Section 1.4 Limits Involving Infinity; Asymptotes(渐近线)

EXAMPLE 5.1 A Simple Limit Revisited

Examine $\lim_{x\to 0} \frac{1}{x}$.

Solution: As we know, as $x\to 0+$, $1/x$ increases without bound, while as $x\to 0-$, $1/x$ decreases without bound, so we say that $\lim_{x\to 0^+} \frac{1}{x}$ and $\lim_{x\to 0^-} \frac{1}{x}$ do not exist. We say that

$$\lim_{x\to 0} \frac{1}{x} \text{ does not exist.}$$

REMARK 5.1

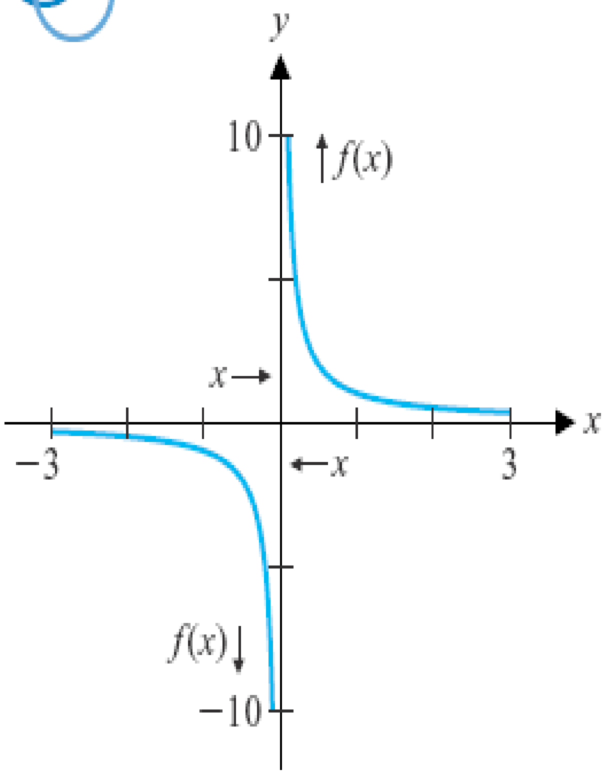


FIGURE 1.32

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \text{ and } \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

It may at first seem contradictory to say that $\lim_{x\to 0^+} \frac{1}{x}$ does not exist and then to write $\lim_{x\to 0^+} \frac{1}{x} = \infty$. Note that since ∞ is *not* a real number, there is no contradiction here. (When we say that a limit "does not exist," we are saying that there is no real number L that the function values are approaching.) We say that $\lim_{x\to 0^+} \frac{1}{x} = \infty$ to indicate that as $x \to 0^+$, the function values are increasing without bound.

EXAMPLE 5.2 A Function Whose One-Sided Limits Are Both Infinite

Evaluate $\lim_{x \to 0} \frac{1}{x^2}$.

Solution: We can see that

$$\lim_{x \to 0^+} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x^2} = \infty.$$

Since both one-sided limits agree, we say that

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

This one concise statement says that the limit does not exist.

REMARK 5.2

Mathematicians try to convey as much information as possible with as few symbols as possible. For instance, we prefer to say $\lim_{x\to 0} \frac{1}{x^2} = \infty$ rather than $\lim_{x\to 0} \frac{1}{x^2}$ does not exist, since the first statement not only says that the limit does not exist, but also says that $\frac{1}{x^2}$ increases without bound as x approaches 0, with $x > 0$ or $x < 0$.

EXAMPLE 5.3 A Case Where Infinite One-Sided Limits Disagree

Evaluate $\lim_{x \to 5} \dfrac{1}{(x-5)^3}$.

EXAMPLE 5.4 Another Case Where Infinite One-Sided Limits Disagree

Evaluate $\lim_{x \to -2} \dfrac{x+1}{(x-3)(x+2)}$.

EXAMPLE 5.5 A Limit Involving a Trigonometric Function

Evaluate $\lim_{x \to \frac{\pi}{2}} \tan x$.

垂直渐近线

Vertical Asymptotes The line $x = c$ is a *vertical asymptote* of the graph of $f(x)$ if either

$$\lim_{x \to c^-} f(x) = +\infty \quad (\text{or} \ -\infty)$$

or

$$\lim_{x \to c^+} f(x) = +\infty \quad (\text{or} \ -\infty)$$

In general, a rational function $R(x) = \frac{p(x)}{q(x)}$ has a vertical asymptote x=c whenever $q(c) = 0$ but $p(c) \neq 0$.

Example Determine all vertical asymptotes of the graph of

$$f(x)=\frac{x^2-9}{x^2+3x}$$

Solution:

Let $p(x)=x^2-9$ and $q(x)=x^2+3x$ be the numerator and denominator of $f(x)$, respectively. The $q(x)=0$ when $x=-3$ and when $x=0$. However, for $x=-3$, we also have $p(-3)=0$ and

