CHAPTER 3

The Poisson Process

3.1. Definition and Basic Properties

Definition 1. A stochastic process $\{N(t), t \geq 0\}$ is said to be a counting process if $N(t)$ represents the total number of "events" that occur by time t. We see that for a counting process $N(t)$ must satisfy:

(i) $N(t) \geq 0$.

(ii) $N(t)$ is integer valued.

(iii) If $s < t$, then $N(s) \leq N(t)$.

(iv) For $s < t, N(t) - N(s)$ equals the number of events that occur in the interval $(s, t]$.

A counting process is said to possess independent increments if the numbers of events that occur in disjoint time intervals are independent. For example, this means that the number of events that occur by time 10 (that is, $N(10)$) must be independent of the number of events that occur between times 10 and 15 (that is, $N(15) - N(10)$).

A counting process is said to possess stationary increments if the distribution of the number of events that occur in any interval of time depends only on the length of the time interval. In other words, the process has stationary increments if the number of events in the interval ($s, s + t$) has the same distribution for all s.

Definition 2. The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate $\lambda, \lambda > 0$, if

(i) $N(0) = 0$.

(ii) The process has independent increments: for each $n \in \mathbb{N}$ and each $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $N(t_1) - N(t_0), N(t_2) - N(t_1), \cdots, N(t_n) - N(t_{n-1})$ are independent.

(iii) The number of events in any interval of length t is Poisson distributed with mean λt. That is, for all $s, t \geq 0$

$$P(N(t+s) - N(s) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \cdots$$

Note that it follows from condition (iii) that a Poisson process has stationary increments and also that

$$E[N(t)] = \lambda t$$

which explains why λ is called the rate of the process.

EXAMPLE. Claims arrive at an insurance company following a Poisson process $N(t)$ with $\lambda = 1$ claim per day.

- **(a):** Calculate the probability that, over five days, the company receives exactly five claims.
- **(b):** Calculate the probability that, over five days, the company receives exactly one claim each day.

SOLUTION. (a) $P(N(5) = 5) = \mathrm{e}^{-5}5^5/5! = 0.17547$.

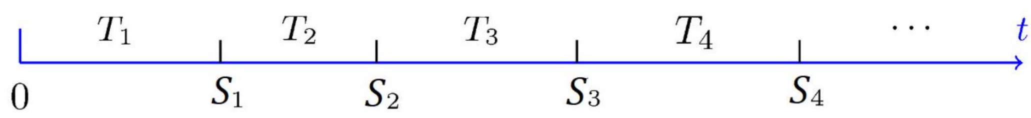
(b)

$$\begin{aligned}
&P(N(1) = 1, N(2) = 2, N(3) = 3, N(4) = 4, N(5) = 5) \\
&\quad = P(N(1) = 1, N(2) - N(1) = 1, N(3) - N(2) = 1, N(4) - N(3) = 1, N(5) - N(4) = 1) \\
&\quad = P(N(1) = 1)P(N(2) - N(1) = 1)P(N(3) - N(2) = 1) \\
&\qquad\quad \cdot P(N(4) - N(3) = 1)P(N(5) - N(4) = 1) \\
&\quad = P(N(1) = 1)^5 \\
&\quad = (\mathrm{e}^{-1})^5 \\
&\quad = 0.006738.
\end{aligned}$$

EXERCISE. Suppose $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda = 2$. Please calculate:

(1) $E(N(2))$;
(2) $E\left[(N(1))^2\right]$;
(3) $E(N(1)N(2)N(3))$.

Construction by exponential interarrival times. Consider a Poisson process, and let us denote the time of the first event by T_1. Further, for $n > 1$, let T_n denote the elapsed time between the $(n-1)$-th and the n-th event. The sequence $\{T_n, n = 1, 2, \cdots\}$ is called the sequence of interarrival times. For instance, if $T_1 = 5$ and $T_2 = 10$, then the first event of the Poisson process would have occurred at time 5 and the second at time 15.



