PT Solution to Assignment 9

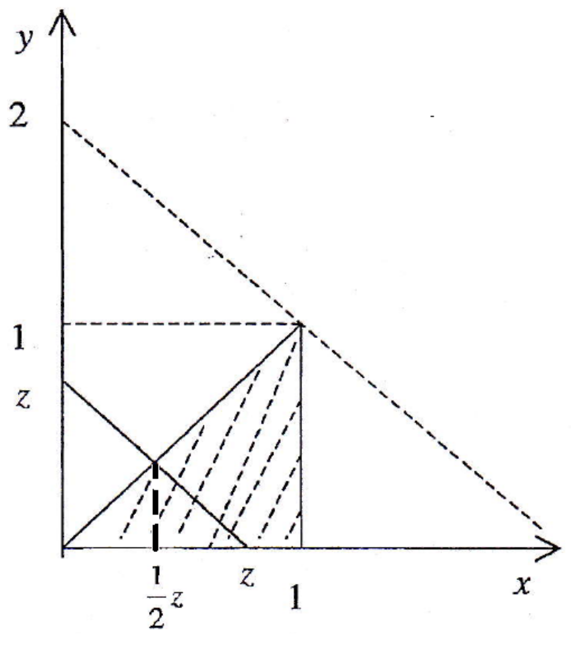
1. The future lifetimes T of a certain population is exponentially distributed with parameter λ where λ is uniformly distributed over (1, 11).

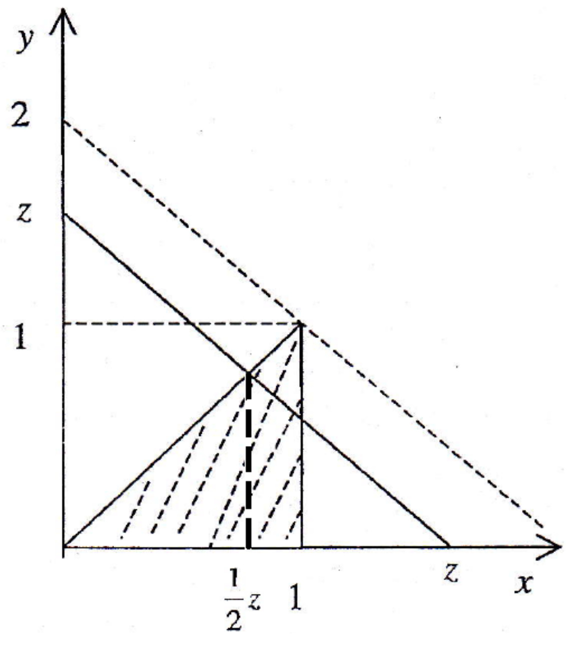
Calculate $P(T > 0.5)$.

Solution

$$\begin{aligned}
P(T > 0.5) &= \int_1^{11} P(T > 0.5 \mid \lambda) f(\lambda) d\lambda \\
&= 0.1\int_1^{11}\left(\int_{0.5}^{\infty} \lambda e^{-\lambda t} dt\right) d\lambda \\
&= 0.1\int_1^{11} e^{-0.5\lambda} d\lambda \\
&= 0.1\left(\frac{-e^{-5.5} + e^{-0.5}}{0.5}\right) \\
&\approx 0.1205
\end{aligned}$$

2. If the joint density function of X and Y is $f_{X,Y}(x,y) = \begin{cases} 8xy & 0 < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$ Find the probability density function of $Z = X + Y$.





Solution

Let F be the distribution function of $Z = X + Y$. Since $0 < X + Y < 2$, $F(z) = 0$ for $z \leqslant 0$ and $F(z) = 1$ for $z \geqslant 2$. For $0 < z \leqslant 1$, we have

$$
\begin{aligned}
F(z) = P(X + Y \leqslant z) &= \int_0^{\frac{1}{2}z} \left(\int_y^{z-y} f_{X,Y}(x, y) dx \right) dy \\
&= \int_0^{\frac{1}{2}z} \left(\int_y^{z-y} 8xy dx \right) dy \\
&= \int_0^{\frac{1}{2}z} y \cdot \left(4(z-y)^2 - 4y^2\right) dy \\
&= 4z \int_0^{\frac{1}{2}z} y \cdot (z - 2y) dy \\
&= 4z^2 \cdot \left(\frac{1}{2} \cdot \left(\frac{1}{2}z\right)^2 - 0 \right) - 8z \left(\frac{1}{3} \left(\frac{1}{2}z\right)^3 - 0 \right) \\
&= \frac{1}{2}z^4 - \frac{1}{3}z^4 = \frac{1}{6}z^4
\end{aligned}
$$

