ASP

Solution to Assignment 8

1. For (X_t), we verify the four conditions in the definition of SBM:

   (a) $X_0 = \frac{1}{c}B_0 = 0$.

   (b) for any $s, t \geq 0$,
   $$X_{t+s} - X_t = \frac{1}{c}B_{c^2(t+s)} - \frac{1}{c}B_{c^2t} = \frac{1}{c}\left(B_{c^2(t+s)} - B_{c^2t}\right).$$
   But $B_{c^2(t+s)} - B_{c^2t} \sim N(0, c^2 s)$. So $X_{t+s} - X_t$ is normal with mean 0 and variance
   $$Var(X_{t+s} - X_t) = \frac{1}{c^2}Var(B_{c^2(t+s)} - B_{c^2t}) = \frac{1}{c^2}c^2 s = s,$$
   i.e.,
   $$X_{t+s} - X_t \sim N(0, s), \tag{1}$$
   showing that (X_t) has stationary increment.

   (c) Suppose $0 \leq t_0 < \cdots < t_n$. Note that for $i = 1, \ldots, n$,
   $$X_{t_i} - X_{t_{i-1}} = \frac{1}{c}B_{c^2t_i} - \frac{1}{c}B_{c^2t_{i-1}} = \frac{1}{c}\left(B_{c^2t_i} - B_{c^2t_{i-1}}\right).$$
   So
   $$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$$
   are mutually independent, since
   $$B_{c^2t_1} - B_{c^2t_0}, B_{c^2t_2} - B_{c^2t_1}, \cdots, B_{c^2t_n} - B_{c^2t_{n-1}}$$
   are mutually independent. This means that (X_t) has independent increment.

   (d) Since the function $t \mapsto B_t$ is a continuous function of t, it follows that $t \mapsto \frac{1}{c}B_{c^2t}$ is also a continuous function of t.

2. Similarly to the previous problem, for $(\tilde{B}_t)$, we verify the four conditions in the definition of SBM:

   (a) $\tilde{B}_0 = B_{t_0} - B_{t_0} = 0$.

   (b) for any $s, t \geq 0$,
   $$\tilde{B}_{t+s} - \tilde{B}_t = (B_{t+s+t_0} - B_{t_0}) - (B_{t+t_0} - B_{t_0}) = B_{t+s+t_0} - B_{t+t_0} \sim N(0, s), \tag{2}$$
   showing that $(\tilde{B}_t)$ has stationary increment.

(c) Suppose $0 \le t_0 < \cdots < t_n$. For $i = 1, \ldots, n$,

$$\tilde{B}_{t_i} - \tilde{B}_{t_{i-1}} = (B_{t_i+t_0} - B_{t_0}) - \big(B_{t_{i-1}+t_0} - B_{t_0}\big) = B_{t_i+t_0} - B_{t_{i-1}+t_0}.$$

So

$$\tilde{B}_{t_1} - \tilde{B}_{t_0}, \tilde{B}_{t_2} - \tilde{B}_{t_1}, \cdots, \tilde{B}_{t_n} - \tilde{B}_{t_{n-1}}$$

are mutually independent, as (B_t) is a SBM and has independent increment.

(d) Since the function $t \mapsto B_t$ is a continuous function of t, it follows that $t \mapsto B_{t+t_0} - B_{t_0}$ is also a continuous function of t.

3.

