ASP

Solution to Assignment 5

1.

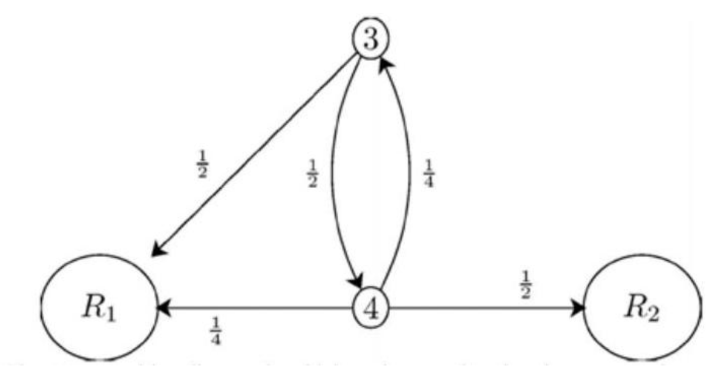
   (a) $\{0, 1, 2\}$ recurrent.

   (b) $\{0, 1, 2, 3\}$ recurrent.

   (c) $\{0, 2\}$ recurrent, $\{1\}$ transient, $\{3, 4\}$ recurrent.

   (d) $\{0, 1\}$ recurrent, $\{2\}$ recurrent, $\{3\}$ transient, $\{4\}$ transient.

2. $\{3, 4\}$ is transient, $\{1, 2\}, \{5, 6, 7\}$ is recurrent. Here, we can replace each recurrent class with one absorbing state. The resulting state diagram is shown in the below figure



The state transition diagram in which we have replaced each recurrent class with one absorbing state. Now we can apply our standard methodology to find probability of absorption in state R_1. In particular, define

$$a_i = P(\text{ absorption in } R_1 \mid X_0 = i), \quad \text{for all } i \in S$$

By the above definition, we have $a_{R_1} = 1$, and $a_{R_2} = 0$. To find the unknown values of a_i 's, we can use the following equations

$$a_i = \sum_k a_k p_{ik}, \quad \text{for } i \in S$$

We obtain

$$\begin{aligned}
a_3 &= \frac{1}{2}a_{R_1} + \frac{1}{2}a_4 \\
&= \frac{1}{2} + \frac{1}{2}a_4 \\
a_4 &= \frac{1}{4}a_{R_1} + \frac{1}{4}a_3 + \frac{1}{2}a_{R_2} \\
&= \frac{1}{4} + \frac{1}{4}a_3
\end{aligned}$$

Solving the above equations, we obtain

$$a_3 = \frac{5}{7}, \quad a_4 = \frac{3}{7}$$

Therefore, if $X_0 = 3$, the chain will end up in class R_1 with probability $a_3 = \frac{5}{7}$.

3. Clearly, $f_{3,3}^1 = 0.2$, $f_{3,3}^{n+1} = 0.4f_{1,3}^n + 0.4f_{2,3}^n, n \geq 1$. We have $f_{3,3}(z) = 0.2z + 0.4zf_{1,3}(z) + 0.4zf_{2,3}(z)$. Similarly,

$$f_{1,3}(z) = 0.3z + 0.4zf_{1,3}(z) + 0.3zf_{2,3}(z) \text{ and } f_{2,3}(z) = 0.3z + 0.6zf_{1,3}(z) + 0.1zf_{2,3}(z$$

$$\begin{pmatrix} 1-0.4z & -0.3z \\ -0.6z & 1-0.1z \end{pmatrix} \begin{pmatrix} f_{1,3}(z) \\ f_{2,3}(z) \end{pmatrix} = \begin{pmatrix} 0.3z \\ 0.3z \end{pmatrix}$$

$$\begin{pmatrix} f_{1,3}(z) \\ f_{2,3}(z) \end{pmatrix} = \begin{pmatrix} 1-0.4z & -0.3z \\ -0.6z & 1-0.1z \end{pmatrix}^{-1} \begin{pmatrix} 0.3z \\ 0.3z \end{pmatrix}$$

$$= \frac{1}{(1-0.4z)(1-0.1z) - 0.18z^2} \begin{pmatrix} 1-0.1z & 0.3z \\ 0.6z & 1-0.4z \end{pmatrix} \begin{pmatrix} 0.3z \\ 0.3z \end{pmatrix}$$

$$= \frac{1}{1-0.5z-0.14z^2} \begin{pmatrix} 0.3z + 0.06z^2 \\ 0.3z + 0.06z^2 \end{pmatrix}$$

$$= \frac{1}{(1-0.7z)(1+0.2z)} \begin{pmatrix} 0.3z + 0.06z^2 \\ 0.3z + 0.06z^2 \end{pmatrix}$$

$$= \frac{1}{1-0.7z} \begin{pmatrix} 0.3z \\ 0.3z \end{pmatrix}$$

And

$$\begin{aligned}
f_{1,3}(z) = f_{2,3}(z) &= \frac{0.3z}{1-0.7z} = 0.3z + (0.3)(0.7)z^2 + (0.3)(0.7)^2z^3 + \cdots \\
f_{3,3}(z) &= 0.2z + 0.4zf_{1,3}(z) + 0.4zf_{2,3}(z) \\
&= 0.2z + 0.4z\frac{0.3z}{1-0.7z} + 0.4z\frac{0.3z}{1-0.7z} \\
&= 0.2z + 2(0.4z)\left(0.3z + (0.3)(0.7)z^2 + (0.3)(0.7)^2z^3 + \cdots\right) \\
&= 0.2z + 0.24z^2 + (0.24)(0.7)z^3 + (0.24)(0.7)^2z^4 + \cdots
\end{aligned}$$

4. There are four communication classes: transient Class $I = \{1, 2\}$, transient Class $II = \{3, 4\}$, transient Class $III = \{5\}$, recurrent Class $IV = \{6, 7, 8\}$. Period of Class I is 1. Period of class II is 2 . Period of class III is ∞. Period of Class IV is 1.