For $1 < z < 2$,

$$
\begin{aligned}
F(z) &= 1 - P(X + Y > z) \\
&= 1 - \int_{\frac{1}{2}z}^{1} dx \left(\int_{z \cdot x}^{x} f_{x,y}(x, y) dy \right) \\
&= 1 - \int_{\frac{1}{2}z}^{1} dx \left(\int_{z \cdot x}^{x} 8xy dy \right) \\
&= 1 - \int_{\frac{1}{2}z}^{1} dx \left(4x \cdot \left(x^2 - (z - x)^2\right)\right) \\
&= 1 - \int_{\frac{1}{2}z}^{1} 4x \cdot (2x - z) \cdot z dx \\
&= 1 - \frac{8}{3}zx^3 \Big|_{\frac{1}{2}z}^{1} + 2z^2x^2 \Big|_{\frac{1}{2}z}^{1} \\
&= 1 - \frac{8}{3}z + \frac{1}{3}z^4 + 2z^2 - \frac{1}{2}z^4 \\
&= 1 - \frac{8}{3}z + 2z^2 - \frac{1}{6}z^4.
\end{aligned}
$$

Thus the density of Z is given by

$$
f(z) = \begin{cases} \frac{2}{3}z^3, & 0 \leq z \leq 1, \\ 4z - \frac{8}{3} - \frac{2}{3}z^3, & 1 < z \leq 2, \\ 0, & \text{otherwise.} \end{cases}
$$

3. Let X and Y be independent random variables and $Z = X + Y$. Using $p_Z(n) = \sum_{k=0}^{n} P(X = k, Y = n - k)$ for nonnegative discrete random variable and

$f_Z(z)=\int_{-\infty}^{\infty} f_X(z-y)f_Y(y)dy$ for continuous random variable, find the probability mass function or probability density function of Z if

(a) X and Y are Gamma distributions with parameters (s,λ) and (t,λ) respectively

(b) X and Y are independent binomial random variables with parameters (n, p) and (m, p) respectively.

(c) X and Y are Normal distributions with parameters $N(\mu_1,\sigma_1^2)$ and $N(\mu_2,\sigma_2^2)$ respectively

(d) X and Y are independent random variables such that

$$p_X(i)=\binom{i+r-1}{r-1}p^r(1-p)^i \quad \text{and} \quad p_Y(j)=\binom{j+s-1}{s-1}p^s(1-p)^j$$

where $r,s\in \boldsymbol{N}$.

[Hint: $(1-x)^{-r}=\sum_{i=0}^{\infty}\binom{r+i-1}{i}x^i$]

What kind of distribution is Z in each case?

Solution

(a) X and Y are Gamma distribution with parameter (s,λ) and (t,λ) respectively

$$\begin{aligned}
f_{X+Y}(a) &= \frac{1}{\Gamma(s)\Gamma(t)}\int_0^a \lambda e^{-\lambda(a-y)}[\lambda(a-y)]^{s-1}\lambda e^{-\lambda y}(\lambda y)^{t-1}\,dy \\
&= Ke^{-\lambda a}\int_0^a (a-y)^{s-1}y^{t-1}dy \\
&= Ke^{-\lambda a}a^{s+t-1}\int_0^1 (1-x)^{s-1}x^{t-1}\,dx \quad \text{by letting } x=\frac{y}{a} \\
&= Ce^{-\lambda a}a^{s+t-1}
\end{aligned}$$

where C is a constant that does not depend on a. But, as the preceding is a density function and thus must integrate to 1, the value of C is determined, and we have

$$f_{X+Y}(a)=\frac{\lambda e^{-\lambda a}(\lambda a)^{s+t-1}}{\Gamma(s+t)}$$

Hence, the result is proved. □

Gamma distributions with parameters $(s+t, \lambda)$

(b)
$$P\{X + Y = k\} = \sum_{i=0}^{n} P\{X = i, Y = k - i\}$$
$$= \sum_{i=0}^{n} P\{X = i\}P\{Y = k - i\}$$
$$= \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} \binom{m}{k - i} p^{k-i} q^{m-k+i}$$

where $q = 1 - p$ and where $\binom{r}{j} = 0$ when $j < 0$. Thus,

$$P\{X + Y = k\} = p^k q^{n+m-k} \sum_{i=0}^{n} \binom{n}{i} \binom{m}{k - i}$$

and the conclusion follows upon application of the combinatorial identity

$$\binom{n + m}{k} = \sum_{i=0}^{n} \binom{n}{i} \binom{m}{k - i}$$

$P(X+Y=k) = \binom{n+m}{k} p^k q^{n+m-k}$.

