Chapter 3 Vector Spaces

Section 3.4 Basis and Dimension

Example Re-visit

There are infinitely many sets of vectors that span the same vector space! e.g.

$$\mathbf{R}^3 = \mathrm{Span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \mathrm{Span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (1,2,3)^T\} = \cdots$$

What is the minimal spanning set for a vector space V?

By "minimal", not too many vectors in a spanning set, also not too few.

Answer: Basis!

Definition (Basis, plural form: bases) The vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ form a *basis* of a vector space V if

1. $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent, and
2. $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a spanning set for V.

In this case, $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a *basis* of V.

Example $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis of $\mathbf{R}^3$.

Reason

1. If $c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $c_1 = c_2 = c_3 = 0$.
2. For any vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbf{R}^3$, if $\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then $\begin{cases} \alpha_1 = a \\ \alpha_2 = b \\ \alpha_3 = c \end{cases}$.

Example Is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (1,1,1)^T\}$ a basis of $\mathbf{R}^3$?
Solution Solving $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 + c_4(1,1,1)^T = \mathbf{0}$, we get $(c_1, c_2, c_3, c_4) = (-\alpha, -\alpha, -\alpha, \alpha)$.
Hence, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (1,1,1)^T$ are linearly dependent.
So, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (1,1,1)^T\}$ is not a basis of $\mathbf{R}^3$.

Example Is $\{\mathbf{e}_1, \mathbf{e}_2\}$ a basis of $\mathbf{R}^3$?
Solution
Let $(a,b,c)^T \in \mathbf{R}^3$. Solving $\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 = (a,b,c)^T$, we have $\begin{cases} \alpha_1 = a \\ \alpha_2 = b \\ 0 = c \end{cases}$.
Since $0 = c$ is inconsistent when $c \neq 0$, there is no solution for $\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 = (a,b,c)^T$ when $c \neq 0$.
Since $\{\mathbf{e}_1, \mathbf{e}_2\}$ is not a spanning set of $\mathbf{R}^3$, $\{\mathbf{e}_1, \mathbf{e}_2\}$ is not a basis of $\mathbf{R}^3$.

Example Is $\{\mathbf{e}_1, \mathbf{e}_2, (1,1,1)^T\}$ a basis of $\mathbf{R}^3$?

Solution

1. If $c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\begin{pmatrix} c_1 + c_3 \\ c_2 + c_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and so $c_1 = c_2 = c_3 = 0$. So, $\mathbf{e}_1, \mathbf{e}_2, (1,1,1)^T$ are linearly independent.

2. Let $(a,b,c)^T \in \mathbf{R}^3$. Solving $\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 (1,1,1)^T = (a,b,c)^T$, we have $\begin{cases} \alpha_1 = a - c \\ \alpha_2 = b - c \\ \alpha_3 = c \end{cases}$. $\{\mathbf{e}_1, \mathbf{e}_2, (1,1,1)^T\}$ is a spanning set for $\mathbf{R}^3$.

Conclusion Since $\mathbf{e}_1, \mathbf{e}_2, (1,1,1)^T$ are linearly independent and form a spanning set for $\mathbf{R}^3$, they form a basis of $\mathbf{R}^3$.

Example (Standard basis) $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is called the standard basis of $\mathbf{R}^n$.

Example (Standard basis) $\{1, x, x^2, \cdots, x^{n-1}\}$ is called the standard basis of P_n.

Example (Standard basis) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is the standard basis of the vector space $\mathbf{R}^{2\times 2}$ of 2×2 matrices.

Example Let $A = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{pmatrix}$. Find a basis of the null space $N(A)$ of A.

Solution The reduced row echelon form of A is $\begin{pmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

$$\begin{aligned} N(A) &= \{\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^T : A\mathbf{x} = \mathbf{0}\} \\ &= \left\{ (x_1, x_2, x_3, x_4, x_5)^T : \begin{pmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} -3\alpha - 7\beta \\ -\alpha - 3\beta \\ \alpha \\ \beta \\ 0 \end{pmatrix} \right\} = \left\{ \alpha \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \end{aligned}$$

$\left\{ \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis of $N(A)$.

Theorem 1 If $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a spanning set for V, then any collection of m vectors in V, $m > n$, is linearly dependent.

Corollary If both $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m\}$ are bases for a vector space V, then $n = m$.

Proof of Corollary Let both $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m\}$ be bases for V. Since $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ span V and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m$ are linearly independent, it follows from the above theorem that $m \leq n$. By the same reasoning, $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m\}$ span V and $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent, so $n \leq m$. The assertion of $m = n$ is proved.

