Chapter 3 Vector Spaces

Section 3.3 Linear independence

Lemma Given n vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$, it is possible to write one of the vectors as a linear combination of the other $n-1$ vectors if and only if there exist scalars c_i, $i = 1, \cdots, n$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

Proof Suppose

$$\mathbf{v}_n = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{n-1}\mathbf{v}_{n-1}.$$

Subtracting $\mathbf{v}_n$ from both sides of this equation, we get

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{n-1}\mathbf{v}_{n-1} - \mathbf{v}_n = \mathbf{0}.$$

Conversely, if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ and at least one of the c_i's, say c_n, is nonzero, then

$$\mathbf{v}_n = (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{n-1}\mathbf{v}_{n-1})/c_n.$$

Definition (Linearly independent) The vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}$ in a vector space V are said to be **linearly independent** if

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{0},$$

implies that all the scalars $c_i = 0$, $i = 1, 2, \cdots, n$.

Definition (Linearly dependent) The vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}$ in a vector space V are said to be **linearly dependent** if there exist scalars c_i ,$i = 1, 2, \cdots, n$, not all zero, such that $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{0}$.

To check whether $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ are linearly independent:

Step 1 Consider the system

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n} = \mathbf{0}.$$

Step 2 Solve for $c_1, c_2, \cdots, c_n$.

- If there is only the trivial solution (i.e. $c_i = 0$ for all i), then the vectors are linearly independent.
- If there are Infinitely many solutions, then the vectors are linearly dependent.

Example Are $(1,2,4)^T, (2,1,3)^T, (4,-1,1)^T$ linearly independent in $\mathbf{R}^3$?

Solution

If $c_1 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\left\{ \begin{array}{llll} 1c_1 & +2c_2 & +4c_3 & = 0 \\ 2c_1 & +1c_2 & -1c_3 & = 0 \\ 4c_1 & +3c_2 & +1c_3 & = 0 \end{array} \right.$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 2 & 1 & -1 & 0 \\ 4 & 3 & 1 & 0 \end{array} \right) \to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -3 & -9 & 0 \\ 0 & -5 & -15 & 0 \end{array} \right) \to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The solutions are $\begin{pmatrix} 2\alpha \\ -3\alpha \\ \alpha \end{pmatrix}$ where α is any real number.

$\Bigg($If $\alpha = 1$, then $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and so $2 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. $\Bigg)$

Therefore, the three vectors are linearly dependent.

Example Is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ linearly independent?

Solution Solving $c \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we have $c = 0$. So $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is linearly independent.

Example Are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ linearly independent?

Solution Solving $c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we have $c_1 = c_2 = 0$. So $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly independent.

Example Are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ linearly independent?

Solution Solving $c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we have $c_1 = -2\alpha$, $c_2 = \alpha$, where α is a real number. So $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ are linearly dependent.

Example Are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ linearly independent?

Solution Solving $c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we have $c_1 = -2\alpha$, $c_2 = \alpha$ and $c_3 = 0$, where α is a real number. So $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly dependent.

Example Show that $(x-1)(x-2),(x-1),1$ are linearly independent in P_3.

Solution Suppose

$$a_1(x-1)(x-2)+a_2(x-1)+a_3=0.$$

Then

$$a_1x^2+(-3a_1+a_2)x+(2a_1+a_3)=0.$$

Comparing the coefficients on both sides, we have

$$\left\{\begin{array}{rrrl} a_1 & & & =0 \\ -3a_1 & +a_2 & & =0 \\ 2a_1 & & +a_3 & =0 \end{array}\right., \quad \text{which gives} \quad \left\{\begin{array}{l} a_1=0 \\ a_2=0 \\ a_3=0 \end{array}\right..$$

Hence, $(x-1)(x-2),(x-1),1$ are linearly independent in P_3.

Theorem Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ be n vectors in $\mathbf{R}^n$ and let $X = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$. The vectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$ will be linearly dependent if and only if X is singular.

Proof The equation

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0}$$

can be rewritten as a matrix equation $X\mathbf{c} = \mathbf{0}$. This equation will have a nontrivial solution if and only if X is singular. Thus, $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ will be linearly dependent if and only if X is singular.

Theorem Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be vectors in a vector space V.
The following are equivalent.

1. Any vector $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ can be written uniquely as a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_n$.
2. $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly independent.

(Negation version) The following are equivalent.

Not 1. There is a vector $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ can be written as more than one linear combinations of $\mathbf{v}_1, \cdots, \mathbf{v}_n$.

Not 2. $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly dependent.

(Negation version) The following are equivalent.

Not 1. There is a vector $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ can be written as more than one linear combinations of $\mathbf{v}_1, \cdots, \mathbf{v}_n$.

Not 2. $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly dependent.

Proof of "Not (1) $\Rightarrow$ Not (2)"

If $\mathbf{x} \in \text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$. Suppose that $\mathbf{x}$ have two expressions of linear combination

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \tag{1}$$

$$\mathbf{x} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n \tag{2}$$

If $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent, then subtracting (2) from (1) yields

$$(c_1 - d_1)\mathbf{v}_1 + (c_2 - d_2)\mathbf{v}_2 + \cdots + (c_n - d_n)\mathbf{v}_n = \mathbf{0}. \tag{3}$$

By the linear independence of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ the coefficients of (3) must all be 0. Hence, $c_i = d_i$, $i = 1, \cdots, n$.

(Negation version) The following are equivalent.

Not 1. There is a vector $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ can be written more than one linear combinations of $\mathbf{v}_1, \cdots, \mathbf{v}_n$.

Not 2. $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly dependent.

Proof Not "(2) $\Rightarrow$ Not (1)" If $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly dependent, then there exist $d_1, \cdots, d_n$, not all 0, such that

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n = \mathbf{0}. \tag{4}$$

Adding (1) and (4), we get

$$\mathbf{x} = (c_1 + d_1)\mathbf{v}_1 + \cdots + (c_n + d_n)\mathbf{v}_n.$$

Since the d_i's are not all 0, thus, if $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly dependent, the representation of a vector as a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is not unique.

Theorem Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be vectors in a vector space V.
The following are equivalent.

1. Any vector $\mathbf{v} \in \mathrm{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ can be written uniquely as a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_n$.
2. $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly independent.

So, if $\mathrm{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\} = V$, The following are equivalent.

1. A vector $\mathbf{v} \in V$ can be written uniquely as a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_n$.
2. $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly independent.

If a) $\mathrm{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\} = V$ and b) $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly independent, then for any vector $\mathbf{v} \in V$,

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

for some unique c_i. And we will call $(c_1, c_2, \cdots, c_n)$ the coordinates of $\mathbf{v}$.