Chapter 1 Matrices and System of Equations

More on Triangular Matrices

Properties on Triangular Matrices

1. The sum of two upper (lower) triangular matrices result in an upper (lower) triangular matrix.

$$\text{Upper} + \text{Upper} = \text{Upper}$$

2. The scalar multiple of an upper (lower) triangular matrix is an upper (lower) triangular matrix.

$$c\ \text{Upper} = \text{Upper}, \quad c \in \mathbf{R}.$$

3. The transpose of an upper (lower) triangular matrix is a lower (upper) triangular matrix.

$$\text{Upper}^T = \text{Lower}$$

Properties on Triangular Matrices

4. The multiplication of two upper (lower) triangular matrices is an upper (lower) triangular matrix.
$$\text{Upper Upper} = \text{Upper}$$

5. The inverse of an upper (lower) triangular matrix is an upper (lower) triangular matrix, given it is invertible.
$$\text{Upper}^{-1} = \text{Upper}, \quad \text{given nonsingular.}$$

6. The k^{th} power of an upper (lower) triangular matrix is an upper (lower) triangular matrix.
$$\text{Upper}^{k} = \text{Upper}, \quad k \text{ is a positive integer.}$$

Specially for diagonal matrices, above properties hold and the resulting matrices stay as diagonal.

Example Find the result for the following matrices after power 3:

Diagonal matrix $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and lower triangular $L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$.

Based on your observation, take a guess on

$$D^k_{n\times n} = \begin{pmatrix} d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & d_n \end{pmatrix}^k = ?$$

Example Find DL and DA:

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Answers:

$$DL = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad \text{and} \quad DA = \begin{pmatrix} 3 & 3 & 3 \\ 2 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix}.$$

Based on your observation, take a guess on

$$\begin{pmatrix} d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & d_n \end{pmatrix} A_{n\times k} = ?$$

Chapter 1 Matrices and System of Equations

Section 1.6 Partitioned Matrices

Definition (Blocks) A matrix can be partitioned into smaller matrices by drawing horizontal lines between the rows and vertical lines between the columns. The smaller matrices are often referred to as *blocks*.

Example

$$A = \left(\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 5 \\ -2 & 4 & -7 & 9 & 3 \\ \hline 7 & 6 & 5 & 9 & 8 \end{array}\right) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where $A_{11} = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & -7 \end{pmatrix}$, $A_{12} = \begin{pmatrix} 4 & 5 \\ 9 & 3 \end{pmatrix}$

$$A_{21} = \begin{pmatrix} 7 & 6 & 5 \end{pmatrix}, A_{22} = \begin{pmatrix} 9 & 8 \end{pmatrix}$$

Example

$$A = \left(\begin{array}{c|c|c|c|c} 1 & 2 & 3 & 4 & 5 \\ -2 & 4 & -7 & 9 & 3 \\ 7 & 6 & 5 & 9 & 8 \end{array}\right) = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{pmatrix}$$

where $\mathbf{a}_1 = \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$, $\mathbf{a}_3 = \begin{pmatrix} 3 \\ -7 \\ 5 \end{pmatrix}$, $\mathbf{a}_4 = \begin{pmatrix} 4 \\ 9 \\ 9 \end{pmatrix}$, $\mathbf{a}_5 = \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix}$.

Example

$$B = \left(\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \hline -2 & 4 & -7 & 9 & 3 \\ \hline 7 & 6 & 5 & 9 & 8 \end{array}\right) = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix}$$

where $\mathbf{b}_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} -2 & 4 & -7 & 9 & 3 \end{pmatrix}$,
$\mathbf{b}_3 = \begin{pmatrix} 7 & 6 & 5 & 9 & 8 \end{pmatrix}$,

Block matrix multiplication If the blocks have the proper dimensions, the block multiplication can be carried out in the same manner as ordinary matrix multiplication.

For example, for $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$,

$$AB = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

Example

$$A=\left(\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 5 \\ -2 & 4 & -7 & 9 & 3 \\ \hline 7 & 6 & 5 & 9 & 8 \end{array}\right)=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B=\left(\begin{array}{cc} 3 & 5 \\ 2 & 6 \\ 1 & 7 \\ \hline -1 & -2 \\ -3 & -4 \end{array}\right)=\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

$$AB=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}=\begin{pmatrix} A_{11}B_1+A_{12}B_2 \\ A_{21}B_1+A_{22}B_2 \end{pmatrix}$$

$$=\left(\begin{array}{c} \begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & -7 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 2 & 6 \\ 1 & 7 \end{pmatrix}+\begin{pmatrix} 4 & 5 \\ 9 & 3 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix} \\ \hline \begin{pmatrix} 7 & 6 & 5 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 2 & 6 \\ 1 & 7 \end{pmatrix}+\begin{pmatrix} 9 & 8 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix} \end{array}\right)=\left(\begin{array}{cc} -9 & 10 \\ -23 & -65 \\ \hline 5 & 56 \end{array}\right)$$

The product AB is the same no matter A or B is treated as partitioned matrices or not.

