2023-24 First Semester

MATH2023 Ordinary and Partial Differential Equations (1002)

Assignment 6 Suggested Solutions

1. (a)

$$r^3 - 2r^2 + r = r(r^2 - 2r + 1) = 0 \to \quad r_1 = 0, \quad r_{2,3} = 1$$

$$Y_H = C_1 + C_2 e^t + C_3 t e^t, \qquad C_i \in \mathbb{R}.$$

$$Y_P = t\ (A_3 t^3 + A_2 t^2 + A_1 t + A_0) + t^2 B e^t$$

(b)

$$r^4 + 4r^2 = r^2(r^2 + 4) = 0 \to \quad r_{1,2} = 0, \quad r_{3,4} = \pm 2i$$

$$Y_H = C_1 + C_2 t + C_3 \cos 2t + C_4 \sin 2t, \qquad C_i \in \mathbb{R}.$$

$$Y_P = (A_1 t + A_0)e^t + t\ (B_1 \cos 2t + B_2 \sin 2t) + t^2 (D_1 t + D_0)$$

(c)

$$r^4 + 2r^3 + 2r^2 = r^2(r^2 + 2r + 2) = 0 \to \quad r_{1,2} = 0, \quad r_{3,4} = -1 \pm i$$

$$Y_H = C_1 + C_2 t + C_3 e^{-t} \cos t + C_4 e^{-t} \sin t, \qquad C_i \in \mathbb{R}.$$

$$Y_P = Ae^t + (B_1 t + B_0)e^{-t} + te^{-t}(D_1 \cos t + D_2 \sin t)$$

(d)

$$r^3 - 3r^2 + r + 5 = 0 \to \quad r_1 = -1, \quad r_{2,3} = 2 \pm i$$

$$Y_H = C_1 e^{-x} + C_2 e^{2x} \cos x + C_3 e^{2x} \sin x, \qquad C_i \in \mathbb{R}.$$

$$Y_P = x(A_1 x + A_0)e^{-x} + xe^{2x}\left[(B_2 x^2 + B_1 x + B_0) \sin x + (D_2 x^2 + D_1 x + D_0) \cos x\right]$$

Remark: The function "roots($\cdots$)" in MATLAB returns roots to a given polynomial.

2. (a) **Characteristic equation** for the corresponding homogeneous equation:

$$r^3 + r = r(r^2 + 1) = 0 \quad \to \quad r_1 = 0, \ r_{2,3} = \pm i$$

General solution for (H):

$$Y_H(t) = C_1 + C_2 \cos t + C_3 \sin t, \qquad C_i \in \mathbb{R}, i = 1, 2, 3$$

Let $y_1 = 1$, $y_2 = \cos t$, $y_3 = \sin t$, assume $Y_P(t) = u_1(x)y_1 + u_2(x)y_2 + u_3(x)y_3$,

$$W(y_1, y_2, y_3)(t) = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = 1 \cdot (\sin^2 t + \cos^2 t) = 1$$

Define

$$W_1(t) = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ \sec t & -\cos t & -\sin t \end{vmatrix} = \sec t, \qquad W_2(t) = \begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & \sec t & -\sin t \end{vmatrix} = -1$$

$$W_3(t) = \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & \sec t \end{vmatrix} = -\tan t$$

Then by the method of variation of parameter, we have

$$u_1 = \int \frac{W_1}{W}\, \mathrm{d}t = -\ln\sqrt{\frac{1+\sin t}{1-\sin t}} + d_1, \qquad u_2 = \int \frac{W_2}{W}\, \mathrm{d}t = -t + d_2$$

$$u_3 = \int \frac{W_3}{W}\, \mathrm{d}t = -\int \frac{\sin(2t)}{\cos(2t)+1}\, \mathrm{d}t = \frac{1}{2}\ln|\cos(2t)+1| + d_3 = \ln|\cos t| + d_3$$

By setting all d_i's to 0, and a particular solution for (N) is

$$Y_P(t) = -\ln\sqrt{\frac{1+\sin t}{1-\sin t}} - t\cos t + \sin t \ln|\cos t|$$

Thus, the **general solution for (N)** is

$$y = C_1 + C_2\cos t + C_3 \sin t - \ln\sqrt{\frac{1+\sin t}{1-\sin t}} - t\cos t + \sin t\ln|\cos t|$$

