FINM3133 Time Series for Finance and Macroeconomics

Solution to Test

1. (a) $E(Y_t) = 0$ and $Var(Y_t) = 2\sigma_e^2$. $Cov(Y_t, Y_{t-k}) = Cov(e_t - e_{t-12}, e_{t-k} - e_{t-12-k}) = -Cov(e_{t-12}, e_{t-k}) = -\sigma_e^2$ when $k = 12$. It is nonzero only for $k = 12$ since, otherwise, all of the error terms involved are uncorrelated. So $\{Y_t\}$ is stationary.

   (b) The autocorrelation function is $\rho_k = \begin{cases} -0.5 & k = 12 \\ 0 & k > 0 \text{ but } k \neq 12 \end{cases}$

2. The autocovariance at lag $k = 1$ is

$$\begin{aligned} \gamma_1 &= Cov(X_t, X_{t-1}) = Cov(0.3X_{t-1} - 0.6X_{t-2} + e_t, X_{t-1}) \\ &= 0.3Cov(X_{t-1}, X_{t-1}) - 0.6Cov(X_{t-2}, X_{t-1}) + Cov(e_t, X_{t-1}) \\ &= 0.3\gamma_0 - 0.6\gamma_1 \end{aligned}$$

since $Cov(e_t, X_{t-1}) = 0$ due to the independence of X_{t-1} and e_t.

Thus, dividing by γ_0 and using the fact that $\rho_0 = 1$, we get

$$\rho_1 = 0.3 - 0.6\rho_1$$

which gives us that

$$\rho_1 = \frac{3}{16}.$$

Similarly, note that the autocovariance at lag $k = 2$ is

$$\begin{aligned} \gamma_2 &= Cov(X_t, X_{t-2}) = Cov(0.3X_{t-1} - 0.6X_{t-2} + e_t, X_{t-2}) \\ &= 0.3Cov(X_{t-1}, X_{t-2}) - 0.6Cov(X_{t-2}, X_{t-2}) + Cov(e_t, X_{t-2}) \\ &= 0.3\gamma_1 - 0.6\gamma_0 \end{aligned}$$

since $Cov(e_t, X_{t-2}) = 0$ due to the independence of X_{t-2} and e_t.

Thus, dividing by γ_0 and using the fact that $\rho_0 = 1$, we obtain

$$\rho_2 = 0.3\rho_1 - 0.6 = 0.3 \times (\frac{3}{16}) - 0.6 = -\frac{87}{160}$$

as we have already derived that $\rho_1 = -\frac{3}{16}$ above. [Note that the value of γ_0 is not required here.]

3. (a) This is a stationary time series process because both the mean and variance are constant. Note that an MA(q) process is always stationary and this model is an MA(1) process.

(b) Yes. The MA(1) model is invertible if and only if $|\theta| < 1$, which is true in this problem.

(c) The mean of this time series is 20.

4. (a) $E(Y_0) = cE(e_0) = 0$ and $E(Y_1) = c_2E(Y_1) + E(e_0) = 0$. Now proceed by induction. Suppose $E(Y_t) = E(Y_{t-1}) = 0$. Then $E(Y_{t+1}) = \phi_1 E(Y_t) + \phi_2 E(Y_{t-1}) + E(e_{t+1}) = 0$ and the result is established by induction on t.

(b) $Var(Y_0) = c_1^2 = Var(Y_1) = c_2^2c_1^2 + 1$ or $c_1 = 1/\sqrt{1-c_2^2}$. Next $Cov(Y_0, Y_1) = Cov(c_1e_0, c_2c_1e_0 + e_1) = Cov(c_1e_0, c_2c_1e_0) = c_2c_1^2$. So $\rho_1 = (c_2c_1^2)/(c_1^2) = c_2$. Finally, from $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$ it can be obtained that $(1-\phi_2)\gamma_1 = \phi_1\gamma_0$, i.e., $\rho_1 = \gamma_1/\gamma_0 = \phi_1/(1-\phi_2)$. Hence, $c_2 = \phi_1/(1-\phi_2)$ and $c_1 = 1/\sqrt{1-c_2^2}$.

