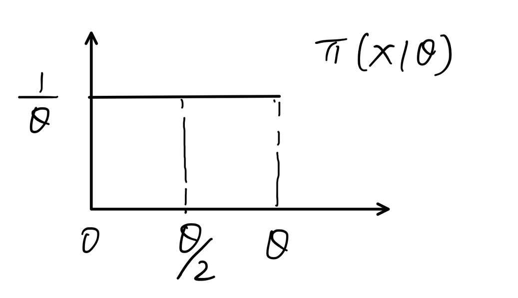
$X_i \sim U(0,\theta)$

↳ MOM to estimate θ

Uniform on $[0,\theta]$



$$\pi(x|\theta) = \begin{cases} \frac{1}{\theta}, & x \in [0,\theta] \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_1 = E(X) = \int_0^\theta x \frac{1}{\theta}\, dx = \frac{1}{\theta}\cdot \frac{1}{2}x^2 \Big|_0^\theta = \frac{\theta}{2}$$

$$\rightarrow \quad \theta = 2\mu_1$$

$$\hat{\mu}_1 = \frac{1}{n}\sum X_i = \bar{X} \quad \rightarrow \quad \hat{\theta}_{MOM} = 2\hat{\mu}_1 = 2\bar{X}$$

↳ MLE of θ :

△ likelihood function :

$$L(\theta) = \prod_{i=1}^n \pi(x_i|\theta) = \left(\frac{1}{\theta}\right)^n$$

monotonic function

△ log-likelihood ,

$$\ell(\theta) = \ln L(\theta) = -n\ln\theta$$

△ Maximize $\ell(\theta)$

$$\frac{d\,\ell(\theta)}{d\theta} = \frac{-n}{\theta} = 0 \quad \rightarrow$$

there is no stationary point !
because $L(\theta)$ or $\ell(\theta)$ is monotonic on $(0,\infty)$

instead , consider a set of new random variables
$X_{(i)}$: the i^{th} smallest value in the sample

i.e. $0 \le X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)} \le \theta$ order statistics

Minimize the value of θ , so that $L(\theta)$ to be as large as possible.

The minimum value of θ it can attain based on a given sample is

$$\hat{\theta}_{MLE} = X_{(n)}$$

△ Fisher information

$$\boxed{I(\theta) \overset{(\text{def. as})}{:=} E\left[\left(\frac{\partial}{\partial\theta}\log\pi(x|\theta)\right)^2\right] \overset{(\text{can be computed as})}{=} E\left[\frac{\partial^2}{\partial\theta^2}\log\pi(x|\theta)\right]}$$

Consider the density function $\pi(x|\theta)$

$$1 = \int_{-\infty}^{\infty} \pi(x|\theta)\, dx$$

take derivative

$$0 = \frac{\partial}{\partial\theta} 1 = \frac{\partial}{\partial\theta}\left[\int_{-\infty}^{\infty}\pi(x|\theta)\,dx\right]$$

$$0 = \int_{-\infty}^{\infty} \frac{\partial}{\partial\theta}\pi(x|\theta)\,dx$$

$$0 = \int_{-\infty}^{\infty} \boxed{\frac{\frac{\partial}{\partial\theta}\pi(x|\theta)}{\pi(x|\theta)}}\, \pi(x|\theta)\,dx$$

$$0 = \int_{-\infty}^{\infty} \boxed{\frac{\partial}{\partial\theta}\log\pi(x|\theta)}\, \pi(x|\theta)\,dx = E\left[\frac{\partial}{\partial\theta}\log\pi(X|\theta)\right] \quad ①$$

take another derivative

$$\frac{\partial}{\partial\theta} 0 = \frac{\partial}{\partial\theta}\int_{-\infty}^{\infty} \underbrace{\frac{\partial}{\partial\theta}\log\pi(x|\theta)}_{u}\ \underbrace{\pi(x|\theta)}_{v}\,dx$$

$$0 = \int_{\mathbb{R}} \underbrace{\frac{\partial^2}{\partial\theta^2}\log\pi(x|\theta)}_{\frac{\partial}{\partial\theta}u} \cdot \underbrace{\pi(x|\theta)}_{v}\,dx + \int_{\mathbb{R}} \underbrace{\frac{\partial}{\partial\theta}\log\pi(x|\theta)}_{u}\cdot \underbrace{\frac{\partial}{\partial\theta}\pi(x|\theta)}_{\frac{\partial v}{\partial\theta}}\,dx$$

$$= E\left[\frac{\partial^2}{\partial\theta^2}\log\pi(X|\theta)\right] + \int_{\mathbb{R}} \frac{\partial}{\partial\theta}\log\pi(x|\theta)\boxed{\frac{\frac{\partial}{\partial\theta}\pi(x|\theta)}{\pi(x|\theta)}}\cdot\pi(x|\theta)\,dx$$

$$= E\left[\frac{\partial^2}{\partial\theta^2}\log\pi(X|\theta)\right] + \int_{\mathbb{R}} \left(\frac{\partial}{\partial\theta}\log\pi(x|\theta)\right)^{\boxed{2}}\cdot\pi(x|\theta)\,dx$$

$$0 = E\left[\frac{\partial^2}{\partial\theta^2}\log\pi(X|\theta)\right] + \underbrace{E\left[\left(\frac{\partial}{\partial\theta}\log\pi(X|\theta)\right)^2\right]}_{\text{def. as } I(\theta)}$$

Thus,

$$I(\theta) := E\left[\left(\frac{\partial}{\partial\theta}\log\pi(X|\theta)\right)^2\right] = -E\left[\frac{\partial^2}{\partial\theta^2}\log\pi(X|\theta)\right]$$

By ① , we can see

$$E\left(\frac{\partial}{\partial\theta}\log\pi(X|\theta)\right) = 0$$

$$Var\left(\frac{\partial}{\partial\theta}\log\pi(X|\theta)\right) = I(\theta) - 0^2 = I(\theta)$$

Here , the function

$$\frac{\partial}{\partial\theta}\log\pi(X|\theta)$$

is called the score function. Recall that it determines the estimating equations for the mle , that is , $\hat{\theta}_{MLE}$ solves

$$\sum_{i=1}^n \frac{\partial\log\pi(X_i|\theta)}{\partial\theta} = 0$$

E.g. (Information for a single Bernoulli R.V.) Let $X \sim$ Bernoulli (θ) , Thus

$$\log\pi(x|\theta) = x\ln\theta + (1-x)\ln(1-\theta)$$

$$\frac{\partial\log\pi(x|\theta)}{\partial\theta} = \frac{x}{\theta} - \frac{1-x}{1-\theta}$$

$$\frac{\partial^2}{\partial\theta^2}\log\pi(x|\theta) = -\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2}$$

$$↳ \quad I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\log\pi(x|\theta)\right] = E\left[\frac{X}{\theta^2} + \frac{1-X}{(1-\theta)^2}\right]$$

$$= \frac{1}{\theta^2}\theta + \frac{1}{(1-\theta)^2}(1-\theta) = \frac{1}{\theta(1-\theta)}$$

$I(\theta)$ gets larger as $\theta \to 1$ or $\theta \to 0$.