PT

Solution to Assignment 12

1. First, observe that

$$\operatorname{Var}[(X+Y)/2] = (0.5)^2 \operatorname{Var}(X+Y) = 0.25[\operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X,Y)]$$

Then,

$$E(X) = \int_0^2 \int_0^2 x\frac{x+y}{8}dxdy = \frac{1}{8}\int_0^2 \frac{8}{3} + 2ydy = \frac{1}{8}\left(\frac{16}{3}+4\right) = \frac{2}{3}+\frac{1}{2} = \frac{7}{6}$$
$$E\left(X^2\right) = \int_0^2 \int_0^2 x^2\frac{x+y}{8}dxdy = \frac{1}{8}\int_0^2 4 + \frac{8}{3}ydy = \frac{1}{8}\left(8+\frac{16}{3}\right) = 1+\frac{4}{6} = \frac{10}{6}$$
$$\operatorname{Var}(X) = 10/6 - (7/6)^2 = 11/36$$

By symmetry, the mean and the variance of Y are the same. Next,

$$E(XY) = \int_0^2 \int_0^2 xy\frac{x+y}{8}dxdy = \frac{1}{8}\int_0^2 \frac{8}{3}y + 2y^2dy = \frac{1}{8}\left(\frac{16}{3}+\frac{16}{3}\right) = \frac{8}{6}$$
$$\operatorname{Cov}(X,Y) = 8/6 - (7/6)(7/6) = -1/36$$

Finally,

$$\operatorname{Var}(X+Y) = 0.25[11/36 + 11/36 + 2(-1/36)] = 5/36 = 10/72$$

2. We have $(x,y) = (wz, z)$ so that

$$\frac{\partial(x,y)}{\partial(w,z)} = \det\begin{bmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial z} \end{bmatrix} = \det\begin{bmatrix} z & w \\ 0 & 1 \end{bmatrix} = z.$$

Therefore for $0 < wz < z < 1$, or $0 < w < 1$ and $0 < z < 1$,

$$f_{W,z}(w,z) = f_{X,Y}(x(w,z), y(w,z)) \cdot \left|\frac{\partial(x,y)}{\partial(w,z)}\right| = 8wz \cdot z \cdot z = 8wz^3,$$

and $f_{w,z}(w,z) = 0$ otherwise.

3. Here (X, Y) are jointly continuous and are related to (R, Θ) by a one-to-one relationship. We use the method of transformations (Theorem 5.1). The function $h(r, \theta)$ is given by

$$\begin{cases} x = h_1(r,\theta) = r\cos\theta \\ y = h_2(r,\theta) = r\sin\theta \end{cases}$$

Thus, we have

$$\begin{aligned} f_{R\Theta}(r,\theta) &= f_{XY}\left(h_1(r,\theta), h_2(r,\theta)\right)|J| \\ &= f_{XY}(r\cos\theta, r\sin\theta)|J| \end{aligned}$$

where

$$J = \det\begin{bmatrix} \frac{\partial h_1}{\partial r} & \frac{\partial h_1}{\partial \theta} \\ \frac{\partial h_2}{\partial r} & \frac{\partial h_2}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

We conclude that

$$\begin{aligned} f_{R\Theta}(r,\theta) &= f_{XY}(r\cos\theta, r\sin\theta)|J| \\ &= \begin{cases} \frac{r}{\pi} & r \in [0,1], \theta \in (-\pi, \pi] \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Note that from above we can write

$$f_{R\Theta}(r,\theta) = f_R(r)f_\Theta(\theta)$$

4.

