{\vec{x}^T\vec{y}}{\|\vec{x}\|\|\vec{y}\|} \ge 0$$

$$\Rightarrow \frac{\vec{x}^T\vec{y}}{\|\vec{x}\|\,\|\vec{y}\|} \le 1$$ (half ly done)

Similarly, by considering $\|\hat{x}+\hat{y}\|^2 \ge 0$, we have

$$\frac{\vec{x}^T\vec{y}}{\|\vec{x}\|\,\|\vec{y}\|} \ge -1$$

Hence. $|\vec{x}^T\vec{y}| \le \|\vec{x}\|\,\|\vec{y}\|$

<u>Method 1</u> consider

$$|\cos\theta| = \frac{|\vec{x}^T\vec{y}|}{\|\vec{x}\|\,\|\vec{y}\|} \le 1$$

Then $|\vec{x}^T\vec{y}| \le \|\vec{x}\|\,\|\vec{y}\|$

<u>Def 5.1.9</u> (Orthogonal Vectors) (in $\mathbb{R}^n$)

For $\vec{x}, \vec{y} \in \mathbb{R}^n$, they are orthogonal iff

$$\langle \vec{x}, \vec{y}\rangle = \vec{x}^T\vec{y} = 0$$

We denoted as $\vec{x} \perp \vec{y}$.

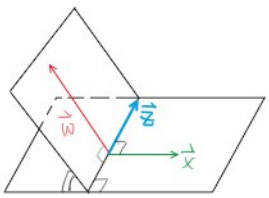
Remarks: ① Two nonzero $\vec{x}, \vec{y} \in \mathbb{R}^n$, if $\theta = \frac{\pi}{2}$, $\vec{x}\perp\vec{y}$ and $\vec{x}^T\vec{y} = 0$.

② The zero vector $\vec{0} \in \mathbb{R}^n$ is $\perp$ to any vector in $\mathbb{R}^n$.

<u>E.g. 5.1.7</u> $\vec{x} = \begin{pmatrix}1\\1\\2\end{pmatrix}$, $\vec{z} = \begin{pmatrix}2\\-4\\1\end{pmatrix}$, $\vec{w} = \begin{pmatrix}0\\1\\4\end{pmatrix}$

$\vec{x}\perp\vec{z}$ ✓? $\vec{x}\perp\vec{w}$ ✗? $\vec{z}\perp\vec{w}$ ✓?

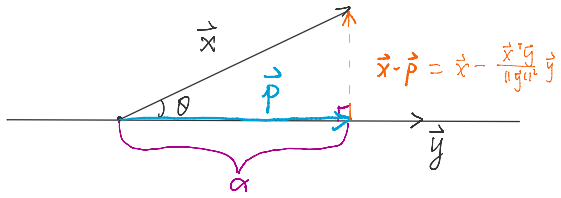
but $\vec{x}$ is not orthogonal to $\vec{w}$.



Def 5.1.10 Scalar projection & vector proj.

Scalar projection of $\vec{x}$ onto $\vec{y}$:

$$\alpha = \|\vec{x}\| \cdot \cos\theta = \|\vec{x}\| \frac{\vec{x}^T\vec{y}}{\|\vec{x}\|\,\|\vec{y}\|} = \frac{\vec{x}^T\vec{y}}{\|\vec{y}\|}$$



$$\vec{x} - \vec{p} = \vec{x} - \frac{\vec{x}^T\vec{y}}{\|\vec{y}\|^2}\vec{y}$$

Vector projection of $\vec{x}$ onto $\vec{y}$:

$$\vec{p} = \alpha \frac{\vec{y}}{\|\vec{y}\|} = \frac{\vec{x}^T\vec{y}}{\|\vec{y}\|} \cdot \frac{1}{\|\vec{y}\|}\vec{y} = \frac{\vec{x}^T\vec{y}}{\|\vec{y}\|^2}\vec{y} = \mathrm{proj}_{\vec{y}}\,\vec{x}$$

