Chapter 5 Systems of First Order Linear Equations

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1 Basic Theory of Systems of First Order Linear Equations

General system of first order ODEs.

$$
\begin{aligned}
x_1'(t) &= F_1(x_1, x_2, \ldots, x_n, t) \\
x_2'(t) &= F_2(x_1, x_2, \ldots, x_n, t) \\
&\vdots \\
x_n'(t) &= F_n(x_1, x_2, \ldots, x_n, t)
\end{aligned}
$$

System of first order linear ODEs

$$
\begin{aligned}
x_1'(t) &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t) \\
x_2'(t) &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + g_2(t) \\
&\vdots \\
x_n'(t) &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + g_n(t)
\end{aligned}
$$

Initial conditions:

$$
x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \ldots, \quad x_n(t_0) = x_n^0.
$$

Theorem 1.1 *(Existence and Uniqueness of Solution to the IVP) If the functions $p_{11}, p_{12}, \ldots,$ $p_{nn}, g_1, \ldots, g_n$ are continuous on an open interval $I: \alpha < t < \beta$, then there exists a unique solution $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ of the system that also satisfies the initial conditions, where t_0 is any point in I, and $x_1^0, \ldots, x_n^0$ are any prescribed numbers. Moreover, the solution exists throughout the interval I.*

The system can be written in matrix form as

$$
\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t).
$$

The system of equations is called **homogeneous** if $\mathbf{g}(t) = \mathbf{0}$,

Consider homogeneous system

$$
\boxed{\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t)} \tag{1.1}
$$

Theorem 1.2 *(Principle of Superposition) If the vector functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions of the system (1.1), then the linear combination $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution for any constants c_1 and c_2.*

Definition 1.3 *The **Wronskian** of n solutions $\mathbf{x}^{(i)}, i = 1, \ldots, n$ of the system (1.1) is*

$$
W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}] = \det[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}].
$$

Theorem 1.4 *If the vector functions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent solutions of the system (1.1) for each point in the interval $\alpha < t < \beta$, then each solution $\mathbf{x} = \phi(t)$ of the system (1.1) can be expressed as a linear combination of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$*

$$\phi(t) = c_1 \mathbf{x}^{(1)}(t) + \cdots + c_n \mathbf{x}^{(n)}(t)$$

in exactly one way.

Definition 1.5 *If the vector functions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent solutions of the system (1.1) for each point in the interval $\alpha < t < \beta$, then we call $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ a **fundamental set of solutions**, and*

$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)}(t) + \cdots + c_n \mathbf{x}^{(n)}(t)$$

the general solution.

Theorem 1.6 *If $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are solutions of the system (1.1) on the interval $\alpha < t < \beta$, then in this interval $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ either is identically zero or else never vanishes.*

Theorem 1.7 *Let*

$$\mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \mathbf{e}^{(n)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

further, let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ be the solutions of the system (1.1) that satisfy the initial conditions

$$\mathbf{x}^{(1)}(t_0) = \mathbf{e}^{(1)}, \quad \ldots, \quad \mathbf{x}^{(n)}(t_0) = \mathbf{e}^{(n)},$$

respectively, where t_0 is any point in $\alpha < t < \beta$. Then $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a fundamental set of solutions of the system.

Theorem 1.8 *Consider the system (1.1), where each element of $\mathbf{P}$ is a real-valued continuous function. If $\mathbf{x} = \mathbf{u}(t) + i\mathbf{v}(t)$ is a complex-valued solution, then its real part $\mathbf{u}(t)$ and its imaginary part $\mathbf{v}(t)$ are also solutions of this equation.*

Theorem 1.9 *The general solution of the inhomogeneous system $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t)$ is*

$$\mathbf{x}(t) = c_1 \mathbf{x}_1 + \cdots + c_n \mathbf{x}_n + \mathbf{x}_p(t),$$

where $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is a fundamental set of solutions of the homogeneous system $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t)$, and $\mathbf{x}_p(t)$ is a particular solution of the inhomogeneous system.

