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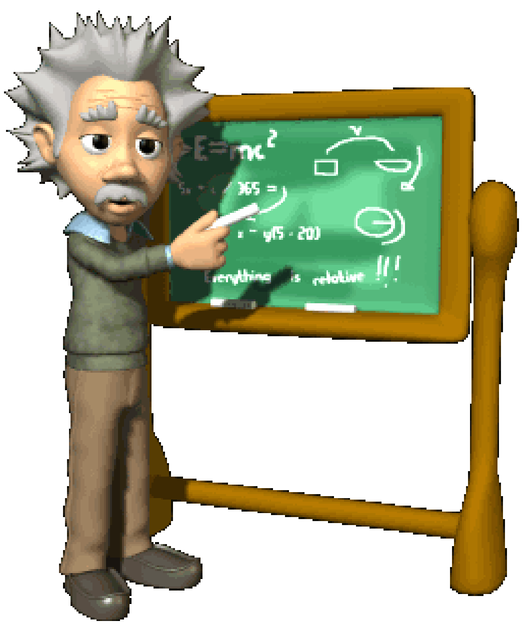
# MATH1073 Calculus Ⅰ



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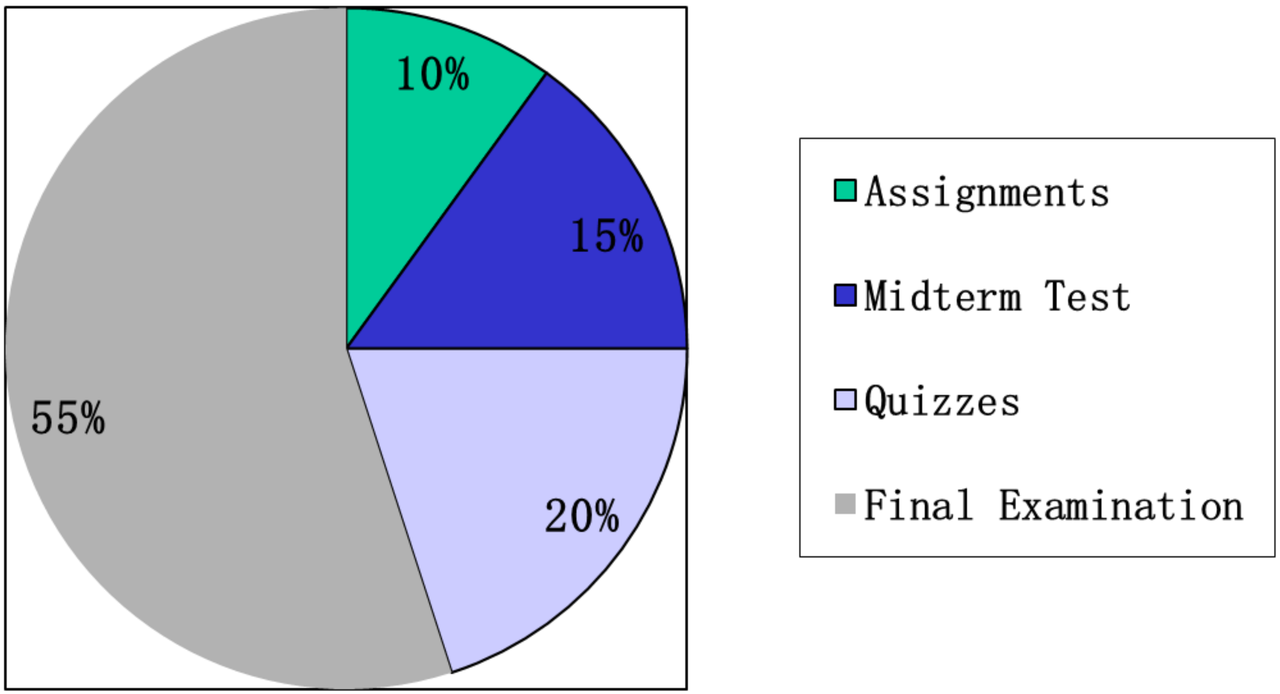
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Assignments
Midterm Test
Quizzes
Final Examination

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# Some notices on this course

- Assignments must be handed in before the deadline. After the deadline, we refuse to accept your assignments!
- For the mid-term test and final examination, you can not bring anything except some stationeries and water! Mobile are not allowed.
- For the final examination, we can not tell you the score before the AR inform the official results. If you have any question on the score, you can check the marked sheet via AR.

# General Information

- Textbook

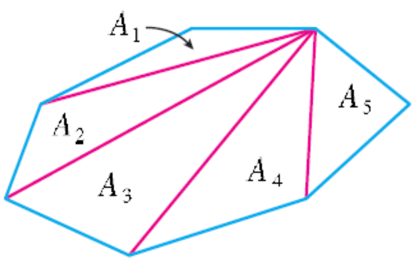
  Calculus-Early Transcendentals

  9th Edition

  James Stewart

- Advantages
  - textbook for two semesters
  - Classical textbook

## The Area Problem




$A = A_1 + A_2 + A_3 + A_4 + A_5$

**FIGURE 1**

The origins of calculus go back at least 2500 years to the ancient Greeks, who found areas using the "method of exhaustion." They knew how to find the area $A$ of any polygon by dividing it into triangles as in Figure 1 and adding the areas of these triangles.

It is a much more difficult problem to find the area of a curved figure. The Greek method of exhaustion was to inscribe polygons in the figure and circumscribe polygons about the figure and then let the number of sides of the polygons increase. Figure 2 illustrates this process for the special case of a circle with inscribed regular polygons.

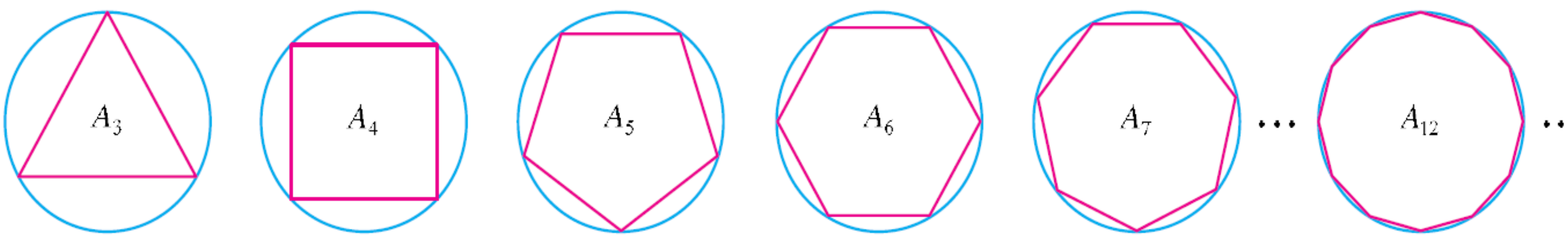



**FIGURE 2**

Let $A_n$ be the area of the inscribed polygon with $n$ sides. As $n$ increases, it appears that $A_n$ becomes closer and closer to the area of the circle. We say that the area of the circle is the limit of the areas of the inscribed polygons, and we write

$$A = \lim_{n \to \infty} A_n$$

**TEC** In the Preview Visual, you can see how areas of inscribed and circumscribed polygons approximate the area of a circle.

The Greeks themselves did not use limits explicitly. However, by indirect reasoning, Eudoxus (fifth century BC) used exhaustion to prove the familiar formula for the area of a circle: $A = \pi r^2$.

We will use a similar idea in Chapter 4 to find areas of regions of the type shown in Figure 3. We will approximate the desired area $A$ by areas of rectangles (as in Figure 4), let the width of the rectangles decrease, and then calculate $A$ as the limit of these sums of areas of rectangles.

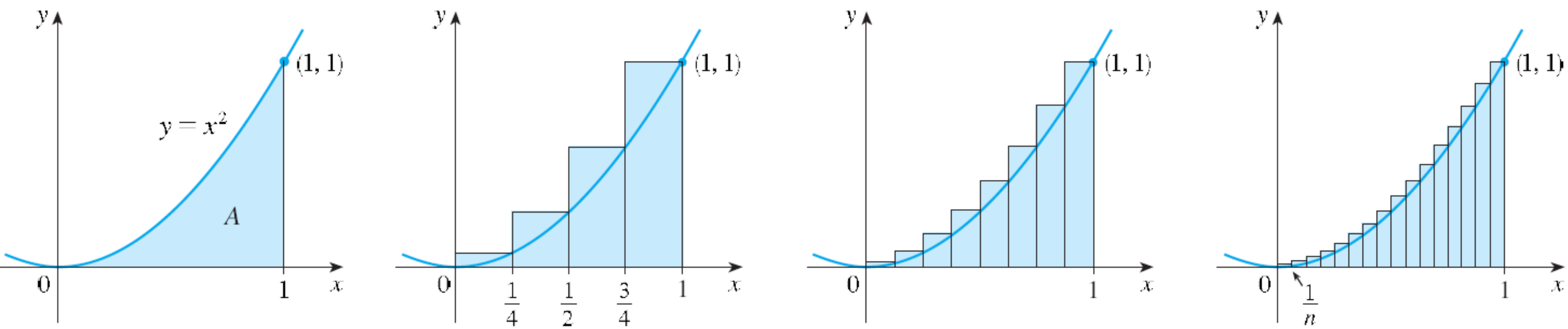



**FIGURE 3**

The area problem is the central problem in the branch of calculus called *integral calculus*. The techniques that we will develop in Chapter 4 for finding areas will also enable us to compute the volume of a solid, the length of a curve, the force of water against a dam, the mass and center of gravity of a rod, and the work done in pumping water out of a tank.

