Caculus II Math 1038 (1002&1003)

Monica CHEN

Week 2:

1. Contact:
    - Lecturer: **Monica** (Dr. Wen CHEN), T3-502-R12 (no key),
        - email: wenchen@uic.edu.cn
        - Lectures: Mon 8-10am 1002; Wed 8-10am 1003; Fri 3-4pm 1002; 5-6pm 1003
        - **Q&A: Fri afternoons**
    - TA: Ms. Mei MING, T3-502-R26-H13
        - Tutorial: 7:00-7:50pm Thursdays

Course of content of Chapter 1 Sequence and Series

1. Sequences
2. Series
    (a) Tests for convergence
    (b) error estimate and radius of convergence
3. Taylor's series & its applications

Review: Calculus I:

1. **limit** of a function
$$\lim_{x\to a} f(x) = L$$
$$\forall \epsilon > 0, \quad \exists \delta > 0,$$
$$s.t. \qquad |x-a| < \delta, \qquad |f(x) - L| < \epsilon$$
(Notations: $\forall$: for all, $\exists$: exists; Greek letters: ϵ: epsilon, δ: delta.)

2. continuity: A function is **continuous** at a number a if
$$\lim_{x\to a} f(x) = f(a)$$

3. differentiation: The **derivative** of a function $f(x)$
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

4. integration: limit of Riemann sum
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k) \triangle x_k$$
where
$$a = x_1 < x_2 < ... < x_n = b$$

5. Improper Integrals
$$\int_a^\infty f(x)dx = \lim_{R\to\infty}\int_a^R f(x)dx$$
If the limit exists, we say that the improper integral converges to a value L. If the limit does not exist, we say that the improper integral diverges: e.g.
$$\int_1^\infty \frac{1}{x^2}dx$$
and
$$\int_1^\infty \frac{1}{x^p}dx$$

Start of Calculus II: Sequences and series

1. Definitions: we use n, k and N to denote positive integers.
   (a) Sequence: a list of numbers in a definite order
   $$\{a_1, a_2, ...\} \qquad \{a_n\} \qquad \{a_n\}_{n=1}^\infty$$
   (b) Limit of a convergent sequence:
   $$\lim_{n\to\infty} a_n = L$$
   (c) Series: the sum of a sequence $\{a_n\}_{n=1}^\infty$
   $$s = \sum_{n=1}^\infty a_n$$
      i. Partial sum:
      $$s_n = \sum_{k=1}^n a_k$$
      ii. Remainder:
      $$R_n = s - s_n = \sum_{k=n+1}^\infty a_k$$
   (d) Limit of a convergent series
   $$\lim_{n\to\infty}\sum_{k=1}^n a_k = L$$
2. Sequences:
   (a) arithmetic sequence: common difference
   (b) geometric sequence: common ratio
   (c) harmonic sequence: $a_n = 1/n$
   (d) Fibonacci sequence $a_{n+2} = a_n + a_{n+1}$
   (e) alternating sequence: absolute convergence
3. Series:
   (a) geometric series
   (b) p-series
   (c) **Taylor's series: power series**
   (d) Fourier series: trigonometric series

4. **Growth rates of sequence** in order:

$$\ln n, \quad n, \quad n \cdot \ln n, \quad n^2, \quad a^n, \quad n! \quad n^n$$

These sequences all go to infinity.

5. Theorems about convergent sequences
   (a) Squeeze Theorem
   (b) Bounded monotonic sequence theorem:
      i. a bounded above monotonically increasing sequence converges;
      ii. a bounded below monotonically decreasing sequence converges.
      iii. a **bounded monotonic** sequence is convergent.
   (c) If a sequence converges to L, then every subsequence converges to L.

6. Test for series divergence
   (a) If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then

$$\lim_{n\to\infty} a_n = 0$$

   Warning! the **converse statement** is not true! counter example: $a_n = 1/n$.
   The contrapositive statement is true, which is the **divergence test**: if

$$\lim_{n\to\infty} a_n \neq 0$$

   then $\sum_{n=1}^{\infty} a_n$ is divergent.

