Applied Stochastic Process

Quiz 2

Date: 11th Dec 2024

Time allowed: 60 minutes **Full mark: 100**

1. (26 points) Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate $\lambda = 0.5$.
   (a) (8 points) Find the probability of no arrivals in $(3, 5]$.
   (b) (8 points) Find the probability that there is exactly one arrival in each of the following intervals: $(0, 1]$, $(1, 2], (2, 3]$, and $(3, 4]$.
   (c) (10 points) Find the probability that there are 1 arrival in $(0, 2]$ and three arrivals in $(1, 4]$.

2. (36 points) Suppose that a fisherman catches fish at random times, according to a Poisson process with rate 4 fish per hour. Suppose that each fish is either a grouper or a snapper, with the probability of being a grouper being $1/4$ (independent of the history up to that time). Let W_g and W_s be the random weights of each grouper and snapper, respectively, (also independent of the history), with means and **standard deviations** (shorted as **SD**):

$$\mathbb{E}[W_g] = 100, \quad SD[W_g] = 20, \quad \mathbb{E}[W_s] = 20, \quad \text{and} \quad SD[W_s] = 10$$

   measured in pounds.
   (a) (10 points) What are the mean and variance of the time until the fisherman catches his fourth fish?
   (b) (8 points) What is the probability that the fisherman catches exactly 4 grouper in a given 2-hour period (along with an unspecified number of snapper)?
   (c) (8 points) What is the probability that the fisherman catches exactly 4 grouper and 5 snapper in a given 2-hour period?
   (d) (10 points) What are the mean and variance of the total weight of all fish caught by the fisherman in a given two-hour period?

3. (18 points) Suppose you roll two fair dice once. Let X be the number on the second die and let Y be the product of the two numbers respectively on the first die and on the second die. Calculate
   (a) (8 points) $\mathbb{E}(Y|X)$;
   (b) (10 points) $\mathbb{E}(X|Y)$.

4. (20 points) Let $\{Y_n\}_{n\in\mathbb{N}}$ be a sequence of independent, identically distributed random variables with

$$\mathbb{P}(Y_1 = -1) = \frac{2}{3} \quad \text{and} \quad \mathbb{P}(Y_1 = 1) = \frac{1}{3}\,.$$

Let

$$S_n = \sum_{j=1}^{n} Y_j, \quad n = 1, 2, ...$$

Prove that the following defined process

$$M_n = 2^{S_n}, \quad n = 1, 2, ...$$

is a martingale with respect to the natural filtration generated by $\{Y_n\}_{n\in\mathbb{N}}$.