Chapter 1 Rate of interest

A common financial transaction is the investment of an amount of money at interest. For example, a person may invest in a saving account at a bank. The initial amount of money $\$K$ is called **principal** and the total amount $\$S$ received after a period of time is called **accumulated value**. The difference between the accumulated value and the principle, $\$S - \$K$, is called **interest**, earned during the period of investment.

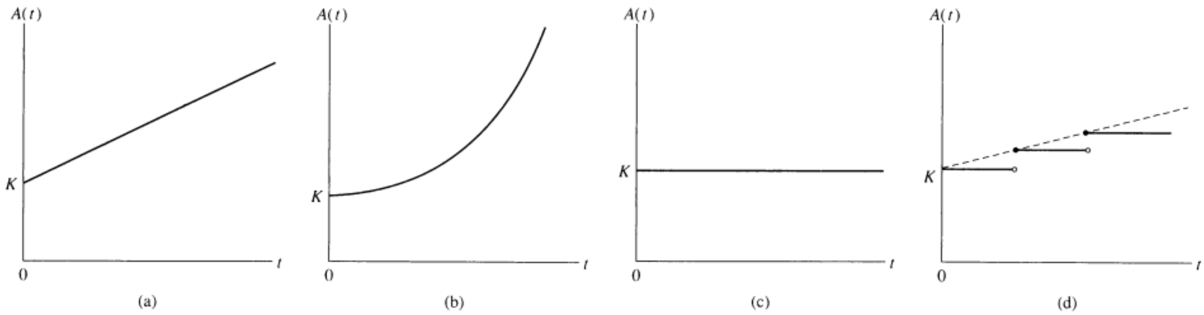
Let t measure time from the date of investment. In theory, time may be measured in many different units, e.g., days, months, years, decades, etc. The unit in which time is measured is called period. The most common measurement period is one year, and this will be assumed unless stated otherwise.

Imagine a fund growing at interest. It would be very convenient to have a function representing the amount in the fund at any time t. The function $a(t)$ is defined as the accumulated value of the fund at time t of an initial investment of \$1 at time 0. $a(t)$ is called the **accumulation function**.

What are the properties of an accumulation function $a(t)$? By definition, $a(0) = 1$, $a(t)$ is continuous and increasing (in most cases).

In general, the original principal will not be one unit but will be some amount $K > 0$. We may also define an amount function $A_K(t)$ as the accumulated value at time t of $\$K$ invested at time 0. Obviously, $A_K(0) = K$. Typically, $A_K(t) = Ka(t)$. However, this relation need not hold in general (Example 1.2). We will assume that $A_K(t) = Ka(t)$ unless stated otherwise. The second and the third properties of $a(t)$ listed above clearly also hold for $A_K(t)$.

Example 1.1



(a) This is so-called "simple interest," where the accumulation function is linear.

(b) This accumulation function is an exponential function. As we shall see, this is referred to as "compound interest," where the fund earns interest on the interest at a constant rate.

(c) This is the accumulation function for money put in a piggy bank.

(d) This is the accumulation function for an account where you are credited with interest only at the end of each interest period.

The **effective rate of interest** i is the amount of money that one unit invested at the beginning of a period will earn during the period, where interest is paid at the end of the period. This definition is equivalent to

$$i = a(1) - a(0) \qquad \text{or} \qquad a(1) = 1 + i.$$

The amount of growth in the t-th year is $a(t) - a(t-1)$. The amount in the fund at the beginning of the t-th year is the same as the amount in the fund at the end of the $(t-1)$-th year, namely, $a(t-1)$. The **effective rate of interest at the t-th year** after investment is:

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)}.$$

The effective rate of interest can also defined as:

$$i_t = \frac{A_K(t) - A_K(t-1)}{A_K(t-1)}$$

We have $a(t) = a(t-1) + a(t-1)i_t$ and $A_K(t) = A_K(t-1) + A_K(t-1)i_t$. The fund at the end of the t-th year is equal to the fund at the beginning of t-th year plus the interest earned during the year.

Consider the investment of one unit such that the amount of interest earned during each period is constant. The accumulated value of 1 at the end of the first period is $1 + i$, at the end of second it is $1 + 2i$, etc. Thus, in general, we have a linear accumulation function

$$a(t) = 1 + it \qquad \text{for } t = 0, 1, 2, \ldots$$

The accruing of interest according to this pattern is called **simple interest**. It is natural to extend the definition to non-integral values of $t > 0$ as well. This is equivalent to the crediting of interest proportionally over any fraction of a period. If this is the case, then the amount function $a(t)$ can be represented by Figure 1.1(a). If interest is accrued only for completed periods with no credit for fractional periods, then the amount function becomes a step function with discontinuities as illustrated by Figure 1.1(d). Unless stated otherwise, it will be assumed that interest is accrued proportionally over fractional periods under simple interest.

The interest for investment $\$K$ in t years starting at time 0 is $A_K(t) - A_K(0)$ $= K(1+it) - K = Kit$.

Simple interest has the property that interest is **NOT** reinvested to earn additional interest. For example, consider an investment of \$100 for two years at 10% simple interest. Under simple interest the investor will receive \$10 at the end of each of the two years.

Let i be the rate of simple interest and let i_n be the effective rate of interest for the n-th period. Then we have

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{(1+in) - (1+i(n-1))}{1+i(n-1)} = \frac{i}{1+i(n-1)}.$$

The effective rate of interest decreases with time.

However, in reality, for the second year the investor has \$110 which could have been invested. Clearly, it would be advantageous to invest the \$110 at 10%, since the investor would then receive \$11 in interest for the second year instead of \$10.

Consequently, you would go into bank, close your account, and then instantly reopen it. But this would be inconvenient for you and for the bank. Therefore, it is sensible for the bank to design an account that grows in a manner where there is no advantage or disadvantage to closing an account and then instantly reopening it. $a(s + t)$ is the amount at $s + t$ for investment \$1 at time 0 without closing and reopening an account. $a(s)a(t)$ is the amount at $s + t$ for investment \$1 at time 0, closing an account and then instantly reopening it at s. We have

$$a(s + t) = a(s)a(t) \qquad \text{for all positive real numbers } t \text{ and } s.$$

Then

$$a'(t) = \lim_{s\to 0}\frac{a(s+t)-a(t)}{s} = \lim_{s\to 0}\frac{a(s)a(t)-a(t)}{s} = a(t)\lim_{s\to 0}\frac{a(s)-1}{s} = a(t)a'(0)$$

$$\int_0^x \frac{a'(t)}{a(t)}dt = \int_0^x a'(0)dt$$

$$a'(0)x = \ln a(t)\big|_0^x = \ln a(x) - \ln a(0) = \ln a(x)$$

$$a'(0) = \ln a(1) = \ln(1+i)$$

$$a(x) = (1+i)^x.$$

Accumulation function $a(t) = (1+i)^t$, for all $t \geq 0$, is called **compound interest accumulation function** at interest rate i. It is an exponential function and can be represented by Figure 1.1(b).

Compound interest can solve this problem by assuming that the interest earned is automatically reinvested to earn additional interest at all times. Consider the investment of 1 which accumulates to $1 + i$ at the end of the first period. This balance of $1 + i$ can be considered as principal at the beginning of the second period and will earn interest of $i(1 + i)$ during the second period. The balance at the end of the second period is $(1+i) + i(1+i) = (1+i)^2$.

Similarly, the balance of $(1+i)^2$ can be considered as principal at the beginning of the third period and will earn interest of $i(1+i)^2$ during the third period. The balance at the end of the third period is $(1+i)^2+i(1+i)^2=(1+i)^3$. Continuing this process infinitely, we obtain

$$a(n)=(1+i)^n \qquad \text{for positive integer } n.$$

In general

$$a(n)=(1+i_n)a(n-1)=(1+i_n)(1+i_{n-1})a(n-2)=\cdots=(1+i_n)(1+i_{n-1})\cdots(1+i_1)$$

for positive integer n.

Furthermore,

$$i_n=\frac{a(n)-a(n-1)}{a(n-1)}=\frac{(1+i)^n-(1+i)^{n-1}}{(1+i)^{n-1}}=\frac{(1+i)-1}{1}=i$$

which is independent of n.

Example 1.2

A bank offers an investment plan with compound interest at an annual effective interest rate of 2% on balances of less than \$2000, 3% on balances of at least \$2000 but less than \$5000, and 4% on balances of at least \$5000. Dr. Wong opens an account with \$1800. Determine the amount function $A_{1800}(t)$ and show that $A_{1800}(t)\neq 1800a(t)$.

Solution

Define t_1 be the time for Dr. Wong's balance to grow to \$2000 and t_2 be the time for the balance to grow from \$2000 to \$5000. Then $1800(1.02)^{t_1}=2000$ and $2000(1.03)^{t_2}=5000$. It follows that

$$t_1=\frac{\ln(2000/1800)}{\ln 1.02}\approx 5.320532174, \quad \text{and} \quad t_2=\frac{\ln(5000/2000)}{\ln 1.03}\approx 30.99891276.$$

Then

$$A_{1800}(t)=\begin{cases} 1800(1.02)^t & \text{if } 0\le t\le t_1; \\ 2000(1.03)^{t-t_1} & \text{if } t_1\le t\le t_1+t_2; \\ 5000(1.04)^{t-t_1-t_2} & \text{if } t_1+t_2\le t. \end{cases}$$

On the other hand, $a(t) = 1.02^t$ for $0 \le t \le t_3$ where $1.02^{t_3} = 2000$, that is for

$$t_3 = \frac{\ln 2000}{\ln 1.02} \approx 383.8330311.$$

Therefore, $A_{1800}(t) \ne 1800a(t)$ if $t > t_1$.

So far, we have been talking about the accumulated value or **future value** (FV) of a fund, i.e., how much is in the fund after *t* years, if we invest a given amount today. Consider the "opposite" question: How much should we invest today in order to have a given amount at the end of *t* year? The amount that we should invest is called the **present value** (PV) due in *t* years.

The process of finding the price that we would be willing to pay for a promise to receive a future amount is called **discounting**. Discounting a future amount means the same thing as finding its present value.

Let $v(t)$ be the amount to invest today in order to have \$1 in *t* year. Since we require that $v(t)$ grows to \$1 in *t* years, we have

$$v(t)a(t) = 1 \quad \text{or} \quad v(t) = \frac{1}{a(t)}.$$

The function $v(t) = 1/a(t)$ is called the **discount function**. In particular, if $a(t) = (1+i)^t$, then the present value of \$1 in *t* years is

$$v(t) = \frac{1}{a(t)} = \frac{1}{(1+i)^t} = (1+i)^{-t}.$$

The present value of \$1 in a year is denoted by *v*, that is $v = (1+i)^{-1}$. The term *v* is often called a **discount factor**.

The effective rate of interest was defined as a measure of interest at the end of the period. We define the **effective rate of discount**, denoted by *d*, as a measure of interest paid at the beginning of the period.

If an investor borrows $\$K$ from the bank for 1 period at an effective rate of discount d, then the investor will have to pay $\$Kd$ in order to receive the use of $\$K$. Therefore, instead of having $\$K$, the investor only has $\$K - \$Kd$ at time 0.

Example 1.3

If A borrows \$100 from a bank for 1 year at an effective rate of interest of 6%, then the bank will give A \$100. At the end of the year, A will repay the bank the original loan of \$100, plus interest of \$6, or a total of \$106.

However, if A borrows \$100 for 1 year at an effective rate of discount of 6%, then the bank will collect its interest of 6% in advance and will give A only \$94. At the end of the year, A will repay \$100.

A paid \$6 interest in both cases. However, A had the use of \$100 for the year in first case while A had the use of \$94 for the year in second case. Thus it is clear that an effective rate of interest of 6% is not the same as an effective rate of discount of 6%.

The effective rate of interest tells us how fast a fund is growing based on the amount in the fund at the beginning of the year. But we could also define a rate of growth based on the amount in the fund at the end of the year. This rate has the symbol d_t and is called the **effective rate of discount at the** t:

$$d_t = \frac{a(t) - a(t-1)}{a(t)} = \frac{A_K(t) - A_K(t-1)}{A_K(t)}.$$

We have

$$a(n-1) = a(n)(1-d_n), \quad a(n-2) = a(n-1)(1-d_{n-1}) = a(n)(1-d_n)(1-d_{n-1}), \cdots$$
$$1 = a(0) = a(n)(1-d_n)(1-d_{n-1}) \cdots (1-d_1) \quad \text{for positive integer } n.$$

It is possible to define simple discount in a manner analogous to the definition of simple interest.

Consider the amount of discount earned during each period is constant. Then, the original principal which will produce an accumulated value of 1 at the end of t periods is

$$v(t) = a^{-1}(t) = 1 - dt. \quad \text{for} \quad 0 \le t < 1/d.$$

$a(t)$ is called the **simple discount accumulation function**. Then

$$d_t = (1-dt)\left(\frac{1}{1-dt} - \frac{1}{1-d(t-1)}\right) = \frac{1-d(t-1)-(1-dt)}{1-d(t-1)} = \frac{d}{1-d(t-1)}.$$

Accumulation function $a(t) = (1-d)^{-t}$, for all $t \ge 0$, is called **compound discount accumulation function** at interest rate d. Then

$$d_t = \frac{a(t)-a(t-1)}{a(t)} = \frac{(1-d)^{-t}-(1-d)^{-(t-1)}}{(1-d)^{-t}} = 1-(1-d) = d$$

is independent of t.

Two rates are said to be **equivalent** if both rates produce the same results over the same period of time. By "same results" we mean that the accumulated value of an investment, or the present value of a future amount, is the same under both rates.

Suppose effective rate of interest i and effective rate of discount d are constant. In order for i and d to be equivalent, we must have

$$1-d = v = \frac{1}{1+i}.$$

Hence

$$i = \frac{d}{1-d}, \quad d = \frac{i}{1+i} = iv, \quad \frac{1}{d} - \frac{1}{i} = 1.$$

Consider the situation that interest is paid more frequently than once per period. Rates of interest and discount in these cases are called **nominal**.

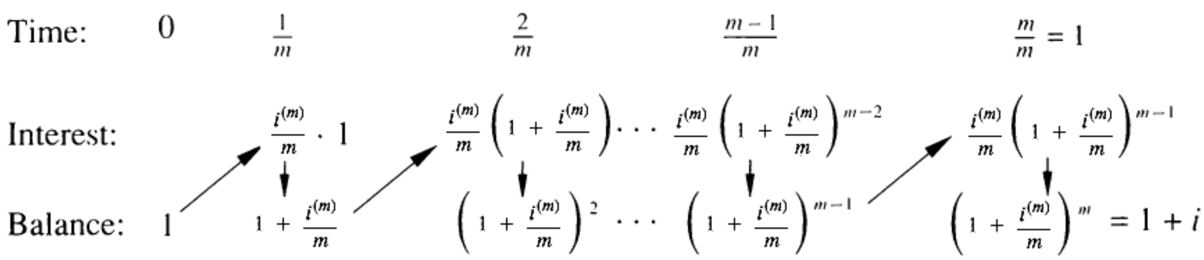
Let $i^{(m)}$ (read "i upper m") be the **nominal rate of interest compounded** m **times a year**.

That is, $\frac{i^{(m)}}{m}$ is the effective rate for $\frac{1}{m}$-th of a year and the accumulated value of \$1 in 1 year is $\left(1+\frac{i^{(m)}}{m}\right)^m$. To find the equivalent effective rate of interest i, we set

$$\left(1+\frac{i^{(m)}}{m}\right)^m = 1+i.$$

Solving for $i^{(m)}$, we have

$$i^{(m)} = m\left((1+i)^{1/m}-1\right).$$



Example 1.4

What nominal rate of interest compounded semiannually is equivalent to a nominal rate of interest of 3% compounded once every two years?

Solution

The given nominal rate is 3% compounded once every 2 years. This means that the effective rate for a two-year period is 6%. Let $i^{(2)}$ be the unknown nominal rate. The accumulated value of \$1 in 2 years is $\left(1+\frac{i^{(2)}}{2}\right)^4$. Setting the 2 results equal

$$\left(1+\frac{i^{(2)}}{2}\right)^4 = 1.06$$

and solving for $i^{(2)}$, we get $2(1.06^{1/4}-1)$.

Let $d^{(m)}$ be the **nominal rate of discount compounded** m **times a year**. This means that $\frac{d^{(m)}}{m}$ is the effective rate of discount for $\frac{1}{m}$-th of a year. Thus, the present value of \$1 due in a year can be expressed as $\left(1-\frac{d^{(m)}}{m}\right)^m$. If d is the equivalent annual effective rate of discount, we can set this expression equal to $1-d$:

$$\left(1-\frac{d^{(m)}}{m}\right)^m = 1-d.$$

Solving for $d^{(m)}$, we get

$$d^{(m)} = m\left(1-(1-d)^{1/m}\right).$$

Time:	0	$\frac{1}{m}$		$\frac{m-2}{m}$	$\frac{m-1}{m}$	$\frac{m}{m}=1$
Discount:	$\frac{d^{(m)}}{m}\left(1-\frac{d^{(m)}}{m}\right)^{m-1}$	$\frac{d^{(m)}}{m}\left(1-\frac{d^{(m)}}{m}\right)^{m-2}$	$\cdots$	$\frac{d^{(m)}}{m}\cdot\left(1-\frac{d^{(m)}}{m}\right)$	$\frac{d^{(m)}}{m}\cdot 1$	
Balance:	$1-d=\left(1-\frac{d^{(m)}}{m}\right)^m$	$\left(1-\frac{d^{(m)}}{m}\right)^{m-1}$	$\cdots$	$\left(1-\frac{d^{(m)}}{m}\right)^2$	$1-\frac{d^{(m)}}{m}$	1

We also have $\left(1-\frac{d^{(m)}}{m}\right)\left(1+\frac{i^{(m)}}{m}\right)=1$. Hence $i^{(m)}=\frac{d^{(m)}}{1-\frac{d^{(m)}}{m}}$ and $d^{(m)}=\frac{i^{(m)}}{1+\frac{i^{(m)}}{m}}$.

Suppose that an amount K is invested for n years at an interest rate of r per annum. If the rate is compounded once per annum, the terminal value of the investment is

$$K(1+r)^n.$$

If the rate is compounded m times per annum, the terminal value of the investment is

$$K\left(1+\frac{r}{m}\right)^{mn}.$$

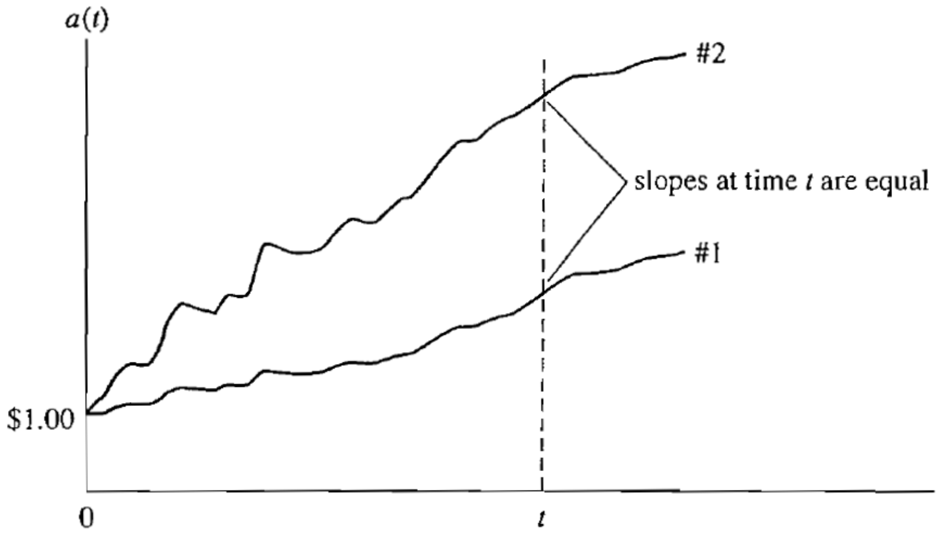
The limit as the compounding frequency, m, tends to infinity is known as continuous compounding. With continuous compounding, it can be shown that an amount K invested for n years at rate r grows to Ke^{rn}.

Effect of the compounding frequency on the value of \$100 at the end of 1 year when the interest rate is 10% per annum.

compounding frequently	Value of \$100 at the end of year
Annually ($m=1$)	110.00
Semiannually ($m=2$)	110.25
Quarterly ($m=4$)	110.3812890625
Monthly ($m=12$)	110.4713067441297242
Weekly ($m=52$)	110.5064792779766422
Daily ($m=365$)	110.5155781616264374
Anytime ($m=\infty$)	110.5170918075647625

Future value of an amount of money at a continuously compounded rate r for n years involves multiplying it by e^{rn}. Present value of it at a continuously compounded rate r for n years involves multiplying by e^{-rn}.

Suppose we want to measure how fast of the growth of $a(t)$ at time t. It is reasonable to consider $a'(t)$. A higher derivative tells us that the fund is increasing by a relatively large amount near time t. A lower derivative tells us that the fund us increasing by a relatively small amount near time t. But, is the derivative a good measure of the growth rate? The answer is NO. To understand this, let us consider the following picture.



Suppose the derivatives of the two $a(t)$ functions are the same at time t. Suppose the amount in fund #2 is double the amount in fund #1 at time t. Which one would you choose?

If fund #1 offers to pay you \$1 in a day on a deposit of \$10000 and fund #2 also offers to pay \$1 in a day on a deposit of \$20000, you would have no problem in concluding that fund #1's effective annual growth rate is twice that of fund #2.

In conclusion, to get growth rate makes sense, we should define how fast a fund is growing at any time dividing by the amount of fund.

Definition 1.5

The force of interest δ_t is defined by

$$\delta_t = \frac{A_K'(t)}{A_K(t)} = \frac{a'(t)}{a(t)}.$$

By simple calculation, we have

$$\delta_t = \frac{1}{a(t)}\frac{d}{dt}a(t) = \frac{d}{dt}\ln a(t)$$
$$\int_0^t \delta_r dr = \ln a(r)\Big|_0^t = \ln a(t) - \ln a(0) = \ln a(t)$$
$$a(t) = e^{\int_0^t \delta_r dr}.$$

Definition of δ_t can be written as $A_K(t)\delta_t = A_K'(t)$. Integrating between 0 and n, we obtain

$$\int_0^n A_K(t)\delta_t dt = \int_0^n A_K'(t)dt = A_K(t)\Big|_0^n = A_K(n) - A_K(0).$$

It has a rather interesting verbal interpretation. The term $A_K(n) - A_K(0)$ is the amount of interest earned over n measurement periods. The differential expression $A_K(t)\delta_t dt$ may be interpreted as the amount of interest earned on $A_K(t)$ at exact time t because of the force of interest δ_t. When this expression is integrated between 0 and n, it gives the total amount of interest earned over the n periods.

For simple interest, $a(t) = 1 + it$. $\delta_t = \frac{a'(t)}{a(t)} = \frac{i}{1+it}$.

For simple discount, $a(t) = (1-dt)^{-1}$. $\delta_t = \frac{a'(t)}{a(t)} = \frac{d(1-dt)^{-2}}{(1-dt)^{-1}} = \frac{d}{1-dt}$.

Example 1.6

A payment of \$300 at the end of 3 years and \$600 at the end of 6 years has the same present value as a payment of \$200 at the end of 2 years and X at the end of 5 years. You are given $\delta_t = \frac{2}{1+t}$. Calculate X.

Solution

$a(n) = e^{\int_0^n \frac{2}{1+t}dt} = e^{2\ln(1+t)}\Big|_0^n = (1+t)^2\Big|_0^n = (1+n)^2 - 1$. Is it correct? No.

It should be $a(n) = e^{\int_0^n \frac{2}{1+t}dt} = e^{2\ln(1+t)|_0^n} = e^{2\ln(1+n)} = (1+n)^2$.

Present value of 1 due in n years is $a(n)^{-1} = (1+n)^{-2}$. We have

$$\frac{300}{16} + \frac{600}{49} = \frac{200}{9} + \frac{X}{36}$$

X = \$315.82.

Example 1.7

The force of interest is $\delta_t = 0.02t$, where t is the number of years from January 1, 2001. If \$1 is invested on January 1, 2003, how much is in the fund on January 1, 2008?

Solution

$a(t) = e^{\int_0^5 0.02t dt} = e^{\left[0.01t^2\right]_0^5} = e^{0.25}$ is not correct because it is the accumulated value on January 1, 2006 of \$1 invested on January 1, 2001. It is true that this is a five-year period but it is not the same five-year period as from January 1, 2003 ($t = 2$) to January 1, 2008 ($t = 7$).

The correct is answer is

$$e^{\int_2^7 0.02t\,dt} = e^{\left[0.01t^2\right]_2^7} = e^{0.49-0.04} = e^{0.45} = \frac{a(7)}{a(2)}.$$

\$1 invested at time 0 will grow to $a(2)$ at time 2 and that $a(2)$ will continue to grow to $a(7)$ at time 7. By proportion, \$1 at time 2 must grow to $a(7)/a(2)$ at time 7.

Theorem 1.8

Let $a(t)$ be the accumulation function. If \$1 is invested at time t_1 and accumulated to time t_2, then accumulated value is

$$e^{\int_{t_1}^{t_2} \delta_t\,dt} = \frac{a(t_2)}{a(t_1)}.$$

The present value is

$$e^{-\int_{t_1}^{t_2} \delta_t\,dt} = \frac{a(t_1)}{a(t_2)}.$$

When the force of interest δ_t is constant, we can drop the subscript t and just call it δ. The formula for the accumulation function simplifies to:

$$a(t) = e^{\int_0^t \delta dr} = e^{\delta t}.$$

This is an exponential function. But the accumulation function under a constant effective rate of interest i is also an exponential function $a(t) = (1+i)^t$. If i is equivalent to δ, then $e^{\delta t} = (1+i)^t$, so that

$$\delta = \ln(1+i).$$

The constant force of interest δ is continuous compounded interest rate. Consider

$$\left(1+\frac{i^{(m)}}{m}\right)^m = 1+i = e^{\delta} = (1-d)^{-1} = \left(1-\frac{d^{(n)}}{n}\right)^{-n}.$$

We have

$$i^{(m)}=m\left(e^{\delta/m}-1\right)=m\left(\frac{\delta}{m}+\frac{\delta^2}{2!m^2}+\frac{\delta^3}{3!m^3}+\cdots\right)=\delta+\frac{\delta^2}{2!m}+\frac{\delta^3}{3!m^2}+\cdots>\delta$$

$\lim_{m\to\infty} i^{(m)}=\delta.$

On the other hand,

$$d^{(n)}=n\left(1-e^{-\delta/n}\right)=n\left(1-\left(1+\frac{\delta}{(-n)}+\frac{\delta^2}{2!(-n)^2}+\frac{\delta^3}{3!(-n)^3}+\cdots\right)\right)=\delta+\frac{\delta^2}{2!(-n)}+\frac{\delta^3}{3!(-n)^2}+\cdots<\delta$$

$\delta=\lim_{n\to\infty} d^{(n)}.$

Since $i^{(m)}=\delta+\dfrac{\delta^2}{2!m}+\dfrac{\delta^3}{3!m^2}+\cdots$, we have $i^{(m)}$ is decreasing as m increase, and $\delta<i^{(m)}<i^{(1)}=i.$

Since $d^{(n)}=\delta+\dfrac{\delta^2}{2!(-n)}+\dfrac{\delta^3}{3!(-n)^2}+\cdots$, we have

$$\frac{d}{dn}d^{(n)}=\frac{\delta^2}{2!(-n)^2}+\frac{\delta^3}{3!(-n)^3}+\cdots>0$$

Therefore, $d^{(n)}$ is increasing as n increase and $d=d^{(1)}<d^{(n)}<\delta.$

In conclusion

$$d<d^{(n)}<\delta<i^{(m)}<i.$$

Example 1.9

Dr. Wong borrows \$1000 with compound interest at an annual effective rate of 10%. He agrees to repay the loan by 2 equally spaced payments of \$525. When should he make these payments?

Solution

PAYMENT:	−\$1,000	\$525	\$525
TIME:	0	T	$2T$
BALANCE:			\$0

Let $t = 0$ be the time of loan. The equation of value at time 0 is

$$1000 = 525v^T + 525v^{2T}, \quad v^T > 0.$$

Then

$$v^T = \frac{-525 + \sqrt{525^2 + 4(525)(1000)}}{2(525)} \approx 0.967910728.$$

$$T = \frac{\ln 0.967910728}{-\ln 1.1} \approx 0.342202894.$$

Consequently, Dr. Wong should make payments at time 0.342202894 and 0.684405787.

Example 1.10

Dr. Wong opens a savings account with a deposit of \$5000. He deposits \$3000 a year later and \$2000 a after that. The account grows by compound interest at a constant annual effective rate i. Just after his \$2000 deposit, his balance is \$11000. Find the effective rate of interest

Solution

PAYMENT:	−\$5,000	−\$3,000	−\$2,000
TIME:	0	1	2
BALANCE:			\$11,000

The equation of value at $t = 2$ is

$$5000(1+i)^2 + 3000(1+i) + 2000 = 11000.$$

This is equivalent to the equation $5(1+i)^2 + 3(1+i) - 9 = 0$. Then

$$1 + i = \frac{-3 + \sqrt{3^2 + 4(5)(9)}}{2(5)} \approx 1.074772708.$$

Hence $i \approx 0.074772708$.

Chapter 2 Forwards and Futures

George and Sue want to speculate on the price of Stock X, which does not pay dividends. Sue thinks that the price will be higher than \$104 in 6 months; George thinks that the price will be lower than \$104. They make a contract that says that Sue is obligates to buy from George, and George is obligates to sell to Sue, 100 shares of this stock in 6 months at a price of \$104, regardless of the actual market price of the stock at that time.

This type of contract is called a **forward contract**. You can see that a forward contract has the following key elements:

- The contract obligates one party to sell and the other party to buy a specified quantity of an asset. The asset on which the contract is based is called the **underlying asset**.
- The contract specifies the date on which the sale will take place. This date is called the **expiration date**, T. (The contract may also specify the time, place, manner of delivery, etc., if appropriate.)
- The contract specifies the price that will be paid on the expiration date. This price is called the **forward price**, F.

Under a forward contract, neither party pays anything to the other when a forward contract is exchanged. Their obligation is to buy or sell the underlying asset on the expiration date.

If the actual market price in 6 months turns out to be more than \$104 per share, say \$114, Sue will be a winner. George would be obligated to sell the stock to her at \$104 per share. Sue could cash out by selling the share for \$114, and she would make a profit of $\$114-\$104=\$10$ per share. George would be a loser; he would have to buy the stock for \$114 a share and sell it to Sue for only \$104. He would lose \$10 per share.

If the market price in 6 months turns out to be less than \$104, say \$94 per share, George would be a winner. He would buy the stock for only \$94 and Sue would be obligated to buy it from

George for \$104. George would make a profit of $\$104 - \$94 = \$10$ per share, while Sue would lose \$10 per share.

When we buy X, we are said to be **long** X, and when we sell X, we are said to be **short** X. The party under the forward contract who is obligated to buy (Sue in the above example) is said to have a **long position**. So we can call the contract from the buyer's point of view a **long forwards**. The long forward makes money if the price of the underlying asset goes up. The party under the forward contract who is obligated to sell (George in the above example) is said to have a **short position**. So we can call the contract from the seller's point of view a **short forwards**. The short forward makes money if the price of the underlying asset goes down.

Generally speaking, we say that someone has a long position in an asset if he earns money when the price of the asset goes up. We say that someone has a short position in an asset if he earns money when the price of the asset goes down.

By the **payoff**, we mean the value of the contract to one of the parties on a particular date. In the above example, Sue's payoff at the end of 6 months would be \$10 if the price of the stock were \$114 at that time. George's payoff would be –\$10 in this case. If the price of the stock were \$94, these payoffs would be reversed.

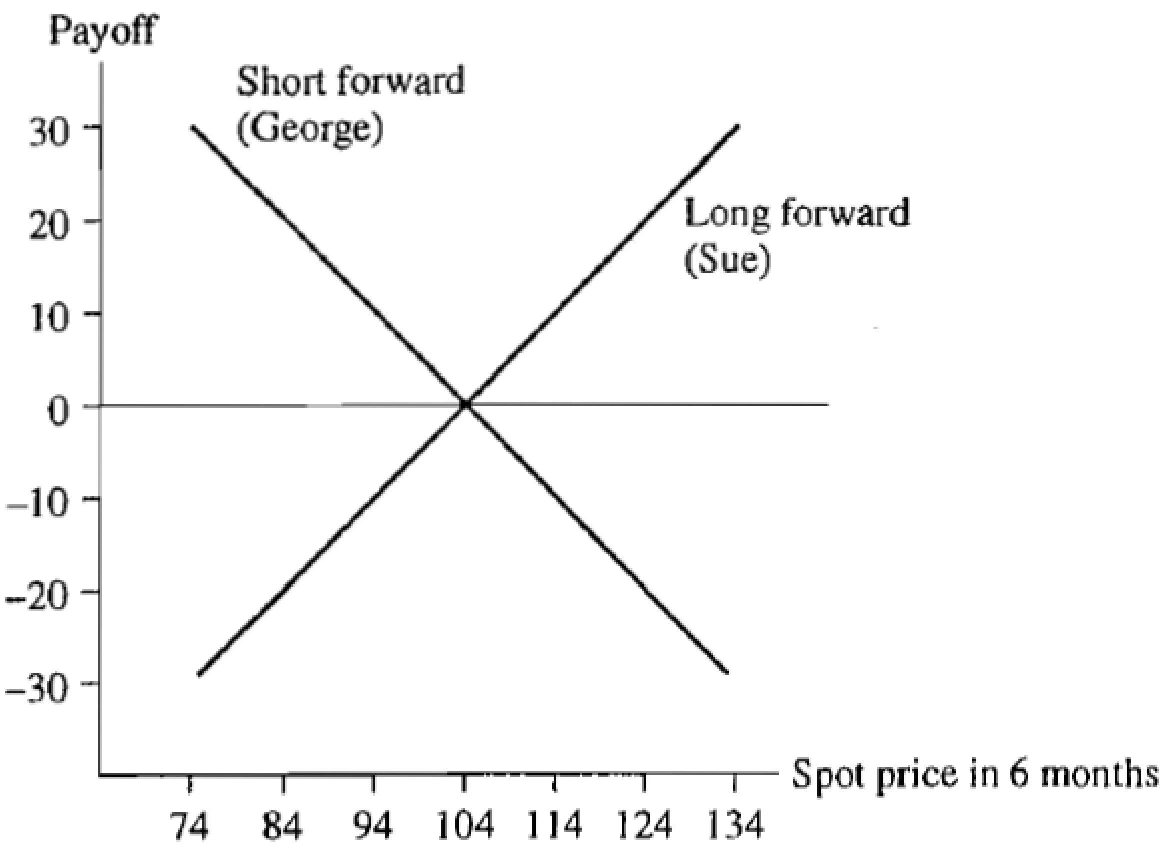
The actual market price of an asset on a particular date T is called the spot price on that date, $S(T)$. So we can express the payoffs as follows:

$$\text{Payoff to long forward} = \text{spot price at expiration} - \text{forward price} = S(T) - F.$$

$$\text{Payoff to short forward} = \text{forward price} - \text{spot price at expiration} = F - S(T).$$

Continuing with the same example that we have been using, the payoffs for some of the possible outcomes are as follows:

Spot Price of Stock In 6 Months	Payoff	
	Long Forward (Sue)	Short Forward (George)
\$74	−\$30	\$30
84	−20	20
94	−10	10
104	0	0
114	10	−10
124	20	−20
134	30	−30



Sue's payoff is negative of George's because the loss of Sue is the gain of George.

We will use the notation $F_{t,T}$ to indicate the price to be paid at time T in a forward agreement made at time t to buy an item at time T.

An **arbitrage** is a set of transactions which when combined have no cost, no possibility of loss and sure profit. One of the fundamental concepts in derivatives pricing is the no-arbitrage principle, which can be loosely stated as "there is no such thing as a free lunch". More formally, in financial term, there are never any opportunities to make an instantaneous risk-free profit. In fact, such opportunities may exist in a real market. But, they cannot last for a significant length of time before prices move to eliminate them. For this reason, we always assume the no-arbitrage principle throughout this notes.

Law of one price says that if there are two portfolios leading to the same end result, they must both have the same cost (or the same cost of two sets of transactions gives the same end result). In a universe with no arbitrage, the law of one price must hold; otherwise one could buy the cheaper portfolio and (short) sell the more expensive portfolio and make a sure profit with no possibility of loss.

A **derivative** can be defined as a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables. Let us determine $F_{0,T}$. The forward agreement is entered at time 0 and provides for transferring the stock at time T.

(i) Forwards on a non-dividend paying stock

There are two ways you can own the stock at time T:

Two Methods for Owning Non-Dividend Paying Stock at Time T

	Method #1: *Buy stock at time 0 and hold it to time T*	Method #2: *Buy forward on stock at time 0 and hold it to time T*
Payment at time 0	S_0	0
Payment at time T	0	$F_{0,T}$

By the principle of no arbitrage, the two ways must have the same cost. The cost at time 0 of the first way is $S(0)$, the price of the stock at time 0. The cost of the second way is a payment of $F_{0,T}$ at time T. Since you can invest at the continuous risk-free rate r, every dollar invested at time 0 becomes e^{rT} dollars at time T. You can pay $F_{0,T}$ at time T by investing $F_{0,T}e^{-rT}$ at time 0.

Suppose $F_{0,T} > S(0)e^{rT}$. At time 0

- borrow the amount $S(0)$ until time T;
- buy one share for $S(0)$;
- enter a short forward contract with forward price $F_{0,T}$.

Then, at time T

- sell the stock for $F_{0,T}$;
- pay $S(0)e^{rT}$ to clear the loan with interest.

This will bring a risk-free profit of $F_{0,T} - S(0)e^{rT} > 0$, contrary to the No-Arbitrage Principle.

Next, suppose that $F_{0,T} < S(0)e^{rT}$. At time 0

- sell short one share for $S(0)$;
- invest the proceeds at the risk-free rate r;
- enter into a long forward contract with forward price $F_{0,T}$.

Then, at time T

- cash the risk-free investment with interest, collecting $S(0)e^{rT}$ dollars;
- buy the stock for $F_{0,T}$ using the forward contract;
- close out the short position in stock by returning it to the owner.

You will end up with a positive amount $S(0)e^{rT} - F_{0,T} > 0$, again a contradiction with the No-Arbitrage Principle. We thus have

$$F_{0,T}e^{-rT} = S(0) \quad \text{or} \quad F_{0,T} = S(0)e^{rT}.$$

If the contract is initiated at time $t \le T$, then

$$F_{t,T}e^{-r(T-t)} = S(t) \quad \text{or} \quad F_{t,T} = S(t)e^{r(T-t)}.$$

(ii) Forwards on a stock with discrete known dividends

Such stock pays known cash amounts at known times. There are the same two ways you can own the stock at time T:

Two Methods for Owning Dividend Paying Stock at Time T

	Method #1: *Buy stock at time 0 and hold it to time T*	Method #2: *Buy forward on stock at time 0 and hold it to time T*
Payment at time 0	S_0	0
Payment at time T	0	$F_{0,T}$

However, if you use the first way, you will also have the stock's dividends accumulated with interest, which you won't have with the second. Therefore, the price of the second way as of time T must be cheaper than the price of the first way as of time T by the accumulated value of the dividends. Equating the 2 costs, and letting CumValue(Div) be the accumulated value at time T of dividends from time 0 to time T, we have

$$F_{0,T} = S(0)e^{rT} - \text{CumValue(Div)}.$$

Suppose $F_{0,T} > S(0)e^{rT} - \text{CumValue(Div)}$. We shall construct an arbitrage strategy. At time 0

- enter into a short forward contract with forward price $F_{0,T}$ and delivery time T;
- borrow $S(0)$ dollars and buy one share.

At time t

- cash the dividend and invest it at the risk-free rate for the remaining time $T-t$.

At time T

- sell the share for $F_{0,T}$;

- pay $S(0)e^{rT}$ to clear the loan with interest and collect $e^{r(T-t)}$ div.

The final balance will be positive:

$$F_{0,T} - S(0)e^{rT} + e^{r(T-t)}\text{div} > 0,$$

a contradiction with the No-Arbitrage Principle. On the other hand, suppose that

$$F_{0,T} < \left(S(0) - e^{-rt}\text{div}\right)e^{rT}.$$

In this case, at time 0

- enter into a long forward contract with forward price $F_{0,T}$ and delivery at time T;
- sell short one share and invest the proceeds $S(0)$ at the risk-free rate.

At time t

- borrow div dollars and pay a dividend to the stock owner.

At time T

- buy one share for $F_{0,T}$ and close out the short position in stock;
- cash the risk-free investment with interest, collecting the amount $S(0)e^{rT}$, and pay $e^{r(T-t)}\text{div}$ to clear the loan with interest.

The final balance will again be positive,

$$-F_{0,T} + S(0)e^{rT} - e^{r(T-t)}\text{div} > 0.$$

Example 2.1

Consider a 10-month forward contract on a stock when the stock price is $50. We assume that the risk-free rate of interest (continuously compounded) is 8% per annum for all maturities. We also assume that dividends of $0.75 per share are expected after 3 months, 6 months, and 9 months. The present value of the dividends, I, is

$$I = 0.75e^{-0.08\times 3/12} + 0.75e^{-0.08\times 6/12} + 0.75e^{-0.08\times 9/12} = 2.162.$$

The variable T is 10 months, so that the forward price, $F_{0,T}$, is given by

$$F_{0,T} = (50 - 2.162)e^{0.08\times 10/12} = \$51.14.$$

(iii) Forwards on a stock with continuous dividends

We will now consider an asset that pays dividends at a continuously compounded rate δ, which get reinvested in the asset. In other words, rather than being paid as cash dividends at certain times, the dividends get reinvested in the asset continuously, so that the investor ends up with additional shares of the asset rather than cash dividends.

If you buy the stock index at time 0 and it pays continuous dividends at the rate δ, you will have $e^{\delta T}$ shares of the index at time T.

So here are two ways to own $e^{\delta T}$ shares of the stock at time T:

Two Methods for Owning $e^{\delta T}$ Shares of Continuous Dividend Paying Stock Index at Time T

	Method #1: *Buy stock index at time 0 and hold it to time T*	Method #2: *Buy $e^{\delta T}$ forwards on stock index at time 0 and hold it to time T*
Payment at time 0	S_0	0
Payment at time T	0	$e^{\delta T} F_{0,T}$

Thus to equate the two ways, you should buy $e^{\delta T}$ units of the forward at time 0. At time T, the accumulated cost of buying the stock at time 0 is $S(0)e^{rT}$, while the cost of $e^{\delta T}$ units of the forward agreement is $F_{0,T}e^{\delta T}$, so we have

$$F_{0,T}e^{\delta T} = S(0)e^{rT} \quad \text{or} \quad F_{0,T} = S(0)e^{(r-\delta)T}.$$

Suppose that $F_{0,T} > S(0)e^{(r-\delta)T}$.

In this case, at time 0

- enter into a short forward contract;
- borrow the amount $S(0)e^{-\delta T}$ to buy $e^{-\delta T}$ shares.

Between time 0 and T collect the dividends paid continuously, reinvesting them in the stock. At time T you will have 1 share, as explained above. At that time

- sell the share for $F_{0,T}$, closing out the short forward position;
- pay $S(0)e^{(r-\delta)T}$ to clear the loan with interest.

The final balance $F_{0,T} - S(0)e^{(r-\delta)T} > 0$ will be your arbitrage profit.

Now suppose that $F_{0,T} < S(0)e^{(r-\delta)T}$.

If this is the case, then at time 0

- take a long forward position;
- sell short a fraction $e^{-\delta T}$ of a share investing the proceeds $S(0)e^{-\delta T}$ risk free.

Between time 0 and T you will need to pay dividends to the stock owner, raising cash by shorting the stock. Your short position in stock will thus increase to 1 share at time T. At that time

- buy one share for $F_{0,T}$ and return it to the owner, closing out the long forward position and the short position in stock;
- receive $S(0)e^{(r-\delta)T}$ from the risk-free investment.

Again you will end up with a positive amount $S(0)e^{(r-\delta)T} - F_{0,T} > 0$, contrary to the no-Arbitrage Principle.

In general, if the contract is initiated at time $t < T$, then

$$F_{t,T} = S(0)e^{(r-\delta)(T-t)}.$$

Futures contracts are similar to forward contracts but there are a number of important differences between them. Here is a summary of their features:

Forwards	**Futures**
The basic feature of the contract is an obligation to buy or sell the underlying asset at a specified price on the expiration date.	Same
Contracts are tailored to the needs of each party.	Contracts are standardized as to expiration dates, sizes, underlying asset or index, etc.
Neither marked-to-market nor settled daily. Settlement is made on expiration date only.	Marked-to-market and settled daily.
Relatively illiquid. Traded over-the-counter. Handled by dealers or brokers. Difficult to settle a contract before expiration date.	Liquid, exchange-traded. Marking-to-market allows for daily settlement.
Credit risk (risk that one of the parties will not fulfill obligation to buy or sell at the specified price) may be a problem.	Marking-to-market and daily settlement minimizes credit risk.
Price limits (limits on changes in the price) are not applicable, since there is no daily marking-to-market or settlement.	The exchange imposes price limits, i.e., if the price changes by a specified percentage or amount, there can be a temporary stop in trading. The rules are complicated.

Throughout this course, we assume the futures price equals the forward price. However, due to marking-to-market, the price may not be equal, as one party or the other can earn interest on the settlement amounts resulting from marking-to-market.

There are several advantages to marking-to-market and daily settlement. As we noted before, the parties have virtually no credit risk if the credits and debits to their accounts follow the changes in the index from day-to-day. Also, it simplifies the procedure for a party to the contract to close that party's position. All that has to be done is for that party to enter into a second futures contract with the opposite position. (E.g., if the original position was long, enter a short futures contract). As soon as a counter-party is found for the second contract, the original party can be released.

There are many ways to buy a stock. One way of buying a stock involve paying and receiving the stock at time 0. It is called an outright purchase. Another way is paying for the stock now and receiving it at time T. This is called a **prepaid forward** or just a **prepay**.

Method of buying stock	Time of payment	Time stock is received	Amount of payment
Outright purchase	0	0	$S(0)$
Prepaid forward contract	0	T	To be determined
Forward contract	T	T	To be determined

Is there any difference between receiving the stock now (outright purchase) and receiving it at time T?

The answer is Yes. It does matter if dividends are paid on the stock between time 0 and time T and you don't get them.

The notation $F_{0,T}^{P}$ is the price of a prepaid forward for the stock bought at time 0 and delivered at time T. Thus we have $F_{0,T}^{P} = S(0)$ in the absence of dividends.

Another way to determine the price is by using the principle of no-arbitrage, i.e., the assumption that derivatives are priced so that it is not possible to make a risk free profit by buying and selling related assets. Suppose the current price of the stock is $S(0)$ = \$100.

If the price of a prepay is more than the current price of the stock, say \$101, you could make a profit of \$1 per share without any risk. Here's what you would do:

(1) Sell a prepay contract to someone for \$101 at time 0.

(2) At the same time, make an outright purchase of the stock for \$100.

(3) Put the \$1 excess that you receive in your pocket.

(4) At time T, you have to give a share of stock to the party that you sold the prepay contract to. But you own a share (see step 2), so you simply give it to that party.

If the price of a prepay is less than the current price of the stock, say \$99, you could make a profit of \$1 per share without any risk. Here's what you would do:

(1) Buy a prepay contract to someone for \$99 at time 0.

(2) At the same time, short the stock by borrowing a share and immediately selling it for \$100.

(3) Put the \$1 excess that you receive in your pocket.

(4) At time T, you have to close the short position by returning a share to the lender of the stock. But you receive a share at that time under the prepay contract (see step 1), so you simply give it to the lender.

But what if there are dividends?

If you had made an outright purchase at the current price, you would have received all of the dividends. If you buy a prepaid forward, you don't get any dividends that were payable before expiration date. It is clear, then, that the price of a prepaid forward should be the current price of the stock minus the present value of the dividends you won't receive.

To take a simple example. Suppose the current price is \$100 and you want to buy a prepaid forward contract with an expiration date that is one year from now. A dividend of \$5 is expected to be paid just before the expiration date. The risk-free interest rate is 6% effective. Then the price of the prepay should be $100-5v=\$95.28$.

Suppose $S(0)$ is the current price of 1 share of stock and you make an outright purchase on 1 share today. Assume that you receive dividends continuously with constant dividend rate δ until time T. You will have $e^{\delta T}$ shares at time T, including the shares purchased by dividends. If the spot price at time T is $S(T)$ per share, the value of your position is $S(T)e^{\delta T}$.

Method of buying the stock	Initial Investment	No. of shares owned at time T	Value at time T
Outright Purchase	$S(0)$	$e^{\delta T}$ (with reinvestment of dividends)	$S(T)e^{\delta T}$
Prepaid forward contract	$F_{0,T}^{P}$	1	$S(T)$

Under an outright purchase, an investment of $S(0)$ grows to $S(T)e^{\delta T}$. Under a prepaid forward contract, an investment of $F_{0,T}^{P}$ grows to $S(T)$. The growth factor under these 2 methods must be the same. (Otherwise, arbitrage would be possible.) By proportion,

$$F_{0,T}^{P}=S(0)e^{-\delta T}.$$

In general, if the forward contract maturing at time T is prepaid at time t, $F_{t,T}^{P}=e^{-r(T-t)}F_{t,T}$.

Underlying asset	Forward price $F_{t,T}$	Prepaid forward price $F_{t,T}^{P}$
Non-dividend-paying stock	$S(t)e^{r(T-t)}$	$S(t)$
Discrete dividend paying stock with known dividend	$S(t)e^{r(T-t)}-\text{CumValue(Divs)}$	$S(t)-\text{PV}_{t,T}\text{(Divs)}$
Discrete dividend paying stock with known yield	$S(t)e^{r(T-t)-(\text{sum of dividend yield})}$	$S(t)e^{-(\text{sum of dividend yield})}$
Continuous dividend paying stock with constant dividend rate δ	$S(t)e^{(r-\delta)(T-t)}$	$S(t)e^{-\delta(T-t)}$
Currency, denominated in currency d for delivery of currency f	$x(t)e^{(r_d-r_f)(T-t)}$	$x(t)e^{-r_f(T-t)}$

Chapter 3 Options

There are two basic types of options: call option and put option.

There are two sides to every option. One side is the investor who has taken the **long** position (i.e., has bought the option). The other side is the investor who has taken a **short** position (i.e., has sold or written the option). The writer of an option receives cash, but has potential liabilities later. The writer's profit or loss is the reverse of that for the purchaser of the option.

There are four possible positions in options markets: a long position in a call, a short position in a call, a long position in a put, and a short position in a put.

A **call option** gives the holder of the option the right, but not the obligation, from the writer of the option to buy the underlying asset by a certain date (**expiration date** T) for a certain price (**strike price**/**exercise price** K).

Exercising a call option means that the call option holder pays the strike price K to buy the underlying asset from the writer of the option at expiration date T.

The writer of the call option is obligated to sell the underlying asset to the holder of the call option if the holder exercises his/her right.

A **put option** gives the holder the right, but not the obligation, to sell an asset to the writer of the put option at **expiration date** T for **strike price**/**exercise price** K.

Exercising a put option means that the put option holder sells the underlying asset at expiration date T for the strike price K.

The writer of the put option is obligated to buy the underlying asset from to the holder of the put option if the holder exercises his/her right.

Options are referred to as in the money, at the money, or out of the money. Let S be the stock price and K be the strike price of an option. A call option is **in the money** when $S > K$, **at the money** when $S = K$, and **out of the money** when $S < K$. A put option is in the money when $S < K$, at the money when $S = K$, and out of the money when $S > K$. Clearly, an option will be exercised only when it is in the money. In the absence of transactions costs, an in-the-money option will always be exercised on the expiration date if it has not been exercised previously.

If K is the strike price and $S(T)$ is the price of the underlying asset at expiration date T, the payoff of a long position in a European call option is

$$\max\{S(T)-K,0\}.$$

This reflects the fact that the option will be exercised if $S(T) > K$ and will not be exercised if $S(T) \leq K$. The payoff of a short position in the European call option is

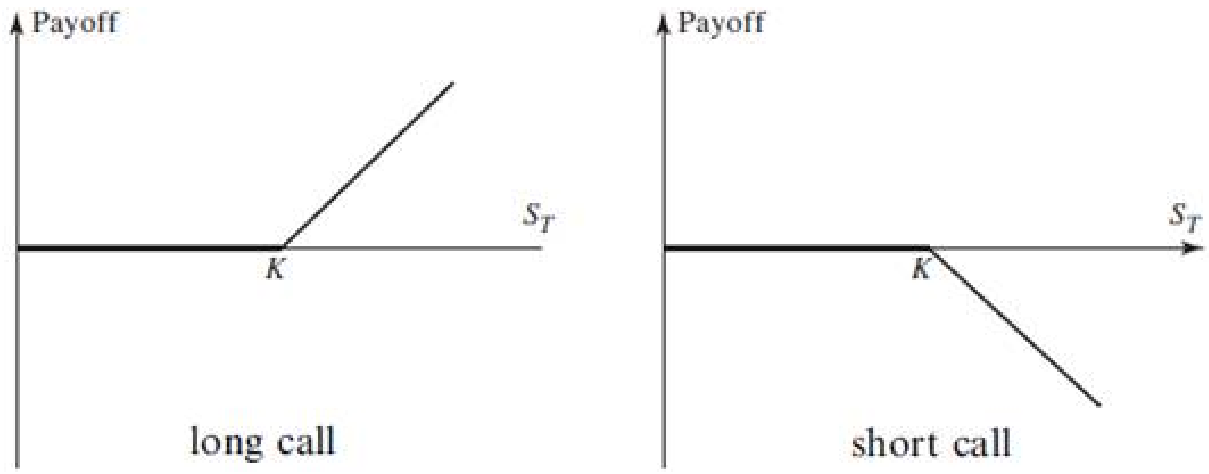
$$-\max\{S(T)-K,0\} = \min\{K-S(T),0\}.$$

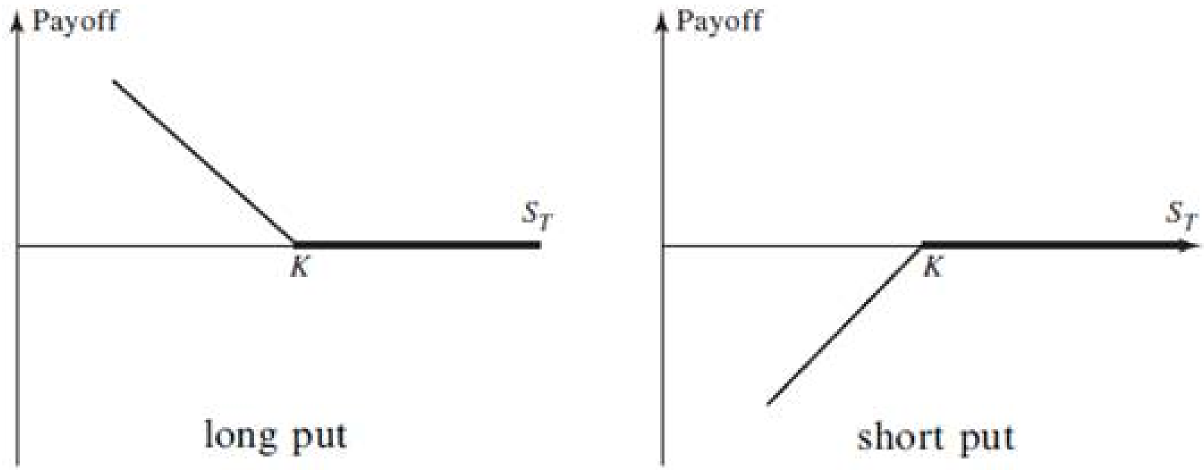
Similarly, the payoff of a long position in a European put option is

$$\max\{K-S(T),0\}$$

and the payoff of a short position in a European put option is

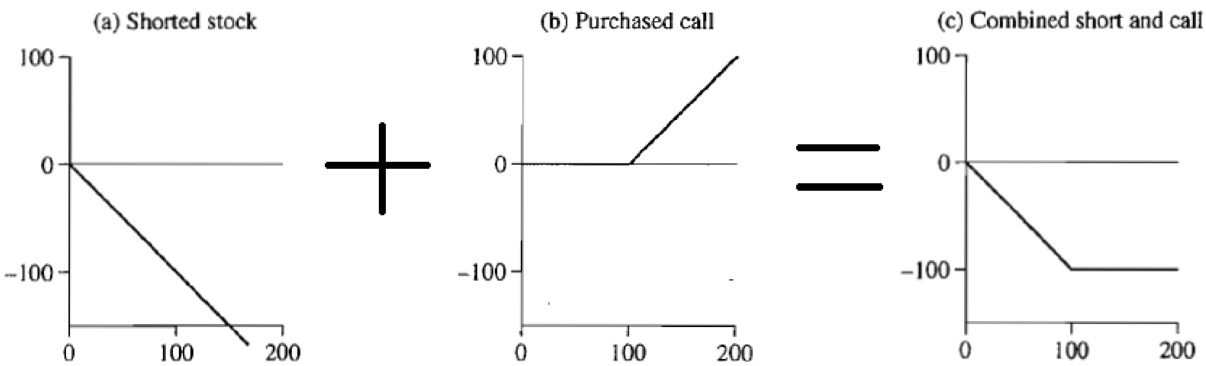
$$-\max\{K-S(T),0\} = \min\{S(T)-K,0\}.$$





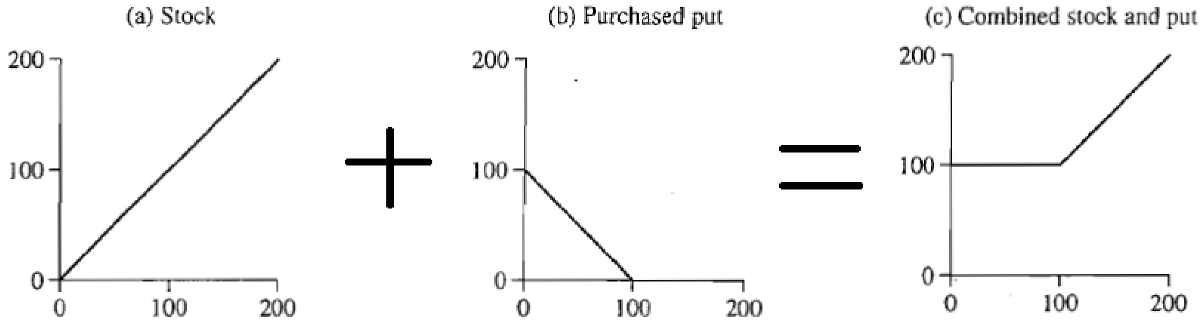
Note that the graph for a long option is the mirror image of the graph for a short option, with the "mirror" being the x-axis. What the option buyer's gain (loss) is what the option seller's loss (gain). This is a zero-sum game.

Suppose you have a short position in a stock and you will have to buy the stock at a future date to close your short position. You could lose an unlimited amount of money if the price of stock increased. However, you can insure yourself against price increases by purchasing a call, with the same stock as the underlying asset.



The call option holder will make money if the price of the asset at expiration is higher than the exercise price, since he will buy the expensive asset for a cheap price. But he will get nothing if the price of the asset at expiration is lower than the exercise price. On the other hand, the call option writer could never make money in either case. The call option writer will not sell the option unless the call option holder pays for it. So if someone wants to own a call option, he has to pay a **premium** C to the call option writer.

Suppose you buy a stock and want to have insurance against the possibility that the price will decline. You can buy a put, with the same stock as the underlying asset. A purchased put has a short position with respect to the stock — the opposite of the long position on the stock itself. So if the stock goes down, you would get a payoff from the long put that would help to offset the loss on the stock itself. If you ask the put option writer to protect your stock against declination of stock price, you need to pay **premium** P to the put option writer.



The profit on a long call is the payoff minus the future value of the call premium C paid for the option. The same for long put.

$$\begin{aligned}\text{Profit on a long call } &= \max\{S(T)-K,0\}-FV(C),\\ \text{Profit on a long put } &= \max\{K-S(T),0\}-FV(P).\end{aligned}$$

The profit on a short call (or put) is negative of the profit on a long call (or put)

$$\begin{aligned}\text{Profit on a short call } &= FV(C)-\max\{S(T)-K,0\},\\ \text{Profit on a short put } &= FV(P)-\max\{K-S(T),0\}.\end{aligned}$$

Example 3.1

The strike price of a 6-month European call option is \$100. The effective rate of interest for a 6-month period is 4%. The premium of call option is \$10.35. What is the payoff and profit of the call at expiration if the spot price of the underlying asset at that time is (a) \$115; (b) \$100; (c) \$90?

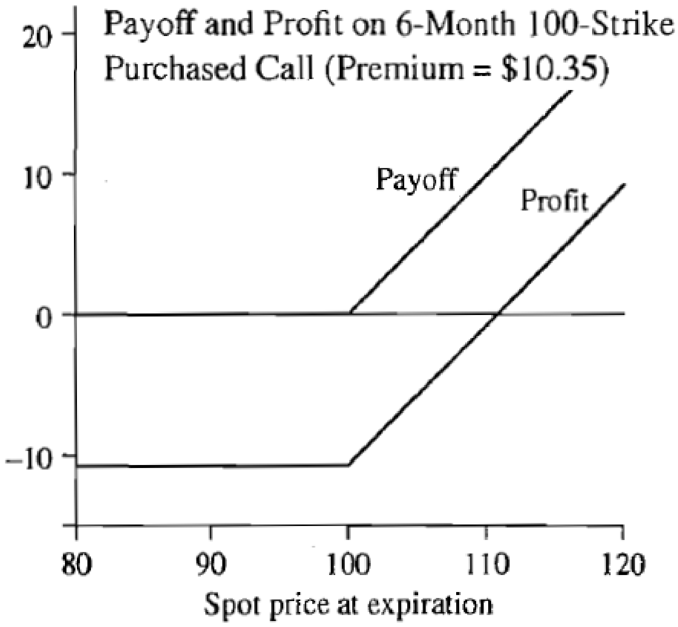
Solution

The future value of premium is 1.04(\$10.35) = \$10.76.

(a) Payoff = $\max\{S(T)-K,0\}=\max\{115-100,0\}=15$. Profit $=15-10.76=4.24$.

(b) Payoff = $\max\{S(T)-K,0\}=\max\{100-100,0\}=0$. Profit $=0-10.76=-10.76$.

(c) Payoff = $\max\{S(T)-K,0\}=\max\{90-100,0\}=0$. Profit $=0-10.76=-10.76$.



Example 3.2

The strike price of a 6-month European put option is \$100. The effective rate of interest for a 6-month period is 4%. The premium of put option is \$6.5. At what spot price at expiration will the put option holder has a 0 profit?

Solution

The future value of the put premium is 1.04(\$6.5) = \$6.76. Let $S(0.5)$ be spot price at expiration for which the profit is 0.

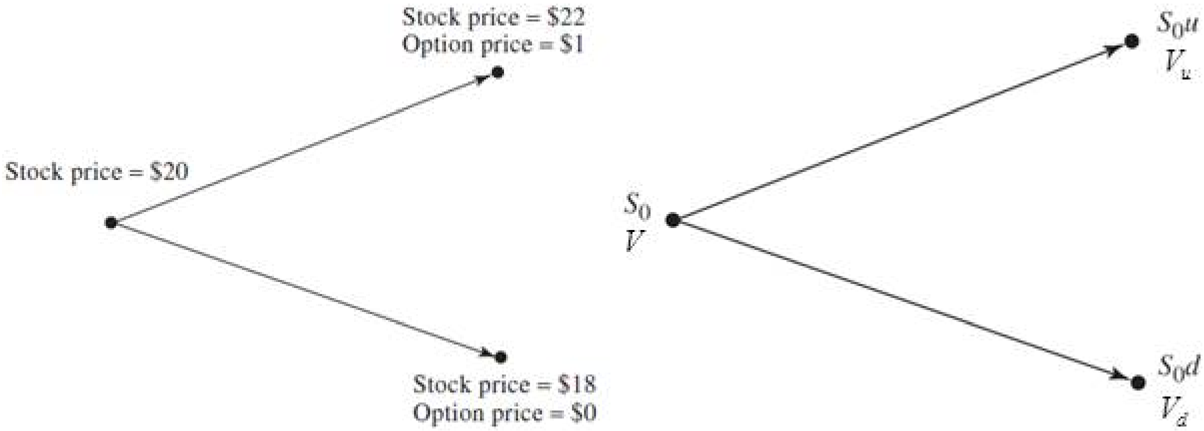
Profit on purchased put = $\max\{100-S(0.5),0\}-6.76=0$.

$S(0.5)=100-6.76=93.24$.

Options can be either American or European, a distinction that has nothing to do with geographical location. **American options** can be exercised at any time up to the expiration date, whereas **European options** can be exercised only at expiration date.

Example 3.3 (One-step binomial model)

A stock price is currently \$20, and it is known that at the end of 3 months it will be either \$22 or \$18. We are interested in valuing a European call option to buy the stock for \$21 in 3 months. This option will have one of two values at the end of the 3 months. If the stock price turns out to be \$22, the value of the option will be \$1; if the stock price turns out to be \$18, the value of the option will be 0.



Consider a portfolio consisting of a long position in Δ (capital Greek letter "delta") shares of the stock and a short position in 1 call option. We calculate the value of Δ that makes the portfolio riskless. If the stock price moves up from \$20 to \$22, the value of the shares is 22Δ and the value of the option is 1, so that the total value of the portfolio is $22\Delta - 1$. If the stock price moves down from \$20 to \$18, the value of the shares is 18Δ and the value of the option is 0, so that the total value of the portfolio is 18Δ. The portfolio is riskless if the value of Δ is chosen so that the final value of the portfolio is the same for both alternatives. This means

$$22\Delta - 1 = 18\Delta \quad \text{or} \quad \Delta = 0.25.$$

A riskless portfolio is long 0.25 shares of stock and short 1 option.

If the stock price moves up to \$22, the value of the portfolio is $22 \times 0.25 - 1 = 4.5$.

If the stock price moves down to \$18, the value of the portfolio is $18 \times 0.25 = 4.5$

Regardless of whether the stock price moves up or down, the value of the portfolio is always 4.5 at the end of the life of the option. This shows that Δ is the number of shares necessary to hedge a short position in 1 call option.

Riskless portfolios must, in the absence of arbitrage opportunities, earn the risk-free rate of interest. Suppose that, in this case, the risk-free rate is 12% per annum. It follows that the value of the portfolio today must be the present value of 4.5, or

$$4.5e^{-0.12\times 3/12} = 4.367.$$

The value of the stock price today is known to be \$20. Suppose the call option price is denoted by C. The value of the portfolio today is

$$20\times 0.25 - C = 5 - C = 4.367, \quad C = 0.633.$$

If the value of the option were more than 0.633, the portfolio would cost less than 4.367 to set up and would earn more than the risk-free rate. If the value of the option were less than 0.633, shorting the portfolio would provide a way of borrowing money at less than the risk-free rate. This shows that, in the absence of arbitrage opportunities, the current value of the option must be 0.633.

We can generalize the no-arbitrage argument just presented by considering a non-dividend paying stock whose price is $S(0)$ and an European (call or put) option on the stock (or any derivative dependent on the stock) whose current price is V. We suppose that the option lasts for time T and that during the life of the call option the stock price can either move up from $S(0)$ to a new level, $S(0)u$, where $u > 1$, or down from $S(0)$ to a new level, $S(0)d$, where $d < 1$. If the stock price moves up to $S(0)u$, payoff from the option is denoted by V_u; if the stock price moves down to $S(0)d$, the payoff from the option is denoted by V_d.

As before, we imagine a portfolio consisting of a long position in Δ shares and a short position in 1 option. We calculate the value of Δ that makes the portfolio riskless. If there is an up movement in the stock price, the value of the portfolio at the end of the life of the option is

$$S(0)u\Delta - V_u.$$

If there is a down movement in the stock price, the value becomes

$$S(0)d\Delta - V_d.$$

The two are equal when

$$S(0)u\Delta - V_u = S(0)d\Delta - V_d \qquad \text{or} \qquad \Delta = \frac{V_u - V_d}{S(0)u - S(0)d}.$$

It shows that Δ is the ratio of the change in the option price to the change in the stock price as we move between the nodes at time T.

In this case, the portfolio is riskless and, for there to be no arbitrage opportunities, it must earn the risk-free continuous interest rate. If we denote the interest rate by r, the present value of the portfolio is

$$(S(0)u\Delta - V_u)e^{-rT} = (S(0)d\Delta - V_d)e^{-rT}.$$

The cost of setting up the portfolio is

$$S(0)\Delta - V.$$

It follows that

$$\begin{aligned}
&S(0)\Delta - V = (S(0)u\Delta - V_u)e^{-rT}.\\
&V = S(0)\Delta\left(1 - ue^{-rT}\right) + V_u e^{-rT}\\
&\quad = S(0)\left(\frac{V_u - V_d}{S(0)u - S(0)d}\right)\left(1 - ue^{-rT}\right) + V_u e^{-rT}\\
&\quad = \frac{V_u\left(1 - ue^{-rT}\right) - V_d\left(1 - ue^{-rT}\right) + V_u e^{-rT}(u-d)}{u-d}\\
&\quad = \frac{V_u\left(1 - de^{-rT}\right) - V_d\left(1 - ue^{-rT}\right)}{u-d}\\
&\quad = e^{-rT}\left(p^* V_u + (1-p^*)V_d\right)
\end{aligned}$$

where $p^* = \dfrac{e^{rT} - d}{u - d}$.

We can replicate an option of premium V with a portfolio of a Δ shares of stock and B in a risk-free bond with interest rate r, that is, $V = S(0)\Delta + B$. We may solve

$$V_u = S(0)u\Delta + Be^{rT} \qquad \text{and} \qquad V_d = S(0)d\Delta + Be^{rT}$$

to get

$$\Delta = \frac{V_u - V_d}{S(0)u - S(0)d} \qquad \text{and} \qquad B = e^{-rT}\frac{uV_d - dV_u}{u-d}.$$

In Example 3.3, $p^* = \dfrac{e^{0.03} - 0.9}{1.1 - 0.9} = 0.6523$ and $B = e^{-0.03}\dfrac{-0.9}{1.1-0.9} = -4.367$. Instead of buying a call option $C = 0.633$, investor can borrow 4.367 from a bank together with $C = 0.633$ and buy 0.25 share of stock. The payoff of the call option is exactly the same as that of long 0.25 share of stock together short 4.367 of bond after 3 months.

