Ordinary Differential Equations

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Chapter 2: First Order Differential Equations

General form

$$F(u, u') = 0$$

Example 1. (Falling object in the air)

The motion of the object is governed by the Newton's law. Let $v(t)$ be the velocity of the object at time t.

$$mv' = mg - \gamma v,$$

where m is the mass of the object, g is the gravitational constant, and γ is the coefficient of air resistant force.

This is a first order linear ODE.

1 Method of integrating factors

Example 2. Find the general solutions of

$$u' + u = 2$$

Idea: combine $u'+u$ into the derivative of another function.

Consider the product rule of differentiation:

$$(fg)' = f'g + fg'.$$

Let $f = u(t)$, $g = e^t$, then

$$[u(t)e^t]' = u'e^t + ue^t = e^t(u'+u).$$

Now, we multiply the original equation by e^t:

$$(u'+u)e^t = 2e^t \Longrightarrow [u(t)e^t]' = 2e^t \Longrightarrow u(t)e^t = \int 2e^t \mathrm{d}t = 2e^t + c$$

Divide by e^t:

$$u(t) = e^{-t}[2e^t + c] = 2 + ce^{-t}.$$

Method of integrating factors:

Consider the 1st order linear ODE (standard form):

$$u'(t) + p(t)u(t) = q(t).$$

Multiply left side by $\mu(t)$:

$$[u'(t) + p(t)u(t)]\mu(t) = u'\mu + pu\mu.$$

We want

$$\mu = g, p\mu = g'$$

i.e.

$$\mu'(t) = p(t)\mu(t) \Longrightarrow \mu(t) = e^{\int p(t)\mathrm{d}t}.$$

Check (by the chain rule and fundamental theorem of calculus)

$$\mu' = e^{\int p(t)\mathrm{d}t}p(t) = \mu(t)p(t).$$

Derivation directly:

$$\mu'(t) = p(t)\mu(t) \Longrightarrow \frac{\mu'(t)}{\mu(t)} = p(t) \Longrightarrow [\ln\mu(t)]' = p(t)$$

$$\Longrightarrow \ln\mu(t) = \int p(t)\mathrm{d}t \Longrightarrow \mu(t) = e^{\int p(t)\mathrm{d}t.}$$

So the original ODE becomes

$$[\mu(t)u(t)]' = \mu(t)q(t) \Longrightarrow \mu(t)u(t) = \int \mu(t)q(t)\mathrm{d}t + c$$

$$\Longrightarrow u(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)\mathrm{d}t + c\right],$$

where

$$\mu(t) = e^{\int p(t)\mathrm{d}t}$$

is called the **integrating factor**.

Example 3. Find the **general solution** of

$$(4+t^2)\frac{dy}{dt} + 2ty = 4t$$

Answer: Rewrite it in the standard form

$$y' + \frac{2t}{4+t^2}y = \frac{4t}{4+t^2}.$$

Here $p(t)=\frac{2t}{4+t^2}$. Then find the integrating factor $\mu(t)$:

$$\mu(t)=e^{\int p(t)\mathrm{d}t}=e^{\ln(4+t^2)}=4+t^2.$$

So the solution is

$$y(t)=\frac{1}{4+t^2}\bigg(\int(4+t^2)\frac{4t}{4+t^2}\mathrm{d}t+c\bigg)=\frac{1}{4+t^2}(2t^2+c)$$

Example 4. Solve the **initial value problem**

$$ty'+2y=4t^2$$
$$\text{initial condition: } y(1)=2$$

Answer: First rewrite the equation into the standard form:

$$y'+\frac{2}{t}y=4t.$$

Find the integrating factor:

$$\mu=e^{\int\frac{2}{t}\mathrm{d}t}=e^{2\ln|t|}=e^{\ln t^2}=t^2.$$

The general solution is

$$y=\frac{1}{t^2}\bigg[\int t^2 4t\mathrm{d}t+c\bigg]=\frac{1}{t^2}[t^4+c]=t^2+\frac{c}{t^2}.$$

Plugging the initial condition:

$$y(1)=2\Longrightarrow 1+\frac{1}{c}=2\Longrightarrow c=1.$$

The solution of the initial value problem is

$$y=t^2+\frac{1}{t^2}.$$

2 Separable equations

Example 5. Solve

$$\frac{dy}{dx}=\frac{x^2}{1-y^2}.$$

This equation is nonlinear. We can separate the variables x and y as follows

$$(1-y^2)\mathrm{d}y=x^2\mathrm{d}x\Longrightarrow\int(1-y^2)\mathrm{d}y=\int x^2\mathrm{d}x$$

$$\Longrightarrow y-\frac{y^3}{3}=\frac{x^3}{3}+c$$

This is an example of **implicit solutions**.

