Section 3.4 Increasing and Decreasing Functions (函数的上升和下降)

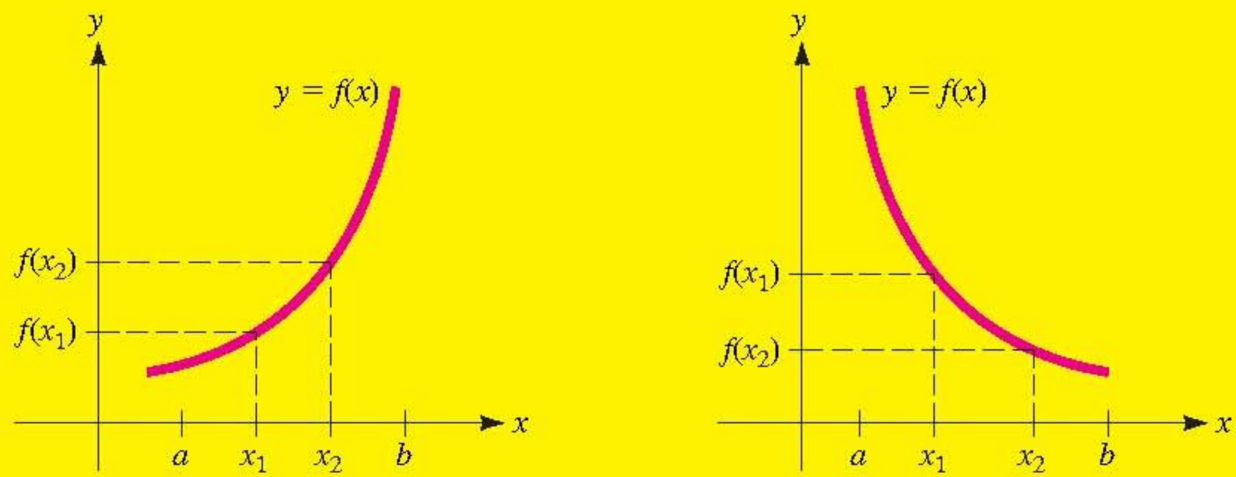
DEFINITION 4.1

A function f is (**strictly**) **increasing** on an interval I if for every $x_1, x_2 \in I$ with $x_1 < x_2$, $f(x_1) < f(x_2)$ [i.e., $f(x)$ gets larger as x gets larger].

A function f is (**strictly**) **decreasing** on the interval I if for every $x_1, x_2 \in I$ with $x_1 < x_2$, $f(x_1) > f(x_2)$ [i.e., $f(x)$ gets smaller as x gets larger].

Monotonic Increasing

Monotonic Decreasing



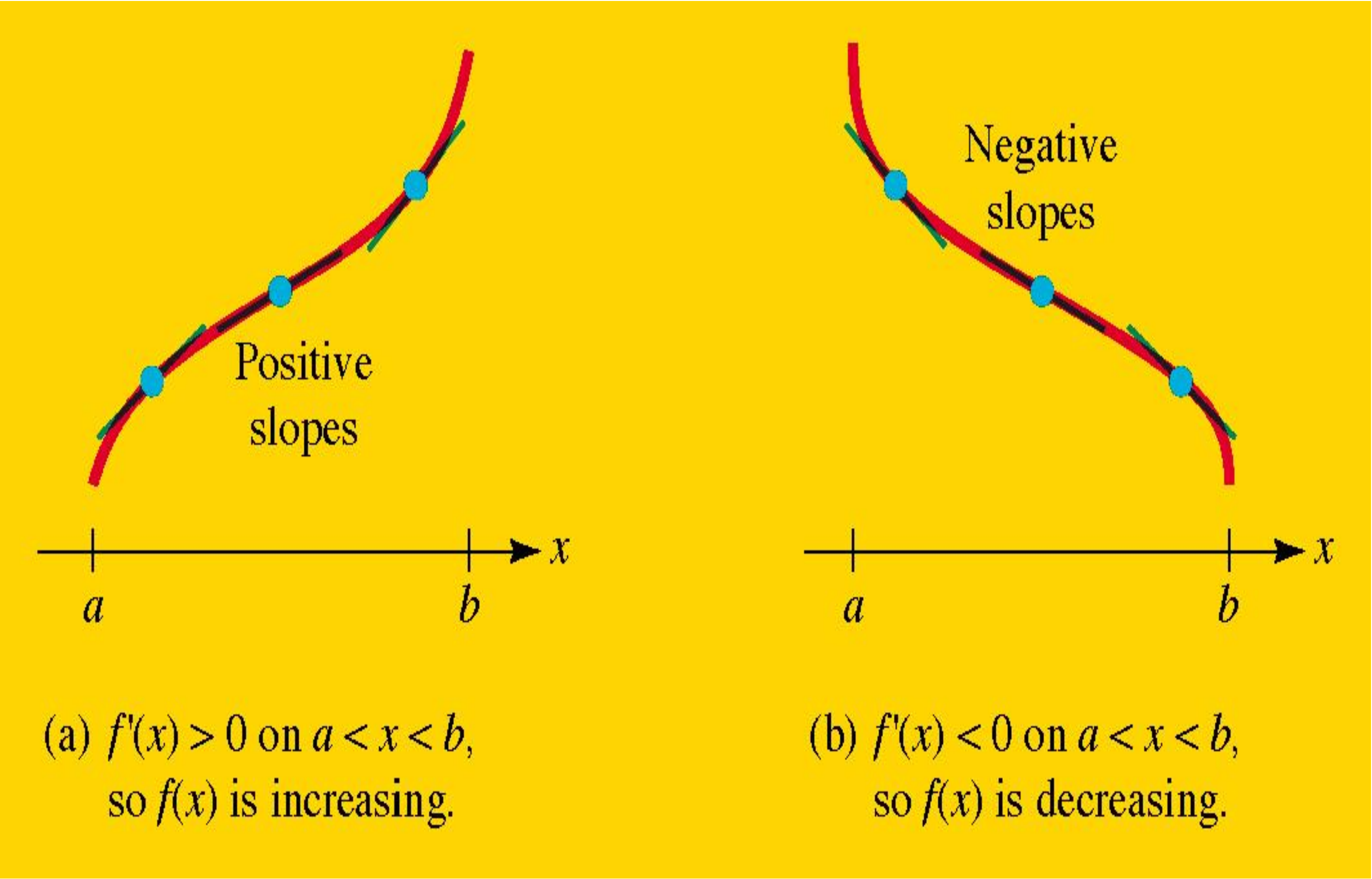
(a) $f(x)$ is increasing on $a < x < b$

(b) $f(x)$ is decreasing on $a < x < b$

Tangent line with positive slope ⟹ *f(x)* will be increasing

⇕

$f'(x) > 0$



$f'(x) < 0$

⇕

Tangent line with negative slope ⟹ *f(x)* will be decreasing

THEOREM 4.1

Suppose that f is differentiable on an interval I.

(i) If $f'(x) > 0$ for all $x \in I$, then f is increasing on I.
(ii) If $f'(x) < 0$ for all $x \in I$, then f is decreasing on I.

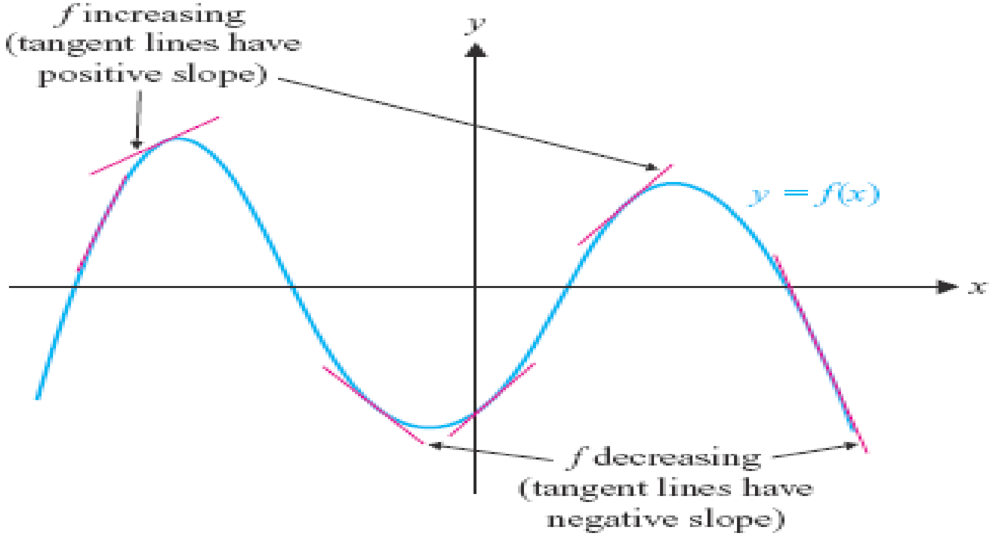


FIGURE 3.43
Increasing and decreasing

How to determine all intervals of increase and decrease for a function ? $\Longleftrightarrow$ How to find all intervals on which the sign of the derivative does not change.

Intermediate value property

A continuous function cannot change sign without first becoming 0.

Procedure for using the derivative to determine intervals of increase and decrease for a function of *f*.

Step 1. Find all values of x for which $f'(x) = 0$ or $\boldsymbol{f'(x)}$ is not defined, and mark these numbers on a number line. This divides the line into a number of open intervals.

Step 2. Choose a test number c from each interval $a<x<b$ determined in the step *1* and evaluate $\boldsymbol{f'(c)}$. Then

If $f'(c)>0$ the function $f(x)$ is increasing on $a<x<b$.

If $f'(c)<0$ the function $f(x)$ is decreasing on $a<x<b$.

Example

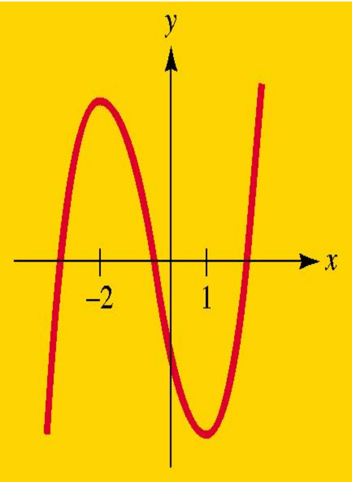
Find the intervals of increase and decrease for the function

$$f(x) = 2x^3 + 3x^2 - 12x - 7$$

Solution:

The derivative of *f(x)* is $f'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1)$

which is continuous everywhere, with $f'(x) = 0$ where $x=1$ and $x=-2$. The number *-2* and *1* divide *x* axis into three open intervals. $x<-2$, $-2<x<1$ and $x>1$.



Interval	Test Number	$f'(c)$	Conclusion	Direction of graph
$x<-2$	-3	$f'(-3)>0$	*f* is increasing	Rising
$-2<x<1$	0	$f'(0)<0$	*f* is deceasing	Falling
$x>1$	2	$f'(2)>0$	*f* is increasing	Rising

Example

Find the interval of increase and decrease for the function

$$f(x) = \frac{x^2}{x-2}$$

Solution:

The function is defined for $x \neq 2$, and its derivative is

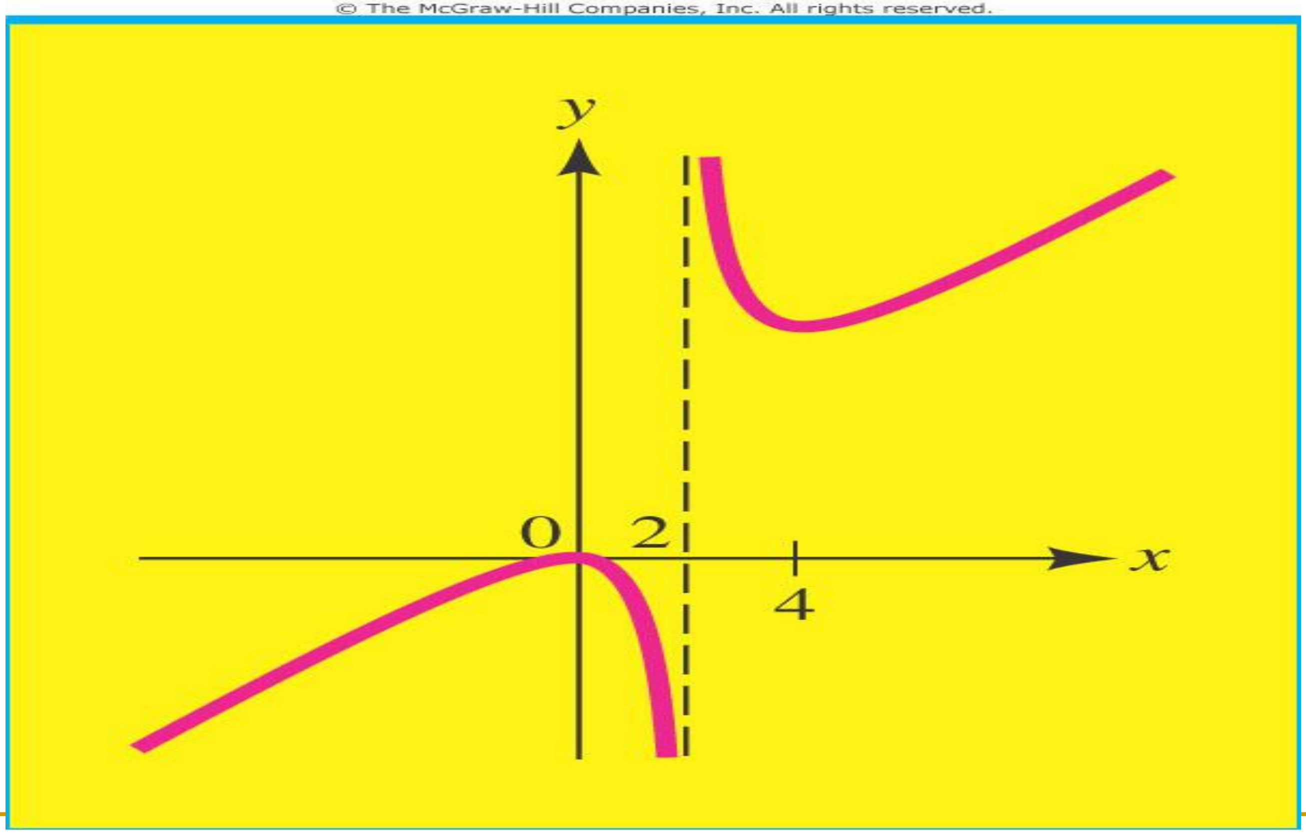
$$f'(x) = \frac{(x-2)(2x) - x^2(1)}{(x-2)^2} = \frac{x(x-4)}{(x-2)^2}$$

which is discontinuous at $x=2$ and has $f'(x)=0$ at $x=0$ and $x=4$. Thus, there are four intervals on which the sign of $f'(x)$ does not change: namely, $x<0$, $0<x<2$, $2<x<4$ and $x>4$. Choosing test numbers in these intervals (say, -2, 1, 3, and 5, respectively), we find that

To be continued

$$f'(-2) = \frac{3}{4} > 0 \qquad f'(1) = -3 < 0 \qquad f'(3) = -3 < 0 \qquad f'(5) = \frac{5}{9} > 0$$

We conclude that $f(x)$ is increasing for $x<0$ and for $x>4$ and that it is decreasing for $0<x<2$ and for $2<x<4$.



THEOREM 4.2 (First Derivative Test)

Suppose that f is continuous on the interval $[a, b]$ and $c \in (a, b)$ is a critical number.

(i) If $f'(x) > 0$ for all $x \in (a, c)$ and $f'(x) < 0$ for all $x \in (c, b)$ (i.e., f changes from increasing to decreasing at c), then $f(c)$ is a local maximum.

(ii) If $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$ (i.e., f changes from decreasing to increasing at c), then $f(c)$ is a local minimum.

(iii) If $f'(x)$ has the *same* sign on (a, c) and (c, b), then $f(c)$ is *not* a local extremum.