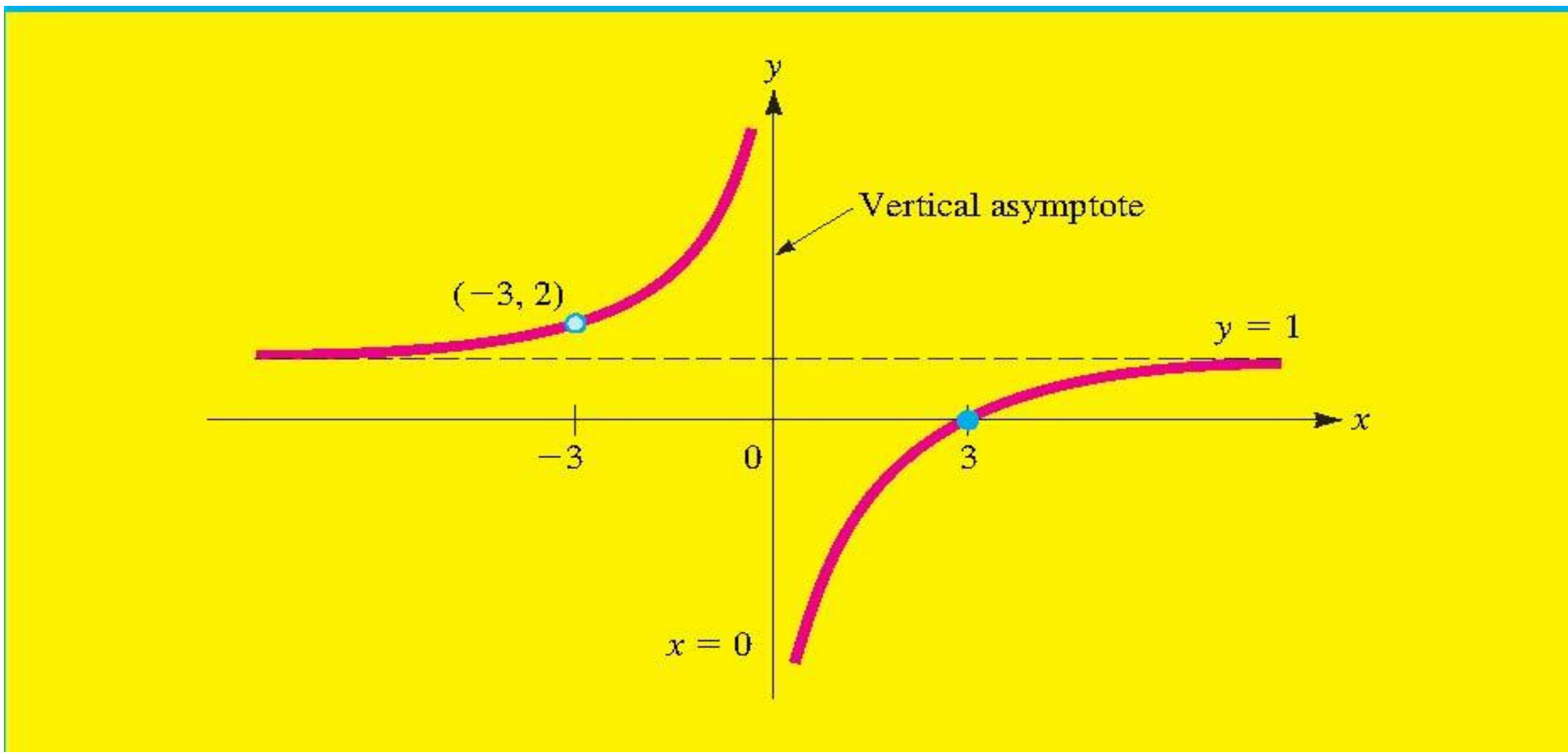
$$\lim_{x\to -3}\frac{x^2-9}{x^2+3x}=\lim_{x\to -3}\frac{x-3}{x}=2$$

This means that the graph of $f(x)$ has a “hole” at the point $(-3,2)$ and $x=-3$ is not a vertical asymptote of the graph.

to be continued

On the other hand, for *x=0* we have *q(0)=0* but *p(0)≠0,* which suggests that the *y* axis is a vertical asymptote for the graph of *f(x)*. This asymptote behavior is verified by noting that

$$\lim_{x\to 0^-} \frac{x^2-9}{x^2+3x} = +\propto \quad \text{and} \quad \lim_{x\to 0^+} \frac{x^2-9}{x^2+3x} = -\propto$$



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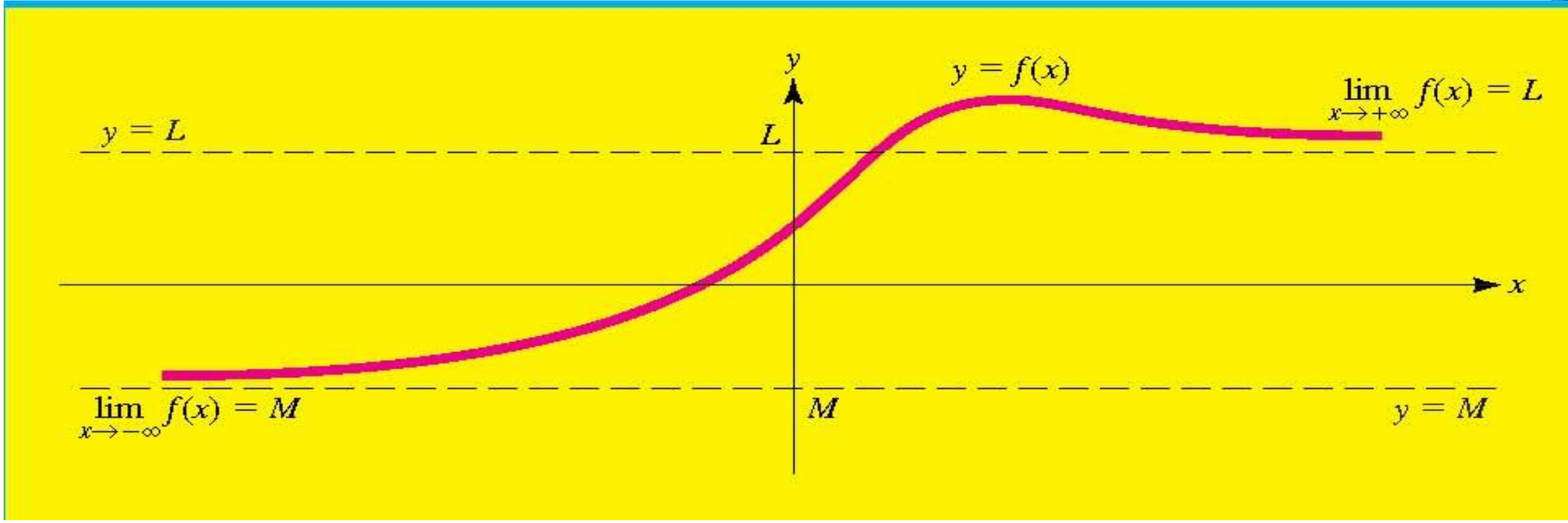
Limits at Infinity （无穷处极限）

Limits at Infinity: If the value of the function *f(x)* approach the number *L* as *x* increases without bound, we write

$$\lim_{x \to +\infty} f(x) = L$$

Similarly, we write $\lim_{x \to -\infty} f(x) = M$

when the functional values *f(x)* approach the number *M* as *x* decreases without bound.



水平渐近线

Horizontal Asymptotes The line $y = b$ is called *horizontal asymptote* of the graph of $f(x)$ if

$$\lim_{x \to -\infty} f(x) = b \quad \text{or} \quad \lim_{x \to +\infty} f(x) = b$$

THEOREM 5.1

For any rational number $t > 0$,

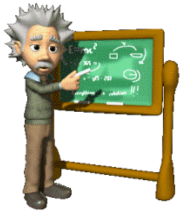
$$\lim_{x \to \pm\infty} \frac{1}{x^t} = 0,$$

where for the case where $x \to -\infty$, we assume that $t = \dfrac{p}{q}$ where q is odd.