2 Homogeneous Linear Systems with Constant Coefficients

Example 2.1. (Decoupled system) Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x}.$$

Answer. Write each component one by one,

$$\begin{aligned} x_1'(t) &= 2x_1(t) \\ x_2'(t) &= -3x_2(t) \end{aligned}$$

Solving them one by one,
$$x_1(t) = c_1 e^{2t}, \quad x_2(t) = c_2 e^{-3t}.$$

Write the solution in vector form,
$$\mathbf{x}(t) = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{-3t} \end{pmatrix} := c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t),$$

where
$$\mathbf{x}_1(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2(t) = e^{-3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now
$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{vmatrix} = e^{-t} \neq 0.$$

So $\mathbf{x}_1, \mathbf{x}_2$ form a fundamental set of solutions, and the general solution is $c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t)$.

Consider the system
$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

We look for solutions of the form
$$\mathbf{x}(t) = e^{\lambda t} \boldsymbol{\xi},$$

where $\boldsymbol{\xi} \neq \mathbf{0}$ is a contant vector.

Example 2.2. Find the general solution of the system
$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x} \Rightarrow \begin{cases} x_1' = x_1 + x_2 \\ x_2' = 4x_1 + x_2 \end{cases}$$

Then plot a **direction field** and draw a **phase portrait**.

Answer. Let $\mathbf{x}(t) = e^{\lambda t} \boldsymbol{\xi}$, then
$$\mathbf{x}' = \lambda e^{\lambda t} \boldsymbol{\xi} = \mathbf{A}\mathbf{x} = \mathbf{A} e^{\lambda t} \boldsymbol{\xi} \quad \Leftrightarrow \quad \mathbf{A}\boldsymbol{\xi} = \lambda \boldsymbol{\xi}.$$

So λ is an eigenvalue and $\boldsymbol{\xi}$ is an associated eigenvector of $\mathbf{A}$.
$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0 \quad \Rightarrow \quad 1-\lambda = \pm 2 \quad \Rightarrow \quad \lambda_1 = -1, \lambda_2 = 3.$$

For $\lambda_1 = -1$,
$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

For $\lambda_2 = 3$,
$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$
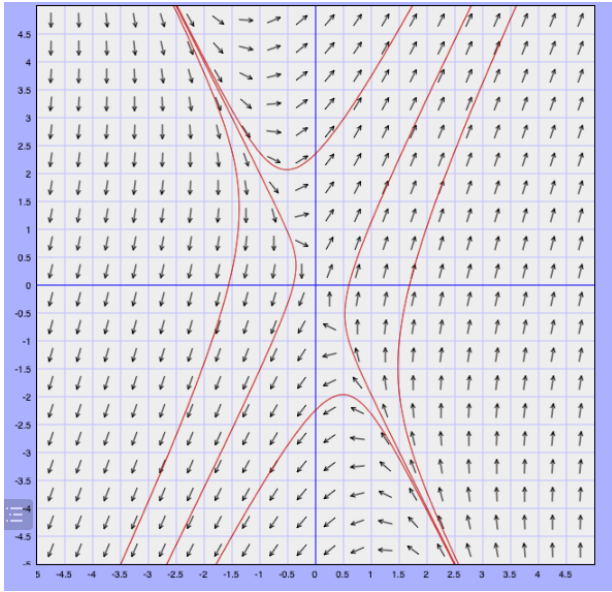
So we have two solutions
$$\mathbf{x}_1(t) = e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{x}_2(t) = e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

and
$$W[\mathbf{x}_1, \mathbf{x}_2](t) = e^{2t} \begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} = 4e^{2t} \neq 0.$$

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So $\mathbf{x}_1, \mathbf{x}_2$ form a fundamental set of solutions, and the general solution is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

- If $c_1 = 1, c_2 = 0$, then $\mathbf{x} = \mathbf{x}_1 = e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, whose graph is a straight line parallel to the vector $(1, -2)^T$. Moreover, $x_1(t) \to 0, x_2(t) \to 0$ as $t \to \infty$.

- If $c_1 = 0, c_2 = 1$, then $\mathbf{x} = \mathbf{x}_2 = e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, whose graph is a straight line parallel to the vector $(1, 2)^T$. Moreover, $x_1(t) \to \infty, x_2(t) \to \infty$ as $t \to \infty$.

- If $c_1 \neq 0, c_2 \neq 0$. Then all solutions converges to $\mathbf{x}_2$ asymptotically as $t \to \infty$.

- The equilibrium solution $\mathbf{x} = \mathbf{0}$ is called a **saddle point**. A saddle point is always unstable.

