PT Assignment 5_Solution

1. Let X be a discrete random variable with probability mass function

$$P_X(k) = \begin{cases} 0.1 & \text{for } k = 0 \\ c & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.2 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $E[X]$ and $Var[X]$.

(b) Let $Y = (X-2)^2$. Find $E[Y]$ by

(i) the formula $\sum (x-2)^2 P_X(x)$.

(ii) probability mass function of Y.

Solution

1 (a) $0.1 + c + 0.3 + 0.2 = 1$

$\therefore c = 0.4$

$E[X] = 0.1 \times 0 + 0.4 \times 1 + 0.3 \times 2 + 0.2 \times 3 = 1.6$

$E[X^2] = 0.1 \times 0^2 + 0.4 \times 1^2 + 0.3 \times 2^2 + 0.2 \times 3^2 = 3.4$

$Var[X] = 3.4 - 1.6^2 = 0.84$

(b)

(i) $E[(x-2)^2] = 0.1 \times (0-2)^2 + 0.4 \times (1-2)^2 + 0.3 \times (2-2)^2 + 0.2 \times (3-2)^2$

$= 0.4 + 0.4 + 0.2$

$= 1$

(ii) Range of $Y = \{(0-2)^2, (1-2)^2, (2-2)^2, (3-2)^2\}$

$= \{0, 1, 4\}$

$P_Y(0) = P_X(2) = 0.3$

$P_Y(1) = P_X(1) + P_X(3) = 0.4 + 0.2 = 0.6$

$P_Y(4) = P_X(0) = 0.1$

$E[Y] = 0 \times 0.3 + 1 \times 0.6 + 4 \times 0.1 = 1$

2. Find the mean and variance of the following distributions:

(a) Discrete Uniform: Let $n \in \boldsymbol{N}$. $P\left(X = \frac{k}{n}\right) = \frac{1}{n}$ for $k = 1, 2, \cdots, n$.

(b) Poisson: Let $\lambda > 0$. $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$ for $k = 0, 1, 2, \cdots$

(c) Binomial: Let $n \in \boldsymbol{N}, 0 \le p \le 1$. $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0, 1, 2, \cdots, n$.

(d) Geometric: Let $0 < p \le 1$. $P(X = k) = (1-p)^k p$ for $k = 0,1,2,\cdots$

(e) Negative Binomial: $P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}, \quad n = r, r+1, \cdots$

Solution

2. (a) $E[X] = \frac{1}{n}\cdot\frac{1}{n} + \frac{2}{n}\cdot\frac{1}{n} + \cdots + \frac{n}{n\cdot n}$

$$= \frac{1}{n}\cdot\frac{n(n+1)}{2n} = \frac{n+1}{2n}$$

$$E[X^2] = \frac{1^2}{n^2}\cdot\frac{1}{n} + \frac{2^2}{n^2}\cdot\frac{1}{n} + \cdots + \frac{n^2}{n^2}\cdot\frac{1}{n}$$

$$= \frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6n^2}$$

$$= \frac{(n+1)(2n+1)}{6n^2}$$

$$Var[X] = \frac{(n+1)(2n+1)}{6n^2} - \frac{(n+1)^2}{(2n)^2}$$

$$= \frac{n+1}{2n^2}\cdot\left[\frac{2n+1}{3} - \frac{n+1}{2}\right]$$

$$= \frac{(n+1)(4n+2-3n-3)}{12n^2}$$

$$= \frac{(n+1)(n-1)}{12n^2}$$

(b)

$$E[X] = 0\,\frac{e^{-\lambda}\lambda^0}{0!} + 1\,\frac{e^{-\lambda}\lambda^1}{1!} + 2\,\frac{e^{-\lambda}\lambda^2}{2!} + 3\,\frac{e^{-\lambda}\lambda^3}{3!} + \cdots$$

$$= e^{-\lambda}\left[\lambda + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + \cdots\right]$$

$$= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right]$$

$$= \lambda e^{-\lambda}\cdot e^{\lambda} = \lambda$$

$$E[X(X-1)] = (2\cdot1)\frac{e^{-\lambda}\lambda^2}{2!} + (3\cdot2)\frac{e^{-\lambda}\lambda^3}{3!} + (4\cdot3)\frac{e^{-\lambda}\lambda^4}{4!} + \cdots$$

$$= e^{-\lambda}\lambda^2\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right]$$

$$= \lambda^2$$

$$\sigma_X^2 = E[X(X-1)] + \mu_X - \mu_X^2 = \lambda^2 - \lambda + \lambda^2 = \lambda$$

