Caculus II Math 1038 (1002&1003)

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Week 5:

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1. Chapter 12 Vector the Geometry of Space
   (a) Definitions and notations
      i. vector $\overrightarrow{v}, \hat{v}, \overline{AB}$: length/magnitude/distance/norm, components of a vector
      ii. scalar c
      iii. coordinate, axes, planes, octant, projection, three dimensional rectangular coordinate system
      iv. surface, solid region,
      v. location vector $\overrightarrow{a} = \langle a_1, a_2 \rangle$, initial point at the origin
      vi. Euclidean space: $\mathbb{R}^n$
      vii. position vector: from origin to the point
      viii. standard basis vectors in $\mathbb{R}^3$ is
      $$\overrightarrow{i} = \langle 1,0,0 \rangle \qquad \overrightarrow{j} = \langle 0,1,0 \rangle \qquad \overrightarrow{k} = \langle 0,0,1 \rangle$$
      ix. unit vector
      $$\overrightarrow{u} = \frac{\overrightarrow{a}}{\|a\|}$$
   (b) Formula
      i. norm of $\overrightarrow{a} = \sqrt{a_1^2 + a_2^2}$
      ii. Distance formula in 3Ds
   (c) Operations
      i. scalar multiplication
      ii. vector addition, substraction

2. **Dot product**: Let $\overrightarrow{a} = (a_1, a_2, a_3)$ and $\overrightarrow{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$. **Dot product** of two vectors $\overrightarrow{a} \cdot \overrightarrow{b} = a_1b_1 + a_2b_2 + a_3b_3$ and
   $$\overrightarrow{a} \cdot \overrightarrow{b} = \|a\| \, \|b\| \cos\theta \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$
   (a) $\overrightarrow{a} \cdot \overrightarrow{b} \leq \|a\| \, \|b\|$ Cauchy-Schwartz Inequality. $\overrightarrow{a} \cdot \overrightarrow{b} = \|a\| \, \|b\|$ if $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel.
   (b) $\|a+b\| \leq \|a\| + \|b\|$ The triangle Inequality. $\|a+b\| = \|a\| + \|b\|$ if $\overrightarrow{a}$ and $\overrightarrow{b}$ have the same direction.
   (c) $\overrightarrow{a} \cdot \overrightarrow{a} = |a|^2$
   (d) Orthoganal: $\overrightarrow{a} \cdot \overrightarrow{b} = 0$
   (e) Projection:
      i. **Scalar projection** (component) of $\overrightarrow{a}$ on $\overrightarrow{b}$:
      $$comp_b a = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{\|b\|} \qquad \text{scalar}$$
      ii. **Vector projection** of $\overrightarrow{a}$ on $\overrightarrow{b}$:
      $$proj_b a = comp_b a \cdot \frac{\overrightarrow{b}}{\|b\|} = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{\|b\|^2} \overrightarrow{b} \qquad \text{vector}$$
      here $\frac{\overrightarrow{b}}{\|b\|}$ is a unit vector with the same direction of $\overrightarrow{b}$

3. **Cross product**: for $\overrightarrow{a} = (a_1, a_2, a_3)$ and $\overrightarrow{b} = (b_1, b_2, b_3)$

   (a) the cross product

   $$\overrightarrow{a} \times \overrightarrow{b} = (a_2 b_3 - b_2 a_3)\overrightarrow{i} - (a_1 b_3 - b_1 a_3)\overrightarrow{j} + (a_1 b_2 - b_1 a_2)\overrightarrow{k}$$

   (b) $\left\| \overrightarrow{a} \times \overrightarrow{b} \right\| = \|a\| \, \|b\| \sin\theta$: **Area of a parallelogram**

   (c) $\overrightarrow{a} \times \overrightarrow{b}$ is **orthogonal** to both $\overrightarrow{a}$ and $\overrightarrow{b}$, so we have $\left(\overrightarrow{a} \times \overrightarrow{b}\right) \cdot \overrightarrow{a} = 0$

   (d) $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0}$ if and only if $\overrightarrow{a}, \overrightarrow{b} \in V_3$ are **parallel**

   (e) For any vector $\overrightarrow{a} \in V^3$, $\overrightarrow{a} \times \overrightarrow{a} = 0$ and $\overrightarrow{a} \times \overrightarrow{0} = \overrightarrow{0}$

