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# Chapter 2
# Differentiation (微分)

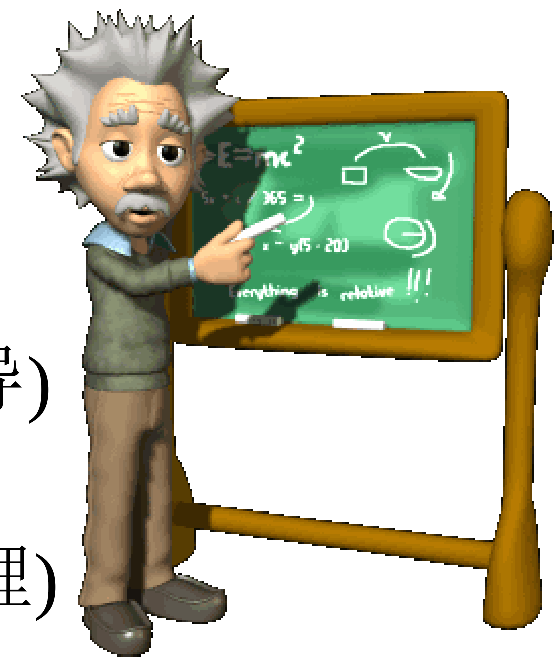
In this Chapter, we will know some important concepts.

- The Derivative (导数)

- The Chain Rule (链锁法则)

- Implicit Differentiation (隐函数求导)

- The Mean Value Theorem (中值定理)



# Section 2.1 Tangent Lines (切线) and Velocity

- Calculus is the mathematics of change, and the primary tool for studying change is a procedure called differentiation.

- In this section, we will introduce this procedure and examine some of its uses, especially in computing rates of change.

- Rate of changes, for example velocity, acceleration, the rate of growth of a population, and many others, are described mathematically by derivatives.

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## Example

If air resistance is neglected, an object dropped from a great height will fall $s(t)=16t^2$ feet in *t* seconds. What is the object's instantaneous velocity after *t=2* seconds?

## Solution:

Average rate of change of *s(t)* over the time period *[2,2+h]* by the ratio

$$V_{ave}=\frac{\text{distance traveled}}{\text{elapsed time}}=\frac{s(2+h)-s(2)}{(2+h)-2}$$

$$=\frac{16(2+h)^2-16(2)^2}{h}=64+16h$$

Compute the instantaneous velocity by the limit

$$V_{ins}=\lim_{h\to 0}V_{ave}=\lim_{h\to 0}(64+16h)=64$$

That is, after 2 seconds, the object is traveling at the rate of 64 feet per second.

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# Rates of Change （变化率）

How to determine instantaneous rate of change or rate of change of *f(x)* at *x=c* ?

Find the average rate of change of *f(x)* as *x* varies from *x=c* to *x=c+h*

$$rate_{ave} = \frac{\Delta f(x)}{\Delta x} = \frac{f(c+h) - f(c)}{(c+h) - c} = \frac{f(c+h) - f(c)}{h}$$

Compute the instantaneous rate of change of *f(x)* at *x=c* by finding the limiting value of the average rate as *h* tends to *0*

$$rate_{ins} = \lim_{h \to 0} rate_{ave} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

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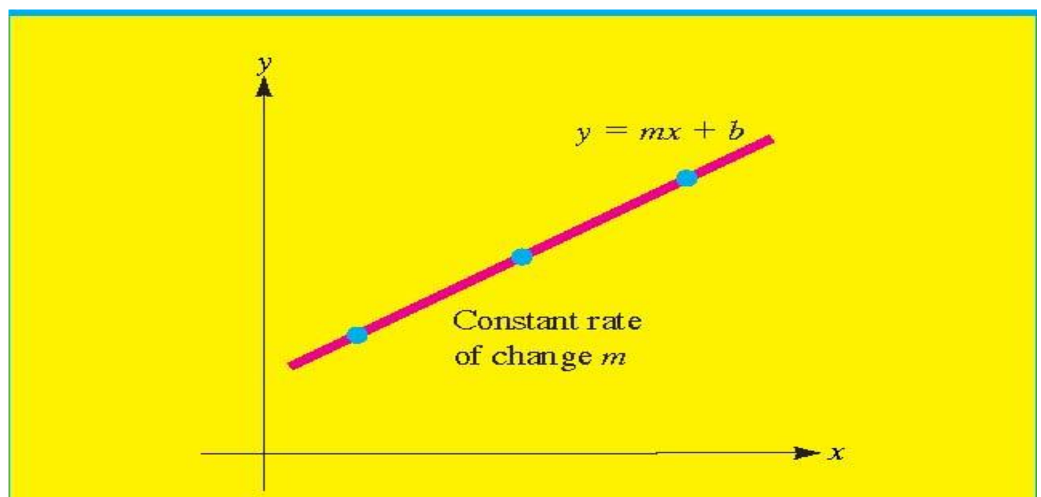
# Rates of Change (Linear function)

