ASP Assignment 4 Solution

1. Since

$$\boldsymbol{P}^3 = \begin{bmatrix} \frac{107}{216} & \frac{109}{216} \\ \frac{109}{192} & \frac{83}{192} \end{bmatrix},$$

so $P[X_3 = 1 | X_0 = 0] = \frac{109}{216}$.

2.

(a). $P(A_k)$.

for $k=1$, $P(A_k)=0$.

for $k \geq 2$, $P(A_k) = P_{0,3} \cdot P_{33}^{k-2} P_{32} = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)^{k-2}\left(\frac{1}{4}\right) = \frac{1}{3}\left(\frac{1}{4}\right)^{k-1}$

(b). $P(B) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \frac{1}{4^n}\right)\frac{1}{4}$

$= \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{1-\frac{1}{4}} = \frac{2}{3} + \frac{1}{9} = \frac{7}{9}$

(c). $P(C) = P_{03}\left[\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots \frac{1}{4^n}\right) \cdot \frac{1}{4}\right] P_{21} = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{18}$

(d) $P(D) = P_{03} P_{32} P_{21} = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{1}{24}$

(e). $P(E) = P_{03} P_{33}^{N-1} = \frac{1}{3}\left(\frac{1}{4}\right)^{N-1}$

3.

- (a) The transition matrix is

$$\begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

(b) It is clear that all states communicate with each other, so this chain is irreducible.

(c) Clearly P[the rat starts at 1 and reaches 4 in less than 5 steps] $= f_{1,4}^1 + f_{1,4}^2 + f_{1,4}^3 + f_{1,4}^4$. Note that $f_{1,4}^1 = \frac{1}{3}$, $f_{1,4}^2 = \frac{1}{3} \times \frac{1}{2} \times 2$, $f_{1,4}^3 = \frac{1}{3} \times \frac{1}{2} \times \frac{1}{3} \times 2$. Moreover, the routes of the rat starting at 1 and first getting to state 4 in four steps are

$$1 \to 2 \to 1 \to 2 \to 4, 1 \to 2 \to 1 \to 3 \to 4, 1 \to 3 \to 1 \to 3 \to 4, 1 \to 3 \to 1 \to 2 \to 4.$$

So $f_{1,4}^1 + f_{1,4}^2 + f_{1,4}^3 + f_{1,4}^4 = \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} \times 2 + \frac{1}{3} \times \frac{1}{2} \times \frac{1}{3} \times 2 + \frac{1}{3} \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} \times 4 = \frac{8}{9}$.

4. Note that

$$\boldsymbol{P}^n = \begin{pmatrix} 0.2 & 0.3 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.3 \end{pmatrix} + (-\frac{1}{3})^n \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0.5 \end{pmatrix} + (-\frac{2}{3})^n \begin{pmatrix} 0.3 & -0.3 & 0.3 & -0.3 \\ -0.2 & 0.2 & -0.2 & 0.2 \\ 0.3 & -0.3 & 0.3 & -0.3 \\ -0.2 & 0.2 & -0.2 & 0.2 \end{pmatrix}.$$

So

$$\begin{aligned} \sum_{n=0}^{\infty} P_{1,1}^n &= \sum_{n=0}^{\infty} \left(0.2 + 0.3 \cdot (-\frac{2}{3})^n\right) \\ &= \sum_{n=0}^{\infty} 0.2 + 0.3 \sum_{n=0}^{\infty} (-\frac{2}{3})^n \\ &= \infty + 0.3 \cdot \frac{1}{1 - (-\frac{2}{3})} = \infty, \end{aligned}$$

which implies that 1 is recurrent.

5.

(a) The chain is irreducible.

(b) If $R > n$, then the chain, after moving to 1 , makes $n-1$ consecutive steps to the right, so

$$P(R > n) = \prod_{i=1}^{n-1} \left(1 - \frac{1}{2 \cdot i^{\alpha}}\right)$$

The product converges to 0 if and only if its logarithm converges to $-\infty$ and that holds if and only if the series

$$\sum_{i=1}^{\infty} \frac{1}{2 \cdot i^{\alpha}}$$

diverges, which is when $\alpha \leq 1$.