How to Price an Option

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1 Random Walk and Brownian Motion

1.1 Random Walk

An ant, who randomly walk alone a straight string, has a equally probability to move ahead or behind for 1 meter for each second. The motion of his one step is

$$Z = \begin{cases} 1, & p = 0.5 \\ -1, & p = 0.5 \end{cases}.$$

A *Random Walk* X_t is a track that the man has already walked, mathematically,

$$X_t = Z_1 + Z_2 + \cdots + Z_t. \tag{1.1}$$

It is clearly that $\mathbb{E}(Z) = 0$, and $\mathrm{Var}(Z) = 1$. So the mean and variance of random walk is

$$\mathbb{E}(X_t) = \mathbb{E}(Z_1) + \mathbb{E}(Z_2) + \cdots + \mathbb{E}(Z_t) = 0, \tag{1.2}$$

$$\mathrm{Var}(X_t) = \mathrm{Var}(Z_1) + \mathrm{Var}(Z_2) + \cdots + \mathrm{Var}(Z_t) = t. \tag{1.3}$$

If the man walks faster and his step is smaller, what will happen? Mathematically, if the man walks m steps and each step go ahead or behind for $1/m$ meters. Then the random walk becomes

$$X_t^{(m)} = \frac{1}{\sqrt{m}} Z_1 + \frac{1}{\sqrt{m}} Z_2 + \cdots + \frac{1}{\sqrt{m}} Z_{mt}. \tag{1.4}$$

For this case, there is also $\mathbb{E}(X_t^{(m)}) = 0$, and $\mathrm{Var}(X_t^{(m)}) = t$. Try to verify this.

1.2 Brownian Motion

A *Brownian Motion* W_t is the random walk with infinity steps. When $m \to 0$, a random walk becomes a Brownian Motion.

If a stochastic process satisfies these properties

- $W_0 = 0$, and W_t is continuous.
- For any $0 \le s < t$, $W_t - W_s$ is normally distributed with 0 mean and $t - s$ variance.
- For any $0 \le t_0 < t_1 < \cdots < t_n$, $W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent.

Then it is a Brownian motion.

Since a Brownian motion is normally distributed, its expectation and variance are

$$\mathbb{E}(W_t) = 0, \tag{1.5}$$

$$\operatorname{Var}(W_t) = E(W_t^2) = t. \tag{1.6}$$

Higher order moments of Brownian motion also can be yielded by normal distribution. For any positive integers k, there is

$$\mathbb{E}(W_t^{2k+1}) = 0, \tag{1.7}$$

$$\mathbb{E}(W_t^{2k}) = \frac{(2n)!}{2^n n!} t^k = (2k-1)!!t^k. \tag{1.8}$$

(Proof of these two properties is seen in Appendix A)

By this property, for any analytic function[1] $f(x)$, if the Taylor expansion of f is

$$f(x+h) = a_0 + a_1 h + a_2 h^2 + \cdots ,$$

then the expectation of $E(f(W_t))$ can be computed as

$$\mathbb{E}(f(W_t)) = a_0 + a_1 \frac{\mathbb{E}(W_t)}{1!} + a_2 \frac{\mathbb{E}(W_t^2)}{2!} + \cdots . \tag{1.9}$$

Example 1.1 Calculate the expectation of e^{aW_t}

$$\begin{aligned}
\mathbb{E}\left(e^{aW_t}\right) &= \int_{-\infty}^{\infty} e^{ax} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\
&= \int_{-\infty}^{\infty} (1 + ax + \tfrac{1}{2}(ax)^2 + \cdots) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\
&= \mathbb{E}(W_t^0) + \frac{a^2}{2!}\mathbb{E}(W_t^2) + \frac{a^4}{4!}\mathbb{E}[W_t^4] \cdots
\end{aligned}$$

[1] If a function has Taylor expansion on a specific interval $I \subset \mathbb{R}$, then it is a analytic function

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{a^{2k}}{(2k)!} \cdot \frac{(2k)!}{2^k k!} t^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(a^2 t)^k}{2^k} \\
&= e^{\frac{1}{2} a^2 t}.
\end{aligned}
$$

Example 1.2 For positive integer k, and $0 = t_0 \leq t_1 < \cdots < t_{2k+1}$, there is

$$
\mathbb{E}\left(W_{t_1} W_{t_2} \cdots W_{t_{2k+1}}\right) = 0. \tag{1.10}
$$

Proof. Let's consider $\mathbb{E}[W_{t_1} W_{t_2} W_{t_3}]$ first, let $P_i = W_{t_i} - W_{t_i - 1}$ and $t_0 = 0$, then

$$
\begin{aligned}
\mathbb{E}[W_{t_1} W_{t_2} W_{t_3}] =& \mathbb{E}[P_1(P_1 + P_2)(P_1 + P_2 + P_3)] \\
=& \mathbb{E}[P_1 P_1 P_1] + \mathbb{E}[P_1 P_1 P_2] + \mathbb{E}[P_1 P_1 P_3] + \mathbb{E}[P_1 P_2 P_1] + \mathbb{E}[P_1 P_2 P_2] + \mathbb{E}[P_1 P_2 P_3]
\end{aligned}
$$

There are $3! = 6$ items, and each item contains at least one odd order moment of increment of Brownian motion, therefore the expectation of each item is 0, thus the whole expectation is 0.

More generally, let $P_i = W_{t_i} - W_{t_{i-1}}$, equation (1.10) can be written as

$$
\begin{aligned}
\mathbb{E}\left(W_{t_1} W_{t_2} \cdots W_{t_{2k+1}}\right) =& E\left[\prod_{i=1}^{2k+1} \sum_{j=1}^{i} P_j\right] \\
=& \sum \mathbb{E}(P_{\alpha_1} P_{\alpha_2} \cdots P_{\alpha_{2k-1}}).
\end{aligned} \tag{1.11}
$$

where $\alpha_n \in I_n = \{1, 2, \ldots, n\}$, for $n \in \{1, 2, \ldots, 2k+1\}$. Each item in this equation contains $2k + 1$ P_i's to be multiplicate. In each item, the number of P_i for any i mustn't be all even number, since $2k + 1$ is odd, summation of a set of even number cannot equal to an odd number. Therefore every item in (1.11) has to be 0. Summation of these items is also zero. □

2 Itô's Lemma

2.1 A Brief Derivation of Itô's Lemma

Since Brownian motion W_t is continuous, a rational question is for a differentiable function $f(t, W_t)$, what is the differentiation of f.

We have known that for a function $f : \mathbb{R}^2 \to \mathbb{R}$, the total differentiation of f is

$$
df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.
$$

This statement is valid for normal calculus, but not suitable for calculus with stochastic terms. Because

$$\int_0^t (dW_t)^2 = t. \tag{2.1}$$

Another expression which may be not strict is $(dW_t)^2 = dt$ (but this is the way how we treat stochastic differentiation). The proof of this can be seen in Appendix B.

For a at least three order differentiable function f, the Taylor expansion is

$$f(W_t + \Delta W_t) = f(W_t) + f'(W_t)\Delta W_t + \frac{1}{2}f''(W_t)\Delta W_t^2 + \mathcal{O}(\Delta W_t^3)$$

As ΔW_t approaches to 0, the differentiation of f is obtained as

$$\begin{aligned} df &= \lim_{\Delta W_t \to 0} f(W_t + \Delta W_t) - f(W_t) \\ &= f'(W_t)dW_t + \frac{1}{2}f''(W_t)(dW_t)^2 \\ &= f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt. \end{aligned} \tag{2.2}$$

For a function $f(t, W_t)$, its differentiation is

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial W_t}dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial W_t^2}(dW_t)^2 \\ &= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W_t^2}\right)dt + \frac{\partial f}{\partial W_t}dW_t, \end{aligned} \tag{2.3}$$

which is famous *Itô's Lemma*.

