2023-24 First Semester
MATH2023 Ordinary Differential Equations (1003)

Assignment 9 Suggested Solutions

1. The eigenvalues: $\lambda_1 = 4$ and $\lambda_2 = 2$, both positive, since

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8 = (\lambda-2)(\lambda-4) = 0$$

The origin is an unstable nodal source.
For λ_1=4,

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \vec{\xi}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \to \quad \vec{\xi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 2$,

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \vec{\xi}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \to \quad \vec{\xi}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The general solution is

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \qquad c_{1,2} \in \mathbb{R}$$

As shown, $\mathbf{x}$ tends to infinity as $t \to \infty$.

2. The eigenvalues are $\lambda = \pm i\sqrt{3}$, since

$$|A-\lambda I| = \begin{vmatrix} -1-\lambda & 1 \\ -4 & 1-\lambda \end{vmatrix} = \lambda^2 + 3 = 0$$

Since the real part of both eigenvalues is 0, **the origin is a stable centre.**
For $\lambda_1 = i\sqrt{3}$,

$$\begin{bmatrix} -1-i\sqrt{3} & 1 \\ -4 & 1-i\sqrt{3} \end{bmatrix} \vec{\xi}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \to \quad \vec{\xi}_1 = \begin{bmatrix} 1 \\ 1+i\sqrt{3} \end{bmatrix}$$

For $\lambda_2 = -i\sqrt{3}$,

$$\vec{\xi}_2 = \overline{\vec{\xi}_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - i \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix}$$

The general solution is

$$\mathbf{x} = c_1 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\sqrt{3}t) - \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix} \sin(\sqrt{3}t) \right) + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(\sqrt{3}t) + \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix} \cos(\sqrt{3}t) \right)$$

$$= c_1 \begin{bmatrix} \cos(\sqrt{3}t) \\ \cos(\sqrt{3}t) - \sqrt{3}\sin(\sqrt{3}t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(\sqrt{3}t) \\ \sin(\sqrt{3}t) + \sqrt{3}\cos(\sqrt{3}t) \end{bmatrix}$$

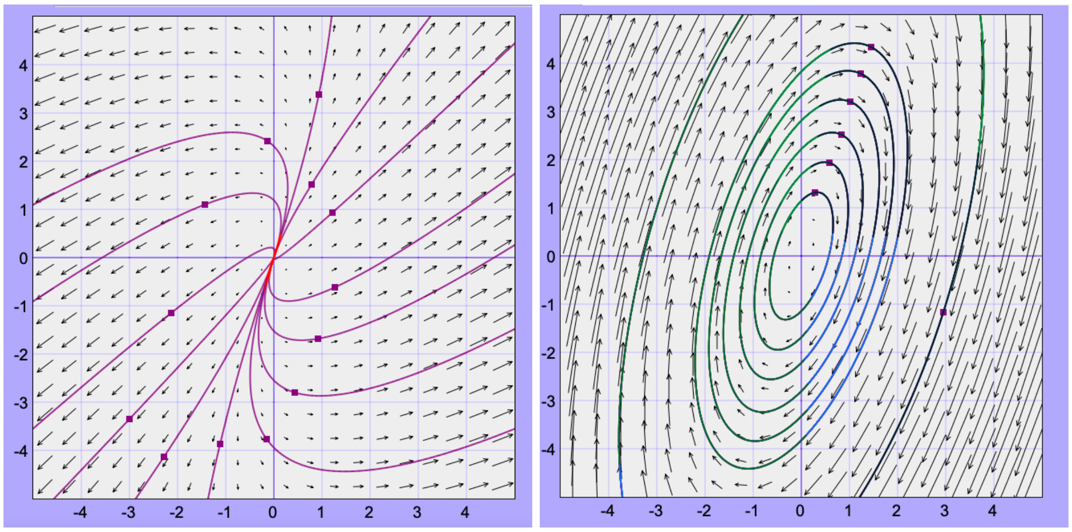
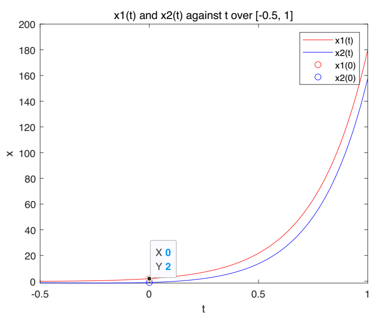