## The Tangent Problem

Consider the problem of trying to find an equation of the tangent line $t$ to a curve with equation $y = f(x)$ at a given point $P$. (We will give a precise definition of a tangent line in Chapter 1. For now you can think of it as a line that touches the curve at $P$ as in Figure 5.) Since we know that the point $P$ lies on the tangent line, we can find the equation of $t$ if we know its slope $m$. The problem is that we need two points to compute the slope and we know only one point, $P$, on $t$. To get around the problem we first find an approximation to $m$ by taking a nearby point $Q$ on the curve and computing the slope $m_{PQ}$ of the secant line $PQ$. From Figure 6 we see that

$$\boxed{1} \qquad m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

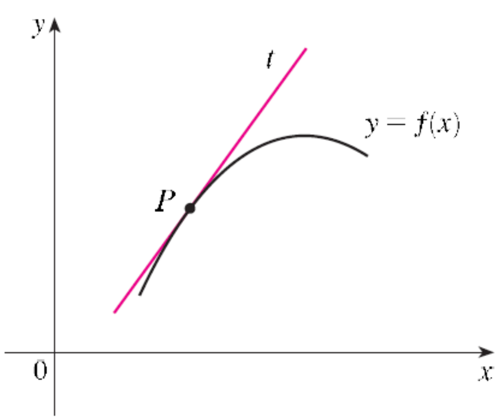



**FIGURE 5**
The tangent line at $P$

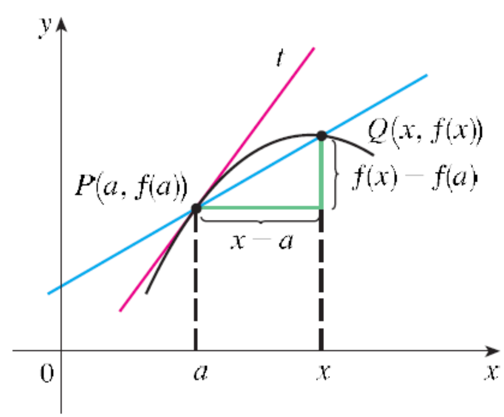



**FIGURE 6**
The secant line at $PQ$

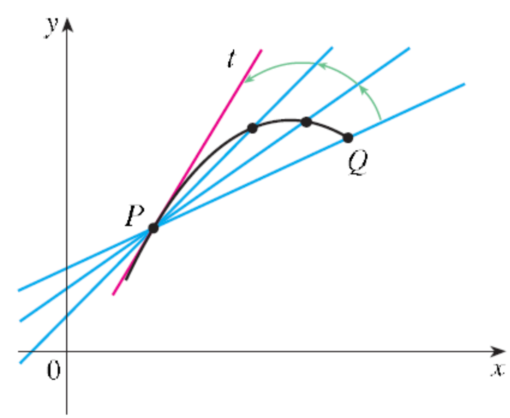



**FIGURE 7**
Secant lines approaching the tangent line

Now imagine that $Q$ moves along the curve toward $P$ as in Figure 7. You can see that the secant line rotates and approaches the tangent line as its limiting position. This means that the slope $m_{PQ}$ of the secant line becomes closer and closer to the slope $m$ of the tangent line. We write

$$m = \lim_{Q \to P} m_{PQ}$$

and we say that $m$ is the limit of $m_{PQ}$ as $Q$ approaches $P$ along the curve. Because $x$ approaches $a$ as $Q$ approaches $P$, we could also use Equation 1 to write

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{2}$$

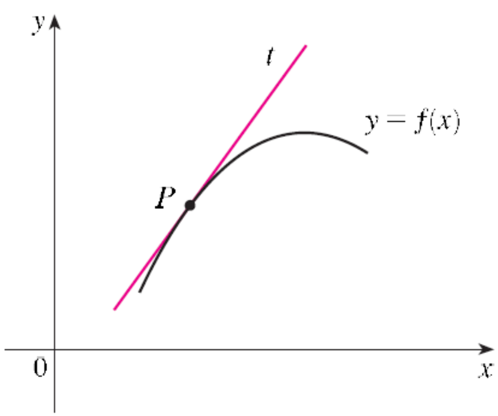



**FIGURE 5**
The tangent line at $P$

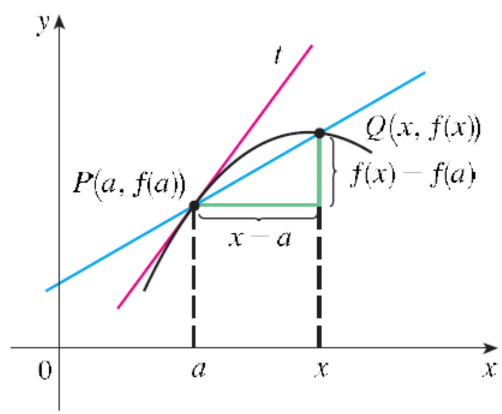



**FIGURE 6**
The secant line at $PQ$

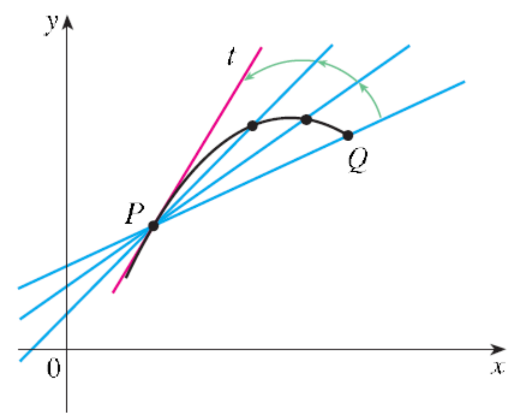



**FIGURE 7**
Secant lines approaching the tangent line

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# Chapter 0 Preliminaries (预备知识)

In this Chapter, we will know some important concepts.

- Math in English
- Essential Functions
- Inverse Functions
- Trigonometric and Inverse Trigonometric Functions

# Section 0.1 Math in English

| Symbol | | Speak |
|---|---|---|
| $+$ | or | plus<br>positive |
| $-$ | or | minus<br>negative |
| $\times$<br>$\cdot$ | or | multiplies<br>times |
| $\div$<br>$/$ | | divided by |

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| Symbol | | Speak |
|---|---|---|
| $=$ | or | equals<br>equal to |
| $\neq$ | or | does not equal<br>not equal |
| $<$ | | less than |
| $\ll$ | | much less than |
| $>$ | | greater than |
| $\gg$ | | much greater than |

| Symbol | | Speak |
|---|---|---|
| ( | or | open parenthesis<br>left parenthesis |
| ) | or | closed parenthesis<br>right parenthesis |
| [ | or | open bracket<br>left bracket |
| ] | or | closed bracket<br>right bracket |
| { | or | open brace<br>left brace |
| } | or | closed brace<br>right brace |

| Symbol | | Speak |
|---|---|---|
| $\lvert a \rvert$ | | absolute value of $a$ |
| $a'$ | | $a$ prime |
| $a''$ | | $a$ double prime |
| $a^n$ | or | $a$ superscript $n$<br>$a$ to the $n$ |
| $\bar{a}$ | | $a$ bar |
| $a^*$ | or | $a$ star<br>$a$ super asterisk |
| $a_n$ | or | $a$ subscript $n$<br>$a$ sub $n$ |
| $\sqrt{a}$ | | square root of $a$ |
| $\sqrt[3]{a}$ | | cube root of $a$ |
| $\sqrt[n]{a}$ | | $n$th root of $a$ |

| **Expression** | **Speak** |
|---|---|
| $\sum_{1}^{N}$ | summation from one to capital *n* |
| $\sum_{i=1}^{\infty} x_i$ | summation from *i* equals one to infinity of *x* sub *i* |
| $\prod$ | product |
| $\prod_{1}^{n}$ | product from one to *n* |
| $\prod_{i=1}^{\infty} y_i$ | product from *i* equals one to infinity of *y* sub *i* |
| $\lim_{x \to a} y = b$ | limit as *x* approaches *a* of *y* equals *b* |
| $\lim_{x \to a-} f(x)$ | limit as *x* approaches *a* minus of *f* of *x* |
| $\int f(x) dx$ | integral of *f* of *x* d *x* |

# Section 0.2 Essential Functions (函数)

1. Linear functions: $f(x) = mx + b$
2. Power functions (幂函数): $f(x) = x^n$
3. Root functions: $f(x) = \sqrt[n]{x}$
4. Exponential functions (指数函数): $f(x) = b^x$
5. Logarithmic functions: $f(x) = \log_b x$
6. Trigonometric functions: $f(x) = \sin x\,; \cos x\,; \tan x$

## The intervals

1. The closed interval (闭区间) *[a,b]* is the set of numbers between *a* and *b*, including *a* and *b* (the endpoints), which is,

$$[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\},$$

2. The open interval *(a,b)* is the set of numbers between *a* and *b*, but not including the endpoints, which is,

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

- Addition and subtraction of real numbers (实数的加法及减法)

II-1: $a+b=b+a$ (commutative law of addition 交换律)

II-2: $(a+b)+c=a+(b+c)$ (associative law 结合律)

II-3: $a+0=a$ (property of zero number)

II-4: For any number $a$, there exists a number $-a$ such that $a+(-a)=0$.

II-5: From $a>b$, we have $a+c>b+c$.

