Caculus II Math 1038 (1002&1003)

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Week 11: Ch15 Multiple integrals

1. Double integral over rectangles

   (a) a closed rectangle $R = [a, b] \times [c, d]$

   (b) partition: subrectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, with base area $\Delta A_{ij} = \Delta x_i \cdot \Delta y_j$

   (c) sample point $\left(x_{ij}^*, y_{ij}^*\right)$ in each subrectangle: average **height** of each subregion $f\left(x_{ij}^*, y_{ij}^*\right)$

   (d) volume of the solid that lies under the graph f and above R using **double sum** (double Riemann sum)

   $$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A_{ij}$$

   write as **double integral**

   $$V = \iint_R f(x, y) dA$$

   (e) Fubini's Theorem: change of order of integration over a **rectangular region**.

   $$\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

   This works only for rectangular region!!! For other general region, we need to make changes for the boundaries.

2. double integrals over general regions

   (a) properties and applications

      i. area of the region

      $$\iint_D 1 dA = A(D)$$

      ii. volume of a solid with area density $\rho(x, y) = f(x, y)$

      $$\iint_D f(x, y) dA$$

      iii. expected value of a random variable in a joint distribution

      iv. union of the region if $D = D_1 \cup D_2$

      $$\iint_D f(x, y) dA = \iint_{D_1} f(x, y) dA + \iint_{D_2} f(x, y) dA$$

   (b) **Type I**: region between two continuous functions of x

   $$D = \left\{(x, y) \middle| a \le x \le b, g_1(x) \le y \le g_2(x)\right\}$$

   $$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

   Example 1 (from the text book): evaluate

   $$\iint_D (x + 2y)\, dA$$

where D is the region bounded by parabolas $y = 2x^2$ and $y = 1 + x^2$.
Solution:
Step 1: find the intersections of the two parabolas by letting

$$2x^2 = 1 + x^2$$

so $x = \pm 1$ and $y = 2$
Step 2: sketch the region D and determine the order of integration. (slicing vertically!)
Step 3: write the double integral as an iterated integral

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dydx$$

Step 4: compute the integral.
Remark: If you treat D as a Type II region and slice it **horizontally**, you will end up with two disjoint pieces when $y > 1$, making it is harder to integrate.

(c) **Type II**: region between two continuous functions of y

$$D = \{(x, y) | h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$$

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

Example 2: evaluate

$$\iint_D xydA$$

where D is the region bounded by the line $y = x - 1$ and parabola $y^2 = 2x + 6$.
Solution:
Step I: find the intersections

$$(x - 1)^2 = 2x + 6$$

so $x = -1$ $y = -2$ and $x = 5$ $y = 4$
Step 2: express D as type II region:

$$D = \left\{(x, y) | \frac{y^2 - 6}{2} \leq x \leq y + 1, -2 \leq y \leq 4\right\}$$

Step 3:

$$\begin{aligned}
\iint_D xydA &= \int_{-2}^{4} \int_{\frac{y^2-6}{2}}^{y+1} xydxdy \\
&= \int_{-2}^{4} \left[\frac{x^2}{2} y\right]_{x=\frac{y^2-6}{2}}^{x=y+1} dy \\
&= \int_{-2}^{4} \left[\frac{(y+1)^2}{2} y - \frac{1}{2}\left(\frac{y^2-6}{2}\right)^2 y\right] dy \\
&= \cdots
\end{aligned}$$

Remark:if we express D as a type I region, we can find the parobola intersect x-axis at $(-3, 0)$ then

$$\iint_D xydA = \int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xydydx + \int_{-1}^{5} \int_{x-1}^{\sqrt{2x+6}} xydydx$$

(d) **Region D descriptions**

i. D is a set

ii. D is bounded by several curves, e.g. lines, parabolas.

(e) **change the order of integration**

i. evaluation of iterated integral using suitable order: sometime one order is **more difficult** or even **impossible**.
ii. sketch the region D
iii. describe the region as boundaries of x and y

3. Polar coordinates

(a) pole, polar axis

(b) infinitesimal "polar rectangle", area element: $dsdr = rd\theta dr$

(c) simple example, the area of an unit circle

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1dydx = \int_0^{2\pi}\int_0^1 1rdrd\theta$$
$$= \int_0^{2\pi} \frac{1}{2} d\theta = \pi$$

(d) Examples

4. Surface Area

(a) S is a surface with equation $z = f(x, y)$

(b) area of a parallelogram $\Delta T_{ij} = |a \times b|$

$$\Delta T = |a \times b| = \sqrt{1 + f_x^2 + f_y^2}\Delta x \Delta y$$

(c) area of the surface, where f_x and f_y are continuou

$$A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2} dA$$

or

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

5. Triple integrals

(a) rectangular boxes:

$$B = \{(x, y, z) | a \le \boldsymbol{x} \le b, c \le \boldsymbol{y} \le d, r \le \boldsymbol{z} \le s\}$$

sub-box:

