ASP Chapter 2: Discrete Markov Chains

Part II

EXAMPLE 1. Assume that the transition matrix is given by

$$\boldsymbol{P} = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.4 & 0.6 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Recall that the n-step transition probabilities are given by powers of $\boldsymbol{P}$. Let us look at some large powers of $\boldsymbol{P}$, beginning with

$$\boldsymbol{P}^4 = \begin{bmatrix} 0.5401 & 0.4056 & 0.0543 \\ 0.5412 & 0.4048 & 0.054 \\ 0.54 & 0.408 & 0.052 \end{bmatrix}$$

Then, to four decimal places,

$$\boldsymbol{P}^8 \approx \begin{bmatrix} 0.5405 & 0.4054 & 0.0541 \\ 0.5405 & 0.4054 & 0.0541 \\ 0.5405 & 0.4054 & 0.0541 \end{bmatrix}$$

and subsequent powers are the same to this precision. The matrix elements appear to converge and the rows become almost identical. Why? What determines the limit? These questions will be answered in the following series of theorems.

We now state the key theorems. Some of these have rather involved proofs (although none is exceptionally difficult), which we will merely sketch or omit altogether.

THEOREM 2 (Proportion of the time spent at i). *Assume that the chain is irreducible and positive recurrent. Let $N_n(i)$ be the number of visits to i in the time interval from 0 through n. Then,*

$$\lim_{n\to\infty} \frac{N_n(i)}{n} = \frac{1}{\mu_{i,i}}$$

in probability.

PROOF. The idea is quite simple: once the chain visits i, it returns, on average, once per $\mu_{i,i}$ time steps, hence the proportion of time spent there is $1/\mu_{i,i}$. We skip a more detailed proof. □

A vector of probabilities, $\pi_i, i \in S$, such that $\sum_{i \in S} \pi_i = 1$ is called an invariant, or stationary, distribution for a Markov chain with transition matrix $\boldsymbol{P}$ if

$$\sum_{i \in S} \pi_i P_{i,j} = \pi_j, \text{ for all } j \in S.$$

In matrix form, if we put π into a row vector $[\pi_0, \pi_1, \ldots]$, then

$$[\pi_0, \pi_1, \ldots] \cdot \boldsymbol{P} = [\pi_0, \pi_1, \ldots].$$

Thus $[\pi_0, \pi_1, \ldots]$ is a left eigenvector of $\boldsymbol{P}$, for eigenvalue 1. More important for us is the following probabilistic interpretation. If π_i is the p.m.f. for X_0, that is, $P(X_0 = i) = \pi_i$, for all $i \in S$, it is also the p.m.f. for X_1 and hence for all other X_n, that is, $P(X_n = i) = \pi_i$, for all n.

THEOREM 3 (Existence and uniqueness of invariant distributions). *An irreducible positive recurrent Markov chain has a unique invariant distribution, which is given by*

$$\pi_i = \frac{1}{\mu_{i,i}}.$$

In fact, an irreducible chain is positive recurrent if and only if a stationary distribution exists.

The formula for π should not be a surprise: if the probability that the chain is at i is always π_i, then one should expect that the proportion of time spent at i, which we already know to be $1/\mu_{i,i}$, to be equal to π_i. We will not, however, go deeper into the proof.

THEOREM 4 (Convergence to invariant distribution). *If a Markov chain is irreducible, aperiodic, and positive recurrent, then, for every $i, j \in S$,*

$$\lim_{n\to\infty} P_{i,j}^n = \pi_j.$$

Recall that $P_{i,j}^n = P(X_n = j \mid X_0 = i)$ and note that the limit is independent of the initial state. Thus, the rows of $\boldsymbol{P}^n$ are more and more similar to the row vector π as n becomes large.

The most elegant proof of this theorem uses coupling, an important idea first developed by a young French probabilist Wolfgang Doeblin in the late 1930s. Start with two independent copies of the chain - two particles moving from state to state according to the transition probabilities - one started from i, the other using the initial distribution π. Under the stated assumptions, they will eventually meet. Afterwards, the two particles move together in unison, that is, they are coupled. Thus, the difference between the two probabilities at time n is bounded above by twice the probability that coupling does not happen by time n, which goes to 0. We will not go into greater detail, but, as we will see in the next example, aperiodicity is necessary.

EXAMPLE 5. A deterministic cycle with $a = 3$ has the transition matrix

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

This is an irreducible chain with the invariant distribution $\pi_0 = \pi_1 = \pi_2 = \frac{1}{3}$ (as it is easy to check). Moreover,

$$\boldsymbol{P}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$\boldsymbol{P}^3 = \boldsymbol{I}, \boldsymbol{P}^4 = \boldsymbol{P}$, etc. Although the chain does spend $1/3$ of the time at each state, the transition probabilities are a periodic sequence of 0's and 1's and do not converge.

Our final theorem is mostly a summary of the results for the special, and for us the most common, case.

THEOREM 6 (Convergence theorem for a finite state space). *Assume that a Markov chain with a finite state space is irreducible. Then:*

1. There exists a unique invariant distribution given by $\pi_i = \frac{1}{\mu_{i,i}}$.

2. For every i, irrespective of the initial state,

$$\frac{1}{n} N_n(i) \to \pi_i,$$

in probability.

