Chap6-Week9

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Chap 6.1 Eigenvalues and eigenvectors

i. Master the skill of finding eigenvalues and corresponding eigenvectors of a square matrix A.
ii. Some useful theorems and properties.

Linear Algebra II by Chiu Fai WONG

Section 6.1 Eigenvectors and Eigenvalues

Definition 6.1.1

Let A be an $\boxed{n \times n}$ matrix. A scalar λ is said to be an **eigenvalue** of A if there exists a $\boxed{\text{nonzero}}$ vector $v \in R^n$ such that $\boxed{Av = \lambda v}$. The vector v is said to be an **eigenvector** corresponding to λ.

Theorem 6.1.2

Let A be an $n \times n$ matrix and λ be a scalar. The following statements are equivalent:

(a) λ is an eigenvalue of A. $A\vec{v} = \lambda\vec{v}$ for some $\vec{v} \neq \vec{0}$ $\Leftrightarrow$ $A\vec{v} = \lambda\vec{v}$, for $\vec{v} \neq \vec{0}$

(b) $(A - \lambda I_n)v = \mathbf{0}$ has a nontrivial solution. $A\vec{v} - \lambda\vec{v} = \underbrace{(A-\lambda I)}_{M}\vec{v} = \vec{0}$ $\Updownarrow$ $M\vec{v} = \vec{0}$ $\Updownarrow$ $\vec{v} \in N(A-\lambda I)$, $\vec{v} \neq \vec{0}$ $\Updownarrow$ $A - \lambda I$ is singular $\Updownarrow$ $\det(A-\lambda I) = 0$

(c) $\det(A - \lambda I_n) = 0$.

Definition 6.1.3

Let A be an $n \times n$ matrix. The polynomial $p(\lambda) = \det(\lambda I_n - A)$ is called the **characteristic polynomial** of A.

Remark 6.1.4

λ is an eigenvalue of A if and only if λ is a root of characteristic polynomial of A.

Any nonzero vector $v \in N(A - \lambda I_n)$ is an eigenvector of A corresponding to eigenvalue λ. In particular, a nonzero vector $v \in N(A)$ is an eigenvector of A corresponding to 0.

Remark 6.1.5 $A\vec{v} = \lambda\vec{v} \Rightarrow A(A\vec{v}) = A\lambda\vec{v} = \lambda(A\vec{v}) = \lambda^2\vec{v} \Rightarrow A^k\vec{v} = \lambda^k\vec{v}$

Let A be an $n \times n$ matrix, λ be an eigenvalue of A corresponding to eigenvector v. Then λ^k is an eigenvalue of A^k corresponding to eigenvector v for positive integer k. Suppose k is a negative integer and A is invertible. Then λ^k is an eigenvalue of A^k corresponding to eigenvector v. Let a be a constant. Then $a\lambda$ is an eigenvalue of aA corresponding to eigenvector v.

prove/disprove by yourselves

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$a \in \mathbb{R}$, $a \neq 0$, $A\vec{v} = \lambda\vec{v} \Rightarrow (aA)\vec{v} = (a\lambda)\vec{v}$
$\Rightarrow A(a\vec{v}) = \lambda(a\vec{v})$ > Same meaning?

E.g. $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

Eigenvalues:

$$0 = \det(\lambda I_n - A) = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \right| = \begin{vmatrix} \lambda-1 & -3 \\ 0 & \lambda-2 \end{vmatrix} = (\lambda-1)(\lambda-2)$$

$\Rightarrow \lambda_1 = 1$, $\lambda_2 = 2$ are both eigenvalues of A.

Corresponding eigenvectors:

For $\lambda_1 = 1$, $(1 \cdot I_2 - A)\vec{v_1} = \vec{0} \rightarrow \begin{bmatrix} 0 & -3 \\ 0 & -1 \end{bmatrix}\vec{v_1} = \vec{0}$, $\vec{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}k$, $k \in \mathbb{R}$

For $\lambda_2 = 2$, $(2 \cdot I_2 - A)\vec{v_2} = \vec{0} \rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}\vec{v_2} = \vec{0}$, $\vec{v_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}k$, $k \in \mathbb{R}$

Discussion: ① Is $\begin{bmatrix} -6 \\ -2 \end{bmatrix}$ an eigenvector corresponding to $\lambda_2 = 2$?

(Hint: check $A\vec{v} \overset{?}{=} \lambda\vec{v}$) ② Is $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ 〃 〃 〃 $\lambda_2 = 2$? $\lambda_1 = 1$?

