2022-23 First Semester
MATH1053 Linear Algebra I

Assignment 3b

Due Date: 1/Nov/2022 (Tuesday), 11:00 in class.

- Write down your **CHN name** and **student ID**. Write neatly on **A4-sized** paper (*staple if necessary*) and **show your steps**.
- **Late submissions or answers without steps won't be graded.**

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1. For each of the following matrices, compute the determinant and state whether the matrix is singular or nonsingular.

   a). $\begin{pmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{pmatrix}$; b). $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{pmatrix}$; c). $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$.

2. Let A be a nonsingular matrix. Show that $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

3. Let A and B be $n \times n$ matrices with $\det(A) = 4$ and $\det(B) = 5$. Find the value of

   a) $\det(AB)$; B) $\det(kA)$, $k \neq 0$; c) $\det(2BA)$; d) $\det(A^{-1}B)$; f) $\det\begin{pmatrix} O & A \\ B & O \end{pmatrix}$

4. Find the value of the determinant of the following matrices.

   (1) $\begin{pmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{pmatrix}$ (2) $\begin{pmatrix} x+a_1 & a_2 & \cdots & a_n \\ a_1 & x+a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & x+a_n \end{pmatrix}$.

5. For any integer $n \geq 2$, let

$$D_n = \begin{pmatrix} x & 0 & 0 & \cdots & 0 & a_0 \\ -1 & x & 0 & \cdots & 0 & a_1 \\ 0 & -1 & x & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & a_{n-2} \\ 0 & 0 & 0 & \cdots & -1 & x+a_{n-1} \end{pmatrix}.$$

   Express the determinant of D_n as a polynomial of x.

6. Consider the distinct real numbers a_0, a_1, $\cdots$, a_n. Define an $(n+1)\times(n+1)$ matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{bmatrix}.$$

Use mathematical induction to show that $\det(A) = \prod_{i>j}(a_i - a_j)$.

7. Find the inverse matrix of the rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

8.

$$\begin{cases} x_1 & +2x_2 & +x_3 = 1 \\ & -x_2 & +x_3 = 2 \\ 2x_1 & +3x_2 & -2x_3 = 3 \end{cases}.$$

(a) Use Cramer's rule to solve the linear system.

(b) Find A^{-1} using the adjoint of A. Then solve the system by computing $\mathbf{x} = A^{-1}\mathbf{b}$.

9. Label the following statements as true or false, and BRIEFLY **state the reason why**.

(a) If all entries of a $k \times k$ matrix A are 7 for $k = 2, 3, \cdots$, then $\det(A) = 7^k$.

(b) If $A^2 - I_n = I_n$, then matrix A must be invertible.

(c) If A is an $n \times n$ matrix such that $A^2 = O$, then matrix $I_n - A$ must be invertible.

(d) There exists an invertible 3×3 matrix A with real entries such that $A^{-1} = -A$.

(e) Matrix $\begin{bmatrix} k & -2 \\ 5 & k-6 \end{bmatrix}$ is invertible for all real numbers k.

10. For square matrices A and B, prove that

$$\det \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \det(A+B)\det(A-B).$$

11. (*Bonus!*) Prove that the matrix $A = I + \mathbf{u}\mathbf{u}^T$ is nonsingular where $\mathbf{u} \in \mathbb{R}^n$ is a column vector.

12. (MATLAB exercise) Building matrices and compute the determinants

    (a) Define a square matrix by yourself and compute the determinant by hand;

    (b) use the command "det( )" in MATLAB to verify your answer in part(a). For example,
    >> A = [1 2 3 2 1;0 0 0 1 -1;1 2 0 -1 1;3 2 1 2 3; 1 0 0 0 0]
    >> det(A)

    (c) Get familiar with the commands "eye(k)", "ones(m,n)", "zeros(m,n)", "magic(k)", "diag", "rand" and "rref".

13. (MATLAB exercise) Building matrices using partitioning

    (a) Matrices in MATLAB can be built up by patching together smaller matrices into a big matrix. The smaller matrices must fit together exactly along rows and columns and not leave any spaces unfilled. For example type the following in MATLAB:
    A=[1,2,3;3,2,1]
    B=[7;8]
    C=[4,5,6]
    D=[A,B;C,0]
    E=[A,zeros(3,7);eye(2,6),A]
    All these cases should have worked fine. Try to understand why.

    (b) Create the following matrix A in MATLAB by defining A1=[1,2;3,4] and A2=3*eye(3) first, then using "blkdiag".

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

14. (MATLAB exercise) Upper(lower) triangular matrix and LU factorization

    (a) Get familiar with the commands "triu", "tril", "lu".

    (b) Create a matrix A=magic(4) and see what are triu(A) and tril(A)?

    (c) Can A=ones(4,4) be LU factorized? If yes, find L and U.