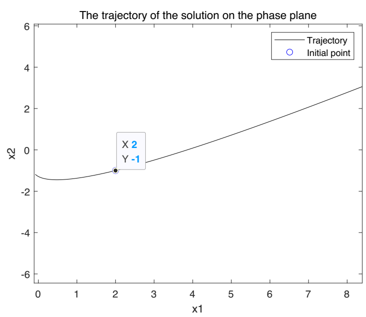
Figure 1: left: Problem 1; right: Problem 2

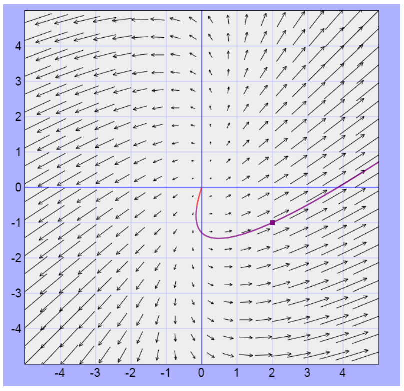
3. The general solution is obtained in Q1. By plugging in the initial conditions,

$$\mathbf{x}(0) = \begin{bmatrix} c_1 + c_2 \\ 3c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \rightarrow \quad c_1 = -1.5, \;\; c_2 = 3.5$$

$$\mathbf{x} = -\frac{3}{2}\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} + \frac{7}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$$







4. (a) The **eigenvalues** of A: $\lambda_1 = \lambda_2 = \lambda_3 = 1$

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 2-\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} = (1-\lambda)[(2-\lambda)(-\lambda)+1]$$
$$= -(\lambda - 1)^3$$

For $\lambda = 1$, an **eigenvector** of A:

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \boldsymbol{\xi} = \mathbf{0}, \quad \boldsymbol{\xi} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(b) From part (a), one solution is

$$\mathbf{x}^{(1)}(t) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^t.$$

(c) Assume the form of the second solution as $\mathbf{x}^{(2)}(t) = t\boldsymbol{\xi} e^t + \boldsymbol{\eta} e^t$. Substitution yields

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ \underline{t\boldsymbol{\xi} e^t} + e^t\underline{(\boldsymbol{\xi}+\boldsymbol{\eta})} &= A(\underline{t\boldsymbol{\xi} e^t} + \underline{\boldsymbol{\eta} e^t}) \end{aligned}$$

By equating the coefficients, we obtain two equations

$$(A - \boldsymbol{I})\boldsymbol{\xi} = \mathbf{0}, \qquad (A - \boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi},$$

which implies $\boldsymbol{\xi}$ is an eigenvector corresponding to $\lambda = 1$, $\boldsymbol{\eta}$ is a generalized eigenvector.

Adopt $\boldsymbol{\xi}$ as in part (a), then a generalized eigenvector can be generated as $\boldsymbol{\eta} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Thus a second solution is $\quad \mathbf{x}^{(2)}(t) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t$ $\quad \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}\eta = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\quad \begin{cases} 2\eta_1 + \eta_2 - \eta_3 = 1 \\ \eta_2 - \eta_3 = 1 \end{cases} \Rightarrow \begin{cases} \eta_1 = 0 \\ \eta_2 = 1 \quad \eta_3 = 0 \end{cases}$

(d) Assume a third solution has the form $\mathbf{x}^{(3)}(t) = \dfrac{t^2}{2}\boldsymbol{\xi} e^t + t\boldsymbol{\eta} e^t + \boldsymbol{\zeta} e^t$. Substitution yields

$A \frac{t^2}{2}\xi e^t = \frac{t^2}{2}\xi e^t \Rightarrow (A-I)\xi = 0$

$A te^t\eta - te^t\eta = te^t\xi \Rightarrow (A-I)\eta = \xi$

$$t\boldsymbol{\xi} e^t + \frac{t^2}{2}\boldsymbol{\xi} e^t + \boldsymbol{\eta} e^t + t\boldsymbol{\eta} e^t + \boldsymbol{\zeta} e^t = A\left(\frac{t^2}{2}\boldsymbol{\xi} e^t + t\boldsymbol{\eta} e^t + \boldsymbol{\zeta} e^t\right)$$

$A\zeta e^t - \zeta e^t = e^t\eta$

$\Downarrow$

$(A-I)\zeta = \eta$

$$\rightarrow \quad \underline{(A - \boldsymbol{I})\boldsymbol{\xi} = \mathbf{0}}, \quad \underline{(A - \boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi}}, \quad \underline{(A - \boldsymbol{I})\boldsymbol{\zeta} = \boldsymbol{\eta}}$$