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

volume element: $\Delta V = \Delta x \Delta y \Delta z$

(b) triple Riemann sum and triple integral

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right)\Delta V = \iiint_B f(x, y, z) dV$$

(c) general bounded region E: Type I

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \le \boldsymbol{z} \le u_2(x, y)\}$$

Type II

$$E = \{(x, y, z) | (y, z) \in D, u_1(y, z) \le \boldsymbol{x} \le u_2(y, z)\}$$

Type III

$$E = \{(x, y, z) | (x, z) \in D, u_1(x, z) \le y \le u_2(x, z)\}$$

(d) Change of the order of integration: Fubini's Thoerem

(e) applications:

i. volume of a solid:

$$\iiint_E 1dV = V(E)$$

ii. "hypervolume"

(f) Fubini's Theorem: change of order of integration over a **rectangular region**. If f is continuous on the rectangular box, then

$$\iiint_B f(x,y,z)dV = \int_a^b \int_c^d \int_r^s f(x,y,)dydxdz = \int_a^b \int_r^s \int_c^d f(x,y,)dydzdx = \cdots$$

This works only for rectangular box!!! For other general region, we need to make changes for the boundaries.

6. triple integrals over general regions

(a) A region $D \subset \mathbb{R}^3$

(b) D is bounded above by a surface $z = H(x,y)$ and below by a surface $z = G(x,y)$, and region $R \subset \mathbb{R}^2$ is Type I region

$$D = \{(x,y,z) \big| (x,y) \in R, H(x,y) \le z \le G(x,y)\}$$

$$\iiint_D f(x,y,z)dV = \iint_R \left[\int_{G(x,y)}^{H(x,y)} f(x,y,z)dz\right] dA = \int_a^b \int_{g(x)}^{h(x)} \left[\int_{G(x,y)}^{H(x,y)} f(x,y,z)dz\right] dydx$$

i. step 1: integrate with respect to z from $z = G(x,y)$ to $z = H(x,y)$, (z is disappeared)

ii. step 2: integrate with resepct to y from $y = g(x)$ to $y = h(x)$ (y is disappeared)

iii. step 3: integrate with respect to x from $x = a$ to $x = b$

7. cylindrical coordinates (r, θ, z):

(a) polar coordinate $(r, \theta)+$ height z

(b) Equations in cylindrical coordinate:

i. cylinder: $\{(r,\theta,z) : r = a, a > 0\}$

ii. vertical half plane $\{(r,\theta,z) : \theta = \theta_0\}$

iii. horizontal plane $\{(r,\theta,z) : z = a\}$

iv. cone: $\{(r,\theta,z) : z = ar, a \neq 0\}$

(c) volume of the wedge: $\Delta V = r\Delta r \cdot \Delta\theta \cdot \Delta z$ where $r\Delta r \cdot \Delta\theta$ is the area of the base polar rectangle and Δz is the height.

(d) triple integral over the region

$$D = \{(r,\theta,z) \big| g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta, H(x,y) \le z \le G(x,y)\}$$

$$\iiint_D f(x,y,z)dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \left[\int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)dz\right] drd\theta$$

8. Spherical coordinates $P = (\rho, \phi, \theta)$

(a) ρ: distance from the origin to P

(b) ϕ: angle between positive z-axis and line OP

(c) θ: angle between the projection of OP and x-axis

(d) $\mathbb{R}^3 = \{(\rho,\phi,\theta) \,| 0 \le \rho < \infty, 0 \le \phi \le \pi, 0 \le \theta \le 2\pi\}$

(e) Transformation:

$$\rho = x^2 + y^2 + z^2$$
$$\tan\theta = \frac{y}{x}$$
$$\sin\phi = \frac{z}{\rho} \qquad \text{or} \qquad \tan\phi = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

spherical to cartesian coordinates:

$$x = r\cos\theta = \rho\sin\phi\cos\theta$$
$$y = r\sin\theta = \rho\sin\phi\sin\theta$$
$$z = r\cos\phi$$

(f) Equations:

i. sphere with radius a center at origin: $\{(\rho,\phi,\theta)\,|\rho = a\}$
ii. sphere with radius a center at $(0,0,a)$: $\{(\rho,\phi,\theta)\,|2a\cos\phi = \rho\}$
iii. cone, rotate about z-axis: $\{(\rho,\phi,\theta)\,|\phi = \phi_0\}$
iv. vertical half plane: $\{(\rho,\phi,\theta)\,|\theta = \theta_0\}$
v. horizontal plane $z = a$: $\left\{(\rho,\phi,\theta)\,|\rho\cos\phi = a, 0 \le \phi \le \frac{\pi}{2}\right\}$
vi. cylinder $\{(\rho,\phi,\theta)\,|\rho\sin\phi = a, 0 \le \phi \le \pi\}$

(g) volume of "spherical box":

$$\Delta V = \rho^2\sin\phi\Delta\rho\Delta\theta\Delta\phi$$