ASP Assignment 1

1. A box contains three marbles: one red, one green, and one blue. Consider an experiment that consists of taking one marble from the box then replacing it in the box and drawing a second marble from the box.
   (a) What is the sample space? If, at all times, each marble in the box is equally likely to be selected, what is the probability of each point in the sample space?
   (b) Repeat (a) when the second marble is drawn without replacing the first marble.

2. Let E, F, G be three events. Find expressions for the events that of E, F, G
   (a) only F occurs,
   (b) both E and F but not G occur,
   (c) at least one event occurs,
   (d) at least two events occur,
   (e) all three events occur,
   (f) none occurs,
   (g) at most one occurs,
   (h) at most two occur.

3. If $\mathbb{P}(E) = 0.9$ and $\mathbb{P}(F) = 0.8$, show that $\mathbb{P}(E \cap F) \geqslant 0.7$. In general, show that

$$\mathbb{P}(E \cap F) \geqslant \mathbb{P}(E) + \mathbb{P}(F) - 1$$

   This is known as Bonferroni's inequality.

4. Show that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) \leqslant \sum_{i=1}^{n} \mathbb{P}\left(E_i\right)$$

   This is known as Boole's inequality. (Hint: Either use mathematical induction, or else show that $\bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} F_i$, where $F_1 = E_1, F_i = E_i \bigcap_{j=1}^{i-1} E_j^c$, and use finite additivity of a probability.)

5. There is a prize-winning ticket in n lottery tickets. These n lottery tickets are supposed to be sold to n different persons randomly.
   (a) If the first $k-1$ customers do not get the prizewinning ticket, find the probability that the k-th customer gets the prize-winning ticket;
   (b) Find the probability that the k-th customer gets the prizewinning ticket.

6. Let X_1 and X_2 be independent Poisson random variables with means λ_1 and λ_2.

   (a) Find the distribution of $X_1 + X_2$.

   (b) Compute the conditional distribution of X_1 given that $X_1 + X_2 = n$.

7. Suppose that $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^\top \sim N(\boldsymbol{a}, \boldsymbol{B})$, where $\boldsymbol{a} = (a_1, a_2, a_3)^\top, \boldsymbol{B} = (b_{ij})_{3\times 3}$. Let

$$\begin{cases} \eta_1 = \frac{\xi_1}{2} - \xi_2 + \frac{\xi_3}{2} \\ \eta_2 = -\frac{\xi_1}{2} - \frac{\xi_3}{2} \end{cases}$$

   Find the distribution of $\boldsymbol{\eta} = (\eta_1, \eta_2)^\top$.

8. Let Z_1, Z_2, Z_3 be independent $N(0,1)$ random variables. Let

$$X_1 = Z_1 + Z_3, \quad X_2 = Z_2 + 4Z_3, \quad X_3 = 2Z_1 - 2Z_2 + rZ_3$$

   where r is a real number.

   (a) Explain why $\mathbf{X} = (X_1, X_2, X_3)$ has a joint normal distribution.

   (b) Find the covariance matrix for $\mathbf{X}$ (in terms of r).

   (c) For what values of r are X_1 and X_3 independent random variables?