(c) $$E[X] = 0\binom{n}{0}p^0(1-p)^n + 1\binom{n}{1}p^1(1-p)^{n-1}$$

$$+ \cdots + k\binom{n}{k}p^k(1-p)^{n-k} + \cdots + n\binom{n}{n}p^n(1-p)^0$$

Since $$k\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{(k-1)!} = n\binom{n-1}{k-1}$$

$$k(k-1)\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{(k-2)!} = n(n-1)\binom{n-2}{k-2}$$

$$E[x] = n\binom{n-1}{0}p(1-p)^{n-1} + \cdots + n\binom{n-1}{k-1}p^k(1-p)^{n-k}$$

$$+ \cdots + n\binom{n-1}{n-1}p^n(1-p)^0$$

$$= np\left[\binom{n-1}{0}(1-p)^{n-1} + \cdots + \binom{n-1}{k-1}p^{k-1}(1-p)^{n-k} + \cdots + \binom{n-1}{n-1}p^{n-1}(1-p)^0\right]$$

$$= np \cdot [p + (1-p)]^{n-1} = np$$

$$E[x(x-1)] = (2\cdot 1)\binom{n}{2}p^2(1-p)^{n-2} + \cdots + k(k-1)\binom{n}{k}p^k(1-p)^{n-k}$$

$$+ \cdots + n(n-1)\binom{n}{n}p^n(1-p)^0$$

$$= n(n-1)p^2\left[\binom{n-2}{0}(1-p)^{n-2} + \cdots + \binom{n-2}{k-2}(1-p)^{n-k}p^{k-2} + \cdots + \binom{n-2}{n-2}p^{n-2}(1-p)^0\right]$$

$$= n(n-1)p^2 \cdot [p + (1-p)]^{n-2} = n(n-1)p^2$$

$$Var[x] = E[x(x-1)] + \mu_x - \mu_x^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= np \cdot [(n-1)p + 1 - np]$$

$$= np(1-p)$$

(d) Recall $\frac{1}{(1-r)^2} = 1 + 2r + 3r^2 + \cdots$ for $|r| < 1$

$$E[x] = 0 + 1(1-p)p + 2(1-p)^2 p + \cdots$$

$$= (1-p)p\,[1 + 2(1-p) + 3(1-p)^2 + \cdots]$$

$$= (1-p)\,p\,\frac{1}{[1-(1-p)]^2}$$

$$= \frac{(1-p)p}{p^2} = \frac{1-p}{p}$$

Recall $\frac{1}{(1-r)^3} = 1 + \binom{3}{1} r + \binom{4}{2} r^2 + \binom{5}{3} r^3 + \cdots \quad |r| < 1$

$$E[x(x-1)] = (2\cdot1)(1-p)^2 p + (3\cdot2)(1-p)^3 p + (4\cdot3)(1-p)^4 p + \cdots$$

$$= 2!\,(1-p)^2 p\left[1 + \frac{3\cdot2}{2!}(1-p) + \left(\frac{4\cdot3}{2!}\right)(1-p)^2 + \cdots\right]$$

$$= 2(1-p)^2 p \cdot \frac{1}{[1-(1-p)]^3}$$

$$= 2\,\frac{(1-p)^2}{p^2}$$

Since $C_n^m = \frac{n!}{m!(n-m)!}$ $\quad C_3^1 = \frac{3!}{2! \times 1} = \frac{3!}{2!}$

$$C_4^2 = \frac{4!}{2! \times 2!} = \frac{4 \times 3}{2!}$$

(e)

Compute the expected value and the variance of a negative binomial random variable with parameters r and p.

Solution We have

$$
\begin{aligned}
E[X^k] &= \sum_{n=r}^{\infty} n^k \binom{n-1}{r-1} p^r (1-p)^{n-r} \\
&= \frac{r}{p} \sum_{n=r}^{\infty} n^{k-1} \binom{n}{r} p^{r+1}(1-p)^{n-r} \quad \text{since} \quad n\binom{n-1}{r-1} = r\binom{n}{r} \\
&= \frac{r}{p} \sum_{m=r+1}^{\infty} (m-1)^{k-1} \binom{m-1}{r} p^{r+1}(1-p)^{m-(r+1)} \quad \begin{array}{c}\text{by setting} \\ m = n+1\end{array} \\
&= \frac{r}{p} E[(Y-1)^{k-1}]
\end{aligned}
$$

where Y is a negative binomial random variable with parameters $r+1, p$. Setting $k = 1$ in the preceding equation yields

$$
E[X] = \frac{r}{p}
$$

Setting $k = 2$ in the equation for $E[X^k]$ and using the formula for the expected value of a negative binomial random variable gives

$$
\begin{aligned}
E[X^2] &= \frac{r}{p} E[Y-1] \\
&= \frac{r}{p}\left(\frac{r+1}{p} - 1\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\text{Var}(X) &= \frac{r}{p}\left(\frac{r+1}{p} - 1\right) - \left(\frac{r}{p}\right)^2 \\
&= \frac{r(1-p)}{p^2}
\end{aligned}
$$