If a stochastic process X_t is governed by

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t,$$

And $Y = f(t, X_t)$, which is a second order differentiable function. Then the differentiation of Y is

$$\begin{aligned} dY &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(dX_t)^2 \\ &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}(adt + bdW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}b^2 dt \\ &= \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial X_t} + \frac{b^2}{2}\frac{\partial^2 f}{\partial X_t^2}\right)dt + b\frac{\partial f}{\partial X_t}dW_t. \end{aligned} \tag{2.4}$$

2.2 Stochastic Differential Equations

A *Stochastic Differential Equations* is a differential equation which contains stochastic terms, such as

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

where W_t is a standard Brownian Motion.

Now Let's solve two specific stochastic differential equations

Example 2.1 Solve of $dX_t = \mu dt + \sigma dW_t$

Solution: The RHS of this SDE doesn't has term of X_t, so it can be directly integrate. Therefore the solution is

$$\begin{aligned} X_t =& \int_0^t \mu dt + \sigma dW_t \\ =& \mu t + \sigma W_t. \end{aligned} \tag{2.5}$$

If there is an additional initial condition $X_0 = a$, then the solution with initial condition is

$$X_t = \mu t + \sigma W_t + a. \tag{2.6}$$

□

Example 2.2 Solve $dS_t = \mu S_t dt + \sigma S_t dW_t$

Solution: Divide S_t to both sides of this SDE, it gives

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

In stochastic calculus, we cannot directly integrate $\frac{dS_t}{S_t}$ to get $\ln S_t$, because the differentiation of stochastic function contains a second partial derivative term. But this gives us a direction to obtain its solution.

Suppose $Y = \ln S_t$, then

$$\begin{aligned} \frac{\partial Y}{\partial t} &= 0, \\ \frac{\partial Y}{\partial W_t} &= \frac{1}{S_t}, \\ \frac{\partial^2 Y}{\partial S_t^2} &= -\frac{1}{S_t^2}. \end{aligned}$$

So,

$$\begin{aligned} dY =& 0 + \frac{1}{S_t} dS_t - \frac{1}{S_t^2} dS_t^2 \\ =& \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) - \frac{1}{2S_t^2}\sigma^2 S_t^2 dt \\ =& (\mu - \frac{\sigma^2}{2}) dt + \sigma dW_t, \end{aligned}$$

which can be directly integrated. Therefore

$$Y = \ln S_t = (\mu - \frac{\sigma^2}{2}) dt + \sigma dW_t$$

$$S_t = e^{\left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t}. \tag{2.7}$$

If it also contains a initial condition $S_0 = a$, then the solution is

$$S_t = ae^{\left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t}. \tag{2.8}$$

□

2.3 Stochastic Integration

Since we have had the differentiation expression of a stochastic function, the next step is to derive the integration. Suppose $f(W_t)$ is second order differentiable, the differentiation of f is

$$df = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt,$$

then

$$f'(W_t)dW_t = df - \frac{1}{2}f''(W_t)dt.$$

Integrate it both sides, it gives

$$\int_a^b f'(W_t)dW_t = f(b) - f(a) - \frac{1}{2}\int_a^b f''(W_t)dt. \tag{2.9}$$

3 Derive the Black-Scholes Equation

Now, we have introduced completely mathematical tools to analyse an option under continuous scenario. We have two assets to construct a equivalent portfolio to an option, *money* and *stock*. The value of money is evaluated by

$$dB = rBdt, \tag{3.1}$$

where r is continuous compounded risk-free interest rate. And the value of stock is

$$dS_t = \mu Sdt + \sigma SdW_t, \tag{3.2}$$

where W_t is a standard Brownian motion. The value of stock is modelled by *Geometric Brownian Motion* (GBM). Assume that the option price equals to the value of such a portfolio

$$\Pi = aB + \Delta S. \tag{3.3}$$

where a and b are all function of time t and stock price S_t. The differentiation form of this portfolio is that

$$d\Pi = d(aB) + d(\Delta S_t)$$

$$
\begin{aligned}
&= Bda + adB + \Delta dS_t + Sd\Delta \\
&= (Bda + Sd\Delta) + (adB + \Delta dS_t).
\end{aligned}
\tag{3.4}
$$

By the self-financing principle, no matter we selling stocks and earning money, or buying stocks and spending money, the total value of the portfolio will not change. It means

$$Bda + Sd\Delta = 0.$$

So,

$$
\begin{aligned}
d\Pi &= adB + \Delta dS_t \\
&= (arB + \mu \Delta S_t)dt + \sigma \Delta S_t dW_t.
\end{aligned}
\tag{3.5}
$$

Suppose $V(t, S_t)$ is the value of an option, by Itô's Lemma, there is

$$dV = \left(\mu S_t \frac{\partial V}{\partial S_t} + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S_t^2} \sigma^2 S_t^2 \right) dt + \frac{\partial V}{\partial S_t} \sigma S_t dW_t. \tag{3.6}$$

The idea is quite similar with what has been done in discrete case, by assuming the option value equals to value of the portfolio, there is

$$V = \Pi = aB + \Delta S_t \tag{3.7}$$

$$dV = d\Pi \tag{3.8}$$

Compare for their differentiation forms of the portfolio and option (i.e. equation (3.5) and equation (3.6)), these two equations can be obtained:

$$\mu S_t \frac{\partial V}{\partial S_t} + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S_t^2} \sigma^2 S_t^2 = raB + \mu \Delta S_t \tag{3.9}$$

$$\frac{\partial V}{\partial S_t} \sigma S_t = \sigma \Delta S_t \tag{3.10}$$

Equation (3.10) shows that $\Delta = \frac{\partial V}{\partial S_t}$. Substitute it into equation (3.7), there is

$$aB = V - \frac{\partial V}{\partial S_t} S \tag{3.11}$$

Combine this equation with equation (3.9), there is

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S_t^2} \sigma^2 S_t^2 + \frac{\partial V}{\partial S_t} r S_t = rV \tag{3.12}$$

Equation (3.12) is the famous *Black-Scholes Equation*.

4 Some Properties of Heat Equation

To seek the solution of Black Scholes equation, we must have an idea about heat equation. But first, let's have a look at a magic function so called *Dirac's Function*, which is defined by

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}, \text{ with } \int_{-\infty}^{\infty} \delta(x)dx = 1. \tag{4.1}$$

One of the properties of this function is

$$\int_{-\infty}^{\infty} \delta(x-a)f(a)dx = f(a). \tag{4.2}$$

Proof.

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(x-a)f(a)dx =& \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{-\varepsilon}^{\varepsilon} + \int_{\varepsilon}^{\infty} \right) \delta(x-a)f(a)dx \\ =& f(a) \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \delta(x-a)dx \\ =& f(a) \end{aligned}$$

□

Now, come back to the *Heat Equation*. Consider a mapping $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$, the heat equation is defined as

$$u_t = cu_{xx}. \tag{4.3}$$

Theorem 4.1 The heat equation with the initial condition

$$\begin{cases} u_t = u_{xx} \\ u(0, x) = \delta(x-a) \end{cases}$$

has unique solution. And the solution is

$$u(t, x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-a)^2}{4t}}. \tag{4.4}$$

Theorem 4.2 For arbitrary one variable function $f(x)$, the heat equation with the initial condition

$$\begin{cases} u_t = u_{xx} \\ u(0, x) = f(x) \end{cases}$$

has unique solution. And the solution is

$$u(t, x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(x')e^{-\frac{(x-x')^2}{4t}} dx'. \tag{4.5}$$

The proof of uniqueness in these two theorems is beyond of this topic, besides this, they can be easily verified by calculating partial derivatives of $u(t, x)$. As for how to obtain the solution of heat equation, see Appendix C to satisfy your curiosity.

5 Transform the Black-Scholes Equation into a Heat Equation

Table 1 shows the variables used in this section.

Variables	Meaning or Definition
V	Price of option
S_t	Price of stock
t	Time variable
T	Exercise date
r	Risk-free continuous compounded rate
σ	Volatility of underlying stock
x	Defined by $S_t = e^x$
τ	Defined by $\tau = \frac{\sigma^2}{2}(T - t)$
k	Defined by $k = 2r/\sigma^2$
a, b	Coefficients to be determined

Table 1: Notions Used in Section 5

The Black-Scholes equation with initial condition

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial W_t^2}\sigma^2 S_t^2 + \frac{\partial V}{\partial S_t} r S_t = rV \\ V(T, S_t) = F(S_t) \end{cases} \tag{5.1}$$

is hard to solve. After three variable substitutions, it will be turned into a heat equation.

