2022-23 First Semester
MATH1063 Linear Algebra II (1003)

Assignment 5

Q1-Q5 Due Date: **31/Mar/2023 (Friday), 09:00 in tutorial class.**
Q6-Q10 Due Date: **4/Apr/2023 (Tuesday), 18:00 in class.**

- Write down your **CHN name** and **student number**. Write neatly on **A4-sized** paper (*staple if necessary*) and **show your steps**.
- **Late submissions or answers without steps won't be graded.**

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1. (Guided proof) Let A be an $m \times n$ matrix. Show that
   (a) if $x \in \mathrm{N}(A)$, then $\mathbf{x}$ must be in $\mathrm{N}(A^T A)$.
   (b) if $x \in \mathrm{N}(A^T A)$, then $A\mathbf{x}$ is in both $\mathrm{Col}(A)$ and $\mathrm{N}(A^T)$.
   (c) If A is of rank n, then $A^T A$ is nonsingular.

2. (Guided proof) Let A be an $m \times n$ matrix. Show that
   (a) if $x \in \mathrm{Col}(A^T A)$, then $\mathbf{x}$ must be in $\mathrm{Col}(A^T)$.
   (b) if $x \in \mathrm{Col}(A^T)$, then $\mathbf{x}$ must be in $\mathrm{Col}(A^T A)$.
   (c) $\mathrm{Col}(A^T A) = \mathrm{Col}(A^T)$.

3. Using least-square method,
   (a) fit a linear function of the form $f(t) = c_0 + c_1 t$ to the data points $(0,0)$, $(0,1)$, $(1,1)$.
   (b) fit a quadratic polynomial to the data points $(0,0)$, $(2,2)$, $(3,6)$, $(4,12)$.
   (c) (Software needed) find the trigonometric function of the form $f(t) = c_0 + c_1 \sin(t) + c_2 \cos(t)$ that best fits the data points $(0,0)$, $(1,1)$, $(2,2)$, $(3,3)$.
   (d) (Software needed) find the equation of the circle that gives the best least squares circle fit to the points $(-1,-2)$, $(0,2.4)$, $(1.1,-4)$, and $(2.4,-1.6)$.
   [Hint: The general equation for a circle is $2xc_1 + 2yc_2 + (r^2 - c_1^2 - c_2^2) = x^2 + y^2$*]*

4. Let S be a subspace of $\mathbb{R}^n$ and $\mathbf{v}$ a vector in $\mathbb{R}^n$. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors with $\mathbf{x} \in S$ and that $\mathbf{v} = \mathbf{x} + \mathbf{y}$. Is it necessarily true that $\mathbf{y}$ is in $S^{\perp}$? Either prove that it is true or find a counter-example.

5. Consider the inner product space $C[a,b]$ with

$$< f, g >= \int_a^b f(x)g(x)\,\mathrm{d}x, \qquad f, g \in C[a,b].$$

   Find the orthogonal projection of f onto g.

(a) $C[-1,1]$, $f(x) = x$ and $g(x) = 1$.

(b) $C[-\pi,\pi]$, $f(x) = x$ and $g(x) = \sin(2x)$.

6. Consider the inner product space $C[0,1]$ with

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\mathrm{d}x, \quad \text{and} \quad \|f\| = \sqrt{\langle f, f\rangle},$$

for any $f, g \in C[0,1]$. Let $S = \text{span}\{1, 2x-1\}$ be a subspace of $C[0,1]$.

(a) Show that the vectors 1 and $2x-1$ are orthogonal.

(b) Compute $\|1\|$ and $\|2x-1\|$.

(c) Find the least squares approximation to $h(x) = \sqrt{x}$ in the subspace S.

7. Let $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T\mathbf{y}$ be the inner product on $\mathbb{R}^n$. Apply the Gram-Schmidt process to find an orthonormal basis for the following subspaces spanned by $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$.

(a) $\mathbf{x}_1 = (1,2)^T, \mathbf{x}_2 = (0,1)^T, \mathbf{x}_3 = (1,-1)^T$ from $\mathbb{R}^2$.

(b) $\mathbf{x}_1 = (1,0,0)^T, \mathbf{x}_2 = (1,1,1)^T, \mathbf{x}_3 = (1,1,-1)^T$ from $\mathbb{R}^3$.

(c) $\mathbf{x}_1 = (4,2,2,1)^T, \mathbf{x}_2 = (2,0,0,2)^T, \mathbf{x}_3 = (1,1,-1,1)^T$ from $\mathbb{R}^4$.

8. Let

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(a) Find an orthonormal basis for $\mathrm{N}(A)$.

(b) Determine the projection matrix Q that projects vectors in $\mathbb{R}^4$ onto $\mathrm{N}(A)$.

9. Let Q be an orthogonal matrix and let $d = \det(Q)$. Show that $|d| = 1$.

10. True or False? If true, explain or prove your answer. If false, state your reasons or give a counter-example to show that the statement is not always true.

(a) If A is an $m \times n$ matrix, then AA^T and A^TA have the same rank.

(b) It is possible to find a nonzero vector $\mathbf{y}$ in the column space of A^T such that $A\mathbf{y} = \mathbf{0}$.

(c) If Q is an orthogonal matrix, then Q^T is also an orthogonal matrix.

(d) If Q is an orthogonal matrix, then $3Q$ is also an orthogonal matrix.