Chapter 3 Vector Spaces

Section 3.2 Subspaces

Definition (Subspace) A subset S of a vector space V over $\mathbf{R}$ is said to be a *subspace* of V if:

(i) S is non-empty, i.e. S contains at least an element.

(ii) $\mathbf{u} + \mathbf{v} \in S$ for any vectors $\mathbf{u}, \mathbf{v} \in S$.

(iii) $\alpha \mathbf{v} \in S$ for any vector $\mathbf{v} \in S$ and scalar $\alpha \in \mathbf{R}$;

Remark Any subspace of a vector space is a vector space itself.

Reason

- (ii) and (iii) are C1, C2.
- A1, A2, A5, A6, A7, A8 hold for all vectors in V, they also hold for the vectors in S.
- Substitute $\alpha = 0$ in (ii), we have A3.
- Substitute $\alpha = -1$ in (ii), we have A4.

Example Show that $S = \left\{ \left(a_1, a_2, a_3\right)^T \in \mathbf{R}^3 : \; a_1 + a_2 + a_3 = 0 \right\}$ is a subspace of $\mathbf{R}^3$.

Solution S is a subset of a vector space $\mathbf{R}^3$.

(i) Since the zero vector $\left(0, 0, 0\right)^T \in S$, S is nonempty.

(ii) Let $\left(a_1, a_2, a_3\right)^T \in S$, $\alpha \in \mathbf{R}$. Then $a_1 + a_2 + a_3 = 0$. Since $\alpha a_1 + \alpha a_2 + \alpha a_3 = 0$,

$$\alpha \left(a_1, a_2, a_3\right)^T = \left(\alpha a_1, \alpha a_2, \alpha a_3,\right)^T \in S.$$

(iii) Let $\mathbf{a}, \mathbf{b} \in V$. Then $a_1 + a_2 + a_3 = 0$ and $b_1 + b_2 + b_3 = 0$. Since $\left(a_1 + b_1\right) + \left(a_2 + b_2\right) + \left(a_3 + b_3\right) = 0$,

$$\mathbf{a} + \mathbf{b} = \left(a_1 + b_1, a_2 + b_2, a_3 + b_3\right)^T \in S.$$

Since S is nonempty and satisfies the two closure conditions, S is a subspace of $\mathbf{R}^3$.

Example Let $S = \{(0,0,0)^T\}$. Is S a subspace of $\mathbf{R}^3$?

Solution

(i) The set S is nonempty, since $x = (0,0,0)^T \in S$.

(ii) Let $\mathbf{x} \in S$ and $\alpha \in \mathbf{R}$. Then $\mathbf{x} = (0,0,0)^T$. Since $\alpha\mathbf{x} = \alpha(0,0,0)^T = (0,0,0)^T \in S$, $\alpha\mathbf{x} \in S$.

(iii) Let $\mathbf{x}, \mathbf{y} \in S$. Then $x = (0,0,0)^T$ and $\mathbf{y} = (0,0,0)^T$. So, $\mathbf{x} + \mathbf{y} = (0,0,0)^T + (0,0,0)^T = (0,0,0)^T \in S$.

Since S is nonempty and satisfies the two closure conditions, S is a subspace of $\mathbf{R}^3$.

Definition (Zero subspace) For any vector space V, $\{\mathbf{0}\}$ is a subspace of V. This subspace is called the zero subspace of V.

Definition (Trivial subspaces, proper subspaces) For any vector space V, $\{\mathbf{0}\}$ and V are subspaces of V. Both $\{\mathbf{0}\}$ and V are called the *trivial subspaces*. All other subspaces are referred to as *proper subspaces*.

Example Let $S = \{A \in \mathbf{R}^{n \times n} | A \text{ is a symmetric matrix}\}$. Show that S is a subspace.

Solution

(i) The set S is nonempty, since $I \in S$.

(ii) Let $A \in S$ and $\alpha \in \mathbf{R}$. Then $A^T = A$. Since

$$(\alpha A)^T = \alpha A^T = \alpha A,$$

$\alpha A \in S$.

(iii) Let $A, B \in S$. Then $A = A^T$ and $B = B^T$. Since

$$(A + B)^T = A^T + B^T = A + B,$$

$A + B \in S$.

Since S is nonempty and satisfies the two closure conditions, S is a subspace of $\mathbf{R}^{n \times n}$.

Definition (Null space) Given $A_{m \times n}$, let $N(A)$ denote the set of all solutions of system of linear equations $A\mathbf{x} = \mathbf{0}$, that is,

$$N(A) = \{x \in \mathbf{R}^n : A\mathbf{x} = \mathbf{0}\},$$

which is called the *null space* of A.

Theorem Let A be an $m \times n$ matrix. Then $N(A)$ is a subspace of $\mathbf{R}^n$.

Proof

(i) The set $N(A)$ is nonempty, since $\mathbf{0} \in N(A)$.

(ii) Let $\mathbf{x} \in N(A)$ and $\alpha \in \mathbf{R}$. Then $A\mathbf{x} = \mathbf{0}$ and

$$A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \alpha\mathbf{0} = \mathbf{0}.$$

So, $\alpha\mathbf{x} \in N(A)$.

(iii) Let $\mathbf{x}$ and $\mathbf{y}$ are elements of $N(A)$. Then $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$. So

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

and $\mathbf{x} + \mathbf{y} \in N(A)$.

It then follows that $N(A)$ is a subspace of $\mathbf{R}^n$.

Example Let $A = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{pmatrix}$. Find the null space $N(A)$ of A.