I. First, let $S_t = e^x$, then $x = \ln S_t$. The partial derivatives of V with respect to x are

$$V_x = \frac{\partial V}{\partial S_t}\frac{dS_t}{dx} = \frac{1}{S_t}\frac{\partial V}{\partial S_t} \tag{5.2}$$

$$\begin{aligned} V_{xx} =& \frac{\partial}{\partial S_t}\left(\frac{1}{S_t}\frac{\partial V}{\partial S_t}\right)\frac{dS_t}{dx} \\ =& -\frac{1}{S_t^2}\frac{\partial V}{\partial S_t} + \frac{1}{S_t}\frac{\partial V_x}{\partial S_t} \end{aligned}$$

$$= -\frac{1}{S_t^2}(-V_x + V_{xx}). \tag{5.3}$$

Substitute V_x and V_{xx} into the equation, there is

$$\begin{cases} V_t + \left(r - \frac{\sigma^2}{2}\right)V_x + \frac{\sigma^2}{2}V_{xx} = rV \\ V(t = T, x) = F(e^x) \end{cases} \tag{5.4}$$

Divide equation (5.4) by $\frac{\sigma^2}{2}$, it gives

$$\frac{2}{\sigma^2}V_t + \left(\frac{2r}{\sigma^2} - 1\right)V_x + V_{xx} = \frac{2r}{\sigma^2}V. \tag{5.5}$$

II. Second, let $\tau = \frac{\sigma^2}{2}(T - t)$, then

$$V_\tau = \frac{\partial V}{\partial t}\frac{dt}{d\tau} = -\frac{2}{\sigma^2}V_t.$$

Substitute it into equation (5.5), there is

$$V_\tau - \left(\frac{2r}{\sigma^2} - 1\right)V_x - V_{xx} + \frac{2r}{\sigma^2}V = 0. \tag{5.6}$$

To make it more convenient to write, introduce $k = 2r/\sigma^2$, then equation (5.6) can be rewritten as

$$\begin{cases} V_\tau - (k-1)V_x - V_{xx} + kV = 0, \\ V(\tau = 0, x) = F(e^x). \end{cases} \tag{5.7}$$

III. Lastly, let $V(\tau, x) = e^{\alpha\tau+\beta x}u(\tau, x)$, partial derivatives of V can be written as

$$V_\tau = e^{\alpha\tau+\beta x}(\alpha u + u_t), \tag{5.8}$$

$$V_x = e^{\alpha\tau+\beta x}(\beta u + u_x), \tag{5.9}$$

$$V_{xx} = e^{\alpha\tau+\beta x}(\beta^2 u + 2\beta u_x + u_{xx}). \tag{5.10}$$

Substitute them into equation (5.7). There is

$$\begin{aligned} e^{\alpha\tau+\beta x}\big((\alpha u + u_t) - (k-1)(\beta u + u_x) - (\beta^2 u + 2\beta u_x + u_{xx}) + ku\big) &= 0 \\ (\alpha u + u_t) - (k-1)(\beta u + u_x) - (\beta^2 u + 2\beta u_x + u_{xx}) + ku &= 0 \\ u_\tau - ((k-1) + 2\beta)u_x - u_{xx} + (\alpha - \beta(k-1) - \beta^2 + k)u &= 0. \end{aligned} \tag{5.11}$$

Setting the u_x and u terms to be zero,

$$\begin{cases} (k-1) + 2\beta = 0 \\ \alpha - \beta(k-1) - \beta^2 + k = 0 \end{cases}$$

The solution of this equation is $\alpha = -\frac{(k+1)^2}{4}$ and $\beta = -\frac{k-1}{2}$. Put u into the previous equation, we get

$$\begin{cases} u_\tau = u_{xx}, \\ u(\tau = 0, x) = e^{-\beta x}F(e^x). \end{cases} \tag{5.12}$$

The BS-Equation is finally transformed into a heat equation.

6 The Solution of BS-Equation (for Vanilla Call and Put Options)

Since the general solution of heat equation has been known, we can directly use it to obtain the solution of BS equation.

$$V = e^{\alpha\tau+\beta x}\int_{-\infty}^{\infty} e^{-\beta x'}F(e^{x'})G(\tau, x-x')dx' \tag{6.1}$$

For call option, its value at exercise day is

$$F(x) = \begin{cases} S_T - K, & S_T \geq K \\ 0, & S_T < K \end{cases}.$$

Therefore, the price of call option C is

$$C = e^{\alpha\tau+\beta x}\int_{-\infty}^{\infty} e^{-\beta x'}\max\{e^{x'} - K\}G(\tau, x-x')dx'. \tag{6.2}$$

For only $x > \ln K$, the option value larger than 0, then

$$\begin{aligned} C =& e^{\alpha\tau+\beta x}\int_{\ln K}^{\infty} e^{-\beta x'}(e^{x'} - K)G(\tau, x-x')dx' \\ =& e^{\alpha\tau+\beta x}\int_{\ln K}^{\infty} e^{(-\beta+1)x'}G(\tau, x-x') - Ke^{-\beta x'}G(\tau, x-x')dx'. \end{aligned}$$

Define

$$I_n = \int_{\ln K}^{\infty} e^{(-\beta+n)x'}G(\tau, x-x')dx', \tag{6.3}$$

Then, the call option value is turned into

$$C = e^{\alpha\tau+\beta x}(I_1 - KI_0). \tag{6.4}$$

Now, focus on I_n,

$$\begin{aligned} I_n =& \int_{\ln K}^{\infty} e^{(-\beta+n)x'}G(\tau, x-x')dx' \\ =& \int_{\ln K}^{\infty} e^{(-\beta+n)x'}\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{(x-x')^2}{4\tau}}dx' \\ =& \frac{1}{\sqrt{4\pi\tau}}\int_{\ln K}^{\infty} e^{(-\beta+n)x'-\frac{(x-x')^2}{4\tau}}dx' \end{aligned}$$

a square term of x' is needed to evaluate this integration. Let $w = (-\beta+n)x' - \frac{(x-x')^2}{4\tau}$, then

$$\begin{aligned} w =& (-\beta+n)x' - \frac{(x-x')^2}{4\tau} \\ =& -\frac{1}{4\tau}\left((x-x')^2 + (\beta-n)4\tau x'\right) \end{aligned}$$

$$
\begin{aligned}
=&-\frac{1}{4\tau}\left[x^2-2xx'+x'^2+(\beta-n)4\tau x'\right]\\
=&-\frac{1}{4\tau}\left[x'^2+2x'(2\tau(\beta-n)-x)+x^2\right]\\
=&-\frac{1}{4\tau}\left[\left(x'+(2\tau(\beta-n)-x)\right)^2-4\tau^2(\beta-n)^2+4\tau x(\beta-n)\right]\\
=&(\beta-n)^2\tau-(\beta-n)x-\frac{1}{4\tau}\left(x'+2\tau(\beta-n)-x\right)^2
\end{aligned}
$$

Substitute the w back into I_n, we get

$$
\begin{aligned}
I_n=&\int_{\ln K}^{\infty}\frac{1}{\sqrt{4\pi\tau}}e^{(\beta-n)^2\tau-(\beta-n)x-\frac{1}{4\tau}\left(x'+2\tau(\beta-n)-x\right)^2}dx'\\
=&e^{(\beta-n)^2\tau-(\beta-n)x}\int_{\ln K}^{\infty}\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{(x'+2\tau(\beta-n)-x)^2}{4\tau}}dx'
\end{aligned}
$$

Let $z=\frac{x'+2\tau(\beta-n)-x}{\sqrt{2\tau}}$, then $dz=dx'/\sqrt{2\tau}$. Then

$$
I_n=e^{(\beta-n)^2\tau-(\beta-n)x}\int_{\frac{\ln K+2\tau(\beta-n)-x}{\sqrt{2\tau}}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz.
$$

The integrated function is p.d.f. of standard normal distribution. Since the function is a even function, therefore

$$
\begin{aligned}
I_n=&e^{(\beta-n)^2\tau-(\beta-n)x}\int_{-\infty}^{\frac{x-\ln K-2\tau(\beta-n)}{\sqrt{2\tau}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\\
=&e^{(\beta-n)^2\tau-(\beta-n)x}\cdot\Phi\left(\frac{x-\ln K-2\tau(\beta-n)}{\sqrt{2\tau}}\right)
\end{aligned}
\tag{6.5}
$$

Back to the value of call option, the first term of equation (6.4) is

$$
\begin{aligned}
e^{\alpha\tau+\beta x}I_1=&e^{\alpha\tau+\beta x}e^{(\beta-1)^2\tau-(\beta-1)x}\Phi\left(\frac{x-\ln K-2\tau(\beta-1)}{\sqrt{2\tau}}\right)\\
=&e^{(\alpha+(\beta-1)^2)\tau+x}\Phi\left(\frac{x-\ln K-2\tau(\beta-1)}{\sqrt{2\tau}}\right)\\
=&e^{\left(-\frac{(k+1)^2}{4}+(-\frac{k-1}{2}-1)^2\right)\tau+x}\Phi\left(\frac{x-\ln K-2\tau(\beta-1)}{\sqrt{2\tau}}\right)\\
=&e^{x}\Phi\left(\frac{x-\ln K-2\tau(\beta-1)}{\sqrt{2\tau}}\right) \qquad (6.6)\\
=&S_t\Phi\left(\frac{\ln S_t-\ln K-\sigma^2(T-t)(-\frac{k+1}{2})}{\sqrt{\sigma^2(T-t)}}\right)\\
=&S_t\cdot\Phi\left(\frac{\ln\frac{S_t}{K}+(r+\frac{\sigma^2}{2})(T-t)}{\sqrt{\sigma^2(T-t)}}\right). \qquad (6.7)
\end{aligned}
$$

The last term of equation (6.4) is

$$
e^{\alpha\tau+\beta x}I_0=e^{\alpha\tau+\beta x}e^{\beta^2\tau-\beta x}\Phi\left(\frac{x-\ln K-2\tau\beta}{\sqrt{2\tau}}\right)
$$

$$=e^{(\alpha+\beta^2)\tau}\Phi\left(\frac{x-\ln K-2\tau\beta}{\sqrt{2\tau}}\right)$$

Since

$$\alpha+\beta^2=-\frac{(k+1)^2}{4}+\frac{(k-1)^2}{4}=-k=-\frac{2r}{\sigma^2}$$

and $\tau=\frac{\sigma^2}{2}(T-t)$, then

$$\begin{aligned} e^{\alpha\tau+\beta x}I_0 =& e^{-r(T-t)}\Phi\left(\frac{x-\ln K-2\tau(\beta-1)}{\sqrt{2\tau}}-\frac{2\tau}{\sqrt{2\tau}}\right) \\ =& e^{-r(T-t)}\Phi\left(\frac{\ln\frac{S_t}{K}+(r-\frac{\sigma^2}{2})(T-t)}{\sqrt{\sigma^2(T-t)}}\right). \end{aligned} \tag{6.8}$$

Therefore, the solution of Black-Scholes Equation for call option is

$$C=S\Phi(d_+)-Ke^{-r(T-t)}\Phi(d_-), \tag{6.9}$$

where

$$d_+ =\frac{\ln\frac{S_t}{K}+(r+\frac{\sigma^2}{2})(T-t)}{\sqrt{\sigma^2(T-t)}} \tag{6.10}$$

$$d_- =d_+-\sqrt{\sigma^2(T-t)} \tag{6.11}$$

$$=\frac{\ln\frac{S_t}{K}+(r-\frac{\sigma^2}{2})(T-t)}{\sqrt{\sigma^2(T-t)}} \tag{6.12}$$

Price of Put Options

You can repeat this approach to obtain the price of put Option. But there is a simpler way. *Call-Put Parity* states that

$$C+Ke^{-r(T-t)}=P+S_t \tag{6.13}$$

Then the price of put option is

$$P =C+Ke^{-r(T-t)}-S_t \tag{6.14}$$

$$\begin{aligned} &=S_t\Phi(d_+)-Ke^{r(T-t)}\Phi(d_-)+Ke^{-r(T-t)}-S_t \\ &=Ke^{r(T-t)}(1-\Phi(d_-))-S_t(1-\Phi(d_+)) \end{aligned}$$

$$=Ke^{-r(T-t)}\Phi(-d_-)-S_t\Phi(-d_+) \tag{6.15}$$

where d_+ and d_- are defined as (6.10) and (6.12).

7 Delta and Other Greek Letters

Delta is a measure to describe what is the sensitivity of option as stock price changing, which is defined as

$$\Delta = \frac{\partial V}{\partial S_t}. \tag{7.1}$$

The delta of call option is

$$\Delta_C = \Phi(d_+) + S_t \frac{\partial}{\partial S_t}\Phi(d_+) - Ke^{r(T-t)}\frac{\partial}{\partial S_t}\Phi(d_-). \tag{7.2}$$

Let's focus on $\frac{\partial}{\partial S_t}\Phi(d_\pm)$, it equals

$$\begin{aligned}
\frac{\partial}{\partial S_t}\Phi(d_\pm) &= \left(\frac{\partial}{\partial d_\pm}\Phi(d_\pm)\right)\frac{\partial d_\pm}{\partial S_t} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_\pm^2} \cdot \frac{1}{S_t\sqrt{\sigma^2(T-t)}} \\
&= \frac{1}{S_t\sigma\sqrt{2\pi(T-t)}}\exp\frac{(\ln\frac{S_t}{K}+r(T-t)\pm\frac{\sigma^2}{2}(T-t))^2}{2\sigma^2(T-t)} \\
&= \frac{1}{S_t\sigma\sqrt{2\pi(T-t)}}\exp\left[-\frac{(\ln\frac{S_t}{K}+r(T-t))^2+(\frac{\sigma^2}{2}(T-t))^2}{2\sigma^2(T-t)}\right]\cdot\exp\left[-\frac{\pm(\ln\frac{S_t}{K}+r(T-t))\sigma^2(T-t)}{2\sigma^2(T-t)}\right]
\end{aligned}$$

Denote

$$A = \frac{1}{S_t\sigma\sqrt{2\pi(T-t)}}\exp -\frac{(\ln\frac{S_t}{K}+r(T-t))^2+(\frac{\sigma^2}{2}(T-t))^2}{2\sigma^2(T-t)},$$

in order to write it more conveniently,

$$\begin{aligned}
\frac{\partial}{\partial S_t}\Phi(d_\pm) &= Ae^{\mp\frac{(\ln\frac{S_t}{K}+r(T-t))\sigma^2(T-t)}{2\sigma^2(T-t)}} \\
&= Ae^{\mp\frac{1}{2}(\ln\frac{S_t}{K}+r(T-t))} \\
&= A\left(\frac{S_t}{K}\right)^{\mp\frac{1}{2}}e^{\mp\frac{r}{2}(T-t)}
\end{aligned} \tag{7.3}$$

Substitute it into equation (7.2), there is

$$\begin{aligned}
\Delta_C &= \Phi(d_+) + S_t A\left(\frac{S_t}{K}\right)^{-\frac{1}{2}}e^{-\frac{r}{2}(T-t)} - Ke^{-r(T-t)}A\left(\frac{S_t}{K}\right)^{+\frac{1}{2}}e^{+\frac{r}{2}(T-t)} \\
&= \Phi(d_+) + A(S_tK)^{\frac{1}{2}}e^{-\frac{r}{2}(T-t)} - A(S_tK)^{\frac{1}{2}}e^{-\frac{r}{2}(T-t)} \\
&= \Phi(d_+)
\end{aligned} \tag{7.4}$$

You can use similar way to obtain the Delta of put option. But *Call-Put Parity* is still a simpler way.

$$\begin{aligned}
\Delta_P &= \frac{\partial}{\partial S_t}(C + Ke^{-r(T-t)} - S_t) \\
&= \Delta_C - 1 \\
&= -\Phi(-d_+)
\end{aligned} \tag{7.5}$$