We shall now determine the distribution of the T_n. Note that the event $\{T_1 > t\}$ takes place if and only if no events of the Poisson process occur in the interval $[0, t]$ and thus,

$$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t}.$$

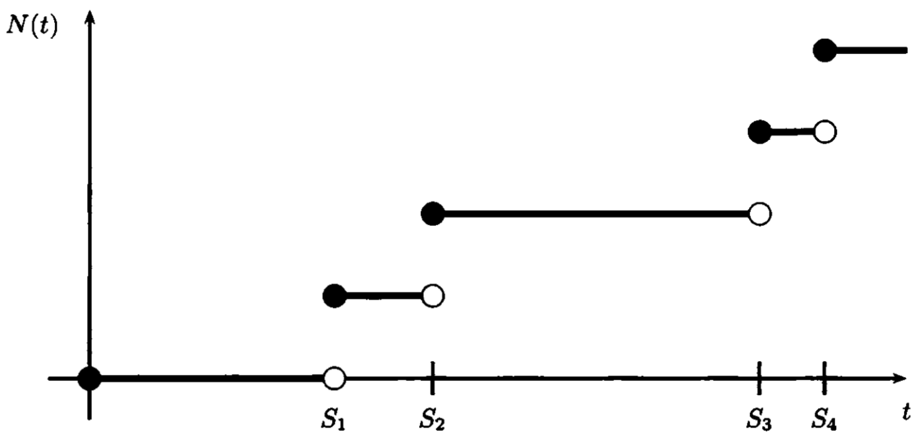
Hence, T_1 has an exponential distribution with mean $1/\lambda$. Now,

$$\begin{aligned} P(T_2 > t \mid T_1 = s) &= P(0 \text{ events in } (s, s+t] \mid T_1 = s) \\ &= P(0 \text{ events in } (s, s+t]) \\ &= e^{-\lambda t} \end{aligned}$$

where the last two equations followed from independent and stationary increments. Therefore, from above equation we conclude that T_2 is also an exponential random variable with mean $1/\lambda$ and, furthermore, that T_2 is independent of T_1. Repeating the same argument yields the following.

THEOREM 3. $T_n, n = 1, 2, \cdots$, *are independent identically distributed exponential random variables having mean* $1/\lambda$.

REMARK 4. Theorem 3 should not surprise us. The assumption of stationary and independent increments is basically equivalent to asserting that, at any point in time, the process probabilistically restarts itself. That is, the process from any point on is independent of all that has previously occurred (by independent increments), and also has the same distribution as the original process (by stationary increments). In other words, the process has no memory, and hence exponential interarrival times are to be expected.



Theorem 3 gives us a way of constructing a Poisson process. Suppose that we start out with a sequence $\{T_n, n \geq 1\}$ of independent identically distributed exponential random variables each having mean $1/\lambda$. Now let us define a counting process by saying that the nth event of this process occurs at time S_n, where $S_n = T_1 + \cdots + T_n, \quad n \geq 1$. The resultant counting process $\{N(t), t \geq 0\}$ will be Poisson with rate λ, where $N(t) := \#\{n : S_n \leq t\}$.

The above defined S_n, the arrival time of the n-th event, also called the waiting time until the nth event. Recall that

$$S_n = \sum_{i=1}^{n} T_i, \quad n \geq 1$$

Since $T_1, \cdots, T_n$ are independent and identically distributed exponential random variables having mean $1/\lambda$,

$$M_{S_n}(u) = E\left[e^{u(T_1+\cdots+T_n)}\right] = E\left[e^{uT_1}\right] \cdots E\left[e^{uT_n}\right] = \left(\frac{\lambda}{\lambda - u}\right)^n \quad \text{for } u < \lambda$$

It follows from the results that S_n has a gamma distribution with parameters n and λ. Its probability density function is

$$f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}.$$

REMARK 5. Since S_n is the arrival time of the n-th event, it follows that $P(S_n \leq t) = P(N(t) \geq n)$. In other words, the n-th event will occur prior to or at time t if and only if the number of events occurring by time t is at least n.

