Caculus II Math 1038 (1002&1003)

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Week 12: Ch15 Multiple integrals 2

1. Triple integrals
   (a) rectangular boxes:
   $$B = \left\{(x, y, z)\middle| a \leq \boldsymbol{x} \leq b, c \leq \boldsymbol{y} \leq d, r \leq \boldsymbol{z} \leq s\right\}$$
   sub-box:
   $$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$
   volume element: $\Delta V = \Delta x \Delta y \Delta z$
   (b) triple Riemann sum and triple integral
   $$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right)\Delta V = \iiint_B f(x, y, z)dV$$
   (c) general bounded region E: Type I
   $$E = \left\{(x, y, z)\middle| (x, y) \in D, u_1(x, y) \leq \boldsymbol{z} \leq u_2(x, y)\right\}$$
   Type II
   $$E = \left\{(x, y, z)\middle| (y, z) \in D, u_1(y, z) \leq \boldsymbol{x} \leq u_2(y, z)\right\}$$
   Type III
   $$E = \left\{(x, y, z)\middle| (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\right\}$$
   (d) Change of the order of integration: Fubini's Thoerem
   (e) applications:
      i. volume of a solid:
      $$\iiint_E 1dV = V(E)$$
      ii. "hypervolume"
   (f) Fubini's Theorem: change of order of integration over a **rectangular region**. If f is continuous on the rectangular box, then
   $$\iiint_B f(x, y, z)dV = \int_a^b\int_c^d\int_r^s f(x, y,)dydxdz = \int_a^b\int_r^s\int_c^d f(x, y,)dydzdx = \cdots$$
   This works only for rectangular box!!! For other general region, we need to make changes for the boundaries.

2. triple integrals over general regions
   (a) A region $D \subset \mathbb{R}^3$
   (b) D is bounded above by a surface $z = H(x, y)$ and below by a surface $z = G(x, y)$, and region $R \subset \mathbb{R}^2$ is Type I region
   $$D = \left\{(x, y, z)\middle|(x, y) \in R, H(x, y) \leq z \leq G(x, y)\right\}$$
   $$\iiint_D f(x, y, z)dV = \iint_R \left[\int_{G(x,y)}^{H(x,y)} f(x, y, z)dz\right]dA = \int_a^b\int_{g(x)}^{h(x)}\left[\int_{G(x,y)}^{H(x,y)} f(x, y, z)dz\right]dydx$$
      i. step 1: integrate with respect to z from $z = G(x, y)$ to $z = H(x, y)$, (z is disappeared)

ii. step 2: integrate with resepct to y from $y = g(x)$ to $y = h(x)$ (y is disappeared)

iii. step 3: integrate with respect to x from $x = a$ to $x = b$

3. cylindrical coordinates (r, θ, z):
   (a) polar coordinate $(r, \theta)+$ height z
   (b) Equations in cylindrical coordinate:
      i. cylinder: $\{(r, \theta, z) : r = a, a > 0\}$
      ii. vertical half plane $\{(r, \theta, z) : \theta = \theta_0\}$
      iii. horizontal plane $\{(r, \theta, z) : z = a\}$
      iv. cone: $\{(r, \theta, z) : z = ar, a \neq 0\}$
   (c) volume of the wedge: $\Delta V = r\Delta r \cdot \Delta\theta \cdot \Delta z$ where $r\Delta r \cdot \Delta\theta$ is the area of the base polar rectangle and Δz is the height.
   (d) triple integral over the region

$$D = \big\{(r, \theta, z)\big|g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta, H(x, y) \leq z \leq G(x, y)\big\}$$

$$\iiint_D f(x, y, z)dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \left[\int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)dz\right] dr d\theta$$