8 Some Expansion of Black-Scholes Equation

8.1 B-S Model when Stock Pays Continuous Dividend

Assume a stock pays a continuous dividend with rate q, which means if you have one share of stock, at time t, you will have e^{qt} shares. The stock price under this process is

$$dS_t = (\mu - q)S_t dt + \sigma_t dW_t.$$

Then the Black-Scholes equation with dividend turns to

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S_t^2} + (r-q)\frac{\partial V}{\partial S_t} - rV = 0. \tag{8.1}$$

Try to derive it as what we have done in chapter 3. For European call/put option, its solution are

$$C = S_t e^{-q(T-t)} N(d_+) - Ke^{-r(T-t)} N(d_-), \tag{8.2}$$
$$P = Ke^{-r(T-t)} N(-d_-) - S_t e^{-q(T-t)} N(-d_+), \tag{8.3}$$

where

$$d_+ = \frac{\log \frac{S_t}{K} + (r + \frac{1}{2}\sigma^2 - q)(T-t)}{\sqrt{\sigma^2(T-t)}}, \tag{8.4}$$
$$d_- = d_+ - \sqrt{\sigma^2(T-t)}. \tag{8.5}$$

8.2 B-S Model when r, σ (and q) are Time-Dependent

If some of parameters like r, σ (and q) are no longer constant, but deterministic, the option price still can be obtained under Black-Scholes Framework.

The Black-Scholes equation with time-dependent $r(t)$ and $\sigma(t)$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(t) S^2 \frac{\partial^2 V}{\partial S_t^2} + r(t)\frac{\partial V}{\partial S_t} - rV = 0. \tag{8.6}$$

for European call/put option are

$$C = S_t N(d_+) - Ke^{-r(T-t)} N(d_-), \tag{8.7}$$
$$P = Ke^{-r(T-t)} N(-d_-) - S_t N(-d_+), \tag{8.8}$$

where

$$d_+ = \frac{\log \frac{S_t}{K} + \int_0^t r(s) + \frac{1}{2}\sigma^2(s) ds}{\sqrt{\int_0^t \sigma^2(s) ds}}, \tag{8.9}$$

$$d_- = d_+ - \left(\int_0^t \sigma^2(s)ds\right)^{\frac{1}{2}}. \tag{8.10}$$

Appendix D shows the proof of this statement. But this proof requires that you to understand the next two chapters.

9 Itô Isometry

Some of stochastic processes may defined as a stochastic integration. In this chapter we will try to evaluate such processes.

Let W_t be a standard Brownian motion, and $f : \mathbb{R}^2 \to \mathbb{R}$ be a second differentiable function. Define stochastic process X_t as

$$X_t = \int_0^t f(t, W_t)dW_t. \tag{9.1}$$

Proposition 9.1 $\mathbb{E}[X_t] = 0$.

Proof. Let's evaluate it by writing it in the form of Riemann sum:

$$\begin{aligned}\mathbb{E}[X_t] &= \mathbb{E}\int_0^t f(t, W_t)dW_t \\ &= \lim_{n\to\infty} \mathbb{E}\left[\sum_{k=1}^{n-1} f(t_k, W_{t_k})\Delta W_k\right]\end{aligned}$$

where $\Delta W_k = W_{t_{k+1}} - W_{t_k}$. By law of total expectation, which is

$$\mathbb{E}[A] = \mathbb{E}\big[\mathbb{E}[A \mid B]\big].$$

There is

$$\mathbb{E}[X_t] = \lim_{n\to\infty} \mathbb{E}[\mathbb{E}[\cdots\mathbb{E}[X_t \mid \mathscr{F}_1]\cdots \mid \mathscr{F}_{n-1}]]$$

where $\mathscr{F}_t$ is the sigma-algebra that defines W_t. Since increment of Brownian motion is also a Brownian motion, for any single time point t_k,

$$\mathbb{E}[f(t_k, W_{t_k})\Delta W_k \mid \mathscr{F}_k] = f(t_k, W_{t_k})\,\mathbb{E}[\Delta W_k] = 0.$$

So,

$$\begin{aligned}\mathbb{E}[X_t] &= \lim_{n\to\infty} \mathbb{E}[\mathbb{E}[\cdots\mathbb{E}[f(t_1, W_{t_1})\Delta W_{t_1} + \cdots + f(t_{n-1}, W_{t_{n-1}})\Delta W_{t_{n-1}} \mid \mathscr{F}_1]\cdots \mid \mathscr{F}_{n-1}]] \\ &= \lim_{n\to\infty} \mathbb{E}[\mathbb{E}[\cdots\mathbb{E}[0 + f(t_2, W_{t_2})\Delta W_{t_2} + \cdots + f(t_{n-1}, W_{t_{n-1}})\Delta W_{t_{n-1}} \mid \mathscr{F}_2]\cdots \mid \mathscr{F}_{n-1}]]\end{aligned}$$

$$
\begin{aligned}
&= \cdots\cdots \\
&= \mathbb{E}[0 + f(t_{n-1}, W_{t_{n-1}})\Delta W_{t_{n-1}}] \\
&= 0.
\end{aligned}
$$

□

As the expectation goes to 0, the variance of X_t is $\mathrm{Var}(X_t) = \mathbb{E}[X_t^2]$. And *Ito Isometry* states that:

Theorem 9.2 If W_t be a standard Brownian motion, and $f : \mathbb{R}^2 \to \mathbb{R}$ be a second differentiable function. The second moment of stochastic process X_t is

$$
\mathbb{E}\left[\left(\int_0^t f(s, W_s)dW_s\right)^2\right] = \int_0^t \mathbb{E}\left[f^2(s, W_s)\right]ds. \tag{9.2}
$$

Proof.

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_0^t f(s, W_s)dW_s\right)^2\right] &= \mathbb{E}\left[\left(\int_0^t f(t', W_{t'})dW_{t'}\right)\left(\int_0^t f(t'', W_{t''})dW_{t''}\right)\right] \\
&= \lim_{m,n\to\infty} \mathbb{E}\left[\sum_{i=1}^{n-1}\sum_{j=1}^{m-1} f(t'_i, W_{t'_i}) f(t''_j, W_{t''_j})\Delta W_i \Delta W_j\right]
\end{aligned}
$$

where ΔW_i and ΔW_j are defined as same as in proof of proposition 9.1. Using law of total expectation, at any time point t_i,

$$
\mathbb{E}[f(t'_i, W_{t'_i}) f(t''_j, W_{t''_j})\Delta W_i \Delta W_j \mid \mathscr{F}_i] = \begin{cases} 0, & i \neq j \\ f^2(t'_i, W_{t'_i})\Delta t, & i = j \end{cases}
$$

So,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t f(s, W_s)dW_s\right)^2\right] \\
&= \lim_{m,n\to\infty} \mathbb{E}\left[\sum_{i=1}^{n-1}\sum_{j=1}^{m-1} f(t'_i, W_{t'_i}) f(t''_j, W_{t''_j})\Delta W_i \Delta W_j\right] \\
&= \lim_{m,n\to\infty} \mathbb{E}\left[\mathbb{E}\left[\cdots \mathbb{E}\left[f^2(t_1, W_{t_1})\Delta t + \sum_{i=2}^{n-1}\sum_{j=1}^{m-1} f(t'_i, W_{t'_i}) f(t''_j, W_{t''_j})\Delta W_i \Delta W_j \,\middle|\, \mathscr{F}_2\right] \cdots \,\middle|\, \mathscr{F}_{n-1}\right]\right] \\
&= \cdots\cdots \\
&= \lim_{m,n\to\infty} \mathbb{E}\left[\sum_{i=1}^{n-2} f^2(t_1, W_{t_1})\Delta t + f(t'_{n-1}, W_{t'_{n-1}})\Delta W_{n-1}\sum_{j=1}^{m-1} f(t''_j, W_{t''_j})\Delta W_j\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\sum_{i=1}^{n-1} f^2(t_1, W_{t_1})\Delta t\right]
\end{aligned}
$$

$$=\mathbb{E}\left[\int_0^t f^2(s,W_s)ds\right]=\int_0^t \mathbb{E}\left[f^2(s,W_s)\right]ds.$$

$\square$

Ito isometry gives us a simpler way to evaluate the second moment of stochastic integrations.

