Risk Management in Finance - Market Risk II – Volatility

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Outline of Market risk II

- Definition of volatility
- Implied Volatilities
- Are Daily Percentage Changes in Financial Variables Normal?
- The Power Law
- Monitoring Daily Volatility
- The Exponentially Weighted Moving Average Model
- The GARCH(1,1) Model
- Choosing Between the Models
- Maximum Likelihood Methods
- Using GARCH(1,1) to Forecast Future Volatility

Definition of Volatility

Definition of Volatility

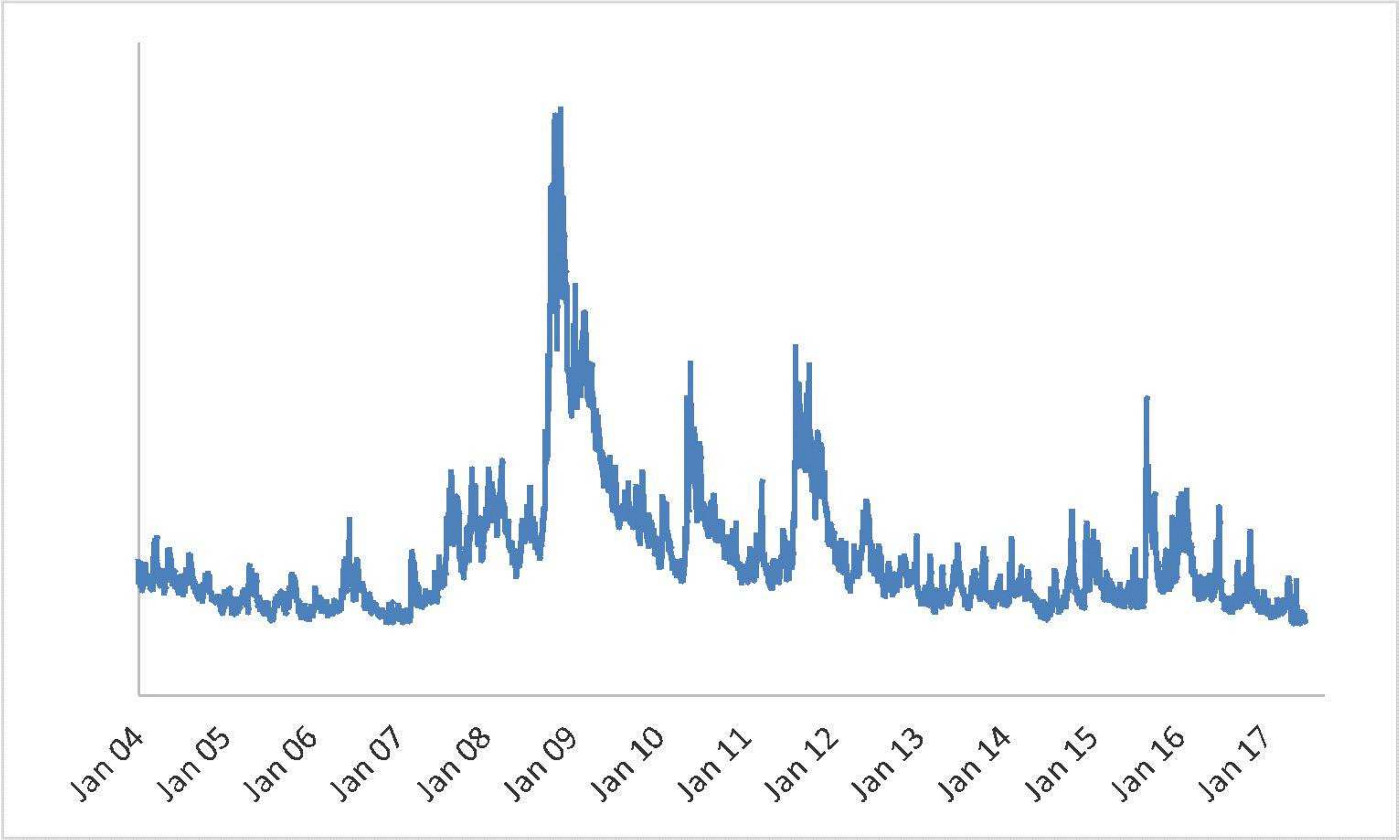
- Suppose that S_i is the value of a variable on day i. The volatility per day is the standard deviation of $\ln(S_j/S_{j-1})$
- Normally days when markets are closed are ignored in volatility calculations (see Business Snapshot 10.1)
- The volatility per year is $\sqrt{252}$ times the daily volatility
- Variance rate is the square of volatility

