2022-23 First Semester
MATH1083 Calculus II (1002&1003)

Assignment 10

Due Date:2pm 15/May/2023(Mon). [Please pay attention to the deadline]

- Write down your **Chinese name** and **student number**. Write neatly on **A4-sized** paper and **show your steps**.
- **Late submissions or answers without details will not be graded.**

1. Evaluate the iterated integrals

   (a)
$$\int_1^5 \int_1^6 \frac{\ln y}{xy} dxdy$$

   Solution:
$$\begin{aligned}\int_1^5 \int_1^6 \frac{\ln y}{xy} dxdy &= \int_1^5 \frac{\ln y}{y} \ln x \Big|_{x=1}^{x=6} dy \\ &= \int_1^5 \frac{\ln y}{y} \ln 6 dy \\ &= \frac{1}{2} \ln 6 (\ln y)^2 \Big|_1^5 \\ &= \frac{1}{2} \ln 6 (\ln 5)^2\end{aligned}$$

   (b)
$$\int_0^1 \int_0^2 ye^{x-y} dxdy$$

   Solution: Method 1: use **integration by parts** to compute the integral
$$\begin{aligned}\int_0^1 \int_0^2 ye^{x-y} dxdy &= \int_0^1 ye^{x-y}\Big|_0^2 dy \\ &= \int_0^1 y\left(e^{2-y} - e^{-y}\right) dy \\ &= \int_0^1 ye^{2-y} dy - \int_0^1 ye^{-y} dy \\ &= -ye^{2-y}\Big|_0^1 + \int_0^1 e^{2-y} dy + ye^{-y}\Big|_0^1 - \int_0^1 e^{-y} dy \\ &= \left[-ye^{2-y} - e^{2-y} + ye^{-y} + e^{-y}\right]\Big|_0^1 \\ &= \left(-2e + 2e^{-1}\right) - \left(-e^2 + 1\right) \\ &= e^2 - 2e - 1 + 2e^{-1}\end{aligned}$$

   Method 2 **[Better!]**: the integrand $ye^{x-y} = ye^{-y} \cdot e^x$, so
$$\begin{aligned}\int_0^1 \int_0^2 ye^{x-y} dxdy &= \int_0^1 ye^{-y} dy \cdot \int_0^2 e^x dx \\ &= \left(-ye^{-y}\Big|_0^1 + \int_0^1 e^{-y} dy\right) \cdot e^x\Big|_0^2 \\ &= \left(-e^{-1} + \left[-e^{-y}\right]\Big|_0^1\right) \cdot \left(e^2 - 1\right) \\ &= \left(1 - 2e^{-1}\right) \cdot \left(e^2 - 1\right) \\ &= e^2 - 2e - 1e^{-1}\end{aligned}$$

(c)

$$\int_0^3 \int_0^{\pi/2} t^2 \sin^3 \phi d\phi dt$$

Solution

$$\begin{aligned}
\int_0^3 \int_0^{\pi/2} t^2 (\sin\phi)^3 \, d\phi dt &= \int_0^3 \int_0^{\pi/2} t^2 \left(-\sin^2\phi\right) d(\cos\phi)\, dt \\
&= \int_0^3 \int_0^{\pi/2} t^2 \left(\cos^2\phi - 1\right) d(\cos\phi)\, dt \\
&= \int_0^3 \int_0^{\pi/2} t^2 \left(\frac{1}{3}\cos^2\phi - \cos\phi\right)\Bigg|_0^{\pi/2} dt \\
&= \frac{2}{3}\int_0^3 t^2 dt \\
&= \frac{2}{9} t^3 \Big|_0^3 \\
&= 6
\end{aligned}$$