(b) $y''' - 3y'' + 3y' - y = t^{-2}e^t$

Characteristic equation for the corresponding homogeneous equation:

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0 \quad \to \quad r_{1,2,3} = 1.$$

General solution for (H):

$$Y_H(t) = (C_0 + C_1 t + C_2 t^2)e^t, \qquad C_i \in \mathbb{R}, i = 0, 1, 2$$

Let $y_1 = e^t$, $y_2 = te^t$, $y_3 = t^2e^t$, assume $Y_P(t) = u_1(x)y_1 + u_2(x)y_2 + u_3(x)y_3$,

$$W(y_1, y_2, y_3)(t) = \begin{vmatrix} e^t & te^t & t^2e^t \\ e^t & (t+1)e^t & (t^2+2t)e^t \\ e^t & (t+2)e^t & (t^2+4t+2)e^t \end{vmatrix} = 2e^{3t}.$$

Define

$$W_1(t) = \begin{vmatrix} 0 & te^t & t^2e^t \\ 0 & (t+1)e^t & (t^2+2t)e^t \\ t^{-2}e^t & (t+2)e^t & (t^2+4t+2)e^t \end{vmatrix} = e^{3t}\begin{vmatrix} 0 & t & t^2 \\ 0 & 1 & 2t \\ t^{-2} & 0 & 2 \end{vmatrix} = e^{3t}$$

$$W_2(t) = \begin{vmatrix} e^t & 0 & t^2e^t \\ e^t & 0 & (t^2+2t)e^t \\ e^t & t^{-2}e^t & (t^2+4t+2)e^t \end{vmatrix} = -2t^{-1}e^{3t}, \quad W_3(t) = e^{3t}\begin{vmatrix} 1 & t & 0 \\ 1 & t+1 & 0 \\ 1 & t+2 & t^{-2} \end{vmatrix} = t^{-2}e^{3t}$$

Then by the method of variation of parameter, we have

$$u_1 = \int \frac{W_1}{W} \, \mathrm{d}t = \frac{t}{2} + d_1, \qquad u_2 = \int \frac{W_2}{W} \, \mathrm{d}t = -\ln|t| + d_2$$

$$u_3 = \int \frac{W_3}{W} \, \mathrm{d}t = \frac{1}{2} \int t^{-2} \, \mathrm{d}t = -\frac{1}{2}t^{-1} + d_3$$

By setting all d_i's to 0, and a particular solution for (N) is

$$Y_P(t) = \frac{1}{2}te^t - te^t \ln|t| - \frac{1}{2}te^t = -te^t \ln|t|$$

Thus, the **general solution for (N)** is

$$y = (C_0 + C_1 t + C_2 t^2)e^t - te^t \ln|t|.$$

Method 2: Let D be the differentiation operator, then

$$y''' - 3y'' + 3y' - y = (D-1)(D-1)(D-1)y = t^{-2}e^t.$$

Denote $(D-1)u(t) = t^{-2}e^t$, $(D-1)w(t) = u(t)$, then $(D-1)y(t) = w(t)$. We solve

$$u' - u = t^{-2}e^t \quad \to \quad (e^{-t}u)' = t^{-2} \quad \to \quad u(t) = -t^{-1}e^t + d_1 e^t$$

Thus,

$$w' - w = -t^{-1}e^t + d_1 e^t \quad \to \quad (e^{-t}w)' = -t^{-1} + d_1 \quad \to \quad w(t) = -e^t \ln t + d_1 te^t + d_2 e^t$$

And finally the general solution can be found as

$$(e^{-t}y)' = -\ln t + d_1 t + d_2 \quad \to \quad y(t) = -te^t \ln t + C_3 t^2 e^t + C_2 te^t + C_1 e^t$$

Notice that a particular solution is $y_P(t) = -te^t \ln t$.

