2022-23 Second Semester
MATH1083 Calculus II Midterm (1002&1003)

Time: 6:30-8pm 13/Apr/2023 (Thu)   Venue: T2-101   **Total score 100 pts**

- Answer all questions using a black/blue ink pen. No pencils
- **Write down** your

Chinese name : ______________ Student NO. ______________

Points awarded

1. **[5pts]** Determine whether each of following **sequence** converges or diverges. If it converges, **find the limit**.

$$\text{(a) } a_n = \frac{1+n}{1+2n} \qquad \text{(b) } a_n = \frac{n\sin n}{n^2+1}$$

Solution: (a) a_n is **monotonically** decreasing and have the lower **bound** $1/2$, so it is **convergent**

$$\lim_{n\to\infty} a_n = \frac{1}{2}$$

(b)

$$\frac{-n}{n^2+1} \le \frac{n\sin n}{n^2+1} \le \frac{n}{n^2+1}$$

the limit of left and right are both 0, so a_n is convergent by the **Squeeze Theorem** and

$$\lim_{n\to\infty} a_n = 0$$

or to prove it is absolutely convergent

$$0 \le \left|\frac{n\sin n}{n^2+1}\right| \le \frac{n}{n^2+1}$$

therefore it is convergent and have limit 0 by the Squeeze Theorem.

2. **[10pts]** Determine whether the following **series** is convergent or divergent, and explain why (which **test** to use?).

$$\text{(a) } \sum_{n=1}^{\infty} \frac{3^n}{n!} \qquad \text{(b) } \sum_{n=1}^{\infty} \frac{n^n}{(\ln n)^n}$$

Solution: (a) let $a_n = \frac{3^n}{n!}$ and applying **Ratio Test**

$$r = \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}}\right| = \left|\frac{3}{n+1}\right| < 1$$

as $n \to \infty$, $r \to 0$, so this series is **convergent**.
(b) let $a_n = \frac{n^n}{(\ln n)^n}$ and applying **Root Test**

$$\sqrt[n]{a_n} = \left| \frac{n}{(\ln n)} \right| \to \infty$$

so this series is **divergent**.

3. **[10pts]** Determine whether the following **series** is **absolutely convergent, conditionally convergent or divergent**, and explain why.

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-\frac{1}{2}}$ (b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{2^{n+1}}$

Solution: (a) since

$$\lim_{n\to\infty} \left| n^{-\frac{1}{2}} \right| = 0$$

so this series is convergent by **Alternating Test**. However,

$$\sum_{n=1}^{\infty} \left| (-1)^{n-1} n^{-\frac{1}{2}} \right| = \sum_{n=1}^{\infty} n^{-\frac{1}{2}}$$

is a p-series where $p = 1/2 < 1$, so series $\sum_{n=1}^{\infty} \left| (-1)^{n-1} n^{-\frac{1}{2}} \right|$ is divergent. Therefore this series is **conditionally convergent**
(b) since

$$\lim_{n\to\infty} \frac{3^n}{2^{n+1}} \neq 0$$

, this series is **divergent**.

4. **[10pts]** Use the **binomial series** to expand the given function as a power series centered at 0, write down the first four terms of this power series and find its radius of convergence

$$\frac{1}{(2+x)^3}$$

Solution:

$$\frac{1}{(2+x)^3} = \frac{1}{8} \frac{1}{\left(1+\frac{x}{2}\right)^3} = \frac{1}{8}\left(1+\frac{x}{2}\right)^{-3}$$
$$= \frac{1}{8} \sum_{n=0}^{\infty} \binom{-3}{n} \left(\frac{x}{2}\right)^n$$

where binomial coefficient

$$\binom{-3}{n} = \frac{-3 \cdot (-4) \cdots (-3-n+1)}{n!}$$

with

$$\binom{-3}{0} = 1 \qquad \binom{-3}{1} = -3, \qquad \binom{-3}{2} = \frac{(-3)(-4)}{2} = 6, \qquad \binom{-3}{3} = \frac{(-3)(-4)(-5)}{3!} = -10$$

so the expansion of this function with the first four terms is as follows

$$\frac{1}{(2+x)^3} = \frac{1}{8}\left[1 - 3\left(\frac{x}{2}\right) + 6\left(\frac{x}{2}\right)^2 - 10\left(\frac{x}{2}\right)^3 \ldots\right]$$

or

$$\frac{1}{(2+x)^3} = \frac{1}{8}\left[1 - \frac{3}{2}x + \frac{3}{2}x - \frac{5}{4}x^3 \ldots\right]$$

or

$$\frac{1}{(2+x)^3} = \frac{1}{8} - \frac{3}{16}x + \frac{3}{16}x^2 - \frac{5}{32}x^3 \ldots$$