A linear function $y(x)=mx+b$ changes at the constant rate $m$ with respect to the independent variable $x$. That is the rate of change of $y(x)$ is given by the slope or steepness of its graph

$$\textbf{Slope} = \textbf{rate of change}$$

$$= \frac{\textbf{change in y}}{\textbf{change in x}}$$

$$= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

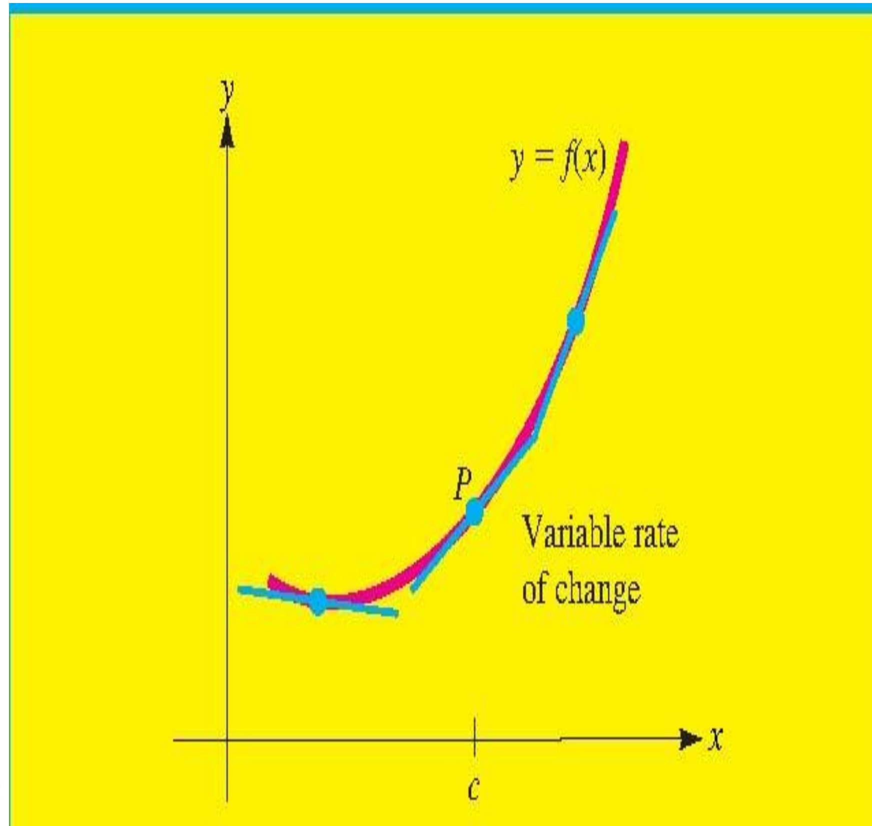



(a) A linear function $L(x) = mx + b$ changes at the constant rate $m$.

# Rates of Change (non-Linear function)

For the function that is nonlinear, the rate of change is not constant but varies with $x$.

In particular, the rate of change at $x=c$ is given by the steepness of the graph of $f(x)$ at the point $(c,f(c))$, which can be measured by the slope of the tangent line to the graph at $p$.




(b) If $f(x)$ is nonlinear, the rate of change at $x = c$ is given by the slope of the tangent line at $P(c, f(c))$.

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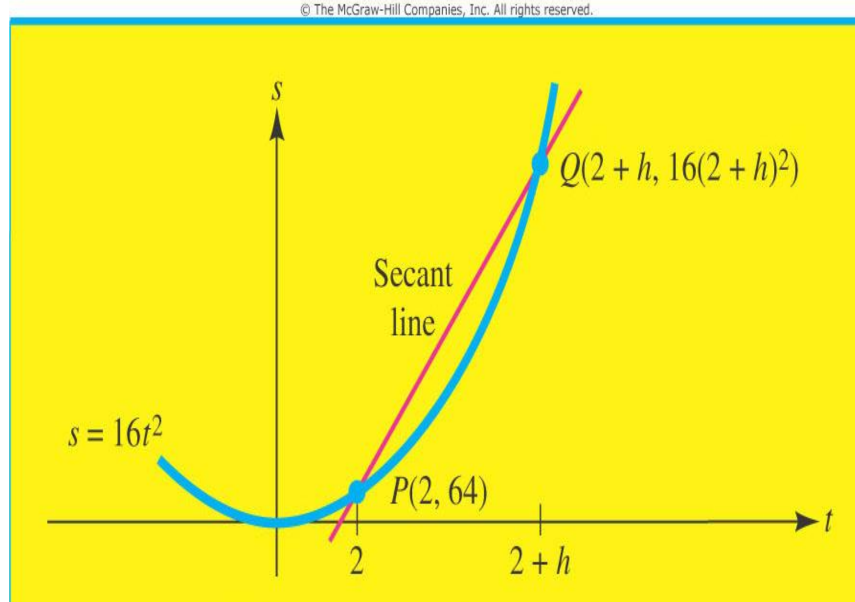
# The slope of secant line （割线的斜率）

The secant line: A line that intersects the curve at the point $x$ and point $x+h$

$$m_{\sec} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}.$$

The average rate of change can be interpreted geometrically as the slope of the secant line from the point $(x,f(x))$ to the point $(x+h,f(x+h))$.






(a) The secant line through $P(2, 64)$ and $Q(2+h, 16(2+h)^2)$.

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# The slope of tangent line（切线的斜率）

**DEFINITION 1.1**

The **slope** $m_{\tan}$ of the tangent line to $y = f(x)$ at $x = a$ is given by

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \qquad (1.2)$$

provided the limit exists.

