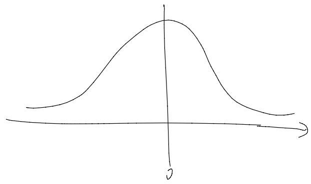
2024年3月8日 8:19

Standard Normal :

$Z \sim N(0,1)$



$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad , \quad z \in (-\infty, \infty)$$

Claim : Let $X = Z^2$. then the distribution of X is
(def)

$$f_X(x) = \frac{1}{\sqrt{2\pi x}} e^{-\frac{x}{2}} = \frac{1}{\sqrt{2}\cdot\Gamma(\frac{1}{2})} x^{\frac{1}{2}-1} e^{-\frac{1}{2}x} \quad , \quad x > 0$$

which is a χ^2- distribution with degree of freedom 1, $\chi^2(1)$.

↳ Transformation : $X = Z^2 \rightarrow Z = \pm\sqrt{X} = \begin{cases} \sqrt{x}, & z_1(x) \\ -\sqrt{x} & z_2(x) \end{cases}$ $\quad Z \rightarrow X$

$(-\infty, \infty) \qquad (0, \infty)$

$$f_X(x) = f_Z(z_1(x))\cdot|J_1| + f_Z(z_2(x))\cdot|J_2| \qquad \text{not one-to-one}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x}{2}}\cdot\left|\frac{\partial\sqrt{x}}{\partial x}\right| + \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}}\left|\frac{\partial(-\sqrt{x})}{\partial x}\right|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}}\cdot\frac{1}{2\sqrt{x}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}}\cdot\frac{1}{2\sqrt{x}}$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi x}} e^{-\frac{x}{2}} & , \quad x > 0 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

At the meanwhile $Y \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$ is

$$f_Y(y) = \frac{(\frac{1}{2})^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} y^{\frac{1}{2}-1} e^{-\frac{1}{2}y} = \frac{1}{\sqrt{2}\cdot\Gamma(\frac{1}{2})} y^{\frac{1}{2}-1} e^{-\frac{1}{2}y} \ , \ y > 0$$

And we can see $\chi^2(1)$ distribution is in fact $\text{Gamma}(\frac{1}{2}, \frac{1}{2})$

$\text{Gamma}(\frac{1}{2}, \frac{1}{2}) \qquad Z^2 , \ Z \sim N(0,1)$

$X \sim \chi^2(1)$

The mgf of $Y \sim \text{Gamma}(\alpha, \beta)$ is

$$M_Y(t) = \frac{\beta^\alpha}{(\beta - t)^\alpha} = \frac{1}{(1 - t/\beta)^\alpha} = (1 - t/\beta)^{-\alpha}$$

$$E(Y) = M'_Y(0) = -\alpha(1 - t/\beta)^{-\alpha-1}\cdot(-\tfrac{1}{\beta})\Big|_{t=0} = \alpha/\beta$$

$$Var(Y) = E(Y^2) - [E(Y)]^2 = M''_Y(0) - [M'_Y(0)]^2 = \alpha/\beta^2$$

For $X \sim \chi^2(1)$, $d.f. = 1$, which is $Gamma(\frac{1}{2}, \frac{1}{2})$

$$E(X) = \frac{1}{2}\Big/\frac{1}{2} = 1 \;(= d.f.), \quad Var(X) = \frac{1}{2}\Big/\frac{1}{2^2} = 2 \;(= 2 \cdot d.f.)$$

What about i.i.d. $Z_i \sim N(0,1)$,

$$X = \sum_{i=1}^{n} Z_i^2 \sim \chi^2(n)$$

When $n=1$, $Z_1^2 \sim \chi^2(1)$, $M_{Z_i^2}(t) = \frac{1}{(1-2t)^{1/2}}$

When $n = r$, $\in Z^+$, Z_i^2 are (mutually) independent, then

$$M_X(t) = E(e^{tZ_1^2 + tZ_2^2 + \cdots + tZ_r^2}) = E(e^{tZ_1^2})\,E(e^{tZ_2^2})\cdots E(e^{tZ_r^2})$$

$$= \left[\frac{1}{(1-2t)^{1/2}}\right]^r = (1-2t)^{\frac{r}{2}}$$

So $X = \sum_{i=1}^{n} Z_i^2 \sim \chi^2(r)$ or $Gamma(\frac{r}{2}, \frac{1}{2})$

Thm For independent X_1 & X_2, if $X_1 \sim \chi^2(r_1)$, $X_2 \sim \chi^2(r_2)$

then $X_1 + X_2$ follows $\chi^2(r_1 + r_2)$

Remarks: If $X \sim N(\mu, \sigma^2)$, then

$$\left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi^2(1)$$

example: $\bar{X} \sim N(\mu, \sigma^2/n)$ for i.i.d $X_i \sim N(\mu, \sigma^2)$

$$\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(1)$$

△ t - distribution

χ^2

$$T = \frac{Z}{\sqrt{U/r}} \quad \rightarrow \quad T^2 = \frac{Z^2}{U/r}$$

where $Z \sim N(0,1)$, $U \sim \chi^2(r)$, Z & U independent

Then T follows a t-distribution with r degrees of freedom

Its pdf can be derived from transformation or mgf's.

