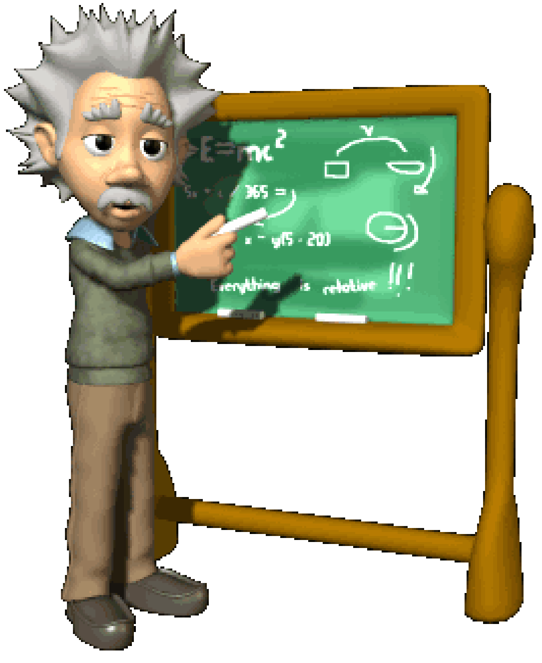
Chapter 6 Integration Techniques (积分技巧)

In this Chapter, we will encounter some important concepts.

- Integration by Parts (分部积分)
- Trigonometric Techniques of Integration (积分中三角函数技巧)
- Improper Integrals (广义积分)



Section 6.1 Review of Formulas and Techniques

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c, \quad \text{for } r \neq -1 \text{ (power rule)}$$

$$\int \frac{1}{x}\,dx = \ln|x| + c, \quad \text{for } x \neq 0$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \sec x \tan x\,dx = \sec x + c$$

$$\int \csc^2 x\,dx = -\cot x + c$$

$$\int \csc x \cot x\,dx = -\csc x + c$$

$$\int e^x\,dx = e^x + c$$

$$\int e^{-x}\,dx = -e^{-x} + c$$

$$\int \tan x\,dx = -\ln|\cos x| + c$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$$

$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1} x + c$$

EXAMPLE 1.1 A Simple Substitution

Evaluate $\int \sin(ax)\,dx$, for $a \neq 0$.

EXAMPLE 1.2 Generalizing a Basic Integration Rule

Evaluate $\displaystyle\int \frac{1}{a^2 + x^2}\,dx$, for $a \neq 0$.

EXAMPLE 1.3 An Integrand That Must Be Expanded

Evaluate $\int (x^2 - 5)^2\,dx$.

EXAMPLE 1.4 An Integral Where We Must Complete the Square

Evaluate $\displaystyle\int \frac{1}{\sqrt{-5 + 6x - x^2}}\,dx$.

Section 6.2 Integration by Parts (分部积分)

If *u(x)* and *v(x)* are both differentiable functions of *x*, then

$$\frac{d}{dx}[u(x)v(x)] = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}$$

$$\Downarrow$$

$$u(x)\frac{dv}{dx} = \frac{d}{dx}[u(x)v(x)] - v(x)\frac{du}{dx}$$

$$\Downarrow$$

Since u(x)v(x) is antiderivative of $\frac{d}{dx}[u(x)v(x)]$ and

$$\int\left[u(x)\frac{dv}{dx}\right]dx = \int\frac{d}{dx}[u(x)v(x)]\,dx - \int\left[v(x)\frac{du}{dx}\right]dx$$

$$= u(x)v(x) - \int\left[v(x)\frac{du}{dx}\right]dx$$

Since $dv = \frac{dv}{dx}dx$ and $du = \frac{du}{dx}dx$

We have

$$\int u\,dv = uv - \int v\,du$$

Integration by parts formula:

$$\int f(x)dx = \int u dv = uv - \int v du$$

Integration by parts: integral $\int f(x)\,dx$

Step 1. Choose functions u and v so that $f(x)dx=udv$. Try to pick u so that du is simpler than u and a dv is easy to integrate

Step 2. Organize the computation of du and v as

$$\begin{array}{ll} u & dv \\ du & v = \int dv \end{array}$$

and substitute into the integration by parts formula

$$\int u\,dv = uv - \int v\,du$$

Step 3. Complete the integration by finding $\int v\,du$ Then

$$\int f(x)\,dx = \int u\,dv = uv - \int v\,du$$

Example

Find $\int x^2 \ln x \, dx$.