THEOREM 5.2

For a polynomial of degree $n > 0$, $p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, we have

$$\lim_{x \to \infty} p_n(x) = \begin{cases} \infty, & \text{if } a_n > 0 \\ -\infty, & \text{if } a_n < 0 \end{cases}.$$

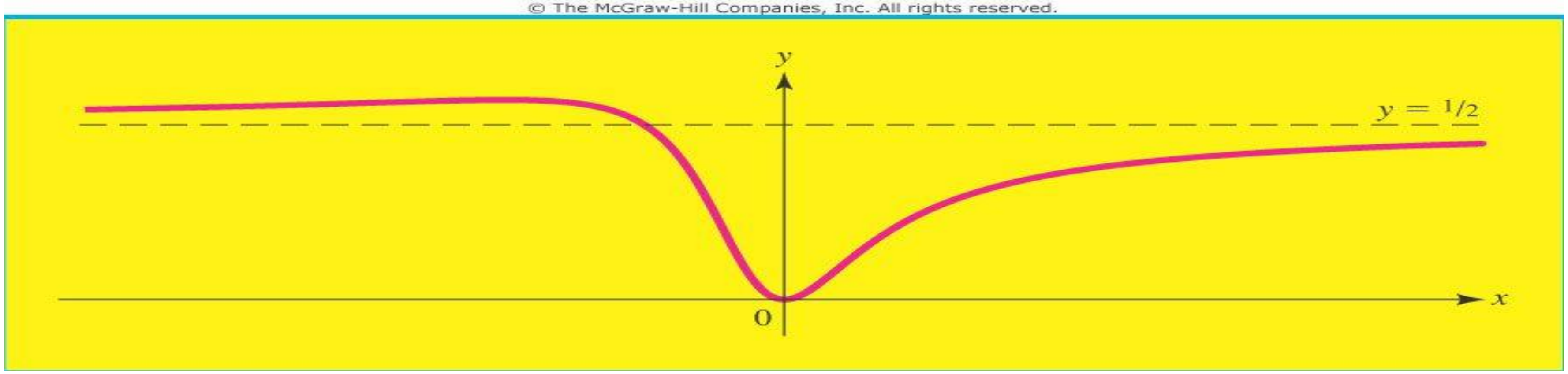


Example

Find $\lim_{x\to+\infty}\frac{x^2}{1+x+2x^2}$

Solution:

$$\lim_{x\to+\infty}\frac{x^2}{1+x+2x^2}=\lim_{x\to+\infty}\frac{x^2/x^2}{1/x^2+x/x^2+2x^2/x^2}=\frac{\lim_{x\to+\infty}1}{\lim_{x\to+\infty}1/x^2+\lim_{x\to+\infty}1/x+\lim_{x\to+\infty}2}=\frac{1}{0+0+2}=0.5$$



Rule of Thumb （经验法则）

Procedure for Evaluating a Limit at Infinity of $f(x)=p(x)/q(x)$

Step 1. Divide each term in $f(x)$ by the highest power x^k that appears in the denominator polynomial $q(x)$.

Step 2. Compute $\lim_{x\to+\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ using algebraic properties of limits and the reciprocal rules.

Exercise

$$\lim_{x\to+\infty}\frac{3x^4-8x^2+2x}{5x^4+1} \quad \text{and} \quad \lim_{x\to+\infty}\frac{\sin(x)}{x}$$

Example Determine all horizontal asymptotes of the graph of

$$f(x) = \frac{x^2}{x^2 + x + 1}$$

Solution:

Dividing each term in the rational function *f(x)* by x^2 (the highest power of x in the denominator), we find that

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x^2}{x^2 + x + 1} = \lim_{x \to +\infty} \frac{x^2 / x^2}{x^2 / x^2 + x / x^2 + 1 / x^2}$$

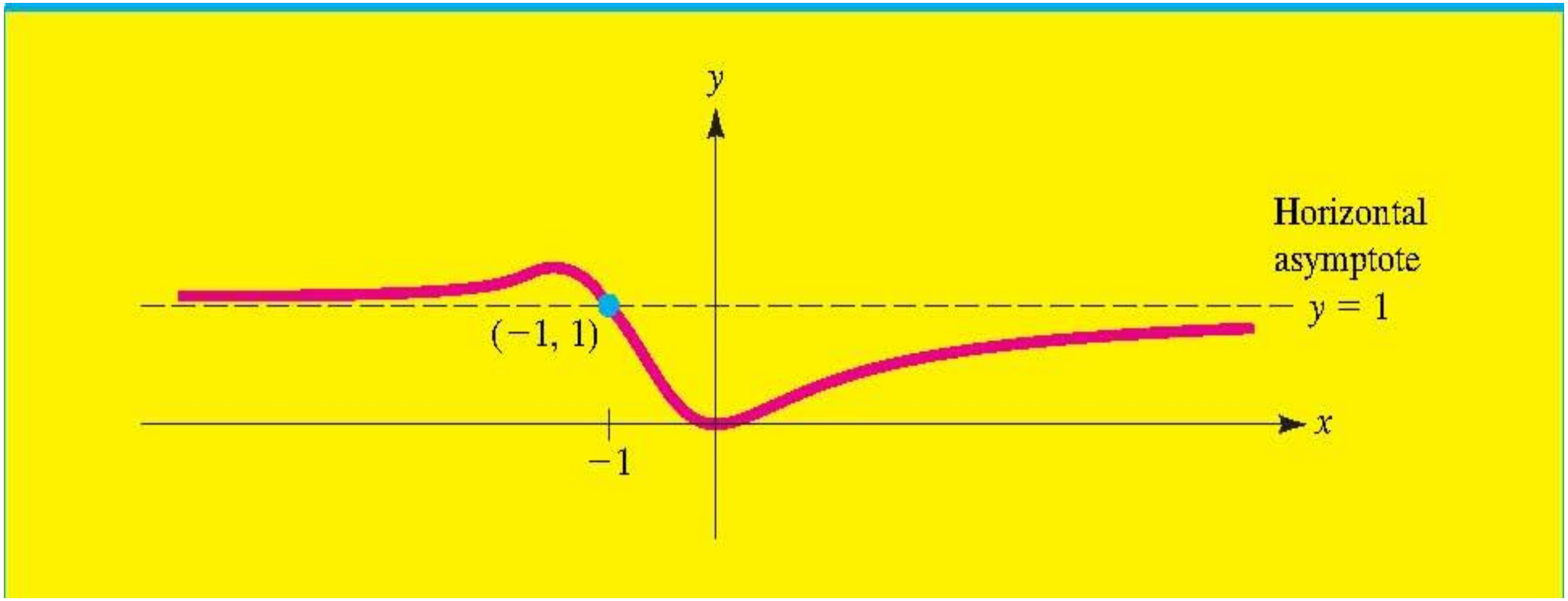
$$= \lim_{x \to +\infty} \frac{1}{1 + 1/x + 1/x^2} = 1 \qquad \textit{reciprocal power rule}$$

and similarly,

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2}{x^2 + x + 1} = 1$$

to be continued

Thus, the graph of $f(x)$ has $y=1$ as a horizontal asymptote.



