AFM Brief solution to Assignment 4

1. Solve the PDE $\begin{cases} \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \\ U_{(t=0,x)} = x^3 \end{cases}$

   Solution:

$$\begin{aligned} U(t,x) &= \int_{-\infty}^{\infty} x'^3 G(x-x')\,\mathrm{d}x' \\ &= \int_{-\infty}^{\infty} (x'-x+x)^3 \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-x')^2}{4t}}\,\mathrm{d}x' \\ &= \int_{-\infty}^{\infty} [(x'-x)^3 + 3x(x'-x)^2 + 3x^2(x'-x) + x^3] \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-x')^2}{4t}}\,\mathrm{d}x' \\ &= x^3 + 3x \int_{-\infty}^{\infty} (x'-x)^2 \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-x')^2}{4t}}\,\mathrm{d}x' \\ &= x^3 + 3x(2t) \\ &= x^3 + 6tx \end{aligned}$$

2. Solve the PDE $\begin{cases} \frac{\partial U}{\partial t} = t^n \frac{\partial^2 U}{\partial x^2} \\ U_{(t=0,x)} = ax^2 + bx + c \end{cases}$, here a, b and c are constants, n is a positive integer.

   Solution:

   We make the change of variables, $\tau = \frac{1}{n+1} t^{n+1}$, $U(t,x) = V(\tau, x)$. Then $\frac{\partial U}{\partial t} = \frac{\partial V}{\partial \tau}\frac{\partial \tau}{\partial t} = t^n \frac{\partial V}{\partial \tau}$ and $\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 V}{\partial x^2}$. Thus, we get that $V(\tau, x)$ satisfies the PDE

$$\begin{cases} \frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x^2} \\ V_{(\tau=0,x)} = ax^2 + bx + c \end{cases}.$$

   Solving the above PDE, we can get that

$$\begin{aligned} V(\tau,x) &= \int_{-\infty}^{\infty} (ax'^2 + bx' + c) G(\tau, x-x')\,\mathrm{d}x' \\ &= \int_{-\infty}^{\infty} [a(x'-x+x)^2 + b(x'-x+x) + c] G(\tau, x-x')\,\mathrm{d}x' \\ &= \int_{-\infty}^{\infty} a(x'-x)^2 G(\tau, x-x')\,\mathrm{d}x' \\ &\quad + \int_{-\infty}^{\infty} (2ax+b)(x'-x) G(\tau, x-x')\,\mathrm{d}x \\ &\quad + \int_{-\infty}^{\infty} (ax^2+bx+c) G(\tau, x-x')\,\mathrm{d}x' \\ &= 2a\tau + ax^2 + bx + c. \end{aligned}$$

Hence

$$U(t,x) = V(\tau,x) = 2a\tau + ax^2 + bx + c = \frac{2a}{n+1}t^{n+1} + ax^2 + bx + c.$$

3. Solve the PDE $\begin{cases} \frac{\partial U}{\partial t} = e^{-t}\frac{\partial^2 U}{\partial x^2} \\ U_{(t=0,x)} = ax^2 + bx + c \end{cases}$, here a, b and c are constants.

   Solution:

   We make the change of variables, $\tau = 1 - e^{-t}$, $U(t,x) = V(\tau,x)$. Then $\frac{\partial U}{\partial t} = \frac{\partial V}{\partial \tau}\frac{\partial \tau}{\partial t} = e^{-t}\frac{\partial V}{\partial \tau}$ and $\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 V}{\partial x^2}$. Thus, we get that $V(\tau,x)$ satisfies the PDE

$$\begin{cases} \frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x^2} \\ V_{(\tau=0,x)} = ax^2 + bx + c \end{cases}.$$

   Solving the above PDE, we can get that

$$\begin{aligned}
V(\tau,x) =& \int_{-\infty}^{\infty} (ax'^2 + bx' + c)G(\tau, x - x')\,\mathrm{d}x' \\
=& \int_{-\infty}^{\infty} [a(x' - x + x)^2 + b(x' - x + x) + c]G(\tau, x - x')\,\mathrm{d}x' \\
=& \int_{-\infty}^{\infty} a(x' - x)^2 G(\tau, x - x')\,\mathrm{d}x' + \int_{-\infty}^{\infty} a2x(x' - x)G(\tau, x - x')\,\mathrm{d}x \\
&+ \int_{-\infty}^{\infty} ax^2 G(\tau, x - x')\,\mathrm{d}x' + \int_{-\infty}^{\infty} b(x' - x)G(\tau, x - x')\,\mathrm{d}x \\
&+ \int_{-\infty}^{\infty} bxG(\tau, x - x')\,\mathrm{d}x + \int_{-\infty}^{\infty} cG(\tau, x - x')\,\mathrm{d}x \\
=& a(2\tau + x^2) + bx + c.
\end{aligned}$$

   Hence

$$U(t,x) = V(\tau,x) = a(2\tau + x^2) + bx + c = 2a(1 - e^{-t}) + ax^2 + bx + c.$$

4. Solve the PDE $\begin{cases} \frac{\partial U}{\partial t} = (2 + \sin t)\frac{\partial^2 U}{\partial x^2} \\ U_{(t=0,x)} = e^{\lambda x} \end{cases}$, where λ is a constant.