Example 9.1 Calculate $\mathbb{E}\left[(\int_0^t W_t\,dW_t)^2\right]$.

Solution: First, we try to use Ito isometry to solve it

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t W_t\,dW_t\right)^2\right]&=\int_0^t \mathbb{E}[W_t^2]dt\\&=\int_0^t t\,dt\\&=\frac{t^2}{2}.\end{aligned}$$

We can also evaluate it directly as what we have done in chapter 2.3. By Ito lemma,

$$d\frac{W_t^2}{2}=W_tdW_t+\frac{1}{2}dt.$$

Integrate it, we get

$$\int_0^t W_s\,dW_s=\frac{W_t^2}{2}-\frac{1}{2}\int_0^t 1\,ds=\frac{1}{2}(W_t^2-t).$$

Then

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t W_t\,dW_t\right)^2\right]=&\mathbb{E}\left[\frac{1}{4}(W_t^2-t)^2\right]\\=&\frac{1}{4}\mathbb{E}[W_t^4-2tW_t^2+t^2]\\=&\frac{1}{4}\mathbb{E}[3t^2-2t^2+t^2]\\=&\frac{1}{2}t^2\end{aligned}$$

We can see that these two methods give us the same answer in this example. But the calculation for Ito isometry is much simpler.

10 Martingale

Definition 10.1 Let M_t be a stochastic process and $E[|M_t|]<\infty$, $\{\mathscr{F}_t\}$ for $t=1,2,\cdot$ (or $t\in R^+$) is a filtration. If

1. M_t is $\mathscr{F}_t$ measurable,
2. For any $0 \leq s < t$, $\mathbb{E}[M_t \mid \mathscr{F}_s] = M_s$,

Then M_t is a *Martingale*.

Example 10.1 Constant process is a martingale.

Proof. Let $M_t = C$. Since $\mathbb{E}[M_t] = \mathbb{E}[C] = C < \infty$, and

$$\mathbb{E}[M_t \mid \mathscr{F}_s] = C = M_s.$$

Therefore M_t is a martingale. □

Example 10.2 Brownian motion is a martingale.

Proof. Let W_t be a standard Brownian motion, then $E[W_t] = 0 < \infty$, and

$$\mathbb{E}[W_t \mid \mathscr{F}_s] = \mathbb{E}[W_t - W_s] + W_s = W_s.$$

Therefore W_t is a martingale. □

Exersice: Prove these processes are martingale

1. $W_t^2 - t$
2. $e^{aW_t - \frac{1}{2}a^2 t}$, where a is a constant.

11 * Martingale Method for Option Pricing

In this chapter, we introduce a different method to evaluate the price of options, which is martingale method.

We used Geometric Brownian motion to describe the price of stock,, which is

$$dS_t = S_t(\tilde{\mu} dt + \sigma dW_t), \tag{11.1}$$

Its solution is

$$S_t = S_0 e^{(\tilde{\mu} - \frac{\sigma^2}{2})t + \sigma W_t} \tag{11.2}$$

In martingale method, we assume that the discounted stock price is a martingale, i.e.

$$\tilde{S}_t = e^{rt} S_t. \tag{11.3}$$

to make the shift rate of $\tilde{S}_t$ zero, $\tilde{\mu}$ must equal to r. That is, in risk neutral measure, the return of stock must be risk free rate.

Let V_t be the price of option. In risk neutral measure, option price is also a martingale. Denote $\tau = T - t$, then

$$V(t, S_t) = \mathbb{E}[e^{-r\tau} V(T, S_T) \mid \mathscr{F}_t] \tag{11.4}$$
$$= \mathbb{E}[e^{-r\tau} F(S_T) \mid \mathscr{F}_t] \tag{11.5}$$

For call options

$$F(S_T) = \max\{S_T - K, 0\}; \tag{11.6}$$

For put options

$$F(S_T) = \max\{K - S_T, 0\}. \tag{11.7}$$

The conditional distribution of S_T given S_t is

$$S_T \mid S_t = S_t e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma(W_T - W_t)}, \tag{11.8}$$

for call options,

$$\mathbb{E}[e^{-r\tau} V(T, S_T) \mid \mathscr{F}_t] = \mathbb{E}\left[e^{-r\tau} F(S_T) \mid \mathscr{F}_t\right]$$
$$= \int_{-\infty}^{\infty} e^{-r\tau} F\left(S_t e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}x}\right) \varphi(x) dx \tag{11.9}$$
$$= \int_{-\infty}^{\infty} e^{-r\tau} \max\left\{S_t e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}x} - K, 0\right\} \varphi(x) dx \tag{11.10}$$

Let $X = \tau^{-\frac{1}{2}}(W_T - W_t) \sim N(0, 1)$, when $S_T - K = 0$, then

$$S_t e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma(W_T - W_t)} - K = 0$$

Solve this equation, we get

$$X = -\frac{\log \frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = -d_2 \tag{11.11}$$

Then

$$\mathbb{E}[e^{-r\tau} V(T, S_T) \mid \mathscr{F}_t] = \int_{-d_2}^{\infty} e^{-r\tau} \left(S_t e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}x} - K\right) \varphi(x) dx$$
$$= \int_{-d_2}^{\infty} \left(S_t e^{-\frac{1}{2}\sigma^2\tau + \sigma\sqrt{\tau}x}\right) \varphi(x) dx - K e^{-r\tau} \Phi(d_2) \tag{11.12}$$

Now let's look at the integration in equation (11.12).

$$I = \int_{-d_2}^{\infty} \left(S_t e^{-\frac{1}{2}\sigma^2\tau + \sigma\sqrt{\tau}x}\right) \varphi(x) dx$$
$$= S_t \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma^2\tau + \sigma\sqrt{\tau} - \frac{x^2}{2}} dx$$
$$= S_t \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sigma\sqrt{\tau})^2}{2}} dx \tag{11.13}$$

Let $u = x - \sigma\sqrt{\tau}$, then

$$\begin{aligned} I =& S_t \int_{-d_2-\sigma\sqrt{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \\ =& S_t \int_{-\infty}^{d_2+\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \\ =& S_t \Phi(d_2 + \sigma\sqrt{\tau}) \end{aligned} \tag{11.14}$$

Therefore, the price of call option is

$$V(t, S_t) = S_t \Phi(d_1) - K e^{-r\tau} \Phi(d_2), \tag{11.15}$$

where

$$d_1 = \frac{\log \frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \tag{11.16}$$

$$d_2 = d_2 - \sigma\sqrt{\tau}. \tag{11.17}$$

The process to derive the put option price is quite similar.

Martingale method could deal with much more sophisticated situations than Black-Scholes equation, like American options and exotic options. In the actual study of stock price volatility, volatility shows some statistical characteristics, such as volatility smile, heavy tail distribution, etc. In view of these phenomena, many researches have improved the assumptions that the underlying assets in the Black-Scholes model obey geometric Brown motion and volatility is constant. In the revision of the distribution of the underlying assets, it is proposed to describe the stock price process by jumping diffusion process. In the revision of volatility, stochastic volatility models like Heston model have appeared. Martingale method is able to price options in many of these models.

