2022-23 First Semester
MATH1063 Linear Algebra II (1003)

Assignment 2 Suggested Solutions

1. (a) $L(1) = 1$, $L(x) = -1 + 2x$ and $L(x^2) = (2x-1)^2 = 1 - 4x + 4x^2$. Thus

$$[L]_\alpha = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}.$$

(b) $L(1) = 1$, $L(x) = (2x-1) - 1 = 2(x-1)$ and $L(x^2) = (2x-1-1)^2 = 4(x-1)^2$.

$$[L]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

(c) The transition matrix S from β to α is

$$S = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad \to \quad S^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix representing L^k with respect to the basis α is

$$\begin{aligned} ([L]_\alpha)^k = S\ ([L]_\beta)^k\ S^{-1} &= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 4^k \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1-2^k & (1-2^k)^2 \\ 0 & 2^k & 2^{k+1}(1-2^k) \\ 0 & 0 & 2^{2k} \end{bmatrix}. \qquad k \in \mathbb{Z}^+ \end{aligned}$$

Verification: $$[L]_\alpha^2 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}^2 = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 4 & -24 \\ 0 & 0 & 16 \end{bmatrix}.$$

2. **Proof**: Let S be a nonsingular matrix so that $B = S^{-1}AS$.

(a) Then $A - \lambda I$ and $B - \lambda I$ are similar since $S^{-1}(A-\lambda I)S = S^{-1}AS - \lambda S^{-1}IS = B - \lambda I$.

(b) Based on part (a), we know similar matrices have the same determinant.

3. **Proof**:

(a) $\forall\,\mathbf{x} \in \ker(B)$, $B\mathbf{x} = \mathbf{0}$. Then $AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$. $\quad \rightarrow \quad \ker(B) \subseteq \ker(AB)$.
Hence, $\ker(B)$ is always contained in $\ker(AB)$. Conversely, it depends. Only when $n \geq k$ and $\mathrm{N}(A) = \{\mathbf{0}\}$, $\ker(AB) = \ker(B)$.

(b) $\forall\,\mathbf{y} \in \mathrm{Col}(AB)$, namely $\mathbf{y} = AB\mathbf{x}$ for some $\mathbf{x}$, then $A(B\mathbf{x}) = \mathbf{y}$ suggests that $\mathbf{y}$ is also in the image of A. $\quad \rightarrow \quad \mathrm{Col}(AB) \subseteq \mathrm{Col}(A)$.
Hence, $\mathrm{Col}(AB)$ is always contained in $\mathrm{Col}(A)$. Only when $k < m$ and $\mathrm{Col}(B) = \mathbb{R}^k$, then $\mathrm{Col}(A) = \mathrm{Col}(AB)$.

4. **Proof**: $c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2 + \cdots + c_kA\mathbf{x}_k = \mathbf{0} \quad \Leftrightarrow \quad A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k) = \mathbf{0}$.

(a) Since A is nonsingular, it follows that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = A^{-1}\mathbf{0} = \mathbf{0}$.
Since $\mathbf{x}_1, \cdots, \mathbf{x}_k$ are linearly independent it follows that $c_1 = c_2 = \cdots = c_k = 0$.
Therefore $A\mathbf{v}_1, A\mathbf{v}_2, \cdots, A\mathbf{v}_k$ are linearly independent.

(b) Since $m > n$ and $\mathrm{rank}(A) = n$, then $\mathrm{Col}(A) = \mathbb{R}^n$ and $\mathrm{N}(A) = \{\mathbf{0}\}$.
It follows that $A\displaystyle\sum_{i=1}^{k} c_i\mathbf{x}_i = \mathbf{0}$ iff $\displaystyle\sum_{i=1}^{k} c_i\mathbf{x}_i = \mathbf{0}$.
$A\mathbf{v}_1, \cdots, A\mathbf{v}_k$ are linearly independent iff $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are linearly independent.

(c) The linear mapping L can be represented by an $m \times n$ matrix A corresponding to certain bases. Based on the discussion in part(a) and (b), we should impose the condition $\ker(L) = \{\mathbf{0}_V\}$, that is, L is one-to-one mapping from V to W.
When $m = n$, the matrix A is nonsingular as in part (a); when $m > n$, A is an $m \times n$ matrix of rank n as in part (b).

5. The angle $\theta = \arccos \dfrac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \arccos \frac{1}{\sqrt{n}}$. Then $\lim\limits_{n\to\infty} \theta = \arccos 0 = \pi/2$.

6. The vector projection of $\mathbf{y}$ onto $\mathbf{x}$ is $\mathbf{p} = \dfrac{\mathbf{y}^T\mathbf{x}}{\mathbf{x}^T\mathbf{x}}\mathbf{x}$. $\quad$ (a). $\mathbf{p} = \dfrac{14}{17}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$; $\quad$ (b). $\mathbf{p} = \dfrac{8}{14}\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$.