MATH2033 Mathematical Statistics
Assignment 2 Suggested Solutions

1. X and Y have a bi-variate normal distribution. with $\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$

   Let $A = \begin{bmatrix} a & b \end{bmatrix}$ and $V = \begin{bmatrix} X & Y \end{bmatrix}^T$, $Z = AV$. Thus, Z has a normal distribution

$$N(A\boldsymbol{\mu}, A\boldsymbol{\Sigma}A^T) = N(0, a^2 + b^2 + 2ab\rho)$$

2. $\boldsymbol{X} \sim N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let

$$\boldsymbol{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix} = A\boldsymbol{X}$$

   where $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Therefore $\boldsymbol{Y} \sim N_2(A\boldsymbol{\mu}, A\boldsymbol{\Sigma}A^T)$. If $\text{Var}(X_1) = \text{Var}(X_2)$, then

$$A\boldsymbol{\Sigma}A^T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) \\ \text{Cov}(X_1, X_2) & \text{Var}(X_2) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Sigma}_x^2 + \boldsymbol{\Sigma}_y^2 + 2\boldsymbol{\Sigma}_{xy} & 0 \\ 0 & \boldsymbol{\Sigma}_x^2 + \boldsymbol{\Sigma}_y^2 + 2\boldsymbol{\Sigma}_{xy} \end{bmatrix}$$

   Thus, Y_1 and Y_2 are independent.

3. Let $Y = A\boldsymbol{X} = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = X_1 - 2X_2 + X_3$ Hence, Y has a normal distribution $N(0, A\boldsymbol{\Sigma}A^T) = N(0, 4)$. Hence, $(Y^2/4) \sim \chi^2(1)$ Thus,

   $P((X_1 - 2X_2 + X_3)^2 > 15.36) = P((X_1 - 2X_2 + X_3)^2/4 > 3.84) = 1 - 0.9499565 = 0.0500435$

4. $Y = 2\beta X$ is a one-to-one transformation mapping $\{x \geq 0\}$ onto $\{y \geq 0\}$.

$$f_Y(y) = f_X(x^{-1})|J| = \frac{\beta^{\frac{r}{2}}}{\Gamma(\frac{r}{2})} \left(\frac{y}{2\beta}\right)^{\frac{r}{2}-1} e^{-\frac{\beta y}{2\beta}} \left|\frac{1}{2\beta}\right| = \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} y^{\frac{r}{2}-1} e^{-\frac{y}{2}}, \quad y \geq 0,$$

   and zero elsewhere. Y follows a $\chi^2(r)$ with $df = r$.

5. Since $F = \frac{U/r_1}{V/r_2}$, then $\frac{1}{F} == \frac{V/r_2}{U/r_1}$. which has an F-distribution with r_2 and r_1 degrees of freedom.

6. Since in the F-table, we can only find the 95% percentile. We may need do some transformation

$$P(F \leq a) = P(\frac{1}{F} \geq \frac{1}{a}) = 1 - P(\frac{1}{F} \leq \frac{1}{a})$$

which equivalent to,

$$P(\frac{1}{F} \leq \frac{1}{a}) = 0.95$$

According to the F-table, $\frac{1}{a} = 4.735063$ $a = 0.2111904$, $b = 3.325835$

7. Note

$$T^2 = \frac{W^2}{V/r} = \frac{W^2/1}{V/r}$$

Since W is $N(0,1)$ distributed, then W^2 is $\chi^2(1)$. Thus T^2 is F-distributed with 1 and r degrees of freedom.