EXAMPLE 4.4 Finding Local Extrema of a Function with Fractional Exponents

Find the local extrema of $f(x) = x^{5/3} - 3x^{2/3}$.

Section 3.5 Concavity (凹凸性) and The Second Derivative Test

Increase and decrease of the slopes are our concern!

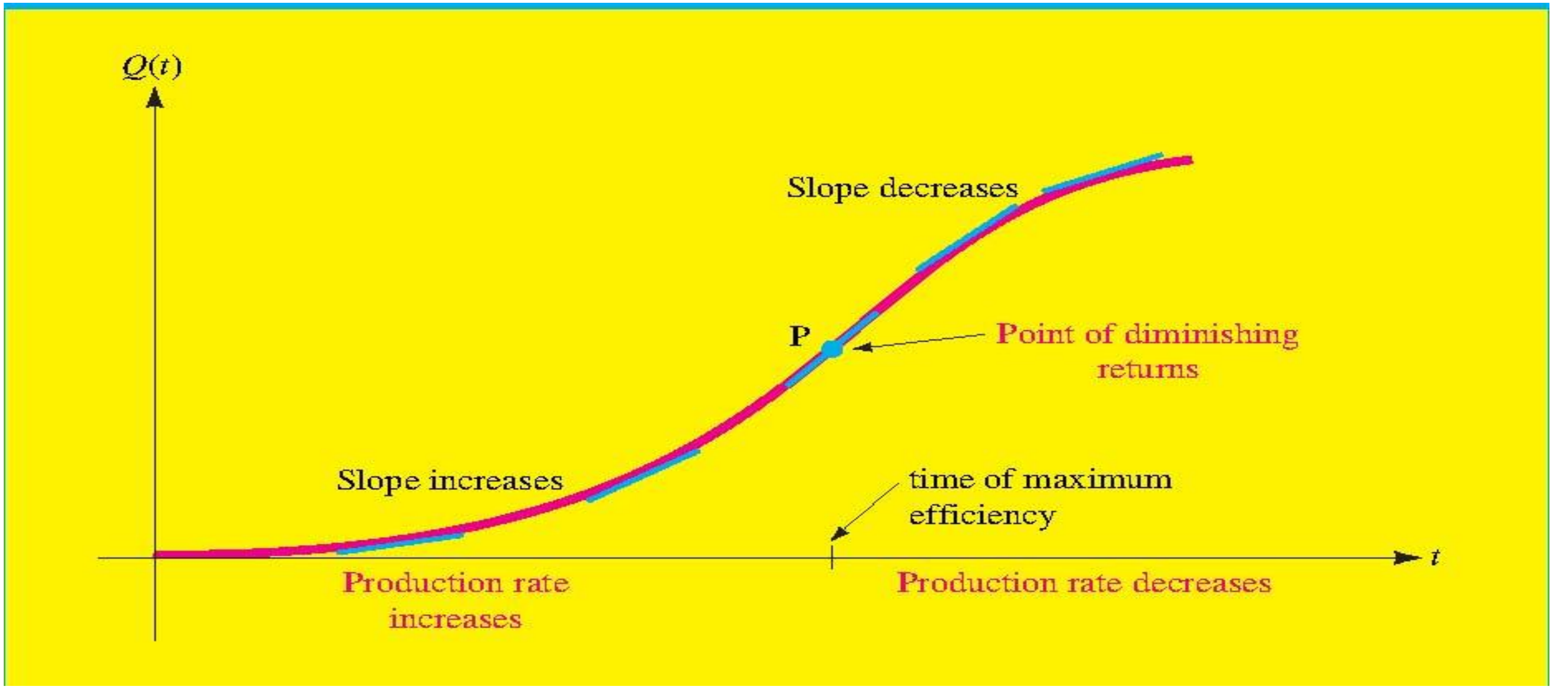
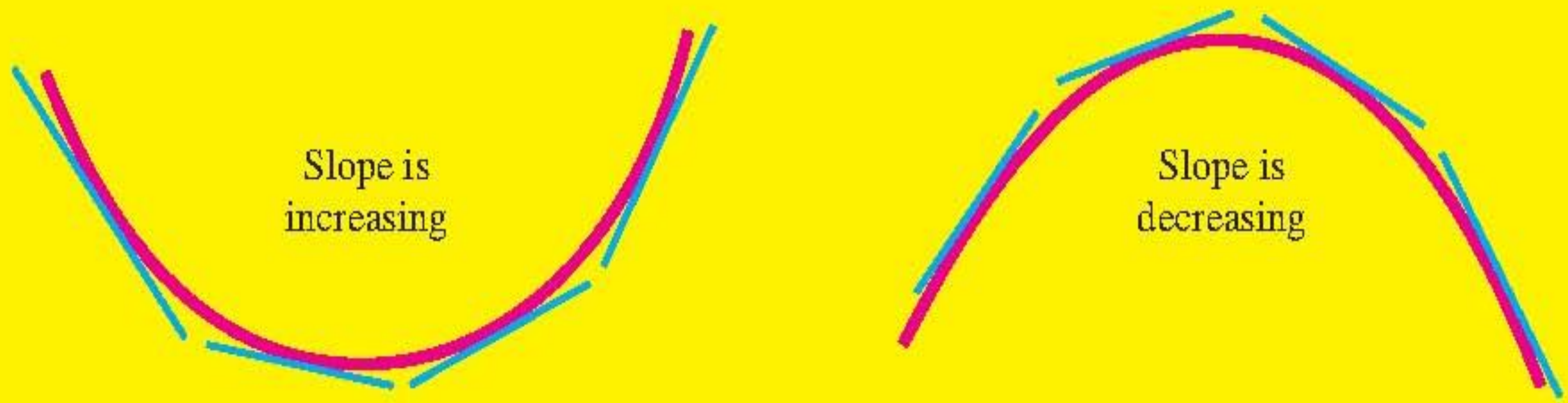


Figure: The output $Q(t)$ of a factory worker t hours after coming to work.

Concavity: If the function $f(x)$ is differentiable on the interval $a<x<b$ then the graph of f is

Concave up on $a<x<b$ if $f'(x)$ is increasing on the interval

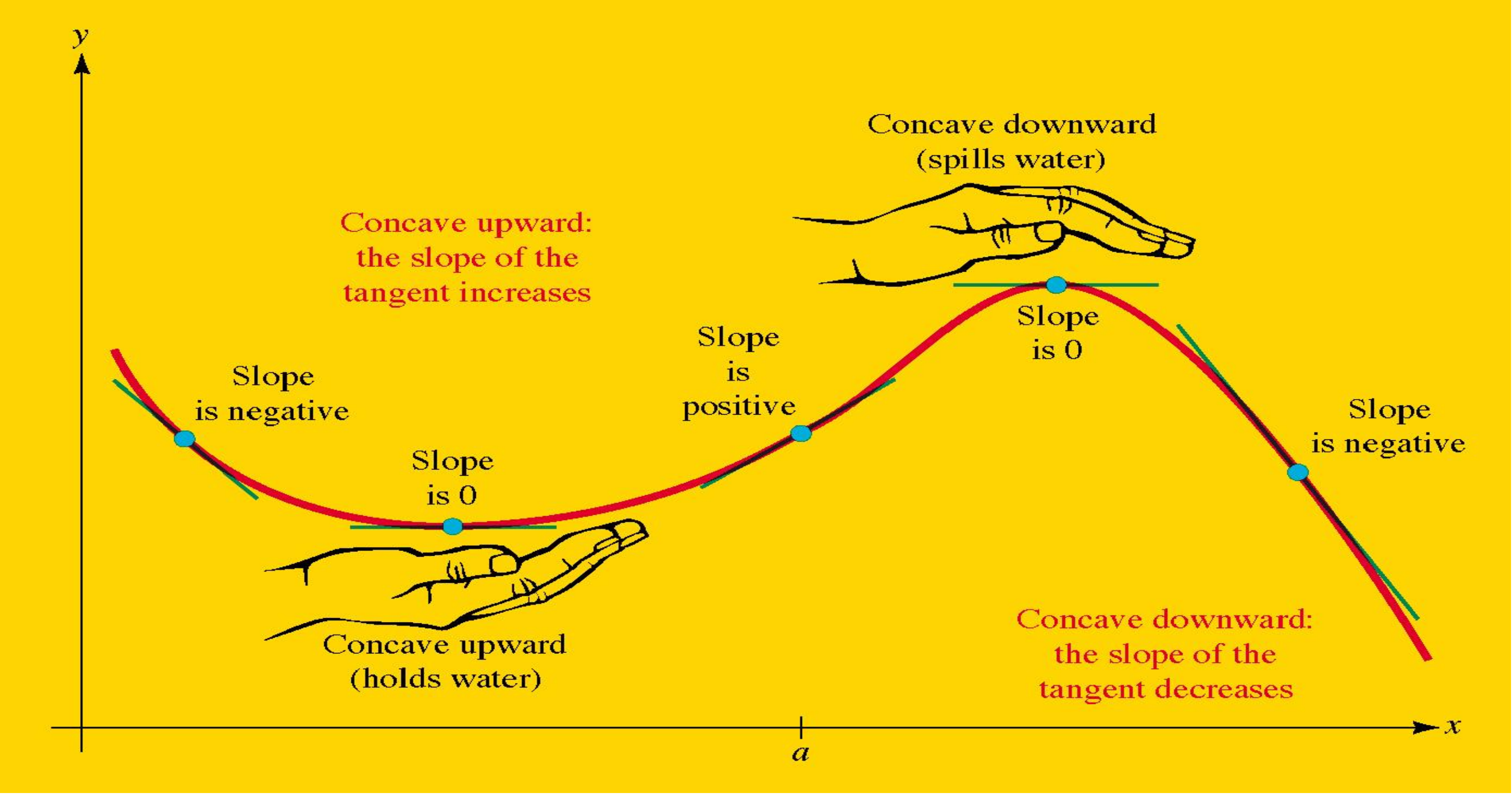
Concave down on $a<x<b$ if $f'(x)$ is decreasing on the interval



(a) Concave upward graph: The graph is above each tangent line.

(b) Concave downward graph: The graph is below each tangent line.

A graph is concave upward on the interval if it lies above all its tangent lines on the interval and concave downward on an Interval where it lies below all its tangent lines.



THEOREM 5.1

Suppose that f'' exists on an interval I.

(i) If $f''(x) > 0$ on I, then the graph of f is concave up on I.
(ii) If $f''(x) < 0$ on I, then the graph of f is concave down on I.

Determining Intervals of Concavity Using the Sign of *f''*

Step 1. Find all values of x for which $f''(x)=0$ or $f''(x)$ is not defined, and mark these numbers on a number line. This divides the line into a number of open intervals.

Step 2. Choose a test number c from each interval $a<x<b$ determined in the step *1* and evaluate $f''(c)$. Then

If $f''(c)>0$, the graph of $f(x)$ is concave upward on $a<x<b$.

If $f''(c)<0$, the graph of $f(x)$ is concave downward on $a<x<b$.

Example

Determine intervals of concavity for the function

$$f(x) = 2x^6 - 5x^4 + 7x - 3$$

Solution:

We find that $f'(x) = 12x^5 - 20x^3 + 7$ and

$$f''(x) = 60x^4 - 60x^2 = 60x^2(x^2 - 1) = 60x^2(x-1)(x+1)$$

The second derivative $f''(x)$ is continuous for all x and $f''(x)=0$ for $x=0$, $x=1$, and $x=-1$. These numbers divide the x axis into four intervals on which $f''(x)$ does not change sign; namely, $x<1$, $-1<x<0$, $0<x<1$, and $x>1$. Evaluating $f''(x)$ at test numbers in each of these intervals (say, at $x=-2$, $x=-1/2$, $x=1/2$, and $x=5$, respectively), we find that

to be continued

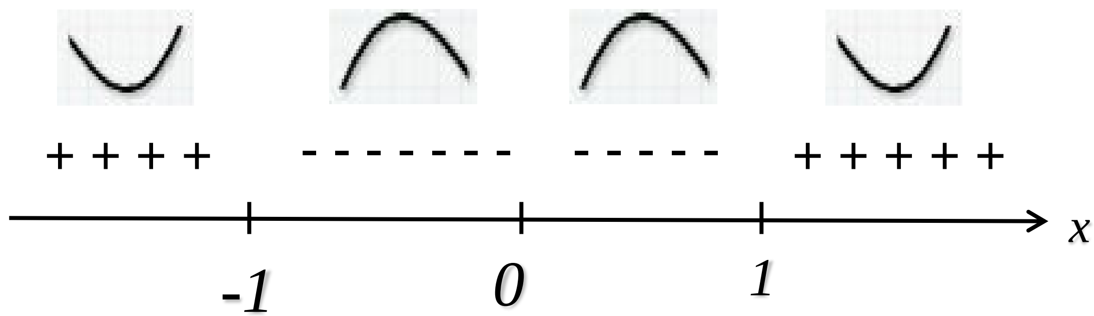
$$f''(-2) = 720 > 0 \qquad f''\left(\frac{-1}{2}\right) = \frac{-45}{4} < 0$$

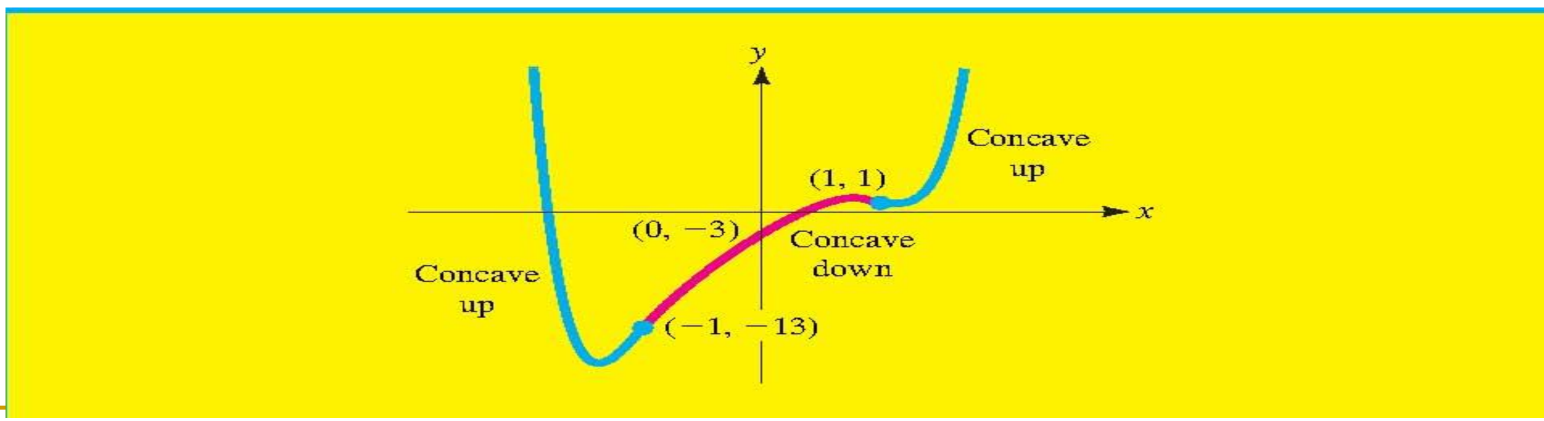
$$f''\left(\frac{1}{2}\right) = \frac{-45}{4} < 0 \qquad f''(5) = 36{,}000 > 0$$

Thus, the graph of *f(x)* is concave up for *x<-1* and for *x>1* and concave down for -1<x<0 and for 0<x<1, as indicated in this concavity diagram.