7. **Test for series convergence**
   (a) Find the **exact sum** of the series:
      i. geometric series:

$$\begin{aligned} s &= a + ar + ar^2 + ... \\ &= a\sum_{n=0}^{\infty} r^n \\ &= \lim_{n\to\infty} a\frac{1-r^n}{1-r} \\ &= \begin{cases} \frac{a}{1-r} & r < 1 \\ \infty & r \geq 1 \end{cases} \end{aligned}$$

      ii. telescoping series, e.g.:

$$\begin{aligned} s &= \sum_{n=0}^{\infty} \frac{1}{n\,(n+1)} \\ &= \sum_{n=0}^{\infty} \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} \cdots \\ &= 1 \end{aligned}$$

   It is usually difficult to find the exact sum of series, so we develop several tests which enable us to determine the convergence without finding the exact sum.
   (b) **Integral Test**: f is a **continuous**, **positive**, **decreasing** function on $[1, \infty)$ and $f(n) = a_n$

$$\int_1^{\infty} f(x)dx \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n$$

   both converge or both diverge.

i. **p-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.
Since (*check 3 conditions*)

A. $f(x) = \frac{1}{x^p}$ is continuous on $[1, \infty)$,

B. $f(x) > 0$, positive

C. $f(x)' = -p\frac{1}{n^{p-1}} < 0$ decreasing

and

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{N\to\infty} \frac{1}{p-1}\left[1 - \frac{1}{x^{p-1}}\right]_{x=1}^{N} = \begin{cases} \frac{1}{p-1} & p > 1 \quad \text{converges} \\ \infty & p \leq 1 \quad \text{diverges} \end{cases}$$

which converges, therefore using integral test

$$\sum_{n=1}^{\infty} \frac{1}{x^n}$$

converges.
when $p = 1$, it is a harmonic series which is divergent.

(c) **Comparison Test**

i. Direct comparison test, e.g. $a_n = \frac{1}{n^2+1}$

$$\frac{1}{n^2+1} < \frac{1}{n^2}$$

Since $\sum \frac{1}{n^2}$ is convergent, the series $\sum \frac{1}{n^2+1}$ is convergent too.

ii. **Limit** comparison test, e.g. $a_n = \frac{1}{n^2-1}$ (we cannot use the direct comparison. However, $\frac{1}{n^2-1}$ and $\frac{1}{n^2}$ have almost the same behaviour at ∞ because the limit of their ratio is a finite constant.)

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = 1$$

Since $\sum \frac{1}{n^2}$ is convergent, the series $\sum \frac{1}{n^2-1}$ is convergent too.

(d) **Ratio Test**

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

i. $0 \leq r < 1$, converges

ii. $r > 1$, diverges

iii. $r = 1$, inconclusive (we cannot make a conclusion, we need another test!)

(e) **Root Test**

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

i. $L < 1$, **absolutely convergent**

ii. $L > 1$, divergent

iii. $L = 1$, inconclusive

(f) Absolute convergence $\sum |a_n|$ implies convergence $\sum a_n$.

(g) Alternating convergence Test. For an alternation series $s = \sum (-1)^n b_n$, where $b_n > 0$

$$\lim_{n\to\infty} b_n = 0$$

then the alternating series is convergent.

8. **Error estimate**

   (a) Remainder estimate for the **integral test and comparison test**. $R_n = s - s_n = \sum_{k=n+1}^{\infty} a_k$

   $$\int_{n+1}^{\infty} f(x)dx \le R_n \le \int_{n}^{\infty} f(x)dx$$

   This can be proved through the *areas under the curves*. Note: $f(x)$ is continuous, positive and decreasing.

   (b) **Alternating series estimation theorem**. If $s = \sum (-1)^n b_n$, where $b_n > 0$ satisfies

   $$b_{n+1} \le b_n \qquad \lim_{n\to\infty} b_n = 0$$

   then

   $$|R_n| = |s - s_n| \le b_{n+1}$$

   (c) Two types of questions:

      i. what is the error estimate for R_n [given n and **find the error** ϵ]

      ii. how many terms are required to ensure the sum is accurate/correct to within a value of ϵ. [given the error ϵ and **find the minimum** n]

9. **Which test should I use? (good question)**

   Quick answer: For each test, do as much exercise as you can. Then you can develop intuitions about which test you should choose.

