2022-23 Second Semester
MATH1083 Calculus II Quiz One (1002&1003)

Time: 7-7:50pm 16/Mar/2023 (Thu)   Venue: T3-201   **Total score 100 pts**

- Answer all questions using a black/blue ink pen. No pencils
- **Write down** your

Chinese name :   Student NO.

Points awarded

1. **[10 pts]** Rank the growth rate of the following sequences **in ascending order** (from the slowest to the fastest)

$$A)n^n, \qquad B)n^{100}\ln n, \qquad C)n!, \qquad D)n^{100}, \qquad E)10^n, \qquad F)e^{2n}$$

Solution: (all correct 10, otherwise 0)

$$D << B << F << E << C << A$$

The growth rates in order: logarithm, linear, polynomial, exponenetial, factorial, n^n. This can be proved by experience/first order derivative/L'Hospital rule/plotting (by a computer)

2. **[10pts]** Determine whether each of following **sequence** converges or diverges. If it converges, **find the limit**.

(a) $a_n = \sqrt[n]{3^n + 4^n}$

(b) $b_n = \cos n\pi$ [Dr Wong's page 2]

Solution (a) **(5pts)**

$$4 = \sqrt[n]{4^n} \leq \sqrt[n]{3^n + 4^n} \leq \sqrt[n]{4^n + 4^n} = 4\sqrt[n]{2} \to 4$$

it is **convergent** by **Squeeze Theorem**

$$\lim_{n\to\infty} a_n = 4$$

[This is a question from Dr Wong's notes on page 7.]

You can also prove the convergence by showing that a_n is monotonically increasing, and with a upper bound of $4\sqrt{2}$, but then still need to find the limit.

(b)**(5pts)** b_n is **divergent** as we can form **two subsequences converges** to different values

$$\lim_{n\to\infty} b_{2n} = 1, \qquad \lim_{n\to\infty} b_{2n+1} = -1$$

Or you can just simply say, the limit does not exist.

3. **[10pts]** Evaluate the series (find the value)

   (a) **(5pts)**

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

   (b) **(5pts)**

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

   Solution: (a) Telescoping series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$$

   [primary school level]

   (b) geometric series

$$s = \frac{2/3}{1-2/3} = 2$$

   [Middle school level]

4. **[15pts]** Determine whether the following **series** converges or diverges, and explain why (which test to use?).

   (a)

$$\sum_{k=1}^{\infty} \frac{1}{k^2+2}$$

   (b)

$$\sum_{k=4}^{\infty} k^{-0.9999}$$

   (c)

$$\sum_{k=2}^{\infty} (-1)^{k+1} \frac{k-1}{k+2}$$

   a) Convergent by **Direct Comparison Test** $\frac{1}{k^2+2} < \frac{1}{k^2}$. [Note: $\lim_{k\to\infty} a_k = 0$ does **NOT** imply $\sum a_k$ is convergent. Counter example: harmonic series.]
   b) Divergent by **Direct Comparison Test** $k^{-0.9999} > \frac{1}{k}$
   c) Divergent by **Divergent test**,

$$\lim_{k\to\infty} (-1)^{k+1} \frac{k-1}{k+2} \neq 0$$

5. **[10pts]** Find the **values of p** for which the series is **convergent**

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

   Solution: Let function

$$f(x) = \frac{1}{x(\ln x)^p}$$

We can find that $f(x)$ (The variable is x. DO **NOT** use n, because n denotes intergers) is continous on $[2,\infty)$, decreasing and positive. We can apply **Integral Test**

$$\int_2^\infty \frac{1}{x(\ln x)^p}dx = \int_2^\infty \frac{1}{(\ln x)^p}d(\ln x) = -p(\ln x)^{-p+1}|_2^\infty$$
$$= p\left[(\ln 2)^{-p+1} - \frac{1}{(\ln R)^{p-1}}\right] \qquad R\to\infty$$

it **converges when** $\frac{1}{(\ln R)^{p-1}} \to 0$ **that means** $p>1$**, and diverges when** $p<1$.
When $p=1$, the series is

$$\sum_{n=2}^\infty \frac{1}{n\ln n}$$

we can use Integral Test, and let $g(x) = 1/x\ln x$ which is continous on $[2,\infty)$, decreasing and positive.

$$\int_2^\infty \frac{1}{x\ln x}dx = \int_2^\infty \frac{1}{\ln x}d(\ln x) = \ln(\ln x)|_2^\infty \to \infty$$

Therefore, **when** $p>1$**, the series converges**.
If you use Ratio Test, it is almost not possible to find the right way to prove.

6. **[10pts]** For series

$$\sum_{n=1}^\infty \frac{1}{n^4}$$

(a) Find the **sum of the first 3 terms** to **approximat**e the sum of the series, and estimate the error.

(b) How many terms are required to ensure that the sum is accuracy to within 10^{-3}.

Solution:**[Dr Wong's Page 15, Theorem 1.3.4 and example on Page16]**

$$\sum_{n=1}^3 \frac{1}{n^4} = 1 + \frac{1}{16} + \frac{1}{81}$$

or

$$\frac{1393}{1296}$$

Using Remainder Estimate for the Integral Test, we found that

$$Error \le \int_3^\infty \frac{1}{x^4}dx = -\frac{1}{3}x^{-3}|_3^\infty = \frac{1}{81}$$

To reach an accuracy within 10^{-3}, we need to find the value of n such that

$$R_n \le \int_n^\infty \frac{1}{x^4}dx = -\frac{1}{3}x^{-3}|_n^\infty = \frac{1}{3n^3} \le 10^{-3}$$

then $n \ge 7$.