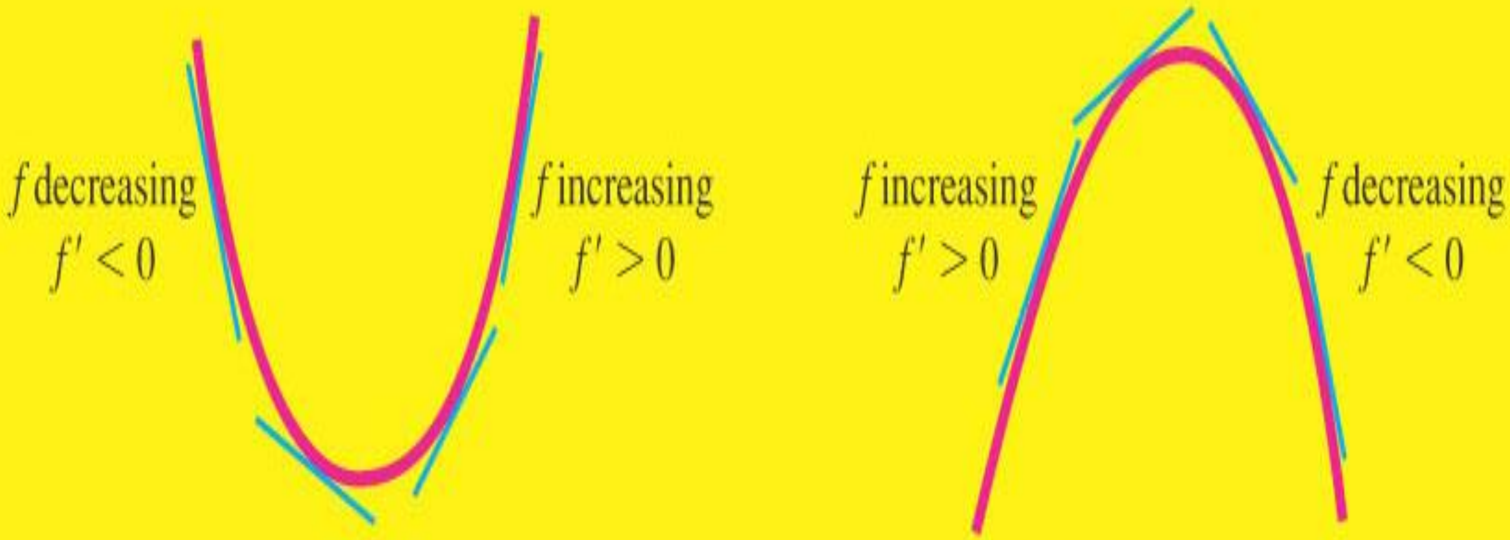
Type of concavity

Sign of $f''(x)$





Note Don’t confuse the concavity of a graph with its “direction” (rising or falling). A function may be increasing or decreasing on an interval regardless of whether its graph is concave upward or concave downward on the interval.

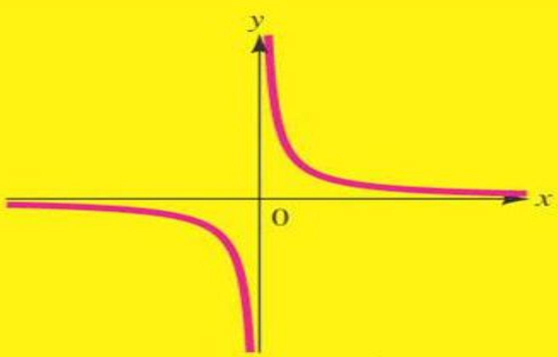


(a) Concave up $f'' > 0$ (b) Concave down $f'' < 0$

DEFINITION 5.2 拐点

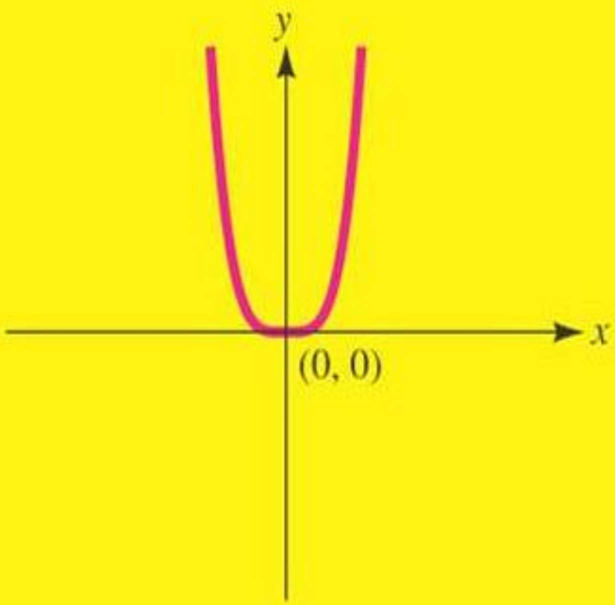
Suppose that f is continuous on the interval (a, b) and that the graph changes concavity at a point $c \in (a, b)$ (i.e., the graph is concave down on one side of c and concave up on the other). Then, the point $(c, f(c))$ is called an **inflection point** of f.

Note: A function can have an inflection point only where it is continuous!!



(a) The graph of $f(x) = \frac{1}{x}$. The concavity changes at $x = 0$ but there is no inflection point since $f(0)$ is not defined.

For example, if *f(x)=1/x*, then $f''(x) = 2/x^3$, so *f''(x)<0* if *x<0* and *f''(x)>0* if *x>0*. The concavity changes from downward to upward at *x=0* but there is no inflection point at *x=0* since *f(0)* is not defined, Knowing that $f(c)$ is defined and that $f''(c) = 0$ does not guarantee that $(c, f(c))$ is an inflection point.

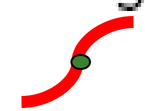
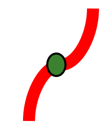


(b) The graph of $f(x) = x^4$ is always concave upward, so (0, 0) is not an inflection point even though $f''(0) = 0$.

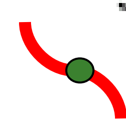
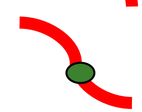
For instance, if $f(x) = x^4$, then $f(0)=0$ and $f''(x) = 12x^2$ so $f''(0)=0$. However, $f''(x)>0$ for any number $x\neq 0$, so the graph of f is always concave upward, and there is no inflection point at $(0,0)$.

Behavior of Graph $f(x)$ at an inflection point $P(c,f(c))$

- **Graph is Rising ($f' > 0$)**

Before P(for $x < c$)	After P(for $x > c$)	shape of graph at P
$f'' > 0$	$f'' < 0$	
$f'' < 0$	$f'' > 0$	

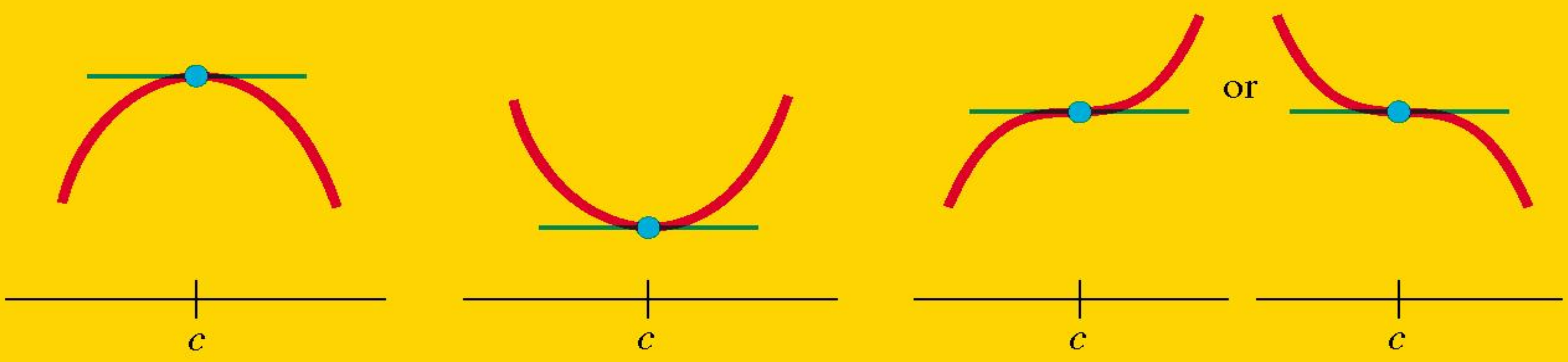
- **Graph is Falling ($f' < 0$)**

Before P(for $x < c$)	After P(for $x > c$)	shape of graph at P
$f'' > 0$	$f'' < 0$	
$f'' < 0$	$f'' > 0$	

THEOREM 5.2 (Second Derivative Test)

Suppose that f is continuous on the interval (a, b) and $f'(c) = 0$, for some number $c \in (a, b)$.

(i) If $f''(c) < 0$, then $f(c)$ is a local maximum.
(ii) If $f''(c) > 0$, then $f(c)$ is a local minimum.



(a) Relative maximum $f'(c) = 0,\ f''(c) < 0$

(b) Relative minimum $f'(c) = 0,\ f''(c) > 0$

(c) No extremum $f'(c) = 0,\ f''(c) = 0$

EXAMPLE 5.4 Using the Second Derivative Test to Find Extrema

Use the Second Derivative Test to find the local extrema of $f(x) = x^4 - 8x^2 + 10$.

REMARK 5.1

If $f''(c) = 0$ or $f''(c)$ is undefined, the Second Derivative Test yields no conclusion. That is, $f(c)$ may be a local maximum, a local minimum or neither. In this event, we must rely solely on first derivative information (i.e., the First Derivative Test) to determine whether $f(c)$ is a local extremum. We illustrate this with example 5.5.

EXAMPLE 5.5 Functions for Which the Second Derivative Test Is Inconclusive

Use the Second Derivative Test to try to classify any local extrema for (a) $f(x) = x^3$, (b) $g(x) = x^4$ and (c) $h(x) = -x^4$.

Section 3.5 Overview of Curve Sketching

A General Procedure for Sketching the Graph of f(x)

1. Find the domain of $f(x)$

2. Find and plot all intercepts. The y intercept (where x=0) is usually easy to find, but x intercepts (where $f(x) = 0$) may require a calculator.

3. Determine all vertical and horizontal asymptotes of the graph. Draw the asymptotes in a coordinate plane.

4. Find $f'(x)$ and use it to determine the critical number of $f(x)$ and intervals of increase and decrease.

to be continued

5. Determine all relative extrema. Plot each relative maximum with a "cap" and each relative minimum with a "cup".

6. Find $f''(x)$ and use it to determine intervals of concavity and points of inflection. Plot each inflection point with a "twist" to suggest the shape of the graph near the point.

7. Plot additional points if needed, and complete the sketch by joining the plotted points in the direction indicated. Be sure to remember that the graph can not cross a vertical asymptote.

Example 13

Sketch the graph of the function

$$f(x) = \frac{x}{(x+1)^2}$$

Solution:

1.and 2. The function is defined for all x except x=-1, and the only intercept is the origin (0,0).

3. The line x=-1 is a vertical asymptote of the graph of f(x) since f(x) decreases indefinitely as x approach -1 from either side;

$$\lim_{x\to 1^-} \frac{x}{(x+1)^2} = \lim_{x\to 1^+} \frac{x}{(x+1)^2} = -\infty$$

Moreover, since

$$\lim_{x\to -\infty} \frac{x}{(x+1)^2} = \lim_{x\to +\infty} \frac{x}{(x+1)^2} = 0$$

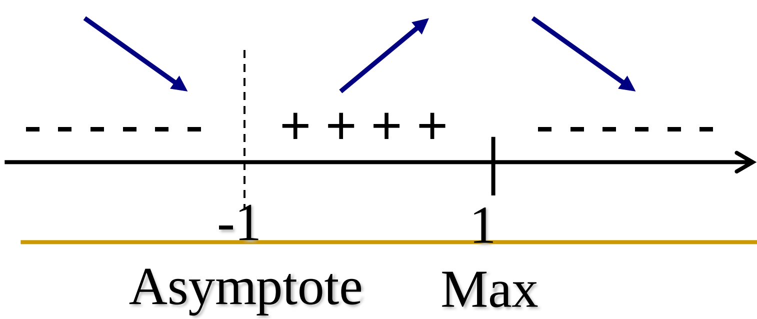
to be continued

the line $y = 0$ (the axis) is a horizontal asymptote. Draw dashed line $x = -1$ and $y = 0$ on a coordinate plane.

4. Compute the derivative of f(x):

$$f'(x) = \frac{(x+1)^2(1) - x[2(x+1)(1)]}{(x+1)^4} = \frac{1-x}{(x+1)^3}$$

Since $f'(1) = 0$, it follows that $x = 1$ is a critical number. Note that even though $f'(1)$ does not exist, $x = -1$ is not a critical number since it is not in the domain of f(x). Place $x = 1$ and $x = -1$ on a number line with a dashed vertical line at $x = -1$ to indicate thee vertical asymptote there. Then evaluate $f'(x)$ at appropriate test numbers (say, at -2, 0, and 3) to obtain the arrow diagram.



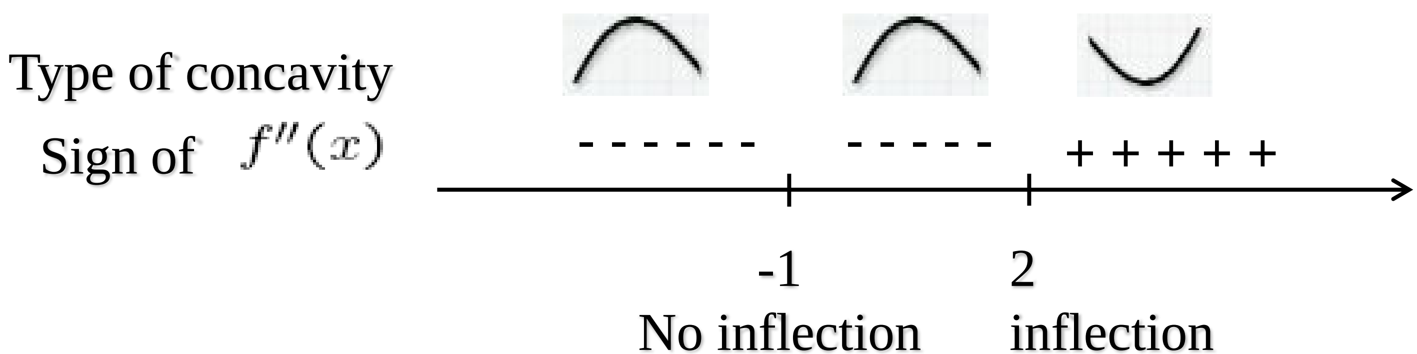
to be continued

5. The arrow pattern in the diagram indicates there is a relative maximum at $x = 1$. Since $f(1) = \frac{1}{4}$, we plot a "cap" at $(1, \frac{1}{4})$.

6.

$$f''(x) = \frac{2(x-2)}{(x+1)^4}$$

Since $f''(x) = 0$ at $x = 2$ and $f''(x)$ does not exist at $x = -1$, plot -1 and 2 on a number line and check the sign of $f''(x)$ on the intervals $x < -1, -1 < x < 2$, and $x > 2$ to obtain the concavity diagram.
Note that the concavity changes at $x = 2$. Since $f(2) = \frac{2}{9}$, plot a "twist" at $(2, \frac{2}{9})$ to indicate the inflection point there.