NOTE: The graph of a function $f(x)$ can never cross a vertical asymptote $x=c$ because at least one of the one-sided limits must be infinite. However, it is possible for a graph to cross its horizontal asymptotes.

✓ Limits involving exponential functions are very important in many applications.

EXAMPLE 5.9 Two Limits of an Exponential Function

Evaluate $\lim_{x \to 0^-} e^{1/x}$ and $\lim_{x \to 0^+} e^{1/x}$.

✓ Inverse trigonometric functions may have horizontal asymptotes.

EXAMPLE 5.10 Two Limits of an Inverse Trigonometric Function

Evaluate $\lim_{x \to \infty} \tan^{-1} x$ and $\lim_{x \to -\infty} \tan^{-1} x$.

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Section 1.5 Formal Definition Of The Limit (极限的正式定义)

HISTORICAL NOTES

Augustin Louis Cauchy (1789–1857) A French mathematician who developed the ε-δ definitions of limit and continuity. Cauchy was one of the most prolific mathematicians in history, making important contributions to number theory, linear algebra, differential equations, astronomy, optics and complex variables. A difficult man to get along with, a colleague wrote, "Cauchy is mad and there is nothing that can be done about him, although right now, he is the only one who knows how mathematics should be done."

So far we have know that

$$\lim_{x\to a} f(x) = L,$$

if *f(x)* gets closer and closer to *L* as *x* gets closer and closer to *a*.

This may seem a bit odd, when you realize that we have never actually *defined* what a limit is.

✓ We begin with the careful examination of an elementary example. You should certainly believe that

$$\lim_{x\to 2}(3x+4)=10.$$

Suppose that you were asked to explain the meaning of this particular limit to a fellow student.

✓ You would probably repeat the intuitive explanation we have used so far: that as *x* gets closer and closer to *2*, *(3x+4)* gets arbitrarily close to *10*.

✓ *Exactly what do we mean by close?*

- For instant, can we force *(3x+4)* to be within distance *1* of *10*?

- To see what values of *x* will guarantee this, we write an inequality that says that *(3x+4)* is within *1* unit of *10*:

$$|(3x+4)-10| < 1.$$

Eliminating the absolute values, we see that this is equivalent to $-1 < 3x-6 < 1.$

- Dividing by *3*, we get $|x-2| < \frac{1}{3}.$

- We see that if *x* is within distance *1/3* of *2*, then *(3x+4)* will be within the specified distance *(1)* of *10*.

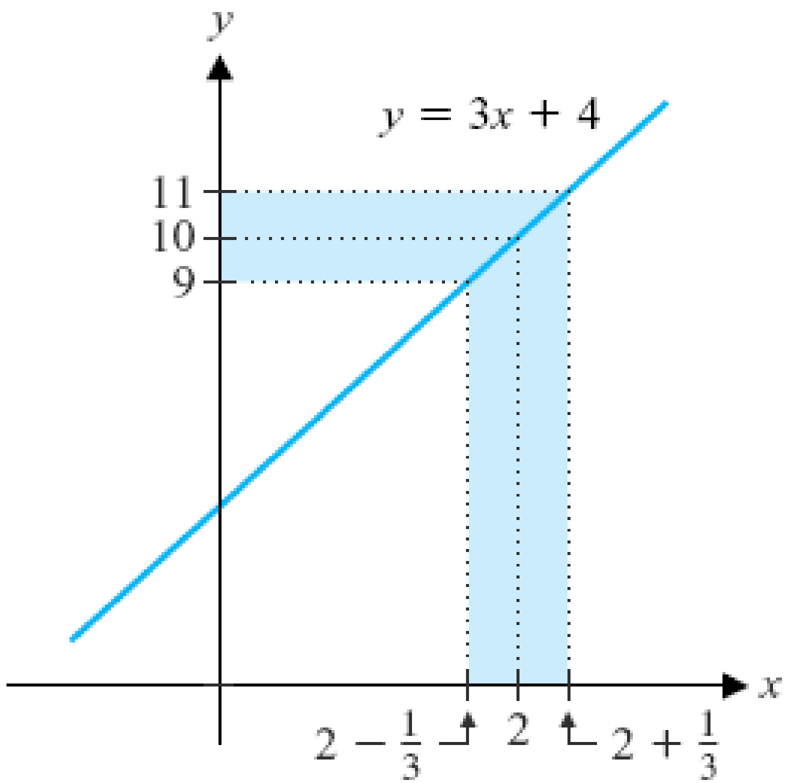

FIGURE 1.44

$2 - \frac{1}{3} < x < 2 + \frac{1}{3}$ guarantees that $|(3x + 4) - 10| < 1$

Does this convince you that you can make (3x+4) as close as you want to 10?

Probably not, but if you used a smaller distance, perhaps you'd be more convinced.

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EXAMPLE 6.1 Exploring a Simple Limit

Find the values of x for which $(3x+4)$ is within distance $\dfrac{1}{100}$ of 10.

Remark: All we've been able to show is that we can make *(3x+4)* pretty close to *10*.

Question: how close do we need to be able to make it? The answer is *arbitrarily close*, *(as close as anyone would ever demand).*

Now we use an unspecified distance, call it ε (*epsilon*, where $\varepsilon > 0$)

EXAMPLE 6.2 Verifying a Limit

Show that we can make $(3x + 4)$ within any specified distance ε of 10 (no matter how small ε is), just by making x sufficiently close to 2.