EXAMPLE 6. Consider a Poisson process with rate λ. Compute (a) E (time of the 10 th event), (b) P (the 10th event occurs 2 or more time units after the 9 th event), (c) P (the 10th event occurs later than time 20), and (d) $P(2$ events in $(1,4]$ and 3 events in $(3,5])$.

The answer to (a) is $\frac{10}{\lambda}$ by Theorem 3. The answer to (b) is $e^{-2\lambda}$, since

$$P(T_{10} \geq 2) = \int_2^\infty \lambda e^{-\lambda t} dt = -e^{-\lambda t}\Big|_2^\infty = e^{-2\lambda}.$$

The answer to (c) is $P(S_{10} > 20) = P(N(20) < 10)$, so we can either write the integral

$$P(S_{10} > 20) = \int_{20}^\infty \lambda e^{-\lambda t} \frac{(\lambda t)^9}{9!} dt$$

or use

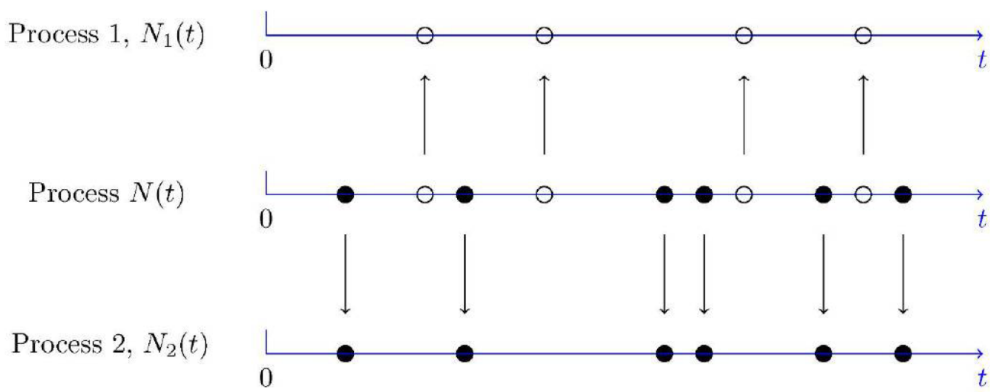
$$P(N(20) < 10) = \sum_{j=0}^{9} e^{-20\lambda} \frac{(20\lambda)^j}{9!}$$

To answer (d), we condition on the number of events in $(3,4]$:

$$\sum_{k=0}^{2} P(2 \text{ events in } (1,4] \text{ and } 3 \text{ events in } (3,5] \mid k \text{ events in } [3,4]) \cdot P(k \text{ events in } (3,4])$$
$$=\sum_{k=0}^{2} P(2-k \text{ events in } (1,3] \text{ and } 3-k \text{ events in } (4,5]) \cdot P(k \text{ events in (3, 4] })$$
$$=\sum_{k=0}^{2} e^{-2\lambda}\frac{(2\lambda)^{2-k}}{(2-k)!}\cdot e^{-\lambda}\frac{\lambda^{3-k}}{(3-k)!}\cdot e^{-\lambda}\frac{\lambda^k}{k!}$$
$$=e^{-4\lambda}\left(\frac{1}{3}\lambda^5+\lambda^4+\frac{1}{2}\lambda^3\right).$$

Transformations. Consider the following example. Customers arrive at a store following a Poisson process with rate λ. Each is either male or female with probability $\frac{1}{2}$. If we look at the arrival of male customers, intuitively, it seems also be a Poisson process. This is indeed the case. The procedure of selecting certain events and forget the other is referred to "thinning" of a Poisson process.