$$AB = \left(\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 5 \\ -2 & 4 & -7 & 9 & 3 \\ \hline 7 & 6 & 5 & 9 & 8 \end{array}\right) \left(\begin{array}{cc} 3 & 5 \\ 2 & 6 \\ 1 & 7 \\ \hline -1 & -2 \\ -3 & -4 \end{array}\right) = \left(\begin{array}{cc} -9 & 10 \\ -23 & -65 \\ \hline 5 & 56 \end{array}\right).$$

$$AB = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -2 & 4 & -7 & 9 & 3 \\ 7 & 6 & 5 & 9 & 8 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 2 & 6 \\ 1 & 7 \\ -1 & -2 \\ -3 & -4 \end{pmatrix} = \begin{pmatrix} -9 & 10 \\ -23 & -65 \\ 5 & 56 \end{pmatrix}.$$

If the blocks have the proper dimensions, the block multiplication can be carried out in the same manner as ordinary matrix multiplication.

Example

Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -2 & 4 & -7 & 9 & 3 \\ 7 & 6 & 5 & 9 & 8 \end{pmatrix}$, $B = \left(\begin{array}{r|r} 3 & 5 \\ 2 & 6 \\ 1 & 7 \\ -1 & -2 \\ -3 & -4 \end{array}\right) = \left(\mathbf{b}_1 \mid \mathbf{b}_2 \right)$. Since A is of size 3×5 and $\mathbf{b}_1$ is of size 5×1, the partition of B can be used to compute AB.

$$AB = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{pmatrix}$$

Example

Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -2 & 4 & -7 & 9 & 3 \\ 7 & 6 & 5 & 9 & 8 \end{pmatrix}, B = \left(\begin{array}{cc} 3 & 5 \\ \hline 2 & 6 \\ \hline 1 & 7 \\ \hline -1 & -2 \\ \hline -3 & -4 \end{array}\right) = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \\ \mathbf{b}_5 \end{pmatrix}.$$

Since A is of size 3×5 and $\mathbf{b}_1$ is of size 1×2, the partition of B cannot be used to compute AB.

Exercise Let $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right)$ and $D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Find $\begin{pmatrix} D & O \\ O & I \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$.

Analysis before writing the solution

Since D has two rows, O has 2 rows.

Since B has 4 rows, $\begin{pmatrix} D & O \\ O & I \end{pmatrix}$ has to have 4 columns in order to have multiplication. So O has two columns and hence is of size 2×2.

Since O has two columns, I has 2 columns and hence is of size 2×2.

Since D has two columns, O has 2 columns.

Since I has 2 columns, O has 2 rows and hence of size 2×2.

Example Let $D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right)$.

Find $\begin{pmatrix} D & O \\ O & I \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$.

Solution

$$\begin{aligned}
\begin{pmatrix} D & O \\ O & I \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} &= \begin{pmatrix} DB_{11} + OB_{21} & DB_{12} + OB_{22} \\ OB_{11} + IB_{21} & OB_{12} + IB_{22} \end{pmatrix} \\
&= \begin{pmatrix} DB_{11} & DB_{12} \\ B_{21} & B_{22} \end{pmatrix} \\
&= \left(\begin{array}{cc|cc} 2 & 2 & 2 & 2 \\ 2 & 4 & 2 & 2 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right)
\end{aligned}$$

Example

Suppose A and C are $m \times m$ matrices, B and D are $n \times n$ matrices. Then

$$\left(\begin{array}{c|c} A & O_{m\times n} \\ \hline O_{n\times m} & B \end{array}\right)\left(\begin{array}{c|c} C & O_{m\times n} \\ \hline O_{n\times m} & D \end{array}\right)=\left(\begin{array}{c|c} AC & O_{m\times n} \\ \hline O_{n\times m} & BD \end{array}\right).$$

In particular, for any positive integer k, we have

$$\left(\begin{array}{cc|ccc} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{array}\right)^k = \left(\begin{array}{c|c} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^k \end{array}\right).$$

Theorem Let A be an $n \times n$ square matrix with $A = \begin{pmatrix} A_{11} & O \\ O & A_{22} \end{pmatrix}$, where A_{11} has a size of $k \times k$ and $k < n$. Then A is nonsingular if and only if A_{11} and A_{22} are nonsingular. In this case we have

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{pmatrix}.$$

Proof of the "if" part If A is nonsingular, then let $B = A^{-1}$ and partition B in the same manner as A. Since $BA = I = AB$,

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} A_{11} & O \\ O & A_{22} \end{pmatrix} = \begin{pmatrix} I_k & O \\ O & I_{n-k} \end{pmatrix} = \begin{pmatrix} A_{11} & O \\ O & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\begin{pmatrix} B_{11}A_{11} & B_{12}A_{22} \\ B_{21}A_{11} & B_{22}A_{22} \end{pmatrix} = \begin{pmatrix} I_k & O \\ O & I_{n-k} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}$$

Namely,

$$B_{11}A_{11} = I_k = A_{11}B_{11}, \qquad B_{22}A_{22} = I_{n-k} = A_{22}B_{22}.$$

Hence, A_{11} and A_{22} are both nonsingular and $A_{11}^{-1} = B_{11}$, $A_{22}^{-1} = B_{22}$.