# PT

## Solution to Assignment 10

1. We have

$$E(X \mid Y = y) = \int_0^\infty x f_X(x \mid Y = y) dx$$

As

$$\begin{aligned} f_Y(y) &= \int_0^\infty \frac{e^{-x/y} e^{-y}}{y} dx \\ &= \frac{e^{-y}}{y} \int_0^\infty e^{-x/y} dx \\ &= \frac{e^{-y}}{y} \left[ -y e^{-x/y} \right]_{x=0}^{x=\infty} \\ &= e^{-y}, \text{ for } y > 0 \end{aligned}$$

we have

$$\begin{aligned} f_X(x \mid Y = y) &= \frac{f(x, y)}{f_Y(y)} \\ &= \frac{e^{-x/y}}{y}, \text{ for } x, y > 0 \end{aligned}$$

and so

$$\begin{aligned} E(X \mid Y = y) &= \int_0^\infty \frac{x}{y} e^{-x/y} dx \\ &= y \int_0^\infty z e^{-z} dz \\ &= y \end{aligned}$$

2. Let $S$ be the speed and $X$ be the loss. Given $S, X$ has an exponential distribution with mean $3X$. Then, noting that the variance of an exponential random variable is the square of the mean, the variance of a uniform random variable is the square of the range divided by 12, and for any random variable the second moment is the variance

plus the square of the mean:

$$
\begin{aligned}
\operatorname{Var}(X) &= \operatorname{Var}[E(X \mid S)] + E[\operatorname{Var}(X \mid S)] \\
&= \operatorname{Var}[3S] + E\left(9S^2\right) \\
&= 9(20-5)^2/12 + 9\left[(20-5)^2/12 + 12.5^2\right] \\
&= 1743.75
\end{aligned}
$$

3. Let $X_i$ be the numbers rolled on the die and $N$ be the number of Tails tossed before first Heads. We know that

$$
\begin{aligned}
EX_1 &= \frac{7}{2}, \\
\operatorname{Var}(X_1) &= \frac{35}{12}.
\end{aligned}
$$

Moreover, $N+1$ is a Geometric $\left(\frac{1}{2}\right)$ random variable, and so

$$
\begin{aligned}
EN &= 2 - 1 = 1 \\
\operatorname{Var}(N) &= \frac{1-\frac{1}{2}}{\left(\frac{1}{2}\right)^2} = 2 \\
E[S] &= E[N]E[X_1] = \frac{7}{2} \\
\operatorname{Var}[S] &= E[N]\operatorname{Var}[X_1] + E[X_1]^2 \operatorname{Var}[N] = \frac{35}{12} + 2\left(\frac{7}{2}\right)^2 = \frac{329}{12}
\end{aligned}
$$

4. Clearly, $E[Y] = \operatorname{Var}[Y] = 1$. We have

$$
\begin{aligned}
&E[X] = E[E[X \mid Y]] = 3E[Y] \\
&\operatorname{Var}[X] = \operatorname{Var}[E[X \mid Y]] + E[\operatorname{Var}[X \mid Y]] = \operatorname{Var}[3Y] + E[2] = 9\operatorname{Var}[Y] + 2 = 9 + 2 = 11
\end{aligned}
$$