Take $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ as in part $(a), (b)$, we know

$$\boldsymbol{\xi} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \rightarrow \quad \boldsymbol{\eta} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \rightarrow \quad \boldsymbol{\zeta} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}.$$

$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}\zeta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\begin{cases} 2\zeta_1 + \zeta_2 - \zeta_3 = 1 \\ \zeta_2 - \zeta_3 = 0 \end{cases} \Rightarrow \begin{cases} \zeta_1 = \frac{1}{2} \\ \zeta_2 = \zeta_3 = 0 \end{cases}$

Thus a third solution could be $\quad \mathbf{x}^{(3)}(t) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \dfrac{t^2}{2} e^t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} te^t + \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix} e^t.$

The general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)}, \quad c_{1,2,3} \in \mathbb{R}.$$

Comments:

i) The form of the general solution is not unique, since the eigenvector is not unique.

ii) In part (d), we adopt the value of previous $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ because they satisfy the same equations in both (c) and (d) under the assumption

$$\mathbf{x}^{(3)}(t) = \frac{t^2}{2}\boldsymbol{\xi} e^{rt} + t\boldsymbol{\eta} e^{rt} + \boldsymbol{\zeta} e^{rt}$$

.

If we change the form of $\mathbf{x}^{(3)}(t)$ to $t^2\boldsymbol{\xi} e^{rt} + t\boldsymbol{\eta} e^{rt} + \boldsymbol{\zeta} e^{rt}$, the relationship among ξ, η and ζ may change accordingly.

5. The coefficient matrix A has eigenpairs $\lambda_1 = 2, \boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = -3, \boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$. Thus

$A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix}$

$= -(1-\lambda)(2+\lambda) - 4$

$= \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2) = 0$

$\lambda_1 = -3 \quad \lambda_2 = 2$

for $\lambda_1 = -3$: $\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{\xi}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

for $\lambda_2 = 2$: $\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} \xi_2^{(1)} \\ \xi_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{\xi}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \underline{\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}^{-1}} := PDP^{-1}.$$

$= \frac{1}{1\times(-4)-1}\begin{pmatrix} -4 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$

Denote $\mathbf{y} = P^{-1}\mathbf{x}$ and $\mathbf{h}(t) = P^{-1}\mathbf{g}(t)$, so the equivalent system is

$x' = Ax + g(t)$
$= PDP^{-1}x + g(t)$
$P^{-1}x' = DP^{-1}x + P^{-1}g(t)$.
$y = P^{-1}x \quad h = P^{-1}g(t)$.
$y' = Dy + h$

$$\mathbf{y}' = D\mathbf{y} + \mathbf{h} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \mathbf{y} + \frac{1}{5} \begin{bmatrix} 4e^{-2t} - 2e^t \\ e^{-2t} + 2e^t \end{bmatrix},$$

which is decoupled, i.e.,

$(y_1' - 2y_1)e^{-2t} = \frac{1}{5}(4e^{-4t} - 2e^{-t})$
$(y_1 e^{-2t})' = \frac{1}{5}(4e^{-4t} - 2e^{-t})$
$y_1 e^{-2t} = -\frac{1}{5}e^{-4t} + \frac{2}{5}e^{-t} + c$
$y_1 = -\frac{1}{5}e^{-2t} + \frac{2}{5}e^{t} + ce^{2t}$

$$\begin{cases} y_1' = 2y_1 + \frac{1}{5}(4e^{-2t} - 2e^t) \\ y_2' = -3y_2 + \frac{1}{5}(e^{-2t} + 2e^t) \end{cases} \rightarrow \begin{cases} y_1(t) = -\frac{1}{5}e^{-2t} + \frac{2}{5}e^t + c_1 e^{2t} \\ y_2(t) = \frac{1}{5}e^{-2t} + \frac{1}{10}e^t + c_2 e^{3t} \end{cases}$$

$y_2' = -3y_2 + \frac{1}{5}(e^{-2t} + 2e^{t})$
$(y_2' + 3y_2)e^{3t} = \frac{1}{5}(e^{t} + 2e^{4t})$
$(y_2 e^{3t})' = \frac{1}{5}(e^{t} + 2e^{4t})$
$y_2 e^{3t} = \frac{1}{5}e^{t} + \frac{1}{10}e^{4t} + c$
$y_2 = \frac{1}{5}e^{-2t} + \frac{1}{10}e^{t} + ce^{-3t}$

