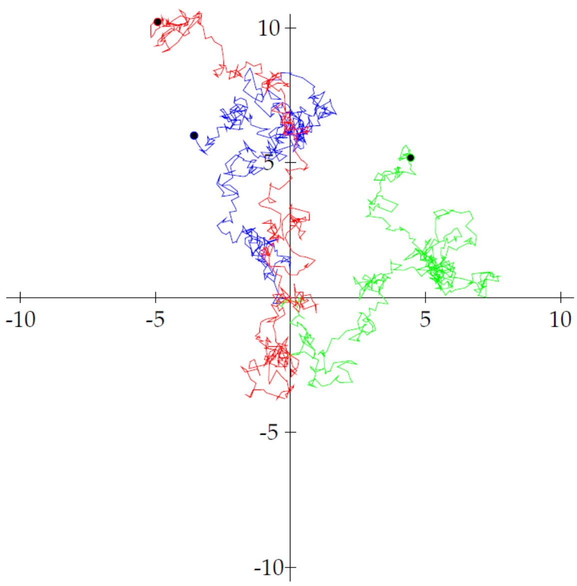
CHAPTER 5

Brownian Motion

Botanist R. Brown described the motion of a pollen particle suspended in fluid in 1828. It was observed that a particle moved in an irregular, random fashion. A. Einstein, in 1905, argued that the movement is due to bombardment of the particle by the molecules of the fluid, he obtained the equations for Brownian motion. In 1900, L. Bachelier used the Brownian motion as a model for movement of stock prices in his mathematical theory of speculation. The mathematical foundation for Brownian motion as a stochastic process was done by N. Wiener in 1931, and this process is also called the Wiener process.

Three 2d Brownian sample paths



We will be spending the rest of the course on Brownian motion.

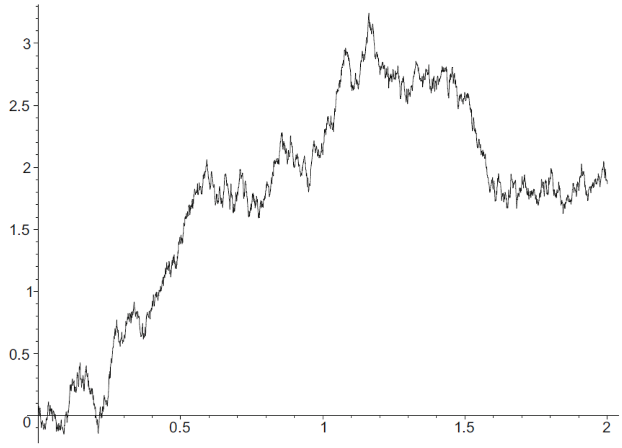


FIGURE 5.1.1. Computer simulation of a Brownian motion.

5.1. Definition of Brownian motion

First of all Brownian motion is a random process in continuous time and continuous space. We will start with dimension one: Brownian motion on the real line.

Here is the definition.

DEFINITION 1. A stochastic process $\{B_t\}$ is called a standard Brownian motion (shorted as SBM) if it satisfies the following:

(1) $B_0 = 0$.
(2) For $s < t$, the distribution of $B_t - B_s$ is normal with mean 0 and variance $t - s$.
(3) For any $n \geq 1$ and $t_0 < \cdots < t_n$,

$$B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}$$

are independent.
(4) The function $t \mapsto B_t$ is a continuous function of t.

THEOREM 2. *The SBM $\{B_t\}$ satisfies the following invariance properties:*

- $(-B_t)_{t\geq 0}$ *is a SBM („Mirror-invariance");*
- $\left(c \cdot B_{t/c^2}\right)_{t\geq 0}$ *ist a SBM all $c > 0$ ("Scaling-invariance");*

- $(B_{t+c} - B_c)_{t\geq 0}$ *is a SBM all* $c > 0$ *("Renewal property").*

EXAMPLE 3. Let $(B(t))_{t\geq 0}$ be a SBM. Compute $P(B(1) > 0 \mid B(1/2) = 1)$.

