Chapter Eight

Index Models

Motivation

- Drawbacks to Markowitz procedure
  - Requires a huge number of estimates to fill the covariance matrix
  - Model does not provide any guidelines for finding useful estimates of these covariances or the risk premiums

Motivation

- Suppose your security analysts can thoroughly analyze 50 stocks. Your input list will include:

$$
\begin{aligned}
n &= \quad\;\; 50 \text{ estimates of expected returns} \\
n &= \quad\;\; 50 \text{ estimates of variances} \\
(n^2 - n)/2 &= \underline{1{,}225} \text{ estimates of covariances} \\
& \quad\; 1{,}325 \text{ total estimates}
\end{aligned}
$$

- Doubling n to 100, number of estimates = 5,150.
- If $n = 3000$, still less than the number of issues included in the Wilshire 5000 index, we need more than 4.5 million estimates.

Motivation

- Introduction of index models
  - Simplifies estimation of the covariance matrix
  - Enhances analysis of security risk premiums

Motivation

- We focus on risk by separating the actual rate of return on any security, i, into the sum of its previously expected value plus an unanticipated surprise:
- $r_i = E(r_i) + unanticipated\ surprise$
  - Due to unexpected developments in issues that are particular to the firm
  - Or to unexpected changes in conditions that affect the broad economy.

A Single-Factor Security Market

$$r_i = E(r_i) + \beta_i m + e_i$$

- *m* = market factor that measures unanticipated developments in the macroeconomy
- For example, it might be the difference between GDP growth and the market's previous expectation of that growth
- As such, it has a mean of zero (over time, surprises will average out to zero), with standard deviation σ_m.

A Single-Factor Security Market

$$r_i = E(r_i) + \beta_i m + e_i$$

- e_i = firm-specific random variable
- Measures only the firm-specific surprise.
- As a surprise, it too has zero expected value.
- Since firm-specific, e_i and e_j are assumed to be mutually uncorrelated.
- Notice m has no subscript because the same common factor affects all securities.
- m and e_i are assumed to be uncorrelated.

A Single-Factor Security Market

$$r_i = E(r_i) + \beta_i m + e_i$$

- β_i = sensitivity coefficient for firm I
- We recognize that some securities will be more sensitive than others to macroeconomic shocks.

A Single-Factor Security Market

$$r_i = E(r_i) + \beta_i m + e_i$$

- $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma^2(e_i)$
- $Cov(r_i, r_j) = \beta_i \beta_j \sigma_m^2$

Single-Index Model

- Because the systematic factor affects the rate of return on all stocks, the rate of return on a broad market index can plausibly proxy for that common factor.
- This approach leads to an equation similar to the single-factor model, which is called a **single-index model** because it uses the market index to stand in for the common factor.

Single-Index Model
(1 of 3)

- **Regression equation**

$$R_i(t) = \alpha_i + \beta_i R_M(t) + e_i(t)$$

- Expected return-beta relationship

$$E(R_i) = \alpha_i + \beta_i E(R_M)$$

Single-Index Model
(2 of 3)

$$E(R_i) = \alpha_i + \beta_i E(R_M)$$

- Message 1: part of a security's risk premium is due to the risk premium of the index.
  - Beta measures the relative sensitivity of the individual security.
  - Securities with high betas have a magnified sensitivity to market risk and will therefore enjoy a greater risk premium as compensation for this risk.
  - We call this the *systematic* risk premium.

Single-Index Model
(3 of 3)

$$E(R_i) = \alpha_i + \beta_i E(R_M)$$

- Message 2: the remainder of the risk premium is given by alpha, a *nonmarket* premium.
  - Alpha may be large if you think a security is underpriced and therefore offers an attractive expected return.
  - For now, let's assume that each security analyst comes up with his or her own estimates of alpha.
  - If managers believe that they can do a superior job of security analysis, then they will be confident in their ability to find stocks with nonzero values of alphas.

Risk and Covariance in the Single-Index Model

- Remember that one of the problems with the Markowitz model is the large number of parameter estimates required to implement it.
- We will see that the index model vastly reduces the number of parameters that must be estimated.

Risk and Covariance in the Single-Index Model

- Total risk = Systematic risk + Firm-specific risk

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma^2(e_i)$$

- Covariance = Product of betas × Market-index risk

$$\text{Cov}(r_i, r_j) = \beta_i \beta_j \sigma_M^2$$

- Correlation = Product of correlations with the market index

$$Corr(r_i, r_j) = \frac{\beta_i \beta_j \sigma_M^2}{\sigma_i \sigma_j} = \frac{\beta_i \sigma_M^2}{\sigma_i \sigma_M} \times \frac{\beta_j \sigma_M^2}{\sigma_j \sigma_M}$$

$$= Corr(r_i, r_M) \times Corr(r_i, r_M)$$

The Set of Estimates Needed for the Single-Index Model

- n estimates of the extra-market expected excess returns, α_i
- n estimates of the sensitivity coefficients, β_i
- n estimates of the firm-specific variances, $\sigma^2(e_i)$
- 1 estimate for the market risk premium, $E(R_M)$
- 1 estimate for the variance of the (common) macroeconomic factor, σ_M^2

For a 50-security portfolio, $(3n + 2) = 152$.