Solution:

Our strategy is to express $\int x^2 \ln x \, dx$ as $\int u \, dv$ by choosing u and v so that $\int v \, du$ is easier to evaluate than $\int u \, dv$. This strategy suggests that we choose

$$u = \ln x \qquad \text{and} \qquad dv = x^2 \, dx$$

since

$$du = \frac{1}{x} \, dx$$

is a simpler expression than $\ln x$, while v can be obtained by the relatively easy integration

$$v = \int x^2 \, dx = \frac{1}{3} x^3$$

to be continued

(For simplicity, we leave the "+ C" out of the calculation until the final step.) Substituting this choice for u and v into the integration by parts formula, we obtain

$$\int x^2 \ln x\,dx = \int \underbrace{(\ln x)}_{u}\underbrace{(x^2\,dx)}_{dv} = \underbrace{(\ln x)}_{u}\underbrace{\left(\frac{1}{3}x^3\right)}_{v} - \int \underbrace{\left(\frac{1}{3}x^3\right)}_{v}\underbrace{\left(\frac{1}{x}\,dx\right)}_{du}$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\,dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\left(\frac{1}{3}x^3\right) + C$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

Example Find $\int xe^{2x}dx$.

Solution:

Although both factors x and e^{2x} are easy to integrate, only x becomes simpler when differentiated. Therefore, we choose $u = x$ and $dv = e^{2x}dx$ and find

$$u = x \qquad dv = e^{2x}dx$$

$$du = dx \qquad v = \frac{1}{2}e^{2x}$$

Substituting into the integration by parts formula, we obtain

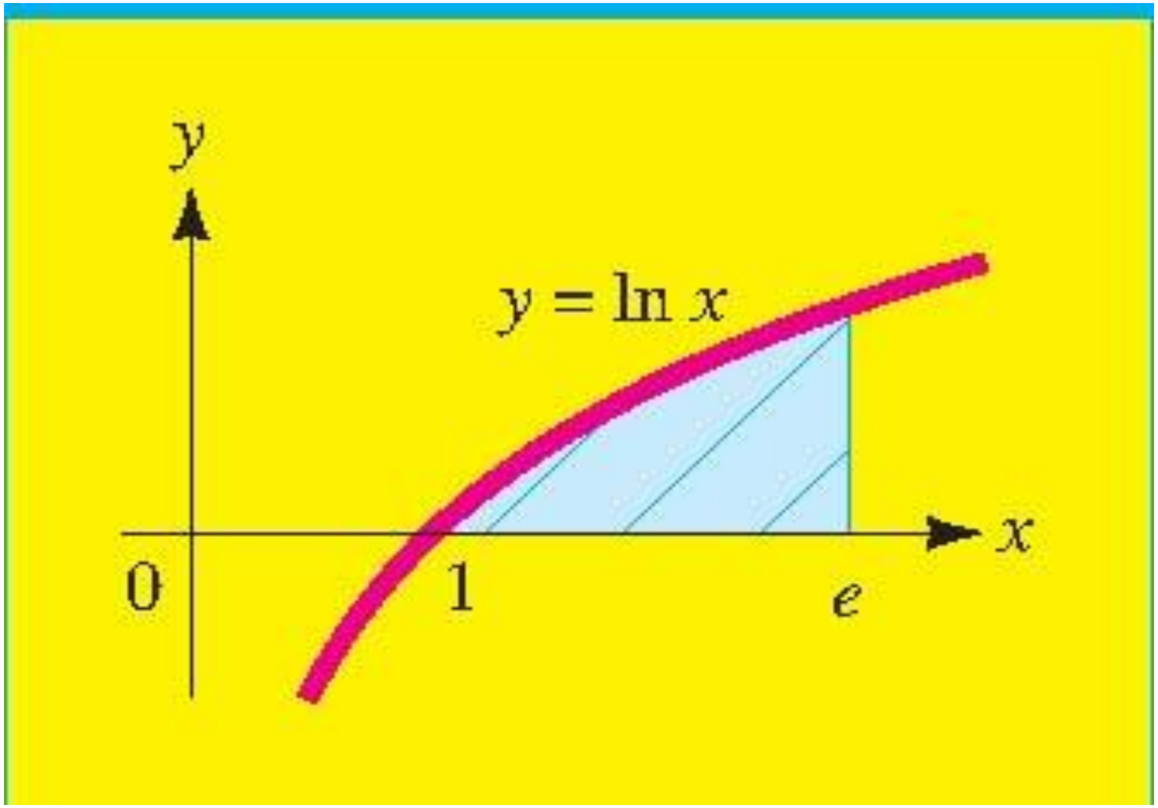
$$\int \underbrace{x}_{u}\underbrace{(e^{2x}dx)}_{dv} = \underbrace{x}_{u}\underbrace{\left(\frac{1}{2}e^{2x}\right)}_{v} - \int \underbrace{\left(\frac{1}{2}e^{2x}\right)}_{v}\underbrace{dx}_{du}$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\left(\frac{1}{2}e^{2x}\right) + C$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\right)e^{2x} + C$$