Risk-neutral means investors do not increase the expected return they require from an investment to compensate for increased risk. That means, for example, there is no difference between having \$20 for sure and having a random amount of money with expected value \$20. A world where investors are risk-neutral is referred to as a risk-neutral world.

A risk-neutral world has two features that simplify the pricing of derivatives:

1. The expected return on a stock (or any other investment) is the risk-free rate.

2. The discount rate used for the expected payoff on an option (or any other instrument) is the risk-free rate.

The parameter $p^* = \dfrac{e^{rT} - d}{u - d}$ should be interpreted as the probability of an up movement in a risk-neutral world, so that $1 - p^*$ is the probability of a down movement in this world. The expression

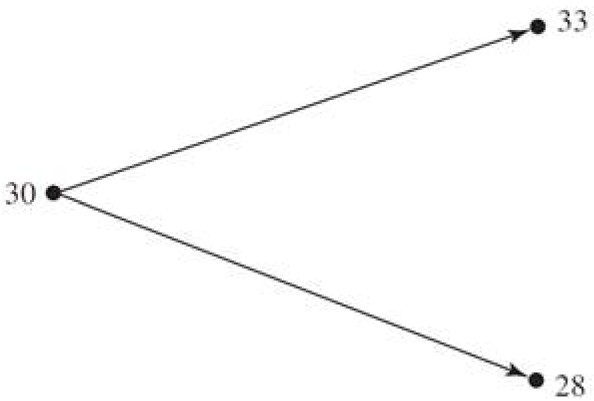
$$E[V(T)] = p^* V_u + (1 - p^*)V_d$$

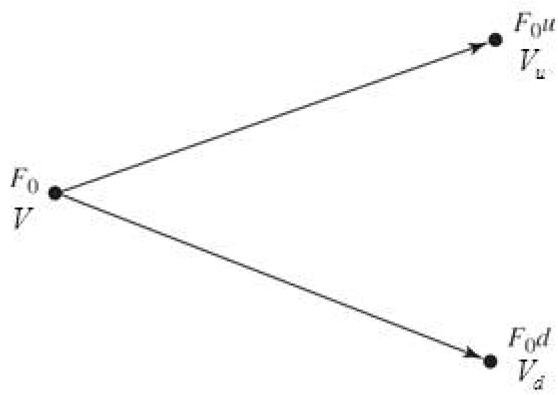
is the expected future payoff from the option in a risk-neutral world and equation $V = e^{-rT}\left(p^* V_u + (1-p^*)V_d\right)$ states that the value of the option today is its expected future payoff in a risk-neutral world discounted at the risk-free rate.

A futures option, or option on futures contract, is an option in which the underlying asset is a futures contract. The buyer has the right, but not the obligation, to enter into a futures contract at a certain futures price by a certain date. Specifically, a call futures option is the right to enter into a long futures contract at a certain price and a put futures option is the right to enter into a short futures contract at a certain price. The payoff of the futures call and put are $\max\{F(T)-K, 0\}$ and $\max\{K-F(T), 0\}$ respectively, where $F(T)$ is the futures price at the time of exercise and K is the strike price.

Example 3.4

Suppose that the current futures price is 30 and that it will move either up to 33 or down to 28 over the next month. We consider a one-month call option on the futures with a strike price of 29 and ignore daily settlement. (If the call option is exercised, the investor will enter a long position of futures.)





If the futures price is 33, the payoff from the option is 4 and the value of the futures contract is 3. If the futures price is 28, the payoff from the option is zero and the value of the futures contract is -2.

To set up a riskless hedge, we consider a portfolio consisting of a short position in 1 call option and a long position in Δ futures contracts. If the futures price moves up to 33, the value of the portfolio is $3\Delta - 4$; if it moves down to 28, the value of the portfolio is -2Δ. The portfolio is riskless when these are the same, that is, when

$$3\Delta - 4 = -2\Delta \qquad \text{or} \qquad \Delta = 0.8.$$

For this value of Δ, we know the portfolio will be worth $3 \times 0.8 - 4 = -1.6$ in one month. Assume a risk-free interest rate of 6%. The value of the portfolio today must be

$$-1.6e^{-0.06\times 1/12} = -1.592.$$

The portfolio consists of one short call option and Δ futures contracts. Because the value of the futures contract today is zero, the value of the option today must be 1.592.

We can generalize this analysis by considering a futures price that starts at $F(0)$ and is anticipated to rise to $F(0)u$ or move down to $F(0)d$ over the time period T. We consider an option maturing at time T and suppose that its payoff is V_u if the futures price moves up and V_d if it moves down.

The riskless portfolio in this case consists of a short position in one option combined with a long position in Δ futures contracts, that is $\Pi(t) = \Delta(F(t) - F(0)) - V_t$. The value of the portfolio at time T is

$$\Delta\left(F(0)u - F(0)\right) - V_u = \Delta\left(F(0)d - F(0)\right)\Delta - V_d.$$

Then

$$\Delta = \frac{V_u - V_d}{F(0)u - F(0)d}.$$

Denoting the risk-free interest rate by r, we obtain the value of the portfolio today as

$$\left[\Delta\left(F(0)u - F(0)\right) - V_u\right]e^{-rT} = \left[\Delta\left(F(0)d - F(0)\right) - V_d\right]e^{-rT}.$$

Another expression for the present value of the riskless portfolio is $\Pi(0) = \Delta(F(0) - F(0)) - V$ $= -V$, where V is the value of the futures option today. It follows that

$$-V = \left[\Delta\left(F(0)u - F(0)\right) - V_u\right]e^{-rT}.$$

Substituting for Δ and simplifying reduces this equation to

$$\begin{aligned} V &= -\left[\Delta\left(F(0)u - F(0)\right) - V_u\right]e^{-rT} \\ &= -\left[\left(u-1\right)\frac{V_u - V_d}{u-d} - V_u\right]e^{-rT} \\ &= -\left[\frac{\left(u-1\right)V_u - (u-d)V_u - \left(u-1\right)V_d}{u-d}\right]e^{-rT} \\ &= \left[\frac{\left(1-d\right)V_u + \left(u-1\right)V_d}{u-d}\right]e^{-rT} \\ &= \left[p^* V_u + (1-p^*)V_d\right]e^{-rT} \end{aligned}$$

where

$$p^* = \frac{1-d}{u-d}.$$

The value of futures option V is equivalent to long Δ futures contracts and long $V = -\Pi(0)$ bonds.

In Example 3.4, u = 1.1, d = 0.9333, r = 0.06, $T = 1/12$, $V_u = 4$, and $V_d = 0$. Then $p^* = \frac{1-0.9333}{1.1-0.9333} = 0.4, \Delta = \frac{4-0}{30(1.1-0.9333)} = 0.8$ and $V = \left[0.4\times 4 + 0.6\times 0\right]e^{-0.06\times 1/12} = 1.592$.

In above, there are two securities (the stock and the bond) and only two possible outcomes of option. It is always possible to replicate portfolio. A tree with more than 2 branches can also be valued with a replicating portfolio, as long as there are $n-1$ different assets together with bond for an n-way branch.

Example 3.5

You are given 2 non-dividend paying stocks, X and Y. The current price of each one is 100. After 1 year, there are 3 possibilities for their prices (trinomial model):

Outcome	Price of *X*	Price of *Y*
1	130	150
2	100	110
3	95	70

Let *C*(*X*, 95, 1) be the price of a European call option on *X* with strike price 95 and 1 year to expiry. Let *P*(*Y*, 90, 1) be the price of a European put option on *X* with strike price 90 and 1 year to expiry. The continuously compounded risk-free rate is 10%. Calculate *C*(*X*, 95, 1) + *P*(*Y*, 90, 1).

Solution

Let Δ_X be the number of shares of *X* to buy and Δ_Y be the number of shares of *Y* to buy. Let *B* be the amount of risk-free bonds to buy. The replicating portfolio is $\Delta_X X(0)+\Delta_Y Y(0)+B$, and must replicate the three outcomes. The call pays 35 and 5 at the top 2 outcomes, and the put pays 20 at the lowest outcome. So

$$\begin{aligned}130\Delta_X+150\Delta_Y+Be^{0.1}&=35\\100\Delta_X+110\Delta_Y+Be^{0.1}&=5\\95\Delta_X+70\Delta_Y+Be^{0.1}&=20\end{aligned}$$

We obtain $\Delta_X=1.8, \Delta_Y=-0.6$ and $Be^{0.1}=-109$. The option portfolio is

$$\Delta_X X(0)+\Delta_Y Y(0)+B=100(1.8)+100(-0.6)-109e^{-0.1}=21.37.$$

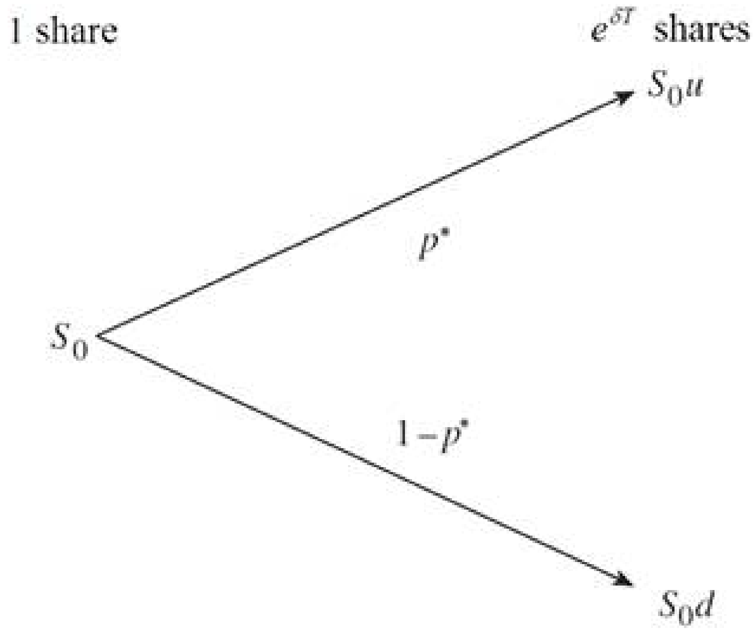
It is also possible to solve this example using risk-neutral probabilities. Let p_i^* be the probability of outcome *i*. Then $p_1^*+p_2^*+p_3^*=1,$ and

$$130p_1^*+100p_2^*+95(1-p_1^*-p_2^*)=100e^{0.1},\quad 150p_1^*+110p_2^*+70(1-p_1^*-p_2^*)=100e^{0.1}$$
$$35p_1^*+5p_2^*=100e^{0.1}-95,\qquad 80p_1^*+40p_2^*=100e^{0.1}-70$$

We have $p_1^*=0.418098,$ $p_2^*=0.176731$ and $p_3^*=0.405171.$ The value of the option portfolio is

$$e^{-0.1}(0.418098(35)+0.176731(5)+0.405171(20))=21.37$$

Let $S(0)$ be the stock price at 0 with continuous dividend rate δ. Let V be the current option price, V_u and V_d the value of the option at the upper and lower nodes respectively. Let B be the risk-free bond with interest rate r and T the expiration date.



The option premium is $V = \Delta S(0) + B$. We have

$$e^{\delta T}\Delta S(0)u + Be^{rT} = V_u \qquad \text{and} \qquad e^{\delta T}\Delta S(0)d + Be^{rT} = V_d$$

Then

$$\Delta = e^{-\delta T}\frac{V_u - V_d}{S(0)(u-d)} \quad \text{and} \quad B = e^{-rT}\frac{uV_d - dV_u}{u-d}$$

It follows that

$$\begin{aligned}
V &= \Delta S(0) + B \\
&= e^{-\delta T}\frac{V_u - V_d}{u-d} + e^{-rT}\frac{uV_d - dV_u}{u-d} \\
&= \frac{e^{-\delta T} - de^{-rT}}{u-d}V_u + \frac{ue^{-rT} - e^{-\delta T}}{u-d}V_d \\
&= e^{-rT}\left(\frac{e^{(r-\delta)T} - d}{u-d}V_u + \frac{u - e^{(r-\delta)T}}{u-d}V_d\right) \\
&= e^{-rT}\left(p^*V_u + (1-p^*)V_d\right)
\end{aligned}$$

where $p^* = \dfrac{e^{(r-\delta)T} - d}{u-d}$ is the risk-neutral probability.

A foreign currency can be regarded as an asset providing a yield at the foreign risk-free rate of interest, r_f.

In general, u, d, p^* may not be constant at any nodes (see Example 3.16).

In practice, when constructing a binomial tree to represent the movements in a stock price, we choose the parameters *u* and *d* to match the expected return rate *r* and the volatility σ of the stock price.

In order to match the expected return rate on the stock with the tree's parameters, we must have

$$p^* S(0)u + (1-p^*)S(0)d = S(0)e^{(r-\delta)\Delta t} \quad \text{or} \quad p^* = \frac{e^{(r-\delta)\Delta t} - d}{u-d}.$$

In order to match the stock price volatility with the tree's parameters, we must therefore have

$$\begin{aligned}
\sigma^2 \Delta t &= p^* u^2 + (1-p^*)d^2 - (p^* u + (1-p^*)d)^2 \\
&= p^* u^2 + (1-p^*)d^2 - (p^*)^2 u^2 - (1-p^*)^2 d^2 - 2p^*(1-p^*)ud \\
&= p^*(1-p^*)u^2 + p^*(1-p^*)d^2 - 2p^*(1-p^*)ud \\
&= p^*(1-p^*)(u-d)^2 \\
&= (u - e^{(r-\delta)\Delta t})(e^{(r-\delta)\Delta t} - d)
\end{aligned}$$

One of the solutions is **Cox-Ross-Rubinstein tree**. The tree is centered on 1,

$$u = e^{\sigma\sqrt{\Delta t}} \qquad \text{and} \qquad d = e^{-\sigma\sqrt{\Delta t}}.$$

By Taylor expansion,

$$\begin{aligned}
u = e^{\sigma\sqrt{\Delta t}} &= 1 + \sigma\sqrt{\Delta t} + \frac{\sigma^2 \Delta t}{2!} + \cdots \\
d = e^{-\sigma\sqrt{\Delta t}} &= 1 - \sigma\sqrt{\Delta t} + \frac{\sigma^2 \Delta t}{2!} - \cdots \\
e^{(r-\delta)\Delta t} &= 1 + (r-\delta)\Delta t + \frac{((r-\delta)\Delta t)^2}{2!} + \cdots
\end{aligned}$$

$$
\begin{aligned}
(u-e^{(r-\delta)\Delta t})(e^{(r-\delta)\Delta t}-d) &= \left(1+\sigma\sqrt{\Delta t}+\frac{\sigma^2\Delta t}{2!}+\cdots-\left(1+(r-\delta)\Delta t+\frac{((r-\delta)\Delta t)^2}{2!}+\cdots\right)\right) \\
&\quad \left(1+(r-\delta)\Delta t+\frac{((r-\delta)\Delta t)^2}{2!}+\cdots-\left(1-\sigma\sqrt{\Delta t}+\frac{\sigma^2\Delta t}{2!}-\cdots\right)\right) \\
&= \left(\sigma\sqrt{\Delta t}+\left(\frac{\sigma^2}{2!}-(r-\delta)\right)\Delta t+\cdots\right)\left(\sigma\sqrt{\Delta t}+\left((r-\delta)-\frac{\sigma^2}{2!}\right)\Delta t+\cdots\right) \\
&\approx \sigma^2\Delta t
\end{aligned}
$$

when higher powers of Δt are ignored.

For binomial tree on futures based on Cox-Ross-Rubinstein tree,

$$
p^* = \frac{1-d}{u-d} = \frac{1-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}} = \frac{1-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}(1-e^{-2\sigma\sqrt{\Delta t}})} = \frac{1}{1+e^{\sigma\sqrt{\Delta t}}},
$$

$$
1-p^* = 1-\frac{1}{1+e^{\sigma\sqrt{\Delta t}}} = \frac{e^{\sigma\sqrt{\Delta t}}}{1+e^{\sigma\sqrt{\Delta t}}} = \frac{1}{1+e^{-\sigma\sqrt{\Delta t}}}.
$$

We cannot make u too low or d too high. The result of a risk-free investment is greater than the lower node and less than the upper node. That is

$$
d < e^{(r-\delta)\Delta t} < u.
$$

Otherwise, arbitrage would be possible:

either $e^{(r-\delta)\Delta t} \le d$, the stock would always be better than the risk-free investment (in which case you would borrow money as much as you could and buy the stock)

or $u \le e^{(r-\delta)\Delta t}$, the risk-free investment would always be better than the stock (in which case you would short the stock as much as possible and buy bonds).

If Δt is small enough, then $d = e^{-\sigma\sqrt{\Delta t}} < e^{(r-\delta)\Delta t} < e^{\sigma\sqrt{\Delta t}} = u.$ But it cannot hold if Δt is large.

Another solution is **forward tree**. The tree is centered on $e^{(r-\delta)\Delta t}$,

$$
u = e^{(r-\delta)\Delta t+\sigma\sqrt{\Delta t}} \qquad \text{and} \qquad d = e^{(r-\delta)\Delta t-\sigma\sqrt{\Delta t}}.
$$

$$\begin{aligned}(u-e^{(r-\delta)\Delta t})(e^{(r-\delta)\Delta t}-d) &= \left(1+(r-\delta)\Delta t+\sigma\sqrt{\Delta t}+\cdots-\left(1+(r-\delta)\Delta t+\cdots\right)\right) \\ &\quad \left(1+(r-\delta)\Delta t+\cdots-\left(1-\left((r-\delta)\Delta t+\sigma\sqrt{\Delta t}\right)+\cdots\right)\right) \\ &= \left(\sigma\sqrt{\Delta t}+\cdots\right)\left(\sigma\sqrt{\Delta t}+\cdots\right) \\ &\approx \sigma^2\Delta t\end{aligned}$$

when higher powers of Δt are ignored.

Clearly, $d=e^{(r-\delta)\Delta t-\sigma\sqrt{\Delta t}}<e^{(r-\delta)\Delta t}<e^{(r-\delta)\Delta t+\sigma\sqrt{\Delta t}}=u$ for any Δt. The risk neutral probability is

$$p^*=\frac{e^{(r-\delta)\Delta t}-d}{u-d}=\frac{e^{(r-\delta)\Delta t}-e^{(r-\delta)\Delta t-\sigma\sqrt{\Delta t}}}{e^{(r-\delta)\Delta t+\sigma\sqrt{\Delta t}}-e^{(r-\delta)\Delta t-\sigma\sqrt{\Delta t}}}=\frac{1-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}=\frac{1-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}(1-e^{-2\sigma\sqrt{\Delta t}})}=\frac{1}{1+e^{\sigma\sqrt{\Delta t}}}$$

and

$$1-p^*=\frac{1}{1+e^{-\sigma\sqrt{\Delta t}}}.$$

The other solution is **lognormal tree**, which is known as Jarrow-Rudd binomial model. The risk neutral probability is 0.5, but not $p^*=\dfrac{e^{(r-\delta)\Delta t}-d}{u-d}$. The tree is centered on $e^{(r-\delta-0.5\sigma^2)\Delta t}$,

$$u=e^{(r-\delta-0.5\sigma^2)\Delta t+\sigma\sqrt{\Delta t}} \qquad \text{and} \qquad d=e^{(r-\delta-0.5\sigma^2)\Delta t-\sigma\sqrt{\Delta t}}.$$

$$\begin{aligned}p^*(1-p^*)(u-d)^2 &= 0.25\left(e^{(r-\delta-0.5\sigma^2)\Delta t+\sigma\sqrt{\Delta t}}-e^{(r-\delta-0.5\sigma^2)\Delta t-\sigma\sqrt{\Delta t}}\right)^2 \\ &= 0.25\left(\begin{array}{l}1+(r-\delta-0.5\sigma^2)\Delta t+\sigma\sqrt{\Delta t}+\dfrac{\left((r-\delta-0.5\sigma^2)\Delta t+\sigma\sqrt{\Delta t}\right)^2}{2!}+\cdots \\ -\left(1+(r-\delta-0.5\sigma^2)\Delta t-\sigma\sqrt{\Delta t}+\dfrac{\left((r-\delta-0.5\sigma^2)\Delta t-\sigma\sqrt{\Delta t}\right)^2}{2!}+\cdots\right)\end{array}\right)^2 \\ &\approx 0.25\left(2\sigma\sqrt{\Delta t}\right)^2 \\ &= \sigma^2\Delta t\end{aligned}$$

when higher powers of Δt are ignored.

This model has the potential of having $u = e^{(r-\delta-0.5\sigma^2)\Delta t+\sigma\sqrt{\Delta t}} < e^{(r-\delta)\Delta t}$ which would violate the no-arbitrage principle, but this would only happen for unusually high values of σ and Δt.

All 3 trees have $\sigma\sqrt{\Delta t} = \frac{1}{2}(\ln u - \ln d)$. The further apart u and d are, the greater the volatility of stock's price.

Example 3.6

A put option on a stock with expiration date Δt is modeled as a 1-period binomial tree based on forward free. The stock price is 75. The strike price is 80. $r = 0.08$, $\delta = 0.02$, $\sigma = 0.3$, $u =$ 1.17939. Determine the premium of the put option.

Solution

$$
\begin{aligned}
&u = e^{(r-\delta)\Delta t+\sigma\sqrt{\Delta t}} = e^{(0.08-0.02)\Delta t+(0.3)\sqrt{\Delta t}} = 1.17939 \\
&0.06\Delta t + 0.3\sqrt{\Delta t} = \ln 1.17939 = 0.165 \\
&\sqrt{\Delta t} = 0.5 \quad \text{or} \quad -5.5 \quad \text{(rejected)} \\
&\Delta t = 0.25 \\
&d = e^{(r-\delta)\Delta t-\sigma\sqrt{\Delta t}} = e^{(0.08-0.02)(0.25)-(0.3)\sqrt{0.5}} = e^{-0.135} = 0.87372
\end{aligned}
$$

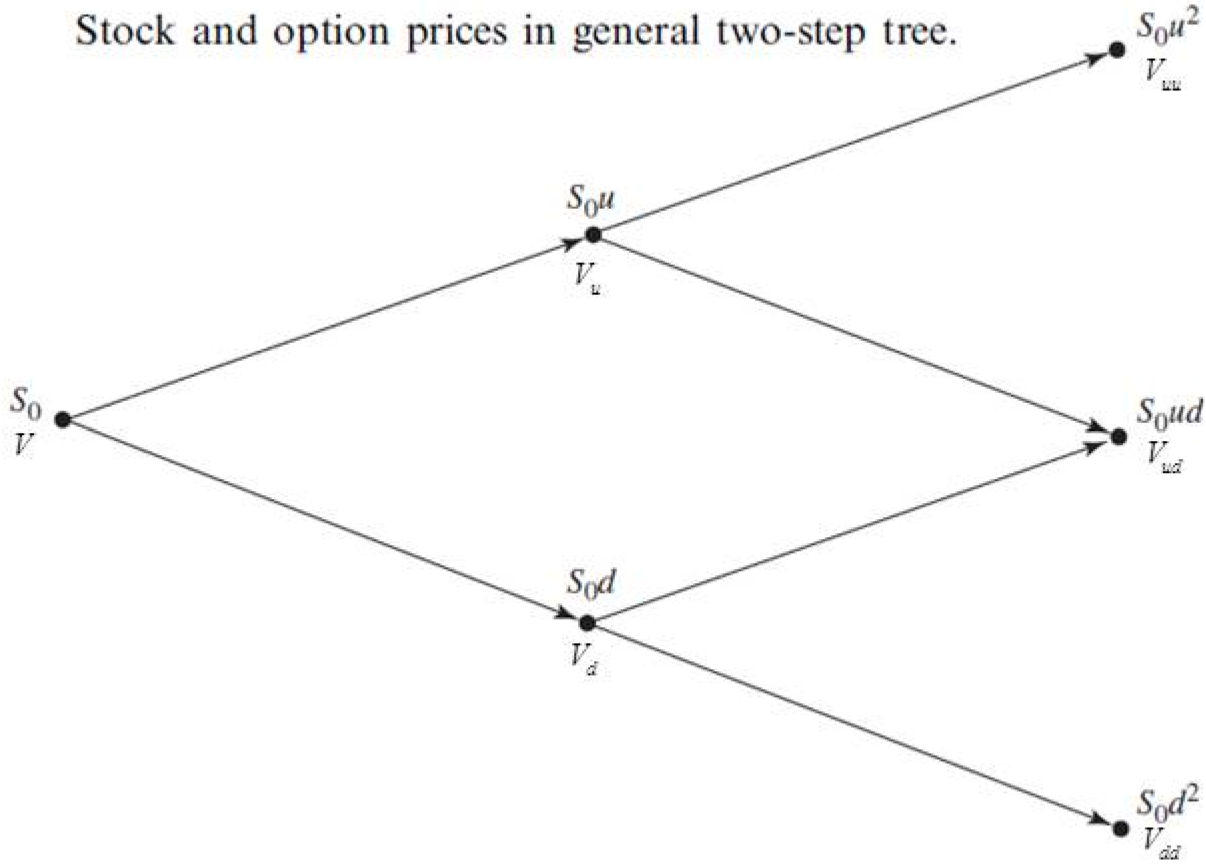
The risk neutral probability is $p^* = \dfrac{1}{1+e^{\sigma\sqrt{\Delta t}}} = \dfrac{1}{1+e^{0.3(0.5)}} = 0.46257.$

The payoff of the put is $\max\{0, 80 - uS\} = \max\{0, 80 - 1.17939(75)\} = 0$ at the upper node and $\max\{0, 80 - dS\} = \max\{0, 80 - 0.87372(75)\} = 14.471$ at the lower node.

The put premium is

$$
e^{-0.08(0.25)}\left(p^*(0) + (1-p^*)(14.471)\right) = e^{-0.02}(1-0.46257)(14.471) = 7.623
$$

We can generalize the binomial tree model of two time steps. The stock price is initially $S(0)$. During each time step, it either moves up to u times its initial value or moves down to d times its initial value. The notation for the value of the option is shown on the tree. (For example, after two up movements the value of the option is V_{uu}.) We suppose that the risk-free interest rate is r and the length of the time step is Δt years.



Because the length of a time step is now Δt rather than T, we have

$$V = e^{-r\Delta t}\left(p^* V_u + (1-p^*)V_d\right) \quad \text{and} \quad p^* = \frac{e^{(r-\delta)\Delta t} - d}{u-d}.$$

Repeating the process, we get

$$V_u = e^{-r\Delta t}\left(p^* V_{uu} + (1-p^*)V_{ud}\right) \quad \text{and} \quad V_d = e^{-r\Delta t}\left(p^* V_{ud} + (1-p^*)V_{dd}\right).$$

Substituting these 2 equations into f, we obtain

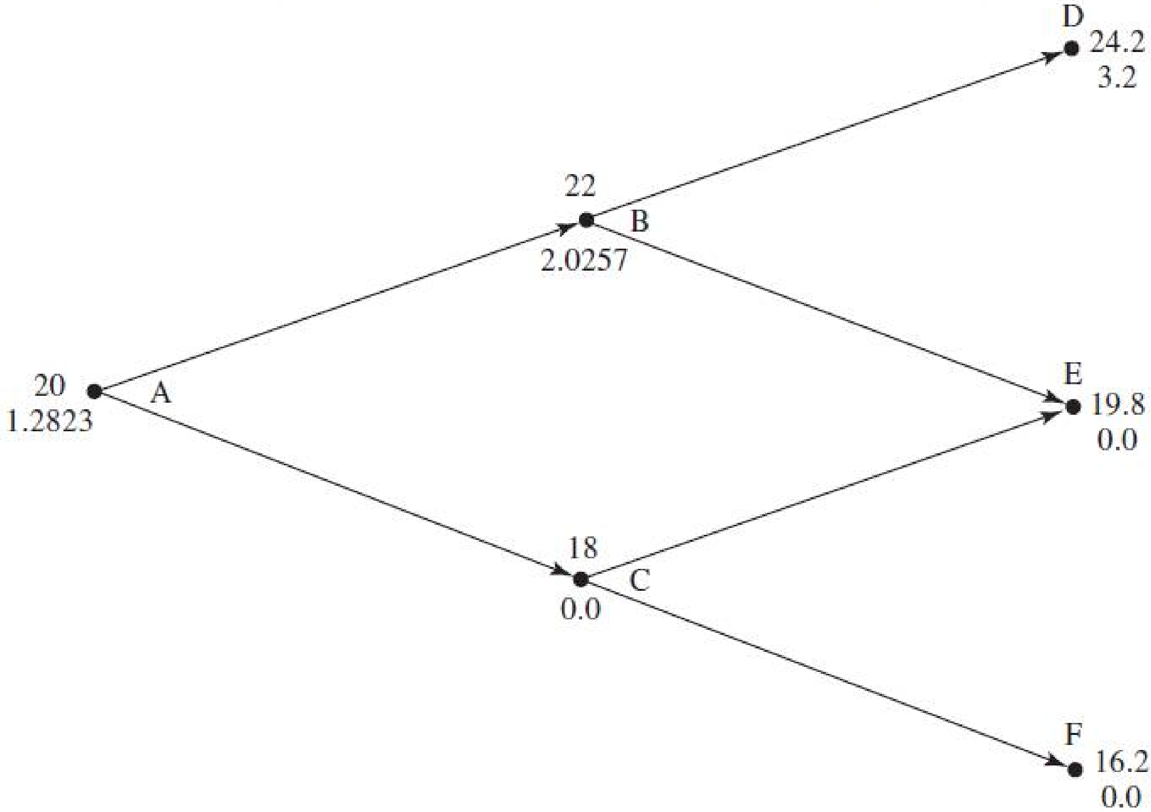
$$V = e^{-r\Delta t}\left(p^* V_u + (1-p^*)V_d\right) = e^{-2r\Delta t}\left(p^{*2} V_{uu} + 2p^*(1-p^*)V_{ud} + (1-p^*)^2 V_{dd}\right).$$

This is consistent with the principle of risk-neutral valuation mentioned earlier. The variables $(p^*)^2, 2p^*(1-p^*)$ and $(1-p^*)^2$ are the probabilities that the upper, middle, and lower final nodes will be reached. The option price is equal to its expected payoff in a risk-neutral world discounted at the risk-free interest rate.

Example 3.7

We can extend the analysis of Example 3.3 to a two-step binomial tree. Here the stock price starts at \$20 and in each of two time steps may go up by 10% or down by 10%. Each time step is 3 months long and the risk-free continuous interest rate is 12% per annum. We consider a 6-month call option with a strike price of \$21.

Stock and option prices in a two-step tree. The upper number at each node is the stock price and the lower number is the option price.



At node D the stock price is 24.2 and the option price is $24.2-21=3.2$; at nodes E and F the option is out of the money and its value is zero. At node C the option price is zero, because node C leads to either node E or node F and at both of those nodes the option price is zero. The value of the option at node B is

$$e^{-0.12\times 3/12}(0.6523\times 3.2+0.3477\times 0)=2.0257.$$

The value of the option at node A is

$$e^{-0.12\times 3/12}(0.6523\times 2.0257+0.3477\times 0)=1.2823.$$

The possible final stock prices are: 24.2, 19.8, and 16.2. In this case, $V_{dd}=0=V_{ud}$, and

$V_{uu} = 3.2$. The premium of call option is

$$V = e^{-2r\Delta t}\left((p^*)^2 V_{uu} + 2p^*(1-p^*)V_{ud} + (1-p^*)^2 V_{dd}\right) = e^{-0.06}(0.6523^2 \times 3.2) = 1.2823.$$

Example 3.8

Suppose you can buy or sell a call option in Example 3.7 for \$1. Construct an arbitrage strategy by delta hedging.

Solution

The call premium C = \$1, which is less than

$$S(0)\Delta + B = \frac{2.0257-0}{22-18}\times 20 + e^{-0.12\times 3/12}\frac{1.1(0)-0.9(2.0257)}{1.1-0.9} = 1.2823$$

at node A. That means the call option is underpriced.

- Long 1 call option, short $\frac{2.0257}{22-18}$ shares of stock and long $e^{-0.12\times 3/12}\frac{0.9(2.0257)}{1.1-0.9}$ bond. Then you have $1.2823-1=0.2823$ at time 0.

case (1): the stock price goes up after first 3 months

The bonds is worth $\frac{0.9(2.0257)}{1.1-0.9}$.

- You need to short sell $\frac{3.2-0}{24.2-19.8}-\frac{2.0257}{22-18}$ shares of stock together with $\frac{2.0257}{22-18}$ shares of shorted stock from node A in order to delta hedge the call option at node B.

Now you have $\underbrace{22\left(\frac{3.2-0}{24.2-19.8}-\frac{2.0257}{22-18}\right)}_{\text{by short selling stocks}} + \underbrace{\frac{0.9(2.0257)}{1.1-0.9}}_{\text{bonds from node A}} = 13.9743$ bonds.

case (1.1) the stock price goes up after another 3 months.

At node D, the call option is worth 3.2, the bond is worth $13.9743e^{0.12\times 3/12} = 14.4$ and the stock is worth $\frac{3.2}{24.2-19.8}\times 24.2 = 17.6$.

- The short position of stock can be closed by bond and call option.

case (1.2) the stock price goes down after another 3 months.

At node E, the call option is worth 0, the bonds is worth 14.4 and the stock is worth $\dfrac{3.2}{24.2-19.8}\times 19.8 = 14.4$.

- The short position of stock can be closed by bond.

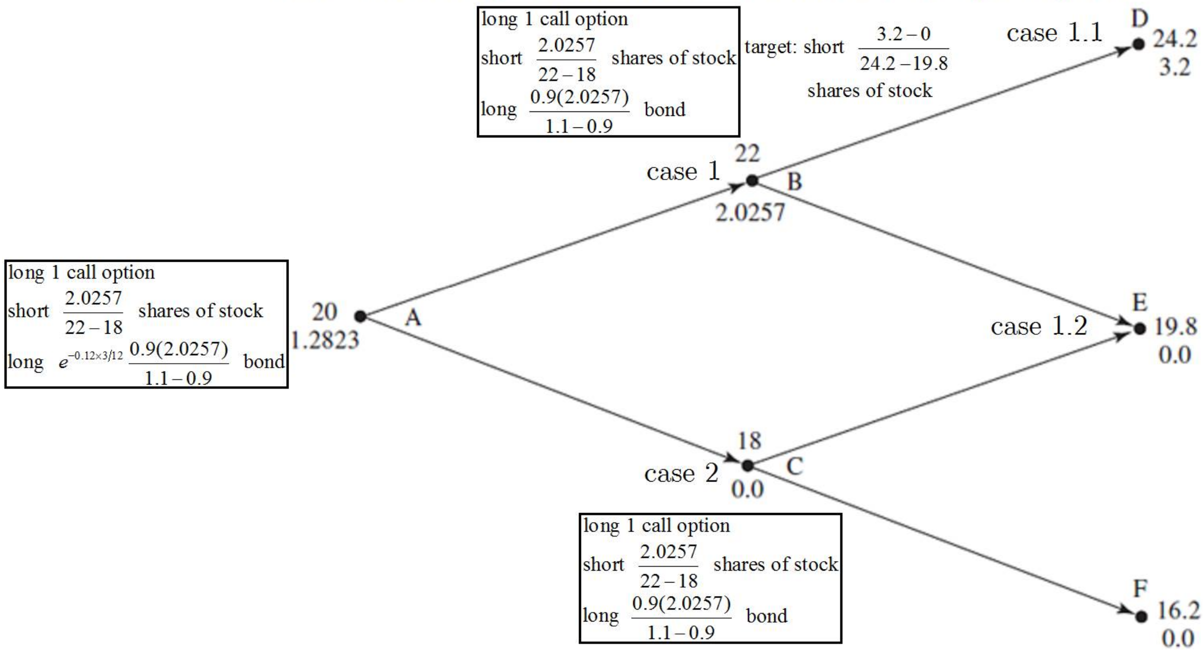
case (2): the stock price goes down after first 3 months

The call option at node C is worth nothing because the call option will be 0 at node E or node F after another 3 months.

The bond is worth $\dfrac{0.9(2.0257)}{1.1-0.9}$ and the stock is worth $\dfrac{2.0257}{22-18}\times 18$ after 3 months.

- The short position of stock can be closed by bond.

Stock and option prices in a two-step tree. The upper number at each node is the stock price and the lower number is the option price.



In an N-step binomial model, the price of a European call and put option on a stock S with strike price K to be exercised after N time steps is given by

$$
\begin{aligned}
C_{Eur}(S(0),K,N\Delta t) &= e^{-rN\Delta t}E\left[\max\left(S(0)u^i d^{N-i}-K,0\right)\right] \\
&= e^{-rN\Delta t}\sum_{i=m}^{N}\binom{N}{i}(p^*)^i(1-p^*)^{N-i}\left(S(0)u^i d^{N-i}-K\right) \\
P_{Eur}(S(0),K,N\Delta t) &= e^{-rN\Delta t}E\left[\max\left(K-S(0)u^i d^{N-i},0\right)\right] \\
&= e^{-rN\Delta t}\sum_{i=0}^{m-1}\binom{N}{i}(p^*)^i(1-p^*)^{N-i}\left(K-S(0)u^i d^{N-i}\right)
\end{aligned}
$$

where r is the risk-free interest rate, p^* is risk neutral probability and m is the least integer such that $S(0)u^m d^{N-m} > K$. Then $m = \max\left\{0, \text{the smallest integer } > \dfrac{\ln(K/S(0)) - N\ln d}{\ln(u/d)}\right\}$.

Denote $\Phi(m,N,p)$ the cumulative binomial distribution with N trials and probability p of success in each trial, that is

$$
\Phi(m,N,p) = \sum_{i=0}^{m}\binom{N}{i}p^i(1-p)^{N-i}.
$$

Then

$$
\begin{aligned}
C_{Eur}(S(0),K,N\Delta t) &= e^{-rN\Delta t}\sum_{i=m}^{N}\binom{N}{i}(p^*)^i(1-p^*)^{N-i}\left(S(0)u^i d^{N-i}-K\right) \\
&= S(0)\sum_{i=m}^{N}\binom{N}{i}\left(p^*ue^{-r\Delta t}\right)^i\left((1-p^*)de^{-r\Delta t}\right)^{N-i} - Ke^{-rN\Delta t}\sum_{i=m}^{N}\binom{N}{i}(p^*)^i(1-p^*)^{N-i} \\
&= S(0)\left[1-\Phi\left(m-1,N,p^*ue^{-r\Delta t}\right)\right] - Ke^{-rN\Delta t}\left[1-\Phi\left(m-1,N,p^*\right)\right] \\
P_{Eur}(S(0),K,N\Delta t) &= e^{-rN\Delta t}\sum_{i=0}^{m-1}\binom{N}{i}(p^*)^i(1-p^*)^{N-i}\left(K-S(0)u^j d^{N-j}\right) \\
&= Ke^{-rN\Delta t}\sum_{i=0}^{m-1}\binom{N}{i}(p^*)^i(1-p^*)^{N-i} - S(0)\sum_{i=0}^{m-1}\binom{N}{i}\left(p^*ue^{-r\Delta t}\right)^i\left((1-p^*)de^{-r\Delta t}\right)^{N-i} \\
&= Ke^{-rN\Delta t}\Phi\left(m-1,N,p^*\right) - S(0)\Phi\left(m-1,N,p^*ue^{-r\Delta t}\right)
\end{aligned}
$$

Theorem 3.9 (Cox-Ross-Rubinstein Formula)

$$
\begin{aligned}
C_{Eur}(S(0),K,N\Delta t) &= S(0)\left[1-\Phi\left(m-1,N,p^*ue^{-r\Delta t}\right)\right] - Ke^{-rN\Delta t}\left[1-\Phi\left(m-1,N,p^*\right)\right] \\
P_{Eur}(S(0),K,N\Delta t) &= Ke^{-rN\Delta t}\Phi\left(m-1,N,p^*\right) - S(0)\Phi\left(m-1,N,p^*ue^{-r\Delta t}\right)
\end{aligned}
$$

Example 3.10

Let $S(0) = 50$, $r = 0.05$, $\Delta t = 1, \delta = 0$, $u = 1.3$ and $d = 0.9$. Find the price of a European call and put with $K = 60$ to be exercised after $N = 3$ time steps by Cox-Ross-Rubinstein Formula.

Solution

$\dfrac{\ln(K/S(0)) - N\ln d}{\ln(u/d)} = 1.35537 < 2 = m.$ The risk neutral probability is $p^* = \dfrac{e^{r\Delta t} - d}{u - d} = 0.3782$

$$\begin{aligned}
C_{Eur}(50,60,3) &= 50\left[1-\Phi\left(1,3,0.467654\right)\right] - e^{-3(0.05)}60\left[1-\Phi\left(1,3,0.3782\right)\right] \\
&= 50(0.45154868489) - e^{-0.15}60(0.32091386446) \approx 6 \\
P_{Eur}(50,60,3) &= e^{-3(0.05)}60\Phi\left(1,3,0.3782\right) - S(0)\Phi\left(1,3,0.467654\right) \\
&= e^{-0.15}60(0.67908613554) - 50(0.54845131511) \\
&\approx 7.6471254568367558439362489
\end{aligned}$$

Let $T = N\Delta t$ be the expiration date of a European call and put options. As $\Delta t \to 0, N \to \infty,$ Cox-Ross-Rubinstein Formula becomes Black-Scholes Formula. (John Hull, Ch 12, Appendix)

Theorem 3.11 (Black-Scholes Formula)

Let σ the volatility parameter of a non-dividend-paying stock. The price of European call and put options with risk free interest rate r, strike price K and exercise time T is given by

$$C = S(0)N(d_1) - Ke^{-rT}N(d_2) \quad \text{and} \quad P = Ke^{-rT}N(-d_2) - S(0)N(-d_1).$$

where $d_1 = \dfrac{\ln\frac{S(0)}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T} = \dfrac{\ln\frac{S(0)}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$

In general, we have

$$C = F_{t,T}^P(S)N(d_1) - F_{t,T}^P(K)N(d_2) \quad \text{and} \quad P = F_{t,T}^P(K)N(-d_2) - F_{t,T}^P(S)N(-d_1)$$

where

$$d_1 = \frac{\ln\left(F_{t,T}^P(S)/F_{t,T}^P(K)\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(F_{t,T}^P(S)/F_{t,T}^P(K)\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

Example 3.12

The price of a non-dividend-paying stock is 100. The stock's volatility is 30%. The continuous compounded risk-free interest rate is 4%. Find the 6-month European call and put options on the stock with strike price 90.

Solution

$$d_1 = \frac{\ln\frac{S(0)}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = 0.6970211553 \text{ and } d_2 = \frac{\ln\frac{S(0)}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = 0.48488912092.$$

$$C = S(0)N(d_1) - Ke^{-rT}N(d_2) = 100(0.7571) - 90e^{-0.04(0.5)}(0.6861) = \$15.1823$$
$$P = Ke^{-rT}N(-d_2) - S(0)N(-d_1) = 90e^{-0.04(0.5)}(1-0.6861) - 100(1-0.7571) = \$3.4001$$

We also have call and put premiums for strike prices \$100 and \$110.

Strike Price	Call	Put
\$90	\$15.1823	\$3.4001
\$100	\$9.3904	\$7.4103
\$110	\$5.4115	\$13.2333

Example 3.13 (Black-Scholes for options on stock paying discrete known dividends)

Consider a European call option on a stock which pays dividends in two months and five months. Each dividend is expected to be \$0.50. The current share price is \$40, the exercise price is \$40, the stock price volatility is 30% per annum, the risk-free rate of interest is 9% per annum, and the time to maturity is six months. The present value of the dividends is

$$0.5e^{-0.09\times 2/12} + 0.5e^{-0.09\times 5/12} = 0.9742$$

The option price can therefore be calculated from the Black-Scholes formula, with $F^P(S) = 40 - 0.9742 = 39.0258$, $K = 40$, $r = 0.09$, $\sigma = 0.3$, and $T = 0.5$:

$$d_1 = \frac{\ln(39.0258/40) + (0.09 + 0.3^2/2)\times 0.5}{0.3\sqrt{0.5}} = 0.2020$$
$$d_2 = \frac{\ln(39.0258/40) + (0.09 - 0.3^2/2)\times 0.5}{0.3\sqrt{0.5}} = -0.0102$$

gives

$$N(d_1) = 0.5800 \quad \text{and} \quad N(d_2) = 0.4959$$

The call price is

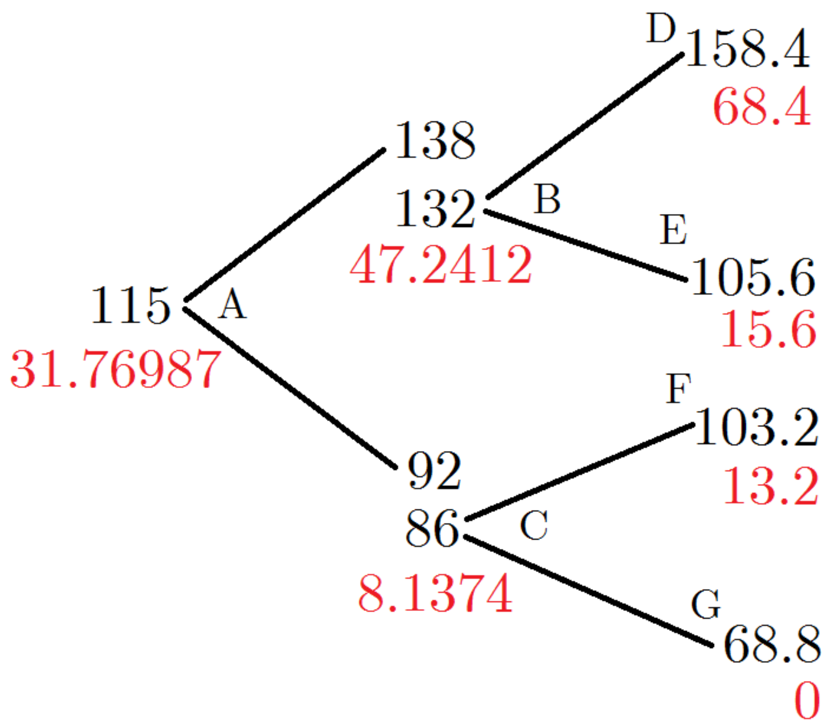
$$C = F^P(S)N(d_1) - Ke^{-rT}N(d_2) = 39.0258 \times 0.5800 - 40e^{-0.09\times0.5} \times 0.4959 = 3.67$$

Example 3.14

Consider a 2-year European call with a strike price of $90 on a discrete dividend paying stock with current price is $115. Suppose that there are 2 time steps of 1 year, and in each time step the stock price either moves up by 20% or moves down by 20%. $6 dividend will be paid after movement of stock price in 1 year. The risk-free interest rate is 6%. Find the premium of the call option.

Solution

The upper number at each node is stock price.
The middle number at B and C is stock price after paying dividend. The lower number is option price



The stock price at node D is $(S(0)u-D)u=S(0)u^2-Du$, and the stock price at node G is $(S(0)d-D)d=S(0)d^2-Dd$.

The stock price at node E is $(S(0)u-D)d=S(0)ud-Dd$, which is not equal to the stock price at node F, $(S(0)d-D)u=S(0)ud-Du$.

Clearly, u = 1.2, d = 0.8. The risk neutral probability is $p^*=\dfrac{e^{0.06}-0.8}{1.2-0.8}=0.6545913663634$.

The value of option at node B is $V_B=e^{-0.06}\left(68.4p^*+15.6(1-p^*)\right)=47.2412$.

The value of option at node C is $V_C=e^{-0.06}\left(13.2p^*+0(1-p^*)\right)=8.1374$.

The value of option at node A is $V_A=e^{-0.06}\left(p^*V_B+(1-p^*)V_C\right)=31.76987$.

The call premium can be also calculated by present value of expected payoff at expiration date.

$$V_A=e^{-0.12}\left(68.4p^{*2}+15.6p^*(1-p^*)+13.2(1-p^*)p^*V_{ud}+0(1-p^*)^2\right)=31.76987$$

Keep in mind that paying dividend will affect the payoff of the option at expiration date but not risk neutral probability p^* at node A.

Example 3.15

Suppose you can buy or sell a call option in Example 3.14 for \$32. Construct an arbitrage strategy by delta hedging.

Solution

The call premium C = \$32, which is more than

$$S(0)\Delta+B=\frac{47.2412-8.1374}{138-92}\times 115+e^{-0.06}\frac{138(8.1374)-92(47.2412)}{138-92}=31.76987$$

at node A. That means the call option is overpriced.

- Short $e^{-0.06}\dfrac{92(47.2412)-138(8.1374)}{138-92}$ bonds and long $\dfrac{47.2412-8.1374}{138-92}$ shares of stock and short 1 call option. Then you have $32-31.76987=0.23013$ at $t=0$.

case (1): the stock price goes up at $t=1$

- You need to buy $\underbrace{\frac{68.4-15.6}{158.4-105.6}}_{1}-\frac{47.2412-8.1374}{138-92}$ shares of stock after paying dividend together with $\frac{47.2412-8.1374}{138-92}$ shares of stock from node A in order to delta hedge the call option at node B. (No matter what is the movement of the stock at $t=2$, call option will be exercised at $t=2$. You need 1 share of stock for exercising).

- Now you short

$$\underbrace{132\left(1-\frac{47.2412-8.1374}{138-92}\right)}_{\text{cost of buying stocks in case (1)}}+\underbrace{\frac{92(47.2412)-138(8.1374)}{138-92}}_{\text{bonds from node A}}-\underbrace{\frac{47.2412-8.1374}{138-92}6}_{\text{dividend paid by stock}}=84.7588$$

bonds.

case (1.1) the stock price goes up at $t=2$.

The call option is exercised. Pay the stock for \$90 and cover short position of bonds, $84.7588e^{0.06}=90$.

case (1.2) the stock price goes down at $t=2$.

Same as case (1.1)

case (2): the stock price goes down at $t=1$.

- You need to buy $\frac{13.2}{103.2-68.8}-\frac{47.2412-8.1374}{138-92}$ (or sell $\frac{47.2412-8.1374}{138-92}-\frac{13.2}{103.2-68.8}$) shares of stock after paying dividend together with $\frac{47.2412-8.1374}{138-92}$ shares of stock from node A in order to delta hedge the call option at node C.

- Now you short

$$\underbrace{86\left(\frac{13.2}{103.2-68.8}-\frac{47.2412-8.1374}{138-92}\right)}_{\text{selling stocks in case (2)}}+\underbrace{\frac{92(47.2412)-138(8.1374)}{138-92}}_{\text{bonds from node A}}-\underbrace{\frac{47.2412-8.1374}{138-92}6}_{\text{dividend paid by stock}}$$

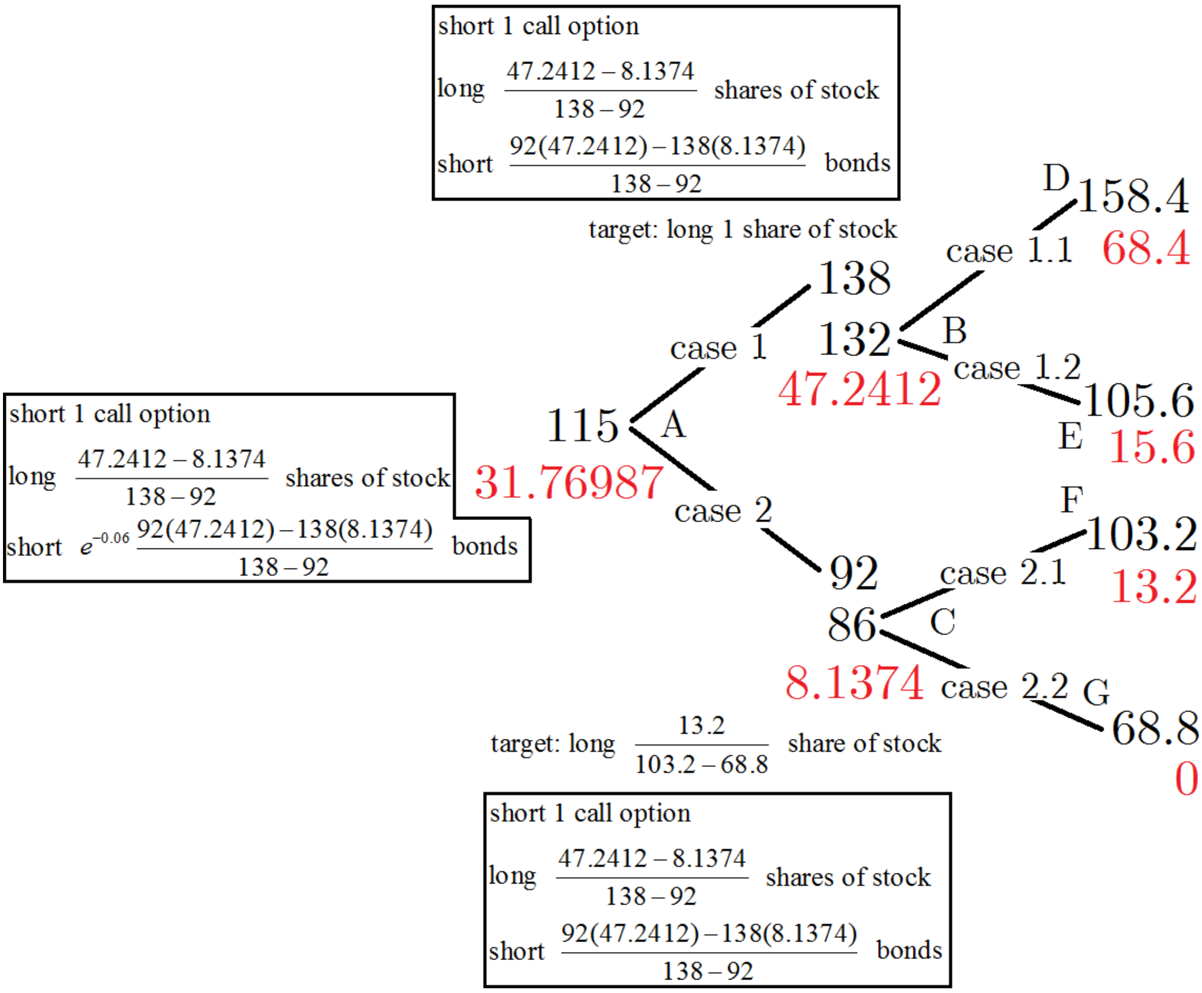
$=24.8626$ bonds.

case (2.1) the stock price goes up at $t = 2$.

Long position of stock, which worth $\frac{13.2}{103.2-68.8}103.2 = 39.6$, can cover short position of option, which worth 13.2 and short position of bonds, which worth $24.8626e^{0.06} = 26.4$.

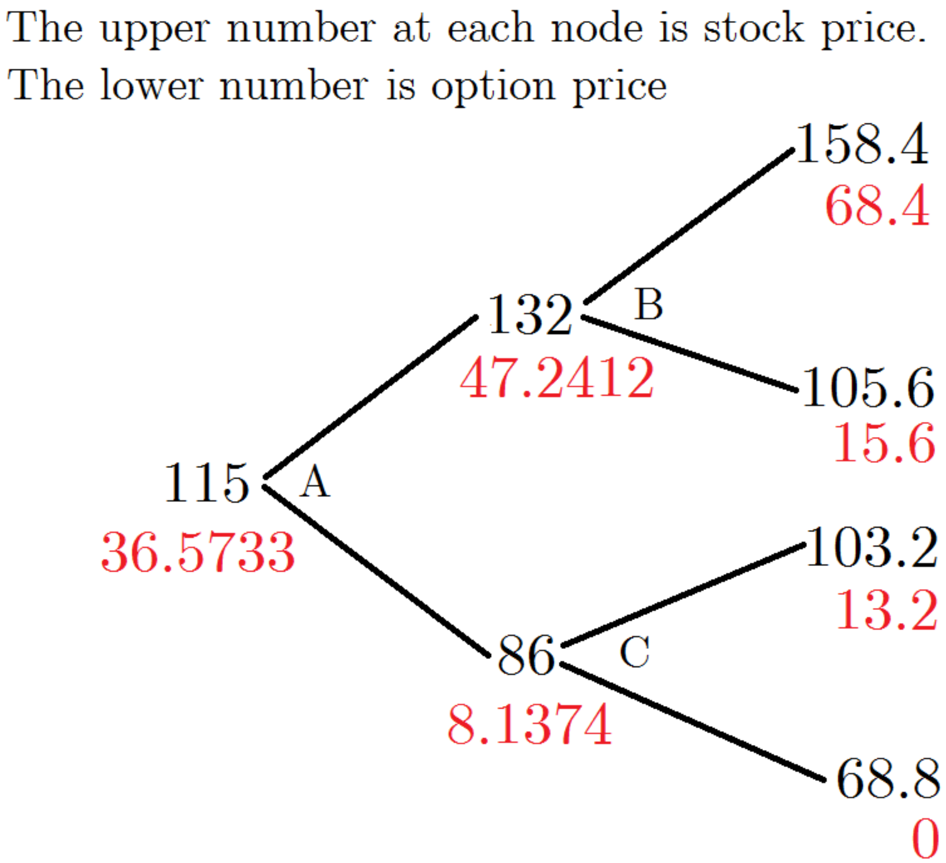
case (2.2) the stock price goes down at $t = 2$.

Long position of stock, which worth $\frac{13.2}{103.2-68.8}68.8 = 26.4$, can cover short position of bonds, which worth $24.8626e^{0.06} = 26.4$.



Example 3.16

Consider the following binomial tree model with risk free interest rate $r = 0.06$.



At nodes B and C, $u = 1.2$, $d = 0.8$. The value of the risk-neutral probability, $p_B^* = p_C^*$, is given by

$$p_B^* = p_C^* = \frac{e^{0.06} - 0.8}{1.2 - 0.8} = 0.6545913663633991.$$

At nodes A, $u = \frac{132}{115}, d = \frac{86}{115}$. The value of the risk-neutral probability, p_A^*, is given by

$$p_A^* = \frac{e^{0.06} - \frac{86}{115}}{\frac{132}{115} - \frac{86}{115}} = 0.7850261489720947.$$

$$V_B = e^{-0.06}\left(68.4p_B^* + 15.6(1 - p_B^*)\right) = 47.2412,$$
$$V_C = e^{-0.06}13.2p_C^* = 8.1374,$$
$$V_A = e^{-0.06}\left(47.2412p_A^* + 8.1374(1 - p_A^*)\right) = 36.5733$$

The call premium can be also calculated by present value of expected payoff at expiration date.

$$V_A = e^{-0.12}\left(68.4p_A^*p_B^* + 15.6p_A^*(1 - p_B^*) + 13.2(1 - p_A^*)p_C^* + 0(1 - p_A^*)(1 - p_C^*)\right) = 36.5733$$

Up to now all the options we have considered have been European. We now move on to consider how American options can be valued using a binomial tree. The procedure is to work back through the tree from the end to the beginning, testing at each node to see whether early exercise is optimal. The value of the option at the final nodes is the same as for the European option. At earlier nodes the value of the option is the maximum of

1. The value given by $V = e^{-r\Delta t}\left(p^* V_u + (1-p^*)V_d\right)$.

2. The payoff from early exercise.

Example 3.17

Consider a 2-year American put with a strike price of \$52 on a stock with current price is \$50. Suppose that there are 2 time steps of 1 year, and in each time step the stock price either moves up by 20% or moves down by 20%. We also suppose that the risk-free interest rate is 7% and continuous dividend rate is 2%. Find the premium of American put option.

Solution

In this case u = 1.2, d = 0.8, $\Delta t = 1$, and r = 0.05. The value of the risk-neutral probability, p^*, is given by

$$p^* = \frac{e^{0.05} - 0.8}{1.2 - 0.8} = 0.6282.$$

The possible final stock prices are: \$72, \$48, and \$32. In this case, $P_{uu} = 0, P_{ud} = 4$, and $P_{dd} = 20$. At node B, the value of the option is

$$e^{-r\Delta t}\left(p^* P_{uu} + (1-p^*)P_{ud}\right) = e^{-0.07}(1-0.6282)4 = 1.3867391,$$

whereas the payoff from early exercise is negative $(=-8)$. Clearly early exercise is not optimal at node B, and the value of the option at this node is 1.3867391. At node C, the value of the option is

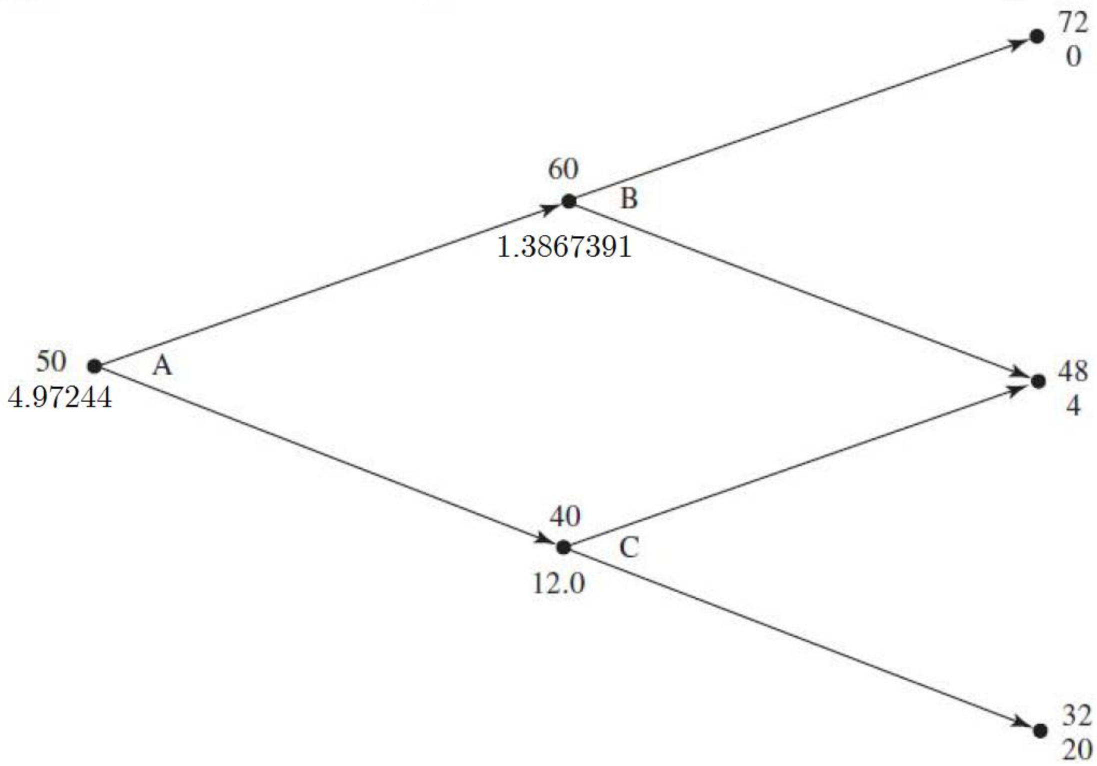
$$e^{-r\Delta t}\left(p^* P_{ud} + (1-p^*)P_{dd}\right) = e^{-0.07}\left((0.6282)4 + (1-0.6282)20\right) = 9.2765317,$$

whereas the payoff from early exercise is 12. In this case, early exercise is optimal and the value of the option at the node is 12. At the initial node A, the value of the option is

$$e^{-0.07}\left((0.6282)1.3867391+(1-0.6282)12\right)=4.97244.$$

and the payoff from early exercise is 2. In this case early exercise is not optimal. The value of the option is therefore \$ 4.97244.

Using a two-step tree to value an American put option. At each node, the upper number is the stock price and the lower number is the option price



Example 3.18

Suppose you can buy or sell a put option in Example 3.17 for \$5. Construct an arbitrage strategy by delta hedging.

Solution

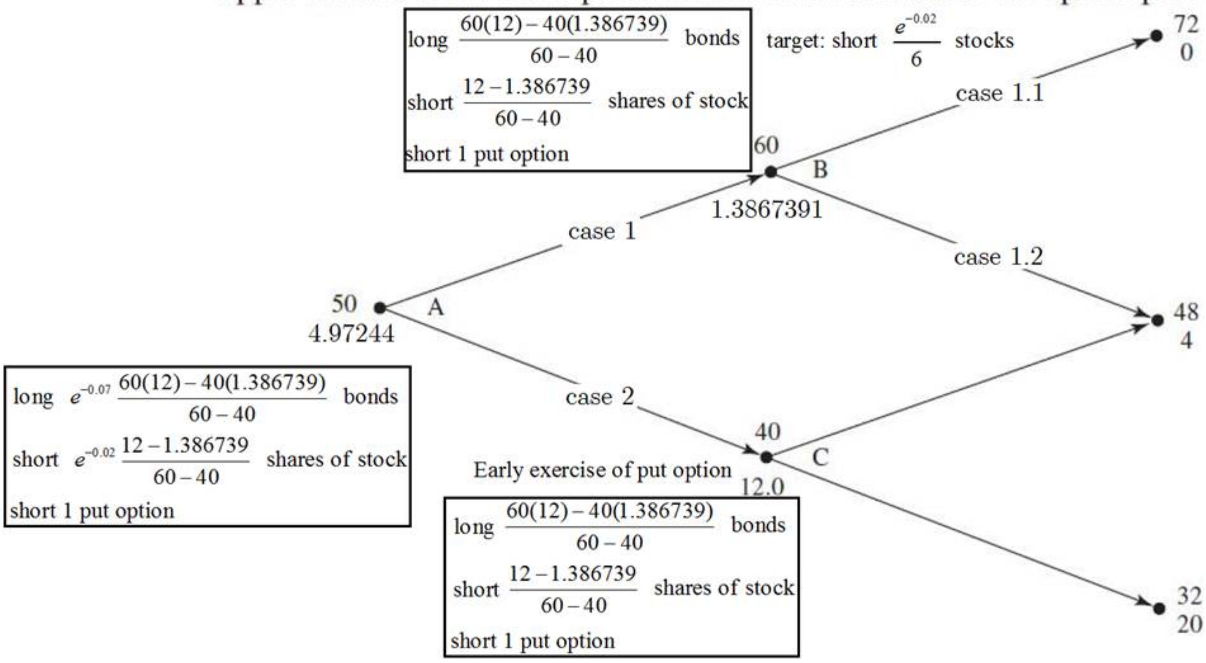
The put premium P = \$5, which is less than

$$S(0)\Delta+B=e^{-0.02}\frac{1.386739-12}{60-40}\times 50+e^{-0.07}\frac{60(12)-40(1.386739)}{60-40}=4.97244$$

at node A. That means the put option is overpriced.

- Long $e^{-0.07}\dfrac{60(12)-40(1.386739)}{60-40}$ bonds and short $e^{-0.02}\dfrac{12-1.386739}{60-40}$ shares of stock and short 1 put option. Then you have $5-4.97244=0.02756$ at t = 0.

Using a two-step tree to value an American put option. At each node, the upper number is the stock price and the lower number is the option price



case (1): the stock price goes up at t = 1

- You need to sell $e^{-0.02}\dfrac{4-0}{72-48}-\dfrac{12-1.386739}{60-40}$ shares of stock together with shorted $\dfrac{12-1.386739}{60-40}$ shares of stock from node A in order to delta hedge the put option at node B.

- Now you long $\underbrace{60\left(\dfrac{e^{-0.02}}{6}-\dfrac{12-1.386739}{60-40}\right)}_{\text{selling stocks in case (1)}}+\underbrace{\dfrac{60(12)-40(1.386739)}{60-40}}_{\text{bonds from node A}}=11.1887257$ bonds

case (1.1) the stock price goes up at t = 2.

Put option will not be exercised. Long position of bonds, which worth $11.1887257e^{0.07}=12$, can cover short position of stock, which worth 72/6 = 12.

case (1.2) the stock price goes down at t = 2.

Long position of bonds, which worth $11.1887257e^{0.07}=12$, can cover short position of stock, which worth 48/6 = 8 and short position of put option, which worth 4.

case (2): the stock price goes down at $t = 1$.

Put option will be exercised. Short position of put, which worth 12, and short position of stock, which worth $40\frac{12-1.386739}{60-40}$ can be covered by long position of bonds, which worth $\frac{60(12)-40(1.386739)}{60-40}=40\frac{12-1.386739}{60-40}+12$.

Example 3.19

Consider the situation in Example 3.14, but suppose that the option is American rather than European. Find the call option premium.

Suppose you can buy or sell a call option for \$32. Construct an arbitrage strategy by delta hedging.