7. **[20pts]** For function

$$f(x) = \ln(3+4x)$$

(a) Find the Taylor series about $a = 0$.

(b) Find the **radius and interval** of convergence of this Taylor series.

(c) Approximate $\ln(3.04)$ by using Taylor polynomial with degree 2 T_2 and estimate the error. $[\ln 3 = 1.09861229]$

Solution: Taylor series:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

here $a = 0$

$$\begin{aligned} f(0) &= \ln 3 \\ f'(x) &= \frac{4}{3+4x} \qquad f'(0) = \frac{4}{3} \\ f''(x) &= \frac{-4^2}{(3+4x)^2} \qquad f''(0) = \frac{-4^2}{3^2} - \frac{16}{9} \\ f'''(x) &= \frac{2\cdot 4^3}{(3+4x)^3} \qquad f'''(0) = 2!\left(\frac{4}{3}\right)^3 = \frac{128}{27} \end{aligned}$$

$$f^{(n)}(x) = (-1)^{n+1}\left(\frac{4}{3}\right)^n (n-1)! \qquad \text{for} \qquad n = 1, 2, \ldots$$

$$\begin{aligned} f(x) &= \ln 3 + \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{4}{3}\right)^n \frac{(n-1)!}{n!} x^n \\ &= \ln 3 + \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{4}{3}\right)^n \frac{1}{n} x^n \end{aligned}$$

(b) We apply Ratio Test

$$\left|\frac{(-1)^{n+2}\left(\frac{4}{3}\right)^{n+1}/n}{(-1)^{n+1}\left(\frac{4}{3}\right)^n/(n-1)}\right| |x| < 1$$

and

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}\left(\frac{4}{3}\right)^{n+1}/n}{(-1)^{n+1}\left(\frac{4}{3}\right)^n/(n-1)}\right| = \frac{4}{3}$$

therefore

$$|x| < \frac{3}{4}$$

and

$$R = \frac{3}{4}$$

then we test the **two endpoints**. Let $x = 3/4$: The Taylor series is

$$\ln 3 + \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$$

which is convergent as it is an **alternating harmonic series**.
Let $x = -3/4$,Taylor series is

$$\ln 3 + \sum_{n=1}^{\infty} (-1)^{2n+1} \frac{1}{n} = \ln 3 - \sum_{n=1}^{\infty} \frac{1}{n}$$

which is a harmonic series, therefore it is divergent. The interval of convergence is

$$\left(-\frac{3}{4}, \frac{3}{4}\right]$$

c) To approximate $\ln 3.04$, we let $x = 0.01$, using Taylor polynomial

$$\ln 3.04 \approx T_2(0.01) = \ln 3 + \frac{4}{3} \cdot 0.01 - \frac{16}{9} 0.01^2$$

Error estimate

$$|R_2(0.01)| \le \left| \frac{f'''(z)}{3!} x^3 \right|$$

where $z \in (0, 0.01)$. Since when $z = 0$

$$\max \left| f'''(z) \right| = \max \left| \frac{2 \cdot 4^3}{(3 + 4 \times 0)^3} \right| = 2 \left(\frac{4}{3}\right)^3$$

then

$$|R_2(0.01)| \le 2 \left(\frac{4}{3}\right)^3 \frac{1}{3!} 0.01^3$$

[This type of questions will appear in your mid-term and final exam again and again...]

8. **[10pts]** Find the limit below **WITHOUT** using L'Hospital Rule.

$$\lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4}$$

Solution: $\sin^2 x = (1 - \cos 2x)/2$ double angle formula, and Taylor series of $\cos 2x$

$$\cos 2x = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} \left[\frac{(2x)^2}{2} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} + \cdots \right]$$

$$\begin{aligned} \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4} &= \lim_{x \to 0} \frac{\frac{1}{2}\left[\frac{(2x)^2}{2} - \frac{(2x)^4}{4!} + \cdots\right] - x^2}{x^4} \\ &= -\frac{1}{3} \end{aligned}$$

OR:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

then

$$\begin{aligned}\sin^2 x &= \left(x - \frac{x^3}{3!} + \cdots\right)^2 \\ &= x^2 - 2\cdot\frac{x^4}{3!} + \left(\frac{x^3}{3!}\right)^2 + \cdots \\ &\approx x^2 - \frac{x^4}{3}\end{aligned}$$

9. **[5pts]**For these two vectors $\overrightarrow{a} = \langle 2, -1, 3\rangle$ and $\overrightarrow{b} = \langle 4, 2, 1\rangle$, find $\overrightarrow{a}\cdot\overrightarrow{b}$, $\overrightarrow{a}\times\overrightarrow{b}$ and $\overrightarrow{b}\times\overrightarrow{a}$

   Solution: $\overrightarrow{a}\cdot\overrightarrow{b} = 8 - 2 + 3 = 9$

   $\overrightarrow{a}\times\overrightarrow{b} = \langle -7, 10, 8\rangle$ and $\overrightarrow{b}\times\overrightarrow{a} = -\overrightarrow{a}\times\overrightarrow{b} = \langle 7, -10, -8\rangle$

   [This is simple definitions of dot and cross product.]