2023-24 First Semester
MATH2023 Ordinary and Partial Differential Equations (1002)

Assignment 2 Suggested Solution

1. Let $M(x,y) = ax - by,\ N(x,y) = bx - cy$, then the DE is **exact** when
$$M_y = -b = N_x = b \quad \to \quad b = 0.$$
Also the DE becomes,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ax}{cy}, c \neq 0 \quad \to \quad \int cy\ \mathrm{d}y = \int ax\ \mathrm{d}x + C$$
i.e. $cy^2 = ax^2 + C$, $C \in \mathbb{R}$.
Remark: *In fact, you can show that any separable equation* $y' = \frac{M(x)}{N(y)}$ *is also exact.*

2. (a) Nonlinear ODE, but not separable or exact, since $N_x - M_y = 2y - 1 \neq 0$. However,
$$\frac{N_x - M_y}{M} = \frac{2y-1}{y}$$
is a function of y alone. There exists an integrating factor $u(y)$ s.t.
$$\frac{1}{u}\mathrm{d}u = \frac{2y-1}{y}\mathrm{d}y \quad \to \quad \ln|u| = 2y - \ln|y| \quad \to \quad u(y) = e^{2y}/y.$$
Now rewrite the given ODE as $e^{2y}\mathrm{d}x + (2xe^{2y} - 1/y)\mathrm{d}y = 0$. It is now exact and easy to find the solution as
$$\Psi(x,y) = xe^{2y} - \ln y = C, \quad C \in \mathbb{R}.$$

(b) Nonlinear ODE, but not separable or exact, since $M_y - N_x = (x+2)\cos(y) - \cos(y) \neq 0$. Consider an integrating factor $u(x)$, then
$$\frac{M_y - N_x}{N} = \frac{(x+1)\cos(y)}{x\cos(y)} = \frac{x+1}{x}$$
is a function of x alone. It means such factor $u(x)$ exists and satisfies
$$\frac{1}{u(x)}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x+1}{x} \quad \to \quad u(x) = xe^x.$$
Multiplying $u(x)$ on both sides of the original equation results in an exact equation
$$e^x(x^2+2x)\sin(y) + x^2e^x\cos(y)y' = 0.$$
To find $\psi(x,y)$, let
$$\begin{aligned} \psi(x,y) &= \int M(x,y)dx + h(y) = x^2e^x\sin(y) + h(y) \\ \frac{\partial\psi(x,y)}{\partial y} &= x^2e^x\cos(y) + h'(y) = N(x,y) \\ \to \quad h'(y) &= 0 \end{aligned}$$
Hence, $h(y) = c$ and the implicit solution is :
$$x^2e^x\sin(y) = C, \qquad C \in \mathbb{R}.$$

3. (a) Let $M(x, y) = 2x - y,\ N(x, y) = -4y - x$, then the DE is an **exact** equation since

$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x},$$

Obtain $\Psi(x, y)$ by integrating $N(x, y)$ with respect to y and compute Ψ_x,

$$\Psi(x, y) = -2y^2 - xy + g(x)$$

$$\frac{\partial \Psi(x, y)}{\partial x} = -y + g'(x) = M(x, y) = 2x - y$$

$$\to g'(x) = 2x \quad \to \quad g(x) = x^2 + C_1, \quad C_1 \in \mathbb{R}.$$

Hence the solution is $\Psi(x, y) = C$, with arbitrary constant C, i.e.

$$-2y^2 - xy + x^2 = C.$$

Imposing the initial condition $y(1) = 3$, $C = -20$. Solution to this IVP is

$$y = \frac{-x + \sqrt{9x^2 + 160}}{4}, \quad x \in \mathbb{R}.$$

(b) First order linear DE, **Standard form:**

$$y' + \frac{2t}{4 - t^2} y = \frac{3t^2}{4 - t^2}, \qquad t \neq \pm 2.$$

Integrating factor:

$$e^{\int \frac{2t}{4-t^2} \mathrm{d}t} = e^{-ln|4-t^2|} = \frac{1}{4 - t^2}.$$

Multiply $u(t)$ to the standard form:

$$\frac{1}{4 - t^2} y' + \frac{2t}{(4 - t^2)^2} y = \frac{3t^2}{(4 - t^2)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{4 - t^2}\right) = \frac{3t^2}{(4 - t^2)^2}$$

Integrate by partial fractions and we yield

$$\frac{y}{4 - t^2} = 3\left(\frac{\ln|t - 2| - \ln|t + 2|}{8} - \frac{1}{4(t + 2)} - \frac{1}{4(t - 2)}\right) + C$$

General solution:

$$y(t) = \frac{3(4 - t^2)}{8} \ln\left|\frac{t - 2}{t + 2}\right| + \frac{3t}{2} + C(4 - t^2), \quad C \in \mathbb{R}.$$

$$y(0) = 0 + 0 + 4C = 4, \qquad \to \quad C = 1$$

Solution to the IVP:

$$y(t) = \frac{3(4 - t^2)}{8} \ln\left|\frac{t - 2}{t + 2}\right| + \frac{3t}{2} + (4 - t^2)$$

(c) This is a separable equation.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{-4t}{y}, \quad y \neq 0 \\ \int y\mathrm{d}y &= \int -4t\mathrm{d}t \\ \frac{1}{2}y^2 &= -2t^2 + C \\ y &= \pm\sqrt{-4t^2 + 2C}\end{aligned}$$