Implied Volatilities

Implied Volatility

- Of the variables needed to price an option the one that cannot be observed directly is volatility
- We can therefore imply volatilities from market prices and vice versa

VIX Index: A Measure of the Implied Volatility of the S&P 500



Are Daily Percentage Changes in Financial Variables normal?

Are Daily Percentage Changes Normally Distributed?

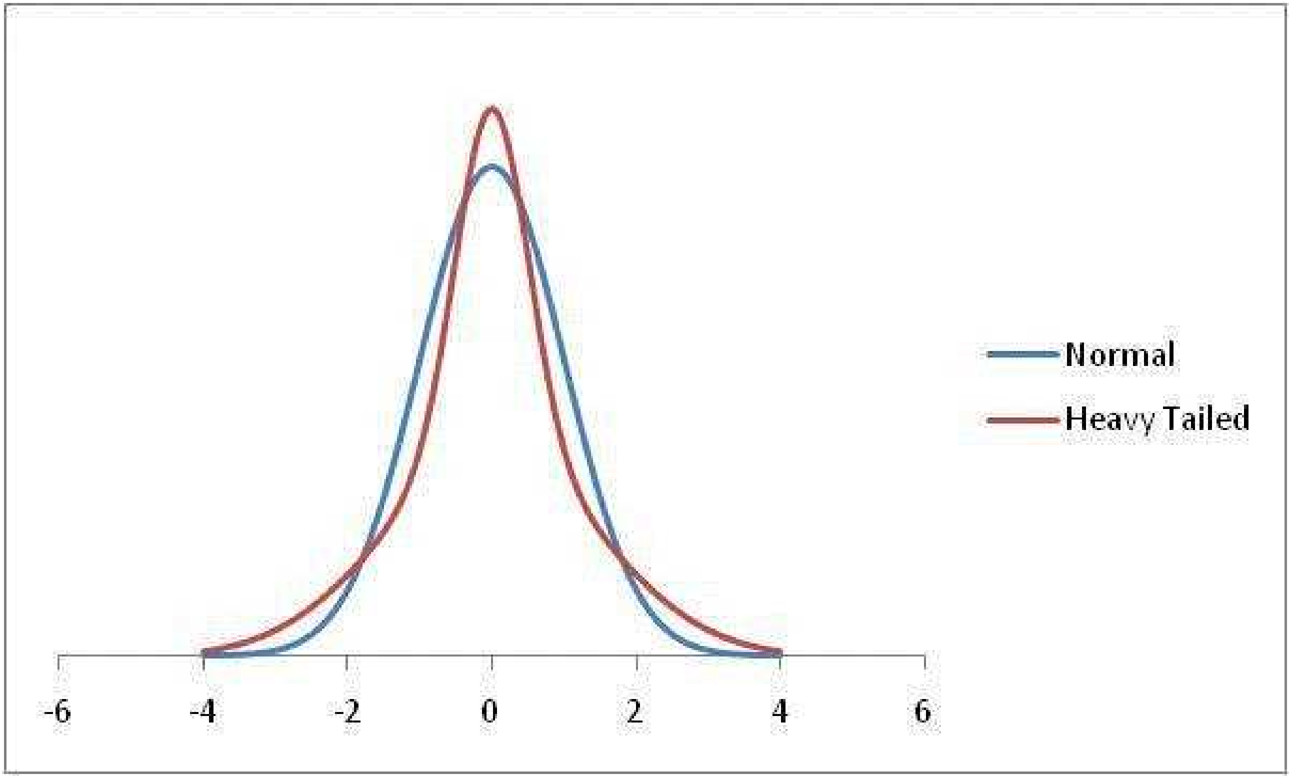
	Real World (%)	Normal Model (%)
$> 1\text{SD}$	23.32	31.73
$> 2\text{SD}$	4.67	4.55
$> 3\text{SD}$	1.30	0.27
$> 4\text{SD}$	0.49	0.01
$> 5\text{SD}$	0.24	0.00
$> 6\text{SD}$	0.13	0.00