2. Evaluate the double integrals as the volume of a solid over region R

(a)

$$\iint_R \sqrt{2} dA, \qquad R = \{(x,y)\,|2 \le x \le 6, -1 \le y \le 5\}$$

Solution:

$$\begin{aligned}
\iint_R \sqrt{2} dA &= \int_{-1}^5 \int_2^6 \sqrt{2} dx dy \\
&= \int_{-1}^5 \sqrt{2} x|_2^6 dy \\
&= \int_{-1}^5 4\sqrt{2} dy \\
&= 4\sqrt{2} y\Big|_{-1}^5 \\
&= 24\sqrt{2}
\end{aligned}$$

(b)

$$\iint_R \left(1 - x^2 y\right) dA, \qquad R = \{(x,y)\,|0 \le x \le 1, 1 \le y \le 2\}$$

Solution:

$$\begin{aligned}
\iint_R \left(1 - x^2 y\right) dA &= \int_0^1 \int_1^2 \left(1 - x^2 y\right) dy dx \\
&= \int_0^1 \left(y - \frac{1}{2}x^2 y^2\right)\Bigg|_1^2 dx \\
&= \int_0^1 \left[\left(2 - 2x^2\right) - \left(1 - \frac{1}{2}x^2\right)\right] dy \\
&= \int_0^1 \left(1 - \frac{3}{2}x^2\right) dx \\
&= \left(x - \frac{1}{2}x^3\right)\Bigg|_0^1 = \frac{1}{2}
\end{aligned}$$

3. Find the volume of the solid enclosed by the surface $z = 1 + x^2 y e^y$ and the planes $z = 0$, $x = \pm 1$ and $y = 0$ and $y = 1$.

Solution:

$$\iint_R \left(1 + x^2 y e^y\right) dx dy, \qquad R = \{(x, y) \,|\, -1 \le x \le 1, 0 \le y \le 1\}$$

so we can calculate the integral

$$\begin{aligned}
\iint_R \left(1 + x^2 y e^y\right) dx dy &= \int_0^1 \int_{-1}^1 \left(1 + x^2 y e^y\right) dx dy \\
&= \int_0^1 \left(x + \frac{1}{3} x^3 y e^y\right) \Big|_{-1}^1 dy \\
&= \int_0^1 \left(2 + \frac{2}{3} y e^y\right) dy \\
&= 2y \Big|_0^1 + \frac{2}{3} y e^y \Big|_0^1 - \frac{2}{3} \int_0^1 e^y dy \\
&= 2 + \frac{2}{3} e - \frac{2}{3}(e - 1) \\
&= \frac{8}{3}
\end{aligned}$$

4. Evaluate the iterated integrals

(a)

$$\int_0^2 \int_0^{y^2} x^2 y dx dy$$

Solution: (a)

$$\begin{aligned}
\int_0^2 \int_0^{y^2} x^2 y dx dy &= \frac{1}{3} \int_0^2 \left[x^3 y\right] \Big|_{x=0}^{x=y^2} dy \\
&= \frac{1}{3} \int_0^2 y^7 dy \\
&= \frac{1}{24} 2^8 = \frac{32}{3}
\end{aligned}$$

5. Evaluate the integrals by reversing the order of integration

(a)

$$\int_0^1 \int_{3y}^3 e^{x^2} dx dy$$

(b)

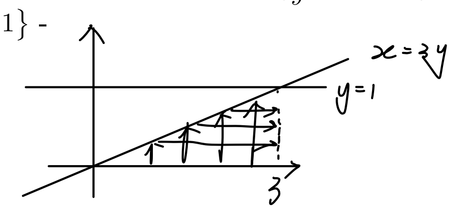
$$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1} dy dx$$

(c)

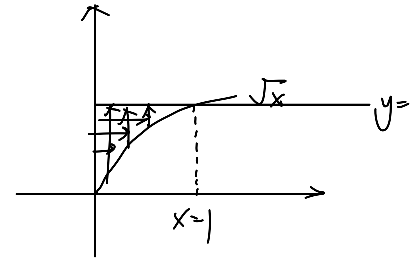
$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1 + \cos^2 x} dx dy$$