Solution: The objective is to determine the range of *x*-values that will guarantee that *(3x+4)* stays within ε of *10*. We have

$$|(3x + 4) - 10| < \varepsilon.$$

This is equivalent to $-\varepsilon < 3x - 6 < \varepsilon.$

Dividing by 3, we get $|x - 2| < \dfrac{\varepsilon}{3}.$

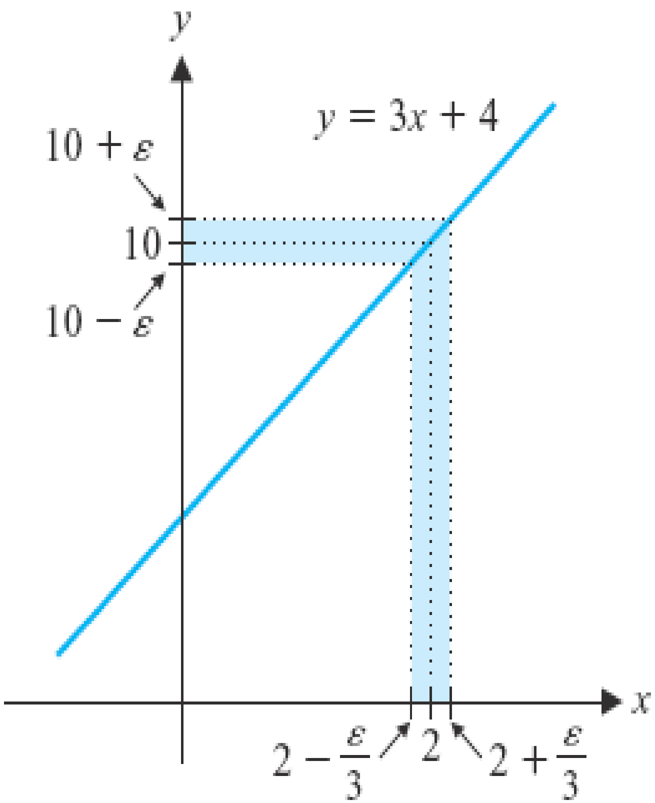


FIGURE 1.45
The range of x-values that keep $|(3x+4)-10| < \varepsilon$

Notice that each of the preceding steps is reversible, so that $|x-2| < \frac{\varepsilon}{3}$. also implies that $|(3x+4)-10| < \varepsilon$.

This says that as long as *x* is within distance $\varepsilon/3$ of *2*, *(3x+4)* will be within the required distance ε of *10*.

Remark: By using an *unspecified* distance, ε , we have verified that we can indeed make *(3x+4)* as close to *10* as might be demanded (*i.e*., arbitrarily close; just name whatever $\varepsilon > 0$ you would like), simply by making x sufficiently close to *2*.

EXAMPLE 6.3 Proving That a Limit Is Correct

Prove that $\lim\limits_{x\to 1} \dfrac{2x^2+2x-4}{x-1} = 6.$

Solution:

First, notice that f is undefined *at* $x = 1$. So, we seek a distance δ (*delta*, $\delta > 0$), such that if x is within distance δ of 1, but $x \neq 1$ (i.e., $0 < |x - 1| < \delta$), then this guarantees that $|f(x) - 6| < \varepsilon$.

Notice that we have specified that $0 < |x - 1|$ to ensure that $x \neq 1$. Further, $|f(x) - 6| < \varepsilon$ is equivalent to

$$-\varepsilon < \frac{2x^2+2x-4}{x-1} - 6 < \varepsilon.$$

Since the numerator factors, this is equivalent to

$$-\varepsilon < \frac{2(x-1)^2}{x-1} < \varepsilon.$$

Since $x \neq 1$, we can cancel two of the factors of $(x-1)$ to yield

$$-\varepsilon < 2(x-1) < \varepsilon$$

or

$$-\frac{\varepsilon}{2} < x-1 < \frac{\varepsilon}{2}, \quad \text{Dividing by 2.}$$

which is equivalent to $|x-1| < \varepsilon/2$. So, taking $\delta = \varepsilon/2$ and working backward, we see that requiring x to satisfy

$$0 < |x-1| < \delta = \frac{\varepsilon}{2}$$

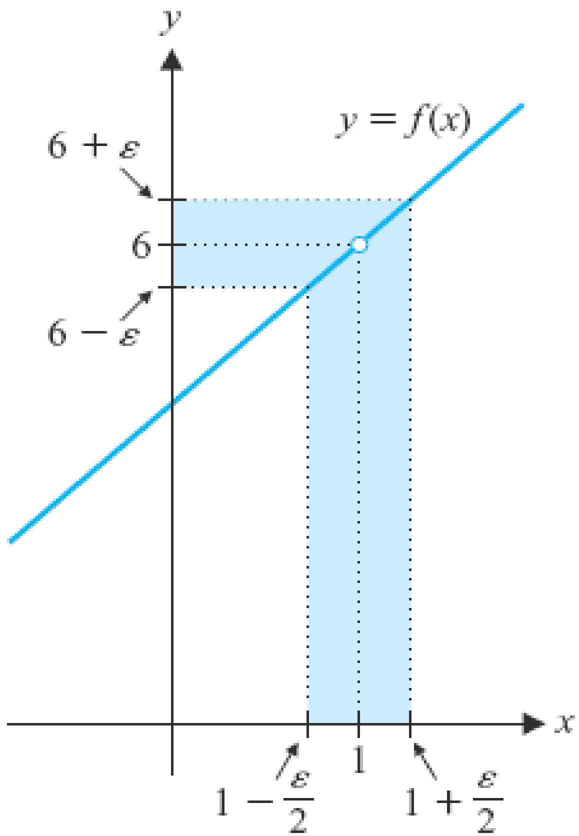


FIGURE 1.46

$0 < |x - 1| < \frac{\varepsilon}{2}$ guarantees that

$$6 - \varepsilon < \frac{2x^2 + 2x - 4}{x - 1} < 6 + \varepsilon.$$

We see that requiring x to satisfy

$$0 < |x - 1| < \delta = \frac{\varepsilon}{2}$$

will guarantee that

$$\left| \frac{2x^2 + 2x - 4}{x - 1} - 6 \right| < \varepsilon.$$

Precise Definition of Limit

DEFINITION 6.1 (Precise Definition of Limit)

For a function f defined in some open interval containing a (but not necessarily *at a* itself), we say

$$\lim_{x \to a} f(x) = L,$$

if given any number $\varepsilon > 0$, there is another number $\delta > 0$, such that $0 < |x - a| < \delta$ guarantees that $|f(x) - L| < \varepsilon$.