The Power Law

Heavy Tails

- Daily exchange rate changes are not normally distributed
  - The distribution has heavier tails than the normal distribution
  - It is more peaked than the normal distribution
- This means that small changes and large changes are more likely than the normal distribution would suggest
- Many market variables have this property, known as excess kurtosis

Normal and Heavy-Tailed Distribution

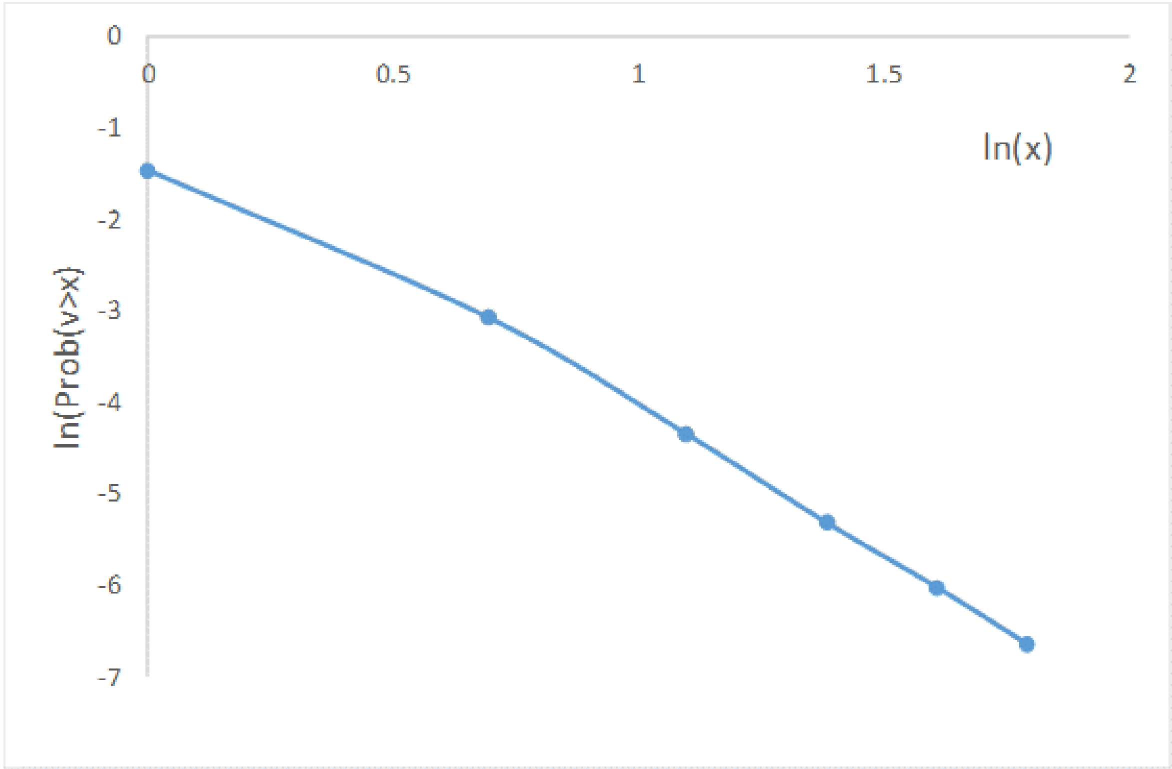


Alternatives to Normal Distributions: The Power Law

$$\text{Prob}(v > x) = Kx^{-\alpha}, \quad \text{for large enough } x > 0.$$

This seems to fit the behavior of the returns on many market variables better than the normal distribution

Log-Log Test for Exchange Rate Data



v is number of standard deviations which the exchange rate moves

Monitoring Daily Volatility

Standard Approach to Estimating Volatility

- Define σ_n as the volatility per day between day $n-1$ and day n, as estimated at end of day $n-1$
- Define S_i as the value of market variable at end of day i
- Define

$$u_i = \ln(S_i/S_{i-1})$$

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i} - \overline{u})^2$$

$$\overline{u} = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}$$

Simplifications Usually Made in Risk Management

- Define u_i as $(S_i - S_{i-1})/S_{i-1}$
- Assume that the mean value of u_i is zero
- Replace $m-1$ by m

