2021-22 First Semester
MATH1083 Calculus II (1003)

Assignment 1

Due Date: 11:30am 28/Feb/2021(Tue).

- Write down your **Chinese name** and **student number**. Write neatly on **A4-sized** paper and **show your steps**.
- **Late submissions or answers without details will not be graded.**

1. Using $\epsilon - \delta$ **definition** to prove that the sequence $\{a_n\}$

$$a_n = \frac{1}{e^n}$$

converges.

Proof: We can first look for the limit

$$\lim_{n\to\infty} \frac{1}{e^n} = 0.$$

For every $\epsilon > 0$, there exist a positive integer $N = \left[\ln \frac{1}{\epsilon}\right] + 1$, such that if $n \geq N$,

$$|a_n| < \epsilon$$

.

[Here we need to solve this inequality:

$$\begin{aligned} e^{-n} &< \epsilon \\ -n &< \ln \epsilon \\ n &> \ln \frac{1}{\epsilon} \end{aligned}$$

]

2. If $\sum a_n$ is convergent and $\sum b_n$ is divergent, show that the series $\sum (b_n - a_n)$ is divergent.
[Hint: proof by **contradiction**]

Answer: Let us assume that the series $\sum (b_n - a_n)$ is convergent.

Since $\sum a_n$ is convergent, so the sum of these two series

$$\sum (b_n - a_n) + \sum a_n = \sum b_n$$

also converges, which contradicts the condition that $\sum b_n$ is divergent.

3. Prove sequence

$$a_n = \frac{2^n n!}{(2n+1)!}$$

is convergent by **squeeze theorem**.

Proof: First, it is an positive sequence, so $a_n > 0$.

$$\begin{aligned} \frac{2^n n!}{(2n+1)!} &= \frac{2^n}{(n+1)(n+2)\cdots(2n+1)} \\ &= \frac{2}{n+1} \cdot \frac{2}{n+2} \cdots \frac{2}{2n} \cdot \frac{2}{2n+1} \\ &< \frac{2}{n+1} \end{aligned}$$

so we have

$$0 < a_n < \frac{2}{n+1}$$

and since

$$\lim_{n\to\infty}\frac{2}{n+1}=0$$

,using the squeeze theorem, we have

$$\lim_{n\to\infty}a_n=0$$

4. Determine whether each imporper integrals is convergent or not, and find the limit if it is convergent.

   (a)
$$\int_1^\infty\frac{1}{\sqrt{x}}dx$$

(a) $\int_1^\infty x^{-\frac{1}{2}}dx=\lim_{n\to\infty}\left[2x^{\frac{1}{2}}\right]_1^n=\infty$ divergent

   (b)
$$\int_1^\infty\frac{1}{a^x}dx,\qquad a>1$$

   (c)
$$\int_1^\infty\frac{1}{a^x}dx,\qquad a<1$$

(c) $\int_1^\infty\frac{1}{a^x}dx=\lim_{n\to\infty}\left[\frac{1}{\ln\frac{1}{a}}\frac{1}{a^x}\right]_1^n$

$=\lim_{n\to\infty}\left[-\frac{1}{\ln a}\left(\frac{1}{a^n}-\frac{1}{a}\right)\right]$ $a<1$

$=\infty$ divergent

   (d)
$$\int_2^\infty\frac{1}{(x-1)(x+2)}dx$$

Solution:

(a,c) divergent.

(b,d) convergent.

$$\begin{aligned}
&\int_1^\infty\frac{1}{a^x}dx\\
=&\lim_{n\to\infty}\left[\frac{1}{\ln(1/a)}\frac{1}{a^x}\right]_2^n\\
=&\lim_{n\to\infty}\left[-\frac{1}{\ln(a)}\left(\frac{1}{a^n}-\frac{1}{a^2}\right)\right]_2^n\\
=&\frac{1}{\ln(a)\,a^2}
\end{aligned}$$

$$\begin{aligned}
&\int_2^\infty\frac{1}{(x-1)(x+2)}dx\\
=&\int_2^\infty\frac{1}{3}\left(\frac{1}{x-1}-\frac{1}{x+2}\right)dx\\
=&\lim_{n\to\infty}\left[\frac{1}{3}\ln\left(\frac{x-1}{x+2}\right)\right]_2^n\\
=&\frac{\ln 4}{3}
\end{aligned}$$