(a) Obviously, $E(X_t) = E(B_t - tB_1) = E(B_t) - tE(B_1) = 0$. So for $s, t \in [0, 1]$,

$$\begin{aligned}
E(X_s X_t) &= E(X_s X_t) - E(X_s)E(X_t) \\
&= Cov(X_s, X_t) \\
&= Cov(B_s - sB_1, B_t - tB_1) \\
&= Cov(B_s, B_t) - sCov(B_1, B_t) - tCov(B_s, B_1) + stCov(B_1, B_1) \\
&= \min(s, t) - st - st + st \\
&= \min(s, t) - st.
\end{aligned}$$

(b) Suppose $t \in [0, 1]$. Since

$$\begin{aligned}
X_t = B_t - tB_1 &= B_t - t(B_1 - B_t + B_t) \\
&= (1 - t)B_t - t(B_1 - B_t) \\
&= (1 - t, -t) \begin{pmatrix} B_t \\ B_1 - B_t \end{pmatrix}
\end{aligned}$$

and $(B_t, B_1 - B_t)^\top$ is a normal random vector, X_t, as a linear transform of a normal random vector, is again normal with mean 0 and variance

$$Var(X_t) = E(X_t X_t) = \min(t, t) - t \cdot t = t - t^2.$$

4.

(a) We have

$$\begin{aligned}
E\left[e^{uB_t}\right] &= \int_{\mathbb{R}} e^{ux} \cdot \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)} dx \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{ux - x^2/(2t)} dx \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-(2t)^{-1}(x-ut)^2 + 2^{-1}u^2 t} dx \\
&= \exp(\frac{1}{2}u^2 t) \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-(x-ut)^2/(2t)} dx.
\end{aligned}$$

Since $\frac{1}{\sqrt{2\pi t}}e^{-u^2/(2t)}$ is the density function of B_t, using the change of variables $y = x - ut$, we get

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-(x-ut)^2/(2t)} dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-y^2/(2t)} dy = 1.$$

So

$$E\left[e^{uB_t}\right] = \exp(\frac{1}{2}u^2 t), \quad \text{for all } u \in \mathbb{R}. \tag{3}$$

(b) Considering the power series expansion of the exponential function on both sides of (3), we have

$$E\left[e^{uB_t}\right] = E\left[\sum_{n=0}^{\infty} \frac{(uB_t)^n}{n!}\right] = \sum_{n=0}^{\infty} \frac{E\left[(B_t)^n\right]}{n!} \cdot u^n$$

and

$$\exp(\frac{1}{2}u^2 t) = \sum_{k=0}^{\infty} \frac{(u^2 t/2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(t/2)^k}{k!} u^{2k}.$$

So (3) implies

$$\sum_{n=0}^{\infty} \frac{E\left[(B_t)^n\right]}{n!} \cdot u^n = \sum_{k=0}^{\infty} \frac{(t/2)^k}{k!} u^{2k}, \quad \text{for all } u \in \mathbb{R}.$$

Comparing the coefficients of the polynomials on both sides, we get

$$\frac{E\left[(B_t)^{2n}\right]}{2n!} = \frac{(t/2)^n}{n!}$$

or

$$E\left[(B_t)^{2n}\right] = \frac{(t/2)^n}{n!} 2n! = \frac{(2n)!}{2^n \cdot n!} t^n.$$

(c) We prove the asseriton by induction. First, we have

$$E\left[(B_t)^2\right] = t,$$

which is equal to $\frac{(2)!}{2^1 \cdot 1!} t^1 = t$. So the assertion is true for $n = 1$. Suppose now that the assertion is true for n, i.e.,

$$E\left[(B_t)^{2n}\right] = \frac{(2n)!}{2^n \cdot n!} t^n. \tag{4}$$

Then by integration by parts,

$$\begin{aligned}
E\left[(B_t)^{2(n+1)}\right] &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} x^{2(n+1)}e^{-\frac{x^2}{2t}}\,dx \\
&= \frac{-t}{\sqrt{2\pi t}}\int_{\mathbb{R}} x^{2n+1}d\left(e^{-\frac{x^2}{2t}}\right) \\
&= \frac{-t}{\sqrt{2\pi t}}x^{2n+1}e^{-\frac{x^2}{2t}}\Big|_{-\infty}^{\infty} + \frac{(2n+1)t}{\sqrt{2\pi t}}\int_{\mathbb{R}} x^{2n}e^{-\frac{x^2}{2t}}\,dx \\
&= 0 + (2n+1)t\cdot E\left[(B_t)^{2n}\right] \\
&\overset{(4)}{=} (2n+1)t\cdot\frac{(2n)!}{2^n\cdot n!}t^n \\
&= \frac{(2n+2)!}{2^{n+1}\cdot(n+1)!}t^{n+1},
\end{aligned}$$

showing that the assertion is also true for $n+1$. By induction, the statement holds for any $n\in\mathbb{N}$.

