AFM Brief solution to Assignment 5

1. Given standard Brownian motion W_t, show whether the following expressions are martingale or not?(λ is a constant)

   (1) $e^{\lambda W_t}$

   (2) $e^{W_t - \frac{1}{2}\lambda^2}$

   (3) $(t + W_t^2)^2$

   (4) $(W_t^2 - t)^2 - 2t^2$

   Solution:

   (1)

$$\begin{aligned} E[e^{\lambda W_t}|\mathcal{F}_s] &= E[e^{\lambda(W_t - W_s + W_s)}|\mathcal{F}_s] \\ &= e^{\lambda W_s} E[e^{\lambda(W_t - W_s)}|\mathcal{F}_s] \\ &= e^{\lambda W_s} e^{\frac{\lambda^2}{2}(t-s)} \end{aligned}$$

   hence $e^{\lambda W_s}$ is not a martingale.

   (2)

$$\begin{aligned} E[e^{W_t - \frac{\lambda^2}{2}}|\mathcal{F}_s] &= e^{-\frac{\lambda^2}{2}} E[e^{W_s + (W_t - W_s)}|\mathcal{F}_s] \\ &= e^{W_s - \frac{\lambda^2}{2}} e^{\frac{(t-s)}{2}} \end{aligned}$$

   hence $e^{W_t - \frac{\lambda^2}{2}}$ is not a martingale except $t = s$.

   (3)

$$\begin{aligned} E[(t + W_t^2)^2 \mid \mathcal{F}_s] &= E[t^2 + 2tW_t^2 + W_t^4 \mid \mathcal{F}_s] \\ &= t^2 + 2tE[W_t^2 \mid \mathcal{F}_s] + E[W_t^4 \mid \mathcal{F}_s] \\ &= t^2 + 2tE[(W_t - W_s + W_s)^2 \mid \mathcal{F}_s] \\ &\quad + E[(W_t - W_s + W_s)^4 \mid \mathcal{F}_s] \\ &= t^2 + 2t\left((t-s) + W_s^2\right) \\ &\quad + E[W_s^4 + 4W_s^3(W_t - W_s) + 6W_s^2(W_t - W_s)^2 \\ &\quad + 4W_s(W_t - W_s)^3 + (W_t - W_s)^4 \mid \mathcal{F}_s] \\ &= t^2 + 2t(t-s) + 2tW_s^2 + W_s^4 \\ &\quad + 6W_s^2(t-s) + 3(t-s)^2 \end{aligned}$$

   Hence, $(t + W_t^2)^2$ is not a martingale.

(4)

$$
\begin{aligned}
E[(W_t^2 - t)^2 - 2t^2 \mid \mathcal{F}_s] &= E[W_t^4 - 2tW_t^2 - t^2 \mid \mathcal{F}_s] \\
&= -t^2 - 2tE[W_t^2 \mid \mathcal{F}_s] + E[W_t^4 \mid \mathcal{F}_s] \\
&= -t^2 - 2tE[(W_t - W_s + W_s)^2 \mid \mathcal{F}_s] \\
&\quad + E[(W_t - W_s + W_s)^4 \mid \mathcal{F}_s] \\
&= -t^2 - 2t\left((t-s) + W_s^2\right) \\
&\quad + E[W_s^4 + 4W_s^3(W_t - W_s) + 6W_s^2(W_t - W_s)^2 \\
&\quad + 4W_s(W_t - W_s)^3 + (W_t - W_s)^4 \mid \mathcal{F}_s] \\
&= -t^2 - 2t(t-s) - 2tW_s^2 + W_s^4 \\
&\quad + 6W_s^2(t-s) + 3(t-s)^2 \\
&= (W_s^2 - s)^2 - 2s^2 + 4s(s - t + W_s^2)
\end{aligned}
$$

Hence, $(W_t^2 - t)^2 - 2t^2$ is not a martingale .

2. Given standard Brownian motion W_t, based on Itô isometry,

$$
E\left[\left(\int_0^T f(t, W_t)dW_t\right)^2\right] \quad = \quad E\int_0^T |f(t, W_t)|^2 dt,
$$

evaluate

(1) $E\left[\left(\int_0^T (t + 2W_t)dW_t\right)^2\right]$

(2) $E\left[\left(\int_0^T e^{W_t^2 - t} dW_t\right)^2\right]$

(3) $E\left[\left(\int_0^t W_s^2 dW_s\right)^2\right]$

(4) $E\left[\int_0^t W_s^2 ds\right]$

Solution:

(1)

$$
\begin{aligned}
E\left[\left(\int_0^T (t + 2W_t)dW_t\right)^2\right] &= E\left[\int_0^T (t + 2W_t)^2 dt\right] \\
&= E\left[\int_0^T (t^2 + 4tW_t + 4W_t^2)dt\right] \\
&= \int_0^T (t^2 + 4t)dt = \frac{1}{3}T^3 + 2T^2.
\end{aligned}
$$

(2)

$$
\begin{aligned}
E\left[\left(\int_0^T e^{-W_t^2} dW_t\right)^2\right] &= \int_0^T E\left[e^{-2W_t^2}\right] dt \\
&= \int_0^T \frac{1}{\sqrt{1+4t}} dt \\
&= \frac{1}{2}(\sqrt{1+4T}-1)
\end{aligned}
$$

(3)

$$
\begin{aligned}
E\left[\left(\int_0^t W_s^2 dW_s\right)^2\right] &= E\left[\int_0^t W_s^4 ds\right] \\
&= \int_0^t E(W_s^4) ds \\
&= \int_0^t (3s^2) ds = t^3
\end{aligned}
$$

(4)

$$
\begin{aligned}
E\left[\int_0^t W_s^2 ds\right] &= \int_0^t E(W_s^2) ds \\
&= \int_0^t s ds \\
&= \frac{1}{2}t^2
\end{aligned}
$$