2021-22 First Semester
MATH1083 Calculus II (1002&1003)

Assignment 9

Due Date:2am 28/Apr/2021(Fri). [Please pay attention to the deadline]

- Write down your **Chinese name** and **student number**. Write neatly on **A4-sized** paper and **show your steps**.
- **Late submissions or answers without details will not be graded.**

1. Use **definition** to find $f_x(x,y)$ and $f_y(x,y)$ for

$$f(x,y) = xy^2 - x^3y$$

Solution:

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)\,y^2 - (x+h)^3\,y - (xy^2 - x^3y)}{h} \\ &= \lim_{h\to 0} \frac{hy^2 - \left(3x^2h + 3xh^2 + h^3\right)y}{h} \\ &= \lim_{h\to 0} y^2 - \left(3x^2 + 3xh + h^2\right)y \\ &= y^2 - 3x^2y \end{aligned}$$

similarly

$$\begin{aligned} f_y(x,y) &= \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k} \\ &= \lim_{h\to 0} \frac{x\,(y+k)^2 - x^3\,(y+k) - (xy^2 - x^3y)}{h} \\ &= \lim_{h\to 0} \frac{x\left(2yk + k^2\right) - x^3k}{k} \\ &= \lim_{h\to 0} 2xy + xk - x^3 \\ &= 2xy - x^3 \end{aligned}$$

2. If $f(x,y) = \sqrt[3]{x^3+y^3}$,

   (a) find $f_x(x,y)$.

$$\frac{\partial f}{\partial x} = \frac{1}{3}\left(x^3+y^3\right)^{-\frac{2}{3}} \cdot 3x^2 = \frac{x^2}{(x^3+y^3)^{\frac{2}{3}}}$$

   (b) find $f_x(0,0)$ and $f_y(0,0)$.
   Since $(0,0)$ is NOT in the domain of $f_x(x,y)$, so we have to get $f_x(0,0)$ **by definition**. As $f(h,0) = h$ $f(0,k) = k$ and $f(0,0)$

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{h-0}{h} = 1 \end{aligned}$$

$$\begin{aligned} f_y(0,0) &= \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} \\ &= \lim_{h\to 0} \frac{k-0}{k} = 1 \end{aligned}$$

3. Determine the set of points at which the function is continuous

$$f(x,y)=\begin{cases}\frac{xy}{x^2+xy+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

Solution:We can prove the limit does not exist at $(0,0)$ by choosing these two paths:1. along x-axis, so $y=0$ which equals 0 and 2. along $y=x$, the $f(x,y)=1/3$. So this function is continuous on $\{(x,y)\in\mathbb{R}^2|\,(x,y)\neq(0,0)\}$

4. Find the equation of the tangent plane to the surfaces at the specified points P

 (a) $z=x\sin(2x+y)$, $P=(1,-1,\sin 1)$

 Solution: $f_x(x,y)=\sin(2x+y)+2x\cos(2x+y)$ and $f_y(x,y)=x\cos(2x+y)$, so

$$f_x(1,-1)=\sin 1+2\cos 1$$

 and

$$f_y(1,-1)=\cos 1$$

 so the equation of the tangent plane is

$$\begin{aligned}z&=\sin 1+f_x(1,-1)(x-1)+f_y(1,-1)(y+1)\\&=\sin 1+(\sin 1+2\cos 1)(x-1)+\cos 1(y+1)\end{aligned}$$

 (a) $xy+yz+zx=11$, $P=(1,2,3)$

 Solution: let $F(x,y,z)=xy+yz+zx-11$, then $F_x=y+z$, $F_y=x+z$ and $F_z=x+y$, so

$$F_x(1,2,3)=5,\qquad F_y(1,2,3)=4,\qquad F_z(1,2,3)=3$$

 so the equation of the tangent plane is

$$F_x(1,2,3)(x-1)+F_y(1,2,3)(y-2)+F_z(1,2,3)(z-3)=0$$

 that is

$$5(x-1)+4(y-2)+3(z-3)=0$$

5. Find all the second partial derivatives of function $f(x,y)=\ln(x^2-y^2)$

 Solution:

$$f_x=\frac{2x}{x^2-y^2},\qquad f_y=\frac{-2y}{x^2-y^2}$$

$$f_{xx}=\frac{2\left(x^2-y^2\right)-4x^2}{\left(x^2-y^2\right)^2},\qquad f_{xy}=\frac{-4xy}{\left(x^2-y^2\right)^2}\qquad f_{yy}=\frac{-2\left(x^2-y^2\right)+4y^2}{\left(x^2-y^2\right)^2}$$

6. Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$ for

$$e^z=xyz$$

 Solution: let $F(x,y,z)=e^z-xyz$, so $F_x=-yz$, $F_y=-xz$ and $F_z=e^z-xy$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{yz}{e^z-xy}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{xz}{e^z-xy}$$

.

7. Find the directional derivative of $f = \sin xe^{2y}$ at the point $P = (0,0)$ in the direction of the point $Q = (1,1)$ and find the direction in which the function changes fastest at the point $R = (0,1)$.

