PT Assignment 9

1. The future lifetimes T of a certain population is exponentially distributed with parameter λ, where λ is uniformly distributed over $(1, 11)$. Calculate $P(T > 0.5)$. [Hint: use $P(X \in A) = \int_{-\infty}^{\infty} P(X \in A \mid \lambda = y) f_\lambda(y) dy$.]

2. If the joint density function of X and Y is $f_{X,Y}(x,y) = \begin{cases} 8xy & 0 < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$ Find the probability density function of $Z = X + Y$.

3. Let X and Y be independent random variables and $Z = X + Y$. Using $p_Z(n) = \sum_{k=0}^{n} P(X = k, Y = n - k)$ for nonnegative discrete random variable and $f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy$ for continuous random variable, find the probability mass function or probability density function of Z if

   (a) X and Y are Gamma distributions with parameters (s, λ) and (t, λ) respectively

   (b) X and Y are binomial random variables with parameters (n, p) and (m, p) respectively.

   (c) X and Y are Normal distributions with parameters $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ respectively

   (d) X and Y are random variables such that

   $$p_X(i) = \begin{pmatrix} i + r - 1 \\ r - 1 \end{pmatrix} p^r (1-p)^i \quad \text{and} \quad p_Y(j) = \begin{pmatrix} j + s - 1 \\ s - 1 \end{pmatrix} p^s (1-p)^j,$$

   where $r, s \in \mathbb{N}$. [Hint: $(1-x)^{-r} = \sum_{i=0}^{\infty} \begin{pmatrix} r + i - 1 \\ i \end{pmatrix} x^i$]

   What kind of distribution is Z in each case?

4. Let X_1, X_2 be independent exponential random variables and their density function are defined by

   $$f_{X_k} = \begin{cases} ke^{-kx} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   Suppose $Y = \max\{X_1, X_2\}$. Find the distribution function of Y. Hence find its probability density function. Find $\mathrm{Var}[Y]$.