Applied Stochastic Process

Solution to Quiz 2

1. (26 points)

   (a) (8 points) If Y is the number of arrivals in $(3, 5]$, then $Y \sim \text{Poisson}(\mu = 0.5 \times 2)$. Therefore,

   $$P(Y = 0) = e^{-1} = 0.3679$$

   (b) (8 points) Let Y_1, Y_2, Y_3 and Y_4 be the numbers of arrivals in the intervals $(0, 1], (1, 2], (2, 3]$, and $(3, 4]$. Then $Y_i \sim \text{Poisson}(0.5)$ and Y_i 's are independent, so

   $$\begin{aligned} P(Y_1 = 1, Y_2 = 1, Y_3 = 1, Y_4 = 1) &= P(Y_1 = 1) \cdot P(Y_2 = 1) \cdot P(Y_3 = 1) \cdot P(Y_4 = 1) \\ &= \left[0.5e^{-0.5}\right]^4 \approx 8.5 \times 10^{-3} \end{aligned}$$

   (c) (10 points) Note that the two intervals $(0, 2]$ and $(1, 4]$ are not disjoint. Thus, we cannot multiply the probabilities for each interval to obtain the desired probability. In particular,

   $$(0, 2] \cap (1, 4] = (1, 2]$$

   Let X, Y, and Z be the numbers of arrivals in $(0, 1], (1, 2]$, and $(2, 4]$ respectively. Then X, Y, and Z are independent, and

   $$\begin{aligned} X &\sim \text{Poisson}(\lambda \cdot 1) \\ Y &\sim \text{Poisson}(\lambda \cdot 1) \\ Z &\sim \text{Poisson}(\lambda \cdot 2) \end{aligned}$$

   Let A be the event that there are one arrival in $(0, 2]$ and three arrivals in $(1, 4]$. We can use the law of total probability to obtain $P(A)$. In particular,

   $$\begin{aligned} P(A) &= P(X + Y = 1 \text{ and } Y + Z = 3) \\ &= \sum_{k=0}^{\infty} P(X + Y = 1 \text{ and } Y + Z = 3 \mid Y = k) P(Y = k) \\ &= P(X = 1, Z = 3 \mid Y = 0) P(Y = 0) + P(X = 0, Z = 2 \mid Y = 1) P(Y = 1) \\ &= P(X = 1, Z = 3) P(Y = 0) + P(X = 0, Z = 2) P(Y = 1) \\ &= P(X = 1) P(Z = 3) P(Y = 0) + P(X = 0) P(Z = 2) P(Y = 1) \\ &= \left(\frac{e^{-\lambda}\lambda^1}{1!}\right) \cdot \left(\frac{e^{-2\lambda}(2\lambda)^3}{3!}\right) \cdot \left(\frac{e^{-\lambda}\lambda^0}{0!}\right) + \left(\frac{e^{-\lambda}\lambda^0}{0!}\right) \cdot \left(\frac{e^{-2\lambda}(2\lambda)^2}{2!}\right) \cdot \left(\frac{e^{-\lambda}\lambda^1}{1!}\right) \\ &= \frac{1}{12}e^{-2} + \frac{1}{4}e^{-2} = \frac{1}{3}e^{-2} \approx 0.0451 \end{aligned}$$

2. (36 points)

(a) (10 points) Let T be the time until catching the fourth fish. Since the times between catching successive fish are i.i.d. exponential random variables with mean $1/\lambda = 1/4$ hour,

$$E[T] = \frac{4}{4} = 1 \quad \text{hour} \quad \text{and} \quad \mathrm{Var}(T) = \frac{4}{4^2} = \frac{1}{4}.$$

(b) (8 points) Let $N_G(t)$ and $N_S(t)$ be the number of grouper and snapper, respectively, caught up to time t. By independent thinning of a Poisson process, the stochastic processes $\{N_G(t) : t \geq 0\}$ and $\{N_S(t) : t \geq 0\}$ are independent Poisson Processes with rates $\lambda_G = \lambda/4 = 1$ and $\lambda_S = 3\lambda/4 = 3$. Hence,

$$P\left(N_G(t+2) - N_G(t) = 4\right) = \frac{e^{-2}2^4}{4!} = \frac{2}{3}e^{-2} \approx 0.0902$$

(c) (8 points) Since the stochastic processes $\{N_G(t) : t \geq 0\}$ and $\{N_S(t) : t \geq 0\}$ are independent Poisson Processes with rates $\lambda_G = \lambda/4 = 1$ and $\lambda_S = 3\lambda/4 = 3$, respectively,

$$\begin{aligned} &P\left(N_G(t+2) - N_G(t) = 4, N_S(t+2) - N_S(t) = 5\right) \\ &\quad = P\left(N_G(t+2) - N_G(t) = 4\right) P\left(N_S(t+2) - N_S(t) = 5\right) \\ &\quad = \left(\frac{e^{-2}2^4}{4!}\right)\left(\frac{e^{-6}6^5}{5!}\right) = \frac{216}{5}e^{-8} = 43.2e^{-8} \approx 0.0145 \end{aligned}$$

