2023-24 First Semester

MATH2023 Ordinary and Partial Differential Equations (1002)

Assignment 4 Suggested Solutions

1. (a) $r^2 - 2r + 2 = 0 \quad \rightarrow \quad r = 1 \pm i$

   $y = C_1 e^t \cos(t) + C_2 e^t \sin(t), \qquad C_1, C_2 \in \mathbb{R}.$

   (b) $4r^2 + 9 = 0 \quad \rightarrow \quad r = \pm\frac{3}{2}i$

   $y = C_1 \cos\frac{3}{2}t + C_2 \sin\frac{3}{2}t \qquad C_1, C_2 \in \mathbb{R}.$

   (c) $r^2 + 5r + 6.25 = 0 \quad \rightarrow \quad r_1 = r_2 = -2.5$

   $y = (C_1 + C_2 t)e^{-2.5t}, \qquad C_1, C_2 \in \mathbb{R}.$

2. (a) The characteristic equation:

$$2r^2 + 3r - 2 = (r+2)(2r-1) = 0 \quad \rightarrow \quad r_1 = -2, r_2 = 1/2 \quad \rightarrow \quad y_1 = e^{-2t}, y_2 = e^{\frac{t}{2}}.$$

The general solution to the D.E. is

$$y(t) = c_1 e^{-2t} + c_2 e^{t/2}, \quad t > 0, \; c_1, c_2 \in \mathbb{R}.$$

The I.C. yield:

$$\begin{cases} 1 = y(0) = c_1 + c_2 \\ -\beta = y'(0) = -2c_1 + c_2/2 \end{cases} \quad \rightarrow \quad \begin{cases} c_1 = (1+2\beta)/5 \\ c_2 = (4-2\beta)/5 \end{cases}.$$

   (b) To detect the local mininum of $y(t)$, we try to analyze on $y'(t)$ and the behavior of $y(t)$. With c_1, c_2 given in part (a),

$$y'(t) = -2c_1 e^{-2t} + 0.5c_2 e^{t/2}.$$

Since $\beta > 0$,

∘ For $\beta = 2$, $c_1 > 0$, $c_2 = 0$, $y'(t) < 0$, and

$$y \rightarrow 0 \;\text{ as }\; t \rightarrow \infty, \qquad \text{there is no minimum;}$$

∘ For $\beta > 2$, $c_1 > 0$, $c_2 < 0$, $y'(t) < 0$, and

$$y \rightarrow -\infty \;\text{ as }\; t \rightarrow \infty, \qquad \text{there is no minimum;}$$

∘ For $0 < \beta < 2$, $c_1 > 0$, $c_2 > 0$, there is a local min at $t_0 = \frac{2}{5}\ln\frac{4c_1}{c_2}$.

$y'(t) = 0, \; -2c_1 e^{-2t} + 0.5c_2 e^{\frac{t}{2}} = 0 \Rightarrow e^{-\frac{5}{2}t} = \frac{1}{4}\frac{c_2}{c_1} \Rightarrow t = \frac{2}{5}\ln\frac{4c_1}{c_2}$

Hence $\beta = 2$ is the smallest value such that the solution has no minimum.

3. Since $y_1(x) = x^{-1/2}\sin x$, then $y_1' = -\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x$.

(a) **Method 1:** Let $y_2 = v(x)y_1$ and plug y_2 into the DE:

$$\begin{aligned} x^2(v''y_1 + 2v'y_1' + \boxed{vy_1''}) + x(v'y_1 + \boxed{vy_1'}) + \boxed{(x^2-0.25)(vy_1)} &= 0 \\ \hline 0 + x^2y_1v'' + (2x^2y_1' + xy_1)v' &= 0 \end{aligned}$$

$\underbrace{v(x^2y_1'' + xy_1' + (x^2-0.25)y_1)}_{=0} + x^2v''y_1 + 2x^2v'y_1' + xv'y_1 = 0$

