SPA

Solution to Assignment 1

1. Let R, G, B respectively represent the red, green and blue marbles.
   (a) The sample space is $\Omega = \{\mathrm{RR}, \mathrm{RG}, \mathrm{RB}, \mathrm{GR}, \mathrm{GG}, \mathrm{GB}, \mathrm{BR}, \mathrm{BG}, \mathrm{BB}\}$. Since all outcomes have the same probability, the probability of each point in the sample space is $\frac{1}{9}$.
   (b) The sample space is $\Omega = \{\mathrm{RG}, \mathrm{RB}, \mathrm{GR}, \mathrm{GB}, \mathrm{BR}, \mathrm{BG}\}$. Since all outcomes have the same probability, the probability of each point in the sample space is $\frac{1}{6}$.
2. The evens can be represented as follows:
   (a) $F \cap E^c \cap G^c$
   (b) $E \cap F \cap G^c$
   (c) $E \cup F \cup G$
   (d) $(E \cap F) \cup (E \cap G) \cup (F \cap G)$
   (e) $E \cap F \cap G$
   (f) $E^c \cap F^c \cap G^c$
   (g) $(E \cap G^c \cap F^c) \cup (E^c \cap G \cap F^c) \cup (E^c \cap G^c \cap F) \cup (E^c \cap F^c \cap G^c)$
   (h) $E^c \cup F^c \cup G^c$.
3. For two events E and F, we have $\mathbb{P}(E) + \mathbb{P}(F) = \mathbb{P}(E \cap F) + \mathbb{P}(E \cup F)$. So

$$\begin{aligned}\mathbb{P}(E \cap F) &= \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cup F)\\ &\geqslant \mathbb{P}(E) + \mathbb{P}(F) - 1.\end{aligned}$$

   In the case where $\mathbb{P}(E) = 0.9$ and $\mathbb{P}(F) = 0.8$, we get $\mathbb{P}(E \cap F) \geqslant 0.9 + 0.8 - 1 = 0.7$.
4. Recall that $F_1 = E_1, F_i = E_i \bigcap_{j=1}^{i-1} E_j^c$, $j = 2, \ldots, n$. Since $F_i \subset E_i$, it is easy to see that $\bigcup_{i=1}^n F_i \subset \bigcup_{i=1}^n E_i$. We now show that $\bigcup_{i=1}^n E_i \subset \bigcup_{i=1}^n F_i$. For this, let's take an arbitrary element $\omega \in \bigcup_{i=1}^n E_i$. Suppose ω first enters the sets E_i at index k, that is, $\omega \notin E_1, E_2, \ldots, E_{k-1}$ but $\omega \in E_k$. Then $\omega \in F_k = E_k \bigcap_{j=1}^{k-1} E_j^c$ and thus

$\omega \in \bigcup_{i=1}^{n} F_i$. Because $\omega \in \bigcup_{i=1}^{n} E_i$ is arbitrary, we have shown that $\bigcup_{i=1}^{n} E_i \subset \bigcup_{i=1}^{n} F_i$. So $\bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} F_i$.

On the other hand, it is not hard to check F_1, F_2, ...F_n are mutually disjoint. So by finite additivity, $\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} F_i\right) = \sum_{i=1}^{n} \mathbb{P}(F_i) \leqslant \sum_{i=1}^{n} \mathbb{P}(E_i)$.

5. We first remark that (a) is equivalent to finding the conditional probability given the condition that the first $k-1$ customers do not get the prizewinning ticket, and (b) is equivalent to finding a probability without any additional condition.

