
△ Sum and Direct Sum

Def 5.2.6 (Sum) Let X and Y be subspaces of W. The sum of X and Y is

$$X + Y = \{\vec{x}+\vec{y} \in W \mid \vec{x} \in X, \vec{y} \in Y\}$$

Remark : If $\{\vec{x}_1, \vec{x}_2, \cdots \vec{x}_k\}$ is a basis of X, $\{\vec{y}_1, \vec{y}_2, \cdots, \vec{y}_m\}$ is a basis of Y.

Then $X+Y = \operatorname{span}\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k, \vec{y}_1, \cdots, \vec{y}_m\}$

and $\dim(X+Y) = \dim X + \dim Y - \dim(X \cap Y)$

Def 5.2.10 (Direct Sum)

Let X and Y be subspaces of S and $X+Y=S$.

If every $\vec{z} \in S$ can be uniquely written as

$$\vec{z} = \underbrace{\vec{x}}_{\in X} + \underbrace{\vec{y}}_{\in Y}$$

then $X+Y$ is a direct sum and we denote it as $X \oplus Y$.

Thm 5.2.11

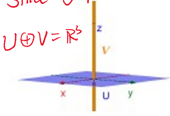
$X + Y$ is a direct sum $\iff$ ① $X+Y=S$
② $[\dim(X+Y) = \dim X + \dim Y]$
and
③ $X \cap Y = \{\vec{0}\}$.

①+②
or
①+③

Example 5.2.8. Consider the following examples of sum of subspaces

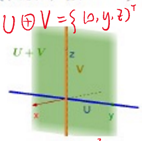
$U=\{(x,y,0)^T|x,y\in\mathbb{R}\}$
$V=\{(0,0,z)^T|z\in\mathbb{R}\}$
$U+V=\mathbb{R}^3$

Since $U\cap V=\{\vec{0}\}$,
$U\oplus V=\mathbb{R}^3$



$U=\{(0,y,0)^T|y\in\mathbb{R}\}$
$V=\{(0,0,z)^T|z\in\mathbb{R}\}$
$U+V=$
$\{(0,y,z)^T|y,z\in\mathbb{R}\}$

$U\oplus V=\{(0,y,z)^T \mid y,z\in\mathbb{R}\}$

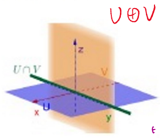


$U\oplus V \neq \mathbb{R}^3$

e.g. $(1,0,0)^T \notin U+V$.

$U=\{(x,y,0)^T|x,y\in\mathbb{R}\}$
$V=\{(0,y,z)^T|y,z\in\mathbb{R}\}$
$U+V=\mathbb{R}^3$

$U\oplus V \neq \mathbb{R}^3$

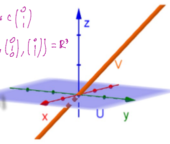


$(1,2,3)^T = \underbrace{(1,1,0)^T}_{\in U}+\underbrace{(0,1,3)^T}_{\in V}$
$= (1,2,0)^T+(0,0,3)^T$
$= \cdots\cdots$

Example 5.2.12.

$\forall\, \vec{x} \in U+V$, $\vec{x} = \begin{pmatrix}a\\b\\0\end{pmatrix}+\begin{pmatrix}0\\c\\c\end{pmatrix}$, $a,b,c\in\mathbb{R}$
$= a\begin{pmatrix}1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\0\end{pmatrix}+c\begin{pmatrix}0\\1\\1\end{pmatrix}$
i.e. $U+V=\operatorname{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}=\mathbb{R}^3$

$U = \{(x,y,0)^T|x,y\in\mathbb{R}\}$
$V = \{(0,a,a)^T|a\in\mathbb{R}\}$
Then $U\oplus V=\mathbb{R}^3$. — $\begin{cases} U+V=\mathbb{R}^3 \ \checkmark \\ \text{unique decomp} \ \checkmark \end{cases}$ $U\cap V=\{\vec{0}\}$



direct sum or sum has nothing to do with the angle.

By Definition 5.2.10:
$(x,y,z)^T = (x,y-z,0)^T + (0,z,z)^T$
where
$(x,y-z,0)^T \in U$ and $(0,z,z)^T\in V$.

By Theorem 5.2.11:
Since $U+V=\mathbb{R}^3$ and $U\cap V=\{\mathbf{0}\}$,
$U\oplus V=\mathbb{R}^3$.

