Chapter Seven

Efficient Diversification

Chapter Overview

- The investment decision is a top-down process
  1. Capital allocation (risky vs. risk-free)
  2. Asset allocation within the risky portfolio across broad asset classes
  3. Security selection of individual assets within each asset class
- Optimal risky portfolio construction
- Efficient diversification
- Long-term vs. short-term investment horizons

Portfolios of Two Risky Assets

- Expected return
  - Weighted average of expected returns on the component securities
- Portfolio risk
  - Variance of the portfolio is a weighted sum of covariances, and each weight is the product of the portfolio proportions of the pair of assets

Portfolios of Two Risky Assets: Expected Return

Consider a portfolio made up of equity (stocks) and debt (bonds)…

$$r_p = w_D r_D + w_E r_E$$

where r_P = *rate of return on portfolio*
w_D = *proportion invested in the bond fund*
w_E = *proportion invested in the stock fund*
r_D = *rate of return on the debt fund*
r_E = *rate of return on the equity fund*

$$E(r_p) = w_D E(r_D) + w_E E(r_E)$$

Portfolios of Two Risky Assets: Risk

- Variance of r_P

$$\sigma_p^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E \mathrm{Cov}\left(r_D, r_E\right)$$

- Bond variance

$$\sigma_D^2$$

- Equity variance

$$\sigma_E^2$$

- Covariance of returns for bond and equity

$$\mathrm{Cov}\left(r_D, r_E\right)$$

Portfolios of Two Risky Assets: Covariance

- Covariance of returns on bond and equity

$$Cov(r_D, r_E) = \rho_{DE}\sigma_D\sigma_E$$

  - ρ_{DE} = Correlation coefficient of returns
  - σ_D = Standard deviation of bond returns
  - σ_E = Standard deviation of equity returns

Portfolios of Two Risky Assets: Correlation Coefficients (1 of 2)

- Range of values for correlation coefficient

$$-1.0 \le \rho \le 1.0$$

  - If ρ = 1.0 → perfectly positively correlated securities
  - If ρ = 0 → the securities are uncorrelated
  - If ρ = - 1.0 → perfectly negatively correlated securities

Portfolios of Two Risky Assets: Correlation Coefficients (2 of 2)

- Portfolio variance is higher when ρ_{DE} is higher.
- When $\rho_{DE} = 1$, there is no diversification

$$\sigma_P = w_E \sigma_E + w_D \sigma_D$$

- When $\rho_{DE} < 1$, the portfolio standard deviation is less than the weighted average.

Portfolios of Two Risky Assets: Correlation Coefficients (2 of 2)

- Assets with negative correlation are particularly effective in reducing total risk.
- With expected return unaffected by correlation, we always prefer to add to our portfolios assets with low or, even better, negative correlation.
- When ρ_{DE} = -1, a perfect hedge is possible

$$w_E = \frac{\sigma_D}{\sigma_D + \sigma_E} = 1 - w_D$$

Portfolios of Two Risky Assets: Example — 50%/50% Split

Table 7.1

Descriptive statistics for two mutual funds

	Debt		Equity
Expected return, $E(r)$	8%		13%
Standard deviation, σ	12%		20%
Covariance, $\text{Cov}(r_D, r_E)$		72	
Correlation coefficient, ρ_{DE}		0.30	

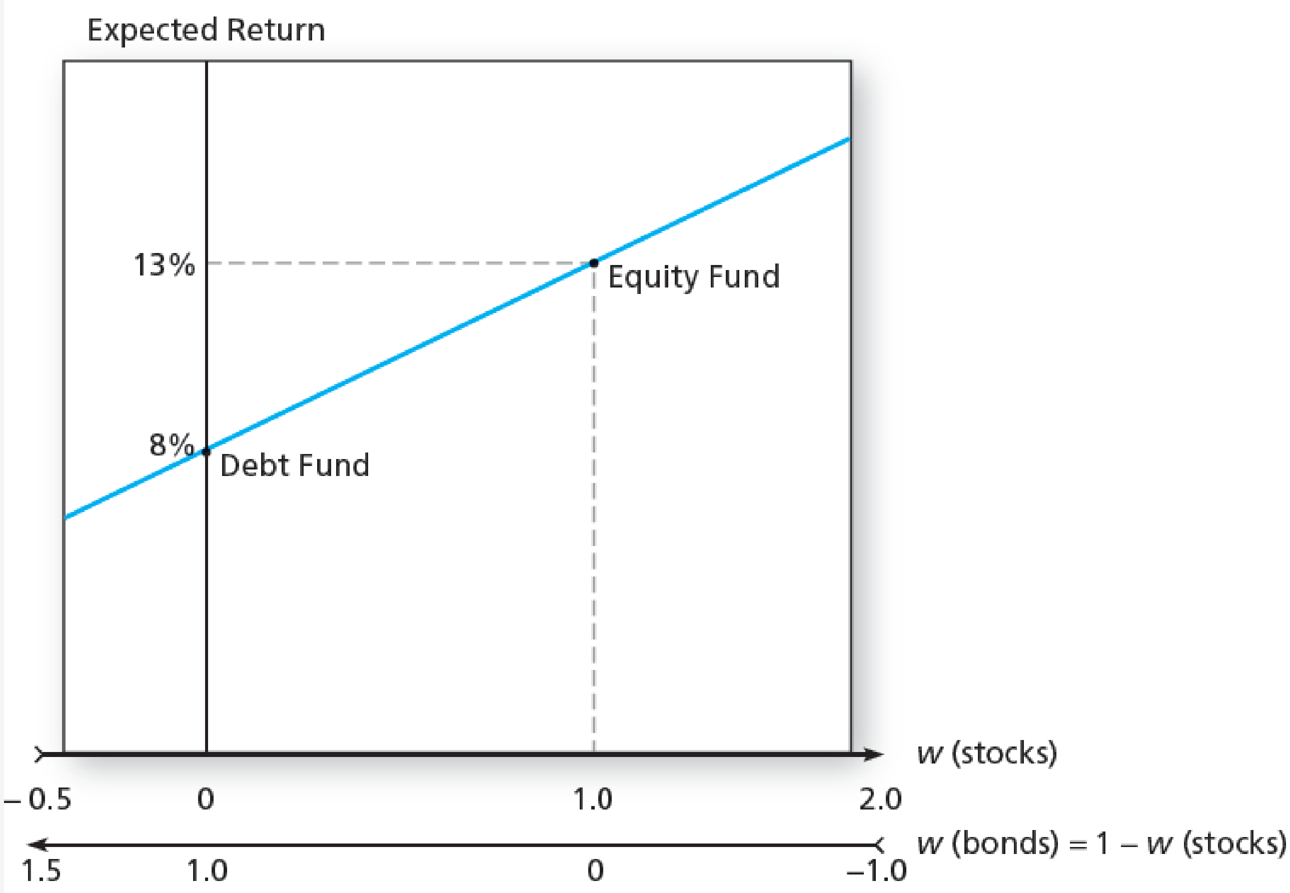
Expected Return:

