PT Assignment 12

1. The joint probability density function of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y}{8} & 0 < x, y < 2 \\ 0 & \text{otherwise.} \end{cases}$$

   Calculate the variance of $(X+Y)/2$.

2. Suppose X and Y have the joint density function

$$f_{X,Y}(x,y) = \begin{cases} 8xy & 0 < x < y < 1 \\ 0 & \text{otherwise .} \end{cases}$$

   What is the joint density function of $W = \frac{X}{Y}$ and $Z = Y$?

3. Consider the unit disc

$$D = \{(x,y) \mid x^2 + y^2 \leq 1\}$$

   Suppose that we choose a point (X, Y) uniformly at random in D. That is, the joint PDF of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{1}{\pi} & (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$

   Let (R, Θ) be the corresponding polar coordinates The inverse transformation is given by

$$\begin{cases} X = R\cos\Theta \\ Y = R\sin\Theta \end{cases}$$

   where $R \geq 0$ and $-\pi < \Theta \leq \pi$. Find the joint PDF of R and Θ. Show that R and Θ are independent.

4. Let $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be a normal random vector with the following mean vector and covariance matrix

$$\mathbf{m} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

   Let also

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \mathbf{AX} + \mathbf{b}$$

   (a) Find $P(0 \leq X_2 \leq 1)$.

   (b) Find the expected value vector of $\mathbf{Y}, \mathbf{m_Y} = E\mathbf{Y}$.

(c) Find the covariance matrix of $\mathbf{Y}, \mathbf{C}_{\mathbf{Y}}$.

(d) Find $P\left(Y_3 \leq 4\right)$.

5. Suppose $\boldsymbol{X} = \left(X_1, X_2, X_3\right)^{\top}$ is a normal random vector with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, where

$$\boldsymbol{\mu} = (2,1,2)^{\top}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Find the joint distribution of $Y_1 = X_1 + X_2 + X_3$ and $Y_2 = X_1 - X_2$.

6. Suppose $\boldsymbol{X}$ is a two-dimensional normal random vector with joint density

$$f\left(x_1, x_2\right) = k^{-1} \exp\left\{-\frac{1}{2}\left(x_1^2 + 2x_2^2 - x_1 x_2 - 3x_1 - 2x_2 + 4\right)\right\}.$$

Find $E(\boldsymbol{X})$ and $\operatorname{Cov}(\boldsymbol{X})$.

7. Let X and Y be jointly normal random variables with parameters $\mu_X = 2, \sigma_X^2 = 4, \mu_Y = 1, \sigma_Y^2 = 9, \rho = -\frac{1}{2}$. Find $P(X + Y > 0 \mid 2X - Y = 0)$.

8. Let $\boldsymbol{X} = \left(X_1, X_2, X_3\right)^T$ be a normal random vector such that $E[\boldsymbol{X}] = (1,2,0)^T$ and covariance matrix of $\boldsymbol{X}$ is

$$C_X = \begin{pmatrix} \frac{54}{49} & -\frac{6}{49} & \frac{24}{49} \\ -\frac{6}{49} & \frac{17}{49} & \frac{30}{49} \\ \frac{24}{49} & \frac{30}{49} & \frac{76}{49} \end{pmatrix}.$$

Express X_3 in terms of X_1 and X_2. Hence find two non-constant independent random variables in terms of X_1 and X_2. Hint:

$$C_X = \begin{pmatrix} \frac{54}{49} & -\frac{6}{49} & \frac{24}{49} \\ -\frac{6}{49} & \frac{17}{49} & \frac{30}{49} \\ \frac{24}{49} & \frac{30}{49} & \frac{76}{49} \end{pmatrix} = \begin{pmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{pmatrix}^T.$$