△ Fundamental Subspaces Theorem

Thm 5.2.13 (The Fundamental Subspaces Theorem)

For any $m \times n$ matrix A, we have

$$N(A) = \mathrm{Col}(A^T)^{\perp} \xrightarrow{A \to A^T} N(A^T) = \mathrm{Col}(A)^{\perp}.$$

idea of proof: $A = \begin{bmatrix} — & \vec{a}_1 & — \\ — & \vec{a}_2 & — \\ & \vdots & \\ — & \vec{a}_m & — \end{bmatrix}_{m\times n}$, $A\vec{x} = \begin{bmatrix} — & \vec{a}_1 & — \\ — & \vec{a}_2 & — \\ & \vdots & \\ — & \vec{a}_m & — \end{bmatrix}_{m\times n} \vec{x}_{n\times 1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$

$N(A) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \}$ $\rightarrow$ $\vec{a}_i \vec{x} = 0$ for $i = 1, 2, \cdots, m$

$\vec{a}_i^{\,T} \perp \vec{x}$

$\mathrm{Col}(A^T) = \mathrm{Row}(A) \perp N(A)$

proof: i) $N(A) \subseteq \mathrm{Col}(A^T)^{\perp}$: $\forall \vec{x} \in N(A)$, then

$\vec{x}$ is orthogonal to vectors in $\mathrm{Row}(A)$, i.e. $\mathrm{Col}(A^T)$

i.e. $\vec{x} \perp$ to any vectors in $\mathrm{Col}(A^T)$, then $\vec{x} \in [\mathrm{Col}(A^T)]^{\perp}$

ii) $\mathrm{Col}(A^T)^{\perp} \subseteq N(A)$: $\forall \vec{y} \in \mathrm{Col}(A^T)^{\perp}$, then

$\vec{y}$ is orthogonal to any vector in $\mathrm{Col}(A^T)$, i.e. $\mathrm{Row}(A)$.

Namely, $\forall \vec{a}_i \in \mathrm{Row}(A)$, $\vec{a}_i^{\,T} \vec{y} = 0$, so $\begin{bmatrix} — & \vec{a}_1 & — \\ & \vdots & \\ — & \vec{a}_m & — \end{bmatrix}_{m\times n} \vec{y} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$, i.e. $\vec{y} \in N(A)$.

$\Rightarrow$ So $N(A) = \mathrm{Col}(A^T)^{\perp}$.

(b) The result also holds for the matrix $B = A^T$, $N(A^T) = \mathrm{Col}(A)^{\perp}$.

Use the result of Thm 5.2.13, we can prove

Thm 5.2.15 If S is a subspace of $\mathbb{R}^n$, then

(Thm 5.2.5-3) $\dim S + \dim S^{\perp} = n$.

Further if $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_r\}$ is a basis of S, $\{\vec{x}_{r+1}, \cdots, \vec{x}_n\}$ is a basis of $S^{\perp}$.

then $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}$ forms a basis of $\mathbb{R}^n$.

Namely, $S \oplus S^{\perp} = \mathbb{R}^n$

Outline of proof: Define a matrix A^T whose columns form a basis of S.

$$A^T = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \\ | & | & & | \end{bmatrix}_{n\times r} \qquad \mathrm{Col}(A^T) = S$$

$$N(A) = \mathrm{Col}(A^T)^{\perp} = S^{\perp}$$

$\dim S + \dim S^{\perp} = \dim(\mathrm{Col}(A^T)) + \dim(N(A))$ # of columns in $A = n$

$= \dim(\mathrm{Col}(A)) + \dim(N(A))$

Refer to the note / textbook for more details.

As a result of Thm 5.2.15 and Thm 5.2.13 :

Thm 5.2.14 For any $m \times n$ matrix A.

$$N(A) \oplus Col(A^T) = \mathbb{R}^n, \quad N(A^T) \oplus Col(A) = \mathbb{R}^m.$$

$N(A) \subseteq \mathbb{R}^n$. $Col(A^T) \subseteq \mathbb{R}^n$

$A_{m\times n}\vec{x}_{n\times 1} = \vec{0}$ $(A^T)_{n\times m}$

proof :
- $N(A)$ and $Col(A^T)$ are subspaces of $\mathbb{R}^n$.
- $\dim N(A) + \dim Col(A^T) = nullity(A) + rank(A) = \#$ of columns in $A = n$
- $N(A) \cap Col(A^T) = \{\vec{0}\}$.

By Thm 5.2.11, $N(A) \oplus Col(A^T) = \mathbb{R}^n$

$N(A^T) \oplus Col(A) = \mathbb{R}^m$

Example $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 0 & 0 \end{bmatrix}$

Then $Col(A) = Span\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$ $Col(A^T) = Span\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right\}$

$N(A) = Span\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$ $N(A^T) = span\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$

Clearly. $N(A) \perp Col(A^T)$ and $N(A^T) \perp Col(A)$

$N(A) \oplus Col(A^T) = \mathbb{R}^3$ and $N(A^T) \oplus Col(A) = \mathbb{R}^2$

Applications : Use matrices to find orthogonal complement.