   Solution:

   We make the change of variables, $\tau = 2t - \cos t + 1$, $U(t,x) = V(\tau,x)$. Then $\frac{\partial U}{\partial t} = \frac{\partial V}{\partial \tau}\frac{\partial \tau}{\partial t} = (2 + \sin t)\frac{\partial V}{\partial \tau}$ and $\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 V}{\partial x^2}$. Thus, we get that $V(\tau,x)$ satisfies the PDE

$$\begin{cases} \frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x^2} \\ V_{(\tau=0,x)} = e^{\lambda x} \end{cases}.$$

   Solving the above PDE, we can get that

$$
\begin{aligned}
V(\tau, x) &= \int_{-\infty}^{\infty} e^{\lambda x'} G(x - x') \, \mathrm{d}x' \\
&= \int_{-\infty}^{\infty} e^{\lambda x'} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(x-x')^2}{4\tau}} \, \mathrm{d}x' \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(x'-x-2\lambda\tau)^2 - 4x\lambda\tau - 4\lambda^2\tau^2}{4\tau}} \, \mathrm{d}x' \\
&= e^{\lambda x + \lambda^2\tau} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{(x'-x-2\lambda\tau)^2}{4t}} \, \mathrm{d}x' \\
&= e^{\lambda x + \lambda^2\tau}
\end{aligned}
$$

Hence

$$U(t, x) = V(\tau, x) = e^{\lambda x + \lambda^2(2t - \cos t + 1)}$$

5. The price of an at-the-money put option with strike price $K = 300$ currently has price \$15. At-the-money means that the current stock price equals the strike price. The option is European style and will mature in 6 months. The interest rate is 3%. What is the price of a call option written on the same stock, with the same strike price and same maturity date?

   Solution:

   According to call-put parity, we have

   $$C = P + S_t - Ke^{-r(T-t)} = 15 + 300 - 300e^{-0.03\times\frac{1}{2}} = 19.4664$$

6. Evaluate $\Delta = \frac{\partial P}{\partial S}$, where P is the Black-Scholes formula of the price of a European put option with no dividend, i.e.,

   $$P(t, S) = Ke^{-r(T-t)}\mathrm{N}(-d_2) - S\mathrm{N}(-d_1)$$

   where $d_1 = \frac{\ln\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sqrt{\sigma^2(T-t)}}$, and $d_2 = \frac{\ln\frac{S}{K} + (r - \frac{\sigma^2}{2})(T-t)}{\sqrt{\sigma^2(T-t)}}$.

   Do not apply the call-put parity. Evaluate the expression $\Delta = \frac{\partial P}{\partial S}$ directly,
   Solution:

$$
\begin{aligned}
\Delta &= \frac{\partial P}{\partial S} = \frac{\partial}{\partial S}\Big[Ke^{-r(T-t)}\mathrm{N}(-d_2) - S\mathrm{N}(-d_1)\Big] \\
&= -\mathrm{N}(-d_1) + S\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \cdot \frac{1}{S\sqrt{\sigma^2(T-t)}} - Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}} \cdot \frac{1}{S\sqrt{\sigma^2(T-t)}} \\
&= -\mathrm{N}(-d_1) + \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{S\sqrt{\sigma^2(T-t)}}[Se^{-\frac{d_1^2}{2}} - Ke^{-r(T-t)}e^{-\frac{d_2^2}{2}}] \\
&= -\mathrm{N}(-d_1) + \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{S}e^{-\frac{1}{2}\{[\frac{\ln\frac{S}{K} + r(T-t)}{\sqrt{\sigma^2(T-t)}}]^2 + [\frac{\sigma^2(T-t)}{2\sqrt{\sigma^2(T-t)}}]^2\}} \\
&\qquad \cdot\{Se^{-\frac{1}{2}(\ln\frac{S}{K} + r(T-t))} - Ke^{-r(T-t)}e^{\frac{1}{2}(\ln\frac{S}{K} + r(T-t))}\} \\
&= -\mathrm{N}(-d_1).
\end{aligned}
$$