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3. An automobile insurance company has a block of one-year car insurance policies. The policies are divided into three classes: A, B, and C. A randomly chosen policy has 40% chance of being in class A, 10% in class B, and 50% in class C. The probability that a policy will produce a claim is 20% in class A, 10% in class B and 5% in class C. A class of policies (i.e., either class A, or class B, or class C) is chosen at random, with probability of being chosen proportional to the random chance of a policy being chosen from class (i.e., 40% for class A, 10% for class B, and 50% for class C) and five policies are selected at random from that class. It turns out that exactly one of the five policies produced a claim. What is the probability that these policies are from class A?

Solution

Let $p_A = 0.20$ be the probability of producing a claim for class A, $p_B = 0.10$ be the same probability for class B, and $p_C = 0.05$ be the corresponding probability for class C. Let us write p for any of these three probabilities. Then the probability of producing exactly one claim among five policies is $\binom{5}{1} \cdot p^1 \cdot (1-p)^4$. By the Bayes' Theorem,

$$\Pr(5 \text{ policies came from A} \mid \text{One claim exactly among 5 policies}) =$$

$$= \frac{\underbrace{\left(\binom{5}{1} \cdot 0.20^1 \cdot 0.80^4\right)}_{\Pr(\text{One claim exactly among 5 policies} \mid 5 \text{ policies came from A})} \cdot \underbrace{0.40}_{\Pr(A)}}{\left(\binom{5}{1} \cdot 0.20^1 \cdot 0.80^4\right) \cdot 0.40 + \left(\binom{5}{1} \cdot 0.10^1 \cdot 0.90^4\right) \cdot 0.10 + \left(\binom{5}{1} \cdot 0.05^1 \cdot 0.95^4\right) \cdot 0.50} \approx$$

$$\approx 0.54895444.$$

4. Let X have the Poisson distribution such that $4p(2) = p(1) + p(0)$. Calculate $P(X \geq 2 \mid X \leq 4)$.

Solution

$$4p(2) = p(1) + p(0)$$

$$4e^{-\lambda}\frac{\lambda^2}{2!} = e^{-\lambda}\lambda + e^{-\lambda}$$

$$2\lambda^2 - \lambda - 1 = 0$$

$$\lambda = 1 \quad \text{or} \quad \lambda = -0.5 \text{ (rejected)}$$

$$\Pr(X \geq 2 | X \leq 4) = \frac{\Pr(\{X \geq 2\} \cap \{X \leq 4\})}{\Pr(X \leq 4)} = \frac{\Pr(2 \leq X \leq 4)}{\Pr(X \leq 4)} =$$

$$= \frac{e^{-1} \cdot \frac{1}{2!} + e^{-1} \cdot \frac{1}{3!} + e^{-1} \cdot \frac{1}{4!}}{e^{-1} \cdot \frac{1}{0!} + e^{-1} \cdot \frac{1}{1!} + e^{-1} \cdot \frac{1}{2!} + e^{-1} \cdot \frac{1}{3!} + e^{-1} \cdot \frac{1}{4!}} = \frac{\frac{1}{2} + \frac{1}{6} + \frac{1}{24}}{1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}} = \frac{17}{65}.$$

5. A card is drawn at random from an ordinary deck of 52 cards and replaced. This is done a total of 5 independent times. What is the conditional probability of drawing the ace of spades exactly 4 times, given that this ace is drawn at least 4 times.

Solution

Probability that exactly 4 aces of spades are drawn is $\binom{5}{4}\left(\frac{1}{52}\right)^4 \cdot \frac{51}{52}$, and the probability that exactly 5 aces of spades are drawn is $\binom{5}{5}\left(\frac{1}{52}\right)^5$, so that the probability sought is

$$\frac{\binom{5}{4}\left(\frac{1}{52}\right)^4 \cdot \frac{51}{52}}{\binom{5}{4}\left(\frac{1}{52}\right)^4 \cdot \frac{51}{52} + \binom{5}{5}\left(\frac{1}{52}\right)^5} = \frac{255}{256}.$$