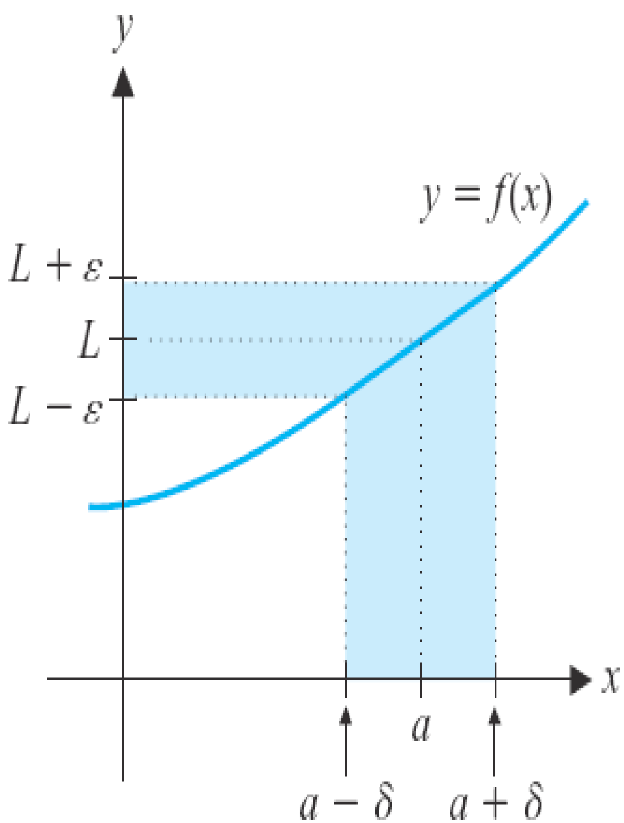


FIGURE 1.47

$a - \delta < x < a + \delta$ guarantees that $L - \varepsilon < f(x) < L + \varepsilon$.

"I remember standing at the blackboard in Room 213 of the mathematics building with Warren Ambrose and suddenly I understood epsilons. I understood what limits were, and all of that stuff that people had been drilling into me became clear…….I could prove the theorems. That afternoon I became a mathematician

-------Paul Halmos"

REMARK 6.1

We want to emphasize that this formal definition of limit is not a new idea. Rather, it is a more precise mathematical statement of the same intuitive notion of limit that we have been using since the beginning of the chapter. Also, we must in all honesty point out that it is rather difficult to explicitly find δ as a function of ε, for all but a few simple examples. Despite this, learning how to work through the definition, even for a small number of problems, will shed considerable light on a deep concept.

EXAMPLE 6.4 Using the Precise Definition of Limit

Use Definiton 6.1 to prove that $\lim_{x\to 2}(x^2+1)=5$.

Solution If this limit is correct, then given any $\varepsilon > 0$, there must be a $\delta > 0$ for which $0 < |x-2| < \delta$ guarantees that

$$|(x^2+1)-5| < \varepsilon.$$

Notice that

$$\begin{aligned} |(x^2+1)-5| &= |x^2-4| && \text{Factoring the difference} \\ &= |x+2||x-2|. && \text{of two squares.} \end{aligned} \tag{6.2}$$

Our strategy is to isolate $|x-2|$ and so, we'll need to do something with the term $|x+2|$. Since we're interested only in what happens near $x = 2$, anyway, we will only consider x's within a distance of 1 from 2, that is, x's that lie in the interval $[1, 3]$ (so that $|x-2| < 1$). Notice that this will be true if we require $\delta \le 1$ and $|x-2| < \delta$. In this case, we have

$$|x+2| \le 5, \qquad \text{Since } x \in [1,3].$$

and so, from (6.2),

$$\begin{aligned} |(x^2+1)-5| &= |x+2||x-2| \\ &\le 5|x-2|. \end{aligned} \tag{6.3}$$

Finally, if we require that

$$5|x-2| < \varepsilon, \tag{6.4}$$

then we will also have from (6.3) that

$$|(x^2+1)-5| \le 5|x-2| < \varepsilon.$$

Of course, (6.4) is equivalent to

$$|x-2| < \frac{\varepsilon}{5}.$$

So, in view of this, we now have two restrictions: that $|x-2|<1$ *and* that $|x-2|<\frac{\varepsilon}{5}$. To ensure that both restrictions are met, we choose $\delta=\min\left\{1,\frac{\varepsilon}{5}\right\}$ (i.e., the **minimum** of 1 and $\frac{\varepsilon}{5}$). Working backward, we get that for this choice of δ,

$$0<|x-2|<\delta$$

will guarantee that

$$|(x^2+1)-5|<\varepsilon,$$

It is important to remember that limits describe the behavior of a function *near* a particular point, not *necessarily* at the point itself.

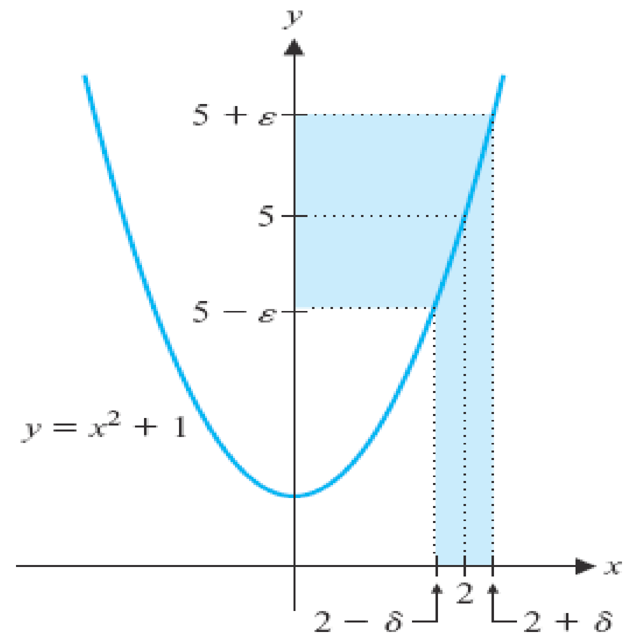


FIGURE 1.48
$0<|x-2|<\delta$ guarantees that $|(x^2+1)-5|<\varepsilon$.