Solve for $\mathbf{y}$, then $\mathbf{x}$:

$$\mathbf{y} = \frac{e^{-2t}}{5}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{e^t}{10}\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{3t} \end{bmatrix}$$

$$\mathbf{x} = P\mathbf{y} = e^{-2t}\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \frac{1}{2}e^t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ -4 \end{bmatrix}, \quad c_{1,2} \in \mathbb{R}.$$

6. For the associated homogeneous equation $y'' + 2y = 0$,

$r^2 + 2 = 0 \quad r = \pm\sqrt{2}i$

$$y_h(x) = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)$$

Assume a particular solution to the non-homogeneous equation is $\underline{y_p(x) = Ax + B}$, then $B = 0$ and $A = 1/2$. The general solution to (N) is

$y'' + 2y = x$

$2(Ax+B) = x \Rightarrow B = 0$
$A = \frac{1}{2}$

$$y(x) = y_h + y_p = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x) + x/2$$

The boundary conditions determine $c_1 = 0$ and $c_2 = -\frac{\pi}{2\sin(\sqrt{2}\pi)}$.

7. Consider

(i) For $\lambda = -k^2 < 0$, where $k = \sqrt{|\lambda|}$, the equation has the general solution

$$y(x) = c_1 e^{kt} + c_2 e^{-kt}$$

Two boundary conditions imply

$$kc_1 - kc_2 = 0, \quad kc_1 e^{k\pi} - kc_2 e^{-k\pi} = 0, \quad \rightarrow \quad c_1 = c_2 = 0.$$

There is no negative eigenvalues.

$r^2+\lambda=0$: $\lambda \begin{cases} =0 & r^2=0 \quad r_1=r_2=0 \quad y=c_1x+c_2 \\ =-k^2<0 & r^2-k^2=0 \quad r_1=k \quad r_2=-k \quad y=c_1e^{kx}+c_2e^{-kx} \\ =k^2>0 & r^2+k^2=0 \quad r_1=ki \quad r_2=-ki \quad y=c_1\cos kx+c_2\sin kx \end{cases}$

(ii) For $\lambda = 0$, the equation has the general solution

$$y(x) = c_1 x + c_2$$

补充(iii) $\lambda=-k^2<0$ $\quad k=\pm\sqrt{-\lambda}$

$y=c_1e^{kx}+c_2e^{kx}$

Two boundary conditions yield $\quad y'(\pi)=y'(0)=c_1=0$, $c_2 \in R$

$$c_1 = 0, \ c_2 \in \mathbb{R}$$

Thus, $\lambda = 0$ is an eigenvalue with eigenfunction $y(x) = 1$. 因 $y(x)=c_2 \Rightarrow y(x)=1$

(iii) For $\lambda = k^2 > 0$, where $k = \sqrt{|\lambda|}$, the equation has the general solution

$$y(x) = c_1 \cos(kx) + c_2 \sin(kx)$$

The boundary condition $y'(0) = 0$ implies $c_2 = 0$. Therefore, $y(x) = c_1 \cos(kx)$. Now the other condition $y'(\pi) = 0$ implies $-kc_1 \sin(k\pi) = 0$. To have $c_1 \neq 0$, we must choose $k = n$, where $n = 1, 2, \cdots$.

This problem has eigenvalues and eigenfunctions

$$\lambda_n = n^2, \qquad y_n(x) = \cos(nx), \quad n = 0, 1, 2, \cdots$$

8. (i) There is no negative eigenvalues in this case.

(ii) For $\lambda = 0$, the equation has the general solution

$$y(x) = c_1 x + c_2$$

Two boundary conditions yield

$$c_1 = 0, \ c_2 = 0$$

Only trivial solution exists. $\lambda = 0$ is NOT an eigenvalue.

(iii) For $\lambda = k^2 > 0$, where $k = \sqrt{|\lambda|}$, the equation has the general solution

$$y(x) = c_1 \cos(kx) + c_2 \sin(kx)$$

The boundary condition $y'(0) = 0$ implies $c_2 = 0$. Therefore, $y(x) = c_1 \cos(kx)$. The other condition $y(L) = 0$ implies $c_1 \cos(kL) = 0$. To have $c_1 \neq 0$, we must choose

$$k = \frac{(2n-1)\pi}{2L},$$

where $n = 1, 2, \cdots$.

This problem has eigenvalues and eigenfunctions

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L} \right)^2, \qquad y_n(x) = \cos\left(\frac{(2n-1)\pi x}{2L} \right), \quad n = 1, 2, \cdots$$