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- Multiplication and division of real numbers (实数的乘法及除法)

III-1: $ab=ba$ (commutative law)

III-2: $(ab)c=a(bc)$ (associative law)

III-3: $a\cdot 1=a$ (property of $1$)

III-4: For any number $a\neq 0$, there exists its reciprocal number $1/a$ such that $a\cdot\frac{1}{a}=1$

III-5: $(a+b)c=ac+bc$ (distributive law 分配律 )

III-6: From $a>b$ *and* $c>0$, we have $a\cdot c>b\cdot c$

## EXAMPLE 1.1 Solving a Linear Inequality

Solve the linear inequality $2x + 5 < 13$.

## EXAMPLE 1.2 Solving a Two-Sided Inequality

Solve the two-sided inequality $6 < 1 - 3x \leq 10$.

## EXAMPLE 1.3 Solving an Inequality Involving a Fraction

Solve the inequality $\dfrac{x-1}{x+2} \geq 0$.

## EXAMPLE 1.4 Solving a Quadratic Inequality

Solve the quadratic inequality

$$x^2 + x - 6 > 0.$$

**DEFINITION 1.1**

The **absolute value** of a real number $x$ is $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}.$

## Triangle inequality (三角不等式)

$$|a+b| \leq |a| + |b|.$$

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## Equations of Lines (直线方程)

**DEFINITION 1.2**

For $x_1 \neq x_2$, the **slope** of the straight line through the points $(x_1, y_1)$ and $(x_2, y_2)$ is the number

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \tag{1.5}$$

When $x_1 = x_2$, the line through $(x_1, y_1)$ and $(x_2, y_2)$ is **vertical** and the slope is undefined.

Notice that a line is **horizontal** if and only if its slope is zero.

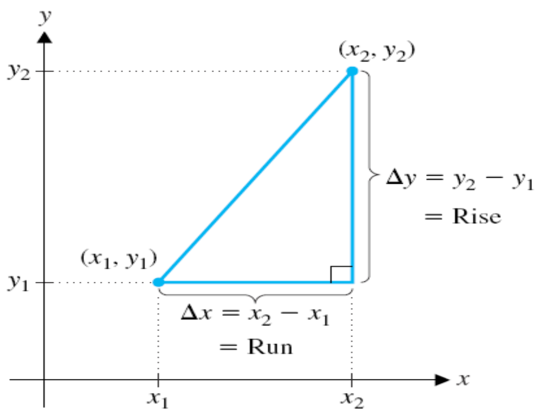


**FIGURE 0.12a**
Slope

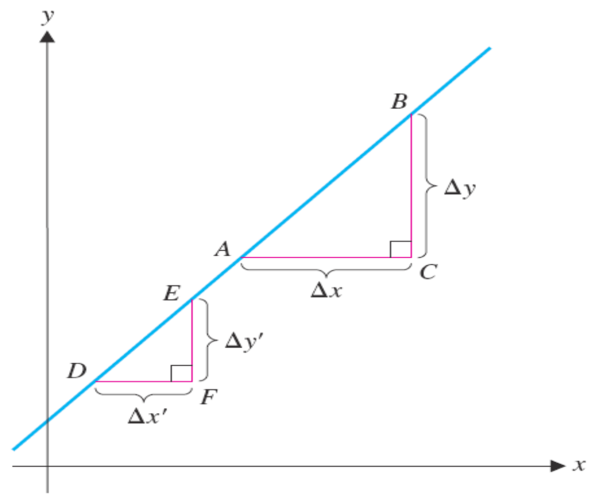


**FIGURE 0.12b**
Similar triangles and slope

## POINT-SLOPE FORM OF A LINE

$$y = m(x - x_0) + y_0. \tag{1.7}$$

a form of the equation called the **slope-intercept form** is more convenient. This has the form

$$y = mx + b,$$

## THEOREM 1.2

Two (nonvertical) lines are **parallel** if they have the same slope. Further, any two vertical lines are parallel. Two (nonvertical) lines of slope $m_1$ and $m_2$ are **perpendicular** whenever the product of their slopes is $-1$ (i.e., $m_1 \cdot m_2 = -1$). Also, any vertical line and any horizontal line are perpendicular.

## EXAMPLE 1.13 Finding the Equation of a Parallel Line

Find an equation of the line parallel to $y = 3x - 2$ and through the point $(-1, 3)$.

## EXAMPLE 1.14 Finding the Equation of a Perpendicular Line

Find an equation of the line perpendicular to $y = -2x + 4$ and intersecting the line at the point $(1, 2)$.

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## DEFINITION 1.4

A **polynomial** is any function that can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, a_2, \ldots, a_n$ are real numbers (the **coefficients** of the polynomial) with $a_n \neq 0$ and $n \geq 0$ is an integer (the **degree** of the polynomial).

## EXAMPLE 1.17 Sample Polynomials

The following are all examples of polynomials:

$f(x) = 2$ (polynomial of degree 0 or **constant**),

$f(x) = 3x + 2$ (polynomial of degree 1 or **linear** polynomial),

$f(x) = 5x^2 - 2x + 1$ (polynomial of degree 2 or **quadratic** polynomial),

$f(x) = x^3 - 2x + 1$ (polynomial of degree 3 or **cubic** polynomial),

# Rational Function (有理函数)

**DEFINITION 1.5**

Any function that can be written in the form

$$f(x) = \frac{p(x)}{q(x)},$$

where $p$ and $q$ are polynomials, is called a **rational** function.

Notice that since $p(x)$ and $q(x)$ are polynomials, they are both defined for all $x$, and so, the rational function $f(x) = \dfrac{p(x)}{q(x)}$ is defined for all $x$ for which $q(x) \neq 0$.

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# Section 0.3 Inverse Functions (反函数)

- **Composition of Functions（复合函数）**: Given functions *f(u)* and *g(x)*, the composition *f(g(x))* is the function of *x* formed by substituting *u=g(x)* for *u* in the formula for *f(u)*.

**Example**

Find the composition function *f(g(x))*, where $f(u)=u^3+1$ and $g(x)=x+1$

**Solution:**

Replace *u* by *x+1* in the formula for *f(u)* to get

$$f(g(x))=(x+1)^3+1=x^3+3x^2+3x+2$$

Question: How about *g(f(x))*?

Note: In general, *f(g(x))* and *g(f(x))* will not be the same.

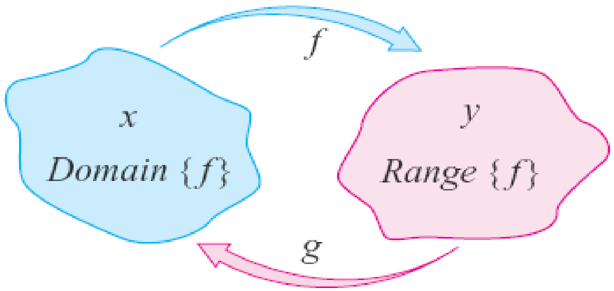



**FIGURE 0.41**
$g(x) = f^{-1}(x)$

# Inverse Functions

**DEFINITION 3.1**

Assume that $f$ and $g$ have domains $A$ and $B$, respectively, and that $f(g(x))$ is defined for all $x \in B$ and $g(f(x))$ is defined for all $x \in A$. If

$$f(g(x)) = x, \quad \text{for all } x \in B \quad \text{and}$$
$$g(f(x)) = x, \quad \text{for all } x \in A,$$

we say that $g$ is the **inverse** of $f$, written $g = f^{-1}$. Equivalently, $f$ is the inverse of $g$, $f = g^{-1}$.

# Section 0.4 Trigonometric (三角函数) and Inverse Trigonometric Functions (反三角函数)

Many phenomena encountered in your daily life involve *waves*.

✓ Radar Signals

✓ Electromagnetic waves

✓ Electrocardiogram

The mathematical description of such phenomena involves periodic functions, the most familiar of which are the trigonometric functions.

## DEFINITION 4.2

The **tangent** function is defined by $\tan x = \dfrac{\sin x}{\cos x}$.

The **cotangent** function is defined by $\cot x = \dfrac{\cos x}{\sin x}$.

The **secant** function is defined by $\sec x = \dfrac{1}{\cos x}$.

The **cosecant** function is defined by $\csc x = \dfrac{1}{\sin x}$.

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

## THEOREM 4.2

For any real numbers $\alpha$ and $\beta$, the following identities hold:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha \tag{4.1}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{4.2}$$

$$\sin^2\alpha = \tfrac{1}{2}(1-\cos 2\alpha) \tag{4.3}$$

$$\cos^2\alpha = \tfrac{1}{2}(1+\cos 2\alpha). \tag{4.4}$$

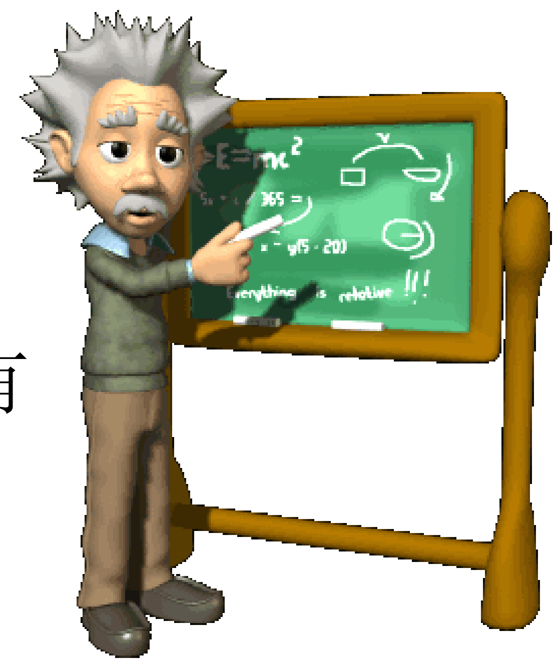
$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

# Chapter 1

# Limits and Continuity (极限和连续)

In this Chapter, we will know some important concepts.