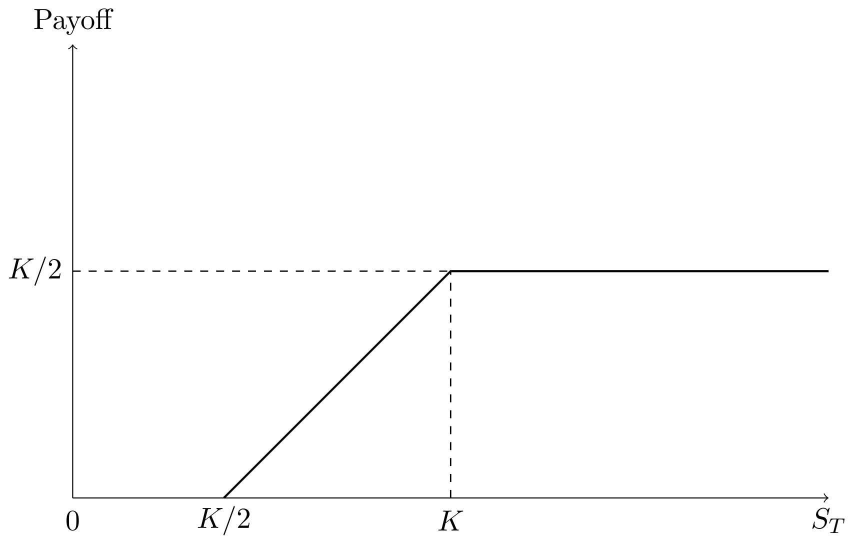

Figure 1:

7. Based on the call-put parity and the solution of Question **6**, calculate $\frac{\partial C}{\partial S}$, where C is the price of the European call option with the same underlying stock, the same strike price and the same maturity date as the put option in Question **6**.

   Solution:

$$\begin{aligned} & C + Ke^{-r(T-t)} = P + S \\ \Rightarrow & \frac{\partial C}{\partial S} = 1 + \frac{\partial P}{\partial S} = 1 - \mathrm{N}(-d_1) = \mathrm{N}(d_1)) \end{aligned}$$

8. What is the value of an option with the payoff given by Figure 1?

   Solution:

   Let $C(K)$ be the value of the call option with strike price K and the same maturity as the option given in the question.

   By inspection, we can get that the payoff given in Figure 1 is the difference of the payoffs of two call options with different strike prices $K/2$ and K, hence the value of such an option is $C(K/2) - C(K)$.

9. What is the value of an option with the payoff given by Figure 2?

   Solution:

   Let $P(K)$ be the value of he put option with strike price K and the same maturity as the option given in the question.

   By inspection, we can get that the payoff given in Figure 2 is a combination of the payoffs of three put options with different strike prices 50, 100 and 150, hence the value of such an option is $P(50) - P(100) + P(150)$.

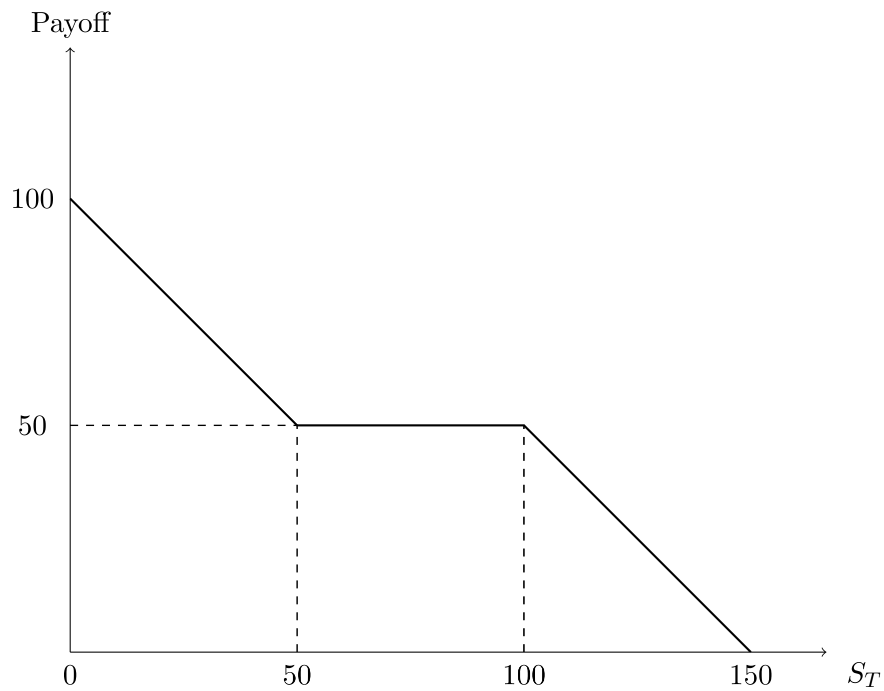

Figure 2: