1.2 Linear and Nonlinear Equation

The general form of ODE is

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0. \tag{1}$$

If $F(\cdots)$ is a linear function of y, y', $\cdots$, $y^{(n)}$, i.e., (1) takes the form

$$\boxed{a_0(x)}y^{(n)} + \boxed{a_1(x)}y^{(n-1)} + \cdots + \boxed{a_{n-1}(x)}y' + a_n(x)y = q(x), \tag{2}$$

then, we say that (1) is a linear ODE. Otherwise, (1) is said to be a nonlinear ODE. For example, $\frac{dy}{dx} = x$ is linear while $\frac{d^2\theta}{dt^2} + \sin\theta = 0$ is nonlinear.

linear function: $\boxed{}$ 都是可以变化的，非常数。其中 $a_m(x)$ 不可以是 y 的导数或 y^m 形式。

例如 ① $\frac{d^2y}{dx^2} + \frac{2dy}{dx}\frac{d^3y}{dx^3} + x = 0 \rightarrow$ nonlinear

linear function: $a_m(x)$ 无此种形式

② $\frac{d^2y}{dx^2} - \sin y = 0 \longrightarrow$ nonlinear

$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots$

linear function: $a_m(x)$ 无此种形式

The equation for continuous compounding of interest and continuous deposits is obtained by taking the limit $\Delta t \to 0$. The resulting differential equation is

$$\frac{dS}{dt} = rS + k \tag{8}$$

which can be solved with the initial condition $S(0) = S_0$, where S_0 is the initial capital.

The solution is

$$S = S_0 e^{rt} + \frac{k}{r} e^{rt}(1 - e^{-rt}). \tag{9}$$

Suppose $S_0 = 0$, then from (9), we have

$$k = \frac{rS(t)}{e^{rt} - 1}. \tag{10}$$

$\frac{dS}{dt} = rS + k$

integrating factor

$\frac{dS}{dt} - rS = k \quad p(t) = -r \quad q(t) = k \quad u(t) = e^{\int p(t)\,dt} = e^{\int -r\,dt} = e^{-rt}$

$$S = \frac{\int u(t)\,q(t)\,dt + C}{u(t)} = \frac{\int e^{-rt} k\,dt + C}{e^{-rt}} = \frac{-\frac{k}{r}e^{-rt} + C}{e^{-rt}}$$

$= e^{rt}\left(-\frac{k}{r}e^{-rt} + C\right)$

when $t = 0$ $\quad S_0 = e^0\left(-\frac{k}{r} + C\right) \quad C = S_0 + \frac{k}{r}$

$S = e^{rt}\left(-\frac{k}{r}e^{-rt} + S_0 + \frac{k}{r}\right) = S_0 e^{rt} + \frac{k}{r}e^{rt} - \frac{k}{r}$

2

(1) $(C)' = 0$, (2) $(x^\mu)' = \mu x^{\mu-1}$,
(3) $(\sin x)' = \cos x$, (4) $(\cos x)' = -\sin x$,
(5) $(\tan x)' = \sec^2 x$, (6) $(\cot x)' = -\csc^2 x$,
(7) $(\sec x)' = \sec x \tan x$, (8) $(\csc x)' = -\csc x \cot x$,
(9) $(a^x)' = a^x \ln a$, (10) $(e^x)' = e^x$,
(11) $(\log_a x)' = \frac{1}{x \ln a}$, (12) $(\ln x)' = \frac{1}{x}$,
(13) $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$,
(14) $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$,
(15) $(\arctan x)' = \frac{1}{1+x^2}$,
(16) $(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$.

1. solution to first order linear ODE $f(x,y)$

$f(x,y) + p(x)y = q(x)$.

$\Rightarrow y' + p(x)y = q(x)$

we have $u(x) = e^{\int p(x)dx}$ $\quad y = \frac{\int u(x)q(x)dx + C}{u(x)}$

integrating factor

2. general solution & particular solution of linear ODE

① $y = \frac{\int u(x)q(x)dx + C}{u(x)}$ $\quad$ C is an arbitrary constant ⇒ general

② Given initial condition, $C = C_1$ ⇒ particular

3. seperable equation

$M(x) + N(y)\frac{dy}{dx} = 0$

$\Rightarrow -M(x)dx = N(y)dy$

$\Rightarrow -\int M(x)dx = \int N(y)dy + C$

4.

Difference Between Linear and Nonlinear Equations → linear equation must have explicit solution nonlinear equation may have implicit solution

Consider

$$y' = f(x,y), \quad y(x_0) = y_0. \tag{1}$$

There are three major questions: existence, uniqueness, valid interval of the solution.

Theorem Suppose that $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous in some rectangle $\alpha \le x \le \beta$, $\gamma \le y \le \delta$, which contains (x_0, y_0). Then (1) has a unique solution which is valid in some interval $\boxed{x_0 - h \le x \le x_0 + h}$ within $\alpha \le x \le \beta$.

Remark: Notice the difference of conditions between linear and nonlinear ODEs. Here, the exact values h is not stated in the theorem. It depends on the differential equation as well as the initial condition.

Example 1 Solve the initial value problem.

$$y' = y^2, \quad y(0) = 1, \tag{2}$$

and determine the interval in which the solution exists.