5. Let $X := B_1, Y := B_2 - B_1, Z := B_3 - B_2$. Then X, Y, Z are mutually independent, $E(X) = E(Y) = E(Z) = 0$ and

$$B_2 = X + Y,\ B_3 = X + Y + Z.$$

So

$$\begin{aligned}
E(B_1^2B_2B_3) &= E\left[X^2(X+Y)(X+Y+Z)\right] \\
&= E\left[X^4 + X^3Y + X^3Z + X^3Y + X^2Y^2 + X^2YZ\right] \\
&= E\left[X^4\right] + E[X^3Y] + E[X^3Z] + E[X^3Y] + E[X^2Y^2] + E[X^2YZ] \\
&= E\left[X^4\right] + E[X^3]E[Y] + E[X^3]E[Z] \\
&\quad + E[X^3]E[Y] + E[X^2]E[Y^2] + E[X^2]E[Y]E[Z] \\
&= E\left[X^4\right] + E[X^2]E[Y^2] \\
&= 3 + 1\cdot 1 = 4,
\end{aligned}$$

where we have used $E\left[X^4\right] = E\left[B_1^4\right] = 3$ as shown in Problem 4.

6. $\left(B_t^2 - t\right)_{t\ge 0}$ is a martingale, since:

   (a) $B_t \sim N(0,t) \Rightarrow E\left(|B_t^2 - t|\right) \le E\left(B_t^2 + t\right) = t + t = 2t < \infty$.

   (b) Note $\mathcal{F}_t = \sigma\left(B_s, 0\le s\le t\right)$. So $B_t^2 - t$ is $\mathcal{F}_t$-measurable. Thus $\left(B_t^2 - t\right)_{t\ge 0}$ is $\mathcal{F}_t$-adapted.

(c) If $s < t$, then

$$\begin{aligned}
\mathbb{E}\left[B_t^2 - t \mid \mathcal{F}_s\right] &= \mathbb{E}\left[(B_t - B_s + B_s)^2 - t \mid \mathcal{F}_s\right] \\
&= \mathbb{E}\left[(B_t - B_s)^2 + 2(B_t - B_s)B_s + B_s^2 - t \mid \mathcal{F}_s\right] \\
&= \mathbb{E}\left[(B_t - B_s)^2 \mid \mathcal{F}_s\right] + \mathbb{E}\left[2(B_t - B_s)B_s \mid \mathcal{F}_s\right] + \mathbb{E}\left[B_s^2 \mid \mathcal{F}_s\right] - t \\
&= t - s + B_s\mathbb{E}\left[2(B_t - B_s) \mid \mathcal{F}_s\right] + B_s^2 - t \qquad [\textit{Taking out known}] \\
&= B_s^2 - s.
\end{aligned}$$

7. (a) Note that $B_t = B_s + B_t - B_s$. So $B_t^2 = B_s^2 + (B_t - B_s)^2 + 2B_s(B_t - B_s)$. Therefore, by taking out what is known and the role of independence, we have

$$E\left(B_t^2|\mathcal{F}_s\right) = B_s^2 + E\left(B_t - B_s\right)^2 + 2B_sE\left(B_t - B_s\right) = B_s^2 + t - s$$