(a) From m and c we have $X_2 \sim N(1,2)$. Thus

$$\begin{aligned} P(0 \le X_2 \le 1) &= \Phi\left(\frac{1-1}{\sqrt{2}}\right) - \Phi\left(\frac{0-1}{\sqrt{2}}\right) \\ &= \Phi(0) - \Phi\left(\frac{-1}{\sqrt{2}}\right) = 0.2602 \end{aligned}$$

(b)

$$\begin{aligned} m_Y = EY &= AEX + b \\ &= \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}. \end{aligned}$$

(c)

$$\begin{aligned}
C_Y &= AC_XA^T \\
&= \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.
\end{aligned}$$

(d) From m_Y and c_Y we have $Y_3 \sim N(3,1)$, thus

$$P(Y_3 \leq 4) = \Phi\left(\frac{4-3}{1}\right) = \Phi(1) = 0.8413$$

5. Note that

$$\begin{aligned}
\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} &= \begin{pmatrix} X_1 + X_2 + X_3 \\ X_1 - X_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = A \cdot \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}.
\end{aligned}$$

This implies that

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left(A\mu, A\Sigma A^\top\right)$$

Where

$$A\mu = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

and

$$A\Sigma A^\top = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}$$

So

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}\right).$$

6. We know that if

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right).$$

then

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)} \cdot \left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2^2)}{\sigma_2^2}\right]\right\}.$$

Comparing with the given form of $f(x_1, x_2)$, especially the coefficients in front of x_1^2, x_2^2 and $x_1 x_2$, we get

$$\frac{1}{(1-\rho^2)\,\sigma_1{}^2} = 1 \tag{1}$$

$$\frac{-2\rho}{(1-\rho^2)\,\sigma_1\sigma_2} = -1 \tag{2}$$

$$\frac{1}{(1-\rho^2)\,\sigma_2{}^2} = 2 \tag{3}$$

$(1) \times (3) \div \left((2)^2\right) \Rightarrow \frac{1}{4\rho^2} = 2 \Rightarrow \rho^2 = \frac{1}{8}$. By (2), we know $\rho > 0$. So $\rho = \frac{\sqrt{2}}{4}$. By (1) and (3), we get $\sigma_1^2 = \frac{8}{7}$ and $\sigma_2^2 = \frac{4}{7}$. So

$$\mathrm{Cov}(\boldsymbol{X}) = \begin{pmatrix} \frac{8}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{4}{7} \end{pmatrix}.$$

Set $\Sigma = \mathrm{Cov}(\boldsymbol{X})$. Then

$$\Sigma^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{pmatrix}.$$

So

$$\begin{aligned}(\boldsymbol{x}-\boldsymbol{\mu})^\top \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu}) &= (x_1-\mu_1, x_2-\mu_2) \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{pmatrix} \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}. \\ &= (x_1-\mu_1)^2 + 2\,(x_2-\mu_2)^2 - (x_1-\mu_1)\,(x_2-\mu_2)\,. \end{aligned}$$

But

$$(\boldsymbol{x}-\boldsymbol{\mu})^\top \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu}) = x_1^2 + 2x_2^2 - x_1 x_2 - 3x_1 - 2x_2 + 4.$$

So, comparing the Coefficients in front of x_1 and x_2,

$$\Rightarrow \begin{cases} -2\mu_1 + \mu_2 = -3 \\ -4\mu_2 + \mu_1 = -2 \end{cases} \Rightarrow \begin{cases} \mu_1 = 2 \\ \mu_2 = 1 \end{cases}$$

Therefore,

$$E(\boldsymbol{X}) = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\text{and } \mathrm{Cov}(\boldsymbol{X}) = \begin{pmatrix} \frac{8}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{4}{7} \end{pmatrix}.$$

7. If we define $U = X + Y$ and $V = 2X - Y$, then note that U and V are jointly normal. We have

$$E[U] = 3, \mathrm{Var}[U] = 7, E[V] = 3, \mathrm{Var}[V] = 37$$

and

$$\begin{aligned}\mathrm{Cov}(U,V) = \mathrm{Cov}(X+Y, 2X-Y) &= 2\,\mathrm{Cov}(X,X) - \mathrm{Cov}(X,Y) + 2\,\mathrm{Cov}(Y,X) - \mathrm{Cov}(Y,Y) \\ &= 2\,\mathrm{Var}(X) + \mathrm{Cov}(X,Y) - \mathrm{Var}(Y) = 8 - 3 - 9 = -4.\end{aligned}$$

Thus,

$$\rho(U,V)=\frac{\operatorname{Cov}(U,V)}{\sigma_U\sigma_V}=\frac{-4}{\sqrt{7\times 37}}$$