Solution

The value of call option at nodes C, D, E, F and G is the same as before. The value of call option at node B is 47.2412, whereas the payoff from early exercise is 48. In this case, early exercise is optimal and the value of the option at node B is 12. At the initial node A, the value of the option is $V_A = e^{-0.06}\left(48p^* + 8.1374(1-p^*)\right) = 32.237649.$

The call premium C = \$32, which is less than

$$S(0)\Delta + B = \frac{48-8.1374}{138-92}\times 115 + e^{-0.06}\frac{138(8.1374)-92(48)}{138-92} = 32.237649$$

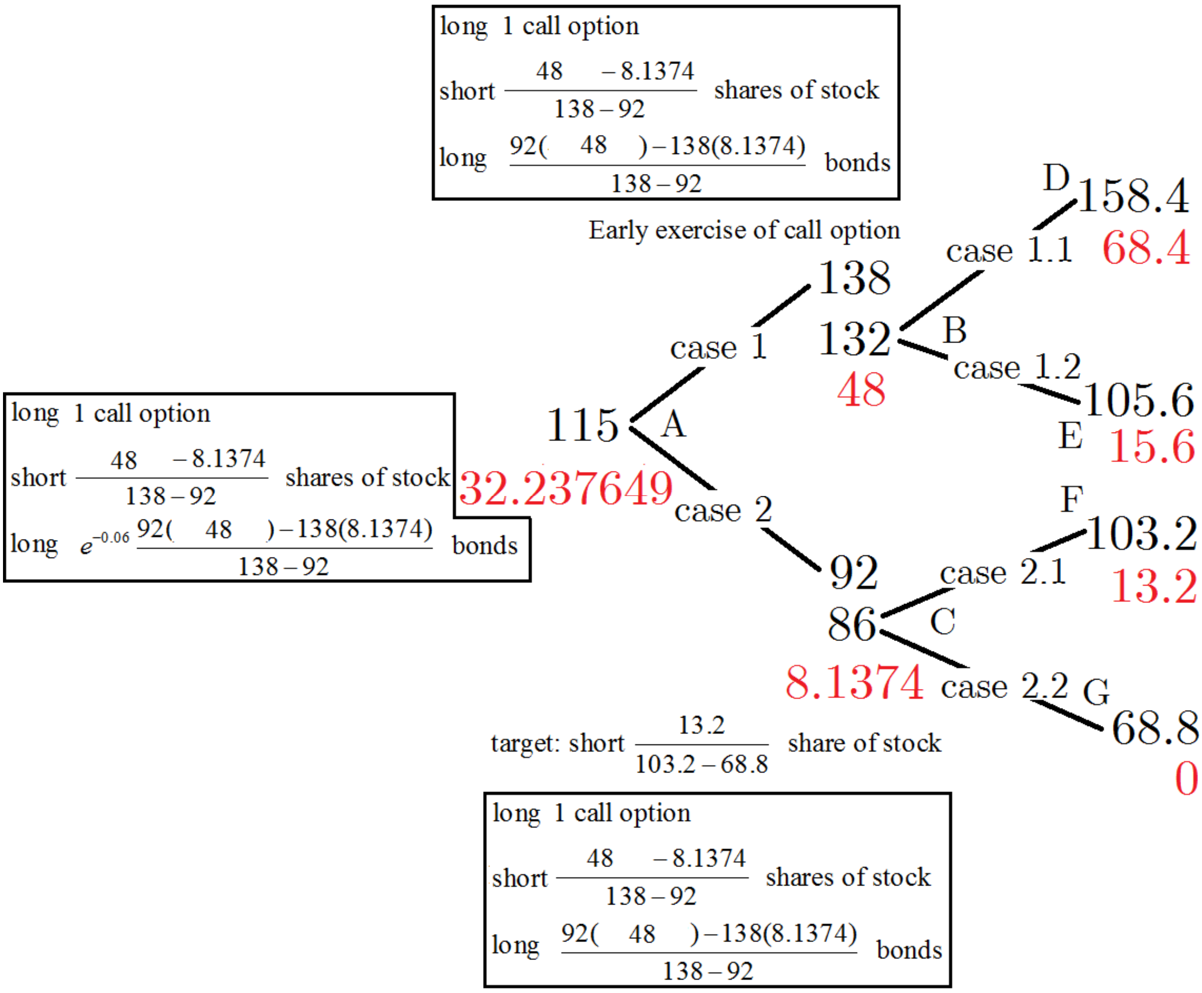
at node A. That means the call option is underpriced.

- Long $e^{-0.06}\frac{92(48)-138(8.1374)}{138-92}$ bonds and short $\frac{48-8.1374}{138-92}$ shares of stock and long 1 call option. Then you have $32.237649-32=0.237649$ at $t = 0$.

case (1): the stock price goes up at $t = 1$

Exercising call option, you will get 48. Together with long position of bonds, you have $\frac{92(48)-138(8.1374)}{138-92}+48=119.5878$ which can cover the short position of stock

$$\frac{48-8.1374}{138-92}138=119.5878$$



case (2): the stock price goes down at $t = 1$.

- You need to sell $\frac{13.2}{103.2-68.8}-\frac{48-8.1374}{138-92}$ (or buy $\frac{48-8.1374}{138-92}-\frac{13.2}{103.2-68.8}$) shares of stock after paying dividend together with shorted $\frac{48-8.1374}{132-86}$ shares of stock from node A in order to delta hedge the call option at node C.

- Now you long $\underbrace{86\left(\frac{13.2}{103.2-68.8}-\frac{48-8.1374}{138-92}\right)}_{\text{buying stocks in case (2)}}+\underbrace{\frac{92(48)-138(8.1374)}{138-92}}_{\text{bonds from node A}}-\underbrace{\frac{48-8.1374}{138-92}6}_{\text{dividend paid by stock}}$

  $= 24.8626$ bonds.

case (2.1) the stock price goes up at $t = 2$.

The value of call option, which worth 13.2 and long position of bonds, which worth $24.8626e^{0.06} = 26.4$ can cover short position of stock, which worth $\frac{13.2}{103.2-68.8}103.2 = 39.6$.

case (2.2) the stock price goes down at $t = 2$.

The call option will not be exercised. Long position of bonds, which worth $24.8626e^{0.06} = 26.4$ can cover short position of stock, which worth $\frac{13.2}{103.2-68.8}68.8 = 26.4$.

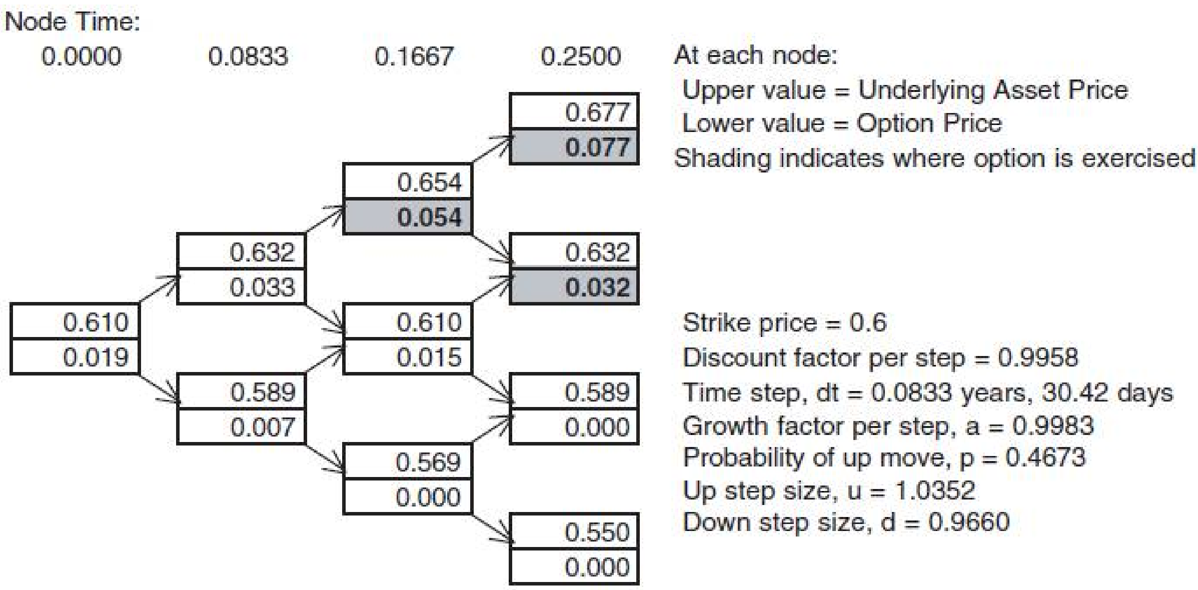
Example 3.20

The Australian dollar is currently worth 0.6100 US dollars and this exchange rate has a volatility of 12%. The Australian risk-free rate is 7% and the US risk-free rate is 5%. Consider a 3-month American call option with a strike price of 0.6000 using a 3-step tree. Using Cox-Ross-Rubinstein tree, we have

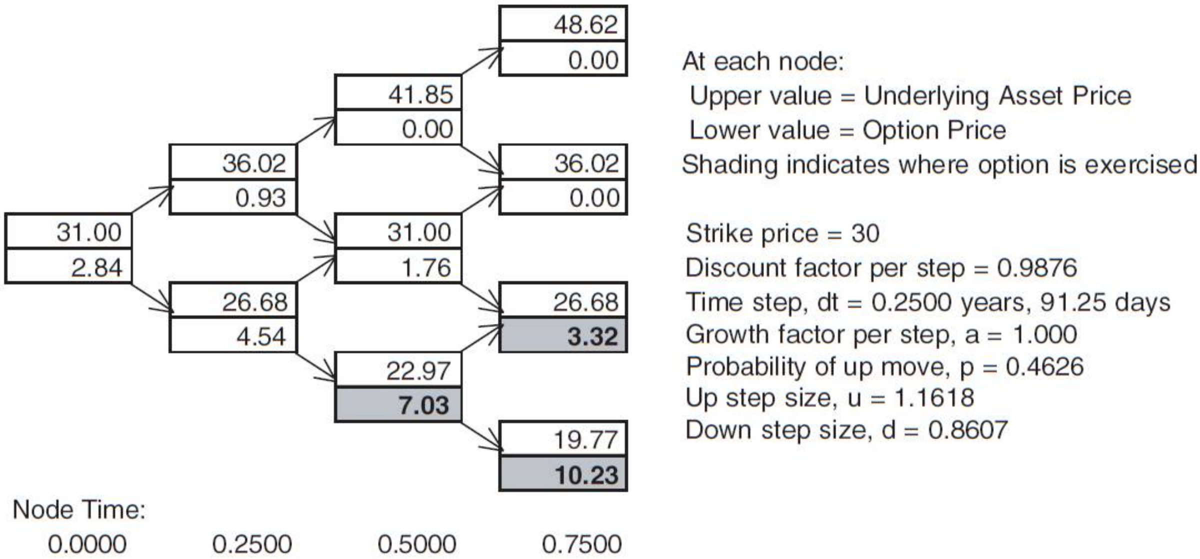
$$\Delta t = 0.08333, \quad u = e^{0.12\sqrt{0.08333}} = 1.0352, \quad d = e^{-0.12\sqrt{0.08333}} = 0.9660$$

$$p^* = \frac{e^{(r-r_f)\Delta t} - d}{u-d} = \frac{e^{(0.05-0.07)0.08333} - 0.9660}{1.0352 - 0.9660} = 0.4673$$

The value of the option is 0.019.



Example 3.21



A futures price is currently 31 and has a volatility of 30%. The risk-free rate is 5%. Consider a 9-month American put option with a strike price of 30 using a three-step tree. Using Cox-Ross-Rubinstein tree, we have

$$\Delta t = 0.25, u = e^{0.3\sqrt{0.25}} = 1.1618, d = 1/u = 1/1.1618 = 0.8607,$$
$$p^* = (1-0.8607)/(1.1618-0.8607) = 0.4626$$

The value of the option is 2.84.

Let α be the expected return on a stock. A stock is a risky investment; the actual price may be greater than or less than the expected price. Therefore, a risk-adverse investor in the stock expects a return greater than the risk-free rate. In other words, $\alpha > r.$ If we consider 1 period binomial tree model on a stock paying dividend at a rate $\delta,$ then

$$puSe^{\delta h} + (1-p)dSe^{\delta h} = Se^{\alpha h}$$

where p is true probability. Solving for p, we have

$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d}.$$

Let V be an option such that $V = \Delta S + B.$ Let γ be its discount rate (or rate of return). This must be the same as the discount rate for the replicating portfolio

$$Ve^{\gamma h} = \Delta S e^{\alpha h} + Be^{rh}$$

After time h,

$$V_u = \Delta e^{\delta h} Su + Be^{rh}, \quad V_d = \Delta e^{\delta h} Sd + Be^{rh}$$

Then

$$\begin{aligned} pV_u + (1-p)V_d &= p\left(\Delta e^{\delta h} Su + Be^{rh}\right) + (1-p)\left(\Delta e^{\delta h} Sd + Be^{rh}\right) \\ &= p\Delta e^{\delta h} Su + (1-p)\Delta e^{\delta h} Sd + Be^{rh} \\ &= \Delta S e^{\alpha h} + Be^{rh} \\ &= Ve^{\gamma h} \end{aligned}$$

We can calculate the rate of return on an option at node # by

$$V_{\#} = e^{-\gamma h}\left(pV_{\#u} + (1-p)V_{\#d}\right) = e^{-rh}\left(p^* V_{\#u} + (1-p^*)V_{\#d}\right)$$

Example 3.22

The price of a stock is 52. Its continuously compounded dividend rate is 10%. An American call option on the stock expires in 6 months. The strike price is 53. The continuously compounded risk-free rate is 3%. The expected return on the stock is 15%. The option is modeled with a 2-period binomial tree in which u = 1.3, d = 0.8. Determine the discount rate for the call option at each node.

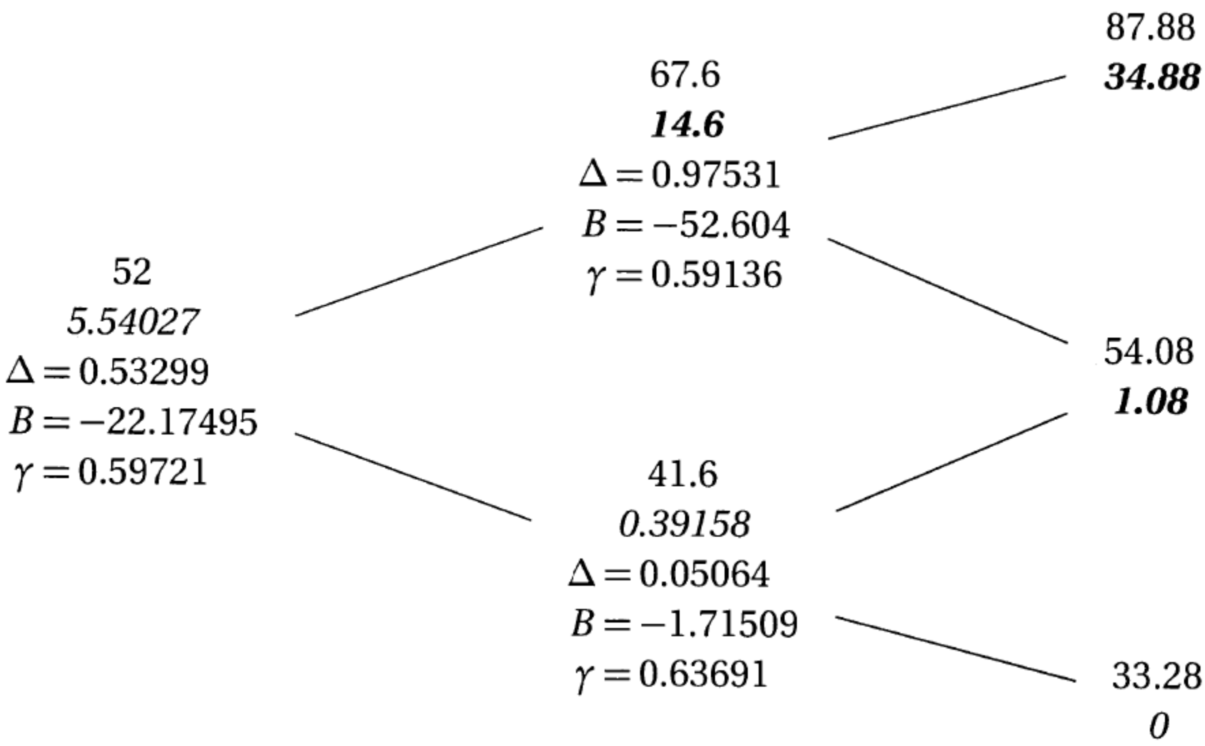
Solution

$r = 0.03$, $\alpha = 0.15$ and $\delta = 0.1$, while h, the period of the binomial tree, is 1/4 of a year. The true probability of an up movement is

$$p = \frac{e^{(\alpha-\delta)h} - d}{u-d} = \frac{e^{0.05(0.25)} - 0.8}{1.3-0.8} = \frac{1.012578 - 0.8}{0.5} = 0.425157,$$

and the risk neutral probability of an up movement is

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.03-0.1)(0.25)} - 0.8}{1.3-0.8} = 0.365304.$$



At the *u* node:

$$e^{-0.0075}\left(0.365304(34.88)+0.634696(1.08)\right)=e^{-0.25\gamma}\left(0.425157(34.88)+0.574843(1.08)\right)$$
$$\gamma = 0.59136$$

At the *d* node:

$$e^{-0.0075}\left(0.365304(1.08)\right)=e^{-0.25\gamma}\left(0.425157(1.08)\right)$$
$$\gamma = 0.63691$$

At the initial node:

$$e^{-0.0075}\left(0.365304(14.6)+0.634696(0.39158)\right)=e^{-0.25\gamma}\left(0.425157(14.6)+0.574843(0.39158)\right)$$
$$\gamma = 0.59721$$

Note that even though the rate of return on the underlying asset is constant, the rate of return on the option varies by period. In fact, the rate of return of an option varies continuously.

A European call option gives the holder the right to buy one share of a stock in a cheap price. No matter what happens, the option can never be worth more than the stock. Hence, the stock price is an upper bound to the option price:

$$C_{Amer} \le S(0), \quad C_{Eur} \le F_{0,T}^{P}(S).$$

If these relationships were not true, an arbitrageur could easily make a riskless profit by buying the stock and selling the call option.

Consider the following two portfolios:

Portfolio A: one European call option with exercise price K and expiration date at time T plus a zero-coupon bond Ke^{-rT}.

Portfolio B: prepaid forward $F_{0,T}^{P}(S)$ at time 0 to get one share of the stock at time T.

In portfolio A, the zero-coupon bond will be worth K at time T. If $S(T) > K$, the call option is exercised at T and portfolio A is worth $S(T)$. If $S(T) \leq K$, the call option expires worthless and the portfolio is worth K. Hence, at time T, portfolio A is worth

$$\max\{S(T), K\}.$$

Portfolio B is worth $S(T)$ at time T. Hence, portfolio A is always worth as much as, and can be worth more than, portfolio B at T.

		$S(T) > K$	$S(T) < K$
Portfolio A	European call option	$S(T) - K$	0
	Zero-coupon bond	K	K
	Total	$S(T)$	K
Portfolio B	One share of the stock	$S(T)$	$S(T)$

In the absence of arbitrage opportunities this must also be true today. The European call option and zero-coupon bond are worth C_{Eur} and Ke^{-rT} respectively today. Hence,

$$C_{Eur} + Ke^{-rT} \geq F_{0,T}^{P}(S) \qquad \text{or} \qquad C_{Eur} \geq F_{0,T}^{P}(S) - Ke^{-rT}.$$

Because the worst that can happen to a call option is that it expires worthless, its value cannot be negative. This means that $C_{Eur} \geq 0$ and therefore

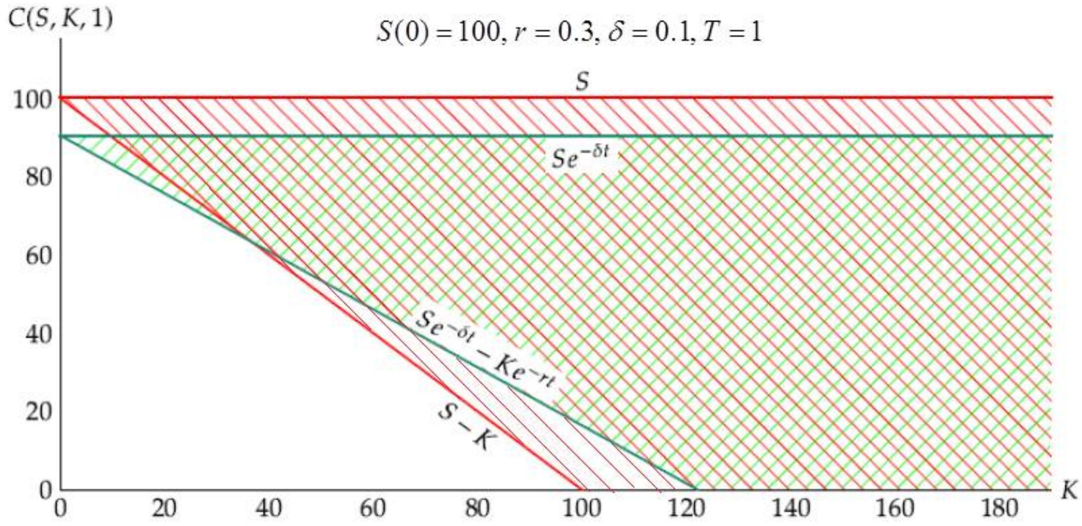
$$\max\{F_{0,T}^{P}(S) - Ke^{-rT}, 0\} \leq C_{Eur} \leq F_{0,T}^{P}(S).$$

For an American call option on any stock, the condition

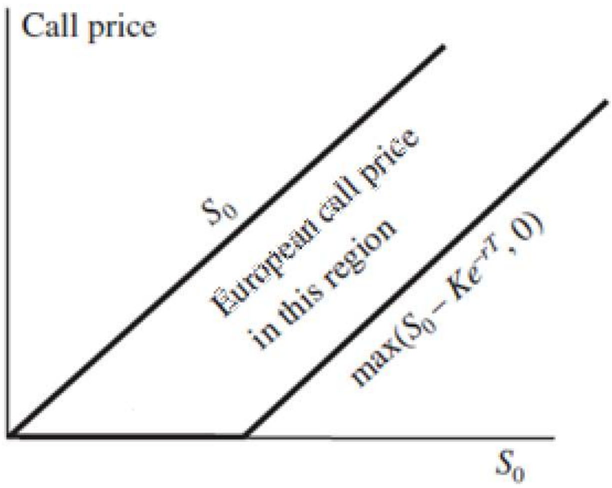
$$\max\{S(0)-K,0\}\le C_{Amer}$$

must apply because the option can be exercised at any time. The bounds for an American call option on any stock are

$$\max\{S(0)-K,0\}\le C_{Amer}\le S(0).$$



Bounds for American and European call option prices. $S = 100$. The lower red boundary line is the $S-K$ bound for an American option, and the lower green line is the $Se^{-\delta t}-Ke^{-rt}$ bound for a European option; however, when the latter is higher than the former, it is a lower bound for an American option as well. The red hatched area shows the possible values for an American call and the green hatched area shows the possible values for a European call.



Bounds for European call options when there are no dividends.

An American put option gives the holder the right to sell one share of a stock for *K*. No matter how low the stock price becomes, the option can never be worth more than *K*. Hence,

$$P_{Amer}\le K.$$

For European put options, we know that at maturity the option cannot be worth more than K. It follows that it cannot be worth more than the present value of K today:

$$P_{Eur} \leq Ke^{-rT}.$$

If this were not true, an arbitrageur could make a riskless profit by writing the put option and investing the proceeds of the sale at the risk-free interest rate r.

Consider the following two portfolios:

Portfolio C: one European put option plus prepaid forward $F_{0,T}^{P}(S)$ to get one share of the stock at time T.

Portfolio D: a zero-coupon bond Ke^{-rT}.

If $S(T) < K$, then the option in portfolio C is exercised at T and the portfolio becomes worth K. If $S(T) \geq K$, then the put option expires worthless and the portfolio is worth $S(T)$ at this time. Hence, portfolio C is worth

$$\max\{S(T), K\}$$

at T. Portfolio D is worth K at T. Hence, portfolio C is always worth as much as, and can sometimes be worth more than, portfolio D at T.

		$S(T)>K$	$S(T)<K$
Portfolio C	European pull option	0	$K-S(T)$
	One share of the stock	$S(T)$	$S(T)$
	Total	$S(T)$	K
Portfolio D	Zero-coupon bond	K	K

It follows that in the absence of arbitrage opportunities portfolio C must be worth at least as much as portfolio D today. Hence,

$$P_{Eur} + F_{0,T}^{P}(S) \geq Ke^{-rT} \qquad \text{or} \qquad P_{Eur} \geq Ke^{-rT} - F_{0,T}^{P}(S).$$

Because the worst that can happen to a put option is that it expires worthless, its value cannot be negative. This means that

$$\max\{Ke^{-rT} - F_{0,T}^{P}(S), 0\} \le P_{Eur} \le Ke^{-rT}.$$

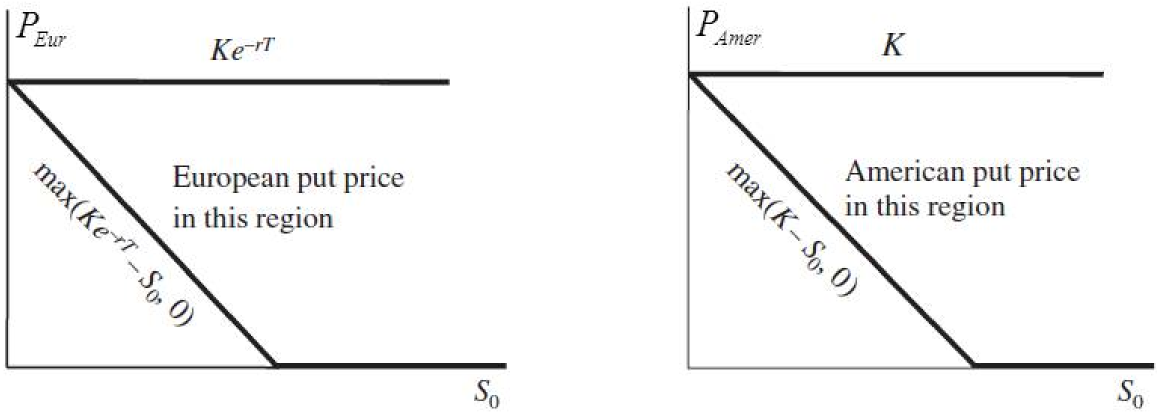
For an American put option on any stock, the condition

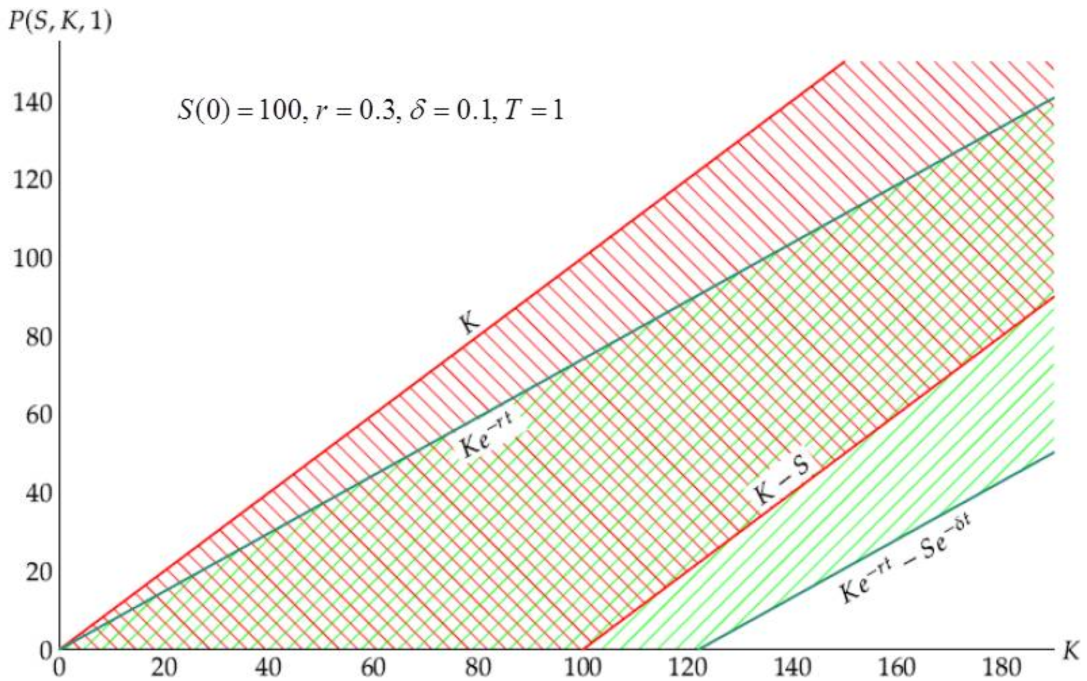
$$\max\{K - S(0), 0\} \le P_{Amer}$$

must apply because the option can be exercised at any time. The bounds for an American put option on any stock are

$$\max\{K - S(0), 0\} \le P_{Amer} \le K.$$

Bounds for European and American put options when there are no dividends.



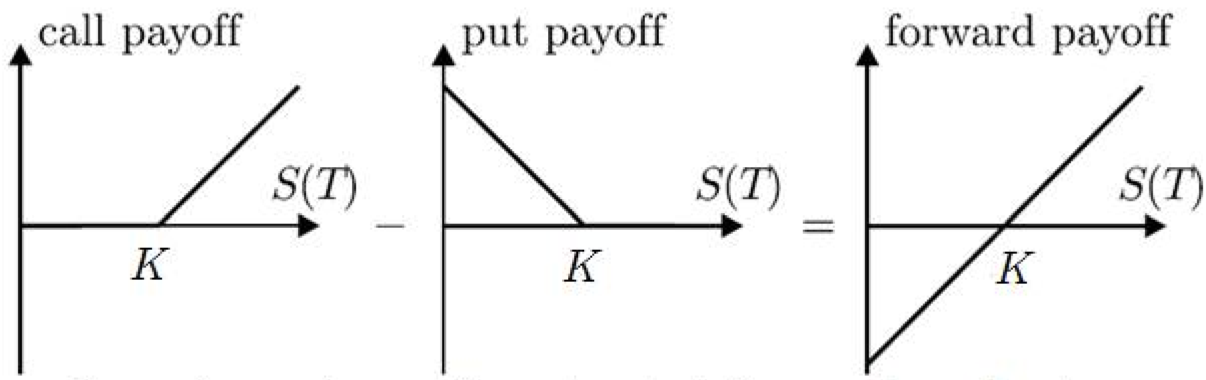


Bounds for American and European put option prices. $S = 100$. The red boundary and hatch lines represent possible values for an American put and the green boundary and hatch lines represent possible values for a European put.

Suppose you bought a European call option and sold a European put option, both having the same underlying asset, the same strike price K and the same expiration date T. You need to pay $C(K,T) - P(K,T)$ at time 0. One of the 2 options is sure to be exercised at time T, unless $S(T) = K$, in which both options are worthless.

- If $S(T) > K$, you would exercise your long call and buy the asset at the strike price K. The purchaser of the put that you wrote would not exercise the option to sell you the stock. Thus, you end up buying the asset for K.
- If $S(T) < K$, you would not exercise your long call. However the purchaser of the put that you wrote would exercise the option to sell you the asset at the strike price. Thus, you still end up buying the asset for K.
- If $S(T) = K$, you can buy the underlying asset in the market.

You can see that no matter what happens to the spot price, you will always end up buying the asset at the strike price of K. But this is exactly what happens under a long forward: you are obligated to buy the asset for the forward price.



Long forward payoff constructed from calls and puts

Because the combined payoff is similar to payoff of a long forward, we call the combination of the long call and short put a **synthetic** long forward contract. However, under a "true" long forward contract, there is no premium.

Therefore, there are 2 ways to receive $S(T)$ at time T:

1. Buy a European call option and sell a European put option, both having the same underlying asset, the same strike price K and the same expiration date T.

2. Enter a long forward contract to buy the asset for $S(T)$, and prepaid $F_{0,T}^{P}$ at time 0.

Portfolio	Cash outflows at time 0	Cash outflows at time T	Asset at time T
1. Synthetic forward	$C(K,T)-P(K,T)$	K	$S(T)$
2. Prepaid forward	$F_{0,T}^{P}$	0	$S(T)$

By the no-arbitrage principle, these 2 portfolios must cost the same. Discounting to time 0, we have

Theorem 3.23 (Put-Call Parity)

$$C(K,T)-P(K,T)+Ke^{-rT}=F_{0,T}^{P}.$$

For a non-dividend paying stock, put-call parity becomes

$$C(K,T)-P(K,T)+Ke^{-rT}=S(0).$$

Example 3.24

A non-dividend paying stock has a price of 50. The price of a 6-month European call and put option with an exercise price of \$48 is \$5 and \$3 respectively. The risk-free interest rate is a constant annual 8%, compounded continuously. State an arbitrage strategy involving buying or selling 1 share of stock and buying or selling puts and calls. Calculate the profit after 6 months.

Solution

First use put-call parity to determine whether the put is underpriced or overpriced.

$$P(48,0.5)=C(48,0.5)+48e^{-0.08\times 1/2}-50=1.1179.$$

Since the put has price \$3, it is overpriced and you should sell it.

At time 0

- Short sell a stock for \$50.
- Buy 1 call for \$5.

- Sell 1 put for \$3.
- Buy a zero coupon bond for $50-5+3=\$48$

After 6 months, \$48 will grow to $\$48e^{0.04}=\$49.959$. You will pay 48 for the stock and the net gain is $\$49.959-\$48=\$1.959$.

If a stock pays discrete dividends, put-call parity becomes

$$C(K,T)-P(K,T)+Ke^{-rT}=S(0)-\mathrm{PV}_{0,T}(\mathrm{Divs}).$$

We can create a synthetic stock by

$$S(0)=\underbrace{C(K,T)-P(K,T)}_{\text{synthetic long forward contract}}+\underbrace{Ke^{-rT}+\mathrm{PV}_{0,T}(\mathrm{Divs})}_{\text{amount to lend}}$$

Example 3.25

Suppose the risk-free rate is 5%. The strike price of call and put option is 40 and the expiration date is 1 year. The dividends of stock are 0.5 at the end of 3 months and at the end of 9 months. Then their present value is

$$0.5e^{-0.05(0.25)}+0.5e^{-0.05(0.75)}=0.97539$$

To create a synthetic stock, we buy a call, sell a put, and lend $0.97539+40e^{-0.05}=39.0246$. At the end of the year, it will have 40 plus accumulated value of the dividends:

$$\left(0.97539+40e^{-0.05}\right)e^{0.05}=40+0.97539e^{0.05}.$$

One of the options will be exercise, so the 40 will be exchanged for 1 share of the stock.

Consider a stock with continuous dividends at rate δ. Put-call parity becomes

$$C(K,T)-P(K,T)+Ke^{-rT}=S(0)e^{-\delta T}.$$

Example 3.26

Let δ be the continuous dividend rate of a stock. You are given the following values for 1-year European call and put options at various strike prices:

Strike Price	Call Premium	Put Premium
40	8.25	1.12
45	5.40	P_2
50	4.15	6.47

Determine P_2.

Solution

By put-call parity,

$$40e^{-r} - S(0)e^{-\delta} = 1.12 - 8.25 = -7.13 \quad \text{and} \quad 50e^{-r} - S(0)e^{-\delta} = 6.47 - 4.15 = 2.32.$$

We have $e^{-r} = 0.945$, $S(0)e^{-\delta} = 44.93$.

Hence $P_2 = C_2 + Ke^{-rT} - S(0)e^{-\delta T} = 5.4 + 45(0.945) - 44.93 = 2.995$.

The options we have discusses so far involve receiving or giving a stock in return for cash. We can generalize to an **exchange option** to receive a stock in return for a different stock. Let $S(t)$ be the value of the underlying asset and $Q(t)$ be the price of strike asset. $F_{t,T}^{P}(Q)$ means a prepaid forward contract to purchase asset Q.

$C(S(t),Q(t),T-t)$ means a call option written at time t which lets the purchaser has the right to receive $S(T)$ in return for $Q(T)$ at time T; in other words, to receive $\max(0,S(T)-Q(T))$.

$P(S(t),Q(t),T-t)$ means a put option written at time t which lets the purchaser has the right to give $S(T)$ in return for $Q(T)$ at time T; in other words, to receive $\max(0,Q(T)-S(T))$.

The put-call parity equation is

$$C(S(t),Q(t),T-t) - P(S(t),Q(t),T-t) = F_{t,T}^{P}(S) - F_{t,T}^{P}(Q).$$

The definitions of calls and puts are mirror images, that means,

$$P(S(t),Q(t),T-t)=C(Q(t),S(t),T-t)$$

and we can write the above equation with just calls:

$$C(S(t),Q(t),T-t)-C(Q(t),S(t),T-t)=F_{t,T}^{P}(S)-F_{t,T}^{P}(Q)$$

and we can write the above equation with just puts:

$$P(Q(t),S(t),T-t)-P(S(t),Q(t),T-t)=F_{t,T}^{P}(S)-F_{t,T}^{P}(Q).$$

Example 3.27

Stock A pays dividends at a continuous rate of 2% and Stock B pays dividends at a continuous rate of 4%. The current price for Stock A is 70 and the current price for Stock B is 30. The continuously compounded risk-free rate is 5%.

A European put option which allows an investor to sell 2 shares of Stock B for 1 share of Stock A at the end of a year is \$11.5. Determine the premium of the European call option which allows the investor to purchase 2 shares of Stock B with 1 share of Stock A at the end of a year.

Solution

$$F_{0,1}^{P}(S)=S(0)e^{-\delta_S T}=2(30)e^{-0.04}=57.64737$$
$$F_{0,1}^{P}(Q)=Q(0)e^{-\delta_Q T}=70e^{-0.02}=68.61391$$
$$C(S,Q,1)=P(S,Q,1)+F_{0,1}^{P}(S)-F_{0,1}^{P}(Q)=11.50+57.64737-68.61391=0.5335$$

Let the domestic currency (HK dollars) be the one the option is denominated in, the one in which the price is expressed. Let the foreign currency (US dollars) be the underlying asset of the option. Let $x(0)$ (say, 7.8) be the spot exchange rate expressed as units of domestic currency (HK dollars) per foreign currency (US dollars). The exchange rate expressed in the foreign currency (US dollars) is $1/x(0)=1/7.8$ units of the foreign currency (US dollars) per 1 unit of the domestic currency (HK dollars). A foreign currency (US dollars) is analogous to a stock paying a known dividend yield. The owner of foreign currency (US dollars) receives a yield δ equal to the risk-free interest rate, r_f, in the foreign currency (US dollars).

Let $C_d(x(0),K,T)$ be the price of call option in the domestic currency (HK dollars) on foreign currency (US dollars) with spot exchange rate $x(0)$ to exercise at exchange rate K at time T, and $P_d(x(0),K,T)$ the corresponding option. The put-call parity equation for currency options (in domestic HK dollars) is

$$C_d(x(0),K,T)-P_d(x(0),K,T)=x(0)e^{-r_fT}-Ke^{-r_dT}.$$

where r_d is risk-free rate of domestic currency (HK dollars). Domestic currency is the same as currency of strike price.

Example 3.28

You are given

(i) The spot exchange rate for US dollars to pounds is \$1.4/£.

(ii) The continuously compounded risk-free rate for dollars is 5%.

(iii) The continuously compounded risk-free rate for pounds is 8%.

A 9-month European put option allows selling £1 at the rate of \$1.5/£. A corresponding 9-month dollar denominated call option is \$0.0223. Determine the premium of the 9-month dollar denominated put option.

Solution

The prepaid forward price for pound is $x(0)e^{-r_fT}=1.4e^{-0.08(0.75)}=1.31847$.

The prepaid forward for the strike asset (dollars) is $Ke^{-r_dT}=1.5e^{-0.05(0.75)}=1.44479$.

$$P(x(0),K,T)=C(x(0),K,T)-x(0)e^{-r_fT}+Ke^{-r_dT}=0.0223-1.31847+1.44479=0.14862.$$

A domestic-denominated call $C_d(x(0),K,T)$ giving a right to purchase 1 foreign currency (US dollars) for K domestic currency (HK dollars) is equivalent to a domestic-denominated put giving a right to sell K domestic currency (HK dollars) for 1 foreign currency (US dollars).

A foreign-denominated put $P_f\left(\frac{1}{x(0)},\frac{1}{K},T\right)$ giving a right to sell 1 domestic currency (HK dollars) for $1/K$ foreign currency (US dollars). We have

$$\underbrace{C_d(x(0),K,T)}_{\text{domestic currency}} = \underbrace{KP_f\left(\frac{1}{x(0)},\frac{1}{K},T\right)}_{\text{foreign currency}}$$

However, the units are different. Express the option prices in domestic currency, we get

$$C_d(x(0),K,T) = Kx(0)P_f\left(\frac{1}{x(0)},\frac{1}{K},T\right).$$

Similarly,

$$P_d(x(0),K,T) = Kx(0)C_f\left(\frac{1}{x(0)},\frac{1}{K},T\right).$$

We still have put-call parity equation

$$C_f\left(\frac{1}{x(0)},\frac{1}{K},T\right)+\frac{1}{K}e^{-r_fT} = P_f\left(\frac{1}{x(0)},\frac{1}{K},T\right)+\frac{1}{x(0)}e^{-r_dT}$$

Example 3.29

The spot exchange rate for dollars into euros is \$1.05/€. A 6-month dollar denominated call option to buy one euro at strike price \$1.1/€ is \$0.04. Determine the premium of the corresponding euro-denominated put option to sell one dollar for euros at the corresponding strike price.

Solution

Domestic currency is US dollars \$. Foreign currency is euros €. $x(0) = 1.05$, $T = 0.5$, $K = 1.1$

$$\underbrace{C_\$(1€,\$1.1,0.5)}_{\text{US dollars}} = \underbrace{1.1P_€\left(\$1,\frac{1}{1.1}€,0.5\right)}_{\text{euros}}.$$

$$\underbrace{P_€\left(\$1,\frac{1}{1.1}€,0.5\right)}_{\text{euros}} = \underbrace{\frac{1}{1.1}C_\$(1€,\$1.1,0.5)}_{\text{US dollars}} = \$\frac{0.04}{1.1} = \frac{0.04}{1.1(1.05)}€.$$

Consider European call and put futures options, both with strike price K and time to expiration T. We can form two portfolios:

Portfolio A: a European call futures option plus an amount of cash equal to Ke^{-rT}.

Portfolio B: a European put futures option plus a long futures contract plus an amount of cash equal to $F(0)e^{-rT}$, where $F(0)$ is the futures price.

In portfolio A, the cash can be invested at the risk-free rate, r, and grows to K at time T. Let $F(T)$ be the futures price at maturity of the option. If $F(T) > K$, the call option in portfolio A is exercised and portfolio A is worth $F(T)$. If $F(T) \le K$, the call is not exercised and portfolio A is worth K. The value of portfolio A at time T is therefore

$$\max\{F(T), K\}.$$

In portfolio B, the cash can be invested at the risk-free rate to grow to $F(0)$ at time T. The put option provides a payoff of $\max\{K - F(T), 0\}$. The futures contract provides a payoff $F(T) - F(0)$. The value of portfolio B at time T is therefore

$$F(0) + \left(F(T) - F(0)\right) + \max\{K - F(T), 0\} = \max\{F(T), K\}.$$

Because the two portfolios have the same value at time T and European options cannot be exercised early, it follows that they are worth the same today. The value of portfolio A today is

$$C + Ke^{-rT}$$

where C is the price of the call futures option. The daily settlement process ensures that the futures contract in portfolio B is worth zero today. Portfolio B is therefore worth

$$P + F(0)e^{-rT}$$

where P is the price of the put futures option. Hence

$$C + Ke^{-rT} = P + F(0)e^{-rT}.$$

The difference between this put-call parity relationship and the one for a non-dividend-paying stock is that the stock price, $S(0)$, is replaced by the discounted futures price, $F(0)e^{-rT}$.

American options are worth at least as much as European options, since you can exercise the American option before expiration date.

$$C_{Amer} \geq C_{Eur}, \quad P_{Amer} \geq P_{Eur}.$$

An American call on a non-dividend-paying stock should not be exercised early.

Consider an American call option on a non-dividend-paying stock with one month to expiration when the stock price is \$70 and the strike price is \$40. The option is deep in the money, and the investor who owns the option might well be tempted to exercise it immediately.

However, if the investor plans to hold the stock obtained by exercising the option for more than one month, this is not the best strategy. A better action is to keep the option and exercise it at the end of the month. Exercising the call option and paying out strike price \$40 one month later is better than that immediately, so that interest is earned on the \$40 for one month. Because the stock pays no dividends, there is no income from the stock. A further advantage of waiting rather than exercising immediately is that there is some chance that the stock price will fall below \$40 in one month. In this case the investor will not exercise in one month and will be glad that the decision to exercise early was not taken! The later the strike price is paid out the better.

Furthermore there are no advantages to exercising early if the investor plans to take the profit from the call option. In this case, the investor is better off selling the option than exercising it. The option will be bought by another investor who wants to buy the stock in a cheap price. Once the option has been exercised and the strike price has been exchanged for the stock price, this insurance vanishes.

No early exercise of American call option can be explained by put-call parity. By put-call parity, at time t, an option that expires at time T satisfies

$$\begin{aligned} C_{Amer}(S(t), K, T-t) &\geq C_{Eur}(S(t), K, T-t) \\ &= P_{Eur}(S(t), K, T-t) + S(t) - Ke^{-r(T-t)} \end{aligned}$$

$$= \underbrace{P_{Eur}(S(t), K, T-t)}_{\substack{\text{value of buying a stock} \\ \text{at a price below } K \text{ at } T}} + \underbrace{(S(t)-K)}_{\text{call exercise value at } t} + \underbrace{K\left(1-e^{-r(T-t)}\right)}_{\text{PV of interest of } K \text{ in } T-t}$$

$$\geq S(t) - K$$

Exercising the American call option at time t is worth $S(t)-K$, so the call option for the remainder of the period $T-t$ is worth more than the exercise value. Early exercise is not rational. The above inequality does not mean you must hold the option until expiration time T. It just means selling the option will give you a higher profit than exercising it early.

The **intrinsic value** of an American option is defined as the maximum of zero and the value the option would have if it were exercised immediately. For a call option, the intrinsic value is therefore $\max\{S-K, 0\}$. For a put option, it is $\max\{K-S, 0\}$. An in-the-money American option must be worth at least as much as its intrinsic value because the holder can realize the intrinsic value by exercising immediately. Often it is optimal for the holder of an in-the-money American option to wait rather than exercise immediately. The option is then said to have **time value**. The total value of an American option can be thought of as the sum of its intrinsic value and its time value.

Suppose you exercise the American call option at time t before expiration date T. You need to pay the exercise price K, receive the stock, and sell it, your net profit is $S(t)-K$ regardless of how the price goes at time T. You also get interest on $S(t)-K$ for the period $T-t$.

On the other hand, you don't exercise the American call option at time t but sell the stock short, your net profit is $S(t)-S(T)$ from the shorted stock, plus $\max\{S(T)-K, 0\}$ from exercise of the American call option at time T. You also get interest on $S(t)$ for the period $T-t$. The interest of $S(t)$ is greater than that of early exercise since $S(t) > S(t)-K$. If $S(T) \geq K$, $S(t)-S(T)+\max\{S(T)-K, 0\} = S(t)-K$, the same as payoff with early exercise. If $S(T) < K$, the payoff is still greater.

Theorem 3.30

The prices of European and American call options on a stock that pays no dividends are equal,

$C_{Amer} = C_{Eur}$, whenever the strike price K and expiry time T are the same for both options.

Usually there is no closed form formula for an American option. An exception is if it is not rational to exercise the option early. Specifically, it is not rational to exercise an American call option on a non-dividend paying stock early. For options which are not rational to exercise early, an American option can be priced with the formula for a European option.

Before going further, here are 2 terms we will use:

Ex-dividend means "after dividend is paid". Cum-dividend means "including dividend; before dividend is paid".

An extension to Theorem 3.30 shows that, when there are discrete dividends, it can only be optimal to exercise at a time immediately before the stock goes ex-dividend. If the stock pays dividends, it may be (not necessary) rational to exercise an American call option early. As an extreme example, if a stock worth 100 is about to pay the entire value of the stock as dividend and the strike price is 60, then the option is worth 40 before the dividend, 0 afterwards.

In fact, for the case of an American option with strike price K on a stock which pays discrete dividend before expiration date T,

$$\begin{aligned} C_{Amer}(S(t),K,T-t) &\geq C_{Eur}(S(t),K,T-t) \\ &= P_{Eur}(S(t),K,T-t) + F^P_{t,T}(S) - Ke^{-r(T-t)} \\ &= P_{Eur}(S(t),K,T-t) + S(t) - \mathrm{PV}_{t,T}(\mathrm{Div}) - Ke^{-r(T-t)} \\ &= \underbrace{P_{Eur}(S(t),K,T-t)}_{\substack{\text{value of buying a stock} \\ \text{at a price below } K \text{ at } T}} + \underbrace{(S(t)-K)}_{\text{call exercise value at } t} + \underbrace{K\left(1-e^{-r(T-t)}\right)}_{\text{PV of interest of } K \text{ in } T-t} - \mathrm{PV}_{t,T}(\mathrm{Div}) \end{aligned}$$

Suppose the option is in the money at t. When you exercise an American call option early, you lose interest of the strike price K on $[t, T]$ and gain dividends $\mathrm{PV}_{t,T}(\mathrm{Div})$ on the stock. The other effect of early exercise is losing a chance of buying a stock at a price below K at T. Its value is $P_{Eur}(S(t),K,T-t)$. Early exercise will not be rational if $\mathrm{PV}_{t,T}(\mathrm{Div}) <$

$K\left(1-e^{-r(T-t)}\right)+P_{Eur}(S(t),K,T-t)$. It does not mean early exercise will be rational if not.

However, for an American call option on a stock with one discrete dividend, it is possible to develop a formula.

Let t be the time at which the dividend is paid, T the final expiry, and D the dividend. $S(t^+)$ will denote the stock price at time t after paying dividend, so the stock price at time t before paying dividend is $S(t^-)=S(t^+)+D$. Then at time t, there are 2 choices:

1. No exercise the option early (at time t). Then the option is worth $C_{Eur}(S(t^+),K,T-t^+)$.

2. Exercise the option early (at time t before paying dividend). Then you will pay K and receive $S(t^+)+D$.

Therefore, the value of the American option at time t (before paying dividend) is

$$\begin{aligned} C_{Amer}(S(t^-),K,T-t^-) &= \max\left\{S(t^+)+D-K, C_{Eur}(S(t^+),K,T-t^+)\right\} \\ &= \max\left\{S(t^+)+D-K, S(t^+)-Ke^{-r(T-t^+)}+P_{Eur}(S(t^+),K,T-t^+)\right\} \\ &= S(t^+)-K+\max\left\{D, K\left(1-e^{-r(T-t^+)}\right)+P_{Eur}(S(t^+),K,T-t^+)\right\} \end{aligned}$$

Suppose the option is in the money at t^-. If $K\left(1-e^{-r(T-t^+)}\right)+P_{Eur}(S(t^+),K,T-t^+)<D$, it is rational to exercise this option early (at time t before paying dividend). If not, it is not rational.

Example 3.31

Consider Example 3.14. The "value of interest of strike price from t = 1 to t = 2" at t = 1 is $90(1-e^{-0.06})=5.2412<6$. It can be exercised (but not necessary) American call at t = 1.

Consider node B. The "value of a chance of buying stock at a price below K at T = 2" at t = 1 is 0 because both stock prices at T = 2 is larger than K. Since $D=6>P_{Eur}(S(t^+),K,T-t)$

$+90(1-e^{-0.06})=5.2412,$ it is optimal to exercise the American call option at node B. The value of American call at node B is $S(t^{+})+D-K=132+6-90=48.$

Consider node C. The "value of a chance of buying stock at a price below K at $T=2$" at $t=1$ is $e^{-0.06}(90-68.8)(1-p^{*})=6.8962=P_{Eur}(S(t^{+}),K,T-t).$ Since $D=6<P_{Eur}(S(t^{+}),K,T-t)$ $+90(1-e^{-0.06})=12.1374,$ it is not optimal to exercise the American call option at node C. The value of American call at node C is $S(t^{+})-K+12.3174=12.3174-4=8.3174.$

Example 3.32

A 3-month American call option on a stock has a strike price of 34. Suppose $r = 0.05$ and $\sigma=0.2.$ The stock will pay a dividend at the end of first month. If its price is 33.5 after paying dividend, it is optimal to exercise the option before the dividend payment. Determine the dividend.

Solution

If the stock price at the end of 1 month is 33.5, then the price of the put is

$$
\begin{aligned}
&d_1=\frac{\ln(33.5/34)+\left(0.05+0.5(0.2)^2\right)(1/6)}{0.2\sqrt{1/6}}=-0.03856\\
&N(-d_1)=N(0.03856)=0.51538\\
&d_2=-0.03856-0.2\sqrt{1/6}=-0.12021\\
&N(-d_2)=N(0.12021)=0.54784\\
&P(33.5,34,1/6)=34e^{-0.05/6}(0.54784)-33.5(0.51538)=1.207
\end{aligned}
$$

For exercise to be optimal, we need

$$D>K\left(1-e^{-r(T-t)}\right)+P(33.5,34,1/6)=34\left(1-e^{-0.05/6}\right)+1.207=1.489$$

Assume that n ex-dividend dates of a stock are anticipated and that they are at times $t_1,t_2,\cdots,t_n,$ with $t_1<t_2<\cdots<t_n.$ The dividends corresponding to these times will be denoted by $D_1,D_2,\cdots,D_n$ respectively.

It is not rational to exercise at time t_i if $\text{PV}_{t_i}(D_i,\cdots,D_k) < K\left(1-e^{-r(t_{k+1}-t_i)}\right)$ for every $k \geq i$.

That is

$$D_i e^{r(t_{i+1}-t_i)} < K\left(e^{r(t_{i+1}-t_i)}-1\right), \quad D_i e^{r(t_{i+2}-t_i)} + D_{i+1}e^{r(t_{i+2}-t_{i+1})} < K\left(e^{r(t_{i+2}-t_i)}-1\right), \quad \cdots$$

Example 3.33

An American call option on a stock has a strike price of 85 and expires in 5 months. The risk free rate is 4%. Volatility is 0.2. A dividend of 0.4 is payable at the end of today, and another dividend of 1 is payable in 3 months. The ex-dividend stock price is 95 after 3 months. What is the decision of a rational investor?

Solution

The present value of the interest on the strike price between today and 3 months later is

$$85\left(1-e^{-0.04(3/12)}\right) = 0.8457641313207154 > 0.4.$$

It is not rational to exercise today. It reduces to 1 dividend case.

$$\begin{aligned}
d_1 &= \frac{\ln(95/85)+\left(0.04+0.5(0.2)^2\right)(1/6)}{0.2\sqrt{1/6}} = 1.4847 \\
N(-d_1) &= N(-1.4847) = 0.068811 \\
d_2 &= 1.4847 - 0.2\sqrt{1/6} = 1.403055 \\
N(-d_2) &= N(-1.403055) = 0.0803 \\
P &= 85e^{-0.04/6}(0.0803) - 95(0.068811) = 0.243103
\end{aligned}$$

$$K\left(1-e^{-r(T-t^+)}\right) + P_{Eur}(S(t^+),K,T-t^+) = 85\left(1-e^{-0.04/6}\right) + 0.243103 = 0.807884968322076 < 1.$$

It is rational to exercise at 3 months.

Example 3.34

Redo Example 3.28 if a dividend of 1 is payable at the end of today, and another dividend of

0.4 is payable in 3 months.

Solution

The present value of the interest on the strike price between today and 3 months later is

$$85\left(1-e^{-0.04(3/12)}\right)=0.8457641313207154<1.$$

It does not mean it is rational to exercise today.

The present value of the interest on the strike price between today and 5 months later is

$$85\left(1-e^{-0.04(5/12)}\right)=1.4049264251625125>1+0.4e^{-0.04(3/12)}=1.3960199334996672.$$

It is not rational to exercise today. It reduces to 1 dividend case.

$$\begin{aligned}
d_1 &= \frac{\ln(95/85)+\left(0.04+0.5(0.2)^2\right)(1/6)}{0.2\sqrt{1/6}}=1.4847\\
N(-d_1) &= N(-1.4847)=0.068811\\
d_2 &= 1.4847-0.2\sqrt{1/6}=1.403055\\
N(-d_2) &= N(-1.403055)=0.0803\\
P &= 85e^{-0.04/6}(0.0803)-95(0.068811)=0.243103
\end{aligned}$$

$$K\left(1-e^{-r(T-t^+)}\right)+P_{Eur}(S(t^+),K,T-t^+)=85\left(1-e^{-0.04/6}\right)+0.243103=0.807884968322076>0.4$$

It is not rational to exercise at 3 months.

Example 3.35

Consider the situation in Example 3.13, but suppose that the option is American rather than European. In this case, $D_1=D_2=0.5, S(0)=K=40, r=0.09, t_1=2/12, t_2=5/12, T=0.5$. Since

$$K\left(1-e^{-r(t_2-t_1)}\right)=40\left(1-e^{-0.09(0.25)}\right)=0.89>0.5,$$

the option should never be exercised immediately before the first ex-dividend date. In addition, since

$$K\left(1-e^{-r(T-t_2)}\right)=40\left(1-e^{-0.09(0.08333)}\right)=0.3<0.5,$$

it is sufficiently deep in the money. The option can be exercised immediately before the second ex-dividend date.

Black's approximation takes the greatest value of the option when it can be (but not necessary) exercised at the end of 5 months or 6 months.

Suppose the option will be exercised after 5 months. The present value of the first dividend is

$$0.5e^{-0.1667\times 0.09}=0.4926.$$

The option price can therefore be calculated from the Black-Scholes formula, with $F^P(S)=40-0.4926=39.5074$, $K=40$, $r=0.09$, $\sigma=0.3$ and $T=5/12$:

$$d_1=\frac{\ln(39.5074/40)+(0.09+0.3^2/2)\times 5/12}{0.3\sqrt{5/12}}=0.2264845371047694$$

$$d_2=\frac{\ln(39.5074/40)+(0.09-0.3^2/2)\times 5/12}{0.3\sqrt{5/12}}=0.0328353697943986$$

gives

$$N(d_1)=0.5895877 \quad\text{and}\quad N(d_2)=0.513097.$$

The call price is

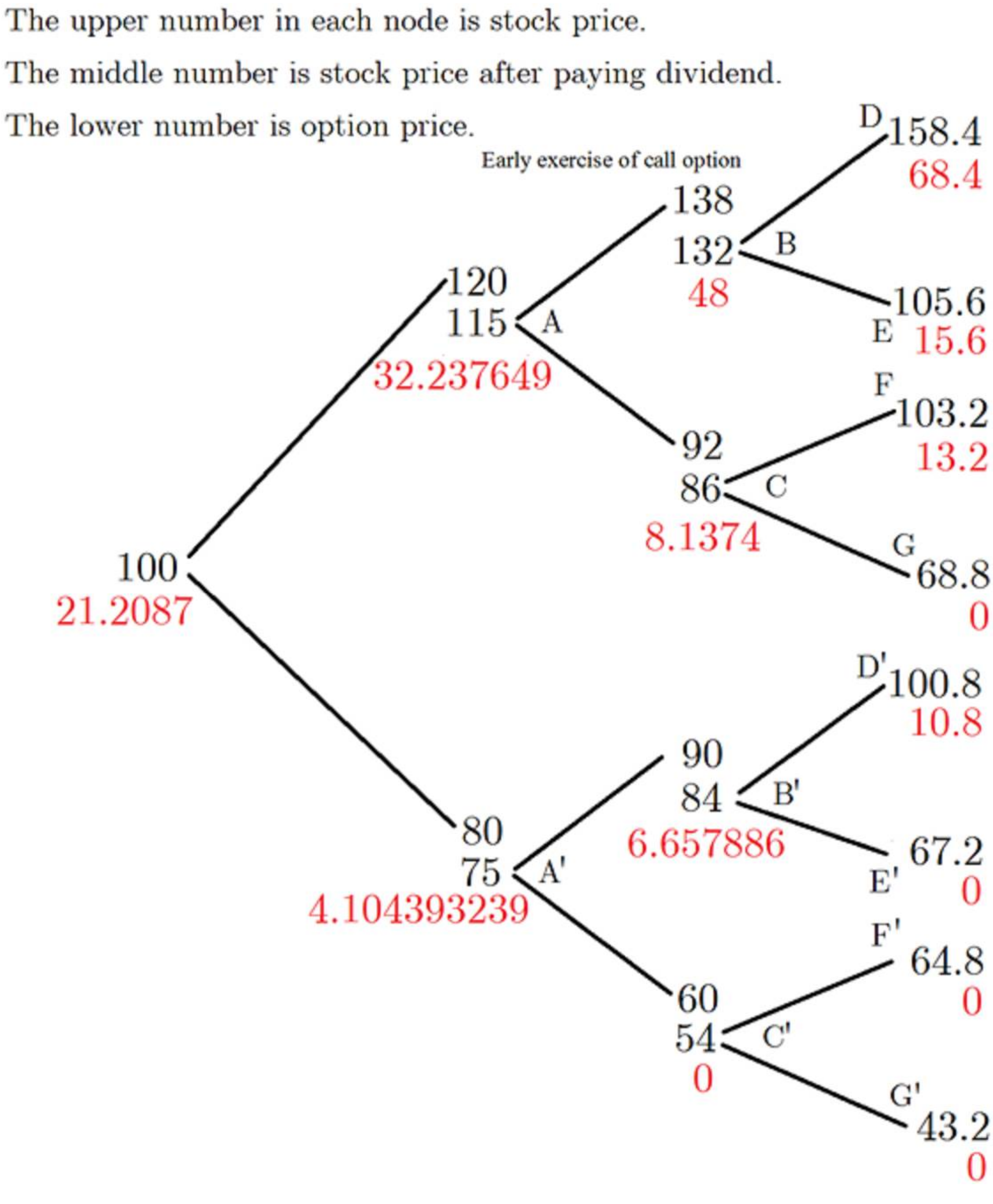
$$C=F^P(S)N(d_1)-Ke^{-rT}N(d_2)=39.5074(0.5895877)-40e^{-0.09\times 5/12}(0.513097)=3.52459$$

Suppose the option will be exercised after 6 months. From Example 3.14, its value is 3.67. Black's approximation, therefore, gives the value of the American call as 3.67.

Indeed, the actual value of the American call is 3.72, which is a little bit large than 3.67. In Black's approximation, the assumption is that the holder has to decide today whether the option will be exercised after 5 months or after 6 months; actually it allows the investor to make decision on early exercise at the 5-month point to depend on the stock price at that time. That is why the actual value is larger than Black's approximation.

Example 3.36

Consider a 3-year American call with a strike price of \$90 on a discrete dividend paying stock with current price is \$100. Suppose that there are 3 time steps of 1 year, and in each time step the stock price either moves up by 20% or moves down by 20%. \$5 and \$6 dividend will be paid after movement of stock price in 1 year and 2 years respectively. The exercise price is \$90 and the continuously risk-free rate of interest is 6% per annum.



The value of "interest of K from $t = 1$ to $t = 2$" at $t = 1$ is the same as the value of "interest of K from $t = 2$ to $t = 3$" at $t = 2$, which is $90(1-e^{-0.06}) = 5.2412$. It lies between 5 and 6.

It is not rational to exercise the American call option in 1 year and can be (but not necessary) exercised in 2 years. It can be also verified by binomial tree model.

The American option will not be exercised at A' because of negative payoff. It also will not be exercised at A because it is worth 32.237649 which is greater than the payoff = 32. It is only optimal to exercise the option at node B in 2 years.

It reduces to 1 dividend case.

Determination of early exercise at node B and node C can be referred to Example 3.31. Determination of early exercise node C' is similar to that of node C.

The 2-year European option is worth $e^{-0.12}\left(48(p^*)^2+2p^*(1-p^*)\right)=18.6428$. The 3-year European option is worth $e^{-0.18}\left(68.4(p^*)^3+(15.6+13.2+10.8)(p^*)^2(1-p^*)\right)$ = 20.92. Hence, the black approximation is 20.92.

The American call premium is $32.237649p^*e^{-0.06}+10.8e^{-0.18}(p^*)^2(1-p^*)=21.2087.$

Consider an American call option with expiration date T on a stock paying dividends at a continuous rate δ. We have

$$\begin{aligned}
C_{Amer}(S(t),K,T-t) &\geq C_{Eur}(S(t),K,T-t) \\
&= P_{Eur}(S(t),K,T-t)+F^P_{t,T}(S)-Ke^{-r(T-t)} \\
&= P_{Eur}(S(t),K,T-t)+S(t)e^{-\delta(T-t)}-Ke^{-r(T-t)} \\
&= \underbrace{P_{Eur}(S(t),K,T-t)}_{\substack{\text{value of buying a stock}\\ \text{at a price below } K \text{ at } T}}+\underbrace{(S(t)-K)}_{\text{call exercise value at } t}+\underbrace{K\left(1-e^{-r(T-t)}\right)}_{\text{PV of interest of } K \text{ in } T-t}-\underbrace{S(t)\left(1-e^{-\delta(T-t)}\right)}_{\text{PV of future dividends in } T-t}
\end{aligned}$$

Example 3.37

For an American call option on a stock, $r=0.05, \delta=0.04, T=0.25, K=60.$ If the value of a 3-month European put option with strike price 60 is 1 and exercise immediately is optimal, what is the lowest possible current value of the stock?

Solution

The present value of future dividends is

$$S(t)\left(1-e^{-\delta(T-t)}\right)=S(0)\left(1-e^{-0.04(0.25)}\right)=0.00995017S(0)$$

The present value of interest on the strike price is

$$K\left(1-e^{-r(T-t)}\right)=60\left(1-e^{-0.05(0.25)}\right)=0.745332$$

Therefore to make early exercise optimal,

$$\begin{aligned}
&S(t)\left(1-e^{-\delta(T-t)}\right)\geq P_{Eur}(S(t),K,T-t)+K\left(1-e^{-r(T-t)}\right)\\
&0.00995017S(0)\geq 1+0.745332\\
&S(0)\geq 175.407
\end{aligned}$$

It can be optimal to exercise an American put option on a non-dividend-paying stock early. Indeed, at any given time during its life, a put option should always be exercised early if it is sufficiently deep in the money.

To illustrate, consider an extreme situation. Suppose that the strike price is \$10 and the stock price is virtually zero. By exercising immediately, an investor makes an immediate gain of \$10. If the investor waits, the gain from exercise might be less than \$10, but it cannot be more than \$10, because negative stock prices are impossible.

Furthermore, receiving \$10 now is preferable to receiving \$10 in the future. It follows that the option should be exercised immediately.

For (discrete) dividend paying stock, we have

$$\begin{aligned}
P_{Amer}(S(t),K,T-t)\geq P_{Eur}&(S(t),K,T-t)\\
&=C_{Eur}(S(t),K,T-t)-F_{t,T}^{P}(S)+Ke^{-r(T-t)}\\
&=C_{Eur}(S(t),K,T-t)-S(t)+\mathrm{PV}_{t,T}(\mathrm{Div})+Ke^{-r(T-t)}\\
&=\underbrace{C_{Eur}(S(t),K,T-t)}_{\substack{\text{value of selling a stock}\\ \text{at a price above } K \text{ at } T}}+\underbrace{(K-S(t))}_{\text{put exercise value at } t}-\underbrace{K\left(1-e^{-r(T-t)}\right)}_{\text{PV of interest of } K \text{ in } T-t}+\mathrm{PV}_{t,T}(\mathrm{Div})
\end{aligned}$$

$$S(t)+P_{Amer}(S(t),K,T-t)\geq C_{Eur}(S(t),K,T-t)+K-K\left(1-e^{-r(T-t)}\right)+\mathrm{PV}_{t,T}(\mathrm{Div}).$$

Suppose the option is in the money at t. By exercising the option early, you

(1) get the cash earlier, so you earn interest on the strike price.

(2) give up a chance for waiting stock price greater than strike price at expiration date. Its value is the European call premium.

(3) give up dividends.

If $C_{Eur}(S(t),K,T-t)+\mathrm{PV}_{t,T}(\mathrm{Div})>K\left(1-e^{-r(T-t)}\right)$, it is not rational to exercise early. If not, it may be rational (but not always) to exercise early.

In conclusion, for European call options on dividend-paying stocks — regardless of whether dividends are discrete or continuous — and for all European put options, no general statement can be made.

An American (call or put) option with expiry T and strike price K must cost at least as much as one the expiry t, $t < T$, and same strike price since you have a choice to exercise the option after t. That is

$$C_{Amer}(S(0),K,t)\leq C_{Amer}(S(0),K,T) \quad \text{and} \quad P_{Amer}(S(0),K,t)\leq P_{Amer}(S(0),K,T).$$

Theorem 3.38

For a (European or American) call option on any stock, the higher the strike price, the lower the premium. For a (European or American) put option on any stock, the higher the strike price, the higher the premium. Algebraically, for $K_2 > K_1$, we have

$$C(S,K_2,T)\leq C(S,K_1,T) \quad \text{and} \quad P(S,K_2,T)\geq P(S,K_1,T).$$

With derivatives

$$\frac{\partial C(S,K,T)}{\partial K} \leq 0 \quad \text{and} \quad \frac{\partial P(S,K,T)}{\partial K} \geq 0.$$

Intuitively, this is obvious. For a call option, you must pay the strike price to get the stock. Increasing the strike price means you must pay more. It will decrease the value of option. For a put option, you receive the strike price upon selling the stock. The more you receive, the higher the value of the option.

Suppose $t < T$. Now we know inequalities of European and American calls

$$C_{Eur}(K,t) = C_{Amer}(K,t) \leq C_{Amer}(K,T) = C_{Eur}(K,T) \quad \text{and} \quad C(Ke^{r(T-t)},T) \leq C(K,T)$$

for a non-dividend-paying stock. What if

$$C(S(0),K,t) \quad \text{VS} \quad C(S(0),Ke^{r(T-t)},T)?$$

Theorem 3.39

Suppose $t < T$. We have inequality of (European or American) calls

$$C(S(0),K,t) \leq C(S(0),Ke^{r(T-t)},T)$$

for a non-dividend-paying stock.

Proof.

It is irrational to exercise both call options early. We only consider both options to be European.

Suppose $C_{Eur}(S(0),K,t) > C_{Eur}(S(0),Ke^{r(T-t)},T)$. Buy $C_{Eur}(S(0),Ke^{r(T-t)},T)$ and sell $C_{Eur}(S(0),K,t)$. You will have profit $C_{Eur}(S(0),K,t) - C_{Eur}(S(0),Ke^{r(T-t)},T)$ at time 0.

Case (i): $S(t) \leq K$. You lose nothing if $C_{Eur}(S(0),K,t)$ is not exercised at time *t*.

Case (ii): $S(t) > K$. Short a stock and exchange it for *K* at time *t*. Invest *K* into a bond, you will get $Ke^{r(T-t)}$ at time *T*. Get a stock by at most $Ke^{r(T-t)}$ and close the short position.

Theorem 3.40

Suppose $t < T$. We have inequality of same style (European/American) puts

$$P(S(0), K, t) \le P(S(0), Ke^{r(T-t)}, T)$$

for a non-dividend paying stock.

Proof

For European puts, by put-call parity,

$$C(S(0), K, t) - P(S(0), K, t) = S(0) - Ke^{-rt}$$
$$C(S(0), Ke^{r(T-t)}, T) - P(S(0), Ke^{r(T-t)}, T) = S(0) - Ke^{r(T-t)}e^{-rT} = S(0) - Ke^{-rt}$$
$$0 \le C(S(0), Ke^{r(T-t)}, T) - C(S(0), K, t) = P(S(0), Ke^{r(T-t)}, T) - P(S(0), K, t)$$

Hence $P(S(0), K, t) \le P(S(0), Ke^{r(T-t)}, T)$.