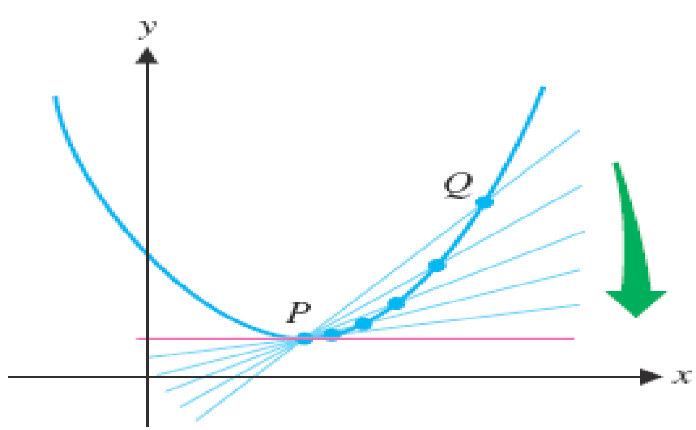
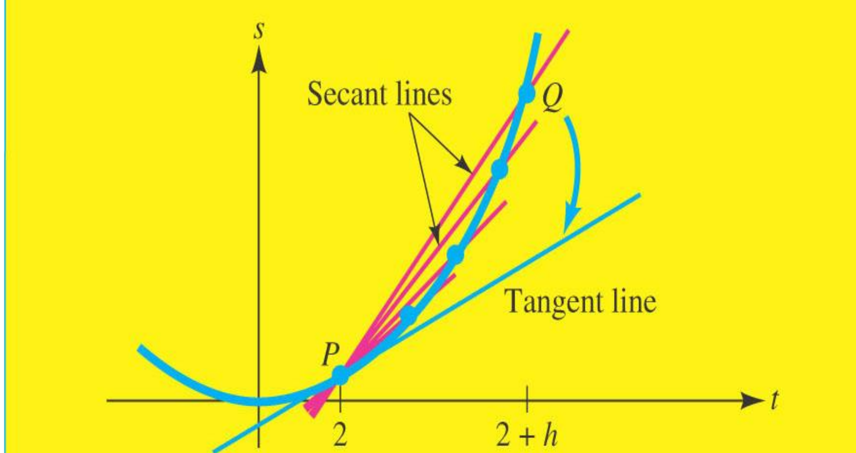



**FIGURE 2.7**
Secant lines approaching the tangent line at the point $P$






(b) As $h \to 0$, the secant line $PQ$ tends toward the tangent line at $P$.

The Tangent line is then the line passing through the point $(a,f(a))$ with slope $m_{\tan}$ and so, the point-slope form of the equation of the tangent line is

$$\boxed{y = m_{\tan}(x - a) + f(a).}$$

## Example

What is the equation of the tangent line to the curve $y=\sqrt{x}$ at the point where $x=4$ ?

## Solution:

According to the definition of derivative, we have

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

Thus, the slope of the tangent line to the curve at the point where $x=4$ is $f'(4)=1/4$ . So the point-slope form is $y-2=\frac{1}{4}(x-4)\Leftrightarrow y=\frac{1}{4}x+1$

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# Section 2.2 The Derivative (导数)

**The Derivative of a Function**: The derivative of the function *f(x)* with respect to *x* is the function *f'(x)* given by

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

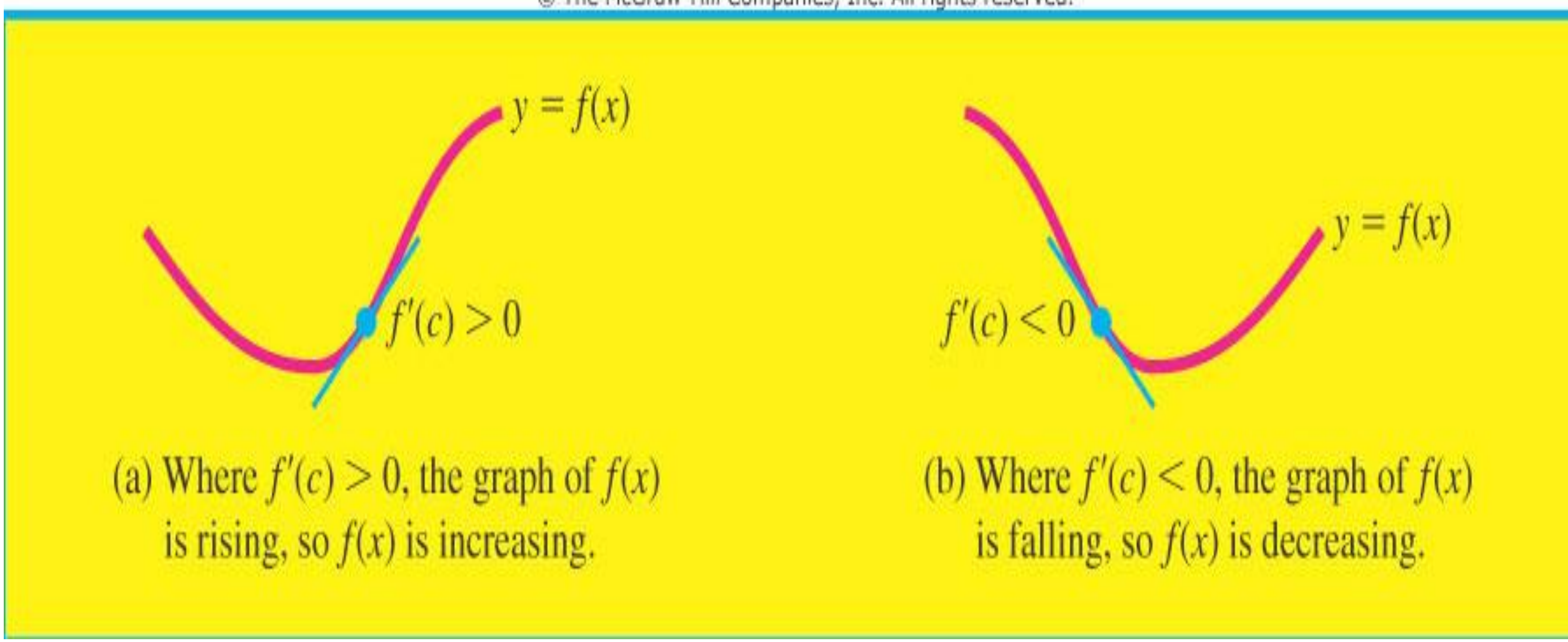
The process of computing the derivative is called differentiation. *f(x)* is differentiable at *x=c* if *f'(x)* exists

**Significance of the Sign of the derivative $f'(x)$**: If the function $f$ is differentiable at $x=c$, then

$f$ is increasing at $x=c$ if $f'(c)>0$ and

$f$ is decreasing at $x=c$ if $f'(c)<0$






(a) Where $f'(c) > 0$, the graph of $f(x)$ is rising, so $f(x)$ is increasing.