4. Spherical coordinates $P = (\rho, \phi, \theta)$
   (a) ρ: distance from the origin to P
   (b) ϕ: angle between positive z-axis and line OP
   (c) θ: angle between the projection of OP and x-axis
   (d) $\mathbb{R}^3 = \{(\rho, \phi, \theta) \,| 0 \leq \rho < \infty, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$
   (e) Transformation:

$$\begin{aligned} \rho &= x^2 + y^2 + z^2 \\ \tan\theta &= \frac{y}{x} \\ \sin\phi &= \frac{z}{\rho} \qquad \text{or} \qquad \tan\phi = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \end{aligned}$$

   spherical to cartesian coordinates:

$$\begin{aligned} x &= r\cos\theta = \rho\sin\phi\cos\theta \\ y &= r\sin\theta = \rho\sin\phi\sin\theta \\ z &= r\cos\phi \end{aligned}$$

   (f) Equations:
      i. sphere with radius a center at origin: $\{(\rho, \phi, \theta) \,| \rho = a\}$
      ii. sphere with radius a center at $(0, 0, a)$: $\{(\rho, \phi, \theta) \,| 2a\cos\phi = \rho\}$
      iii. cone, rotate about z-axis: $\{(\rho, \phi, \theta) \,| \phi = \phi_0\}$
      iv. vertical half plane: $\{(\rho, \phi, \theta) \,| \theta = \theta_0\}$
      v. horizontal plane $z = a$: $\left\{(\rho, \phi, \theta) \,| \rho\cos\phi = a, 0 \leq \phi \leq \frac{\pi}{2}\right\}$
      vi. cylinder $\{(\rho, \phi, \theta) \,| \rho\sin\phi = a, 0 \leq \phi \leq \pi\}$
   (g) volume of "spherical box":

$$\Delta V = \rho^2 \sin\phi \Delta\rho\Delta\theta\Delta\phi$$

5. Change of variables

(a) Transformation from uv-plane to xy-plane

$$T(u,v) = (x,y)$$

where $x = g(u,v)$ or $x = x(u,v)$ and $y = h(u,v)$ or $y(u,v)$. If g and h have continuous first order partial derivatives, then T is a C^1 transformation.

(b) domain S and range R: subset of $\mathbb{R}^2$

(c) T is one-to-one: $T(P) = T(Q) \Rightarrow P = Q$

(d) R: image of S

(e) inverse transformation T^{-1} from xy-plane to uv-plane

$$T^{-1}(x,y) = (u,v)$$

(f) vector equation in xy-plane

$$\vec{r}(u,v) = g(u,v)\vec{i} + h(u,v)\vec{j}$$

tangent vector at (x_0, y_0)

$$\vec{r}_u(u,v) = g_u(u_0,v_0)\vec{i} + h_u(u_0,v_0)\vec{j} = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j}$$

$$\vec{r}_v(u,v) = g_v(u_0,v_0)\vec{i} + h_v(u_0,v_0)\vec{j} = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j}$$

(g) approximation of the area of R by a parallelogram which is a tangent plane formed by $\Delta u \cdot r_u$ and $\Delta v \cdot r_v$

$$|(\Delta u \cdot r_u) \times (\Delta v \cdot r_v)| = |r_u \times r_v|\, \Delta u \Delta v = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \Delta v$$

(h) Jacobian determinant of a transformation of two variables

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

(i) areas in S and R

$$\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v$$

(j) double integral of f over R and double integral of f over S

$$\iint_R f(x,y)\,dA \approx \sum\sum f(x_i,y_j)\Delta A_{ij} = \sum\sum f(g(u_i,v_j),h(u_i,v_j))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u_i \Delta v_j = \iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

6. **Change of variables**

(a) Problem-solving strategy

i. sketch the region R in the xy-plane and write the equations of the curves of the boundaries

ii. choose the transformation T depending on the region R and integrand $f(x,y)$. e.g. parallel lines/curves.