Definition 6

An ODE in the form of

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{f(x)}{g(y)}$$

is called ***separable.***

We can solve it as follows

$$\int f(x)\mathrm{d}x=\int g(y)\mathrm{d}y$$

Example 7. Solve the initial value problem

$$\frac{dy}{dx}=\frac{3x^2+4x+2}{2(y-1)}, \quad y(0)=-1$$

Solution:

$$\int (3x^2+4x+2)\mathrm{d}x=\int 2(y-1)\mathrm{d}y$$

$$\Rightarrow \quad x^3+2x^2+2x=y^2-2y+c.$$

Plugging the initial condition

$$y(0)=-1\Rightarrow 0=1+2+c\Rightarrow c=-3.$$

The solution to the initial value problem is

$$x^3+2x^2+2x=y^2-2y-3.$$

Question 1. What is the domain and range of the solution?

3 Exact Equations

3.1 Motivation and definition

Suppose $\psi(x,y)=c$ is an solution of some ODE. Taking $d/\mathrm{d}x$ on both sides of the solution.

$$\frac{d}{\mathrm{d}x}\psi(x,y)=\frac{d}{\mathrm{d}x}c\Rightarrow\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow M(x,y)+N(x,y)y'=0,$$

where

$$M(x,y)=\partial_x\psi, \quad N(x,y)=\partial_y\psi.$$

Example 8. Solve $2x+y^2+2xyy'=0$.

Answer. Guess the solution. Let $\psi=x^2+y^2x$. Then

$$\psi_x=2x+y^2, \quad \psi_y=2xy.$$

So

$$0=\psi_x+\psi_y y'=\frac{d}{\mathrm{d}x}\psi(x,y)$$

So the solution is

$$\psi(x,y)=c.$$

Definition

An ODE of the form

$$M(x,y)+N(x,y)y'=0 \;\; \text{or} \;\; M(x,y)\mathrm{d}x+N(x,y)\mathrm{d}y=0$$

is called ***exact*** *if there exists $\psi(x,y)$ such that*

$$\psi_x=M, \quad \psi_y=N.$$

The solution of the equation is

$$\psi(x,y)=c,$$

where c is an arbitrary constant.

3.2 Theorem and method

Theorem 9

Suppose an ODE can be written in the form

$$M(x,y)+N(x,y)y'=0 \;\; or \;\; M(x,y)\mathrm{d}x+N(x,y)\mathrm{d}y=0 \tag{1}$$

where the functions M, N, M_y and N_x are all continuous in the rectangular region $R=[a,b]\times[c,d]$. Then Eq. (1) is an exact differential equation ***if and only if***

$$M_y(x,y)=N_x(x,y), \forall (x,y)\in R.$$

Proof. "$\Longrightarrow$". Suppose Eq. (1) is exact. Then there exists a $\psi(x,y)$ such that

$$\psi_x=M, \quad \psi_y=N.$$

Then

$$M_y=\psi_{xy}, \quad N_x=\psi_{yx}.$$

Since M_y, N_x are continuous, we have ψ_{xy} and ψ_{yx} are continuous. So

$$\psi_{xy}=\psi_{yx}.$$

i.e.

$$M_y=N_x.$$

"$\Longleftarrow$" Suppose $M_y=N_x$. We want to find a function $\psi(x,y)$ such that $\psi_x=M$ and $\psi_y=N$. Let

$$\psi=\int M(x,y)\mathrm{d}x+h(y).$$

Then $\psi_x=M$, and

$$\psi_y=\partial_y\int M(x,y)\mathrm{d}x+h'(y).$$

We want $\psi_y=N$, that is

$$h'(y)=N(x,y)-\partial_y\int M(x,y)\mathrm{d}x.$$

We need the RHS to be independent of x. That is

$$\frac{\partial}{\partial x}\left[N(x,y)-\partial_y\int M(x,y)\mathrm{d}x\right]=0.$$