Def (6.1.1) Eigenvalue and Eigenvectors

☑ $A\vec{v} = \lambda\vec{v}$

eigenvector ($\vec{v}$), eigenvalue (λ) "eigen-pair"

☑ Square Matrix $A_{n \times n}$ ☑ Nonzero $\vec{v} \in \mathbb{R}^n$

Thm (6.1.2) proof: Denote $M = A - \lambda I_n$, then

(b) $M\vec{v} = \vec{0}$ has nontrivial (nonzero) solutions

(c) $\Leftrightarrow N(M) \neq \{\vec{0}\} \Leftrightarrow M$ singular $\Leftrightarrow \det(M) = 0$.

(a) $A\vec{v} = \lambda\vec{v}$ for some $\vec{v} \neq \vec{0}$

$\Leftrightarrow A\vec{v} - \lambda\vec{v} = A\vec{v} - \lambda I_n\vec{v} = \vec{0}$

(b) $\Leftrightarrow (A - \lambda I_n)\vec{v} = \vec{0}$ for some $\vec{v} \neq \vec{0}$.

Remarks (6.1.4): ① λ is an eigenvalue of A

Characteristic Equation of A

$\Leftrightarrow \lambda$ is a solution of $\det(A - \lambda I_n) = 0$.

(2a) Any nonzero vectors $\vec{v} \in N(A - \lambda I_n)$ is an eigenvector of A corresponding to λ.

(2b) A nonzero $\vec{v} \in N(A)$ is an eigenvector of A to $\lambda = 0$.

Exercise: $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$

Eigenvalues: $\det(\lambda I - A) = (\lambda - 4)(\lambda + 2) = 0$

$\lambda_1 = 4$, $\lambda_2 = -2$

Eigenvectors: For $\lambda_1 = 4$, , i.e.

$\begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} b = -2a \\ a \in \mathbb{R} \end{cases}$, $\vec{v_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}a$, $a \neq 0$.

For $\lambda_2 = -2$,

$(\lambda_2 I - A)\vec{v_2} = \begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} a = b \\ b \in \mathbb{R} \end{cases}$, $\vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}a$, $a \neq 0$

Review: $A_{n \times n}\vec{v} = \lambda\vec{v}$

for some scalar $\lambda \in \mathbb{R}$ and some nonzero vector $\vec{v}$.

$\vec{v}$ must be nonzero

$\Downarrow$

$(A - \lambda I_n)\vec{v} = \vec{0}$ has nontrivial solutions

$\Downarrow$

$(A - \lambda I_n)$ must be singular.

$\Downarrow$

$\det(A - \lambda I_n) = 0$

Chap6.1 Eigenvalues and Eigenvectors

1. Be familiar with the eigenvalues and eigenvectors, master the skill in finding eigenpairs
   a. Characteristic equation
   b. Eigenspace, algebraic multiplicity of an eigenvalue
   c. Eigenvalues, determinant and trace
2. Diagonalization
   a. With n distinct eigenvalues
   b. With <n distinct eigenvalues
3. Non-diagonalizable? (Generalized eigenvectors and Jordan form)

Thm (Extra) Eigenvalues for a triangular matrix

Let A be a triangular matrix, then the eigenvalues of A are diagonal elements.

$(A-\lambda I_n)$ must be singular.

$\Downarrow$

$\det(A-\lambda I_n) = 0$

- Solve $\det(A-\lambda I)=0$ for λ to find the eigenvalues

  Characteristic equation: $\det(A-\lambda I)=0$

  Characteristic polynomial: $\det(\lambda I - A)$

- Substitute λ_i back to $(A-\lambda_i I)\vec{v}=\vec{0}$ to find eigenvectors

Why diagonalization?

If $A = PDP^{-1}$, $A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$

$A^n =$

$D^k = \begin{bmatrix} \lambda_1^k & & \\ & \lambda_2^k & \\ & & \ddots \ \lambda_n^k \end{bmatrix}$ is much easier to compute than A^n.

Theorem 6.1.6 (later)

Suppose $p(\lambda) = \det(\lambda I_n - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$. Then $a_0 = (-1)^n \det(A)$ and $a_{n-1} = -Tr(A)$.

Definition 6.1.7

An $n\times n$ matrix A is said to be **diagonalizable** if there exists a nonsingular matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

The k-th diagonal entry of D is an eigenvalue of A corresponding to the eigenvector, the k-th column vector of P.

Theorem 6.1.8

An $n\times n$ matrix A is diagonalizable $\boxed{\text{if and only if}}$ ($\Leftrightarrow$) A has n linearly independent eigenvectors.