$P_{Amer}(S(0), Ke^{r(T-t)}, T)$ has longer expiry and higher strike price than $P(S(0), K, t)$. So as result.

Example 3.41

For a non-dividend paying stock with price 21:

(i) The continuously compounded risk-free rate is 4%.

(ii) A 3-month European put option on the stock with strike price 20 costs 1.00.

(iii) A 6-month European put option on the stock with strike price 20.30 costs 0.90.

You take advantage of arbitrage. Assume that you sell one 3-month European put option and follow the optimal strategy, and that the stock price is 19 after 3 months and 21 after 6 months.

Determine your net profit after 6 months.

Solution

A 6-month European option with strike price $20e^{0.04(0.25)} = 20.2010$ is worth more than a 3-month European option with strike price 20. A higher price makes a put option worth even more, so the 6-month option is certainly more than 1.00.

You sell one 3-month put option and buy one 6-month put option, for an immediate gain on 0.10. After 3 months. You pay 20, which you borrow, and get a stock. After 6 months you sell the stock for 21 and your option is worthless.

Adding up the 3 cash flows accumulated at 4% interest, the net profit is

$$0.1e^{0.04(0.5)} - 20e^{0.04(0.25)} + 21 = 0.9010.$$

Theorem 3.42

Consider European call option and put option on any stock. For $K_2 > K_1$,

$$C(S,K_1,T) - C(S,K_2,T) \le (K_2 - K_1)e^{-rT} \quad \text{and} \quad P(S,K_2,T) - P(S,K_1,T) \le (K_2 - K_1)e^{-rT}$$

With derivatives

$$\frac{\partial C(S,K,T)}{\partial K} \ge -e^{-rT} \quad \text{and} \quad \frac{\partial P(S,K,T)}{\partial K} \le e^{-rT}.$$

For American call and put and $K_2 > K_1$,

$$C(S,K_1,T) - C(S,K_2,T) \le K_2 - K_1 \quad \text{and} \quad P(S,K_2,T) - P(S,K_1,T) \le K_2 - K_1.$$

With derivatives

$$\frac{\partial C(S,K,T)}{\partial K} \ge -1 \quad \text{and} \quad \frac{\partial P(S,K,T)}{\partial K} \le 1$$

Proof

Intuitively, this is clear. If the strike price of a European call option is decreased by $K_2 - K_1$, the best way is that $C(S,K_1,T)$ pays $K_2 - K_1$ more than $C(S,K_2,T)$ at expiration date *T*. It can be paid $(K_2 - K_1)e^{-rT}$ at time 0. That is the most additional amount it can pay, and it may not even pay that much additional if, for example, it is worthless at expiry or worth less than $K_2 - K_1$ at expiry. The same logic works for put options.

For American option, it can be exercise any time before *T*. It is better to pay $K_2 - K_1$ at 0.

Example 3.43

Two 1-year European call options on the same stock are priced as follows:

Strike price	Premium
40	10
45	4

The continuously compounded risk-free rate is 0.08. You take advantage of arbitrage by buying one 45-strike call and selling c 40-strike calls, where c is possible value that results in no possibility of loss. Suppose the minimum profit of portfolio is 0 and the stock price is 46 after one year. Determine your profit including interest.

Solution

Your initial gain is $10c-4$. The payoff function is $\max\{S(1)-45,0\}-c\max\{S(1)-40,0\}$.

The profit function is $\max\{S(1)-45,0\}-c\max\{S(1)-40,0\}+(10c-4)e^{0.08}$.

Clearly, $c \le 1$. Otherwise, the profit is negative if $S(1)$ is large.

$$\begin{aligned}&\max\{S(1)-45,0\}-c\max\{S(1)-40,0\}+(10c-4)e^{0.08}\\&=\begin{cases}S(1)-45-c(S(1)-40)+(10c-4)e^{0.08} & \text{if } S(1)>45\\ -c(S(1)-40)+(10c-4)e^{0.08} & \text{if } 40<S(1)\le 45\\ (10c-4)e^{0.08} & \text{if } S(1)\le 40\end{cases}\end{aligned}$$

The profit is minimum when $S(1) = 45$. Suppose the profit is 0 in this case.

$$-5c+(10c-4)e^{0.08}=0. \quad c=\frac{4e^{0.08}}{10e^{0.08}-5}=0.743.$$

1 year later, the profit is $46-45-0.743(46-40)+(7.43-4)e^{0.08}=0.257$.

Theorem 3.44

Consider (European or American) call options and put options on any stock. For $K_3 > K_2 > K_1$,

$$\frac{C(S,K_3,T)-C(S,K_2,T)}{K_3-K_2} \geq \frac{C(S,K_2,T)-C(S,K_1,T)}{K_2-K_1}$$
$$\frac{P(S,K_3,T)-P(S,K_2,T)}{K_3-K_2} \geq \frac{P(S,K_2,T)-P(S,K_1,T)}{K_2-K_1}$$

Equivalently,

$$C(S,K_2,T) \leq \frac{(K_2-K_1)C(S,K_3,T)+(K_3-K_2)C(S,K_1,T)}{K_3-K_1}$$
$$P(S,K_2,T) \leq \frac{(K_2-K_1)P(S,K_3,T)+(K_3-K_2)P(S,K_1,T)}{K_3-K_1}$$

In terms of derivatives,

$$\frac{\partial^2 C(S,K,T)}{\partial K^2} \geq 0 \quad \text{and} \quad \frac{\partial^2 P(S,K,T)}{\partial K^2} \geq 0.$$

Proof

I only show you the proof for call option. The proof for put is similar. I only consider European option. Everything applies equally well to American option, since if one option is exercise, the other 2 options could be exercise at the same time.

Suppose there is mispricing,

$$C(S,K_2,T) > \frac{(K_2-K_1)C(S,K_3,T)+(K_3-K_2)C(S,K_1,T)}{K_3-K_1}.$$

Sell the K_2-strike call, buy $\frac{K_3-K_2}{K_3-K_1}$ numbers of K_1-strike calls and buy $\frac{K_2-K_1}{K_3-K_1}$ numbers of K_3-strike calls. So you have an initial gain.

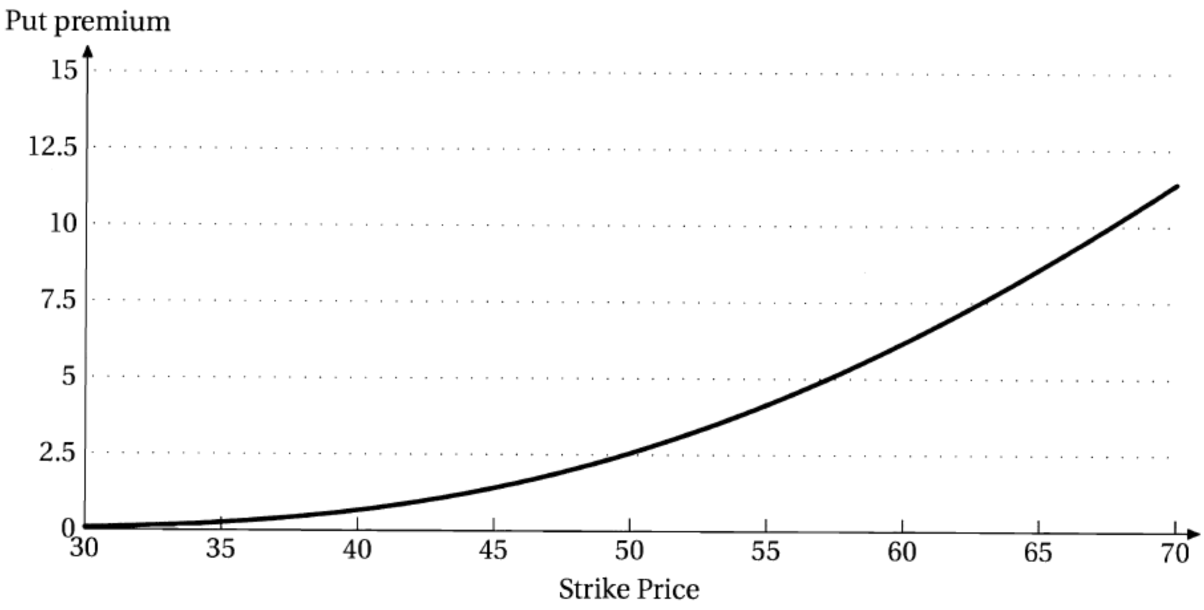
If $S(T) \leq K_2$, so you don't pay anything and have at least the initial gain.

If $K_2 < S(T) \leq K_3$, the K_2-strike call holder pays you K_2. Exercising K_1-strike calls, you pay $\frac{K_3-K_2}{K_3-K_1}K_1$ to receive $\frac{K_3-K_2}{K_3-K_1}$ shares of stock. Since $K_2 - \frac{K_3-K_2}{K_3-K_1}K_1 = \frac{K_2-K_1}{K_3-K_1}K_3 > \frac{K_2-K_1}{K_3-K_1}S(T)$, spend $\frac{K_2-K_1}{K_3-K_1}S(T)$ to buy $\frac{K_2-K_1}{K_3-K_1}$ shares of stock. Give $\frac{K_3-K_2}{K_3-K_1}+\frac{K_2-K_1}{K_3-K_1}=1$ share of stock to K_2 -strike call holder.

If $K_3 < S(T)$, the K_2-strike call holder pays you K_2. Exercising K_1-strike calls and K_3-

strike calls, you pay $\frac{K_3-K_2}{K_3-K_1}K_1$ to get $\frac{K_3-K_2}{K_3-K_1}$ shares of stock and $\frac{K_2-K_1}{K_3-K_1}K_3$ to get $\frac{K_2-K_1}{K_3-K_1}$ shares of stock. That means, you pay $\frac{K_3-K_2}{K_3-K_1}K_1+\frac{K_2-K_1}{K_3-K_1}K_3=K_2$ to get $\frac{K_3-K_2}{K_3-K_1}+\frac{K_2-K_1}{K_3-K_1}=1$ share of stock. Give 1 share of stock to K_2-strike call holder.

Call premium as a function of strike price. Assumes $S=50$.



Put premium as a function of strike price. Assumes $S=50$.

Example 3.45

For three 6-month American call options on a stock:

(i) One with strike price 45 costs for 6.30

(ii) One with strike price 44 costs for 7.00

(iii) One with strike price 40 costs for 9.50

Verify the option with strike price 44 is overpriced based on the convexity property of option premium. You therefore sell it. Determine the maximal and the minimal amount of the other 2 options you should buy to guarantee a profit anytime in 6 months.

Solution

$$\frac{(K_2-K_1)C(S,K_3,T)+(K_3-K_2)C(S,K_1,T)}{K_3-K_1}=\frac{(44-40)6.3+(45-44)9.5}{45-40}=6.94<C(S,K_2,T)$$

$C(S,K_2,T)$ is overpriced. Let x be the number of 40-strike calls you buy and y the number of 45-strike calls you buy. There is no guarantee of future gain since it is possible that none of the option will be exercised. To assure initial net gain, you need

$$9.5x+6.3y\le 7. \tag{1}$$

If $S(t)\le 44$ anytime in 6 months, the 44-strike option will not be exercised. You will at least gain $(7-9.5x-6.3y)e^{0.5r}$ after 6 months.

If the 44-strike option is exercised, then you need to exercise 40-strike or possibly 45-strike option. If the 44-strike option is exercised at t when $44<S(t)\le 45$, you need

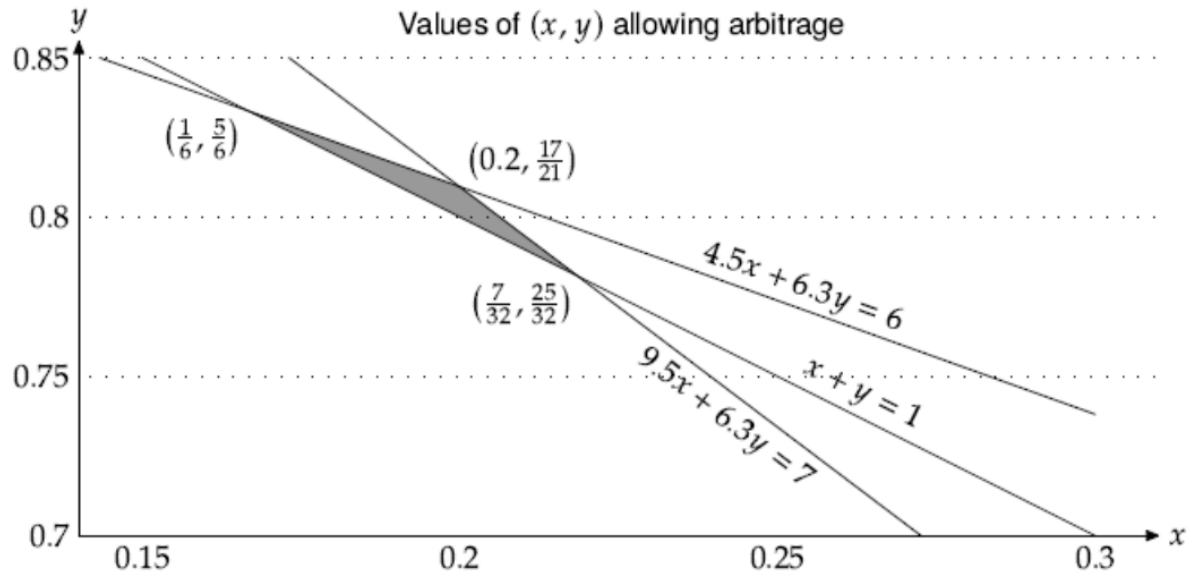
$$x(S(t)-40)+(7-9.5x-6.3y)e^{rt}\ge S(t)-44.$$

The worst possible case is $S(0) = 45$. You have $4.5x+6.3y\le 6.$ (2)

If the 44-strike option is exercised at t when $45 < S(t)$, then you have $(x+y-1)$ share of stocks. Since the stock price can be arbitrarily high, you need

$$x+y\ge 1. \tag{3}$$

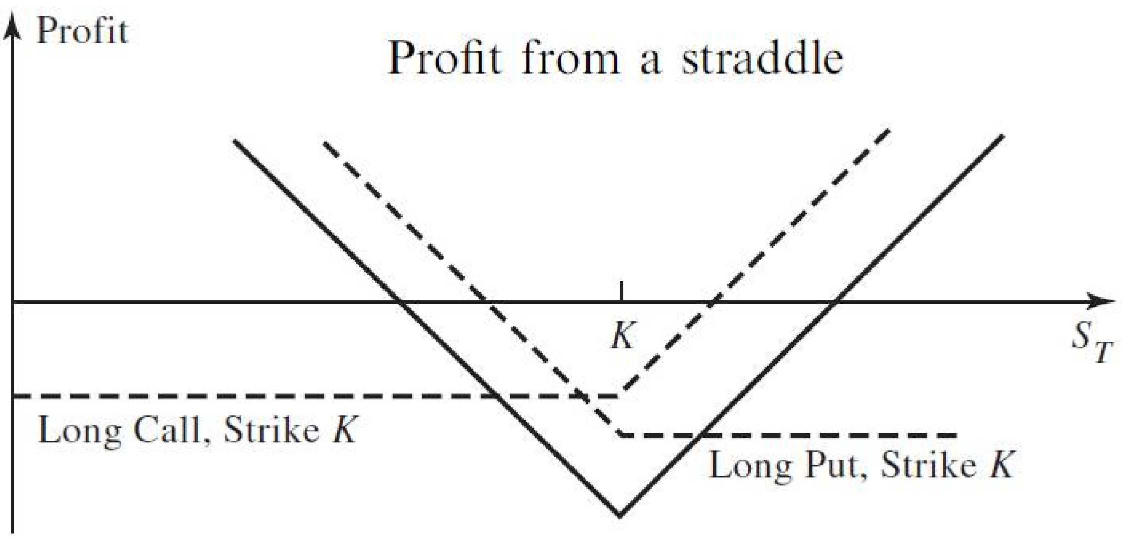
Calculate the intersections of the 3 lines induced by inequalities (1), (2) and (3), x may be between $1/6$ and $7/32$, while y may be between $25/32$ and $5/6$.



The examples in the rest of this chapter are based on Example 3.12.

Suppose you believe that the price of a stock 6 months from now will be very different from its current price. You want to make a profit if your belief turns out to be true. You are willing to make less of a profit, or even suffer a loss, if the stock price in 6 months is very close to its current price. What should the general shape of profit graph be? It can be a V-shaped.

A straddle involves buying a European call and put with the same strike price K and expiration date T.



The payoff of a straddle is $\max\{S(T)-K,0\}+\max\{K-S(T),0\}=|S(T)-K|$ and the profit of a straddle is

$$|S(T)-K|-FV(C)-FV(P).$$

Example 3.46

Tom invests in a 6-month straddle with strike price K = \$100. For what range of spot prices at expiration will Tom have a loss?

Solution

$$\text{Profit} = |S(T)-K|-FV(C)-FV(P) = |S(T)-100|-e^{0.02}(9.3904+7.4103)<0$$
$$|S(T)-100|<e^{0.02}(9.3904+7.4103)=17.4863495739740569$$
$$82.5136504260259431<S(T)<117.4863495739740569$$

Suppose your opinion about the future price of an asset is the complete opposite of an investor who buys a straddle: You believe that the future price will be very close to the current. It is obvious that you would want a payoff or profit graph that is the mirror image of the graph for a straddle, i.e, you would want a graph that has an inverted V shape. This is called a written or short straddle, which consists of a short call and a short put with the same strike price and expiration date.

Example 3.47

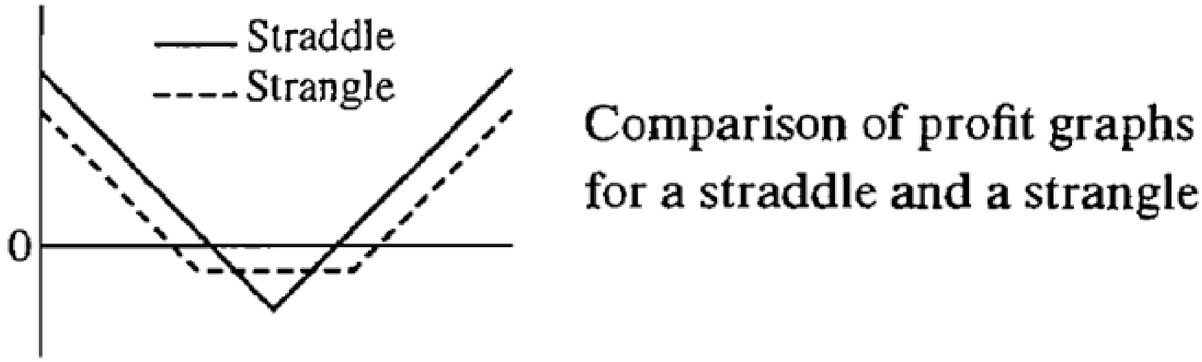
Suppose Jim is the writer of the straddle purchased by Tom in Example 3.46. Jim's profit is \$8 at expiration of the straddle in 6 months. What was the spot price at expiration?

Solution

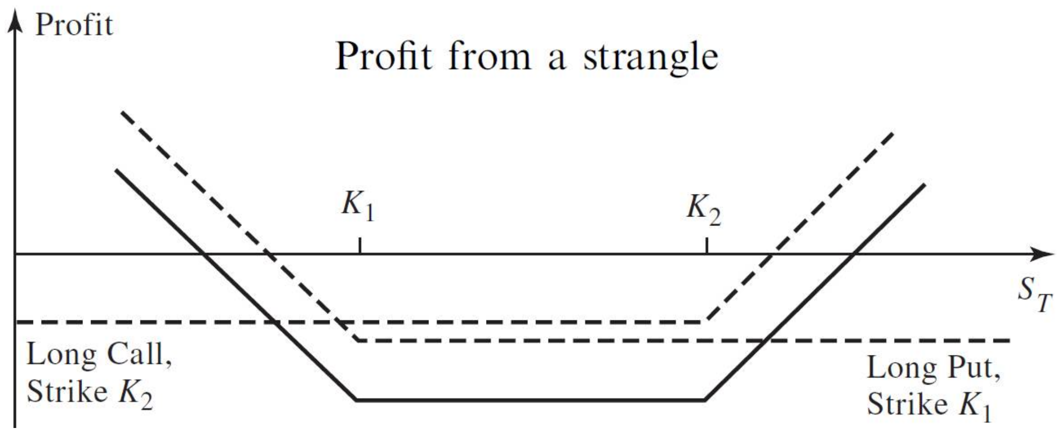
$$\text{Profit} = -|S(T)-K|+FV(C)+FV(P)=8$$
$$|S(T)-100|=17.4863495739740569-8=9.4863495739740569$$
$$S(T)=90.5136504260259431 \quad \text{or} \quad 109.4863495739740569$$

The cost of straddle in Example 3.46 is \$17.4863495739740569. That means the option holder has a potential loss of \$17.4863495739740569. He would prefer to reduce his losses somewhat for a range of spot prices at expiration that are close to the strike price, even giving up some of the potential profits for spot prices outside of this range.

A strangle is a similar strategy to a straddle, but we want to keep our cost lower. Flattening the base of V will solve this problem.



A strangle involves buying a European call with strike price K_2 and put with strike price K_1 and same expiration date T where $K_1 \le K_2$. It will produce a flat-bottomed V with a lower premium than a straddle.



The profit pattern obtained with a strangle depends on how close together the strike prices K_1 and K_2 are. The farther they are apart, the less the downside risk.

The payoff of a strangle is $\max\{S(T)-K_2,0\}+\max\{K_1-S(T),0\}$ and its profit is

$$\max\{S(T)-K_2,0\}+\max\{K_1-S(T),0\}-FV(C)-FV(P).$$

Example 3.48

Consider a 90-strike long put and a 110-strike long call strangle. What is the profit at a spot price at expiration of (a) \$80; (b) \$90; (c) \$100; (d) \$110; (e) \$120? For what spot price at expiration would the profit be 0?

Solution

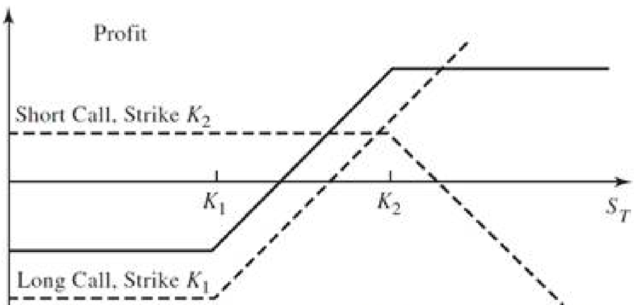
The total premium paid for strangle is \$5.4115 + \$3.4001 = \$ 8.8116. The cost at the end of 6 months is $e^{0.02} \times \$8.8116 = \$9.1712$. This is only about 52% of \$17.486349574 for a straddle.

The profit for this strangle is

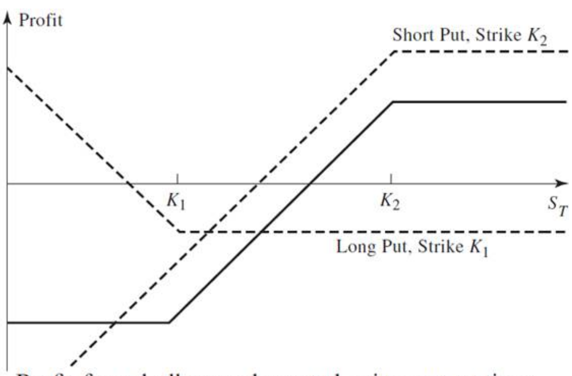
$$\max\{S(T)-110,0\}+\max\{90-S(T),0\}-9.1712.$$

Using this formula, we find the respective profits to be (a) 0.8288; (b), (c) and (d) -9.1712; (e) \$0.8288. The profit would be 0 for spot prices at expiration of \$80.8288 and \$119.1712. These answers reflect the fact that the profit graph is flat-bottomed between \$90 and \$110, and it is symmetrical about \$100.

An investor who enters into a bull spread is hoping that the stock price will increase. This can be created by buying a European call option on a stock with a certain strike price K_1 and selling a European call option on the same stock with a higher strike price K_2. Both options have the same expiration date. A bull spread strategy limits the investor's upside as well as downside risk. The strategy can be described by saying that the investor has a call option with a strike price equal to K_1 and has chosen to give up some upside potential by selling a call option with strike price K_2 $(K_2 > K_1)$. In return for giving up the upside potential, the investor gets the price of the option with strike price K_2.



Profit from bull spread created using call options.



Profit from bull spread created using put options.

The profits from the two option positions taken separately are shown by the dashed lines. The profit from the whole strategy is the sum of the profits given by the dashed lines and is indicated by the solid line. Because a call price always decreases as the strike price increases, the value of the option sold is always less than the value of the option bought. A bull spread, when created from calls, therefore requires an initial investment.

Stock price range	Payoff from long call option	Payoff from short call option	Total payoff
$S(T) \le K_1$	0	0	0
$K_1 < S(T) < K_2$	$S(T) - K_1$	0	$S(T) - K_1$
$K_2 \le S(T)$	$S(T) - K_1$	$-(S(T) - K_2)$	$K_2 - K_1$

Example 3.49

What is the financing cost of 90-110 bull spread created by call options at the end of 6 months? For what spot price at expiration would the profit be 0?

Solution

Buy a 90-strike call and write a 110-strike call, the financing cost would be $e^{0.02}(\$15.1823 - \$5.4115) = \$10.17$. From the shape of the profit graph, the profit would be 0 for some spot prices between \$90 and \$110. For this spread of spot prices, we have:

$$\text{Profit} = \max\{0, S(T) - 90\} - \max\{0, S(T) - 110\} - FV(\text{Premiums}) = S(T) - 90 - 10.17$$

Setting this equal to 0, we find that $S(T)$ = \$100.17.

Bull spreads can also be created by buying a European put with a low strike price K_1 and selling a European put with a high strike price K_2 $(K_2 > K_1)$. Unlike bull spreads created from calls, those created from puts involve a positive cash flow to the investor (ignoring margin requirements) and a payoff that is either negative or zero.

Stock price range	Payoff from long put option	Payoff from short put option	Total payoff
$S(T) \le K_1$	$K_1 - S(T)$	$-(K_2 - S(T))$	$K_1 - K_2$
$K_1 < S(T) < K_2$	0	$S(T) - K_2$	$S(T) - K_2$
$K_2 \le S(T)$	0	0	0

Example 3.50

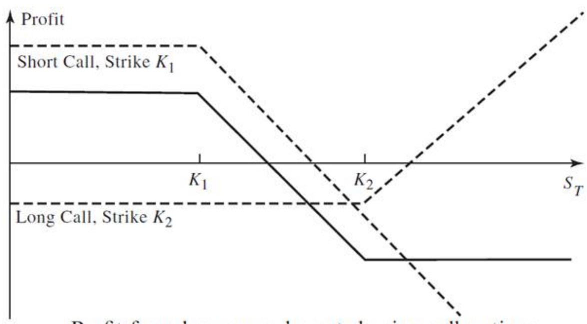
Compare the payoffs and profits with those of a bull spread using all calls and all puts with $K_1 = 90, K_2 = 100.$

Solution

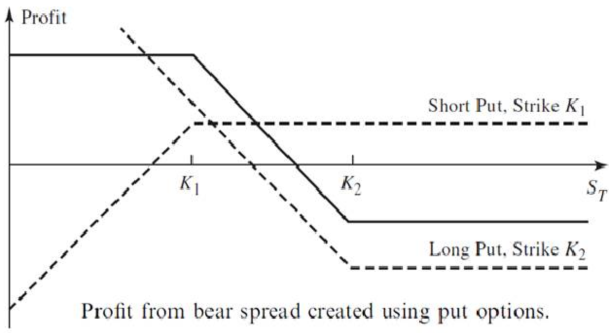
The payoff of bull spread using all puts is $K_2 - K_1 = 100 - 90 = \$10$ less than that of bull spread using all calls. The financing cost using calls is $e^{0.02}(\$15.1823 - \$9.3904) \approx \$6.$ The financing cost using puts is $e^{0.02}(\$3.4001 - \$7.4103) \approx -\$4.$

$$\begin{aligned}\text{Profit of bull spread using calls} &= \text{payoff bull spread using calls} - \text{financing cost using calls}\\ &= \text{payoff bull spread using puts} + 10 - 6\\ &= \text{payoff bull spread using puts} + 4\\ &= \text{payoff bull spread using puts} - \text{financing cost using puts}\\ &= \text{profit of bull spread using puts}\end{aligned}$$

An investor who enters into a bear spread is hoping that the stock price will decline. It is the opposite of a bull spread and the graph is a mirror image. The investor buys a call with a high strike price and sells a call with a low strike price. Bear spreads created with calls involve an initial cash inflow. A bear spread created from puts involves an initial cash outflow because the price of the put sold is less than the price of the put purchased.



Profit from bear spread created using call options.



Profit from bear spread created using put options.

For example, to create a 100-110 bear spread, you would write a 100-strike call and buy a 110-strike call. (Once again, this could be done by using puts instead of calls).

Stock price range	Payoff from long call option	Payoff from short call option	Total payoff
$S(T) \le K_1$	0	0	0
$K_1 < S(T) < K_2$	0	$-(S(T) - K_1)$	$-(S(T) - K_1)$
$K_2 \le S(T)$	$S(T) - K_2$	$-(S(T) - K_1)$	$-(K_2 - K_1)$

Stock price range	Payoff from long put option	Payoff from short put option	Total payoff
$S(T) \le K_1$	$K_2 - S(T)$	$-(K_1 - S(T))$	$K_2 - K_1$
$K_1 < S(T) < K_2$	$K_2 - S(T)$	0	$K_2 - S(T)$
$K_2 \le S(T)$	0	0	0

The butterfly spread and the written straddle have the same general strategy, but the investor in the butterfly spread thinks:

"If I write a straddle, I will make a profit if the price of the underlying asset stays close to its current price. But if the price changes a lot, I will be hit with very large losses. (The loss is theoretically unlimited as the spot price at expiration increases.) While I believe the price won't change too much, I want to protect myself in case I am wrong: I want to flatten out my losses for big derivations in the price." (That is where the "wings" of the butterfly come in).

Suppose $K_1 < K_2 < K_3$ and $K_1 + K_3 = 2K_2$. A butterfly spread can be created by

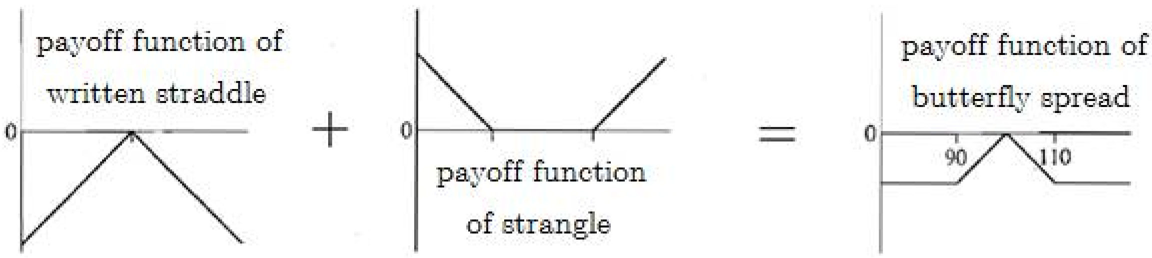
(i) buying a K_3 call, buying a K_1 put and selling a K_2 straddle; or

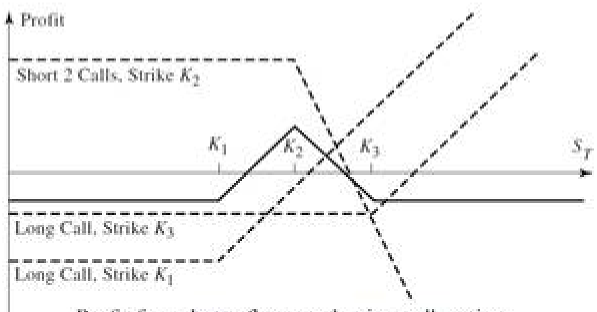
(ii) buying a K_1 call, buying a K_3 call and selling two K_2 call options; or

(iii) buying a K_1 put, buying a K_3 put and selling two K_2 put options; or

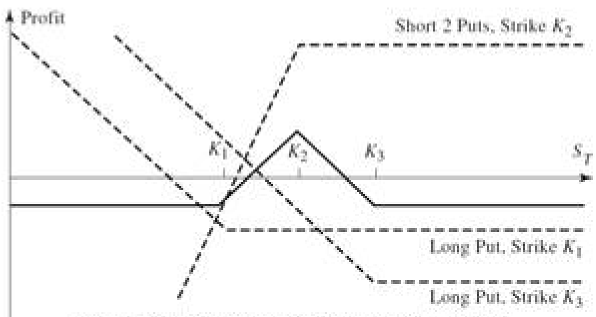
(iv) buying a bull spread with strike K_1 and K_2, and buying a bear spread with strike K_2 and K_3.

Suppose K_2 is close to the current stock price. A butterfly spread leads to a profit if the stock price stays close to K_2, but gives rise to a small loss if there is a significant stock price move in either direction. It is therefore an appropriate strategy for an investor who feels that large stock price moves are unlikely. The strategy requires a small investment initially.





Profit from butterfly spread using call options

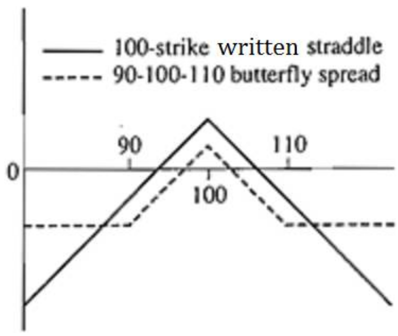


Profit from butterfly spread using put options

Example 3.51

For what spot price at expiration would the profit be the same under a 90-100-110 butterfly spread and a 100-strike written straddle?

Solution



We know from the graph that there are actually two spot prices at expiration for which the profit are the same, and that one is less than \$90 and the other one is greater than \$110. These points are equally distant from \$100.

The financing cost for 90-100-110 butterfly spread by (i) at expiration is $e^{0.02}(5.41+3.4-9.39-7.41)=-8.113$. The financing cost for 100-strike written straddle at expiration is $-e^{0.02}(9.39+7.41)=-17.49$.

The profit under the 90-100-110 butterfly spread for $S(T)$ < \$90 is $-10+8.113=-1.887$. The profit under the written 100-straddle for $S(T)$ < \$100 is $-(100-S(T))+17.49$. We have

$$-1.887=-82.51+S(T), \quad S(T)=\$80.623.$$

This is \$19.377 below \$100, so the other spot price we are looking for is \$119.377.

The maximum profit in Example 3.51 is earned when the spot price at expiration is $K_2=100$. The profit is symmetric on either side of K_2.

Suppose $K_1<K_2<K_3$ and K_2 is not necessary the midpoint of K_1 and K_3. Suppose you want your maximum profit to occur at a spot price at expiration that is to the left or right of

midpoint of K_1 and K_3. The following combination of options is called an asymmetric butterfly spread:

(i) buying *n* bull spreads with strike K_1 and K_2, and buying *m* bear spreads with strike K_2 and K_3.

(ii) *n* and *m* are selected so that the payoff for $S(T) \leq K_1$ and that for $K_3 \leq S(T)$ are equal.

Let us work out the relation between *m* and *n* for an asymmetric butterfly spread constructed by call bull and bear spreads.

When $K_3 \leq S(T)$, the payoff of a call bull spreads with strike K_1 and K_2 is $K_2 - K_1$. The payoff of a call bear spreads with strike K_2 and K_3 is $-(K_3 - K_2)$.

The payoff of the asymmetric butterfly spread at 2 extreme are $n(K_2 - K_1) - m(K_3 - K_2) = 0$

The weighting of K_1-strike long calls, K_2-strike short calls, K_3-strike long calls is $n : n + m : m = K_3 - K_2 : K_3 - K_1 : K_2 - K_1$.

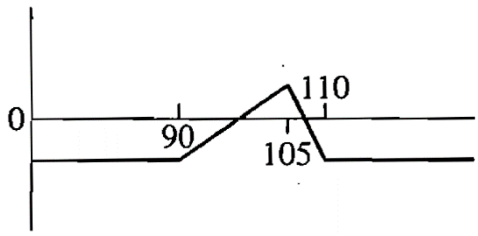
If $m = n = 1$, then $K_1 + K_3 = 2K_2$. It is a symmetric butterfly spread.

Example 3.52

Create a 90-105-110 asymmetric butterfly spread using a 105-strike short call. Find its net premium if the premium for a 105-strike call is \$7.19.

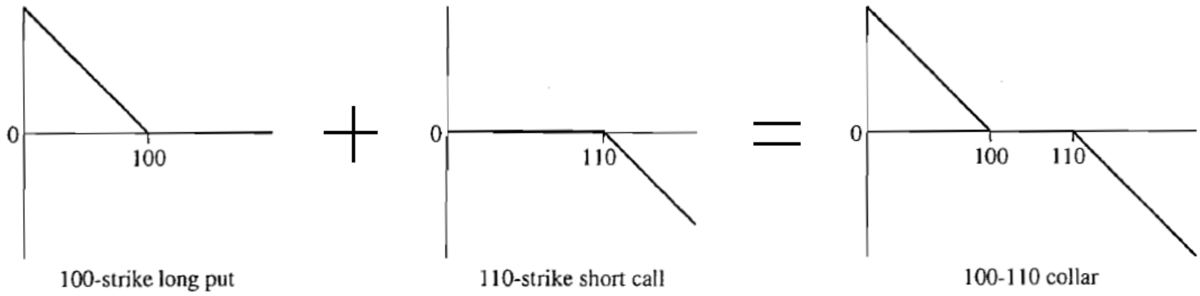
Solution

The weighting of 90-strike long calls, 105-strike short calls, 110-strike long calls is 0.25 : 1 : 0.75. The net premium is $(0.25)(\$15.18) + (0.75)(\$5.41) - \$7.19 = \$0.6625$.



In Example 3.52, you would write 1 unit of a 105-strike put, buy 0.25 units of a 90-strike put and buy 0.75 units of 110-strike put. The total payoffs would be the same as the total payoffs using 3 calls.

Suppose you want your payoff to be constant over a range of spot prices at expiration. For spot prices outside this range, you want your payoff to increase as the spot price at expiration decreases and decrease as the spot price at expiration increases. The below figure shows the payoff graph looks like. This position is called a **collar**. The range of constant payoff is called the **collar width**. A collar can be created by using a combination of a long put and a short call at a higher strike price.



Example 3.53

Consider the collar created by a combination of a long 100-strike put and a short 110 strike call. For what spot price at expiration is the profit equal to the negative of the profit for a spot price at expiration of \$95?

Solution

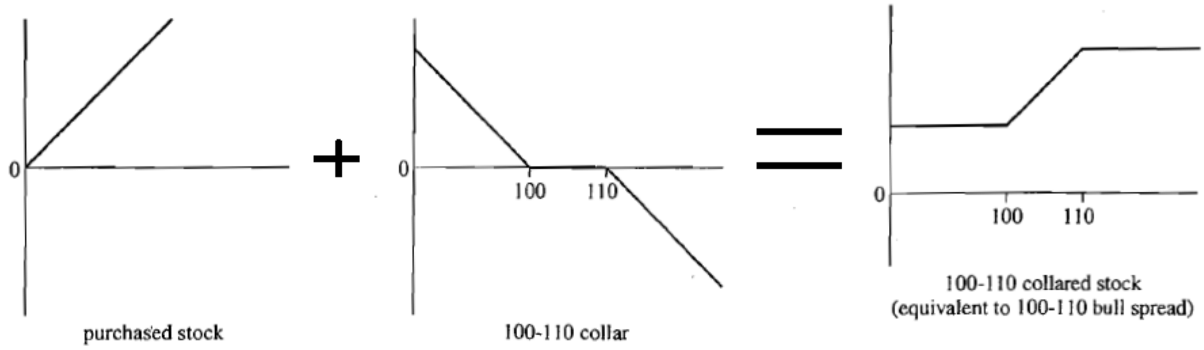
Firstly, the profit for a spot price at expiration of \$95. The payoff is $100-95=\$5$. The finance cost is $e^{0.02}(7.41-5.41)=\$2.0816$. The profit is $5-2.0816=\$2.9184$.

Let $S(T)$ be the spot price for which the profit is $-\$2.9184$. Clearly, $S(T)$ is greater than \$110. The profit is:

$$-(S(T)-110)-2.0816=-2.9184, \quad S(T)=\$110.8368.$$

Suppose you own a share of stock and believe that its price will go up to 100 at least. However you want to have insurance in case it goes down. You may do this by buying a put with a strike price of 100. But you are concerned about the financing cost. You can reduce this cost if we write a call at a higher strike price, say 110. (By writing the call, you receive the premium to offset the premium paid for the put. But you need to give up potential profits at spot price greater than 110). The combination of the 100-strike long put and 110-strike short call is the collar that we just considered above. (The strike of put is less than the strike of call).

The combination of owning a stock and buying a collar with the stock as the underlying asset is called a **collar stock**.



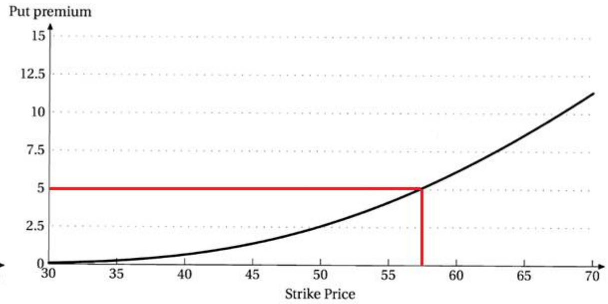
What does the shape of the graph of the above collar stock look like?

A 100-110 bull spread. But the financing cost of the collar stock is more than that of 100-110 bull spread.

Since you buy a put and sell a call at different strike prices to create a collar, the question arises as to whether we can determine strike prices for which the premiums for the put and the call are equal. If we do it in this way, your financing cost for the collar would be 0. (We pay the put premium and receive the call premium). This is called a **zero-cost collar**.

Example 3.54

Call premium as a function of strike price. Assumes $S = 50$.



Put premium as a function of strike price. Assumes $S = 50$.

The above 2 pictures are call and put premiums against strike price on the same underlying asset and expiration date.

The call premium is 5 when its strike price is 66 and the put premium is also 5 when its strike price is 57.5.

We can construct a 57.5-66 zero-cost collar with collar width $66-57.5=8.5$.

Ratio spread is different number of options at different strike prices are bought and sold.

Example 3.55

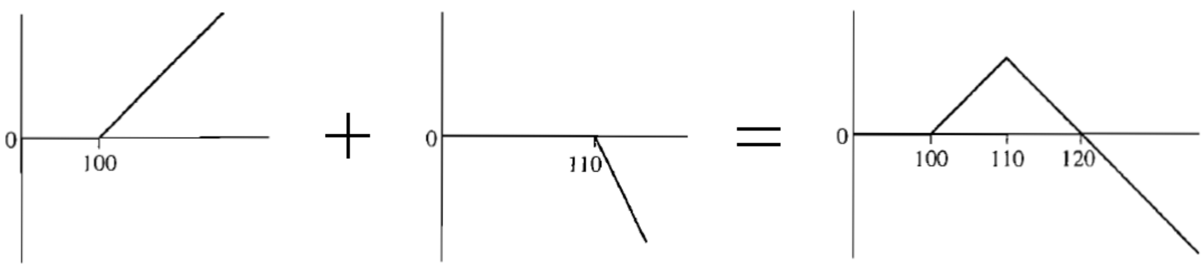
Consider a ratio spread which consists of long one 100-strike call and short two 110-strike calls. Find the range of stock price at expiration so that the ratio spread has a profit.

Solution

The financing cost of the ratio spread at expiration is $e^{0.02}(9.39-2(5.41))=-1.458888$. That means you receive 1.458888 at $t=0$ if you have such ratio spread. Its profit function is

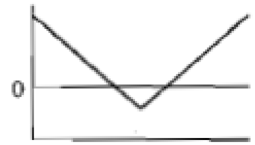
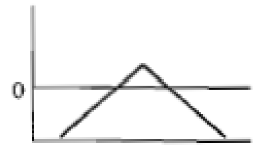
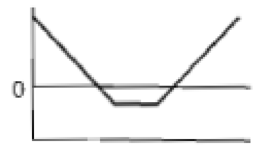
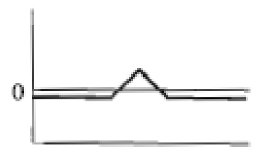
$$\begin{aligned}
&\max\{S(T)-100,0\}-2\max\{S(T)-110\}+1.458888 \\
&=\begin{cases} 1.458888 & \text{if } S(T)\le 100 \\ S(T)-100+1.458888>0 & \text{if } 100<S(T)\le 110 \\ S(T)-100-2(S(T)-110)+1.458888=-S(T)+121.458888 & \text{if } 110\le S(T) \end{cases}
\end{aligned}$$

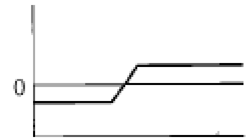
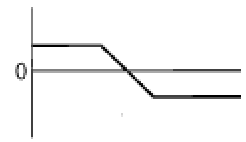
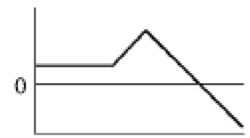
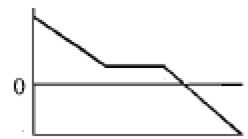
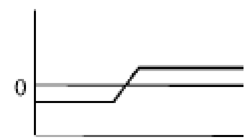
The required range of stock price at expiration is from 0 to 121.458888.



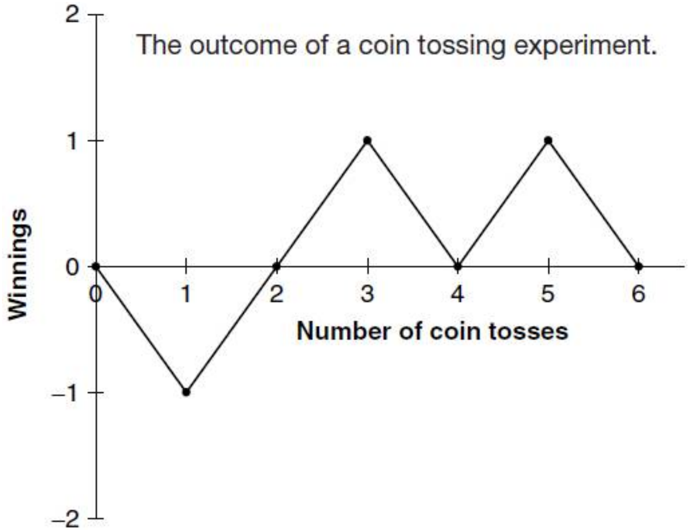
One 100-strike long call   Two 110-strike short calls   Ratio spread

This strategy is very similar to strategy of butterfly spread, but the investor thinks he needs insurance only for the downside of spot prices at expiration.

Option Position	ⓘ Memory Alert! (see text for more details)	Combination of Options	Comments	Profit Graph
Straddle	"To straddle" is to take both sides of an issue at once. In finance, this means that you believe the stock price will go either up or down. Thus, you want a V-shaped profit graph.	Long put plus long call	A bet that the stock price will be volatile. High financing cost.	
Written Straddle	Opposite position to a straddle: You believe the stock price will not change. You want an inverted V-shaped profit graph.	Short put plus short call	A bet that the stock price will be stable.	
Strangle	"Strangle" suggests the shape of a cord around the throat, i.e., a flat-bottomed V. Your strategy is similar to a straddle (you think that the stock price will change) but you want to lower your financing cost.	Out-of-the-money options: long put plus long call	Lower losses than the straddle for spot prices at expiration close to current price, but lower profits at other spot prices.	
Butterfly Spread	The name of this position suggests an inverted V with "wings." Your strategy is similar to a written straddle, but you want insurance for large changes in price.	Start with a written straddle and add a lower-strike long put and a higher-strike long call (i.e., buy a strangle).	Can also be created by using all calls or all puts.	
Asymmetric Butterfly Spread	The name of this position suggests a butterfly spread but with the peak profit to the left or right of the current price. Otherwise, your strategy is similar to the butterfly spread.	Example: For a 90–105–110 asymmetric butterfly, one unit of 105-strike short call plus 0.25 units of 90-strike long call and 0.75 units of 110-strike long call.	Can also be created using all puts. Weights are determined by location of the peak between the end points of the spread.	

Bull Spread	You are a "bull" (you think the price will go up) but you are willing to have an increase in profits only for a "spread" of spot prices at expiration. (See profit graph at the right.)	Long call at one strike price and short call at a higher strike price.	Could also be created by using puts.	
Bear Spread	The opposite of a bull spread; the graph is the mirror image.	Short call at one strike price and long call at a higher strike price.	Could also be created by using puts.	
Box Spread	So-called because the premiums for the calls and puts come from a "box." Your strategy is to receive a guaranteed payoff, regardless of changes in the market price.	Synthetic long forward and synthetic short forward at a higher strike price.	Could also be described as a combination of a bull spread and a bear spread, or a combination of a purchased and written strangle.	
Ratio Spread	The name of this position tells you that an unequal number of options at different strike prices are purchased and written.	Example: One 100-strike long call and two 110-strike short calls.	For this example, the strategy is a stable price but with insurance for decreases in the price.	
Collar	The name of this position suggests that you want to put a collar around your profit (constant profit for a range of spot prices at expiration). Outside of this range, you are a bear. (Remember the collar is around the neck of a bear).	Long put and short call at a higher strike price.	There are an infinite number of zero-cost collars (i.e., where the premiums for the long put and short call are equal).	
Collared Stock	If you buy a stock and buy a collar, you have a "collard stock," which insures your position for low spot prices at expiration.	Stock plus collar	Equivalent to a bull spread. Could be used if you think the price will rise, but you can't (or don't want to) sell the stock immediately.	

Toss a fair coin. Every time you throw a head I give you \$1, every time you throw a tail you give me \$1. The below figure shows how much money you have after six tosses. In this experiment the sequence was *THHTHT*.



Let X_i, either \$1 or −\$1, be the outcome of the *i*-th toss, then

$$E[X_i] = 0 = E[X_i X_j] \quad \text{for } i \neq j \qquad \text{and} \qquad Var[X_i] = E[X_i^2] = 1.$$

Define Z_n the total amount of money you have won up to and including the *n*-th toss so that

$$Z_0 = 0 \qquad \text{and} \qquad Z_n = \sum_{i=1}^{n} X_i \quad \text{if} \quad n > 0.$$

The outcome of the first $n-1$ tosses does not affect the outcome of the *n*-th toss. The distribution of the value of the random variable Z_n conditional upon all of the past events only depends on the previous value Z_{n-1}. This is the **Markov property**.

The **quadratic variation** of the random walk is defined by

$$\sum_{i=1}^{n} (Z_i - Z_{i-1})^2.$$

Because you either win or lose \$1 after each toss, the quadratic variation is always n:

$$\sum_{i=1}^{n}(Z_i - Z_{i-1})^2 = \sum_{i=1}^{n} X_i^2 = n.$$

If we restrict the time allowed for the n tosses to a period t, so that each toss will take a time t/n. Furthermore, the size of each bet will not be \$1 but $\sqrt{t/n}$. Its quadratic variation is

$$\sum_{i=1}^{n}\left(Z_i - Z_{i-1}\right)^2 = n\left(\sqrt{\frac{t}{n}}\right)^2 = t.$$

To speed up the game, the limiting process for this random walk as the time steps go to zero is called **Standard Arithmetic Brownian motion**, and denoted by $Z(t)$. It is independent of t.

The following are the properties of Arithmetic Brownian motion:

(i) $Z(b) - Z(a) \sim N(0, b-a)$.

(ii) The conditional distribution of a future state $Z(t + s)$ given the present $Z(s)$ and the past $Z(u)$, $0 < u < s$, depends only on the present $Z(s)$. Moreover, $E[Z(t+s) \mid Z(s)] = Z(s)$.

(iii) The 2 normal random variables $Z(b) - Z(a)$ and $Z(d) - Z(c)$ are independent if $[a, b] \cap [c, d] = \phi$, no overlapping between $[a, b]$ and $[c, d]$.

(iv) The distribution of $Z(t+s) - Z(t)$ does not depend on t.

(v) Furthermore, we have $(dZ(t))^2 = dt$ and $dZ(t) \sim N(0, dt)$.

Methods from ordinary calculus may be used in stochastic calculus, but caution is needed. For example, the product rule of ordinary calculus says $d(uv) = udv + vdu$. This is derived from the fact that

$$(u + \Delta u)(v + \Delta v) - uv = u\Delta v + v\Delta u + \Delta u \Delta v$$

and $\Delta u \Delta v$ is close to 0. In stochastic calculus,

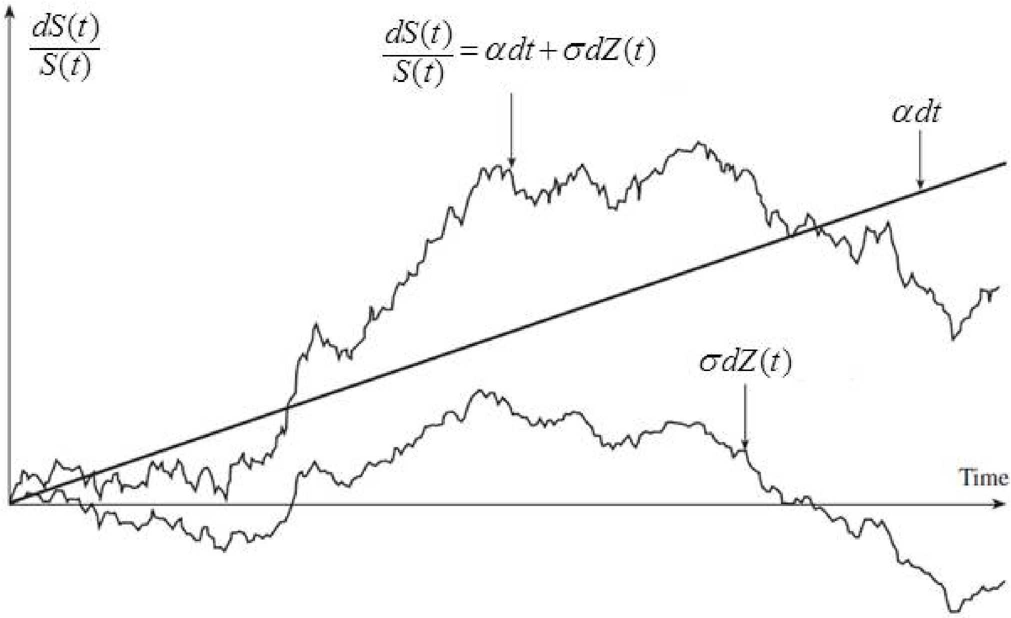
$$dZ(t) \times dZ(t) = dt, \quad dZ(t) \times dt = 0, \quad dt \times dZ(t) = 0, \quad dt \times dt = 0.$$

Thus the product rule only works if at least one of the factors is non-stochastic.

Let $S(t)$ be stock price at time t. The variable dS is the change in the stock price S in a small time interval dt. $dS(t)$ consists of 2 parts, risk free part $\alpha S(t)dt$ and risky part $\sigma S(t)dZ(t)$.

$$dS(t) = \alpha S(t)dt + \sigma S(t)dZ(t).$$

The above equation does not refer to the past history of asset price; the next asset price $S + dS$ only depends on today's price $S(t)$.



Here α is called the continuous compounded annual rate of return and σ is called volatility.

$$E[dS] = E[\alpha Sdt + \sigma SdZ(t)] = \alpha Sdt$$
$$Var[dS] = Var[\alpha Sdt + \sigma SdZ(t)] = Var[\sigma SdZ(t)] = \sigma^2 S^2 dt$$

If $\sigma = 0$, then $dS(t) = \alpha S(t)dt$. The solution for differential equation $\frac{dS(t)}{dt} = \alpha S(t)$ is

$$S(t) = S(0)e^{\alpha t}.$$

Example 4.1

Consider a stock that pays no dividends, has a volatility of 30% per annum, and provides an expected return of 15% per annum with continuous compounding. In this case, $\alpha = 0.15$ and $\sigma = 0.30$. The process for the stock price is

$$\frac{dS(t)}{S(t)} = 0.15dt + 0.30dZ(t)$$

If S is the stock price at a particular time and dS is the increase in the stock price in the next small interval of time,

$$\frac{dS(t)}{S(t)} = 0.15dt + 0.30\sqrt{dt}Z$$

where Z is a standard normal distribution. Consider a time interval of 1 week, or 0.0192 year, so that $dt = 0.0192$. Then

$$\frac{dS(t)}{S(t)} = 0.15 \times 0.0192 + 0.30\sqrt{0.0192}Z \quad \text{or} \quad dS(t) = 0.00288S(t) + 0.0416S(t)Z.$$

The initial stock price is assumed to be \$100. For the first period, Z is sampled as 0.52.

$$dS(t) = 0.00288(100) + 0.0416(100)(0.52) = 2.45.$$

Therefore, at the beginning of the second time period, the stock price is \$102.45. The value of Z sampled for the next period is 1.44. The change during the second time period is

$$dS(t) = 0.00288(102.45) + 0.0416(102.45)(1.44) = 6.43.$$

So, at the beginning of the next period, the stock price is \$108.88, and so on.

Simulation of stock price when $\alpha = 0.15$ and $\sigma = 0.30$ during 1-week periods.

Stock price at start of period	*Random sample for Z*	*Change in stock price during period*
100.00	0.52	2.45
102.45	1.44	6.43
108.88	−0.86	−3.58
105.30	1.46	6.70
112.00	−0.69	−2.89
109.11	−0.74	−3.04
106.06	0.21	1.23
107.30	−1.10	−4.60
102.69	0.73	3.41
106.11	1.16	5.43
111.54	2.56	12.20

Definition 4.2

Partition an interval [0, T] in into n equal subintervals $[t_{i-1}, t_i]$ such that $t_i = \frac{iT}{n}$. Let $Z(t)$ be the Brownian on [0, T]. $Y(t_i)$ is a random variable which depends on the history of $Z(t_1), \cdots, Z(t_i), Y(t_0), \cdots, Y(t_{i-1})$ and is independent from the increment $Z(t_{i+1}) - Z(t_i)$. Furthermore, we may think of $Y(t_i)$ as the number of shares of stock we hold over a time interval $[t_i, t_{i+1})$. A stochastic integral is defined as

$$\int_0^T Y(t)dZ(t) = \lim_{n\to\infty} \sum_{i=0}^{n-1} Y(t_i)\big(Z(t_{i+1}) - Z(t_i)\big).$$

Remark 4.3

In general, $\lim_{n\to\infty} \sum_{i=0}^{n-1} Y(t_i)\big(Z(t_{i+1}) - Z(t_i)\big) \neq \lim_{n\to\infty} \sum_{i=0}^{n-1} Y(t_{i+1})\big(Z(t_{i+1}) - Z(t_i)\big)$. Say, for instance,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=0}^{n-1} Z(t_i)\big(Z(t_{i+1}) - Z(t_i)\big) &= \lim_{n\to\infty} \frac{1}{2} \sum_{i=0}^{n-1} \Big[\big(Z(t_{i+1})^2 - Z(t_i)^2\big) - \big(Z(t_{i+1}) - Z(t_i)\big)^2\Big] \\
&= \lim_{n\to\infty} \frac{1}{2} \sum_{i=0}^{n-1} \big(Z(t_{i+1})^2 - Z(t_i)^2\big) - \lim_{n\to\infty} \frac{1}{2} \underbrace{\sum_{i=0}^{n-1} \big(Z(t_{i+1}) - Z(t_i)\big)^2}_{\text{quadratic variation}} \\
&= \frac{1}{2} Z(T)^2 - \frac{1}{2} T.
\end{aligned}$$

However,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=0}^{n-1} Z(t_{i+1})\big(Z(t_{i+1}) - Z(t_i)\big) &= \lim_{n\to\infty} \frac{1}{2} \sum_{i=0}^{n-1} \Big[\big(Z(t_{i+1})^2 - Z(t_i)^2\big) + \big(Z(t_{i+1}) - Z(t_i)\big)^2\Big] \\
&= \lim_{n\to\infty} \frac{1}{2} \sum_{i=0}^{n-1} \big(Z(t_{i+1})^2 - Z(t_i)^2\big) + \lim_{n\to\infty} \frac{1}{2} \underbrace{\sum_{i=0}^{n-1} \big(Z(t_{i+1}) - Z(t_i)\big)^2}_{\text{quadratic variation}} \\
&= \frac{1}{2} Z(T)^2 + \frac{1}{2} T.
\end{aligned}$$

Remark 4.4

$$\int_a^b dX(t) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(X\left(a + \frac{i(b-a)}{n}\right) - X\left(a + \frac{(i-1)(b-a)}{n}\right) \right) = X(b) - X(a).$$

Example 4.5

Calculate $\int_0^T Z(t)dZ(t).$

Solution

$\int_0^T Z(t)dZ(t) = \left[\frac{1}{2}Z(t)^2\right]_0^T = \frac{1}{2}Z(T)^2$ which is incorrect. If not, their expectation are equal.

$$\begin{aligned} E\left[\int_0^T Z(t)dZ(t)\right] &= \lim_{n\to\infty}\sum_{i=0}^{n-1} E\left[Z(t_i)\left(Z(t_{i+1})-Z(t_i)\right)\right] \\ &= \lim_{n\to\infty}\sum_{i=0}^{n-1} E\left[Z(t_i)\right]E\left[Z(t_{i+1})-Z(t_i)\right] \\ &= 0. \end{aligned}$$

However $E\left[\frac{1}{2}Z(T)^2\right] = \frac{1}{2}E\left[Z(T)^2\right] = \frac{1}{2}\left(Var\left[Z(T)\right] + E\left[Z(T)\right]^2\right) = \frac{T}{2}.$ The correct answer is

$$\int_0^T Z(t)dZ(t) = \frac{1}{2}Z(T)^2 - \frac{1}{2}T$$

as in the first summation of Remark 4.3.

The derivative of a stochastic integral can be calculated as in ordinary calculus. If the upper limit is t and lower limit is constant, then the derivative of the integral is the integrand including the "d" part evaluated at t,

$$d\left(\int_0^t X(s)dZ(s)\right) = X(t)dZ(t).$$

If $dS(t) = \sigma S(t)dZ(t) + \alpha S(t)dt,$ then $S(T) - S(0) = \int_0^T dS(t) = \int_0^T \sigma S(t)dZ(t) + \int_0^T \alpha S(t)dt.$

Conversely,

$$\begin{aligned} S(T) - S(0) &= \int_0^T dS(t) = \int_0^T \sigma S(t)dZ(t) + \int_0^T \alpha S(t)dt \\ dS(T) - dS(0) &= d\int_0^T \sigma S(t)dZ(t) + d\int_0^T \alpha S(t)dt \\ dS(T) &= \sigma S(T)dZ(T) + \alpha S(T)dT. \end{aligned}$$

Example 4.6

Let $f(S)$ be a smooth function of S. From the Taylor series expansion, we can write

$$df = \frac{df}{dS}dS + \frac{1}{2}\cdot\frac{d^2 f}{dS^2}(dS)^2 + \cdots$$

Assume $\dfrac{dS(t)}{S(t)} = \sigma dZ(t) + \alpha dt.$ We have

$$dS(t) = \sigma S(t)dZ(t) + \alpha S(t)dt, \quad (dS(t))^2 = \sigma^2 S(t)^2 (dZ(t))^2 + 2\alpha\sigma S(t)^2 dZ(t)dt + \alpha^2 S(t)^2 (dt)^2.$$

Since $(dZ(t))^2 = dt$ and $dZ(t)dt = (dt)^2 = 0,$ we have $(dS(t))^2 = \sigma^2 S(t)^2 dt.$

$$\begin{aligned} df &= \frac{df}{dS}dS + \frac{1}{2}\frac{d^2 f}{dS^2}(dS)^2 + \cdots \\ &= \frac{df}{dS}(\alpha S dt + \sigma S dZ) + \frac{1}{2}\frac{d^2 f}{dS^2}\left(\sigma^2 S^2 (dZ)^2 + 2\sigma\alpha S^2 dt dZ + \alpha^2 S^2 (dt)^2\right) + \cdots \\ &= \sigma S\frac{df}{dS}dZ + \left(\alpha S\frac{df}{dS} + \frac{1}{2}\sigma^2 S^2\frac{d^2 f}{dS^2}\right)dt \end{aligned}$$

In particular, if $f(S) = \ln S$, then

$$\frac{df}{dS} = \frac{1}{S}, \quad \frac{d^2 f}{dS^2} = -\frac{1}{S^2}.$$

Hence, $d\ln S(t) = \sigma dZ(t) + \left(\alpha - \frac{1}{2}\sigma^2\right)dt$ is a normal distribution with mean $\left(\alpha - \frac{1}{2}\sigma^2\right)dt$ and variance $\sigma^2 dt.$ We have

$$\begin{aligned} &\int_0^T d\ln S(t) = \int_0^T \sigma dZ(t) + \int_0^T \left(\alpha - \frac{1}{2}\sigma^2\right)dt \\ &\ln S(T) - \ln S(0) = \sigma\left(Z(T) - Z(0)\right) + \left(\alpha - \frac{1}{2}\sigma^2\right)T \\ &S(T) = S(0)e^{(\alpha - 0.5\sigma^2)T + \sigma Z(T)} \end{aligned}$$

$\dfrac{S(t)}{S(0)} = e^{\sigma Z(t) + \left(\alpha - \frac{1}{2}\sigma^2\right)t}$ is a lognormal distribution with parameters $\left(\left(\alpha - 0.5\sigma^2\right)t, \sigma^2 t\right).$

Since $E[S(t)] = S(0)e^{\alpha t}$, that is why α is called the continuous compounded annual rate of return.

Remark 4.7

Note that $\dfrac{dS(t)}{S(t)} = \sigma dZ(t) + \alpha dt \neq d\ln S(t) = \sigma dZ(t) + \left(\alpha - \dfrac{1}{2}\sigma^2\right)dt$ because $(dZ(t))^2 = dt$.

Example 4.8

Let $S(t)$ be the price of a non-dividend paying stock at time t. $S(0) = 80$, $\alpha = 0.15, \sigma = 0.3$. Calculate $P(S(4) > 150)$.

Solution

$S(4)/S(0)$ is a lognormal random variable with parameters $\left(\left(0.15 - 0.5(0.3)^2\right)(4), 0.3^2(4)\right)$ = (0.42, 0.36).

$$P\left(\frac{S(4)}{S(0)} > \frac{150}{80}\right) = P\left(\ln\frac{S(4)}{S(0)} > \ln\frac{150}{80}\right) = 1 - N\left(\frac{\ln(15/8) - 0.42}{0.6}\right) = 0.36404.$$

Theorem 4.9 (Ito's Lemma)

Consider $f(Z(t), t)$, a function of normal random variable $Z(t)$ and time t. We can expand $f(Z(t) + dZ(t), t + dt)$ in a Taylor series about $(Z(t), t)$ to get

$$\begin{aligned} df &= \frac{\partial f}{\partial Z}dZ + \frac{\partial f}{\partial t}dt + \frac{1}{2}\left(\frac{\partial^2 f}{\partial Z^2}(dZ)^2 + 2\frac{\partial^2 f}{\partial Z \partial t}dZdt + \frac{\partial^2 f}{\partial t^2}(dt)^2\right) + \cdots \\ &= \frac{\partial f}{\partial Z}dZ + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial Z^2}dt. \end{aligned}$$

If $f(S(t), t)$ is a function of stock price $S(t)$ and time t, and $dS(t) = \sigma S(t)dZ(t) + \alpha S(t)dt$, then

$$\begin{aligned} df &= \frac{\partial f}{\partial S}dS + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 + \cdots \\ &= \frac{\partial f}{\partial S}(\alpha S dt + \sigma S dZ) + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\left(\sigma^2 S^2 (dZ)^2 + 2\sigma\alpha S^2 dt dZ + \alpha^2 S^2 (dt)^2\right) + \cdots \\ &= \sigma S\frac{\partial f}{\partial S}dZ + \left(\alpha S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right)dt \end{aligned}$$

Example 4.10

Given that $X(t) = \mu t + \sigma Z(t)$. Calculate $dX(t)$ using Ito's Lemma.

Solution

$$\frac{\partial X}{\partial Z} = \sigma, \quad \frac{\partial^2 X}{\partial Z^2} = 0, \quad \frac{\partial X}{\partial t} = \mu$$

$$dX = \frac{\partial X}{\partial Z} dZ + \frac{\partial X}{\partial t} dt + \frac{1}{2}\frac{\partial^2 X}{\partial Z^2} dt = \sigma dZ + \mu dt.$$

Example 4.11

Given that $S(t) = S(0)e^{(\alpha - 0.5\sigma^2)t + \sigma Z(t)}$. Calculate $dS(t)$ using Ito's Lemma.

Solution

$$\frac{\partial S}{\partial Z} = \sigma S(0)e^{(\alpha - 0.5\sigma^2)t + \sigma Z(t)} = \sigma S(t), \quad \frac{\partial^2 S}{\partial Z^2} = \sigma^2 S(t), \quad \frac{\partial S}{\partial t} = \left(\alpha - \frac{1}{2}\sigma^2\right) S(t)$$

$$dS = \frac{\partial S}{\partial Z} dZ + \frac{\partial S}{\partial t} dt + \frac{1}{2}\frac{\partial^2 S}{\partial Z^2} dt = \sigma S(t) dZ + \left(\alpha - \frac{1}{2}\sigma^2\right) S(t) dt + \frac{1}{2}\sigma^2 S(t) dt = \sigma S(t) dZ + \alpha S(t) dt.$$

Remark 4.12

Let $S(t)$ be the stock at time t. Let α be the continuously compounded expected rate of return, and σ be the volatility. The following are equivalent:

(i) $\dfrac{dS(t)}{S(t)} = \sigma dZ(t) + \alpha dt.$

(ii) $d \ln S(t) = \sigma dZ(t) + \left(\alpha - \dfrac{1}{2}\sigma^2\right) dt.$

(iii) $S(t) = S(0)e^{(\alpha - 0.5\sigma^2)t + \sigma Z(t)}.$

(iv) $S(T) = S(0) + \int_0^T \sigma S(t) dZ(t) + \int_0^T \alpha S(t) dt.$

If $S(t)$ pays continuous dividends at a rate δ, then change α into $\alpha - \delta$.

