PT Assignment 4

1. Let X be a random variable on a given probability space, and let $a \in \mathbb{R}$. Show that aX is a random variable.

2. A random variable X has distribution function F. What is the distribution function of $Y = aX + b$, where a and b are real constants?

3. A fair coin is tossed n times. Show that, under reasonable assumptions, the probability of exactly k heads is $\binom{n}{k}\left(\frac{1}{2}\right)^n$. What is the corresponding quantity when heads appears with probability p on each toss?

4. Each toss of a coin results in a head with probability p. The coin is tossed until the first head appears. Let X be the total number of tosses. What is $\mathbb{P}(X > m)$? Find the distribution function of the random variable X.

5. Show that if F and G are distribution functions and $0 \leq \lambda \leq 1$, then $\lambda F + (1-\lambda)G$ is a distribution function. Is the product FG a distribution function? [**Hint**: use the following characterization of distribution functions: F is the distribution function of some random variable if and only if it satisfies (a), (b), and (c) of Lemma 7 in Chapter 2.]

6. Let F be a distribution function and r a positive integer. Show that the following are distribution functions:

   (a) $F(x)^r$,

   (b) $1 - \{1 - F(x)\}^r$.

7. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}, P(\omega_1) = P(\omega_2) = P(\omega_3) = 1/3$, and define X, Y, and Z as follows:

$$X(\omega_1) = 1, X(\omega_2) = 2, X(\omega_3) = 3$$
$$Y(\omega_1) = 2, Y(\omega_2) = 3, Y(\omega_3) = 1$$
$$Z(\omega_1) = 3, Z(\omega_2) = 1, Z(\omega_3) = 2.$$

   Show that these three random variables have the same probability mass function. Find the probability mass functions of $X + Y, Y + Z$, and $Z + X$.

8. Let X be a discrete random variable with probability mass function

$$P_X(k) = \begin{cases} 0.1 & \text{for } k = 0 \\ c & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.2 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine c.

(b) Plot the distribution function of X.

(c) Find $P(X \leq 2 \mid X \geq 1)$.