Applied Stochastic Process

Midterm Test

Date: 20 November 2024

Time allowed: 90 minutes **Full mark: 100**

Name: ______________ **Id:** ___________________ **Score:** __________

1. (12 points)
   (a) (4 points) Recall the definition of a σ-field.
   (b) (4 points) Give two examples of σ-fields.
   (c) (4 points) Give one example of a set system that is *not* a σ-field.

2. (10 points) Consider a gambler's ruin chain with $N = 4$. That is, if $1 \le i \le 3$, $P_{i,i+1} = 0.4$, and $P_{i,i-1} = 0.6$, but the endpoints are absorbing states: $P_{0,0} = 1$ and $P_{4,4} = 1$. Compute $P^3_{1,4}$ and $P^3_{1,0}$.

3. (16 points) Draw the picture of the following two Markov chains, respectively. Specify the classes of each of the Markov chains, and determine whether they are transient or recurrent. (No need to give any steps.)
   (a) (8 points)

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

   (b) (8 points)

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{5} & \frac{2}{5} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4. (16 points) Consider a Markov chain $\{S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$ with following transition matrix

$$\boldsymbol{P} = \begin{bmatrix} 0 & \frac{3}{7} & \frac{4}{7} & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{9} & 0 & \frac{4}{9} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

(a) (4 points) Draw the transition graph of this Markov chain;

(b) (12 points) Find the equivalence classes for this Markov chain. Which class is recurrent? Which class is transient? Calculate the period of each class.

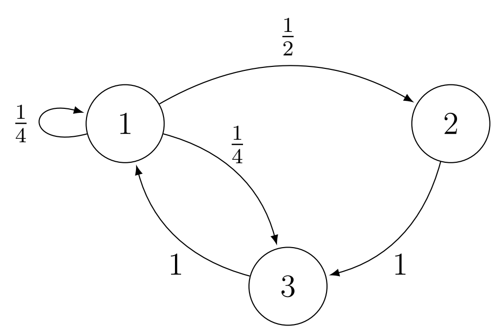
5. (18 points)

(a) (6 points) Consider a Markov chain with the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Describe the family of all invariant distributions for the Markov chain.

(b) (12 points) Consider a Markov chain with the following transition diagram



i. (6 points) Does the Markov chain have a unique invariant distribution, if so find it?

ii. (6 points) Find the expected time to reach state 2 (for the first time) if the chain started at state 1. (Hint: Consider expected times needed to reach state 2 starting from different states:

$$t_i = \mathbb{E}(\exists n \geq 0 : X_n = 2 \mid X_0 = i), \quad i = 1, 2, 3.$$

Then

$$t_i = 1 + \sum_{k=1}^{3} t_k p_{i,k}, \qquad i = 1, 3.)$$

6. (15 points) Suppose $N(t)$ is a Poisson process with rate 3. Let S_n denote the time of the nth arrival. Find

   (a) $P(N(2) = 5)$.

   (b) $P(N(1) = 1 \mid N(3) = 4)$.

   (c) $E(S_{12})$.

   (d) $E(S_{12} \mid N(2) = 5)$.

   (e) $E(N(5) \mid N(2) = 5)$.

7. (13 points) People arrive at the Durham Farmer's market at rate 15 per hour. 4/5 's are vegetarians, and 1/5 are meat eaters. Vegetarians spend an average of \$7 with a standard deviation of 3. Meat eaters spend an average of \$15 with a standard deviation of 8.

   (a) (5 points) Compute the probability that in the first 20 minutes exactly three vegetarians and two meat eaters arrive. You do not have to simplify your answer.

   (b) (8 points) Find the mean and standard deviation of the amount of money spent during the four hours the market is open.