(b)

$$\begin{aligned}
B_t^3 =&(B_s + B_t - B_s)^3 \\
=&3B_s^2\left(B_t - B_s\right) + B_s^3 + \left(B_t - B_s\right)^3 + 3B_s\left(B_t - B_s\right)^2
\end{aligned}$$

Hence,

$$E[B_t^3|\mathcal{F}_s] = B_s^3 + 3(t-s)B_s$$

(c)

$$\begin{aligned}
B_t^4 &= 3B_s^2\left(B_t - B_s\right)^2 + 3B_s^3\left(B_t - B_s\right) + B_s^4 + B_s^3\left(B_t - B_s\right) \\
&+ \left(B_t - B_s\right)^4 + B_s\left(B_t - B_s\right)^3 + 3B_s^2\left(B_t - B_s\right)^2 + 3B_s\left(B_t - B_s\right)^3 \\
&= 6B_s^2\left(B_t - B_s\right)^2 + 4B_s^3\left(B_t - B_s\right) + B_s^4 + \left(B_t - B_s\right)^4 + 4B_s\left(B_t - B_s\right)^3
\end{aligned}$$

$$\begin{aligned}
E[B_t^4|\mathcal{F}_s] &= 6B_s^2E\left(B_t - B_s\right)^2 + 4B_s^3E\left(B_t - B_s\right) + B_s^4 + E\left(B_t - B_s\right)^4 + 4B_sE\left(B_t - B_s\right)^3 \\
&= 6(t-s)B_s^2 + B_s^4 + 3(t-s)^2,
\end{aligned}$$

where we have used that if $\xi \sim N(0, \sigma^2)$, then $E[\xi^4] = 3\sigma^4$, thus $E\left(B_t - B_s\right)^4 = 3(t-s)^2$.

(d) Notice that $e^{4B_t - 2} = e^{-2}e^{4(B_t - B_s + B_s)} = e^{-2}e^{4B_s}e^{4(B_t - B_s)}$, then

$$E[e^{4B_t - 2}|\mathcal{F}_s] = e^{-2}e^{4B_s}E[e^{4(B_t - B_s)}] = e^{-2}e^{4B_s}e^{8(t-s)}$$

8.

(a) Yes. In fact, for any $n \in \mathbb{N}$ and

$$0 \leqslant t_1 < t_2 < \cdots < t_n,$$

we have

$$\begin{aligned}
(Y_{t_1}, \cdots, Y_{t_n})^\top &= \left(t_1 B_{1/t_1}, t_2 B_{1/t_2}, \cdots, t_n \cdot B_{1/t_n}\right)^\top \\
&= \begin{pmatrix} 0 & \cdots & \cdots & t_1 \\ 0 & \cdots & t_2 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ t_n & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} B_{1/t_n} \\ B_{1/t_{n-1}} \\ \vdots \\ B_{1/t_1} \end{pmatrix}.
\end{aligned}$$

(b) Since $\left(B_{1/t_n}, \cdots, B_{1/t_1}\right)^\top$ is multi-dimensional normal, $(Y_{t_1}, \cdots, Y_{t_n})^\top$ is also normal. So (Y_t) is a Gaussian process.

$$\begin{aligned}
Cov\,(Y_s, Y_t) &= Cov\left(sB_{1/s}, tB_{1/t}\right) \\
&= st \cdot Cov\left(B_{1/s}, B_{1/t}\right)
\end{aligned}$$

If $s \leq t$, $Cov\,(Y_s, Y_t) = s \cdot t \cdot \frac{1}{t} = s$, and if $s > t$ $Cov\,(Y_s, Y_t) = st \cdot \frac{1}{s} = t$. So

$$Cov(Y_s, Y_t) = \min(s, t)$$

(c) Yes. Similar to (a), we can easily verify that (Y_t) has independent increments. It remains to show that Y_t has normal increments. For $0 \leq s < t$, we have

$$\begin{aligned}
Y_t - Y_s &= t \cdot B_{1/t} - s \cdot B_{1/s} \\
&= t \cdot B_{1/t} - s \cdot \left(B_{1/s} - B_{1/t} + B_{1/t}\right) \\
&= t \cdot B_{1/t} - s \cdot \left(B_{1/s} - B_{1/t}\right) - s \cdot B_{1/t} \\
&= (t-s)B_{1/t} - s\left(B_{1/s} - B_{1/t}\right) \\
&= (t-s, -s) \cdot \begin{pmatrix} B_{1/t} \\ B_{1/s} - B_{1/t} \end{pmatrix}.
\end{aligned}$$