SOLUTION. We have
$P(B(1) > 0 \mid B(1/2) = 1)$

$= P(B(1) - B(1/2) > -1 \mid W(1/2) = 1)$

$= P(B(1) - B(1/2) > -1)$ [by independent increments]

$= P(B(1/2) > -1)$ [by stationary increments]

$= P\left(\frac{B(1/2)}{1/\sqrt{2}} > -\sqrt{2}\right)$

$= \Phi(\sqrt{2})$

$\approx 0.9213,$
where $\Phi(x) = P(N(0,1) \leq x)$ is the standard normal distribution function.

5.2. Understanding Brownian motion

Let $\{B_t\}$ be a standard Brownian motion. We let $\Delta > 0$ be chosen, then

$$B_{t+\Delta} - B_t \sim N(0, \Delta).$$

If Z denotes an $N(0,1)$-distributed random variable, then

$$\sqrt{\Delta}Z \sim N(0, \Delta).$$

THEOREM 4. *For each* $t \geq 0$*, with probability one the trajectories of Brownian motion are not differentiable at t.*

PROOF. Consider $\frac{B_{t+\Delta}-B_t}{\Delta} \overset{d}{=} \frac{\sqrt{\Delta}Z}{\Delta} = \frac{Z}{\sqrt{\Delta}}$, for some standard Normal random variable Z. Thus the ratio converges to ∞ in distribution, since $\mathbb{P}\left(\left|\frac{Z}{\sqrt{\Delta}}\right| > K\right) \to 1$ for any K, as $\Delta \to 0$, precluding existence of the derivative at t. $\square$

While nondifferentiable paths may seem surprising, a little thought about our assumptions will show why we could not expect to have differentiable paths. Suppose that B_t were differentiable at t. Then, we could determine the derivative by observing $B_s, 0 \leq s \leq t$. This would then tell us something about the increment $B_{t+\Delta t} - B_t$ for $\Delta t > 0$. However, our assumptions tell us that $B_{t+\Delta t} - B_t$ is independent of $B_s, 0 \leq s \leq t$.

5.2.1. Brownian motion as a continuous martingale. The definition of a martingale in continuous time is essentially the same as in discrete time. Suppose we have an increasing filtration $\{\mathcal{F}_t\}$ of information and integrable random variables M_t such that for each t, M_t is $\mathcal{F}_t$-measurable. (We say that M_t is *adapted* to the filtration if M_t is $\mathcal{F}_t$-measurable for each t.) Then, M_t is a *martingale* with respect to $\{\mathcal{F}_t\}$ if for each $s < t$,

$$\mathbb{E}\left[M_t \mid \mathcal{F}_s\right] = M_s.$$

As in the discrete case, if the filtration is not mentioned explicitly then one assumes that $\mathcal{F}_t$ is the information contained in $\{M_s : s \leq t\}$.

Let $\{B_t\}$ be a standard Brownian motion. Set $\mathcal{F}_t := \sigma\left(B_s : 0 \leq s \leq t\right)$.Then $\left(\mathcal{F}_t\right)_{t\geq 0}$ is the filtration generated by the Brownian motion.

Theorem 5. *The following are all $\left(\mathcal{F}_t\right)_{t\geq 0}$-martingales:*

- $\left(B_t\right)_{t\geq 0}$;
- $\left(B_t^2 - t\right)_{t\geq 0}$;
- $\left(\mathrm{e}^{aB_t - \frac{1}{2}a^2 t}\right)_{t\geq 0}$, *where $a \in \mathbb{R}$ is a constant.*

Proof. In the first case, we have:

a) $B_t \sim N(0, t) \Rightarrow E\left(B_t\right) = 0$.

b) Note $\mathcal{F}_t = \sigma\left(B_s, 0 \leq s \leq t\right)$. So B_t is $\mathcal{F}_t$-measurable.

c) If $s < t$, then

$$\mathbb{E}\left[B_t \mid \mathcal{F}_s\right] = \mathbb{E}\left[B_s \mid \mathcal{F}_s\right] + \mathbb{E}\left[B_t - B_s \mid \mathcal{F}_s\right] = B_s + \mathbb{E}\left[B_t - B_s\right] = B_s. \tag{5.1}$$

d) $W_t = -B_t$ is also continuous in t, since B_t is so. □

A martingale M_t is called a *continuous martingale* if with probability one the function $t \mapsto M_t$ is a continuous function. The word continuous in continuous martingale refers not just to the fact that time is continuous but also to the fact that the paths are continuous functions of t. One can have martingales in continuous time that are not continuous martingales. One example is to let N_t be a Poisson process with rate λ and

$$M_t = N_t - \lambda t.$$

Then using the fact that the increments are independent we see that for $s < t$,

$$\begin{aligned} \mathbb{E}\left[M_t \mid \mathcal{F}_s\right] &= \mathbb{E}\left[M_s \mid \mathcal{F}_s\right] + \mathbb{E}\left[N_t - N_s \mid \mathcal{F}_s\right] - \lambda(t-s) \\ &= M_s + \mathbb{E}\left[N_t - N_s\right] - \lambda(t-s) = M_s. \end{aligned}$$

5.2.2. Brownian motion as a Markov process. A continuous time process X_t is called Markov if for every t, the conditional distribution of $\{X_s : s \geq t\}$ given $\{X_r : r \leq t\}$ is the same as the conditional distribution given X_t. In other words, the future of the process is conditionally independent of the past given the present value.

Brownian motion is a Markov process. Indeed, if B_t is a standard Brownian motion, and

$$Y_s = B_{t+s}, \quad 0 \leq s < \infty,$$

then the conditional distribution of $\{Y_s\}$ given $\mathcal{F}_t$ is that of a Brownian motion with initial condition $Y_0 = B_t$. Indeed, if

$$\hat{B}_s = B_{t+s} - B_t,$$

then $\hat{B}_s$ is a Brownian motion that is independent of $\mathcal{F}_t$. There is a stronger notion of this called the *strong Markov property* that we will discuss in Section 5.3.

5.2.3. Brownian motion as a Gaussian process. Recall that a process is called Gaussian if all its finite-dimensional distributions are multivariate Normal.

EXAMPLE 6. Let random variables X and Y be independent Normal with distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$. Then the distribution of $(X, X+Y)^\top$ is bivariate Normal with mean vector $(\mu_1, \mu_1 + \mu_2)^\top$ and covariance matrix $\begin{bmatrix} \sigma_1^2 & \sigma_1^2 \\ \sigma_1^2 & \sigma_1^2 + \sigma_2^2 \end{bmatrix}$.

To see this let $Z = (Z_1, Z_2)^\top$ have standard Normal components, then it is easy to see that

$$(X, X+Y)^\top = \boldsymbol{\mu} + AZ,$$

where $\boldsymbol{\mu} = (\mu_1, \mu_1 + \mu_2)^\top$, and matrix $A = \begin{bmatrix} \sigma_1 & 0 \\ \sigma_1 & \sigma_2 \end{bmatrix}$. The result follows by the definition of the general Normal distribution as a linear transformation of standard Normals.

Similarly to the above example, the following representation

$$\begin{aligned} &(B(t_1), B(t_2), \ldots, B(t_n)) \\ &= (B(t_1), B(t_1) + (B(t_2) - B(t_1)), \ldots, B(t_{n-1})) + (B(t_n)) - B(t_{n-1})) \end{aligned}$$

shows that this vector is a linear transformation of the standard Normal vector, hence it has a multivariate Normal distribution.

Let $Y_1 = B(t_1)$, and for $k > 1, Y_k = B(t_k) - B(t_{k-1})$. Then by the property of independence of increments of Brownian motion, Y_k 's are independent. They also have Normal distribution, $Y_1 \sim N(0, t_1)$, and $Y_k \sim N(0, t_k - t_{k-1})$. Thus $(B(t_1), B(t_2), \ldots, B(t_n))$ is a linear transformation of $(Y_1, Y_2, \ldots, Y_n)$. But $Y_1 = \sqrt{t_1} Z_1$, and $Y_k = \sqrt{t_k - t_{k-1}} Z_k$, where Z_k 's are independent standard Normal. Thus $(B(t_1), B(t_2), \ldots, B(t_n))$ is a linear transformation of $(Z_1, \ldots, Z_n)$. Finding the matrix A of this transformation is left as an exercise.

To obtain the joint density function of $B(t_1), B(t_2), \cdots, B(t_n)$ for $t_1 < t_2 < \cdots < t_n$, note first that the set of equalities

$$B(t_1) = x_1, B(t_2) = x_2, \cdots, B(t_n) = x_n$$

is equivalent to

$$B(t_1) = x_1, B(t_2) - B(t_1) = x_2 - x_1, \cdots, B(t_n) - B(t_{n-1}) = x_n - x_{n-1}.$$

However, by the independent increment assumption it follows that $B(t_1), B(t_2) - B(t_1), \cdots, B(t_n) - B(t_{n-1})$, are independent and, by the stationary increment assumption, that $B(t_k) - B(t_{k-1}) \sim N(0, t_k - t_{k-1})$ is normal with mean 0 and variance $t_k - t_{k-1}$. Hence, the joint density of $B(t_1), B(t_2), \cdots, B(t_n)$ is

$$\begin{aligned} f(x_1, x_2, \cdots, x_n) &= f_{B(t_1)}(x_1) f_{B(t_2)-B(t_1)}(x_2 - x_1) \cdots f_{B(t_n)-B(t_{n-1})}(x_{n-1} - x_n) \\ &= \frac{1}{\sqrt{2\pi t_1}} e^{-\frac{x_1^2}{2t_1}} \frac{1}{\sqrt{2\pi(t_2 - t_1)}} e^{-\frac{(x_2-x_1)^2}{2(t_2-t_1)}} \cdots \frac{1}{\sqrt{2\pi(t_n - t_{n-1})}} e^{-\frac{(x_n - x_{n-1})^2}{2(t_n - t_{n-1})}} \\ &= \frac{1}{(2\pi)^{n/2}\sqrt{t_1(t_2 - t_1)\cdots(t_n - t_{n-1})}} e^{-\frac{1}{2}\left[\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{(t_2-t_1)} + \cdots + \frac{(x_n-x_{n-1})^2}{(t_n-t_{n-1})}\right]}. \end{aligned}$$

Recall the following definition:

- The covariance function of the process $X(t)$ is defined by

$$\begin{aligned} \gamma(s,t) = \mathrm{Cov}(X(t), X(s)) &= \mathrm{E}(X(t) - \mathrm{E}X(t))(X(s) - \mathrm{E}X(s)) \\ &= \mathrm{E}(X(t)X(s)) - \mathrm{E}X(t)\mathrm{E}X(s) \end{aligned}$$

The next result characterizes Brownian motion as a particular Gaussian process.

THEOREM 7. *A Brownian motion is a Gaussian process with zero mean function, and covariance function* $\min(t,s)$.

PROOF. Since the mean of the Brownian motion is zero,

$$\gamma(s,t) = \mathrm{Cov}(B(t), B(s)) = \mathrm{E}(B(t)B(s))$$

If $t < s$ then $B(s) = B(t) + B(s) - B(t)$, and

$$\mathrm{E}(B(t)B(s)) = \mathrm{E}B^2(t) + \mathrm{E}(B(t)(B(s) - B(t))) = \mathrm{E}B^2(t) = t.$$

where we used independence of increments property. Similarly if $t > s$, $\mathrm{E}(B(t)B(s)) = s$. Therefore

$$\mathrm{E}(B(t)B(s)) = \min(t, s)$$

$\square$

EXAMPLE 8. We find the distribution of $B(1) + B(2) + B(3) + B(4)$. Consider the random vector $\boldsymbol{X} = (B(1), B(2), B(3), B(4))^\top$. Since Brownian motion is a Gaussian process, all its finite-dimensional distributions are Normal, in particular $\boldsymbol{X}$ has a multivariate Normal distribution with mean vector zero and covariance matrix given $\Sigma = (\sigma_{ij})$ by $\sigma_{ij} = \mathrm{Cov}\,(X_i, X_j)$. For example, $\mathrm{Cov}\,(X_1, X_3) = \mathrm{Cov}((B(1), B(3)) = 1$. So

$$\Sigma = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Now, let $\mathbf{a} = (1, 1, 1, 1)$. Then

$$\boldsymbol{a}\boldsymbol{X} = X_1 + X_2 + X_3 + X_4 = B(1) + B(2) + B(3) + B(4),$$

$\boldsymbol{a}\boldsymbol{X}$ has a Normal distribution with mean zero and variance $\boldsymbol{a}\Sigma\boldsymbol{a}^T$, and in this case the variance is given by the sum of the elements of the covariance matrix. Thus $B(1)+B(2)+B(3) + B(4)$ has a Normal distribution with mean zero and variance 30. Alternatively, we can calculate the variance of the sum by the formula

$$\begin{aligned}&\mathrm{Var}\,(X_1 + X_2 + X_3 + X_4) \\ &\quad = \mathrm{Cov}\,(X_1 + X_2 + X_3 + X_4, X_1 + X_2 + X_3 + X_4) = \sum_{i,j} \mathrm{Cov}\,(X_i, X_j) = 30.\end{aligned}$$

5.3. Computations for Brownian motion

We will discuss some methods for computing probabilities for Brownian motions. For ease, we will assume that $\{B_t\}$ is a standard Brownian motion starting at the origin with respect to a filtration $\{\mathcal{F}_t\}$. If we are interested in probabilities about the Brownian motion at one time t, we need only use the normal distribution. Often, it is easier to scale to the standard normal. For example,

$$\begin{aligned}\mathbb{E}\left[|B_t|\right] = \mathbb{E}\left[t^{1/2}\left|B_1\right|\right] &= \frac{t^{1/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x|e^{-x^2/2}dx \\ &= \sqrt{\frac{2t}{\pi}}\int_0^{\infty}xe^{-x^2/2}dx \\ &= \sqrt{\frac{2t}{\pi}}\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}\left\{B_t \geq r\right\} = \mathbb{P}\left\{\sqrt{t}B_1 \geq r\right\} &= \mathbb{P}\left\{B_1 \geq r/\sqrt{t}\right\} \\ &= 1 - \Phi(r/\sqrt{t}) \\ &= \int_{r/\sqrt{t}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx,\end{aligned}$$

where Φ denotes the distribution function for a standard normal. If we are considering probabilities for a finite number of times, we can use the joint normal distribution. Often it is easier to use the Markov property as we now illustrate. We will compute

$$\mathbb{P}\left\{B_1 > 0, B_2 > 0\right\}.$$

The events $\{B_1 > 0\}$ and $\{B_2 > 0\}$ are not independent; we would expect them to be positively correlated. We compute by considering the possibilities at time 1,

$$\begin{aligned}\mathbb{P}\left\{B_1 > 0, B_2 > 0\right\} &= \int_0^{\infty}\mathbb{P}\left\{B_2 > 0 \mid B_1 = x\right\}d\mathbb{P}\left\{B_1 = x\right\} \\ &= \int_0^{\infty}\mathbb{P}\left\{B_2 - B_1 > -x\right\}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\ &= \int_0^{\infty}\int_{-x}^{\infty}\frac{1}{2\pi}e^{-\left(x^2+y^2\right)/2}dydx \\ &= \int_0^{\infty}\int_{-\pi/4}^{\pi/2}\frac{e^{-r^2/2}}{2\pi}d\theta r dr = \frac{3}{8}\end{aligned}$$

One needs to review polar coordinates to do the fourth equality. Note that

$$\mathbb{P}\{B_2 > 0 \mid B_1 > 0\} = \frac{\mathbb{P}\{B_1 > 0, B_2 > 0\}}{\mathbb{P}\{B_1 > 0\}} = \frac{3}{4},$$

which confirms our intuition that the events are positively correlated.

For more complicated calculations, we need to use the *strong Markov property*. We say that a random variable T taking values in $[0, \infty]$ is a stopping time (with respect to the filtration $\{\mathcal{F}_t\}$) if for each t, the event $\{T \leq t\}$ is $\mathcal{F}_t$-measurable. In other words, the decision to stop can use the information up to time t but cannot use information about the future values of the Brownian motion.

- If $x \in \mathbb{R}$ and

$$T = \min\{t : B_t = x\},$$

then T is a stopping time.

- Constants are stopping times.

- If S, T are stopping times then

$$S \wedge T = \min\{S, T\}$$

and

$$S \vee T = \max\{S, T\}$$

are stopping times.

THEOREM 9. *(Strong Markov Property). If T is a stopping time with $\mathbb{P}\{T < \infty\} = 1$ and*

$$Y_t = B_{T+t} - B_T,$$

then Y_t is a standard Brownian motion. Moreover, Y is independent of

$$\{B_t : 0 \leq t \leq T\}$$

Let us apply this theorem, to prove a very useful tool for computing probabilities.

PROPOSITION 10. *(Reflection Principle). If B_t is a standard Brownian motion with $B_0 = 0$, then for every $a > 0$,*

$$\mathbb{P}\left\{\max_{0\le s\le t} B_s \ge a\right\} = 2\mathbb{P}\{B_t > a\} = 2[1 - \Phi(a/\sqrt{t})].$$

To derive the reflection principle, let

$$T_a = \min\{s : B_s \ge a\} = \min\{s : B_s = a\}.$$

The second equality holds because B_s is a continuous function of s. Then

$$\mathbb{P}\left\{\max_{0\le s\le t} B_s \ge a\right\} = \mathbb{P}\{T_a \le t\} = \mathbb{P}\{T_a < t\}.$$

The second equality uses the fact that $\mathbb{P}\{T_a = t\} \le \mathbb{P}\{B_t = a\} = 0$. Since $B_{T_a} = a$,

$$\begin{aligned}\mathbb{P}\{B_t > a\} &= \mathbb{P}\{T_a < t, B_t > a\}\\ &= \mathbb{P}\{T_a < t\}\,\mathbb{P}\{B_t - B_{T_a} > 0 \mid T_a < t\}\end{aligned}$$

We now appeal to the Strong Markov Property to see that

$$\mathbb{P}\{B_t - B_{T_a} > 0 \mid T_a < t\} = 1/2.$$

This gives the first equality of the proposition and the second follows from

$$\mathbb{P}\{B_t > a\} = \mathbb{P}\left\{B_1 > a/\sqrt{t}\right\} = 1 - \Phi(a/\sqrt{t}).$$

EXAMPLE 11. Let $a > 0$ and let $T_a = \inf\{t : B_t = a\}$. The random variable T_a is called a passage time. We will find the density of T_a. To do this, we first find its distribution function

$$F(t) = \mathbb{P}\{T_a \le t\} = \mathbb{P}\left\{\max_{0\le s\le t} B_s \ge a\right\} = 2[1 - \Phi(a/\sqrt{t})]$$

The density is obtained by differentiating

$$f(t) = F'(t) = -2\Phi'\left(\frac{a}{\sqrt{t}}\right)\left(-\frac{a}{2t^{3/2}}\right) = \frac{a}{t^{3/2}\sqrt{2\pi}}e^{-\frac{a^2}{2t}}, \quad 0 < t < \infty.$$

5.4. Processes derived from Brownian motion

The purpose of this section is to get some feeling for the distributional and pathwise properties of Brownian motion.

Various Gaussian and non-Gaussian stochastic processes of practical relevance can be derived from Brownian motion. Below we introduce some of those processes which will find further applications in the course of this book. As before, $B = (B_t, t \in [0, \infty))$ denotes a standard Brownian motion.

Brownian motion with drift.

DEFINITION 12. A stochastic process B_t is called a Brownian motion with drift m and variance (parameter) σ^2 starting if it satisfies the following.

- $B_0 = 0$.
- For $s < t$, the distribution of $B_t - B_s$ is normal with mean $m(t-s)$ and variance $\sigma^2(t-s)$.
- If $s < t$, the random variable $B_t - B_s$ is independent of the values B_r for $r \leq s$.
- The function $t \mapsto B_t$ is a continuous function of t.

DEFINITION 13. Note that If $m = 0, \sigma^2 = 1$, then B_t is simply a standard Brownian motion. Recall that if Z has a $N(0,1)$ distribution and $Y = \sigma Z + m$, then Y has a $N\left(m, \sigma^2\right)$ distribution. Given that it is easy to show the following.

- If B_t is a standard Brownian motion and

$$Y_t = \sigma B_t + mt$$

  then Y_t is a Brownian motion with drift m and variance σ^2.

Brownian bridge. Consider the process

$$X_t = B_t - tB_1, \quad 0 \leq t \leq 1$$

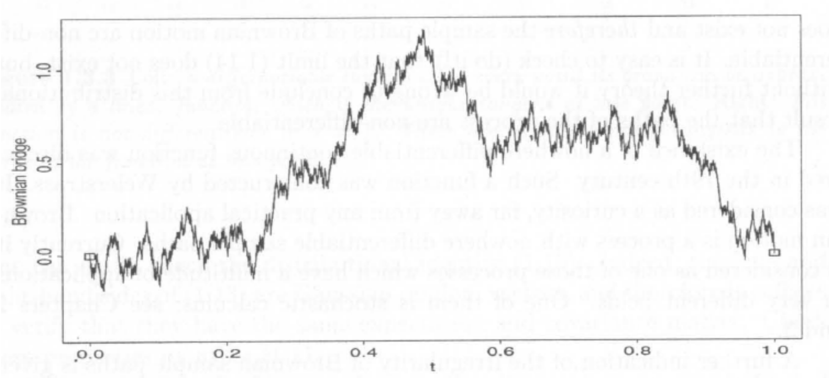


FIGURE 5.4.1. A sample path of the Brownian bridge.

Obviously,

$$X_0 = B_0 - 0B_1 = 0 \quad \text{and} \quad X_1 = B_1 - 1B_1 = 0$$

For this simple reason, the process X bears the name (standard) Brownian bridge or tied down Brownian motion.

A glance at the sample paths of this "bridge" (see above Figure) may or may not convince you that this name is justified.

Geometric Brownian Motion. With the fundamental discovery of Bachelier in 1900 that prices of risky assets (stock indices, exchange rates, share prices, etc.) can be well described by Brownian motion, a new area of applications of stochastic processes was born. However, Brownian motion, as a Gaussian process, may assume negative values, which is not a very desirable property of a price. In their celebrated papers from 1973, Black, Scholes and Merton suggested another stochastic process as a model for speculative prices.

DEFINITION 14. (Geometric Brownian motion) The process suggested by Black, Scholes and Merton is given by

$$X_t = \mathrm{e}^{\mu t + \sigma B_t}, \quad t \geq 0,$$

i.e. it is the exponential of Brownian motion with drift.

Clearly, X is not a Gaussian process (why?). For the purpose of later use, we calculate the expectation of geometric Brownian motion. For readers, familiar with probability theory, you may recall that for an $N(0,1)$ random variable Z,

$$E\mathrm{e}^{\lambda Z} = \mathrm{e}^{\lambda^2/2}, \quad \lambda \in \mathbb{R}.$$

It is easily derived as shown below:

$$\begin{aligned} E\mathrm{e}^{\lambda Z} &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \mathrm{e}^{\lambda z} \mathrm{e}^{-z^2/2} dz \\ &= \mathrm{e}^{\lambda^2/2} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \mathrm{e}^{-(z-\lambda)^2/2} dz \\ &= \mathrm{e}^{\lambda^2/2}. \end{aligned}$$

Here we used the fact that $(2\pi)^{-1/2} \exp\left\{-(z-\lambda)^2/2\right\}$ is the density of an $N(\lambda,1)$ random variable. It follows immediately that

$$E[X_t] = \mathrm{e}^{\mu t} E\mathrm{e}^{\sigma B_t} = \mathrm{e}^{\mu t} E\mathrm{e}^{\sigma t^{1/2} B_1} = \mathrm{e}^{\left(\mu + 0.5\sigma^2\right)t}.$$

Geometric Brownian motion is useful in the modeling of stock prices over time if

(i) the percentage changes are independent and identically distributed.