Online tool for direction field and phase portrait: `https://aeb019.hosted.uark.edu/pplane.html`

How to draw a phase portrait for $\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

phase portrait (saddle point)
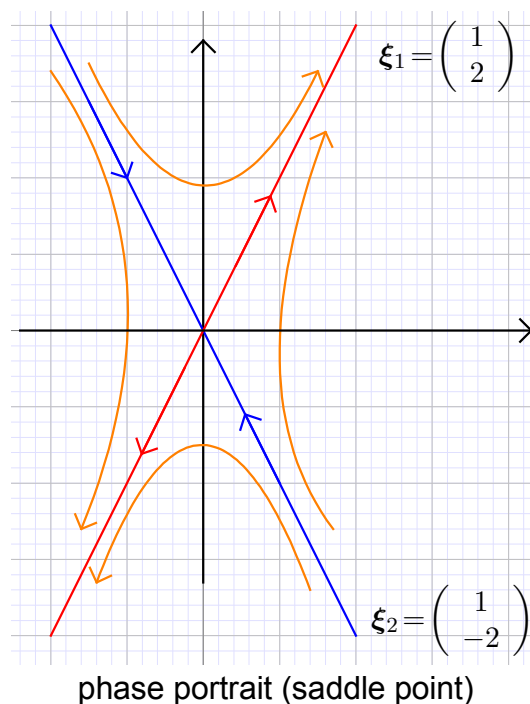
Example 2.3. Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \mathbf{x}.$$

Then plot a **direction field** and draw a **phase portrait**.

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Answer. First, find eigenvalues,

$$\begin{vmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{vmatrix} = (3+\lambda)(2+\lambda) - 2 = \lambda^2 + 5\lambda + 4 \Rightarrow \lambda_1 = -1, \lambda_2 = -4.$$

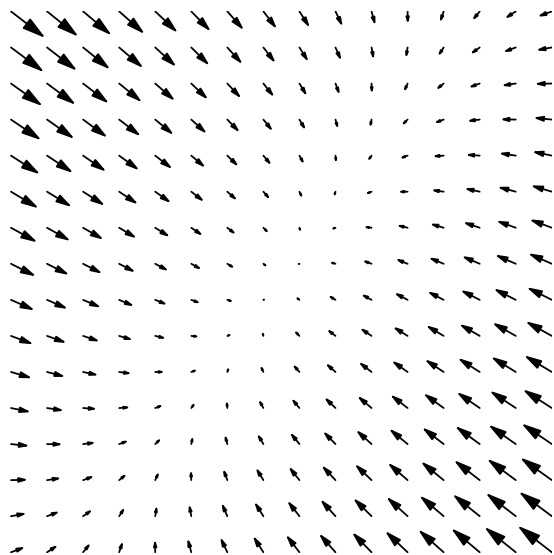Find the eigenvectors.

For $\lambda_1 = -1$

$$\begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \rightarrow \boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix},$$

For $\lambda_2 = -4$
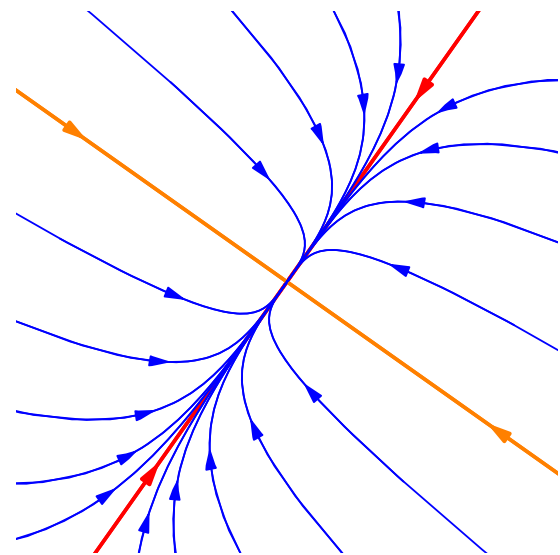
$$\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \rightarrow \boldsymbol{\xi}_2 = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}.$$

A fundamental set of solutions are

$$y_1 = e^{-t}\boldsymbol{\xi}_1 = e^{-t}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \quad y_2 = e^{-4t}\boldsymbol{\xi}_2 = e^{-4t}\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}.$$

direction field

phase portrait (node)

In this case, the equilibrium solution $\mathbf{x} = \mathbf{0}$ is called a **node**. If both eigenvalues are negative, as in this example, the node is **asymptotically stable**. If both eigenvalues are positive, then we still have a node, but it's **unstable**.