This gives

$$\sigma_n^2 = \frac{1}{m} \sum_{i=1}^{m} u_{n-i}^2$$

Weighting Scheme

Instead of assigning equal weights to the observations we can set

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

vvhere

$$\sum_{i=1}^{m} \alpha_i = 1$$

ARCH(m) Model

In an ARCH(m) model we also assign some weight to the long-run variance rate V_L:

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

vvhere

$$\gamma + \sum_{i=1}^{m} \alpha_i = 1$$

The Exponentially Weighted Moving Average Model

EWMA Model

- In an exponentially weighted moving average model, the weights assigned to the u^2 decline exponentially as we move back through time

- This leads to

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2$$

Attractions of EWMA

- Relatively little data needs to be stored
- We need only remember the current estimate of the variance rate and the most recent observation on the market variable
- Tracks volatility changes
- $\lambda = 0.94$ has been found to be a good choice across a wide range of market variables

The GARCH(1,1) Model

GARCH (1,1)

In GARCH (1,1) we assign some weight to the long-run average variance rate

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2.$$

Since welghts must sum to 1,

$$\gamma + \alpha + \beta = 1.$$

GARCH (1,1) continued

Setting $\omega = \gamma V_L$ the GARCH (1,1) model is

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

and

$$V_L = \frac{\omega}{1 - \alpha - \beta}.$$

Example

- Suppose

$$\sigma_n^2 = 0.000002 + 0.13u_{n-1}^2 + 0.86\sigma_{n-1}^2$$

- The long-run variance rate is 0.0002 so that the long-run volatility per day is 1.4%

Example continued

- Suppose that the current estimate of the volatility is 1.6% per day and the most recent percentage change in the market variable is 1%.
- The new variance rate is

$$0.000002 + 0.13 \times 0.0001 + 0.86 \times 0.000256 = 0.00023336$$

- The new volatility is 1.53% per day

Choosing Between the models

GARCH (p,q)

$$\sigma_n^2 = \omega + \sum_{i=1}^{p} \alpha_i u_{n-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{n-j}^2$$

Other Models

- Many other GARCH models have been proposed
- For example, we can design a GARCH models so that the weight given to u_i^2 depends on whether u_i is positive or negative

Maximum Likelihood methods

Maximum Likelihood Methods

- In maximum likelihood methods we choose parameters that maximize the likelihood of the observations occurring

Case 1

- We observe that a certain event happens one time in ten trials. What is our estimate of the proportion of the time, p, that it happens?
- The probability of the outcome is

$$p(1-p)^9$$

- We maximize this to obtain a maximum likelihood estimate: $p = 0.1$

Case 2

Estimate the variance of observatlons from a normal distribution with mean zero:

- Maximize:

$$\prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi v}}\exp\left(\frac{-u_i^2}{2v}\right)\right]$$

- Same as maximizing:

$$\sum_{i=1}^{n}\left[-\ln(v)-\frac{u_i^2}{v}\right]$$

- Maximum value when $v = \frac{1}{n}\sum_{i=1}^{n} u_i^2$.

Application to EWMA and GARCH

We choose parameters that maximize

$$\prod_{i=1}^{m} \frac{1}{\sqrt{2\pi v_i}} \exp\left(-\frac{u_i^2}{2v_i}\right)$$

or

$$\sum_{i=1}^{m} \left[-\ln(v_i) - \frac{u_i^2}{v_i}\right].$$

S&P 500 Excel Application

- Start with trial values of parameters (λ for EWMA and ω, α, and β for GARCH(1,1))
- Update variances
- Calculate