(b) Where $f'(c) < 0$, the graph of $f(x)$ is falling, so $f(x)$ is decreasing.

**EXAMPLE 2.3** Finding the Derivative of a Simple Rational Function

If $f(x) = \dfrac{1}{x}$ $(x \neq 0)$, find $f'(x)$.

**EXAMPLE 2.4** The Derivative of the Square Root Function

If $f(x) = \sqrt{x}$ (for $x \geq 0$), find $f'(x)$.

# Alternative Derivative Notation

Given a function $y=f(x)$ all of the following are equivalent and represent the derivative of $f(x)$ with respect to $x$.

$$f'(x)=y'=\frac{df}{dx}=\frac{dy}{dx}=\frac{d}{dx}\left(f(x)\right)=\frac{d}{dx}(y)$$

If we want to evaluate the derivative at $x=a$ all of the following are equivalent

$$f'(a)=y'\big|_{x=a}=\left.\frac{df}{dx}\right|_{x=a}=\left.\frac{dy}{dx}\right|_{x=a}$$

# Differentiability and Continuity

**Continuity of a Differentiable Function**: If the function $f(x)$ is differentiable at $x=c$, then it is also continuous at $x=c$.

Notice that a continuous function may not be differentiable

**EXAMPLE 2.7** Showing That a Function Is Not Differentiable at a Point

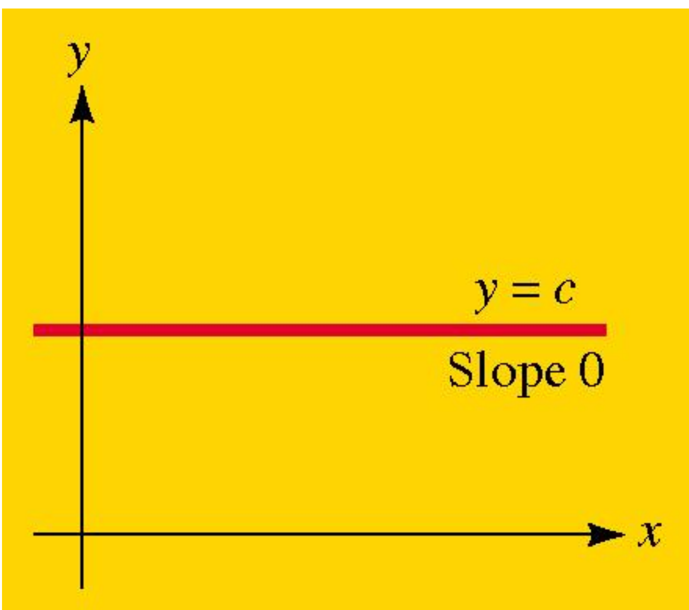
Show that $f(x) = \begin{cases} 4 & \text{if } x < 2 \\ 2x & \text{if } x \geq 2 \end{cases}$ is not differentiable at $x = 2$.

# Section 2.3 Techniques of Differentiation

**The Constant Rule:** For any constant c, we have

$$\frac{d}{dx}[c] = 0$$

That is, the derivative of a constant is zero




**Proof:** Since $f(x+h)=c$ for all $x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = 0$$

**The General Power Rule** For any real number $n$

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

In words, to find the derivative of $x^n$, reduce the exponent $n$ of $x$ by $1$ and multiply your new power of $x$ by original exponent.

## EXAMPLE 3.2 Using the General Power Rule

Find the derivative of $\frac{1}{x^{19}}$, $\sqrt[3]{x^2}$ and $x^{\pi}$.

**The constant Multiple Rule** If $c$ is a constant and $f(x)$ is differentiable then so is $cf(x)$ and

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

That is, the derivative of a multiple is the multiple of the derivative

For Example

$$\frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^4) = 3(4x^3) = 12x^3$$

$$\frac{d}{dx}\left(\frac{-7}{\sqrt{x}}\right) = \frac{d}{dx}(-7x^{-1/2}) = -7\left(-\frac{1}{2}x^{-3/2}\right) = \frac{7}{2}x^{-3/2}$$

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**The Sum Rule**: If $f(x)$ and $g(x)$ are differentiable then so is the sum of $s(x)=f(x)+g(x)$ and

$$\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}[f(x)]+\frac{d}{dx}[g(x)]$$

That is, the derivative of a sum is the sum of the separate derivative

## EXAMPLE 3.4 Rewriting a Function before Computing the Derivative

Find the derivative of $f(x)=\dfrac{4x^2-3x+2\sqrt{x}}{x}$.

# Higher Order Derivatives(高阶导数)

**The *n*th Derivative:** For any positive integer *n*, the *n*th derivative of a function is obtained from the function by differentiating successively *n* times. If the original function is $y=f(x)$ the *n*th derivative is denoted by

$$\frac{d^n y}{dx^n} \quad \text{or} \quad f^{(n)}(x)$$

## EXAMPLE 3.6 Computing Higher Order Derivatives

If $f(x) = 3x^4 - 2x^2 + 1$, compute as many derivatives as possible.

