2023-24 First Semester

MATH2043 Ordinary Differential Equations (1002)

Assignment 8 Suggested Solutions

1. i Since $Q(x)/P(x)$ and $R(x)/P(x)$ have singularities at x_0, hence a singular point. Further

$$x\frac{Q(x)}{P(x)} = x\frac{0}{x^2} = 0, \qquad x^2\frac{R(x)}{P(x)} = x^2\frac{1}{4x^2} = \frac{1}{4}$$

Since both $x\frac{Q(x)}{P(x)}$ and $x^2\frac{R(x)}{P(x)}$ are analytic at $x=0$, so $x=0$ is a regular singular point.

ii Assume that there is a solution: $y = \sum_{n=0}^{\infty} a_n x^{r+n}$, with $a_0 \neq 0$. Substitute y into the D.E.:

$$x^2\sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n-2} + (x^2+\frac{1}{4})\left[\sum_{n=0}^{\infty} a_n x^{r+n}\right]$$

$$= \sum_{n=0}^{\infty} a_n x^{r+n+2} + \sum_{n=0}^{\infty}\left[a_n(r+n)(r+n-1)+\frac{1}{4}a_n\right]x^{r+n}$$

$$= a_0\left[r(r-1)+\frac{1}{4}\right]x^r + a_1\left[(r+1)r+\frac{1}{4}\right]x^{r+1} + \sum_{n=2}^{\infty}\left[a_{n-2}+a_n(r+n)(r+n-1)+\frac{1}{4}a_n\right]x^{r+n} = 0$$

Since $a_0 \neq 0$, the indicial equation is:

$$r^2+(p_0-1)r+q_0 = 0, \quad \to r^2 - r + \frac{1}{4} = 0$$

$$\to r_1 = r_2 = 1/2$$

And the recurrence relations are:

$$a_1\cdot(\frac{3}{2}\cdot\frac{1}{2}+\frac{1}{4}) = 0, \text{ and } a_{n-2}+a_n(r+n)(r+n-1)+\frac{1}{4}a_n = 0, \text{ for } n=2,3,\cdots$$

$$\to \quad a_1 = 0, \ a_n = \frac{-1}{(n+r-1/2)^2}a_{n-2}, \quad n=2,3,\cdots$$

iii For $r=1/2$, we have $a_1=0$ and $a_n = -\frac{a_{n-2}}{n^2}, \quad n=2,3,\cdots$.
With $a_0=1$,

$$\begin{aligned} y_1(x) &= 1\cdot x^{1/2} - \frac{x^{2+1/2}}{(2!)^2} + \frac{x^{4+1/2}}{(4!)^2(2!)^2} - \frac{x^{6+1/2}}{(6!)^2(4!)^2(2!)^2} + \cdots \\ &= \sum_{k=0}^{\infty}(-1)^k\frac{1}{2^{2k}(k!)^2}x^{2k+\frac{1}{2}} \end{aligned}$$

2. (a) Since $Q(x)/P(x)$ and $R(x)/P(x)$ have singularities at both $x_0 = \pm 1$, they are singular. Further

$$p(x) = (x-1)\frac{Q(x)}{P(x)} = \frac{x}{x+1}, \qquad q(x) = (x-1)^2\frac{R(x)}{P(x)} = \frac{-\alpha^2(x-1)}{x+1}$$

Both $p(x)$ and $q(x)$ are analytic at $x=1$, it is regular singular.
Similarly, both $(x+1)\frac{Q(x)}{P(x)}$ and $(x+1)^2\frac{R(x)}{P(x)}$ are analytic at -1, and hence regular singular.

(b) When $\alpha = 0$, the point $x_0 = 1$ is regular singular to $(1-x^2)y'' - xy' = 0$. Assume a solution has the form

$$y_1(x) = (x-1)^{r} \sum_{n=0}^{\infty} a_n (x-1)^n, \qquad \text{with } a_0 \neq 0.$$

Substitute y into the D.E.:

$$\begin{aligned}
&(1-x^2)\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n(x-1)^{r+n-2} - x\sum_{n=0}^{\infty}(r+n)a_n(x-1)^{r+n-1} \\
&= -(x-1)(x-1+2)\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n(x-1)^{r+n-2} - (x-1+1)\sum_{n=0}^{\infty}(r+n)a_n(x-1)^{r+n-1} \\
&= -\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n} - 2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-1} \\
&\quad -\sum_{n=0}^{\infty}(r+n)a_n x^{r+n} - \sum_{n=0}^{\infty}(r+n)a_n(x-1)^{r+n-1} \\
&= -\sum_{n=0}^{\infty}(r+n)^2 a_n x^{r+n} - \sum_{n=0}^{\infty}(r+n)(2n+2r-1)a_n x^{r+n-1} \\
&= -r(2r-1)a_0 x^{r-1} - \sum_{n=0}^{\infty}\left[(r+n)^2 a_n + (r+n+1)(2r+2n+1)a_{n+1}\right]x^{r+n} = 0
\end{aligned}$$

Since $a_0 \neq 0$, the indicial equation is:

$$r^2 + (p_0-1)r + q_0 = 0 \qquad \text{or} \qquad r(2r-1) = 0 \;\rightarrow\; r_1 = \frac{1}{2}, r_2 = 0.$$

And the recurrence relation is

$$(r+n)^2 a_n + (r+n+1)(2r+2n+1)a_{n+1} = 0, \qquad \text{for } n = 0,1,2,\cdots$$

$$a_n = -\frac{(r+n-1)^2}{(n+r)(2r+2n-1)}a_{n-1}, \quad n = 1,2,\cdots$$

At the larger exponent $r_1 = 1/2$, we have

$$\begin{aligned}
a_n &= (-1)^n \frac{[(r+n-1)!]^2}{2^n(n+r)!(r+n-1/2)!}a_0 = (-1)^n \frac{[(n-1/2)!]^2}{2^n(n+1/2)!n!}a_0 \\
&= (-1)^n \frac{(2n-1)^2(2n-3)^2\cdots 1^2}{2^n(2n+1)!}a_0
\end{aligned}$$

A solution is

$$y_1(x) = (x-1)^{1/2}\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)^2(2n-3)^2\cdots 1^2}{2^n(2n+1)!}(x-1)^n\right].$$

3. The matrix A has eigenpairs $\lambda_1 = 2$, $\boldsymbol{\xi}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ and $\lambda_2 = -3$, $\boldsymbol{\xi}_2 = \begin{pmatrix}1\\-4\end{pmatrix}$. The general solution is

$$\mathbf{x}_H(t) = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\-4\end{pmatrix}e^{-3t}.$$