- Limits (极限)
- Continuity (连续)
- Limits Involving Infinity (与无穷有关的极限)
- Formal Definition of the Limit



If $f(x)$ approaches $L$ as $x$ tends toward $c$ from the left $(x<c)$, we write

$$\lim_{x\to c^-} f(x) = L$$

where $L$ is called the limit from the left (or left-hand limit)

Likewise if $f(x)$ approaches $M$ as $x$ tends toward $c$ from the right $(x>c)$, then $\lim_{x\to c^+} f(x) = M$

$M$ is called the limit from the right (or right-hand limit.)

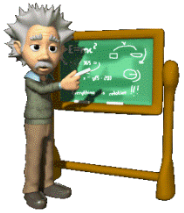
- As a start, we consider the functions

$$f(x) = \frac{x^2 - 4}{x - 2} \quad \text{and} \quad g(x) = \frac{x^2 - 5}{x - 2}.$$

For first function *f(x)*, we compute some values of the function for *x* close to *2*, as in the following tables.

| $x$ | $f(x) = \frac{x^2 - 4}{x - 2}$ |
|---|---|
| 1.9 | 3.9 |
| 1.99 | 3.99 |
| 1.999 | 3.999 |
| 1.9999 | 3.9999 |

| $x$ | $f(x) = \frac{x^2 - 4}{x - 2}$ |
|---|---|
| 2.1 | 4.1 |
| 2.01 | 4.01 |
| 2.001 | 4.001 |
| 2.0001 | 4.0001 |



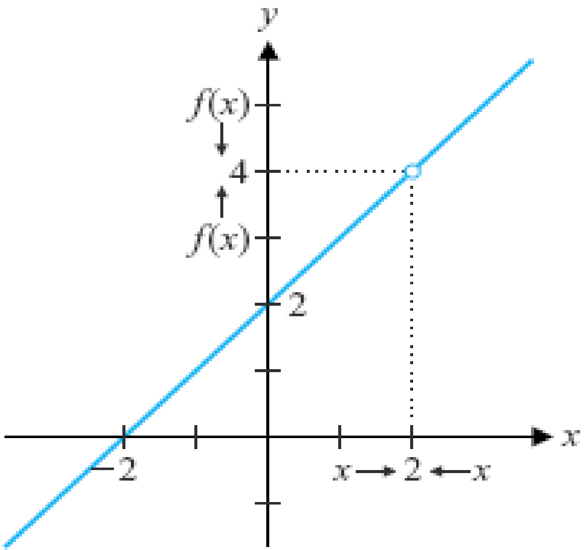



**FIGURE 1.7a**

$$y = \frac{x^2 - 4}{x - 2}$$

Notice that the table and the graph both suggest that as *x approaches 2 from two sides* (left side and right side), *f(x) gets closer and closer to 4*, written

$$\lim_{x \to 2} f(x) = 4.$$

It is important to remember that limits describe the behavior of a function *near* a particular point, not *necessarily* at the point itself.

For the second function *g(x)*, we have following tables and figures

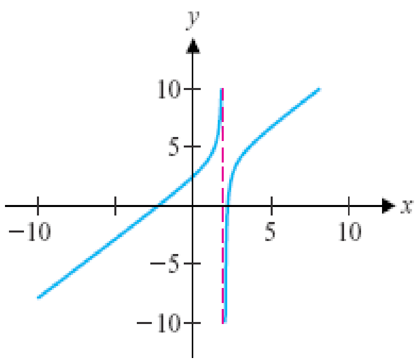



**FIGURE 1.7b**

$y = \dfrac{x^2 - 5}{x - 2}$

| $x$ | $g(x) = \dfrac{x^2 - 5}{x - 2}$ |
|---|---|
| 1.9 | 13.9 |
| 1.99 | 103.99 |
| 1.999 | 1003.999 |
| 1.9999 | 10,003.9999 |

| $x$ | $g(x) = \dfrac{x^2 - 5}{x - 2}$ |
|---|---|
| 2.1 | −5.9 |
| 2.01 | −95.99 |
| 2.001 | −995.999 |
| 2.0001 | −9995.9999 |

Since neither limit exists, we say that the *limit of g(x) as x approaches 2 does not exist*, written

$$\lim_{x \to 2^-} g(x) \text{ does not exist.} \qquad \lim_{x \to 2^+} g(x) \text{ does not exist.}$$

$$\lim_{x \to 2} g(x) \text{ does not exist.}$$

**Existence of a Limit**: The two-sided limit $\lim_{x\to 2} f(x)$ exists if and only if the two one-sided limits $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$ exist and are equal, and then

$$\lim_{x\to 2} f(x) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

For the function

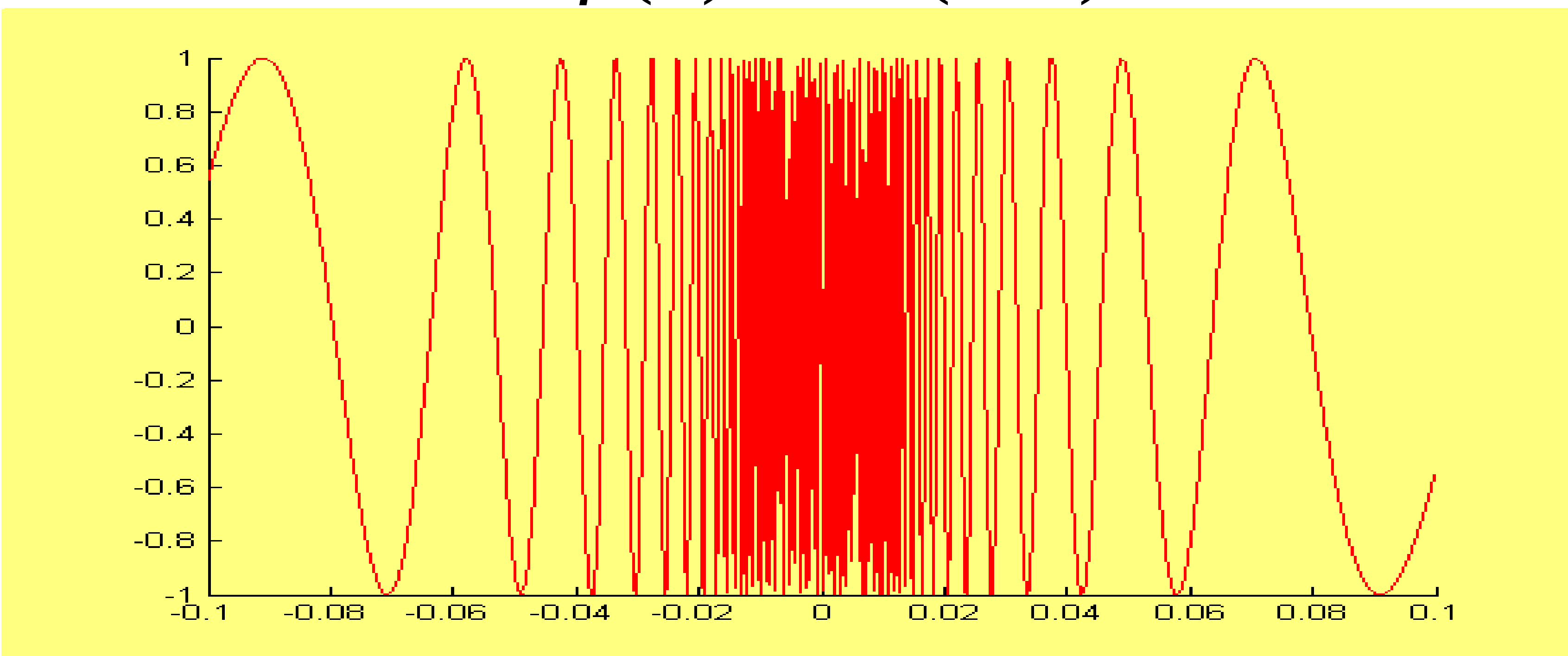
$$f(x) = \begin{cases} 1 - x^2 & \text{if } \; x < 2 \\ 2x + 1 & \text{if } \; x \geq 2 \end{cases}$$

evaluate the one-sided limits $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$

# Nonexistent One-sided Limits

A simple example is provided by the function

$$f(x) = \sin(1/x)$$




As *x* approaches *0* from either the left or the right, *f(x)* oscillates between -*1* and *1* infinitely often. Thus neither one-sided limit at *0* exists.

## EXAMPLE 2.2 A Limit Where Two Factors Cancel

Evaluate $\lim_{x \to -3} \dfrac{3x+9}{x^2-9}$.

| $x$ | $\dfrac{3x+9}{x^2-9}$ |
|---|---|
| −3.1 | −0.491803 |
| −3.01 | −0.499168 |
| −3.001 | −0.499917 |
| −3.0001 | −0.499992 |

| $x$ | $\dfrac{3x+9}{x^2-9}$ |
|---|---|
| −2.9 | −0.508475 |
| −2.99 | −0.500835 |
| −2.999 | −0.500083 |
| −2.9999 | −0.500008 |

Since the function approaches the same value as $x \to -3$ both from the right and from the left, we write

$$\lim_{x \to -3} \frac{3x+9}{x^2-9} = -\frac{1}{2}.$$

# EXAMPLE 2.4 Approximating the Value of a Limit

Evaluate $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$.