A Proof of Formula (1.7) and (1.8)

Proof for (1.7):

$$E[W_t^{2n+1}] = \int_{-\infty}^{\infty} x^{2n+1} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx.$$

Denote $\varphi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$, it is obviously a even function. And $y = x^{2n+1}$ is a odd function. An even function times an odd function is an odd function. and for odd function $f(x)$

$$\int_{-n}^{n} f(x)dx = 0.$$

So

$$\mathbb{E}[W_t^{2n+1}] = \lim_{n\to\infty} \int_{-n}^{n} x^{2n+1}\varphi(x)dx = 0.$$

□

Proof for (1.8) Let $\lambda = \frac{1}{2t}$,

$$\begin{aligned}
\mathbb{E}[W_t^{2n}] &= \int_{-\infty}^{\infty} x^{2n} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} x^{2n} e^{-\lambda x^2} dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} (-1)^n \frac{\partial^n}{\partial \lambda^n} e^{-\lambda x^2} dx \\
&= \frac{(-1)^n}{\sqrt{2\pi t}} \frac{\partial^n}{\partial \lambda^n} \int_{-\infty}^{\infty} e^{-\lambda x^2} dx \\
&= \frac{(-1)^n}{\sqrt{2\pi t}} \frac{\partial^n}{\partial \lambda^n} \sqrt{\frac{\pi}{\lambda}} \\
&= \frac{(-1)^n}{\sqrt{2t}} (-\tfrac{1}{2})(-\tfrac{3}{2}) \cdots (-\tfrac{2n-1}{2}) \lambda^{-n-\frac{1}{2}} \\
&= \frac{(-1)^{2n}}{\sqrt{2t}} \frac{\prod_{i=1}^{n}(2i-1)}{2^n} \lambda^{-n-\frac{1}{2}} \\
&= \frac{1}{\sqrt{2t}} \frac{(2n-1)!!}{2^n} (2t)^{n+\frac{1}{2}} \\
&= (2n-1)!! t^n.
\end{aligned}$$

□

B Proof of Formula (2.1)

To prove this equation, we need *Markov Inequality*. It says for random variable $X > 0$ which has finite expectation, i.e. $E(X) < \infty$, then for any $a > 0$, there is

$$P(X > a) \leq \frac{\mathbb{E}(X)}{a}. \tag{B.1}$$

Furthermore,

$$P(X^2 > a^2) \leq \frac{\mathbb{E}(X^2)}{a^2}. \tag{B.2}$$

You can search for it on Internet if you wish.

We cannot directly integrate a stochastic process since it is indeterministic, but we can treat it by proving such a fact, for any $\delta > 0$,

$$P\left[\int_0^t (dW_t)^2 - t > \delta\right] = 0. \tag{B.3}$$

We can represent a integration by a Riemann sum, so

$$\int_0^t (dW_t)^2 = \lim_{n\to\infty}\sum_{i=0}^{n}(W_i - W_{i-1})^2. \tag{B.4}$$

Define $Z_i = \frac{W_i - W_{i-1}}{\sqrt{t/n}}$, then

$$E(Z_i) = \sqrt{\frac{n}{t}}(E(W_i) - E(W_{i-1}) = 0;$$

$$E(Z_i^2) = \mathrm{Var}(Z_i) = \frac{t}{n}\mathrm{Var}(W_i - W_{i-1}) = 1.$$

We can see Z_i is a standard normal distribution. Using the notation of Z_i,

$$\int_0^t (dW_t)^2 - t = \frac{t}{n}\lim_{n\to\infty}\sum_{i=1}^{n}(Z_i^2 - 1) \tag{B.5}$$

Now come back to the probability,

$$\begin{aligned} I =& P\left[\int_0^t (dW_t)^2 - t > \delta\right] \\ =& P\left[\frac{t}{n}\lim_{n\to\infty}\sum_{i=1}^{n}(Z_i^2 - 1) > \delta\right] \end{aligned} \tag{B.6}$$

By inequation (B.2),

$$\begin{aligned} I &\leq \lim_{n\to\infty}\frac{\mathbb{E}[(\frac{t}{n}\sum_{i=1}^{n}(Z_i^2-1))(\frac{t}{n}\sum_{i=1}^{n}(Z_i^2-1))]}{\delta^2} \\ &= \lim_{n\to\infty}\frac{\frac{t^2}{n^2}\mathbb{E}[\sum_{i,j=1}^{n}(Z_i^2 Z_j^2) - (Z_i^2 + Z_j^2) + 1]}{\delta^2} \\ &= \lim_{n\to\infty}\frac{t^2}{n^2\delta^2}\left(\mathbb{E}\sum_{i,j=1}^{n} Z_i^2 Z_j^2 - n\right) \\ &= \lim_{n\to\infty}\frac{t^2}{n^2\delta^2}\left(\sum_{i=1}^{n}\mathbb{E}(Z_i^4) + \sum_{i\neq j}\mathbb{E}(Z_i^2 Z_j^2) - n\right). \end{aligned} \tag{B.7}$$

Since Z_i is standard normal distribution and for $i \neq j$, Z_i and Z_j are independent. Therefore

$$\begin{aligned} P\left[\int_0^t (dW_t)^2 - t > \delta\right] &= \lim_{n\to\infty} \frac{t^2}{n^2\delta^2}(3n-n) \\ &= \lim_{n\to\infty} \frac{2t^2}{n\delta^2} \\ &= 0. \end{aligned} \tag{B.8}$$

Therefore we can say

$$\int_0^t (dW_t)^2 = t. \tag{B.9}$$

□

Under mean of integration, we can write that

$$(dW_t)^2 = dt. \tag{B.10}$$

C Fourier Transformation and Solution of Heat Equation

C.1 Fourier Transformation

There are various way to derive the general solution of heat equation. Here, method of Fourier transformation will be introduced.

Definition C.1 (Fourier Transformation) Let $f(x)$ is absolutely integrable in $\mathbb{R}$, the function $g(\lambda) = \mathcal{F}[f(x)]$ is defined as

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx \tag{C.1}$$

is called Fourier transformation of f.

The Inverse Fourier Transformation of function $g(\lambda)$ is defined by

$$\mathcal{F}^{-1}[g(\lambda)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\lambda)e^{i\lambda x}d\lambda, \tag{C.2}$$

because of the fact of

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{i\lambda(x-\xi)}d\xi d\lambda. \tag{C.3}$$

Obviously, Fourier transformation is a linear operator.

Proposition C.1 (Linearity of Fourier Transformation) Let f and g are all absolutely integrable on $\mathbb{R}$, and $a, b \in \mathbb{R}$ The Fourier Transformation of $af(x) + bg(x)$ is

$$\mathcal{F}[af(x) + bg(x)] = a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)]. \tag{C.4}$$

This can be easily verified by linearity of integration.

Let's introduce a concept, which is deeply connected with Fourier transformation, so called Convolution.

Definition C.2 For two function $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, the convolution of these two function is defined by

$$(f_1 * f_2)(x) = \int_{-\infty}^{\infty} f_1(t) f_2(x-t) dt. \tag{C.5}$$

One important property of convolution is commutative, i.e.

$$f_1 * f_2 = f_2 * f_1. \tag{C.6}$$

The relation between convolution and multiplication of Fourier transformation is shown by the proposition below.

Proposition C.2 (Convolution Theorem (1)) The Fourier transformation of a convolution of two functions equals to the multiplication of Fourier transformation of these two function, mathematically,

$$\mathcal{F}[f_1 * f_2] = \mathcal{F}[f_1] \cdot \mathcal{F}[f_2]. \tag{C.7}$$

Proof. By definition,

$$\begin{aligned}\mathcal{F}[f_1 * f_2] &= \int_{-\infty}^{\infty} e^{-i\lambda x} dx \int_{-\infty}^{\infty} f_1(t) f_2(x-t) dt \\ &= \int_{-\infty}^{\infty} f_1(t) dt \int_{-\infty}^{\infty} f_2(x-t) e^{-i\lambda x} dx\end{aligned}$$

Let $\xi = x - t$, then $x = t + \xi$, $dx = d\xi$,

$$\begin{aligned}\mathcal{F}[f_1 * f_2] &= \int_{-\infty}^{\infty} f_1(t) dt \int_{-\infty}^{\infty} f_2(\xi) e^{-i\lambda(t+\xi)} d\xi \\ &= \int_{-\infty}^{\infty} e^{-i\lambda t} f_1(t) dt \int_{-\infty}^{\infty} e^{-i\lambda \xi} f_2(\xi) d\xi \\ &= \mathcal{F}[f_1] \cdot \mathcal{F}[f_2].\end{aligned}$$

□

And you can verify that inverse Fourier transformation also has this property (but the coefficient is no longer 1). Let $g_i(\lambda) = \mathcal{F}[f_i]$, for $i = 1, 2$, we get

$$\frac{1}{2\pi}\mathcal{F}^{-1}[g_1 * g_2] = \mathcal{F}^{-1}[g_1] \cdot \mathcal{F}^{-1}[g_2] = f_1 \cdot f_2.$$

Apply Fourier transformation to both sides of this equation, it gives

Proposition C.3 (Convolution Theorem (2))

$$\mathcal{F}[f_1 \cdot f_2] = \frac{1}{2\pi}\mathcal{F}[f_1] * \mathcal{F}[f_2]. \tag{C.8}$$

Equation Equation (C.7) and Equation (C.8) are the *convolution theorem*.

