FINM3133 Time Series for Finance and Macroeconomics

Chapter 2 Solution

1. (a) $E(Y_t) = 5 + 2t$

   (b) Let α_k be the autocovariance function of Yt.

$$\begin{aligned}\alpha_k &= Cov(Y_{t-k}, Y_t) \\ &= Cov(5 + 2t - 2k + X_{t-k}, 5 + 2t + X_t) \\ &= Cov(X_{t-k}, X_t) \\ &= \gamma_k\end{aligned}$$

   (c) Since its mean function is not constant, Y_t is not stationary.

2. (a) $Cov(Y_tt - k, Y_t) = Cov(X_{t-k} + a, X_t + b) = Cov(X_{t-k}, X_t)$, for any constant a, b. Since X_t is stationary, $Cov(X_{t-k}, X_t)$ is free of t for all lags.

   (b) Y_t is not stationary, because the mean function of Y_t is not constant. For example,

$$E(Y_{t-1}) \neq E(Y_t)$$

   for any t.

3. (a) Since Y_t is stationary,

$$E(W_t) = E(Y_t) - E(Y_{t-1}) = 0$$

   which is constant

$$\begin{aligned}Cov(W_{t-k}, W_t) &= Cov(Y_{t-k} - Y_{t-k-1}, Y_t - Y_{t-1}) \\ &= 2\gamma_k - \gamma_{k+1} - \gamma_{k-1}\end{aligned}$$

   which is free of t. Therefore W_t is also stationary.

   (b)

$$U_t = \Delta^2 Y_t = \Delta[Y_t - Y_{t-1}]$$

   $Y_t - Y_{t-1}$, we know, is stationay. Then, U_t is also stationary, according to the conclusion of (a).

4. (a) Normally, we assume the white noise has zero-mean. In this case, $e_t \sim N(0, \sigma)$. Its moment generating function is $M(t) = E(e^{te_t}) = e^{\frac{1}{2}\sigma^2 t^2}$. Its third moment is obviously 0, because the distribution of e^t is even function, e_t^3 is odd. Its fourth moment is $E(e_t^4) = \frac{d^4 M(0)}{dt^4} = 3\sigma^4$

$$Y_t = e_t - \theta(e_{t-1})^2$$

It is obvious that the correlation is 0, if their lag is larger than 2.Because e^t is iid.

The variance of Y_t

$$
\begin{aligned}
Var(Y_t) &= Var(e_t - \theta(e_{t-1})^2) \\
&= Var(e_t) + \theta^2 Var(e_{t-1}^2) \\
&= \sigma^2 + \theta^2[E(e_{t-1}^4) - E(e_{t-1}^2)^2] \\
&= \sigma^2 + \theta^2(3\sigma^4 - \sigma^4) \\
&= \sigma^2 + 2\theta^2\sigma^4
\end{aligned}
$$

The covariance at lag 1:

$$
\begin{aligned}
Cov(Y_t, Y_{t-1}) &= Cov(e_t - \theta e_{t-1}^2, e_{t-1} - \theta e_{t-2}^2) \\
&= Cov(-\theta e_{t-1}^2, e_{t-1}) \\
&= -\theta E(e_{t-1}^3) \\
&= 0
\end{aligned}
$$

Therefore, its autocorrelation function for Y_t

$$
\rho_k = \begin{cases} 1 & k = 0 \\ 0 & k > 0 \end{cases}
$$

(b) Its mean is a constant. $E(Y_t) = E(e_t - \theta(e_{t-1})^2) = -\theta\sigma_e^2$ Its variance is finite and its autocorrelation doesn't vary from t.

5.

$$
\begin{aligned}
Var(\bar{Y}) &= Var(\frac{1}{n}\sum_{1}^{n} Y_t) \\
&= \frac{1}{n^2} Cov[\sum_{t=1,n} Y_t, \sum_{s=1}^{n} Y_s] = \frac{1}{n^2}\sum_{t=1}^{n}\sum_{s=1}^{n}\gamma_{t-s}
\end{aligned}
$$

Since the series is stationary, now make the change of variable $t - s = k$ and t = j. The range of the summation {1 ≤ t ≤ n, 1 ≤ s ≤ n} is transformed into {1 ≤ j ≤ n, 1 ≤ j − k ≤ n} = {k + 1 ≤ j ≤ n + k, 1 ≤ j ≤ n} which may be written {k > 0, k + 1 ≤ j ≤ n} ∪ {k ≤ 0, 1 ≤ j ≤ n + k}. Thus

$$
\begin{aligned}
Var(\bar{Y}) &= \frac{1}{n^2}[\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}\gamma_k + \sum_{k=-n+1}^{0}\sum_{j=1}^{n+k}\gamma_k] \\
&= \frac{1}{n^2}[\sum_{k=1}^{n-1}(n-k)\gamma_k + \sum_{k=-n+1}^{0}(n+k)\gamma_k] \\
&= \frac{1}{n}\sum_{-n+1}^{n-1}(1 - \frac{|k|}{n})\gamma_k
\end{aligned}
$$

Notice in the second expression, $\gamma_k = \gamma_{-k}$

6. (a) Substitute $Y_{t-1} = \theta_0 + Y_{t-2} + e_{t-1}$ into $Y_t = \theta_0 + Y_{t-1} + e_t$ and repeat until you get e_1.

   (b) $E(Y_t) = E(t\theta_0 + e_t + e_{t-1} + ... + e_1) = t\theta_0$

   (c) The autocovariance function for Y_t

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov(t\theta_0 + e_t + e_{t-1} + ... + e_1, (t-k)\theta_0 + e_{t-k} + e_{t-k-1} + ... + e_1) \\ &= Cov(e_t + e_{t-1} + ... + e_1, e_{t-k} + e_{t-k-1} + ... + e_1) \\ &= Var(e_{t-k} + e_{t-k-1} + ... + e_1) \\ &= (t-k)\sigma_e^2 \end{aligned}$$

7. (a) $\mu_1 = E(Y_1) = E(e_1) = 0$ Then $E(Y_t) = E(Y_{t-1} + e_t) = E(Y_{t-1}) + E(e_t) = E(Y_{t-1})$ or $\mu_t = \mu_{t-1}$ for $t > 1$ and the result follows by induction.

   (b) $Var(Y_1) = \sigma_e^2$ is immediate. Then $Var(Y_t) = Var(Y_{t-1} + e_t) = Var(Y_{t-1}) + Var(e_t) = Var(Y_{t-1}) + \sigma_e^2$. Recursion or induction on t yields $Var(Y_t) = t\sigma_e^2$.

   (c) $Cov(Y_t, Y_s) = Cov(Y_t, Y_t + e_{t-1} + e_{t-2} + ... + e_s) = Cov(Y_t, Y_t) = Var(Y_t) = t\sigma_e^2$ and hence the result.