$$E(r_p) = w_D E(r_D) + w_E E(r_E)$$
$$= .50 \times 8\% + .50 \times 13\% = 10.5\%$$

Variance:

$$\sigma_p^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E \text{Cov}\left(r_D, r_E\right)$$
$$= .50^2 \times 12^2 + .50^2 \times 20^2 + 2 \times .5 \times .5 \times 72 = 172$$
$$\sigma_P = \sqrt{172} = 13.23\%$$

Portfolio Expected Return

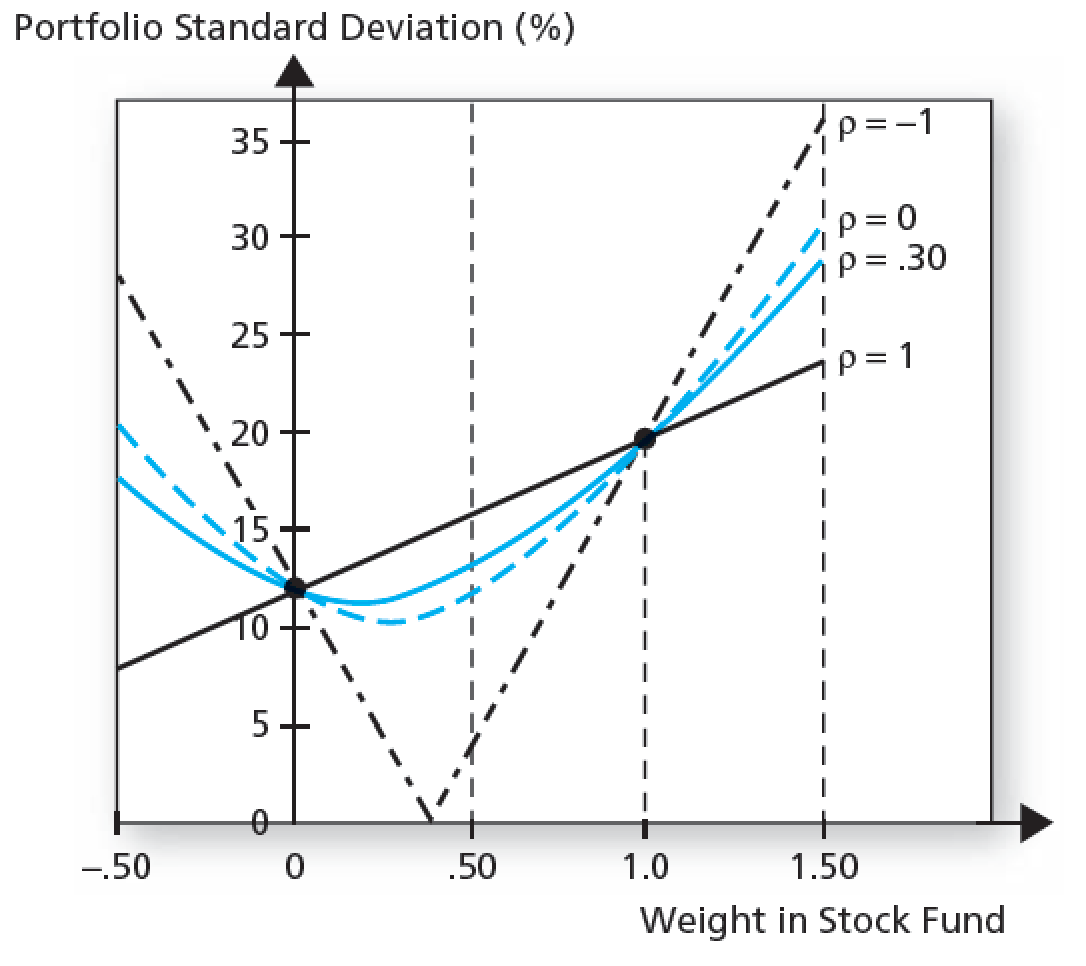


Computation of Portfolio Variance from the Covariance Matrix

A. Bordered Covariance Matrix		
Portfolio Weights	w_D	w_E
w_D	$\text{Cov}(r_D, r_D)$	$\text{Cov}(r_D, r_E)$
w_E	$\text{Cov}(r_E, r_D)$	$\text{Cov}(r_E, r_E)$

B. Border-Multiplied Covariance Matrix		
Portfolio Weights	w_D	w_E
w_D	$w_D w_D \text{Cov}(r_D, r_D)$	$w_D w_E \text{Cov}(r_D, r_E)$
w_E	$w_E w_D \text{Cov}(r_E, r_D)$	$w_E w_E \text{Cov}(r_E, r_E)$
$w_D + w_E = 1$	$w_D w_D \text{Cov}(r_D, r_D) + w_E w_D \text{Cov}(r_E, r_D)$	$w_D w_E \text{Cov}(r_D, r_E) + w_E w_E \text{Cov}(r_E, r_E)$
Portfolio variance	$w_D w_D \text{Cov}(r_D, r_D) + w_E w_D \text{Cov}(r_E, r_D) + w_D w_E \text{Cov}(r_D, r_E) + w_E w_E \text{Cov}(r_E, r_E)$	

