2022-23 First Semester
MATH1063 Linear Algebra II (1003)

Assignment 5 Suggested Solutions

1. (a) If $\mathbf{x} \in \mathrm{N}(A)$, $A\mathbf{x} = \mathbf{0}$, then $A^TA\mathbf{x} = A^T\mathbf{0} = \mathbf{0}$ and $x \in \mathrm{N}(A^TA)$. Thus, $\mathrm{N}(A) \subseteq \mathrm{N}(A^TA)$.

   (b) If $\mathbf{x} \in \mathrm{N}(A^TA)$, $A^TA\mathbf{x} = A^T(A\mathbf{x}) = \mathbf{0}$. Then $A\mathbf{x} \in \mathrm{N}(A^T)$.
   On the other hand, $A\mathbf{x} = \sum_i x_i\mathbf{a}_i \in \mathrm{Col}(A)$, so $A\mathbf{x} \in \mathrm{Col}(A) \cap \mathrm{N}(A^T)$.
   Since $\mathrm{Col}(A) \perp \mathrm{N}(A^T)$, so $\mathrm{Col}(A) \cap \mathrm{N}(A^T) = \{\mathbf{0}\}$, then $A\mathbf{x} = \mathbf{0}$ and it implies that $\mathbf{x} \in \mathrm{N}(A)$ and $\mathrm{N}(A^TA) \subseteq \mathrm{N}(A)$.
   Based on (a) and (b). $\mathrm{N}(A^TA) = \mathrm{N}(A)$.

   (c)
$$\begin{cases} \mathrm{rank}(A) + \text{Nullity of } A = \# \text{ of columns in } A = n \\ \mathrm{rank}(A^TA) + \text{Nullity of } A^TA = \# \text{ of columns in } A^TA = n \end{cases}$$
   From (c), we know A and A^TA have the same rank.
   When A has rank n, A^TA is also of rank n and therefore nonsingular.

2. (a) If $\mathbf{x} \in \mathrm{Col}(A^TA)$, then $\mathbf{x} = A^TA\mathbf{y}$ for some $\mathbf{y} \in \mathbb{R}^n$. That is, $\mathbf{x} = A^TA\mathbf{y} = A^T(A\mathbf{y})$, then $\mathbf{x} \in \mathrm{Col}(A^T)$. Thus, $\mathrm{Col}(A^TA) \subseteq \mathrm{Col}(A^T)$.

   (b) If $\mathbf{x} \in \mathrm{Col}(A^T)$, then
$$\mathbf{x}^T\mathbf{z} = 0 \qquad \text{for any } \mathbf{z} \in \mathrm{N}(A)$$
   Since $\mathrm{N}(A) = \mathrm{N}(A^TA)$ and $\mathrm{N}(A^TA)^\perp = \mathrm{Col}((A^TA)^T) = \mathrm{Col}(A^TA)$, then it implies that $\mathbf{x}$ is orthogonal to any vectors in $\mathrm{N}(A^TA)$ and hence $\mathbf{x} \in \mathrm{Col}(A^TA)$.
   Then $\mathrm{Col}(A^T) \subseteq \mathrm{Col}(A^TA)$.

   (c) Based on (a) and (b), $\mathrm{Col}(A^TA) = \mathrm{Col}(A^T)$.

3. (a) Fit the points into the model and denote the coefficient matrix as A:
$$\begin{cases} 0 = c_0 + c_1 \cdot 0 \\ 1 = c_0 + c_1 \cdot 0 \\ 1 = c_0 + c_1 \cdot 1 \end{cases} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \rightarrow \quad (A^TA)^{-1}A^T\mathbf{b} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
   since $\mathrm{rank}(A) = 2$. So $f(t) = \frac{1}{2} + \frac{1}{2}t$.

   (b) To fit the quadratic function $f(t) = c_0 + c_1t + c_2t^2$, denote $\mathbf{c} = [c_0, c_1, c_2]'$, then
$$A\mathbf{c} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 6 \\ 12 \end{bmatrix} = \mathbf{b},$$

since $\text{rank}(A) = 3$, we find $A^T A = \begin{bmatrix} 4 & 9 & 29 \\ 9 & 29 & 99 \\ 29 & 99 & 353 \end{bmatrix}$, $\quad A^T\mathbf{b} = \begin{bmatrix} 20 \\ 70 \\ 254 \end{bmatrix}$ and

$$\rightarrow \quad (A^T A)^{-1} A^T \mathbf{b} = \frac{1}{440} \begin{bmatrix} 436 & -306 & 50 \\ -306 & 571 & -135 \\ 50 & -135 & 35 \end{bmatrix} \begin{bmatrix} 20 \\ 70 \\ 254 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

So $f(t) = -t + t^2$.