Example 4.13

Let $Y(t) = 0.5Z(t)^2$. By Ito's Lemma, we have

$$\frac{\partial Y}{\partial Z} = Z, \quad \frac{\partial^2 Y}{\partial Z^2} = 1, \quad \frac{\partial Y}{\partial t} = 0$$
$$dY(t) = \frac{\partial Y}{\partial Z} dZ(t) + \frac{1}{2}\frac{\partial^2 Y}{\partial Z^2} dt + \frac{\partial Y}{\partial t} dt = Z(t)dZ(t) + \frac{1}{2}dt$$
$$\int_0^T Z(t)dZ(t) = \int_0^T dY(t) - \int_0^T \frac{1}{2}dt = Y(T) - Y(0) - \frac{T}{2} = \frac{Z(T)^2}{2} - \frac{T}{2}.$$

Example 4.14

Suppose a stock $S(t)$ pays a continuous dividends at a rate δ. The risk free interest rate is r. The forward price of the stock at 0 is $F_{0,t}(S(t)) = S(0)e^{(r-\delta)t}$, which is not random. Clearly,

$$\frac{dF_{0,t}(S(t))}{F_{0,t}(S(t))} = (r-\delta)dt.$$

However, the forward price $F_{t,T}(S(T))$ of the stock at t with T fixed is random. We have

$$F_{t,T}(S(T)) = S(t)e^{(r-\delta)(T-t)}.$$

On the other hand, we also have $dS = \sigma S dZ + (\alpha - \delta)S dt$. By Ito's Lemma,

$$\frac{\partial F_{t,T}}{\partial t} = -(r-\delta)S(t)e^{(r-\delta)(T-t)}, \quad \frac{\partial F_{t,T}}{\partial S} = e^{(r-\delta)(T-t)}, \quad \frac{\partial^2 F_{t,T}}{\partial S^2} = 0$$
$$dF_{t,T} = \sigma S \frac{\partial F_{t,T}}{\partial S} dZ + \left((\alpha-\delta)S\frac{\partial F_{t,T}}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F_{t,T}}{\partial S^2} + \frac{\partial F_{t,T}}{\partial t} \right) dt$$
$$= \sigma S(t)e^{(r-\delta)(T-t)} dZ + \left((\alpha-\delta)S(t)e^{(r-\delta)(T-t)} - (r-\delta)S(t)e^{(r-\delta)(T-t)} \right) dt$$
$$\frac{dF_{t,T}}{F_{t,T}} = \sigma dZ + \left((\alpha-\delta) - (r-\delta) \right) dt = \sigma dZ + (\alpha - r)dt.$$

So the forward follows geometric Brownian motion with the same volatility as the stock, and the growth rate $\alpha - r$, which is independent of δ. The buyer of the forward earns the risk premium $\alpha - r$ on the stock without making any investment in the stock.

Let $S(t)$ be the risky asset paying no dividend and follows

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dZ(t).$$

Let $B(t)$ be the risk-free asset and the continuous compounded risk-free interest rate is r. We have

$$\frac{dB(t)}{B(t)} = rdt.$$

Let us analyze a portfolio consisting of a risky asset and a risk-free asset in which the risky asset is a constant proportion, φ, of the portfolio. This implies continuous rehedging.

$$W(t) = \varphi W(t) + (1-\varphi)W(t) = \underbrace{\varphi\frac{W(t)}{S(t)}}_{\text{number of shares of stock}} S(t) + \underbrace{(1-\varphi)\frac{W(t)}{B(t)}}_{\text{number of bonds}} B(t)$$

$$dW(t) = \varphi\frac{W(t)}{S(t)}dS(t) + (1-\varphi)\frac{W(t)}{B(t)}dB(t)$$

$$\begin{aligned}
\frac{dW(t)}{W(t)} &= \varphi\frac{dS(t)}{S(t)} + (1-\varphi)\frac{dB(t)}{B(t)} \\
&= \varphi(\alpha dt + \sigma dZ(t)) + (1-\varphi)rdt \\
&= \big(\varphi\alpha + (1-\varphi)r\big)dt + \varphi\sigma dZ(t).
\end{aligned}$$

The solution to this differential equation is

$$\frac{W(t)}{W(0)} = e^{\left(\varphi\alpha+(1-\varphi)r-0.5\varphi^2\sigma^2\right)t+\varphi\sigma Z(t)}.$$

Since $\left(\dfrac{S(t)}{S(0)}\right)^{\varphi} = e^{\varphi(\alpha-0.5\sigma^2)t+\varphi\sigma Z(t)}$, we have

$$\begin{aligned}
\frac{W(t)}{W(0)} &= e^{\left(\varphi\alpha+(1-\varphi)r-0.5\varphi^2\sigma^2\right)t+\varphi\sigma Z(t)} \\
&= e^{\varphi(\alpha-0.5\sigma^2)t+\varphi\sigma Z(t)} e^{\left(\varphi\alpha+(1-\varphi)r-0.5\varphi^2\sigma^2\right)t-\varphi(\alpha-0.5\sigma^2)t} \\
&= \left(\frac{S(t)}{S(0)}\right)^{\varphi} e^{\left((1-\varphi)r-0.5\varphi^2\sigma^2+0.5\varphi\sigma^2\right)t} \\
&= \left(\frac{S(t)}{S(0)}\right)^{\varphi} e^{(1-\varphi)\left(r+0.5\varphi\sigma^2\right)t}.
\end{aligned}$$

If the risky asset pays dividends, they get reinvested in the portfolio. If the portfolio pays dividends, this will lower the rate of appreciation of the portfolio. Let δ_S and δ_W be the continuous dividend rates of the stock and the portfolio. The Ito processes for $S(t)$ and $W(t)$ are

$$\frac{dS(t)}{S(t)} = (\alpha - \delta_S)dt + \sigma dZ(t), \quad \frac{dW(t)}{W(t)} = \left(\varphi\alpha - \delta_W + (1-\varphi)r\right)dt + \varphi\sigma dZ(t).$$

Then

$$\left(\frac{S(t)}{S(0)}\right)^{\varphi} = e^{\varphi\left((\alpha-\delta_S)-0.5\sigma^2\right)t+\varphi\sigma Z(t)}, \quad \frac{W(t)}{W(0)} = e^{\left(\varphi\alpha-\delta_W+(1-\varphi)r-0.5\varphi^2\sigma^2\right)t+\varphi\sigma Z(t)}$$

$$= e^{\varphi\left((\alpha-\delta_S)-0.5\sigma^2\right)t+\varphi\sigma Z(t)} e^{\left(\varphi\alpha-\delta_W+(1-\varphi)r-0.5\varphi^2\sigma^2\right)t-\varphi\left((\alpha-\delta_S)-0.5\sigma^2\right)t}$$

$$= \left(\frac{S(t)}{S(0)}\right)^{\varphi} e^{\left(\varphi\delta_S-\delta_W+(1-\varphi)r-0.5\varphi^2\sigma^2+0.5\varphi\sigma^2\right)t}.$$

Example 4.15

The stock $S(t)$ follows $d\ln S(t) = 0.055dt + 0.3dZ(t)$ and pays continuous dividends at a rate of 2%. The continuously compounded risk-free interest rate is 5%. A portfolio $W(t)$ consists of 75% in the stock and 25% in a risk-free asset. It is continuously rebalanced and pays continuous dividends of 4%. Find $\dfrac{dW(t)}{W(t)}$.

Solution

$$\sigma_W = \varphi\sigma = (0.75)(0.3) = 0.225. \quad \alpha_S - \delta_S - 0.5(0.3^2) = 0.055, \alpha_S = 0.12$$
$$\varphi\alpha_S - \delta_W + (1-\varphi)r = (0.75)(0.12) - 0.04 + (0.25)(0.05) = 0.0625$$
$$\frac{dW(t)}{W(t)} = \left(\varphi\alpha - \delta_W + (1-\varphi)r\right)dt + \varphi\sigma dZ(t) = 0.0625dt + 0.225dZ(t).$$

Assume 2 stocks $S_1(t)$ and $S_2(t)$ depend on the same $Z(t)$. Their continuous dividend rates are δ_1 and δ_2 respectively. Then

$$dS_1(t) = (\alpha_1 - \delta_1)S_1(t)dt + \sigma_1 S_1(t)dZ(t), \quad dS_2(t) = (\alpha_2 - \delta_2)S_2(t)dt \pm \sigma_2 S_2(t)dZ(t).$$

$+\sigma_2 S_2(t)dZ(t)$ in $dS_2(t)$ if S_1 and S_2 are positive correlated and $-\sigma_2 S_2(t)dZ(t)$ if S_1 and S_2 are negative correlated.

Let r be the continuously compounded risk-free interest rate. A risk-free portfolio can be created by buying c_1 shares of S_1 and buying (or selling) c_2 shares of S_2 as follows:

$$\begin{aligned} d\left(c_1S_1(t)+c_2S_2(t)\right) &= c_1dS_1(t)+c_2dS_2(t) \\ &= \left(c_1(\alpha_1-\delta_1)S_1(t)+c_2(\alpha_2-\delta_2)S_2(t)\right)dt+\left(c_1\sigma_1S_1(t)\pm c_2\sigma_2S_2(t)\right)dZ(t). \end{aligned}$$

If $c_1\sigma_1S_1(t)\pm c_2\sigma_2S_2(t)=0,$ then $c_1S_1(t)+c_2S_2(t)$ is a risk free portfolio. We have

$$d\left(c_1S_1(t)+c_2S_2(t)\right)=r\left(c_1S_1(t)+c_2S_2(t)\right)dt.$$

Hence

$$\begin{gathered} c_1(\alpha_1-\delta_1)S_1(t)+c_2(\alpha_2-\delta_2)S_2(t)=r\left(c_1S_1(t)+c_2S_2(t)\right) \quad \text{or} \\ c_1(\alpha_1-\delta_1-r)S_1(t)+c_2(\alpha_2-\delta_2-r)S_2(t)=0. \end{gathered}$$

Solving $c_1\sigma_1S_1(t)\pm c_2\sigma_2S_2(t)=0$ and $c_1(\alpha_1-\delta_1-r)S_1(t)+c_2(\alpha_2-\delta_2-r)S_2(t)=0,$ we have

$$\frac{\alpha_1-\delta_1-r}{\sigma_1}=\pm\frac{\alpha_2-\delta_2-r}{\sigma_2}.$$

$\frac{\alpha-r}{\sigma}$ is called **Sharpe** ratio of an asset. The dividend rate δ is not subtracted from α.

Example 4.16

For 2 non-dividend paying stocks, $S_1(0)=40, S_2(0)=60,$

$$dS_1(t)=0.07S_1(t)dt+0.3S_1(t)dZ(t) \quad \text{and} \quad dS_2(t)=0.04S_2(t)dt-\sigma S_2(t)dZ(t).$$

The continuously compounded risk-free rate is 0.05. A risk-free portfolio consist of 100 shares of S_1 and x shares of S_2. Find x and σ.

Solution

$$c_1(\alpha_1-\delta_1-r)S_1(0)+c_2(\alpha_2-\delta_2-r)S_2(0)=0 \quad\Rightarrow\quad 100(0.07-0.05)40+x(0.04-0.05)60=0$$

$$x=\frac{400}{3}$$

$$\frac{\alpha_1-r}{\sigma_1}=-\frac{\alpha_2-r}{\sigma_2} \quad\Rightarrow\quad \frac{0.07-0.05}{0.3}=-\frac{0.04-0.05}{\sigma} \quad\Rightarrow\quad \sigma=0.15.$$

Suppose $S(t)$ satisfies $\dfrac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma dZ(t)$ and $S(T) = S(0)e^{(\alpha-\delta-0.5\sigma^2)T+\sigma Z(T)}$.

Consider $S(T)^a = S(0)^a e^{a(\alpha-\delta-0.5\sigma^2)T+a\sigma Z(T)}$. When $a = -1$, it represents a currency exchange from the other party's viewpoint. When $a = 2$, $S(T)^2$ can be viewed as a financial derivative which pays x shares of a stock at expiry, where x is the price of the stock at expiry. The financial derivative pays the square of the value of the stock.

$\left(S(T)/S(0)\right)^a$ is a lognormal distribution with parameters $\left((\alpha - \delta - 0.5\sigma^2)aT, a^2\sigma^2 T\right)$. So the expected value of $S(T)^a$ is

$$E[S(T)^a] = S(0)^a e^{a(\alpha-\delta-0.5\sigma^2)T+0.5a^2\sigma^2 T} = S(0)^a e^{\left[a(\alpha-\delta)+0.5a(a-1)\sigma^2)\right]T}.$$

The forward price of an asset can be calculated as the expected price of the asset in a risk neutral world. In a risk neutral world, α is replaced by r:

$$S(0)^a e^{\left[a(r-\delta)+0.5a(a-1)\sigma^2)\right]T}.$$

Its forward price and prepaid price is

$$F_{0,T}\left(S(T)^a\right) = S(0)^a e^{\left[a(r-\delta)+0.5a(a-1)\sigma^2)\right]T} = \left[F_{0,T}\left(S(T)\right)\right]^a e^{0.5a(a-1)\sigma^2 T}.$$

$$F_{0,T}^P\left(S(T)^a\right) = e^{-rT}F_{0,T}\left(S(T)^a\right) = e^{-rT}S(0)^a e^{\left[a(r-\delta)+0.5a(a-1)\sigma^2\right]T}.$$

Example 4.17

Suppose the forward price of $S(2)$ and $\sqrt{S(2)}$ is 30 and 5 respectively. Determine the forward price of $\sqrt[3]{S(2)}$.

Solution

$$F_{0,T}\left(\sqrt{S(2)}\right) = \left[F_{0,T}\left(S(T)\right)\right]^a e^{0.5a(a-1)\sigma^2 T} = \sqrt{F_{0,T}\left(S(2)\right)}e^{0.5(0.5)(-0.5)\sigma^2(2)}.$$

$$e^{(\sigma^2)} = \left(\frac{\sqrt{F_{0,T}(S(2))}}{F_{0,T}\left(\sqrt{S(2)}\right)}\right)^4 = \left(\frac{\sqrt{30}}{5}\right)^4 = 1.44$$

$$F_{0,T}\left(\sqrt[3]{S(2)}\right) = \sqrt[3]{F_{0,T}(S(2))}e^{0.5(1/3)(-2/3)\sigma^2(2)} = \sqrt[3]{30}e^{(-2/9)\sigma^2} = \sqrt[3]{30}(1.44)^{(-2/9)} = 2.8654.$$

Let $C(S,t) = S(t)^a$. By Ito's Lemma,

$$\frac{\partial C}{\partial S} = aS^{a-1}, \quad \frac{\partial^2 C}{\partial S^2} = a(a-1)S^{a-2}, \quad \frac{\partial C}{\partial t} = 0$$

$$dC = \sigma S\frac{\partial C}{\partial S}dZ(t) + \left((\alpha-\delta)S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t}\right)dt$$

$$= a\sigma S^a dZ(t) + \left((\alpha-\delta)aS^a + \frac{1}{2}\sigma^2 a(a-1)S^a\right)dt$$

$$\frac{dC}{C} = a\sigma dZ(t) + \left((\alpha-\delta)a + \frac{1}{2}\sigma^2 a(a-1)\right)dt.$$

Alternatively,

$$d\ln C = ad\ln S = a\sigma dZ(t) + a\left(\alpha - \delta - \frac{1}{2}\sigma^2\right)dt$$

$$\frac{dC}{C} = a\sigma dZ(t) + \left((\alpha-\delta)a - \frac{1}{2}\sigma^2 a + \frac{1}{2}a^2\sigma^2\right)dt = a\sigma dZ(t) + \left((\alpha-\delta)a + \frac{1}{2}\sigma^2 a(a-1)\right)dt.$$

Let γ be the expected return of $S(T)^a$. Then

$$\frac{\gamma - \delta_{S^a} - r}{a\sigma} = \frac{\alpha - \delta_S - r}{\sigma} \text{ if } a > 0 \text{ and } \frac{\gamma - \delta_{S^a} - r}{a\sigma} = -\frac{\alpha - \delta_S - r}{\sigma} \text{ if } a < 0.$$

Example 4.18

Assume $S(T)$ is the price of a stock following geometric Brownian motion with parameters α, δ and σ having their usual meanings. Let $Y(t) = F_{t,T}\left(S(T)^a\right)$. Determine the Ito process followed by $Y(t)$.

Solution

Since $Y(t) = F_{t,T}\left(S(T)^a\right) = S(t)^a e^{\left[a(r-\delta)+0.5a(a-1)\sigma^2\right](T-t)}$,

$$\frac{\partial Y}{\partial S} = aS(t)^{a-1} e^{\left[a(r-\delta)+0.5a(a-1)\sigma^2\right](T-t)} = \frac{aY}{S}$$

$$\frac{\partial^2 Y}{\partial S^2} = a(a-1)S(t)^{a-2} e^{\left[a(r-\delta)+0.5a(a-1)\sigma^2\right](T-t)} = \frac{a(a-1)Y}{S^2}$$

$$\frac{\partial Y}{\partial t} = -S(t)^a \left[a(r-\delta)+0.5a(a-1)\sigma^2\right] e^{\left[a(r-\delta)+0.5a(a-1)\sigma^2\right](T-t)} = -\left[a(r-\delta)+0.5a(a-1)\sigma^2\right]Y$$

$$\begin{aligned}
dY &= \frac{\partial Y}{\partial S}dS + \frac{1}{2}\frac{\partial^2 Y}{\partial S^2}(dS)^2 + \frac{\partial Y}{\partial t}dt \\
&= \frac{aY}{S}dS + \frac{a(a-1)Y}{2S^2}(dS)^2 - \left[a(r-\delta)+0.5a(a-1)\sigma^2\right]Ydt \\
&= aY\left((\alpha-\delta)dt + \sigma dZ(t)\right) + \frac{a(a-1)Y}{2}\sigma^2 dt - \left[a(r-\delta)+0.5a(a-1)\sigma^2\right]Ydt \\
&= a(\alpha - r)Ydt + a\sigma Y dZ(t)
\end{aligned}$$

$$\frac{dF_{t,T}\left(S^a\right)}{F_{t,T}\left(S^a\right)} = a(\alpha-r)dt + a\sigma dZ(t) = a\frac{dF_{t,T}\left(S\right)}{F_{t,T}\left(S\right)}.$$

Suppose that we have an option whose value $V(S, t)$ depends only on S and t. It is not necessary to specify whether V is a call or a put; indeed, V can be the value of a whole portfolio of different options. Using Ito's lemma, we can write

$$dV = \sigma S\frac{\partial V}{\partial S}dZ + \left(\alpha S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt.$$

Now construct a portfolio of one option and $-\Delta$ of the underlying asset S. This number is as yet unspecified. The value of this portfolio is

$$\Pi = V - \Delta S.$$

The change $d\Pi$ in the value of the portfolio in the time interval dt is given by

$$\begin{aligned}
d\Pi &= dV - \Delta dS \\
&= \sigma S\frac{\partial V}{\partial S}dZ + \left(\alpha S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt - \Delta(\sigma S dZ + \alpha S dt)
\end{aligned}$$

$$=\sigma S\left(\frac{\partial V}{\partial S}-\Delta\right)dZ+\left(\alpha S\left(\frac{\partial V}{\partial S}-\Delta\right)+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+\frac{\partial V}{\partial t}\right)dt.$$

We can eliminate the random component by choosing

$$\Delta=\frac{\partial V}{\partial S}.$$

This results in a portfolio is totally deterministic:

$$d\Pi=\left(\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+\frac{\partial V}{\partial t}\right)dt.$$

Because this equation does not involve dZ, the portfolio must be riskless during time dt. The portfolio must instantaneously earn the same rate of return as other risk-free securities. If it earned more than this return, arbitrageurs could make a riskless profit by borrowing money to buy the portfolio; if it earned less, they could make a riskless profit by shorting the portfolio and buying risk-free securities. It follows that

$$d\Pi=r\Pi dt=r\left(V-S\frac{\partial V}{\partial S}\right)dt$$

where r is the risk-free interest rate. Hence

$$d\Pi=\left(\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+\frac{\partial V}{\partial t}\right)dt=r\left(V-S\frac{\partial V}{\partial S}\right)dt$$
$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0.$$

This is the **Black-Scholes partial differential equation**.

Example 4.19

If V is a function of t only but not of $S(t)$, then the Black-Scholes partial differential equation becomes $\frac{\partial V}{\partial t}=rV$. The solution $V(t)=V(0)e^{rt}$ is a risk-free bond.

On the other hand, let $V(t)=V(0)e^{rt}$ is a risk-free bond. Then $\frac{\partial V}{\partial t}=rV(0)e^{rt}=rV(t)$ and

$\frac{\partial V}{\partial S} = \frac{\partial^2 V}{\partial S^2} = 0.$ $V(t)$ satisfies Black-Scholes partial differential equation.

Let $V = S(t)$. $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial S^2} = 0, \frac{\partial V}{\partial S} = 1.$ Clearly, stock $S(t)$ satisfies Black-Scholes partial differential equation.

Their combination $aS(t) + be^{rt}$ also satisfy Black-Scholes partial differential equation where a and b are real constants.

We consider a European call, with value at t denoted by $C(S, t)$, with exercise price K, expiration date T on a non-dividend paying stock. The Black-Scholes partial differential equation for the call is

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC + \frac{\partial C}{\partial t} = 0.$$

We must next consider final and boundary conditions, for otherwise the partial differential equation does not have a unique solution.

At $t = T$, the payoff of a call is

$$C(S,T) = \max(S(T) - K, 0).$$

If $S(t) = 0$, the call option is worthless even if there is a long time to expiry. We have

$$C(0, t) = 0.$$

As the asset price increases without bound, the option will be exercised and the strike price is less and less important. Thus as $S \to \infty$, the value of the option is close to that of the asset.

$$C(S,t) \sim S - Ke^{-r(T-t)} \sim S \quad \text{as} \quad S \to \infty.$$

In order to get rid of $S\frac{\partial C}{\partial S}$ and $S^2\frac{\partial^2 C}{\partial S^2}$ in Black-Scholes equation, we set

$$S(t) = Ke^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad C(S,t) = Kv(x,\tau) = Kv\left(\ln\frac{S(t)}{K}, \frac{\sigma^2}{2}(T-t)\right).$$

The partial derivatives of V with respect to S and t expressed in terms of partial derivatives of v in terms of x and τ are:

$$\frac{\partial C}{\partial t}=-\frac{\sigma^2 K}{2}\frac{\partial v}{\partial \tau},\quad \frac{\partial C}{\partial S}=\frac{K}{S}\frac{\partial v}{\partial x},\quad \frac{\partial^2 C}{\partial S^2}=-\frac{K}{S^2}\frac{\partial v}{\partial x}+\frac{K}{S^2}\frac{\partial^2 v}{\partial x^2}.$$

Placing these expressions into the Black-Scholes partial differential equation and simplifying, we have

$$\frac{\partial v}{\partial \tau}=\frac{\partial^2 v}{\partial x^2}+(k-1)\frac{\partial v}{\partial x}-kv,\quad -\infty<x<\infty,\quad 0\le\tau\le\frac{\sigma^2}{2}T$$

where $k=2r/\sigma^2$. The initial condition becomes

$$v(x,0)=\frac{1}{K}C(S,T)=\max(e^x-1,0),\quad -\infty<x<\infty.$$

One more change of variables is needed in order to eliminate v and $\frac{\partial v}{\partial x}$ of the last equation. To this end set

$$v(x,\tau)=e^{ax+b\tau}u(x,\tau),$$

where we'll pick a and b later. Computing the partials of v in terms of x and τ, we have

$$\begin{aligned}
\frac{\partial v}{\partial \tau}&=be^{ax+b\tau}u(x,\tau)+e^{ax+b\tau}\frac{\partial u}{\partial \tau},\\
\frac{\partial v}{\partial x}&=ae^{ax+b\tau}u(x,\tau)+e^{ax+b\tau}\frac{\partial u}{\partial x},\\
\frac{\partial^2 v}{\partial x^2}&=a^2e^{ax+b\tau}u(x,\tau)+2ae^{ax+b\tau}\frac{\partial u}{\partial x}+e^{ax+b\tau}\frac{\partial^2 u}{\partial x^2}.
\end{aligned}$$

Placing these expressions into the partial differential equation which v satisfies,

$$\begin{aligned}
bu+\frac{\partial u}{\partial \tau}&=a^2u+2a\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}+(k-1)\left(au+\frac{\partial u}{\partial x}\right)-ku,\\
\frac{\partial u}{\partial \tau}&=\frac{\partial^2 u}{\partial x^2}+(2a+(k-1))\frac{\partial u}{\partial x}+(a^2+(k-1)a-k-b)u.
\end{aligned}$$

We can obtain an equation with no u term by choosing

$$b=a^2+(k-1)a-k$$

while the choice

$$2a+(k-1)=0$$

eliminates the $\partial u/\partial x$ term as well. These equation for *a* and *b* give

$$a=-\frac{1}{2}(k-1)=\frac{\sigma^2-2r}{2\sigma^2},\quad b=-\frac{1}{4}(k+1)^2=-\left(\frac{\sigma^2+2r}{2\sigma^2}\right)^2.$$

Let $v(x,\tau)=e^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau}u(x,\tau)$. We then have the heat equation

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial x^2},\quad -\infty<x<\infty,\quad 0\le\tau\le\frac{\sigma^2}{2}T,$$

with initial condition

$$u(x,0)=e^{-\alpha x}v(x,0)=\max\left(e^{\frac{1}{2}(k+1)x}-e^{\frac{1}{2}(k-1)x},0\right).$$

Define $U(x,s)=\mathcal{L}_s\left(u(x,\tau)\right):=\int_0^\infty e^{-s\tau}u(x,\tau)d\tau$, where $\mathcal{L}_s$ is the Laplace operator with parameter *s*. Taking Laplace transform to $\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial x^2}$, we have $\mathcal{L}_s\left(\frac{\partial u}{\partial \tau}\right)=\mathcal{L}_s\left(\frac{\partial^2 u}{\partial x^2}\right)$. Then

$$sU(x,s)-u(x,0)=U_{xx}(x,s).$$

Consider the second order differential equation

$$U_{xx}(x,s)-sU(x,s)=-u(x,0).$$

Its characteristic equation is $r^2-s=0$. The roots are $r_1=\sqrt{s}, r_2=-\sqrt{s}$.

The homogeneous solution is $U_h(x,s)=c_1e^{x\sqrt{s}}+c_2e^{-x\sqrt{s}}$. Before calculating particular solution, we need Definition 4.20 and Theorem 4.21.

Definition 4.20

Define error function $erf(x)=\frac{1}{\sqrt{\pi}}\int_{-x}^{x}e^{-t^2}dt=\frac{2}{\sqrt{\pi}}\int_{0}^{x}e^{-t^2}dt$ and complementary error function $erfc(x)=1-erf(x)=\frac{2}{\sqrt{\pi}}\int_{x}^{\infty}e^{-t^2}dt.$

Theorem 4.21

Suppose $\frac{\partial u}{\partial\tau}=\frac{\partial^2 u}{\partial x^2}$ with $u(x,0)=u(\infty,\tau)=0, u(0,\tau)=1$. Then $u(x,\tau)=\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{\tau}}}^{\infty}e^{-t^2}dt.$

Furthermore, $\mathcal{L}_s\left(\frac{1}{2\sqrt{\pi\tau}}e^{-\frac{x^2}{4\tau}}\right)=\frac{e^{-x\sqrt{s}}}{2\sqrt{s}}$ for $x>0$.

Solution

Let $\xi=\frac{x}{\sqrt{\tau}}$ such that $u(x,\tau)=U(\xi)$.

$$\frac{\partial u}{\partial\tau}=U'(\xi)\frac{d\xi}{d\tau}=U'(\xi)\left(-\frac{1}{2}x\tau^{-\frac{3}{2}}\right)=-\frac{1}{2\tau}\xi U'(\xi)$$
$$\frac{\partial u}{\partial x}=U'(\xi)\frac{d\xi}{dx}=U'(\xi)\frac{1}{\sqrt{\tau}}$$
$$\frac{\partial^2 u}{\partial x^2}=U''(\xi)\frac{d\xi}{dx}\frac{1}{\sqrt{\tau}}=U''(\xi)\frac{1}{\tau}$$
$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial x}\quad\Rightarrow\quad U''(\xi)\frac{1}{\tau}=-\frac{1}{2\tau}\xi U'(\xi)\quad\Rightarrow\quad\frac{U''(\xi)}{U'(\xi)}=-\frac{1}{2}\xi\quad\Rightarrow\quad\int\frac{U''(\xi)}{U'(\xi)}d\xi=\int-\frac{1}{2}\xi d\xi$$
$$U'(\xi)=Ce^{-\frac{\xi^2}{4}}$$
$$U(\xi)=C\int_0^{\xi}e^{-\frac{s^2}{4}}ds+D$$
$$u(x,0)=u(\infty,\tau)=0, u(0,\tau)=1\quad\Rightarrow\quad U(0)=1, U(\infty)=0.$$

Recall $\int_0^{\infty}e^{-\frac{s^2}{4}}ds=\sqrt{\pi}$. We have $D=1$, $C=-\frac{1}{\sqrt{\pi}}$, $U(\xi)=\frac{1}{\sqrt{\pi}}\int_{\xi}^{\infty}e^{-\frac{s^2}{4}}ds$ and

$$u(x,\tau)=\frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{\tau}}}^{\infty}e^{-\frac{s^2}{4}}ds\overset{t=\frac{s}{2}}{=}\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{\tau}}}^{\infty}e^{-t^2}dt=erfc\left(\frac{x}{2\sqrt{\tau}}\right).$$

$$U(x,s)=c_1e^{x\sqrt{s}}+c_2e^{-x\sqrt{s}}.$$
$$U(\infty,s)=\mathcal{L}_s\left(u(\infty,\tau)\right)=0 \quad\Rightarrow\quad c_1=0$$
$$U(0,s)=\mathcal{L}_s\left(u(0,\tau)\right)=\int_0^\infty e^{-s\tau}d\tau=\frac{1}{s} \quad\Rightarrow\quad c_2=\frac{1}{s}$$
$$\frac{e^{-x\sqrt{s}}}{s}=U(x,s)=\mathcal{L}_s\left(erfc\left(\frac{x}{2\sqrt{\tau}}\right)\right)$$

Since $\mathcal{L}_s\left(f'(\tau)\right)=s\mathcal{L}_s\left(f(\tau)\right)-\lim_{\tau\to0^+}f(\tau)$,

$$\mathcal{L}_s\left(\frac{d}{d\tau}erfc\left(\frac{x}{2\sqrt{\tau}}\right)\right)=s\mathcal{L}_s\left(erfc\left(\frac{x}{2\sqrt{\tau}}\right)\right)-erfc(\infty) \quad \text{if } x>0$$
$$\mathcal{L}_s\left(\left(\frac{x}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{\sqrt{\tau^3}}\right)\frac{-2}{\sqrt{\pi}}e^{-\frac{x^2}{4\tau}}\right)=e^{-x\sqrt{s}}$$
$$\mathcal{L}_s\left(\frac{-x\tau}{2\sqrt{\pi\tau^3}}e^{-\frac{x^2}{4\tau}}\right)=\frac{d}{ds}\mathcal{L}_s\left(\frac{x}{2\sqrt{\pi\tau^3}}e^{-\frac{x^2}{4\tau}}\right)=\frac{d}{ds}e^{-x\sqrt{s}}=\frac{-xe^{-x\sqrt{s}}}{2\sqrt{s}}$$

We have

$$\mathcal{L}_s\left(\frac{1}{2\sqrt{\pi\tau}}e^{-\frac{x^2}{4\tau}}\right)=\frac{e^{-x\sqrt{s}}}{2\sqrt{s}}.$$

Let us come back to particular solution. Let $y_1(x)=e^{x\sqrt{s}}$ and $y_2(x)=e^{-x\sqrt{s}}$.

The Wronskian of $y_1(x)$ and $y_2(x)$ is

$$W(y_1,y_2)=y_1y_2'-y_1'y_2=-\sqrt{s}e^{x\sqrt{s}}e^{-x\sqrt{s}}-\sqrt{s}e^{x\sqrt{s}}e^{-x\sqrt{s}}=-2\sqrt{s}.$$

By variation of parameters in ordinary differential equations, the particular solution is

$$U_p(x,s)=u_1(x)y_1(x)+u_2(x)y_2(x)=u_1(x)e^{x\sqrt{s}}+u_2(x)e^{-x\sqrt{s}}$$

where $u_1'(x)=\dfrac{y_2(x)u(x,0)}{W(y_1,y_2)(x)}=-\dfrac{e^{-x\sqrt{s}}u(x,0)}{2\sqrt{s}}$ and $u_2'(x)=-\dfrac{y_1(x)u(x,0)}{W(y_1,y_2)(x)}=\dfrac{e^{x\sqrt{s}}u(x,0)}{2\sqrt{s}}$. Then

$$u_1(x)=-\int\frac{e^{-x\sqrt{s}}u(x,0)}{2\sqrt{s}}dx=-\int_\infty^x\frac{e^{-\xi\sqrt{s}}u(\xi,0)}{2\sqrt{s}}d\xi=\int_x^\infty\frac{e^{-\xi\sqrt{s}}u(\xi,0)}{2\sqrt{s}}d\xi$$

and

$$u_2(x) = \int \frac{e^{x\sqrt{s}}u(x,0)}{2\sqrt{s}}dx = \int_{-\infty}^{x} \frac{e^{\xi\sqrt{s}}u(\xi,0)}{2\sqrt{s}}d\xi.$$

$$\begin{aligned}
U_p(x,s) &= u_1(x)e^{x\sqrt{s}} + u_2(x)e^{-x\sqrt{s}} \\
&= e^{x\sqrt{s}}\int_x^{\infty} \frac{e^{-\xi\sqrt{s}}u(\xi,0)}{2\sqrt{s}}d\xi + e^{-x\sqrt{s}}\int_{-\infty}^{x} \frac{e^{\xi\sqrt{s}}u(\xi,0)}{2\sqrt{s}}d\xi \\
&= \int_x^{\infty} \frac{e^{-(\xi-x)\sqrt{s}}u(\xi,0)}{2\sqrt{s}}d\xi + \int_{-\infty}^{x} \frac{e^{-(x-\xi)\sqrt{s}}u(\xi,0)}{2\sqrt{s}}d\xi \\
&= \int_{-\infty}^{\infty} \frac{e^{-|\xi-x|\sqrt{s}}u(\xi,0)}{2\sqrt{s}}d\xi.
\end{aligned}$$

Thus, to get $u(x,\tau)$, it has to be done inverse Laplace operation

$$\begin{aligned}
u(x,\tau) &= \mathcal{L}_s^{-1}\left(U(x,s)\right) \\
&= \mathcal{L}_s^{-1}\left(\int_{-\infty}^{\infty} \frac{e^{-|\xi-x|\sqrt{s}}u(\xi,0)}{2\sqrt{s}}d\xi\right) \\
&= \int_{-\infty}^{\infty} \mathcal{L}_s^{-1}\left(\frac{e^{-|\xi-x|\sqrt{s}}}{2\sqrt{s}}\right)u(\xi,0)d\xi \\
&= \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} e^{-\frac{(\xi-x)^2}{4\tau}}u(\xi,0)d\xi.
\end{aligned}$$

It is convenient to make the change of variable $x_1 = \dfrac{\xi - x}{\sqrt{2\tau}}$, so that

$$\begin{aligned}
u(x,\tau) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x_1\sqrt{2\tau}+x,0)e^{-\frac{x_1^2}{2}}dx_1 \\
&= \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)(x+x_1\sqrt{2\tau})}e^{-\frac{x_1^2}{2}}dx_1}_{I_1} - \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k-1)(x+x_1\sqrt{2\tau})}e^{-\frac{x_1^2}{2}}dx_1}_{I_2}
\end{aligned}$$

$$\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)(x+x_1\sqrt{2\tau})}e^{-\frac{x'^2}{2}}dx_1 \\
&= \frac{e^{\frac{1}{2}(k+1)x}}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{4}(k+1)^2\tau}e^{-\frac{1}{2}\left(x_1-\frac{1}{2}(k+1)\sqrt{2\tau}\right)^2}dx_1 \\
&= \frac{e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}-\frac{1}{2}(k+1)\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\rho^2}d\rho \quad \left[\rho = x_1 - \frac{1}{2}(k+1)\sqrt{2\tau}\right] \\
&= e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}N(d_1).
\end{aligned}$$

where $d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}$, and $N(d_1)$ is the cumulative distribution function for the normal distribution.

Similarly, the procedure to obtain I_2 is the same used to get I_1, only changes $k+1$ to $k-1$. We obtain

$$I_2 = e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau} N(d_2)$$

where $d_2 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}$.

$$\begin{aligned}
v(x,\tau) &= e^{\alpha x+\beta t}u(x,\tau) \\
&= e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau}\left(e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}N(d_1) - e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau}N(d_2)\right) \\
&= e^x N(d_1) - e^{-k\tau}N(d_2) \\
&= \frac{S(t)}{K}N(d_1) - e^{-\frac{2r}{\sigma^2}\frac{\sigma^2}{2}(T-t)}N(d_2) \\
&= \frac{S(t)}{K}N(d_1) - e^{-r(T-t)}N(d_2).
\end{aligned}$$

Theorem 4.22 (Black-Scholes for options on non-dividend paying stock)

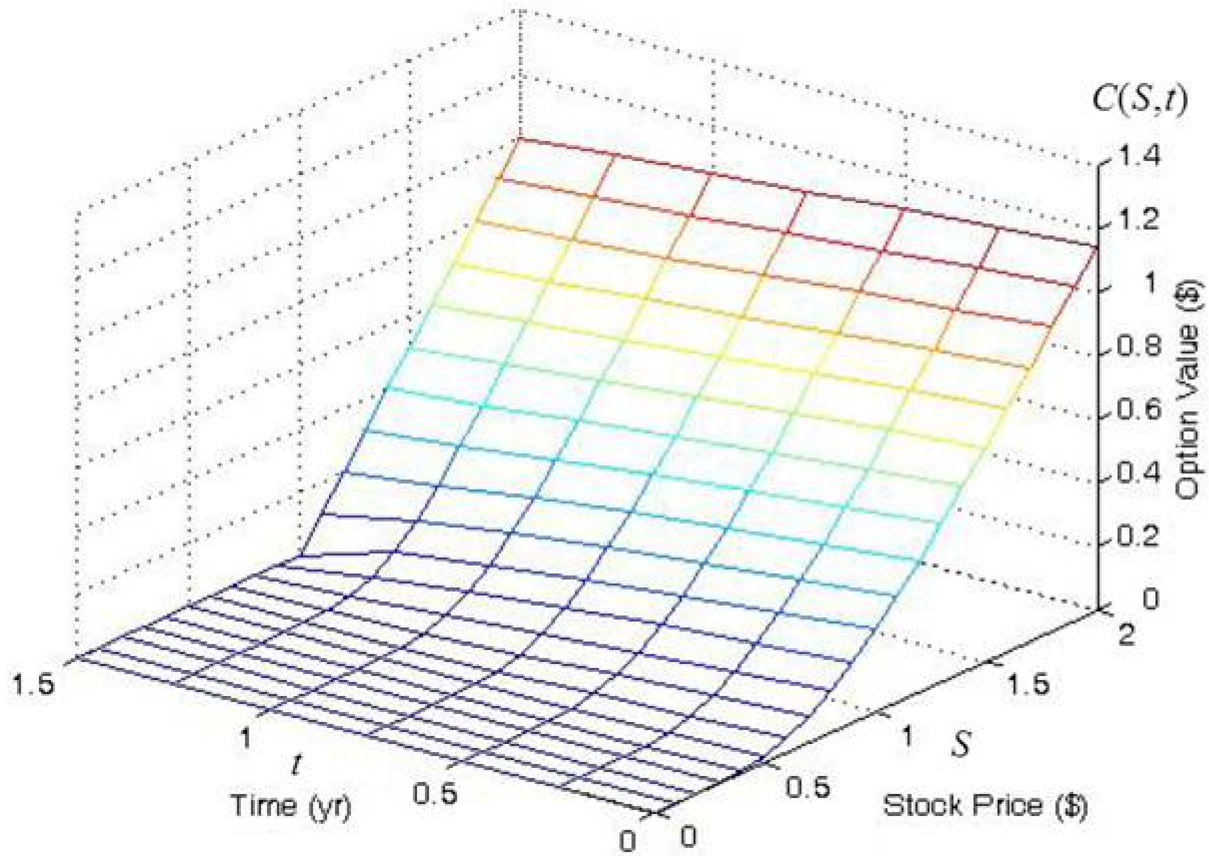
$$C(S,t) = Kv(x,\tau) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$$

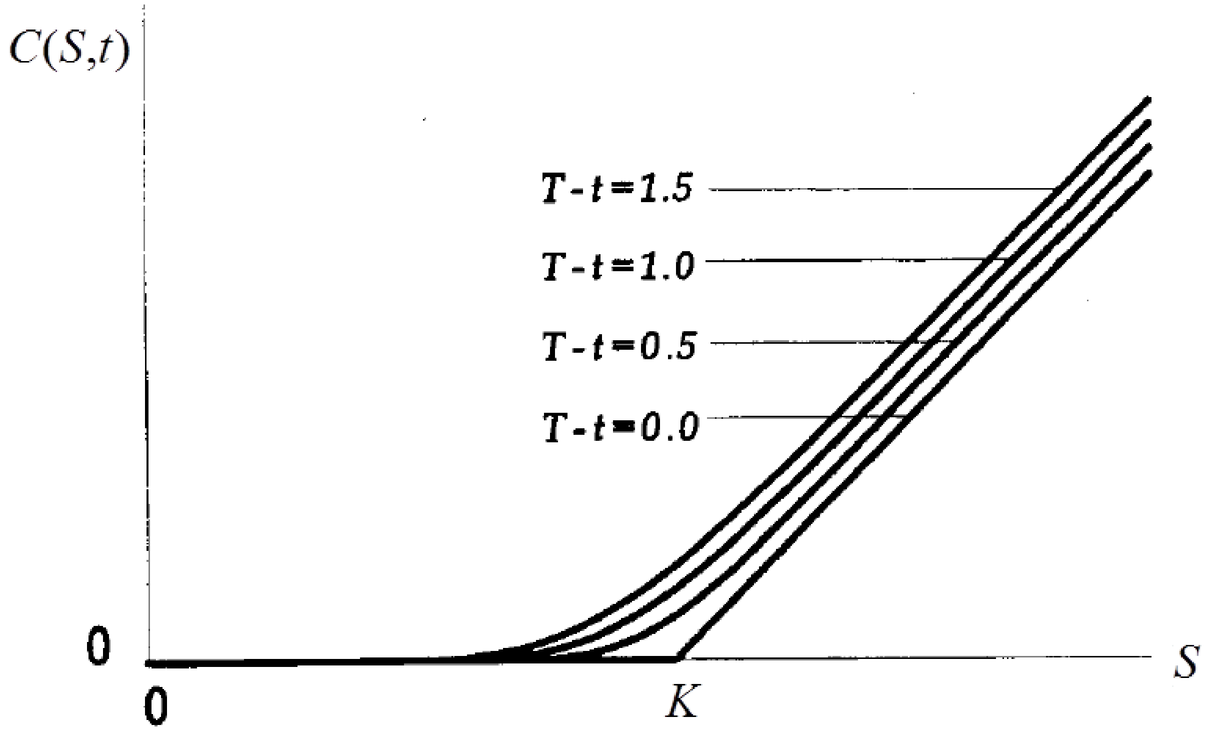
where $d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau} = \frac{\ln\frac{S(t)}{K}}{\sigma\sqrt{T-t}} + \frac{1}{2}\left(\frac{2r}{\sigma^2}+1\right)\sigma\sqrt{T-t} = \frac{\ln\frac{S(t)}{K} + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$

and $d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\frac{S(t)}{K} + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$.

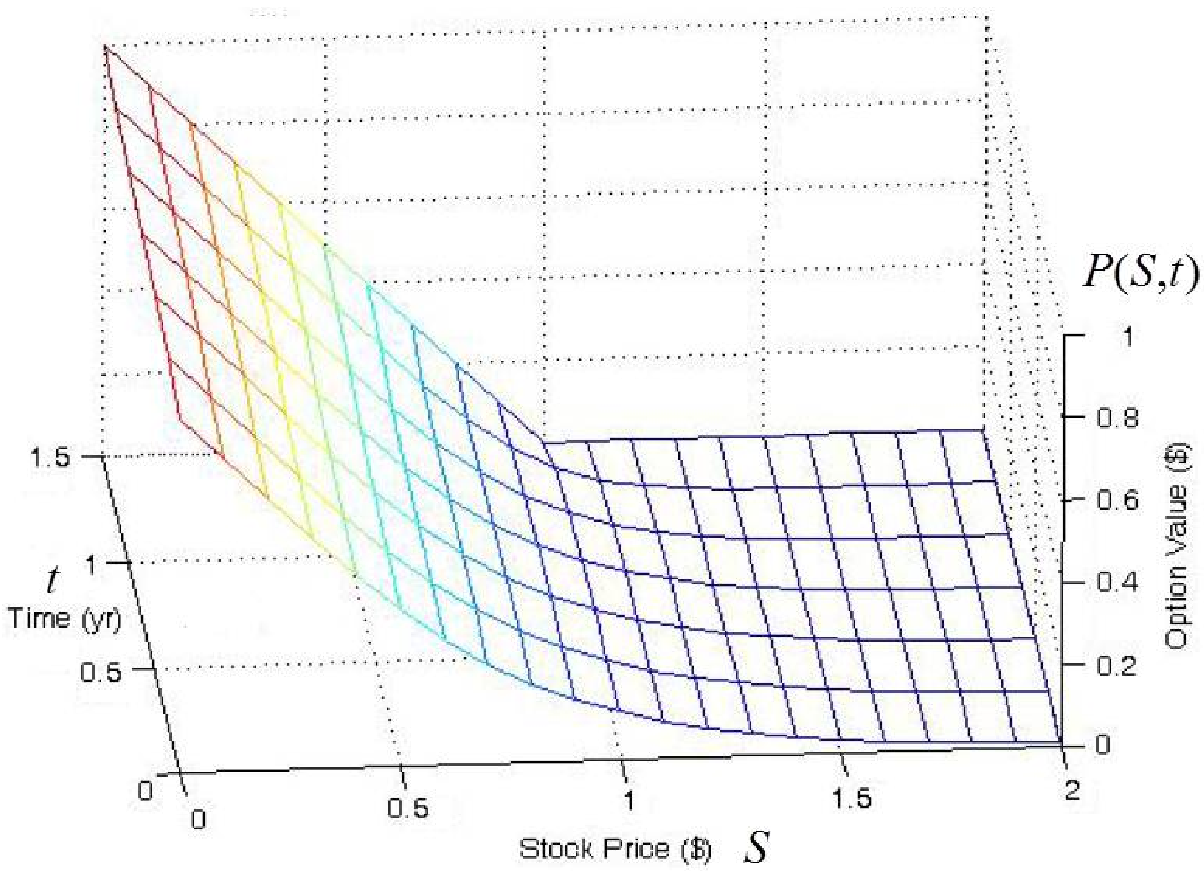
By put-call parity equation,

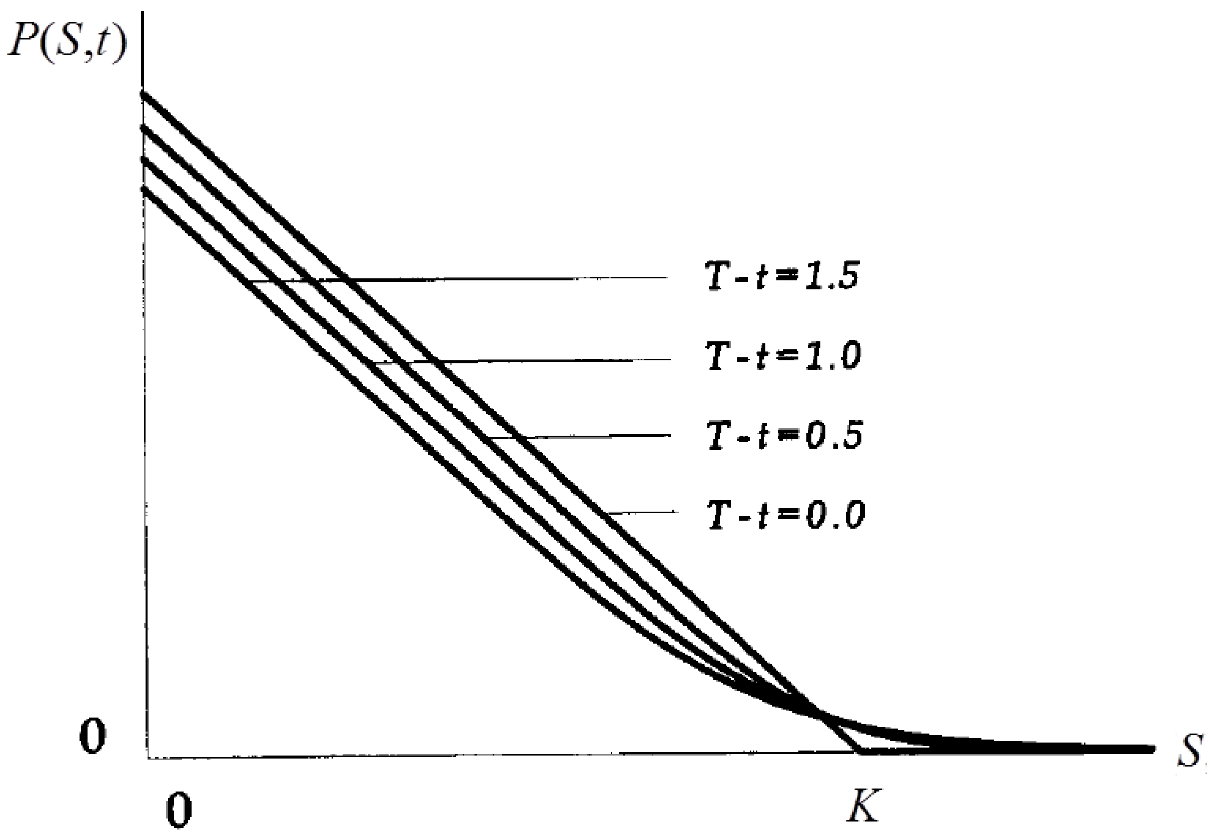
$$\begin{aligned}
P(S,t) &= C(S,t) + Ke^{-r(T-t)} - S(t) \\
&= \left(S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)\right) + Ke^{-r(T-t)} - S(t) \\
&= Ke^{-r(T-t)}N(-d_2) - S(t)N(-d_1).
\end{aligned}$$





European call value $C(S, t)$ as a function of S and t ($r = 0.1$, $\sigma = 0.4$, $K = 1$, $T = 1.5$)





European put value $P(S, t)$ as a function of S and t ($r = 0.1$, $\sigma = 0.4$, $K = 1$, $T = 1.5$)

Suppose the stock S pays continuous dividends with rate δ. In each time step dt, the asset price must fall by the amount of the dividend payment, $\delta S dt$, in addition to the usual oscillations. It follows the random walk for the asset price is modified to

$$dS = \sigma S dZ + (\alpha - \delta) S dt.$$

We construct a portfolio Π consisting of long one option and short Δ of the underlying asset S. Since we hold $-\Delta$ of the underlying asset S, our portfolio changes by an amount

$$-\delta S \Delta dt,$$

i.e., the dividends of S. Thus

$$d\Pi = dV - \Delta dS - \delta S \Delta dt.$$

The analysis proceeds exactly as before but with the addition of an extra term $-\delta S \Delta dt$. We find that

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) S \frac{\partial V}{\partial S} - rV = 0.$$

For a call option, the final condition is still $C(S,T) = \max(S - K, 0)$, and the boundary condition at $S = 0$ remains as $C(0, t) = 0$. The only change to the boundary conditions is

$$C(S,t) \sim Se^{-\delta(T-t)} \quad \text{as} \quad S \to \infty.$$

This is because in the limit $S \to \infty$, the option becomes equivalent to the asset but without its dividend income.

Consider $C_1(S,t) = C(S,t)e^{\delta(T-t)}$. Then $C_1(S,T) = C(S,T) = \max(S-K,0)$, $C_1(0,T) = 0$ and $C_1(S,t) \sim S$ as $S \to \infty$. Then

$$\begin{aligned}
&\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-\delta) S \frac{\partial C}{\partial S} - rC = 0 \\
\Rightarrow\ &\frac{\partial C_1}{\partial t} + \delta C_1 + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_1}{\partial S^2} + (r-\delta) S \frac{\partial C_1}{\partial S} - rC_1 = 0 \\
\Rightarrow\ &\frac{\partial C_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_1}{\partial S^2} + (r-\delta) S \frac{\partial C_1}{\partial S} - (r-\delta) C_1 = 0.
\end{aligned}$$

The value of $C_1(S,t)$ is just a European call on a non-dividend paying stock with interest rate $r-\delta$.

Theorem 4.23 (Black-Scholes for options on stock paying continuous dividend)

$$\begin{aligned}
C(S,t) &= C_1(S,t)e^{-\delta(T-t)} \\
&= e^{-\delta(T-t)}\left(S(t)N(d_1) - Ke^{-(r-\delta)(T-t)}N(d_2)\right) \\
&= e^{-\delta(T-t)}S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \\
P(S,t) &= C(S,t) + Ke^{-r(T-t)} - e^{-\delta(T-t)}S(t) \quad \text{[put-call parity]} \\
&= e^{-\delta(T-t)}S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) + Ke^{-r(T-t)} - e^{-\delta(T-t)}S(t) \\
&= Ke^{-r(T-t)}N(-d_2) - e^{-\delta(T-t)}S(t)N(-d_1)
\end{aligned}$$

where $d_1 = \dfrac{\ln\frac{S(t)}{K} + (r-\delta+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t} = \dfrac{\ln\frac{S(t)}{K} + (r-\delta-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$.

Remark 4.24

In risk neutral world, $\alpha = r$. Then

$$P(S(T)\le K) = P\left(\ln\frac{S(T)}{S(t)} \le \ln\frac{K}{S(t)}\right) = N\left(\frac{\ln(K/S(t)) - \left(\alpha-\delta-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) = N(-d_2),$$

$$P(S(T) > K) = 1 - N(-d_2) = N(d_2).$$

Theorem 4.25 (Black-Scholes for option on currency)

For a currency option, the domestic risk-free rate is *r* and the foreign risk-free rate is r_f. The values for a call and a put option are

$$C = x(t)e^{-r_f(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2), \quad P = Ke^{-r(T-t)}N(-d_2) - x(t)e^{-r_f(T-t)}N(-d_1)$$

where $d_1 = \dfrac{\ln\dfrac{x(t)}{K} + (r-r_f+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t} = \dfrac{\ln\dfrac{x(t)}{K} + (r-r_f-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$

and $x(t)$ is the current exchange rate, the number of units of domestic currency per unit of foreign currency.

Example 4.26

The currency exchange rate for yen in US dollars is 0.009, $\sigma = 0.05$. The continuously compounded risk-free rate for yen and US dollars is 2% and 4% respectively. Calculate the price for a 1-year European dollar-denominated call option on yen with a strike price of 0.01.

Solution

$$d_1 = \frac{\ln(0.009/0.01) + 0.04 - 0.02 + \frac{1}{2}(0.05^2)}{0.05} = -1.68221, \quad N(d_1) = 0.04626$$
$$d_2 = -1.68221 - 0.05 = -1.73221, \quad N(d_2) = 0.04162$$
$$C = 0.009e^{-0.02}(0.04626) - 0.01e^{-0.04}(0.04162) = 0.0000082.$$

Suppose a stock $S(t)$ pays a continuous dividends at a rate δ. The risk free interest rate is r. In Example 4.14, the future price of the stock at t with T fixed is

$$F_{t,T}(S(T)) = S(t)e^{(r-\delta)(T-t)}$$

and

$$\frac{dF_{t,T}}{F_{t,T}} = \sigma dZ + \big((\alpha - \delta) - (r - \delta)\big)dt = \sigma dZ + (\alpha - r)dt.$$

We have $(dF_{t,T})^2 = \sigma^2 F_{t,T}^2 dt$. Options on futures have a value that depends on $F_{t,T}$ and t, i.e., of the form $V(F_{t,T}, t)$. By Ito's Lemma, it gives

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2}\right)dt + \frac{\partial V}{\partial F}dF.$$

We take the usual hedge portfolio $\Pi = V - \Delta F$. The choice $\Delta = \partial V/\partial F$ makes the portfolio instantaneously risk-free, so

$$d\Pi = dV - \Delta dF = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2}\right)dt + \frac{\partial V}{\partial F}dF - \frac{\partial V}{\partial F}dF = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2}\right)dt.$$

The cost of setting up the portfolio Π is just V, since it costs nothing to enter into a futures contract. Therefore, $d\Pi = rVdt$, and we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV = 0.$$

Since the equation is identical to the Black-Scholes equation when the asset pays dividends at a continuous risk-free rate r, we have the following theorem:

Theorem 4.27 (Black-Scholes for options on Future)

Let F be the price of future contract on an underlying asset S, C the price of call option on future contract, P the price of put option on future contract, T_1 the duration to expiry of the option ($\le T_2$, the duration to expiry of the future), K the strike price, r the risk-free rate and σ the volatility of the future. Then

$$C = F(t)e^{-r(T_1-t)}N(d_1) - Ke^{-r(T_1-t)}N(d_2), \quad P = Ke^{-r(T_1-t)}N(-d_2) - F(t)e^{-r(T_1-t)}N(-d_1)$$

where $F(t) = S(t)e^{(r-\delta)(T_2-t)}$,

$$d_1 = \frac{\ln(F(t)/K) + \frac{1}{2}\sigma^2(T_1-t)}{\sigma\sqrt{T_1-t}}, \; d_2 = d_1 - \sigma\sqrt{T_1-t} = \frac{\ln(F(t)/K) - \frac{1}{2}\sigma^2(T_1-t)}{\sigma\sqrt{T_1-t}}.$$

Example 4.28

The current stock price is 40. The stock pays dividends proportional to its price at a continuous rate of 0.02. The annual volatility of the stock is 0.3. The continuously compounded risk-free rate is 0.06. Determine the premium for a 3-month European put option on a 1-year futures contract on the stock with strike price 45.

Solution

The 1-year futures price of the stock is $F = S(0)e^{(r-\delta)T_2} = 40e^{(0.06-0.02)(1)} = 41.6324.$

For the put premium, use $T_1 = 0.25,$ the period of the option.

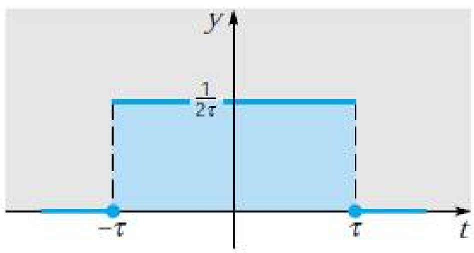
$$d_1 = \frac{\ln(41.6324/45) + \frac{1}{2}(0.3)^2(0.25)}{0.3\sqrt{0.25}} = -0.44356, \quad N(-d_1) = N(0.44356) = 0.67132$$
$$d_2 = -0.44356 - 0.3\sqrt{0.25} = -0.59356, \quad N(-d_2) = N(0.59356) = 0.72360$$
$$P = 45e^{-0.06(0.25)}(0.72360) - 41.6324e^{-0.06(0.25)}(0.67132) = 4.54.$$

Let us value a European call $C_d(S,t)$ and put $P_d(S,t)$ with one dividend payment of dividend yield D at time t_d. A discrete dividend payment results in a jump in the value of the underlying asset across the dividend date.

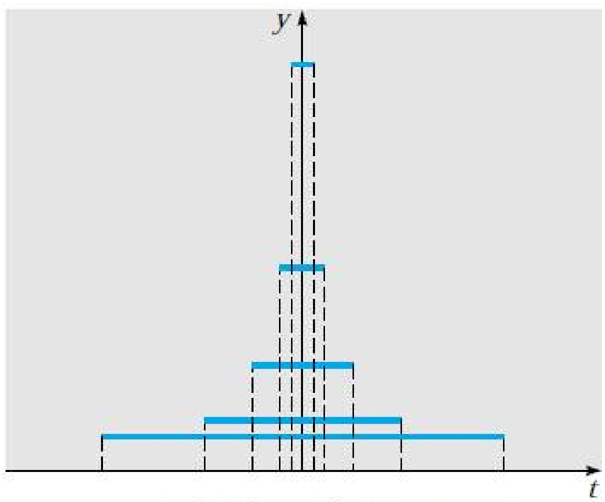
Let $d_\tau(t) = \begin{cases} 1/2\tau & -\tau < t < \tau \\ 0 & \text{otherwise.} \end{cases}$ Then $\int_{-\infty}^{\infty} d_\tau(t)dt = \int_{-\tau}^{\tau} \frac{1}{2\tau}dt = 1.$

Define delta "function" by $\delta(t) = \lim_{\tau \to 0^+} d_\tau(t).$ We have

$$\delta(t) = \lim_{\tau \to 0^+} d_\tau(t) = 0 \quad \text{for } t \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1.$$



Graph of $y = d_\tau(t)$.



Graphs of $y = d_\tau(t)$ as $\tau \to 0$.

The modified stochastic differential equation for discrete dividend payment is

$$dS(t) = \sigma S(t)dZ(t) + (\alpha S(t) - DS(t)\delta(t - t_d))dt,$$
$$d\ln S(t) = \sigma dZ(t) + \left(\alpha - \frac{1}{2}\sigma^2 - D\delta(t - t_d)\right)dt.$$

Integrating across the dividend date, we find that

$$\int_{t_d^-}^{t_d^+} d\ln S(t) = \int_{t_d^-}^{t_d^+} \sigma dZ(t) + \int_{t_d^-}^{t_d^+} \left(\alpha - \frac{1}{2}\sigma^2\right)dt - D\int_{t_d^-}^{t_d^+} \delta(t - t_d)dt.$$

Since t_d^- and t_d^+ differ only infinitesimally, the only nonzero term on the right-hand side is the one containing the delta function and hence we obtain

$$\ln\left(\frac{S(t_d^+)}{S(t_d^-)}\right) = \int_{S(t_d^-)}^{S(t_d^+)} d\ln S(t) = -D\int_{t_d^-}^{t_d^+} \delta(t - t_d)dt = -D,$$
$$S(t_d^+) = S(t_d^-)e^{-D}.$$

Thus $S(t) = \begin{cases} S(0)e^{(\alpha - 0.5\sigma^2)t + \sigma Z(t)} & t < t_d, \\ S(0)e^{(\alpha - 0.5\sigma^2)t + \sigma Z(t) - D} & t \geq t_d. \end{cases}$

Across the dividend date, the value of the asset changes discontinuously. However the option holder does not receive the dividend. The value of the option must be continuous as a function of the time across the dividend date, that is

$$V(S(t_d^-), t_d^-) = V(S(t_d^+), t_d^+) = V(S(t_d^-)e^{-D}, t_d^+).$$

At expiration date T, $C(e^{-D}S(0), T, K) = \max\{e^{-D}S(T) - K, 0\}$ and $P(e^{-D}S(0), T, K) = \max\{K - e^{-D}S(T), 0\}$. Thus we have the following theorem:

Theorem 4.29 (Black-Scholes for options on stock paying discrete dividend of known yield)

Suppose a stock pays one dividend of yield D at time t_d. Then a European call $C_d(S, t)$ and put $P_d(S, t)$ with strike price K is

$$C_d(S(0), K) = C(e^{-D}S(0), K) \quad \text{and} \quad P_d(S(0), K) = P(e^{-D}S(0), K).$$

Example 4.30 (Black-Scholes for options on stock paying discrete unknown dividends)

Consider a European call option on a stock when there are ex-dividend dates in two months and five months. The dividend rate on ex-dividend date in two months is 2% and five months is 3%. The current share price is \$40, the exercise price is \$40, the stock price volatility is 30% per annum, the risk-free rate of interest is 9% per annum, and the time to maturity is six months. The option price can therefore be calculated from the Black-Scholes formula, with

$e^{-(0.02+0.03)}S(0) = e^{-0.05}40 = 38.04917698,\ \ K = 40,\ r = 0.09,\ \ \sigma = 0.3,$ and $T = 0.5$:

$$d_1 = \frac{\ln(38.0491769800285604/40) + (0.09 + 0.3^2/2)\times 0.5}{0.3\sqrt{0.5}} = 0.0824957911384305$$
$$d_2 = \frac{\ln(38.0491769800285604/40) + (0.09 - 0.3^2/2)\times 0.5}{0.3\sqrt{0.5}} = -0.1296362432175337$$

gives

$$N(d_1) = 0.53287 \quad \text{and} \quad N(d_2) = 0.448427.$$

The call price is

$$C = e^{-0.05}S(0)N(d_1) - Ke^{-rT}N(d_2) = 38.04917698(0.53287) - 40e^{-0.09\times 0.5}(0.448427) = 3.12746.$$

Theorem 4.31 (General Black-Scholes Formula)

In general, we have

$$C = F_{t,T}^P(S)N(d_1) - F_{t,T}^P(K)N(d_2), \quad P = F_{t,T}^P(K)N(-d_2) - F_{t,T}^P(S)N(-d_1)$$

where

$$d_1 = \frac{\ln\left(F_{t,T}^P(S)/F_{t,T}^P(K)\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(F_{t,T}^P(S)/F_{t,T}^P(K)\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

You may refer to Example 3.13 as Black-Scholes for options on stock paying discrete known dividends.

Start with option prices and a pricing model (such as Black-Scholes), and back out σ from the option prices. Volatility estimated in this way is called **implied volatility**.

Example 4.36

The price of a non-dividend paying stock is 50. The expiration date and strike price of a European call is t and $50e^{0.04t}$ respectively. The continuously compounded risk-free rate is 0.04.

Time to expiry	Option Price
3 months	3.98
1 year	5.96
2 years	7.14

Calculate the implied volatilities of options for these 3 periods using Black-Scholes model.

Solution

For all options:

$$d_1 = \frac{\ln(50/50e^{0.04t}) + 0.04t + 0.5\sigma^2 t}{\sigma\sqrt{t}} = 0.5\sigma\sqrt{t}$$
$$d_2 = d_1 - \sigma\sqrt{t} = -0.5\sigma\sqrt{t} = -d_1$$
$$C(S, K, t) = 50N(d_1) - 50e^{0.04t}e^{-0.04t}N(d_2) = 50N(d_1) - 50\left(1 - N(d_1)\right) = 100N(d_1) - 50.$$

For a 3-month option, $t = 0.25$.

$$100N(d_1) - 50 = 3.98, \quad N(d_1) = 0.5398, \quad d_1 = 0.1, \quad 0.5\sigma\sqrt{0.25} = 0.1, \quad \sigma = 0.4.$$

For a 1-year option, $t = 1$.

$$100N(d_1) - 50 = 5.96, \quad N(d_1) = 0.5596, \quad d_1 = 0.15, \quad 0.5\sigma = 0.15, \quad \sigma = 0.3.$$

For a 2-year option, $t = 2$.

$$100N(d_1) - 50 = 7.14, \quad N(d_1) = 0.5714, \quad d_1 = 0.18, \quad 0.5\sigma\sqrt{2} = 0.18, \quad \sigma = 0.255.$$

Chapter 5 Greek Symbols and Hedging

Definition 5.1

Delta (Δ) of an option is defined as the rate of change of the option price with respect to the price of the underlying asset. It is the slope of the curve that relates the option price to the underlying asset price. In general,

$$\Delta = \frac{\partial V}{\partial S}$$

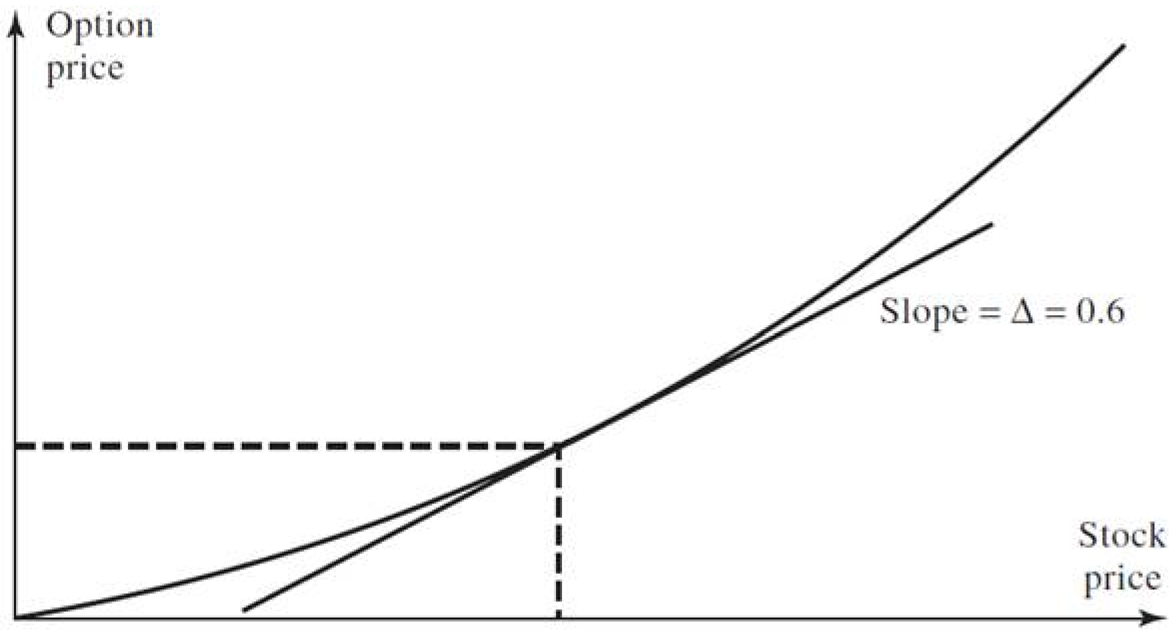
where V is the price of option and S is the stock price.

The Black-Scholes differential equation can be derived by setting up a riskless portfolio consisting of a position in an option on a stock and a position in the stock. Expressed in terms of Δ, the portfolio is

$$-1 \text{ option and } \Delta \text{ shares of the stock.}$$

Suppose that the delta of a call option on a stock is 0.6. This means that when the stock price changes by a small amount, the option price changes by about 60% of that amount. Imagine an investor who has sold 20 call options—that is, options on 20 shares of stock. The investor's position could be hedged by buying $0.6 \times 20 = 12$ shares of stock. The gain (loss) on the stock position would then tend to offset the loss (gain) on the option position. For example, if the stock price goes up by \$1 (producing a gain of \$12 on the stocks purchased), the option price will tend to go up by $0.6 \times \$1 = \$0.6$ (producing a loss of $20 \times \$0.6 = \$12$ to the options writer); if the stock price goes down by \$1 (producing a loss of \$12 on the shares purchased), the option price will tend to go down by \$0.60 (producing a gain of \$12 to the options writer).