Theorem 6.1.9

If $\lambda_1,\cdots,\lambda_k$ are distinct eigenvalues of an $n\times n$ matrix A corresponding to eigenvectors $v_1,\cdots,v_k$ then $\{v_1,\cdots,v_k\}$ is linearly independent. If A has n distinct eigenvalues then A is diagonalizable.

Example 6.1.10

Find all eigenvectors and eigenvalues of $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$. Hence dingonalize A.

Solution

eigenvalues

The characteristic polynomial is

$$\det(\lambda I_2 - A) = \det\begin{pmatrix} \lambda-1 & -3 \\ -4 & \lambda-2 \end{pmatrix} = (\lambda-2)(\lambda-1) - 12 = \lambda^2 - 3\lambda - 10 = (\lambda-5)(\lambda+2)$$

The eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -2$. Two distinct eigenvalues

$\Rightarrow A_{2\times 2}$ is diagonalizable.

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Thm (Extra) Eigenvalues for a triangular matrix

Let A be a triangular matrix, then the eigenvalues of A are diagonal elements.

Proof: (WLOG, assume A as an upper triangular matrix)

$A = \begin{bmatrix} a_{11} & * & \cdots & * \\ & a_{22} & & \vdots \\ & & \ddots & * \\ 0 & & & a_{nn} \end{bmatrix}_{n\times n}$, $A-\lambda I_n = \begin{bmatrix} a_{11}-\lambda & * & \cdots & * \\ & a_{22}-\lambda & & \vdots \\ & & \ddots & * \\ 0 & & & a_{nn}-\lambda \end{bmatrix}$, $|A-\lambda I_n| = \prod_{i=1}^{n}(a_{ii}-\lambda) = 0$.

Exercise Eigenvalues by observation

$A = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 0 & & & \\ * & 1 & & \\ \vdots & & \ddots & \\ * & \cdots & * & n \end{bmatrix}$, $D = \left[\begin{array}{cc|c} 0 & 1 & 4 \\ 1 & 0 & 3 \\ \hline 0 & 0 & 2 \end{array}\right]$

Remark (6.1.4) If $(\lambda, \vec{v})$ is an eigenpair of $A_{n\times n}$, then so is $(\lambda, c\vec{v})$, $\forall c \neq 0$. $A(c\vec{v}) = \lambda(c\vec{v})$

$\hookrightarrow$ ∞-ly many eigenvec's corresponding to an eigenvalue.

Question

If $A\vec{v}_1 = \lambda\vec{v}_1$ and $A\vec{v}_2 = \lambda\vec{v}_2$, how about $\vec{v}_1 + \vec{v}_2$?

Def (6.1.21) Eigenspace E_{λ_i}

$E_{\lambda_i} = N(A-\lambda_i I_n)$: The eigenspace of A corresponding to

$= \{\vec{0}\} \cup \{\vec{v} \in \mathbb{R}^n \mid A\vec{v} = \lambda_i \vec{v}$, for some $\vec{v} \neq \vec{0}\}$

Def (6.1.7) Diagonalizable

$$A_{n\times n} = PDP^{-1}$$

where

$D = \begin{bmatrix} \ddots & & 0 \\ & \lambda_j & \\ 0 & & \ddots \end{bmatrix}_{n\times n}$ and $P = \begin{bmatrix} | & \cdots & \vec{v}_j & \cdots & | \end{bmatrix}_{n\times n}$

k^{th} diagonal entry $\leftarrow$ $k = 1, 2, \cdots, n$ $\rightarrow$ k^{th} column

Thm (6.1.8)

$A_{n\times n}$ diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors

Thm (6.1.9) Distinct eigenvalues $\Rightarrow$ Linearly independent eigenvectors.

$\Uparrow$ $\Downarrow$

Corollary If $A_{n\times n}$ has n distinct eigenvalues, then A is diagonalizable.

Proof by contradiction:

For $\lambda \neq \beta$, $A\vec{v} = \lambda\vec{v}$, $A\vec{\eta} = \beta\vec{\eta}$. If $\vec{v} = c\vec{\eta}$ for some $c \neq 0$, then ...

eigenvectors

Consider $\lambda_1 = 5$.

$$N(A-5I_2)=N\begin{pmatrix}1-5 & 3\\ 4 & 2-5\end{pmatrix}=N\begin{pmatrix}-4 & 3\\ 4 & -3\end{pmatrix}=span\left\{\begin{pmatrix}3\\4\end{pmatrix}\right\}.$$

Eigenvector $v_1=\begin{pmatrix}3\\4\end{pmatrix}$.