For a general $n \times n$ system

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x},$$

If we have n linearly independent eigenvectors, then a fundamental set of solution is

$$y_i = e^{\lambda_i t}\boldsymbol{\xi}_i, \quad i = 1, \dots, n,$$

even though some eigenvalues are repeated.

An $n \times n$ symmetric matrices always have n linearly indepdent eigenvectors (even better, the eigenvectors for distinct eigenvalues are orthogonal).

Example 2.4. Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}.$$

Answer. First, find eigenvalues

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda)$$

$$= -\lambda(\lambda^2 - 1) + 2(\lambda + 1) = (\lambda + 1)(-\lambda^2 + \lambda + 2) = (\lambda + 1)^2(-\lambda + 2)$$

$$\Rightarrow \quad \lambda_{1,2} = -1, \quad \lambda_3 = 2.$$

Next, find eigenvectors.
For $\lambda_{1,2} = -1$,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow x_1 + x_2 + x_3 = 0 \rightarrow \boldsymbol{\xi} = \begin{pmatrix} -s - t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{\xi}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{\xi}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda_3 = 2$,

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 3 & -3 \\ 0 & -3 & 3 \\ 1 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \boldsymbol{\xi}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

So a fundamental set of solutions are

$$y_1 = e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad y_2 = e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad y_3 = e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

3 Complex Eigenvalues

If the matrix $\mathbf{A}$ is real-valued, then the complex eigenvalues appear in conjugate pairs. Let's consider the complex eigenvalues

$$\lambda \pm i\mu.$$

with eigenvectors

$$\mathbf{u} \pm i\mathbf{v}.$$

We have complex-valued solutions

$$e^{(\lambda + i\mu)t}(\mathbf{u} + i\mathbf{v}) = e^{\lambda t}(\cos \mu t + i \sin \mu t)(\mathbf{u} + i\mathbf{v})$$

$$= e^{\lambda t}[(\mathbf{u}\cos \mu t - \mathbf{v}\sin \mu t) + i(\mathbf{v}\cos \mu t + \mathbf{u}\sin \mu t)].$$

So fundamental set of real-valued solutions are

$$y_1 = e^{\lambda t}(\mathbf{u}\cos \mu t - \mathbf{v}\sin \mu t), \quad y_2 = e^{\lambda t}(\mathbf{v}\cos \mu t + \mathbf{u}\sin \mu t).$$

(You should check the Wronskian).

Example 3.1. Find a fundamental set of real-valued solutions of the system

$$\mathbf{x}' = \begin{pmatrix} -\dfrac{1}{2} & 1 \\ -1 & -\dfrac{1}{2} \end{pmatrix} \mathbf{x}.$$

Plot a phase portrait and graphs of components of typical solutions.

Answer. First, find eigenvalues,

$$\begin{vmatrix} -\dfrac{1}{2} - \lambda & 1 \\ -1 & -\dfrac{1}{2} - \lambda \end{vmatrix} = \left(-\dfrac{1}{2} - \lambda\right)^2 + 1 \quad \Rightarrow \quad \lambda_{1,2} = -\dfrac{1}{2} \pm i.$$

Then find eigenvectors. For $\lambda_2 = -\dfrac{1}{2} - i$,

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \to \begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix} \to \boldsymbol{\xi}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix} \to \boldsymbol{\xi}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

So the complex solutions are

$$y = e^{\left(-\frac{1}{2} + i\right)t} \begin{pmatrix} -i \\ 1 \end{pmatrix} = e^{-\frac{1}{2}t}(\cos t + i \sin t)\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$= e^{-\frac{1}{2}t}\begin{pmatrix} \sin t - i \cos t \\ \cos t + i \sin t \end{pmatrix} = e^{-\frac{1}{2}t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + i e^{-\frac{1}{2}t}\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}.$$