Solution: (a) sketch the region $D = \{(x, y) \,|3y \le x \le 3, 0 \le y \le 1\}$, the intersection of $x = 3y$ and $x = 3$ is at $(3, 1)$, so we can write the region $D = \left\{(x, y) \,|0 \le x \le 3, \frac{1}{3}x \le y \le 1\right\}$ -

$$\int_0^1 \int_{3y}^3 e^{x^2} dx dy$$



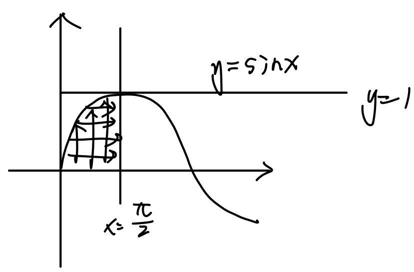
(b) $D = \{(x, y) \,|\, \sqrt{x} \le y \le 1, 0 \le x \le 1\}$ the intersection of $y = \sqrt{x}$ and $y = 1$ is $(1, 1)$, so the region can also be written as $D = \left\{(x, y) \,|0 \le y \le 1, 0 \le x \le y^2\right\}$



$$
\begin{aligned}
\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+1}dydx &= \int_0^1 \int_0^{y^2} \sqrt{y^3+1}dxdy \\
&= \int_0^1 \sqrt{y^3+1}y^2 dy \\
&= \frac{1}{3}\int_0^1 \sqrt{y^3+1}d\left(y^3+1\right) \\
&= \frac{2}{9}\left(y^3+1\right)^{3/2}\Big|_0^1 = \frac{2}{9}\left(2\sqrt{2}-1\right)
\end{aligned}
$$

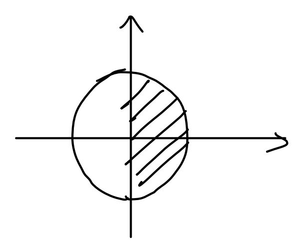
(c)The region $D = \{(x, y) \,|\, \arcsin y \le x \le \pi/2, 0 \le y \le 1\}$, the region can be written as $D = \{(x, y) \,|0 \le x \le \pi/2, 0 \le y \le \text{si}$

$$
\begin{aligned}
\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\sqrt{1+\cos^2 x}dxdy &= \int_0^{\frac{\pi}{2}} \int_0^{\sin x} \cos x\sqrt{1+\cos^2 x}dydx \\
&= \int_0^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}\sin xdx \\
&= -\int_0^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}d\cos x \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \sqrt{1+\cos^2 x}d\left(\cos^2 x+1\right) \\
&= -\frac{1}{3}\left(1+\cos^2 x\right)^{\frac{3}{2}}\Big|_0^{\frac{\pi}{2}} \\
&= \frac{1}{3}\left(2\sqrt{2}-1\right)
\end{aligned}
$$



6. Evaluate $\iint_D e^{-x^2-y^2}dxdy$, where D is the region bounded by the semicircle $x = \sqrt{4-y^2}$ and y-axis.
Solution: The region D is a semicircle of radius 2 in the first and forth quadrant. In polar coordinate $x = 2\cos\theta$, $y = 2\sin\theta$, thus the disk is

$$
D = \left\{(r, \theta) \,|-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, 0 \le r \le 2\right\}
$$



then the double integral

$$
\begin{aligned}
\iint_D e^{-x^2-y^2}dxdy &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 e^{-r^2}rdrd\theta \\
&= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 e^{-r^2}d\left(r^2\right)d\theta \\
&= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-e^{-r^2}\Big|_{r=0}^{r=2}\right]d\theta \\
&= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[1-e^{-4}\right]d\theta \\
&= \frac{1}{2}\left(1-e^{-4}\right)\theta\Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \\
&= \frac{\pi}{2}\left(1-e^{-4}\right)
\end{aligned}
$$

7. Use polar coordinates to find the volume of the given solid under the paraboloid $z = x^2+y^2$ and above the disk $x^2+y^2 \le 25$.