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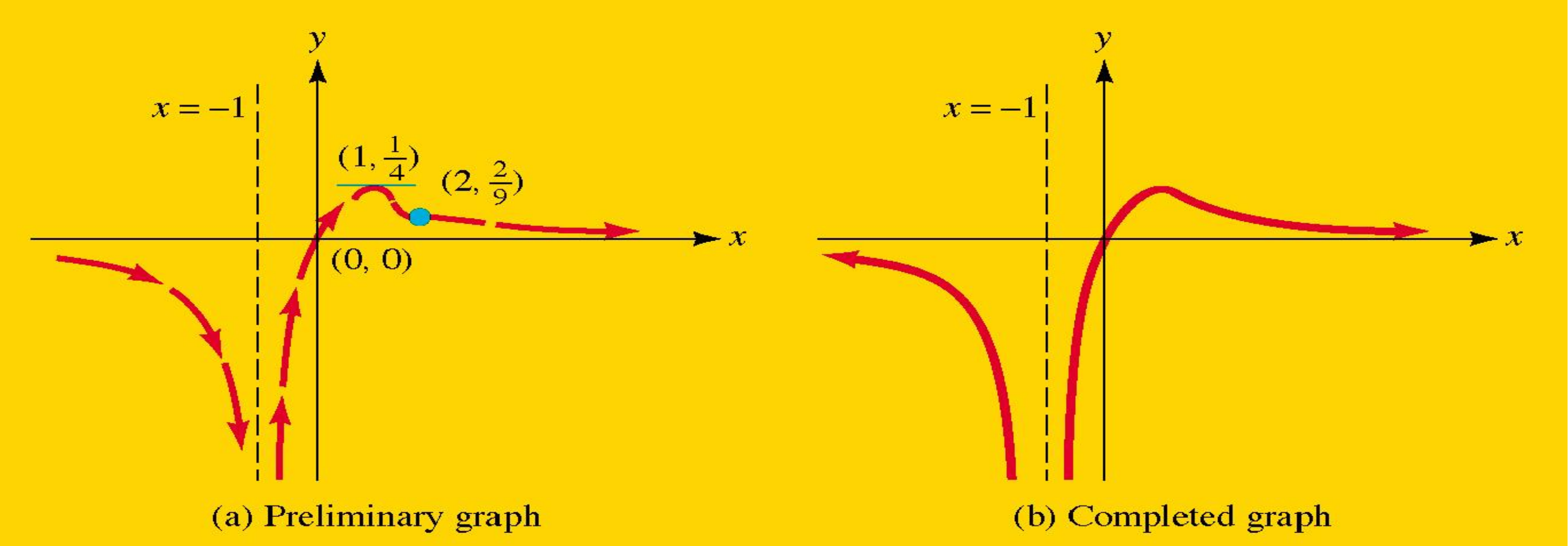
7. The vertical asymptote (dashed line) breaks the graph into two parts. join the features in each separate part by a smooth curve to obtain the completed graph.



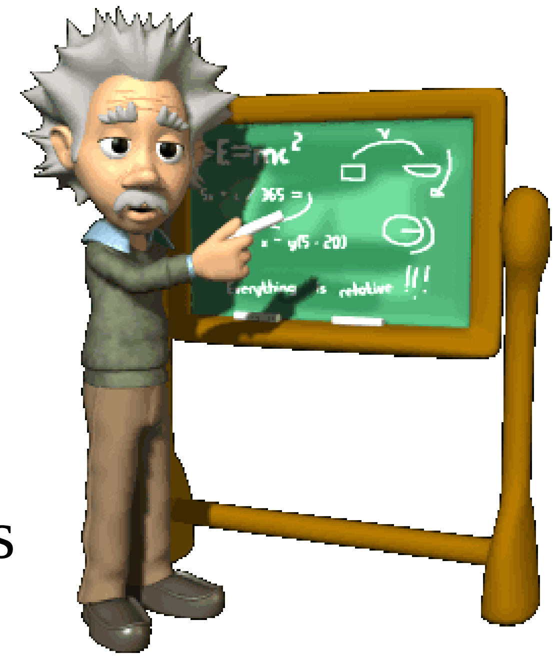
Exercise

Sketch the graph of $f(x) = \dfrac{3x^2}{x^2 + 2x - 15}$

Chapter 4 Integration (积分)

In this Chapter, we will encounter some important concepts.

- Antiderivatives
- Sums and Sigma Notation (求和记号)
- The Definite Integral (定积分)
- The Fundamental Theorem of Calculus (微积分基本定理)
- Integration by Substitution (换元积分法)



Section 4.1 Antiderivative

Antidifferentiation: A function *F(x)* is said to be an antiderivative of *f(x)* if

$$F'(x) = f(x)$$

for every *x* in the domain of *f(x)*. The process of finding antiderivatives is called antidifferentiation or indefinite integration.

EXAMPLE 1.1 Finding Several Antiderivatives of a Given Function

Find an antiderivative of $f(x) = x^2$.

THEOREM 1.1

Suppose that F and G are both antiderivatives of f on an interval I. Then,

$$G(x) = F(x) + c,$$

for some constant c.

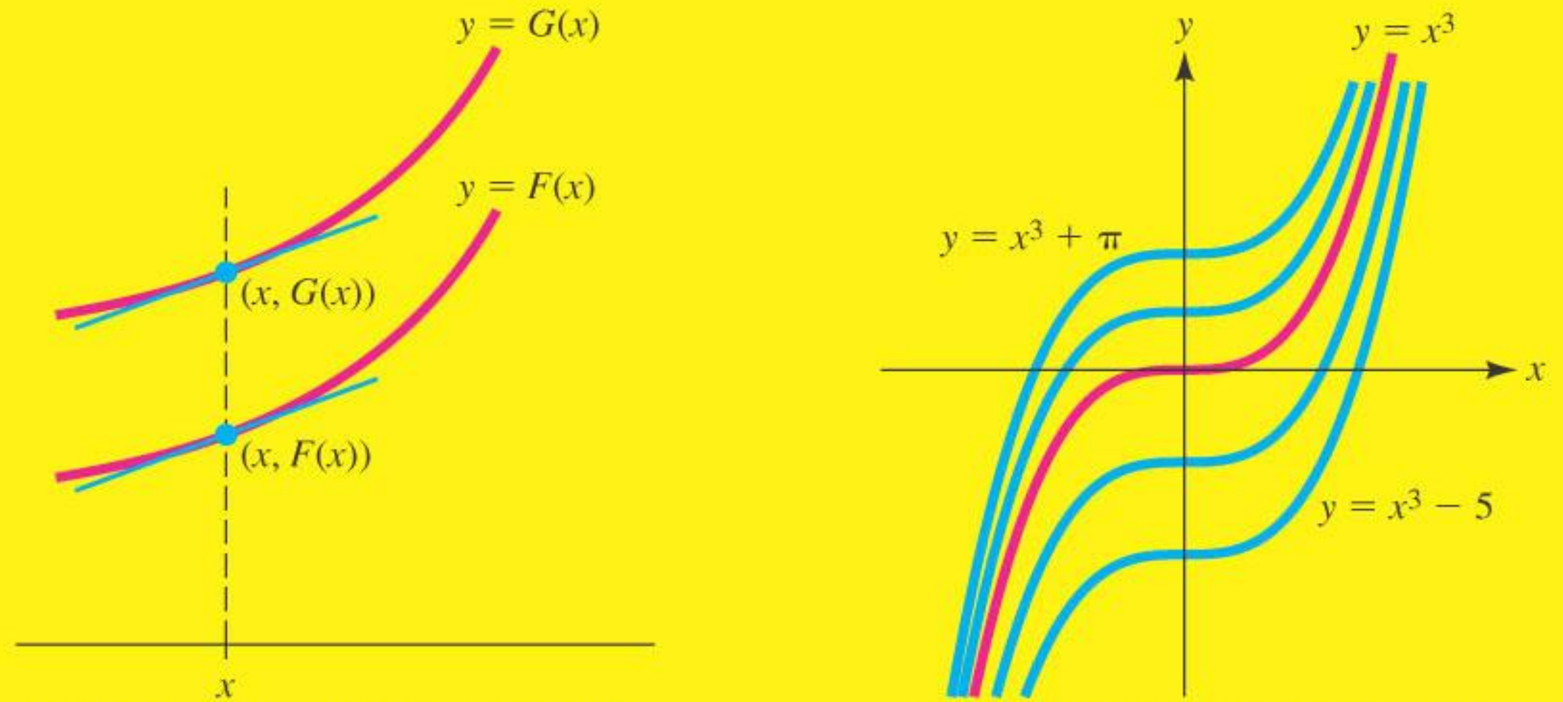
DEFINITION 1.1

Let F be any antiderivative of f. The **indefinite integral** of $f(x)$ (with respect to x), is defined by

$$\int f(x)\,dx = F(x) + c,$$

where c is an arbitrary constant (the **constant of integration**).

The process of computing an integral is called ***integration***. Here, *f(x)* is called the ***integrand*** and the term *dx* identifies *x* as the ***variable of integration***.



(a) If $F'(x) = G'(x)$, the tangent lines at $(x, F(x))$ and $(x, G(x))$ are parallel

(b) Graphs of some members of the family of antiderivatives of $f(x) = 3x^2$

THEOREM 1.2 (Power Rule)

For any rational power $r \neq -1$,

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + c. \tag{1.2}$$

EXAMPLE 1.4 Using the Power Rule

Evaluate $\int x^{17} dx$.

EXAMPLE 1.5 The Power Rule with a Negative Exponent

Evaluate $\displaystyle\int \frac{1}{x^3} \, dx$.

EXAMPLE 1.6 The Power Rule with a Fractional Exponent

Evaluate (a) $\displaystyle\int \sqrt{x} \, dx$ and (b) $\displaystyle\int \frac{1}{\sqrt[3]{x}} \, dx$.

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c, \ \text{for } r \neq -1 \text{ (power rule)}$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \csc^2 x\,dx = -\cot x + c$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$$

$$\int \sec x \tan x\,dx = \sec x + c$$

$$\int \csc x \cot x\,dx = -\csc x + c$$

$$\int e^x\,dx = e^x + c$$

$$\int e^{-x}\,dx = -e^{-x} + c$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$$

$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1} x + c$$

THEOREM 1.3

Suppose that $f(x)$ and $g(x)$ have antiderivatives. Then, for any constants, a and b,

$$\int [af(x) + bg(x)]\, dx = a\int f(x)\, dx + b\int g(x)\, dx. \qquad (1.3)$$

EXAMPLE 1.7 An Indefinite Integral of a Sum

Evaluate $\int (3\cos x + 4x^8)\, dx$.

EXAMPLE 1.8 An Indefinite Integral of a Difference

Evaluate $\displaystyle\int \left(3e^x - \frac{2}{1+x^2}\right) dx$.

THEOREM 1.4

For $x \neq 0$, $\dfrac{d}{dx} \ln |x| = \dfrac{1}{x}$.

COROLLARY 1.1

For $x \neq 0$,

$$\int \frac{1}{x}\, dx = \ln |x| + c.$$

COROLLARY 1.2

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c, \tag{1.4}$$

provided $f(x) \neq 0$.

EXAMPLE 1.10 The Indefinite Integral of a Fraction of the Form $\dfrac{f'(x)}{f(x)}$

Evaluate $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx$.

Section 4.2 Sums and Sigma Notation (求和记号)

We use the Greek capital letter sigma, ∑, as a symbol for *sum* and write the sum in **summation notation** as

$$\sum_{i=1}^{20} i^2 = 1^2 + 2^2 + 3^2 + \cdots + 20^2,$$

to indicate that we add together terms of the form i^2, starting with *i=1* and ending with *i=20*. The variable *i* is called the **index of summation**

THEOREM 2.1

If n is any positive integer and c is any constant, then

(i) $\sum_{i=1}^{n} c = cn$ **(sum of constants),**

(ii) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ **(sum of the first n positive integers)** and

(iii) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ **(sum of the squares of the first n positive integers).**

THEOREM 2.2

For any constants c and d,

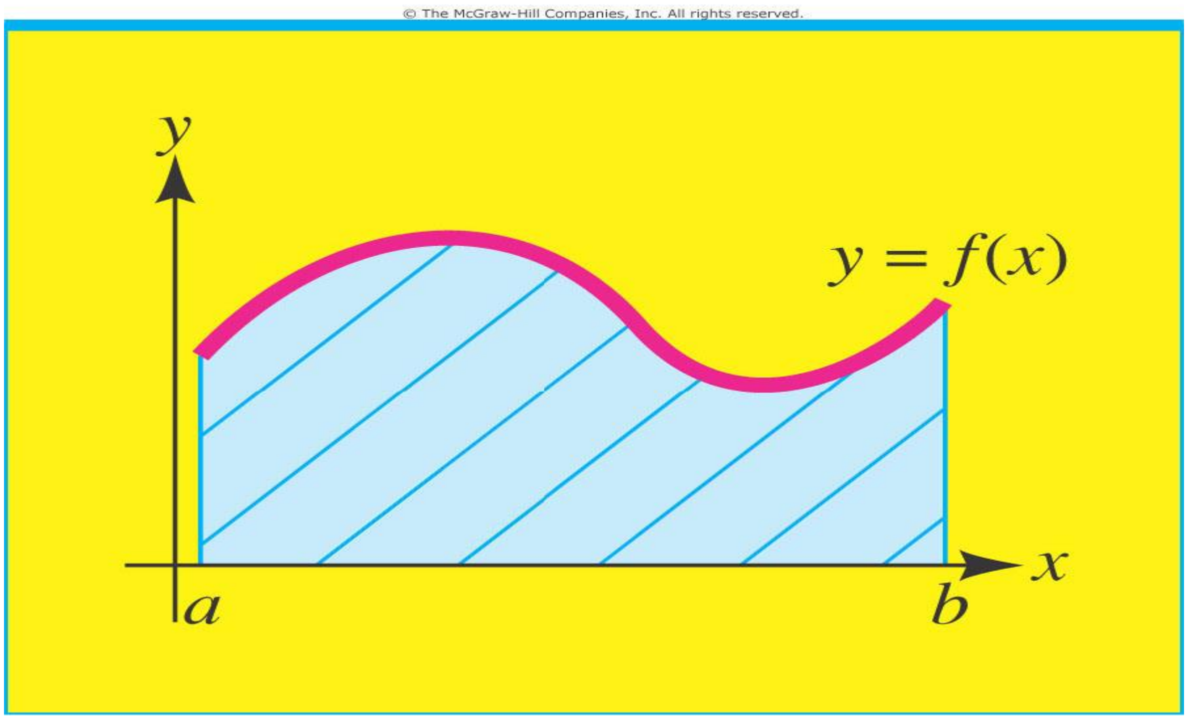
$$\sum_{i=1}^{n}(ca_i + db_i) = c\sum_{i=1}^{n} a_i + d\sum_{i=1}^{n} b_i.$$

Principle of Mathematical Induction (数学归纳法)

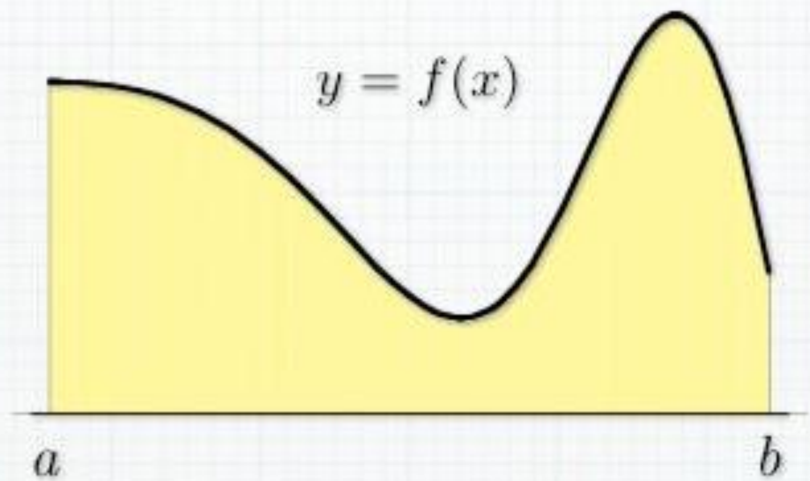
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$ (sum of the squares of the first n positive integers).

