FINM3133 Time Series for Finance and Macroeconomics

Chapter 6 Solution

1. Verify

$$Var(r_j) \approx \frac{1}{n}[\frac{(1+\phi^2)(1-\phi^{2j})}{1-\phi^2} - 2j\phi^{2j}]$$

We know that r_j approximate to a normal distribution with mean ρ_j and variance c_{jj}/n.

$$\begin{aligned}
c_{jj} &= \sum_{k=-\infty}^{\infty} (\rho_{k+j}\rho_{k+j} + \rho_{k-j}\rho_{k+j} - 2\rho_j\rho_k\rho_{k+j} - 2\rho_j\rho_k\rho_{k+j} + 2\rho_j\rho_j\rho_k^2) \\
&= \sum_{k=-\infty}^{\infty} (\rho_{k+j}^2 + \rho_{k-j}\rho_{k+j} - 4\rho_j\rho_k\rho_{k+j} + 2\rho_j^2\rho_k^2) \\
&= (1+2\rho_j^2)\sum_{k=-\infty}^{\infty}\rho_k^2 + \sum_{k=-\infty}^{\infty}\rho_{k-j}\rho_{k+j} - 4\rho_j\sum_{k=-\infty}^{\infty}\rho_k\rho_{k+j}
\end{aligned}$$

For AR(1) process, we have $\rho_k = \phi^k$. Next we need to sum these infinite series.

$$\sum_{k=-\infty}^{\infty}\rho_{k^2} = 1 + 2\sum_{k=1}^{\infty}\phi^{2k} = 1 + 2\frac{\phi^2}{1-\phi^2} = \frac{1+\phi^2}{1-\phi^2}$$

$$\begin{aligned}
\sum_{k=-\infty}^{\infty}\rho_{k-j}\rho_{k+j} &= \sum_{k=-\infty}^{-j-1}\phi^{j-k-k-j} + \sum_{k=-j}^{j-1}\phi^{j-k+k+j} + \sum_{k=j}^{\infty}\phi^{k-j+k+j} \\
&= \sum_{k=-\infty}^{-j-1}\phi^{-2k} + \sum_{k=-j}^{j-1}\phi^{2j} + \sum_{k=j}^{\infty}\phi^{2k} \\
&= \frac{\phi^{2(j+1)}}{1-\phi^2} + 2j\phi^{2j} + \frac{\phi^{2j}}{1-\phi^2} \\
&= \phi^{2j}(\frac{1+\phi^2}{1-\phi^2}) + 2j\phi^{2j}
\end{aligned}$$

$$\begin{aligned}
\sum_{k=-\infty}^{\infty}\rho_k\rho_{k+j} &= \sum_{k=-\infty}^{-j}\phi^{-k}\phi^{-k-j} + \sum_{k=-j+1}^{0}\phi^{-k}\phi^{k+j} + \sum_{k=1}^{\infty}\phi^{2k} \\
&= \frac{\phi^j}{1-\phi^2} + j\phi^j + \phi^j(\frac{\phi^2}{1-\phi^2}) \\
&= (\frac{1+\phi^2}{1-\phi^2})\phi^j + j\phi^j
\end{aligned}$$

Thus,

$$
\begin{aligned}
c_{jj} &= (1+2\phi^{2j})\frac{1+\phi^2}{1-\phi^2} + \phi^{2j}(\frac{1+\phi^2}{1-\phi^2}) + 2j\phi^{2j} - 4\phi^j[(\frac{1+\phi^2}{1-\phi^2})\phi^j + j\phi^j] \\
&= \frac{(1+\phi^2)(1-\phi^{2j})}{1-\phi^2} - 2j\phi^{2j}
\end{aligned}
$$

Therefore, $Var(r_j) = c_{jj}/n$, the result is just as the result.

2. For sample ACF, we use $2/\sqrt{n} = 0.2$ as a threshold to detect significant correlations. Obviously, only the first three lags may regard as significant. Thus, MA(2) or MA(3) would be possible. we can employ an hypothesis test to check whether $rho_3 = 0$, whether MA(2) is enough to fit the series.

   $$Var(r_3) \approx c_{33}/100 = (1 + 3[(-0.49)2 + (0.31)2/100 = 0.016724])$$

   The test statistics,
   $$z = \frac{r_3 - 0}{\sqrt{(0.016724)}} = -1.62$$
   shows insignigicance to reject the null hypothesis. Thus, both MA(2) and MA(3) would be possible.

3. Notethat $r_2/r_1 = 0.78$, $r_3/r_2 = 0.81$, $r_4/r_3 = 0.81$,and $r_5/r_4 = 0.76$ and we do not have $r_k \approx r_1^k$.This would seem to rule out an AR(1) model but support an ARMA(1,1) with $\phi \approx 0.8$.

4. (a) The lag one autocorrelation for Series A will be strongly positive since neighboring points in time are almost uni- versally on the same side of the mean. The scale is not relevant. The lag one autocorrelation for Series B, on the other hand, will be strongly negative since neighboring points in time are almost universally on opposite sides of the mean.

   (b) The lag two autocorrelation for Series A will also be positive again since points two apart in in time are almost uni- versally on the same side of the mean. The lag two autocorrelation for Series B will be (strongly) positive since points two apart in time are almost uniersally on the same side of the mean.