3. If the chain is also aperiodic, then, for all i *and* j,

$$P^n_{i,j} \to \pi_j.$$

4. If the chain is periodic with period d*, then, for every pair* $i, j \in S$*, there exists an integer* $r, 0 \le r \le d-1$*, so that*

$$\lim_{m \to \infty} P^{md+r}_{i,j} = d\pi_j$$

and so that $P^n_{i,j} = 0$ *for all* n *such that* $n \neq r \bmod d$.

EXAMPLE 7. We begin by our first example, Example 1. Suppose that the states are labelled as 1,2, and 3. That was clearly an irreducible and aperiodic chain. The invariant distribution $[\pi_1, \pi_2, \pi_3]$ is given by

$$0.7\pi_1 + 0.4\pi_2 = \pi_1$$

$$0.2\pi_1 + 0.6\pi_2 + \pi_3 = \pi_2$$

$$0.1\pi_1 = \pi_3$$

This system has infinitely many solutions and we need to use

$$\pi_1 + \pi_2 + \pi_3 = 1$$

to get the unique solution

$$\pi_1 = \frac{20}{37} \approx 0.5405, \pi_2 = \frac{15}{37} \approx 0.4054, \pi_3 = \frac{2}{37} \approx 0.0541.$$

EXAMPLE 8. The general two-state Markov chain. Here $S = \{1, 2\}$ and

$$\boldsymbol{P} = \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$$

and we assume that $0 < \alpha, \beta < 1$.

$$\begin{aligned} \alpha\pi_1 + \beta\pi_2 &= \pi_1 \\ (1-\alpha)\pi_1 + (1-\beta)\pi_2 &= \pi_2 \\ \pi_1 + \pi_2 &= 1 \end{aligned}$$

and, after some algebra,

$$\pi_1 = \frac{\beta}{1+\beta-\alpha}$$
$$\pi_2 = \frac{1-\alpha}{1-\beta+\alpha}.$$

Here are a few common follow-up questions:

- Start the chain at 1. In the long run, what proportion of time does the chain spend at 2 ? Answer: π_2 (and this does not depend on the starting state).
- Start the chain at 2. What is the expected return time to 2? Answer: $\frac{1}{\pi_2}$.

Computing the invariant distribution amounts to solving a system of linear equations. Nowadays this is not difficult to do, even for enormous systems; still, it is worthwhile to observe that there are cases when the invariant distribution is easy to identify.

We call a square matrix with nonnegative entries doubly stochastic if the sum of the entries in each row and in each column is 1.

PROPOSITION 9 (Invariant distribution in a doubly stochastic case). *If the transition matrix for an irreducible Markov chain with a finite state space S is doubly stochastic, its (unique) invariant measure is uniform over S.*

PROOF. Assume that $S = \{1, \dots, m\}$, as usual. If $[1, \dots, 1]$ is the row vector with m 1's, then $[1, \dots, 1]\boldsymbol{P}$ is exactly the vector of column sums, thus $[1, \dots, 1]$. This vector is preserved by right multiplication by $\boldsymbol{P}$, as is $\frac{1}{m}[1, \dots, 1]$. This vector specifies the uniform p.m.f. on S. $\square$

EXAMPLE 10. Simple random walk on a circle. Pick a probability $p \in (0, 1)$. Assume that a points labeled $0, 1, \dots, a-1$ are arranged on a circle clockwise. From i, the walker moves to $i + 1$ (with a identified with 0) with probability p and to $i - 1$ (with -1 identified with $a - 1$) with probability $1 - p$. The transition matrix is

$$\boldsymbol{P} = \begin{bmatrix} 0 & p & 0 & 0 & \dots & 0 & 0 & 1-p \\ 1-p & 0 & p & 0 & \dots & 0 & 0 & 0 \\ 0 & 1-p & 0 & p & \dots & 0 & 0 & 0 \\ & & & \dots & & & & \\ 0 & 0 & 0 & 0 & \dots & 1-p & 0 & p \\ p & 0 & 0 & 0 & \dots & 0 & 1-p & 0 \end{bmatrix}$$

and is doubly stochastic. Moreover, the chain is aperiodic if a is odd and otherwise periodic with period 2. Therefore, the proportion of time the walker spends at any state is $\frac{1}{a}$, which is also the limit of $P^n_{i,j}$ for all i and j if a is odd. If a is even, then $P^n_{i,j} = 0$ if $(i - j)$ and n have a different parity, while if they are of the same parity, $P^n_{i,j} \to \frac{2}{a}$.

Assume that we change the transition probabilities a little: assume that, only when the walker is at 0, she stays at 0 with probability $r \in (0, 1)$, moves to 1 with probability $(1-r)p$, and to $a-1$ with probability $(1-r)(1-p)$. The other transition probabilities are unchanged. Clearly, now the chain is aperiodic for any a, but the transition matrix is no longer doubly stochastic. What happens to the invariant distribution?

The walker spends a longer time at 0; if we stop the clock while she stays at 0, the chain is the same as before and spends an equal proportion of time at all states. It follows that our perturbed chain spends the same proportion of time at all states except 0, where it spends a Geometric$(1 - r)$ time at every visit. Therefore, π_0 is larger by the factor $\frac{1}{1-r}$ than other π_i. Thus, the row vector with invariant distributions is

$$\frac{1}{\frac{1}{1-r}+a-1}\begin{bmatrix}\frac{1}{1-r} & 1 & \cdots & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{1+(1-r)(a-1)} & \frac{1-r}{1+(1-r)(a-1)} & \cdots & \frac{1-r}{1+(1-r)(a-1)}\end{bmatrix}.$$

Thus, we can still determine the invariant distribution if only the self-transition probabilities $P_{i,i}$ are changed.