THEOREM 7 (Thinning of a Poisson process). *Each event in a Poisson process $N(t)$ with rate λ is independently a Type I event with probability p; the remaining events are Type II. Let $N_1(t)$ and $N_2(t)$ be the numbers of Type I and Type II events in $[0,t]$. These are independent Poisson processes with rates λp and $\lambda(1-p)$.*



Splitting a Poisson process into two independent Poisson Processes

***Sketch of the proof*.** Note that $N(t) = N_1(t) + N_2(t)$.

$$P(N_1(t) = i, N_2(t) = j)$$
$$= P(N_1(t) = i, N_2(t) = j, N(t) = i + j)$$
$$= P(N_1(t) = i, N_2(t) = j \mid N(t) = i + j)\, P(N(t) = i + j)$$

Then

$$P(N_1(t) = i, N_2(t) = j)$$
$$= \binom{i+j}{i} p^i (1-p)^j e^{-\lambda t} \frac{(\lambda t)^{i+j}}{(i+j)!} = e^{-\lambda t p} \frac{(\lambda t p)^i}{i!} e^{-\lambda t(1-p)} \frac{(\lambda t(1-p))^j}{j!}.$$

Hence,

$$P(N_1(t) = i)$$
$$= \sum_{j=0}^{\infty} P(N_1(t) = i, N_2(t) = j) = e^{-\lambda t p} \frac{(\lambda t p)^i}{i!} \sum_{j=0}^{\infty} e^{-\lambda t(1-p)} \frac{(\lambda t(1-p))^j}{j!} = e^{-\lambda t p} \frac{(\lambda t p)^i}{i!}.$$

and similarly,

$$P(N_2(t) = j) = e^{-\lambda t(1-p)} \frac{(\lambda t(1-p))^j}{j!}.$$

Finally,

$$P(N_1(t) = i, N_2(t) = j) = e^{-\lambda t p} \frac{(\lambda t p)^i}{i!} e^{-\lambda t(1-p)} \frac{(\lambda t(1-p))^j}{j!} = P(N_1(t) = i)\, P(N_2(t) = j).$$

So $N_1(t)$ and $N_2(t)$ are independent of each other. This completes the sketch of the proof.

EXAMPLE 8. Customers arrive at a store at a rate of 10 per hour. Each is either male or female with probability $\frac{1}{2}$. Assume that you know that exactly 10 women entered within some hour (say, 10 to 11am). (a) Compute the probability that exactly 10 men also entered. (b) Compute the probability that at least 20 customers have entered during this hour.

Male and female arrivals are independent Poisson processes, with parameter $\frac{1}{2} \cdot 10 = 5$, so the answer to (a) is

$$e^{-5} \frac{5^{10}}{10!}$$

The answer to (b) is

$$\sum_{k=10}^{\infty} P(k \text{ men entered }) = \sum_{k=10}^{\infty} e^{-5} \frac{5^k}{k!} = 1 - \sum_{k=0}^{9} e^{-5} \frac{5^k}{k!}.$$

Example 9. Assume that cars arrive at a rate of 10 per hour. Assume that each car will pick up a hitchhiker with probability $\frac{1}{10}$. You are second in line. What is the probability that you will have to wait for more than 2 hours?

Cars that pick up hitchhikers are a Poisson process with rate $10 \cdot \frac{1}{10} = 1$. For this process,

$$P(T_1 + T_2 > 2) = P(N(2) \leq 1) = e^{-2}(1+2) = 3e^{-2}.$$

3.2. Compound Poisson Processes

In this section we will embellish our Poisson process by associating an independent and identically distributed (i.i.d.) random variable Y_i with each arrival. By independent we mean that the Y_i are independent of each other and of the Poisson process of arrivals. To explain why we have chosen these assumptions, we begin with two examples for motivation.

Example 10. Consider the McDonald's restaurant on Route 13 in the southern part of Ithaca. Assume that between 12:00 and 1:00 cars arrive according to a Poisson process with rate λ. Let Y_i be the number of people in the i th vehicle. It seems reasonable to assume that the Y_i are i.i.d. and independent of the Poisson process of arrival times.

Example 11. Messages arrive at a computer to be transmitted across the Internet. If we imagine a large number of users writing emails on their laptops (or tablets or smart phones), then the arrival times of messages can be modeled by a Poisson process. If we let Y_i be the size of the ith message, then again it is reasonable to assume $Y_1, Y_2, \ldots$ are i.i.d. and independent of the Poisson process of arrival times.