(c) $f(t) = c_0 + c_1 \sin(t) + c_2 \cos(t)$. Denote $\mathbf{c} = [c_0, c_1, c_2]'$, then

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & \sin 1 & \cos 1 \\ 1 & \sin 2 & \cos 2 \\ 1 & \sin 3 & \cos 3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \quad \rightarrow \quad \hat{\mathbf{c}} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 1.5 \\ 0.1 \\ -1.41 \end{bmatrix},$$

since $\text{rank}(A) = 3$. So $f(t) = 1.5 + 0.1 \sin(t) - 1.41 \cos(t)$.

(d) The general equation for a circle is $2xc_1 + 2yc_2 + (r^2 - c_1^2 - c_2^2) = x^2 + y^2$. Denote $r^2 - c_1^2 - c_2^2 = c_3$, then

$$\begin{bmatrix} 2x_1 & 2y_1 & 1 \\ 2x_2 & 2y_2 & 1 \\ 2x_3 & 2y_3 & 1 \\ 2x_4 & 2y_4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1^2 + y_1^2 \\ x_2^2 + y_2^2 \\ x_3^2 + y_3^2 \\ x_4^2 + y_4^2 \end{bmatrix}, \text{ i.e. } \begin{bmatrix} -2 & -4 & 1 \\ 0 & 4.8 & 1 \\ 2.2 & -8 & 1 \\ 4.8 & -3.2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5.76 \\ 17.21 \\ 8.32 \end{bmatrix}.$$

By using MATLAB, we have

$$\hat{\mathbf{c}} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 0.575 & -0.643 & 6.68 \end{bmatrix}^T.$$

So $r = \sqrt{c_3 + c_1^2 + c_2^2} \approx 2.73$ and the equation should be $(x - c_1)^2 + (y - c_2)^2 = r^2$. The least squares circle centers at $(0.58, -0.64)$ with radius 2.73 (rounded to two decimal places).

4. No, it is not necessarily true, since $\mathbf{y}$ is only orthogonal to one vector $\mathbf{x} \in S$ but not necessarily orthogonal to all vectors in S.
Consider $S = \text{span}\{(1,2,0)', (0,1,0)'\}$, $\mathbf{x} = (1,2,0)' \in S$, $\mathbf{y} = (-2,1,0)'$ and $\mathbf{v} = (-1,3,0)'$. Then $\mathbf{x} \perp \mathbf{y}$, but $\mathbf{y} \notin S^\perp$ since $\mathbf{y}$ is not orthogonal to $(0,1,0)' \in S$.

5. (a) $\text{Proj}_g f = \dfrac{\langle f, g \rangle}{\langle g, g \rangle} g = \dfrac{\int_{-1}^{1} x \mathrm{d}x}{\int_{-1}^{1} 1 \mathrm{d}x} \cdot 1 = 0$, suggesting $f \perp g$ in this inner product space.

(b) $\text{Proj}_g f = \dfrac{\langle f, g \rangle}{\langle g, g \rangle} g = -\sin(2x)$, since

$$\langle f, g \rangle = \int_{-\pi}^{\pi} x \sin(2x) \mathrm{d}x = -\frac{1}{2} x \cos(2x)|_{-\pi}^{\pi} + \int_0^{\pi} \cos(2x) \mathrm{d}x = -\pi$$

and $\langle g, g \rangle = \int_{-\pi}^{\pi} \sin^2(2x) \mathrm{d}x = \int_0^{\pi} 1 - \sin(4x) \mathrm{d}x = \pi$.

6. (a) They are orthogonal in this inner product space since

$$\langle 1, 2x-1\rangle = \int_0^1 1\cdot(2x-1)\ \mathrm{d}x = x^2 - x\Big|_0^1 = 0.$$

(b) The norm induced by the inner product gives

$$\|1\|^2 = \langle 1, 1\rangle = \int_0^1 1\cdot 1\ \mathrm{d}x = 1;$$

$$\|2x-1\|^2 = \int_0^1 (2x-1)^2\ \mathrm{d}x = \frac{1}{3}.$$

Therefore

$$\|1\| = 1 \quad \text{and} \quad \|2x-1\| = \frac{1}{\sqrt{3}}.$$

(c) Due to the previous results, we have $\{1, \sqrt{3}(2x-1)\}$ as an orthonormal basis for S. Then the least squares approximation to $h(x) = x^{1/2}$ from S is given by

$$\hat{h}(x) = c_1\cdot 1 + c_2\cdot\sqrt{3}(2x-1)$$

where

$$c_1 = \langle 1, x^{1/2}\rangle = \int_0^1 \sqrt{x}\ \mathrm{d}x = \frac{2}{3};$$

$$c_2 = \langle \sqrt{3}(2x-1), x^{1/2}\rangle = \int_0^1 \sqrt{3}(2x-1)\sqrt{x}\ \mathrm{d}x = \frac{2\sqrt{3}}{15}.$$