Example

Find the area of the region bounded by the curve $y=\ln x$, the x axis, and lines $x=1$ and $x=e$.



Solution:

The region is shown in Figure 6.1. Since $\ln x \geq 0$ for $1 \leq x \leq e$, the area is given by the definite integral

$$A = \int_1^e \ln x\,dx$$

To evaluate this integral using integration by parts, think of $\ln x\,dx$ as $(\ln x)(1\ dx)$ and use

$$u = \ln x \qquad dv = 1\,dx$$

$$du = \frac{1}{x}\,dx \qquad v = \int 1\ dx = x$$

Thus, the required area is

$$\begin{aligned}
A &= \int_1^e \ln x\,dx = x \ln x\Big|_1^e - \int_1^e x\left(\frac{1}{x}\,dx\right) \\
&= x \ln x\Big|_1^e - \int_1^e 1\,dx = (x \ln x - x)\Big|_1^e \\
&= [e \ln e - e] - [1 \ln 1 - 1] \\
&= [e(1) - e] - [1(0) - 1] \qquad\qquad \textit{ln}\ \mathrm{e} = 1\ \textit{and ln}\ 1 = 0 \\
&= 1
\end{aligned}$$

Repeated Application of Integration by parts

Example

Find $\int x^2 e^{2x}\,dx.$

Solution:

Since the factor e^{2x} is easy to integrate and x^2 is simplified by differentiation, we choose

$$u = x^2 \qquad dv = e^{2x}\,dx$$

so that

$$du = 2x\,dx \qquad v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$$

to be continued

Integrating by parts, we get

$$\int x^2 e^{2x}\,dx = x^2\left(\frac{1}{2}e^{2x}\right) - \int\left(\frac{1}{2}e^{2x}\right)(2x\,dx)$$

$$= \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\,dx$$

The integral $\int x e^{2x}\,dx$ that remains can also be obtained using integration by parts. Indeed, in Example 6.1.2, we found that

$$\int x e^{2x}\,dx = \frac{1}{2}\left(x - \frac{1}{2}\right)e^{2x} + C$$

Thus,

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\,dx$$

$$= \frac{1}{2}x^2 e^{2x} - \left[\frac{1}{2}\left(x - \frac{1}{2}\right)e^{2x}\right] + C$$

$$= \frac{1}{4}(2x^2 - 2x + 1)e^{2x} + C$$

EXAMPLE 2.1 Integration by Parts

Evaluate $\int x \sin x \, dx$.

EXAMPLE 2.3 An Integrand with a Single Term

Evaluate $\int \ln x \, dx$.

EXAMPLE 2.4 Repeated Integration by Parts

Evaluate $\int x^2 \sin x \, dx$.

EXAMPLE 2.5 Repeated Integration by Parts with a Twist

Evaluate $\int e^{2x} \sin x \, dx$.