Binomial random variables with parameters $(n + m, p)$

(c) To begin, let X and Y be independent normal random variables with X having mean 0 and variance σ^2 and Y having mean 0 and variance 1. We will determine the density function of $X + Y$ by utilizing Equation (3.2). Now, with

$$c = \frac{1}{2\sigma^2} + \frac{1}{2} = \frac{1 + \sigma^2}{2\sigma^2}$$

we have

$$\begin{aligned}
f_X(a - y)f_Y(y) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(a - y)^2}{2\sigma^2}\right\} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} \\
&= \frac{1}{2\pi\sigma} \exp\left\{-\frac{a^2}{2\sigma^2}\right\} \exp\left\{-c\left(y^2 - 2y\frac{a}{1 + \sigma^2}\right)\right\}
\end{aligned}$$

Hence, from Equation (3.2),

$$\begin{aligned}
f_{X+Y}(a) &= \frac{1}{2\pi\sigma} \exp\left\{-\frac{a^2}{2\sigma^2}\right\} \exp\left\{\frac{a^2}{2\sigma^2(1 + \sigma^2)}\right\} \\
&\quad \times \int_{-\infty}^{\infty} \exp\left\{-c\left(y - \frac{a}{1 + \sigma^2}\right)^2\right\} dy \\
&= \frac{1}{2\pi\sigma} \exp\left\{-\frac{a^2}{2(1 + \sigma^2)}\right\} \int_{-\infty}^{\infty} \exp\{-cx^2\}\, dx \\
&= C \exp\left\{-\frac{a^2}{2(1 + \sigma^2)}\right\}
\end{aligned}$$

where C does not depend on a. But this implies that $X + Y$ is normal with mean 0 and variance $1 + \sigma^2$.

Now, suppose that X_1 and X_2 are independent normal random variables with X_i having mean μ_i and variance $\sigma_i^2, i = 1, 2$. Then

$$X_1 + X_2 = \sigma_2\left(\frac{X_1 - \mu_1}{\sigma_2} + \frac{X_2 - \mu_2}{\sigma_2}\right) + \mu_1 + \mu_2$$

But since $(X_1 - \mu_1)/\sigma_2$ is normal with mean 0 and variance σ_1^2/σ_2^2, and $(X_2 - \mu_2)/\sigma_2$ is normal with mean 0 and variance 1, it follows from our previous result that $(X_1 - \mu_1)/\sigma_2 + (X_2 - \mu_2)/\sigma_2$ is normal with mean 0 and variance $1 + \sigma_1^2/\sigma_2^2$, implying that $X_1 + X_2$ is normal with mean $\mu_1 + \mu_2$ and variance $\sigma_2^2(1 + \sigma_1^2/\sigma_2^2) = \sigma_1^2 + \sigma_2^2$.

(d) $(1-x)^{-r} = \sum_{i=0}^{\infty} \binom{r+i-1}{i} x^i \quad (1-x)^{-s} = \sum_{j=0}^{\infty} \binom{s+j-1}{j} x^j \quad (1-x)^{-(r+s)} = \sum_{n=0}^{\infty} \binom{r+s+n-1}{n} x^n$

$$\binom{n+r+s-1}{n} = \sum_{i=0}^{n} \binom{r+i-1}{i}\binom{n-i+s-1}{n-i}$$

$$\begin{aligned}
p_{X+Y}(n) &= \sum_{k=0}^{n} P(X=k, Y=n-k) \\
&= \sum_{k=0}^{n} P(X=k)P(Y=n-k) \\
&= \sum_{k=0}^{n} \binom{k+r-1}{r-1} p^r (1-p)^k \binom{n-k+s-1}{s-1} p^s (1-p)^{n-k} \\
&= p^{r+s}(1-p)^n \sum_{k=0}^{n} \binom{k+r-1}{r-1}\binom{n-k+s-1}{s-1} \\
&= \binom{n+r+s-1}{r+s-1} p^{r+s}(1-p)^n
\end{aligned}$$

Negative binomial distribution with *n* failure before $(r + s)$-th success.

4. Let X_1, X_2 be independent exponential random variables and their density function are defined by

$$f_{X_k} = \begin{cases} ke^{-kx} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $Y = \max\{X_1, X_2\}$. Find the cumulative distribution function of Y. Hence find its probability density function. Find $Var[Y]$.

Solution

$$
\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(\max\{X_1, X_2\} \le y) \\
&= P(X_1 \le y, X_2 \le y) \\
&= P(X_1 \le y)P(X_2 \le y) \\
&= \begin{cases} \left(1-e^{-y}\right)\left(1-e^{-2y}\right) & y \ge 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
f_Y(y) = \begin{cases} 2e^{-2y}\left(1-e^{-y}\right) + e^{-y}\left(1-e^{-2y}\right) = e^{-y} + 2e^{-2y} - 3e^{-3y} & y \ge 0 \\ 0 & \text{otherwise} \end{cases}
$$

$$
E[Y] = \int_0^\infty y\left(e^{-y} + 2e^{-2y} - 3e^{-3y}\right)dy = 1 + \frac{1}{2} - \frac{1}{3} = \frac{7}{6}
$$

$$
E[Y^2] = \int_0^\infty y^2\left(e^{-y} + 2e^{-2y} - 3e^{-3y}\right)dy = 2\left(1 + \frac{1}{2^2} - \frac{1}{3^2}\right) = \frac{41}{18}
$$

$$
Var[Y] = \frac{11}{12}
$$