Since $y(0) = \pm\sqrt{2C} = y_0, \quad \rightarrow \quad C = y_0^2/2$, then

$$y = \begin{cases} \sqrt{y_0^2 - 4t^2}, & \text{if } y_0 > 0; \\ -\sqrt{y_0^2 - 4t^2}, & \text{if } y_0 < 0. \end{cases}, \quad |t| < |y_0|/2$$

4. (a) Proof: The equation $\mu M + \mu N y' = 0$ is exact if $(\mu M)_y = (\mu N)_x$, i.e.

$$\mu_y M - \mu_x N = \mu(N_x - M_y). \tag{1}$$

Assume μ is a function depending *only* on the quantity $t = xy$, we denote $\mu = \mu(t)$. It follows that

$$\mu_x = \frac{\mathrm{d}\mu}{\mathrm{d}t}\frac{\partial t}{\partial x} = \mu' y, \quad \mu_y = \frac{\mathrm{d}\mu}{\mathrm{d}t}\frac{\partial t}{\partial y} = \mu' x.$$

Then (1) becomes

$$\frac{\mathrm{d}\mu}{\mathrm{d}t}(xM - yN) = \mu(N_x - M_y) \quad \rightarrow \quad \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{N_x - M_y}{xM - yN} = R.$$

Moreover, $\mu = \int R(t)\mathrm{d}t$, where $t = xy$.

□

(b) Since $\frac{N_x - M_y}{xM - yN} = \frac{1}{xy}$ depends only on xy, then there exists an integrating factor μ s.t.

$$\frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{N_x - M_y}{xM - yN} = \frac{1}{t}, \quad \text{where } t = xy.$$

Solve this separable equation and we have $\mu = xy$. Now rewrite the given ODE as

$$\left(3x^2y + 6x\right)\mathrm{d}x + \left(x^3 + 3y^2\right)\mathrm{d}y = 0.$$

The solution is $x^3y + 3x^2 + y^3 = C$, $C \in \mathbb{R}$.

5. (a) The balance increases at a rate of rS per year, and decreases at a constant rate of k per year. The governing equation can be

$$\frac{\mathrm{d}S}{\mathrm{d}t} = rS - k, \qquad k > 0.$$

(b) Solve the first order linear differential equation in part (a) as a separable eqn.

$$\int \frac{1}{rS - k}\mathrm{d}S = \int 1\mathrm{d}t + C_1$$

$$\frac{1}{r}\ln|rS-k| = t + C_1$$
$$rS = e^{rt}\cdot e^{rC_1} + k$$

The general solution is

$$S(t) = \frac{C}{r}e^{rt} + \frac{k}{r}, \qquad C \in \mathbb{R}.$$

Impose the initial condition $S(0) = S_0$ and solve the IVP, we obtain the solution as

$$S(t) = \left(S_0 - \frac{k}{r}\right)e^{rt} + \frac{k}{r}.$$

(c) Note that the balance will remain at a constant level when $S'(t) = 0$, i.e. $S' = rS - k = 0$ and $S(t) = \frac{k}{r}$. Comparing with the solution obtained in part (b), if the withdrawal rate is $k_0 = rS_0$, then $S(t)$ will remain constant as $\frac{k_0}{r}$.

(d) Based on the conclusion in Problem 7, $f'(S)\big|_{S=k_0/r} = \frac{\mathrm{d}}{\mathrm{d}S}(rS-k)\Big|_{S=k_0/r} = r > 0$, then $S = \frac{k_0}{r}$ is an unstable equilibrium solution.
From the other side, we can also see that when the initial value $S(0)$ varies, the solution in part (b) $\lim\limits_{t\to\infty} S(t)$ would tend to $\pm\infty$ unless $S(0) = S_0$.

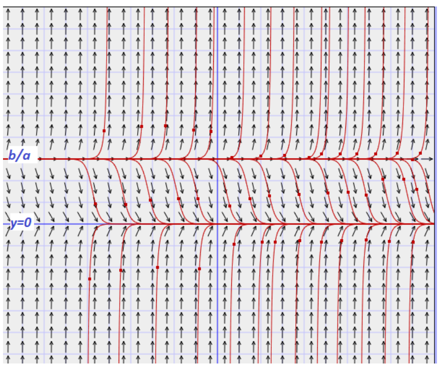
(e) Let $S(T_0) = 0$. Solve for T_0 and we have $T_0 = \frac{1}{r}\ln\left[\frac{k}{k-rS_0}\right]$ where $k > rS_0$.

(f) Setting $T_0 = T$ and solving for k in Part (e), it results in $e^{rT} = \frac{k}{k-rS_0}$ and

$$k = \frac{rS_0e^{rT}}{e^{rT}-1}.$$

(g) In part (f), let $k = 2000$, $r = .08$, and $T = 20$. The required investment becomes $S_0 = \$19,952.6$.

6. The critical points are 0 and b/a. From the phase portrait we see that 0 is asymptotically stable and b/a is unstable. Thus, if an initial population satisfies $P_0 > b/a$, the population becomes unbounded as t increases, most probably in finite time, i.e. $P(t) \to \infty$ as $t \to T$. If $0 < P_0 < b/a$, then the population eventually dies out, that is, $P(t) \to 0$ as $t \to \infty$. Since population $P > 0$ we do not consider the case $P_0 < 0$.



7. If $f'(y_1) < 0$ then the slope of f is negative at y_1 and thus $f(y) > 0$ for $y < y_1$ and $f(y) < 0$ for $y > y_1$ since $f(y_1) = 0$. Hence y_1 is an asymptotically stable critical point. A similar argument will yield the result for $f'(y_1) > 0$.