Remark: As $r \to \infty$, T will be looked like a normal distribution

△ F-distribution

$$F = \frac{U/r_1}{V/r_2} \quad \rightarrow \quad \frac{1}{F}$$

where U and V are independent. $U \sim \chi^2(r_1)$, $V \sim \chi^2(r_2)$.

Then F follows a F-dist with d.f. r_1 and r_2

order matters.

Prove Student's Thm:

② $\bar{X}$ & S^2 are independent:

Consider $\bar{X}$ and $\vec{X} - \bar{X}$ by letting

$$\vec{Y} = \begin{bmatrix} \bar{X} \\ \hline \vec{X} - \bar{X} \end{bmatrix} = \begin{bmatrix} \frac{1}{n} \vec{1}_{1\times n} \\ \hline (I_n - \frac{1}{n} \vec{1}_{n\times n}) \end{bmatrix} \vec{X} = A\vec{X}$$

then $\vec{Y} \sim N_{n+1}(A\vec{\mu}, \sigma^2 A I_n A^T)$ since $\vec{X} \sim N_n(\vec{\mu}, \sigma^2 I_n)$.

Consider the covariance of $\bar{X}$ and $\vec{X} - \bar{X}$ in $\sigma^2 AA^T$,

$$\sigma^2 AA^T = \sigma^2 \begin{bmatrix} A_{11} \\ \hline \end{bmatrix} \begin{bmatrix} A_{11}^T & A_{21}^T \end{bmatrix} = \begin{bmatrix} 1\times 1 & (\Sigma_{12})_{1\times n} \\ \Sigma_{21} & \end{bmatrix}$$

$$\sigma^2 AA^T = \sigma^2 \begin{bmatrix} A_{11} \\ \hdashline A_{21} \end{bmatrix} \left[A_{11}^T \;\vdots\; A_{21}^T \right] = \left[\begin{array}{c|c} 1\times 1 & (\Sigma_{12})_{1\times n} \\ \hline \Sigma_{21} & n\times n \end{array} \right]$$

$$\Sigma_{12} = A_{11} A_{21}^T = \left(\tfrac{1}{n}\vec{1}_{1\times n}\right)\left(I_n - \tfrac{1}{n} 1_{n\times n}\right)^T = \tfrac{1}{n}\vec{1}_{1\times n} - \tfrac{1}{n}[n, n, \cdots, n] = \vec{0}_{1\times n}$$

Thus, $\bar{X}$ and $\vec{X}-\bar{X}$ are independent since $\Sigma_{12} = 0$.

$\bar{X}$ and S^2 〃 〃 since S^2 is a function of $\vec{X}-\bar{X}$.

③. Consider

$$\frac{(n-1)\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left[(X_i-\mu+\mu-\bar{X})^2\right]$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}\left[(X_i-\mu)^2 + 2(X_i-\mu)(\mu-\bar{X}) + (\mu-\bar{X})^2\right]$$

$$= \frac{1}{\sigma^2}\left[\sum_{i=1}^{n}(X_i-\mu)^2 + 2n(\bar{X}-\mu)(\mu-\bar{X}) + n(\mu-\bar{X})^2\right]$$

$$= \frac{1}{\sigma^2}\left[\sum_{i=1}^{n}(X_i-\mu)^2 - n(\bar{X}-\mu)^2\right]$$

$$= \underbrace{\sum_{i=1}^{n}\frac{(X_i-\mu)^2}{\sigma^2}}_{\sim \chi^2(n)} - \underbrace{\frac{(\bar{X}-\mu)^2}{\sigma^2/n}}_{\sim \chi^2(1)}$$

independent

By mgf,

$$M_{\frac{(n-1)S^2}{\sigma^2}}(t) = \frac{(1-2t)^{-\frac{n}{2}}}{(1-2t)^{-\frac{1}{2}}} = (1-2t)^{-\frac{n-1}{2}}$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$