Chapter 3 Vector Spaces

Section 3.1 Definition and examples

Definition (Vector space) Let V be a set of objects (called *elements*) on which the operations of *addition* and *scalar multiplication* are defined that satisfy the two closure properties and eight axioms below.

- C1 For each $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} \in V$
- C2 For each $\alpha \in \mathbf{R}$ and each $\mathbf{y} \in V$, $\alpha\mathbf{y} \in V$.
- A1 $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$, for all $\mathbf{x}, \mathbf{y} \in V$;
- A2 $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$;
- A3 There exists a zero element $\mathbf{0} \in V$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in V$;
- A4 For each element $\mathbf{x} \in V$, there exists an additive inverse $\mathbf{y} \in V$ such that $\mathbf{x} + \mathbf{y} = \mathbf{0}$;
- A5 $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ for each scalar α and any $\mathbf{x}$ and $\mathbf{y}$ in V.
- A6 $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ for any scalars α and β and any $\mathbf{x}$ in V.
- A7 $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for any scalars α and β and any $\mathbf{x} \in V$.
- A8 $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$.

Example The Vector Space $\mathbf{R}^n$ (Euclidean Vector Space)

Let $\mathbf{R}^n = \{(a_1, a_2, a_3, \cdots, a_n)^T | a_i \in \mathbf{R}\}$.

Define the standard addition and scalar multiplication on $\mathbf{R}^n$ by

$$(a_1, a_2, \cdots, a_n) + (b_1, b_2, \cdots, b_n) = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n)$$

and

$$\alpha(a_1, a_2, \cdots, a_n) = (\alpha a_1, \alpha a_2, \cdots, \alpha a_n)$$

for any $(a_1, a_2, \cdots, a_n), (b_1, b_2, \cdots, b_n) \in \mathbf{R}^n$ and $\alpha \in \mathbf{R}$.

$\mathbf{R}^n$ is a vector space. In this case, the zero vector is $(0, \cdots, 0)$.

Notation ($\mathbf{e}_i$) Denote $\mathbf{e}_i$ to be the vector with entry 1 in the ith position and entries 0 in other positions.

Example The Vector Space P_n

Let P_n be the set of all polynomials of degree less than n. That is, $P_n = \{a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 | a_i \in \mathbf{R}\}$.

Define the standard addition and scalar multiplication of polynomials by

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots (a_{n-1} + b_{n-1})x^{n-1}$$

and

$$\alpha p(x) = \alpha a_0 + \alpha a_1 x + \cdots + \alpha a_{n-1}x^{n-1}$$

for any $p(x) = a_0 + \cdots + a_{n-1}x^{n-1},\ q(x) = b_0 + \cdots + b_{n-1}x^{n-1} \in P_n$ and $\alpha \in \mathbf{R}$.

Then P_n is a vector space.

Example

Let $\mathbf{R}^{m \times n}$ be the set of all $m \times n$ matrices with real entries.

The addition and scalar multiplication in $\mathbf{R}^{m \times n}$ are defined as the usual matrix addition and scalar multiplication defined in Section 1.3.

We have check in Section 1.3 that $\mathbf{R}^{m \times n}$ satisfies all axioms of a vector space, and hence is a vector space. In particular, the additive identity is the zero matrix.

Example

Let V be the set of infinite sequences. That is $V = \{(a_1, a_2, a_3, \cdots)|a_i \in \mathbf{R}\}$.
For example, $(1, -1, 1, -1, \cdots)$, $(1, 2, 3, 4, \cdots)$,$(1, 4, 7, 11, \cdots)$ are in V.

Define $\mathbf{a} + \mathbf{b}$ and $\alpha\mathbf{a}$ respectively by

$$(\mathbf{a} + \mathbf{b})_i = a_i + b_i \quad \text{and} \quad (\alpha\mathbf{a})_i = \alpha a_i$$

For example, $2(1, -1, 1, -1, \cdots) = (2, -2, 2, -2, \cdots)$

$$(1, 2, 3, 4, \cdots) + (1, 1, 1, 1, \cdots) = (2, 3, 4, 5, \cdots).$$

In this case, the zero vector is $(0, 0, 0, 0, \cdots)$.
It can be verified that all the vector space axioms hold. V is a vector space.

Example

Let $C[a, b]$ be the set of all continuous functions with domain $[a, b]$ and range $\mathbf{R}$.

The sum $f + g$ of two functions in $C[a, b]$ is defined by

$$(f + g)(x) = f(x) + g(x)$$

for all x in $[a, b]$.

For any scalar α, the scalar multiplication αf in $C[a, b]$ is defined by

$$(\alpha f)(x) = \alpha f(x)$$

for all x in $[a, b]$.

Then $C[a, b]$ is a vector space.

