2.3 investment

(1) interest is compounding continuously at rate r.

(2) deposits ($k>0$) or withdraws ($k<0$) take place at a constant rate k

$$\frac{ds}{dt} = rs + k$$

$$\Rightarrow \frac{ds}{dt} - rs = k \qquad u(t) = e^{-rt}$$

$$S(t) = \frac{\int e^{-rt} k\,dt + C}{e^{-rt}} = \frac{-\frac{k}{r}e^{-rt} + C}{e^{-rt}} = Ce^{rt} - \frac{k}{r}$$

$S(0) = S_0$

$$\Rightarrow S_0 = C - \frac{k}{r} \Rightarrow C = S_0 + \frac{k}{r}$$

$$\Rightarrow S(t) = (S_0 + \frac{k}{r})e^{rt} - \frac{k}{r} = S_0 e^{rt} + \frac{k}{r}(e^{rt} - 1)$$

2.5 population dynamics

1. **autonomous equation** 自治方程

$$\frac{dy}{dt} = f(y)$$

independent variable (t) doesn't appear

$f(y)=0$ critical points

2. let $y=\phi(t)$ be the population of given species at time t

(1) exponential growth

$$\frac{dy}{dt} = ry \qquad y(0) = y_0$$

$r>0$ rate of growth / $r<0$ rate of decline

进一步修正

$$\Rightarrow y = y_0 e^{rt}$$ 没有环境阻力情况下的种群的增长模型．但遇到环境阻力后，增长到一定阶段，种群数量受到限制．所以需修正

(2) **logistic growth** : growth rate depends on the population

$$\frac{dy}{dt} = h(y)y$$

y很小时，$h(y) \approx r$

$y\uparrow$，$h(y)\downarrow$

y足够大时，$h(y)<0$

$$\Rightarrow h(y) = r - ay \ (a>0)$$

$\Rightarrow$ **logistic equation**:

$$\frac{dy}{dt} = (r-ay)y$$

$$= r(1-\frac{y}{K})y, \quad K = \frac{r}{a}$$

Saturation level 环境中所能容纳种群的最大值

intrinsic growth rate 内禀自然增长率：环境无限制时稳定年龄结构的种群能达到的最大增长率

consider the simplest type logistic equation : $\frac{dy}{dt} = 0$ 相当于 $\frac{dy}{dt} = f(y) = 0 \Rightarrow$ critical points: $f(y)=0$

$$\Rightarrow r(1-\frac{y}{K})y = 0$$

$\Rightarrow$ **equilibrium solution** $y = \phi_1(t) = 0 \qquad y = \phi_2(t) = K$

解释

随时间t增长，种群数量无变化   随时间t增长，种群数量增加到K.

$\Rightarrow$ **direction filed** 方向场 方程斜率相对的图像

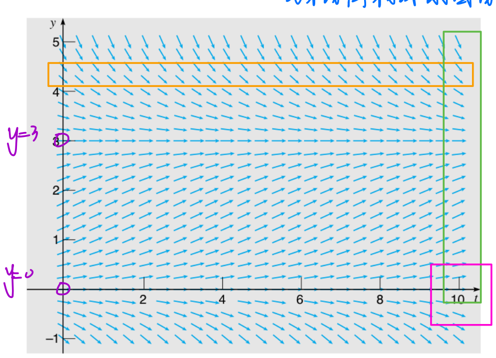


FIGURE 2.5.2 Direction field for $dy/dt = r(1 - y/K)y$ with $r = 1/2$ and $K = 3$.

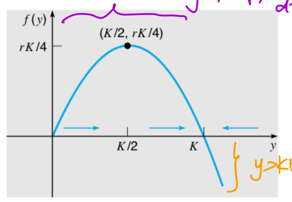
① 根据 equilibrium solution. $\frac{dy}{dt} = 0 \Rightarrow \phi_1(t) = 0$, $\phi_2(t) = 3$

$y=0$与$y=3$斜率都是水平的．

② 观察 $\boxed{}$ 可知，水平方向上斜率不变、所以 $\frac{dy}{dt}$ 只与y相关．而与t无关

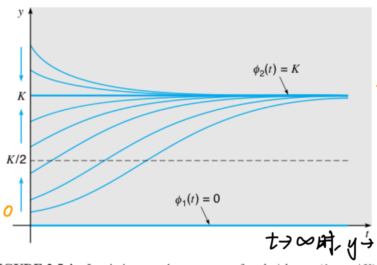
③ 观察 $\boxed{}$ 与 $\boxed{}$ 可知，在$t\to\infty$时，y都趋向于$y=3$，除了$y=0$.