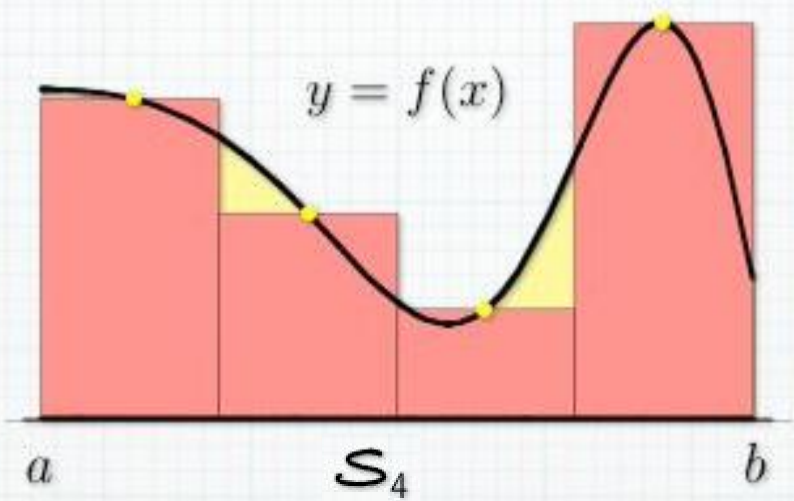
Section 4.3 Area (面积)

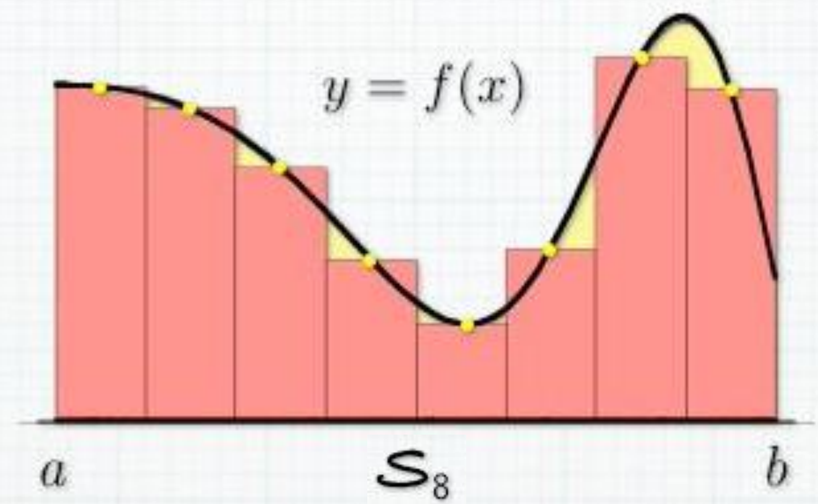
Our goal in this section is to show how area under a curve can be expressed as a limit of a sum of terms called a **definite integral**.

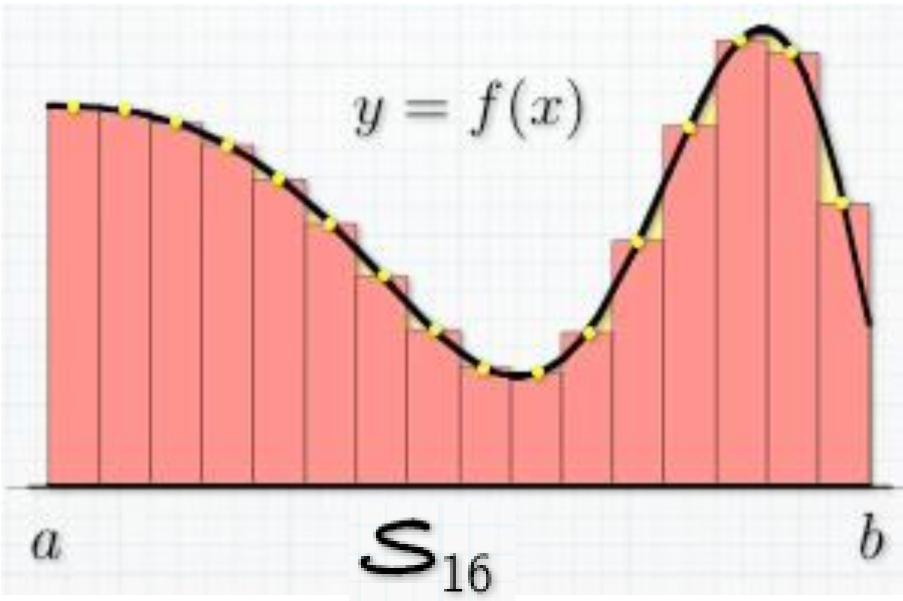


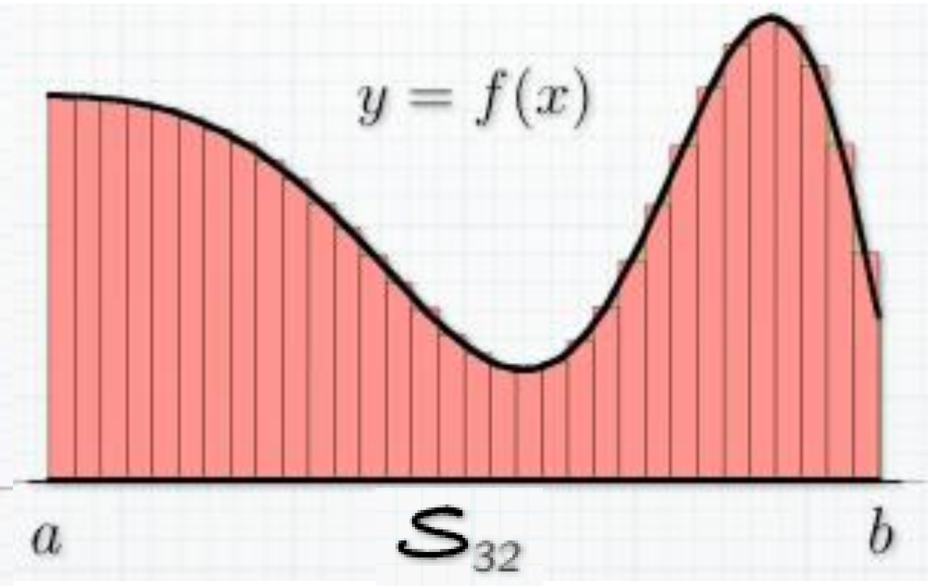
Approximation by Sums of Rectangle Areas

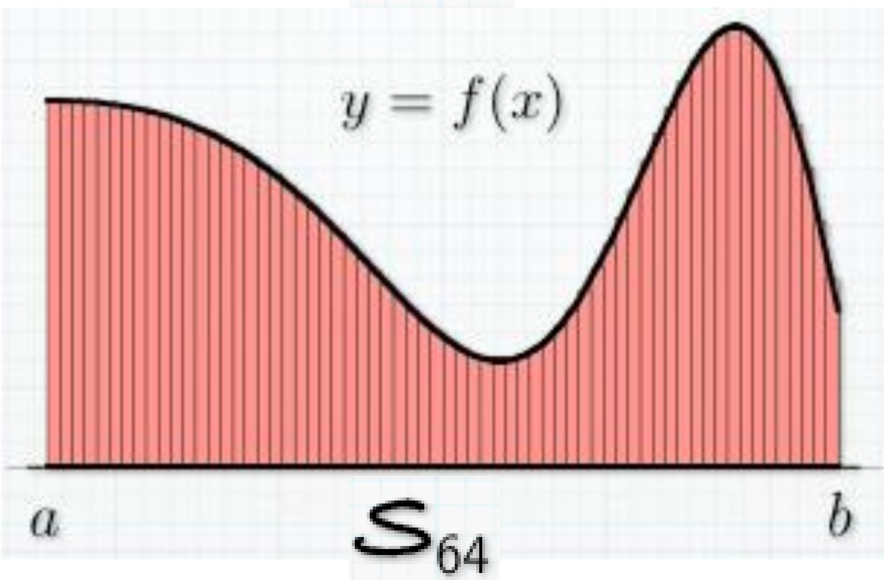










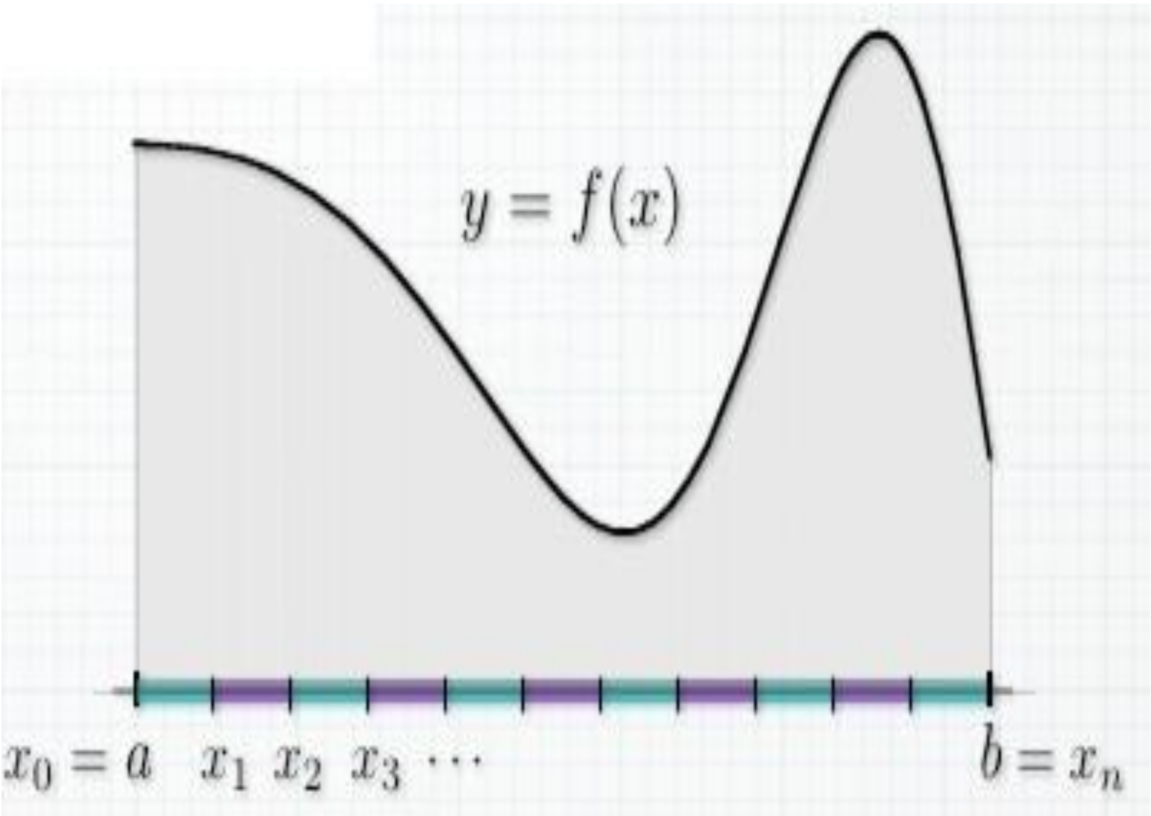


Area $\approx \mathcal{S}_n$ for large n

$$\text{Area} = \lim_{n \to \infty} \mathcal{S}_n$$

General Set-up of an S_n (uniform grid)

All rectangles have same width.



- *n* subintervals:

$$[x_{i-1}, x_i],\ i = 1, 2, \ldots, n$$

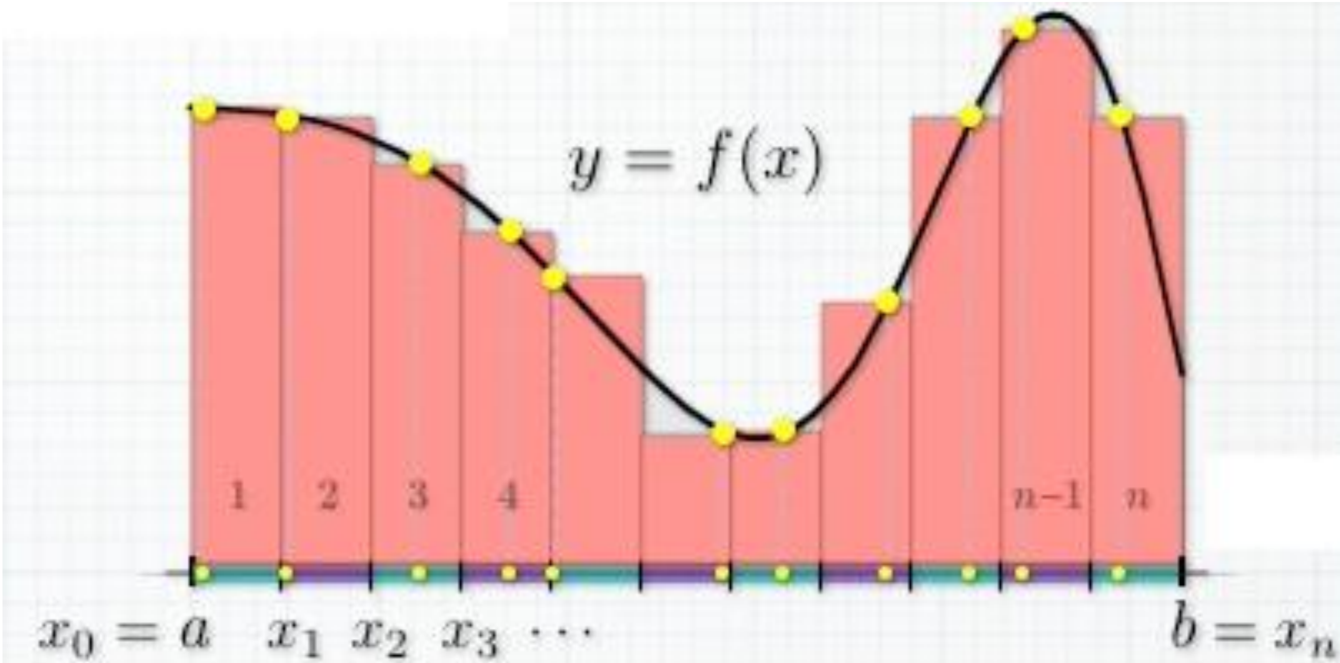
- Subinterval width

$$\Delta x = \frac{b-a}{n}$$

- Formula for x_i:

$$x_i = a + i\,\Delta x$$

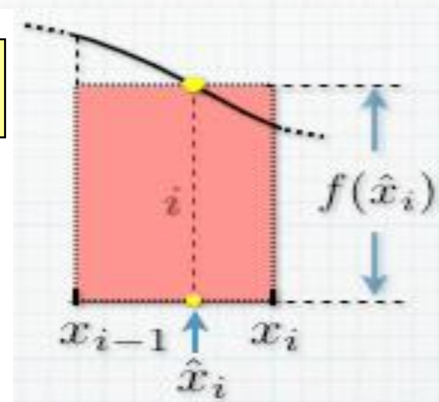
General Set-up of an $\mathcal{S}_n$ (uniform grid)



Choice of *n* evaluation points

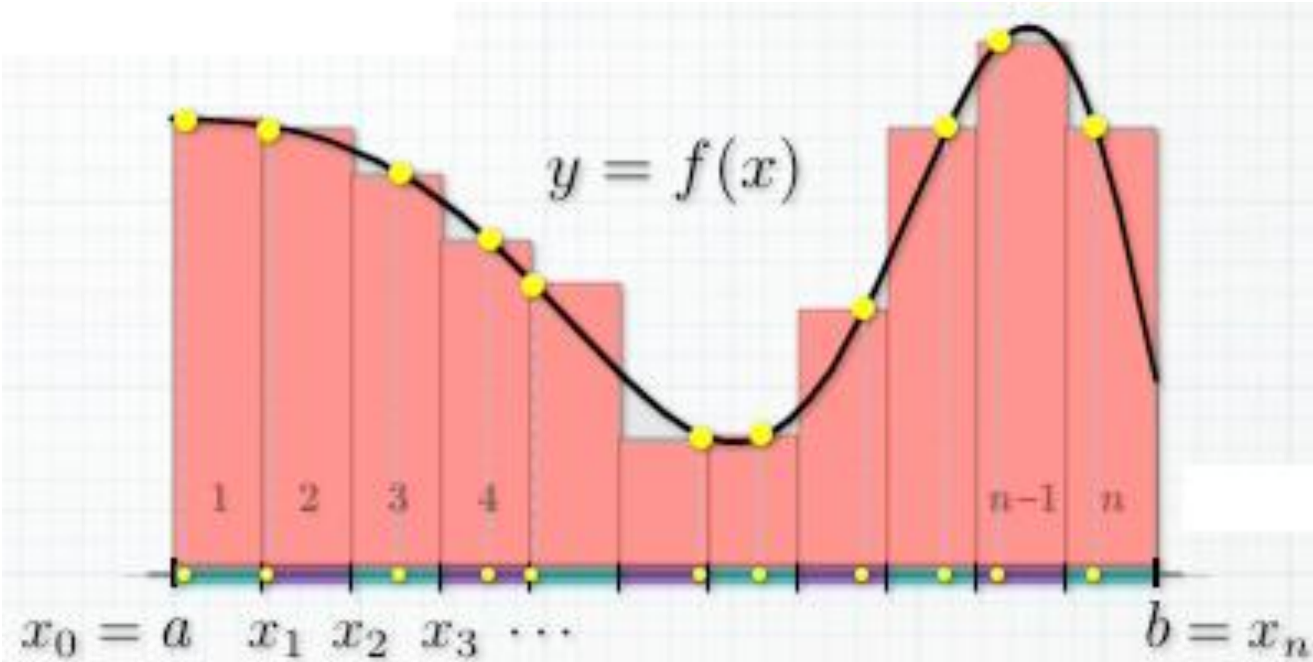
$$\hat{x}_1,\ \hat{x}_2,\ \hat{x}_3,\ \ldots,\ \hat{x}_n$$

where $x_{i-1} \le \hat{x}_i \le x_i$



area of the i^{th} rectangle

$$f(\hat{x}_i)\,\Delta x$$



area under the curve $\approx S_n = f(\hat{x}_1)\,\Delta x + f(\hat{x}_2)\,\Delta x + \cdots + f(\hat{x}_n)\,\Delta x$

$$= \sum_{i=1}^{n} f(\hat{x}_i)\,\Delta x$$

The sum of $f(\hat{x}_i)\,\Delta x$ as i goes from 1 to n

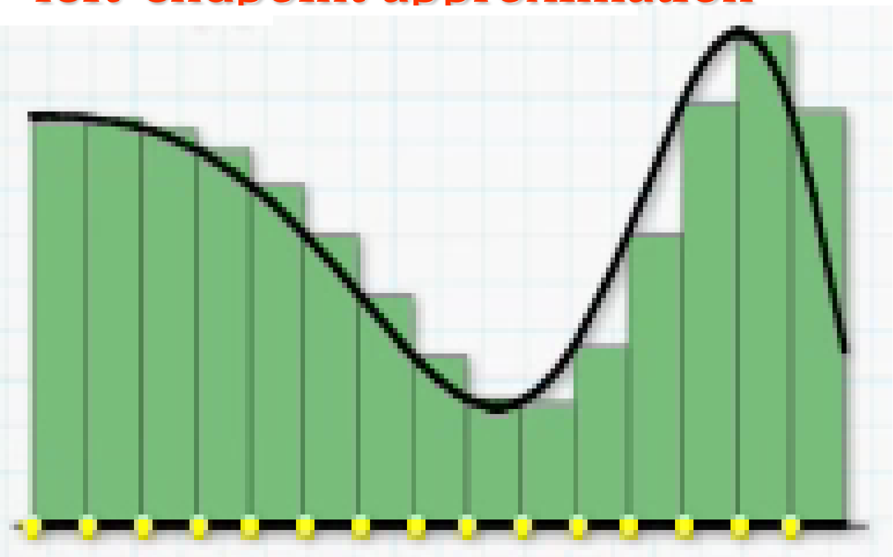
Convenient Choices of $\hat{x}_i$

Right-endpoint approximation

$$\hat{x}_i = x_i = a + i\,\Delta x$$

$$S_n = \sum_{i=1}^{n} f(a + i\,\Delta x)\,\Delta x$$

left-endpoint approximation

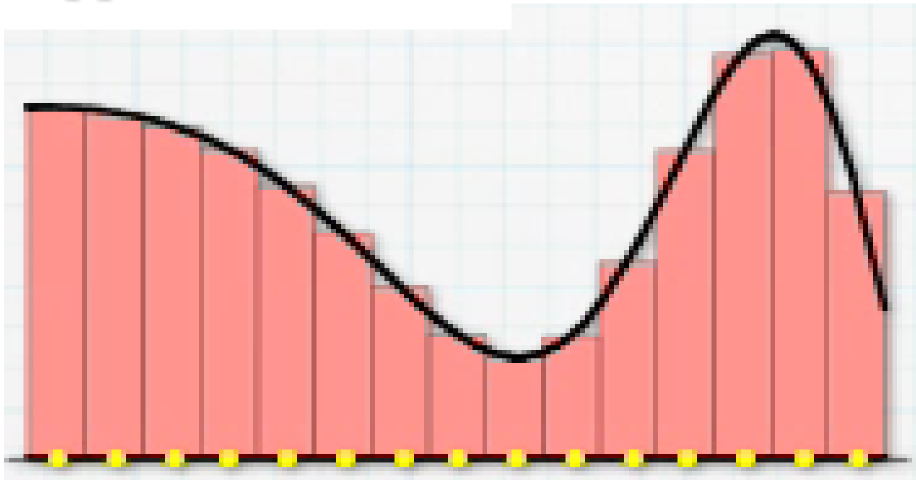


$$\hat{x}_i = x_{i-1} = a + (i-1)\,\Delta x$$

$$S_n = \sum_{i=1}^{n} f(a + (i-1)\Delta x)\,\Delta x$$

Convenient Choices of $\hat{x}_i$

Midpoint Approximation

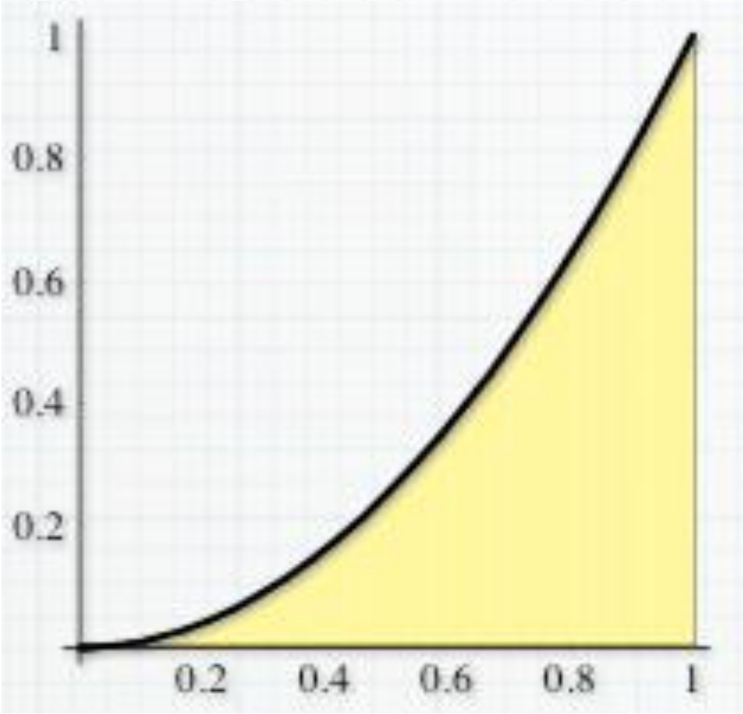


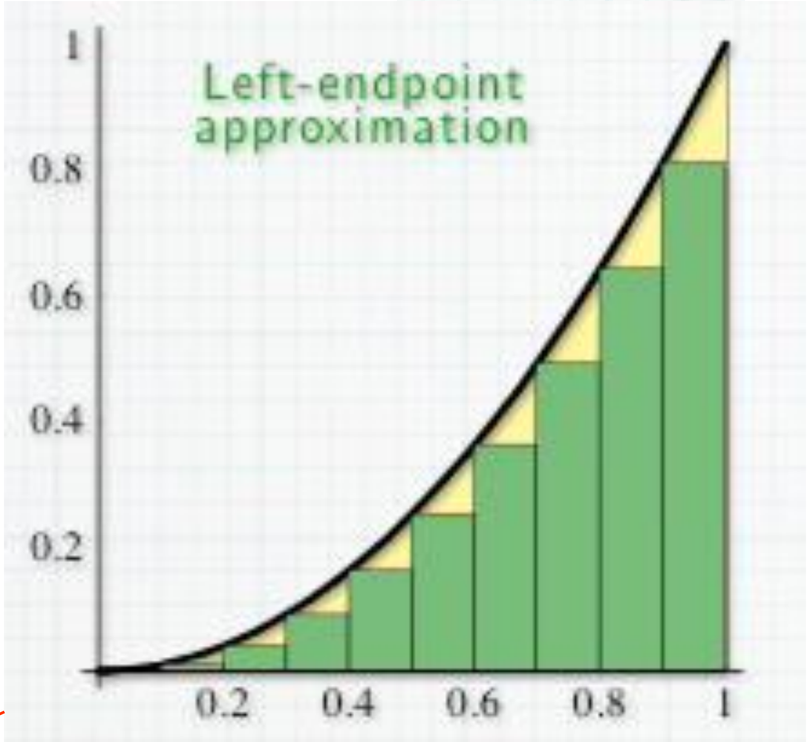
$$\hat{x}_i = \frac{x_{i-1} + x_i}{2} = a + \left(i - \tfrac{1}{2}\right)\Delta x$$

$$S_n = \sum_{i=1}^{n} f\left(a + \left(i - \tfrac{1}{2}\right)\Delta x\right)\Delta x$$

Example 11

$f(x) = x^2$ on $[0, 1]$. $\quad n = 10 \quad \Delta x = \frac{1}{10}$

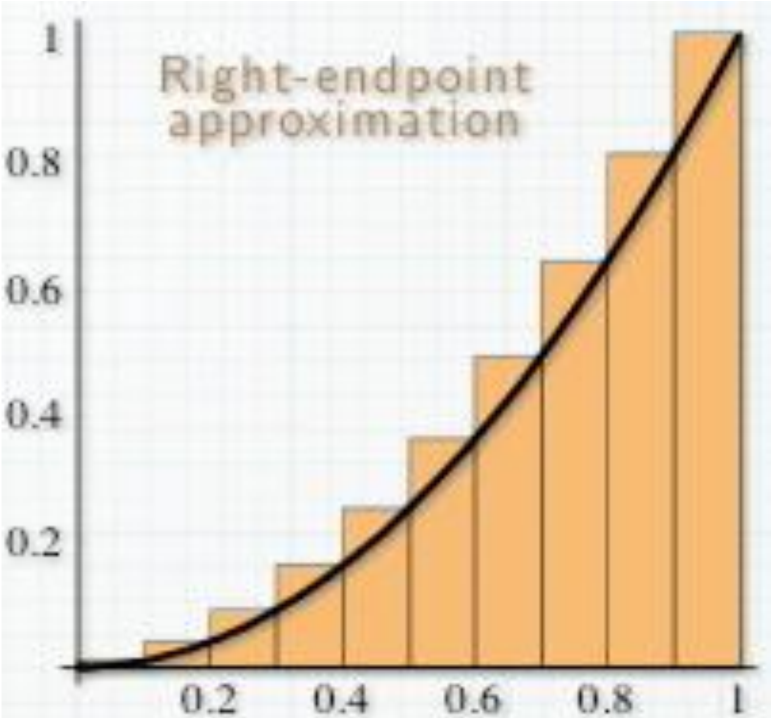


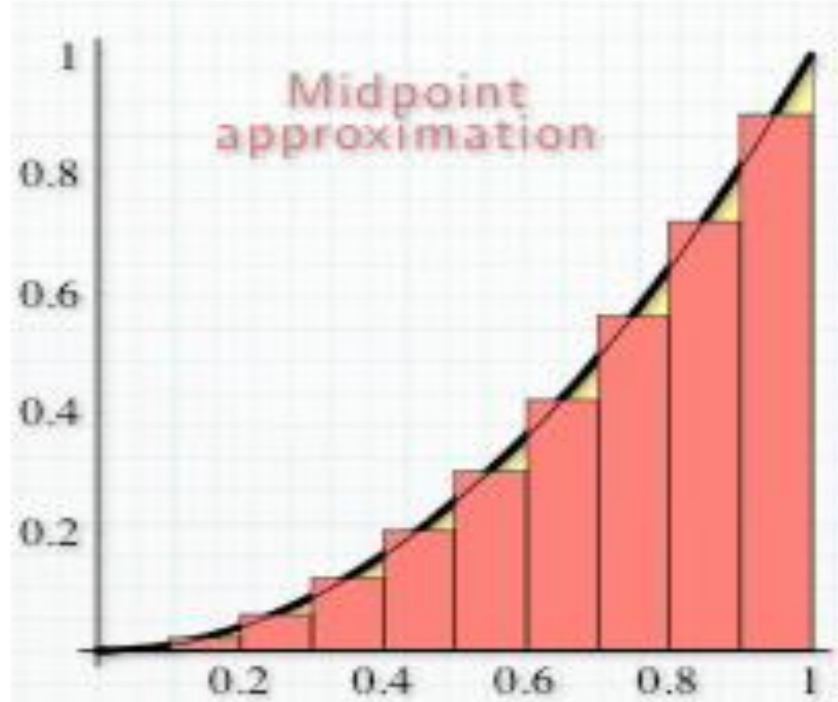


$$\hat{x}_i = (i-1)\,\frac{1}{10}$$

$$S_{10} = \sum_{i=1}^{10} \left((i-1)\frac{1}{10}\right)^2 \frac{1}{10} = 0.285$$

to be continued





$$\hat{x}_i = i\,\frac{1}{10}$$

$$\mathcal{S}_{10} = \sum_{i=1}^{10} \left(i\,\frac{1}{10}\right)^2 \frac{1}{10} = 0.39$$

$$\hat{x}_i = \left(i - \frac{1}{2}\right) \frac{1}{10}$$

$$\mathcal{S}_{10} = \sum_{i=1}^{10} \left(\left(i - \tfrac{1}{2}\right)\tfrac{1}{10}\right)^2 \frac{1}{10} = 0.33$$

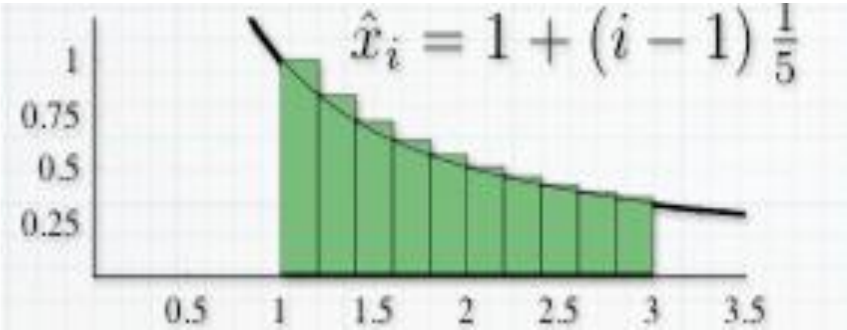
Example 12

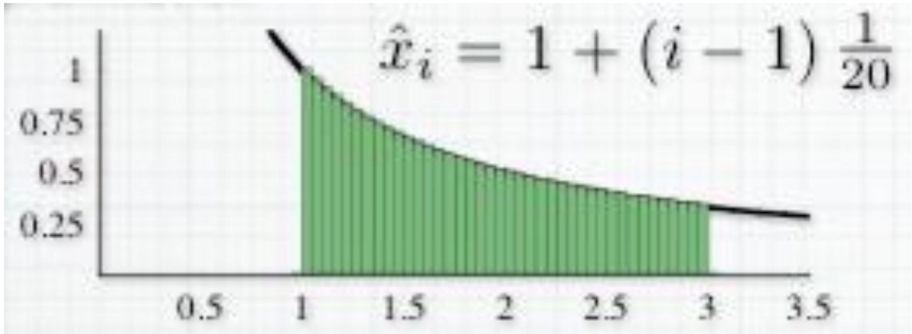
$f(x) = 1/x$ on $[1,3]$.