   (f) for standard basis vector

   $$\begin{array}{ccc} \overrightarrow{i} \times \overrightarrow{j} = \overrightarrow{k} & \overrightarrow{j} \times \overrightarrow{k} = \overrightarrow{i} & \overrightarrow{k} \times \overrightarrow{i} = \overrightarrow{j} \\ \overrightarrow{j} \times \overrightarrow{i} = -\overrightarrow{k} & \overrightarrow{k} \times \overrightarrow{j} = -\overrightarrow{i} & \overrightarrow{i} \times \overrightarrow{k} = -\overrightarrow{j} \end{array}$$

   (g) Triple product $\overrightarrow{a} \cdot \left(\overrightarrow{b} \times \overrightarrow{c}\right)$, if $\overrightarrow{a} \cdot \left(\overrightarrow{b} \times \overrightarrow{c}\right) = 0$, then the three vectors are **coplanar**.

4. **Distance**:

   (a) the geometric interpretation of magnitude

      i. 1D: length $|\overrightarrow{a}|$

      ii. 2D: areas of a **parallelogram** $A = \left|\overrightarrow{a} \times \overrightarrow{b}\right|$

      iii. 3D: volumn of **parallelepiped** $V = \left|\overrightarrow{a} \cdot \left(\overrightarrow{b} \times \overrightarrow{c}\right)\right|$

   (b) Let P be a point not on the line L that pass through the points Q and R. $\overrightarrow{a} = \overrightarrow{QR}$ and $\overrightarrow{b} = \overrightarrow{QP}$
   The distance d from a point P to line L is

   $$d = \frac{\left|\overrightarrow{a} \times \overrightarrow{b}\right|}{|\overrightarrow{a}|}$$

   (c) Let P be a point not on the plane L that pass through the points Q, $R$and S. $\overrightarrow{a} = \overrightarrow{QR}$ and $\overrightarrow{b} = \overrightarrow{QP}$
   The distance d from a point to the plane is

   $$d = \frac{\left|\overrightarrow{a} \cdot \left(\overrightarrow{b} \times \overrightarrow{c}\right)\right|}{\left|\overrightarrow{a} \times \overrightarrow{b}\right|}$$

5. **Equations (vector equation and parametric equation)**

   (a) Lines L: a point and a direction or two points

      i. 2D in the xy plane: a point on the line and the direction (slope): $y = ax + b$

      ii. 3D: a point $P_0(x_0, y_0, z_0)$ on L, $P = (x, y, z)$ is an arbitrary point in L. Let $\overrightarrow{r_0}$ and $\overrightarrow{r}$ be the position vectors of P_0 and P, let $\overrightarrow{a} = t\overrightarrow{v}$, we have the **vector equation of** L:

      $$\overrightarrow{r} = \overrightarrow{r_0} + \overrightarrow{a} = \overrightarrow{r_0} + t\overrightarrow{v}, \qquad t \in \mathbb{R}$$

      each **parameter** t gives the position vector $\overrightarrow{r}$ of a point on L.
      If we use the **componenet form** for each vector

      $$\overrightarrow{v} = \langle a, b, c \rangle, \qquad \overrightarrow{r} = \langle x, y, z \rangle \quad \text{and} \quad \overrightarrow{r_0} = \langle x_0, y_0, z_0 \rangle$$

      $$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

then these 3 equations are the **parametric equations of L which go through** $P_0(x_0, y_0, z_0)$ and parallel to the direction $\langle a, b, c\rangle$

$$x = x_0 + ta, \qquad y = y_0 + tb, \qquad z = z_0 + tc$$

and the **symmetric equations** of L

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$$

Remark: vector equation and parametric equations are NOT unique.

iii. Line segment: from $\overrightarrow{r_0}$ to $\overrightarrow{r_1}$ is given by vector equation

$$\overrightarrow{r} = (1 - t)\,\overrightarrow{r_0} + t\overrightarrow{r_1}, \qquad \mathbf{0 \le \boldsymbol{t} \le 1}$$

iv. skew lines: that do not intersect and are not parallel.