E.g. 5.2.17 Read by yourselves.

Exercise $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$. Find $Col(A)$, $N(A)$, $Col(A^T)$, $N(A^T)$

$Col(A) = span\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \mathbb{R}^2$, $N(A) = span\left\{\begin{pmatrix} -2 \\ -5 \\ 1 \end{pmatrix}\right\}$

$Col(A^T) = span\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}\right\}$, $N(A^T) = \{\vec{0}\}$

$N(A)^\perp = Col(A^T)$, $N(A^T)^\perp = Col(A)$, $N(A) \oplus Col(A^T) =$

E.g. 5.2.18 Let $S = span\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}\right\}$ be a subspace of $\mathbb{R}^3$.

Find a basis of $S^\perp$.

$S^\perp = span\left\{\begin{pmatrix} -2 \\ -5 \\ 1 \end{pmatrix}\right\}$

Example Find a basis for the orthogonal complement of the solution space of

$\underbrace{\text{orthogonal complement}}_{Col(A^T)}$ of the $\underbrace{\text{solution space}}_{N(A)}$

$$\begin{cases} x_1 + 2x_3 = 0 \\ x_2 + 5x_3 = 0 \end{cases} \longleftrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}}_{A} \vec{x} = \vec{0}$$

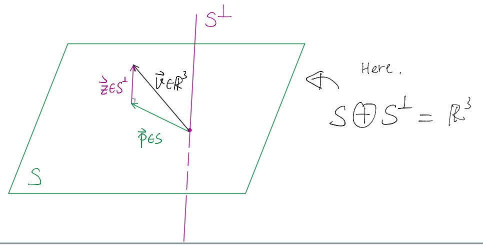
$$Col(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} \right\}$$

Corollary 5.2.16 (Othogonal Decomposition Theorem)

Let S be a subspace of $\mathbb{R}^n$, then $S \oplus S^\perp = \mathbb{R}^n$.

Namely, for any $\vec{v} \in \mathbb{R}^n$, there exists a unique $\vec{p} \in S$ and $\vec{z} \in S^\perp$ s.t.

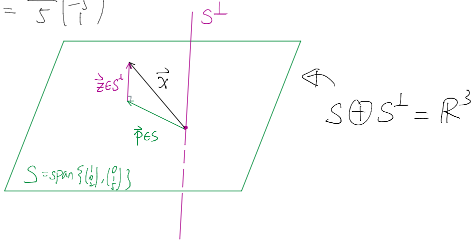
$$\vec{v} = \underbrace{\vec{p}}_{\in S} + \underbrace{\vec{z}}_{\in S^\perp}$$



Example Let $S = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} \right\}$, then $S^\perp = \text{Span}\left\{ \begin{pmatrix} -2 \\ -5 \\ 1 \end{pmatrix} \right\}$.

Write $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ as the sum of $\vec{p}$ and $\vec{z}$, where $\vec{p} \in S$, $\vec{z} \in S^\perp$.

$$\vec{z} = \text{proj}_{S^\perp} \vec{x} = \text{proj}_{\begin{pmatrix} -2 \\ -5 \\ 1 \end{pmatrix}} \vec{x} = \frac{(1,1,1)\begin{pmatrix} -2 \\ -5 \\ 1 \end{pmatrix}}{(-2,-5,1)\begin{pmatrix} -2 \\ -5 \\ 1 \end{pmatrix}} \begin{pmatrix} -2 \\ -5 \\ 1 \end{pmatrix} = \frac{-1}{5}\begin{pmatrix} -2 \\ -5 \\ 1 \end{pmatrix}$$



$$\vec{p} = \vec{x} - \vec{z} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} -2 \\ -5 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix}$$

orthogonal projection of $\vec{x}$ onto a subspace S

Verification : $\vec{p} \cdot \vec{z} = 0$, $\rightsquigarrow$ $\vec{p} \perp \vec{z}$ and $\vec{p} \in S$.

$$\vec{p} = \frac{3}{5}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} \in S.$$

Question : Can we find $\vec{p}$ directly ? (Find the projection of a vector onto a subspace)

Answer : Yes ! ① Orthogonal basis — Sec 5.6 Gram-Schmidt Process

② Non-orthogonal basis — Sec 5.3 Projection Matrix

△ Projection Matrix

Idea: $\vec{x} = \underbrace{\vec{p}}_{\in S} + \underbrace{\vec{z}}_{\in S^\perp}$, then $\vec{z} = \vec{x} - \vec{p}$, $\vec{z} \perp \vec{p}$

Define a matrix $A_{m \times n}$ whose columns span S, then $S = Col A$ and $S^\perp = N(A^T)$.