The delta of the trader's short position in 20 options is $0.6 \times (-20) = -12.$ The delta of 12 share of the stock is 12. The delta of the trader's overall position is, therefore, 0. The delta of the stock position offsets the delta of the option position. A position with a delta of zero is referred to as delta neutral.



Recall that Black-Scholes formulas for the prices of a European call option and a European put option on a dividend-paying stock with a dividend yield at rate δ are

$$C = S(t)e^{-\delta\tau}N(d_1) - Ke^{-r\tau}N(d_2) \quad \text{and} \quad P = Ke^{-r\tau}N(-d_2) - S(t)e^{-\delta\tau}N(-d_1)$$

where all symbols follow usual notation, $\tau = T - t$, the time of expiry,

$$d_1 = \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{\tau} = \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r - \delta - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

We want to know the rate of change of option price with respect to stock price, time, volatility, risk-free interest rate and dividend rate. To make the following derivatives more easily, we need Theorem 5.2 in advanced.

Theorem 5.2

$$S(t)e^{-\delta\tau}N'(d_1) = Ke^{-r\tau}N'(d_2) \quad \text{where} \quad N'(d_1) = \frac{\partial N(d_1)}{\partial d_1} \quad \text{and} \quad N'(d_2) = \frac{\partial N(d_2)}{\partial d_2}.$$

Proof

$$
\begin{aligned}
N'(d_1) &= \frac{\partial N(d_1)}{\partial d_1} = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}, \\
N'(d_2) &= \frac{\partial N(d_2)}{\partial d_2} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(d_1 - \sigma\sqrt{\tau}\right)^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} e^{d_1\sigma\sqrt{\tau}} e^{-\frac{\sigma^2\tau}{2}} \\
&= N'(d_1) e^{\ln\left(\frac{S(t)}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)\tau} e^{-\frac{\sigma^2\tau}{2}} \\
&= N'(d_1) \frac{S(t)}{K} e^{(r-\delta)\tau}.
\end{aligned}
$$

Hence $S(t)e^{-\delta\tau}N'(d_1) = Ke^{-r\tau}N'(d_2).$

Theorem 5.3

For a dividend-paying stock with a continuous dividend yield at rate δ, delta of European call and put option are

$$
\Delta_C = e^{-\delta\tau}N(d_1) > 0 \qquad \text{and} \qquad \Delta_P = e^{-\delta\tau}\left(N(d_1) - 1\right) = -e^{-\delta\tau}N(-d_1) < 0.
$$

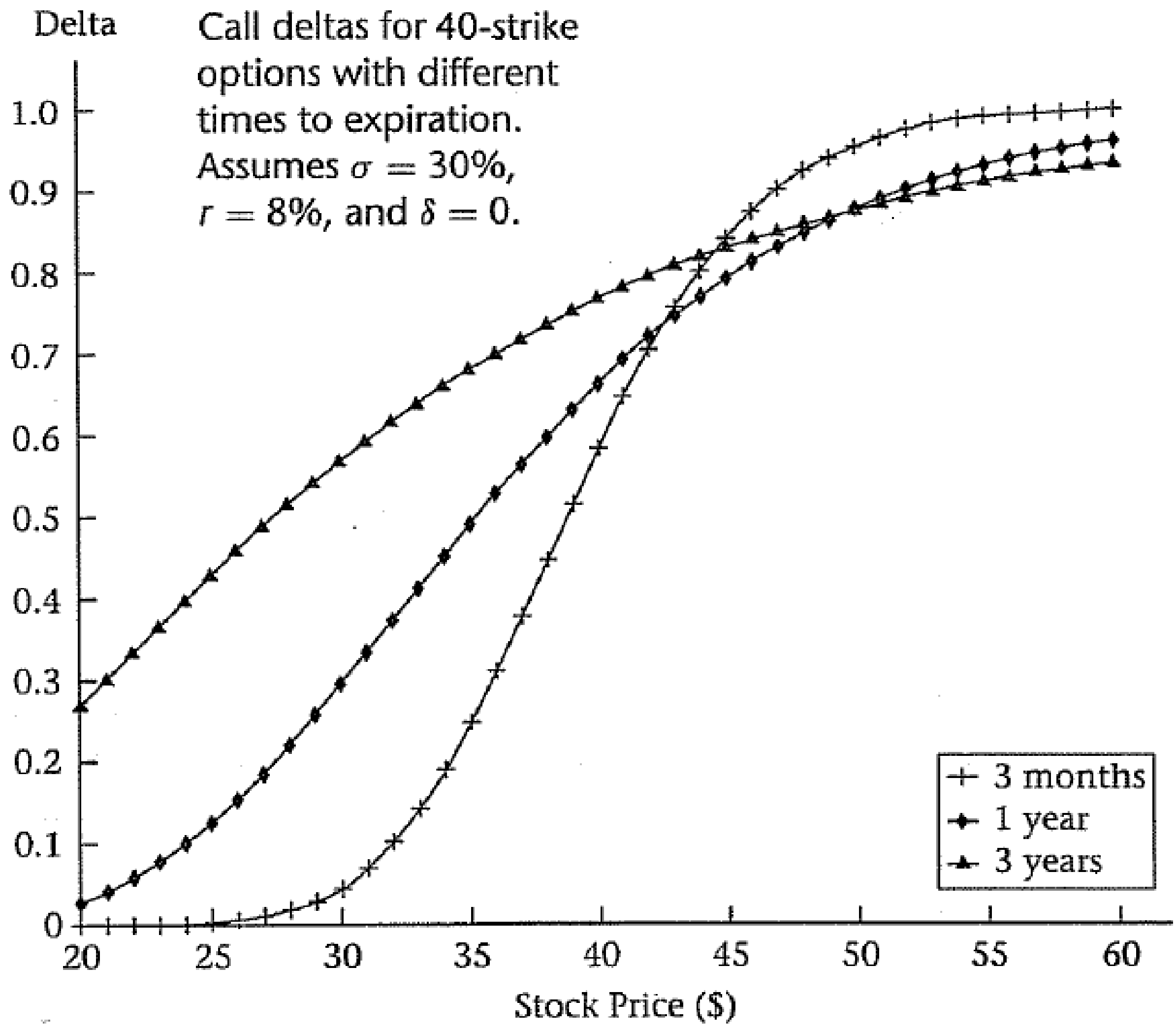
Proof

$$
\begin{aligned}
\Delta_C &= \frac{\partial C}{\partial S} \\
&= e^{-\delta\tau}N(d_1) + S(t)e^{-\delta\tau}\frac{\partial N(d_1)}{\partial S} - Ke^{-r\tau}\frac{\partial N(d_2)}{\partial S} \\
&= e^{-\delta\tau}N(d_1) + S(t)e^{-\delta\tau}\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} - Ke^{-r\tau}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S} \\
&= e^{-\delta\tau}N(d_1) + S(t)e^{-\delta\tau}N'(d_1)\frac{1}{S(t)\sigma\sqrt{\tau}} - Ke^{-r\tau}N'(d_2)\frac{1}{S(t)\sigma\sqrt{\tau}} \\
&= e^{-\delta\tau}N(d_1).
\end{aligned}
$$

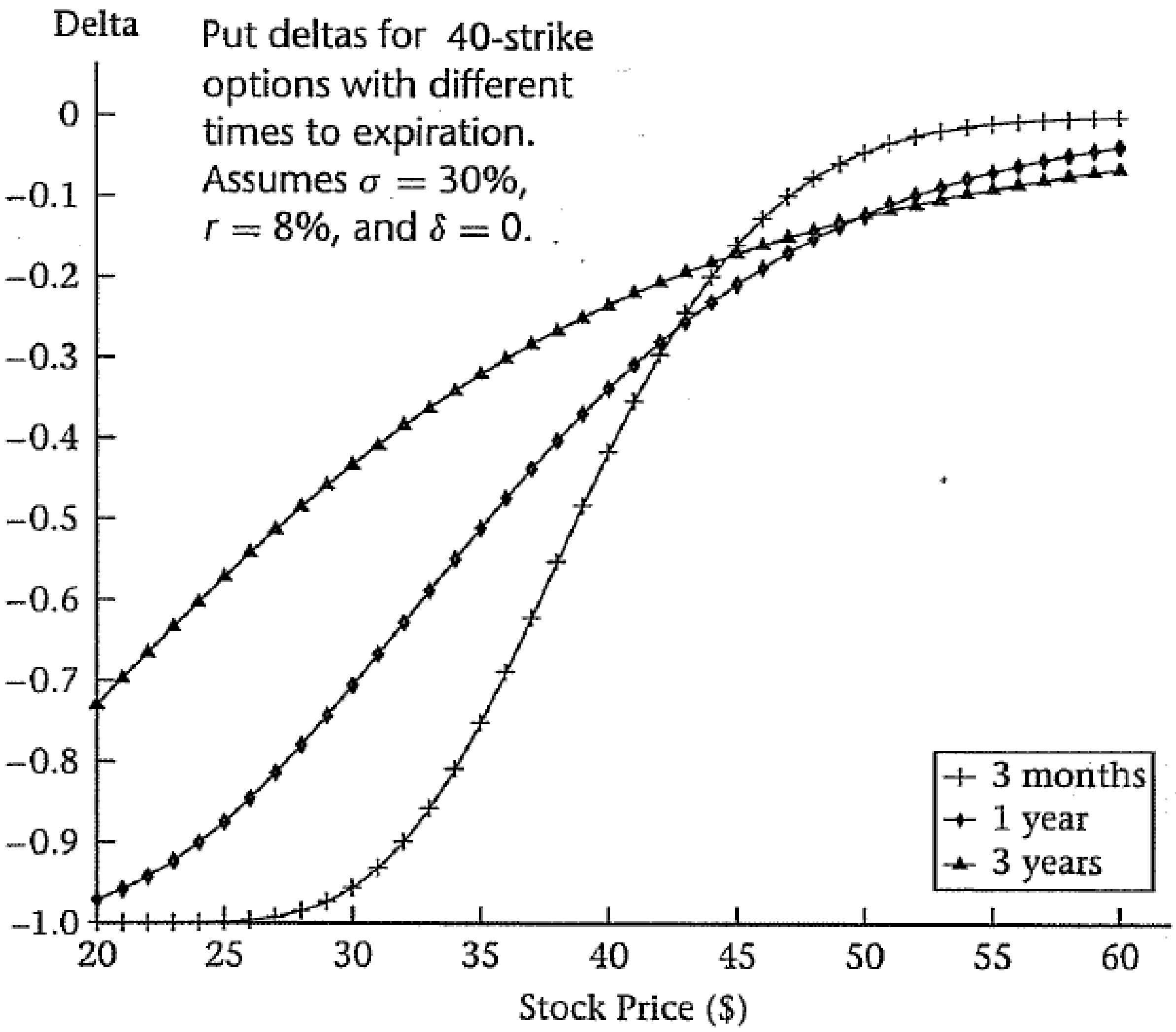
By put-call parity equation,

$$\Delta_P = \frac{\partial P}{\partial S} = \frac{\partial(C - e^{-\delta\tau}S(t) + Ke^{-r\tau})}{\partial S} = e^{-\delta\tau}N(d_1) - e^{-\delta\tau} = e^{-\delta\tau}(N(d_1) - 1).$$

For a non-dividend paying stock, $\Delta_C = N(d_1)$ and $\Delta_P = N(d_1) - 1 = -N(-d_1)$.



The above figure represents the behavior of delta for 3 options with different times to expiration. It illustrates that an in-the-money option will be more sensitive to the stock than an out-of-the-money option. If an option is deep in-the-money, it is likely to be exercise and hence the option should behave much like a leverage position in a full share. Delta approaches 1 in this case and the option is equivalent to 1 share of stock. If the option is out-of-the-money, it is unlikely to be exercised and the option has a low price, behaving like a position with very few shares. In this case delta is approximately 0 and equivalent to 0 share of stock.

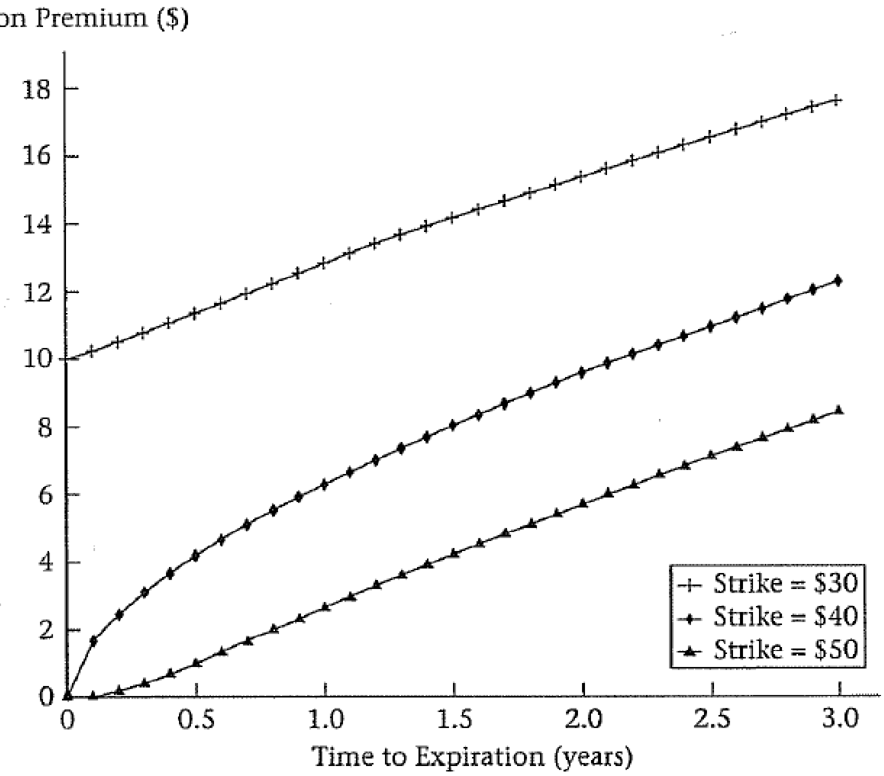


If a call option is exercised at some future time, the payoff will be the amount by which the stock price exceeds the strike price. Call options therefore become more valuable as the stock price increases and less valuable as the strike price increases. For a put option, the payoff on exercise is the amount by which the strike price exceeds the stock price. Put options therefore behave in the opposite way from call options: they become less valuable as the stock price increases and more valuable as the strike price increases.

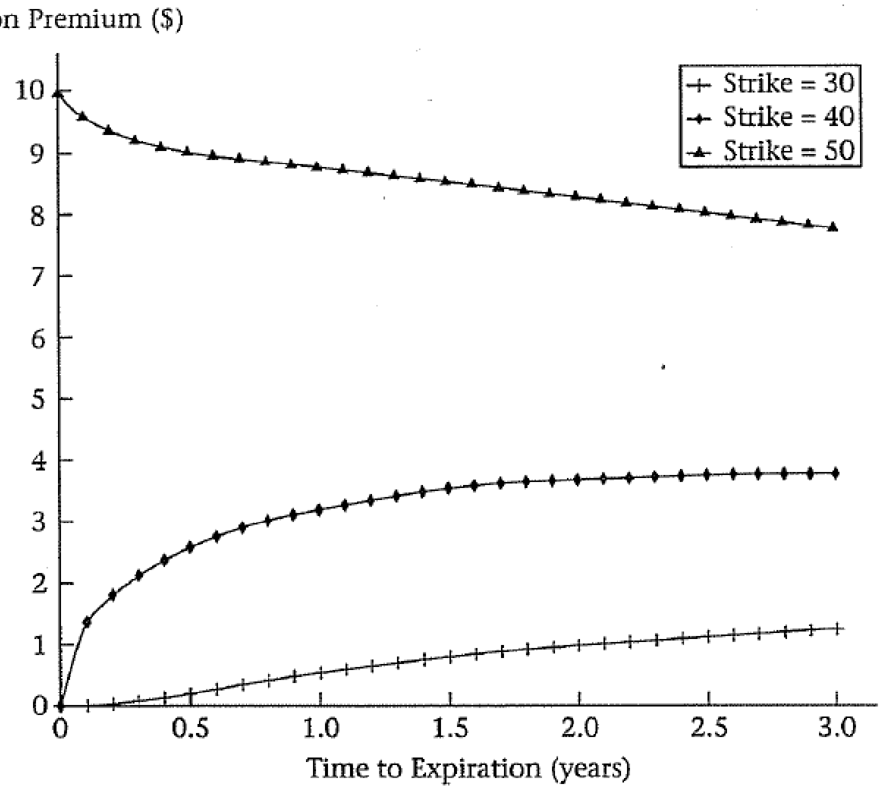
Theorem 5.4

For a dividend-paying stock with a continuous dividend yield at rate δ,

$$\frac{\partial C}{\partial K} = -e^{-r\tau} N(d_2) < 0 \qquad \text{and} \qquad \frac{\partial P}{\partial K} = e^{-r\tau} N(-d_2) > 0.$$



Call prices for options with different strikes at different times to expiration. Assumes $S = \$40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.



Put prices for options with different strikes at different times to expiration. Assumes $S = \$40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.

Proof

$$
\begin{aligned}
\frac{\partial C}{\partial K} &= S(t)e^{-\delta\tau}\frac{\partial N(d_1)}{\partial K} - e^{-r\tau}N(d_2) - Ke^{-r\tau}\frac{\partial N(d_2)}{\partial K} \\
&= S(t)e^{-\delta\tau}\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial K} - e^{-r\tau}N(d_2) - Ke^{-r\tau}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial K} \\
&= S(t)e^{-\delta\tau}N'(d_1)\frac{1}{\sigma\sqrt{\tau}}\left(-\frac{1}{K}\right) - e^{-r\tau}N(d_2) - Ke^{-r\tau}N'(d_2)\frac{1}{\sigma\sqrt{\tau}}\left(-\frac{1}{K}\right) \\
&= -e^{-r\tau}N(d_2).
\end{aligned}
$$

By put-call parity equation,

$$
\frac{\partial P}{\partial K} = \frac{\partial(C - e^{-\delta\tau}S(t) + Ke^{-r\tau})}{\partial K} = \frac{\partial C}{\partial K} + e^{-r\tau} = e^{-r\tau} - e^{-r\tau}N(d_2) = e^{-r\tau}N(-d_2).
$$

(refer to Theorem 3.33 and 3.37)

Definition 5.5

The gamma of an option, Γ, is defined as the rate of change of delta respected to the rate of change of underlying asset price:

$$
\Gamma = \frac{\partial\Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}
$$

where V is the option price and S is the underlying asset price.

Theorem 5.6

For a dividend-paying stock with a continuous dividend yield at rate δ, gamma of European call and put option are

$$
\Gamma_C = \Gamma_P = \frac{e^{-\delta\tau}}{S(t)\sigma\sqrt{\tau}}N'\left(d_1\right) > 0.
$$

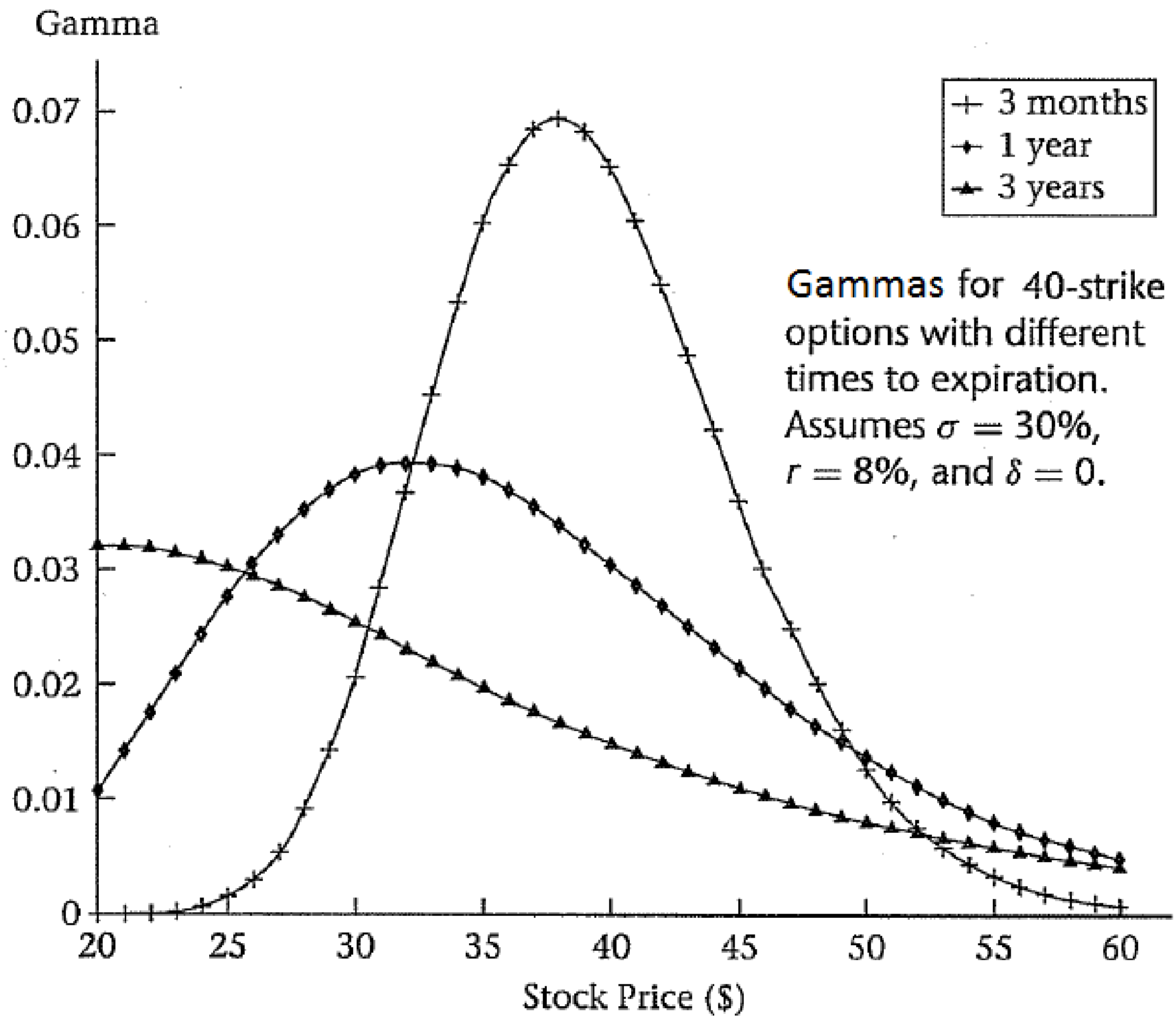
Proof

$$
\Gamma_C = \frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial C}{\partial S}\right) = \frac{\partial\left(e^{-\delta\tau}N(d_1)\right)}{\partial S} = e^{-\delta\tau}\frac{\partial N\left(d_1\right)}{\partial d_1}\frac{\partial d_1}{\partial S} = \frac{e^{-\delta\tau}}{S(t)\sigma\sqrt{\tau}}N'\left(d_1\right).
$$

By put-call parity equation,

$$\Gamma_P = \frac{\partial^2 P}{\partial S^2} = \frac{\partial^2 (C - e^{-\delta\tau} S(t) + Ke^{-r\tau})}{\partial S^2} = \frac{\partial^2 C}{\partial S^2} = \frac{e^{-\delta\tau}}{S(t)\sigma\sqrt{\tau}} N'(d_1).$$

For a non-dividend paying stock, $\Gamma_C = \Gamma_P = \dfrac{1}{S(t)\sigma\sqrt{\tau}} N'(d_1) = \dfrac{1}{S(t)\sigma\sqrt{2\pi\tau}} e^{-\frac{d_1^2}{2}}$.



Gamma of a call or put will not change too much if the option is deep in-the-money or deep out-of-the-money. It is because little change of stock price will not change too much the number of share of stock equivalent to the option. On the other hand, the value of gamma attains its maximum when the option is at the money and short expiration date since the number of share of stock equivalent to the option vary more intensely (from 0 to 1 share) with respect to the change of S when the option is at the money.

Definition 5.7

The theta of an option, θ, is defined as the rate of change of the option price respected to the passage of time:

$$\theta = \frac{\partial V}{\partial t}$$

where V is the option price and t is the passage of time.

If $\tau = T - t$, theta θ can also be defined as minus one times the rate of change of the option price respected to time to maturity. The derivation of such transformation is easy and straight forward:

$$\theta = \frac{\partial V}{\partial t} = \frac{\partial V}{\partial \tau}\frac{\partial \tau}{\partial t} = (-1)\frac{\partial V}{\partial \tau}$$

where $\tau = T - t$ is time to maturity.

Theorem 5.8

For a dividend-paying stock with a continuous dividend yield at rate δ, theta of European call and put option are

$$\begin{aligned}\theta_C &= \delta S(t)e^{-\delta\tau}N(d_1) - \frac{S(t)e^{-\delta\tau}\sigma}{2\sqrt{\tau}}N'(d_1) - rKe^{-r\tau}N(d_2),\\ \theta_P &= rKe^{-r\tau}N(-d_2) - \delta S(t)e^{-\delta\tau}N(-d_1) - \frac{S(t)e^{-\delta\tau}\sigma}{2\sqrt{\tau}}N'(d_1).\end{aligned}$$

Proof

$$\begin{aligned}\theta_C &= -\frac{\partial C}{\partial \tau}\\ &= \delta S(t)e^{-\delta\tau}N(d_1) - S(t)e^{-\delta\tau}\frac{\partial N(d_1)}{\partial \tau} + \left(-r\right)Ke^{-r\tau}N(d_2) + Ke^{-r\tau}\frac{\partial N(d_2)}{\partial \tau}\\ &= \delta S(t)e^{-\delta\tau}N(d_1) - S(t)e^{-\delta\tau}\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial \tau} - rKe^{-r\tau}N(d_2) + Ke^{-r\tau}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial \tau}\end{aligned}$$

$$= \delta S(t)e^{-\delta\tau}N(d_1) - S(t)e^{-\delta\tau}N'(d_1)\left(-\frac{\ln\left(\frac{S(t)}{K}\right)}{2\sigma\tau^{3/2}} + \frac{r-\delta+\frac{\sigma^2}{2}}{2\sigma\sqrt{\tau}} \right)$$

$$-rKe^{-r\tau}N(d_2) + Ke^{-r\tau}N'(d_2)\left(-\frac{\ln\left(\frac{S(t)}{K}\right)}{2\sigma\tau^{3/2}} + \frac{r-\delta-\frac{\sigma^2}{2}}{2\sigma\sqrt{\tau}} \right)$$

$$= \delta S(t)e^{-\delta\tau}N(d_1) - S(t)e^{-\delta\tau}N'(d_1)\frac{\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}N(d_2)$$

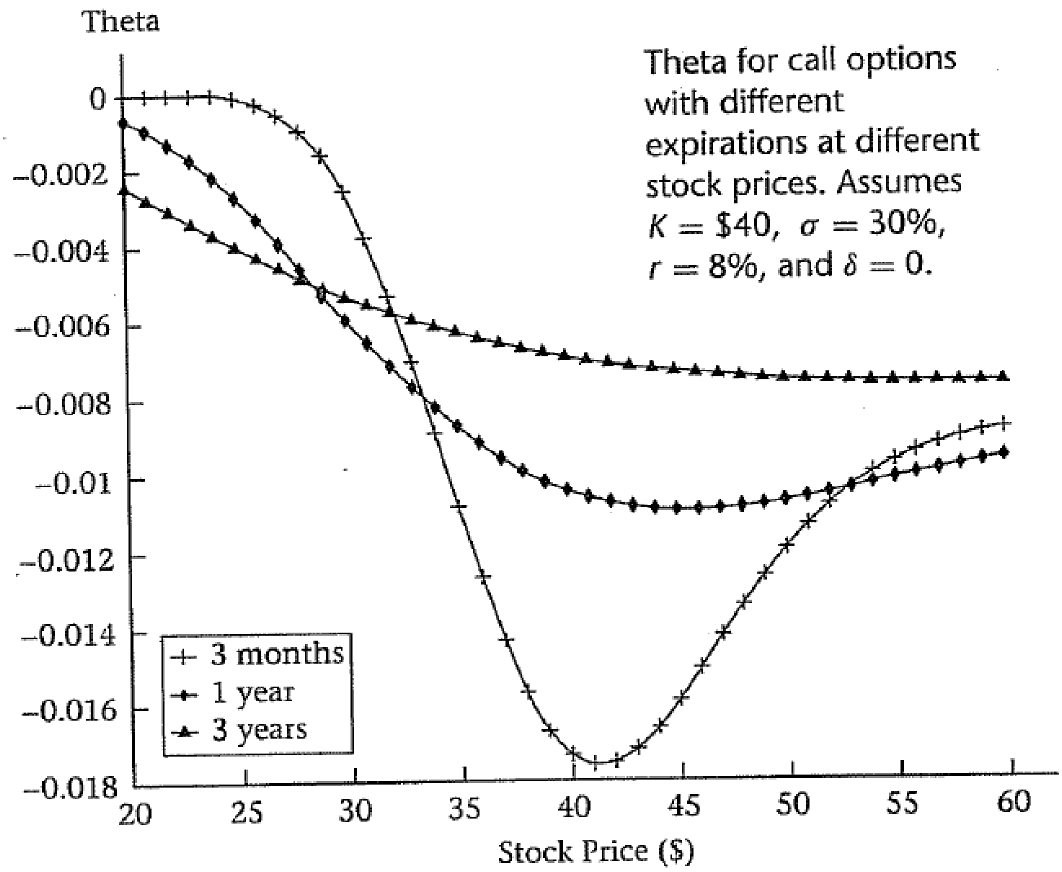
By put-call parity equation,

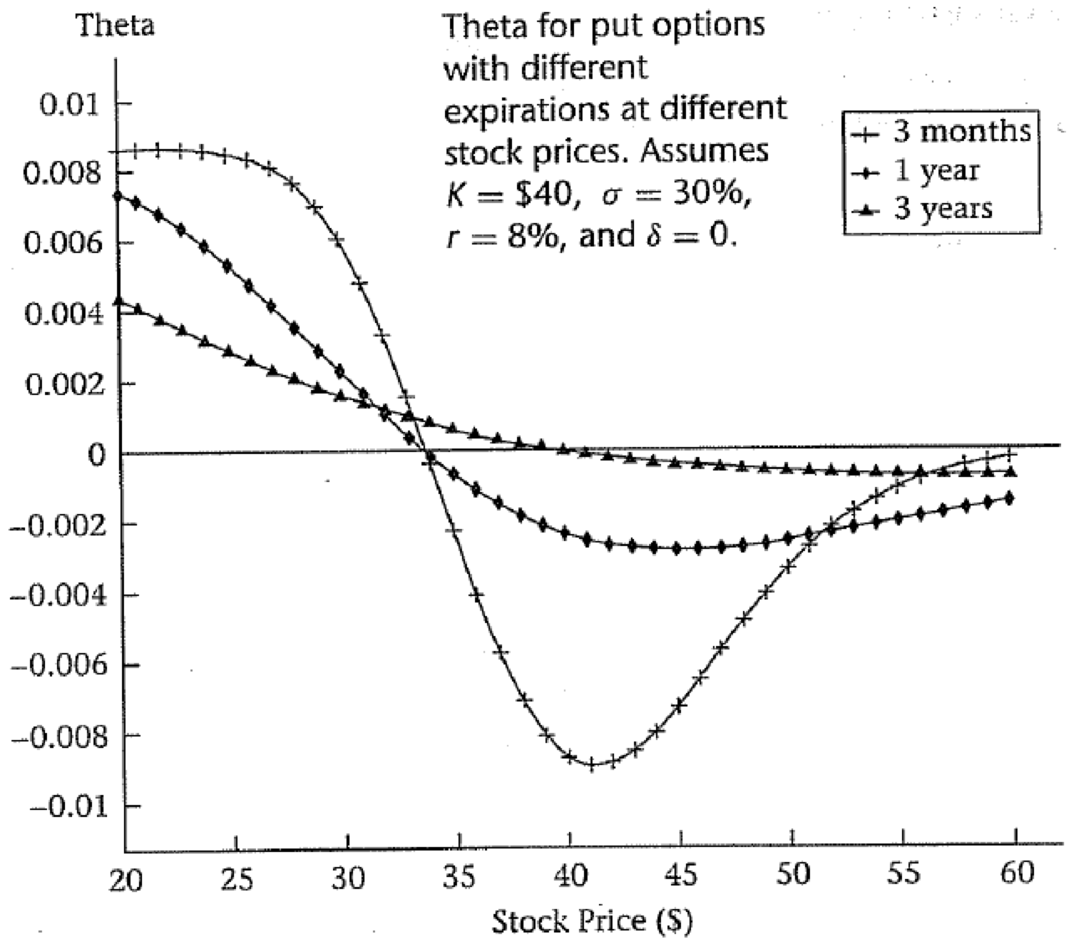
$$\begin{aligned}
\theta_P &= -\frac{\partial P}{\partial \tau} \\
&= -\frac{\partial(C - e^{-\delta\tau}S(t) + Ke^{-r\tau})}{\partial\tau} \\
&= -\frac{\partial C}{\partial\tau} - \delta e^{-\delta\tau}S(t) + rKe^{-r\tau} \\
&= \delta S(t)e^{-\delta\tau}N(d_1) - \frac{S(t)e^{-\delta\tau}\sigma}{2\sqrt{\tau}}N'(d_1) - rKe^{-r\tau}N(d_2) - \delta e^{-\delta\tau}S(t) + rKe^{-r\tau} \\
&= rKe^{-r\tau}N(-d_2) - \delta S(t)e^{-\delta\tau}N(-d_1) - \frac{S(t)e^{-\delta\tau}\sigma}{2\sqrt{\tau}}N'(d_1)
\end{aligned}$$

For a non-dividend paying stock, $\theta_C = -\frac{S(t)\sigma}{2\sqrt{\tau}}N'(d_1) - rKe^{-r\tau}N(d_2) < 0$ (refer to Theorem 3.32) and $\theta_P = -\frac{S(t)\sigma}{2\sqrt{\tau}}N'(d_1) + rKe^{-r\tau}N(-d_2)$.

European call and put options usually (but not always) become more valuable as the time to expiration increases.

If the Black-Scholes formula is differentiated with respect to *t* without adjustment, θ is the change in option per year. Usually, when theta is quoted, time is measured in days. The formula for θ must be divided by 365.


Theta
Theta for call options with different expirations at different stock prices. Assumes $K = \$40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.
0
−0.002
−0.004
−0.006
−0.008
−0.01
−0.012
−0.014
−0.016
−0.018
3 months
1 year
3 years
20
25
30
35
40
45
50
55
60
Stock Price ($)


Theta
Theta for put options with different expirations at different stock prices. Assumes $K = \$40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.
3 months
1 year
3 years
0.01
0.008
0.006
0.004
0.002
0
−0.002
−0.004
−0.006
−0.008
−0.01
20
25
30
35
40
45
50
55
60
Stock Price ($)

Definition 5.9

The vega of an option, ν, is defined as the rate of change of the option price respected to the volatility of the underlying asset:

$$\nu = \frac{\partial V}{\partial \sigma}$$

where V is the option price and σ is volatility of the stock price.

Theorem 5.10

For a dividend-paying stock with a continuous dividend yield at rate δ, vega of European call and put option are

$$\nu_C = \nu_P = S(t)e^{-\delta\tau}\sqrt{\tau}N'(d_1) > 0.$$

Proof

$$\begin{aligned}
\nu_C &= \frac{\partial C}{\partial \sigma} \\
&= S(t)e^{-\delta\tau}\frac{\partial N(d_1)}{\partial \sigma} - Ke^{-r\tau}\frac{\partial N(d_2)}{\partial \sigma} \\
&= S(t)e^{-\delta\tau}\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial \sigma} - Ke^{-r\tau}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial \sigma} \\
&= S(t)e^{-\delta\tau}N'(d_1)\left(\frac{\sigma^2\tau^{3/2} - \left[\ln\frac{S(t)}{K} + \left(r - \delta + \frac{\sigma^2}{2}\right)\tau\right]\sqrt{\tau}}{\sigma^2\tau}\right) \\
&\quad - Ke^{-r\tau}N'(d_2)\left(\frac{-\sigma^2\tau^{3/2} - \left[\ln\frac{S(t)}{K} + \left(r - \delta - \frac{\sigma^2}{2}\right)\tau\right]\sqrt{\tau}}{\sigma^2\tau}\right) \\
&= S(t)e^{-\delta\tau}N'(d_1)\left(\frac{\frac{\sigma^2\tau^{3/2}}{2} - \left[\ln\frac{S(t)}{K} + (r-\delta)\tau\right]\sqrt{\tau}}{\sigma^2\tau}\right)
\end{aligned}$$

$$-S(t)e^{-\delta\tau}N'(d_1)\left(\frac{-\frac{\sigma^2\tau^{3/2}}{2}-\left[\ln\frac{S(t)}{K}+(r-\delta)\tau\right]\sqrt{\tau}}{\sigma^2\tau}\right)$$

$$=S(t)e^{-\delta\tau}N'(d_1)\left(\frac{\sigma^2\tau^{3/2}}{\sigma^2\tau}\right)$$

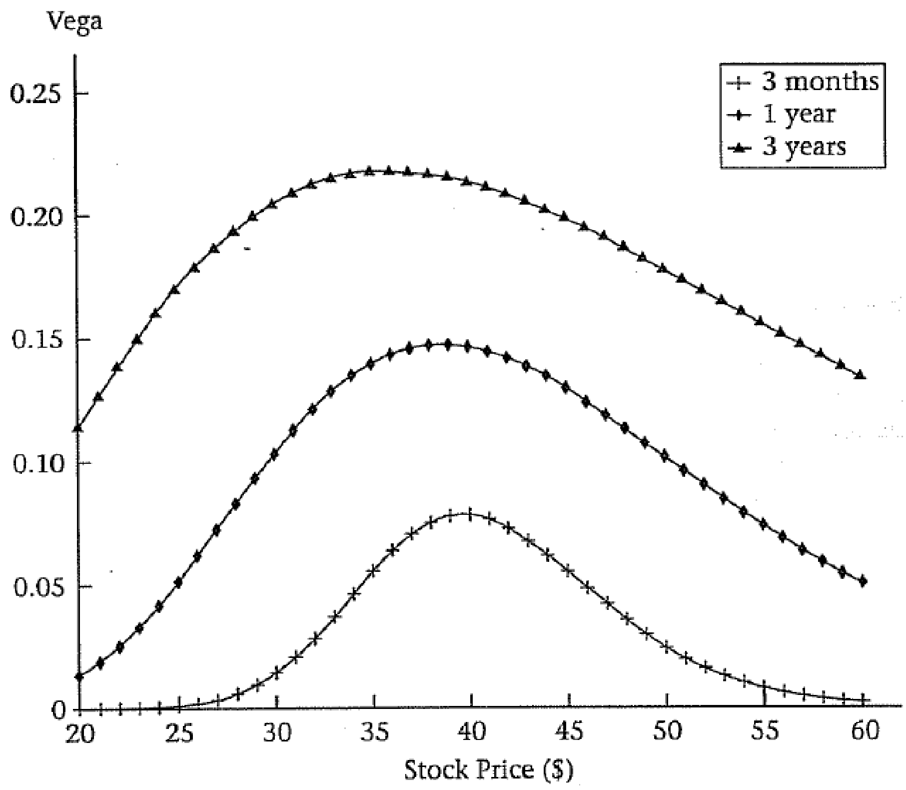
$$=S(t)e^{-\delta\tau}\sqrt{\tau}\cdot N'(d_1).$$

By put-call parity equation,

$$v_P=\frac{\partial P}{\partial\sigma}=\frac{\partial(C-e^{-\delta\tau}S(t)+Ke^{-r\tau})}{\partial\sigma}=\frac{\partial C}{\partial\sigma}=S(t)e^{-\delta\tau}\sqrt{\tau}N'(d_1).$$

For a non-dividend paying stock, $v_C=v_P=S(t)\sqrt{\tau}N'(d_1)$.

Vegas for 40-strike options with different times to expiration. Assumes $\sigma=30\%$, $r=8\%$, and $\delta=0$.



Higher volatility raises the probability and the amount by which a stock price will move above (for a call) or below (for a put) the strike price, and thus makes the option more valuable. The effect is significant when the option is at-the-money. Greater volatility is especially valuable to longer lived options, which have more opportunity to oscillate. Vega, the derivative of option price with respect to volatility, is always positive. When the time is close to the maturity date T and the option is deep in-the-money or deep out-of-the-money, ν becomes smaller, which is due to that the period of time in which the volatility σ can affect the option value become smaller.

Definition 5.11

The rho of an option is defined as the rate of change of the option price respected to the interest rate:

$$\rho = \frac{\partial V}{\partial r}$$

where V is the option price and r is interest rate.

Theorem 5.12

For a dividend-paying stock with a continuous dividend yield at rate δ, rho of European call and put option are

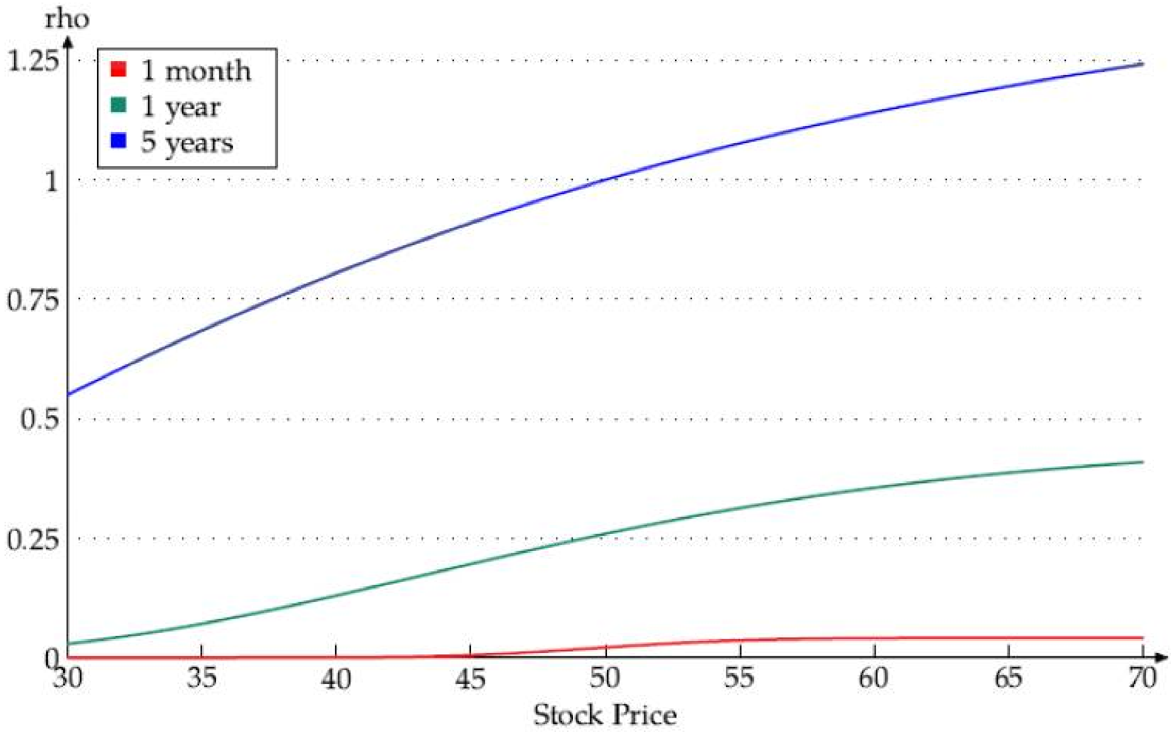
$$\rho_C = K\tau \cdot e^{-r\tau} N(d_2) > 0 \qquad \text{and} \qquad \rho_P = -K\tau \cdot e^{-r\tau} N(-d_2) < 0.$$

Proof

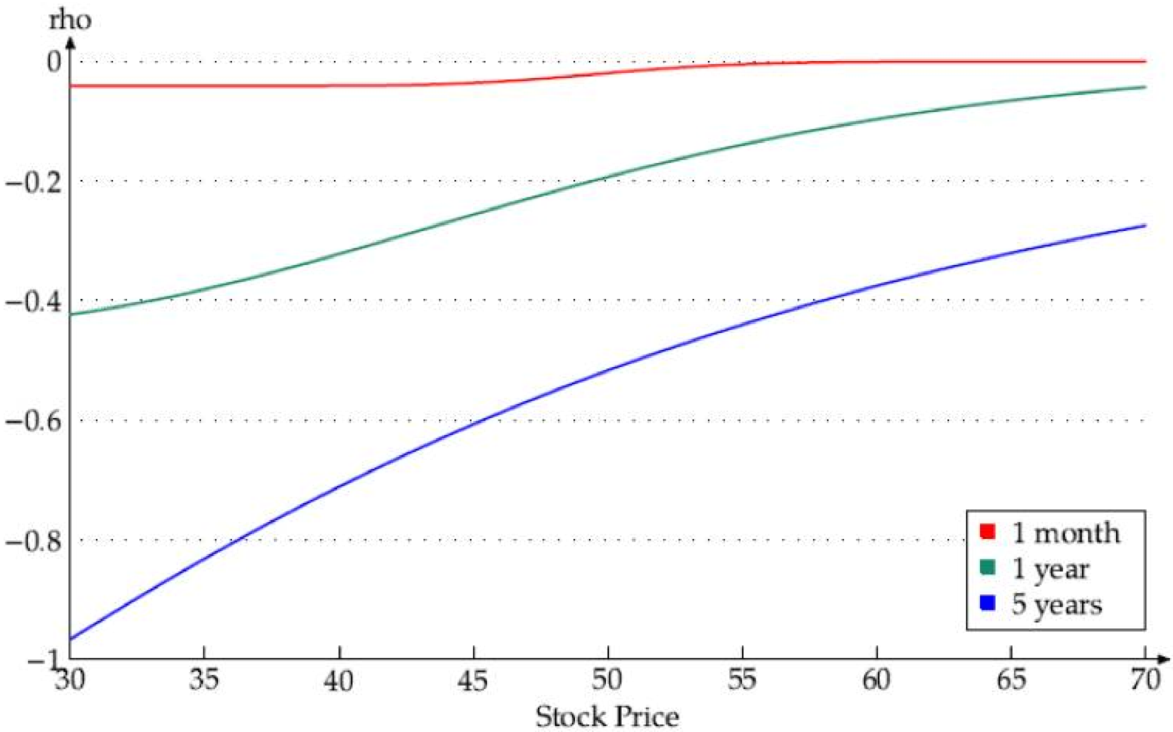
$$\begin{aligned}
\rho_C &= \frac{\partial C}{\partial r} \\
&= S(t)e^{-\delta\tau} \frac{\partial N(d_1)}{\partial r} - (-\tau) K e^{-r\tau} N(d_2) - K e^{-r\tau} \frac{\partial N(d_2)}{\partial r} \\
&= S(t)e^{-\delta\tau} N'(d_1) \frac{\partial d_1}{\partial r} + K\tau e^{-r\tau} N(d_2) - K e^{-r\tau} N'(d_2) \frac{\partial d_2}{\partial r} \\
&= S(t)e^{-\delta\tau} N'(d_1) \left(\frac{\sqrt{\tau}}{\sigma} \right) + K\tau e^{-r\tau} N(d_2) - K e^{-r\tau} N'(d_2) \left(\frac{\sqrt{\tau}}{\sigma} \right) \\
&= K\tau e^{-r\tau} N(d_2).
\end{aligned}$$

By put-call parity equation,

$$\rho_P = \frac{\partial P}{\partial r} = \frac{\partial(C - e^{-\delta\tau}S(t) + Ke^{-r\tau})}{\partial r} = \rho_C - \tau Ke^{-r\tau} = K\tau \cdot e^{-r\tau}N(d_2) - \tau Ke^{-r\tau} = -\tau Ke^{-r\tau}N(-d_2).$$



Call rho as a function of stock price. Assumes $K = 50$, $\sigma = 0.30$, $r = 0.10$, $\delta = 0$.



Put rho as a function of stock price. Assumes $K = 50$, $\sigma = 0.30$, $r = 0.10$, $\delta = 0$.

Call options become more valuable as r is increased, since the strike's value depreciates more. The longer-lived the option, the less the strike price is worth as r grows. Therefore, rho of a call is positive.

For put options, put premium decreases with higher r, since higher r leads to a lower present value for the strike price. Therefore, rho of a put is negative.

Rho is closed to 0 if the option is deep out-of-the-money. It is because it has a higher chance of not being exercise. Interest rate will not take any effect to the option.

Definition 5.13

The psi of an option is defined as the rate of change of the option price respected to the dividend rate:

$$\psi = \frac{\partial V}{\partial \delta}$$

where V is the option price.

Theorem 5.14

For a dividend-paying stock with a continuous dividend yield at rate δ, psi of European call and put option are

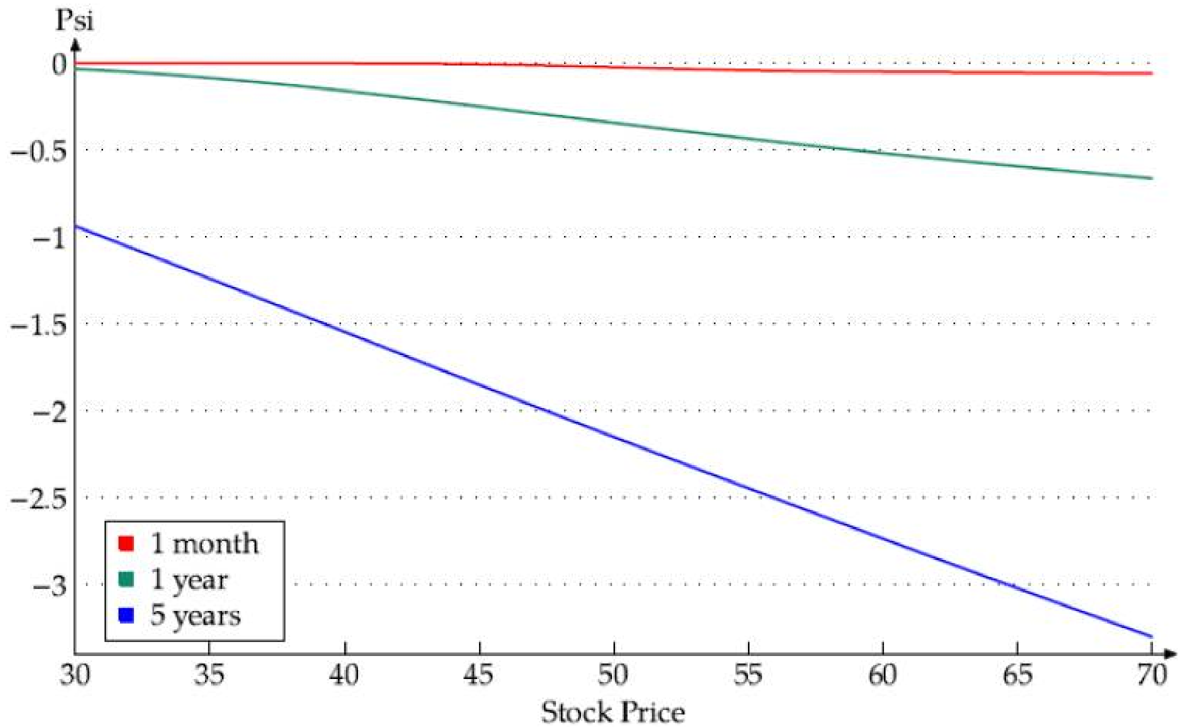
$$\psi_C = -S(t)\tau e^{-\delta\tau} N(d_1) < 0 \qquad \text{and} \qquad \psi_P = S(t)\tau e^{-\delta\tau} N(-d_1) > 0.$$

Proof

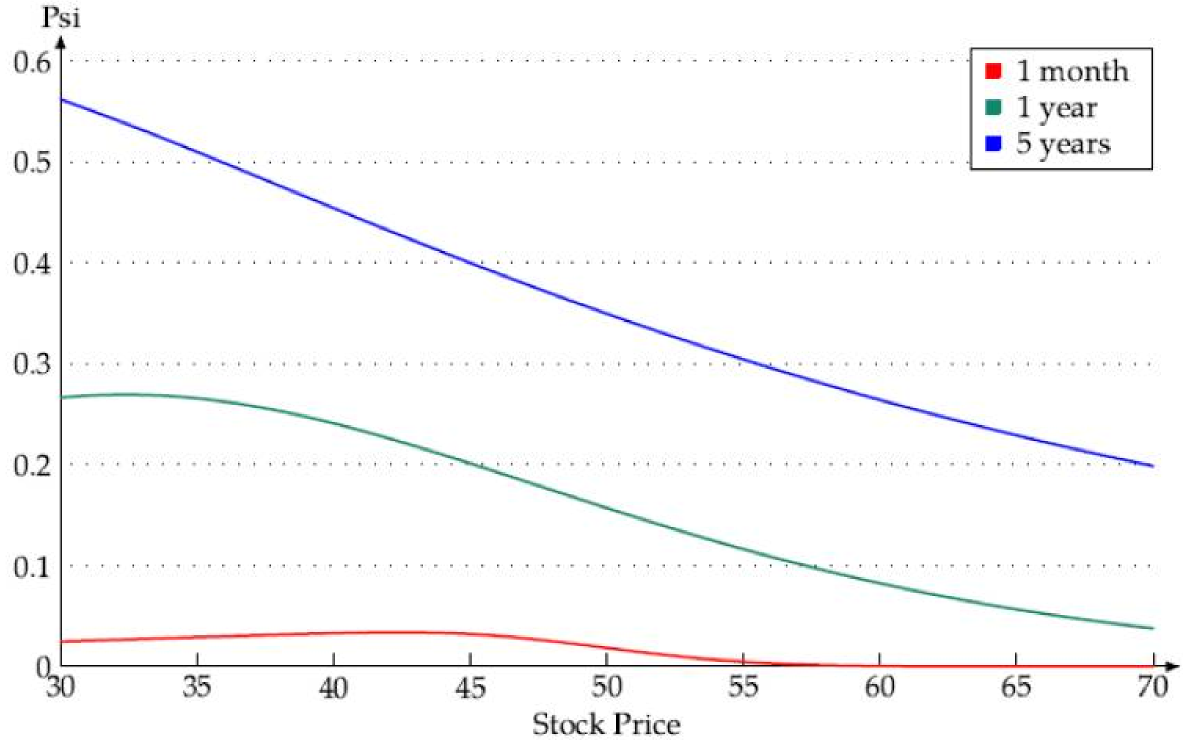
$$\begin{aligned}
\psi_C &= \frac{\partial C_t}{\partial \delta} \\
&= S(t)e^{-\delta\tau}\frac{\partial N(d_1)}{\partial \delta} - S(t)\tau e^{-\delta\tau}N(d_1) - Ke^{-r\tau}\frac{\partial N(d_2)}{\partial \delta} \\
&= S(t)e^{-\delta\tau}\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial \delta} - S(t)\tau e^{-\delta\tau}N(d_1) - Ke^{-r\tau}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial \delta} \\
&= -S(t)e^{-\delta\tau}N'(d_1)\frac{\sqrt{\tau}}{\sigma} - S(t)\tau e^{-\delta\tau}N(d_1) + Ke^{-r\tau}N'(d_2)\frac{\sqrt{\tau}}{\sigma} \\
&= -S(t)\tau e^{-\delta\tau}N(d_1).
\end{aligned}$$

By put-call parity equation,

$$\psi_P = \frac{\partial P}{\partial \delta} = \frac{\partial (C - e^{-\delta\tau} S(t) + K e^{-r\tau})}{\partial \delta} = \psi_C + \tau e^{-\delta\tau} S(t) = -S(t)\tau e^{-\delta\tau} N(d_1) + \tau e^{-\delta\tau} S(t) = S(t)\tau e^{-\delta\tau} N(-d_1).$$



Call psi as a function of stock price. Assumes $K = 50$, $\sigma = 0.30$, $r = 0.10$, $\delta = 0$.



Put psi as a function of stock price. Assumes $K = 50$, $\sigma = 0.30$, $r = 0.10$, $\delta = 0$.

Change in dividend return works exactly the opposite from change in risk-free interest rate. Call premiums decline with increasing dividend rates, since dividends reduce the value of the stock. Therefore, psi of a call, derivative of a call option with respect to interest rate, is negative.

For put option the opposite is true. Psi of a put is positive.

Psi is closed to 0 if the option is deep out-of-the-money. It is because it has a higher chance of not being exercise. Dividend rate will not take any effect to the option.

Black-Scholes partial differential equation $\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-\delta)S\frac{\partial V}{\partial S}-rV=0$ gives us the relation between Greek symbols

$$\theta+(r-\delta)\Delta S+0.5S^2\sigma^2\Gamma=rV.$$

Example 5.15

The price of a non-dividend paying stock is 40. Volatility is 0.5. Risk-free interest rate is 8%. Δ, Γ, θ of a call option is 0.7171, 0.0098, -3.0295 per year respectively. Determine the price of the call option using Black-Scholes equation.

Solution

By the Black-Scholes equation,

$$rC(S)=\theta+rS\Delta+\tfrac{1}{2}\sigma^2S^2\Gamma$$
$$0.08C(S)=-3.0295+(0.08)(40)(0.7171)+\tfrac{1}{2}(0.5^2)(40^2)(0.0098)=1.2252.$$

Therefore, $C(S)=\frac{1.2252}{0.08}=15.32.$

The solution to the Black-Scholes partial differential equation $V_t+(r-\delta)V_S S(t)+\frac{1}{2}S(t)^2\sigma^2V_{SS}=rV$ depends on the boundary value of V. If you have a formula which matches the boundary value (the value at expiry) and satisfies Black-Scholes partial differential equation, that formula is a pricing formula.

Example 5.16

Let $S(t)$ be the price of a stock and $F_{t,T}^P(S)$ a prepaid forward of $S(t)$ maturing at time T. Let us verify Black-Scholes equation for the prepaid forward with boundary condition 1 share of stock. The formula for a prepaid forward is

$$F_{t,T}^P(S) = S(t)e^{-\delta(T-t)}.$$

We have

$$\frac{\partial F_{t,T}^P}{\partial S} = e^{-\delta(T-t)}, \quad \frac{\partial^2 F_{t,T}^P}{\partial S^2} = 0, \quad \frac{\partial F_{t,T}^P}{\partial t} = -S(t)(-\delta)e^{-\delta(T-t)},$$
$$-S(t)(-\delta)e^{-\delta(T-t)} + (r-\delta)S(t)e^{-\delta(T-t)} + \frac{1}{2}S(t)^2\sigma^2 0 = rS(t)e^{-\delta(T-t)} = rF_{t,T}^P(S).$$

The prepaid forward $F_{t,T}^P(S)$ and $e^{-\delta(T-t)}$ shares of stock satisfy Black-Scholes equation with the boundary condition of 1 share of stock $F_{T,T}^P(S) = S(T)$.

Example 5.17

Let us verify Black-Scholes equation for call option with boundary condition $\max\{S(T)-K, 0\}$.

If $S(T) > K$, then $d_1 = \dfrac{\ln(S(T)/K) + (r-\delta+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \to \infty, d_2 = d_1 - \sigma\sqrt{T-t} \to \infty$ as $t \to T^-$. Hence $N(d_1), N(d_2) \to N(\infty) = 1$ as $t \to T^-$. We have

$$C = S(t)e^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \to S(T) - K \quad \text{as} \quad t \to T^-.$$

If $S(T) < K$, then $d_1 = \dfrac{\ln(S(T)/K) + (r-\delta+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \to -\infty, d_2 = d_1 - \sigma\sqrt{T-t} \to -\infty$ as $t \to T^-$. Hence $N(d_1), N(d_2) \to N(-\infty) = 0$ as $t \to T^-$. We have

$$C = S(t)e^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \to 0 \quad \text{as} \quad t \to T^-.$$

If $S(T) = K$, then $d_1 = \dfrac{\ln(S(T)/K) + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \to 0, d_2 = d_1 - \sigma\sqrt{T-t} \to 0$ as $t \to T^-$. Hence $N(d_1), N(d_2) \to N(0) = 1/2$ as $t \to T^-$. We have

$$C = S(t)e^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \to S(T)/2 - K/2 = 0 \quad \text{as} \quad t \to T^-.$$

$$\begin{aligned}
&C = S(t)e^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \\
&C_S = \Delta = e^{-\delta(T-t)}N(d_1) \\
&C_{SS} = \Gamma = \frac{e^{-\delta(T-t)}N'(d_1)}{S\sigma\sqrt{T-t}} \\
&C_t = \theta = \delta S(t)e^{-\delta(T-t)}N(d_1) - S(t)e^{-\delta(T-t)}N'(d_1)\frac{\sigma}{2\sqrt{(T-t)}} - rKe^{-r(T-t)}N(d_2)
\end{aligned}$$

$$\begin{aligned}
S(t)(r-\delta)C_S + \frac{S^2\sigma^2 C_{SS}}{2} + C_t &= S(t)(r-\delta)e^{-\delta(T-t)}N(d_1) + \frac{S(t)^2\sigma^2}{2}\frac{e^{-\delta(T-t)}N'(d_1)}{S(t)\sigma\sqrt{T-t}} \\
&\quad + \delta S(t)e^{-\delta(T-t)}N(d_1) - S(t)e^{-\delta(T-t)}N'(d_1)\frac{\sigma}{2\sqrt{(T-t)}} - rKe^{-r(T-t)}N(d_2) \\
&= S(t)re^{-\delta(T-t)}N(d_1) - rKe^{-r(T-t)}N(d_2) \\
&= rC
\end{aligned}$$

Assume the stock price S will not change too much after time h. Then choose an option V with $\theta < 0$ and $\Gamma > 0$. Let r be the risk-free rate. A delta hedged portfolio Π consists of selling an option V, buying Δ shares of stock S, and borrowing the money $\Delta S(0) - V(0)$ from a bank:

$$\Pi = \overbrace{-V}^{\text{selling an option}} + \overbrace{\Delta S}^{\text{buying } \Delta \text{ share of stock}} - \overbrace{(\Delta S(0) - V(0))}^{\text{borrowing money from a bank}}$$

(If an investor chooses an option V with $\theta > 0$ and $\Gamma < 0$, then he should buy an option, sell Δ shares of stock and buy $\Delta S(0) - V(0)$ bond.)

$$\Pi = \overbrace{V}^{\text{buying an option}} - \overbrace{\Delta S}^{\text{selling } \Delta \text{ share of stock}} + \overbrace{(\Delta S(0) - V(0))}^{\text{buying bond}}$$

(If the stock price S will change a lot after h, then try opposite strategy of the above portfolios.)

Suppose a stock pays continuous dividends at a rate δ. Then, after time h,

$$\Pi(h) = -V(h) + \Delta e^{\delta h}S(h) - e^{rh}(\Delta S(0) - V(0)).$$

$$
\begin{aligned}
\text{Profit} &= \Pi(h)-\Pi(0) \\
&= -\left(V(h)-V(0)\right)+\Delta e^{\delta h}(S(h)-S(0))-\left(e^{rh}-1\right)\left(\Delta S(0)-V(0)\right)+\left(e^{\delta h}-1\right)\Delta S(0) \\
&\approx -\overbrace{\left(V(h)-V(0)\right)}^{\text{change in the value of the option}}+\Delta\overbrace{\left(S(h)-S(0)\right)}^{\text{change in the price of the stock}}-\overbrace{\left(e^{rh}-1\right)\left(\Delta S(0)-V(0)\right)}^{\text{Interest on the borrowed money}}+\overbrace{\left(e^{\delta h}-1\right)\Delta S(0)}^{\text{dividend}}.
\end{aligned}
$$

The first two terms in above equation are called "mark-to-market", since the option and stock are not usually sold; their values are just recalculated.

Example 5.18

A market-maker sells 100 30-day European call options on a non-dividend paying stock and delta-hedges them. You are given that initially (day 0):

$$S(0)=200, \quad r=0.05, \quad \sigma=0.4, \quad K=200.$$

Calculate the market-maker's overnight profit:

1. If the stock's price stays the same on the next day (day 1).
2. If the stock's price increases to 205 on the next day (day 1).
3. If the stock's price decreases to 195 on the next day (day 1).

Solution

Initially

$$
\begin{gathered}
d_1=\frac{\left(0.05+0.5(0.4)^2\right)(30/365)}{0.4\sqrt{30/365}}=0.09317 \\
\Delta(0)=N(d_1)=N(0.09317)=0.53712 \\
d_2=0.09317-0.4\sqrt{30/365}=-0.02150 \\
N(d_2)=N(-0.02150)=0.49142 \\
C=200(0.53712)-200e^{-0.05(30/365)}(0.49142)=9.542
\end{gathered}
$$

The initial investment by the market-maker is the cost of Δ_{30} shares of stock minus the call premium, or

$$100\left((0.53712)(200)-9.542\right)=9788.$$

The interest cost for one day is $\left(e^{0.05/365}-1\right)(9788)=1.3$.

1. If the stock's price remains the same, then the value of the call after one day is

$$d_1=\frac{\left(0.05+0.5(0.4)^2\right)(29/365)}{0.4\sqrt{29/365}}=0.09161$$
$$\Delta(1/365)=N(d_1)=N(0.09161)=0.53650$$
$$d_2=0.09161-0.4\sqrt{29/365}=-0.02114$$
$$N(d_2)=N(-0.02114)=0.49157$$
$$C=200(0.53650)-200e^{-0.05(29/365)}(0.49157)=9.376$$

The mark-to-market profit on the call is $100(9.542-9.376)=16.6$. Subtracting the interest cost, the net profit is $16.6-1.3=15.3$. Profit occurred because the option's value decays with time.

2. If the stock's price increase to 205, then the value of the call after one day is

$$d_1=\frac{\ln(205/200)+\left(0.05+0.5(0.4)^2\right)(29/365)}{0.4\sqrt{29/365}}=0.31061$$
$$\Delta(1/365)=N(d_1)=N(0.31061)=0.62195$$
$$d_2=0.31061-0.4\sqrt{29/365}=0.19787$$
$$N(d_2)=N(0.19787)=0.57842$$
$$C=205(0.62195)-200e^{-0.05(29/365)}(0.57842)=12.274$$

The mark-to-market profit on the call is $100(9.542-12.274)=-273.2$. Gain on the stock is $100(0.53712)(205-200)=268.6$. Total profit, including interest, is $-273.2+268.6-1.3$ $=-5.9$.

3. If the stock's price decrease to 195, then the value of the call after one day is

$$d_1=\frac{\ln(195/200)+\left(0.05+0.5(0.4)^2\right)(29/365)}{0.4\sqrt{29/365}}=-0.13294$$
$$\Delta(1/365)=N(d_1)=N(-0.13294)=0.44712$$
$$d_2=-0.13294-0.4\sqrt{29/365}=-0.24569$$
$$N(d_2)=N(-0.24569)=0.40296$$
$$C=195(0.44712)-200e^{-0.05(29/365)}(0.40296)=6.916$$

The mark-to-market profit on the call is $100(9.542-6.916)=262.6$. Gain on the stock is $100(0.53712)(195-200)=-268.6$. Total profit, including interest, is $262.6-268.6-1.3$ $=-7.3$.

Delta hedging did not prevent profits or losses because delta itself keeps changing. To minimize the risk, the market-maker would rehedge frequently. To rehedge, the market-maker buys or sells stock so that he owns Δ shares of stock for each option, where Δ is revised at the rehedging date. Assuming that the position is always fully financed, the "cash flow" at the time of rehedging where cash flow includes mark-to-market gains and losses equals the overnight profit or loss.

Example 5.19

In Example 5.18, calculate the purchase or sale of stock necessary on the first day to re-delta hedge the position of the stock price is

1. 200 on day 1.

2. 205 on day 1.

3. 195 on day 1.

Solution

1. $\Delta(1/365)=N(0.09161)=0.53650$, whereas $\Delta(0)=0.53712$, so the market-maker **sells** $100(0.53712-0.53650)=0.062$ shares.

2. $\Delta(1/365)=N(0.31061)=0.62195$, whereas $\Delta(0)=0.53712$, so the market-maker **buys** $100(0.62195-0.53712)=8.48$ shares.

3. $\Delta(1/365)=N(-0.13294)=0.44712$, whereas $\Delta(0)=0.53712$, so the market-maker **sells** $100(0.53712-0.44712)=9.00$ shares.