5. Use **Integral Test** to determine whether the series is convergent or not.

$$\sum_{n=2}^{\infty}\frac{\tan^{-1}n}{1+n^2}$$

$\sum_{n=2}^{\infty}\frac{\tan^{-1}n}{1+n^2}$ $\int_2^\infty\frac{\tan^{-1}x}{1+x^2}dx$

Solution: Let

$$f(x)=\frac{\tan^{-1}x}{1+x^2}$$

then $f(x)$ is continous on $[1,\infty)$, positive and decreasing. The improper integral

$$
\begin{aligned}
&\int_1^\infty \frac{\tan^{-1} x}{1+x^2}dx \\
&= \int_1^\infty \tan^{-1} x d(\tan^{-1} x) \\
&= \frac{1}{2}\left[(\tan^{-1} x)^2\right]_1^\infty \\
&= \frac{1}{2}\left(\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right) = \frac{3\pi^2}{32}
\end{aligned}
$$

is convergent, therefore by Integral Test, the series is convergent.

6. For the series

$$s = \sum_{n=1}^\infty \frac{1}{n^4}$$

(a) Estimate the error if we use s_{10} as an approximation to s.
(b) Find a value of n, so that s_n is within 9×10^{-9} of the sum.

Solution:
(a)

$$R_{10} \le \int_{10}^\infty \frac{1}{x^4} dx \; \frac{1}{3 \cdot (10)^3}$$

(b)

$$R_n \le \int_n^\infty \frac{1}{x^4} dx \; \frac{1}{3 \cdot (n)^3} < 9 \times 10^{-9}$$

so we have

$$n^3 > \frac{1}{27} \times 10^9$$
$$n > \frac{1000}{3}$$

$n =$ 334

7. For the alternating series

$$s = \sum_{n=1}^\infty \frac{\cos n\pi}{\sqrt{n}}$$

(a) Determine whether the series absolutely convergent, conditionally convergent or divergent.
(b) Is the 100-th partial sum s_{100} an overestimate or underestimate? and explain why.

(a) $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ $p=\frac{1}{2}<1$

$\therefore$ diverge

suppose $a_n = \frac{1}{\sqrt{n}}$

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = 1$

$\lim_{n\to\infty} a_n = 0$

$\therefore \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converge

$\therefore \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ conditionally converge

Solution:
(a)

$$s = \sum_{n=1}^\infty \frac{\cos n\pi}{\sqrt{n}} = \sum_{n=1}^\infty \frac{(-1)^n}{\sqrt{n}}$$

it is conditionally convergent as $p = 1/2$.
(b) Let

$$s = \sum_{n=1}^\infty (-1)^n b_n = -b_1 + b_2 - b_3 + b_4 - \cdots$$

where $b_n = \frac{1}{\sqrt{n}} > 0$
Since $b_{101} = \frac{1}{\sqrt{101}}$ and $b_{102} = \frac{1}{\sqrt{102}}$, so $b_{101} - b_{102} > 0$ and $b_{2n-1} - b_{2n} > 0$ for all n. Then we have

$$
\begin{aligned}
s_{100} = \sum_{n=1}^{100} \frac{\cos n\pi}{\sqrt{n}} &= s + b_{101} - b_{102} + b_{103} - b_{104} \ldots \\
&= s + \left(\frac{1}{\sqrt{101}} - \frac{1}{\sqrt{102}}\right) + \left(\frac{1}{\sqrt{102}} - \frac{1}{\sqrt{103}}\right) + \\
&\ge s
\end{aligned}
$$

therefore s_{100} is an overestimate of s.

8. Use the **Ratio Test** to determine whether the series

$$1 - \frac{2!}{1 \cdot 3} + \frac{3!}{1 \cdot 3 \cdot 5} - \frac{4!}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots + (-1)^{n+1} \frac{n!}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)}$$

is convergent or divergent.

Solution:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)(2n+1)}}{\frac{n!}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)}} = \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1$$

So this series is convergent by Ratio Test.

9. Use the **Root Test** to determine whether the series

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$$

is convergent or divergent.

Solution: Since

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(1 + \frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n} = e > 1$$

So this series is divergent by Root Test.