iii. determine the **new limit** of the integration in uv-plane

iv. find the Jacobian $J(u,v)$

v. replace the variables in the integrand $f(x,y) \to f(x(u,v), y(u,v))$

vi. replace $dydx$ or $dxdy$ by $|J(u,v)|\, dudv$

(b) **Example 1** : transformation T from the $r\theta$-plane (polar rectangle) to the xy-plane (rectangle) given by

$$x = g(r,\theta) = r\cos\theta, \qquad y = h(r,\theta) = r\sin\theta$$

the Jacobian of T

$$J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

(c) **Example 2** : use transformation to evaluate double integral with regions bounded by "parallel curves" (Dr. Wong's notes example 4.8.6)

(d) **Example 3:** evaluate double integral

$$\iint_R (x-y)\, dydx$$

where R is the parallelogram joining the points $(1,2)$, $(3,4)$, $(4,3)$ and $(6,5)$ make appropriate changes of variables and write the resulting integral.
Step 1, understand region R, the sides of the parallelogram are

$$y = x+1 \qquad y = x-1$$

$$y = \frac{1}{3}x + \frac{5}{3} \qquad y = \frac{1}{3}x + 3$$

Step 2: which can be written as

$$x - y = -1 \qquad x - y = 1$$

$$x - 3y = -5 \qquad x - 3y = -9$$

Remark: the region R can be discribed as: the **region bounded by the lines**:

$$y = x+1 \quad y = x-1, \quad y = \frac{1}{3}x + \frac{5}{3} \quad y = \frac{1}{3}x + 3$$

change of variables Let $u = x - y$ and $v = x - 3y$, and we have

$$x = \frac{3u - v}{2} \qquad y = \frac{u - v}{2}$$

so the transformation

$$T(u,v) = \left(\frac{3u-v}{2}, \frac{u-v}{2}\right)$$

Step 3:the new **limits on the integral** would be

$$-1 \leq u \leq 1 \qquad -9 \leq v \leq -5$$

Step 4: compute the determinant of Jacobian $J(u,v)$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Step 5: change integrand $f(x,y) = x - y = \frac{3u-v}{2} - \frac{u-v}{2} = u$
Step 6: by the transformation, the integral changes to

$$\begin{aligned} \iint_R (x-y)\, dydx &= \int_{-9}^{-5} \int_{-1}^{1} u \cdot |J(u,v)|\, dudv \\ &= \int_{-9}^{-5} \int_{-1}^{1} \frac{u}{2} dudv \end{aligned}$$

(e) **Example 4** : use transformation to evaluate double integral (textbook example)

$$\iint_R e^{\frac{x+y}{x-y}}\, dA$$

where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$.
Step1: [Define inverse transformation T^{-1} for u and v] Let $u = x + y$ and $v = x - y$,
Step 2: transformation T from uv-plane to xy-plane :

$$x = \frac{1}{2}(u+v), \qquad y = \frac{1}{2}(u-v)$$

Step 3: compute the Jacobian of T

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Step 4: [find region of S in uv-plane] From the four vertices $A = (1,0)$, $B = (2,0)$, $C = (0,-2)$, and $D = (0,-1)$ form the region R on the lines:

$$y = 0, \qquad x - y = 2, \qquad x = 0, \qquad x - y = 1$$

the image in the uv-plane are

$$u = v, \qquad v = 2, \qquad u = -v, \qquad v = 1$$

Thus the region

$$S = \{(u,v)\,|1 \le v \le 2, -v \le u \le v\}$$

Step 5: compute the integral on S

$$\begin{aligned}
\iint_R f(x,y)dA &= \iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \\
&= \int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} \cdot \frac{1}{2} \cdot dudv \\
&= \int_1^2 \frac{v}{2} \int_{-v}^{v} e^{\frac{u}{v}} \cdot d\left(\frac{u}{v}\right) dv \\
&= \int_1^2 \frac{v}{2} \left[e^{\frac{u}{v}} \Big|_{-v}^{v} \right] dv \\
&= \int_1^2 \frac{v}{2} \left[e - e^{-1} \right] dv \\
&= \frac{1}{4}\left(e - e^{-1}\right) v^2 \Big|_1^2 = \frac{3}{4}\left(e - e^{-1}\right)
\end{aligned}$$