Chap6_Week12

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§ 6.4 Hermitian Matrix (Cont.)

Thm If the eigenvalues of a Hermitian matrix A are distinct, then $\exists$ a unitary matrix U that diagonalizes A.

E.g. $A = \begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix}$

$\lambda_1 = 0$, $\lambda_2 = 3$

$\vec{x}_1 = \begin{bmatrix} -1 \\ 1+i \end{bmatrix}$ $\quad \vec{x}_2 = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$

$\vec{x}_1^H = \begin{pmatrix} -1 \\ 1-i \end{pmatrix}^T$

Let

$\vec{u}_1 = \frac{1}{\|\vec{x}_1\|}\vec{x}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1+i \end{pmatrix}$

Since $\|\vec{x}_1\| = \sqrt{\vec{x}_1^H \vec{x}_1} = \sqrt{(-1, 1-i)\begin{pmatrix} -1 \\ 1+i \end{pmatrix}} = \sqrt{1+(1-i)(1+i)} = \sqrt{3}$

$\vec{u}_2 = \frac{1}{\|\vec{x}_2\|}\vec{x}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1-i \\ 1 \end{pmatrix}$

$\|\vec{x}_2\| = \sqrt{(1+i, 1)\begin{pmatrix} 1-i \\ 1 \end{pmatrix}} = \sqrt{(1+i)(1-i)+1} = \sqrt{3}$

The unitary matrix U:

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & 1-i \\ 1+i & 1 \end{pmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{or} \quad D = U^H A U = \cdots$$

So $A = \frac{1}{3}\begin{bmatrix} -1 & 1-i \\ 1+i & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1-i \\ 1+i & 1 \end{bmatrix}$

Thm (Schur's Thm)

For each $n \times n$ matrix A, there exits a unitary matrix U s.t.

$T = U^H A U$ is upper triangular.

(proof in the textbook) 数学归纳法 + 构造法

Remark: The factorization $A = UTU^H$ is called the Schur decomposition of A. In the case A is Hermitian, the matrix T will be diagonal.

Thm (Spetral Thm)

If A is Hermitian, then there exists a unitary matrix U s.t.

$$A = UDU^H$$

where D is a diagonal matrix.

proof $A \in \mathbb{C}^{n \times n}$, then A has the Schur decomposition as

$$A = UTU^H$$

Since A is Hermitian, $A = A^H$, then

$$T^H = (U^H A U)^H = U^H A^H U = U^H A U = T$$

Because T is upper triangular, so T must be diagonal with $\overline{t_{ii}} = t_{ii}$, for $i = 1, 2, \ldots, n$

Hence T must be a real-entried diagonal matrix.

Thm (Real Schur's Thm)*

If $A \in \mathbb{R}^{n \times n}$, then A can be factorized as

$$A = QTQ^T$$

where Q is an orthogonal matrix and T is an block upper triangular matrix (in Schur form)

$$T = \begin{bmatrix} B_1 & * & * \cdots & * \\ & B_2 & & \\ & 0 & \ddots & * \\ & & & B_j \end{bmatrix}$$

B_i's are either 1×1 or 2×2 matrices.

Thm (Real Spectral Thm)

If $A \in \mathbb{R}^{n \times n}$, A is real symmetric iff. A is orthogonally diagonalizable. $A = QDQ^T$.

Example $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

Find the Schur's decomposition of A and B.

Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = \lambda_3 = 2$

Eigenvectors: for $\lambda = 4$

$(A - 4I)\vec{u} = \vec{0}$

$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(B - 4I)\vec{x}_1 = \vec{0}$

rank 2
nullity 1

$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$(A-4I)\vec{u}=\vec{0}$$

rank 2 → nullity 1

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} a=0 \\ c=0 \\ b\in\mathbb{R} \end{cases} \rightarrow \vec{v}_1=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\alpha,\ \alpha\neq 0$$

$$E_{\lambda_A=4}=\text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

rank 2 nullity 1

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} d \\ e \\ f \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} d=0 \\ e=2f \\ f\in\mathbb{R} \end{cases} \rightarrow \vec{x}_1=\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\alpha,\ \alpha\neq 0$$

$$E_{\lambda_B=4}=\text{span}\left\{\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\right\}.$$

Eigenvectors for $\lambda=2$

$$(A-2I)\vec{v}_2=\vec{0}$$

rank 2 nullity 1

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\vec{v}_2=\vec{0}$$

$$\vec{v}_2=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\beta,\quad \beta\neq 0$$

$$E_{\lambda_A=2}=\text{span}\{\vec{e}_3\}$$

$$(B-2I)\vec{x}_2=\vec{0}$$

rank 1 nullity 2

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}\vec{x}_2=\vec{0}$$

$$\rightarrow \vec{x}_2=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\beta+\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\gamma,$$ β,γ are not both 0.

$$E_{\lambda_B=2}=\text{span}\{\vec{e}_1,\vec{e}_3\}$$

(Notice that A is deficient and B is diagonalizable.)