# Acceleration

What information does the second derivative of a function give us? Graphically, we get a property called *concavity*, which we develop in Chapter 3. One important application of the second derivative is acceleration, which we briefly discuss now.

You are probably familiar with the term **acceleration,** which is **the instantaneous rate of change of velocity.** Consequently, if the velocity of an object at time $t$ is given by $v(t)$, then the acceleration is

$$a(t) = v'(t) = \frac{dv}{dt}.$$

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# Section 2.4 The Product and Quotient Rules(乘法和除法准则)

The derivative of a product of functions is not the product of separate derivative!! Similarly, the derivative of a quotient of functions is not the quotient of separate derivative.

Suppose we have two function $f(x)=x^3$ and $g(x)=x^6$

$$(fg)' = (x^3x^6)' = (x^9)' = 9x^8 \qquad f'(x)g'(x) = (3x^2)(6x^5) = 18x^7$$

$$(fg)' \neq f'g'$$

$$\left(\frac{f}{g}\right)' = \left(\frac{x^3}{x^6}\right)' = \left(\frac{1}{x^3}\right)' = (x^{-3})' = -3x^{-4} = -\frac{3}{x^4} \qquad \frac{f'(x)}{g'(x)} = \frac{3x^2}{6x^5} = \frac{1}{2x^3}$$

$$\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}$$

**The Product Rule** If the two functions *f(x)* and *g(x)* are differentiable at *x*, then we have the derivative of the product $P(x)=f(x)g(x)$ is

$$\frac{d}{dx}\left[f(x)g(x)\right]=f(x)\frac{d}{dx}[g(x)]+g(x)\frac{d}{dx}[f(x)]$$

or equivalently,

$$(fg)'=fg'+gf'$$

## EXAMPLE 4.1 Using the Product Rule

Find $f'(x)$ if $f(x)=(2x^4-3x+5)\left(x^2-\sqrt{x}+\frac{2}{x}\right)$.

**The Quotient Rule** If the two functions $f(x)$ and $g(x)$ are differentiable at x, then the derivative of the quotient $Q(x)=f(x)/g(x)$ is given by

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}[f(x)]-f(x)\frac{d}{dx}[g(x)]}{g^2(x)} \qquad \text{if } g(x)\neq 0$$

or equivalently, $$\left(\frac{f}{g}\right)'=\frac{gf'-fg'}{g^2}$$

**EXAMPLE 4.3** Using the Quotient Rule

Compute the derivative of $f(x)=\dfrac{x^2-2}{x^2+1}$.

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A Word of Advice: The quotient rule is somewhat cumbersome, so don't use it unnecessarily.

**Example**

Differentiate the function $y=\frac{2}{3x^2}-\frac{x}{3}+\frac{4}{5}+\frac{x+1}{x}$ .

**Solution:**

Don't use the quotient rule! Instead, rewrite the function as

$$y=\frac{2}{3}x^{-2}-\frac{1}{3}x+\frac{4}{5}+1+x^{-1}$$

and then apply the power rule term by term to get

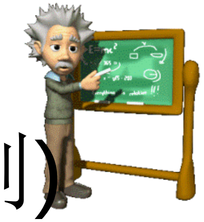
$$\frac{dy}{dx}=\frac{2}{3}(-2x^{-3})-\frac{1}{3}+0+0+(-1)x^{-2}$$

$$=-\frac{4}{3}x^{-3}-\frac{1}{3}-x^{-2}=-\frac{4}{3x^3}-\frac{1}{3}-\frac{1}{x^2}$$

# Applications

**EXAMPLE 4.5** Investigating the Rate of Change of Revenue

Suppose that a product currently sells for \$25, with the price increasing at the rate of \$2 per year. At this price, consumers will buy 150 thousand items, but the number sold is decreasing at the rate of 8 thousand per year. At what rate is the total revenue changing? Is the total revenue increasing or decreasing?



# Section 2.5 The Chain Rules(链锁准则)

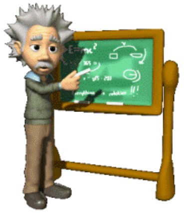
Suppose the total manufacturing cost at a certain factory is a function of the number of units produced, which in turn is a function of the number of hours the factory has been operating. If $C$, $q$, $t$, denote the cost, units produced and time respectively, then

$$\frac{dC}{dq} = \begin{bmatrix} \text{rate of change of cost} \\ \text{with respect to output} \end{bmatrix} \quad \text{(dollars per unit)}$$

$$\frac{dq}{dt} = \begin{bmatrix} \text{rate of change of output} \\ \text{with respect to time} \end{bmatrix} \quad \text{(units per hour)}$$

The product of these two rates is the rate of change of cost with respect to time that is

$$\frac{dC}{dt} = \frac{dC}{dq}\frac{dq}{dt} \qquad \text{(dollars per hour)}$$



The Chain Rule: If $y=f(u)$ is a differentiable function of $u$ and $u=g(x)$ is in turn a differentiable function of $x$, then the composite function $y=f(g(x))$ is a differentiable function of $x$ whose derivative is given by the product

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

or, equivalently, by

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

Note: One way to remember the chain rule is to pretend the derivative *dy/du* and *du/dx* are quotients and to "cancel" *du*.

## EXAMPLE 5.2 Using the Chain Rule on a Radical Function

Find $\dfrac{d}{dt}(\sqrt{100+8t})$.

## EXAMPLE 5.4 A Derivative Involving Multiple Chain Rules

Find the derivative of $f(x) = (\sqrt{x^2+4} - 3x^2)^{3/2}$.

