2023-24 First Semester

MATH2023 Ordinary and Partial Differential Equations (1002)

Assignment 4

Due Date: **23/Oct/2023(Monday), on or before 10:00, in lecture.**

- Write down your **CHN name** and **student ID**. Write neatly on **A4-sized** paper (*staple if necessary*) and **show your steps**.
- For online students, hand in your homework in **one pdf file** on iSpace.
- **Late submissions or answers without details will not be graded.**

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1. Find the general solution of the given differential equations.

   (a) $y'' - 2y' + 2y = 0$.

   (b) $4y'' + 9y = 0$.

   (c) $y'' + 5y' + 6.25y = 0$.

2. Consider the initial value problem

   $$2y'' + 3y' - 2y = 0, \quad y(0) = 1, \; y'(0) = -\beta, \qquad \text{with } \beta > 0.$$

   (a) Solve the initial value problem.

   (b) Determine the smallest value of β for which the solution has no minimum point.

3. Find a second linearly independent solution of

   $$x^2y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0,$$

   where one solution is given as $y_1(x) = x^{-1/2}\sin x$.

4. Determine the **form** of a particular solution to the following non-homogeneous equations, **do not attempt to solve them**.

   (a) $y'' - 10y' + 34y = te^{5t}\sin(3t) + t^3$.

   (b) $y'' - 3y' = 2t^4 + t^2e^{3t} + \sin 3$.

   (c) $y'' - 4y' + 4y = \cos t + 4t^2e^{2t} + te^t \sin 2t$.

5. Find the solution of the given initial value problem.

(a) $y'' + 4y = t^2 + 3e^t, \quad y(0) = \dfrac{7}{5}, \quad y'(0) = \dfrac{3}{5};$

(b) $y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1.$

⋆ **Change of Variables:**

6. Consider the following differential equation

$$x^2 y'' + 2xy' - 6y = 0, \quad x > 0.$$

Define a new independent variable $t = \ln x$, then $x = e^t$. Denote $u(t) = y(e^t) = y(x)$.

(a) Re-write the differential equation in terms of u and t only,

(b) Solve the equation for u first and then find out $y(x)$ by substitution.

⋆ **Reduction of order by changing variables for special equations:**

7. (Equations with Dependent Variable y Missing.) Solve the following ODE.

$$t^2 y'' + 2ty' - 1 = 0, \qquad t > 0.$$

Guides: For a 2$^{\text{nd}}$ order DE of the form $y'' = f(t, y')$, by letting $v = y'$, we can reduce the order of the equation and make it into $\frac{\mathrm{d}v}{\mathrm{d}t} = f(t, v)$:
Step 1. Let $v = y'$, then $\dfrac{\mathrm{d}v}{\mathrm{d}t} = y''$.
Step 2. Then the original DE becomes a first order equation of the form $v' = f(t, v)$.
Step 3. Solve for v first, then solve for y.

8. (Equations with Independent Variable x Missing.) Solve the following ODE.

$$y'' + y(y')^3 = 0.$$

Guides: Consider a 2$^{\text{nd}}$ order DE of the form $y'' = f(y, y')$. We can use change of variables to transform the equation into $\frac{\mathrm{d}v}{\mathrm{d}y} = f(v, y)$:
Let $v = \mathrm{d}y/\mathrm{d}t$ and think of y as the independent variable, then by the chain rule $\mathrm{d}v/\mathrm{d}t = (\mathrm{d}v/\mathrm{d}y)(\mathrm{d}y/\mathrm{d}t) = v(\mathrm{d}v/\mathrm{d}y)$. Hence the original DE can be written as $v(\mathrm{d}v/\mathrm{d}y) = f(y, v)$, which is a first order equation. Solve v as a function of y, then solve y as a function of t.