THEOREM 6.1

Suppose that for a real number a, $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$. Then,

$$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2.$$

PROOF

Since $\lim_{x\to a} f(x) = L_1$, we know that given any number $\varepsilon_1 > 0$, there is a number $\delta_1 > 0$ for which

$$0 < |x - a| < \delta_1 \text{ guarantees that } |f(x) - L_1| < \varepsilon_1. \tag{6.7}$$

Likewise, since $\lim_{x\to a} g(x) = L_2$, we know that given any number $\varepsilon_2 > 0$, there is a number $\delta_2 > 0$ for which

$$0 < |x - a| < \delta_2 \text{ guarantees that } |g(x) - L_2| < \varepsilon_2. \tag{6.8}$$

Now, in order to get

$$\lim_{x\to a}[f(x) + g(x)] = (L_1 + L_2),$$

we must show that, given any number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \text{ guarantees that } |[f(x) + g(x)] - (L_1 + L_2)| < \varepsilon.$$

Notice that

$$\begin{aligned} |[f(x) + g(x)] - (L_1 + L_2)| &= |[f(x) - L_1] + [g(x) - L_2]| \\ &\le |f(x) - L_1| + |g(x) - L_2|, \end{aligned} \tag{6.9}$$

by the triangle inequality. Of course, both terms on the right-hand side of (6.9) can be made arbitrarily small, from (6.7) and (6.8). In particular, if we take $\varepsilon_1 = \varepsilon_2 = \dfrac{\varepsilon}{2}$, then as long as

$$0 < |x - a| < \delta_1 \quad and \quad 0 < |x - a| < \delta_2,$$

we get from (6.7), (6.8) and (6.9) that

$$
\begin{aligned}
|[f(x)+g(x)]-(L_1+L_2)| &\le |f(x)-L_1|+|g(x)-L_2| \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,
\end{aligned}
$$

as desired. Of course, this will happen if we take

$$0<|x-a|<\delta=\min\{\delta_1,\delta_2\}. \quad \blacksquare$$

Exercise 1

Prove Theorem (*The Properties of Limits*):

Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and let *c* be any constant. The following then apply:

I. $$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

II. $$\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$$

III. $$\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)]$$

IV. $$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0$$

Exercise 2

Prove the following theorem:

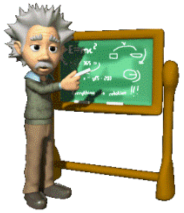
THEOREM 3.3

Suppose that $\lim_{x\to a} f(x) = L$ and n is any positive integer. Then,

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L},$$

where for n even, we assume that $L > 0$.

Exercise 3



Prove Squeeze Theorem:

THEOREM 3.5 (Squeeze Theorem)

Suppose that

$$f(x) \le g(x) \le h(x)$$

for all x in some interval (c, d), except possibly at the point $a \in (c, d)$ and that

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

for some number L. Then, it follows that

$$\lim_{x \to a} g(x) = L, \text{ also.}$$

Precise Definition of Limits Involving Infinity

Recall that we have discussed Limits Involving Infinity so far:

I. Infinite Limits

$$\lim_{x \to a} f(x) = +\infty \quad \text{or} \quad \lim_{x \to a} f(x) = -\infty$$

II. Limits at Infinity

$$\lim_{x \to +\infty} f(x) = L$$

$$\lim_{x \to -\infty} f(x) = L$$

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DEFINITION 6.2

For a function f defined in some open interval containing a (but not necessarily at a itself), we say

$$\lim_{x \to a} f(x) = \infty,$$

if given any number $M > 0$, there is another number $\delta > 0$, such that $0 < |x - a| < \delta$ guarantees that $f(x) > M$. (See Figure 1.53 for a graphical interpretation of this.)

- Can you give the precise definition for

$$\lim_{x \to a} f(x) = -\propto ?$$

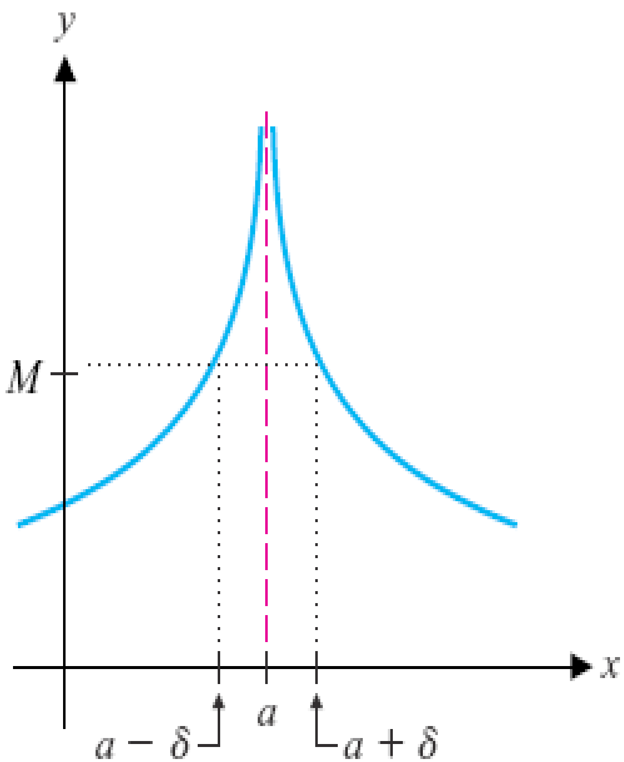

FIGURE 1.53

$$\lim_{x \to a} f(x) = \infty$$

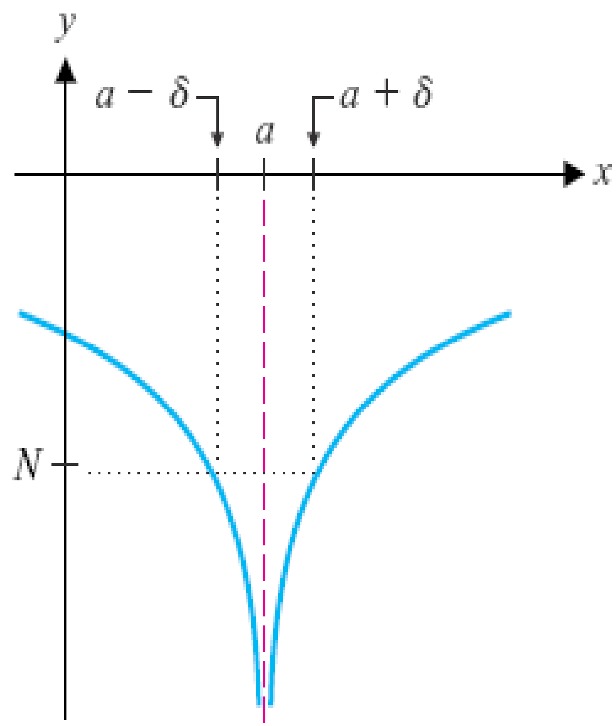

FIGURE 1.54

$$\lim_{x \to a} f(x) = -\infty$$

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EXAMPLE 6.8 Using the Definition of Limit Where the Limit Is Infinite

Prove that $\lim\limits_{x \to 0} \dfrac{1}{x^2} = \infty$.