We conclude that given $V=0, U$ is normally distributed with

$$E[U\mid V=0]=\mu_U+\rho(U,V)\sigma_U\frac{0-\mu_V}{\sigma_V}=3+\frac{4}{\sqrt{7\times 37}}\sqrt{7}\frac{3}{\sqrt{37}}=3.324$$

$$\operatorname{Var}[U\mid V=0]=\left(1-\rho(U,V)^2\right)\sigma_U^2=\left(1-\frac{16}{259}\right)\times 7=6.568$$

Thus

$$P(X+Y>0\mid 2X-Y=0)=P(U>0\mid V=0)=1-\Phi(-1.3)=0.9032$$

8. Diagonalize $\boldsymbol{C}_X$, we have

$$\boldsymbol{C}_X=\begin{pmatrix}\frac{54}{49} & -\frac{6}{49} & \frac{24}{49}\\ -\frac{6}{49} & \frac{17}{49} & \frac{30}{49}\\ \frac{24}{49} & \frac{30}{49} & \frac{76}{49}\end{pmatrix}=\begin{pmatrix}\frac{3}{7} & -\frac{6}{7} & \frac{2}{7}\\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7}\\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7}\end{pmatrix}\begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\frac{3}{7} & -\frac{6}{7} & \frac{2}{7}\\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7}\\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7}\end{pmatrix}^T$$

Let $\begin{pmatrix}Y_1\\ Y_2\\ Y_3\end{pmatrix}=\begin{pmatrix}\frac{3}{7} & -\frac{6}{7} & \frac{2}{7}\\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7}\\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7}\end{pmatrix}^T\begin{pmatrix}X_1\\ X_2\\ X_3\end{pmatrix}=\begin{pmatrix}\frac{3}{7}X_1+\frac{2}{7}X_2+\frac{6}{7}X_3\\ -\frac{6}{7}X_1+\frac{3}{7}X_2+\frac{2}{7}X_3\\ \frac{2}{7}X_1+\frac{6}{7}X_2-\frac{3}{7}X_3\end{pmatrix}$. Then $\boldsymbol{C}_Y=\begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$. We obtain

$$0=\operatorname{Var}\left(Y_3\right)=\operatorname{Var}\left(\frac{2}{7}X_1+\frac{6}{7}X_2-\frac{3}{7}X_3\right)$$

Hence $\frac{2}{7}X_1+\frac{6}{7}X_2-\frac{3}{7}X_3=c$ Since $E[\boldsymbol{X}]=(1,2,3)^T$, that is $E\left[X_1\right]=1, E\left[X_2\right]=2, E\left[X_3\right]=0$, then $c=\frac{2}{7}E\left[X_1\right]+\frac{6}{7}E\left[X_2\right]-\frac{3}{7}E\left[X_3\right]=2$ Therefore

$$\frac{2}{7}X_1+\frac{6}{7}X_2-\frac{3}{7}X_3=2 \quad \text{ and } \quad X_3=\frac{2}{3}X_1+2X_2-\frac{14}{3}$$

By Theorem 4.49, $\operatorname{Cov}\left(Y_1, Y_2\right)=0$ implies that Y_1, Y_2 are independent.

$$Y_1=\frac{3}{7}X_1+\frac{2}{7}X_2+\frac{6}{7}X_3=\frac{3}{7}X_1+\frac{2}{7}X_2+\frac{6}{7}\left(\frac{2}{3}X_1+2X_2-\frac{14}{3}\right)=X_1+2X_2-4$$

$$Y_2=-\frac{6}{7}X_1+\frac{3}{7}X_2+\frac{2}{7}X_3=-\frac{6}{7}X_1+\frac{3}{7}X_2+\frac{2}{7}\left(\frac{2}{3}X_1+2X_2-\frac{14}{3}\right)=-\frac{2}{3}X_1+X_2-\frac{4}{3}$$

Hence X_1+2X_2-4 and $-\frac{2}{3}X_1+X_2-\frac{4}{3}$ are independent.