AFM Midterm Solutions

Question 1

Given the moments of Brownian motion W_u:

$$E(W_u^{2k}) = \frac{(2k)!}{k!2^k}u^k$$

for $u > 0$ and $k = 1, 2, 3...$

(a) Compute $E[(W_t^2 + 7)^2]$:

$$(W_t^2 + 7)^2 = W_t^4 + 14W_t^2 + 49$$
$$E[W_t^4] = 3t^2, \quad E[W_t^2] = t$$
$$E[(W_t^2 + 7)^2] = 3t^2 + 14t + 49$$

(b) Compute $E[(W_{12} - W_5 + 7)^3]$:

$$Z = W_{12} - W_5 \sim N(0, 7)$$
$$(Z + 7)^3 = Z^3 + 21Z^2 + 147Z + 343$$
$$E[(Z + 7)^3] = 21 \times 7 + 343 = 490$$

(c) Compute $E[W_s^4 W_t^2]$ for $t \geq s \geq 0$:

$$W_t = (W_t - W_s) + W_s, \quad Z = W_t - W_s \sim N(0, t - s)$$
$$W_s^4 W_t^2 = W_s^6 + 2W_s^5 Z + W_s^4 Z^2$$
$$E[W_s^4 W_t^2] = 15s^3 + 3s^2(t - s) = 12s^3 + 3ts^2$$

Question 2

1. Take $F(t, W_t) = f(t)e^{\lambda W_t}$. Then,

$$\frac{\partial F}{\partial t} = f'(t)e^{\lambda W_t}, \ \frac{\partial F}{\partial W_t} = \lambda f(t)e^{\lambda W_t}, \ \frac{\partial^2 F}{\partial W_t^2} = \lambda^2 f(t)e^{\lambda W_t}.$$

As

$$dF(t, W_t) = \left[\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W_t^2}\right]dt + \frac{\partial F}{\partial W_t}dW_t,$$

$$dX_t = \left[\frac{g'(t)}{g(t)} + \frac{\lambda^2}{2}\right]X_t dt + \lambda X_t dW_t,$$

We have, $ae^{-at} = \frac{f'(t)}{f(t)} + \frac{\lambda^2}{2}$ and $b = \lambda$.

Hence,

$$d(ln(f(t)) = ae^{-at} - \frac{b^2}{2}$$

$$ln(f(t)) - ln(f(0)) = 1 - e^{-at} - \frac{b^2}{2}t, f(0) = c$$

$$f(t) = ce^{1-e^{-at}-\frac{b^2}{2}t}$$

Question 3

A stochastic process is generated by the equation

$$dX_t = X_t(adt + bdW_t)$$

with the initial condition of $X_0 = 9$. Which equation governs the process $Y(t, X_t) = e^{\lambda t} X_t^n$?

According to Ito's lemma, we have

$$dY_t = \left[\frac{\partial Y}{\partial t} + \mu \frac{\partial Y}{\partial X_t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 Y}{\partial X_t^2}\right] dt + \sigma X_t \frac{\partial Y}{\partial X_t} dW_t.$$

Since

$$\frac{\partial Y}{\partial t} = \lambda Y, \ \frac{\partial Y}{\partial X_t} = \frac{n}{X_t} Y, \ \frac{\partial^2 Y}{\partial X_t^2} = \frac{n(n-1)}{X_t^2} Y$$

we have

$$dY_t = \left[\lambda + an + \frac{n(n-1)b^2}{2}\right] Y_t dt + bnY_t dW_t.$$

with the initial condition of $Y_0 = 9^n$.

Question 4

Solve the SDE:

$$dS_t = S_t(\lambda \cos t dt + \sigma dW_t), \quad S_0 = x$$

Take

$$Y_t = ln(S_t),$$

apply Ito's lemma to

$$Y_t.$$

Since

$$\frac{\partial Y}{\partial t} = 0, \ \frac{\partial Y}{\partial S_t} = \frac{1}{S_t}, \ \frac{\partial^2 Y}{\partial X_t^2} = -\frac{1}{S_t^2}$$

$$dln(S_t) = dY_t = (\lambda \cos t - \frac{1}{2}\sigma^2)dt + \sigma dW_t.$$

We have

$$lnS_t - lnS_0 = \lambda \sin t - \frac{1}{2}\sigma^2 t + \sigma W_t$$

$$S_t = x \exp\left(\lambda \sin t - \frac{1}{2}\sigma^2 t + \sigma W_t\right)$$

Question 5

Heat equation:

$$U_t = U_{xx}, \quad U(x,0) = 3 + 7x$$

$$U(t,x) = \int_{-\infty}^{\infty} (3 + 7x')G(x - x')\,\mathrm{d}x'$$
$$= 3 + 7\int_{-\infty}^{\infty} (x' - x + x)G(x - x')\,\mathrm{d}x'$$
$$= 3 + 7\int_{-\infty}^{\infty} (x' - x)G(x - x')\,\mathrm{d}x' + 7x\int_{-\infty}^{\infty} G(x - x')\,\mathrm{d}x'$$
$$= 3 + 0 + 7x$$
$$= 3 + 7x$$

Solution:

$$U(x,t) = 3 + 7x$$

Question 6

Heat equation:

$$U_t = (3 + 7t)U_{xx}, \quad U(x,0) = 11x^2$$

We choose

$$\tau = 3t + \frac{7}{2}t^2$$

Then

$$V_\tau = V_{xx} \quad V(\tau,0) = 11x^2$$
$$V(t,x) = \int_{-\infty}^{\infty} 11x'^2 G(x - x')\,\mathrm{d}x'$$
$$= 11\int_{-\infty}^{\infty} (x' - x + x)^2 G(x - x')\,\mathrm{d}x'$$
$$= 11\int_{-\infty}^{\infty} (x'-x)^2 G(x-x')\,\mathrm{d}x' + 22x\int_{-\infty}^{\infty} (x-x')G(x-x')\,\mathrm{d}x' + 11x^2\int_{-\infty}^{\infty} G(x-x')\,\mathrm{d}x'$$
$$= 22\tau + 0 + 11x^2$$
$$= 11x^2 + 66t + 77t^2$$
$$V(x,t) = 11x^2 + 66t + 77t^2$$