Example Let $V = \{(a,1) : a \in \mathbf{R}\} \subset \mathbf{R}^2$. Define a new addition $+_{new}$ and scalar multiplication $\cdot_{new}$ of V by

$$(a,1) +_{new} (b,1) := (a+b,1)$$

and

$$\alpha \cdot_{new} (a,1) := (\alpha a, 1)$$

for any $(a,1),(b,1) \in V$ and $\alpha \in \mathbf{R}$.

Example Let $V = \{(a, 1) : a \in \mathbf{R}\} \subset \mathbf{R}^2$. Define a new addition $+_{new}$ and scalar multiplication $\cdot_{new}$ of V by

$$(a, 1) +_{new} (b, 1) := (a + b, 1)$$

and

$$\alpha \cdot_{new} (a, 1) := (\alpha a, 1)$$

for any $(a, 1), (b, 1) \in V$ and $\alpha \in \mathbf{R}$.
In this case, the zero vector is $(0, 1)$, but not $(0, 0)$.
All the vector space axioms hold.
Thus, V over $\mathbf{R}$, with $+_{new}$ and $\cdot_{new}$, is a vector space.

Example Let $V = \{(x, y)^T | x, y \in \mathbf{R}\}$. Define addition componentwise, that is

$$(x_1, x_2)^T + (y_1, y_2)^T = (x_1 + y_1, x_2 + y_2)^T$$

and define scalar multiplication by $\alpha \in \mathbf{R}$ to be

$$\alpha(x_1, x_2)^T = (\alpha x_1, 0)^T.$$

Determine whether or not V under these operations is a vector space.

Example Let $V = \{(x, y)^T | x, y \in \mathbf{R}\}$. Define addition componentwise, that is

$$(x_1, x_2)^T + (y_1, y_2)^T = (x_1 + y_1, x_2 + y_2)^T$$

and define scalar multiplication by $\alpha \in \mathbf{R}$ to be

$$\alpha(x_1, x_2)^T = (\alpha x_1, 0)^T.$$

Determine whether or not V under these operations is a vector space.

Solution Let $\mathbf{x} = (2, 3)^T$. By the definition of scalar multiplication, $1\mathbf{x} = 1(2, 3)^T = (1 \times 2, 0)^T = (2, 0)^T \neq \mathbf{x}$. So axiom A8 (the existence of a multiplicative identity) does not hold. V is not a vector space under these prescribed operations.

Example Let V denote the set of all polynomials of degree n, $n \geq 1$,

$$V = \{a_0 + a_1 x + \cdots + a_n x^n : a_0, \cdots, a_n \in \mathbf{R},\ a_n \neq 0\}.$$

Define the standard addition and scalar multiplication of polynomials by

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots (a_n + b_n)x^n$$

and

$$\alpha p(x) = \alpha a_0 + \alpha a_1 x + \cdots + \alpha a_n x^n$$

for any $p(x) = a_0 + \cdots + a_n x^n,\ q(x) = b_0 + \cdots + b_n x^n \in V$ and $\alpha \in \mathbf{R}$. Determine whether or not V under these operations is a vector space.

Solution
Since $(1 + x^n) + (-x^n) = 1 \notin V$, the addition is not closed. That is, C1 is not satisfied. So V, with the standard addition and scalar multiplication of polynomials, is NOT a vector space.

Theorem If V is a vector space and $\mathbf{x}$ is any element of V, then

(i) $0\mathbf{x} = \mathbf{0}$

(ii) If $\mathbf{x} + \mathbf{y} = \mathbf{0}$ and $\mathbf{x} + \mathbf{z} = \mathbf{0}$, then $\mathbf{y} = \mathbf{z}$. That is, the additive inverse of $\mathbf{x}$ is unique.

(iii) $\mathbf{x} + (-1)\mathbf{x} = \mathbf{0}$. That is, the additive inverse of $\mathbf{x}$ is $(-1)\mathbf{x}$.

Proof A proof can be found in the textbook.

Example In $\mathbf{R}^3$,

(i) $0(1, 2, 3)^T = (0, 0, 0)^T$

(ii) If $(1, 2, 3)^T + (y_1, y_2, y_3)^T = (0, 0, 0)^T$ and $(1, 2, 3)^T + (z_1, z_2, z_3)^T = (0, 0, 0)^T$, then $(y_1, y_2, y_3)^T = (z_1, z_2, z_3)^T$.

(iii) $(1, 2, 3)^T + (-1)(1, 2, 3)^T = (1, 2, 3)^T + (-1, -2, -3)^T = (1 - 1, 2 - 2, 3 - 3)^T = (0, 0, 0)^T$.

Remark Since the additive inverse of $\mathbf{x}$ is unique, we write $-\mathbf{x}$ for the additive inverse of $\mathbf{x}$.