Separate variables and take integration w.r.t. x:

$$\begin{aligned} x^{3/2}\sin x\frac{\mathrm{d}v'}{\mathrm{d}x} &= \left(-2x^{3/2}\cos x\right)v' \\ \int \frac{1}{v'}\mathrm{d}v' &= -2\int\frac{\cos x}{sinx}\mathrm{d}x \\ \ln|v'| &= -2\ln\sin x + C \\ v' &= \frac{C}{(\sin x)^2} \end{aligned}$$

$\tan' x = \frac{1}{\cos^2 x} = \sec^2 x$

$\cot' x = -\frac{1}{\sin^2 x} = -\csc^2 x$

$\sec' x = \tan x\sec x$

$\csc' x = -\cot x\csc x$

$$\to v(x) = -C_1\cot x + C_2$$

Choose $C_1 = -1$, $C_2 = 0$, then $v(x) = \cot x$ and

$$y_2 = vy_1 = (\cot x)(x^{-1/2}\sin x) = x^{-1/2}\cos x.$$

(b) **Method 2:** Let $y_2 = v(x)y_1$, by the Abel's Theorem,

$$\begin{aligned} W(y_1,y_2)(x) &= C\exp\left(-\int p(x)\ \mathrm{d}x\right) = \frac{C}{x} \\ W(y_1,y_2)(x) &= \begin{vmatrix} y_1 & vy_1 \\ y_1' & v'y_1 + vy_1' \end{vmatrix} = v'y_1^2 \\ \to \qquad v'y_1^2 &= \frac{C}{x} \end{aligned}$$

Take $C = 1$, we have $v(x) = -\cot x$ and $y_2 = vy_1 = -x^{-1/2}\cos x$.

4. (a) $y'' - 10y' + 34y = \underline{te^{5t}\sin(3t)} + \underline{t^3}$

$(A_1t + \boxed{B_1)e^{5t}\sin 3t} + (A_2t + \boxed{B_2)e^{5t}\cos 3t}$

$C_3t^3 + C_2t^2 + C_1t + C_0$

项数重合，为了线性独立

$t(A_1t + B_1)e^{5t}\sin 3t + t(A_2t + B_2)e^{5t}\cos 3t$

$r^2 - 10r + 34 = 0$

$(r-5)^2 = -9$

$r_1 = -3i + 5 \quad r_2 = 3i + 5$

$$\Rightarrow Y_H(t) = \boxed{c_1e^{5t}\sin(3t) + c_2e^{5t}\cos(3t)}, \quad c_1, c_2 \in \mathbb{R}.$$

Then we may assume a particular solution to (N) has the form:

$$Y_P(t) = t(A_1t + B_1)e^{5t}\sin(3t) + t(A_2t + B_2)e^{5t}\cos(3t) + (C_3t^3 + C_2t^2 + C_1t + C_0).$$

(b) $y'' - 3y' = \underline{2t^4} + \underline{t^2e^{3t}} + \underline{\sin 3}$

$A_4t^4 + A_3t^3 + A_2t^2 + A_1t + \boxed{A_0}$

$(B_2t^2 + B_1t + \boxed{B_0)e^{3t}}$

项数重合，为了线性独立

$t(A_4t^4 + A_3t^3 + A_2t^2 + A_1t + A_0) + t(B_2t^2 + B_1t + B_0)e^{3t}$

$r^2 - 3r = 0$

$r(r-3) = 0$

$r_1 = 0 \quad r_2 = 3$

$$\Rightarrow Y_H(t) = \boxed{c_1} + \boxed{c_2e^{3t}}, \quad c_1, c_2 \in \mathbb{R}.$$

Then we may assume a particular solution to (N) has the form:

$$Y_P(t) = t\left(A_4t^4 + A_3t^3 + A_2t^2 + A_1t + A_0\right) + t(B_2t^2 + B_1t + B_0)e^{3t}.$$