Portfolio Standard Deviation



The Minimum-Variance Portfolio

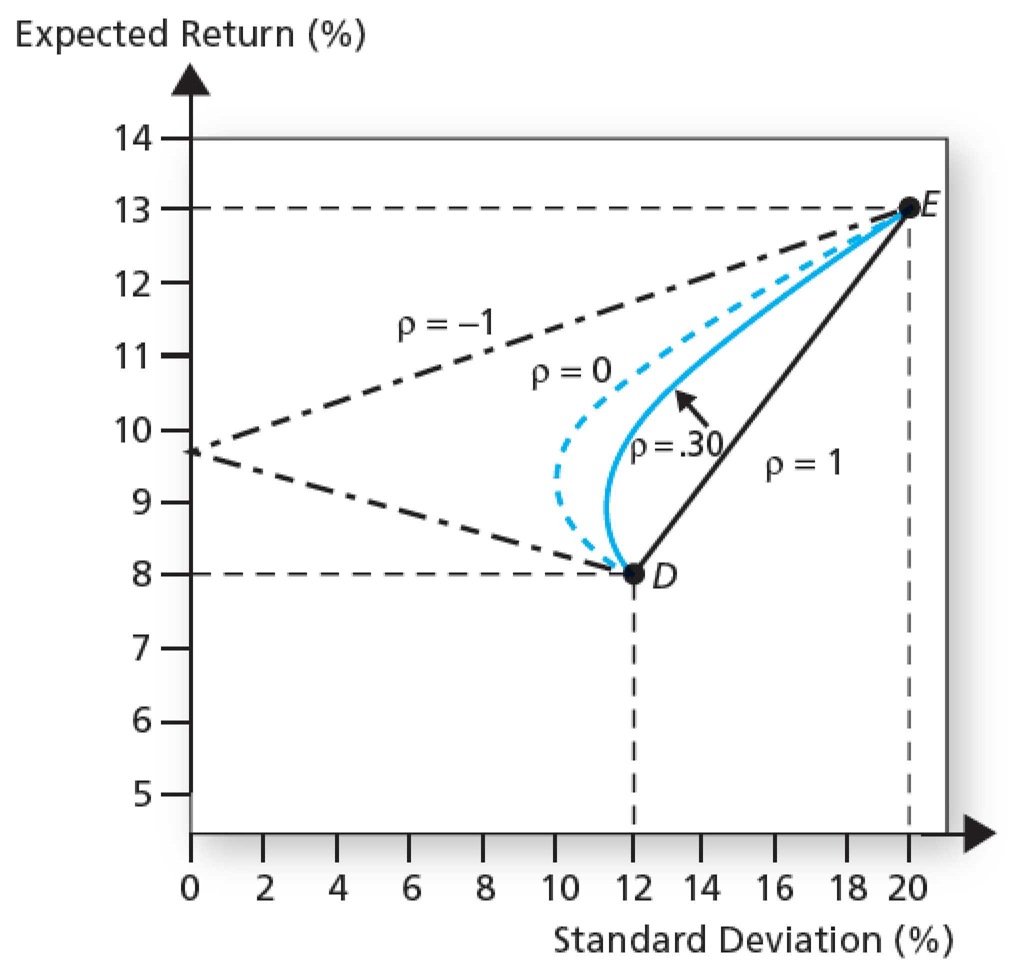
- Weights of the minimum-variance portfolio

$$w_{Min}(D) = \frac{\sigma_E^2 - \rho_{DE}\sigma_D\sigma_E}{\sigma_D^2 + \sigma_E^2 - 2\rho_{DE}\sigma_D\sigma_E},$$

$$w_{Min}(E) = 1 - w_{Min}(D) = \frac{\sigma_D^2 - \rho_{DE}\sigma_D\sigma_E}{\sigma_D^2 + \sigma_E^2 - 2\rho_{DE}\sigma_D\sigma_E}$$

- For a pair of assets,
  - When $\rho_{DE} < \sigma_D/\sigma_E$, volatility will initially fall when we start with all bonds and begin to move into stocks.
  - When $\rho_{DE} > \sigma_D/\sigma_E$, volatility will increase monotonically from the low-risk asset (bonds) to the high-risk asset (stocks). Even in this case, however, there is a positive (if small) benefit from diversification.