Theorem 1 If $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a spanning set for V, then any collection of m vectors in V, $m > n$, is linearly dependent.

Proof Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m$ be m vectors in V, where $m \geq n$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ span V, we have $\mathbf{u}_i = a_{i1}\mathbf{v}_1 + \cdots + a_{in}\mathbf{v}_n$ and a linear combination

$$
\begin{aligned}
c_1\mathbf{u}_1 + \cdots + c_m\mathbf{u}_m =& c_1 \sum_{j=1}^{n} a_{1j}\mathbf{v}_j + \cdots + c_m \sum_{j=1}^{n} a_{kj}\mathbf{v}_j \\
=& \sum_{i=1}^{m} \left[c_i \sum_{j=1}^{n} a_{ij}\mathbf{v}_j \right] \\
=& \sum_{j=1}^{n} \left[\sum_{i=1}^{m} a_{ij} c_i \right] \mathbf{v}_j .
\end{aligned}
$$

Proof (continuity) Now consider the system of equations with unknown c_i's as follows

$$\sum_{i=1}^{m} a_{ij} c_i = 0, j = 1, 2, \cdots, n.$$

This is a homogeneous system with more unknowns than equations. Therefore, by a theorem in Sc 1.2 (An $m \times n$ homogeneous system of linear equations has a nontrivial solution if $m < n$.) the system must have a nontrivial solution, denoted by $c_1^*, \cdots, c_m^*$. But then

$$c_1^* \mathbf{u}_1 + \cdots + c_m^* \mathbf{u}_m = \sum_{j=1}^{n} 0 \mathbf{v}_j.$$

Hence, $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m$ are linearly dependent.

Definition (Dimension) Let V be a vector space. If V has a basis consisting of n vectors, we say that V has *dimension* n and write $\dim V = n$.

Example $\mathbf{R}^n$ is of dimension n.

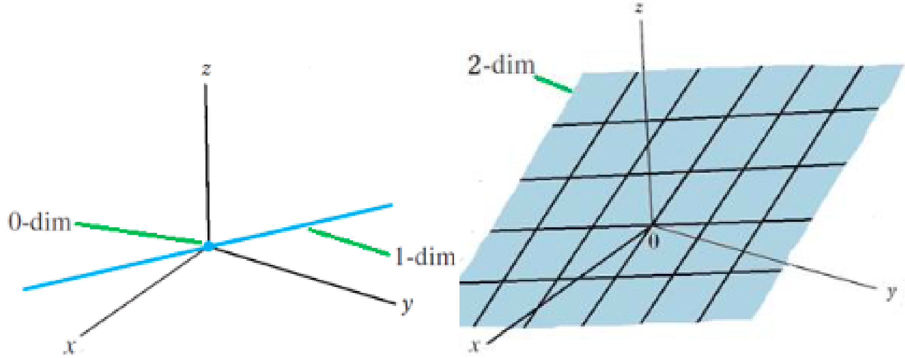
Example $\mathbf{R}^{m\times n}$ is of dimension mn.

Example P_n is of dimension n.

Example The subspace $\{\mathbf{0}\}$ of a vector space V is said to have dimension 0. (Some books would say that the basis of $\{\mathbf{0}\}$ is $\emptyset$.)

Example The subspaces of $\mathbf{R}^3$ can be classified by dimension.

- 0-dimensional subspaces: The zero subspace.
- 1-dimensional subspaces: Any subspace spanned by a single nonzero vector. Such subspaces are lines through the origin.
- 2-dimensional subspaces: Any subspace spanned by two linearly independent vectors. Such subspaces are planes through the origin.
- 3-dimensional subspaces: $\mathbf{R}^3$


z
0-dim
1-dim
y
x
z
2-dim
0
y
x

Definition (Finite-dimensional / Infinite-dimensional) V is said to be *finite-dimensional* if a basis of V has only finitely many elements. Otherwise, V is said to be *infinite-dimensional*.

Example $\mathbf{R}^n$, $\mathbf{R}^{m\times n}$, P_n are all finite-dimensional.

Example $\{1, x, x^2, \cdots\}$ is a basis of vector space P of all polynomials. So P is of infinite-dimensional.

Example The vector space of all continuous functions is infinite-dimensional.

Example The vector space of infinite sequences is infinite-dimensional.

Theorem 2 If V is a vector space of dimension $n > 0$, then

(I) any set of n linearly independent vectors spans V;

(II) any n vectors that span V are linearly independent.

