ASP Assignment 2

1. Let $\Omega = \{1,2,3\}$, $\mathcal{F}_1 = \big\{\{1\},\{2,3\},\Omega,\emptyset\big\}$ and $\mathcal{F}_2 = \big\{\{1,2\},\{3\},\Omega,\emptyset\big\}$. Show that:
   (a) $\mathcal{F}_1$ and $\mathcal{F}_2$ are both σ-fields.
   (b) $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ-field.

2. Let $\Omega = \{1,2,3,4,5\}$ and $\mathcal{U} := \big\{\{1,2,3\},\{3,4,5\}\big\}$. Determine $\sigma(\mathcal{U})$.

3. Let $\mathcal{B}(\mathbb{R})$ be the Borel σ-field. Show that

   (a) $\emptyset \in \mathcal{B}(\mathbb{R}), \mathbb{R} \in \mathcal{B}(\mathbb{R})$,

   (b) $\{c\} \in \mathcal{B}(\mathbb{R}), \quad \forall c \in \mathbb{R}$,

   (c) For all $a \in \mathbb{R}, b \in \mathbb{R}$ with $a < b$,

   $$[a,b], [a,b), (a,b) \in \mathcal{B}(\mathbb{R}),$$

   (d) $\mathbb{N} \in \mathcal{B}(\mathbb{R}), \mathbb{Q} \in \mathcal{B}(\mathbb{R})$, where $\mathbb{N}$ is the set of all natural numbers and $\mathbb{Q}$ the set of all rational numbers.

4. Let $e_1, e_2, \ldots$ be a sequence of independent, identically distributed random variables each with zero mean and variance σ_e^2. Let $Y_1 = \theta_0 + e_1$, and then for $t > 1$ define Y_t recursively by $Y_t = \theta_0 + Y_{t-1} + e_t$. Here θ_0 is a constant. The process $\{Y_t : t = 1, 2, \ldots\}$ is called a random walk with drift.

   (a) Show that Y_t may be rewritten as $Y_t = t\theta_0 + e_t + e_{t-1} + \cdots + e_1$.

   (b) Find the mean function for $\{Y_t\}$.

   (c) Find the autocovariance function for $\{Y_t\}$.

   (d) Does $\{Y_t\}$ have independent and stationary increments? Explain.

5. Suppose that ξ, η are uncorrelated random variables with $\mathbb{E}(\xi) = \mathbb{E}(\xi) = 0$ and $\mathrm{Var}(\xi) = \mathrm{Var}(\eta) = \sigma^2$. Let $\lambda > 0$ be a constant. Define

   $$X_n := \xi\cos(\lambda n) + \eta\sin(\lambda n).$$

   Find $\mathbb{E}(X_n)$ and $\mathrm{Cov}(X_n, X_m)$.

6. Suppose that ξ, η are independent random variables and $\xi, \eta \sim N(0,1)$. Let $\theta > 0$ be a constant. Define

   $$X_t := \xi\cos(\theta t) + \eta\sin(\theta t), \quad t \in \mathbb{R}.$$

   Determine the finite-dimensional distributions of the process $\{X_t : t \in \mathbb{R}\}$.