Consider $\lambda_2=-2$.

$$N(A+2I_2)=N\begin{pmatrix}1+2 & 3\\ 4 & 2+2\end{pmatrix}=N\begin{pmatrix}3 & 3\\ 4 & 4\end{pmatrix}=span\left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\}.$$

Eigenvector $v_2=\begin{pmatrix}1\\-1\end{pmatrix}$.

$$\begin{pmatrix}1 & 3\\ 4 & 2\end{pmatrix}=\begin{pmatrix}3 & 1\\ 4 & -1\end{pmatrix}\begin{pmatrix}5 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}3 & 1\\ 4 & -1\end{pmatrix}^{-1}=(\vec{v}_1\ \vec{v}_2)\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}(\vec{v}_1\ \vec{v}_2)^{-1}$$

Definition 6.1.11

Let λ be an eigenvalue of A with characteristic polynomial $f(x)$. The $\boxed{\textbf{algebraic multiplicity}}$ of λ is the largest positive integer k for which $(x-\lambda)^k$ is a factor of $f(x)$, i.e., there are exactly k copies of eigenvalue λ.

Moreover, the geometric multiplicity of λ_i is the dimension of E_{λ_i}.

Theorem 6.1.12 $\quad A=S^{-1}BS$

If two $n\times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Hint: check $p_A(\lambda)=\det(\lambda I-A)=\cdots=p_B(\lambda)$.

Theorem 6.1.13 (later)

Let A be an $n\times n$ matrix and $\lambda_1,\cdots,\lambda_n$ be all eigenvalues of A. Then

$$Tr(A)=\lambda_1+\cdots+\lambda_n \text{ and } \det(A)=\lambda_1\cdots\lambda_n.$$

$\triangle$ Sum or Product of Eigenvalues

Thm (6.1.6) & (6.1.13)

Suppose $p(\lambda)=\det(\lambda I_n-A)=\lambda^n+C_{n-1}\lambda^{n-1}+\cdots+C_1\lambda+C_0$.

Then $C_{n-1}=-Tr(A)$ and $C_0=(-1)^n\det(A)$

proof: Take $\lambda=0$ in $p(\lambda)$

$\det(-A)=C_0=(-1)^n\det(A)$.

As for $C_{n-1}=-Tr(A)$, use the determinant of λI_n-A.

Hence if $\lambda_1,\lambda_2,\cdots,\lambda_n$ are eigenvalues of A, then

$$\begin{aligned}p(\lambda)&=(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)\\&=\lambda^n+(-\Sigma\lambda_i)\lambda^{n-1}+\cdots+(-1)^n\lambda_1\cdots\lambda_n\end{aligned}$$

Comparing to Thm 6.1.6 we have

$$\sum_{i=1}^n\lambda_i=Tr(A),\quad \prod_{i=1}^n\lambda_i=\det(A).$$

Exercise $A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$. $\begin{cases}\lambda_1+\lambda_2=2+2\\ \lambda_1\lambda_2=3\end{cases}\rightarrow\begin{cases}\lambda_1=1\\ \lambda_2=3\end{cases}$

$B=\begin{bmatrix}2 & 0 & 0\\ -1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$. $\begin{cases}\lambda_1=2\\ \lambda_2+\lambda_3=2+3+3-\lambda_1\\ \lambda_1\lambda_2\lambda_3=2\cdot(9-1)=16\end{cases}\rightarrow\begin{cases}\lambda_1=2\\ \lambda_2=2\\ \lambda_3=4\end{cases}$

Conclusion on diagonalizable matrices:

$A_{n\times n}$ is diagonalizable if $\begin{cases}A \text{ has } n \text{ distinct eigenvalues (see A)}\\ almu(\lambda_i)=gemu(\lambda_i) \text{ for each } i \text{ (See B)}\end{cases}$

Question True or False?

$\lambda_1=5,\ \vec{v}_1=\begin{pmatrix}3\\4\end{pmatrix}$; $\lambda_2=-2,\ \vec{v}_2=\begin{pmatrix}1\\-1\end{pmatrix}$

① $A=\begin{pmatrix}3 & -2\\ 4 & 2\end{pmatrix}\begin{pmatrix}5 & \\ & -2\end{pmatrix}\begin{pmatrix}3 & -2\\ 4 & 2\end{pmatrix}^{-1}$?

② $A=\begin{pmatrix}1 & 3\\ -1 & 4\end{pmatrix}\begin{pmatrix}-2 & \\ & 5\end{pmatrix}\begin{pmatrix}1 & 3\\ -1 & 4\end{pmatrix}^{-1}$?

③ $A=\begin{pmatrix}3 & -2\\ 4 & 2\end{pmatrix}\begin{pmatrix}5 & \\ & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}3 & -2\\ 4 & 2\end{pmatrix}^{-1}$?