Solution: $f(x,y) = y^2$ and $\frac{\partial f}{\partial y} = 2y$. Thus, according to the theorem, there is a unique solution which is valid in some interval containing $x = 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 \rightarrow \frac{1}{y^2}\mathrm{d}y = \mathrm{d}x,$$
$$-\frac{1}{y} = x + C \rightarrow y = -\frac{1}{x+C}. \tag{3}$$

y^2 and $2y$ are continuous on $(-\infty, \infty)$

Then using the initial value condition $y(0) = 1$, one has

$$1 = -\frac{1}{C} \rightarrow C = -1. \tag{4}$$

$x_0 = 0$. $x \neq 1 \Rightarrow$ valid interval $(-\infty, 1)$

Thus, the solution is

$$y = \frac{1}{1-x}. \tag{5}$$

We require that $x \neq 1$, i. e., $-\infty < x < 1$ is the interval in which the solution is valid. If we take $y(0) = -1$, then one has $C = 1$ and the solution to the initial value problem is

$x_0 = 0$. $x \neq -1 \Rightarrow$ valid interval $(-1, \infty)$

$$y = -\frac{1}{1+x}, \tag{6}$$

and the valid interval is then $(-1, \infty)$. This example shows that the valid intervals depend on the initial conditions.

5. Applications of modeling with first order ODE(For reading only)

Example 1 A body of constant mass m is projected vertically upward from the surface of the earth with an initial velocity v_0. The gravitational acceleration of the earth is assumed to be constant. During the motion, the body is subjected to an air resistance which is proportional to the magnitude of the velocity, say, $k|v|$. Find

a) The time at which the maximum height is reached;

b) The maximum height attained by the body.

也就是求 $V=0$ 时刻

a: (gravitation acceleration)

$V(0) = V_0$

受力分析: $k|V|$, mg

we have $F = -mg - k|V| = ma = m\frac{dV}{dt}$

$V(0) = V_0$

then $\frac{dV}{dt} = -g - \frac{k|V|}{m} = -g - \frac{kV}{m}$ (V恒>0)

① integrating factor

$\frac{dV}{dt} + \frac{k}{m}V = -g$ $\quad p(t) = \frac{k}{m}$ $\quad q(t) = -g \Rightarrow u(t) = e^{\int p(t)\,dt} = e^{\frac{k}{m}t}$

$$V = \frac{\int u(t)q(t)\,dt + C}{u(t)} = \frac{\int e^{\frac{k}{m}t}(-g)\,dt + C}{e^{\frac{k}{m}t}}$$

$$= \frac{-\frac{mg}{k}e^{\frac{k}{m}t} + C}{e^{\frac{k}{m}t}}$$

$V(0) = V_0 = -\frac{mg}{k} + C \rightarrow C = V_0 + \frac{mg}{k}$

Thus. $V = -\frac{mg}{k} + \left(V_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t}$

② seperable equation $M(x)\,dx = -N(y)\,dy$

$$\frac{dv}{\frac{k}{m}v+g} = -dt \Rightarrow -t+c_1 = \left(\ln\left|\frac{k}{m}v+g\right|\right)\frac{m}{k} \Rightarrow \left|\frac{k}{m}v+g\right| = e^{\frac{m}{k}(-t+c_1)}$$

$$\Rightarrow \frac{k}{m}v+g = e^{\frac{m}{k}(-t+c_1)} = Ce^{-\frac{m}{k}t} \Rightarrow V = \frac{m}{k}\left(Ce^{-\frac{m}{k}t}-g\right)$$

c_1 and C are arbitrary constant!

we have $V(0) = V_0$,

then $V_0 = \frac{m}{k}(C-g) \Rightarrow C = \frac{k}{m}V_0+g \Rightarrow V = \frac{m}{k}\left(\left(\frac{k}{m}V_0+g\right)e^{-\frac{mt}{k}}-g\right) = \left(V_0+\frac{mg}{k}\right)e^{-\frac{mt}{k}}-\frac{mg}{k}$

a 问求 $V=0$ 时刻的 t

$$V = \frac{dx}{dt} = -\frac{mg}{k}+\left(V_0+\frac{mg}{k}\right)e^{-\frac{k}{m}t}$$

also we have $x(0) = 0$.

then $x = -\frac{mg}{k}t+\left(-\frac{m}{k}\right)\left(V_0+\frac{mg}{k}\right)e^{-\frac{k}{m}t}+c'$

$x(0) = -\frac{m}{k}\left(V_0+\frac{mg}{k}\right)+c' \Rightarrow c' = \frac{m}{k}\left(V_0+\frac{mg}{k}\right) = \frac{mV_0}{k}+\frac{m^2g}{k^2}$

Thus, $x = -\frac{mg}{k}t+\left(-\frac{m}{k}\right)\left(V_0+\frac{mg}{k}\right)e^{-\frac{k}{m}t}+\frac{mV_0}{k}+\frac{m^2g}{k^2}$

$= \frac{m}{k}\left(V_0+\frac{mg}{k}\right)\left(1-e^{-\frac{k}{m}t}\right)-\frac{mg}{k}t$

when $V=0$, $0 = \left(\frac{m}{k}+V_0\right)e^{-\frac{mt}{k}}-\frac{mg}{k} \Rightarrow t = \frac{m}{k}\ln\left(\frac{V_0k}{mg}+1\right)$

then $x = \frac{mV_0}{k}-\frac{m^2g}{k^2}\ln\left(\frac{V_0k}{mg}+1\right)$

6. Exact Equations and Integrating Factors

$\longrightarrow$ $M(x,y)+N(x,y)\frac{dy}{dx} = 0$, $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$ exact equation

Consider

$$M(x,y)+N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{1}$$

Exact Equations: If we can find a function $\Psi(x,y)$ such that

$$M(x,y) = \frac{\partial \Psi}{\partial x}, \quad N(x,y) = \frac{\partial \Psi}{\partial y}. \tag{2}$$

Then, the ODE becomes

$$\frac{\partial \Psi}{\partial x}+\frac{\partial \Psi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \to \frac{\mathrm{d}\Psi(x,y)}{\mathrm{d}x} = 0 \to \Psi(x,y) = C. \tag{3}$$

In this case, (1) is called an exact ODE.

Question: Under which condition this $\Psi(x,y)$ exists?

Theorem Suppose that $M, N, \frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous in rectangle $R: \alpha < x < \beta, \nu < y < \delta$. Then (1) is an exact ODE in R (i. e., there exists a $\Psi(x,y)$ such that (2) is satisfied) if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \quad \text{in } R \tag{4}$$

Proof (i) Necessary Condition

Suppose that (1) is an exact ODE, i. e., (2) is satisfied, we have

$$M(x,y)=\frac{\partial\Psi}{\partial x},\quad N(x,y)=\frac{\partial\Psi}{\partial y}\to\frac{\partial M}{\partial y}=\frac{\partial^2\Psi}{\partial y\partial x},\quad \frac{\partial N}{\partial x}=\frac{\partial^2\Psi}{\partial y\partial x},\to\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}. \tag{5}$$

So (4) is satisfied.

(ii) Sufficient Condition

Suppose that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\text{ in }R. \tag{6}$$

Next, we consider a function $\Psi(x,y)$ defined by

$$\boxed{\Psi(x,y)=\int M(x,y)\mathrm{d}x+\int\left[N(x,y)-\int\frac{\partial M(x,y)}{\partial y}\mathrm{d}x\right]\mathrm{d}y.} \tag{7}$$

$\psi(x,y)$: 已知 $M(x,y)+N(x,y)\frac{dy}{dx}=0$

对 $M(x,y)=\frac{\partial\psi}{\partial x}$ 或 $N(x,y)=\frac{\partial\psi}{\partial y}$ 积分

Then we have

$$\frac{\partial\Psi}{\partial x}=M(x,y)+\int\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]\mathrm{d}y=M(x,y),$$
$$\frac{\partial\Psi}{\partial y}=\int\frac{\partial M}{\partial y}\mathrm{d}x+N(x,y)-\int\frac{\partial M}{\partial y}\mathrm{d}x=N(x,y). \tag{8}$$

如对 $M(x,y)=\frac{\partial\psi}{\partial x}$ 积分，

$M(x,y)=\frac{\partial\psi}{\partial x}\Rightarrow\psi=\int M(x,y)dx+h(y)$

Thus, for such a $\Psi(x,y)$, (2) is satisfied, i. e., (1) is an exact ODE.

$$\frac{\partial\psi}{\partial y}=\frac{\partial\int M(x,y)dx}{\partial y}+h'(y)=N(x,y)$$

$$h'(y)=-\frac{\partial\int M(x,y)dx}{\partial y}+N(x,y)$$

h(y)只与y有关，在此处充当的作用相当于C。如果没有h(y)，$\frac{\partial\psi}{\partial y}=\frac{\partial\int M(x,y)dx}{\partial y}\neq N(x,y)$

$$h(y)=\int\left(N(x,y)-\frac{\partial\int M(x,y)dx}{\partial y}\right)dy+C_1$$

C_1为任意实数。为了简便，令C=0

$$h(y)=\int\left(N(x,y)-\frac{\partial\int M(x,y)dx}{\partial y}\right)dy$$

$$\Rightarrow\psi=\int M(x,y)dx+\int\left(N(x,y)-\frac{\partial\int M(x,y)dx}{\partial y}\right)dy$$

Example 1 Find the solution of the ODE

$$(y\cos x+2xe^y)+(\sin x+x^2e^y-1)y'=0 \tag{9}$$

找到 $M(x,y)$ 与 $N(x,y)$ ⇒ 判断 $\frac{\partial M(x,y)}{\partial y}$ 与 $\frac{\partial N(x,y)}{\partial x}$ 是否相等 ⇒ 若相等

⇒ 对 $M(x,y)=\frac{\partial\psi}{\partial x}$ 或 $N(x,y)=\frac{\partial\psi}{\partial y}$ 积分（哪个好积积哪个）

⇒ ① 如果对 $M(x,y)=\frac{\partial\psi}{\partial x}$ 积分，

$\psi=\int M(x,y)dx+h(y)$ $\quad h'(y)=N(x,y)-\frac{\partial\int M(x,y)dx}{\partial y}$

找出 h(y)

h(y)只可与y相关，若包含x，则没有满足的 $\psi(x,y)$

② 如果对 $N(x,y)=\frac{\partial\psi}{\partial y}$ 积分。

$\psi=\int N(x,y)dy+h(x)$ $\quad h'(x)=M(x,y)-\frac{\partial\int N(x,y)dy}{\partial x}$

找出 h(x)

h(x)只可与x相关，若包含y，则没有满足的 $\psi(x,y)$

Integrating factor It is sometimes possible to convert a differential equation not exact into an exact equation by multiplying the equation with a suitable integrating factor. Consider

$$M(x,y)+N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}=0. \tag{22}$$

Suppose that it is not exact. Now, we multiplying it by an integrating factor $u(x,y)$ which gives

$$uM+uN\frac{\mathrm{d}y}{\mathrm{d}x}=0. \tag{23}$$

The purpose is trying to make this new equation to be exact, i. e.,

$$\frac{\partial(uM)}{\partial y}=\frac{\partial(uN)}{\partial x} \rightarrow M\frac{\partial u}{\partial y}-N\frac{\partial u}{\partial x}+u\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=0. \tag{24}$$

But (24) is not so easy to be solved, except in special cases.

→ $M(x,y)+N(x,y)\frac{dy}{dx}=0$

$\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$

integrating factor $u(x,y)$:

① $u(x,y)=u(x)$

$\frac{1}{u}\frac{du}{dx}=\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$

② $u(x,y)=u(y)$

$\frac{1}{u}\frac{du}{dy}=-\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}$

Example 3 Find the solution of the ODE

$$(3xy+y^2)+(x^2+xy)y'=0. \tag{29}$$

找到 $M(x,y)$ 与 $N(x,y)$ ⇒ 判断 $\frac{\partial M(x,y)}{\partial y}$ 与 $\frac{\partial N(x,y)}{\partial x}$ 是否相等 ⇒ 若不相等

⇒ 如果 $u(x,y)=u(x)$，那么 $\ln|u|=\int\left(\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}\right)dx$

⇒ 如果 $u(x,y)=u(y)$，那么 $\ln|u|=\int-\left(\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}\right)dy$

⇒ 找出u后再转化成 exact equation 求解

3 second order linear equation

1. Homogeneous equations with constant coefficient

(1) general form of second order ODE $\frac{d^2y}{dx^2}=f(x,y,\frac{dy}{dx})$

general form of second order linear ODE $\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=g(x)$

initial value problem $\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=g(x)$ $y(x_0)=y_0$ $y'(x_0)=y_1$

(2) Homogeneous $\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=0$ $g(x)=0$

Nonhomogeneous $\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=C$ $g(x)\neq 0$

(3) Characteristic equation of $y''(x)+by'(x)+cy=0$ when $p(x)=b$ $q(x)=c$ $g(x)=0$

— Three solutions $r^2+br+c=0$ (令 $y=e^{rx}$ $r^2e^{rx}+bre^{rx}+ce^{rx}=0$)

① 2 distinct real roots: r_1, r_2 $y_1=e^{r_1x}$ $y_2=e^{r_2x}$

general solution $y(x)=c_1e^{r_1x}+c_2e^{r_2x}$ (c_1, c_2 are arbitrary constants)

initial value $y(x_0)=y_0$ $y'(x_0)=y_1$ 代入 $y(x)=c_1e^{r_1x}+c_2e^{r_2x}$ 可得 c_1 与 c_2 的值.

② $r_1=r_2$

③ r_1与r_2为共轭复数, 如 $r_1=a+bi$ $r_2=a-bi$

2. Fundamental Solutions of Linear Homogeneous Equations

此定理只存在于 second order linear ODE

Theorem If y_1 and y_2 are two solutions to the differential equations $y''+p(x)y'+q(x)y=0$, then the linear combination $C_1y_1+C_2y_2$ is also a solution for any values of C_1 and C_2.

(1) $y=c_1y_1+c_2y_2$ 中 C_1与C_2是唯一解(给定 $y(x_0)=a_0$, $y'(x_0)=a_1$)

$$\begin{aligned} c_1y_1(x_0)+c_2y_2(x_0)&=a_0 \\ c_1y_1'(x_0)+c_2y_2'(x_0)&=a_1 \end{aligned} \Rightarrow \begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

Wronskian for $y_1(x)$ and $y_2(x)$ is $W(y_1,y_2)=\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$

所以 $W(y_1,y_2)\neq 0$时, c_1 & c_2 有唯一解

(2) Fundamental set of solutions

$W(y_1,y_2)|_{x=x_0}\neq 0$, y_1 and y_2 are fundamental set of solutions

(3) General Solution

$W(y_1, y_2)|_{x=x_0} \neq 0$, $y = c_1 y_1 + c_2 y_2 = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is general solution.

where $c_1 = \dfrac{\begin{vmatrix} a_0 & y_2(x_0) \\ a_1 & y_2'(x_0) \end{vmatrix}}{W_0}$ $\quad c_2 = \dfrac{\begin{vmatrix} y_1(x_0) & a_0 \\ y_1'(x_0) & a_1 \end{vmatrix}}{W_0}$ $\quad W_0 = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}$

3. Linear Independence and Wronskian

(1) **Theorem** Suppose that f and g are differentiable on an open interval I. If $W(f,g) \neq 0$ for a point x_0 in I, then f and g are linear independent. Equivalently, this theorem states that if f, g are linear dependent, then $W(f,g) = 0$ for every point x in I.

① $k_1, k_2 = 0$. $W(f,g) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} \neq 0$ for one point in interval $I \Rightarrow f(x)$ and $g(x)$ are independent.

② $k_1, k_2 \neq 0$. $W(f,g) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = 0$ for one point in interval $I \Rightarrow f(x)$ and $g(x)$ are dependent.

$y'' + p(x)y' + q(x)y = 0$ $\quad W(f,g) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$

$W(f,g) = f(x)g'(x) - f'(x)g(x)$

$$\frac{d W(f,g)}{dx} = f(x)g''(x) + f'(x)g'(x) - f'(x)g'(x) - f''(x)g(x)$$
$$= f(x)g''(x) - f''(x)g(x)$$

Since we have $f''(x) + p(x)f'(x) + q(x)f(x) = 0 \quad (a)$

$g''(x) + p(x)g'(x) + q(x)g(x) = 0 \quad (b)$

From $f(x)(b) - g(x)(a)$. we get

$$f(x)g''(x) + f(x)p(x)g'(x) + f(x)q(x)g(x) - g(x)f''(x) - g(x)p(x)f'(x) - g(x)q(x)f(x) = 0$$

$$\Rightarrow f(x)g''(x) - g(x)f''(x) + p(x)\left(f(x)g'(x) - g(x)f'(x)\right) = 0$$

$$\frac{d W(f,g)}{dx} + p(x)W(f,g) = 0 \Rightarrow W(f,g) = c \cdot \boxed{e^{-\int p(x)dx}} \Rightarrow \text{never equal to } 0.$$

arbitrary constant

Thus, if on one point $W(f,g) \neq 0$, it means $C \neq 0$. then $W(f,g) \neq 0$ for all interval I

if on one point $W(f,g) = 0$, it means $C = 0$. then $W(f,g) = 0$ for all interval I

Summary

Following five statements are equivalent, regarding $y'' + p(x)y' + q(x) = 0$ with two solutions y_1 and y_2:

1. $W(y_1, y_2)|_{x=x_0} \neq 0$;
2. $W(y_1, y_2) \neq 0$ for all x;
3. y_1 and y_2 are linear independent.
4. y_1 and y_2 are a fundamental set of solutions;
5. $C_1 y_1 + C_2 y_2$ is the general solution;

4. **Complex roots of the characteristic equations** of homogenous equation

$y''+by'+cy=0$ ($p(x)=b$, $q(x)=c$: $y''+p(x)y'+q(x)y=0$)

$\Rightarrow y=e^{rx} \Rightarrow r^2+br+c=0$

if r_1 and r_2 are complex conjugate roots $\Rightarrow$ $r_1=\lambda+i\mu$ $r_2=\lambda-i\mu$

令 $r=A+Bi$

{ 代入 $r^2+br+c=0$ 计算 / 或直接用 $i^2=-1$ 运算 }

$\Rightarrow y_1=e^{r_1x}=e^{(\lambda+i\mu)x}=e^{\lambda x}(\cos\mu x+i\sin\mu x)$ (a)

$y_2=e^{r_2x}=e^{(\lambda-i\mu)x}=e^{\lambda x}(\cos\mu x-i\sin\mu x)$ (b)

$e^{i\mu x}$ Taylor expasion at $x=0$:

$e^0+\frac{e^0(i\mu x)^1}{1!}+\frac{e^0(i\mu x)^2}{2!}+\cdots+\frac{e^0(i\mu x)^n}{n!}$

$=1+i\mu x-\frac{\mu^2x^2}{2!}-\frac{i\mu^3x^3}{3!}+\frac{\mu^4x^4}{4!}+\frac{i\mu^5x^5}{5!}+\cdots$

$=(1-\frac{\mu^2x^2}{2!}+\frac{\mu^4x^4}{4!}-\frac{\mu^6x^6}{6!}+\cdots)+(i\mu x-\frac{i\mu^3x^3}{3!}+\frac{i\mu^5x^5}{5!}-\frac{i\mu^7x^7}{7!}+\cdots)$

$=\cos\mu x+i\sin\mu x$

$\Rightarrow \frac{(a)+(b)}{2}$ $y_{11}=\frac{1}{2}(y_1+y_2)=e^{\lambda x}\cos\mu x$

$\frac{(a)-(b)}{2i}$ $y_{22}=\frac{1}{2i}(y_1-y_2)=e^{\lambda x}\sin\mu x$

$$\Rightarrow W(y_{11},y_{22})=\begin{vmatrix} y_{11} & y_{22} \\ y_{11}' & y_{22}' \end{vmatrix}=\begin{vmatrix} e^{\lambda x}\cos\mu x & e^{\lambda x}\sin\mu x \\ \lambda e^{\lambda x}\cos\mu x-e^{\lambda x}\mu\sin\mu x & \lambda e^{\lambda x}\sin\mu x+e^{\lambda x}\mu\cos\mu x \end{vmatrix}$$

$=\mu e^{2\lambda x}$ never equal to 0

Thus, y_{11} and y_{22} are a fundamental set of solutions. $y=c_1y_{11}+c_2y_{22}$ is general solution $\Rightarrow$ $\boxed{y=c_1e^{\lambda x}\cos\mu x+c_2e^{\lambda x}\sin\mu x}$

5. **Repeated Roots: Reduction of Order**

$y''+by'+cy=0 \Rightarrow r^2+br+c=0$

if $b^2-4c=0$. $r_1=r_2=-\frac{b}{2}$ $\Rightarrow y_1=e^{-\frac{b}{2}x}$. $y_2=V(x)y_1=V(x)e^{-\frac{b}{2}x}$

we have $y_2''+by_2'+cy_2=0$

$\Rightarrow v(x)''y_1+2V(x)'y_1'+\underbrace{V(x)y_1''+bV(x)y_1'+cV(x)y_1}_{=0}+bV(x)'y_1=0$

$\Rightarrow V(x)''y_1+2V(x)'y_1'+bV(x)'y_1=0$

$2y_1'+by_1=-bV(x)e^{-\frac{b}{2}x}+bV(x)e^{-\frac{b}{2}x}=0$

$\Rightarrow V(x)''y_1=0 \Rightarrow V(x)''=0 \Rightarrow V(x)=mx+n$

arbitrary constant, suppose $m=1$, $n=0$

$\Rightarrow V(x)=x \Rightarrow y_2=xe^{-\frac{b}{2}x}$

Thus, $\boxed{y = C_1 e^{-\frac{b}{2}x} + C_2 x e^{-\frac{b}{2}x}}$ is general solution.

6. Nonhomogeneous Equations and Method of Undetermined Coefficients

$y'' + p(x)y' + q(x)y = g(x)$ ⇐ Nonhomogeneous equation

首先参考 $g(x)=0$ 时， $y''+p(x)y'+q(x)y=0 \Rightarrow y = c_1y_1 + c_2y_2$

那么有 $y''+p(x)y'+q(x)y=g(x) \Rightarrow \underset{\text{general solution}}{y(x)} = c_1y_1 + c_2y_2 + \underset{\text{particular solution}}{Y(x)}$

☆一般情况下，只有 $g(x)$ 为 $e^{\alpha x}$, $\sin\beta x$, $\cos\beta x$ 或 $P_n(x)$ 时才能找出 $Y(x)$

$g(x)=P_n(x)$: $y''+by'+cy = P_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ (a)

Assume that $Y(x) = \underline{A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n}$ (b)

Substitude (b) to (a), we have

$$\begin{cases} cA_0 = a_0 \\ cA_1 + nbA_0 = a_1 \\ \cdots \\ cA_{n-1} + 2bA_{n-2} + 6A_{n-3} = a_{n-1} \\ cA_n + bA_{n-1} + 2A_{n-2} = a_n \end{cases}$$

展开即可，只为推理，不需要背

☆ ① if $c \neq 0 \Rightarrow 0$ is not a root of characteristic equation ($r^2+br+c=0$)

unique solution: $Y(x) = A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n$

② if $c=0$, $b\neq 0 \Rightarrow 0$ is a single root of characteristic equation ($r^2+br=0$)

unique solution: $Y(x) = x(A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n)$

③ if $c=0$, $b=0 \Rightarrow 0$ is a double root of characteristic equation ($r^2=0$)

unique solution: $Y(x) = x^2(A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n)$

只针对多项式，若 $Y(x) = A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n + Be^{\alpha x}$

① $c\neq 0$, $Y(x)$ 不变

② $c=0$, $b\neq 0 \Rightarrow Y(x) = x(A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n) + Be^{\alpha x}$

③ $c=0$, $b=0 \Rightarrow Y(x) = x^2(A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n) + Be^{\alpha x}$

$\boxed{\text{对多项式外的元素无影响}}$

$g(x)=e^{\alpha x}P_n(x)$: suppose we have $y=u(x)e^{\alpha x}$

substitude $y=u(x)e^{\alpha x}$ to $y''+by'+cy'=e^{\alpha x}P_n(x)$

$\Rightarrow P_n(x)=u''+(2\alpha+b)u'+(\alpha^2+b\alpha+c)u$

$Y(x)=(A_0x^n+A_1x^{n-1}+\cdots+A_{n-1}x+A_n)e^{\alpha x}$

① if $\alpha^2+b\alpha+c\neq 0 \Rightarrow$ 0 is not a root of characteristic equation $[r^2+(2\alpha+b)r+(\alpha^2+b\alpha+c)=0]$

unique solution: $Y(x)=(A_0x^n+A_1x^{n-1}+\cdots+A_{n-1}x+A_n)e^{\alpha x}$

② if $\alpha^2+b\alpha+c=0$, $2\alpha+b\neq 0 \Rightarrow$ 0 is a single root of characteristic equation $[r^2+(2\alpha+b)r=0]$

$r^2+br+c=0$ 解有一个与α重合

unique solution: $Y(x)=x(A_0x^n+A_1x^{n-1}+\cdots+A_{n-1}x+A_n)e^{\alpha x}$

③ if $\alpha^2+b\alpha+c=0$, $2\alpha+b=0 \Rightarrow$ 0 is a double root of characteristic equation $[r^2=0]$

$r^2+br+c=0$ 解两个都与α重合

unique solution: $Y(x)=x^2(A_0x^n+A_1x^{n-1}+\cdots+A_{n-1}x+A_n)e^{\alpha x}$

$g(x)=e^{\alpha x}\cos\beta x\, P_n(x)$: 已知有 $e^{i\beta}=\cos\beta x+i\sin\beta x$
$e^{-i\beta}=\cos\beta x-i\sin\beta x$ $\Rightarrow \cos\beta x=\dfrac{e^{i\beta}+e^{-i\beta}}{2}$

substitude $\cos\beta x=\dfrac{e^{i\beta}+e^{-i\beta}}{2}$ to $y''+by'+cy=e^{\alpha x}\cos\beta x\,P_n(x)$

$$\Rightarrow y''+by'+cy=\frac{e^{(\alpha+i\beta)x}+e^{(\alpha-i\beta)x}}{2}P_n(x)$$

分成两部分 $\begin{cases} y''+by'+cy=\dfrac{e^{(\alpha+i\beta)x}}{2}P_n(x) & \text{转变为 } g(x)=e^{\alpha x}P_n(x) \\ y''+by'+cy=\dfrac{e^{(\alpha-i\beta)x}}{2}P_n(x) \end{cases}$

$$Y(x)=(A_0x^n+A_1x^{n-1}+\cdots+A_{n-1}x+A_n)e^{(\alpha+i\beta)x}+(B_0x^n+B_1x^{n-1}+\cdots+B_{n-1}x+B_n)e^{(\alpha-i\beta)x}$$

① if $r_1=\alpha+i\beta$ is not a root of characteristic equation $r^2+br+c=0$
$r_2=\alpha-i\beta$.

unique solution: $Y(x)=(A_0x^n+A_1x^{n-1}+\cdots+A_{n-1}x+A_n)e^{(\alpha+i\beta)x}$
$+(B_0x^n+B_1x^{n-1}+\cdots+B_{n-1}x+B_n)e^{(\alpha-i\beta)x}$

② if $r_1=\alpha+i\beta$, $r_2=\alpha-i\beta$ are single root of characteristic equation