Let

$$a_n = \binom{-3}{n}\left(\frac{x}{2}\right)^n$$

if this series is convergent, applying the **Ratio Test**, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\binom{-3}{n+1}\left(\frac{x}{2}\right)^{n+1}}{\binom{-3}{n}\left(\frac{x}{2}\right)^n}\right| = \left|\frac{-3-n}{n+1}\right|\left|\frac{x}{2}\right| < 1$$

(or some they actually looked at the ratio of

$$\left|\frac{a_n}{a_{n-1}}\right| = \left|\frac{-2-n}{n}\right|\left|\frac{x}{2}\right|$$

this is also correct.)Since

$$\left|\frac{-3-n}{n+1}\right| \to 1$$

when we have

$$\left|\frac{x}{2}\right| < 1$$

hence, $|x| < 2$, the **radius of convergence is 2**.

5. **[15pts]** For function

$$f(x) = \sqrt[3]{x}$$

   (a) **(7pts)** **Approximate** the function by Taylor Polynomial of degree 2 $T_2(x)$ at $a = 8$.

   (b) **(5pts)** How accurate is the approximation when $6 \le x \le 10$ (Estimate $R_2(x)$) [$6^{-8/3} = 0.0084$, $7^{-8/3} = 0.0056$, $8^{-8/3} = 0.0039$, $6^{-5/3} = 0.0504$, $7^{-5/3} = 0.0390$, $8^{-5/3} = 0.0313$].

   (c) **(3pts)** Using $T_2(x)$ to evaluate $\sqrt[3]{10}$ and estimate its error $R_2(10)$.

Solution: (a) **Taylor Polynomial of degree 2:**

$$f(x) \approx T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

here $a = 8$

$$\begin{aligned}
f(8) &= \sqrt[3]{8} = 2 \\
f'(x) &= \frac{1}{3}x^{-\frac{2}{3}} \qquad f'(8) = \frac{1}{12} \\
f''(x) &= -\frac{2}{9}x^{-\frac{5}{3}} \qquad f''(8) = -\frac{1}{144} \\
f'''(x) &= \frac{10}{27}x^{-\frac{8}{3}}
\end{aligned}$$

Thus the Polynomial is

$$\sqrt[3]{x} \approx T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

(b) Using Taylor's Theorem

$$|R_2(x)| \leq \left| \frac{f'''(z)}{3!}(x-8)^3 \right| \leq \frac{|f'''(z)|}{3!}|x-8|^3$$

since $6 \leq z \leq 10$, then $|f'''(z)| = \frac{10}{27}z^{-\frac{8}{3}}$ reaches its maximum value when $z = 6$ and $\max|x-8| = 2$

$$\max|f'''(z)| = \frac{10}{27} \cdot \frac{1}{6^{8/3}} = \frac{10}{27} \times 0.0084$$

$$|R_2(x)| \leq \frac{\max|f'''(z)|}{3!}|x-8|^3 = \frac{1}{6} \times \frac{10}{27} \times 0.0084 \times 2^3$$

(c) for $x = 10$ $x - 8 = 2$

$$\sqrt[3]{10} \approx 2 + \frac{2}{12} - \frac{4}{288} = \frac{155}{72}$$

for $x = 10$, there exist $z \in (8, 10)$

$$|R_2(10)| \leq \frac{\max|f'''(z)|}{3!}|x-8|^3 = \frac{1}{6} \times \frac{10}{27} \times 0.0039 \times 2^3$$

$|f'''(z)|$ attains its maximum when $z = 8$ or they can just simply use this estimate from (b) for $z \in (6, 10)$ where $|f'''(z)|$ attains its maximum when $z = 6$ as

$$|R_2(10)| \leq \frac{\max|f'''(z)|}{3!}|x-8|^3 = \frac{1}{6} \times \frac{10}{27} \times 0.0084 \times 2^3$$

However, it is **WRONG** to use $z = 10$, because then you will get the minimum of $|f'''(z)|$.