Let's check:

$$\frac{\partial}{\partial x}\left[N(x,y)-\partial_y\int M(x,y)\mathrm{d}x\right] \;=\; N_x-\partial_y\partial_x\int M\mathrm{d}x=N_x-M_y=0.$$

□

Example 10. Solve the ODE

$$(y\cos x+2xe^y)+(\sin x+x^2e^y-1)y'=0.$$

Answer:

$$\begin{aligned} M_y &= \cos x+2xe^y \\ N_x &= \cos x+2xe^y \end{aligned}$$

So $M_y=N_x$, and the equation is exact.

Next, let

$$\psi=\int M dx=\int y(\cos x)+2xe^y\,dx=y(\sin x)+x^2e^y+h(y).$$

Then

$$\psi_y = \sin x + x^2 e^y + h'(y) = N = \sin x + x^2 e^y - 1$$

$$\Longrightarrow \quad h'(y) = -1 \quad \Rightarrow \quad h(y) = -y.$$

So the solution is

$$\psi = y(\sin x) + x^2 e^y - y = c.$$

Exercise. Solve the above equation, but using $\psi = \int N dy + h(x)$ first.

Question. What is the relationship between separable and exact equations?

3.3 Integrating factors

Sometimes we can multiply a function to a non-exact equation to make it exact. Take a function $\mu(x, y) \neq 0$,

$$\begin{aligned} M(x, y)\, dx + N(x, y)\, dy &= 0 \\ \mu(x, y)[M(x, y)\, dx + N(x, y)\, dy] &= 0 \\ \mu(x, y) M(x, y)\, dx + \mu(x, y) N(x, y)\, dy &= 0 \\ \tilde{M}(x, y)\, dx + \tilde{N}(x, y)\, dy &= 0 \end{aligned}$$

where $\tilde{M}(x, y) = \mu(x, y) M(x, y)$, $\tilde{N}(x, y) = \mu(x, y) N(x, y)$. Then let

$$\tilde{M}_y = \mu_y M + \mu M_y, \quad \tilde{N}_x = \mu_x N + \mu N_x.$$

We want $\tilde{M}_y = \tilde{N}_x$, i.e.

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

Let's choose μ such that $\mu_y = 0$. Then the above equation reduces to

$$\mu M_y = \mu_x N + \mu N_x \quad \Leftrightarrow \mu_x = \frac{M_y - N_x}{N} \mu.$$

If the function $(M_y - N_x)/N$ is a function of x only, then we can solve μ as a separable equation. Here μ is called an integrating factor.

Example 11. Solve the ODE

$$(3xy + y^2) + (x^2 + xy) y' = 0.$$

Answer: It's first order, nonlinear, and not separable. Check if it's exact:

$$M_y = 3x + 2y, \quad N_x = 2x + y.$$

Not exact!. Next, try integrating factors.

$$\frac{M_y - N_x}{N} = \frac{x + y}{x^2 + xy} = \frac{1}{x}$$

is a function of x only! Let

$$\mu'(x) = \frac{1}{x} \mu \quad \Rightarrow \quad \mu(x) = x.$$

Then multiply μ to the original equation:

$$x(3xy + y^2) + x(x^2 + xy) y' = 0$$

$$(3x^2 y + xy^2) + (x^3 + x^2 y) y' = 0$$

Double check the new equation is exact! Then solve it as usual (exercise).

Similarly, if $(N_x - M_y)/M$ is a function of y only, then we can use the integrating factor $\mu(y)$ solving

$$\mu' = \frac{N_x - M_y}{M}\mu.$$

4 Direction fields

Consider the first order ODE:

$$y' = f(t, y)$$

Draw small arrows as a vector $(1, f(t, y))$ at many points (t, y)

Online plotter:

https://aeb019.hosted.uark.edu/dfield.html

Example. Consider

$$y' = \frac{y\cos x}{1 + 3y^3}$$

5 The Existence and Uniqueness Theorem

5.1 Linear equations

Theorem

Consider the initial value problem

$$y' + p(t)y = q(t), \quad y(t_0) = y_0.$$

If p, q are continuous on an interval $I = [a, b]$ containing t_0, then the IVP has a unique solution on I.