EXAMPLE 2.7 Integration by Parts for a Definite Integral

Evaluate $\int_1^2 x^3 \ln x \, dx$.

Section 6. 3 Trigonometric Techniques of Integration

✓ Integrals Involving Powers of Trigonometric Functions

We first consider integrals of the form

$$\int \sin^m x \cos^n x \, dx,$$

where *m* and *n* are positive integers.

Case 1: *m* or *n* is an odd positive integer

If *m* is odd, first isolate one factor of *sinx*. (you will need this for *du*) Then, replace any factors of $\sin^2 x$ with $1 - \cos^2 x$ and make the Substitution *u=cosx*. If *n* is odd, the procedure is similar.

EXAMPLE 3.2 An Integrand with an Odd Power of Sine

Evaluate $\int \cos^4 x \sin^3 x \, dx$.

EXAMPLE 3.3 An Integrand with an Odd Power of Cosine

Evaluate $\int \sqrt{\sin x} \cos^5 x \, dx$.

Case 2: *m* and *n* are both even positive integers

We can use the half-angle formulas for sine and cosine to reduce the powers in the integrand.

NOTES

Half-angle formulas

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

EXAMPLE 3.4 An Integrand with an Even Power of Sine

Evaluate $\int \sin^2 x\,dx$.

Solution Using the half-angle formula, we can rewrite the integral as

$$\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx.$$

We can evaluate this last integral by using the substitution $u = 2x$, so that $du = 2\,dx$. This gives us

$$\int \sin^2 x\,dx = \frac{1}{2}\left(\frac{1}{2}\right)\int \underbrace{(1-\cos 2x)}_{1-\cos u}\underbrace{2\,dx}_{du} = \frac{1}{4}\int (1-\cos u)\,du$$

$$= \frac{1}{4}(u-\sin u)+c = \frac{1}{4}(2x-\sin 2x)+c. \quad \text{Since } u = 2x.$$

EXAMPLE 3.5 An Integrand with an Even Power of Cosine

Evaluate $\int \cos^4 x\,dx$.

Our next aim is to devise a strategy for evaluating integrals of the form

$$\int \tan^m x \sec^n x \, dx,$$

where *m* and *n* are integers.

Case 1: *m* is an odd positive integer

First, isolate one factor of *secxtanx*. (you'll need this for *du*.) Then, replace any factors of $\tan^2 x$ with $\sec^2 x - 1$ and make the substitution $u = \sec x$.

EXAMPLE 3.6 An Integrand with an Odd Power of Tangent

Evaluate $\int \tan^3 x \sec^3 x\, dx$.

Solution Looking for terms that are derivatives of other terms, we rewrite the integral as

$$\int \tan^3 x \sec^3 x\, dx = \int \tan^2 x \sec^2 x (\sec x \tan x)\, dx$$
$$= \int (\sec^2 x - 1) \sec^2 x (\sec x \tan x)\, dx,$$

where we have used the Pythagorean identity

$$\tan^2 x = \sec^2 x - 1.$$

You should see the substitution now. We let $u = \sec x$, so that $du = \sec x \tan x\, dx$ and hence,

$$\int \tan^3 x \sec^3 x\, dx = \int \underbrace{(\sec^2 x - 1) \sec^2 x}_{(u^2-1)u^2} \underbrace{(\sec x \tan x)\, dx}_{du}$$
$$= \int (u^2 - 1)u^2 du = \int (u^4 - u^2)\, du$$
$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + c = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + c. \quad \text{Since } u = \sec x.$$

Case 2: *n* is an even positive integer

First, isolate one factor of $\sec^2 x$. (You'll need this for *du*.) Then, replace any remaining factors of $\sec^2 x$ with $1 + \tan^2 x$ and make the Substitution $u = \tan x$.

EXAMPLE 3.7 An Integrand with an Even Power of Secant

Evaluate $\int \tan^2 x \sec^4 x \, dx$.