## THEOREM 5.2

If $f$ is differentiable at all $x$ and has an inverse function $g(x) = f^{-1}(x)$, then

$$g'(x) = \frac{1}{f'(g(x))},$$

provided $f'(g(x)) \neq 0$.

## EXAMPLE 5.5 The Derivative of an Inverse Function

Given that the function $f(x) = x^5 + 3x^3 + 2x + 1$ has an inverse function $g(x)$, compute $g'(7)$.

# Section 2.6 Derivatives of Trigonometric Functions (三角函数求导)

**THEOREM 6.1**

$$\frac{d}{dx}\sin x = \cos x.$$

**LEMMA 6.1**

$$\lim_{\theta\to 0}\sin\theta = 0.$$

## LEMMA 6.2

$$\lim_{\theta \to 0} \cos \theta = 1.$$

## LEMMA 6.3

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

## LEMMA 6.4

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0.$$

## THEOREM 6.2

$$\frac{d}{dx}\cos x = -\sin x.$$

## THEOREM 6.3

$$\frac{d}{dx}\tan x = \sec^2 x.$$

The derivatives of the remaining trigonometric functions are left as exercises. The derivatives of all six trigonometric functions are summarized below.

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

## EXAMPLE 6.1 A Derivative That Requires the Product Rule

Find the derivative of $f(x) = x^5 \cos x$.

## EXAMPLE 6.2 Computing Some Routine Derivatives

Compute the derivatives of (a) $f(x) = \sin^2 x$ and (b) $g(x) = 4 \tan x - 5 \csc x$.

## EXAMPLE 6.3 The Derivatives of Some Similar Trigonometric Functions

Compute the derivative of $f(x) = \cos x^3$, $g(x) = \cos^3 x$ and $h(x) = \cos 3x$.

## EXAMPLE 6.4 A Derivative Involving the Chain Rule and the Quotient Rule

Find the derivative of $f(x) = \sin\left(\frac{2x}{x+1}\right)$.

# Section 2.7 Derivatives of Exponential and Logarithmic Functions (指数函数和对数函数求导)

**Exponential function:** If $b$ is a positive number other than *1* ($b>0$, $b\neq 1$), there is a unique function called the exponential function with base $b$ that is defined by

$$f(x)=b^x \quad \text{for every real number } x$$

NOTE: Such function can be used to describe exponential and logistic growth and a variety of other important quantities.

**Example** Sketch the graph of $f(x) = 2^x$ and $g(x) = \left(\dfrac{1}{2}\right)^x$

**Solution:**

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| -2 | $f(-2) = 2^{-2} = \dfrac{1}{4}$ | $g(-2) = \left(\dfrac{1}{2}\right)^{-2} = 4$ |
| -1 | $f(-1) = 2^{-1} = \dfrac{1}{2}$ | $g(-1) = \left(\dfrac{1}{2}\right)^{-1} = 2$ |
| 0 | $f(0) = 2^0 = 1$ | $g(0) = \left(\dfrac{1}{2}\right)^{0} = 1$ |
| 1 | $f(1) = 2$ | $g(1) = \dfrac{1}{2}$ |
| 2 | $f(2) = 4$ | $g(2) = \dfrac{1}{4}$ |

$$\lim_{x\to-\infty} 2^x = 0 \qquad \lim_{x\to-\infty} \left(\frac{1}{2}\right)^x = +\infty$$

$$\lim_{x\to+\infty} 2^x = +\infty \qquad \lim_{x\to+\infty} \left(\frac{1}{2}\right)^x = 0$$