(ii) the probabilities concerning the ratio of the price a time t in the future to the present price will not depend on the present price. (Thus, for instance, the model implies that the probability of a given security doubles in price in the next month is the same no matter whether the present price is 10 or 25).

Suppose the stock price at t is $S(t) = se^{\mu t + \sigma B(t)}$, where $B(t)$ is standard Brownian motion and $s > 0$ is a constant. So $S(t) = s$ is the initial price of the stock. Let us calculate the covariance of $S\left(t_1\right)$ and $S\left(t_2\right)$. Assume $t_2 > t_1$.

$$\operatorname{Cov}\left(S\left(t_1\right), S\left(t_2\right)\right) = E\left[S\left(t_1\right) S\left(t_2\right)\right] - E\left[S\left(t_1\right)\right] E\left[S\left(t_2\right)\right].$$

$S\left(t_1\right)$ and $S\left(t_2\right)$ cover the overlapping time interval $[0, t_1]$ and $[0, t_2]$, and are not independent. Let $Q_1 = S\left(t_1\right)/S(0) = e^{\mu t_1 + \sigma B(t_1)}$, and $Q_2 = S\left(t_2\right)/S\left(t_1\right) = e^{\mu(t_2 - t_1) + \sigma B(t_2) - \sigma B(t_1)}$.

Then Q_1 and Q_2 cover non-overlapping time intervals and are independent.

$$\begin{aligned} E\left[S\left(t_1\right)S\left(t_2\right)\right] &= s^2 E\left[Q_1^2 Q_2\right] \\ &= s^2 E\left[Q_1^2\right] E\left[Q_2\right] \\ &= s^2 e^{2\mu t_1 + 2\sigma^2 t_1} e^{\mu\left(t_2 - t_1\right) + 0.5\sigma^2\left(t_2 - t_1\right)} \\ &= s^2 e^{\mu\left(t_2 + t_1\right) + \sigma^2\left(1.5 t_1 + 0.5 t_2\right)} \end{aligned}$$

Meanwhile, $E\left[S\left(t_1\right)\right] = s e^{\mu t_1 + 0.5\sigma^2 t_1}$ and $E\left[S\left(t_2\right)\right] = s e^{\mu t_2 + 0.5\sigma^2 t_2}$. So the covariance and correlation of $S\left(t_1\right)$ and $S\left(t_2\right)$ is

$$\begin{aligned} \operatorname{Cov}\left(S\left(t_1\right), S\left(t_2\right)\right) &= s^2\left(e^{\mu\left(t_1+t_2\right)+\sigma^2\left(1.5 t_1+0.5 t_2\right)} - e^{\mu\left(t_1+t_2\right)+0.5\sigma^2\left(t_1+t_2\right)}\right) \\ &= s^2 e^{\left(\mu+0.5\sigma^2\right)\left(t_1+t_2\right)}\left(e^{\sigma^2 t_1} - 1\right), \end{aligned}$$

$$\begin{aligned} \rho_{S\left(t_1\right), S\left(t_2\right)} &= \frac{\operatorname{Cov}\left(S\left(t_1\right), S\left(t_2\right)\right)}{\sigma_{S\left(t_1\right)}\sigma_{S\left(t_2\right)}} \\ &= \frac{s^2 e^{\left(\mu+0.5\sigma^2\right)\left(t_1+t_2\right)}\left(e^{\sigma^2 t_1} - 1\right)}{\sqrt{s^2 e^{2\left(\mu+0.5\sigma^2\right)t_1}\left(e^{\sigma^2 t_1}-1\right) s^2 e^{2\left(\mu+0.5\sigma^2\right)t_2}\left(e^{\sigma^2 t_2}-1\right)}} \\ &= \sqrt{\frac{e^{\sigma^2 t_1} - 1}{e^{\sigma^2 t_2} - 1}}. \end{aligned}$$