Example. Consider

$$ty' + 2y = 4t^2, \quad y(1) = 2.$$

Solve it by integrating factors,

$$y' + \frac{2}{t}y = 4t \quad \Rightarrow \quad \mu = \exp\left[\int \frac{2}{t}\,dt\right] = t^2.$$

$$y = \frac{1}{t^2}\left[\int 4t^3\,dt + c\right] = \frac{1}{t^2}[t^4 + c] = t^2 + \frac{c}{t^2}.$$

Plugging $y(1) = 2$, we obtain $c = 1$. The solution is

$$y = t^2 + \frac{1}{t^2}.$$

Now, $p(t) = \frac{2}{t}$, $q(t) = 4t$. So p, q are continuous in $(-\infty, 0) \cup (0, \infty)$. But $1 \in (0, \infty)$ only, so we know from the theorem the IVP has a unique solution in $(0, \infty)$, which is

$$y = t^2 + \frac{1}{t^2}, \quad t \in (0, \infty).$$

If the initial condition is changed to $y(-1) = 2$, then the solution is

$$y = t^2 + \frac{1}{t^2}, \quad t \in (-\infty, 0).$$

5.2 Nonlinear equations

Theorem

Consider the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0.$$

If f and $\partial_y f$ are continuous on a rectangular domain $R = [a, b] \times [c, d]$ containing the point (t_0, y_0). Then the IVP has a unique solution in some interval I containing t_0.

Example. Consider the IVP.

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$$

It is separable. Let's solve it first,

$$2(y-1)dy = (3x^2 + 4x + 2)dx \quad \Rightarrow \quad y^2 - 2y = x^3 + 2x^2 + 2x + c$$

$$y(0) = -1 \quad \Rightarrow \quad c = 3.$$

The solution is

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

$$y = \frac{2 \pm \sqrt{4 + 4(x^3 + 2x^2 + 2x + 3)}}{2} = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4} = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.$$

Here f and $\partial_y f$ are continuous everywhere except $y = 1$.

Example. Consider

$$y' = y^{1/3}, \quad y(0) = 0 \quad (t \geq 0)$$

First, let's solve it as a separable equation.

$$y^{-1/3}\,dy = dt \quad \Rightarrow \quad \frac{3}{2}y^{2/3} = t + c$$

Plugging $y(0) = 0$ yields $c = 0$. So

$$y = \pm\left(\frac{2}{3}t\right)^{3/2}$$

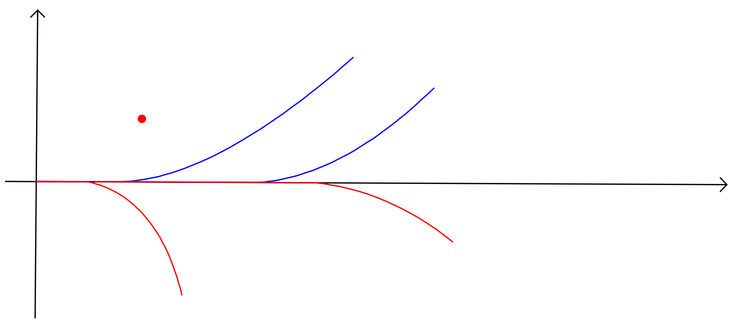
are two solutions. In addition

$$y = 0$$

is also a solution. In fact, we have infinitely many solutions defined as

$$y = \begin{cases} 0, & t < t_0 \\ \left[\frac{2}{3}(t - t_0)\right]^{3/2}, & t \geqslant t_0 \end{cases}, \text{ or } y = \begin{cases} 0, & t < t_0 \\ -\left[\frac{2}{3}(t - t_0)\right]^{3/2}, & t \geqslant t_0 \end{cases}$$

for any $t_0 > 0$. (**Exercise:** check y is continuous and differentiable at $t = t_0$.)



In fact,

$$f = y^{1/3}, \quad \partial_y f = \frac{1}{3} y^{-2/3}.$$

So $\partial_y f$ is discontinuous near $(0,0)$. So there exists no rectangle R containing $(0,0)$ such that $f, \partial_y f$ are both continuous in R. So we can't gaurantee the existence and uniqueness of solution for the IVP.