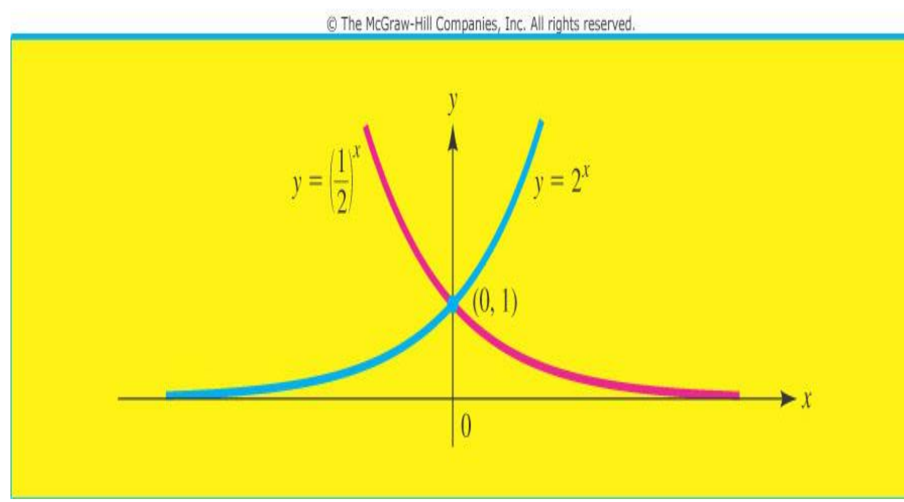
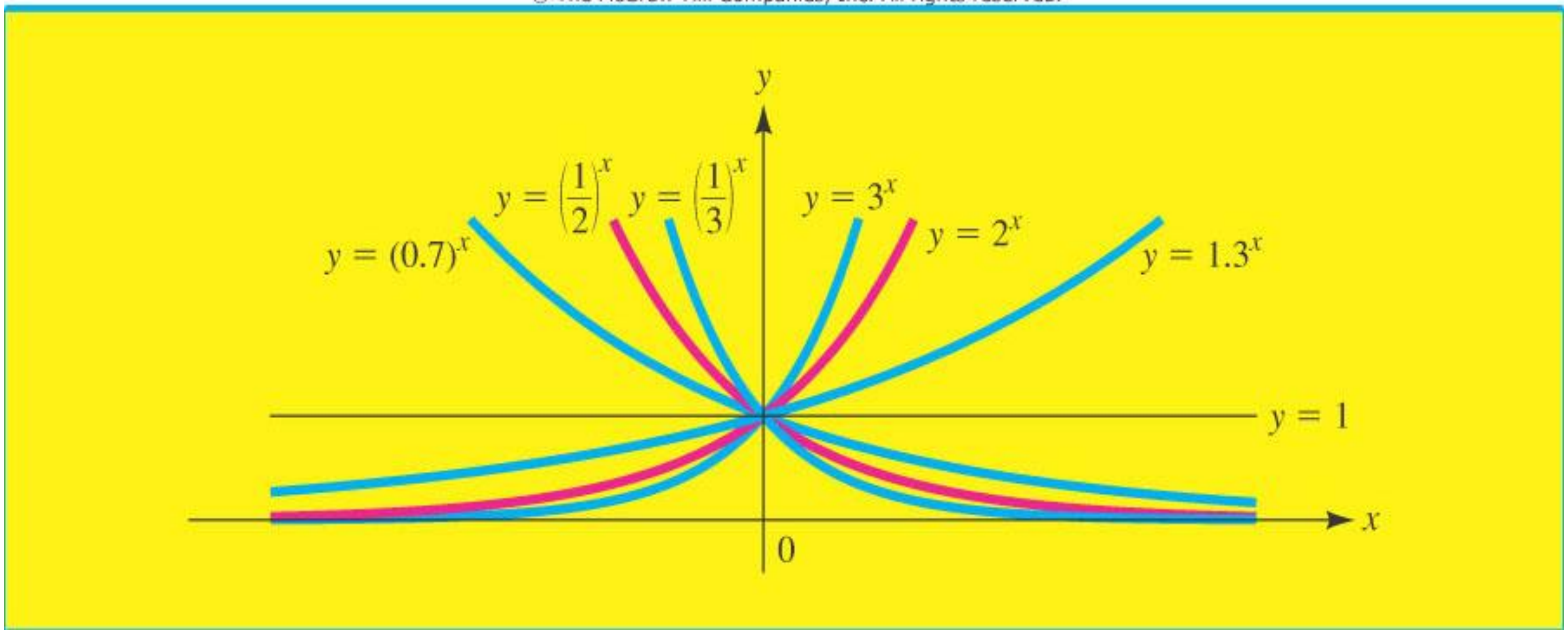


Figure below shows graphs of various members of the family of exponential functions $y = b^x$






NOTE: Students often confuse the *power* function $p(x) = x^b$ with the *exponential* function $f(x) = b^x$

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# Properties of $f(x) = b^x$

1. $f(0) = 1$ The function will always take the value of 1 at $x = 0$.

2. $f(x) \neq 0 \quad f(x) > 0$, the range of an exponential function is $(0, +\infty)$

3. The domain of an exponential function is $(-\infty, +\infty)$.

4. If $0 < b < 1$ then,

   a. $f(x) \to 0 \quad$ as $\quad x \to +\infty$
   b. $f(x) \to +\infty \quad$ as $\quad x \to -\infty$

5. If $b > 1$ then,

   a. $f(x) \to +\infty \quad$ as $\quad x \to +\infty$
   b. $f(x) \to 0 \quad$ as $\quad x \to -\infty$

**Basic Properties of Exponential Function** For bases $a, b(a > 0, b > 0)$ and any real number $x, y$, we have

- The equality rule $b^x = b^y$ if and only if $x = y$
- The **product rule** $b^x b^y = b^{(x+y)}$
- The **quotient rule** $\frac{b^x}{b^y} = b^{x-y}$
- The **power rule** $(b^x)^y = b^{xy}$
- The **multiplication rule** $(ab)^x = a^x b^x$
- The **division rule** $(\frac{a}{b})^x = \frac{a^x}{b^x}$

**Definition:** The natural exponential function is

$$f(x) = e^x,$$

Where

$$e = \lim_{n \to +\infty} (1 + \frac{1}{n})^n = 2.71828$$

| n | 10 | 100 | 1000 | 10,000 | 100,000 |
|---|---|---|---|---|---|
| | 2.59374 | 2.70481 | 2.71692 | 2.711815 | 2.71827 |

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# Logarithmic Functions

**Logarithmic Function** If x is a positive number, then the **logarithm** of $x$ to the base $b$ $(b > 0)$ $b \neq 1$, denoted $\log_b x$, is the number $y$ such that $b^y = x$; that is

$$y = \log_b x \quad \text{if and only if} \quad b^y = x \quad \text{for} \quad x > 0$$

**Properties of Logarithms** Let $b$ $(b > 0)$ $b \neq 1$ be any logarithmic base. Then.

$$\log_b 1 = 0 \quad \text{and} \log_b b = 1$$

and if $u$ and $v$ are any positive numbers, we have

| | |
|---|---|
| The **equality rule** | $\log_b u = \log_b v$ if and only if $u = v$ |
| The **product rule** | $\log_b(uv) = \log_b u + \log_b v$ |
| The **power rule** | $\log_b u^r = r \log_b u$ for any real number r |
| The **quotient rule** | $\log_b(\frac{u}{v}) = \log_b u - \log_b v$ |

