# ASP Assignment 8

1. Let $(B_t)_{t\geq 0}$ be a **SBM** (standard Brownian motion) and $c > 0$. Show that the process
$$X_t = \frac{1}{c} B_{c^2 t},\ t \geq 0$$
is a SBM.

2. Let $(B_t)_{t\geq 0}$ be a SBM and fix $t_0 \geq 0$. Prove that
$$\tilde{B}_t := B_{t+t_0} - B_{t_0},\ t \geq 0$$
is a SBM.

3. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and $(X_t)_{t\geq 0}$ be the process defined as
$$X_t = B_t - tB_1,\ 0 \leq t \leq 1.$$
The process $(X_t)_{0\leq t\leq 1}$ is called a Brownian Bridge.
   (a) Compute the means $E(X_t)$ and the covariances $E(X_s X_t)$, where $s, t \geq 0$.
   (b) What is the distribution of $X_t$?

4. Let $(B_t)_{t\geq 0}$ be a SBM.
   (a) Prove that the moment generating function of $B_t$ is given by
$$E\left[e^{uB_t}\right] = \exp\left(\frac{1}{2}u^2 t\right), \quad \text{for all } u \in \mathbb{R}.$$
   (b) Use the power series expansion of the exponential function on both sides, compare the terms with the same power of $u$ and deduce that
$$E[B_t^4] = 3t^2$$
and more generally that
$$E\left[B_t^{2k}\right] = \frac{(2k)!}{2^k \cdot k!} t^k,\ k \in \mathbb{N}.$$
   (c) If you feel uneasy about the lack of rigour in the method in b), you can proceed as follows: first note that
$$E[f(B_t)] = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(x) e^{-\frac{x^2}{2t}}\, dx$$
for all functions $f$ such that the integral on the right converges. Then apply this to $f(x) = x^{2k}$ and use integration by parts and induction on $k$.

5. Let $(B_t)_{t\geq 0}$ be a SBM. Find $E(B_1^2 B_2 B_3)$.

6. Suppose $B_t$ is a standard Brownian motion and let $\mathcal{F}_t$ be its corresponding filtration. Let
$$M_t = B_t^2 - t.$$
Show that $M_t$ is a martingale with respect to $\mathcal{F}_t$.

7. Let $B_t$ be a standard Brownian motion and let $\{\mathcal{F}_t\}$ denote the usual filtration. Suppose $s < t$. Compute the following.
   (a) $E\left[B_t^2 \mid \mathcal{F}_s\right]$
   (b) $E\left[B_t^3 \mid \mathcal{F}_s\right]$
   (c) $E\left[B_t^4 \mid \mathcal{F}_s\right]$
   (d) $E\left[e^{4B_t-2} \mid \mathcal{F}_s\right]$

8. Let $B_t$ be a standard Brownian motion and let
$$Y(t) = tB(1/t), \quad t > 0$$
and $Y(0) = 0$.
   (a) Is $Y(t)$ a Gaussian process?
   (b) Compute $\mathrm{Cov}(Y(s), Y(t))$.
   (c) Does $Y(t)$ have the distribution of a standard Brownian motion?

9. Let $B_t$ be a standard Brownian motion. Find the following probabilities. If you cannot give the answer precisely give it up to at least three decimal places using a table of the normal distribution.
   (a) $\mathbb{P}\{B_3 \geq 1/2\}$
   (b) $\mathbb{P}\{B_1 \leq 1/2, B_3 > B_1 + 2\}$
   (c) $\mathbb{P}(E)$, where $E$ is the event that the path stays below the line $y = 6$ up to time $t = 10$.
   (d) $\mathbb{P}\{B_4 \leq 0 \mid B_2 \geq 0\}$.

10. Suppose $B_t$ is a standard Brownian motion and let $\mathcal{F}_t$ be its corresponding filtration. Let
$$M_t = e^{\sigma B_t - \frac{\sigma^2 t}{2}}.$$
Show that $M_t$ is a martingale with respect to $\mathcal{F}_t$.

11. Let $B_t$ be a standard (one-dimensional) Brownian motion starting at 0 and let
$$M = \max\{B_t : 0 \leq t \leq 1\}.$$
Find the density for $M$ and compute its expectation and variance.

12. Let $B_t$ be 2-dimensional Brownian motion, that is

$$B_t = (X_t, Y_t),$$

where $(X_t)$ and $(Y_t)$ are independant standard 1-dimensional Brownian motions. Put

$$D_\rho = \left\{ x = (x_1, x_2) \in \mathbf{R}^2 : \ |x| = \sqrt{x_1^2 + x_2^2} < \rho \right\} \qquad \text{for } \rho > 0.$$

Compute

$$\mathbb{P}\left[B_t \in D_\rho\right].$$