Thus, $\vec{z} \in N(A^T)$ and $A^T(\vec{x} - \vec{p}) = \vec{0}$,

$$A^T\vec{x} = A^T\vec{p}, \quad \text{where } \vec{p} \text{ is } \text{proj}_S \vec{x}.$$

Since $\vec{p} \in Col(A)$, then $\vec{p} = A\vec{y}$ for some $\vec{y} \in \mathbb{R}^n$

i.e. $$\boxed{A^T\vec{x} = \underset{n\times n}{A^TA}\, \underset{n\times 1}{\vec{y}} \quad \text{— the normal equation}}$$

Here, the vector $\vec{p}$, as the orthogonal projection of $\vec{x}$ onto S, must be unique.

But for $\vec{p} = A\vec{y}$, the vector $\vec{y}$ could be unique or infinitely many. (It's always possible to find a $\vec{y}$ by solving $A^TA\vec{y} = A^T\vec{x}$)

Specially, when A^TA is of rank n, the normal equation has a unique solution.

$$\vec{y} = (A^TA)^{-1}A^T\vec{x} \quad \rightarrow \quad proj_S \vec{x} = \vec{p} = A\vec{y} = \underbrace{A(A^TA)^{-1}A}_{\text{projection matrix}}\vec{x}$$

△ Projection Matrix (To find the orthogonal projection of $\vec{b}$ onto W)

<u>Def.</u> 5.3.6 If $A_{m\times n}$ is of rank n, then the projection of $\vec{b}$ onto $Col(A)$.

$$\underbrace{\hat{b}}_{proj_{Col(A)}\vec{b}} = A\hat{y} = \underbrace{A(A^TA)^{-1}A^T}_{\text{projection matrix}}\vec{b}$$

<u>Example</u>: $\vec{b} = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $S = span\left\{\begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}0\\1\\5\end{pmatrix}\right\}$, $S^\perp = span\left\{\begin{pmatrix}-2\\-5\\1\end{pmatrix}\right\}$. Find $proj_S\vec{b}$.

Define a matrix $A = \begin{bmatrix}1&0\\0&1\\2&5\end{bmatrix}_{3\times 2}$, then $S = Col(A)$ and $S^\perp = N(A^T)$.

Since $rank(A) = 2$, then A^TA is nonsingular and

$$proj_S\vec{b} = proj_{Col(A)}\vec{b} = A(A^TA)^{-1}A^T\vec{b} = \underbrace{\begin{pmatrix}1&0\\0&1\\2&5\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}5&10\\10&26\end{pmatrix}^{-1}\begin{pmatrix}1&0&2\\0&1&5\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}}_{\vec{y}} = \begin{pmatrix}1&0\\0&1\\2&5\end{pmatrix}\begin{pmatrix}3/5\\0\end{pmatrix} = \frac{3}{5}\begin{pmatrix}1\\0\\2\end{pmatrix}.$$

<u>Example</u>: $\vec{b} = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $S = span\left\{\begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}2\\0\\4\end{pmatrix}, \begin{pmatrix}0\\1\\5\end{pmatrix}\right\}$, $S^\perp = span\left\{\begin{pmatrix}-2\\-5\\1\end{pmatrix}\right\}$. Find $proj_S\vec{b}$.

Notice that $\left\{\begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}2\\0\\4\end{pmatrix}, \begin{pmatrix}0\\1\\5\end{pmatrix}\right\}$ is not a basis of S, but when we define $A = \begin{bmatrix}1&2&0\\0&0&1\\2&4&5\end{bmatrix}$, we still have $S = Col(A)$.

The normal equation: $A^TA\vec{y} = A^T\vec{b}$, i.e. $\begin{bmatrix}5&10&10\\10&20&20\\10&20&26\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}3\\6\\6\end{bmatrix}$

$\rightarrow \vec{y} = \begin{bmatrix}-2\\1\\0\end{bmatrix}\alpha + \begin{bmatrix}3/5\\0\\0\end{bmatrix}$, $\alpha \in \mathbb{R}$ (infinitely many solutions for $\vec{y}$)

$proj_S\vec{b} = A\vec{y} = \begin{bmatrix}0\\0\\0\end{bmatrix} + \frac{3}{5}\begin{bmatrix}1\\0\\2\end{bmatrix} = \frac{3}{5}\begin{bmatrix}1\\0\\2\end{bmatrix}$ (but the projection of $\vec{b}$ onto S is unique)

△ Special Matrices AA^T and A^TA

For any $m\times n$ matrix A, AA^T is of size $m\times m$
A^TA is of size $n\times n$

both are symmetric matrices with good properties and closely related to A.