E.g. 5.1.11 $\vec{x} = \begin{pmatrix}1\\1\\2\end{pmatrix}$, $\vec{y} = \begin{pmatrix}2\\-1\\0\end{pmatrix}$

vector projection of $\vec{x}$ onto $\vec{y}$: $\mathrm{proj}_{\vec{y}}\,\vec{x} = \frac{\vec{x}^T\vec{y}}{\|\vec{y}\|^2}\vec{y} = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\-1\\0\end{pmatrix}$

vector proj. of $\vec{y}$ onto $\vec{x}$: $\mathrm{proj}_{\vec{x}}\,\vec{y} = \frac{\vec{y}^T\vec{x}}{\|\vec{x}\|^2}\vec{x} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\2\end{pmatrix}$

** △ Normal vector to a plane in $\mathbb{R}^3$.

plane equation: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

normal vector: $\vec{n} = (a, b, c)^T$

Randomly pick two points in the plane ℓ

Then $\overrightarrow{PP_0}$ is a vector in the plane ℓ

$P(x_1, y_1, z_1)$

① $a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)=0$

$P_0(x_2, y_2, z_2)$

② $a(x_2-x_0)+b(y_2-y_0)+c(z_2-z_0)=0$

$\overrightarrow{PP_0} = (x_2-x_1, y_2-y_1, z_2-z_1)$

$\vec{n}$ as a normal vector to the plane ℓ, $\vec{n}$ should be orthogonal to any vector in ℓ. Namely, $\vec{n} \perp \overrightarrow{PP_0}$

Verify:

$$\vec{n}^T \overrightarrow{PP_0} = (a, b, c)\begin{pmatrix} x_2-x_1 \\ y_2-y_1 \\ z_2-z_1 \end{pmatrix}$$

$$= a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)$$

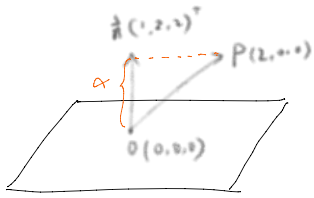
$$= 0$$

Since ② - ① :

$$a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0$$

<u>E.g. 5.1.13</u> Find the distance from $P(2,0,0)$ to the plane $2y+x+2z=0$.

$\vec{n} = (1, 2, 2)^T$, notice that $O(0,0,0)$ is on the plane



The scalar projection of $\overrightarrow{OP}$ onto $\vec{n}$:

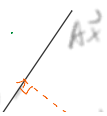
$$\alpha = \frac{\overrightarrow{OP}^T \vec{n}}{\|\vec{n}\|} = \frac{(2,0,0)\begin{pmatrix}1\\2\\2\end{pmatrix}}{\sqrt{1^2+2^2+2^2}} = \frac{2}{3}$$

<u>E.g. 5.1.12</u> $A\vec{x} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \vec{b}$ has no solution.

Can we find an $\tilde{x}$ s.t. $A\tilde{x}$ is the st vector to $\vec{b}$ for all $\vec{x}$?

$A\tilde{x}$: a vector projection of $\vec{b}$ onto $A\vec{x}$ (or $\begin{pmatrix}1\\2\end{pmatrix}$) .

$A\tilde{x} = \text{proj}_{A\vec{x}}\vec{b} = \frac{(3\ 2)\begin{pmatrix}1\\2\end{pmatrix}}{\ }\begin{pmatrix}1\\2\end{pmatrix} = \frac{7}{5}\begin{pmatrix}1\\2\end{pmatrix}$

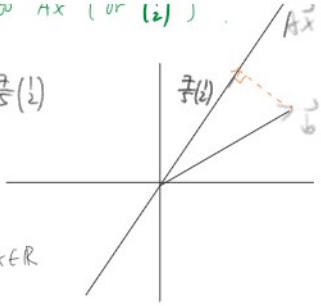


$A\tilde{x}$: a vector projection of b onto Ax (or $\begin{pmatrix}1\\2\end{pmatrix}$).

$$A\tilde{x} = \text{proj}_{\begin{pmatrix}1\\2\end{pmatrix}}\vec{b} = \frac{(3\ 2)\begin{pmatrix}1\\2\end{pmatrix}}{5}\begin{pmatrix}1\\2\end{pmatrix} = \frac{7}{5}\begin{pmatrix}1\\2\end{pmatrix}$$

Solve $\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \frac{7}{5}\begin{pmatrix}1\\2\end{pmatrix}$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}\frac{7}{5}\\0\end{bmatrix} + \begin{bmatrix}-2\\1\end{bmatrix}\alpha, \quad \alpha \in \mathbb{R}$$



△ Section 5.2 Orthogonal Subspaces (in $\mathbb{R}^n$)

Def 5.2.1 (orthogonal subspaces)

Two subspaces X and Y of $\mathbb{R}^n$ are said to be orthogonal if $\vec{x}^T\vec{y} = 0$, for any $\vec{x} \in X$ and $\vec{y} \in Y$.