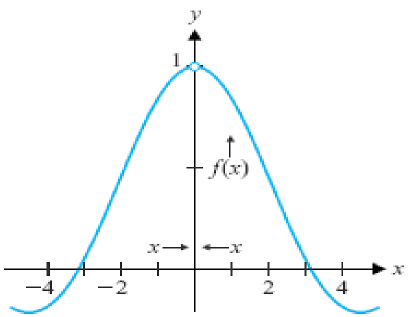



FIGURE 1.11
$\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

| $x$ | $\dfrac{\sin x}{x}$ |
|---|---|
| 0.1 | 0.998334 |
| 0.01 | 0.999983 |
| 0.001 | 0.99999983 |
| 0.0001 | 0.9999999983 |
| 0.00001 | 0.999999999983 |

| $x$ | $\dfrac{\sin x}{x}$ |
|---|---|
| −0.1 | 0.998334 |
| −0.01 | 0.999983 |
| −0.001 | 0.99999983 |
| −0.0001 | 0.9999999983 |
| −0.00001 | 0.999999999983 |

The graph and the tables of values lead us to the conjectures:

$$\lim_{x \to 0^+} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \to 0^-} \frac{\sin x}{x} = 1,$$

from which we conjecture that

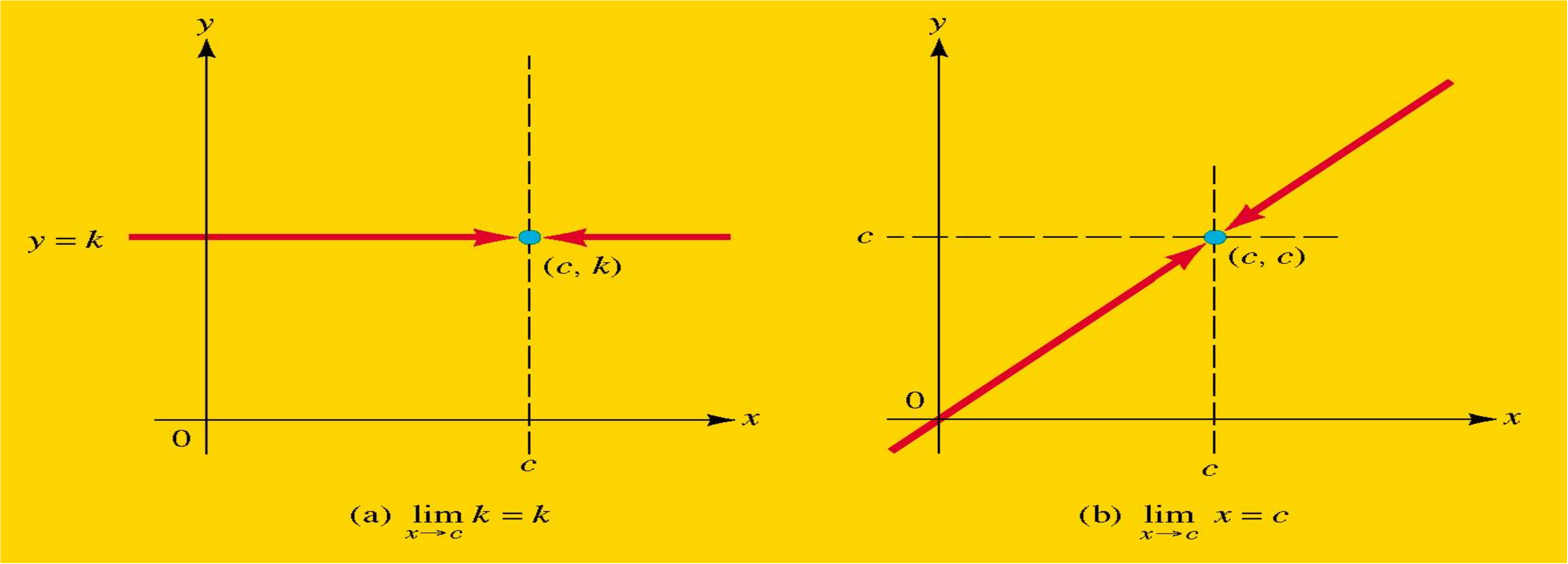
$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

# Section 1.2 Computation Of Limits

For any constant *k*,

$$\lim_{x\to c} k = k \quad (3.1) \quad \text{and} \quad \lim_{x\to c} x = c \quad (3.2)$$

That is, the limit of a constant is the constant itself, and the limit of *f(x)*=*x* as *x* approaches *c* is *c*.



## THEOREM 3.1

Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and let $c$ be any constant. The following then apply:

(i) $\lim_{x\to a}[c \cdot f(x)] = c \cdot \lim_{x\to a} f(x)$,

(ii) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$,

(iii) $\lim_{x\to a}[f(x) \cdot g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$ and

(iv) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \left(\text{if } \lim_{x\to a} g(x) \neq 0\right)$.

## EXAMPLE 3.1 Finding the Limit of a Polynomial

Apply the rules of limits to evaluate $\lim_{x \to 2}(3x^2 - 5x + 4)$.

**Solution** We have

$$\lim_{x \to 2}(3x^2 - 5x + 4) = \lim_{x \to 2}(3x^2) - \lim_{x \to 2}(5x) + \lim_{x \to 2} 4 \qquad \text{By Theorem 3.1 (ii).}$$

$$= 3 \lim_{x \to 2} x^2 - 5 \lim_{x \to 2} x + 4 \qquad \text{By Theorem 3.1 (i).}$$

$$= 3 \cdot (2)^2 - 5 \cdot 2 + 4 = 6. \qquad \text{By (3.4).}$$

## EXAMPLE 3.2 Finding the Limit of a Rational Function

Apply the rules of limits to evaluate $\lim\limits_{x\to 3} \dfrac{x^3 - 5x + 4}{x^2 - 2}$.

**Solution** We get

$$\lim_{x\to 3} \frac{x^3 - 5x + 4}{x^2 - 2} = \frac{\lim\limits_{x\to 3}(x^3 - 5x + 4)}{\lim\limits_{x\to 3}(x^2 - 2)} \qquad \text{By Theorem 3.1 (iv).}$$

$$= \frac{\lim\limits_{x\to 3} x^3 - 5\lim\limits_{x\to 3} x + \lim\limits_{x\to 3} 4}{\lim\limits_{x\to 3} x^2 - \lim\limits_{x\to 3} 2} \qquad \text{By Theorem 3.1 (i) and (ii).}$$

$$= \frac{3^3 - 5\cdot 3 + 4}{3^2 - 2} = \frac{16}{7}. \qquad \text{By (3.4).}$$

**THEOREM 3.2 Limits of Polynomials and Rational Functions:** If *p(x)* and *q(x)* are polynomials, then

$$\lim_{x\to c} p(x) = p(c)$$

and

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \qquad \text{if} \quad q(c) \neq 0$$

**THEOREM 3.3**

Suppose that $\lim_{x\to a} f(x) = L$ and $n$ is any positive integer. Then,

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L},$$

where for $n$ even, we assume that $L > 0$.

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# Indeterminate Form

If $\lim\limits_{x\to c} f(x)=0$ and $\lim\limits_{x\to c} g(x)=0$, then $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}$ is said to be indeterminate. The term indeterminate is used since the limit may or may not exist.

**Example** Find the Limit by Factoring and Rationalizing

(a) Find $\lim\limits_{x\to 1}\dfrac{x^2-1}{x^2-3x+2}$ (b) Find $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{x-1}$

**Solution:**

a. $$\lim_{x\to 1}\frac{x^2-1}{x^2-3x+2}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(x-2)}=\lim_{x\to 1}\frac{x+1}{x-2}=\frac{2}{-1}=-2$$

b. $$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to 1}\frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{(x-1)\left(\sqrt{x}+1\right)}=\lim_{x\to 1}\frac{x-1}{(x-1)\left(\sqrt{x}+1\right)}$$

$$=\lim_{x\to 1}\frac{1}{\sqrt{x}+1}=\frac{1}{2}$$

## THEOREM 3.4

For any real number $a$, we have

(i) $\lim_{x \to a} \sin x = \sin a$,

(ii) $\lim_{x \to a} \cos x = \cos a$,

(iii) $\lim_{x \to a} e^x = e^a$ and

(iv) $\lim_{x \to a} \ln x = \ln a$, for $a > 0$.

(v) $\lim_{x \to a} \sin^{-1} x = \sin^{-1} a$, for $-1 < a < 1$,

(vi) $\lim_{x \to a} \cos^{-1} x = \cos^{-1} a$, for $-1 < a < 1$,

(vii) $\lim_{x \to a} \tan^{-1} x = \tan^{-1} a$, for $-\infty < a < \infty$ and

(viii) if $p$ is a polynomial and $\lim_{x \to p(a)} f(x) = L$, then $\lim_{x \to a} f(p(x)) = L$.

## EXAMPLE 3.6 Evaluating a Limit of an Inverse Trigonometric Function

Evaluate $\lim_{x \to 0} \sin^{-1}\left(\frac{x+1}{2}\right)$.