Note 12. One may not be able to find all solutions to nonlinear equations using one method.

Example. Consider

$$\frac{dy}{dt} = y^2, \quad y(0) = y_0 \neq 0$$

We can first solve it as a separable eqn.

$$y^{-2} dy = dt \quad \Rightarrow \quad -y^{-1} = t + c \quad \Rightarrow \quad y = -\frac{1}{t+c} \quad \Rightarrow \quad y = -\frac{1}{t - \frac{1}{y_0}}.$$

Now $f = y^2, \partial_y f = 2y$ are continuous everywhere. However, the solution is not defined for every t. For example, if $y_0 > 0$,then the solution is defined only in $\left(-\infty, \frac{1}{y_0}\right)$.

6 Applications

6.1 Falling object in the air

$$m v' = m g - \gamma v,$$

where v is the velocity, m, g, γ are constants.

- Analyze the solutions using direction field.
- Solve it by integrating factors.

  $$v' + \frac{\gamma}{m} v = g$$

  integrating factor

  $$\mu = e^{\int \gamma / m} = e^{\frac{\gamma}{m} t}$$

  $$v(t) = e^{-\frac{\gamma}{m} t} \left[\int g e^{\frac{\gamma}{m} t} dt + c \right] = e^{-\frac{\gamma}{m} t} \left[\frac{g m}{\gamma} e^{\frac{\gamma}{m} t} + c \right] = \frac{g m}{\gamma} + c e^{-\frac{\gamma}{m} t}.$$

  If the initial condition is $v(0) = v_0$. Then $c = v_0 - g m / \gamma$. So the solution of the IVP is

  $$v(t) = \frac{g m}{\gamma} + \left[v_0 - \frac{g m}{\gamma} \right] e^{-\frac{\gamma}{m} t}.$$

  So

  $$\lim_{t \to \infty} v(t) = \frac{g m}{\gamma}.$$

  All other solutions converge to the **equilibrium solution** $v = g m / \gamma$ as $t \to \infty$. This equilibrium solution is a **stable** one.

6.2 Compound interest with deposits/withdrawals

Assume the annual interest rate is r. The continuous rate of deposit/withdrawal is k. Then the ODE model for the total balance $u(t)$ is

$$u' = r u + k.$$

integrating factor

$$\mu = e^{-rt}$$

$$u = e^{rt}\left[\int ke^{-rt} + c\right] = e^{rt}\left[-\frac{k}{r}e^{-rt} + c\right] = -\frac{k}{r} + ce^{rt}.$$

If the initial condition is $u(0) = u_0$, then $c = u_0 + k/r$. So the solution of the IVP is

$$u = -\frac{k}{r} + \left(u_0 + \frac{k}{r}\right)e^{rt}.$$

The equilibrium solution is $u = -\frac{k}{r}$, and it is an **unstable** one since all other solutions diverge from it as $t \to \infty$.

6.3 Population dynamics

6.3.1 Exponential growth

$$y' = ry,$$

The solution is

$$y = y_0 e^{rt}$$

where $y_0 = y(0)$.

- If $r > 0$, we have exponential growth
- If $r < 0$, we have exponential decay, such as radioactive decay.

6.3.2 Logistic growth

$$y' = (r - ay)y.$$

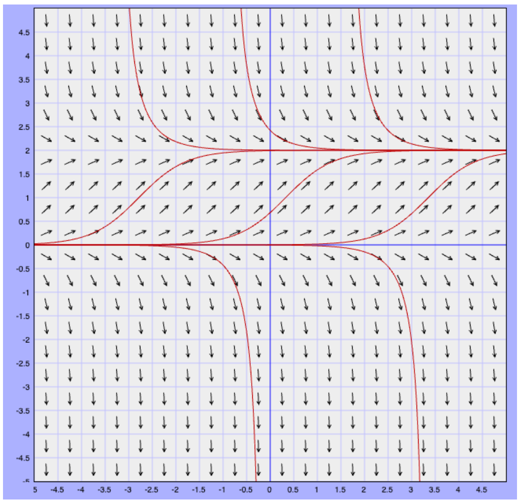
Note that the right-hand-side depends on y only. In general, ODE of the form

$$y' = f(y)$$

is called **autonomous**. There are two equilibrium solutions

$$y = 0, \quad y = \frac{r}{a}.$$

From the direction field we can tell the equilibrium solution $y = 0$ is unstable, while the solution $y = \frac{r}{a}$ is stable.