Portfolio Expected Return as a Function of Standard Deviation

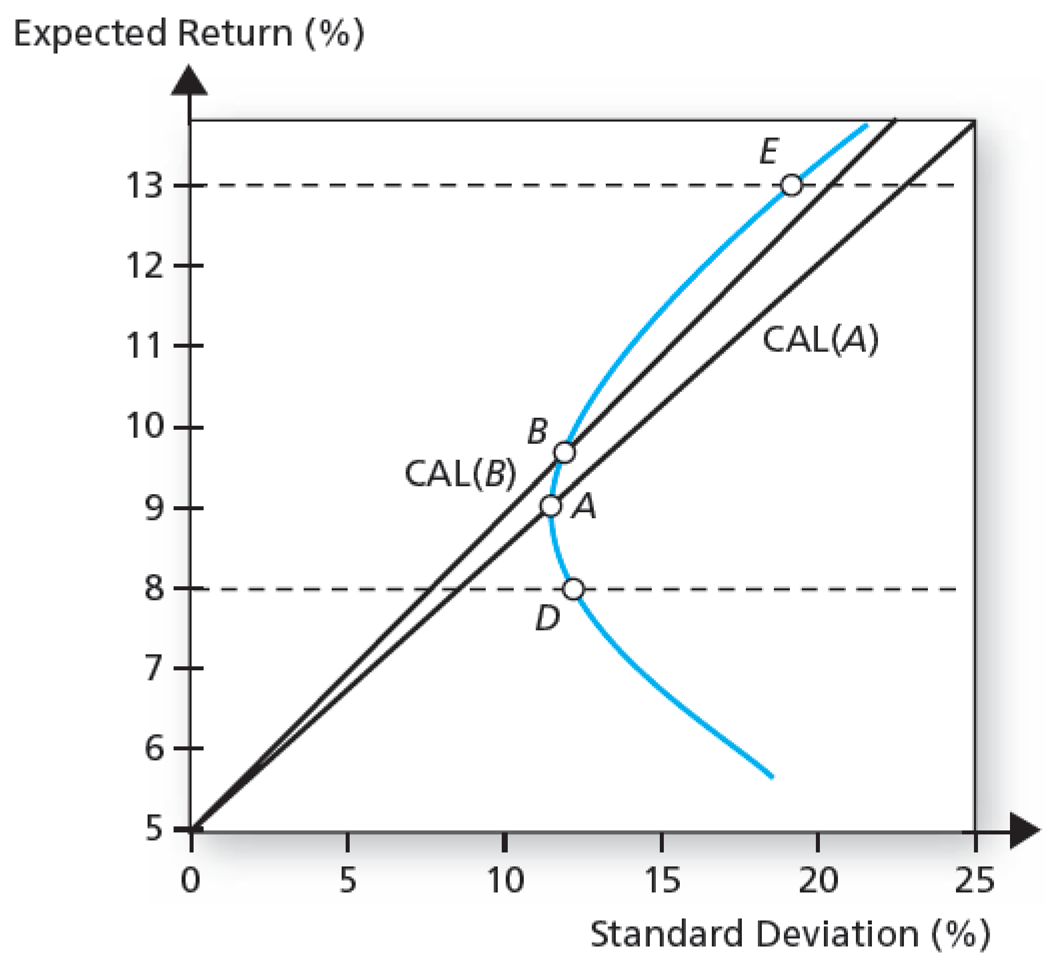


The Minimum-Variance Portfolio

- The **minimum-variance portfolio** has a standard deviation *smaller than that of either of the individual component assets*

- Risk reduction depends on the correlation:
  - If ρ = +1.0, no risk reduction is possible
  - If ρ = 0, σ_P may be less than the standard deviation of either component asset
  - If ρ = -1.0, a riskless hedge is possible

The Opportunity Set of the Debt and Equity Funds and Two Feasible CALs



Portfolio A

$$E(r_A) = 8.9\%$$

$$\sigma_A = 11.45\%$$

Portfolio B

$$E(r_B) = 9.5\%$$

$$\sigma_B = 11.70\%$$

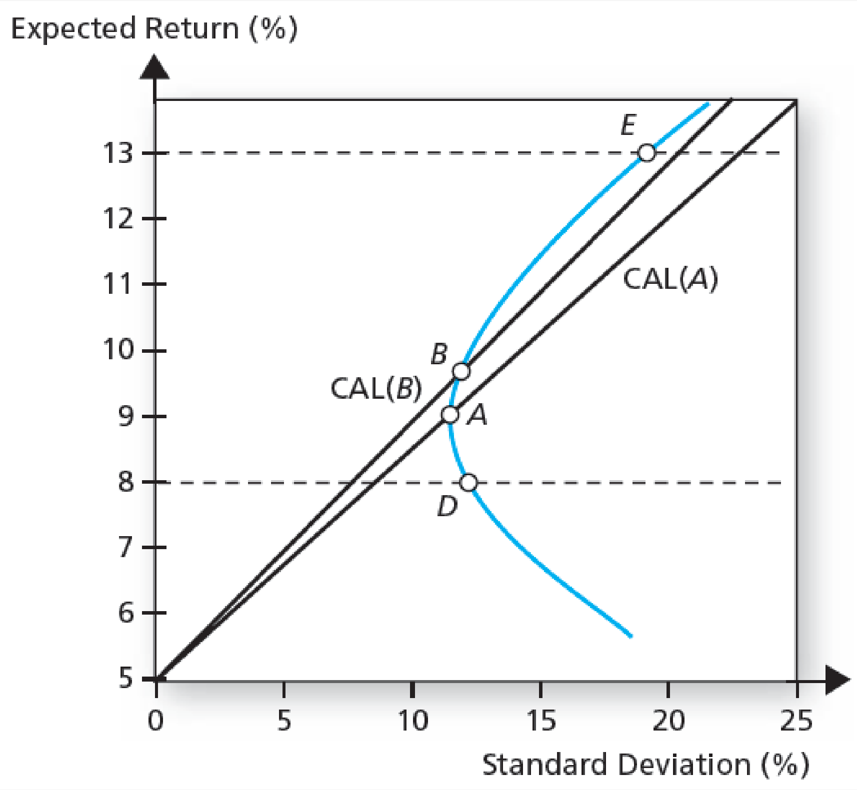
The Sharpe Ratio

- Objective is to find the weights w_D and w_E that result in the highest slope of the CAL

- Thus, our *objective function* is the Sharpe ratio:

$$S_p = \frac{E(r_p) - r_f}{\sigma_p}$$

The Sharpe Ratio: Example



Portfolio A

$$E(r_A) = 8.9\%$$

$$\sigma_A = 11.45\%$$

$$S_A = \frac{E(r_A) - r_f}{\sigma_A} = \frac{8.9\% - 5\%}{11.45\%} = .34$$

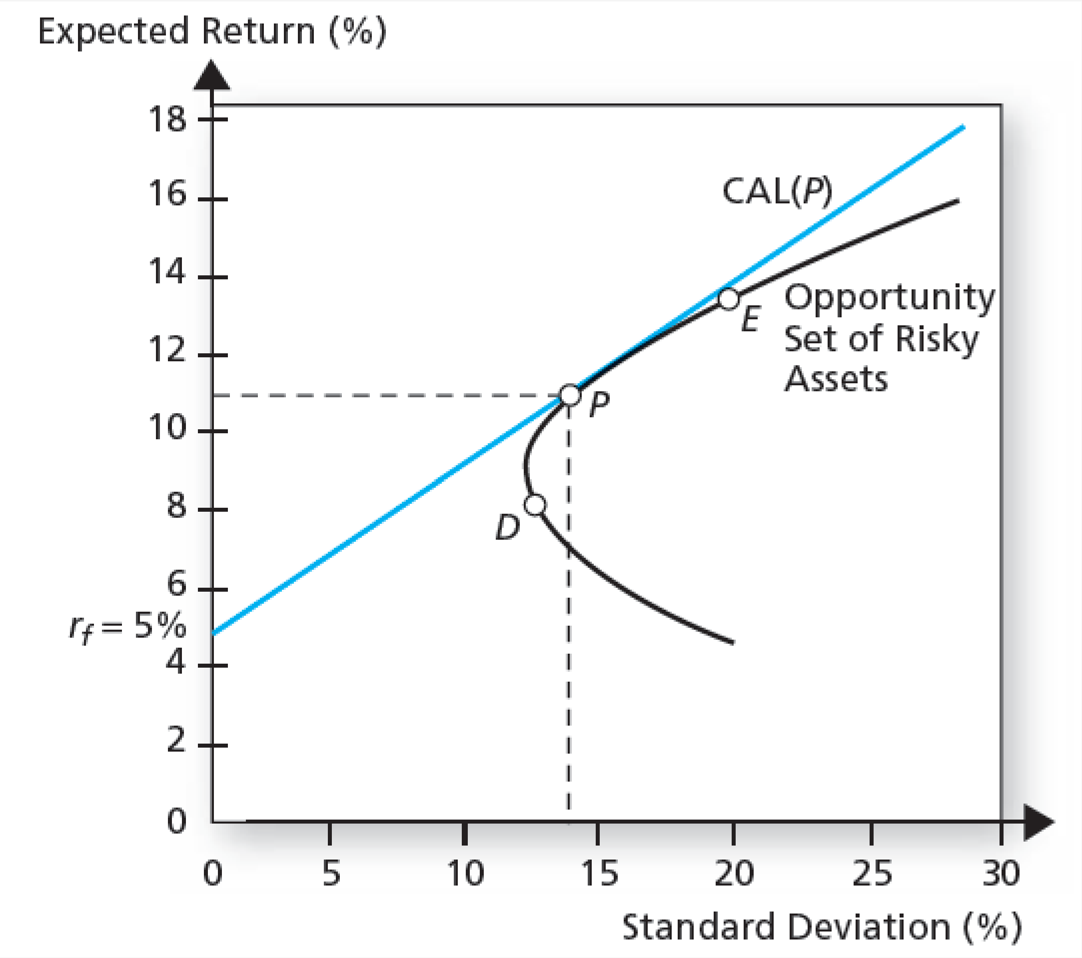
Portfolio B

$$E(r_B) = 9.5\%$$

$$\sigma_B = 11.70\%$$

$$S_B = \frac{E(r_B) - r_f}{\sigma_B} = \frac{9.5\% - 5\%}{11.70\%} = .38$$

Debt and Equity Funds with the Optimal Risky Portfolio



Optimal Risky Portfolio

$$E(r_P) = 11\%$$

$$\sigma_P = 14.2\%$$

$$S_P = \frac{E(r_P) - r_f}{\sigma_P} = \frac{11\% - 5\%}{14.2\%} = .42$$

Optimal Risky Portfolio

- Optimization problem

$$\max_{w_i} \frac{E(r_p) - r_f}{\sigma_p}$$

$$\text{s.t. } w_D + w_E = 1$$

- Optimal risky portfolio

$$w_D = \frac{E(R_D)\sigma_E^2 - E(R_E)Cov(R_D, R_E)}{E(R_D)\sigma_E^2 + E(R_E)\sigma_D^2 - [E(R_D) + E(R_E)]Cov(R_D, R_E)}, w_E = 1 - w_D$$

where *R* denotes the excess return.

Optimal Risky Portfolio

- Using the data in Table 7.1

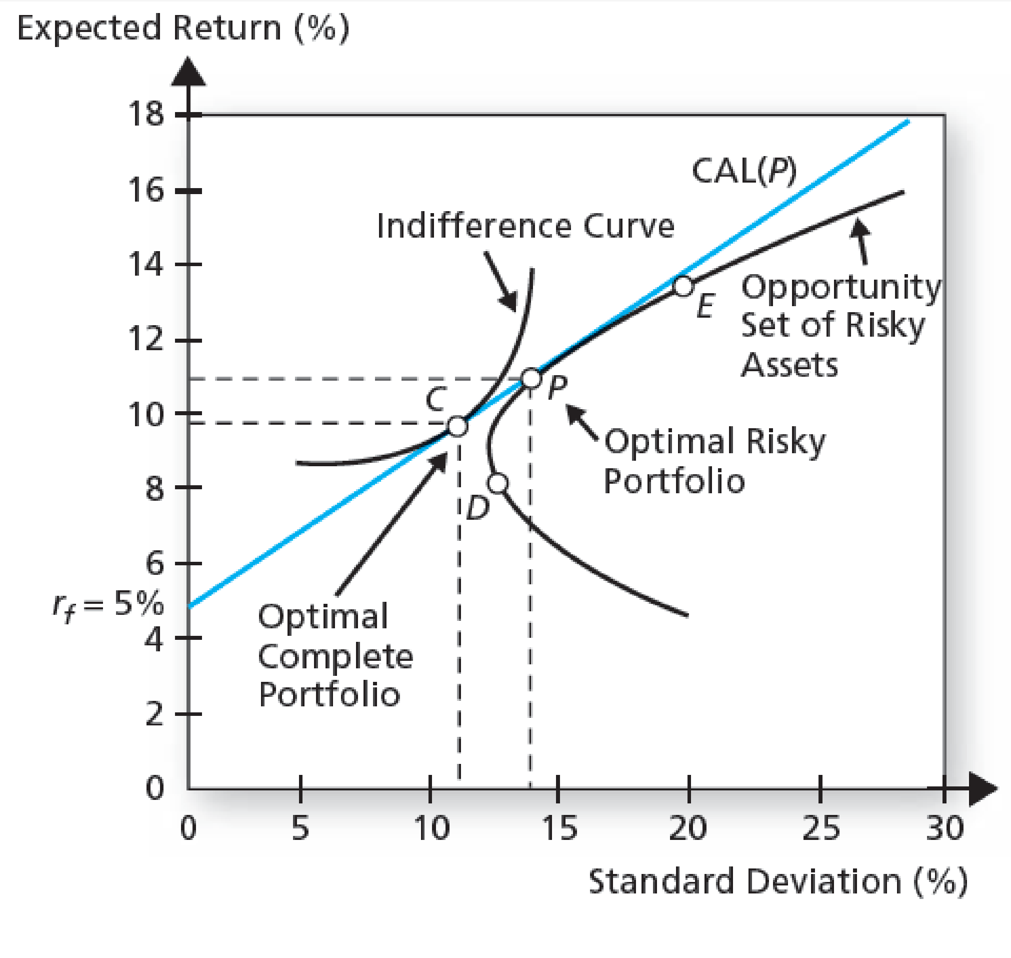
$$w_D = \frac{(8-5)400-(13-5)72}{(8-5)400+(13-5)144-(8-5+13-5)72} = .40,$$

$$w_E = 1 - .40 = .60$$

$$E(r_P) = (.4 \times 8) + (.6 \times 13) = 11\%$$

$$\sigma_P = [(.4^2 \times 144) + (.6^2 \times 400) + (2 \times .4 \times .6 \times 72)]^{1/2} = 14.2\%$$