**Solution** By Theorem 3.4, we have

$$\lim_{x \to 0} \sin^{-1}\left(\frac{x+1}{2}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

## EXAMPLE 3.7 A Limit of a Product That Is Not the Product of the Limits

Evaluate $\lim_{x\to 0}(x \cot x)$.

**Idea:**

Your first reaction might be to say that this is a limit of a product and it must be the product of the limits:

$$\lim_{x\to 0}(x\cot x) = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0}\cot x\right)$$

This is incorrect!

$$= 0 \cdot ? = 0,$$

Since $\lim_{x\to 0}\cot x$ does not exist, we can not directly use Theorem 3.1.

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# Squeeze Theorem (两面夹定理)

**THEOREM 3.5** (Squeeze Theorem)

Suppose that

$$f(x) \le g(x) \le h(x)$$

for all $x$ in some interval $(c, d)$, except possibly at the point $a \in (c, d)$ and that

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

for some number $L$. Then, it follows that

$$\lim_{x \to a} g(x) = L, \text{ also.}$$

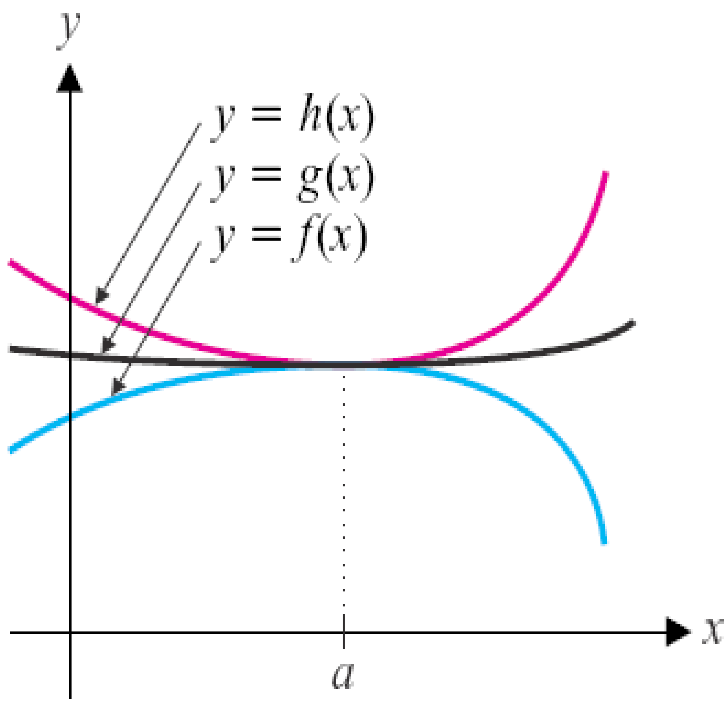



**FIGURE 1.18**
The Squeeze Theorem

The challenge in using the Squeeze Theorem is in finding appropriate functions *f* and *h* that bound a given function *g* from below and above, respectively, and that have the same limit as $x \to a$.

**EXAMPLE 11** Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

**SOLUTION** First note that we **cannot** use

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x}$$

because $\lim_{x \to 0} \sin(1/x)$ does not exist (see Example 1.5.4).

Instead we apply the Squeeze Theorem, and so we need to find a function $f$ smal than $g(x) = x^2 \sin(1/x)$ and a function $h$ bigger than $g$ such that both $f(x)$ and $h(x)$ approach 0. To do this we use our knowledge of the sine function. Because the sine any number lies between $-1$ and 1, we can write.

$$\boxed{4} \qquad -1 \leqslant \sin \frac{1}{x} \leqslant 1$$

Any inequality remains true when multiplied by a positive number. We know that $x^2 \geqslant 0$ for all $x$ and so, multiplying each side of the inequalities in (4) by $x^2$, we get

$$-x^2 \leqslant x^2 \sin \frac{1}{x} \leqslant x^2$$

as illustrated by Figure 8. We know that

$$\lim_{x \to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x \to 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

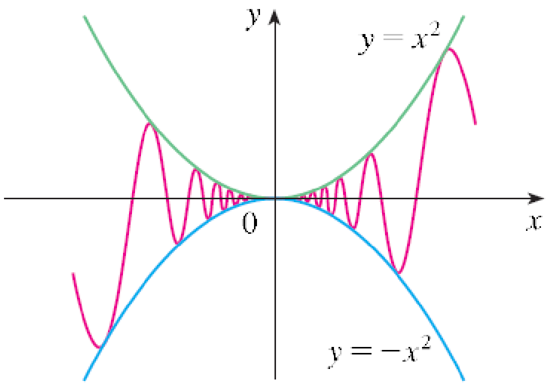
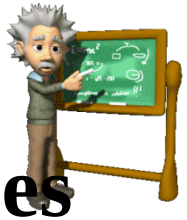
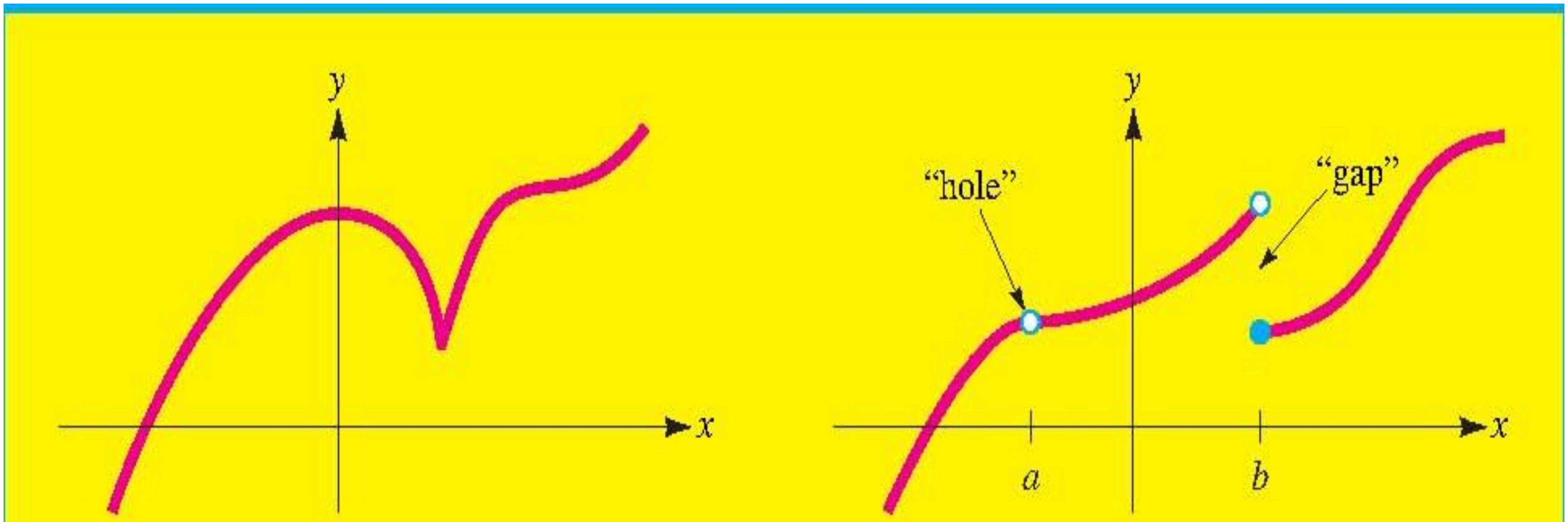



**FIGURE 8**
$y = x^2 \sin(1/x)$



# Section 1.3 Continuity and Its Consequences

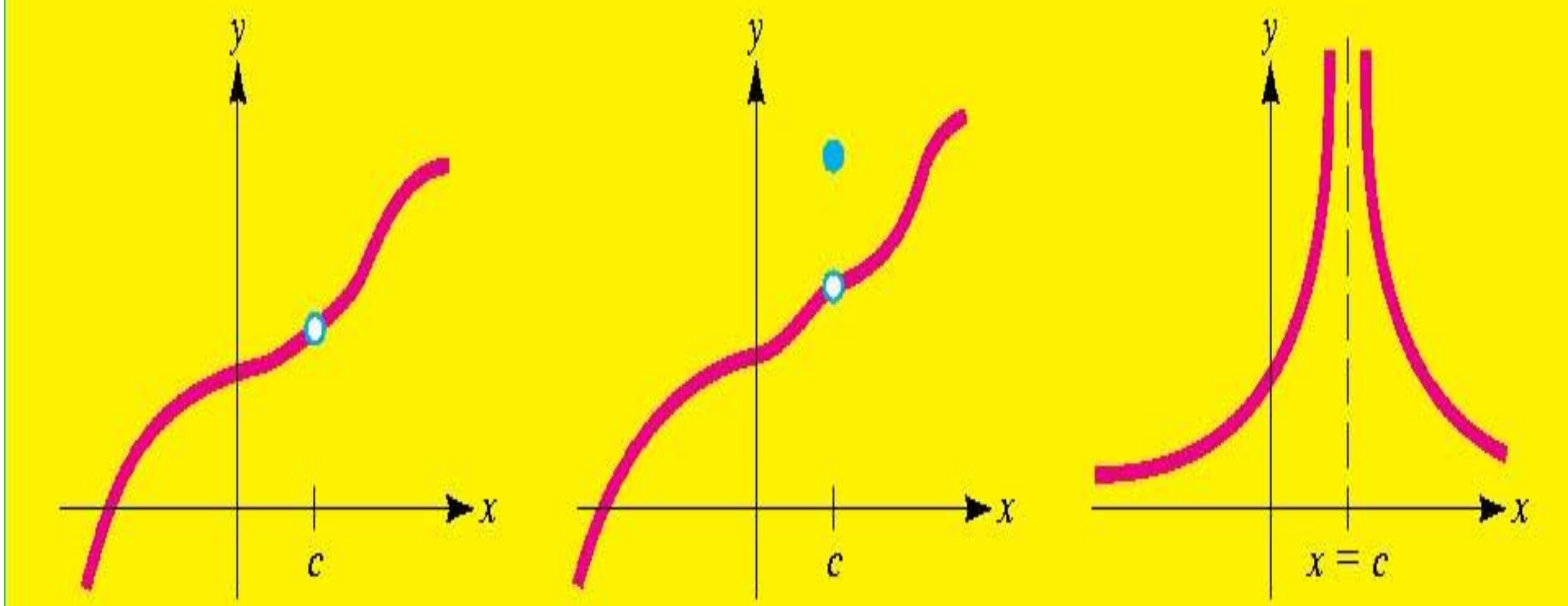
- A continuous function is one whose graph can be drawn without the "pen" leaving the paper. (no holes or gaps )