Case 3: *m* is an even positive integer and *n* is an odd positive integer

Replace any factors of $\tan^2 x$ with $\sec^2 x - 1$ and then use a special reduction formula to evaluate integrals of the form $\int \sec^n x \, dx$.

EXAMPLE 3.8 An Unusual Integral

Evaluate the integral $\int \sec x \, dx$.

✓ Trigonometric Substitution

If an integral contains a term of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$ for some a>0, you can often evaluate the integral by making a substitution involving a trigonometric function.

NOTE

Terms of the form $\sqrt{a^2 - x^2}$ can also be simplified using the substitution $x = a\cos\theta$, using a different restriction for θ.

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\sin\theta)^2} = \sqrt{a^2 - a^2\sin^2\theta}$$

$$= a\sqrt{1 - \sin^2\theta} = a\sqrt{\cos^2\theta} = a\cos\theta,$$

EXAMPLE 3.9 An Integral Involving $\sqrt{a^2 - x^2}$

Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{4 - x^2}}\,dx.$

EXAMPLE 3.10 An Integral Involving $\sqrt{a^2 + x^2}$

Evaluate the integral $\displaystyle\int \frac{1}{\sqrt{9 + x^2}}\,dx.$

EXAMPLE 3.11 An Integral Involving $\sqrt{x^2 - a^2}$

Evaluate the integral $\displaystyle\int \frac{\sqrt{x^2 - 25}}{x}\,dx$, for $x > 5$.

Section 6.4 Integration of Rational Functions Using Partial Fractions(部分分式)

Partial fractions decomposition (部分分式分解)

The three factors $a_1x + b_1$, $a_2x + b_2$, and $a_3x + b_3$ are all distinct, then we can write

$$\boxed{\frac{a_1x + b_1}{(a_2x + b_2)(a_3x + b_3)} = \frac{A}{a_2x + b_2} + \frac{B}{a_3x + b_3},}$$

for some choice of constants A and B to be determined. Notice that the partial fractions on the right-hand side are very easy to integrate.

EXAMPLE 4.1 Partial Fractions: Distinct Linear Factors

Evaluate $\int \frac{1}{x^2 + x - 2}\, dx.$

If the degree of *P(x)*<*n* and the factors $(a_i x + b_i)$, for $i = 1, 2, \ldots, n$ are all distinct, then we can write

$$\boxed{\frac{P(x)}{(a_1 x + b_1)(a_2 x + b_2)\cdots(a_n x + b_n)} = \frac{c_1}{a_1 x + b_1} + \frac{c_2}{a_2 x + b_2} + \cdots + \frac{c_n}{a_n x + b_n},}$$

for some constants $c_1, c_2, \ldots, c_n$.

EXAMPLE 4.2 Partial Fractions: Three Distinct Linear Factors

Evaluate $\int \frac{3x^2 - 7x - 2}{x^3 - x}\, dx.$

When the denominator of a rational expression contains repeated linear factors, and if *P(x)* is less than *n*, then we have

$$\frac{P(x)}{(ax+b)^n} = \frac{c_1}{ax+b} + \frac{c_2}{(ax+b)^2} + \cdots + \frac{c_n}{(ax+b)^n},$$

for constants $c_1, c_2, \ldots, c_n$ to be determined.

EXAMPLE 4.4 Partial Fractions with a Repeated Linear Factor

Use a partial fractions decomposition to find an antiderivative of

$$f(x) = \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}.$$

Brief Summary of Integration Techniques

Integration by Substitution: $\displaystyle\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$

Integration by Parts: $\displaystyle\int u\,dv = uv - \int v\,du$

Trigonometric Substitution:

What to look for:

1. Terms like $\sqrt{a^2 - x^2}$: Let $x = a\sin\theta\ (-\pi/2 \le \theta \le \pi/2)$, so that $dx = a\cos\theta\,d\theta$ and $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta$; for example,

$$\int \frac{\overbrace{x^2}^{\sin^2\theta}}{\underbrace{\sqrt{1-x^2}}_{\cos\theta}}\underbrace{dx}_{\cos\theta\,d\theta} = \int \sin^2\theta\,d\theta.$$