Index Model and Diversification

- Suppose that we choose an equally weighted portfolio of n securities, and the excess rate of return on each security is given by

$$R_i = \alpha_i + \beta_i R_M + e_i$$

- Then the excess return on the portfolio would be

$$R_p = \frac{1}{n}\sum_{i=1}^{n}\alpha_i + (\frac{1}{n}\sum_{i=1}^{n}\beta_i)R_M + \frac{1}{n}\sum_{i=1}^{n}e_i$$

- Thus, $\beta_p = \frac{1}{n}\sum_{i=1}^{n}\beta_i$, $e_p = \frac{1}{n}\sum_{i=1}^{n}e_i$.

Index Model and Diversification

$$R_p = \frac{1}{n}\sum_{i=1}^{n}\alpha_i + (\frac{1}{n}\sum_{i=1}^{n}\beta_i)R_M + \frac{1}{n}\sum_{i=1}^{n}e_i$$

- Thus, $\beta_p = \frac{1}{n}\sum_{i=1}^{n}\beta_i$, $e_p = \frac{1}{n}\sum_{i=1}^{n}e_i$.
- $\sigma_p^2 = \beta_p^2\sigma_M^2 + \sigma^2(e_p)$.
- The systematic component of the portfolio variance, $\beta_p^2\sigma_M^2$, depends on the average beta and σ_M^2 and will persist regardless of the extent of portfolio diversification.

