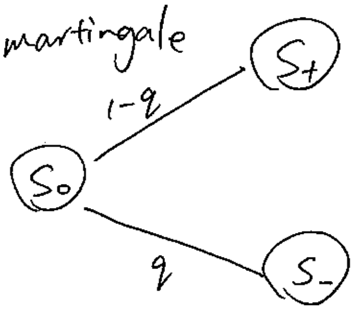
2017.2.14

risk neutral probability

martingale



$$q = \frac{S_+ - S_0 e^{rT}}{S_+ - S_-}$$

$$V_0 = e^{-rT}\left[q V_- + (1-q) V_+ \right]$$

$$V^i = e^{-r(t_{i+1}-t_i)}\left[q V_-^{i+1} + (1-q) V_+^{i+1} \right]$$

$t \to t + \Delta t$

$t_i \to t_{i+1}$

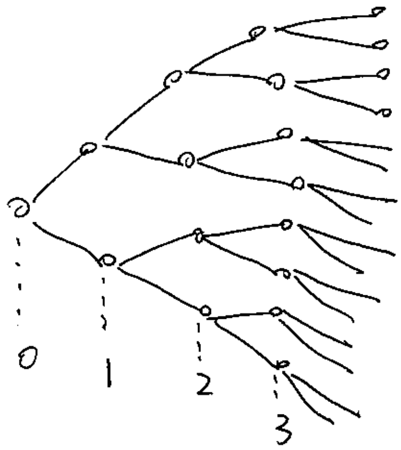
$$\frac{V^i}{e^{rt_i}} = q\,\frac{V_-^{i+1}}{e^{rt_{i+1}}} + (1-q)\,\frac{V_+^{i+1}}{e^{rt_{i+1}}}$$

if let

$$\tilde{V}^j = \frac{V^j}{e^{rt_j}}$$

then $\tilde{V}^i = q\,\tilde{V}_-^{i+1} + (1-q)\,\tilde{V}_+^{i+1}$

$\tilde{V}^i = E\left[\tilde{V}^j, j > i \mid \mathcal{F}_i \right]$ $\quad \mathcal{F}_i$ Filtration

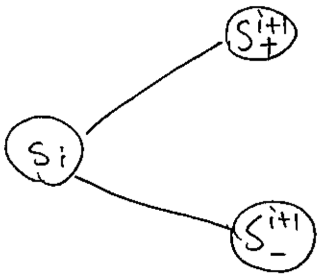


which is also the definition of martingale for a random process.

$$M_s = E_q\left[M_t \mid \mathcal{F}_s \right], \quad s \le t$$

Is option price a martingale?
No, discounted option price is a martingale.

- The discounted wealth should be martingale for eliminating arbitrage opportunities.



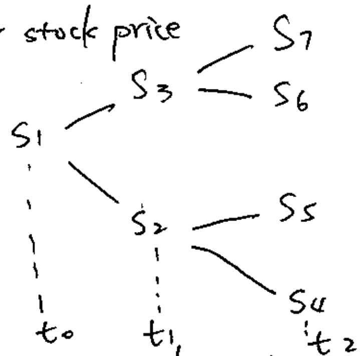
$$\frac{S^i}{e^{rt_i}} = q\,\frac{S_-^{i+1}}{e^{rt_{i+1}}} + (1-q)\,\frac{S_+^{i+1}}{e^{rt_{i+1}}}$$

$$\Rightarrow q = \frac{\dfrac{S_+^{i+1}}{e^{rt_{i+1}}} - \dfrac{S_i}{e^{rt_i}}}{\dfrac{S_+^{i+1}}{e^{rt_{i+1}}} - \dfrac{S_-^{i+1}}{e^{rt_{i+1}}}}$$

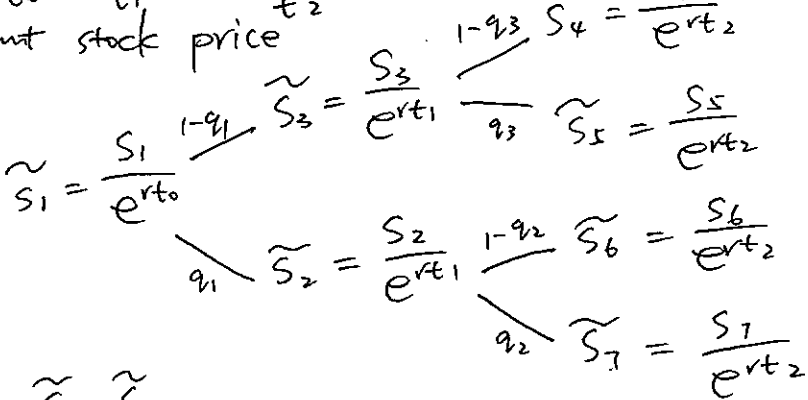
$$= \frac{S_+^{i+1} - e^{r(t_{i+1}-t_i)}\,S^i}{S_+^{i+1} - S_-^{i+1}}$$