Solution Given any (large) number $M > 0$, we need to find a distance $\delta > 0$ such that if x is within δ of 0 (but not equal to 0) then

$$\frac{1}{x^2} > M. \tag{6.5}$$

Since both M and x^2 are positive, (6.5) is equivalent to

$$x^2 < \frac{1}{M}.$$

Taking the square root of both sides and recalling that $\sqrt{x^2} = |x|$, we get

$$|x| < \sqrt{\frac{1}{M}}.$$

So, for any $M > 0$, if we take $\delta = \sqrt{\frac{1}{M}}$ and work backward, we have that $0 < |x - 0| < \delta$ guarantees that

$$\frac{1}{x^2} > M,$$

DEFINITION 6.4

For a function f defined on an interval (a, ∞), for some $a > 0$, we say

$$\lim_{x \to \infty} f(x) = L,$$

if given any $\varepsilon > 0$, there is a number $M > 0$ such that $x > M$ guarantees that

$$|f(x) - L| < \varepsilon.$$

(See Figure 1.55 for a graphical interpretation of this.)

- Similarly, Can you give the precise definition for

$$\lim_{x \to -\infty} f(x) = L?$$

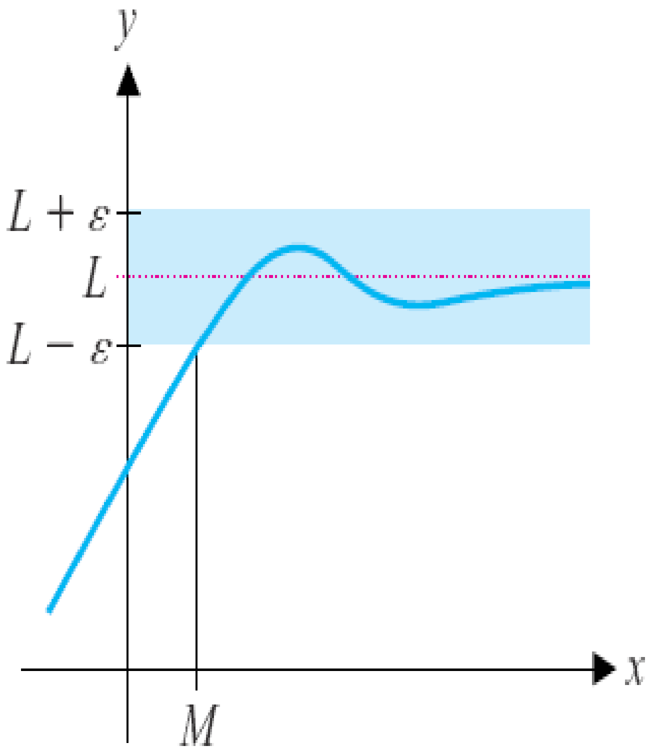


FIGURE 1.55

$$\lim_{x \to \infty} f(x) = L$$

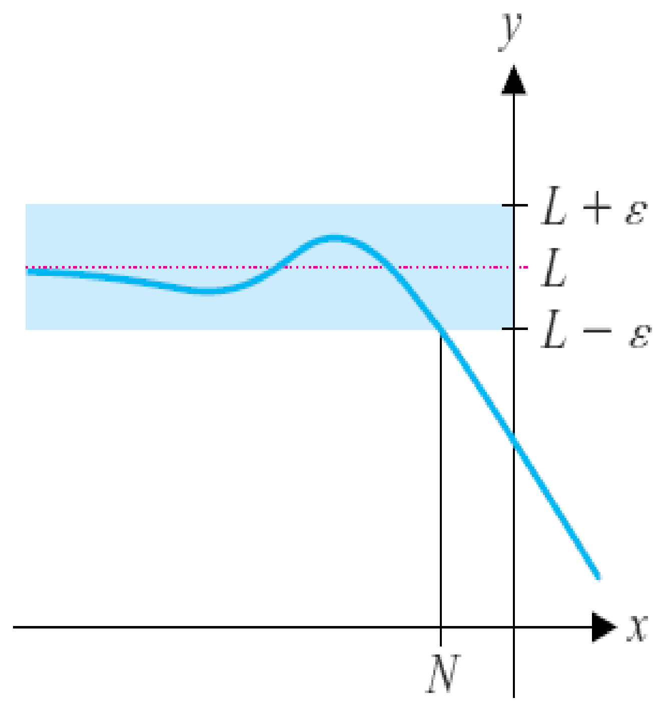


FIGURE 1.56

$$\lim_{x \to -\infty} f(x) = L$$

EXAMPLE 6.9 Using the Definition of Limit Where x Is Becoming Infinite

Prove that $\lim_{x \to -\infty} \dfrac{1}{x} = 0$.

Solution Here, we must show that given any $\varepsilon > 0$, we can make $\dfrac{1}{x}$ within ε of 0, simply by making x sufficiently large in absolute value and negative. So, we need to determine those x's for which

$$\left| \frac{1}{x} - 0 \right| < \varepsilon$$

or

$$\left| \frac{1}{x} \right| < \varepsilon. \tag{6.6}$$

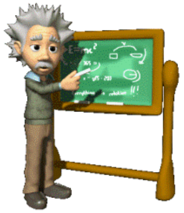
Since $x < 0$, $|x| = -x$, and so (6.6) becomes

$$\frac{1}{-x} < \varepsilon.$$

Dividing both sides by ε and multiplying by x (remember that $x < 0$ and $\varepsilon > 0$, so that this will change the direction of the inequality), we get

$$-\frac{1}{\varepsilon} > x.$$

So, if we take $N = -\frac{1}{\varepsilon}$ and work backward, we have satisfied the definition and thereby proved that the limit is correct. ∎



Exercise 4

Prove the Theorem:

THEOREM 5.1

For any rational number $t > 0$,

$$\lim_{x \to \pm\infty} \frac{1}{x^t} = 0,$$

where for the case where $x \to -\infty$, we assume that $t = \frac{p}{q}$ where q is odd.