$r^2+br+c=0$ 解与 r_1, r_2 重合

unique solution: $Y(x)=x(A_0x^n+A_1x^{n-1}+\cdots+A_{n-1}x+A_n)e^{(\alpha+i\beta)x}$
$+x(B_0x^n+B_1x^{n-1}+\cdots+B_{n-1}x+B_n)e^{(\alpha-i\beta)x}$

③ if $r=\alpha+i\beta$ or $r=\alpha-i\beta$ is double root of characteristic equation

不可能存在 $\alpha+i\beta\neq\alpha-i\beta$ 不存在 repeated root

rewrite particular solution:

$$Y(x)=\underbrace{(A_0x^n+A_1x^{n-1}+\cdots+A_{n-1}x+A_n)e^{(\alpha+i\beta)x}}_{y_1}+\underbrace{(B_0x^n+B_1x^{n-1}+\cdots+B_{n-1}x+B_n)e^{(\alpha-i\beta)x}}_{y_2}$$

we have

$$A_ke^{(\alpha+i\beta)x}+B_ke^{(\alpha-i\beta)x}$$
$$=e^{\alpha x}[(A_k+B_k)\cos\beta x+i(A_k-B_k)\sin\beta x]$$
$$=e^{\alpha x}(C_k\cos\beta x+D_k\sin\beta x)\Leftarrow\begin{cases}C_k=A_k+B_k\\D_k=i(A_k-B_k)\end{cases}$$

Then ① $Y(x)=e^{\alpha x}(C_0\cos\beta x+D_0\sin\beta x)x^n+e^{\alpha x}(C_1\cos\beta x+D_1\sin\beta x)x^{n-1}$
$+\cdots+e^{\alpha x}(C_n\cos\beta x+D_n\sin\beta x)$.

② $Y(x)=e^{\alpha x}(C_0\cos\beta x+D_0\sin\beta x)x^{n+1}+e^{\alpha x}(C_1\cos\beta x+D_1\sin\beta x)x^n$
$+\cdots+e^{\alpha x}(C_n\cos\beta x+D_n\sin\beta x)x$

可以理解为 $y''+by'+cy=g_1(x)+g_2(x)$

$$\begin{cases}y''+by'+cy=g_1(x)\Rightarrow \text{particular solution } Y_1(x)=y_1\\ y''+by'+cy=g_2(x)\Rightarrow \text{particular solution } Y_2(x)=y_2\end{cases}$$

$\Rightarrow Y=Y_1(x)+Y_2(x)$

这也解释了为什么对多项式之外的因素没影响

例题: $y''+2y'+y=e^{-x}[(5-2x)\cos x-(3+3x)\sin x]$ $\alpha=-1$ $\beta=1$

$r^2+2r+1=0$ $(r+1)^2=0$ $r_1=r_2=-1$

$r_1'=-1+i\neq r_1$ $r_2'=-1-i\neq r_2$

Thus, r_1' and r_2' are not solution of $r^2+2r+1=0$. $\Rightarrow$ condition ①

$\Rightarrow Y(x)=(A_0x+A_1)e^{(-1+i)x}+(B_0x+B_1)e^{(-1-i)x}$

Summary

$y'' + p(x)y' + q(x) = g(x)$

(1) $g(x) = 0 \Leftrightarrow$ Homogenous function

① y_1, y_2 are solutions of $y'' + p(x)y' + q(x)y = 0$

$y = c_1 y_1 + c_2 y_2$, $y(x_0) = a_0$ $y'(x_0) = a_1$, then we have

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = c \cdot e^{-\int p(x)dx} \neq 0$$

$\Rightarrow$ 判断c是否为0：如果在$x=x_0$时有$W=0$ 则$c=0$，因为$e^{-\int p(x)dx}$永远大于0

如果在$x=x_0$时有$W\neq0$ 则$c\neq0$，因为$e^{-\int p(x)dx}$永远大于0

法一：

$$c_1 = \frac{\begin{vmatrix} a_0 & y_2(x_0) \\ a_1 & y_2'(x_0) \end{vmatrix}}{W(y_1, y_2)} \qquad c_2 = \frac{\begin{vmatrix} y_1(x_0) & a_0 \\ y_1'(x_0) & a_1 \end{vmatrix}}{W(y_1, y_2)}$$

法二：

已知y_1为 $y'' + p(x)y' + q(x)y = 0$ 的解，那么有$y_2 = v(x)y_1$

把 $y_2 = v(x)y_1$ 代入 $y'' + p(x)y' + q(x)y = 0$

$\Rightarrow v''y_1 + v'[2y_1' + p(x)y_1] = 0$, suppose $u = \frac{dv}{dt}$

$\Rightarrow \underline{u'y_1 + u[2y_1' + p(x)y_1]} = 0$

根据题意解开

② 同理当 $y'' + by'' + cy = 0$ for $p(x) = b$ $q(x) = c$

y_1, y_2 are solutions of $y'' + by'' + cy = 0$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$$c_1 = \frac{\begin{vmatrix} a_0 & y_2(x_0) \\ a_1 & y_2'(x_0) \end{vmatrix}}{W(y_1, y_2)} \qquad c_2 = \frac{\begin{vmatrix} y_1(x_0) & a_0 \\ y_1'(x_0) & a_1 \end{vmatrix}}{W(y_1, y_2)}$$

When $y'' + by' + cy = 0 \Leftrightarrow r^2 + br + c = 0$

(i) $b^2 - 4c \neq 0$ $\quad r_1 \neq r_2$ then we have $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