Steps for solving problems (call, put options)

① tree for stock price



② discount stock price



③ $$q_1 = \frac{\tilde{S}_3 - \tilde{S}_1}{\tilde{S}_3 - \tilde{S}_2}$$

$$q_2 = \frac{\tilde{S}_5 - \tilde{S}_2}{\tilde{S}_5 - \tilde{S}_4}$$

$$q_3 = \frac{\tilde{S}_7 - \tilde{S}_3}{\tilde{S}_7 - \tilde{S}_6}$$

(4) $$F(S) = \begin{cases} \max\{S-K, 0\} & \text{for call} \\ \max\{K-S, 0\} & \text{for put} \end{cases}$$

$$\tilde{V}_7 = \frac{F(S_7)}{e^{rt_2}}$$

$$\tilde{V}_6 = \frac{F(S_6)}{e^{rt_2}}$$

$$\tilde{V}_5 = \frac{F(S_5)}{e^{rt_2}}$$

$$\tilde{V}_4 = \frac{F(S_4)}{e^{rt_2}}$$

(5)

$\tilde{V}_1 \xrightarrow{1-q_1} \tilde{V}_3$, $\tilde{V}_1 \xrightarrow{q_1} \tilde{V}_2$

$\tilde{V}_3 \xrightarrow{1-q_3} \tilde{V}_7$, $\tilde{V}_3 \xrightarrow{q_3} \tilde{V}_6$

$\tilde{V}_2 \xrightarrow{1-q_2} \tilde{V}_5$, $\tilde{V}_2 \xrightarrow{q_2} \tilde{V}_4$

(6)

$V_1 = e^{rt_0}\tilde{V}_1$

$V_3 = e^{rt_1}\tilde{V}_3$

$V_2 = e^{rt_1}\tilde{V}_2$

$V_7 = e^{rt_2}\tilde{V}_7$

$V_6 = e^{rt_2}\tilde{V}_6$

$V_5 = e^{rt_2}\tilde{V}_5$

$V_4 = e^{rt_2}\tilde{V}_4$

We consider a special case:

(1)

S_0

$S_0 u$, $S_0 d$

$S_0 u \cdot u$, $S_0 u \cdot d$, $S_0 d \cdot u$, $S_0 d \cdot d$

$0 \quad 1 \quad 2 \quad \cdots\, n \;\cdots\; N$

$\underbrace{\quad}_{\Delta t} \quad \underbrace{\quad}_{\Delta t} \quad \cdots$

$$S_n = S_0 \cdot X_1 X_2 \cdots X_n = S_0\, d^i u^{n-i}$$

where $X_j = \begin{cases} u \\ d \end{cases}$

assume $u \cdot d = 1$

② $\tilde{S}_n = e^{-rn\Delta t} \cdot S_0 d^i u^{n-i}$

③
$$
\begin{tikzcd}
 & \tilde{S}_n^i \cdot u \cdot e^{-r\Delta t} \\
\tilde{S}_n^i \arrow[ur, "1-q_n^i"] \arrow[dr, "q_n^i"'] & \\
 & \tilde{S}_n^i \cdot d \cdot e^{-r\Delta t}
\end{tikzcd}
$$

$$\tilde{S}_n^i = (1-q_n^i)\tilde{S}_n^i u e^{-r\Delta t} + q_n^i \tilde{S}_n^i d e^{-r\Delta t}$$

$$\Rightarrow q_n^i = \frac{u - e^{r\Delta t}}{u-d}$$

④ $F(S_N^i) = \max\{S_N^i - K, 0\}$ for call

$$= \max\{S_0 d^i u^{N-i} - K, 0\}$$

$$\tilde{F}(S_N^i) = \max\{S_0 d^i u^{N-i} - K, 0\} \cdot e^{-rN\Delta t}$$

⑤ $\tilde{V}_0 = \sum_{i=0}^{N} \frac{N!}{i!(N-i)!} q^i (1-q)^{N-i} \cdot \tilde{F}(S_N^i)$

$$\tilde{V}_k = E[\tilde{V}_N \mid \mathcal{F}_k]$$

$$= E[F(S_N^i)e^{-rN\Delta t} \mid \mathcal{F}_k]$$

$$= \sum_{i=0}^{N-k} \frac{(N-k)!}{i!(N-k-i)!} q^i (1-q)^{N-k-i} F(S_N^i) \cdot e^{-rN\Delta t}$$

$$V_k = e^{rk\Delta t}\tilde{V}_k$$

face value

$$= e^{-r(N-k)\Delta t} \sum_{i=0}^{N-k} \frac{(N-k)!}{i!(N-k-i)!} q^i (1-q)^{N-k-i} F(S_N^i)$$