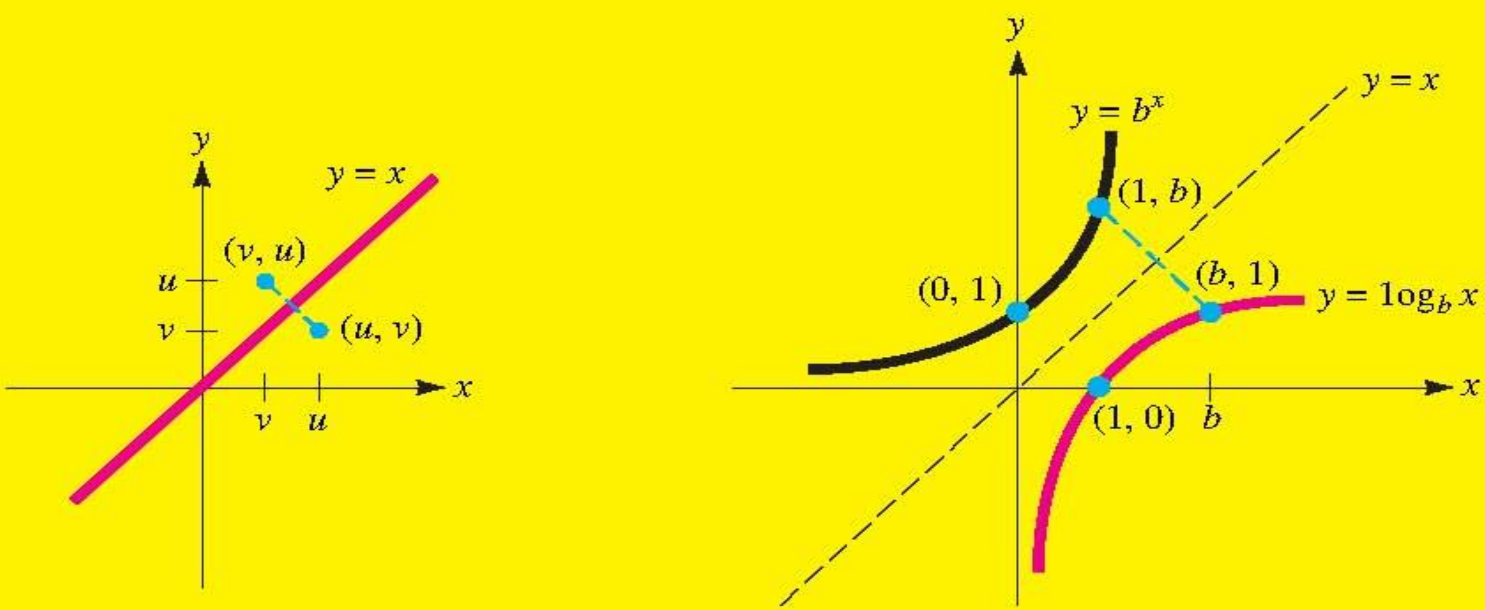


**TABLE 4.1** Comparison of Properties of Exponential and Logarithmic Functions

| Exponential Property | Logarithmic Property |
|---|---|
| $b^x b^y = b^{x+y}$ | $\log_b (xy) = \log_b x + \log_b y$ |
| $\dfrac{b^x}{b^y} = b^{x-y}$ | $\log_b \left(\dfrac{x}{y}\right) = \log_b x - \log_b y$ |
| $b^{xp} = (b^x)^p$ | $\log_b x^p = p \log_b x$ |

# Relationship Between the Graph of $y = \log_b x$ and $y = b^x$

The graphs of $y = log_b x$ and $y = b^x$ are mirror images of one another in the line $y = x$. Therefore, the graph of $y = log_b x$ can be obtained by reflecting the graph of $y = b^x$ in the line $y = x$.




(a) The point $(v, u)$ is the mirror image of $(u, v)$ in the line $y = x$.

(b) The graphs of $y = \log_b x$ and $y = b^x$ are reflections of one another in the line $y = x$.

- **Natural Logarithm** The logarithem $\log_e x$ is called the natural logarithm of x and is denoted by $\ln x$; that is, for $x > 0$

$$y = \ln x \quad \text{if and only if } e^y = x$$

- **The Inverse Relationship between $e^x$ and $\ln x$**

$$e^{\ln x} = x \quad \text{for } x > 0 \quad \text{and} \quad \ln e^x = x \text{ for all } x$$

- **Conversion Formula for Logarithms** If $a$ and $b$ are pos-titive numbers with $b \neq 1$, then

$$\log_b a = \frac{\ln a}{\ln b}$$

**THEOREM 7.3**

For $x > 0$,

$$\frac{d}{dx}(\ln x) = \frac{1}{x}. \tag{7.4}$$

**THEOREM 7.2**

$$\frac{d}{dx}e^x = e^x.$$

Taking the derivatives of some complicated functions can be simplified by using logarithms. This is called **logarithmic differentiation**.

**Example** Differentiate the function $f(x)=\frac{\sqrt[3]{x+1}}{(1-3x)^4}$

**Solution:** Take logarithms of both sides of the expressions of $f$

$$\ln f(x) = \ln\left[\frac{\sqrt[3]{x+1}}{(1-3x)^4}\right] = \ln \sqrt[3]{x+1} - \ln(1-3x)^4$$

$$= \frac{1}{3}\ln(x+1) - 4\ln(1-3x)$$

to be continued

**THEOREM 7.1**

For any constant $a > 0$,

$$\frac{d}{dx}a^x = a^x \ln a. \qquad (7.3)$$

## EXAMPLE 7.6 Logarithmic Differentiation

Find the derivative of $f(x) = x^x$, for $x > 0$.