Determination of the Optimal Complete Portfolio



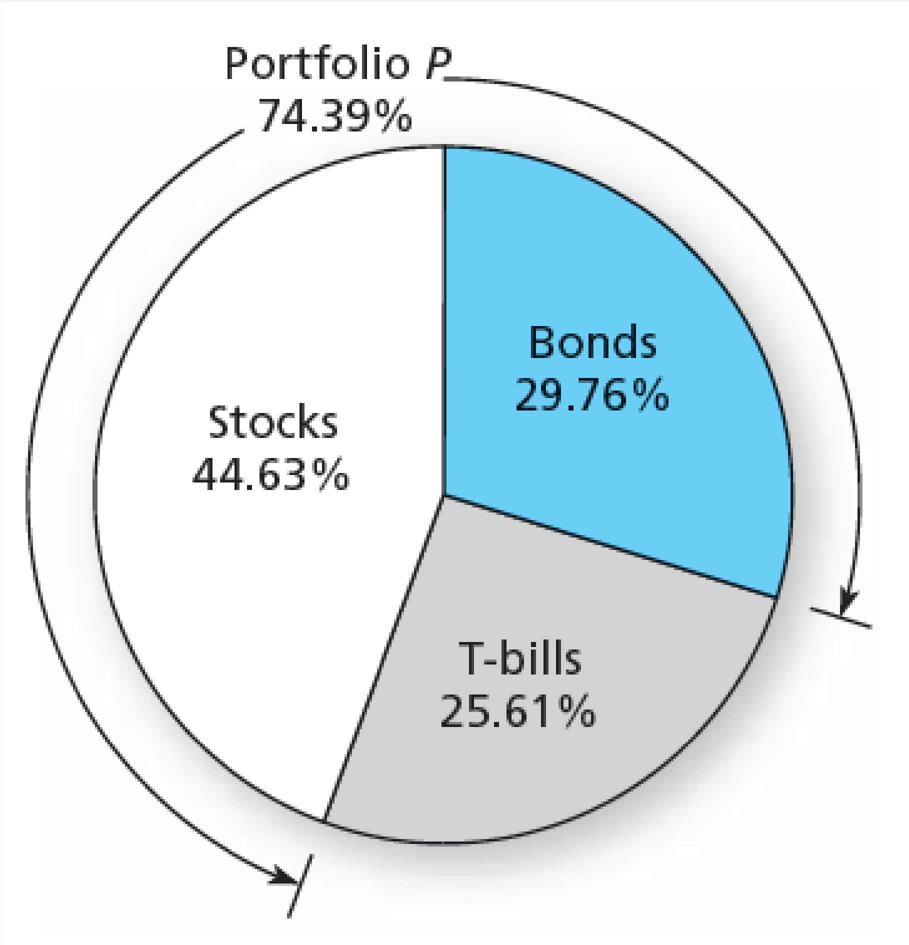
Optimal Allocation to P

$$A = 4$$

$$y = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

$$= \frac{11\% - 5\%}{4 \times (14.2\%)^2} = .7439$$

The Proportions of the Optimal Complete Portfolio



Overall Portfolio

$$E(r_P) = 11\% \quad y = .7439$$

$$\sigma_P = 14.2\% \quad r_f = 5\%$$

$$E(r_{Overall}) = y \times E(r_p) + (1-y) \times r_f$$

$$= .7439 \times 11\% + .2561 \times 5\%$$

$$= 9.46\%$$

$$\sigma_{Overall} = .7439 \times 14.2\% = 10.56\%$$

$$S_{Overall} = \frac{9.46\% - 5\%}{10.56\%} = .42$$

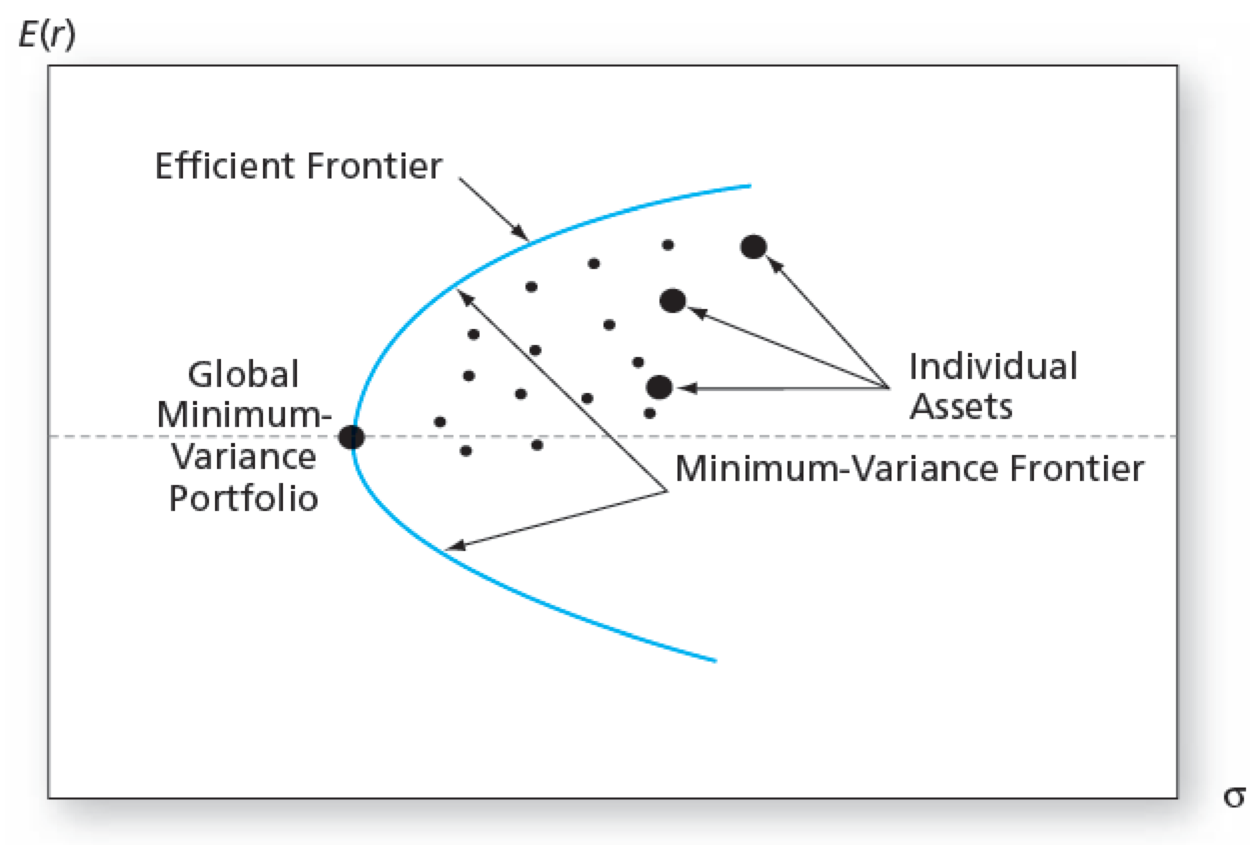
Steps to Arrive the Complete Portfolio

1. Specify the return characteristics of all securities (expected returns, variances, covariances)
2. Establish the risky portfolio (asset allocation):
   a. Calculate the optimal risky portfolio, *P*.
   b. Calculate the properties (i.e., $E(r_p)$ and σ_P) of portfolio *P* using the weights determined in step (a)