3. **Method of undetermined coefficients:**
   Based on Q(1a), we assume a particular solution to (N) has the form

$$Y_P = t\,(A_3 t^3 + A_2 t^2 + A_1 t + A_0) + t^2 Be^t.$$

Then

$$\begin{aligned}
Y_P' &= (4A_3 t^3 + 3A_2 t^2 + 2A_1 t + A_0) + B(2t + t^2)e^t \\
Y_P'' &= (12A_3 t^2 + 6A_2 t + 2A_1) + B(2 + 4t + t^2)e^t \\
Y_P''' &= (24A_3 t + 6A_2) + B(6 + 6t + t^2)e^t
\end{aligned}$$

By substituting Y_P into (N), we have

$$\begin{aligned}
(24A_3 t + 6A_2) + B(6 + 6t + t^2)e^t - 2\left[(12A_3 t^2 + 6A_2 t + 2A_1) + B(2 + 4t + t^2)e^t\right] \\
+(4A_3 t^3 + 3A_2 t^2 + 2A_1 t + A_0) + B(2t + t^2)e^t &= t^3 + 2e^t
\end{aligned}$$

By equating the coefficients for corresponding terms,

$$\begin{cases} 6A_2 - 4A_1 + A_0 &= 0 \\ 24A_3 - 12A_2 + 2A_1 &= 0 \\ -24A_3 + 3A_2 &= 0 \\ 4A_3 &= 1 \\ 6B - 4B &= 2 \end{cases} \quad \to \quad \begin{cases} A_0 &= 24 \\ A_1 &= 9 \\ A_2 &= 2 \\ A_3 &= 1/4 \\ B &= 1 \end{cases}$$

A particular solution to (N) is

$$Y_P = \frac{1}{4}t^4 + 2t^3 + 9t^2 + 24t + t^2e^t.$$

Method of variation of parameters:

We assume $Y_P = u_1 + u_2e^t + u_3te^t$, then by the variation of parameter,

$$\begin{bmatrix} 1 & e^t & te^t \\ 0 & e^t & (1+t)e^t \\ 0 & e^t & (2+t)e^t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t^3 + 2e^t \end{bmatrix}$$

$$W(1, e^t, te^t)(t) = \begin{vmatrix} 1 & e^t & te^t \\ 0 & e^t & (1+t)e^t \\ 0 & e^t & (2+t)e^t \end{vmatrix} = e^{2t}.$$

Hence

$$W_1(t) = \begin{vmatrix} 0 & e^t & te^t \\ 0 & e^t & (1+t)e^t \\ t^3+2e^t & e^t & (2+t)e^t \end{vmatrix} = (t^3 + 2e^t)e^{2t}, \quad u_1 = \int \frac{W_1}{W}\ \mathrm{d}t = \frac{1}{4}t^4 + 2e^t$$

$$W_2(t) = -(t^3 + 2e^t)(1+t)e^t, \quad u_2 = \int \frac{W_2}{W}\ \mathrm{d}t = (t^4 + 5t^3 + 15t^2 + 30t + 30)e^{-t} - (2t + t^2)$$

$$W_3(t) = (t^3 + 2e^t)e^t, \quad u_3 = \int \frac{W_3}{W}\ \mathrm{d}t = -(t^3 + 3t^2 + 6t + 6)e^{-t} + 2t$$

$$Y_P = \frac{1}{4}t^4 + 2t^3 + 9t^2 + 24t + 30 + (t^2 - 2t + 2)e^t.$$

Since $y = 30 + (-2t + 2)e^t$ is a solution to (H), the answers obtained by two methods coincide.

Method of reduction of order:

Let $w(t) = y'(t)$, then the equation becomes

$$w'' - 2w' + w = t^3 + 2e^t.$$

Solve the associated homogeneous problem, we have $w_H(t) = e^t(C_0 + C_1t)$. Then

$$y_H(t) = \int w_H(t)\ \mathrm{d}t = c_0e^t + c_1te^t + c_2.$$

By reduction of order, assume a particular solution

$$w_p(t) = u(t)e^t,$$ homegeneous solution中一解

then

$$e^t u'' + (2e^t - 2e^t)u' = t^3 + 2e^t \quad \rightarrow \quad u = \int \left[\int t^3 e^{-t} + 2 \ \mathrm{d}t \right] \mathrm{d}t$$

$$u(t) = \int \left((-t^3 - 3t^2 - 6t - 6)e^{-t} + 2t \right) \mathrm{d}t = (t^3 + 6t^2 + 18t + 24)e^{-t} + t^2$$