Since $\left(B_{1/t}, B_{1/s} - B_{1/t}\right)^\top \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{t} & 0 \\ 0 & \frac{1}{s} - \frac{1}{t} \end{pmatrix}\right)$, $Y_t - Y_s$ is normal with mean 0 and

$$\begin{aligned}
\mathrm{Var}\,(Y_t - Y_s) &= (t-s)^2 \cdot \frac{1}{t} + s^2 \cdot \left(\frac{1}{s} - \frac{1}{t}\right) \\
&= (t-s) \cdot \frac{(t-s)+s}{t} = t - s.
\end{aligned}$$

9. (a)

$$P\left(B_3 \geq 1/2\right) = P\left(\sqrt{3}B_1 \geq \frac{1}{2}\right) = P\left(B_1 \geq \sqrt{3}/6\right) = 1 - \Phi(\sqrt{3}/6) \approx 0.386$$

(b)

$$P\left(B_1 \leq 1/2, B_3 \geq B_1 + 2\right) = P(B_1 \leq \frac{1}{2}, B_3 - B_1 \geq 2) = \Phi(1/2)(1 - \Phi(\sqrt{2})) = 0.055$$

(c)

$$1 - P\big(\max_{0 \leq s \leq 10} B_s \geq 6\big) = 1 - 2P\left(B_{10} \geq 6\right) = 2\Phi(\sqrt{6}/10) - 1 \approx 0.196$$

(d) Note that

$$P\left(B_4 \leqslant 0 \mid B_2 \geqslant 0\right\} = \frac{P\left(B_2 \geqslant 0, B_4 \leq 0\right)}{P\left(B_2 \geqslant 0\right)}$$

and $P\left(B_2 \geqslant 0\right) = \frac{1}{2}$. So

$$\begin{aligned} P\left(B_2 \geqslant 0, B_4 \leq 0\right) &= \int_0^\infty P\left(B_4 \leqslant 0 \mid B_2 = x\right\} dP\left(B_2 = x\right) \\ &= \int_0^\infty P\left(B_4 - B_2 < -x\right\} \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}} dx \\ &= \int_0^{+\infty} \int_{-\infty}^{-x} \frac{1}{4\pi} e^{-\frac{x^2+y^2}{4}} dy dx \\ &= \int_0^{+\infty} \int_{-\pi/4}^{-\frac{\pi}{2}} \frac{1}{4\pi} e^{-\frac{r^2}{4}} r d\theta dr \\ &= 1/8 \end{aligned}$$

Therefore, $P\left(B_4 \leqslant 0 \mid B_2 \geqslant 0\right\} = 1/4$

10. Let $\mathcal{F}_t = \sigma\left(B_s : 0 \leqslant s \leqslant t\right)$. Then $M_t = e^{\sigma B_t - \frac{\sigma^2 t}{2}}$ can be written as $M_t = \phi\left(t, B_t\right)$. So M_t is adapted w.r.t. $\mathcal{F}_t$. Because $B_t \sim N(0, t)$,

$$E\left[|M_t|\right] = E\left[\left|e^{\sigma B_t - \frac{\sigma^2 t}{2}}\right|\right] = E\left[e^{\sigma B_t - \frac{\sigma^2 t}{2}}\right] = e^{-\frac{\sigma^2 t}{2} + \frac{\sigma^2 t}{2}} = e^0 = 1 < \infty.$$