We denote as $X \perp Y$.

"cap" = intersection

Thm 5.2.2 If $X \perp Y$, then $X \cap Y = \{\vec{0}\}$. $(\subset \mathbb{R}^n)$

proof: Assume that there exists a nonzero $\vec{z} \in X \cap Y$.

then $\vec{z} \in X$ and $\vec{z} \in Y$. Since $X \perp Y$, then

$\vec{z}^T\vec{z} = z_1^2 + \cdots + z_n^2 = 0 \rightarrow \vec{z} = \vec{0}$, contradiction!

E.g. Let $S = \left\{\begin{pmatrix}0\\a\\0\end{pmatrix} \mid a \in \mathbb{R}\right\}$ and $V = \left\{\begin{pmatrix}0\\0\\b\end{pmatrix} \mid b \in \mathbb{R}\right\}$ be subspaces of $\mathbb{R}^3$. $S \perp V$?

Yes. Consider arbitrary vectors $\vec{x} = \begin{pmatrix}0\\a\\0\end{pmatrix} \in S$ and $\vec{y} = \begin{pmatrix}0\\0\\b\end{pmatrix} \in V$, since

$\vec{x}^T\vec{y} = (0, a, 0)\begin{pmatrix}0\\0\\b\end{pmatrix} = 0$ for any $a, b \in \mathbb{R}$

Then $\vec{x} \perp \vec{y}$ and $S \perp V$.

E.g. The wall and the floor $\perp$?

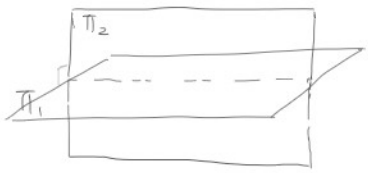
No.

$S = \text{span}\{\vec{e}_1, \vec{e}_2\}$

$V = \text{span}\{\vec{e}_2, \vec{e}_3\}$

Take $\vec{e}_2 \in S$, and $\vec{e}_2 \in V$.

$\vec{e}_2^T\vec{e}_2 = 1 \neq 0$.



Def 5.2.3 (orthogonal complement)

Let Y be a subspace of $\mathbb{R}^n$. The orthogonal complement of Y is

$$Y^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x}^T\vec{y} = 0, \ \forall \vec{y} \in Y\}$$

$Y^\perp$: "Y perp"

E.g. 5.2.4 $X = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$, $Y = \text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$

orthogonal subspaces? Yes. since $\forall \begin{pmatrix}a\\0\\0\end{pmatrix} \in X$ and $\begin{pmatrix}0\\b\\0\end{pmatrix} \in Y$, $a, b \in \mathbb{R}$.

$(a, 0, 0)\begin{pmatrix}0\\b\\0\end{pmatrix} = 0$. so $X \perp Y$.

orthogonal complements?

Let $X^\perp = \left\{ \begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; (a\ b\ c)\vec{x} = 0,\ \forall \vec{x} \in X \right\}$, and $\vec{x} = \begin{pmatrix}d\\0\\0\end{pmatrix} \in X$, $d \in \mathbb{R}$.

$(a, b, c)\begin{pmatrix}d\\0\\0\end{pmatrix} = ad = 0$ for any $d \in \mathbb{R}$, then $a = 0$, $b, c \in \mathbb{R}$

$X^\perp = \left\{ \begin{pmatrix}0\\b\\c\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; b, c \in \mathbb{R} \right\} = \text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