④ $A=\begin{pmatrix}1 & 3\\ -1 & 4\end{pmatrix}\begin{pmatrix}0 & -2\\ 5 & 0\end{pmatrix}\begin{pmatrix}1 & 3\\ -1 & 4\end{pmatrix}^{-1}$?

Check $A=PDP^{-1}$ or $AP=PD$.

↳ Such diagonalization is NOT unique.

Exercise Diagonalizable or not?

$A=\begin{bmatrix}5 & 0\\ 0 & 2\end{bmatrix}$, $B=\begin{bmatrix}1 & 7 & 9\\ 0 & -2 & 3\\ 0 & 0 & 3\end{bmatrix}$, $C=\begin{bmatrix}0 & & \\ * & 1 & \\ \vdots & \ddots & \ddots \\ * & \cdots\ * & n\end{bmatrix}$, $D=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$

E.g. $A=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$. $\lambda_1=\lambda_2=2$

For $\lambda=2$, $\vec{0}=(A-2I)\vec{v}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\vec{v}\rightarrow E_2=span\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$

$P=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $D=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$, then $A=PDP^{-1}$, diagonalizable.

Def (6.1.11) algebraic multiplicity geometric multiplicity

E.g. $A=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\rightarrow \det(\lambda I-A)=(\lambda-2)^2=0 \rightarrow$ corresponding eigenvector(s) $\vec{v}_1=\begin{pmatrix}1\\0\end{pmatrix},\ \vec{v}_2=\begin{pmatrix}0\\1\end{pmatrix}$

$almu(\lambda=2)=2$ $\quad gemu(\lambda=2)=2$

$B=\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix}\rightarrow\det(\lambda I-B)=(\lambda-2)^2=0\rightarrow (2I-B)\vec{v}=\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}\vec{v}=\vec{0}$

$almu(\lambda=2)=2$ $\quad \vec{v}=\begin{bmatrix}1\\0\end{bmatrix}$, $gemu(\lambda=2)=1$.

Remark: The algebraic multiplicity and the geometric multiplicity of an eigenvalue are not always equal.

E.g. $A=\begin{bmatrix}2 & 0 & 0\\ -1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$.

Eigenvalues: $\det(A-\lambda I_n)=(2-\lambda)[(3-\lambda)^2-1]=(2-\lambda)(3-\lambda-1)(3-\lambda+1)=0$.

$\lambda_1=\lambda_2=2$, $\lambda_3=4$ $\quad almu(2)=\quad$, $almu(4)=$

Eigenvectors:

For $\lambda_1=2$, $E_2=N(A-2I_3)=span\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$, $gemu(2)=$

$\vec{0}=(A-2I_3)\vec{v}=\begin{pmatrix}0 & 0 & 0\\ -1 & 1 & 1\\ -1 & 1 & 1\end{pmatrix}\vec{v}\rightarrow\begin{cases}-x_1+x_2+x_3=0\\ x_2\in\mathbb{R}\\ x_3\in\mathbb{R}\end{cases}\rightarrow\vec{v}=\begin{pmatrix}x_2+x_3\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}x_2+\begin{pmatrix}1\\0\\1\end{pmatrix}x_3$

For $\lambda_3=4$, $E_4=N(A-4I_3)=span\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$, $gemu(4)=$

$\vec{0}=(A-4I)\vec{v}=\begin{pmatrix}-2 & 0 & 0\\ -1 & -1 & 1\\ -1 & 1 & -1\end{pmatrix}\vec{v}\rightarrow\begin{cases}x_1=0\\ -x_2+x_3=0\\ x_3\in\mathbb{R}\end{cases}\rightarrow\vec{v}=\begin{pmatrix}0\\x_3\\x_3\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix}x_3$

Diagonalize A:

Let $D=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 4\end{bmatrix}$, $P=\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}$ ($\vec{v}_1\ \vec{v}_2\ \vec{v}_3$) $\quad \vec{v}_1,\vec{v}_2\in E_2$, linearly independent

or, $D=\begin{bmatrix}2 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 2\end{bmatrix}$, $P=\begin{bmatrix}1 & 0 & 2\\ 1 & 2 & 1\\ 0 & 2 & 1\end{bmatrix}$ ($\vec{v}_1\ 2\vec{v}_3\ \vec{v}_1+\vec{v}_2$) $\quad$ as long as P has n linearly independent cols.

Thm (6.1.26)-(b)

For each i, $almu(\lambda_i)=gemu(\lambda_i)$ $\Leftrightarrow$ $A_{n\times n}$ is diagonalizable.