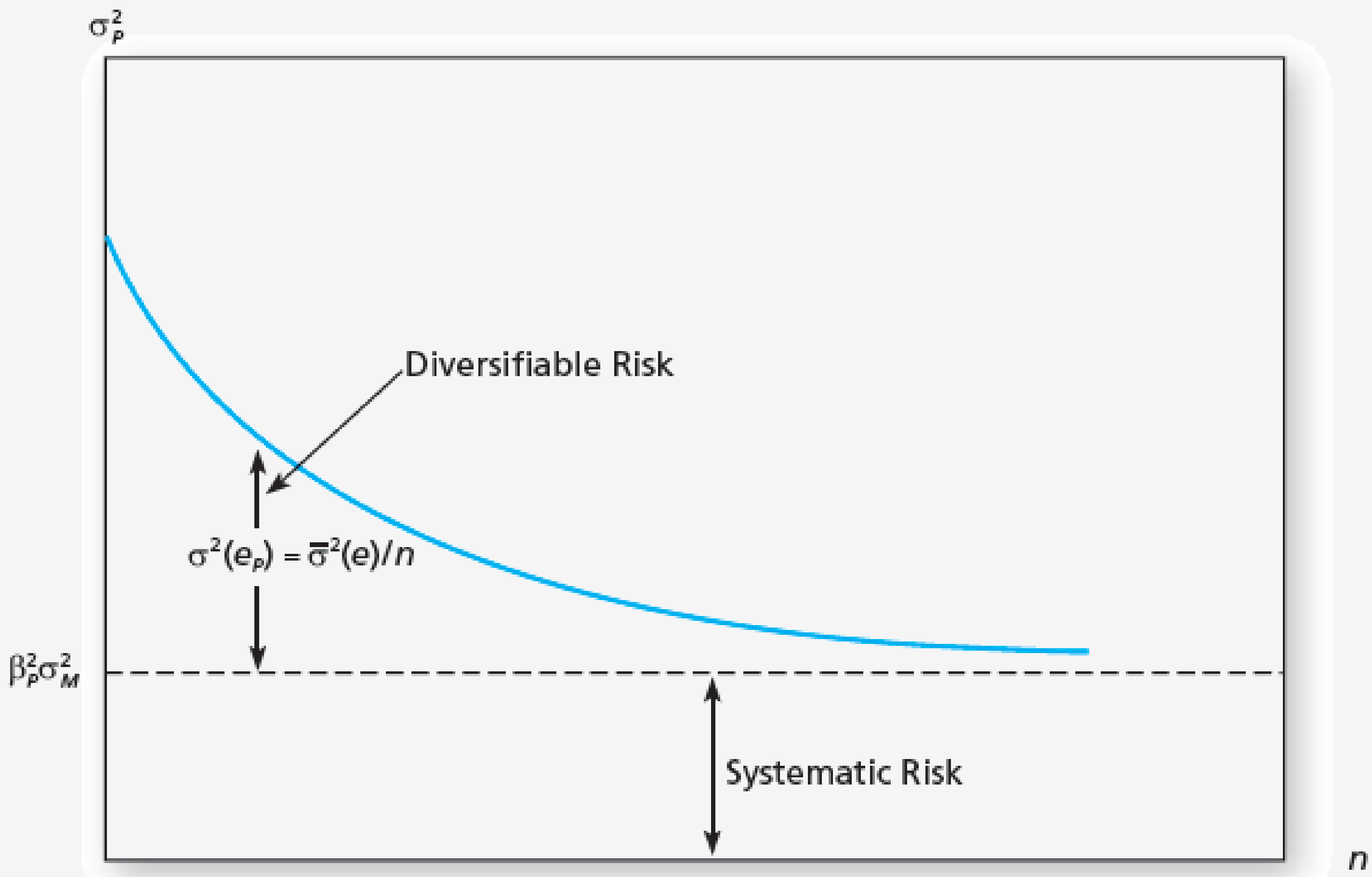
Index Model and Diversification

- Variance of the equally-weighted portfolio of firm-specific components:

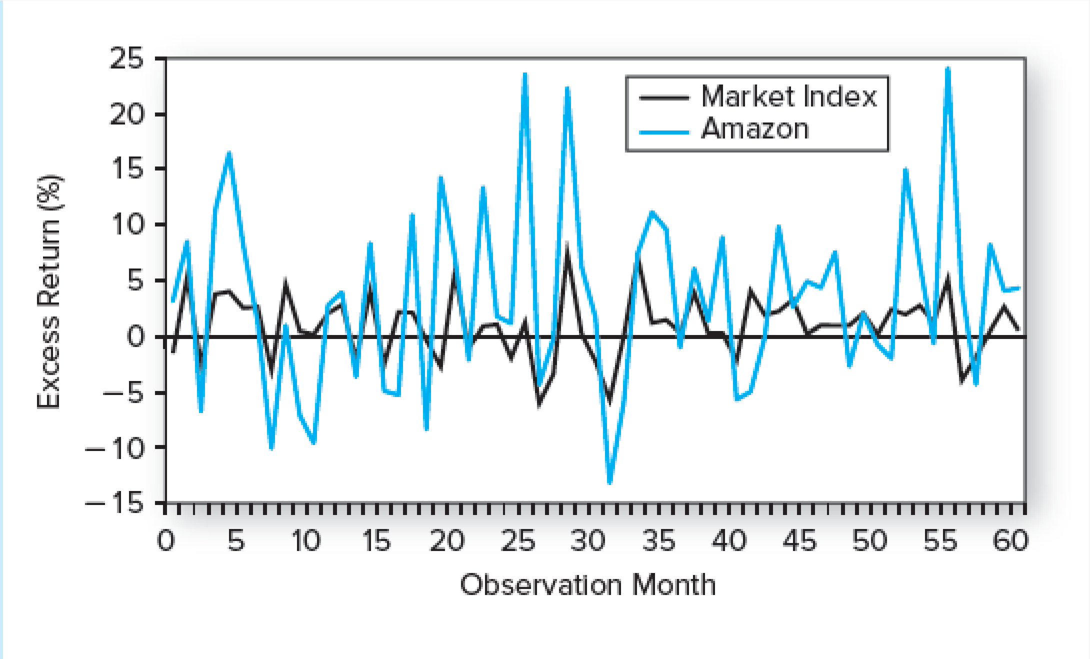
$$\sigma^2(e_P) = Var\left(\frac{1}{n}\sum_{i=1}^{n} e_i\right) = \sum_{i=1}^{n}\left(\frac{1}{n}\right)^2 \sigma^2(e_i) = \frac{1}{n}\sum_{i=1}^{n}\frac{\sigma^2(e_i)}{n} = \frac{1}{n}\overline{\sigma}^2(e)$$

- When n gets large, $\sigma^2(e_p)$ becomes negligible
- As diversification increases, the total variance of a portfolio approaches the systematic variance