Thm 5.2.5 Let S be a subspace of $\mathbb{R}^n$. Then

1. $S^\perp$ is also a subspace of $\mathbb{R}^n$.
   ① nonempty: Consider $\vec{0} \in \mathbb{R}^n$, since $\vec{0} \perp \vec{x}$, for any $\vec{x} \in S$, $\vec{0} \in S^\perp$.
   ② Closure of addition: $\forall\ \vec{u}, \vec{w} \in S^\perp$, then $\vec{u}^T\vec{x} = 0$ and $\vec{w}^T\vec{x} = 0$ for any $\vec{x} \in S$,
   $(\vec{u} + \vec{w})^T\vec{x} = \vec{u}^T\vec{x} + \vec{w}^T\vec{x} = 0$. $\Rightarrow$ $(\vec{u} + \vec{w}) \in S^\perp$
   ③ Closure under Scalar multiplication: $\forall\ \vec{u} \in S^\perp$, $c \in \mathbb{R}$, $(c\vec{u})^T\vec{x} = c\vec{u}^T\vec{x} = 0$, for any $\vec{x} \in S$ $\Rightarrow$ $c\vec{u} \in S^\perp$.
2. $S^\perp$ is the largest one among all subspaces of $\mathbb{R}^n$ that are $\perp$ to S.
   Reasoning by contradiction: Assume that there exists $\vec{x} \in \mathbb{R}^n$, s.t. $\vec{x} \notin S^\perp$ but $\vec{x}$ is orthogonal to any vector in S.

3. $\dim S + \dim S^\perp = n$ $\quad S \cap S^\perp = \{\vec{0}\}$ } prove later

4. $S \oplus S^\perp = \mathbb{R}^n$ (direct sum)

5. $(S^\perp)^\perp = S$.

idea: Method 1: ① $(S^\perp)^\perp \subseteq S$
② $S \subseteq (S^\perp)^\perp$ } $\Rightarrow (S^\perp)^\perp = S$

Method 2: ① $(S^\perp)^\perp \subseteq S$
② $\dim((S^\perp)^\perp) = \dim S$ } $\Rightarrow (S^\perp)^\perp = S$

proof: Method 2 ① $\forall\ \vec{x} \in (S^\perp)^\perp$, $\vec{y} \in S^\perp$, then

$$\vec{x}^T\vec{y} = 0$$

then $\vec{x} \in S$ due to the fact that $S \perp S^\perp$. $\Rightarrow (S^\perp)^\perp \subseteq S$.

② $\dim S + \dim S^\perp = n$

since $S^\perp$ is also a subspace of $\mathbb{R}^n$, then

$\therefore \quad \dim S + \dim S^{\perp} = n$

Since $S^{\perp}$ is also a subspace of $\mathbb{R}^n$, then

$\dim(S^{\perp}) + \dim(S^{\perp})^{\perp} = n \qquad \Rightarrow \quad \dim(S^{\perp})^{\perp} = \dim S.$

△ Sum and Direct Sum

Def 5.2.6 (Sum) Let X and Y be subspaces of W. The sum of X and Y is

$$X + Y = \{\vec{x}+\vec{y} \in W \mid \vec{x} \in X,\ \vec{y} \in Y\}$$

Remark: If $\{\vec{x}_1, \vec{x}_2, \cdots \vec{x}_k\}$ is a basis of X, $\{\vec{y}_1, \vec{y}_2, \cdots, \vec{y}_m\}$ is a basis of Y.

Then $\quad X+Y = \text{span}\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k, \vec{y}_1, \cdots, \vec{y}_m\}$ — spanning set but not a basis

and $\quad \dim(X+Y) = \dim X + \dim Y - \dim(X \cap Y)$

$\vec{x} = \sum_{i=1}^{k} a_i \vec{x}_i$

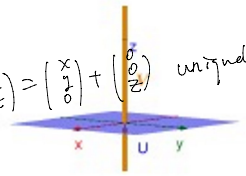
$\vec{y} = \sum_{j=1}^{m} b_j \vec{y}_j$

$\vec{x}+\vec{y} = \sum_{i=1}^{k} a_i \vec{x}_i + \sum_{j=1}^{m} b_j \vec{y}_j$

Example 5.2.8. Consider the following examples of sum of subspaces

$U = \{(x, y, 0)^T \mid x, y \in \mathbb{R}\}$
$V = \{(0, 0, z)^T \mid z \in \mathbb{R}\}$
$U + V = \mathbb{R}^3$

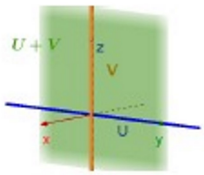
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$ uniquely.