$A_1\cos t + A_2\sin t$ $\quad (B_2t^2+\boxed{B_1t+B_0)e^{2t}}$ 重复→ $t^2(B_2t^2+B_1t+B_0)e^{2t}$

(c) $y'' - 4y' + 4y = \underline{\cos t} + \underline{4t^2e^{2t}} + \underline{te^t\sin 2t}$ $\quad (C_1t+C_0)e^t\sin 2t + (D_1t+D_0)e^t\cos t$

$r^2-4r+4=0$ $\quad y_1=e^{2t}$

$(r-2)^2=0$ $\quad y_2=e^{2t}t \Rightarrow$

$r_1=r_2=2$

$$Y_H(t) = \boxed{c_1e^{2t} + c_2te^{2t},} \quad c_1, c_2 \in \mathbb{R}.$$

Then we may assume the form as:

$$Y_P(t) = A_1\cos t + A_2\sin t + t^2(B_2t^2+B_1t+B_0)e^{2t} + (C_1t+C_0)e^t\sin 2t + (D_1t+D_0)e^t\cos 2t.$$

Remarks: **Sort** the terms in g before making any assumptions on the form of Y_P. Otherwise, you might involve unnecessary unknowns.

In 1(b), $g(t)$ is actually $t^2e^{3t} + (2t^4 + \sin 3)$. Thus, the initial guess of Y_P should be

$$\left(A_4t^4 + A_3t^3 + A_2t^2 + A_1t + A_0\right) + (B_2t^2 + B_1t + B_0)e^{3t},$$

instead of

$$\left(A_4t^4 + A_3t^3 + A_2t^2 + A_1t + A_0\right) + (B_2t^2 + B_1t + B_0)e^{3t} + C.$$

5. (a) To solve the corresponding **homogeneous** equation with characteristic eqn.

$y''+4y=\underline{t^2}+\underline{3e^t}$ $\quad B_0e^t$

$A_2t^2+A_1t+A_0$

$$r^2 + 4 = 0, \quad \to \quad r = \pm 2i$$

$$Y_H(t) = C_1\sin(2t) + C_2\cos(2t)$$

Based on the form of $g(t)$, **assume** that a particular solution to (N) has the **form**

$$Y_P(t) = At^2 + Bt + C + De^t$$

then $Y_P' = 2At + B + De^t$ and $Y_P'' = 2A + De^t$. **Substitute** Y_P into (N):

$$2A + De^t + 4(At^2 + Bt + C + De^t) = t^2 + 3e^t$$

$$\begin{cases} 4A = 1, \quad 5D = 3 \\ 4B = 0, \quad 2A + 4C = 0 \end{cases} \to \begin{cases} A = 1/4, \quad D = 3/5 \\ B = 0, \quad C = -1/8 \end{cases} \to \quad Y_P(t) = \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$$

Hence the **general solution for (N)** is

$$Y(t) = Y_H + Y_P = C_1\sin(2t) + C_2\cos(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$$

Set in the **initial conditions**:

$$\begin{cases} Y(0) = C_2 - 1/8 + 3/5 = 7/5 \\ Y'(0) = 2C_1 + 3/5 = 3/5 \end{cases} \to \begin{cases} C_1 = 0 \\ C_2 = 37/40 \end{cases}$$

Solution to the IVP is

$$Y(t) = \frac{37}{40}\cos(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t.$$

$y''-2y'+y = te^t+4$ $\boxed{(A_1t+A_0)e^t}$ B_0

$r^2-2r+1 = (r-1)^2=0$ $\boxed{y_1 = e^t \quad y_2 = te^t}$ 重复 $\Rightarrow t^2(A_1t+A_0)e^t$