Solution: first compute the gradient vector

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \cos xe^{2y}\overrightarrow{i} + 2\sin xe^{2y}\overrightarrow{j}$$

$$\nabla f(0,0) = \langle 1,0\rangle = \overrightarrow{i}$$

the direction $\overrightarrow{u} = \frac{PQ}{|PQ|} = \frac{\sqrt{2}}{2}\overrightarrow{i} + \frac{\sqrt{2}}{2}\overrightarrow{j}$, so

$$D_{\overrightarrow{u}}f(0,0) = \nabla f(3,1)\cdot\overrightarrow{u} = \left(1\overrightarrow{i} + 0\overrightarrow{j}\right)\cdot\left(\frac{\sqrt{2}}{2}\overrightarrow{i} + \frac{\sqrt{2}}{2}\overrightarrow{j}\right) = \frac{\sqrt{2}}{2}$$

Since

$$\nabla f(0,1) = \langle 1,0\rangle = \overrightarrow{i}$$

The direction in which the function changest fastest at $(0,1)$ is

$$\overrightarrow{u} = \frac{\nabla f(0,1)}{|\nabla f(0,1)|} = \overrightarrow{i}$$

with the maximum rate of change

$$|\nabla f(0,1)| = 1$$

8. Find the absolute maximum and minimum values of $f(x,y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$ on the set $D = \{(x,y)\,|0 \le x \le 2, -1 \le y \le 3\}$

Solution: [We need to compare all the: 1. critical points, 2. boundaries, 3. corners.]
To find the critical points of $f(x,y)$, let

$$f_x = 8x - 2y - 8 = 0$$
$$f_y = -2x + 12y + 2 = 0$$

The solution to the system is $x = 1$ and $y = 0$, so $(1,0)$ is a critical point of f, and $f(1,0) = -1$. On the boundary we compare the four line segments:
$L1$: connecting $(0,-1)$ and $(0,3)$, the equation is $x(t) = 0$, $y(t) = t$ for $-1 \le t \le 3$, so

$$f(x,y) = f(t) = 6t^2 + 2t + 3$$

with

$$f'(t) = 12t + 2$$

which attains maximum value when $t = -\frac{1}{6}$, and $f(0,-1/6) = \frac{5}{2}$
$L2$: connecting $(2,-1)$ and $(2,3)$, the equation is $x(t) = 2$, $y(t) = t$ for $-1 \le t \le 3$, so

$$f(x,y) = f(t) = 16 - 4t + 6t^2 - 16 + 2t + 3 = 6t^2 - 2t + 3$$

so $f'(x) = 12t - 2$. $f(x,y)$ attains its maximum when $t = 1/6$ and $f(2,1/6) = 17/6$.
$L3$: connecting $(0,-1)$ and $(2,-1)$, the equation is $x(t) = t$, $y(t) = -1$ for $0 \le t \le 2$, so

$$f(x,y) = f(t) = 4t^2 + 2t + 6 - 8t - 2 + 3 = 4t^2 - 6t + 7$$

with $f'(t) = 8t - 6$ so when $t = 3/4$, $f(3/4,-1) = 19/4$
$L4$: connecting $(0,3)$ and $(2,3)$, the equation is $x(t) = t$, $y(t) = 3$ for $0 \le t \le 2$, so

$$f(x,y) = f(t) = 4t^2 - 6t + 54 - 8t + 6 + 3 = 4t^2 - 14t + 63$$

with $f'(t) = 7/4$ so $f(7/4,3) = 48\frac{5}{7}$.
Then we find the values of $f(x,y)$ at the corners of its domain:

$$f(0,-1) = 7, \qquad f(0,3) = 63, \qquad f(2,-1) = 11, \qquad f(2,3) = 49$$

So the absolute maximum is $f(0,3) = 63$ and the absolute minimum is $f(1,0) = -1$.

9. Use the method of Lagrange multipliers to find the minimum value of

$$f(x, y) = xy$$

subject to the constraint

$$g(x, y) = 4x^2 + y^2 - 8 = 0$$

Solution: we solve the following equation:

$$\nabla f(x, y) = \lambda \nabla f(x, y)$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= y = \lambda 8x = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} &= x = \lambda 2y = \lambda \frac{\partial g}{\partial y} \end{aligned}$$

so $x = 2\lambda y = 16\lambda^2 x$, so $\lambda = \pm\frac{1}{4}$, so $y = \pm 2x$, subsititute this to $g(x, y) = 8x^2 - 8 = 0$, therefore $x = \pm 1$ and $y = \pm 2$. We can calculate the values at all these points:

$$f(1, 2) = 2, \qquad f(1, -2) = -2, \qquad f(-1, 2) = -2, \qquad f(-1, -2) = 2$$

so at points $(1, 2)$ and $(-1, -2)$ we have the maximum value of f which is 2, and at the points $(1, -2)$ and $(-1, 2)$ we have the minimum value of f which is -2.