$n = 10 \quad \Delta x = \frac{3-1}{10} = \frac{1}{5}$

$n = 40 \quad \Delta x = \frac{3-1}{40} = \frac{1}{20}$

left-endpoint approximation



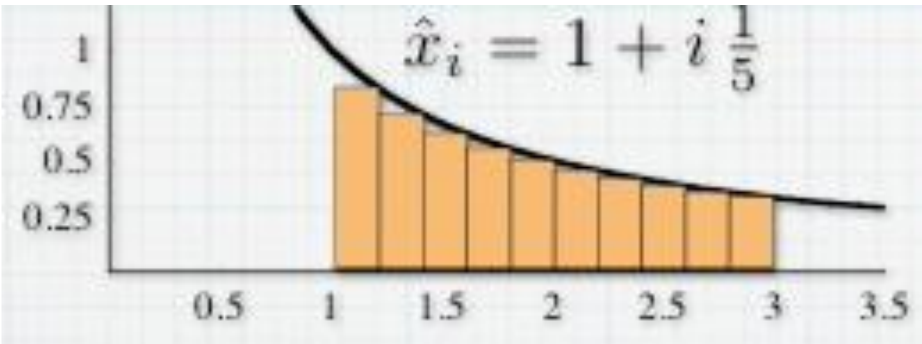


$$S_{10} = \sum_{i=1}^{10} \frac{1}{1+(i-1)\frac{1}{5}} \frac{1}{5} = \sum_{i=1}^{10} \frac{1}{5+(i-1)} \approx 1.168$$

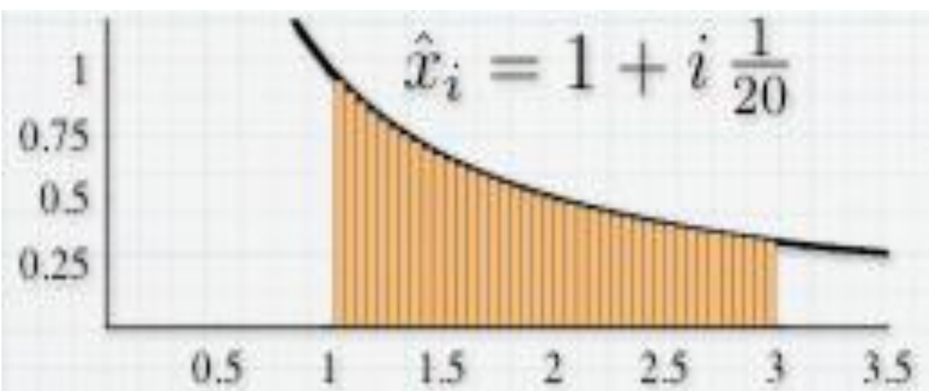
$$S_{40} = \sum_{i=1}^{40} \frac{1}{1+(i-1)\frac{1}{20}} \frac{1}{20} = \sum_{i=1}^{40} \frac{1}{20+(i-1)} \approx 1.115$$

to be continued

Right-endpoint approximation



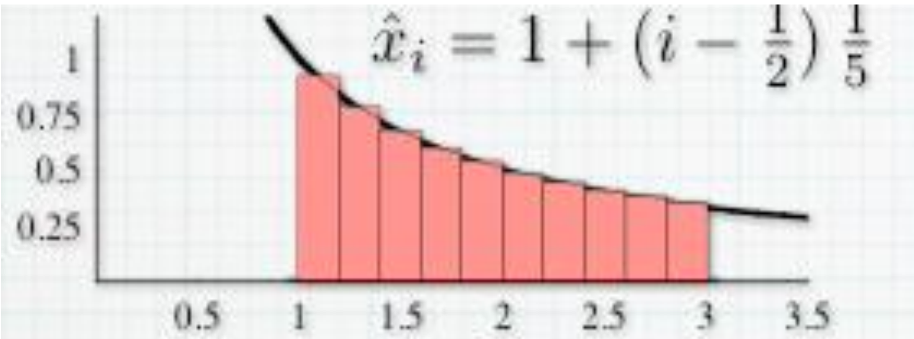
$$S_{10} = \sum_{i=1}^{10} \frac{1}{1+i\frac{1}{5}}\,\frac{1}{5} \approx 1.035$$



$$S_{40} = \sum_{i=1}^{40} \frac{1}{1+i\frac{1}{20}}\,\frac{1}{20} \approx 1.082$$

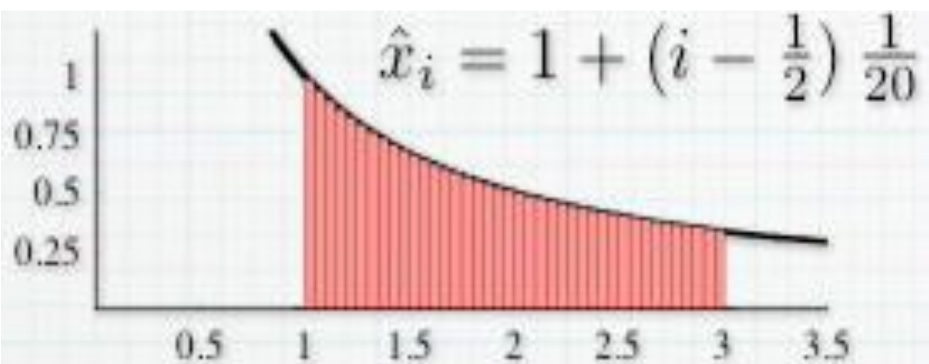
to be continued

Midpoint Approximation



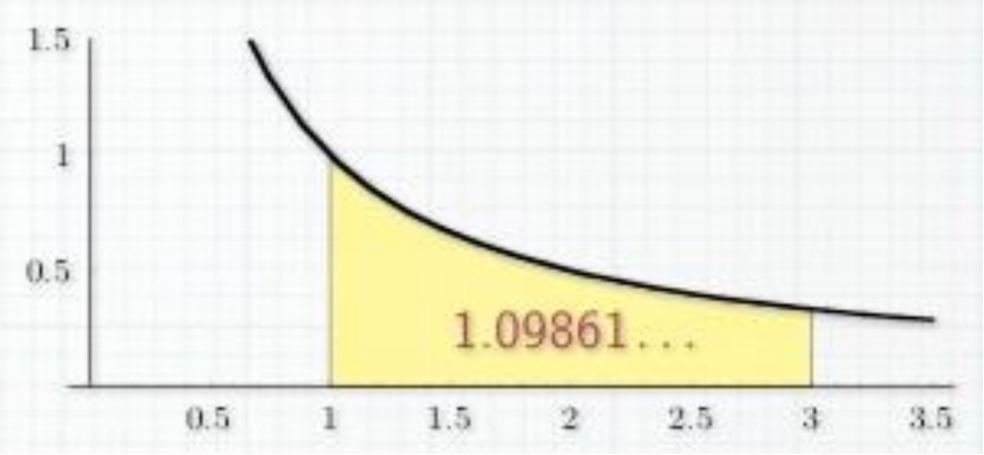
$$S_{10} = \sum_{i=1}^{10} \frac{1}{1+\left(i-\frac{1}{2}\right)\frac{1}{5}}\,\frac{1}{5}$$

$$= \sum_{i=1}^{10} \frac{1}{5+\left(i-\frac{1}{2}\right)} \approx 1.097$$



$$S_{40} = \sum_{i=1}^{40} \frac{1}{1+\left(i-\frac{1}{2}\right)\frac{1}{20}}\,\frac{1}{20} \approx 1.0985$$

$$S_{200} = 1.098608585 \qquad S_{400} = 1.098611363$$



Area Under a Curve Let $f(x)$ be continuous and satisfy $f(x) \geq 0$ on the interval $a \leq x \leq b$. Then the region under the curve $y=f(x)$ over the interval $a \leq x \leq b$ has area

$$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} [f(x_1) + f(x_2) + \ldots + f(x_n)]\Delta x = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\Delta x$$

Where x_j is the point chosen from the jth subinterval if the Interval $a \leq x \leq b$ is divided into n equal parts, each of length $\Delta x = \frac{b-a}{n}$

Riemann sum Let $f(x)$ be a function that is continuous on the interval $a \le x \le b$. Subdivide the interval $a \le x \le b$ into n equal parts, each of width $\Delta x = \frac{b-a}{n}$, and choose a number x_k from the kth subinterval for $k=1, 2, \ldots, n$. Form the sum

$$\left[f(x_1) + f(x_2) + \cdots\cdots + f(x_n) \right] \Delta x$$

Called a **Riemann sum**（黎曼和）.

Note: $f(x) \ge 0$ is not required

Section 4.4 The Definite Integral (定积分)

The Definite Integral the definite integral of f on the interval $a \le x \le b$, denoted by $\int_a^b f(x)dx$ is the limit of the Riemann sum as $n \to +\infty$; that is

$$\int_a^b f(x)\,dx = \lim_{n \to +\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\,\Delta x$$

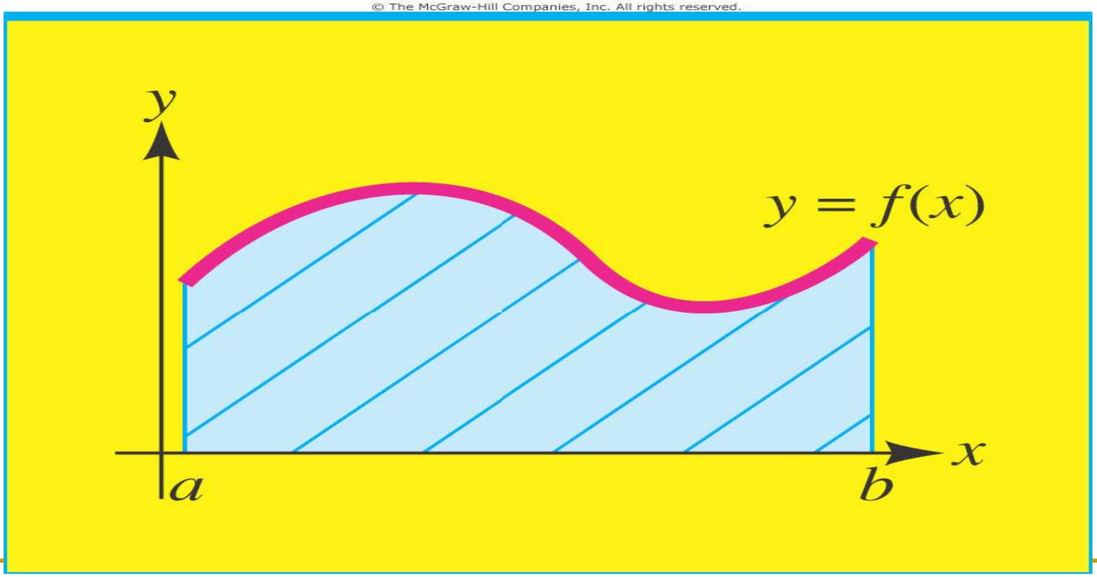
The function *f(x)* is called the **integrand**, and the numbers *a* and *b* are called the **lower and upper limits of integration**, respectively. The process of finding a definite integral is called **definite integration**.

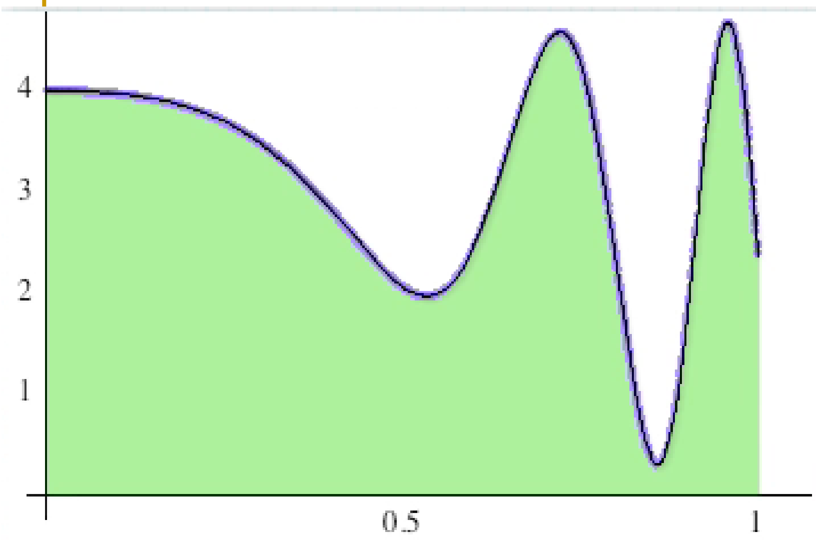
Note: if f(x) is continuous on $a \le x \le b$, the limit used to define integral $\int_a^b f(x)dx$ exist and is same regardless of how the subinterval representatives xk are chosen.

Area as a Definite Integral: If $f(x)$ is continuous and $f(x) \geq 0$ on the interval $a \leq x \leq b$, then the region R under the curve $y = f(x)$ over the interval

$a \leq x \leq b$ has area A given by the definite integral

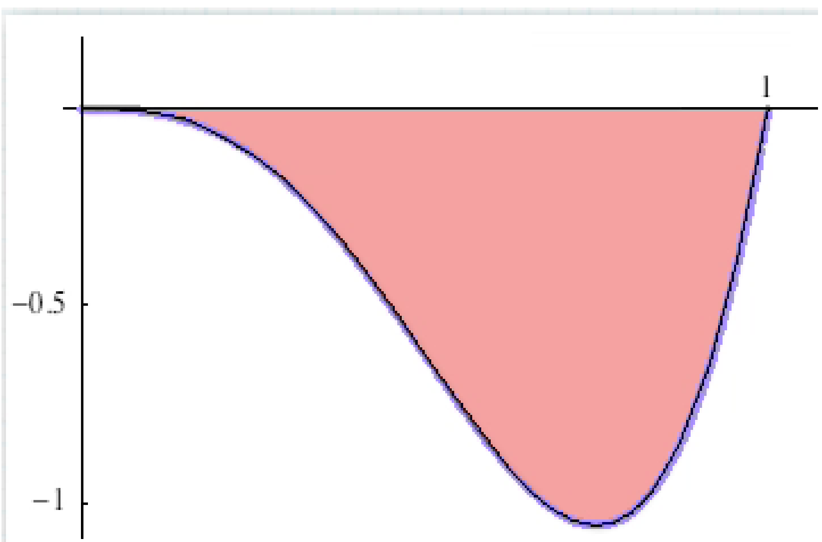
$$A = \int_a^b f(x)\,dx$$





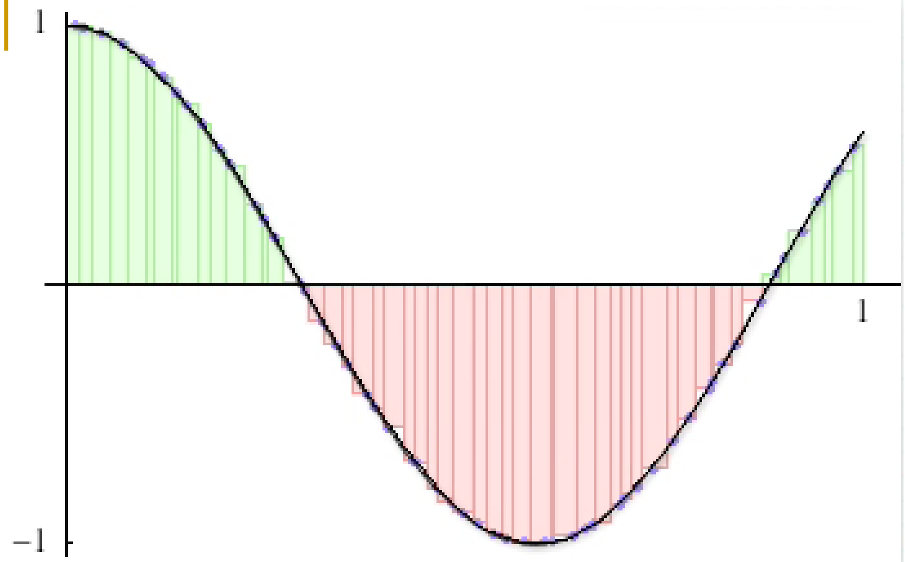
If f(x) is continuous and f(x)≥0 for all x in [a,b],then $\int_a^b f(x)dx \geq 0$