Solution The reduced row echelon form of A is $\begin{pmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. So,

$$
\begin{aligned}
N(A) &= \{\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^T : A\mathbf{x} = \mathbf{0}\} \\
&= \left\{ (x_1, x_2, x_3, x_4, x_5)^T : \begin{pmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} -3\alpha - 7\beta \\ -\alpha - 3\beta \\ \alpha \\ \beta \\ 0 \end{pmatrix} \right\} = \left\{ \alpha \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.
\end{aligned}
$$

Definition (Linear Combination) Let $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}$ be vectors in a vector space V.

$$y = \alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \cdots + \alpha_n \mathbf{v_n}, \qquad \alpha_i \in \mathbf{R}$$

is called a **linear combination** of $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}$.

Definition (Span) The set of all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}$

$$\{\alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \cdots + \alpha_n \mathbf{v_n} : \quad \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbf{R}\}$$

is called the **span** of $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}$ and denoted by

$$\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}.$$

Theorem If $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are elements of a vector space V, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a subspace of V.

Proof

(i) $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ contains $\mathbf{v}_1$ and hence nonempty.

(ii) Let β be a scalar and let $\mathbf{v} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n$ be an arbitrary element of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$. Since

$$\beta\mathbf{v} = (\beta\alpha_1)\mathbf{v}_1 + \cdots + (\beta\alpha_n)\mathbf{v}_n,$$

it follows that $\beta\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$.

(iii) Let $\mathbf{v} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n$ and $\mathbf{w} = \beta_1\mathbf{v}_1 + \cdots + \beta_n\mathbf{v}_n$. Then

$$\mathbf{v} + \mathbf{w} = (\alpha_1 + \beta_1)\mathbf{v}_1 + \cdots + (\alpha_n + \beta_n)\mathbf{v}_n.$$

So, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a subspace of V.

Definition (Spanning set) The set $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ is a **spanning set** for V if and only if every vector in V can be written as a linear combination of $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$.

To prove $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ spans a vector space V:

Step 1 For **arbitrary y** $\in V$, consider the system

$$\mathbf{y} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n}, \quad a_i \in \mathbf{R}$$

Step 2 **Solve** for $a_1, a_2, \cdots, a_n$

- If **consistent**, then $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ spans V.
- If **inconsistent**, then $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ can not span V!

Example Is $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ a spanning set for $\mathbf{R}^2$?

Solution Let $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbf{R}^2$. Solve $\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$. Since there is no solution of $\begin{cases} \alpha = a \\ 0 = b \end{cases}$, $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ is not a spanning set for $\mathbf{R}^2$.

Example Is $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ a spanning set for $\mathbf{R}^2$?

Solution Let $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbf{R}^2$. Solving $\alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$, we have $\begin{cases} \alpha_1 = a \\ \alpha_2 = b \end{cases}$. So $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a spanning set for $\mathbf{R}^2$.

Example Is $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}\right\}$ a spanning set for $\mathbf{R}^2$?

Solution Let $\begin{pmatrix}a\\b\end{pmatrix} \in \mathbf{R}^2$. Solve $\alpha_1 \begin{pmatrix}1\\0\end{pmatrix} + \alpha_2 \begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$. Since there is no solution of $\begin{cases}\alpha_1 + 2\ \ \alpha_2 = a\\ \qquad\qquad 0 = b\end{cases}$, $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}\right\}$ is not a spanning set for $\mathbf{R}^2$.

Example Is $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ a spanning set for $\mathbf{R}^2$?

Solution Let $\begin{pmatrix}a\\b\end{pmatrix} \in \mathbf{R}^2$. Solve $\alpha_1 \begin{pmatrix}1\\0\end{pmatrix} + \alpha_2 \begin{pmatrix}2\\0\end{pmatrix} + \alpha_3 \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$, we have $\begin{cases}\alpha_1 = a - 2\beta\\ \alpha_2 = \beta\\ \alpha_3 = b\end{cases}$, where β is any real number. So $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is a spanning set for $\mathbf{R}^2$.

Example Let P_3 be the set of all polynomials of degree less than 3. Is $\{x^2, x + x^2, -x - x^2\}$ a spanning set of P_3?

Solution

Let $ax^2 + bx + c \in P_3$. We wish to find $\alpha_1, \alpha_2, \alpha_3$ such that

$$\begin{aligned} ax^2 + bx + c &= \alpha_1(x^2) + \alpha_2(x + x^2) + \alpha_3(-x - x^2) \\ &= (\alpha_1 + \alpha_2 - \alpha_3)x^2 + (\alpha_2 - \alpha_3)x \end{aligned}$$

Setting $\begin{cases} \alpha_1 & +\alpha_2 & -\alpha_3 & = a \\ & \alpha_2 & -\alpha_3 & = b \\ & & 0 & = c \end{cases}$.

The system is consistent only if $c = 0$. The system is inconsistent if $c \neq 0$. So $\{x^2, 1 + x, -1 - x\}$ is not a spanning set for P_3.

Extra Exercises*

Which of the following are spanning sets for $\mathbf{R}^3$?

(a) $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ in $\mathbf{R}^3$.

(b) $\{\mathbf{i}, \mathbf{j}, \mathbf{k}, (1, 2, 3)^T\}$ in $\mathbf{R}^3$.

(c) $\{(1, 1, 1)^T, (1, 1, 0)^T, (1, 0, 0)^T\}$.

(d) $\{(1, 0, 1)^T, (0, 1, 0)^T\}$