Solution: In polar coordinate, let $x = r\cos\theta$ and $y = r\sin\theta$, then the disk

$$D = \{(r,\theta)\,|0 \le \theta \le 2\pi, 0 \le r \le 5\}$$

and the height of the solid is $z = x^2 + y^2 = r^2$, so the volume

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^5 r^2 \cdot r dr d\theta \\ &= \frac{1}{4}\int_0^{2\pi}\left[r^4\Big|_{r=0}^{r=5}\right] d\theta \\ &= \frac{625}{4}\int_0^{2\pi} d\theta \\ &= \frac{625\pi}{2} \end{aligned}$$

8. Evaluate the iterated integral by **converting to polar coordinates**

(a)

$$\int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x+y)\, dx dy$$

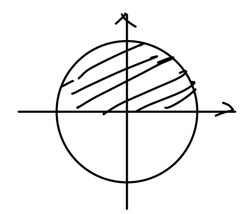
(b)

$$\int_0^1 \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2 dx dy$$

Solution: (a) Let $x = r\cos\theta$ and $y = r\sin\theta$, then the region

$$D = \{(r,\theta)\,|0 \le \theta \le \pi, 0 \le r \le a\}$$ $-\sqrt{a^2-y^2} \le x \le \sqrt{a^2-y^2}$

(a semi circle in the first and second quadrant)



$$\begin{aligned} \int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x+y)\, dx dy &= \int_0^{\pi}\int_0^a (2r\cos\theta + r\sin\theta)\cdot r dr d\theta \\ &= \int_0^{\pi}\int_0^a \left(2r^2\cos\theta + r^2\sin\theta\right)\cdot dr d\theta \\ &= \int_0^{\pi}\left[\frac{2}{3}r^3\cos\theta + \frac{1}{3}r^3\sin\theta\right]\Bigg|_{r=0}^{r=a}\cdot d\theta \\ &= \int_0^{\pi}\left(\frac{2}{3}a^3\cos\theta + \frac{1}{3}a^3\sin\theta\right) d\theta \\ &= \left[\frac{2}{3}a^3\sin\theta - \frac{1}{3}a^3\cos\theta\right]\Bigg|_{\theta=0}^{\theta=\pi} \\ &= \left(0 + \frac{1}{3}a^3\right) - \left(0 - \frac{1}{3}a^3\right) \\ &= \frac{2}{3}a^3 \end{aligned}$$

(b) the region for the lower bound $x = \sqrt{3}y$ it is a staight line $y = \frac{1}{\sqrt{3}}x$ which means the starting angle is $\theta = \frac{\pi}{6}$

$$D = \left\{(r,\theta)\,|\,0 \le \theta \le \frac{\pi}{6}, 0 \le r \le 1\right\}$$

$\sqrt{3}y \le x$

$\tan\theta \le \frac{\sqrt{3}}{3}$

$0 < \theta \le \frac{\pi}{6}$

$$
\begin{aligned}
\int_0^1 \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2 dxdy &= \int_0^{\frac{\pi}{6}} \int_0^1 r\cos\theta \cdot (r\sin\theta)^2 \cdot r dr d\theta \\
&= \int_0^{\frac{\pi}{6}} \cos\theta \cdot \sin^2\theta \int_0^1 r^4 \cdot dr d\theta \\
&= \frac{1}{5} \int_0^{\frac{\pi}{6}} \cos\theta \cdot \sin^2\theta d\theta \\
&= \frac{1}{5} \int_0^{\frac{\pi}{6}} \sin^2\theta d(\sin\theta) \\
&= \frac{1}{15} \left[\sin^3\theta\right] \Big|_{\theta=0}^{\theta=\frac{\pi}{6}} \\
&= \frac{1}{120}
\end{aligned}
$$