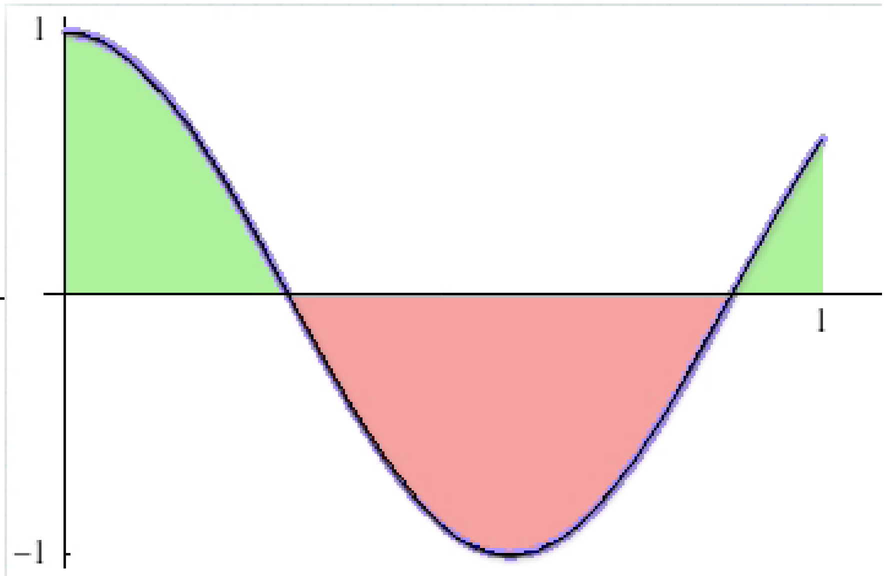
and equals the area of the region bounded by the graph f and the x-axis between x=a and x=b



If f(x) is continuous and f(x)≤0 for all x in [a,b],then $\int_a^b f(x)dx \leq 0$

And $-\int_a^b f(x)dx$ equals the area of the region bounded by the graph f and the x-axis between x=a and x=b



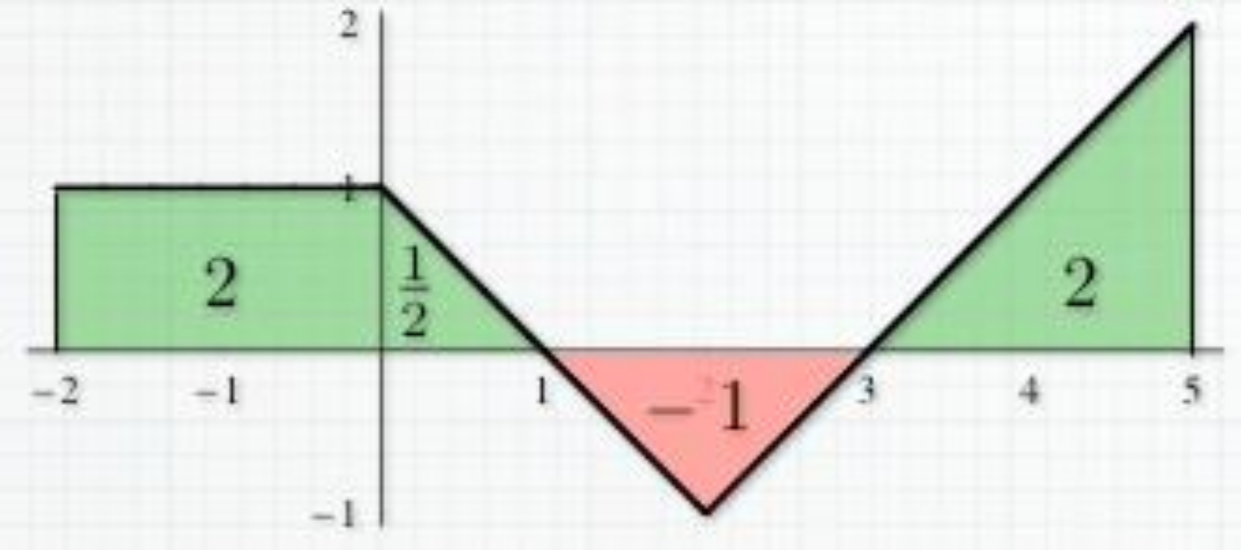


$\int_a^b f(x)dx$ equals the difference between the area under the graph of *f* above the *x*-axis and the area above the graph of f below the x-axis between *x*=*a* and *x*=*b*.

This is the net area of the region bounded by the graph of *f* and the *x*-axis between *x*=*a* and *x*=*b*.

Example

Let $f(x) = \begin{cases} 1, & \text{if } x < 0 \\ 1 - x, & \text{if } 0 \leq x < 2. \\ x - 3, & \text{if } 2 \leq x \end{cases}$ Find $\displaystyle\int_{-2}^{5} f(x)\,dx$.



$$\int_{-2}^{5} f(x)\,dx = 2 + \frac{1}{2} - 1 + 2 = \frac{7}{2}$$

EXAMPLE 4.2 A Riemann Sum for a Function with Positive and Negative Values

For $f(x) = \sin x$ on $[0, 2\pi]$, give an area interpretation of $\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$.

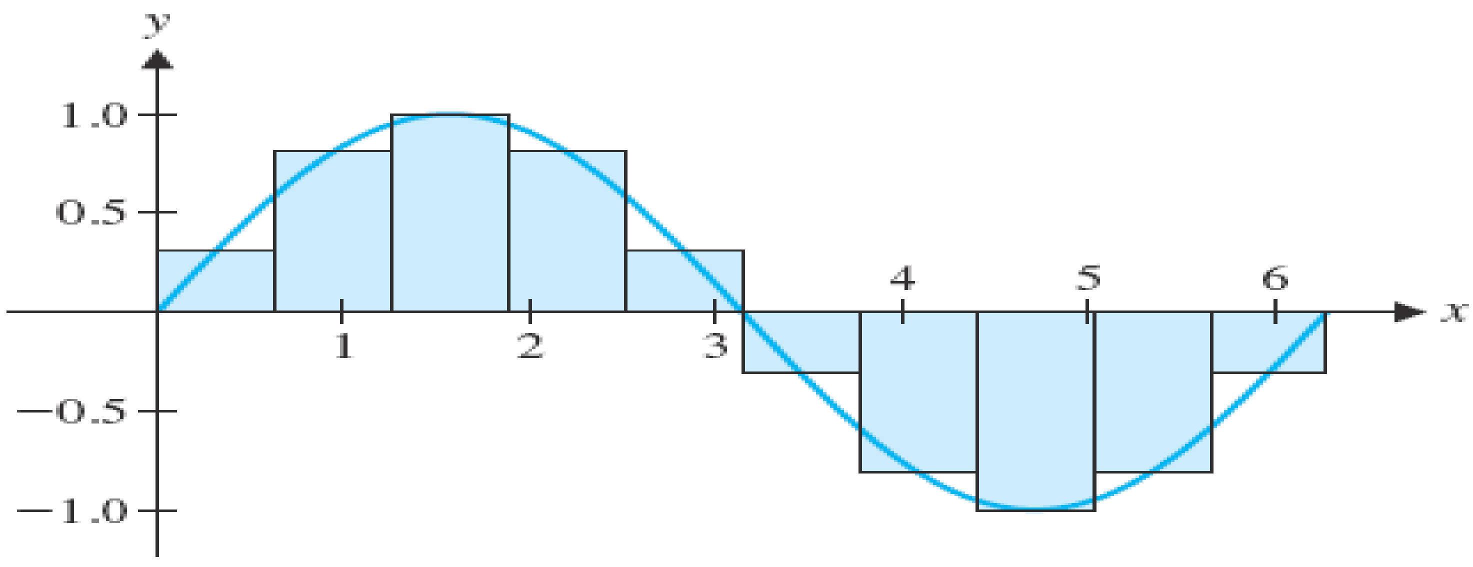


FIGURE 4.17a
Ten rectangles

EXAMPLE 4.3 Using Riemann Sums to Compute a Definite Integral

Compute $\int_0^2 (x^2 - 2x)\,dx$ exactly.

DEFINITION 4.2

Suppose that $f(x) \geq 0$ on the interval $[a, b]$ and A_1 is the area bounded between the curve $y = f(x)$ and the x-axis for $a \leq x \leq b$. Further, suppose that $f(x) \leq 0$ on the interval $[b, c]$ and A_2 is the area bounded between the curve $y = f(x)$ and the x-axis for $b \leq x \leq c$. The **signed area** between $y = f(x)$ and the x-axis for $a \leq x \leq c$ is $A_1 - A_2$, and the **total area** between $y = f(x)$ and the x-axis for $a \leq x \leq c$ is $A_1 + A_2$ (see Figure 4.19).

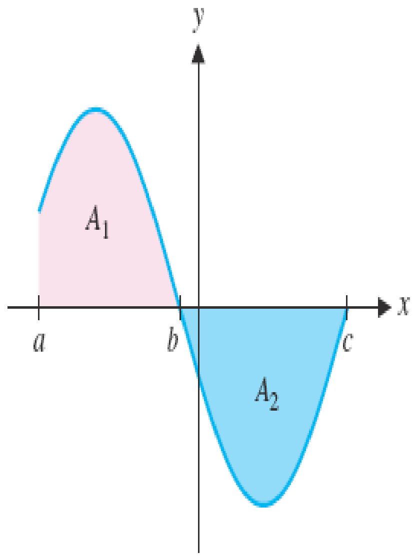


FIGURE 4.19
Signed area

EXAMPLE 4.4 Relating Definite Integrals to Signed Area

Compute three related integrals: $\int_0^2 (x^2 - 2x)\,dx$, $\int_2^3 (x^2 - 2x)\,dx$ and $\int_0^3 (x^2 - 2x)\,dx$, and interpret each in terms of area.

THEOREM 4.1

If f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$.

THEOREM 4.2

If f and g are integrable on $[a, b]$, then the following are true.

(i) For any constants c and d, $\int_a^b [cf(x) + dg(x)]\,dx = c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx$ and

(ii) For any c in $[a, b]$, $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

THEOREM 4.3

Suppose that $g(x) \le f(x)$ for all $x \in [a, b]$ and that f and g are integrable on $[a, b]$. Then,

$$\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx.$$

- The Average Value of Function

The average value of a Function Let $f(x)$ be a function that is continuous on the interval $a \le x \le b$. Then the average value V of $f(x)$ over $a \le x \le b$ is given by the definite integral

$$V = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Average value of a Function

Suppose that *f* is continuous on *[a,b]* , what is average value of the function *f(x)* over the interval $a \le x \le b$?

1. Subdivide the interval $a \le x \le b$ *into n equal parts*

$$[x_{i-1}, x_i],\ i = 1, 2, \ldots, n \qquad \Delta x = \frac{b-a}{n}$$

2. Choose x_j *from the jth subinterval for j=1,2…,n. Then the average of corresponding functional value* $f(x_1), f(x_2), \ldots f(x_n)$ *is*

$$V_n = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

$$V_n = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

$$= \frac{b-a}{b-a}\left[\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}\right]$$

$$= \frac{1}{b-a}[f(x_1) + f(x_2) + \cdots + f(x_n)]\left(\frac{b-a}{n}\right)$$

$$= \frac{1}{b-a}[f(x_1) + f(x_2) + \cdots + f(x_n)]\,\Delta x$$

$$= \frac{1}{b-a}\sum_{j=1}^{n} f(x_j)\,\Delta x$$

Riemann Sum

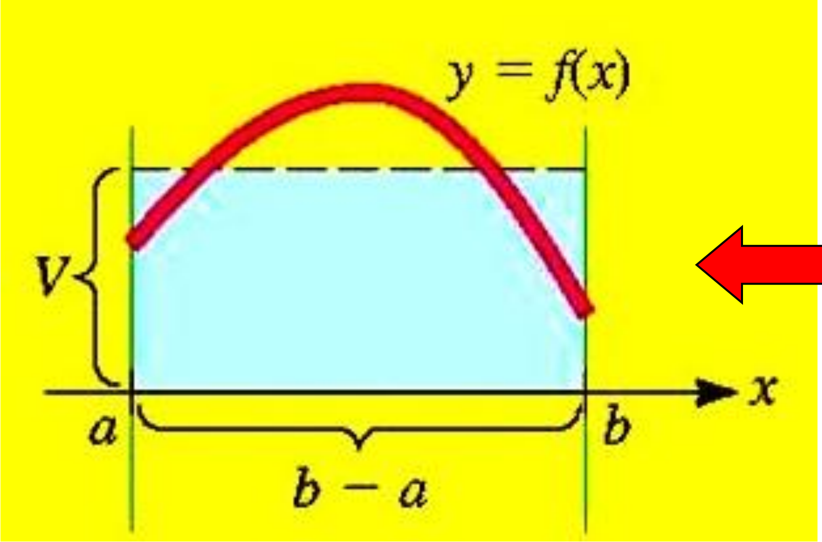
3. *Refine the partition of the interval a≤x≤b by taking more and more subdivision Points. Then V_n becomes more and more like the average value of V over the interval [a,b].The average value V can be think as the limit of the Riemann sum V_n as $n\to\infty$. That is ,*

$$V = \lim_{n\to+\infty} V_n = \lim_{n\to+\infty} \frac{1}{b-a}\sum_{j=1}^{n} f(x_j)\Delta x$$

$$= \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$V = \frac{1}{b-a}\int_a^b f(x)\,dx \Longrightarrow (b-a)V = \int_a^b f(x)\,dx$$

Geometric Interpretation of Average Value The average value V of $f(x)$ over an interval $a \le x \le b$ where $f(x)$ is continuous and satisfies $f(x) \ge 0$ is equal to the height of a rectangle whose base is the interval and whose area is the same as the area under the curve $y=f(x)$ over $a \le x \le b$.



The rectangle with base $a \le x \le b$ and height V has the same area as the region under the curve $y=f(x)$ over $a \le x \le b$.

Integral Mean Value Theorem (积分中值定理)

THEOREM 4.4 (Integral Mean Value Theorem)

If f is continuous on $[a, b]$, then there is a number $c \in (a, b)$ for which

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Section 4.5 The Fundamental Theorem of Calculus(微积分基本定理)

THEOREM 5.1 (Fundamental Theorem of Calculus, Part I)

If f is continuous on $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{5.1}$$

REMARK 5.1

We will often use the notation

$$F(x)\Big|_a^b = F(b) - F(a).$$

This enables us to write down the antiderivative before evaluating it at the endpoints.

EXAMPLE 5.2 Computing a Definite Integral Exactly

Compute $\int_1^4 \left(\sqrt{x} - \frac{1}{x^2}\right) dx$.

EXAMPLE 5.3 Using the Fundamental Theorem to Compute Areas

Find the area under the curve $f(x) = \sin x$ on the interval $[0, \pi]$.

EXAMPLE 5.4 A Definite Integral Involving an Exponential Function

Compute $\int_0^4 e^{-2x}\, dx$.

EXAMPLE 5.5 A Definite Integral Involving a Logarithm

Evaluate $\int_{-3}^{-1} \frac{2}{x}\, dx$.

THEOREM 5.2 (Fundamental Theorem of Calculus, Part II)

If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$, on $[a, b]$.

EXAMPLE 5.7 Using the Fundamental Theorem, Part II

For $F(x) = \int_1^x (t^2 - 2t + 3)\,dt$, compute $F'(x)$.

EXAMPLE 5.8 Using the Chain Rule and the Fundamental Theorem, Part II

If $F(x) = \int_2^{x^2} \cos t\,dt$, compute $F'(x)$.

EXAMPLE 5.9 An Integral with Variable Upper and Lower Limits

If $F(x) = \int_{2x}^{x^2} \sqrt{t^2 + 1}\,dt$, compute $F'(x)$.