Suppose $s > 0$, then

$$
\begin{aligned}
E\left[M_{t+s} \mid M_t\right] &= E\left[e^{\sigma B_{t+s}-\frac{\sigma^2(t+s)}{2}} \,\middle|\, \mathcal{F}_t\right] \\
&= E\left[e^{\sigma(B_{t+s}-B_t+B_t)-\frac{\sigma^2 t}{2}-\frac{\sigma^2 s}{2}} \,\middle|\, \mathcal{F}_t\right] \\
&= e^{B_t-\frac{\sigma^2 t}{2}} \cdot E\left[e^{\sigma(B_{t+s}-B_t)-\frac{\sigma^2 s}{2}} \,\middle|\, \mathcal{F}_t\right] \\
&= M_t \cdot E\left[e^{\sigma(B_{t+s}-B_t)-\frac{\sigma^2 s}{2}}\right] = M_t.
\end{aligned}
$$

Hence M_t is a martingale w.r.t. $\mathcal{F}_t$.

11. Consider the cumulative distribution function of M,

$$
\begin{aligned}
F_M(m) = P(M \leq m) = 1 - P(M > m) = 1 - P\left(\max_{0 \leq t \leq 1} B_t > m\right) \\
= 1 - 2P\left(B_1 > m\right) = 2\Phi(m) - 1.
\end{aligned}
$$

Note that $F_M(0) = 2\phi(0) - 1 = 0$, i.e. $P(M \leqslant 0) = 0$. So the probability density function of M is

$$
f_M(m) = F_M'(m) = \begin{cases} \sqrt{\frac{2}{\pi}} e^{-\frac{m^2}{2}}, m > 0; \\ 0, \text{ otherwise.} \end{cases}
$$

Then

$$
E(M) = \int_0^\infty m \cdot \sqrt{\frac{2}{\pi}} e^{-\frac{m^2}{2}} dm = -\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{m^2}{2}} d\left(-\frac{m^2}{2}\right) = -\left.\sqrt{\frac{2}{\pi}} e^{-\frac{m^2}{2}}\right|_0^\infty = \sqrt{\frac{2}{\pi}},
$$

$$
\begin{aligned}
E\left(M^2\right) &= \int_0^\infty m^2 \cdot \sqrt{\frac{2}{\pi}} e^{-\frac{m^2}{2}} dm = \sqrt{\frac{2}{\pi}} \int_0^\infty -m de^{-\frac{m^2}{2}} = \sqrt{\frac{2}{\pi}}\left[-\left. me^{-\frac{m^2}{2}}\right|_0^\infty + \int_0^\infty e^{-\frac{m^2}{2}} dm\right] \\
&= 0 + 2\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{m^2}{2}} dm = 2 \cdot 1/2 = 1,
\end{aligned}
$$

and thus $\mathrm{Var}(M) = E\left(M^2\right) - E^2(M) = 1 - \frac{2}{\pi}$.

12. Since (X_t) and (Y_t) are independent standard 1-dimensional Brownian motions, the probability density function of B_t is $f(x, y) = \frac{1}{2\pi t} e^{-\frac{x^2+y^2}{2t}}$. Then the probability

$$
\begin{aligned}
P\left(B_t \in D_\rho\right) &= \iint_{\sqrt{x^2+y^2}<\rho} f(x, y) dx dy = \iint_{\sqrt{x^2+y^2}<\rho} \frac{1}{2\pi t} e^{-\frac{x^2+y^2}{2t}} dx dy \\
&= \int_0^\rho \int_0^{2\pi} \frac{1}{2\pi t} e^{-\frac{r^2}{2t}} \cdot r dr d\theta = -\left. e^{-\frac{r^2}{2t}}\right|_0^\rho \\
&= 1 - e^{-\frac{\rho^2}{2t}}.
\end{aligned}
$$