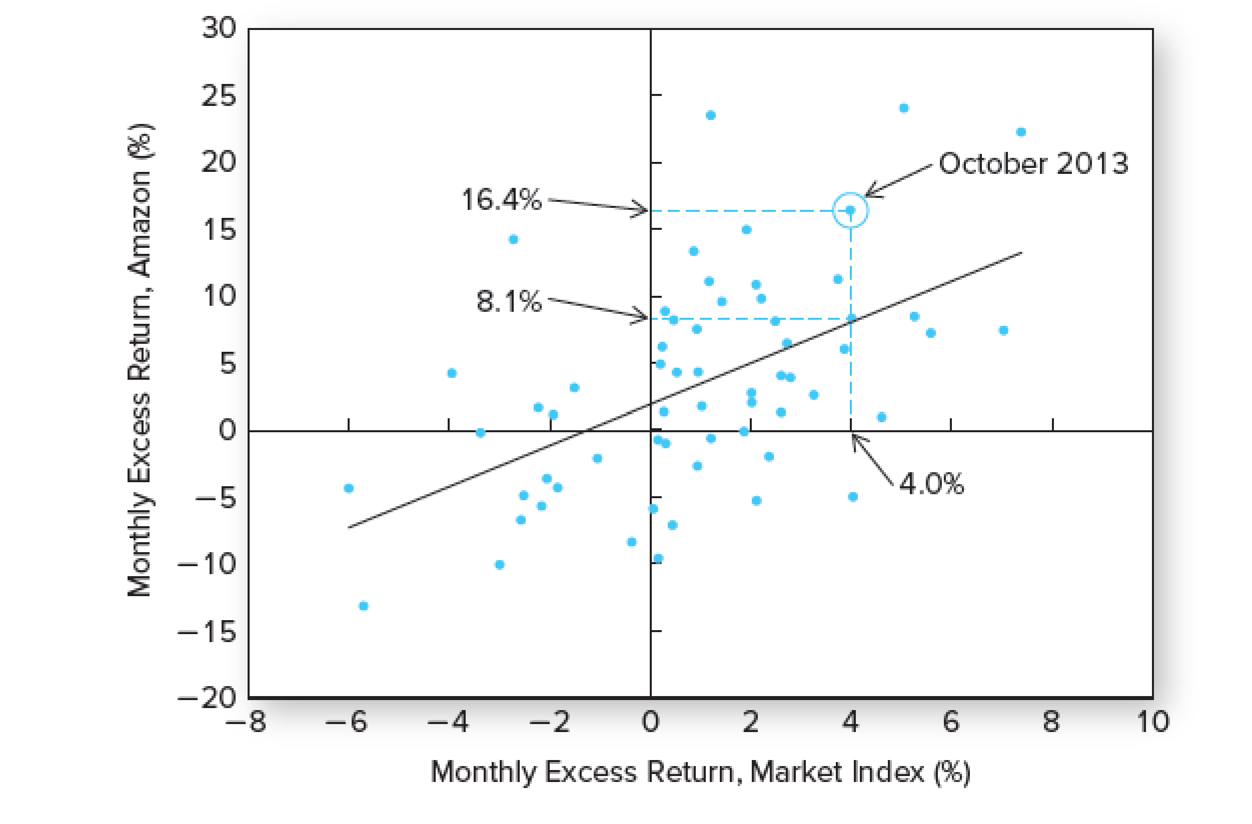
The Variance of an Equally Weighted Portfolio with Risk Coefficient, β_p



Excess Monthly Returns on Amazon and the Market Index



Scatter Diagram



Security Characteristic Line (SCL)

Excess return of security *i*

$$R_i(t) = \alpha_i + \beta_i R_{S\&P500}(t) + e_i(t)$$

Expected excess return when the market excess return is zero

Sensitivity of security *i*'s return to changes in the return of the market

Expected excess return of the market

Zero-mean, firm-specific surprise in security *i*'s return in month *t*. (the *residual*)

Summary

- Single-index model

$$R_i(t) = \alpha_i + \beta_i R_M(t) + e_i(t)$$

- Total risk = Systematic risk + Firm-specific risk $\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma^2(e_i)$
- Covariance = Product of betas × Market-index risk $\mathrm{Cov}(r_i, r_j) = \beta_i \beta_j \sigma_M^2$
- Correlation = Product of correlations with the market index

$$\mathrm{Corr}(r_i, r_j) = \frac{\beta_i \beta_j \sigma_M^2}{\sigma_i \sigma_j} = \frac{\beta_i \sigma_M^2}{\sigma_i \sigma_M} \times \frac{\beta_j \sigma_M^2}{\sigma_j \sigma_M}$$

$$= \mathrm{Corr}(r_i, r_M) \times \mathrm{Corr}(r_i, r_M)$$

Summary

- The index model has been estimated for stocks A and B with the following results:
  $R_A = 0.03 + 0.7R_M + e_A$.
  $R_B = 0.01 + 0.9R_M + e_B$.
  $\sigma_M = 0.35; \sigma(e_A) = 0.20; \sigma(e_B) = 0.10$.

  The covariance between the returns on stocks A and B is

  A) 0.0384.
  B) 0.0406.
  C) 0.1920.
  D) 0.0772.
  E) 0.4000.