   Let $A_i = \{$ the i-th customer gets the prizewinning ticket $\}$. Then the condition in (a) is $A_1^c \cap A_2^c \cap \cdots \cap A_{k-1}^c$. We can obtain by a direct application of classical probability model

$$\mathbb{P}\left(A_k \mid A_1^c \cap A_2^c \cap \cdots \cap A_{k-1}^c\right) = \frac{1}{n-k+1}.$$

   As for (b), $A_k = A_1^c \cap A_2^c \cap \cdots \cap A_{k-1}^c \cap A_k$ obviously holds. So by the product rule, we have

$$\begin{aligned}
\mathbb{P}(A_k) &= \mathbb{P}\left(A_1^c \cap A_2^c \cap \cdots \cap A_{k-1}^c \cap A_k\right) \\
&= \mathbb{P}(A_1^c)\,\mathbb{P}(A_2^c \mid A_1^c)\,\mathbb{P}(A_3^c \mid A_1^c \cap A_2^c) \cdots \mathbb{P}\left(A_k \mid A_1^c \cap A_2^c \cap \cdots \cap A_{k-1}^c\right) \\
&= \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{n-3}{n-2} \cdots \frac{n-k+1}{n-k+2} \cdot \frac{1}{n-k+1} \\
&= \frac{1}{n}.
\end{aligned}$$

   $\mathbb{P}(A_k) = 1/n$ implies that the probability of one customer getting the prizewinning ticket has nothing to do with when he or she buys a lottery ticket.

6. Suppose $X_1 \sim Poi(\lambda_1), X_2 \sim Poi(\lambda_2)$, and X_1 and X_2 are independent.

(a) Then

$$
\begin{aligned}
\mathbb{P}\left(X_1+X_2=k\right) &= \sum_{j=0}^{k}\mathbb{P}(X_1+X_2=k, X_2=j)\\
&= \sum_{j=0}^{k}\mathbb{P}(X_1=k-X_2|X_2=j)\mathbb{P}(X_2=j)\\
&= \sum_{j=0}^{k}\mathbb{P}(X_1=k-j|X_2=j)\mathbb{P}(X_2=j)\\
&= \sum_{j=0}^{k}\mathbb{P}(X_1=k-j)\frac{\lambda_2^j}{j!}e^{-\lambda_2}\\
&= \sum_{j=0}^{k}\frac{\lambda_1^{k-j}}{(k-j)!}e^{-\lambda_1}\frac{\lambda_2^j}{j!}e^{-\lambda_2}\\
&= e^{-\lambda_2}e^{-\lambda_1}\frac{1}{k!}\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}\lambda_1^{k-j}\lambda_2^j\\
&= e^{-\lambda_2}e^{-\lambda_1}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda_1^{k-j}\lambda_2^j\\
&= \frac{(\lambda_1+\lambda_2)^k}{k!}e^{-(\lambda_1+\lambda_2)}.
\end{aligned}
$$

So $X_1+X_2 \sim Poi(\lambda_1+\lambda_2)$.

(b) It holds

$$
\begin{aligned}
\mathbb{P}(X_1=k|X_1+X_2=n) &= \frac{\mathbb{P}(X_1=k, X_1+X_2=n)}{\mathbb{P}(X_1+X_2=n)}\\
&= \frac{\mathbb{P}(X_1=k, X_2=n-k)}{\mathbb{P}(X_1+X_2=n)}\\
&= \frac{\mathbb{P}(X_1=k)\mathbb{P}(X_2=n-k)}{\mathbb{P}(X_1+X_2=n)}\\
&= \frac{\frac{\lambda_1^k}{k!}e^{-\lambda_1}\frac{\lambda_2^{n-k}}{(n-k)!}e^{-\lambda_2}}{\frac{(\lambda_1+\lambda_2)^n}{n!}e^{-(\lambda_1+\lambda_2)}}\\
&= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-k}.
\end{aligned}
$$

Therefore, conditioning on $X_1+X_2=n$, X_1 has a binomial distribution with parameters n and $\lambda_1/(\lambda_1+\lambda_2)$.

7. Note that $\boldsymbol{\eta} = A\boldsymbol{\xi}$, where $A = \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$. By the property of n-dimensional

normal random vectors, we know that $\boldsymbol{\eta} \sim N(A\boldsymbol{a}, A\boldsymbol{B}A^\top)$.