2022-23 Second Semester
MATH1063 Linear Algebra II (1003)

Assignment 9 Suggested Solutions

1. (a) **Solution:**

$$f_x = (y^2+1)(-2x)e^{-x^2}; \qquad f_y = 2ye^{-x^2};$$

$$f_{xy} = (-2x)(2y)e^{-x^2}; \qquad f_{xx} = (-2)(y^2+1)e^{-x^2} + 4x^2(y^2+1)e^{-x^2}; \qquad f_{yy} = 2e^{-x^2}$$

$$f_y = f_x = 0 \quad \to \quad x = y = 0 \quad \to \quad \text{critical point is } (0,0).$$

$H(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$ is indefinite with $\lambda = \pm 2$. $\to$ f has a saddle point at $(0,0)$.

(b) **Solution:**

$$f_x = x^2 + y^2 - 4y; \qquad f_y = 2xy - 4x;$$

$$f_{xy} = 2y - 4; \qquad f_{xx} = 2x; \qquad f_{yy} = 2x$$

$$\begin{cases} \dfrac{\partial f}{\partial y} = 0 \\ \dfrac{\partial f}{\partial x} = 0 \end{cases} \to \begin{cases} x = 0 \\ y = 0, 4 \end{cases} \text{ or } \begin{cases} x = \pm 2 \\ y = 2 \end{cases} \to \text{critical points:} (0,0), (0,4), (2,2) \text{ and } (-2,2).$$

For each stationary point (x_0, y_0), we determine the eigenvalues of

$$H(x_0, y_0) = \begin{pmatrix} 2x_0 & 2y_0 - 4 \\ 2y_0 - 4 & 2x_0 \end{pmatrix}$$

$H(0,0) = \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix}$ is indefinite with $\lambda = \pm 4$, thus f has a saddle point at $(0,0)$.

$H(0,4) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$ is indefinite with $\lambda = \pm 4$, i.e. f has a saddle point at $(0,4)$.

$H(2,2) = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ is positive definite and f has a **local** minimum at $(2,2)$.

$H(-2,2) = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$ is negative definite and f has a **local** maximum at $(-2,2)$.

(c) **Solution:**

$$f_x = \sin(y); \qquad f_y = x\cos(y);$$

$$f_{xy} = \cos(y); \qquad f_{xx} = 0; \qquad f_{yy} = -x\sin(y)$$

$$\begin{cases} \dfrac{\partial f}{\partial y} = 0 \\ \dfrac{\partial f}{\partial x} = 0 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = n\pi, \end{cases} \quad n \in \mathbb{Z} \ \rightarrow \ \text{critical points:}(0, n\pi).$$

For each stationary point (x_0, y_0), we determine the eigenvalues of

$$H(x_0, y_0) = \begin{pmatrix} 0 & \cos(y_0) \\ \cos(y_0) & -x_0 \sin(y_0) \end{pmatrix}$$

$H(0, n\pi) = \begin{pmatrix} 0 & (-1)^n \\ (-1)^n & 0 \end{pmatrix}$ is indefinite with $\lambda = \pm 1$, thus f has saddle points at $(0, n\pi)$, where $n \in \mathbb{Z}$.

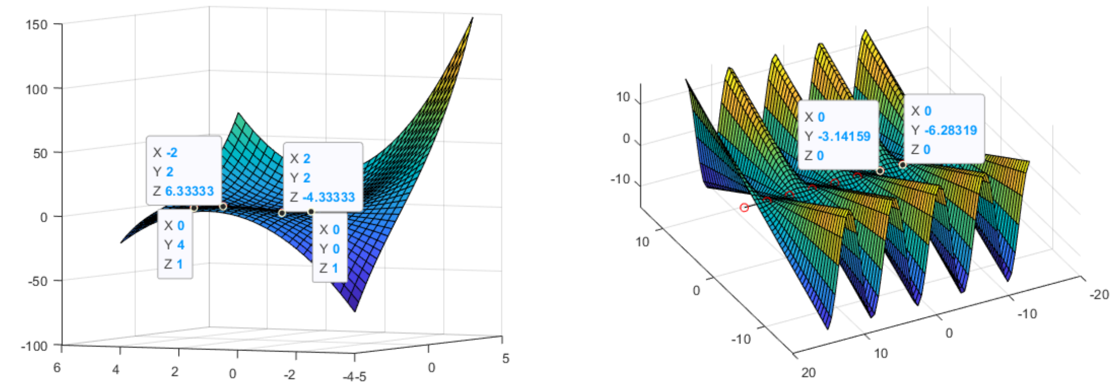

Figure 1: Left: Critical points in (b) Right: Saddle points $(0, n\pi)$ in (c)

2. **Proof:** There exists an orthogonal matrix Q so that A can be orthogonally diagonalized by Q as $A = QDQ^T$. Define an $n \times n$ diagonal matrix Σ whose diagonal entries are the cubic roots of the diagonal entries of D. Let $B = Q\Sigma Q^T$, then

$$B^3 = (Q\Sigma Q^T)(Q\Sigma Q^T)(Q\Sigma Q^T) = Q\Sigma^3 Q^T = QDQ^T = A$$

3. **Proof:**

   (a) The matrix A is real and symmetric, $\det(A - \lambda I_n) = (1 - \lambda)^2 - \frac{1}{4} = 0$ suggesting that the eigenvalues are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{3}{2}$. Hence, A is positive definite.
   For any nonzero $\vec{x} = [x_1, x_2]' \in \mathbb{R}^2$,

$$\vec{x}^T A \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 - x_1 x_2 + x_2^2$$

$$\vec{x}^T B \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 - x_1 x_2 + x_2^2 = \vec{x}^T A \vec{x}$$

(b) For any nonzero $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{x}^T B\mathbf{x} = x_1^2 - x_1x_2 + x_2^2 = (x_1 - \frac{1}{2}x_2)^2 + \frac{3}{4}x_2^2 > 0$, thus B is positive definite.

However, $\mathbf{x}^T B^2\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2 \geq 0.$

The equality sign holds when $x_1 = x_2$, which shows that B^2 is not positive definite.

Remark: Notice that the eigenvalues of B^2 are $\lambda_{1,2} = 1 > 0$, but B^2 is not positive definite. The eigenvalue test fails for non-symmetric matrices.

4. **Proof:** If A is an $m \times n$ matrix with rank n, then $\mathrm{N}(A) = \{\mathbf{0}\}$. For any $\mathbf{x} \neq \mathbf{0}$, we have $A\mathbf{x} \neq \mathbf{0}$ and

$$\mathbf{x}^T A^T A\mathbf{x} = \|A\mathbf{x}\|^2 > 0.$$

Hence $A^T A$ is positive definite.

5. **Solution:** Let Q be an orthogonal matrix and D be a diagonal matrix so that $A = QDQ^T$. If A is positive definite, then the diagonal entries of D, which are eigenvalues of A, are all positive. Take $\sigma_i = \sqrt{d_{ii}}$ and let the matrix Σ be a diagonal matrix with σ_i as its i-th diagonal entry. Define the matrix $B = \Sigma^T Q^T$, then

$$B^T B = Q\Sigma\Sigma^T Q^T = QDQ^T = A.$$