(ii) $b^2 - 4c = 0$ $\quad r_1 = r_2$ then we have repeated root, $y = c_1 e^{r_1 x} + x c_2 e^{r_2 x}$

(iii) $b^2 - 4c < 0$ 根为共轭复数 $r_1 = \alpha + \beta i$ $r_2 = \alpha - \beta i$, then we have

$y = c_1 e^{\alpha x} \cos\beta x + c_2 e^{\alpha x} \sin\beta x$

(2) $g(x) \neq 0 \Leftrightarrow$ Nonhomogenous equation

① $g(x) = e^{\alpha x}$

$\Rightarrow Y(x) = Ae^{\alpha x}$

如果α的值与 $y''+by'+cy=0 \Leftrightarrow r^2+br+c=0$ 中某一r的值重合，则

$Y(x) = xAe^{\alpha x}$

因为 $y = c_1y_1+c_2y_2+Y(x)$ 当 $r^2+br+c=0$ 时，有 $r_1 \neq r_2$，$y_1=e^{r_1x}$ $y_2=e^{r_2x}$，

如果 $\alpha=r_1$ or $\alpha=r_2$，那么 $y_1, y_2, Y(x)$ 就不是 linear independent

而 equation solution 要求 linear independent

② $g(x) = \sin\beta x$ or $\cos\beta x$

$\Rightarrow Y(x) = A\sin\beta x + B\cos\beta x$

因为存在 $(\cos x)' = -\sin x$ $(\sin x)' = \cos x$，所以 $Y(x) = A\sin x + B\cos x$

③ $g(x) = p_n(x)$

$\Rightarrow Y(x) = A_0x^n + A_1x^{n-1} + \cdots + A_n$ （最高次项n与原 $g(x)=p_n(x)$ 中的n一致）

$r^2+br+c=0$

(i) $c\neq 0$ $b\neq 0$，$Y(x) = A_0x^n + A_1x^{n-1} + \cdots + A_n$

(ii) $c=0$，$b\neq 0$，$Y(x) = (A_0x^n + A_1x^{n-1} + \cdots + A_n)x$

(iii) $c=0$，$b=0$ $Y(x) = (A_0x^n + A_1x^{n-1} + \cdots + A_n)x^2$

④ $g(x) = \begin{cases} p_n(x) \\ e^{\alpha x}p_n(x) \\ e^{\alpha x}p_n(x)\sin\beta x \text{ or } e^{\alpha x}p_n(x)\cos\beta x \end{cases}$

$\Rightarrow Y(x) = \begin{cases} A_0x^n + A_1x^{n-1} + \cdots + A_n & (a) \\ e^{\alpha x}(A_0x^n + A_1x^{n-1} + \cdots + A_n) & (b) \\ e^{\alpha x}(A_0x^n + A_1x^{n-1} + \cdots + A_n)\cos\beta + e^{\alpha x}(B_0x^n + B_1x^{n-1} + \cdots + B_n)\sin\beta & (c) \end{cases}$

(a) 见③

(b) $r^2+br+c=0$

(i) $r_1\neq r_2$ 时 若有 $\alpha\neq r_1$ 且 $\alpha\neq r_2$ $Y(x) = e^{\alpha x}(A_0x^n + A_1x^{n-1} + \cdots + A_n)$

(ii) $r_1=r_2$ 时 $\alpha\neq r_1$ 则有 $y = c_1e^{r_1x} + xc_2e^{r_2x} + e^{\alpha x}(A_0x^n + A_1x^{n-1} + \cdots + A_n)$

(iii) $r_1=r_2$ 时 $\alpha=r_1$ 则有 $y = c_1e^{r_1x} + xc_2e^{r_2x} + e^{\alpha x}(A_0x^n + A_1x^{n-1} + \cdots + A_n)$

(c) (i) $r^2+br+c=0$ $r_1=\alpha+\beta i$ $r_2=\alpha-\beta i$ 并非 $r^2+br+c=0$ 解

$$Y(x)=e^{\alpha x}(A\sin\beta x+B\cos\beta x)$$

(ii) $r^2+br+c=0$ $r_1=\alpha+\beta i$ $r_2=\alpha-\beta i$ 为 $r^2+br+c=0$ 解

$$Y(x)=x e^{\alpha x}(A\sin\beta x+B\cos\beta x)$$

(iii) $r=\alpha+\beta i$ 或 $r=\alpha-\beta i$ 不可能是 $r^2+br+c=0$ 的解

(如果是 double root $r_1=r_2 \Rightarrow \beta=0$)

⑤ $g(x)\neq \begin{cases} P_n(x) \\ e^{\alpha x}P_n(x) \\ e^{\alpha x}\cos\beta P_n(x) \text{ or } e^{\alpha x}\sin\beta P_n(x) \end{cases}$ (其中 $y_1(x)$ 与 $y_2(x)$ 为 homogeneous equation 的解)

令 $Y(x)=u_1(x)y_1(x)+u_2(x)y_2(x)$

$$u_1'=\frac{-y_2(x)g(x)}{W(y_1,y_2)} \qquad u_2'=\frac{y_1(x)g(x)}{W(y_1,y_2)}$$

$$u_1=-\int\frac{y_2(x)g(x)}{W(y_1,y_2)}dx \qquad u_2=\int\frac{y_1(x)g(x)}{W(y_1,y_2)}dx$$

其中 $u_1'=\frac{-y_2(x)g(x)}{W(y_1,y_2)}$，$u_2'=\frac{y_1(x)g(x)}{W(y_1,y_2)}$ 的 $g(x)$ 是在 y'' 系数化为1后的新 $g(x)$