(d) (10 points) The total weight of all fish caught up to time t, denoted by $W(t)$ is a compound Poisson process, i.e.,

$$W(t) = \sum_{i=1}^{N(t)} X_i$$

where $\{X_i : i \geq 1\}$ is a sequence of i.i.d. random variables distributed as X, where

$$E[X] = \frac{E\left[W_g\right]}{4} + \frac{3E\left[W_s\right]}{4} = \frac{100}{4} + \frac{3(20)}{4} = 25 + 15 = 40$$

and

$$E\left[X^2\right] = \frac{E\left[W_g^2\right]}{4} + \frac{3E\left[W_s^2\right]}{4} = \frac{100^2 + 20^2}{4} + \frac{3\left(20^2 + 10^2\right)}{4} = 2600 + 375 = 2975$$

Then

$$E[W(t)] = E[N(t)]E[X] = \lambda t E[X] \text{ and } \mathrm{Var}(W(t)) = E[N(t)]E\left[X^2\right] = \lambda t E\left[X^2\right]$$

so that

$$E[W(2)] = 8E[X] = 320 \text{ and } \mathrm{Var}(W(2)) = 8E\left[X^2\right] = 23{,}800$$

3. (18 points) Y is the product of the two numbers on the first die and on the second die, respectively. $\Omega_Y = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, 36\}$. Let $Z = \frac{Y}{X}$ be the number on the first die. Z is independent of X.

(a) (8 poitns)

$$E(Y \mid X) = E(ZX|X) = XE(Z|X) = XE(Z) = X\sum_{j=1}^{6} j \cdot \frac{1}{6} = \frac{21}{6}X = 3.5X$$

(b) (10 points) For $P(Y = y) > 0$,

$$\begin{aligned}
E(X \mid Y = y) &= \sum_{j=1}^{6} jP(X = j \mid Y = y) = \sum_{j=1}^{6} j\frac{P(X = j, Y = y)}{P(Y = y)} \\
&= \sum_{j=1}^{6} j\frac{P(X = j, Zj = y)}{P(Y = y)} = \sum_{j=1}^{6} j\frac{P(X = j)P(Z = y/j)}{P(Y = y)} \\
&= \sum_{j=1}^{6} j\frac{\frac{1}{6} \cdot \frac{1}{6} \cdot \mathbb{1}_{\{ij=y \text{ for some } i=1,2,\cdots,6\}}}{\#\{(a, b) : ab = y, a, b = 1, 2, \cdots, 6\} \cdot \frac{1}{6} \cdot \frac{1}{6}} \\
&= \frac{\sum_{j=1}^{6} j \cdot \mathbb{1}_{\{ij=y \text{ for some } i=1,2,\cdots,6\}}}{\#\{(a, b) : ab = y, a, b = 1, 2, \cdots, 6\}}
\end{aligned}$$

where $\#\{(a, b) : ab = y\}$ is the number of pairs (a, b) satisfying $ab = y$. Thus $E(X \mid Y) = \frac{\sum_{j=1}^{6} j \cdot \mathbb{1}_{\{ij=Y \text{ for some } i=1,2,\cdots,6\}}}{\#\{(a,b):ab=Y,a,b=1,2,\cdots,6\}}$. To be specific,

$$\begin{aligned}
E(X \mid Y = y) &= \frac{\sum_{j=1}^{6} j \cdot \mathbb{1}_{\{ij=y \text{ for some } i=1,2,\cdots,6\}}}{\#\{(a, b) : ab = y, a, b = 1, 2, \cdots, 6\}} \\
&= \begin{cases}
\frac{1}{2}(1 + y), & y = 1, 2, 3, 5; \\
\frac{7}{3}, & y = 4; \\
3, & y = 6, 8, 9; \\
\frac{7}{2}, & y = 10; \\
4, & y = 12, 15, 16; \\
\frac{9}{2}, & y = 18, 20; \\
5, & y = 24, 25; \\
\frac{11}{2}, & y = 30; \\
6, & y = 36.
\end{cases}
\end{aligned}$$

4. (20 points) Denote $(\mathcal{F}_n)_{n\geq 1}$ as the natural filtration generated by $\{Y_n\}_{n\in\mathbb{N}}$.

$$M_n = 2^{S_n} = 2^{\sum_{j=1}^{n} Y_j},$$

thus $(M_n)_{n\geq 1}$ is adapted with respect to $(\mathcal{F}_n)_{n\geq 1}$. Moreover,

$$E(|M_n|) = E(|2^{S_n}|) = E(2^{S_n}) = E(2^{\sum_{j=1}^n Y_j}) = \Pi_{j=1}^n E(2^{Y_j}) = (\frac{1}{2}\times\frac{2}{3}+2\times\frac{1}{3})^n = 1 < \infty.$$

Also,

$$\begin{aligned}
E(M_{n+1} \mid \mathcal{F}_n) &= E(2^{S_{n+1}} \mid \mathcal{F}_n) = E(2^{S_n+Y_{n+1}} \mid \mathcal{F}_n) \\
&= M_n E(2^{Y_{n+1}} \mid \mathcal{F}_m) = M_n E(2^{Y_{n+1}}) \\
&= M_n(\frac{1}{2} \times \frac{2}{3} + 2 \times \frac{1}{3}) \\
&= M_n \cdot 1 = M_n
\end{aligned}$$

Hence $(M_n)_{n\geq 1}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 1}$.