Thus

$$\hat{h}(x) = \frac{4}{5}\left(x + \frac{1}{3}\right).$$

7. (a) Note that span$\{\mathbf{x}_1, \mathbf{x}_2\} = \mathbb{R}^2$, we only need to perform G-S process on $\mathbf{x}_1$ and $\mathbf{x}_2$.
Let $\mathbf{v}_1 = \mathbf{x}_1 = (1, 2)^T$,
Let $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\langle \mathbf{x}_2, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 = (0, 1)^T - \frac{2}{5}(1, 2)^T = \frac{1}{5}(-2, 1)^T$.
$\mathbf{u}_1 = \mathbf{v}_1/\|\mathbf{v}_1\| = \frac{1}{\sqrt{5}}(1, 2)^T$, $\mathbf{u}_2 = \frac{1}{5\sqrt{5}}(-2, 1)^T$.
So an orthonormal basis is $\left\{\frac{1}{\sqrt{5}}(1, 2)^T, \frac{1}{\sqrt{5}}(-2, 1)^T\right\}$.

(b) So an orthonormal basis is $\left\{(1, 0, 0)^T, \frac{1}{\sqrt{2}}(0, 1, 1)^T, \frac{1}{\sqrt{2}}(0, 1, -1)^T\right\}$.

(c) Let $\mathbf{v}_1 = \mathbf{x}_1 = (4, 2, 2, 1)^T$,
Let $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\langle \mathbf{x}_2, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 = (2, 0, 0, 2)^T - \frac{10}{25}(4, 2, 2, 1)^T = \frac{1}{5}(2, -4, -4, 8)^T$,
Let $\mathbf{v}_3 = \mathbf{x}_3 - \dfrac{\langle \mathbf{x}_3, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 - \dfrac{\langle \mathbf{x}_3, \mathbf{v}_2\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\mathbf{v}_2 = (0, 1, -1, 0)^T$.
$\mathbf{u}_1 = \mathbf{v}_1/\|\mathbf{v}_1\| = \frac{1}{5}(4, 2, 2, 1)^T$, $\mathbf{u}_2 = \frac{1}{5}(1, -2, -2, 4)^T$, $\mathbf{u}_3 = \frac{1}{\sqrt{2}}(0, 1, -1, 0)^T$.
So an orthonormal basis is $\left\{(1, 0, 0)^T, \frac{1}{\sqrt{2}}(0, 1, 1)^T, \frac{1}{\sqrt{2}}(0, 1, -1)^T\right\}$.

8. (a) $\begin{cases} \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 &= 0 \\ -\frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 &= 0 \end{cases} \rightarrow \begin{cases} x_1 + x_2 &= 0 \\ x_3 + x_4 &= 0 \end{cases}$

So $\mathrm{N}(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$. Note that $(1,-1,0,0)^T$ and $(0,0,1,-1)^T$ are orthogonal, normalization gives $\{\frac{1}{\sqrt{2}}(1,-1,0,0)^T, \frac{1}{\sqrt{2}}(0,0,1,-1)^T\}$.

(b) Let $B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$, then $\mathrm{Col}(B) = \mathrm{N}(A)$ and the projection matrix Q that projects vectors in $\mathbb{R}^4$ onto $\mathrm{N}(A)$ should be

$$Q = B(B^TB)^{-1}B^T = BB^T = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

since the columns in B are orthonormal vectors.

9. Since Q is an orthogonal matrix, then $QQ^T = I$. Hence we have

$$1 = \det(I) = \det(Q)\det(Q^T) = \det(Q)\det(Q) = d^2 \quad \rightarrow \quad |d| = 1.$$

10. (a) True. Because $\mathrm{rank}(A^T) = \mathrm{rank}(A)$ and $\mathrm{rank}(A) = \mathrm{rank}(A^TA)$. Different null spaces though.

(b) False. $\mathbf{y} \in \mathrm{Col}(A^T) \cap \mathrm{N}(A) = \{\mathbf{0}\}$.

(c) True. Q is orthogonal, then $Q^T = Q^{-1}$, $(Q^T)^{-1} = Q = (Q^T)^T$. Q^T is also orthogonal.

(d) False. $(3Q)(3Q)^T = 9QQ^T = 9I \neq I$.