(b) **General solution to (H):**

$$Y_H = C_1e^t + C_2te^t.$$

Based on $g(t)$, assume $Y_P = e^t(At+B)+C$. **Take** Y_H **into consideration**, the form is **adjusted** as

$$Y_P = t^2e^t(At+B)+C.$$

Substitution gives

$$e^t\left[At^3+(6A+B)t^2+(6A+4B)t+2B\right] - 2e^t\left[At^3+(3A+B)t^2+2Bt\right] + e^t\left[At^3+Bt^2\right]+C \quad = \quad te^t+4$$

$$\rightarrow \quad A = 1/6, \quad B=0, \quad C=4$$

The **general solution to (N)** is

$$Y = Y_H+Y_P = C_1e^t + C_2te^t + \frac{1}{6}t^3e^t+4.$$

Set in IC's:

$$\begin{cases} Y(0) = C_1+4 = 1 \\ Y'(0) = C_1+C_2 = 1 \end{cases} \quad \rightarrow \quad \begin{cases} C_1 = -3 \\ C_2 = 4 \end{cases}$$

Solution to the IVP is

$$Y(t) = -3e^t + 4te^t + \frac{1}{6}t^3e^t + 4.$$

6. Denote $t=\ln x$ and $u(t)=y(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}t}\frac{1}{x} = u'\frac{1}{x},$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\cdot\frac{1}{x}\right) = \frac{\mathrm{d}^2u}{\mathrm{d}t^2}\frac{\mathrm{d}t}{\mathrm{d}x}\cdot\frac{1}{x} + \frac{\mathrm{d}u}{\mathrm{d}t}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right) = (u''-u')\frac{1}{x^2}.$$

After substitution, the original DE becomes a linear DE with constant coefficients,

$$u''+(2-1)u'-6u=0, \quad \rightarrow \quad u(t) = c_1e^{2t}+c_2e^{-3t}.$$

Substitute $t=\ln x$ back and we yield the general solution for y is

$$y(x) = c_1x^2 + c_2x^{-3}, \quad c_1, c_2 \in \mathbb{R}.$$

Remarks: In fact, any equations in the form of

$$x^2y''+\alpha xy'+\beta y = 0$$

are called **Euler equations**. Substituting $t=\ln x$ and $u(t)=y(x)$, the Euler equation can be transformed into a DE with constant coefficients

$$u''+(\alpha-1)u'+\beta u = 0,$$

which can be solved easily.

7. Let $v = \frac{\mathrm{d}y}{\mathrm{d}t}$, then $v' = y''$ and the equation becomes a first order linear ODE

$$t^2 v' + 2tv = 1, \qquad t > 0.$$

By taking the integrating factor as t^2, we have

$$(t^2 v)' = 1 \qquad \to \quad t^2 v = t + c_1$$

Thus, $v = t^{-2}(t + c_1)$ where $c_1 \in \mathbb{R}$. Solve for y:

$$v = \frac{dy}{dt} = \frac{1}{t} + \frac{c_1}{t^2} \qquad \to \quad y = \ln t + \frac{c_1}{t} + c_2$$

where $c_{1,2}$ are arbitrary constants.

8. Let $v = \frac{\mathrm{d}y}{\mathrm{d}t}$, then

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}y} v,$$

The original equation becomes

$$v \frac{dv}{dy} + y v^3 = 0,$$

which is a first order separable equation about unknown function $v(y)$.

$$\frac{dv}{dy} = -y v^2, \qquad \text{for } v \neq 0$$

$$\int -\frac{dv}{v^2} = \int y \ \mathrm{d}y + c \qquad \to \quad \frac{1}{v} = \frac{y^2 + c_1}{2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v = \frac{2}{y^2 + c_1},$$

which is also separable.

$$\int \frac{y^2 + c_1}{2} \ \mathrm{d}y = \int 1 \ \mathrm{d}t + c_2$$

$$\frac{1}{6} y^3 + c_1 y + c_2 = t, \qquad c_1, c_2 \in \mathbb{R}.$$

For $v = 0$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ and the solution is $y = c$.