3. Allocate funds between the risky portfolio and the risk-free asset (capital allocation):
   a. Calculate the fraction of the complete portfolio allocated to portfolio *P* (the risky portfolio) and the risk-free asset.
   b. Calculate the share of the complete portfolio invested in each asset.

Markowitz Portfolio Optimization Model (1 of 6)

- Security selection– generalize the portfolio construction problem to the case of many risky securities and a risk-free asset
  - Determine the risk-return opportunities available
    - **Minimum-variance frontier** of risky assets
  - All portfolios that lie on the minimum-variance frontier from the global minimum-variance portfolio and upward provide the best risk-return combinations
    - **Efficient frontier** of risky assets is the portion of the frontier that lies above the global minimum-variance portfolio

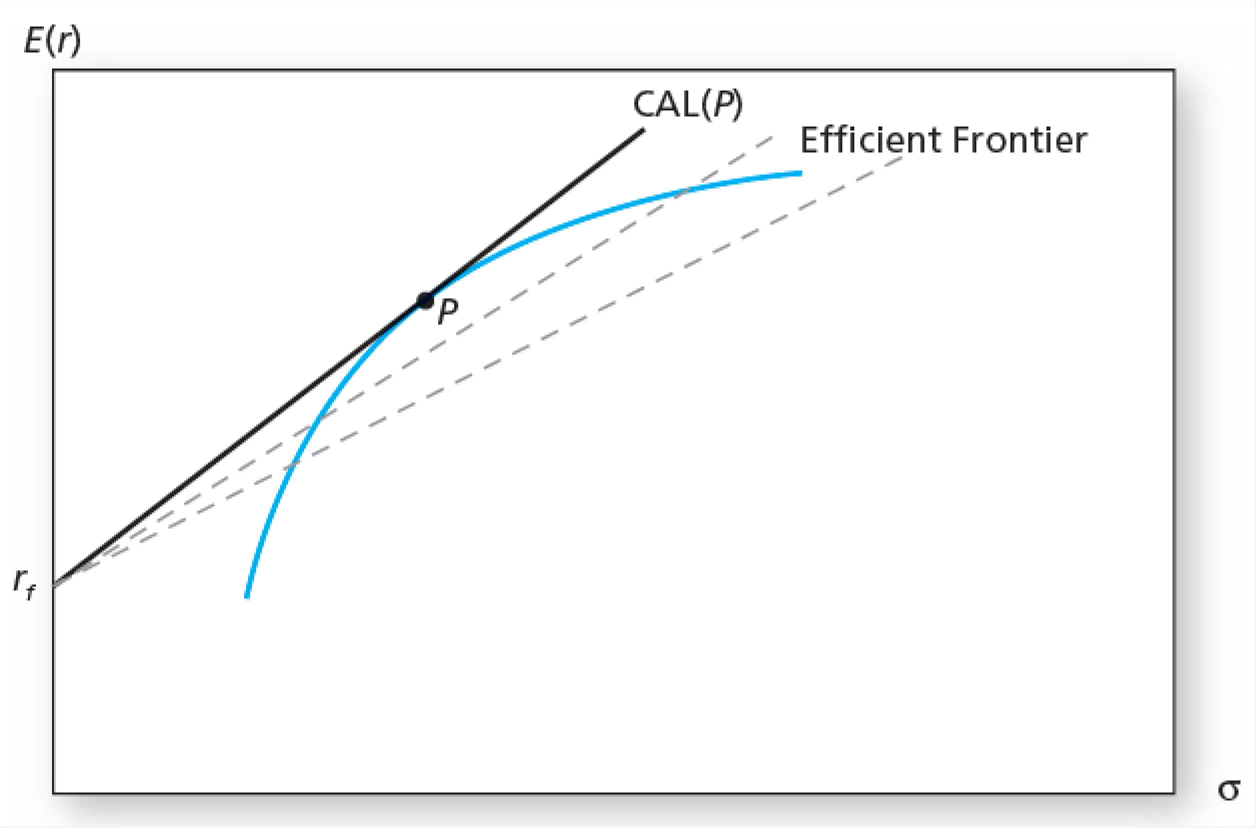
The Minimum-Variance Frontier of Risky Assets