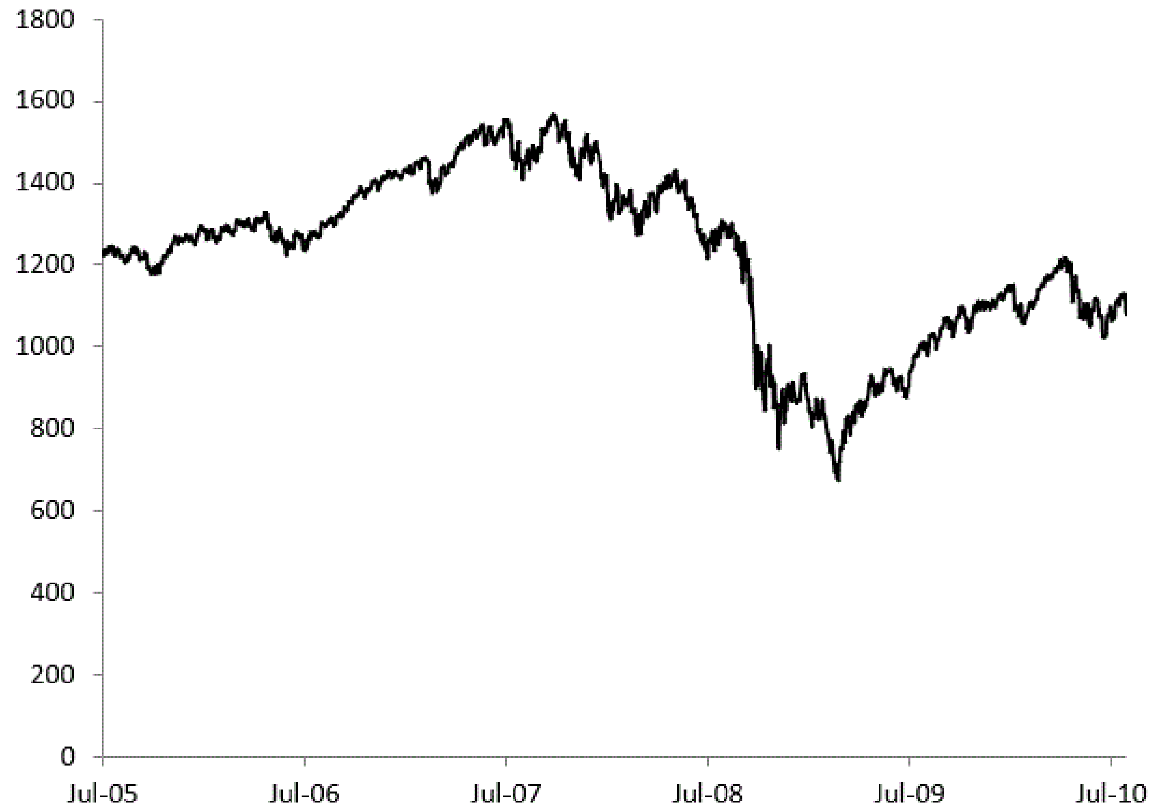
$$\sum_{i=1}^{m}\left[-\ln(v_i)-\frac{u_i^2}{v_i}\right]$$

- Use solver to search for values of parameters that maximize this objective function
- For efficient operation of Solver: set up spreadsheet so that ensure that search is over parameters that are of same order of magnitude and test alternative starting conditions