Fourier transformation also has a properties to transform a differential equation to a algebraic equation, say

Proposition C.4 If the Fourier transformations of $f(x)$ and $f'(x)$ exist, and $f(x) \to 0$ as $|x| \to \infty$, there is

$$\mathcal{F}[f'(x)] = i\lambda\mathcal{F}[f(x)]. \tag{C.9}$$

Proof. By integration by part,

$$\begin{aligned}
\mathcal{F}[f'(x)] =& \int_{-\infty}^{\infty} f'(x)e^{-i\lambda x}dx \\
=& f(x)e^{-i\lambda x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -i\lambda f(x)e^{-i\lambda x}dx \\
=& i\lambda \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx \\
=& i\lambda\mathcal{F}[f(x)].
\end{aligned}$$

$\square$

A series of videos on *Bilibili* illustrates how to derive the concept of Fourier transformation very well. See here.

C.2 General Solution of Heat Equation

Now we shall start to find the general solution of

$$\begin{cases} u_t = u_{xx} \\ u(t=0, x) = \phi(x). \end{cases} \tag{C.10}$$

Let $\tilde{u}$ and $\tilde{\phi}$ be the Fourier transformations of u and ϕ, i.e.

$$\tilde{u} = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} u(x,t)e^{-i\lambda x}dx. \tag{C.11}$$

$$\tilde{\phi} = \mathcal{F}[\phi(x)] = \int_{-\infty}^{\infty} \phi(x)e^{-i\lambda x}dx. \tag{C.12}$$

Then

$$\begin{aligned}\tilde{u}'(t) =& \int_{-\infty}^{\infty} u_t e^{-i\lambda x}dx \\ =& \int_{-\infty}^{\infty} u_{xx} e^{-i\lambda x}dx \\ =& -\lambda^2 \tilde{u},\end{aligned} \tag{C.13}$$

with the initial condition

$$\tilde{u}(\lambda, 0) = \tilde{\phi}(\lambda). \tag{C.14}$$

The solution of this ODE is

$$\tilde{u}(\lambda, t) = \tilde{\phi}(\lambda)e^{-\lambda^2 t}. \tag{C.15}$$

So,

$$u(x,t) = \phi(x) * \mathcal{F}^{-1}[e^{-\lambda^2 t}]. \tag{C.16}$$

Now let's focus on the inverse Fourier transformation term,

$$\begin{aligned}\mathcal{F}^{-1}[e^{-\lambda^2 t}] =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\lambda^2 t}e^{-i\lambda x}d\lambda \\ =& \frac{1}{2\pi}e^{-\frac{x^2}{4t}}\int_{-\infty}^{\infty} e^{-t(\lambda-\frac{ix}{2t})^2}d\lambda \\ =& \frac{1}{2\pi}e^{-\frac{x^2}{4t}}\int_{-\infty}^{\infty} e^{-t\lambda^2}d\lambda \\ =& \frac{1}{2\pi}e^{-\frac{x^2}{4t}}t^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-y^2}dy \quad \text{(by changing of variable)} \\ =& \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}} := G_t(x),\end{aligned}$$

which is the p.d.f. of Normal distribution with $\mu = 0$ and $\sigma^2 = 2t$. So the general solution of equation (C.10) is

$$u(t,x) = \phi(x) * G_t(x) \tag{C.17}$$

$$= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \phi(x')e^{-\frac{(x-x')^2}{4t}}dx'. \tag{C.18}$$

Lemma C.5 For any $f(t)$ such that $\int_0^t f(s)ds > 0$, solution of equation with initial condition

$$\begin{cases} u_t = f(t)u_{xx} \\ u(t=0,x) = \phi(x) \end{cases} \tag{C.19}$$

is

$$u(t,x) = \frac{1}{\sqrt{4\pi \int_0^t f(t')dt'}} \int_{-\infty}^{\infty} \phi(x') e^{-\frac{(x-x')^2}{4\int_0^t f(t')dt'}} dx' \tag{C.20}$$

Proof. Let substitute t by τ first, assume that τ is a function of t, i.e. $\tau = g(t)$. Then

$$u_t = u_\tau \cdot g'(t) = f(t)u_{xx}. \tag{C.21}$$

To cancel the term of $f(t)$, let $g'(t) = f(t)$. By defining $\tau = \int_0^t f(t')dt$, there is

$$u_\tau = u_{xx}. \tag{C.22}$$

The general solution of this equation has been obtained as

$$u(\tau,x) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} \phi(x') e^{-\frac{(x-x')^2}{4\tau}} dx'.$$

Substituting τ into the form of t, then the general solution of equation (C.19) is

$$u(t,x) = \frac{1}{\sqrt{4\pi \int_0^t f(t')dt'}} \int_{-\infty}^{\infty} \phi(x') e^{-\frac{(x-x')^2}{4\int_0^t f(t')dt'}} dx' \tag{C.23}$$

□

D Proof of Price Formula when $r(t)$ and $\sigma(t)$ Are Time-Dependent

Under risk neutral measure, consider a stock, whose price S_t is governed by such a stochastic process

$$dS_t = r(t)dt + \sigma(t)dW_t. \tag{D.1}$$

By Itô Lemma, the logarithm of stock price $\log S_t$ is

$$d\log S_t = \left[r(t) - \frac{1}{2}\sigma^2(t)\right]dt + \sigma(t)dW_t. \tag{D.2}$$

Integrate both sides of this equation,

$$\log S_t = \int_0^t r(s) - \frac{1}{2}\sigma^2(s)ds + \int_0^t \sigma(s)dW_s. \tag{D.3}$$

Look at the second part of this equation, denote this part as $I = \int_0^t \sigma(s)dW_s$. Since $\sigma(t)$ is deterministic, as proposition 9.1 states,

$$\mathbb{E}\left[\int_0^t \sigma(s)dW_s\right] = 0. \tag{D.4}$$

And its variance can be evaluated by Itô isometry, which is

$$\mathbb{E}\left[\left(\int_0^t \sigma(s)dW_s\right)^2\right] = \int_0^t \sigma^2(s)ds. \tag{D.5}$$

Define

$$\bar{r} = t^{-1}\int_0^t r(s)ds, \quad \bar{\sigma}^2 = t^{-1}\int_0^t \sigma^2(s)ds,$$

for any $t > 0$, then $\log S_t$ is normal distributed with parameters $\mu = (\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)t$ and $\sigma^2 = \bar{\sigma}^2 t$. Base on this, stochastic process

$$dS'_t = \bar{r}S'_t dt + \bar{\sigma}S'_t dW_t \tag{D.6}$$

is equivalent to process (D.1), since two SDE have the same solution. If a stock is governed by process (D.6), by martingale method, call option price is

$$C(t, S_t) = \mathbb{E}[e^{-\bar{r}\tau}C(T, S_T) \mid \mathscr{F}_t] \tag{D.7}$$

$$= S_t\Phi(d_1) - Ke^{-\int_t^T r(s)ds}\Phi(d_2). \tag{D.8}$$

And put option price is

$$P(t, S_t) = Ke^{-\int_t^T r(s)ds}\Phi(-d_2) - S_t\Phi(-d_1), \tag{D.9}$$

where

$$d_1 = \frac{\log\frac{S_t}{K} + \int_t^T r(s) + \frac{1}{2}\sigma^2(s)ds}{\sqrt{\int_t^T \sigma^2(s)ds}}, \tag{D.10}$$

$$d_2 = \frac{\log\frac{S_t}{K} + \int_t^T r(s) - \frac{1}{2}\sigma^2(s)ds}{\sqrt{\int_t^T \sigma^2(s)ds}}. \tag{D.11}$$

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