$\Rightarrow f(y)=r(1-\frac{y}{K})y$ $\quad 0<y<K$时, $\frac{dy}{dt}=f(y)>0$



$y>K$时, $\frac{dy}{dt}=f(y)<0$

FIGURE 2.5.3 $f(y)$ versus y for $dy/dt = r(1 - y/K)y$.



$y>K$时 $\frac{dy}{dt}<0$

$0<y<K$时 $\frac{dy}{dt}>0$

$t\to\infty$时, $y\to K \Rightarrow K$ is the upper bound

FIGURE 2.5.4 Logistic growth: y versus t for $dy/dt = r(1 - y/K)y$.

y接近0或K时, $f(y)$接近0

$\Rightarrow \frac{d^2y}{dt^2}=f'(y)y'=f'(y)f(y)$. 观察 Figure 2.5.4

$0<y<\frac{K}{2}$ concave up $\frac{d^2y}{dt^2}>0$, $f(y)>0 \Rightarrow f'(y)>0$

$y>K$ concave up $\frac{d^2y}{dt^2}>0$, $f(y)<0 \Rightarrow f'(y)<0$

$\frac{K}{2}<y<K$ concave down $\frac{d^2y}{dt^2}<0$, $f(y)>0 \Rightarrow f'(y)<0$

$y=\frac{K}{2}$ inflection point

$0<y<\frac{K}{2}$ concave up

$\frac{K}{2}<y<K$ concave down

$\Rightarrow$ Solve $\frac{dy}{dt}=r(1-\frac{y}{K})y \quad y(0)=y_0$

$$\left(\frac{1}{y}+\frac{\frac{1}{K}}{1-\frac{y}{K}}\right)dy=r\,dt$$

$$\Rightarrow \ln|y|-\ln|1-\frac{y}{K}|=rt+C_1$$

we have $0<y<K$ (因K为种群最大值)

$$\ln y-\ln(1-\frac{y}{K})=rt+C_1$$

$$\Rightarrow \ln\frac{y}{1-\frac{y}{K}}=rt+C_1$$

$$\Rightarrow \frac{y}{1-\frac{y}{K}}=Ce^{rt} \qquad C=e^{C_1}$$

$t=0$

$$\Rightarrow \frac{y_0}{1-\frac{y_0}{K}}=C$$

$$\Rightarrow y=\frac{y_0K}{y_0+(K-y_0)e^{-rt}}$$

$y_0=0$与$y_0=K$时, 存在 equilibrium solution $y=\phi_1(x)=0 \quad y=\phi_2(x)=K$

解释: $y_0=0$, $y(t)=0$ for all t

unstable equilibrium

$y_0>0$, $\lim_{t\to\infty}y(t)=\frac{y_0K}{y_0}=K$

asymptotically stable

$y_0=K$, $y(t)=K$ for all t

$\Rightarrow$ **asymptotically stable solution**: 任意$y_0>0$, 都有 $\lim_{t\to\infty}y(t)=K=y=\phi_2(t)$. 那么$\phi_2(t)=K$为 $\frac{dy}{dt}=r(1-\frac{y}{K})y$ 的 asymptotically stable solution.

$\Rightarrow y=K$ is an asymptotically stable equilibrium or critical point

不管初始种群数量是多少(即便很大), 随时间推移, 种群数量最终会维持K的平衡状态.

$\Rightarrow$ **unstable equilibrium solution**: only $y_0=0$, $y(t)=0$ for all t

$\Rightarrow y=0$ is an unstable equilibrium or critical point

当且仅当初始种群数量为0时, 才有最终种群数量为0.