Proof of (I) Suppose that $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly independent and $\mathbf{x}$ is any other vector in V. Since V has dimension n, it has a basis consisting of n vectors and these vectors span V. It follows that the set of $\{\mathbf{v}_1, \cdots, \mathbf{v}_n, \mathbf{x}\}$ must be linearly dependent. Thus, there exist scalars $c_1, \cdots, c_n, c_{n+1}$, not all zero, such that

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n + c_{n+1}\mathbf{x} = \mathbf{0} \qquad (\sharp)$$

The scalar $c_{n+1} \neq 0$, otherwise from $(\sharp)$ $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly dependent. Hence, $(\sharp)$ can be solved for $\mathbf{x}$:

$$\mathbf{x} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n,$$

where $\alpha_i = -c_i/c_{n+1}$. Since $\mathbf{x}$ is arbitrary, the assertion (I) follows.

Proof of (II) Suppose that $\mathbf{v}_1, \cdots, \mathbf{v}_n$ span V. If $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly dependent, then one of the $\mathbf{v}_i$'s, say, $\mathbf{v}_n$, can be written as a linear combination of the others. It follows from Theorem 1 that $\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}$ will still span V. If $\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}$ are linearly dependent, we can eliminate another vector and still have a spanning set. We can continue eliminating vectors in this way until we arrive at a linearly independent spanning set with $k < n$ elements. But this contradicts $\dim V = n$. Therefore, $\mathbf{v}_1, \cdots, \mathbf{v}_n$ must be linearly independent.

Theorem 3 If V is a vector space of dimension $n > 0$, then

(i) no set of fewer than n vectors can span V;

(ii) any subset of fewer than n linearly independent vectors can be extended to form a basis for V;

(iii) any spanning set containing more than n vectors can be pared down to form a basis for V.

Proof of (i) and (ii) Statement (i) follows by the same reasoning that was used to prove part (I) of Theorem 2.

To prove (ii), suppose that $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are linearly independent with $k < n$. It follows from (i) that $\text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is a proper subspace of V, and hence there exists a vector $\mathbf{v}_{k+1} \in V$ that is not in $\text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$. It then follows that $\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{v}_{k+1}$ must be linearly independent. If $k+1 < n$, then, in the same manner, $\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{v}_{k+1}$ can be extended to a set of $k+2$ linearly independent vectors. This extension process may be continued until a set $\{\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{v}_{k+1}, \cdots, \mathbf{v}_n\}$ of n linearly independent vectors is obtained.

Proof of (iii) Suppose that $\mathbf{v}_1, \cdots, \mathbf{v}_m$ span V and $m > n$. Then, by Theorem 1, $\mathbf{v}_1, \cdots, \mathbf{v}_m$ must be linearly dependent. It follows that one of the vectors, say, $\mathbf{v}_m$, can be written as a linear combination of the others. Hence, if $\mathbf{v}_m$ is eliminated from the set, the remaining $m-1$ vectors will still span V. If $m-1 > n$, we can continue to eliminate vectors in this manner until we arrive at a spanning set containing n elements.

Extra Exercise* Determine if the following set of vectors form a basis for $\mathbf{R}^3$.

1. $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in $\mathbf{R}^3$.
2. $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (1, 2, 3)^T\}$ in $\mathbf{R}^3$.
3. $\{(1, 0, 0)^T, (0, 1, 1)^T\}$ in $\mathbf{R}^3$.

Answer:

(a). Yes, (b). No, (c). No

Extra Example* Let
$S = \{(2,-3,5)^T, (8,-12,20)^T, (1,0,2)^T, (0,2,-1)^T, (7,2,0)^T\}$.
$\text{Span}(S) = \mathbf{R}^3$. Find a basis of $\mathbf{R}^3$ providing S.

Answer:

1. Since $(8,-12,20)^T = 4(2,-3,5)^T$, then remove it

$$\text{Span}\{(2,-3,5)^T, (1,0,2)^T, (0,2,-1)^T, (7,2,0)^T\} = \mathbf{R}^3.$$

2. Since $(7,2,0)^T = 2(2,-3,5)^T + 3(1,0,2)^T + 4(0,2,-1)^T$, then remove it

$$\text{Span}\{(2,-3,5)^T, (1,0,2)^T, (0,2,-1)^T\} = \mathbf{R}^3.$$

3. Assume $\alpha_1(2,-3,5)^T + \alpha_2(1,0,2)^T + \alpha_3(0,2,-1)^T = (0,0,0)^T$, the only solution is $\alpha_1 = \alpha_2 = \alpha_3 = 0$, linear independent.
   Therefore, $\{(2,-3,5)^T, (1,0,2)^T, (0,2,-1)^T\}$ is a basis for $\mathbf{R}^3$.