2022-23 Second Semester
MATH1083 Calculus II Quiz Two (1002&1003)

Time: 9:00-10:00pm 4/May/2023 (Thu) Venue: T3-201 **Total score 100 pts**

1. **[20 pts]** Find the partial derivatives $f_x(x,y)$ and $f_y(x,y)$, for functions

   (a)
   $$f(x,y) = x\cos(2y)$$

   (b)
   $$f(x,y) = e^{2x}\cdot\ln\left(x^2y\right)$$

   Solution:(a)
   $$\frac{\partial f}{\partial x} = \cos(2y)$$
   $$\frac{\partial f}{\partial y} = -2x\sin(2y)$$

   (b)
   $$\frac{\partial f}{\partial x} = 2e^{2x}\ln\left(x^2y\right) + e^{2x}\frac{2}{x}$$
   $$\frac{\partial f}{\partial y} = e^{2x}\frac{1}{y}$$

2. **[25pts]** Let
   $$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

   (a) Determine whether f is continuous at $(0,0)$ and justify your answer. (Hint: to investigate $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$)

   (b) When $(x,y) \neq (0,0)$, find the partial derivatives $f_x(x,y)$ and $f_y(x,y)$

   (c) At $(x,y) = (0,0)$, find the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ by definition.

   Solution (a) **(10pts)**
   $$0 \leq \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = |x|\left|\frac{y}{\sqrt{x^2+y^2}}\right| \leq |x| \to 0$$

   the limit exist by **Squeeze Theorem**
   $$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0 = f(0,0)$$

   So $f(x,y)$ is continuous at $(0,0)$
   (b) **(10pts)** When $(x,y) \neq (0,0)$, applying Quotient Rule
   $$f_x(x,y) = \frac{\sqrt{x^2+y^2}\cdot y - \frac{x}{\sqrt{x^2+y^2}}\cdot xy}{x^2+y^2} = \frac{y^3}{(x^2+y^2)^{3/2}}$$

and

$$f_y(x,y) = \frac{\sqrt{x^2+y^2}\cdot x - \frac{y}{\sqrt{x^2+y^2}}\cdot xy}{x^2+y^2} = \frac{x^3}{(x^2+y^2)^{3/2}}$$

(c)**(5pts)** Since $(0,0)$ is not in the domain, so we have to find the partial derivatives by definition. $f(h,0)=0$, $f(0,k)=0$ and $f(0,0)=0$

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{0-0}{k} = 0$$

3. **[15pts]** Find the equation of tangent plane to the surface by the function $f(x,y)=\sin(2x)\cos(4y)$ at the point $\left(\pi/8, \pi/16, \frac{1}{2}\right)$.

   Solution: first find the partial derivatives:

$$f_x(x,y) = 2\cos(2x)\cos(4y) \qquad f_y(x,y) = -4\sin(2x)\sin(4y)$$

   then

$$\begin{aligned} f\left(\frac{\pi}{8},\frac{\pi}{16}\right) &= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = \frac{1}{2} \\ f_x\left(\frac{\pi}{8},\frac{\pi}{16}\right) &= 2\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = 1 \\ f_y\left(\frac{\pi}{8},\frac{\pi}{16}\right) &= -4\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) = -2 \end{aligned}$$

   so the equation of the tangent surface at the point is

$$\begin{aligned} z &= f\left(\frac{\pi}{8},\frac{\pi}{16}\right) + f_x\left(\frac{\pi}{8},\frac{\pi}{16}\right)\left(x-\frac{\pi}{8}\right) + f_y\left(\frac{\pi}{8},\frac{\pi}{16}\right)\left(y-\frac{\pi}{16}\right) \\ &= \frac{1}{2} + \left(x-\frac{\pi}{8}\right) - 2\left(y-\frac{\pi}{16}\right) \\ &= \frac{1}{2} + x - 2y \end{aligned}$$

4. **[20pts]** If $z=f(x,y)=\frac{1}{x}+\ln y$ where $x(s,t)=e^s\cos t$ and $y(s,t)=e^{-s}\sin t$, use Chain Rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ when $s=0$ and $t=\pi/4$

   Solution: when $s=0$, $t=\pi/4$, $x=e^0\cos\pi/4=\sqrt{2}/2$ and $y=e^0\sin\pi/4=\sqrt{2}/2$

   (1pt each: 6 pts)

$$\frac{\partial z}{\partial x} = -\frac{1}{x^2} \qquad \frac{\partial z}{\partial y} = \frac{1}{y}$$

$$\begin{aligned} \frac{\partial x}{\partial s} &= e^s\cos t, & \frac{\partial y}{\partial s} &= -e^{-s}\sin t \\ \frac{\partial x}{\partial t} &= -e^s\sin t, & \frac{\partial y}{\partial t} &= e^{-s}\cos t \end{aligned}$$

(4pts for each chain rule formula, 3pts for each correct substitution)

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= -\frac{1}{x^2}e^s\cos t - \frac{1}{y}e^{-s}\sin t \\ &= -\sqrt{2} - 1\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= \frac{1}{x^2}e^s\sin t + \frac{1}{y}e^{-s}\sin t \\ &= \sqrt{2} + 1\end{aligned}$$

5. **[20pts]** Use Lagrange multipliers to find the **maximum and minimum** values of

$$f(x,y) = x^3 + y^3$$

subject to

$$\frac{1}{x^3} + \frac{1}{y^3} = \frac{1}{4}$$

Solution: [**There is a problem with this question: I forgot to mention the domain for x and y.** In this case f has either absolute maximum or minimum! 16 is a just local minimum of f. If I add one condition $x > 0 \quad y > 0$, then the min is 16 at $(2,2)$.]

$$\nabla f = \left\langle 3x^2, 3y^2 \right\rangle$$

Let

$$g(x,y) = \frac{1}{x^3} + \frac{1}{y^3} - \frac{1}{4} = 0$$

$$\nabla g = \left\langle -3\frac{1}{x^4}, -3\frac{1}{y^4} \right\rangle$$

so we need to solve the system of equations:

$$\begin{cases} 3x^2 = -2\lambda \frac{1}{x^4} & (1) \\ 3y^2 = -2\lambda \frac{1}{y^4} & (2) \\ \frac{1}{x^3} + \frac{1}{y^3} = \frac{1}{4} & (3) \end{cases}$$

from Equation (1) and Equation (2), we have

$$\lambda = -\frac{3}{2}x^6 \qquad \lambda = -\frac{3}{2}y^6$$

so we have $x = \pm y$ and from Equation (3) we get the solution

$$x = 2 \qquad y = 2$$

and therefore $\lambda = -96$, so

$$f(2,2) = 16$$

16 is only the local minimum since $D(2,2) = f_{xx}f_{yy} - f_{xy}^2 = 36xy > 0$. Both the absolute max and min do not exist.