Having introduced the Y_i 's, it is natural to consider the sum of the Y_i 's we have seen up to time t:

$$S(t) = Y_1 + \cdots + Y_{N(t)}$$

where we set $S(t) = 0$ if $N(t) = 0$. In Example 10, $S(t)$ gives the number of customers that have arrived up to time t. In Example 11, $S(t)$ represents the total number of bytes in all of the messages up to time t. In either case it is interesting to know the mean and variance of $S(t)$.

THEOREM 12. *Let $Y_1, Y_2, \ldots$ be independent and identically distributed, let N be an independent Poisson process with rate $\lambda > 0$, and let $S(t) = Y_1 + \cdots + Y_{N(t)}$ with $S(t) = 0$ when $N(t) = 0$.*
(i) If $E|Y_i| < \infty$, then $ES(t) = \lambda t EY_i$.
(ii) If $EY_i^2 < \infty$, then $\mathrm{Var}(S(t)) = \lambda t EY_i^2$.

PROOF. Conditioning on $N(t) = n, S(t) = X_1 + \cdots + X_n$ has $ES(t) = nEY_i$. Breaking things down according to the value of $N(t)$,

$$\begin{aligned} ES &= \sum_{n=0}^{\infty} E(S \mid N(t) = n) \cdot P(N(t) = n) \\ &= \sum_{n=0}^{\infty} nEY_i \cdot P(N(t) = n) = EN(t) \cdot EY_i = \lambda t EY_i. \end{aligned}$$

For the second formula we note that conditioning on $N(t) = n, S(t) = X_1 + \cdots + X_n$ has $\mathrm{var}(S(t)) = n \,\mathrm{var}\,(Y_i)$ and hence,

$$E\left(S(t)^2 \mid N = n\right) = n \,\mathrm{var}\,(Y_i) + (nEY_i)^2$$

Computing as before we get

$$ES(t)^2 = \sum_{n=0}^{\infty} E\left(S(t)^2 \mid N(t) = n\right) \cdot P(N(t) = n)$$

$$= \sum_{n=0}^{\infty} \left\{n \cdot \operatorname{var}(Y_i) + n^2 (EY_i)^2\right\} \cdot P(N(t) = n)$$

$$= (EN(t)) \cdot \operatorname{var}(Y_i) + EN(t)^2 \cdot (EY_i)^2$$

To compute the variance now, we observe that

$$\begin{aligned}
\operatorname{Var}(S(t)) &= E\left[S(t)^2\right] - [ES(t)]^2 \\
&= (EN(t)) \cdot \operatorname{var}(Y_i) + EN(t)^2 \cdot (EY_i)^2 - (EN(t) \cdot EY_i)^2 \\
&= (EN(t)) \cdot \operatorname{var}(Y_i) + \operatorname{var}(N(t)) \cdot (EY_i)^2 \\
&= \lambda t \left[\operatorname{var}(Y_i) + (EY_i)^2\right] \\
&= \lambda t EY_i^2.
\end{aligned}$$

where in the last step we have used $\operatorname{var}(N) = EN^2 - (EN)^2$ to combine the second and third terms. □

For a concrete example of the use of Theorem 12 consider

EXAMPLE 13. Insurance claims are made at times distributed according to a Poisson process with rate λ; the successive claim amounts are independent exponential random variables having distribution mean $\mu > 0$, and are independent of the claim arrival times.

Let C_i denote the amount of the ith claim. The total cost of all claims made up to time t, call it $D(t)$, is defined

$$D(t) = \sum_{i=1}^{N(t)} C_i$$

with the convention that if $N(t) = 0$, the above sum is defined to be 0.

According to Theorem 12, the expected value of $D(t)$ is

$$E[D(t)] = \lambda t EC_i = \lambda t \mu,$$

and the variance of $D(t)$ is

$$\mathrm{Var}(D(t)) = \lambda t E C_i^2 = \lambda t \left[\mathrm{Var}(C_i) + (EC_i)^2 \right] = 2\lambda t \mu^2.$$