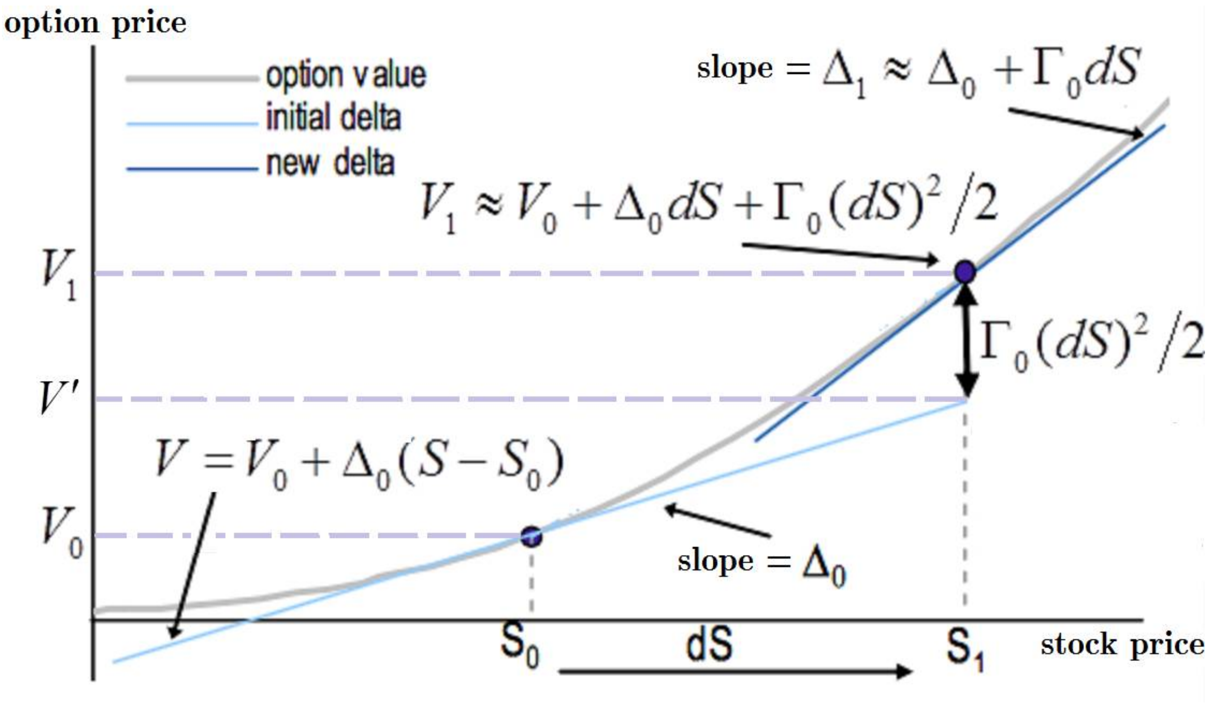
A better approximation of the change on option value than we got using only Δ would be obtained if we brought in the second derivative Γ. By Taylor's series,

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2}f''(x_0)(x-x_0)^2+\text{error term.}$$

If we let $\varepsilon=S(t+h)-S(t)$ be the change in stock price, $x_0=S(t), x=S(t+h)$, $f(x_0)=V(S(t))$, the option price, $f'(x_0)=\Delta$ and $f''(x_0)=\Gamma$, this becomes

$$V(S(t+h))=\underbrace{V(S(t))+\Delta\varepsilon+\frac{1}{2}\Gamma\varepsilon^2}_{\textbf{delta-gamma approximation}}+\text{error term.}$$

If gamma is small, delta changes slowly, and adjustments to keep a portfolio delta neutral need to be made only relatively infrequently. However, if gamma is highly negative or highly positive, delta is very sensitive to the price of the underlying asset. It is then quite risky to leave a delta-neutral portfolio $(\Delta=0)$ unchanged for any length of time. When the stock price moves from S_0 to S_1, delta hedging assumes that the option price moves from V_0 to V', but in fact it moves from V_0 to V_1. The difference between V_1 and V' leads to a hedging error. The size of the error depends on the curvature of the relationship between the option price and the stock price. Gamma measures this curvature.



Example 5.20

Consider Example 5.18. Find Γ. Estimate the change in the value of call option if the stock price immediately change to (a) 205 and (b) 195.

Solution

$$\Gamma = \frac{1}{S(t)\sigma\sqrt{2\pi\tau}} e^{-\frac{d_1^2}{2}} = \frac{1}{200(0.4)\sqrt{30/365}\sqrt{2\pi}} e^{-\frac{0.09317^2}{2}} = 0.017318923932812932545$$

(a) In Example 5.18, $\Delta = 0.53712$. If the stock price is 205, the estimated change is

$$0.53712(5) + 0.5(0.01732)(5^2) = 2.902.$$

The actual price of the call option after the change in the price of stock is

$$d_1 = \frac{\ln(205/200) + \left(0.05 + 0.5(0.4)^2\right)(30/365)}{0.4\sqrt{30/365}} = 0.30850$$
$$N(d_1) = N(0.30850) = 0.62115$$
$$d_2 = 0.30850 - 0.4\sqrt{30/365} = 0.19382$$
$$N(d_2) = N(0.19382) = 0.57684$$
$$C(205, 200, 30/365) = 205(0.62115) - 200e^{-0.05(30/365)}(0.57684) = 12.440$$

and the original option price was 9.542. The true change is $12.440 - 9.542 = 2.898$.

(b) The predicted change is

$$0.53712(-5) + 0.5(0.01732)(-5)^2 = -2.469.$$

The actual price of the call option after the change in the price of stock is

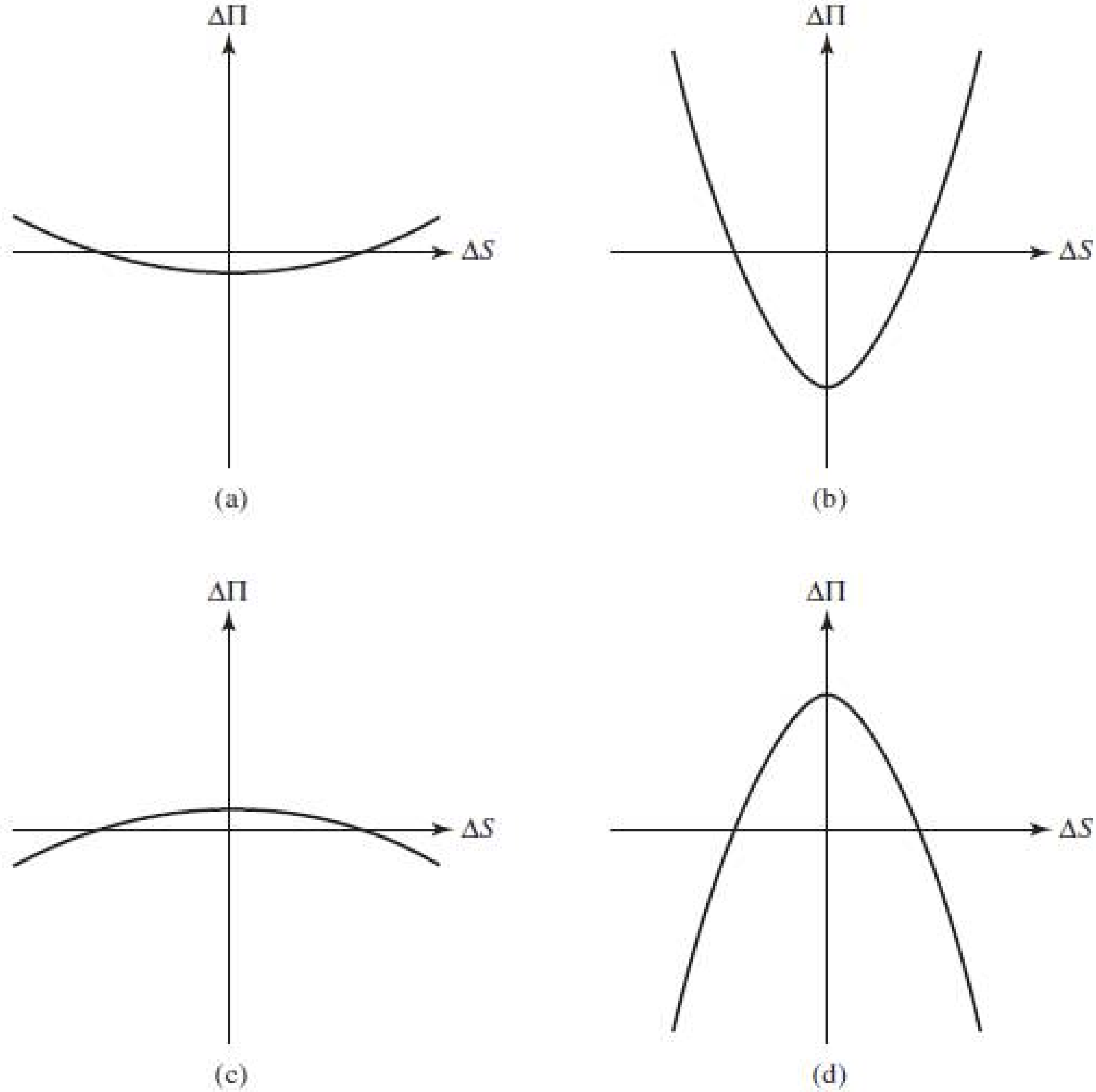
$$d_1 = \frac{\ln(195/200) + \left(0.05 + 0.5(0.4)^2\right)(30/365)}{0.4\sqrt{30/365}} = -0.12760$$
$$N(d_1) = N(-0.12760) = 0.44923$$
$$d_2 = -0.12760 - 0.4\sqrt{30/365} = -0.24228$$
$$N(d_2) = N(-0.24228) = 0.40428$$
$$C(195, 200, 30/365) = 195(0.44923) - 200e^{-0.05(30/365)}(0.40428) = 7.076$$

So the true change is $7.076 - 9.542 = -2.466$.

To consider time effect, we would add θh to the formula for a change in time h. Using Ito's Lemma, the **delta-gamma-theta approximation** of the option price is

$$V(S(t+h)) \approx V(S(t)) + \Delta\varepsilon + \frac{1}{2}\Gamma\varepsilon^2 + h\theta$$

Relationship between $\Delta\Pi$ and ΔS in time Δt for a delta-neutral portfolio with (a) slightly positive gamma, (b) large positive gamma, (c) slightly negative gamma, and (d) large negative gamma.



Consider a delta-neutral portfolio $(\Delta = 0)$. We have

$$\Pi(S(t+h)) - \Pi(S(t)) \approx \frac{1}{2}\Gamma\varepsilon^2 + h\theta.$$

If gamma is positive and large, then theta is negative and large. When gamma is positive, change in stock prices result in higher value of the portfolio. This means that when there is no change in stock prices, the value of the portfolio declines as time approaches to expiration date. As a result, the theta is negative. On the other hand, when gamma is negative and large, change in stock prices result in lower portfolio value. This means that when there is no stock price changes, the value of the portfolio increases as time approaches to expiration and theta is positive. This gives us a trade-off between θ and Γ.

Example 5.21

Consider Example 5.18. Find θ. Calculate the 1-day holding period profit on the call if the stock's price is 205 after 1 day.

Solution

$$\theta = -\frac{S(0)\sigma}{2\sqrt{T}} N'(d_1) - rKe^{-rT} N(d_2) = -60.314602683(\text{per year}) = -0.1652454868(\text{per day})$$

The profit you make on the call is: the change in call premium minus 1 day's interest on the call premium,

$$(C_{29} - C_{30}) - C_{30}\left(e^{r/365} - 1\right) = C_{29} - C_{30}e^{r/365} = 12.274 - 9.542e^{0.05/365} = 2.7306927871792481.$$

Indeed, the profit of an option after time h is $C(S(t+h)) - C(S(t)) - C(S(t))\left(e^{rh} - 1\right)$

$= C(S(t+h)) - C(S(t))e^{rh}$ which can be approximated by $\Delta\varepsilon + \frac{1}{2}\Gamma\varepsilon^2 + \theta h - rhC(S(t)).$

$$\begin{aligned}
&\Delta\varepsilon + \frac{1}{2}\Gamma\varepsilon^2 + \theta h - rhC(S(t)) \\
&= 0.53712(5) + 0.5(0.0173189239328129325 45)5^2 - 0.1652454868 - 9.542(0.05/365) \\
&= 2.7355339390724904239357877
\end{aligned}$$

Example 5.22

Consider Example 5.18. Estimate the change in the option price 1 day later if the stock price is (a) 200, (b) 205, and (c) 195 at that time.

Solution

(a) The stock price change $\varepsilon = 0,$ so the estimated change is $\theta = -0.1652.$ The actual change is $9.376 - 9.542 = -0.166.$

(b) The estimated change is

$$C(S(1/365)) - C(S(0)) \approx \Delta\varepsilon + \frac{1}{2}\Gamma\varepsilon^2 + h\theta = 0.53712(5) + \frac{1}{2}(0.01732)(5^2) - 0.1652 = 2.7369$$

The actual change is $12.274 - 9.542 = 2.732.$

(c) The estimated change is

$$C(S(1/365)) - C(S(0)) \approx 0.53712(-5) + \frac{1}{2}(0.01732)(-5^2) - 0.1652 = -2.6343$$

The actual change is $6.916 - 9.542 = -2.626.$

As long as, we are analyzing changes over small intervals h, we may as well approximate $e^{rh} - 1$ as rh and $e^{\delta h} - 1$ as δh (the first term of their Taylor series).

If the delta hedged portfolio Π is

$$\Pi = \overbrace{-V}^{\text{selling an option}} + \overbrace{\Delta S}^{\text{buying }\Delta\text{ share of stock}} - \overbrace{\left(\Delta S(0) - V(0)\right)}^{\text{borrowed money from a bank}}$$

then

$$\begin{aligned}
\text{Profit} &= -\left(V(h) - V(0)\right) + \Delta\left(S(h) - S(0)\right) - \left(e^{rh} - 1\right)\left(\Delta S(0) - V(0)\right) + \left(e^{\delta h} - 1\right)\Delta S(0) \\
&\approx -\overbrace{\frac{1}{2}\Gamma\varepsilon^2}^{\text{change of stock price effect}} - \overbrace{h\theta}^{\text{time decay of option}} - \overbrace{rh\left(\Delta S(0) - V(0)\right)}^{\text{interest effect}} + \overbrace{\delta h \Delta S(0)}^{\text{dividend effect}} \quad \left[\text{by delta-gamma-theta approximation}\right] \\
&= \frac{1}{2}\Gamma\sigma^2 S(0)^2 h - \frac{1}{2}\Gamma\varepsilon^2 \qquad \left[\text{by Black-Scholes Equation}\right]
\end{aligned}$$

Suppose $\Gamma > 0.$ The market-maker makes money if $|\varepsilon| = |S(h) - S(0)| < \sigma S(0)\sqrt{h}$ and lose money otherwise. He will make the largest money about $\frac{1}{2}\Gamma\sigma^2 S(0)^2 h$ if $S(h) = S(0)$.

If the delta hedged portfolio Π is

$$\Pi = \overbrace{V}^{\text{buying an option}} + \overbrace{-\Delta S}^{\text{selling } \Delta \text{ share of stock}} + \overbrace{\left(\Delta S(0) - V(0)\right)}^{\text{buying bond}}$$

then $\text{Profit} \approx \frac{1}{2}\Gamma\varepsilon^2 - \frac{1}{2}\Gamma\sigma^2 S(0)^2 h.$ Suppose $\Gamma < 0.$ We have the same conclusion as above.

Example 5.23

In Example 5.18, approximate the range of stock price on the next day for which the market-maker will make a profit. What is the maximum profit?

Solution

Since Γ of a call option is positive, the market-maker will make a profit on the next day if

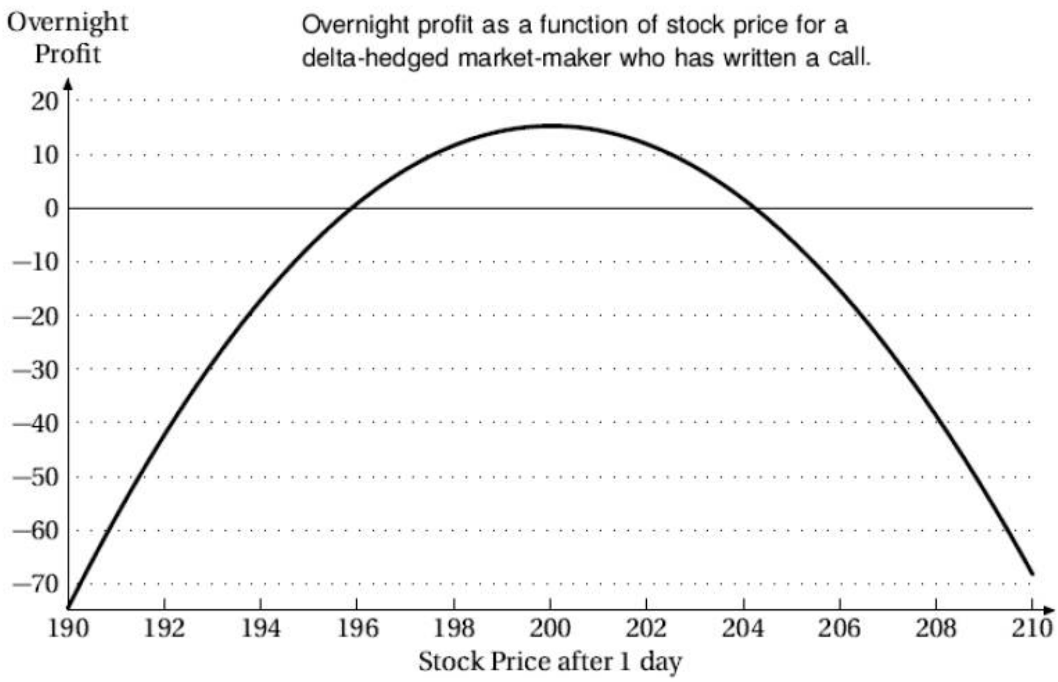
$$\frac{1}{2}\Gamma\sigma^2 S(0)^2 h > \frac{1}{2}\Gamma\varepsilon^2 \quad \text{or} \quad S(0) - \sigma S(0)\sqrt{h} < S(1/365) < S(0) + \sigma S(0)\sqrt{h}$$

Clearly,

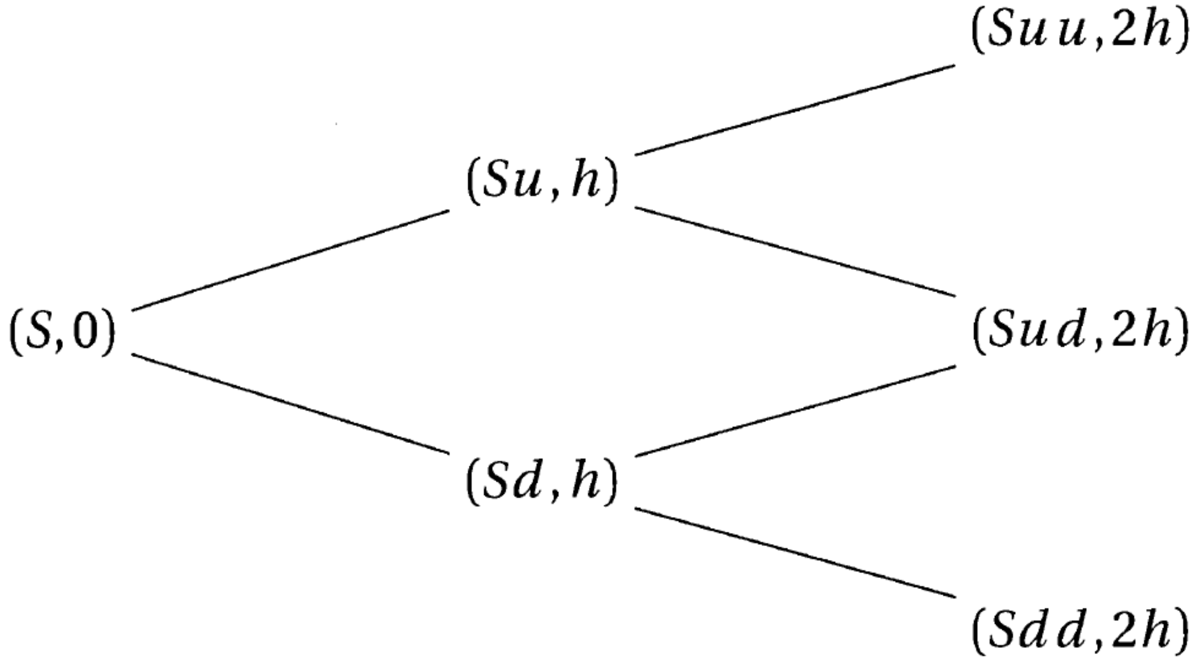
$$S(0) + \sigma S(0)\sqrt{h} = 200 + \frac{0.4(200)}{\sqrt{365}} = 204.1874, \quad S(0) - \sigma S(0)\sqrt{h} = 200 - \frac{0.4(200)}{\sqrt{365}} = 195.8126$$

So if the range of the stock price is (195.8126, 204.1874), the market-maker will make a profit.

The maximum profit is $\frac{100}{2}\Gamma\sigma^2 S(0)^2 h = \frac{100}{2}(0.017319)\frac{0.4^2(200)^2}{365} = 15.1837.$



How can we calculate delta, gamma and theta in Binomial tree model?



(a, b) means the node where the stock price is a and the time is b.

We have already known that

$$\Delta(S,0) = \frac{V_u - V_d}{S(u-d)} e^{-\delta h}.$$

Gamma is the derivative of delta with respect to stock price. To approximate it, we need 2 deltas. We cannot calculate gamma at the initial node of a binomial tree where there is only 1 delta; we need to calculate it after 1 period, where there are 2 nodes. We approximate the derivative by taking the difference in the deltas divided by the difference in the stock prices at the 2 nodes:

$$\Gamma(S,0) \approx \frac{\Delta(Su,h) - \Delta(Sd,h)}{Su - Sd} = e^{-\delta h}\left(\frac{V_{uu} - V_{ud}}{Su^2 - Sud} - \frac{V_{du} - V_{dd}}{Sud - Sd^2}\right)\Big/ S(u-d).$$

It is a good approximation of $\Gamma(S,0)$ if h is small.

Theta is backed out of the delta-gamma-theta approximation, looking 2 periods ahead at the middle node. Let $\varepsilon = Sud - S$ be the change in stock price from the beginning to the middle node 2 period later. We have

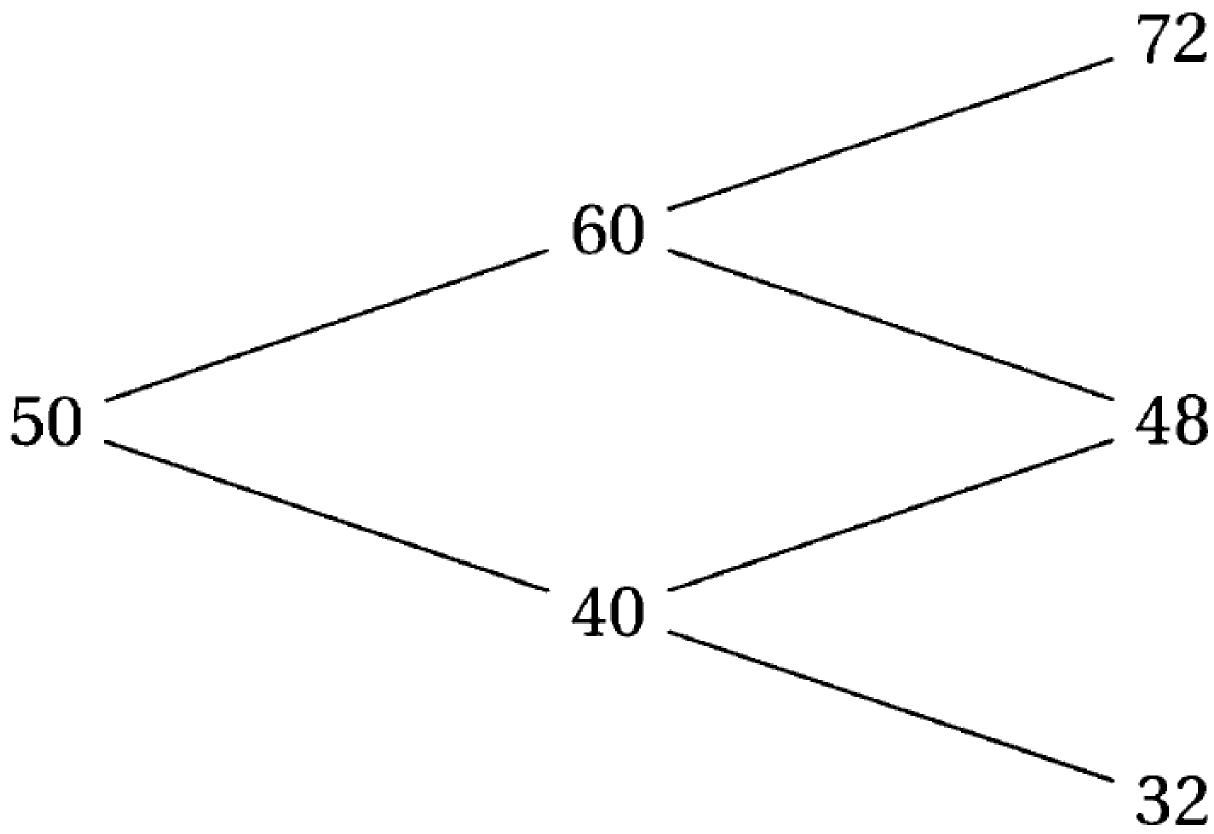
$$V(Sud,2h) = V(S,0) + \Delta(S,0)\varepsilon + \tfrac{1}{2}\Gamma(S,0)\varepsilon^2 + 2h\theta(S,0).$$

So backing out $\theta(S,0)$,

$$\theta(S,0)=\frac{V(Sud,2h)-V(S,0)-\Delta(S,0)\varepsilon-\frac{1}{2}\Gamma(S,0)\varepsilon^2}{2h}.$$

Example 5.24

For a European call option on a non-dividend paying stock with expiry in 2 months, the initial stock price is 50. Stock prices are modeled with a binomial tree having periods of 1 month with $u = 1.2$, $d = 0.8$, as indicated in below figure.



The strike price of the call option is 50. The continuously compounded risk-free interest rate is 0.06. Calculate $\Delta(S,0)$, $\Gamma(S,0)$, and $\theta(S,0)$ for this option.

Solution

We'll need to first calculate the option values at the nodes, then the Δ's. At the final nodes, the option values are $72-50=22$ at the upper node and 0 at the other 2 nodes. The change in the stock price over the 2 periods, ε, is the stock at the middle ending node (48) minus the starting stock price (50), or $\varepsilon = 48 - 50 = -2$. Then

$$p^* = \frac{e^{0.06/12} - 0.8}{1.2 - 0.8} = 0.512531, \quad V_u = e^{-0.06/12}(0.512531)(22) = 11.2195, \quad V_d = 0,$$

$$V_0 = e^{-0.06/12}(0.512531)(11.2195) = 5.7216, \quad \Delta(Su,h) = \frac{22}{72-48} = 0.916667,$$

$$\Delta(Sd,h) = 0, \quad \Delta(S,0) = \frac{11.2195}{60-40} = 0.560973, \quad \Gamma(S,0) = \frac{0.916667}{20} = 0.045833$$

Now we calculate theta. Since $V(Sud, 2h) = 0$,

$$\begin{aligned}\theta(S,0) &= \frac{V(Sud,2h) - V(S,0) - \Delta(S,0)\varepsilon - \frac{1}{2}\Gamma(S,0)\varepsilon^2}{2h} \\ &= \frac{0 - 5.7216 - (0.560973)(-2) - (0.5)(0.045833)(4)}{2/12} \\ &= -28.1482\end{aligned}$$

All of the Greeks are additive. For a portfolio with several derivatives on the same stock, the Greek of the portfolio is the sum of the Greeks of the derivatives.

If a portfolio $\Pi = \sum_{i=1}^{n} w_i V_i$ consists of a quantity w_i of option V_i $(1 \le i \le n)$, then the delta of the portfolio $\Delta = \frac{\partial \Pi}{\partial S}$ can be calculated from the deltas of the individual options

$$\Delta = \sum_{i=1}^{n} w_i \Delta_i$$

where $\Delta_i = \frac{\partial V_i}{\partial S}$ is the delta of the i-th option. The formula can be used to calculate the position in the underlying asset necessary to make the delta of the portfolio zero. When this position has been taken, the portfolio is referred to as being delta neutral.

Example 5.25

A non-dividend paying stock has price 50 and volatility 30%. The continuously compounded risk-free rate is 10%. You purchase a bull spread consisting of buying a 90-day European call with strike price 50 and selling a 90-day European call with strike price 55.

(a) Calculate Δ for the spread.

(b) Calculate the profit on the bull spread if the stock price is 52 after 30 days.

Solution

(a) The premium for the 50-strike call is

$$d_1 = \frac{\left(0.1+0.5(0.3)^2\right)(90/365)}{0.3\sqrt{90/365}} = 0.24001, \quad N(d_1) = N(0.24001) = 0.59484$$
$$d_2 = 0.24001 - 0.3\sqrt{90/365} = 0.09104, \quad N(d_2) = N(0.09104) = 0.53627$$
$$C_{90}(50) = 50(0.59484) - 50e^{-0.1(90/365)}(0.53627) = 3.582, \quad \Delta_{90}(50) = 0.59484.$$

The premium for the 55-strike call is

$$d_1 = \frac{\ln(50/55) + \left(0.1+0.5(0.3)^2\right)(90/365)}{0.3\sqrt{90/365}} = -0.39979, \quad N(d_1) = N(-0.39979) = 0.34465$$
$$d_2 = -0.39979 - 0.3\sqrt{90/365} = -0.54876, \quad N(d_2) = N(-0.54876) = 0.29158$$
$$C_{90}(55) = 50(0.34465) - 55e^{-0.1(90/365)}(0.29158) = 1.5862, \quad \Delta_{90}(55) = 0.34465.$$

The cost of the portfolio is $3.582 - 1.5862 = 1.9958$. Δ for the portfolio is $0.59484 - 0.34465 = 0.25019$.

After 30 days, the value of the 50-strike call is

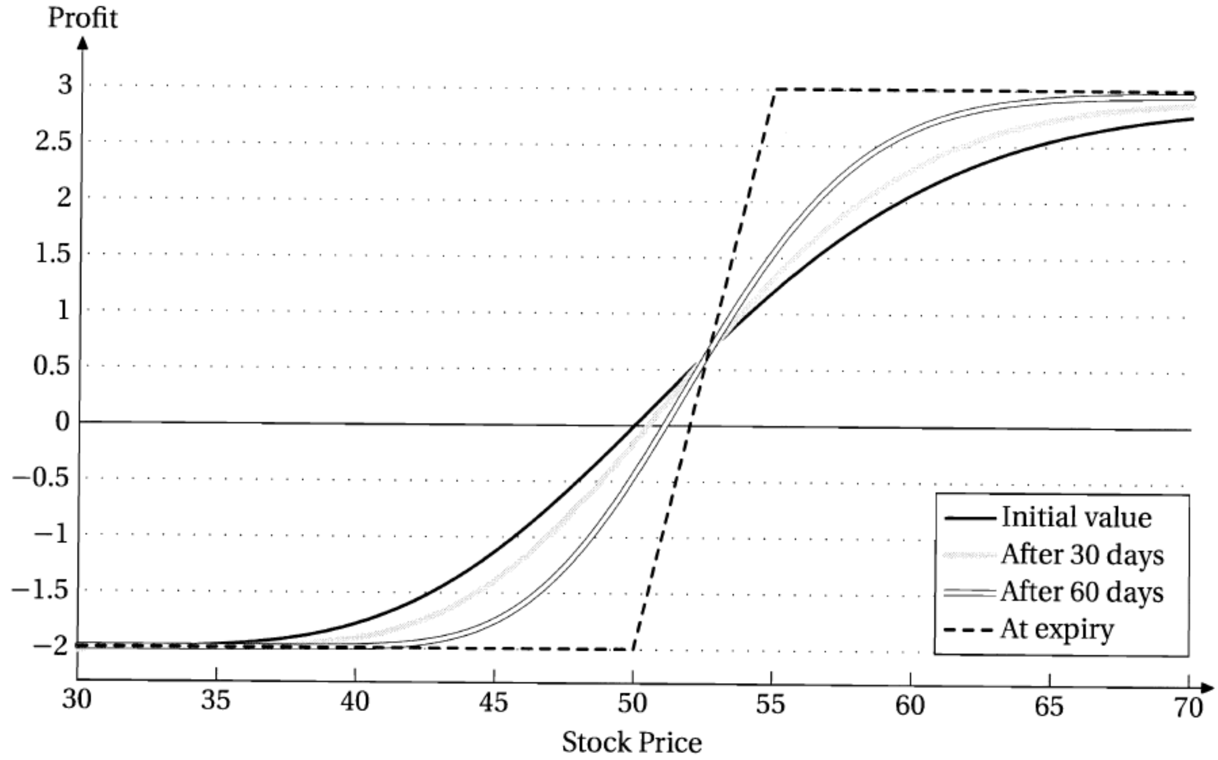
$$d_1 = \frac{\ln(52/50) + \left(0.1+0.5(0.3)^2\right)(60/365)}{0.3\sqrt{60/365}} = 0.51842, \quad N(d_1) = N(0.51842) = 0.69792$$
$$d_2 = 0.51842 - 0.3\sqrt{60/365} = 0.39678, \quad N(d_2) = N(0.39678) = 0.65424$$
$$C_{60}(50) = 52(0.69792) - 50e^{-0.1(60/365)}(0.65424) = 4.113.$$

After 30 days, the value of the 55-strike call is

$$d_1 = \frac{\ln(52/55) + \left(0.1+0.5(0.3)^2\right)(60/365)}{0.3\sqrt{60/365}} = -0.26517, \quad N(d_1) = N(-0.26517) = 0.39544$$

$$d_2 = -0.26517 - 0.3\sqrt{60/365} = -0.38681, \quad N(d_2) = N(-0.38681) = 0.34945$$
$$C_{60}(55) = 52(0.39544) - 55e^{-0.1(60/365)}(0.34945) = 1.656.$$

The portfolio is worth $4.113 - 1.656 = 2.457$. Profit is $2.457 - 1.9958e^{0.1(30/365)} = 0.445$.



In order to match multiple Greeks, one needs more types of assets than just the underlying stock as hedging investments. So multiple hedging is a matter of solving simultaneous linear equations, one for each Greek. Use a consistent sign convention: if you buy some assets and sell others, use positive signs for those you buy and negative signs for you sell.

Example 5.26 (delta hedge a call option by another call option)

You sell a 1-year European call option on a non-dividend paying stock with strike price 40, and delta-hedge it with a 1-year European call option on the same stock with strike price 45. The stock price is 40. Its volatility is 0.3. The continuously compounded risk-free rate is 0.03. Determine the number of 45-strike call options to buy to delta hedge the 40-strike call option you wrote.

Solution

Delta for the 40-strike option is

$$d_1 = \frac{0.03 + 0.5(0.3)^2}{0.3} = 0.25, \qquad \Delta_{40} = N(0.25) = 0.59871$$

Delta for the 45-strike option is

$$d_1 = \frac{\ln(40/45) + 0.03 + 0.5(0.3)^2}{0.3} = -0.14261, \qquad \Delta_{45} = N(-0.14261) = 0.44330$$

Let x be the number of 45-strike call options you buy so that the delta of the portfolio $\Pi = -C_{40} + xC_{45} + (\overbrace{C_{40} - xC_{45}}^{\text{bond}})$ is zero. We have

$$\Delta : -\Delta_{40} + \Delta_{45}x = 0$$
$$-0.59871 + 0.44330x = 0$$

Therefore, $x = 0.59871/0.44330 = 1.3506$, and you need to buy 1.3506 45-strike call options.

Delta neutrality provides protection against relatively small stock price moves between rebalancing. Gamma neutrality $(\Gamma = 0)$ provides protection against larger movements in this stock price between hedge rebalancing.

Example 5.27 (delta-gamma hedge)

The price of a non-dividend paying stock is 50. $r = 0.05$, $\sigma = 0.25$. You write and sell a 1-year European call option with strike price 50. You would like to delta-gamma hedge this call using a call option with strike price 60.

Strike price	Premium	Δ	Γ
50	6.168	0.6274	0.03027
60	2.5127	0.3430	0.02941

(a) Determine the number of shares of stock, the number of 60-strike options and bond to buy or sell to delta-gamma hedge a 50-strike option.

(b) Consider the portfolio in (a), calculate the market-maker's overnight profit if the stock's price increases to 55 on the next day.

Solution

(a) Let x_1 be the number of shares of stock purchased, x_2 the number of 60-strike options purchased and sold $x_1S + x_2C_{60} - C_{50}$ bond. We have

$$\begin{aligned}
&\Pi = x_1S + x_2C_{60} - C_{50} - (\overbrace{x_1S + x_2C_{60} - C_{50}}^{\text{bond}}),\\
&\begin{cases}\Delta : x_1 + x_2\Delta_{60} - \Delta_{50} = 0 & \Rightarrow \quad x_1 + 0.3430x_2 - 0.6274 = 0\\ \quad \Gamma : x_2\Gamma_{60} - \Gamma_{50} = 0 & \Rightarrow \quad 0.02941x_2 - 0.03027 = 0\end{cases}\\
&x_2 = \tfrac{0.03027}{0.02941} = 1.0293, \quad x_1 = 0.6274 - (0.3430)(1.0293) = 0.2743,\\
&x_1S + x_2C_{60} - C_{50} = 0.2743(50) + 1.0293(2.5127) - 6.168 = 10.13332211
\end{aligned}$$

(b) After 1 day, the value of the 50-strike call is

$$\begin{aligned}
&d_1 = \frac{\ln(55/50) + \left(0.05 + 0.5(0.25)^2\right)(364/365)}{0.25\sqrt{364/365}} = 0.7063185315120627, \quad N(d_1) = 0.76\\
&d_2 = 0.7063185315120627 - 0.25\sqrt{364/365} = 0.456661232152945875, \quad N(d_2) = 0.676\\
&C_{50}(\tfrac{364}{365}) = 55(0.76) - 50e^{-0.05(364/365)}(0.676) = 9.6440408276553261
\end{aligned}$$

After 1 days, the value of the 60-strike call is

$$d_1 = \frac{\ln(55/60) + \left(0.05 + 0.5(0.25)^2\right)(364/365)}{0.25\sqrt{364/365}} = -0.0239687753761537, \quad N(d_1) = 0.49$$

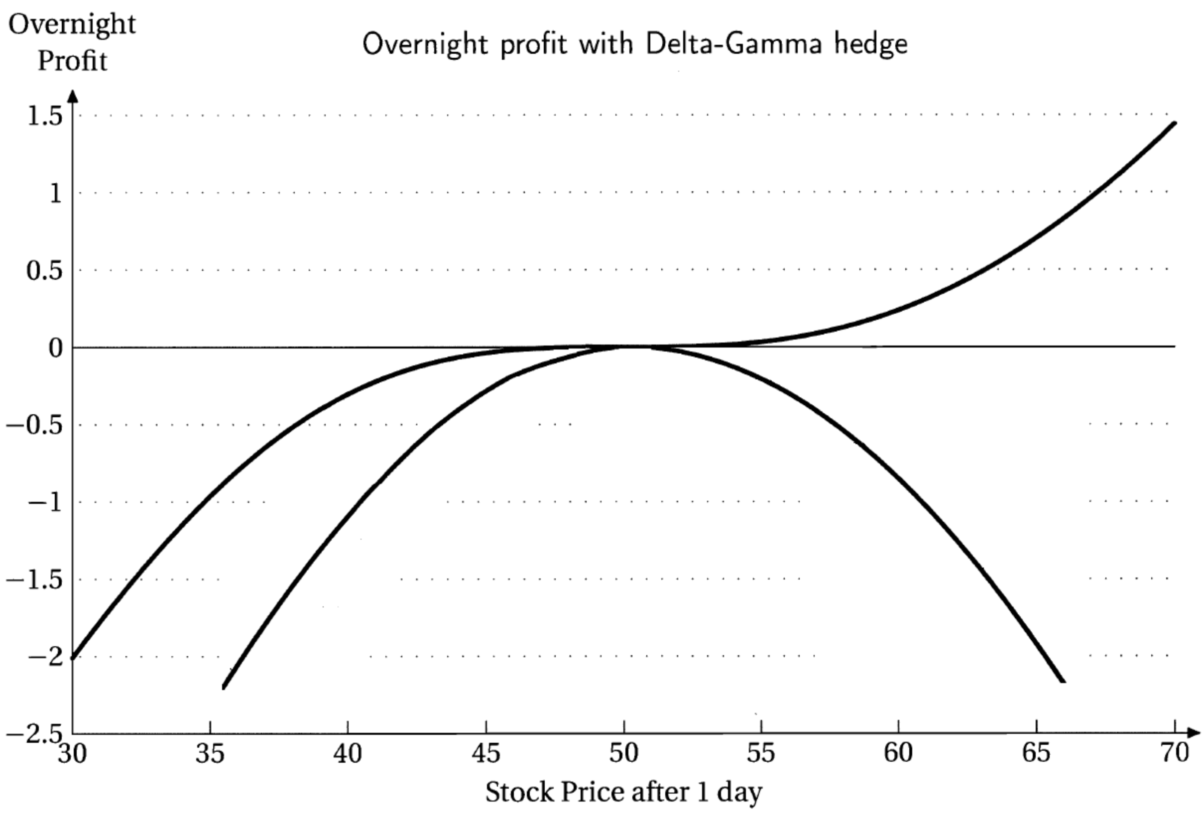
$$\begin{aligned}
&d_2 = -0.0239687753761537 - 0.25\sqrt{364/365} = -0.273626074735270525, \quad N(d_2) = 0.392186\\
&C_{60}(\tfrac{364}{365}) = 55(0.49) - 60e^{-0.05(364/365)}(0.392186) = 4.5634017681091687
\end{aligned}$$

The market-maker's overnight profit is

$$\begin{aligned}
&\Pi(h) - \Pi(0)\\
&= x_1\left(S(h) - S(0)\right) + x_2\left(C_{60}(h) - C_{60}(0)\right) - \left(C_{50}(h) - C_{50}(0)\right) - \left(e^{rh} - 1\right)\left(x_1S + x_2C_{60} - C_{50}\right)\\
&= 5(0.2743) + 1.0293\left(4.5634 - 2.5127\right) - \left(9.644 - 6.168\right) - 10.13332211\left(e^{0.05(1/365)} - 1\right)\\
&= 0.0048972886
\end{aligned}$$

If we consider portfolio of delta hedge $\Pi = \Delta_{50}S - C_{50} - (\overbrace{\Delta_{50}S - C_{50}}^{\text{bond}})$, then the market-maker's overnight profit if the stock's price increases to 55 on the next day is

$$\begin{aligned}\text{Profit} &= \Delta_{50}\left(S(h)-S(0)\right)-\left(C_{50}(h)-C_{50}(0)\right)-\left(e^{rh}-1\right)\left(\Delta_{50}S(0)-C_{50}(0)\right)\\ &= 0.6274(5)-(9.644-6.168)-\left(e^{0.05(1/365)}-1\right)(0.6274(50)-6.168)\\ &= -0.3424525652387956834\end{aligned}$$



Delta hedging protects the portfolio when $49.34572 = S(0)-S(0)\sigma\sqrt{\frac{1}{365}} \le S(\frac{1}{365}) \le S(0)+S(0)\sigma\sqrt{\frac{1}{365}} = 50.65428$. However, from the graph, delta-gamma hedging protects the portfolio in a larger range than delta hedging.

Gamma neutrality protects against large changes in the price of the underlying asset between hedge rebalancing. Vega neutrality protects against a variable σ.

Example 5.28

Consider a portfolio that is delta neutral, with a gamma of -5000 and a vega of -8000. The options shown in the table below can be traded.

	Delta	*Gamma*	*Vega*
Portfolio	0	−5000	−8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

To make the portfolio gamma and vega neutral, both Option 1 and Option 2 can be used. If w_1 and w_2 are the quantities of Option 1 and Option 2 that are added to the portfolio, we require

$$-5000+0.5w_1+0.8w_2=0 \qquad \text{and} \qquad -8000+2w_1+1.2w_2=0.$$

The solution to these equations is $w_1=400, w_2=6000.$ The portfolio can therefore be made gamma and vega neutral by including 400 of Option 1 and 6000 of Option 2. The delta of the portfolio, after the addition of the positions in the two traded options, is $400\times 0.6+6000\times 0.5=3240.$ Hence, 3240 units of the asset would have to be sold to maintain delta neutrality.

Option elasticity is percentage change in the value of an option relative to the percentage change in the value of a stock. If the change in stock price is $\varepsilon,$ then the change in option price is $\varepsilon\Delta.$ The percentage change in stock price and option price is $\frac{\varepsilon}{S(t)}\times 100\%$ and $\frac{\varepsilon\Delta}{V(t)}\times 100\%$ respectively. So the elasticity (denoted by Ω) is

$$\Omega=\frac{\varepsilon\Delta/V(t)}{\varepsilon/S(t)}=\frac{\Delta S(t)}{V(t)}.$$

Let γ be the discount rate for the option (P.66). Since $V=\Delta S+B,$ after time h, we have

$$Ve^{\gamma h}=\Delta Se^{\alpha h}+Be^{rh} \quad \text{and} \quad e^{\gamma h}=\frac{\Delta S}{V}e^{\alpha h}+\frac{B}{V}e^{rh}=\frac{\Delta S}{V}e^{\alpha h}+\left(1-\frac{\Delta S}{V}\right)e^{rh}=\Omega e^{\alpha h}+\left(1-\Omega\right)e^{rh}.$$

If we differentiate the above equations with respect to h at 0, we have

$$\gamma V = \alpha\Delta S + rB \qquad \text{and} \qquad \gamma - r = \Omega(\alpha - r).$$

$$\begin{aligned} dV(t) &= \Delta dS(t) + dB(t) \\ &= \Delta\big((\alpha-\delta)S(t)dt + \sigma S(t)dZ(t)\big) + rB(t)dt \\ &= \big(\Delta(\alpha-\delta)S(t) + rB(t)\big)dt + \Delta\sigma S(t)dZ(t) \end{aligned}$$

$$\frac{dV(t)}{V(t)} = \left(\frac{\Delta(\alpha-\delta)S(t)+rB(t)}{V(t)}\right)dt + \frac{\Delta\sigma S(t)}{V(t)}dZ(t)$$

Hence, $\sigma_{\text{option}} = \left|\frac{\Delta S(t)}{V(t)}\right|\sigma_{\text{stock}} = |\Omega|\sigma_{\text{stock}}$.

If $\delta = 0$, then $\frac{dV(t)}{V(t)} = \left(\frac{\alpha\Delta S(t)+rB(t)}{V(t)}\right)dt + \frac{\Delta\sigma S(t)}{V(t)}dZ(t) = \gamma dt \pm |\Omega|\sigma_{\text{stock}}dZ(t).$

For a call, $\Omega \geq 1$. $\gamma \geq \alpha$ since a call option is a leveraged stock portfolio. (B is negative, meaning that you borrow money to purchase stock). Elasticity Ω is high when the call is out of the money and the call is short-lived, since the value of the call is very small compared to the value of the underlying asset. For a put, $\Omega \leq 0$. $\gamma \leq r$ since you invest money at the risk-free rate and sell an asset on which you must pay a discount rate (equivalent to a leveraged position in selling stock). Elasticity Ω is low for short-lived puts far out of the money, for the same reason as for call: the value of the put is very small compared to the value of the underlying asset.

The Sharpe ratio of a call option on a stock is

$$\frac{\gamma - r}{\sigma_{\text{call option}}} = \frac{\Omega(\alpha - r)}{\Omega\sigma_{\text{stock}}} = \frac{\alpha - r}{\sigma_{\text{stock}}},$$

the same as for the underlying stock. For a put option, it would be negative that of the underlying stock,

$$\frac{\gamma - r}{\sigma_{\text{put option}}} = \frac{\Omega(\alpha - r)}{-\Omega\sigma_{\text{stock}}} = -\frac{\alpha - r}{\sigma_{\text{stock}}}.$$

To calculate the elasticity of a portfolio $\Pi = \sum_{i=1}^{n} n_i V_i$, take a weighted average of the elasticity of the options in it. That is

$$\Omega_\Pi = \frac{S\Delta_\Pi}{\Pi} = \frac{S\sum n_i \Delta_i}{\Pi} = \sum_{i=1}^{n}\left(\frac{n_i V_i}{\Pi}\right)\frac{S\Delta_i}{V_i} = \sum_{i=1}^{n} w_i \Omega_i$$

where $w_i = \dfrac{n_i V_i}{\Pi}$.

Example 5.29

$S(0) = 55$, $\delta = 0.025, \sigma = 0.2, r = 0.04$. A portfolio has 3 European call options on this stock:

The first allows purchase of 100 shares of the stock at the end of 3 months at strike price 55.

The second allows purchase of 200 shares of the stock at the end of 6 months at strike price 60.

The third allows purchase of 200 shares of the stock at the end of 1 year at strike price 65.

Calculate elasticity for this portfolio.

Solution

For the first option,

$$\begin{aligned}
d_1 &= \frac{\ln(55/55) + (0.04 - 0.025 + 0.5(0.2^2))(0.25)}{0.2\sqrt{0.25}} = 0.0875 \\
N(d_1) &= N(0.0875) = 0.53486 \\
\Delta &= e^{-\delta T} N(d_1) = e^{-0.00625}(0.53486) = 0.53153 \\
d_2 &= d_1 - 0.2\sqrt{0.25} = 0.0875 - 0.1 = -0.0125 \\
N(d_2) &= N(-0.0125) = 0.49501 \\
C &= (55)(0.53153) - 55(e^{-0.01})(0.49501) = 2.279
\end{aligned}$$

For the second option,

$$\begin{aligned}
d_1 &= \frac{\ln(55/60) + (0.04 - 0.025 + 0.5(0.2^2))(0.5)}{0.2\sqrt{0.5}} = -0.49152 \\
N(d_1) &= N(-0.49152) = 0.31153 \\
\Delta &= e^{-\delta T} N(d_1) = e^{-0.0125}(0.31153) = 0.30766
\end{aligned}$$

$$d_2 = d_1 - 0.2\sqrt{0.5} = -0.49152 - 0.14142 = -0.63294$$
$$N(d_2) = N(-0.63294) = 0.26339$$
$$C = (55)(0.30766) - 60(e^{-0.02})(0.26339) = 1.431$$

For the third option,

$$d_1 = \frac{\ln(55/65) + (0.04 - 0.025 + 0.5(0.2^2))}{0.2} = -0.66027$$
$$N(d_1) = N(-0.66027) = 0.25454$$
$$\Delta = e^{-\delta T} N(d_1) = e^{-0.025}(0.25454) = 0.24826$$
$$d_2 = d_1 - 0.2 = -0.66027 - 0.2 = -0.86027$$
$$N(d_2) = N(-0.86027) = 0.19482$$
$$C = (55)(0.24826) - 65(e^{-0.04})(0.19482) = 1.487$$

Delta for the portfolio is the weighted sum

$$\Delta_{\text{portfolio}} = 100(0.53153) + 200(0.30766) + 200(0.24826) = 164.3$$

and the value of the portfolio is

$$100(2.279) + 200(1.431) + 200(1.487) = 811.5,$$

making the elasticity $\dfrac{S\Delta}{C} = \dfrac{55(164.3)}{811.5} = 11.14.$

Alternatively, we could calculate the elasticity of each call and then take their weighted average.

	Call premium C	Elasticity $S\Delta/C$	Weight w	Elasticity times weight Ωw
#1	2.279	$\frac{(55)(0.53153)}{2.279} = 12.83$	$\frac{(100)(2.279)}{811.5} = 0.28084$	$(12.83)(0.28084) = 3.60$
#2	1.431	$\frac{(55)(0.30766)}{1.431} = 11.82$	$\frac{(200)(1.431)}{811.5} = 0.3527$	$(11.82)(0.3527) = 4.17$
#3	1.487	$\frac{(55)(0.24825)}{1.487} = 9.18$	$\frac{(200)(1.487)}{811.5} = 0.3665$	$(9.18)(0.3665) = 3.37$
Total			1	$\boxed{\mathbf{11.14}}$

Example 5.30

A financial institution has sold a European call option on 100,000 shares of a non-dividend-paying stock. Suppose $S(0)=49,\quad K=50,\quad r=0.05,\quad \sigma=0.2,\quad T=20/52$.

$$d_1=\frac{\ln(49/50)+\left(0.05+0.5(0.2)^2\right)(20/52)}{0.2\sqrt{20/52}}=0.0541813519238358$$

$$\Delta=N(d_1)=N(0.05366)=0.521604661$$

$$d_2=0.0541813519238358-0.2\sqrt{20/52}=-0.06985338266537266$$

$$N(d_2)=N(-0.0702)=0.47215517884$$

$$C=49(0.521604661)-50e^{-0.05(20/52)}(0.47215517884)=2.4005273201595353$$

The price of the option is about \$240,000.

One strategy open to the financial institution is to do nothing. This is sometimes referred to as a naked position. It is a strategy that works well if the stock price is below \$50 at the end of the 20 weeks. The option then costs the financial institution nothing and it makes a profit of \$240,000. A naked position works less well if the call is exercised because the financial institution then has to buy 100,000 shares at the market price prevailing in 20 weeks to cover the call. The cost to the financial institution is 100,000 times the amount by which the stock price exceeds the strike price. For example, if after 20 weeks the stock price is \$60, the option costs the financial institution \$1,000,000. This is considerably greater than the future value of the option $\$240,000e^{0.05\times(20/52)}=244,660$.

As an alternative to a naked position, the financial institution can adopt a covered position. This involves buying 100,000 shares as soon as the option has been sold. If the option is exercised, this strategy works well, but in other circumstances it could lead to a significant loss. For example, if the stock price drops to \$40, the financial institution loses \$900,000 on its stock position. This is considerably greater than the future value of the option $\$240,000e^{0.05\times(20/52)}=244,660$.

Neither a naked position nor a covered position provides a good hedge. If the assumptions underlying the Black-Scholes formula hold, the cost to the financial institution should always be \$240,000 on average for both approaches. But on any one occasion the cost is liable to range from zero to over \$1,000,000. A good hedge would ensure that the cost is always close to \$240,000.

Tables 1 and 2 provide two examples of the operation of delta hedging for Example 5.30. The hedge is assumed to be adjusted or rebalanced weekly. The initial value of delta for the option being sold is calculated in Example 5.30 as 0.522. This means that the delta of the short option position is initially $-52,200$. As soon as the option is written, \$2,557,800 must be borrowed to buy 52,200 shares at a price of \$49. The rate of interest is 5%. An interest cost of approximately \$2,500 is therefore incurred in the first week.

Table 1 Simulation of delta hedging. Option closes in the money and cost of hedging is \$263,300.

Week	*Stock price*	*Delta*	*Shares purchased*	*Cost of shares purchased (\$000)*	*Cumulative cost including interest (\$000)*	*Interest cost (\$000)*
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	48.12	0.458	(6,400)	(308.0)	2,252.3	2.2
2	47.37	0.400	(5,800)	(274.7)	1,979.8	1.9
3	50.25	0.596	19,600	984.9	2,966.6	2.9
4	51.75	0.693	9,700	502.0	3,471.5	3.3
5	53.12	0.774	8,100	430.3	3,905.1	3.8
6	53.00	0.771	(300)	(15.9)	3,893.0	3.7
7	51.87	0.706	(6,500)	(337.2)	3,559.5	3.4
8	51.38	0.674	(3,200)	(164.4)	3,398.5	3.3
9	53.00	0.787	11,300	598.9	4,000.7	3.8
10	49.88	0.550	(23,700)	(1,182.2)	2,822.3	2.7
11	48.50	0.413	(13,700)	(664.4)	2,160.6	2.1
12	49.88	0.542	12,900	643.5	2,806.2	2.7
13	50.37	0.591	4,900	246.8	3,055.7	2.9
14	52.13	0.768	17,700	922.7	3,981.3	3.8
15	51.88	0.759	(900)	(46.7)	3,938.4	3.8
16	52.87	0.865	10,600	560.4	4,502.6	4.3
17	54.87	0.978	11,300	620.0	5,126.9	4.9
18	54.62	0.990	1,200	65.5	5,197.3	5.0
19	55.87	1.000	1,000	55.9	5,258.2	5.1
20	57.25	1.000	0	0.0	5,263.3	

Table 2 Simulation of delta hedging. Option closes out of the money and cost of hedging is $256,600.

Week	*Stock price*	*Delta*	*Shares purchased*	*Cost of shares purchased ($000)*	*Cumulative cost including interest ($000)*	*Interest cost ($000)*
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	49.75	0.568	4,600	228.9	2,789.2	2.7
2	52.00	0.705	13,700	712.4	3,504.3	3.4
3	50.00	0.579	(12,600)	(630.0)	2,877.7	2.8
4	48.38	0.459	(12,000)	(580.6)	2,299.9	2.2
5	48.25	0.443	(1,600)	(77.2)	2,224.9	2.1
6	48.75	0.475	3,200	156.0	2,383.0	2.3
7	49.63	0.540	6,500	322.6	2,707.9	2.6
8	48.25	0.420	(12,000)	(579.0)	2,131.5	2.1
9	48.25	0.410	(1,000)	(48.2)	2,085.4	2.0
10	51.12	0.658	24,800	1,267.8	3,355.2	3.2
11	51.50	0.692	3,400	175.1	3,533.5	3.4
12	49.88	0.542	(15,000)	(748.2)	2,788.7	2.7
13	49.88	0.538	(400)	(20.0)	2,771.4	2.7
14	48.75	0.400	(13,800)	(672.7)	2,101.4	2.0
15	47.50	0.236	(16,400)	(779.0)	1,324.4	1.3
16	48.00	0.261	2,500	120.0	1,445.7	1.4
17	46.25	0.062	(19,900)	(920.4)	526.7	0.5
18	48.13	0.183	12,100	582.4	1,109.6	1.1
19	46.63	0.007	(17,600)	(820.7)	290.0	0.3
20	48.12	0.000	(700)	(33.7)	256.6	

The portfolio is not permanently riskless. It is riskless only for an infinitesimally short period of time. As S and t change, $\Delta = \frac{\partial \Pi}{\partial S}$ also changes. To keep the portfolio riskless, it is therefore necessary to frequently change the relative proportions of the derivative and the stock in the portfolio.

In Table 1 the stock price falls by the end of the first week to $48.12. The delta of the option declines to 0.458, so that the new delta of the option position is $-45,800$. This means that $52,200-45,800=6,400$ of the shares initially purchased are sold to maintain the hedge. The strategy realizes \$308,000 in cash, and the cumulative borrowings at the end of week 1 are reduced to \$2,252,300. During the second week, the stock price reduces to \$47.37, delta declines again, and so on. Toward the end of the life of the option, it becomes apparent that the option will be exercised and the delta of the option approaches 1. By week 20, therefore, the hedger has a fully covered position. The hedger receives \$5 million for the stock held, so that the total cost of writing the option and hedging it is \$263,300.

Table 2 illustrates an alternative sequence of events such that the option closes out of the money. As it becomes clear that the option will not be exercised, delta approaches zero. By week 20 the hedger has a naked position and has incurred costs totaling \$256,600.

In Tables 1 and 2, the costs of hedging the option, when discounted to the beginning of the period, are close to but not exactly the same as the Black-Scholes price of \$240,000. If the hedging worked perfectly, the cost of hedging would, after discounting, be exactly equal to the Black-Scholes-Merton price for every simulated stock price path. The reason for the variation in the cost of hedging is that the hedge is rebalanced only once a week. As rebalancing takes place more frequently, the variation in the cost of hedging is reduced. Of course, the examples in Tables 1 and 2 are idealized in that they assume that the volatility is constant and there are no transaction costs.

The delta-hedging procedure in Tables 1 and 2 creates the equivalent of a long position in the option. This neutralizes the short position the financial institution created by writing the option. As the tables illustrate, delta hedging a short position generally involves selling stock just after the price has gone down and buying stock just after the price has gone up. It might be termed a buy-high, sell-low trading strategy! The cost of \$240,000 comes from the average difference between the price paid for the stock and the price at which it is sold.

Delta-hedging is a dynamic strategy; the delta hedge is only instantaneously risk-free, and it requires a continuous rebalancing of the portfolio and the ratio of the holdings in the asset and the derivative product. The delta-hedge position must be monitored continually, and in practice it can suffer from losses due to transaction costs.

Suppose a market-maker writes an option and delta-hedges it in every period of length h years. Let $R_{h,i}$ be the profit in the i-th period. Suppose the profit in each period is independent. If the change of stock $\varepsilon = \sigma S(0)\sqrt{h}X_i$ and X_i is a standard normal random variable, then

$$R_{h,i} = \tfrac{1}{2}S(0)^2\sigma^2(1-X_i^2)\Gamma h.$$

Since $M_{X_i}(t) = e^{\frac{1}{2}t^2} = 1+\frac{1}{2}t^2+\frac{1}{2}\left(\frac{1}{2}t^2\right)^2+\cdots = 1+\frac{1}{2!}t^2+\frac{3}{4!}t^4+\cdots$, then $E[X_i^2]=1$ and $E[X_i^4]=3$.

$$\begin{aligned}
E[R_{h,i}] &= \tfrac{1}{2}S(0)^2\sigma^2\Gamma h E\left[1-X_i^2\right] = 0,\\
Var[R_{h,i}] &= \left(\tfrac{1}{2}S(0)^2\sigma^2\Gamma h\right)^2 Var\left[1-X_i^2\right]\\
&= \left(\tfrac{1}{2}S(0)^2\sigma^2\Gamma h\right)^2 Var\left[X_i^2\right]\\
&= \left(\tfrac{1}{2}S(0)^2\sigma^2\Gamma h\right)^2 \left(E\left[X_i^4\right]-E\left[X_i^2\right]^2\right)\\
&= \left(\tfrac{1}{2}S(0)^2\sigma^2\Gamma h\right)^2 (3-1)\\
&= \tfrac{1}{2}\left(S(0)^2\sigma^2\Gamma h\right)^2.
\end{aligned}$$

If we rehedge every h year, that means we will rehedge $\frac{1}{h}$ times per year.

$$\begin{aligned}
&\text{Annual expected profit} = \tfrac{1}{h}E[R_{h,i}] = 0,\\
&\text{Annual variance of profit} = \tfrac{1}{h}Var[R_{h,i}] = \tfrac{1}{2}\left(S(0)^2\sigma^2\Gamma\right)^2 h.
\end{aligned}$$

The annual variance is proportional to h. The smallest h, the smallest the variance.

Example 5.31

The price of a stock is 40. Its volatility is 0.26. Γ of a call option is 0.0131. A delta-hedged market maker writes the call option and rehedges once a month. Calculate the annual variance of the market-maker's profit.

Solution

$$\text{Annual variance of profit} = \tfrac{1}{h}Var[R_{h,i}] = \tfrac{1}{2}\left(40^2(0.26)^2(0.0131)\right)^2\tfrac{1}{12} = 0.0836.$$

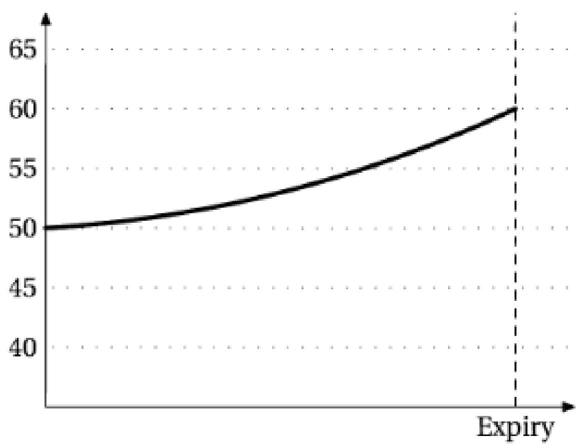
Chapter 6 Exotic Options

A **barrier** option has a payoff which depends on whether over the lifetime of the option the underlying asset hit the barrier. There are 3 basic types: **knock-out options**, **knock-in options**, and **rebate options**.

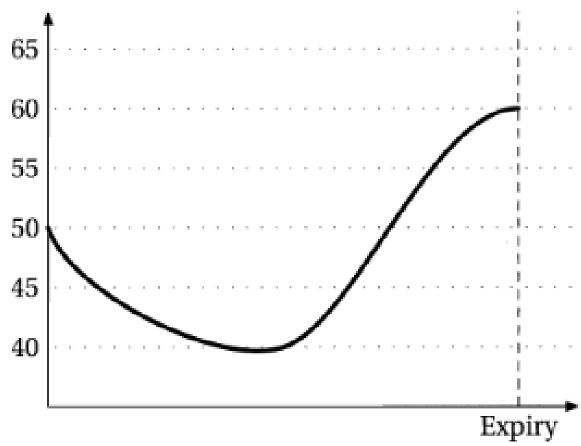
Knock-out option behaves like a European option, except that it does not pay if the underlying asset hits a specified barrier. 2 possibilities are **up-and-out**, meaning that if the price rises to the barrier then the option doesn't pay, and **down-and-out**, meaning that if the price falls to the barrier then the option doesn't pay.

Suppose the stock price is 50. Consider a knock-out barrier call option with strike price 50 and barrier 65 in below figure; this would be an up-and-out option. Figure (a) and (b) will pay 10 since the stock price never hit 65. Figure (c) will not pay since the stock price hit 65 in the middle, whereas figure (d) will not pay even though it didn't hit a barrier because the final price isn't more than the strike price.

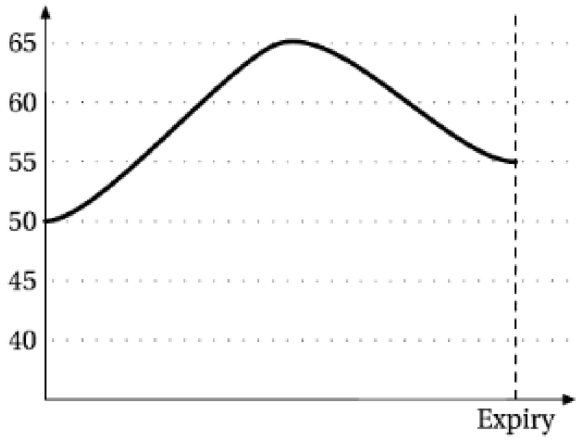
Now consider a knock-out barrier call option with strike price 50 and barrier 40; this would be a down-and-out option. Figure (a) will pay 10 and figure (c) will pay 5, but figure (b) will not pay since the stock price hit 40, while figure (d) will not pay both because the stock price went down to 40 at some point and because the final price doesn't exceed the strike price.



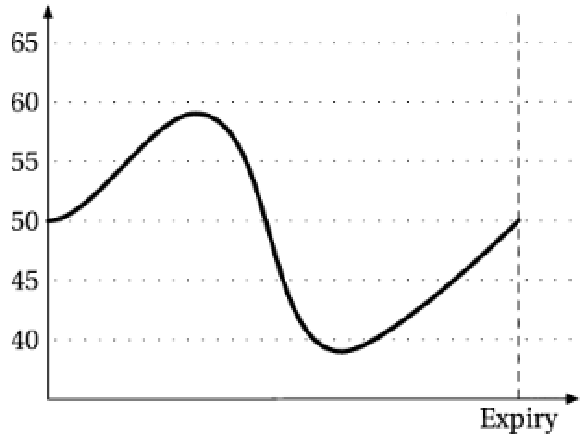
(a)



(b)



(c)



(d)

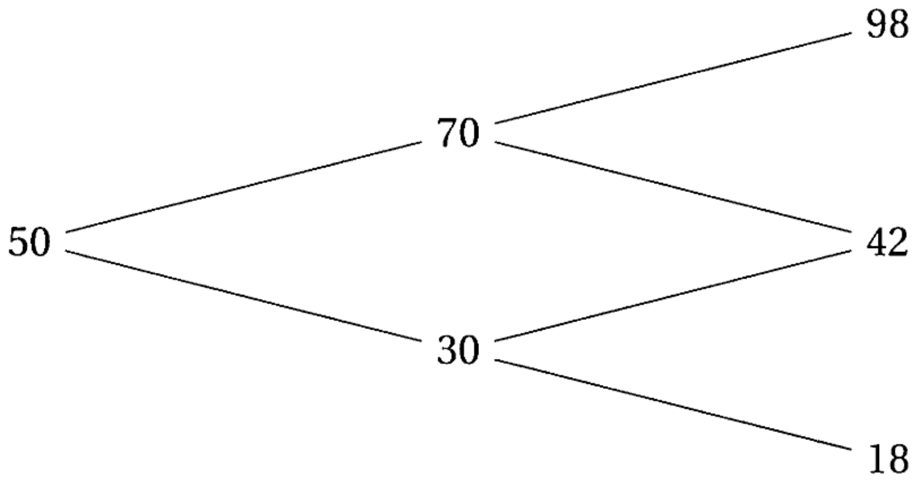
Knock-in option pays only if the underlying asset hits the barrier. There still will only be a payment if the final price is above (call) or below (put) the strike price. 2 possibilities are **up-and-in**, meaning that the underlying asset's price must rise to the barrier in order for the option to qualify for payment, and **down-and-in**, meaning that the underlying asset's price must fall to the barrier in order for the option to qualify for payment.

Consider a knock-in barrier call option with strike price 50 and barrier 65; this would be an up-and-in option. The situation in figure (c) will pay 5, since the stock hit the barrier and the final price is 55, 5 more than the strike price. The situations in figure (a), (b) and (d) will not pay because the stock price never rose to 65. If the barrier were 55, all situations would pay except for figure (d), where even though it hit the barrier, the final price is the strike price.

Now consider a knock-in barrier call option with strike price 50 and barrier 45; this would be a down-and-in option. The situation in figure (b) will pay 10 since the barrier was hit. The situations in figure (a), (c) and (d) will not pay. The barrier was hit in figure (d), but the final price is not higher than the strike price.

Example 6.1

Consider the following binomial tree model for stock prices. Each period is 1 year.



Suppose that $r = 0.08$ and $\delta = 0.02$. A 2-year down-and-in barrier put option with barrier 35 and strike price 45 is priced using this tree. Assume the price movements within each period are monotonic. Calculate the price of the option.

Solution

$u = 1.4$ and $d = 0.6$ at all nodes. Then $p^* = \dfrac{e^{0.08-0.02} - 0.6}{1.4 - 0.6} = 0.577296$. The option only pays if the stock first goes down to 30, but then pays at both the 42 and 18 nodes. The payoffs are 3 and 27 respectively at these nodes. So we calculate the expected value of the 2 paths:

$$P = e^{-2(0.08)}\left(3(1-p^*)p^* + 27(1-p^*)^2\right) = e^{-0.16}\left(3(0.422704)(0.577296) + 27(0.422704)^2\right) = 4.735.$$

A **rebate** option pays a fixed amount if the barrier is hit at any time during the duration of the option. The payment may be made either immediately when the barrier is hit or at expiry of the option (the latter is a deferred rebate). There can be “up rebates” (payment if the price goes up to K) or “down rebates” (payment if the price goes down to K).

Example 6.2

The price of a non-dividend paying stock is 50. Its volatility is 0.3. The continuously compounded risk-free rate is 0.05. A 1-year rebate option pays 10 at a barrier of 40. A 4-period Cox-Ross-Rubinstein binomial tree is used to model the option. Assume the price movements within each period are monotonic. Determine the risk neutral price of the rebate option.

