2023-24 First Semester

MATH2043 Ordinary Differential Equations (1002)

Assignment 10

Due Date: **21/Dec/2023(Thursday), on or before 16:00, in tutorial class.**

- Write down your **CHN name** and **student ID**. Write neatly on **A4-sized** paper (*staple if necessary*) and **show your steps**.
- **Late submissions or answers without details will not be graded.**

In each of Problem 1 and 2,

(a) Find approximate values of the solution of the given initial value problem at $t = 0.1$ and 0.2 using the Euler's method with $h = 0.1$.

(b) Find approximate values of the solution of the given initial value problem at $t = 0.1$ and 0.2 using the backward Euler's method with $h = 0.1$.

1. $y' = 3y - 4t, \quad y(0) = 1.$
2. $y' = t^2 + y^2, \quad y(0) = 1.$
3. (*Optional!*) Use softwares to find approximations of the solution to problem 1 at $t = 2$ using the Euler's method with $h_1 = 0.1$, $h_2 = 0.01$ and $h_3 = 0.001$. Compare the approximations by computing the absolute errors $|\hat{y}_n - y(2)|$.
4. (**Convergence of Euler's Method.**) It can be shown that, under suitable conditions on $f(t,y)$, the numerical approximation generated by the Euler method for the initial value problem $y = f(t,y)$, $y(t_0) = y_0$ converges to the exact solution as the step size h decreases. This is illustrated by the following example. Consider the initial value problem

$$y' = 1 - t + y, \quad y(t_0) = y_0$$

   (a) Find the exact solution $y = \phi(t)$.

   (b) Using the Euler's formula, show that

$$\hat{y}_k = (1+h)\hat{y}_{k-1} + h - ht_{k-1}, \quad k = 1, 2, \cdots$$

   (c) Noting that $\hat{y}_1 = (1+h)(y_0 - t_0) + t_1$, show by induction that

$$\hat{y}_n = (1+h)^n (y_0 - t_0) + t_n \qquad (i)$$

   for each positive integer n.

   (d) Consider a fixed point $t > t_0$ and for a given n choose $h = (t - t_0)/n$. Then $t_n = t$ for every n. Note also that $h \to 0$ as $n \to \infty$. By substituting for h in Eq. (i) and letting $n \to \infty$, show that $y_n \to \phi(t)$ as $n \to \infty$.