FINM3133 Time Series for Finance and Macroeconomics

Chapter 4 Solution

1. For MA(1) process $Y_t = e_t - \theta_1 e_{t-1}$,

$$\rho_1 = \frac{-\theta_1}{1+\theta_1^2} = \frac{-1}{\frac{1}{\theta_1}+\theta_1}$$

   Since $\frac{1}{\theta_1}+\theta_1 \in (-\infty,-2]\cup[2,\infty)$, $\rho_1 \in [-0.5,0)\cup(0,0.5]$. Especially, $\rho_1 = 0.5, -0.5$ when $\theta_1 = -1, 1$.

2. For MA(1) process $Y_t = e_t - \theta e_{t-1}$, replace θ with $1/\theta$. The autocorrelation function for the process become

$$\rho = \frac{-\frac{1}{\theta}}{1+\frac{1}{\theta^2}} = \frac{-\theta}{1+\theta^2}$$

3. (a) For AR(1) process $Y_t = \phi Y_{t-1} + e_t$, $Cov(Y_t, Y_{t-k}) = \phi^k \frac{\sigma_e^2}{1-\phi^2}$
   The autocovariance function of W_t

$$\begin{aligned}
\gamma_k &= Cov(W_t, W_{t-k}) \\
&= Cov(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-k-1}) \\
&= Cov(Y_t, Y_{t-k}) + Cov(Y_t, -Y_{t-k-1}) + Cov(-Y_{t-1}, Y_{t-k}) + Cov(-Y_{t-1}, -Y_{t-k-1}) \\
&= [\phi^k - \phi^{k+1} - \phi^{k-1} + \phi^k]\frac{\sigma_e^2}{1-\phi^2} \\
&= -[\frac{1-\phi}{1+\phi}]\phi^{k-1}\sigma_e^2
\end{aligned}$$

   (b)

$$\begin{aligned}
Var(W_t) &= Var(Y_t - Y_{t-1}) \\
&= Var(Y_t) + Var(Y_{t-1}) - 2Cov(Y_t, Y_{t-1}) \\
&= 2(1-\phi)\frac{\sigma_e^2}{1-\phi^2} \\
&= \frac{2\sigma_e^2}{1+\phi}
\end{aligned}$$

4. (a) MA(1) process has nonzero correlation only at lag 1. Could be positive or negative but must be between -0.5 and +0.5.

(b) MA(2) process has nonzero correlation only at lags1 and 2.

(c) AR(1) process has exponentially decaying autocorrelations starting from lag 0. If $\phi > 0$, then all autocorrelations are positive. If $\phi < 0$, then autocorrelations alternate negative, positive negative, etc.

(d) Autocorrelations of AR(2) process can have several patterns but if the roots of the charcteristic equation are complex numbers, then the pattern will be a cosine with a decaying magnitude.

(e) ARMA(1,1) has exponentially decaying autocorrelations starting from lag 1but not from lag zero.

5. (a)

$$Y_t = 0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2}$$

Multiply Y_{t-k} on both sides,

$$Y_tY_{t-k} = 0.8Y_{t-1}Y_{t-k} + e_tY_{t-k} + 0.7e_{t-1}Y_{t-k} + 0.6e_{t-2}Y_{t-k}$$

For $k > 2$, $E(e_tY_{t-k}) = E(e_{t-1}Y_{t-k}) = E(e_{t-2}Y_{t-k}) = 0$ Take expectation on both sides,

$$\gamma_k = 0.8\gamma_{k-1}$$

which is,

$$\rho_k = 0.8\rho_{k-1}$$

(b) When k =2,

$$Y_tY_{t-2} = 0.8Y_{t-1}Y_{t-2} + e_tY_{t-2} + 0.7e_{t-1}Y_{t-2} + 0.6e_{t-2}Y_{t-2}$$

Take expectation on both sides,

$$\begin{aligned}\gamma_2 &= 0.8\gamma_1 + E\{0.6e_{t-2}[0.8Y_{t-3} + e_{t-2} + 0.7e_{t-3} + 0.6e_{t-4}]\} \\ &= 0.8\gamma_1 + 0.6\sigma_e^2\end{aligned}$$

which is,

$$\rho_2 = 0.8\rho_1 + 0.6\sigma_e^2/\gamma_0$$

6. Notice that these coefficients decrease exponentially in magnitude at rate 0.5 while alternating in sign. Furthermore, the coefficients have nearly died out by θ_6. Thus, an AR(1) process with $\phi = -0.5$ would be nearly the same process.

7. Notice that these coefficients decrease exponentially in magnitude at rate 0.5 while alternating in sign but starting at $\theta_1 = 1$. Furthermore, the coefficients have nearly died out by θ_7, which accord with the behaviour of an ARMA(1,1) process. Equating $\psi_1 = \phi - \theta =$ -1 and $\psi_2 = (\phi - \theta)\phi =$ 0.5 yields $\phi = -0.5$ and $\theta =$ 0.5 in the ARMA(1,1) model that is nearly the same.