(a) A continuous graph

(b) A graph with "holes" or "gaps" is not continuous

# A “hole “ at $x=c$

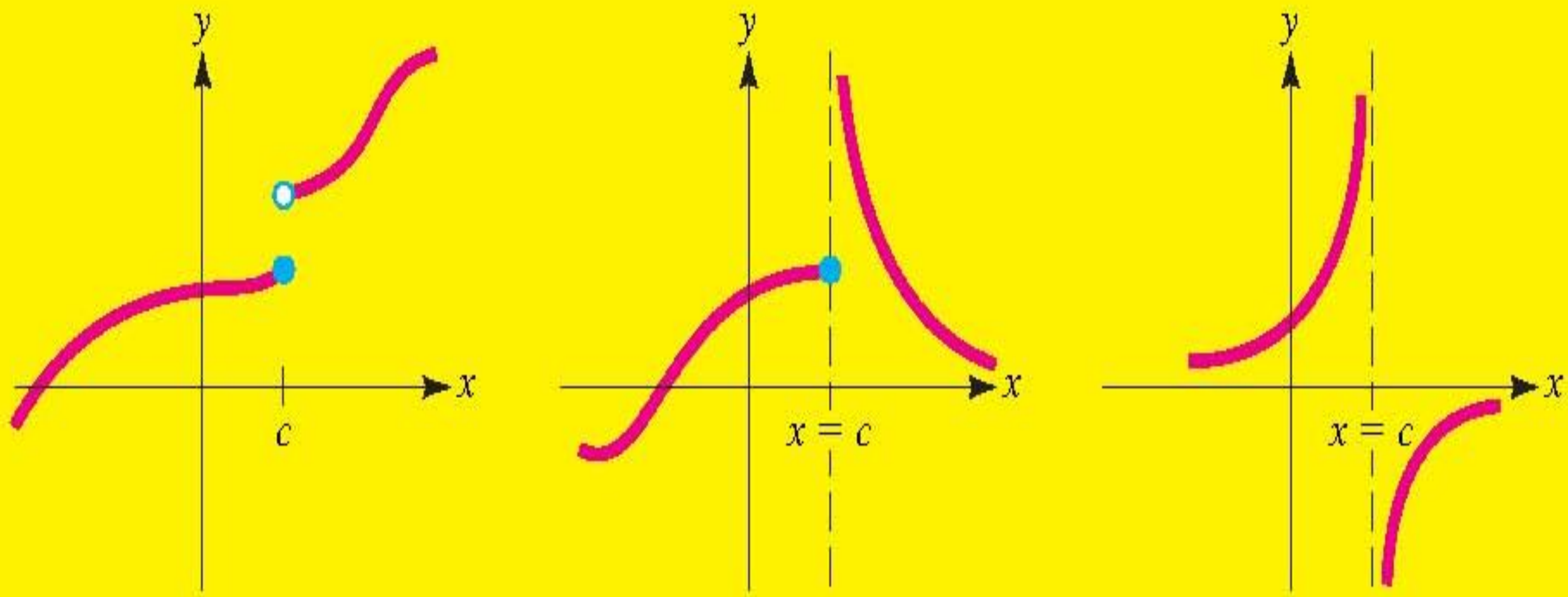



(a) $f(c)$ is not defined

(b) $\lim_{x\to c} f(x) \neq f(c)$

(c) $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = +\infty$

# A "gap" at $x=c$




(a) A finite gap: $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$

(b) An infinite gap: $\lim_{x\to c^-} f(x)$ is finite but $\lim_{x\to c^+} f(x) = +\infty$

(c) An infinite gap: $\lim_{x\to c^-} f(x) = +\infty$ and $\lim_{x\to c^+} f(x) = -\infty$

Definition 4.1

**Continuity(连续)**: A function $f$ is continuous at $c$ if all three of these conditions are satisfied:

a. $f(c)$ is defined

b. $\lim_{x \to c} f(x)$ exists

c. $\lim_{x \to c} f(x) = f(c)$

If $f(x)$ is not continuous at $c$, it is said to have a discontinuity there.

## EXAMPLE 4.1 Finding Where a Rational Function Is Continuous

Determine where $f(x) = \dfrac{x^2 + 2x - 3}{x - 1}$ is continuous.

**Solution** Note that

$$f(x) = \frac{x^2 + 2x - 3}{x - 1} = \frac{(x-1)(x+3)}{x-1} \qquad \text{Factoring the numerator.}$$

$$= x + 3, \text{ for } x \neq 1. \qquad \text{Canceling common factors.}$$

This says that the graph of $f$ is a straight line, but with a hole in it at $x = 1$, as indicated in Figure 1.23. So, $f$ is discontinuous at $x = 1$, but continuous elsewhere. ■

## EXAMPLE 4.2 Removing a Discontinuity

Make the function from example 4.1 continuous everywhere by redefining it at a single point.

- When we can remove a discontinuity by defining the function at that point, we call the discontinuity ***removable***.
- *Not all discontinuities are removable.*

**EXAMPLE 4.3** Nonremovable Discontinuities

Find all discontinuities of $f(x) = \dfrac{1}{x^2}$ and $g(x) = \cos\left(\dfrac{1}{x}\right)$.

# Continuity Polynomials and Rational Functions

- If *p(x)* and *q(x)* are polynomials, then

$$\lim_{x \to c} p(x) = p(c)$$

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \quad \text{if } q(c) \neq 0$$

A polynomial or a rational function is continuous wherever it is defined

## THEOREM 4.1

All polynomials are continuous everywhere. Additionally, $\sin x$, $\cos x$, $\tan^{-1} x$ and $e^x$ are continuous everywhere, $\sqrt[n]{x}$ is continuous for all $x$, when $n$ is odd and for $x > 0$, when $n$ is even. We also have $\ln x$ is continuous for $x > 0$ and $\sin^{-1} x$ and $\cos^{-1} x$ are continuous for $-1 < x < 1$.

## THEOREM 4.2

Suppose that $f$ and $g$ are continuous at $x = a$. Then all of the following are true:

(i) $(f \pm g)$ is continuous at $x = a$,
(ii) $(f \cdot g)$ is continuous at $x = a$ and
(iii) $(f/g)$ is continuous at $x = a$ if $g(a) \neq 0$.

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## THEOREM 4.3

Suppose that $\lim_{x \to a} g(x) = L$ and $f$ is continuous at $L$. Then,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L).$$

## COROLLARY 4.1

Suppose that $g$ is continuous at $a$ and $f$ is continuous at $g(a)$. Then, the composition $f \circ g$ is continuous at $a$.

## EXAMPLE 4.5 Continuity for a Composite Function

Determine where $h(x) = \cos(x^2 - 5x + 2)$ is continuous.

# Continuity on an Interval

- *A function f(x) is said to be continuous on an open interval $a<x<b$* if it is continuous at each point $x=c$ in that interval.

- *$f$ is continuous on closed interval $a\leq x\leq b$*, if it continuous on the open interval $a<x<b$ and

$$\lim_{x\to a^+} f(x) = f(a) \qquad \lim_{x\to b^-} f(x) = f(b)$$

- If $f$ is continuous on all of $(-\infty, \infty)$ , we simply say that $f$ is continuous everywhere.

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## EXAMPLE 4.6 Continuity on a Closed Interval

Determine the interval(s) where $f$ is continuous, for $f(x) = \sqrt{4 - x^2}$.

## EXAMPLE 4.7 Interval of Continuity for a Logarithm

Determine the interval(s) where $f(x) = \ln(x - 3)$ is continuous.