Now let's solve the equation:

$$\frac{dy}{(r-ay)y}=dt \quad\Rightarrow\quad \int\frac{dy}{(r-ay)y}=\int dt \quad\Rightarrow$$

$$\frac{1}{(r-ay)y}=\frac{A}{y}+\frac{B}{r-ay}=\frac{A(r-ay)+By}{y(r-ay)} \quad\Rightarrow\quad 1=A(r-ay)+By$$

$$y=0\Rightarrow A=1/r,\quad y=r/a\Rightarrow B=a/r$$

So

$$\int\frac{dy}{(r-ay)y}=\int\frac{1}{ry}+\frac{a/r}{r-ay}dy=\frac{1}{r}\ln|y|+\frac{a}{r}\left(\frac{1}{-a}\right)\ln|r-ay|=\frac{1}{r}\ln|y|-\frac{1}{r}\ln|r-ay|$$

$$=\frac{1}{r}\ln\frac{|y|}{|r-ay|}=\int dt=t+c$$

$$\Rightarrow\quad \frac{|y|}{|r-ay|}=e^{r(t+c)}=ce^{rt}\quad\Rightarrow\frac{y}{r-ay}=ce^{rt}.$$

$$\Rightarrow\quad y=\frac{rce^{rt}}{1+ace^{rt}}=\frac{rc}{e^{-rt}+ac}=\frac{r}{\frac{1}{c}e^{-rt}+a}.$$

Suppose the initial condition is $y(0)=y_0$, then

$$c=\frac{y_0}{r-ay_0}\quad\Rightarrow\quad y=\frac{r}{\frac{r-ay_0}{y_0}e^{-rt}+a}=\frac{ry_0}{ay_0+(r-ay_0)e^{-rt}}=\boxed{\frac{Ky_0}{y_0+(K-y_0)e^{-rt}}},$$

where $K=\frac{r}{a}$. Note that $y'=(r-ay)y=r\left(1-\frac{y}{K}\right)y$

1. If $0<y_0<K$, then $y(t)$ is an increasing function, and $\lim_{t\to\infty}y(t)=K$, but $y(t)<K$ for all $t>0$. Moreover, $\lim_{t\to\infty}y'(t)=0$, and

   $$y''=\frac{d}{dt}\frac{dy}{dt}=\frac{d}{dt}f(y)=f'(y)\frac{dy}{dt}=f'(y)f(y),$$

   where

   $$f(y)=r\left(1-\frac{y}{K}\right)y,\quad f'(y)=r\left(1-\frac{2y}{K}\right)$$

   a. If $0<y<\frac{K}{2}$, then $y''>0$, so the graph is concave up.

   b. If $\frac{K}{2}<y<K$, then $y''<0$, so the graph is concave down.

2. If $y_0>K$, then $y(t)$ is an decreasing function, and $\lim_{t\to\infty}y(t)=K$, but $y(t)>K$ for all $t>0$. Moreover, $\lim_{t\to\infty}y'(t)=0$, and $y''(t)>0$ for all t.

7 Euler's method

Consider a general 1st order ODE

$$y'=f(t,y).$$

Take (t_0,y_0), then

$$y'(t_0)=f(t_0,y_0)$$

$$y'(t_0)=\lim_{h\to0}\frac{y(t_0+h)-y(t_0)}{h}\approx\frac{y(t_1)-y(t_0)}{t_1-t_0}$$

if $|t_1-t_0|$ is small. So

$$y(t_1)\approx y(t_0)+(t_1-t_0)y'(t_0)=y(t_0)+(t_1-t_0)f(t_0,y_0)$$

Let

$$y_1 = y_0 + (t_1 - t_0) f(t_0, y_0)$$

So $y_1 \approx y(t_1)$. Repeat this process, we obtain an algorithm: For a sequence of $t_0, t_1, t_2, \ldots$

$$\boxed{y_{k+1} = y_k + (t_{k+1} - t_k) f(t_k, y_k)}$$

This sequence of $y_0, y_1, y_2, \ldots$ is an approximation of the true values $y(t_0), y(t_1), y(t_2), \ldots$