Markowitz Portfolio Optimization Model (2 of 6)

- Security selection (continued)
  - Search for the CAL with the highest Shape ratio (i.e., the steepest slope)
  - Individual investor chooses the appropriate mix between the optimal risky portfolio *P* and T-bills
  - Everyone invests in *P*, regardless of their degree of risk aversion
    - More risk averse investors put less in *P*
    - Less risk averse investors put more in *P*

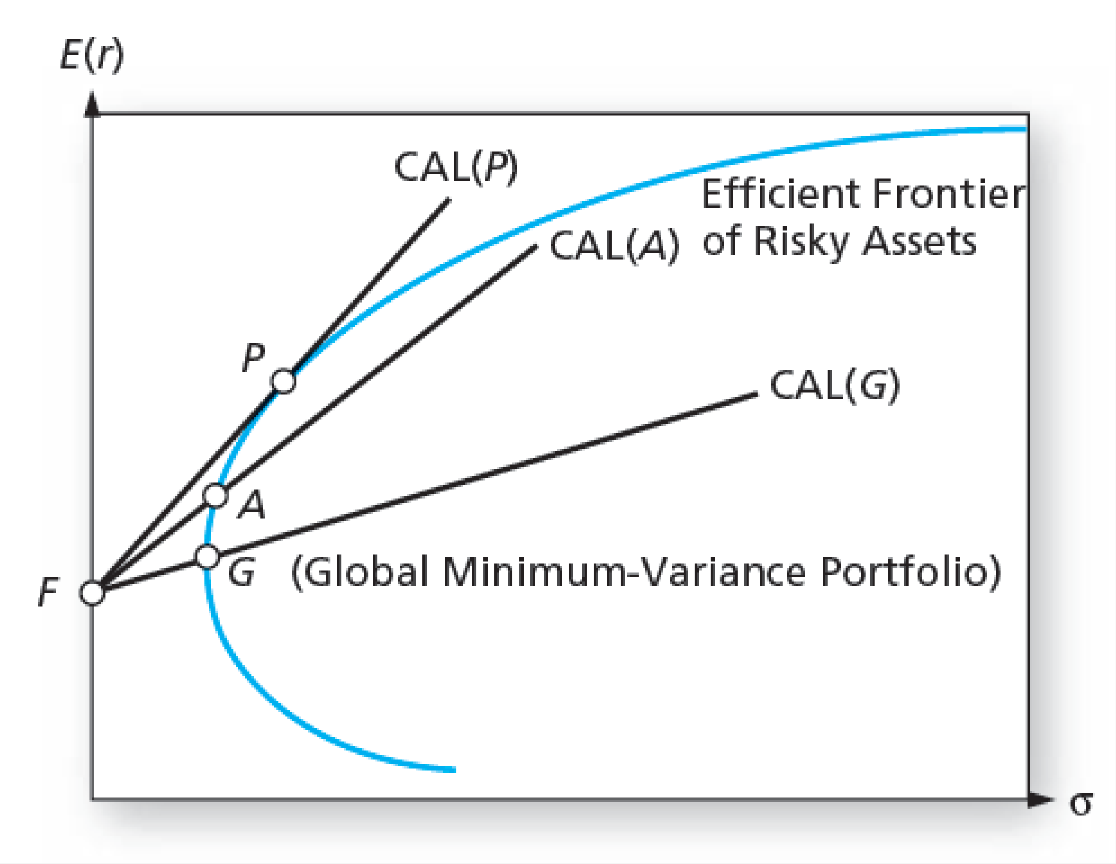
The Efficient Frontier of Risky Assets with the Optimal CAL



Markowitz Portfolio Optimization Model (3 of 6)

- Capital allocation and the **separation property**
  - Portfolio choice problem may be separated into two independent tasks
    - Determination of the optimal risky portfolio is purely technical
    - Allocation of the complete portfolio to risk-free versus the risky portfolio depends on personal preference

Capital Allocation Lines with Various Portfolios from the Efficient Set



Markowitz Portfolio Optimization Model (4 of 6)

- The power of diversification
  - Recall:

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathrm{Cov}\left(r_i, r_j\right)$$

  - Assume we define the average variance and average covariance of the securities as:

$$\bar{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\sigma_i^2$$

$$\overline{\mathrm{Cov}} = \frac{1}{n(n-1)}\sum_{\substack{j=1 \\ j \neq i}}^{n}\sum_{i=1}^{n}\mathrm{Cov}\left(r_i, r_j\right)$$

Markowitz Portfolio Optimization Model (5 of 6)

- The power of diversification (continued)
  - We can then express portfolio variance as

$$\sigma_p^2 = \frac{1}{n}\bar{\sigma}^2 + \frac{n-1}{n}\text{Cov}$$

  - Portfolio variance can be driven to zero if the average covariance is zero
  - The risk of a highly diversified portfolio depends on the covariance of the returns of the component securities

Risk Reduction of Equally Weighted Portfolios

Universe Size n	Portfolio Weights $w = 1/n$ (%)	$\rho = 0$ Standard Deviation (%)	$\rho = 0$ Reduction in σ	$\rho = .40$ Standard Deviation (%)	$\rho = .40$ Reduction in σ
1	100	50.00	14.64	50.00	8.17
2	50	35.36		41.83	
5	20	22.36	1.95	36.06	0.70
6	16.67	20.41		35.36	
10	10	15.81	0.73	33.91	0.20
11	9.09	15.08		33.71	
20	5	11.18	0.27	32.79	0.06
21	4.76	10.91		32.73	
100	1	5.00	0.02	31.86	0.00
101	0.99	4.98		31.86	

Markowitz Portfolio Optimization Model (6 of 6)

- Optimal portfolios and non-normal returns
  - Fat-tailed distributions can result in extreme values of VaR and ES
    - Practice way to estimate values of VaR and ES in the presence of fat tails is called *bootstrapping*
  - If other portfolios provide sufficiently better VaR and ES values than the mean-variance efficient portfolio, we may prefer these when faced with fat-tailed distributions