A fundamental set of real solutions are

$$y_1 = e^{-\frac{1}{2}t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \quad y_2 = e^{-\frac{1}{2}t}\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}.$$
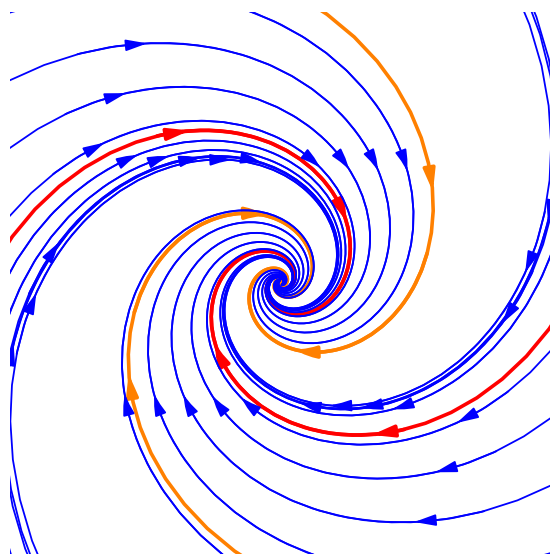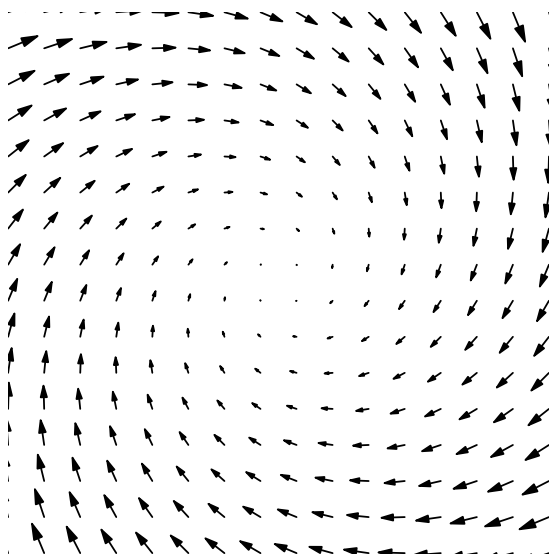
phase portrait (spiral)

In this case, the equilibrium solution $\mathbf{x} = \mathbf{0}$ is called **spiral**. A spiral point is asyptotically stable if the real part of the eigenvalue is negative, and unstable if the real part is positive.

Question 1. The system

$$\mathbf{x}' = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \mathbf{x}$$

contains a parameter α. Describe how the solutions depend qualitatively on α; in particular, find the critical values of α at which the qualitative behavior of the trajectories in the phase plane changes markedly.

Answer. Find the eigenvalues,

$$\begin{vmatrix} \alpha - \lambda & 2 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 - \alpha\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}.$$

1. If $|\alpha| > 4$, then $\lambda_{1,2}$ are both real and distinct.
   a. If $\alpha > 4$, then both eigenvalues are positive. So we have an unstable node.
   b. If $\alpha < -4$, then both eigenvalues are negative. So we have a stable node.
2. If $|\alpha| < 4$, then both eigenvalues are complex.
   a. If $0 < \alpha < 4$, then the real part is positive. So have an unstable spiral.
   b. If $-4 < \alpha < 0$, then the real part is negative. So have a stable spiral.
   c. If $\alpha = 0$, then the real part is zero. The solutions are ellipses, and we have a **center**. A center is stable (but not asymptotically stable).
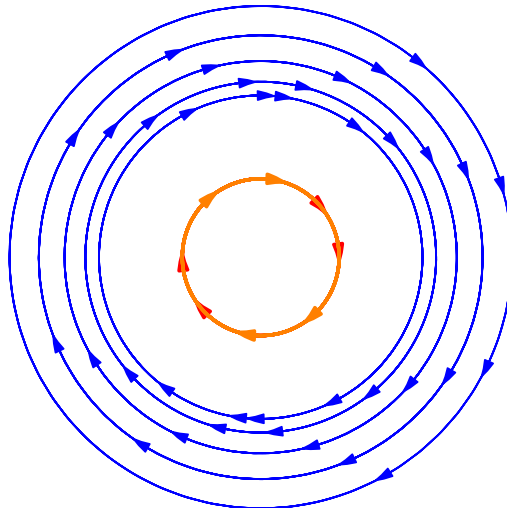
phase portrait (center)

3. If $\alpha = 4$, then both eigenvalues are positive and equal, and we have an unstable **improper node**.
4. If $\alpha = -4$, then both eigenvalues are positive and equal, and we have an asymptotically stable **improper node**.