Hence, $w_p(t) = u(t)e^t = (t^3 + 6t^2 + 18t + 24) + t^2 e^t$ and

$$y(t) = \int w(t) \ \mathrm{d}t = \int e^t(C_0 + C_1 t) + (t^3 + 6t^2 + 18t + 24) + t^2 e^t \ \mathrm{d}t$$

$$= e^t(c_0 + c_1 t) + (\frac{1}{4}t^4 + 2t^3 + 9t^2 + 24t + c_2) + (t^2 - 2t + 2)e^t$$

or just $y(t) = e^t(c_0 + c_1 t) + c_2 + (\frac{1}{4}t^4 + 2t^3 + 9t^2 + 24t) + t^2 e^t$.

Comparing to previous methods, we can see $y = e^t(c_0 + c_1 t) + c_2$ is the general solution to (H).

4. The characteristic equation to the associated homogeneous eqn. is

$$r^3 - 2r^2 + r = r(r^2 - 2r + 1) = 0 \quad \rightarrow \quad r_1 = 0, r_2 = r_3 = 1.$$

The **general solution to (H)**:

$$Y_h = c_1 + c_2 e^x + c_3 x e^x, \qquad c_{1,2,3} \in \mathbb{R}.$$

Assume a particular solution to (N) is

$$Y_p = Ax + x^2(Bx + C)e^x$$

Substituting Y_p into (N), we have

$$A + (2B + 6C)e^x + 6Cxe^x = xe^x + 5$$

Thus, $A = 5, B = -1/2, C = 1/6$ and the **general solution to (N)** is

$$y = c_1 + c_2 e^x + c_3 x e^x + 5x - \frac{1}{2}x^2 e^x + \frac{1}{6}x^3 e^x.$$

Set in the initial conditions to determine $C_{1,2,3}$,

$$\begin{cases} c_1 + c_2 & = 2 \\ c_2 + c_3 + 5 & = 2 \\ c_2 + 2c_3 - 1 & = -1 \end{cases} \quad \rightarrow \quad \begin{cases} c_1 & = 8 \\ c_2 & = -6 \\ c_3 & = 3 \end{cases}$$

Solution to this IVP is

$$y = 8 - 6e^x + 3xe^x + 5x - \frac{1}{2}x^2 e^x + \frac{1}{6}x^3 e^x.$$

5.

$$x^3y''' + x^2y'' - 2xy' + 2y = 2x^4, \qquad x > 0, \qquad y_1(x) = x$$

$$y''' + x^{-1}y'' - 2x^{-2}y' + 2x^{-3}y = 2x$$

Let a particular solution $y(x) = u(x)y_1$ and substitute it back into (N). We get

$$\begin{aligned} y_1u''' + (3y_1' + x^{-1}y_1)u'' + (3y_1'' + 2x^{-1}y_1' - 2x^{-2}y_1)u' &= 2x \\ xu''' + 4u'' + (2x^{-1} - 2x^{-1})u' &= 2x \\ u''' + 4x^{-1}u'' &= 2 \end{aligned}$$

Set $w(x) = u''(x)$, then

$$w' + 4x^{-1}w = 2 \quad \rightarrow \quad (x^4w)' = 2x^4 \quad \rightarrow \quad w(x) = \frac{2x}{5} + c_1x^{-4}$$

We integrate twice on $w(x)$ and set all arbitrary constants as zero to obtain $u(x) = \frac{x^3}{15} + c_1x^{-2} + c_2x + c_3$. Thus, a particular solution to the non-homogeneous problem is

$$y_p(x) = u(x)y_1(x) = \frac{x^4}{15} + c_1x^{-1} + c_2x^2 + c_3x, \qquad c_{1,2,3} \in \mathbb{R}.$$

Note that the form $c_1x^{-1} + c_2x^2 + c_3x$ *serves as the general solution to (H).*