S&P 500 Excel Application II

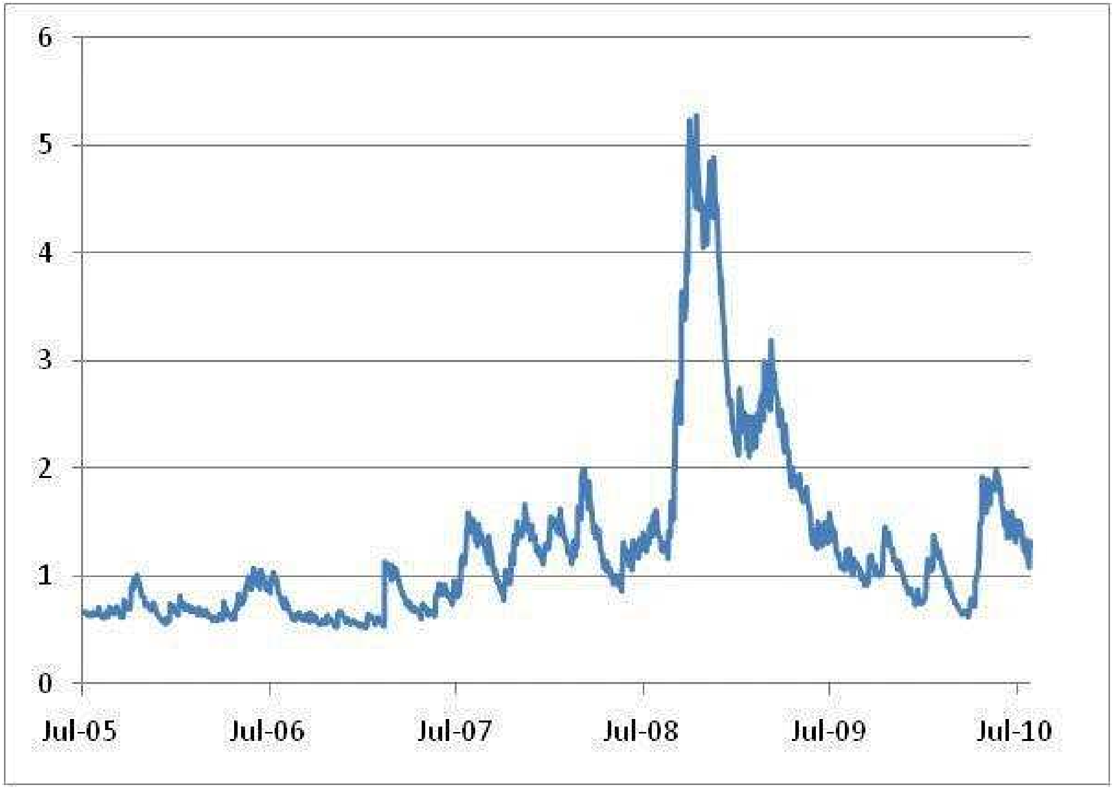
Date	*Day*	S_i	$u_i=(S_i-S_{i-1})/S_{i-1}$	$v_i=\sigma_i^2$	$-\ln(v_i)-u_i^2/v_i$
18-Jul-2005	1	1221.13			
19-Jul-2005	2	1229.35	0.006731		
20-Jul-2005	3	1235.20	0.004759	0.00004531	9.5022
21-Jul-2005	4	1227.04	−0.006606	0.00004447	9.0393
…….	…..	…….	………..	………….	…………
13-Aug-2010	1279	1079.25	−0.004024	0.00016327	8.6209
Total					10,228.2349

The S&P 500



1800
1600
1400
1200
1000
800
600
400
200
0
Jul-05
Jul-06
Jul-07
Jul-08
Jul-09
Jul-10

The GARCH Estimate of Volatility of the S&P 500



$$\omega = 0.0000013465,\ \alpha = 0.083394,\ \beta = 0.910116$$

Variance Targeting

- One way of implementing GARCH(1,1) that increases stability is by using variance targeting
- The long-run average variance equal to the sample variance
- Only two other parameters then have to be estimated

How Good is the Model?

- The Ljung-Box statistic tests for autocorrelation
- We compare the autocorrelation of the u_i^2 with the autocorrelatlon of the u_i^2/σ_i^2

Using GARCH(1,1) to Forecast Future Volatility

Forecasting Future Volatility

A few lines of algebra shows that

$$E[\sigma_{n+k}^2] = V_L + (\alpha + \beta)^k (\sigma_n^2 - V_L)$$

The variance rate for an option expiring on day m is

$$\frac{1}{m} \sum_{k=0}^{m-1} E\left[\sigma_{n+k}^2\right]$$

Forecasting Future Volatility continued

Define

$$a = \ln \frac{1}{\alpha + \beta}.$$

The estimated volatility per annum for an option lasting T days is

$$\sqrt{252 \left(V_L + \frac{1 - e^{-aT}}{aT} [V(0) - V_L] \right)}.$$

S&P Example

- $\omega = 0.0000013465,\ \alpha = 0.083394,\ \beta = 0.91016$
- $a = \ln \dfrac{1}{0.083394 + 0.910116} = 0.006511$

Option Life (days)	10	30	50	100	500
Est. Volatility (% per annum)	27.36	27.10	26.87	26.35	24.32

Results for S&P 500

When instantaneous volatility changes by 1%

Option Life (days)	10	30	50	100	500
Volatility increase (%)	0.97	0.92	0.87	0.77	0.33