   *Slow answer: (Read book chapter **11.7 Strategy for testing series**)*

   For **sequences**, it is easy to test:

   (a) if you think it is convergent, you can look for $\lim a_n$. If the limit exist (e.g. $a_n = 1+1/n$, $\lim_{n\to\infty} a_n = 1$,), then it is **convergent**.
   If the limit does not exist (e.g. $a_n = \sin n$) or goes to ∞ (e.g. $a_n = n^2$), this sequence is **divergent**.

   (b) If you think it is **divergent**, you can look for two **subsequences** which does **NOT** converge to the same limit. e.g. $a_n = (-1)^n$, with

   $$\lim_{n\to\infty} a_{2n} = 1, \qquad \lim_{n\to\infty} a_{2n+1} = -1$$

   (c) We can apply the **Squeeze Theorem**, if you think it is convergent and it is not too hard to find two sequences (or **one sequence** and one **constant** L which equals to the limit) such that one is **smaller** and one is **larger** and both converge to the same value L. e.g. $a_n = \sin n/n^2$, or $\ln n/n^2$. Since

   $$0 \le \frac{\ln n}{n^2} \le \frac{1}{n} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1}{n} = 0$$

   So this $a_n = \ln n/n^2$ is **convergent**.

   For series, we should first be familiar with conditions for geometric series and p-series to be convergent. Let's look at positive series in which each $a_n \ge 0$.

   (a) [**Test for divergent**] If the $\lim a_n \ne 0$, it is easy, the series $\sum a_n$ is going to explode (divergent) as you are keep adding non-zero values to the series.
   If the $\lim a_n = 0$, some series can also be divergent, e.g. p-series when $p \le 1$. One example is the harmonic series $a_n = 1/n$, however $\sum 1/n$ goes to ∞ at *logarithm* speed.

   (b) [**Integral test**] to write $f(n) = a_n$ and check if $f(x)$ satisfies these 3 conditions:
   1, $f(n)$ is continuous on $[1, \infty)$, 2. $f(n) \ge 0$, 3. $f(x)$ is decreasing. If it satisfies then we determine the convergence of the series by the convergence of the improper integral $\int_1^\infty f(x)dx$.e.g. it is hard to evaluate $\sum 1/n^3$, but we know $\int_1^\infty 1/x^3 dx$ converges, then the series converges too. To do the integration, you might need to use some integration techniques, such as *change of variables*, *integration by parts*. Sometimes, you probably cannot find the integal easily, then you should consider applying the comparison.

(c) [**Comparison test**] This is an easy test, if you can construct inequality using geometric series or p-series. e.g. We can compare with a geometric series

$$s = \sum_{n=2}^{\infty} \frac{2^n}{3^n - n}$$

we can compare such as series with a geometric series

$$\frac{2^n}{3^n - n} < \left(\frac{2}{3}\right)^n$$

so

$$s = \sum_{n=2}^{\infty} \frac{2^n}{3^n - n} < \sum_{n=2}^{\infty} \left(\frac{2}{3}\right)^n \qquad \text{which is convergent}$$

therefore, s is convergent.
Another example: compare with a p-series

$$s = \sum_{n=2}^{\infty} \frac{2n}{n^3 + n^2}$$

Since

$$\frac{2n}{n^3 + n^2} < \frac{2n}{n^3} = \frac{2}{n^2}$$

so

$$s = \sum_{n=2}^{\infty} \frac{2n}{n^3 + n^2} < \sum_{n=2}^{\infty} \frac{2}{n^2} \qquad \text{which is convergent}$$

therefore, s is convergent.

(d) [**Alternating Test**] The above are all about positive series. For alternating series, the test is easier. You will just need to check the limit of a_n. If

$$\lim_{n \to \infty} a_n = 0$$

then the alternating series is convergent.e.g. the alternating harmonic series

$$s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$$

is **convergent**.

(e) [**Ratio Test**] To test the series where the ratio between two items is easy to obtain, such as exponenetial and factorial functions, e.g.The convergent series

$$a_n = \frac{2^n}{n!}$$

It is easy to compute the ratio

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1} < 1$$

(f) [**Root Test**] To test the series where the root of the item is easy to obtain. e.g.

$$a_n = \left(\frac{2n+1}{3n+1}\right)^n$$