LAIIQuiz 3-4

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Quiz 3

Q1 $y = \beta_0 + \beta_1 x$

x	-2	2	4
y	2	0	-4

Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}$

Since $\text{rank}(A) = 2$, $\hat{x} = (A^TA)^{-1}A^T\vec{b} = \frac{1}{56}\begin{bmatrix} 32 \\ -54 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 8 \\ -13 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 24 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{3\times 24 - 4\times 4}\begin{bmatrix} 24 & -4 \\ -4 & 3 \end{bmatrix} = \frac{1}{56}\begin{bmatrix} 24 & -4 \\ -4 & 3 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ -20 \end{bmatrix}$$

Q2 proof: Since $\dim[Col(A^T)] = \dim[Col(A)] = \text{rank}(A)$, we only need to argue $\{A\vec{x}_1, A\vec{x}_2, \cdots, A\vec{x}_r\}$ are linearly independent and are from $Col(A)$.

i) $A\vec{x}_i \in Col(A)$, $\forall\ \vec{x}_i \in \mathbb{R}^n$

ii) Let

$$c_1A\vec{x}_1 + c_2A\vec{x}_2 + \cdots + c_rA\vec{x}_r = \vec{0}$$
$$A(c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_r\vec{x}_r) = \vec{0}$$

Hence $\sum_{i=1}^{r} c_i\vec{x}_i \in Col(A^T)$ and $\sum_{i=1}^{r} c_i\vec{x}_i \in N(A)$.

Then

$$c_1\vec{x}_1 + \cdots + c_r\vec{x}_r = \vec{0},$$

since $Col(A^T) \perp N(A)$ and $Col(A^T) \cap N(A) = \{\vec{0}\}$

We have $c_1 = c_2 = \cdots = c_r = 0$

because $\{\vec{x}_1, \cdots, \vec{x}_r\}$ form a basis of $Col(A^T)$.

Q3 A is symmetric then A is orthogonally diagonalizable by an orthogonal matrix Q s.t.

$$A = QDQ^T = Q\begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix}Q^T = \lambda QIQ^T = \lambda I.$$

Quiz 4

Q1 $\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 - b^2 = (a-b-\lambda)(a+b-\lambda)$

$\lambda_1 = a-b$, $\lambda_2 = a+b$

$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $D = \begin{bmatrix} a-b & 0 \\ 0 & a+b \end{bmatrix}$

Q2 Unitary: $UU^H = I$
Hermitian: $U = U^H$

$\Rightarrow U^2 = I$

$\Leftrightarrow U = QDQ^H$

$I = U^2 = QD^2Q^H$, hence eigenvalues of D^2 and I are the same.

$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$

Method 2: Let λ be an eigenvalue of U with eigenvector $\vec{x}$.

$$U\vec{x} = \lambda\vec{x}$$

$$\left.\begin{aligned} U^2\vec{x} &= U(\lambda\vec{x}) = \lambda^2\vec{x} \\ U^2\vec{x} &= U^HU\vec{x} = I\vec{x} = \vec{x} \end{aligned}\right\} \quad (\lambda^2 - 1)\vec{x} = \vec{0}$$

$\Rightarrow \lambda^2 = 1$, $\lambda = \pm 1$.

Q3 proof: A is singular $\Leftrightarrow$ $A\vec{x} = \vec{0} = 0\vec{x}$ for some $\vec{x} \neq \vec{0}$

$\Leftrightarrow$ 0 is an eigenvalue of A with eigenvector $\vec{x}$.