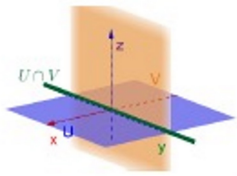
$U+V = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x, y, z \in \mathbb{R} \right\}$

$= \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \mathbb{R}^3$

$U = \{(0, y, 0)^T \mid y \in \mathbb{R}\}$
$V = \{(0, 0, z)^T \mid z \in \mathbb{R}\}$
$U + V =$
$\{(0, y, z)^T \mid y, z \in \mathbb{R}\}$



$U = \{(x, y, 0)^T \mid x, y \in \mathbb{R}\}$ — $\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$
$V = \{(0, y, z)^T \mid y, z \in \mathbb{R}\}$ — $\text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$
$U + V = \mathbb{R}^3 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$



$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$

$= \cdots$

Thm 5.2.9 Let X and Y be subspaces of W. Then

$$\dim(X+Y) \leq \dim X + \dim Y$$

Def 5.2.10 (Direct Sum)

Let X and Y be subspaces of S and $X+Y=S$.

If every $\vec{z} \in S$ can be uniquely written as

$$\vec{z} = \underbrace{\vec{x}}_{\in X} + \underbrace{\vec{y}}_{\in Y}$$

then $X+Y$ is a direct sum and we denote it as $X \oplus Y$.

Rule of Thumb for direct sum:

Thm 5.2.11

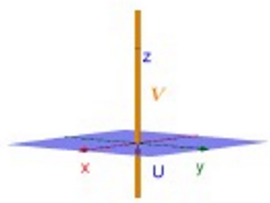
$X+Y$ is a direct sum $\iff$ $\dim(X+Y) = \dim X + \dim Y$ and $X \cap Y = \{\vec{0}\}$.

Example 5.2.8. Consider the following examples of sum of subspaces

$U = \{(x, y, 0)^T | x, y \in \mathbb{R}\}$

$V = \{(0, 0, z)^T | z \in \mathbb{R}\}$

$U + V = \mathbb{R}^3$



$\forall \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$,

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \to \begin{cases} x = a \\ y = b \\ z = c \end{cases}$

unique solution! $U \oplus V = \mathbb{R}^3$

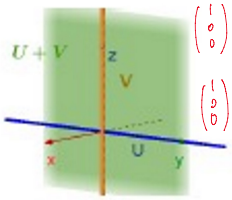
$U = \{(0, y, 0)^T | y \in \mathbb{R}\}$

$V = \{(0, 0, z)^T | z \in \mathbb{R}\}$

$U + V =$

$\{(0, y, z)^T | y, z \in \mathbb{R}\}$

not a direct sum, since for $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \notin U+V$



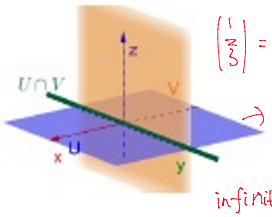
$U = \{(x, y, 0)^T | x, y \in \mathbb{R}\}$

$V = \{(0, y, z)^T | y, z \in \mathbb{R}\}$

$U + V = \mathbb{R}^3$

not a direct sum since for

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ w \\ z \end{pmatrix}$



$\to \begin{cases} x = 1 \\ y + w = 2 \\ z = 3 \end{cases}$ or $\begin{cases} x = 1 \\ y = 2 - w \\ z = 3 \\ w \in \mathbb{R} \end{cases}$

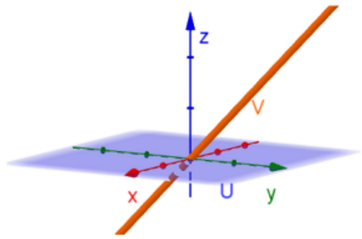
infinitely many solutions.

Example 5.2.12.

$U = \{(x, y, 0)^T | x, y \in \mathbb{R}\}$

$V = \{(0, a, a)^T | a \in \mathbb{R}\}$

Then $U \oplus V = \mathbb{R}^3$.



By Definition 5.2.10:

$(x, y, z)^T = (x, y - z, 0)^T + (0, z, z)^T$

where

$(x, y - z, 0)^T \in U$ and $(0, z, z)^T \in V$.

By Theorem 5.2.11:

Since $U + V = \mathbb{R}^3$ and $U \cap V = \{\mathbf{0}\}$,

$U \oplus V = \mathbb{R}^3$.

$\circledcirc$