(b) Planes: a normal direction $\overrightarrow{n}$ and a point P_0 in the plane

i. simple planes: e.g.$z = 0$, x-y plane $x = 0$: a y-z plane.

ii. Let $P_0(x_0, y_0, z_0)$ a point on the plane, and $P = (x, y, z)$ is an arbitrary point in the plane, and $\overrightarrow{r_0}$ and $\overrightarrow{r}$ be the position vectors of P_0 and P, then

$$\overrightarrow{P_0P} = \overrightarrow{r} - \overrightarrow{r_0}$$

we have **vector equation of the plane**

$$\overrightarrow{n} \cdot (\overrightarrow{r} - \overrightarrow{r_0}) = 0$$

or

$$\overrightarrow{n} \cdot \overrightarrow{r} = \overrightarrow{n} \cdot \overrightarrow{r_0}$$

iii. The **scalar equation** of the plane through point with normal vector $\langle a, b, c\rangle$

$$\langle a, b, c\rangle\,\langle x - x_0, y - y_0, z - z_0\rangle = 0$$

$$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0$$

or

$$ax + by + cz + d = 0$$

where $d = -ax_0 - by_0 - cz_0$. It is a linear equation in x, y and z.

iv. Two planes are **parallel** if their normal vectors are parallel: $\overrightarrow{n_1} = c\overrightarrow{n_2}$

v. If two planes are not parallel, they intersect in a straight line L and the **angle between the two plane** is defined as the acute angle θ between their normal vectors

$$\cos\theta = \frac{\overrightarrow{n_1} \cdot \overrightarrow{n_2}}{|\overrightarrow{n_1}|\,|\overrightarrow{n_2}|}$$

The line L is perpendicular to both of the normal vectors, therefore the direction of the line $\overrightarrow{v} = \overrightarrow{n_1} \times \overrightarrow{n_2}$

6. Distances again

(a) Distance D from a point $P_1(x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$. Let $P_0(x_0, y_0, z_0)$ a point on the plane and $\overrightarrow{b} = \overrightarrow{P_0P_1}$

$$\overrightarrow{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$$

from the equation of the plane, the normal vector $\overrightarrow{n} = \langle a, b, c\rangle$

The distantce$D$ is the scalar projection (component) of $\overrightarrow{b}$ onto $\overrightarrow{n}$:

$$\begin{aligned} D &= \frac{\left|\overrightarrow{n} \cdot \overrightarrow{b}\right|}{|\overrightarrow{n}|} \\ &= \frac{|a\,(x_1 - x_0) + b\,(y_1 - y_0) + c\,(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

Note that the point $P_0(x_0, y_0, z_0)$ is on the plane, so $ax_0 + by_0 + cz_0 + d = 0$ and $d = -\,(ax_0 + by_0 + cz_0)$.

(b) Distance between two parallel planes: Find a point $P_1(x_1, y_1, z_1)$ on the other plane.

(c) Distance between two skew lines: They lies in two parallel planes. They have a common normal vector $\overrightarrow{n}$ which is orthogonal to both $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$, so

$$\overrightarrow{n} = \overrightarrow{v_1} \times \overrightarrow{v_2}$$

7. Cylinders and Quadric surfaces

(a) A cylinder is a surface that consists of all lines that parallel to a given line and pass through a given plane curve

i. Parabolic cylinder: $z = x^2$

ii. circular cylinder: $x^2 + y^2 = 1$

(b) Quadratic surfaces: a graph of a second-degree equation in x, y and z

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where capital letters are constants. By translation or rotation, it can be one of th e two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0$$

or

$$Ax^2 + By^2 + Iz = 0$$

(c) Traces in xy-plane, let $z = 0$. Traces in yz-plane, $x = 0$.

(d) **Ellipsoid** centered at origin

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

this is a **standard form** (RHS=1)

(e) **Elliptic paraboloid**:

$$z = 4x^2 + y^2$$

(f) **Hyperbolic paraboloid**

$$z = y^2 - x^2$$

(g) **Hyperboliod of ONE sheet**: 1 trace of ellipse ($z = 0$) + 2 traces of hyperbolas (x or $y = 0$)

$$\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$$

(h) **Hyperboliod of TWO sheet**: two minus signs indicate two sheet

$$-\frac{x^2}{4} - y^2 + \frac{z^2}{4} = 1$$

(i) Equation of a sphere wth center $C(h, k, l)$:

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

8. English words:
2D noun -se/-bola, adj -ic, 3D noun -oid
ellipse, elliptic, ellipsoid
parabola, parabolic, paraboloid
hyperbola, hyperbolic, hyperboliod
circle, circular, sphere

9. other vocabulary
Parallel, parallelogram, parallelepiped
cylinder, cylindric