Schur's decomp: $A=UTU^H$ ① Form U ② Compute $T=U^HAU$

Take B as an example:

① Finding U: Matrix B has $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ linearly independent eigenvectors

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let $\vec{u}_1=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, since $\vec{u}_2=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\vec{u}_1\perp\vec{u}_2$ and $\|\vec{u}_1\|=\|\vec{u}_2\|=1$

Take

$$\vec{v}_3=\underbrace{\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}}_{\vec{x}_3}-\underbrace{\text{proj}_{\vec{u}_1}\vec{x}_3-\text{proj}_{\vec{u}_2}\vec{x}_3}_{\text{proj}_{\text{span}\{\vec{u}_1,\vec{u}_2\}}\vec{x}_3}=\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}-\frac{\vec{x}_3^T\vec{u}_1}{\|\vec{u}_1\|^2}\vec{u}_1-\frac{\vec{x}_3^T\vec{u}_2}{\|\vec{u}_2\|^2}\vec{u}_2$$

$$\vec{v}_3=\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}-\vec{0}-\frac{1}{1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

let $\vec{u}_3=\frac{1}{\|\vec{v}_3\|}\vec{v}_3=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Let $U=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$$T=U^HBU=\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 4 & 0 \end{bmatrix}U$$

$$=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$ ← upper triangular

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$ ← upper triangular

The Schur decomposition of A: left as exercise

$\forall A \in \mathbb{C}^{n\times n}$ $\xrightarrow[\text{decomp}]{\text{Schur}}$ $A = UTU^H$

U: unitary, T: upper triangular

A is unitarily similar to T.

When A Hermitian $A = A^H$ ↓

spectral decomp.

$A = UDU^H$

U: unitary, D: diagonal

$\xrightarrow[\substack{A = A^T \\ \text{real symmetric}}]{A \in \mathbb{R}^{n\times n}}$ $A = QDQ^T$

Q: orthogonal

Diagonalization: $\mathbb{R}^{n\times n}$

$\forall A \in \mathbb{R}^{n\times n}$

diagonalizable → $A = PDP^{-1}$

P: nonsingular, D: diagonal

deficient → $A = CJC^{-1}$

C: nonsingular, J: Jordan form.

SVD
(Singular Value decomp)

$$A_{m\times n} = U_{m\times m} \Sigma_{m\times n} V_{n\times n}^T$$
$(m \neq n)$

where U and V are both orthogonal matrices.

We name the diagonal entries of Σ as "singular values" of A.

WLOG, we denote $m > n$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}_{m\times n} = \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{m\times m}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ \hline & O_{(m-n)\times n} & \end{bmatrix}_{m\times n}}_{\Sigma} \underbrace{\begin{bmatrix} & & \\ & & \end{bmatrix}_{n\times n}}_{V^T}$$

Where the columns of U are eigenvectors of AA^T

the columns of V are eigenvectors of A^TA

the diagonal entries of Σ are $\sigma_i = \sqrt{\lambda_i}$, with λ_i as eigenvalues of A^TA.

Consider

$$\begin{aligned} A^TA &= V\,\Sigma^T U^T U \Sigma V^T \\ &= V\,\Sigma^T \Sigma\, V^T \\ &= \underset{\text{orthogonal}}{V} \underset{\text{diagonal}}{\begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix}_{n\times n}} V^T \end{aligned}$$

$$\Sigma^T = \begin{bmatrix} \underset{(\Sigma_1^T)_{n\times n}}{\diagdown} & \Big| & \underset{n\times(m-n)}{0} \end{bmatrix}_{n\times m} \qquad \Sigma = \begin{bmatrix} \diagdown \\ \hline O_{(m-n)\times n} \end{bmatrix}_{m\times n} \begin{matrix} (\Sigma_1)_{n\times n} \\ \\ \end{matrix}$$

$$\rightarrow \Sigma^T\Sigma = \Sigma_1^T\Sigma_1 + O_{n\times(m-n)}\,O_{(m-n)\times n}$$

$$= \Sigma_1^T\Sigma_1 \underset{\substack{\text{since} \\ \Sigma_1 \text{ diagonal}}}{=} \Sigma_1^2 = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix}$$

A^TA is symmetric and diagonalized by an orthogonal matrix V.

Thus, the columns of V are eigenvectors of A^TA

the diagonal entries of Σ_1^2 are eigenvalues of A^TA

Consider

$$\begin{aligned} AA^T &= U\,\Sigma V^T V \Sigma^T U^T \\ &= U\,\Sigma\Sigma^T U^T \\ &= \underset{\text{orthogonal}}{U} \underset{\text{diagonal}}{\left[\begin{array}{c|c} \Sigma_1^2 & 0 \\ \hline 0 & 0 \end{array}\right]} U^T \end{aligned}$$

$$\rightarrow \Sigma\Sigma^T = \begin{bmatrix} \Sigma_1 \\ \hline O_{(m-n)\times n} \end{bmatrix}_{m\times n} \left[\begin{array}{c|c} \Sigma_1^T & 0^T \end{array}\right]_{n\times m}$$

$$= \left[\begin{array}{c|c} \Sigma_1\Sigma_1^T & \Sigma_1 O_{n\times(m-n)} \\ \hline O_{(m-n)\times n}\Sigma_1^T & O_{(m-n)\times(m-n)} \end{array}\right]_{m\times m}$$

$$= \left[\begin{array}{ccc|c} \sigma_1^2 & & & \\ & \ddots & & 0 \\ & & \sigma_n^2 & \\ \hline & 0 & & 0 \end{array}\right]_{m\times m}$$

AA^T is symmetric and diagonalized by an orthogonal matrix U.

Thus, the columns of U are eigenvectors of AA^T

the diagonal entries of Σ_1^2 are eigenvalues of AA^T