<u>Thm 5.2.19</u> $N(A^TA) = N(A)$, $N(AA^T) = N(A^T)$

and $Col(A^TA) = Col(A^T)$, $Col(AA^T) = Col(A)$

proof: i) To prove $N(A)\subseteq N(A^TA)$:

fill up the proof by yourself

$\forall\ \vec{x} \in N(A)$, $A^TA\vec{x} = A^T\vec{0} = \vec{0} \rightarrow \vec{x} \in N(A^TA)$

ii) To prove $N(A^TA) \subseteq N(A)$

△ $(A^TA)_{n\times n}$ $A_{m\times n}$

$rank(A^TA) = rank(A)$

$= n - nullity(A^TA) = n - nullity(A)$

∴ thm 4 $col(A^TA) = col(A^T)$

ii) To prove $N(A^TA) \subseteq N(A)$

$\forall \vec{y} \in N(A^TA)$, $A^TA\vec{y} = \vec{0}$

$\vec{y}^TA^TA\vec{y} = \vec{y}^T\vec{0} = 0$

$\|A\vec{y}\|^2 = (A\vec{y})^TA\vec{y} = 0 \Rightarrow A\vec{y} = \vec{0} \rightarrow \vec{y} \in N(A)$.

$= n - \text{nullity}(A^TA) \qquad = n - \text{nullity}(A)$

$\triangle$ proof of $\text{Col}(A^TA) = \text{Col}(A^T)$

Method 1 : $\text{Col}(A^TA) \subseteq \text{Col}(A^T)$

$\text{Col}(A^T) \subseteq \text{Col}(A^TA)$

Method 2 : $\text{Col}(A^TA) \subseteq \text{Col}(A^T)$

$\dim(\text{Col}(A^TA)) = \dim(\text{Col}(A^T))$

Corollary 5.2.20 For any $m \times n$ matrix A,

1. A^TA is nonsingular $\iff$ $\text{rank}(A) = n = \text{rank}(A^T)$

(2. AA^T is nonsingular $\iff$ $\text{rank}(A^T) = m = \text{rank}(A)$)

proof : 1. $(A^TA)_{n\times n}$ nonsingular $\underset{\text{since}}{\iff}$ $n = \text{rank}(A^TA) = \text{rank}(A^T) = \text{rank}(A)$

$\text{Col}(A^TA) = \text{Col}(A^T)$

2. Set $A = A^T$ in part 1).

Thm 5.2.21 Suppose A is an $m\times n$ matrix of rank r.

(a) If $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r\}$ is a basis of $\text{Col}(A^T)$, then $\{A\vec{v}_1, \cdots, A\vec{v}_r\}$ forms a basis of $\text{Col}(A)$.

(b) Let $A = B^T$ in part (a), then we have part (b).

proof : To prove $\{A\vec{v}_1, \cdots, A\vec{v}_r\}$ form a basis of $\text{Col}(A)$:

linear independent + r vectors in $\text{Col}(A)$

Let $c_1A\vec{v}_1 + c_2A\vec{v}_2 + \cdots + c_rA\vec{v}_r = \vec{0}$, $c_1, c_2, \cdots, c_r$ are constant

then $A(c_1\vec{v}_1 + \cdots + c_r\vec{v}_r) = \vec{0}$, which implies $(c_1\vec{v}_1 + \cdots + c_r\vec{v}_r) \in N(A)$

On the other hand, $\sum_{i=1}^{r} c_i\vec{v}_i \in \text{Col}(A^T)$. $N(A) \cap \text{Col}(A^T) = \{\vec{0}\}$

$\Rightarrow \sum_{i=1}^{r} c_i\vec{v}_i = \vec{0}$

Since $\{\vec{v}_1, \cdots, \vec{v}_r\}$ are linearly independent, $c_1 = c_2 = \cdots = c_r = 0$.

Exercise $A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 \end{bmatrix}$, $\text{Col}(A^T) = \text{span}\left\{\begin{pmatrix}2\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\1\\-2\end{pmatrix}\right\}$

Then $\text{Col}(A) = \text{span}\left\{A\begin{pmatrix}2\\0\\0\\0\end{pmatrix}, A\begin{pmatrix}0\\-1\\1\\-2\end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix}2\\4\\0\end{pmatrix}, \begin{pmatrix}-6\\0\\6\end{pmatrix}\right\}$.