2. Terms like $\sqrt{x^2+a^2}$: Let $x = a\tan\theta$ ($-\pi/2 < \theta < \pi/2$), so that $dx = a\sec^2\theta\, d\theta$ and $\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\theta + a^2} = a\sec\theta$; for example,

$$\int \frac{\overbrace{x^3}^{27\tan^3\theta}}{\underbrace{\sqrt{x^2+9}}_{3\sec\theta}} \underbrace{dx}_{3\sec^2\theta\, d\theta} = 27\int \tan^3\theta \sec\theta\, d\theta.$$

3. Terms like $\sqrt{x^2-a^2}$: Let $x = a\sec\theta$, for $\theta \in [0, \pi/2) \cup (\pi/2, \pi]$, so that $dx = a\sec\theta\tan\theta\, d\theta$ and $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\tan\theta$; for example,

$$\int \underbrace{x^3}_{8\sec^3\theta} \underbrace{\sqrt{x^2-4}}_{2\tan\theta} \underbrace{dx}_{2\sec\theta\tan\theta\, d\theta} = 32\int \sec^4\theta \tan^2\theta\, d\theta.$$

Partial Fractions:

What to look for: rational functions; for example,

$$\int \frac{x+2}{x^2-4x+3}\,dx = \int \frac{x+2}{(x-1)(x-3)}\,dx = \int \left(\frac{A}{x-1} + \frac{B}{x-3}\right) dx.$$

Section 6.6 Improper Integrals (广义积分)

✓ Improper Integrals with a Discontinuous Integrand

Recall that in Chapter 4, we defined the definite integral by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x,$$

where c_i was taken to be any point in the subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$. If $f(x) \to \infty$ or $f(x) \to -\infty$ at some point in *[a,b]*, then the limit defining $\int_a^b f(x)\,dx$ is meaningless. In this case, we call this integral an **improper integral**.

DEFINITION 6.1

If f is continuous on the interval $[a, b)$ and $|f(x)| \to \infty$ as $x \to b^-$, we define the improper integral of f on $[a, b]$ by

$$\int_a^b f(x)\,dx = \lim_{R \to b^-} \int_a^R f(x)\,dx.$$

Similarly, if f is continuous on $(a, b]$ and $|f(x)| \to \infty$ as $x \to a^+$, we define the improper integral

$$\int_a^b f(x)\,dx = \lim_{R \to a^+} \int_R^b f(x)\,dx.$$

In either case, if the limit exists (and equals some value L), we say that the improper integral **converges** (to L). If the limit does not exist, we say that the improper integral **diverges.**

EXAMPLE 6.1 An Integrand That Blows Up at the Right Endpoint

Determine whether $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$ converges or diverges.

Solution Based on the work we just completed,

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \lim_{R\to 1^-} \int_0^R \frac{1}{\sqrt{1-x}}\,dx = 2$$

and so, the improper integral converges to 2. ■

EXAMPLE 6.2 A Divergent Improper Integral

Determine whether the improper integral $\int_{-1}^0 \frac{1}{x^2}\,dx$ converges or diverges.

EXAMPLE 6.3 A Convergent Improper Integral

Determine whether the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges or diverges.

EXAMPLE 6.4 A Divergent Improper Integral

Determine whether the improper integral $\int_1^2 \frac{1}{x-1}\,dx$ converges or diverges.

DEFINITION 6.2

Suppose that f is continuous on the interval $[a, b]$, except at some $c \in (a, b)$, and $|f(x)| \to \infty$ as $x \to c$. Again, the integral is improper and we write

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

If *both* $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ converge (to L_1 and L_2, respectively), we say that the improper integral $\int_a^b f(x)\,dx$ **converges,** also, (to $L_1 + L_2$). If *either* of the improper integrals $\int_a^c f(x)\,dx$ or $\int_c^b f(x)\,dx$ diverges, then we say that the improper integral $\int_a^b f(x)\,dx$ **diverges,** also.

EXAMPLE 6.5 An Integrand That Blows Up in the Middle of an Interval

Determine whether the improper integral $\int_{-1}^{2} \frac{1}{x^2}\,dx$ converges or diverges.