Solution

The period h = 0.25. In the Cox-Ross-Rubinstein binomial tree $u = e^{\sigma\sqrt{h}} = e^{0.15} = 1.1618$ and $d = e^{-0.15} = 0.8607$. The risk neutral probability of an up movement is

$$p^* = \frac{e^{rh} - d}{u - d} = \frac{e^{0.05(0.25)} - 0.8607}{1.1618 - 0.8607} = 0.5043.$$

The barrier of 40 can be reached in 3 ways:

1. $dd**$, since $50(0.8607^2) = 37.0409$. Its price is $10e^{-0.05(0.5)}(1-0.5043)^2 = 2.3965168.$

2. $uddd$, since $50(1.1618)(0.8607^3) = 37.0409$. Its price is $10e^{-0.05}(0.5043)(1-0.5043)^3 =$ 0.584293427

3. $dudd$. Its price is $10e^{-0.05}(0.5043)(1-0.5043)^3 = 0.584293427.$

The risk neutral price of the rebate option is

$$2.3965168 + 2(0.584293427) = 3.5651.$$

A barrier option (but not rebate) must be worth less than one without a barrier. In some cases, the barrier has no effect and the values are equal.

For example, suppose that the stock price is 50. Consider a put option with strike price 40. In order for the option to pay even without barrier, the price of the stock must decline below 40. A down-and-in barrier of 45 would have no effect on the value of the put, since the stock price falling below 40 at expiry must hit the price of 45.

Another down-and-in barrier of 35 put option means that the stock must decline to 35 at some time during the option duration and then be worth less than 40 at expiry in order for the option to pay. The lower the barrier, the less the value of the put option.

There is a parity relationship for barrier options having the same barrier:

Knock-in option + Knock-out option = Ordinary option

There are 4 types of **all-or-nothing** options:

1. **asset-or-nothing** call option has payoff

$$\begin{cases} S(T) & \text{if } S(T) > K \\ 0 & \text{otherwise} \end{cases}$$

at expiry time T. Let us denote this option by $S(T)\big|S(T) > K$. Its price at time 0 is $F_{0,T}^{P}(S)N(d_1)$.

2. **asset-or-nothing** put option has payoff

$$\begin{cases} S(T) & \text{if } S(T) < K \\ 0 & \text{otherwise} \end{cases}$$

at expiry time T. Let us denote this option by $S(T)\big|S(T) < K$. Its price at time 0 is $F_{0,T}^{P}(S)N(-d_1)$.

3. **cash-or-nothing** call option has payoff

$$\begin{cases} 1 & \text{if } S(T) > K \\ 0 & \text{otherwise} \end{cases}$$

at expiry time T. Let us denote this option by $1\big|S(T) > K$. Its price at time 0 is $e^{-rT}N(d_2)$.

4. **cash-or-nothing** put option has payoff

$$\begin{cases} 1 & \text{if } S(T) < K \\ 0 & \text{otherwise} \end{cases}$$

at expiry time T. Let us denote this option by $1\big|S(T) < K$. Its price at time 0 is $e^{-rT}N(-d_2)$.

Clearly,

$$\left(S(T)\big|S(T) > K\right) + \left(S(T)\big|S(T) < K\right) = F_{0,T}^{P}(S), \quad \left(1\big|S(T) > K\right) + \left(1\big|S(T) < K\right) = e^{-rT},$$
$$C = \left(S(T)\big|S(T) > K\right) - \left(K\big|S(T) > K\right), \quad P = \left(K\big|S(T) < K\right) - \left(S(T)\big|S(T) < K\right).$$

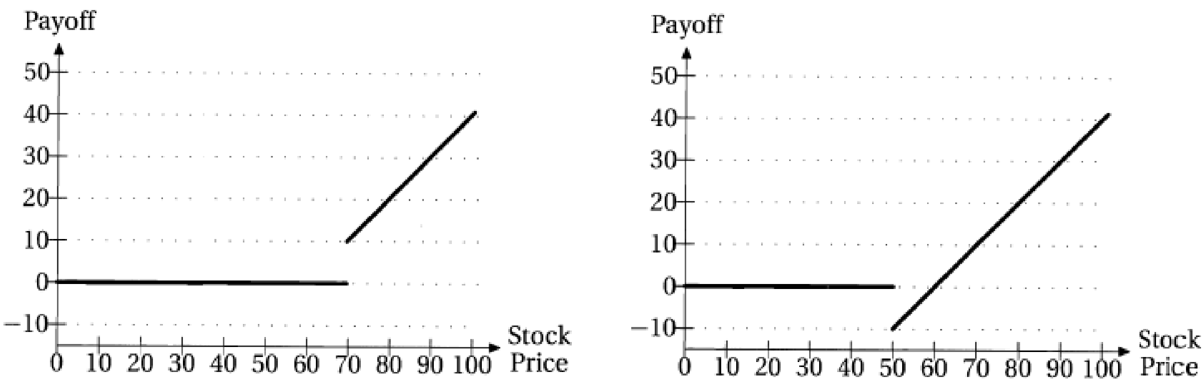
In an ordinary option, the strike price K serves 2 purposes:

1. The option is not exercised unless the final stock price is above (for a call option) or below (for a put option) the strike price.

2. The payment, if exercised, is $S(T)-K$ (for a call option) or $K-S(T)$ for a put option.

These 2 purposes can be split. We can select a **payment trigger**, say K_2. The option **must be exercised** when the price is above (call) or below (put) the trigger, and cannot be exercised otherwise. Let K_1 be the strike price, so that the option pays $S(T)-K_1$ (call) or $K_1-S(T)$ (put) if exercised.

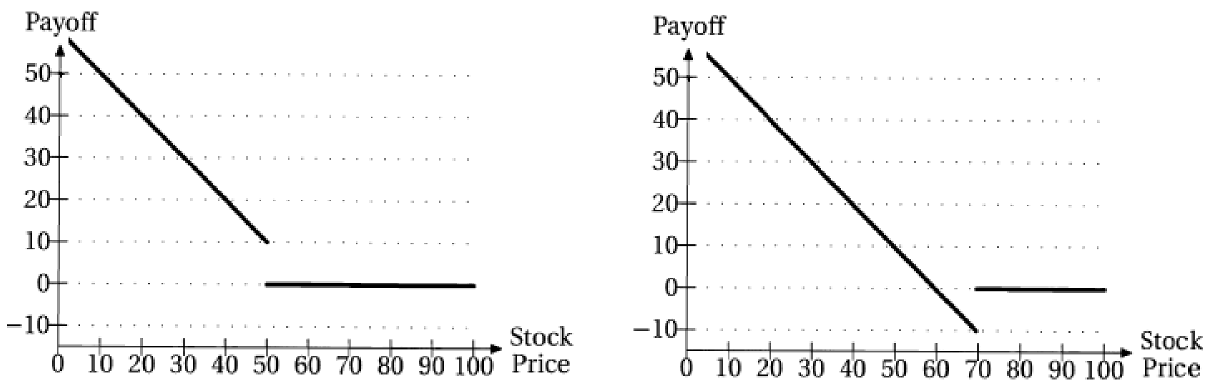
An option with a trigger and a strike price where the two are unequal is called a **gap option**.



(a) Trigger is 70; strike price is 60.

(b) Trigger is 50; strike price is 60.

Payoff on gap call options



(c) Trigger is 50; strike price is 60.

(d) Trigger is 70; strike price is 60.

Payoff on gap put options

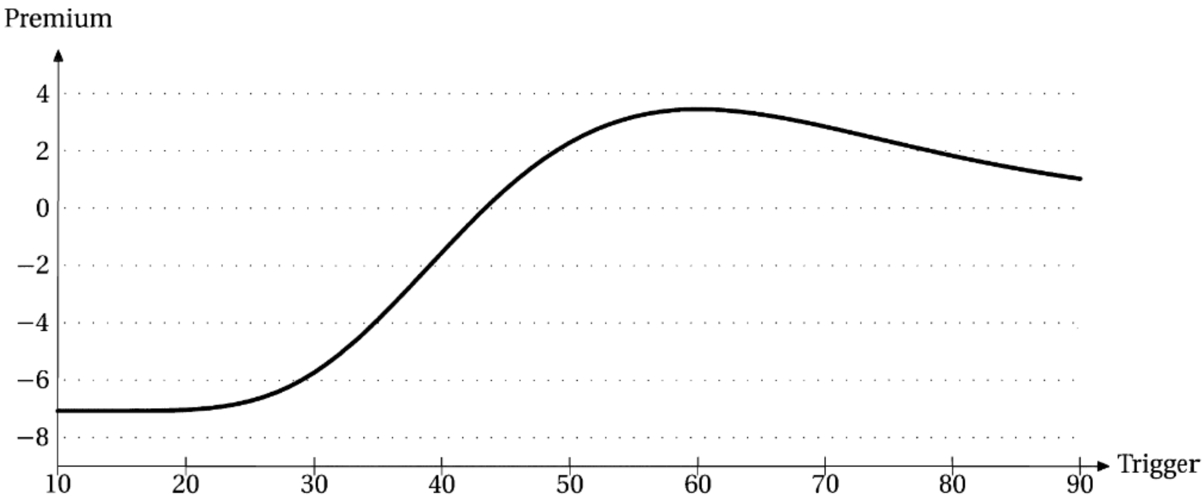
Suppose a gap option on a stock has a trigger 70 and strike 60. Then the option pays

$$\begin{cases} S(T)-60 & \text{if } S(T)>70 \\ 0 & \text{otherwise.} \end{cases}$$

There is a discontinuity for stock price 70, since nothing is paid if $S(T) < 70$ but 10 is paid if $S(T) = 70$. The payoff is illustrated in figure (a).

If the trigger is less than the strike price, negative payoffs are possible. For example, if the trigger is 50 and the strike is 60, then if the stock price is 54, the purchaser of the gap call option receive $54-60=-6$, or in other words pays 6 to the seller of the option. The payoff is illustrated in figure (b). Such an option may have a negative premium. In this case, the gap call option holder receives money from the seller when buying this option because the option holder has a chance to pay money during expiration date.

A gap put option is similar. If the trigger is 50 and the strike is 60, the payoffs are illustrated in figure (c). If the trigger is 70 and the strike is 60, the payoffs are illustrated in figure (d) and will be negative when the stock price is between 60 and 70.



Gap call option premiums with varying payment triggers. Assumes $S=50$, $K_1=60$, $r=0.05$, $\sigma=0.3$, $T=1$, $\delta=0$.

Let us consider varying payment triggers on a European option. The highest premium occurs when the trigger equals the strike price. When the trigger exceeds the strike price, as in figure (a), the purchaser does not get payoffs when the stock price is between the strike price and trigger, so that the option is worth less (although it still has a positive value). When the trigger is less than the strike price, as in figure (b), the purchaser gets additional payoffs that are not provided for an ordinary call option when the stock price is between the strike price and the trigger. However the additional payoffs are negative. As a result, the gap option is worth less than the ordinary option, and may even have a negative premium. The same argument with simple modifications works for gap put options.

Based on the pricing of all-or-nothing options, you can price gap options using a modified version of the Black-Scholes price formula. A gap call option is $C = \left(S(T) \middle| S(T) > K_2\right)$ $-\left(K_1 \middle| S(T) > K_2\right)$, while a gap put option is $P = \left(K_1 \middle| S(T) < K_2\right) - \left(S(T) \middle| S(T) < K_2\right)$, so the Black-Scholes price is computed by using the trigger K_2 to determines whether a payoff occurs (when computing d_1 and d_2), and using the strike price K_1 to determine the size of payment. Thus

$$\begin{gathered} C(S,K_1,K_2,\sigma,r,t,T,\delta) = F_{t,T}^P(S)N(d_1) - K_1 e^{-r(T-t)}N(d_2), \\ P(S,K_1,K_2,\sigma,r,t,T,\delta) = K_1 e^{-r(T-t)}N(-d_2) - F_{t,T}^P(S)N(-d_1) \end{gathered}$$

where $d_1 = \dfrac{\ln\left(F_{t,T}^P(S)/K_2\right) + \left(r + 0.5\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = \dfrac{\ln\left(F_{t,T}^P(S)/K_2\right) + \left(r - 0.5\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$.

Another way to look at a gap option is as an ordinary option plus cash-or-nothing options. Thus a gap call option is

$$C(S,K_1,K_2) = C(S,K_2,K_2) - (K_1 - K_2)\big| S > K_2 = C(S,K_2,K_2) - (K_1 - K_2)e^{-r(T-t)}N(d_2)$$

and a gap put option is

$$P(S,K_1,K_2) = P(S,K_2,K_2) + (K_1 - K_2)\big| S < K_2 = P(S,K_2,K_2) + (K_1 - K_2)e^{-r(T-t)}N(-d_2).$$

Delta for all-or-nothing options on stock with continuous yield rate δ	Call	Put
Cash-or-nothing	$\frac{e^{-\delta\tau}}{K\sigma\sqrt{2\pi\tau}}e^{-\frac{d_1^2}{2}}$	$-\frac{e^{-\delta\tau}}{K\sigma\sqrt{2\pi\tau}}e^{-\frac{d_1^2}{2}}$
Asset-or-nothing	$e^{-\delta\tau}N(d_1)+\frac{e^{-\delta\tau}}{\sigma\sqrt{2\pi\tau}}e^{-\frac{d_1^2}{2}}$	$e^{-\delta\tau}N(-d_1)-\frac{e^{-\delta\tau}}{\sigma\sqrt{2\pi\tau}}e^{-\frac{d_1^2}{2}}$

Delta hedging is not effective for gap options because the payoff on a gap option is discontinuous at the trigger. The delta for a gap call is

$$\begin{aligned}\frac{\partial C}{\partial S} &= e^{-\delta(T-t)}N(d_1)-(K_1-K_2)e^{-r(T-t)}\frac{\partial N(d_2)}{\partial S}\\ &= e^{-\delta(T-t)}N(d_1)-(K_1-K_2)e^{-r(T-t)}\frac{e^{-d_2^2/2}}{\sqrt{2\pi}S(t)\sigma\sqrt{T-t}}\end{aligned}$$

and the delta for a gap put is

$$\begin{aligned}\frac{\partial P}{\partial S} &= e^{-\delta(T-t)}(N(d_1)-1)+(K_1-K_2)e^{-r(T-t)}\frac{\partial N(-d_2)}{\partial S}\\ &= e^{-\delta(T-t)}N(-d_1)-(K_1-K_2)e^{-r(T-t)}\frac{e^{-d_2^2/2}}{\sqrt{2\pi}S(t)\sigma\sqrt{T-t}}\end{aligned}$$

Example 6.3

The initial price of a non-dividend paying stock is 45. Its volatility $\sigma = 0.2$. The continuously compounded risk-free interest rate is 5%. A market-maker sells a European gap call option with trigger 40 and strike price 50 and 1 year to expiry. Calculate

(a) the price of the option,

(b) the number of shares of stock to buy to delta-hedge this option.

Solution

(a) $d_1 = \dfrac{\ln(45/40)+(0.05+0.5(0.2)^2)}{0.2} = 0.9389151782819175$

$d_2 = 0.9389151782819175 - 0.2 = 0.7389151782819175.$

$N(d_1) = 0.82611, \quad N(d_2) = 0.77002$

$C = 45(0.82611) - 50e^{-0.05}(0.77002) = 0.5516659273$

(b) $\dfrac{\partial C}{\partial S} = 0.82611 - 10e^{-0.05}\dfrac{e^{-0.73892^2/2}}{45(0.2)\sqrt{2\pi}} = 0.50520.$

Note that delta may be greater than 1 for a gap option. In the above example, if the strike price is 30, then $\dfrac{\partial C}{\partial S} = 0.82611 + 10e^{-0.05}\dfrac{e^{-0.73892^2/2}}{45(0.2)\sqrt{2\pi}} = 1.46793 > 1.$

A **compound option** is an option whose underlying asset is another option expires later. Suppose the second option has expiration *T* and strike price *K*. Then a compound option sold at time 0 would have an expiration $t < T$ and a strike price *x*, which means you could buy (call) or sell (put) the second option at time *t* for price *x*. Depending on whether the first option is a put or a call and the second option is a put or a call, there are 4 possibilities:

1. $\text{PutOnCall}(S, K, x, \sigma, r, t, T, \delta) = P(C(S, K, T-t), x, t)$ allows selling a call option at *t*.
2. $\text{PutOnPut}(S, K, x, \sigma, r, t, T, \delta) = P(P(S, K, T-t), x, t)$ allows selling a put option at *t*.
3. $\text{CallOnCall}(S, K, x, \sigma, r, t, T, \delta) = C(C(S, K, T-t), x, t)$ allows buying a call option at *t*.
4. $\text{CallOnPut}(S, K, x, \sigma, r, t, T, \delta) = C(P(S, K, T-t), x, t)$ allows buying a put option at *t*.

By Put-Call parity, we have

$$\begin{aligned}
\text{CallOnCall}(S, K, x, \sigma, r, t, T, \delta) - \text{PutOnCall}(S, K, x, \sigma, r, t, T, \delta) &= C(S, K, T) - xe^{-rt}, \\
\text{CallOnPut}(S, K, x, \sigma, r, t, T, \delta) - \text{PutOnPut}(S, K, x, \sigma, r, t, T, \delta) &= P(S, K, T) - xe^{-rt}.
\end{aligned}$$

Bermudan option is somewhere between a European and an American option; it allows early exercise, but only at specified times. In a sense, an American call option on a stock with discrete dividends is something like a Bermudan option, since the only rational times to exercise it early right before dividend payments. Thus you can already express a Bermudan call option on a discrete dividend-paying stock in terms of standard or compound options.

Consider an American option on a stock with 1 discrete dividend. Let t be the time at which the dividend is paid, T the final expiry, and D the dividend. Therefore, the value of the American option at time t (before paying dividend) is

$$\begin{aligned}C_{Amer}(S(t^-),K,T-t^-) &= \max\left\{S(t^+)+D-K, C_{Eur}(S(t^+),K,T-t^+)\right\}\\ &= \max\left\{S(t^+)+D-K, S(t^+)-Ke^{-r(T-t^+)}+P_{Eur}(S(t^+),K,T-t^+)\right\}\\ &= S(t^+)+D-K+\max\left\{0, P_{Eur}(S(t^+),K,T-t^+)+K\left(1-e^{-r(T-t^+)}\right)-D\right\}\end{aligned}$$

Clearly, $\max\left\{0, P_{Eur}(S(t^+),K,T-t^+)+K\left(1-e^{-r(T-t^+)}\right)-D\right\}$ is the payoff of $C\left(P_{Eur}(S(t^+),K,T-t^+), D-K\left(1-e^{-r(T-t^+)}\right), t\right)$ at time t. Discounted to time 0, the value of $\max\left\{0, P_{Eur}(S(t^+),K,T-t^+)+K\left(1-e^{-r(T-t^+)}\right)-D\right\}$ is $\text{CallOnPut}\left(S,K,D-K\left(1-e^{-r(T-t^+)}\right),t,T\right)$, $S(t^+)+D-K$ is $S(0)-Ke^{-rt}$. Thus

$$C_{Amer}(S,K,T) = S(0)-Ke^{-rt}+\text{CallOnPut}\left(S,K,D-K\left(1-e^{-r(T-t^+)}\right),t,T\right).$$

The formula assumes that $D-K\left(1-e^{-r(T-t^+)}\right)>0$. If not, it is not rational to exercise this option early, so its value is the same as that for a European option.

This formula means that such an American call option is equivalent to long a stock, short Ke^{-rt} bonds and long a CallOnPut.

To interpret this formula, it helps to relate the 3 possible outcomes of American call option with the 3 possible outcomes of compound option:

case 1. $P_{Eur}(S(t^+),K,T-t^+) < D-K\left(1-e^{-r(T-t^+)}\right)$

On the left, dividend is larger than the present value of strike price from [t, T] plus the value of a chance of buying a stock below K at T. It is rational to exercise the American option at t^-

and you will get $S(t^+)+D-K$ at t^+.

On the right, $C\left(P_{Eur}(S(t^+),K,T-t^+),D-K\left(1-e^{-r(T-t^+)}\right),t\right)$ is worth 0 at t and will not be exercised. You have 1 share of stock $S(t^+)$ with dividend D but owe K bonds at t^+.

case 2. $P_{Eur}(S(t^+),K,T-t^+)>D-K\left(1-e^{-r(T-t^+)}\right)$

On the left, it is not rational to exercise the American option at t^-.

On the right, exercise $C\left(P_{Eur}(S(t^+),K,T-t^+),D-K\left(1-e^{-r(T-t^+)}\right),t\right)$, you still have 1 share of stock, with dividend D, owe K bonds and pay $D-K\left(1-e^{-r(T-t^+)}\right)$ to get $P_{Eur}(S(t^+),K,T-t^+)$ (a chance of buying a stock below K at T) at t^+. That means, you have 1 share of stock, owe $Ke^{-r(T-t^+)}$ bonds and get $P_{Eur}(S(t^+),K,T-t^+)$.

case 2.1 $S(T)>K$

On the left, exercise the American option and get $S(T)-K$.

On the right, $P_{Eur}(S(t^+),K,T-t^+)$ is worth 0 at T. You have 1 share of stock, owe K bonds.

case 2.2 $S(T)\le K$

On the left, do not exercise the American option and get 0.

On the right, exercise $P_{Eur}(S(t^+),K,T-t^+)$ at T. Pay 1 share of stock for K and close the short position of bonds. You get 0.

Example 6.4

A stock with price 50 will pay a dividend of 5 at time 0.25. An American call option on the stock with strike 45 will expire at time 0.4. $\sigma = 0.3$, $r = 0.06$ and CallOnPut(50, 45, 4.5968, 0.25, 0.4) = 0.1748.

(a) Determine whether it is optimal to exercise the option if the stock price is 45.5 before the payment of the dividend.

(b) Determine the price of the option.

Solution

$$\begin{aligned}
d_1 &= \frac{\ln((45.5-5)/45)+(0.06+0.5(0.3)^2)(0.15)}{0.3\sqrt{0.15}} = -0.77124, \quad N(0.77124)=0.77972 \\
d_2 &= -0.77124-0.3\sqrt{0.15} = -0.88743, \quad N(0.88743)=0.81258 \\
P(40,45,0.3,0.06,0.15,0) &= 45e^{-0.15(0.06)}N(0.88743)-40.5N(0.77124) \\
&= 44.5968(0.81258)-40.5(0.77972) \\
&= 4.6598
\end{aligned}$$

which is greater than $5-45\left(1-e^{-0.06(0.15)}\right)=4.5968$. So the put option should be purchased, which is equivalent to not exercising the option.

(b) $C_{Amer}(S,K,T)=50-45e^{-0.25(0.06)}+0.1748=5.8448.$

Example 6.5

The initial price of a non-dividend paying stock is 80. The continuous compounded risk free interest rate is 0.04. The following prices for compound options with first strike $x=75\left(1-e^{-0.01}\right)=0.746262$ expiring in t = 3 months, second strike K = 75 expiring in T = 6 months, on this stock:

$$\begin{aligned}
&\text{PutOnCall}(S,K,x,\sigma,r,t,T,\delta)=P(C(S,K,T-t),x,t)=0.0225 \\
&\text{PutOnPut}(S,K,x,\sigma,r,t,T,\delta)=P(P(S,K,T-t),x,t)=0.1035 \\
&\text{CallOnPut}(S,K,x,\sigma,r,t,T,\delta)=C(P(S,K,T-t),x,t)=3.3578.
\end{aligned}$$

Bermudan options with strike price 75 and expiry in 6 months allow early exercise only at the end of 3 months. Calculate the value of

(a) Bermudan call option,

(b) Bermudan put option.

Solution

(a) Since there are no dividends, early exercise does not take effect. The Bermudan call option is worth the same as European call option.

$$\begin{aligned}
&\text{CallOnPut}(S,K,x,\sigma,r,t,T,\delta)-\text{PutOnPut}(S,K,x,\sigma,r,t,T,\delta)=P(S,K,T)-xe^{-rt}\\
&3.3578-0.1035=P(S,K,T)-0.746262e^{-0.04(0.25)}\\
&P(S,K,T)=3.993136569\\
&C(S,K,T)=P(S,K,T)+S(0)-Ke^{-rT}=3.993136569+80-75e^{-0.04(0.5)}=10.478236
\end{aligned}$$

The value of Bermudan call is 10.478236.

(b)
$$\begin{aligned}
V(0.25)&=\max\{75-S(0.25),P(S,75,0.25)\}\\
&=\max\left\{75-S(0.25),C(S,75,0.25)+75e^{-0.04(0.25)}-S(0.25)\right\}\\
&=75-S(0.25)+\max\left\{0,C(S,75,0.25)-75\left(1-e^{-0.01}\right)\right\}
\end{aligned}$$

$$\begin{aligned}
&V(0)=75e^{-0.01}-S(0)+\text{CallOnCall}\left(S,75,75\left(1-e^{-0.01}\right)\right)\\
&\text{CallOnCall}(S,K,x,\sigma,r,t,T,\delta)-\text{PutOnCall}(S,K,x,\sigma,r,t,T,\delta)=C(S,K,T)-xe^{-rt}\\
&\text{CallOnCall}\left(S,75,75\left(1-e^{-0.01}\right)\right)=0.0225+10.478236-0.746262e^{-0.04(0.25)}=9.7619\\
&V(0)=74.2537-80+9.7619=4.0156
\end{aligned}$$

Chooser option allows investor to choose, at time $t\le T$, to take either a European call or European put option expiring at time *T*, both with the same strike price *K*.

Let *S* be a stock with continuous dividend rate δ. Let *V* be chooser option.

$$\begin{aligned}
V(t)&=\max\left(C(S,K,T-t),P(S,K,T-t)\right)\\
&=C(S,K,T-t)+\max\left(0,P(S,K,T-t)-C(S,K,T-t)\right)\\
&=C(S,K,T-t)+\max\left(0,Ke^{-r(T-t)}-S(t)e^{-\delta(T-t)}\right)\quad\text{by put-call parity}\\
&=C(S,K,T-t)+e^{-\delta(T-t)}\max\left(0,Ke^{-(r-\delta)(T-t)}-S(t)\right)
\end{aligned}$$

When discounting to time 0, $\max\left\{0,Ke^{-(r-\delta)(T-t)}-S(t)\right\}$ becomes $P\left(S,Ke^{-(r-\delta)(T-t)},t\right)$ and $C(S,K,T-t)$ becomes *C*(*S*, *K*, *T*). We have

$$V(0)=C(S,K,T)+e^{-\delta(T-t)}P\left(S,Ke^{-(r-\delta)(T-t)},t\right).$$

On the other hand, by put-call parity,

$$\begin{aligned}
&\left[C(S,K,T)+e^{-\delta(T-t)}P(S,Ke^{-(r-\delta)(T-t)},t)\right]-\left[P(S,K,T)+e^{-\delta(T-t)}C(S,Ke^{-(r-\delta)(T-t)},t)\right] \\
&=C(S,K,T)-P(S,K,T)-e^{-\delta(T-t)}\left(C(S,Ke^{-(r-\delta)(T-t)},t)-P(S,Ke^{-(r-\delta)(T-t)},t)\right) \\
&=S(0)e^{-\delta T}-Ke^{-rT}-e^{-\delta(T-t)}\left(S(0)e^{-\delta t}-Ke^{-(r-\delta)(T-t)}e^{-rt}\right)=0
\end{aligned}$$

Hence $V(0)=P(S,K,T)+e^{-\delta(T-t)}C(S,Ke^{-(r-\delta)(T-t)},t).$

Example 6.6

A stock pays dividend with continuous rate $\delta=0.02.$ Its initial price is 50 and volatility $\sigma=0.2.$ The continuously compounded risk-free interest rate is 6%. A chooser option allows you to choose between European call and put options with strike price 50 expiring 9 months from now. You must make the choice at the end of 3 months.

(a) Find the price of the chooser option.

(b) Suppose the price of another chooser option $V(0)$ is 5. Construct an arbitrage strategy involving buying or selling a chooser option V.

Solution

(a) The price of a 9-month call is

$$\begin{aligned}
&d_1=\frac{\left(0.06-0.02+0.5(0.2)^2\right)(0.75)}{0.2\sqrt{0.75}}=0.25981, \quad N(d_1)=N(0.25981)=0.60249 \\
&d_1=0.25981-0.2\sqrt{0.75}=0.08660, \quad N(d_2)=N(0.08660)=0.53451 \\
&C(S,50,0.75)=50e^{-0.02(0.75)}(0.60249)-50e^{-0.06(0.75)}(0.53451)=4.1265
\end{aligned}$$

The price of a 3-month put with strike price $Ke^{-0.5(0.06-0.02)}=49.0099$ is

$$\begin{aligned}
&d_1=\frac{\ln\left(50/50e^{-0.02}\right)+\left(0.06-0.02+0.5(0.2)^2\right)(0.25)}{0.2\sqrt{0.25}}=0.35, \quad N(-d_1)=N(-0.35)=0.36317 \\
&d_1=0.35-0.2\sqrt{0.25}=0.25, \quad N(-d_2)=N(-0.25)=0.40129 \\
&P(S,50e^{-0.02},0.25)=50e^{-0.02}e^{-0.06(0.25)}(0.40129)-50e^{-0.02(0.25)}(0.36317)=1.3065
\end{aligned}$$

The chooser option is worth $4.1265+e^{-0.02(0.5)}(1.3065)=5.42.$

(b) Option V is underprice. Long V and short $C(S,50,0.75)+e^{-0.02(0.5)}P(S,50e^{-0.02},0.25)$ at time 0. You get 0.42. At time 0.25 (3 months),

case 1. $S(0.25)>50e^{-0.02}$

$P(S,50e^{-0.02},0.25)$ is worth 0. Choose European call option for V (with strike price 50 and expiration date another 6 months later) to cover the short position of $C(S, 50, 0.75)$.

case 2. $S(0.25)\le 50e^{-0.02}$

$P(S,50e^{-0.02},0.25)$ is exercised. Choose European put option for V (with strike price 50 and expiration date another 6 months later). You still short $C(S, 50, 0.75)$, long $e^{-0.02(0.5)}$ shares of stock, short $e^{-0.02(0.5)}50e^{-0.02}=50e^{-0.06(0.5)}$ bonds. Another 6 months later,

case 2.1 $S(0.75)\le 50$

$C(S, 50, 0.75)$ is worth 0. You long 1 share of stock and short 50 bonds which can be covered by the European put option after exercising.

case 2.2 $S(0.75) > 50$

The European put option is worth 0. You long 1 share of stock and short 50 bonds which can be covered by the European call option $C(S, 50, 0.75)$ after exercising.

An **exchange option** (P.76) lets you exchange an asset to another asset, instead of cash, as payment at expiration date T if the option is exercise. Let S be the price of the asset that may be received and Q the price of the asset that may be exchanged for it at time T. The payoff is $\max\{S(T)-Q(T),0\}$.

Let $\sigma_S=\sqrt{Var(\ln S(1))}$ and $\sigma_Q=\sqrt{Var(\ln Q(1))}$ be the volatilities of the asset S and Q respectively. The volatility of the exchange option σ depends on both assets and their correlation ρ,

$$\sigma=\sqrt{Var\ln\left(S(1)/Q(1)\right)}=\sqrt{Var\left(\ln S(1)-\ln Q(1)\right)}=\sqrt{\sigma_S^2+\sigma_Q^2-2\rho\sigma_S\sigma_Q}\,.$$

Theorem 6.7 (Black-Scholes for exchange options)

$$C(S,Q,t) = F_{t,T}^P(S)N(d_1) - F_{t,T}^P(Q)N(d_2) = P(Q,S,t),$$
$$P(S,Q,t) = F_{t,T}^P(Q)N(-d_2) - F_{t,T}^P(S)N(-d_1) = C(Q,S,t)$$

where

$$d_1 = \frac{\ln\left(F_{t,T}^P(S)/F_{t,T}^P(Q)\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(F_{t,T}^P(S)/F_{t,T}^P(Q)\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

Example 6.8

Your boss will offer you an annual bonus of the minimum of $200S_1$ and $300S_2$, either one payable at the end of 1 year. $S_1(0) = 50, S_2(0) = 30, \delta_{S_2} = 0.02, \sigma_{S_1} = 0.3, \sigma_{S_2} = 0.25, \rho = 0.8$ and S_1 pays a discrete dividend of 5 at the end of 6 months. The continuously compounded risk-free rate is 0.04. Find the current value of the bonus.

Solution

The bonus you were offered is

$$\min\left(200S_1(1), 300S_2(1)\right) = 100\min\left(2S_1(1), 3S_2(1)\right) = 100\left[2S_1(1) - \max\left(2S_1(1) - 3S_2(1), 0\right)\right].$$

Discounting at time 0, this is

$$\begin{aligned} 100\left[2F_{0,1}^P(S_1) - C\left(2S_1, 3S_2\right)\right] &= 100\left[2F_{0,1}^P(S_1) - 2F_{0,1}^P(S_1)N(d_1) + 3F_{0,1}^P(S_2)N(d_2)\right] \\ &= 100\left[2F_{0,1}^P(S_1)N(-d_1) + 3F_{0,1}^P(S_2)N(d_2)\right] \end{aligned}$$

where $d_1 = \dfrac{\ln\left(F_{0,1}^P(2S_1)/F_{0,1}^P(3S_2)\right) + \frac{1}{2}\sigma^2}{\sigma} = \dfrac{\ln\left(2F_{0,1}^P(S_1)/3F_{0,1}^P(S_2)\right) + \frac{1}{2}\sigma^2}{\sigma}$ and $d_2 = d_1 - \sigma$.

$$F_{0,1}^P(S_1) = 50 - 5e^{-0.04(0.5)} = 45.099, \quad F_{0,1}^P(S_2) = 30e^{-0.02} = 29.40596$$
$$\sigma = \sqrt{\sigma_{S_1}^2 + \sigma_{S_2}^2 - 2\rho\sigma_{S_1}\sigma_{S_2}} = \sqrt{0.3^2 + 0.25^2 - 2(0.8)(0.3)(0.25)} = 0.180278$$
$$d_1 = \frac{\ln\left(2F_{0,1}^P(S_1)/3F_{0,1}^P(S_2)\right) + \frac{1}{2}\sigma^2}{\sigma} = \frac{\ln\frac{2(45.099)}{3(29.40596)} + \frac{1}{2}(0.0325)}{0.180278} = 0.213268859$$

$$d_2 = d_1 - \sigma = 0.033$$
$$N(-d_1) = 0.41555863, \quad N(d_2) = 0.51316$$
$$100\left[2F_{0,1}^P(S_1) - C\left(2S_1, 3S_2\right)\right] = 100\left[2(45.099)(0.41555863) + 3(29.40596)(0.51316)\right]$$
$$= 8275.244460954$$

A **forward start option** is a prepaid forward on an option. At time *t*, you will receive an option whose strike price *K* is $cS(t)$ and expiration date is *T*. Then at time *t*,

$$C(t) = S(t)e^{-\delta(T-t)}N(d_1) - cS(t)e^{-r(T-t)}N(d_2), \quad P(t) = cS(t)e^{-r(T-t)}N(-d_2) - S(t)e^{-\delta(T-t)}N(-d_1)$$

where $d_1 = \dfrac{-\ln c + (r - \delta + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t} = \dfrac{-\ln c + (r - \delta - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$.

The prepaid forward price of $S(t)$ is $F_{0,t}^P(S)$, so the price of the option at time 0 is

$$C(0) = F_{0,t}^P(S)e^{-\delta(T-t)}N(d_1) - cF_{0,t}^P(S)e^{-r(T-t)}N(d_2),$$
$$P(0) = cF_{0,t}^P(S)e^{-r(T-t)}N(-d_2) - F_{0,t}^P(S)e^{-\delta(T-t)}N(-d_1).$$

Example 6.9

A forward start option will, in 6 months, give its owner a 1-year European put option with a strike price equal to the stock price at the end of 6 months. The European put option is on a stock that pays 1 at the end of 3 months. The stock's volatility is 30%. The current price of the stock is 40. The continuously compounded risk-free interest rate is 8%. Determine the price of the forward start option.

Solution

After 6 months, the price of the option is

$$d_1 = \frac{0.08 + 0.5(0.3)^2}{0.3} = 0.41667, \quad N(-d_1) = 0.33846$$
$$d_2 = 0.41667 - 0.3 = 0.11667, \quad N(-d_2) = 0.45356$$
$$P(0.5) = S(0.5)\left(e^{-0.08}(0.45356) - 0.33846\right) = 0.08023S(0.5)$$

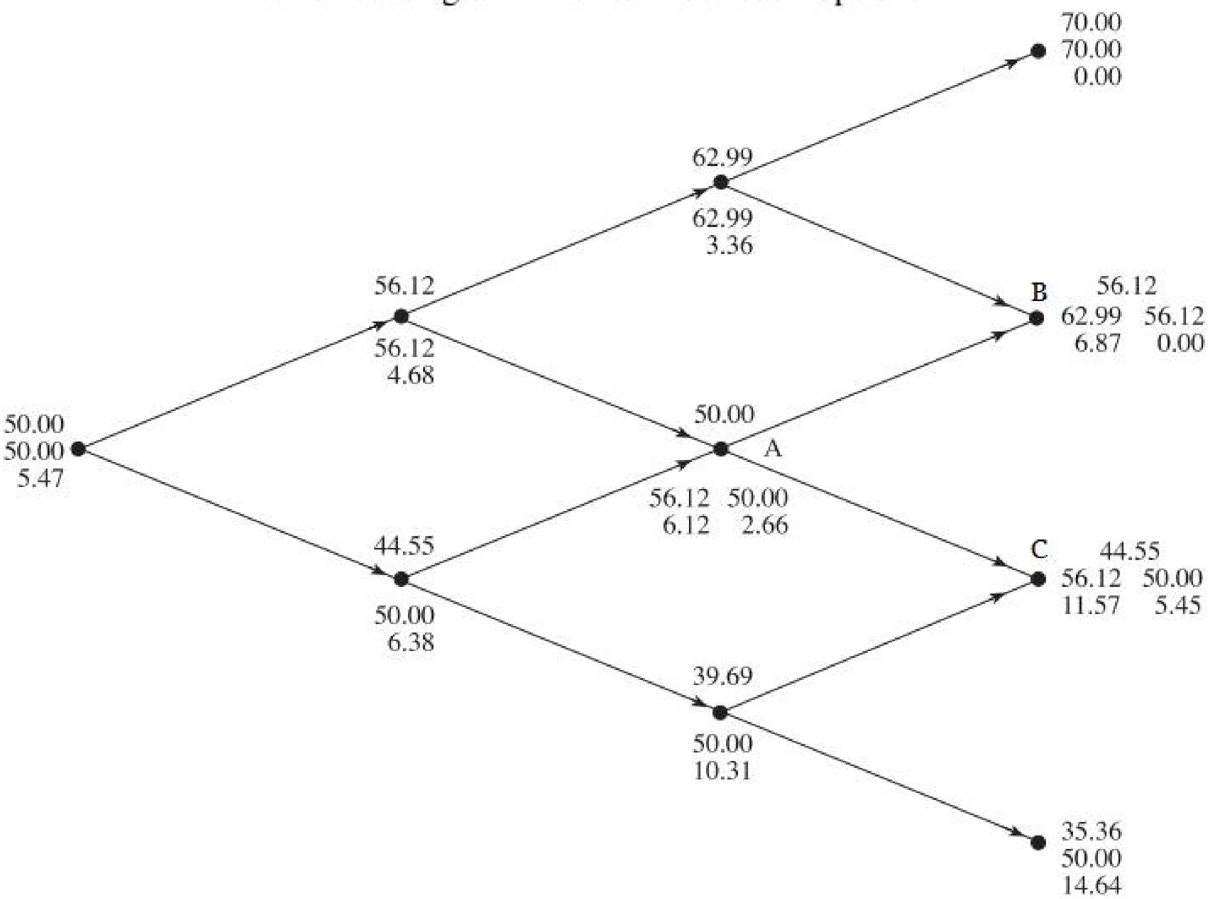
$F_{0,0.5}^P(S) = 40 - e^{-0.08/4} = 39.02$, so the answer is 0.08023(39.02) = 3.1305746.

A **lookback option** has a payoff which depends not only on the asset price at expiry but also on the maximum or the minimum of the underlying asset price over some time prior to expiry.

Example 6.10

Consider a 3-month American lookback put option on a non-dividend-paying stock. The payoff is the amount by which the maximum stock price observed during the option's life exceeds the asset price at the time of exercise. Suppose $T = 1/4$, $S(0) = 50$, $r = 0.1$ and $\sigma = 0.4$. Consider a 3-period Cox-Ross-Rubinstein binomial tree. Then $\Delta t = \frac{1}{12}$, $u = e^{\sigma\sqrt{\Delta t}} = 1.1224$, $d = u^{-1} =$ 0.8909, and $p^* = \dfrac{e^{r\Delta t} - d}{u - d} = 0.5073$.

Tree for valuing an American lookback option.



The top number at each node is the stock price. The next row of numbers shows the alternative values of F at the node. The final row of numbers shows the corresponding values of v.

In general there are $i + 1$ nodes at time $i\Delta t$ in the tree. We denote the lowest node at time $i\Delta t$ by (i, 0), the second lowest by (i, l), and so on. The value of S at node (i, j) is $S(0)u^j d^{i-j}$ (j = 0, 1, ..., i); at node B [that is, node (3, 2)], the value of S is $50(1.1224)^2(0.8909) = 56.12$.

Let $F(t,S) = \max\{S(u) \mid u \leq t\}$ and $v(S,F,t) = \max\{F(t,S) - S(t), 0\}$, payoff of option if exercising at t. We denote the k-th value of F at node (i, j) by $F_{i,j,k}$, and define $v_{i,j,k}$ as the value of the security at node (i, j) when F has this value.

At node A [that is, node (2, l)], the F-value, the maximum stock price to date, is $F_{2,1,1} = 56.12$ and $F_{2,1,2} = 50$. Similarly, for node B [that is, node (3, 2)] and node C [that is, node (3, l)], we obtain $F_{3,2,1} = 62.99; F_{3,2,2} = 56.12$ and $F_{3,1,1} = 56.12; F_{3,1,2} = 50$.

We suppose that the k-th value of F at node (i, j) leads to the k_u-th value of F at node (i + 1, j + 1) when there is an up movement in the stock price, and to the k_d-th value of F at node (i + 1, j) when there is a down movement in stock price. For a European-style derivative security, this means that

$$v_{i,j,k} = e^{-r\Delta t}\left(p^* v_{i+1,j+1,k_u} + (1-p^*)v_{i+1,j,k_d}\right).$$

If the derivative can be exercised at node (i, j), the value in the above equation must be compared with the early exercise value, and $v_{i,j,k}$ must be set equal to the greater of the two.

The payoff at node B when $F_{3,2,1} = 62.99$ is the excess of 62.99 over the current stock price:

$$v_{3,2,1} = 62.99 - 56.12 = 6.87.$$

Similarly $v_{3,2,2} = 0$. At node C we obtain

$$v_{3,1,1} = 11.57 \qquad \text{and} \qquad v_{3,1,2} = 5.45.$$

Consider now the situation at node A when $F_{2,1,2} = 50$. Clearly it is not worth exercising as the payoff from doing so would be zero.

If there is an up-movement so that we move from node A to node B, F changes from 50 to 56.12. This means that $k_u = 2$. If there is a down-movement, so that we move from node A to node C, F stays at $F_{3,1,2} = 50$. This means that $k_d = 2$. The value of being at node A when $F_{2,1,2} = 50$ is

$$v_{2,1,2} = e^{-0.1/12}\left(0.5073 v_{3,2,2} + 0.4927 v_{3,1,2}\right) = e^{-0.1/12}\left(0 \times 0.5073 + 5.45 \times 0.4927\right) = 2.66.$$

A similar calculation for the situation where the value of F at node A is $F_{2,1,1} = 56.12$ gives $k_u = 2$ and $k_d = 1$. The value of the derivative security at node A, without early exercise, is

$$e^{-0.1/12}\left(0.5073 v_{3,2,2} + 0.4927 v_{3,1,1}\right) = e^{-0.1/12}\left(0 \times 0.5073 + 11.57 \times 0.4927\right) = 5.65.$$

In this case, early exercise is optimal, as it gives a value of $v_{2,1,1} = 6.12$.

Working back through the tree,

$$\begin{aligned}
v_{2,2,1} &= e^{-0.1/12}\left(0.5073 v_{3,3,1} + 0.4927 v_{3,2,1}\right) = e^{-0.1/12}\left(0 \times 0.5073 + 6.87 \times 0.4927\right) = 3.36 \\
v_{2,0,1} &= 50 - 39.69 = 10.31 \\
&> e^{-0.1/12}\left(0.5073 v_{3,1,2} + 0.4927 v_{3,0,1}\right) = e^{-0.1/12}\left(5.45 \times 0.5073 + 14.64 \times 0.4927\right) = 9.8951 \\
v_{1,1,1} &= e^{-0.1/12}\left(0.5073 v_{2,2,1} + 0.4927 v_{2,1,1}\right) = e^{-0.1/12}\left(3.36 \times 0.5073 + 6.12 \times 0.4927\right) = 4.68 \\
v_{1,0,1} &= e^{-0.1/12}\left(0.5073 v_{2,1,2} + 0.4927 v_{2,0,1}\right) = e^{-0.1/12}\left(2.66 \times 0.5073 + 10.31 \times 0.4927\right) = 6.38 \\
v_{0,0,1} &= e^{-0.1/12}\left(0.5073 v_{1,1,1} + 0.4927 v_{1,0,1}\right) = e^{-0.1/12}\left(4.68 \times 0.5073 + 6.38 \times 0.4927\right) = 5.47.
\end{aligned}$$

Chapter 7 Finite Difference Methods

Finite difference methods to solve partial differential equations are based on the simple idea of approximating each partial derivative by a difference quotient. This transforms the functional equation into a set of algebraic equations. As in many numerical algorithms, the starting point is a finite series approximation. Under suitable continuity and differentiability conditions, Taylor's Theorem states that

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2!}h^2 f''(x) + \cdots$$

If we neglect the terms of order h^2 and higher, we get

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h).$$

This is the forward approximation for the derivative; indeed, the derivative is just defined as a limit of the difference quotient above as $h \to 0^+$. There are alternative ways to approximate first-order derivatives. By similar reasoning, we may write

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2!}h^2 f''(x) - \cdots,$$

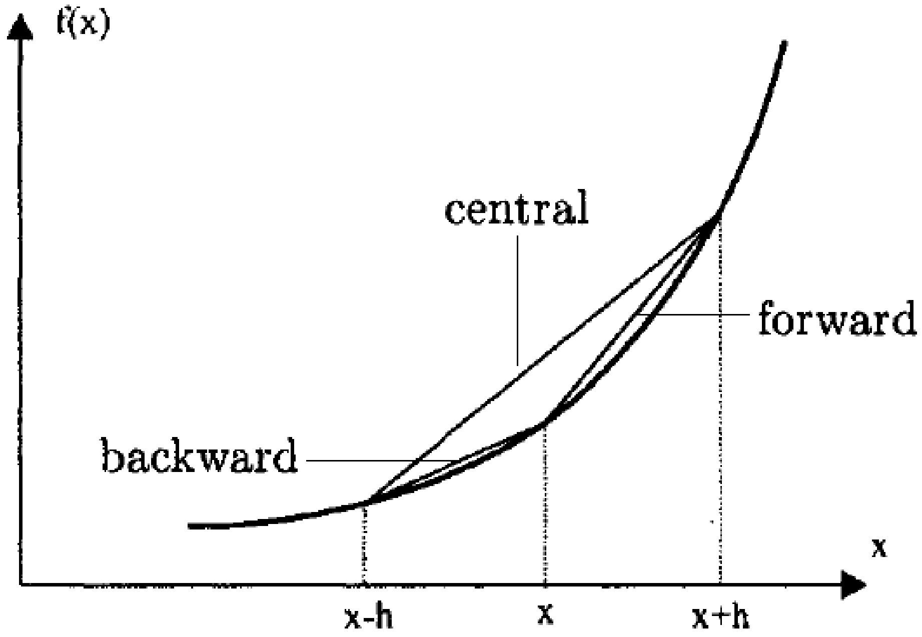
from which we obtain the backward approximation,

$$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h).$$

In both cases we get a truncation error of order $O(h)$. A better approximation can be obtained by subtracting the second Taylor equation from the first Taylor equation and rearranging:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2).$$

This is the central or symmetric approximation, and for small h it is better approximation, since the truncation error is $O(h^2)$. However, this does not imply that forward and backward approximations must be disregarded; they may be useful to come up with efficient numerical schemes, depending on the type of boundary conditions.



Graphical illustration of forward, backward, and central approximations of a derivative.

The reasoning may be extended to second-order derivatives, too. This is obtained by adding the first and second Taylor equations, which yields

$$f(x+h)+f(x-h)=2f(x)+h^2 f''(x)+O(h^4),$$

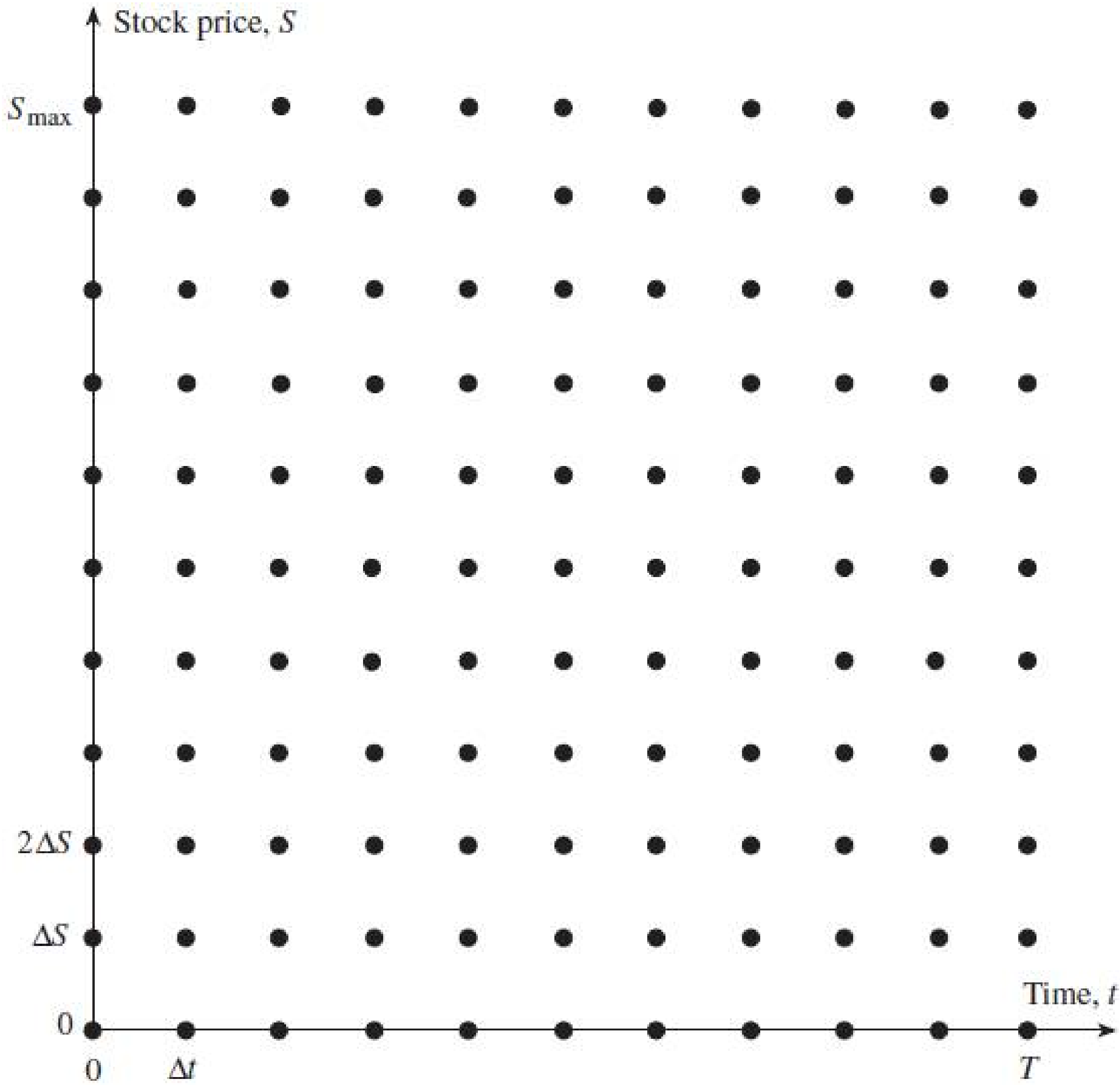
and rearranging yields

$$f''(x)=\frac{f(x+h)-2f(x)+f(x-h)}{h^2}+O(h^2)\approx\frac{1}{h}\left(\frac{f(x+h)-f(x)}{h}-\frac{f(x)-f(x-h)}{h}\right).$$

Finite difference methods value a derivative by solving the differential equation that the derivative satisfies. The differential equation is converted into a set of difference equations, and the difference equations are solved iteratively.

To illustrate the approach, we consider how it might be used to value a European option on a stock paying a dividend yield of δ. The differential equation that the option must satisfy is

$$\frac{\partial V}{\partial t}+(r-\delta)S\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}=rV.$$



Suppose that the expiration date of the option is T. We divide this into N equally spaced intervals of length $\Delta t = T/N$. A total of $N + 1$ times are therefore considered

$$0, \quad \Delta t, \quad 2\Delta t, \quad \cdots, \quad N\Delta t = T.$$

Suppose that $S_{\max}$ is a stock price sufficiently high. We define $\Delta S = S_{\max}/M$ and consider a total of $M + 1$ equally spaced stock prices:

$$0, \quad \Delta S, \quad 2\Delta S, \quad \cdots, \quad M\Delta S = S_{\max}.$$

The time points and stock price points define a grid consisting of a total of $(M + 1)(N + 1)$ points. The (i, j) point on the grid is the point that corresponds to time $i\Delta t$ and stock price $j\Delta S$. We will use the variable $V_{i,j}$ to denote the value of the option at the (i, j) point.

There are 2 types of finite difference methods on S:

(i) Implicit finite difference method,

(ii) Explicit finite difference method.

(i) Implicit finite difference method on S:

For an interior point (i, j) on the grid, $\partial V / \partial S$ can be approximated as

$$\frac{\partial V_{i,j}}{\partial S} \approx \frac{V_{i,j+1} - V_{i,j}}{\Delta S} \qquad \text{(forward approximation)}$$

or as

$$\frac{\partial V_{i,j}}{\partial S} \approx \frac{V_{i,j} - V_{i,j-1}}{\Delta S} \qquad \text{(backward approximation)}$$

We use a more symmetrical approximation by averaging the two

$$\frac{\partial V_{i,j}}{\partial S} \approx \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta S} \qquad \text{(central approximation)}$$

A finite difference approximation for $\partial^2 V / \partial S^2$ at the (i, j) point is

$$\frac{\partial^2 V_{i,j}}{\partial S^2} \approx \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{\Delta S^2}.$$

We will use a forward difference approximation for $\partial V / \partial t$,

$$\frac{\partial V_{i,j}}{\partial t} \approx \frac{V_{i+1,j} - V_{i,j}}{\Delta t}.$$

Substituting central approximation of $\frac{\partial V_{i,j}}{\partial S}$, approximations of $\frac{\partial^2 V_{i,j}}{\partial S^2}$ and $\frac{\partial V_{i,j}}{\partial t}$ into Black-Scholes partial differential equation, the difference equation is

$$\frac{V_{i+1,j} - V_{i,j}}{\Delta t} + (r-\delta) j\Delta S \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta S} + \frac{1}{2}\sigma^2 j^2 \Delta S^2 \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{\Delta S^2} = rV_{i,j}$$

for $j = 1, 2, \cdots, M-1$ and $i = 1, 2, \cdots, N-1$. Rearranging terms, we obtain

$$a_j V_{i,j-1} + b_j V_{i,j} + c_j V_{i,j+1} = V_{i+1,j}$$

where

$$\begin{aligned}
a_j &= \tfrac{1}{2}(r-\delta) j\Delta t - \tfrac{1}{2}\sigma^2 j^2 \Delta t \\
b_j &= 1 + r\Delta t + \sigma^2 j^2 \Delta t \\
c_j &= -\tfrac{1}{2}(r-\delta) j\Delta t - \tfrac{1}{2}\sigma^2 j^2 \Delta t
\end{aligned}$$

Example 7.1

Consider a 5-month American put option on a non-dividend-paying stock when the stock price is \$50, the strike price is \$50, the risk-free interest rate is 10% per annum, and the volatility is 40% per annum. Suppose that we divide the life of the option into 5 intervals of length 1 month for the purposes of constructing a binomial tree.

The value of the put option at time T is $\max\{K - S(T), 0\}$. Hence

$$V_{N,j} = \max\{K - j\Delta S, 0\}, \quad i = 0,1,\cdots,M.$$

The value of the put option when the stock price is zero is K. Hence,

$$V_{i,0} = K, \quad i = 0,1,\cdots,N.$$

We assume that the put option is worth zero when $S = S_{\max}$, so that

$$V_{i,M} = 0, \quad i = 0,1,\cdots,N.$$

There are therefore $M-1$ simultaneous equations

$$\begin{cases}
a_{M-1} V_{N-1,M-2} + b_{M-1} V_{N-1,M-1} + c_{M-1} \overbrace{V_{N-1,M}}^{0} = V_{N,M-1} \\
a_{M-2} V_{N-1,M-3} + b_{M-2} V_{N-1,M-2} + c_{M-2} V_{N-1,M-1} = V_{N,M-2} \\
\vdots \\
a_2 V_{N-1,1} + b_2 V_{N-1,2} + c_2 V_{N-1,3} = V_{N,2} \\
a_1 \underbrace{V_{N-1,0}}_{K} + b_1 V_{N-1,1} + c_1 V_{N-1,2} = V_{N,1}
\end{cases}$$

that can be solved for the $M-1$ unknowns: $V_{N-1,1},\cdots,V_{N-1,M-1}$. After this has been done, each value of $V_{N-1,j}$ is compared with $K-j\Delta S$. If $V_{N-1,j}<K-j\Delta S$, early exercise at time $T-\Delta t$ is optimal and $V_{N-1,j}$ is set equal to $K-j\Delta S$. The nodes corresponding to time $T-2\Delta t$ are handled in a similar way, and so on. Eventually, $V_{0,1},\cdots,V_{0,M-1}$ are obtained. If $M=2N$, the option price is $V_{0,N}$.

Values of 20, 10, and 5 were chosen for M, N and ΔS respectively. Thus, the option price is evaluated at \$5 stock price intervals between \$0 and \$100 and at half-month time intervals throughout the life of the option. The option price given by the grid is \$4.07.

Grid to value American option using implicit finite difference methods.

Stock price (dollars)	*Time to maturity (months)*										
	5	*4.5*	*4*	*3.5*	*3*	*2.5*	*2*	*1.5*	*1*	*0.5*	*0*
100	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
95	0.02	0.02	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
90	0.05	0.04	0.03	0.02	0.01	0.01	0.00	0.00	0.00	0.00	0.00
85	0.09	0.07	0.05	0.03	0.02	0.01	0.01	0.00	0.00	0.00	0.00
80	0.16	0.12	0.09	0.07	0.04	0.03	0.02	0.01	0.00	0.00	0.00
75	0.27	0.22	0.17	0.13	0.09	0.06	0.03	0.02	0.01	0.00	0.00
70	0.47	0.39	0.32	0.25	0.18	0.13	0.08	0.04	0.02	0.00	0.00
65	0.82	0.71	0.60	0.49	0.38	0.28	0.19	0.11	0.05	0.02	0.00
60	1.42	1.27	1.11	0.95	0.78	0.62	0.45	0.30	0.16	0.05	0.00
55	2.43	2.24	2.05	1.83	1.61	1.36	1.09	0.81	0.51	0.22	0.00
50	4.07	3.88	3.67	3.45	3.19	2.91	2.57	2.17	1.66	0.99	0.00
45	6.58	6.44	6.29	6.13	5.96	5.77	5.57	5.36	5.17	5.02	5.00
40	10.15	10.10	10.05	10.01	10.00	10.00	10.00	10.00	10.00	10.00	10.00
35	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00
30	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00
25	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00
20	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00
15	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00
10	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00
5	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00
0	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00

(ii) Explicit finite difference method on S:

The advantage of implicit finite difference method is that it always converges to the solution of the differential equation as S and t approach zero. One of its disadvantages is that $M-1$ simultaneous equations have to be solved in order to calculate the $V_{i,j}$ from the $V_{i+1,j}$. The method can be simplified if the values of $\partial V_{i,j}/\partial S$ and $\partial^2 V_{i,j}/\partial S^2$ are assumed to be the same as at point $(i+1, j)$, that means

$$\frac{\partial V_{i,j}}{\partial S} \approx \frac{V_{i+1,j+1} - V_{i+1,j-1}}{2\Delta S} \quad \text{and} \quad \frac{\partial^2 V_{i,j}}{\partial S^2} \approx \frac{V_{i+1,j+1} - 2V_{i+1,j} + V_{i+1,j-1}}{\Delta S^2}.$$

The difference equation is

$$\frac{V_{i+1,j} - V_{i,j}}{\Delta t} + (r-\delta)j\Delta S\frac{V_{i+1,j+1} - V_{i+1,j-1}}{2\Delta S} + \frac{1}{2}\sigma^2 j^2 \Delta S^2 \frac{V_{i+1,j+1} - 2V_{i+1,j} + V_{i+1,j-1}}{\Delta S^2} = rV_{i,j}$$

or

$$a_j^* V_{i+1,j-1} + b_j^* V_{i+1,j} + c_j^* V_{i+1,j+1} = V_{i,j}$$

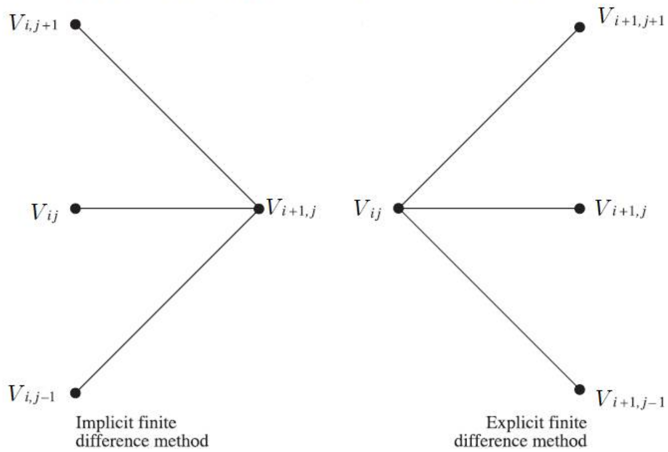
where

$$\begin{aligned} a_j^* &= \frac{1}{1+r\Delta t}\overbrace{\left(-\tfrac{1}{2}(r-\delta)j\Delta t + \tfrac{1}{2}\sigma^2 j^2 \Delta t\right)}^{P(j\Delta S \to (j-1)\Delta S)} \\ b_j^* &= \frac{1}{1+r\Delta t}\overbrace{(1-\sigma^2 j^2 \Delta t)}^{P(j\Delta S \to j\Delta S)} \\ c_j^* &= \frac{1}{1+r\Delta t}\underbrace{\left(\tfrac{1}{2}(r-\delta)j\Delta t + \tfrac{1}{2}\sigma^2 j^2 \Delta t\right)}_{P(j\Delta S \to (j+1)\Delta S)} \end{aligned}$$

The implicit method gives a relationship between 3 different values of the option at time $i\Delta t$ (i.e., $V_{i,j-1}, V_{i,j}$ and $V_{i,j+1}$) and 1 value of the option at time $(i+1)\Delta t$ (i.e., $V_{i+1,j}$).

The explicit method gives a relationship between 1 value of the option at time $i\Delta t$ (i.e., $V_{i,j}$) and 3 different values of the option at time $(i+1)\Delta t$ (i.e., $V_{i+1,j-1}, V_{i+1,j}$ and $V_{i+1,j+1}$).

Difference between implicit and explicit finite difference methods.



Example 7.2

The put option price in Example 7.1 using explicit finite difference method is 4.26.

Grid to value American option using explicit finite difference methods.

Stock price (dollars)	*Time to maturity (months)*										
	5	*4.5*	*4*	*3.5*	*3*	*2.5*	*2*	*1.5*	*1*	*0.5*	*0*
100	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
95	0.06	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
90	−0.11	0.05	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
85	0.28	−0.05	0.05	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
80	−0.13	0.20	0.00	0.05	0.00	0.00	0.00	0.00	0.00	0.00	0.00
75	0.46	0.06	0.20	0.04	0.06	0.00	0.00	0.00	0.00	0.00	0.00
70	0.32	0.46	0.23	0.25	0.10	0.09	0.00	0.00	0.00	0.00	0.00
65	0.91	0.68	0.63	0.44	0.37	0.21	0.14	0.00	0.00	0.00	0.00
60	1.48	1.37	1.17	1.02	0.81	0.65	0.42	0.27	0.00	0.00	0.00
55	2.59	2.39	2.21	1.99	1.77	1.50	1.24	0.90	0.59	0.00	0.00
50	4.26	4.08	3.89	3.68	3.44	3.18	2.87	2.53	2.07	1.56	0.00
45	6.76	6.61	6.47	6.31	6.15	5.96	5.75	5.50	5.24	5.00	5.00
40	10.28	10.20	10.13	10.06	10.01	10.00	10.00	10.00	10.00	10.00	10.00
35	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00
30	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00
25	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00
20	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00
15	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00
10	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00
5	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00
0	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00

When geometric Brownian motion is used for the underlying asset price, it is computationally more efficient to use finite difference methods with $\ln S$ rather than S as the underlying variable. Define $Z = \ln S$. We have

$$\frac{\partial V}{\partial S} = \frac{1}{S}\frac{\partial V}{\partial Z}, \quad \frac{\partial^2 V}{\partial S^2} = -\frac{1}{S^2}\frac{\partial V}{\partial Z} + \frac{1}{S^2}\frac{\partial^2 V}{\partial Z^2}.$$

The Black-Scholes partial differential equation

$$\frac{\partial V}{\partial t} + (r-\delta)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

becomes

$$\frac{\partial V}{\partial t} + \left(r - \delta - \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial Z} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial Z^2} = rV.$$

The grid then evaluates the derivative for equally spaced values of Z rather than for equally spaced values of S.

There are 2 types of finite difference methods on Z:

(i) Implicit finite difference method,

(ii) Explicit finite difference method.

(i) Implicit finite difference method on Z:

The difference equation for the implicit method becomes

$$\frac{V_{i+1,j} - V_{i,j}}{\Delta t} + \left(r - \delta - \frac{\sigma^2}{2}\right)\frac{V_{i,j+1} - V_{i,j-1}}{2\Delta Z} + \frac{1}{2}\sigma^2 \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{\Delta Z^2} = rV_{i,j}$$

for $j = 1, 2, \cdots, M-1$ and $i = 1, 2, \cdots, N-1$. Rearranging terms, we obtain

$$\alpha_j V_{i,j-1} + \beta_j V_{i,j} + \gamma_j V_{i,j+1} = V_{i+1,j}$$

where

$$\alpha_j = \frac{\Delta t}{2\Delta Z}\left(r-\delta-\frac{\sigma^2}{2}\right)-\frac{\Delta t}{2\Delta Z^2}\sigma^2$$
$$\beta_j = 1+r\Delta t+\frac{\Delta t}{\Delta Z^2}\sigma^2$$
$$\gamma_j = -\frac{\Delta t}{2\Delta Z}\left(r-\delta-\frac{\sigma^2}{2}\right)-\frac{\Delta t}{2\Delta Z^2}\sigma^2$$

(ii) Explicit finite difference method on Z:

The difference equation for the explicit method becomes

$$\frac{V_{i+1,j}-V_{i,j}}{\Delta t}+\left(r-\delta-\frac{\sigma^2}{2}\right)\frac{V_{i+1,j+1}-V_{i+1,j-1}}{2\Delta Z}+\frac{1}{2}\sigma^2\frac{V_{i+1,j+1}-2V_{i+1,j}+V_{i+1,j-1}}{\Delta Z^2}=rV_{i,j}$$

for $j=1,2,\cdots,M-1$ and $i=1,2,\cdots,N-1$. Rearranging terms, we obtain

$$\alpha_j^* V_{i+1,j-1}+\beta_j^* V_{i+1,j}+\gamma_j^* V_{i+1,j+1}=V_{i,j}$$

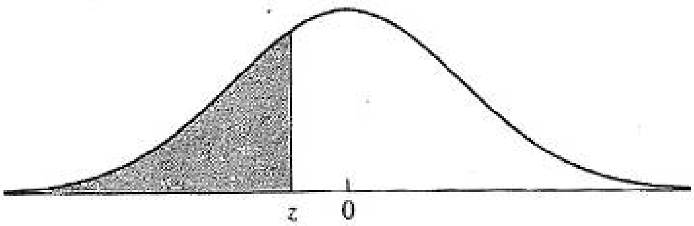
where

$$\alpha_j^* = \frac{1}{1+r\Delta t}\overbrace{\left(-\frac{\Delta t}{2\Delta Z}\left(r-\delta-\frac{\sigma^2}{2}\right)+\frac{\Delta t}{2\Delta Z^2}\sigma^2\right)}^{P(S\to Se^{-\Delta Z})}$$
$$\beta_j^* = \frac{1}{1+r\Delta t}\overbrace{\left(1-\frac{\Delta t}{\Delta Z^2}\sigma^2\right)}^{P(S\to S)}$$
$$\gamma_j^* = \frac{1}{1+r\Delta t}\underbrace{\left(\frac{\Delta t}{2\Delta Z}\left(r-\delta-\frac{\sigma^2}{2}\right)+\frac{\Delta t}{2\Delta Z^2}\sigma^2\right)}_{P(S\to Se^{\Delta Z})}$$

These movements in Z correspond to the stock price changing from S to $Se^{-\Delta Z}, S$ and $Se^{\Delta Z}$ respectively. In most cases, a good choice for ΔZ is $\sigma\sqrt{3\Delta t}$. If we set $\Delta Z=\sigma\sqrt{\Delta t}$, then the trinomial tree of explicit finite difference method on Z is reduced to Cox-Ross-Rubinstein binomial tree.

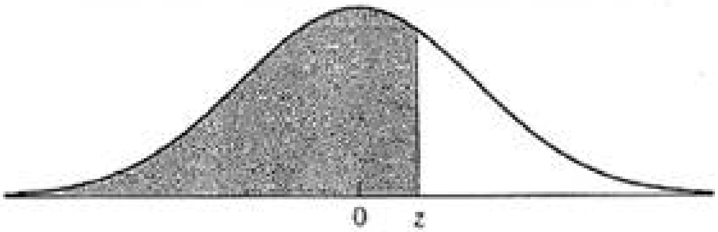
standard normal distribution table

Cumulative normal distribution (z table)



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−3.6	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
−3.5	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Cumulative normal distribution (continued)



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999