PT

Solution to Assignment 2

1. Clearly, the conditions (1) and (2) in the definition of a σ-algebra are satisfied. We now show that $\mathcal{A}$ is closed under countable union. Let $A_1, A_2, \ldots$ be in $\mathcal{A}$. We distinguish between the following cases:

   (a) There exists k such that $A_k = \Omega$. In this case, $\cup_{i=1}^{\infty} A_i = \Omega \in \mathcal{A}$.

   (b) There exists no k such that $A_k = \Omega$, but there are i and j such that $A_i = A$ and $A_j = A^c$. In this case, $\cup_{i=1}^{\infty} A_i = \Omega \in \mathcal{A}$.

   (c) (a) and (b) don't hold, but there is i such that $A_i = A$. In this case, $\cup_{i=1}^{\infty} A_i = A \in \mathcal{A}$.

   (d) (a) and (b) don't hold, but there is j such that $A_j = A^c$. In this case, $\cup_{i=1}^{\infty} A_i = A^c \in \mathcal{A}$.

   (e) All the above cases don't apply, that is, $A_i = \emptyset$ for all i. In this case, $\cup_{i=1}^{\infty} A_i = \emptyset \in \mathcal{A}$.

   So $\mathcal{A}$ is a σ-algebra.

2. We have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{10} = \frac{11}{15}$$

$$P\left(A^c \cap B^c\right) = P\left((A \cup B)^c\right) = 1 - P(A \cup B) = 1 - \frac{11}{15} = \frac{4}{15}$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{10} - \frac{1}{15} - \frac{1}{20} + \frac{1}{30} = \frac{17}{20}$$

$$P\left(A^c \cap B^c \cap C^c\right) = P\left((A \cup B \cup C)^c\right) = 1 - P(A \cup B \cup C) = \frac{3}{20}$$

$$P\left(A^c \cap B^c \cap C\right) = P\left((A \cup B)^c \cap C\right) = P(C \backslash (A \cup B))$$

$$= P(C) - P(C \cap (A \cup B))$$

$$= P(C) - (P(C) + P(A \cup B) - P(C \cup A \cup B))$$

$$= P(A \cup B \cup C) - P(A \cup B)$$

$$= \frac{17}{20} - \frac{11}{15} = \frac{7}{60}$$

$$P\left(\left(A^c \cap B^c\right) \cup C\right) = P\left(A^c \cap B^c\right) + P(C) - P\left(\left(A^c \cap B^c\right) \cap C\right)$$

$$= \frac{4}{15} + \frac{1}{5} - \frac{7}{60} = \frac{7}{20}.$$

3.

(a) $\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 12!}{17!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{17 \cdot 16 \cdot 15 \cdot 14 \cdot 13} = \frac{2}{221}$.

(b) Because there are 9 nonblue balls, the probability is $\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 12!}{17!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{17 \cdot 16 \cdot 15 \cdot 14 \cdot 13} = \frac{9}{442}$.

(c) Because there are 3! possible orderings of the different colors and all possibilities for the final 3 balls are equally likely, the probability is $\frac{3! \cdot 4 \cdot 8 \cdot 5 \cdot 14!}{17!} = \frac{3! \cdot 4 \cdot 8 \cdot 5}{17 \cdot 16 \cdot 15} = \frac{4}{17}$.

(d) The probability that the red balls are in a specified 4 spots is $\frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot 13!}{17!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{17 \cdot 16 \cdot 15 \cdot 14}$. Because there are 14 possible locations of the red balls where they are all together, the probability is $\frac{14 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{17 \cdot 16 \cdot 15 \cdot 14} = \frac{1}{170}$.

4.

(a) $P(S \cup F \cup G) = (28 + 26 + 16 - 12 - 4 - 6 + 2)/100 = 1/2$ The desired probability is $1 - 1/2 = 1/2$.

(b) Use the Venn diagram below to obtain the answer 32/100.

(c) since 50 students are not taking any of the courses, the probability that neither one is taking a course is $\binom{50}{2} / \binom{100}{2} = 49/198$ and so the probability that at least one is taking a course is 149/198.

5.

(a) $\binom{4}{2} / \binom{52}{2} \approx .0045$

(b) $13 \binom{4}{2} / \binom{52}{2} = 1/17 \approx .0588$

6.

(a) $\binom{7}{5} / \binom{10}{5} = 1/12 \approx .0833$

(b) $\binom{7}{4}\binom{3}{1}/\binom{10}{5} + 1/12 = 1/2$

7.

(a) $\dfrac{3 \cdot 4 \cdot 4 \cdot 3}{\binom{14}{4}} = .1439$

(b) $\dfrac{\binom{4}{2}\binom{4}{2}}{\binom{14}{4}} = .0360$

(c) $\dfrac{\binom{8}{4}}{\binom{14}{4}} = .0699$

8. The complement is the union of the three events $A_i = \{$ couple i sits together $\}, i = 1, 2, 3$. Moreover,

$$P(A_1) = \frac{2}{5} = P(A_2) = P(A_3),$$
$$P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_2 \cap A_3) = \frac{3! \cdot 2! \cdot 2!}{5!} = \frac{1}{5},$$
$$P(A_1 \cap A_2 \cap A_3) = \frac{2! \cdot 2! \cdot 2! \cdot 2!}{5!} = \frac{2}{15}.$$

For $P(A_1 \cap A_2)$, for example, pick a seat for husband_3. In the remaining row of 5 seats, pick the ordering for couple 1, couple 2, and wife_3, then the ordering of seats within each of couple 1 and couple 2. Now, by inclusion-exclusion,

$$P(A_1 \cup A_2 \cup A_3) = 3 \cdot \frac{2}{5} - 3 \cdot \frac{1}{5} + \frac{2}{15} = \frac{11}{15},$$

and our answer is $\frac{4}{15}$.