**Example 4.8**

Discuss the continuity of the function $f(x) = \dfrac{x+2}{x-3}$ on the open interval $-2<x<3$ and on the closed interval $-2\leq x\leq 3$

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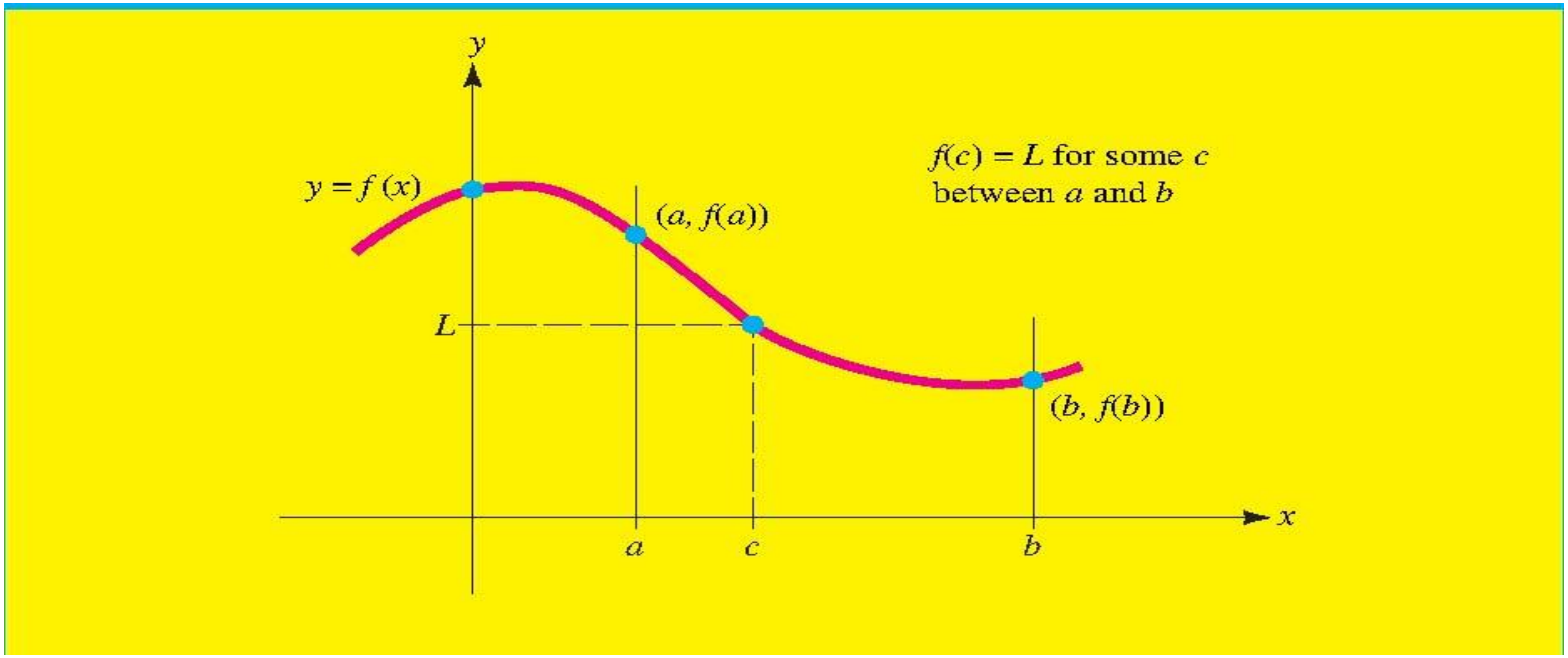
# Theorem 4.4 The Intermediate Value Property

- Suppose that $f(x)$ is continuous on the interval $a \leq x \leq b$ and $L$ is a number between $f(a)$ and $f(b)$, then there exists a number $c$ between $a$ and $b$, such that $f(c)=L$.



## HISTORICAL NOTES

**Karl Weierstrass (1815–1897)** A German mathematician who proved the Intermediate Value Theorem and several other fundamental results of the calculus. Weierstrass was known as an excellent teacher whose students circulated his lecture notes throughout Europe, because of their clarity and originality. Also known as a superb fencer, Weierstrass was one of the founders of modern mathematical analysis.

The condition: *f(x)* is continuous on the closed interval *[a,b]*, can not be relaxed.

Can you find an example?

In following Corollary, we see an immediate and useful application of the Intermediate Value Theorem

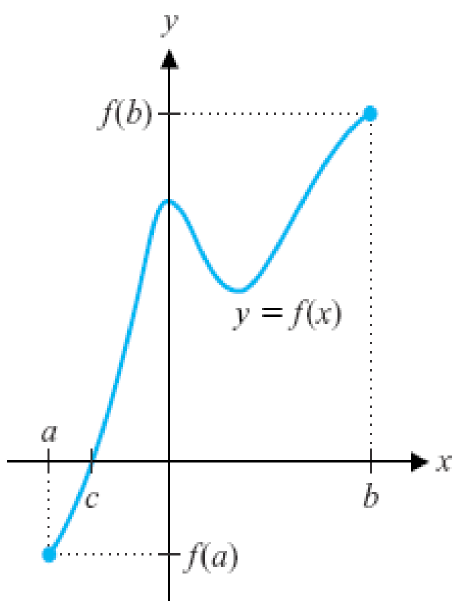
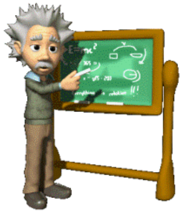



**FIGURE 1.30**
Intermediate Value Theorem where $c$ is a zero of $f$

**Corollary 4.2**

If $f$ is continuous on the closed interval $[a,b]$, and $f(a)$ and $f(b)$ have opposite signs [i.e. $f(a)f(b)<0$], then there exists a number $c$ in $(a,b)$ where $f(c)=0$.



Remark:

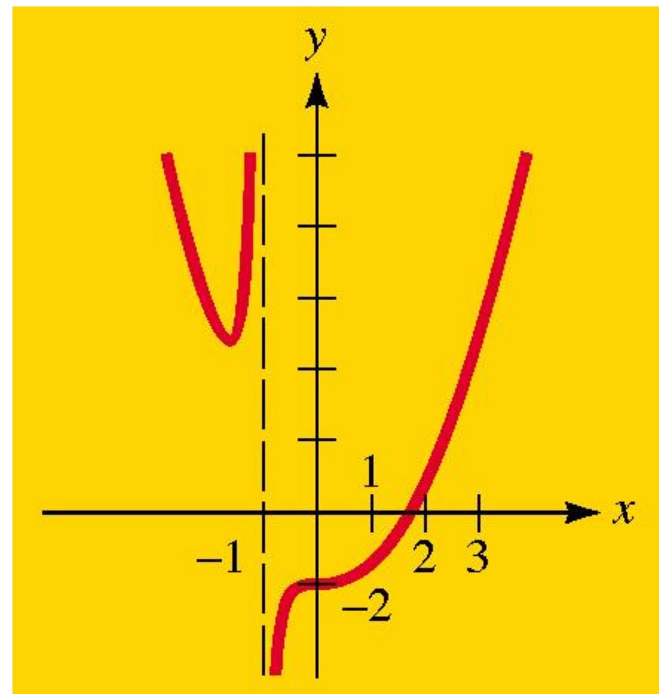
Notice that The Intermediate Value Theorem and Corollary 4.2 are example of *existence theorems*; they tell you that there exists a number $c$ satisfying some condition, but they do not tell you what $c$ is and where $c$ is.

## Example

Show that the equation $x^2 - x - 1 = \dfrac{1}{x+1}$ has a solution for $1<x<2$

## Solution:

Let $f(x) = x^2 - x - 1 - \dfrac{1}{x+1}$. Then $f(1)=-3/2$ and $f(2)=2/3$. Since $f(x)$ is continuous for $1 \leq x \leq 2$, it follows from the intermediate value property that the graph must cross the x axis somewhere between $x=1$ and $x=2$.



# The Method of Bisections (二分法)

- The method of Bisections can help us locate the zeroes of a function.

**EXAMPLE 4.9** Finding Zeros by the Method of Bisections

Find the zeros of $f(x) = x^5 + 4x^2 - 9x + 3$.

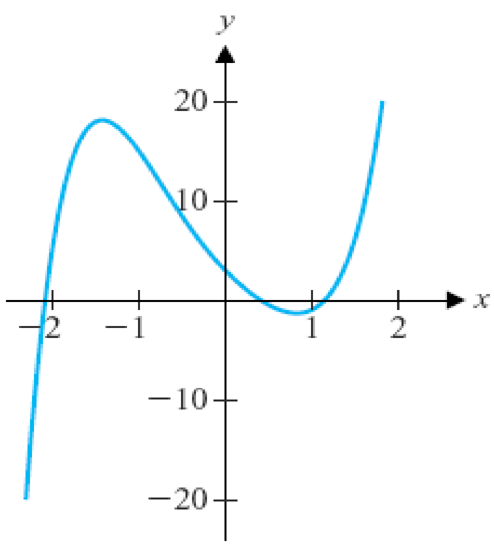



**FIGURE 1.31**
$y = x^5 + 4x^2 - 9x + 3$

| $a$ | $b$ | $f(a)$ | $f(b)$ | Midpoint | $f$(midpoint) |
|---|---|---|---|---|---|
| 0 | 1 | 3 | −1 | 0.5 | −0.469 |
| 0 | 0.5 | 3 | −0.469 | 0.25 | 1.001 |
| 0.25 | 0.5 | 1.001 | −0.469 | 0.375 | 0.195 |
| 0.375 | 0.5 | 0.195 | −0.469 | 0.4375 | −0.156 |
| 0.375 | 0.4375 | 0.195 | −0.156 | 0.40625 | 0.015 |
| 0.40625 | 0.4375 | 0.015 | −0.156 | 0.421875 | −0.072 |
| 0.40625 | 0.421875 | 0.015 | −0.072 | 0.4140625 | −0.029 |
| 0.40625 | 0.4140625 | 0.015 | −0.029 | 0.41015625 | −0.007 |
| 0.40625 | 0.41015625 | 0.015 | −0.007 | 0.408203125 | 0.004 |