5. (a) Using the difference operator ∇, we have

$$
\begin{aligned}
& 2X_t - 7X_{t-1} + 9X_{t-2} - 5X_{t-3} + X_{t-4} \\
= \; & 2X_t - 2X_{t-1} - 5X_{t-1} + 5X_{t-2} + 4X_{t-2} - 4X_{t-3} - X_{t-3} + X_{t-4} \\
= \; & 2(X_t - X_{t-1}) - 5(X_{t-1} - X_{t-2}) + 4(X_{t-2} - X_{t-3}) - (X_{t-3} - X_{t-4}) \\
= \; & 2\nabla X_t - 5\nabla X_{t-1} + 4\nabla X_{t-2} - \nabla X_{t-3} \\
= \; & 2(\nabla X_t - \nabla X_{t-1}) - 3(\nabla X_{t-1} - \nabla X_{t-2}) + (\nabla X_{t-2} - \nabla X_{t-3}) \\
= \; & 2\nabla^2 X_t - 3\nabla^2 X_{t-1} + \nabla^2 X_{t-2} \\
= \; & 2(\nabla^2 X_t - \nabla^2 X_{t-1}) - (\nabla^2 X_{t-1} - \nabla^2 X_{t-1}) \\
= \; & 2\nabla^3 X_t - \nabla^3 X_{t-1}.
\end{aligned}
$$

(b) i. From

$$Y_t = 3.2Y_{t-1} - 3.6Y_{t-2} + 1.6Y_{t-3} - 0.2Y_{t-4} + e_t + 0.3e_{t-1}$$

we have

$$Y_t - Y_{t-1} = 2.2Y_{t-1} - 2.2Y_{t-2} - 1.4Y_{t-2} + 1.4Y_{t-3} + 0.2Y_{t-3} - 0.2Y_{t-4} + e_t + 0.3e_{t-1}$$

$$\nabla Y_t = 2.2\nabla Y_{t-1} - 1.4\nabla Y_{t-2} + 0.2\nabla Y_{t-3} + e_t + 0.3e_{t-1}.$$

Furthermore,

$$\nabla Y_t - \nabla Y_{t-1} = 1.2\nabla Y_{t-1} - 1.2\nabla Y_{t-2} - 0.2\nabla Y_{t-2} + 0.2\nabla Y_{t-3} + e_t + 0.3e_{t-1}$$

$$\nabla^2 Y_t = 1.2\nabla^2 Y_{t-1} - 0.2\nabla^2 Y_{t-2} + e_t + 0.3e_{t-1}.$$

Finally, we have

$$\nabla^2 Y_t - \nabla^2 Y_{t-1} = 0.2\nabla^2 Y_{t-1} - 0.2\nabla^2 Y_{t-2} + e_t + 0.3e_{t-1}.$$

$$\nabla^3 Y_t = 0.2\nabla^3 Y_{t-1} + e_t - (-0.3)e_{t-1}.$$

Thus, we have $d = 3, p = 1, q = 1$, that is, the time series is an ARIMA(1,3,1) with $\phi = 0.2$ and $\theta = -0.3$.

ii. From

$$Y_t = 2.9Y_{t-1} - 2.8Y_{t-2} + 0.9Y_{t-3} + e_t - e_{t-2}$$

we have

$$Y_t - Y_{t-1} = 1.9Y_{t-1} - 1.9Y_{t-2} - 0.9Y_{t-2} + 0.9Y_{t-3} + e_t - e_{t-2}$$

$$\nabla Y_t = 1.9\nabla Y_{t-1} - 0.9\nabla Y_{t-2} + e_t - e_{t-2}.$$

Finally, we get

$$\nabla Y_t - \nabla Y_{t-1} = 0.9\nabla Y_{t-1} - 0.9\nabla Y_{t-2} + e_t - e_{t-1}$$

$$\nabla^2 Y_t = 0.9\nabla^2 Y_{t-1} + e_t - e_{t-2}.$$

Thus, we have $d = 2, p = 1, q = 2$, that is, the time series is an ARIMA(1,2,2) with $\phi = 0.9$ and $\theta_1 = 0, \theta_2 = 1$.

6. Use $2/\sqrt{n} = 0.258$ as a guide, the ACF dies off and the PACF cuts off at lag 2. So an AR(2) model can be identified.