6. **[5pts]** Find the limit below **WITHOUT** using L'Hospital Rule.

$$\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x^2}$$

Solution: Taylor series of e^{2x}

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \cdots$$

$$\begin{aligned}
\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x^2} &= \lim_{x \to 0} \frac{\frac{(2x)^2}{2} + \frac{(2x)^3}{3!} + \cdots}{x^2} \\
&= 2
\end{aligned}$$

7. **[15pts=4+5+2+2+2]** Find

(a) the **parametric equation and** the **symmetric equation** for the line through point $P = (1,0,-3)$ and parallel to $\overrightarrow{a} = 3\overrightarrow{i} - 2\overrightarrow{j} + \overrightarrow{k}$.

Solution: parametric equation: for $t \in \mathbb{R}$

$$x = 1 + 3t$$
$$y = -2t$$
$$z = 3 + t$$

Symmetric equation
$$\frac{x-1}{3} = \frac{y}{-2} = z + 3$$

(b) the **equation for the plane containing** points $P = (1,1,-2)$, $Q = (0,2,1)$ and $R = (-1,-1,0)$.

Solution: first we identify two vectors on the plane:

$$\overrightarrow{PQ} = (0,2,1) - (1,1,-2) = \langle -1,1,3 \rangle$$
$$\overrightarrow{QR} = (-1,-1,0) - (0,2,1) = \langle -1,-3,-1 \rangle$$

or
$$\overrightarrow{PR} = (-1,-1,0) - (1,1,-2) = \langle -2,-2,2 \rangle$$

then we can find the **normal vector** of the plane

$$\overrightarrow{n} = \overrightarrow{PQ} \times \overrightarrow{QR} = \langle 8,-4,4 \rangle$$

so the **equation of the plane** is

$$8(x-1) - 4(y-1) + 4(z+2) = 0$$

or
$$8x - 4y + 4z + 4 = 0$$

or any multiples of the equation above, such as

$$2x - y + z + 1 = 0$$

(c) the **disctance** between the two parallel plane by $2x + y - z = 2$ and $2x + y - z = 8$.

Solution: Method 1 (use distance formula from Section 12.5 Example 8)

$a = 2$, $b = 1$, $c = -1$ and $k = 8 - 2 = 6$

$$d = \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2 + b^2 + c^2}} = \frac{6}{\sqrt{6}} = \sqrt{6}$$

Method 2: point $A(1,0,0)$ lies in the first plane $2x+y-z=2$ and point $B(4,0,0)$ lies in the second plane $2x+y-z=8$. Then

$$\overrightarrow{BA} = (4,0,0)-(1,0,0) = \langle 3,0,0\rangle$$

and the normal direction of both planes is

$$\overrightarrow{n} = \langle 2,1,-1\rangle$$

with unit normal

$$\overrightarrow{u} = \frac{1}{|\overrightarrow{n}|}\langle 2,1,-1\rangle = \frac{1}{\sqrt{6}}\langle 2,1,-1\rangle$$

The the distance is the scalar projection of vector $\overrightarrow{BA}$ onto the unit normal:

$$d = \text{Proj}_{\overrightarrow{u}}\overrightarrow{BA} = \overrightarrow{BA}\cdot\overrightarrow{n} = \frac{6}{\sqrt{6}} = \sqrt{6}$$

(d) Determine whether the following **line**

$$\begin{aligned} x &= 1+2t \\ y &= -2+3t \\ z &= -1+4t \end{aligned}$$

intersects with the given **plane**

$$x+2y-2z=1$$

If they do intersect, determine whether the line is contained **in the plane** or intersects it **in a single point**. Finally, if the line intersects the plane in a single point, determine this point of intersection.

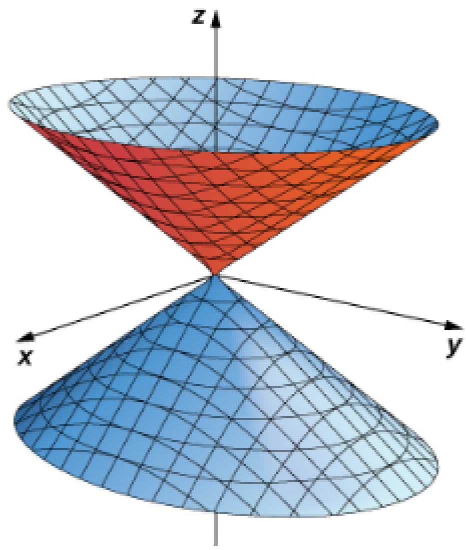
Solution: Method 1: Substitute the equation of line in the plane:

$$\begin{aligned} 1+2t+2(-2+3t)-2(-1+4t) &= 1 \\ -1 &= 1 \end{aligned}$$

which is false. Therefore the line **NEITHER contains NOR intersects** with the plane.