Solution From Definition 6.2, we have

$$\int_{-1}^{2} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{2} \frac{1}{x^2}\,dx.$$

✓ Improper Integrals with an Infinite Limit of Integration

DEFINITION 6.3

If f is continuous on the interval $[a, \infty)$, we define the **improper integral** $\int_a^\infty f(x)\,dx$ to be

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx.$$

Similarly, if f is continuous on $(-\infty, a]$, we define

$$\int_{-\infty}^a f(x)\,dx = \lim_{R\to-\infty} \int_R^a f(x)\,dx.$$

In either case, if the limit exists (and equals some value L), we say that the improper integral **converges** (to L). If the limit does not exist, we say that the improper integral **diverges.**

EXAMPLE 6.6 An Integral with an Infinite Limit of Integration

Determine whether the improper integral $\int_1^\infty \frac{1}{x^2}\,dx$ converges or diverges.

EXAMPLE 6.7 A Divergent Improper Integral

Determine whether $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ converges or diverges.

EXAMPLE 6.8 A Convergent Improper Integral

Determine whether $\int_0^\infty xe^{-x}\,dx$ converges or diverges.

EXAMPLE 6.9 An Integral with an Infinite Lower Limit of Integration

Determine whether $\int_{-\infty}^{-1} \frac{1}{x}\,dx$ converges or diverges.

A final type of improper integral is $\int_{-\infty}^{\infty} f(x)\,dx$, defined as follows.

DEFINITION 6.4

If f is continuous on $(-\infty, \infty)$, we write

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx, \quad \text{for any constant } a,$$

where $\int_{-\infty}^{\infty} f(x)\,dx$ converges if and only if *both* $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$ converge. If either one diverges, the original improper integral also diverges.

EXAMPLE 6.11 An Integral with Two Infinite Limits of Integration

Determine whether $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$ converges or diverges.

EXAMPLE 6.12 An Integral with Two Infinite Limits of Integration

Determine whether $\int_{-\infty}^{\infty} e^{-x}\,dx$ converges or diverges.

EXAMPLE 6.13 An Integral That Is Improper for Two Reasons

Determine the convergence or divergence of the improper integral $\displaystyle\int_0^{\infty} \frac{1}{(x-1)^2}\,dx$.

✓ A Comparison Test

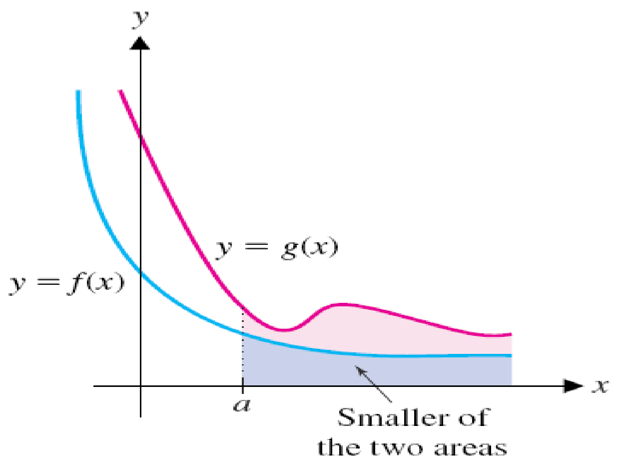


FIGURE 6.12
The Comparison Test

THEOREM 6.1 (Comparison Test)

Suppose that f and g are continuous on $[a, \infty)$ and $0 \leq f(x) \leq g(x)$, for all $x \in [a, \infty)$.

(i) If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges, also.

(ii) If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges, also.

EXAMPLE 6.14 Using the Comparison Test for an Improper Integral

Determine the convergence or divergence of $\int_0^\infty \frac{1}{x+e^x}\,dx$.

EXAMPLE 6.15 Using the Comparison Test for an Improper Integral

Determine the convergence or divergence of $\int_0^\infty e^{-x^2}\,dx$.

EXAMPLE 6.16 Using the Comparison Test: A Divergent Integral

Determine the convergence or divergence of $\int_1^\infty \frac{2+\sin x}{\sqrt{x}}\,dx$.