ASP Assignment 6

1. A Markov chain on states $0, 1, 2$, has the transition matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{5}{6} & 0 & \frac{1}{6} \end{bmatrix}.$$

   (a) Determine the invariant distribution.

   (b) Determine $\lim_{n\to\infty} P^n_{1,0}$. Why does it exist?

2. Roll a fair die n times and let S_n be the sum of the numbers you roll. Determine, with proof, $\lim_{n\to\infty} \mathbb{P}\left(S_n \bmod 13 = 0\right)$. (Hint: Consider S_n mod 13. This is a Markov chain with states $0, 1 \ldots, 12$.)

3. Consider the Markov chain with states $0, 1, 2, 3, 4$, which transition from state $i > 0$ to one of the states $0, \ldots, i-1$ with equal probability, and transition from 0 to 4 with probability 1. Show that all P^n_{ij} converge as $n \to \infty$ and determine the limits.

4. Show by example that chains which are not irreducible may have many different stationary distributions.

5. Suppose $\{N(t),\, t \geq 0\}$ is a Poisson process with rate λ. Let S_n be the arrival time of the n-th event. Please calculate:

   (a) $E(S_4)$;

   (b) $E(N(4) - N(2) \,|\, N(1) = 3)$.

6. Suppose $\{N(t),\, t \geq 0\}$ is a Poisson process with rate $\lambda = 2$. Please calculate:

   (a) $P(N(1) \leq 2)$;

   (b) $P(N(1) = 1,\, N(2) = 3)$.

7. Suppose $\{N(t),\, t \geq 0\}$ is a Poisson process with rate $\lambda = 2$. Please calculate:

   (a) $P(N(3) = 6 \,|\, N(1) = 2)$;

   (b) $E(N(1)N(2))$.