4 Repeated Eigenvalues

Suppose λ is a repeated eigenvalue of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with eigenvector $\boldsymbol{\xi}$. So one solution is

$$y_1 = e^{\lambda t}\boldsymbol{\xi}.$$

Consider a second solution of the form

$$y_2 = te^{\lambda t}\boldsymbol{\xi} + e^{\lambda t}\boldsymbol{\eta} = e^{\lambda t}(t\boldsymbol{\xi} + \boldsymbol{\eta}).$$

What should be $\boldsymbol{\eta}$? Plugging y_2 into the equation,

$$y_2' = \lambda e^{\lambda t}(t\boldsymbol{\xi} + \boldsymbol{\eta}) + e^{\lambda t}\boldsymbol{\xi} = \mathbf{A}\, y_2 = e^{\lambda t} A(t\boldsymbol{\xi} + \boldsymbol{\eta})$$

$$\Rightarrow \underline{A(t\boldsymbol{\xi} + \boldsymbol{\eta}) = \lambda(t\boldsymbol{\xi} + \boldsymbol{\eta}) + \boldsymbol{\xi}} \quad \Rightarrow \quad \mathbf{A}\boldsymbol{\eta} = \lambda\boldsymbol{\eta} + \boldsymbol{\xi} \quad \Rightarrow \quad \boxed{(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}} \quad \Rightarrow$$

$$\boxed{(\mathbf{A} - \lambda\mathbf{I})^2\boldsymbol{\eta} = 0}$$

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We call $\boldsymbol{\eta}$ a **generalized eigenvector**.

Example 4.1. Find a fundamental set of solutions of

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

and draw a phase portrait for this system.

Answer. Eigenvalues,

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 \quad \Rightarrow \quad \lambda_{1,2} = 2.$$

Eigenvectors

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \rightarrow \boldsymbol{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So one solution is

$$y_1 = e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We seek a second solution in the form

$$y_2 = e^{2t} (t\boldsymbol{\xi} + \boldsymbol{\eta}),$$

and $\boldsymbol{\eta}$ can be found as follows.

$$(\mathbf{A} - \lambda\mathbf{I})\,\boldsymbol{\eta} = \boldsymbol{\xi} \quad \Rightarrow \quad \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \eta_1 + \eta_2 = -1 \quad \Rightarrow \quad \boldsymbol{\eta} = \begin{pmatrix} -1-s \\ s \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Note that we can take $s = 0$. So a second is

$$y_2 = e^{2t} \left[t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] = e^{2t} \begin{pmatrix} t-1 \\ -t \end{pmatrix}.$$
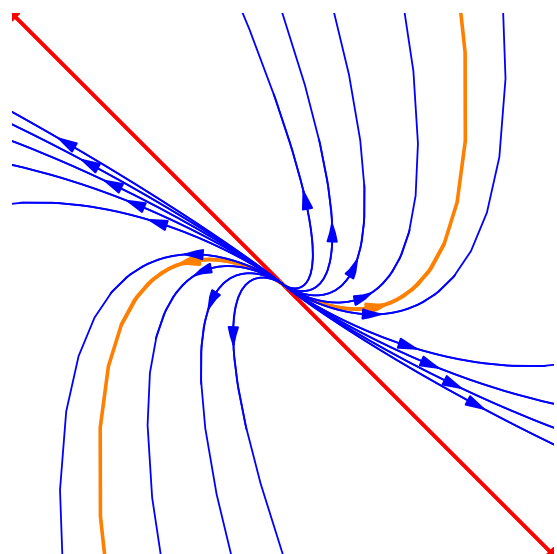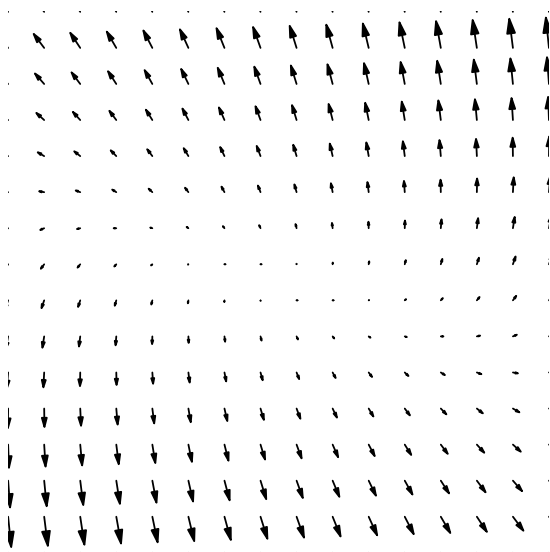
phase portrait (improper node)