Method 2: find the distance from any point on the line to the point, and it is a non-zero constant.

(e) Which is the equation of the quadric surface: **A**



A. $x^2+y^2-z^2=0$, B. $-x^2+y^2+z^2=0$, C. $x^2-y^2+z^2=0$, D. $x^2+y^2-z=0$,
E. $-x+y^2+z^2=0$, F. $x^2-y+z^2=1$, G. $x^2+y^2-z^2=1$, H. $x^2-y^2+z^2=1$

8. **[5pts]**Find the length of the curve

$$\overrightarrow{r}(t)=\sqrt{2}t\,\overrightarrow{i}+e^t\overrightarrow{j}+e^{-t}\overrightarrow{k} \qquad 0\le t\le 1$$

Solution: $f'(t)=\sqrt{2}$, $g'(t)=e^t$ and $h'(t)=-e^{-t}$

$$\begin{aligned}
L &= \int_0^1 \sqrt{[f'(t)]^2+[g'(t)]^2+[h'(t)]^2}dt \\
&= \int_0^{\pi/4} \sqrt{2+e^{2t}+e^{-2t}}dt \\
&= \int_0^{\pi/4} \left(e^t+e^{-t}\right)dt \\
&= \left[e^t-e^{-t}\right]_0^1 = e-e^{-1}
\end{aligned}$$

9. **[10pts]** For the curve

$$\overrightarrow{r}(t)=\langle 3\sin t,4t,3\cos t\rangle\,,t>0$$

(a) Find the unit tangent and unit normal vectors $\overrightarrow{T}(t)$ and $\overrightarrow{N}(t)$

(b) Find the curvature.

Solution: a) $\overrightarrow{r}'(t)=\langle 3\cos t,4,-3\sin t\rangle$ $|\overrightarrow{r}'(t)|=5$, the **unit tangent vector**

$$\overrightarrow{T}(t)=\frac{\overrightarrow{r}'(t)}{|\overrightarrow{r}'(t)|}=\frac{1}{5}\langle 3\cos t,4,-3\sin t\rangle$$

so $\overrightarrow{T}'(t)=\frac{1}{5}\langle -3\sin t,0,-3\cos t\rangle$ and $\left|\overrightarrow{T}'(t)\right|=\frac{3}{5}$, the principal **unit normal vector**:

$$\overrightarrow{N}(t)=\frac{\overrightarrow{T}'(t)}{\left|\overrightarrow{T}'(t)\right|}=\langle -\sin t,0,-\cos t\rangle$$

The **curvature**

$$\kappa = \left|\frac{\overrightarrow{T}'(t)}{\overrightarrow{r}'(t)}\right| = \left|\frac{\frac{3}{5}}{5}\right| = \frac{3}{25}$$

10. **[10pts]** Find the limit if it exists or show that the limit does not exist

(a)

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+3y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4}$$

Solution: (a) Path 1: along x-axis, $y=0$,

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+3y^2} = 0$$

Path 2:along $y=x$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2}{x^2+3x^2} = \frac{1}{2}$$

therefore, **the limit does not exist**.

(b) Using Squeeze Theorem (be careful of the **absolute sign**!)

$$0 \le \left|\frac{xy^4}{x^4+y^4}\right| = \left|\frac{y^4}{x^4+y^4}\right| |x| \le |x| \to 0$$

so

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4} = 0$$

11. **[5pts]** Find the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ of the function $f(x,y,z) = x^3y^2e^z$ and their values at point $(1,2,0)$.

Solution: **(1pt for each correct answer)**

$$\frac{\partial f}{\partial x} = 3x^2y^2e^{4z}, \qquad \frac{\partial f}{\partial x} = 2x^3ye^{4z}, \qquad \frac{\partial f}{\partial x} = x^3y^2e^z$$

$$\frac{\partial f}{\partial x}(1,2,0) = 12, \qquad \frac{\partial f}{\partial x}(1,2,0) = 4, \qquad \frac{\partial f}{\partial x}(1,2,0) = 4$$