2022-23 Second Semester
MATH1063 Linear Algebra II (1003)

Assignment 8 Suggested Solutions

1. (a) For $A = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$, $\det(A - \lambda I_n) = \lambda^2 - 4^2 = 0$.

   $\rightarrow \quad \lambda_1 = -4, \quad \lambda_2 = 4$, eigenvectors: $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

   Let $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} -4 & 0 \\ 0 & 4 \end{bmatrix}$, then $A = QDQ^T$ is orthogonally diagonalizable.

   (b) For $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$, $\det(A - \lambda I_n) = -(3-\lambda)^2(3+\lambda) = 0$.

$$\rightarrow \quad \lambda_1 = \lambda_2 = 3, \quad \lambda_3 = -3.$$

   For $\lambda = 3$, eigenvectors: $\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

   For $\lambda = -3$, eigenvector: $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

   Let $P = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$, then $A = PDP^{-1}$. Notice that A is diagonalizable, yet P is not orthogonal.

   Remark: Since $\vec{v}_1$ and $\vec{v}_2$ are both eigenvectors relating to $\lambda = 3$, their linear combinations are also eigenvectors of A corresponding to $\lambda = 3$. By performing Gram-Schmidt process on $\{\vec{v}_1, \vec{v}_2\}$, we obtain two orthonormal eigenvectors $\left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \frac{\sqrt{5}}{2\sqrt{6}} \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{4}{5} \end{bmatrix} \right\}$, then use them to form an orthogonal matrix Q such that $A = QDQ^T$ is orthogonally diagonalizable.

2. If A has a singular value decomposition $U\Sigma V^T$, then A^T has a singular value decomposition as $V\Sigma^T U^T$. The matrices Σ and Σ^T have the same nonzero diagonal elements. Thus A and A^T have the same nonzero singular values.

3. The eigenvalues of A^TA are $\lambda_1 = 16$ and $\lambda_2 = 4$ with $\lambda_1 \geq \lambda_2$ since

$$A^TA = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}, \quad \det(A^TA - \lambda I) = (16-\lambda)(4-\lambda) = 0.$$

Thus the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 4$ and $\sigma_2 = \sqrt{\lambda_2} = 2$.
For $\lambda_1 = 16$, an eigenvector is $\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\lambda_2 = 4$, an eigenvector is $\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Let $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{4\sqrt{2}} \begin{pmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

Since $\mathbf{u}_3$ and $\mathbf{u}_4$ are orthonormal vectors in $\mathrm{N}(A^T)$, take $\mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{u}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

Thus,

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

4. Columns in V which are related to nonzero singular values form an orthonormal basis for $\mathrm{Col}(A^T)$: $\mathbf{v}_1 = \frac{1}{3}(2,2,1)^T, \mathbf{v}_2 = \frac{1}{3}(-2,1,2)^T$.
Columns in V that belong to zero singular values form an orthonormal basis for $\mathrm{N}(A)$: $\mathbf{v}_3 = \frac{1}{3}(1,-2,2)^T$.
Columns in U which belong to nonzero singular values form an orthonormal basis for $\mathrm{Col}(A)$: $\mathbf{u}_1 = \frac{1}{2}(1,1,1,1)^T, \mathbf{u}_2 = \frac{1}{2}(1,-1,-1,1)^T$.
Columns in U which belong to zero singular values form an orthonormal basis for $\mathrm{N}(A^T)$: $\mathbf{u}_3 = \frac{1}{2}(1,-1,1,-1)^T, \mathbf{u}_4 = \frac{1}{2}(1,1,-1,-1)^T$.

5. If A is a symmetric matrix, then $A^TA = A^2$. Thus the eigenvalues of A^TA are $\lambda_1^2, \lambda_2^2, \cdots, \lambda_n^2$. The singular values of A are the positive square roots of the eigenvalues of A^TA.

6. If σ is a singular value of A, then let $\mathbf{x}$ be an eigenvector of A^TA corresponding to its eigenvalue σ^2. It follows that

$$A^TA\mathbf{x} = \sigma^2\mathbf{x} \quad \rightarrow \quad \mathbf{x}^T(A^TA\mathbf{x}) = \mathbf{x}^T(\sigma^2\mathbf{x}) = \sigma^2\mathbf{x}^T\mathbf{x}$$

$$\|A\mathbf{x}\|^2 = \sigma^2\|\mathbf{x}\|^2 \quad \rightarrow \quad \sigma = \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$$