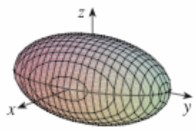
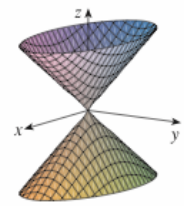
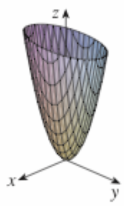
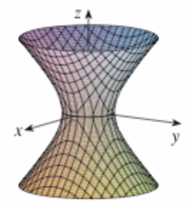
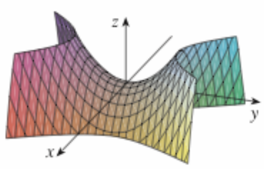
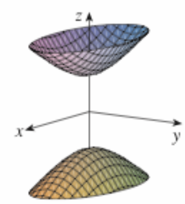
Table 1 Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid  	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	**Cone**  	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid  	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	**Hyperboloid of One Sheet**  	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid  	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	**Hyperboloid of Two Sheets**  	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

2021-22 First Semester
MATH1083 Calculus II (1003)

Assignment 6

Due Date: 11:30am 28/Mar/2021(Tue).

- Write down your **Chinese name** and **student number**. Write neatly on **A4-sized** paper and **show your steps**.
- **Late submissions or answers without details will not be graded.**

1. Match the equation with its graph

21–28 Match the equation with its graph (labeled I–VIII). Give reasons for your choices.

21. $x^2 + 4y^2 + 9z^2 = 1$

22. $9x^2 + 4y^2 + z^2 = 1$

23. $x^2 - y^2 + z^2 = 1$

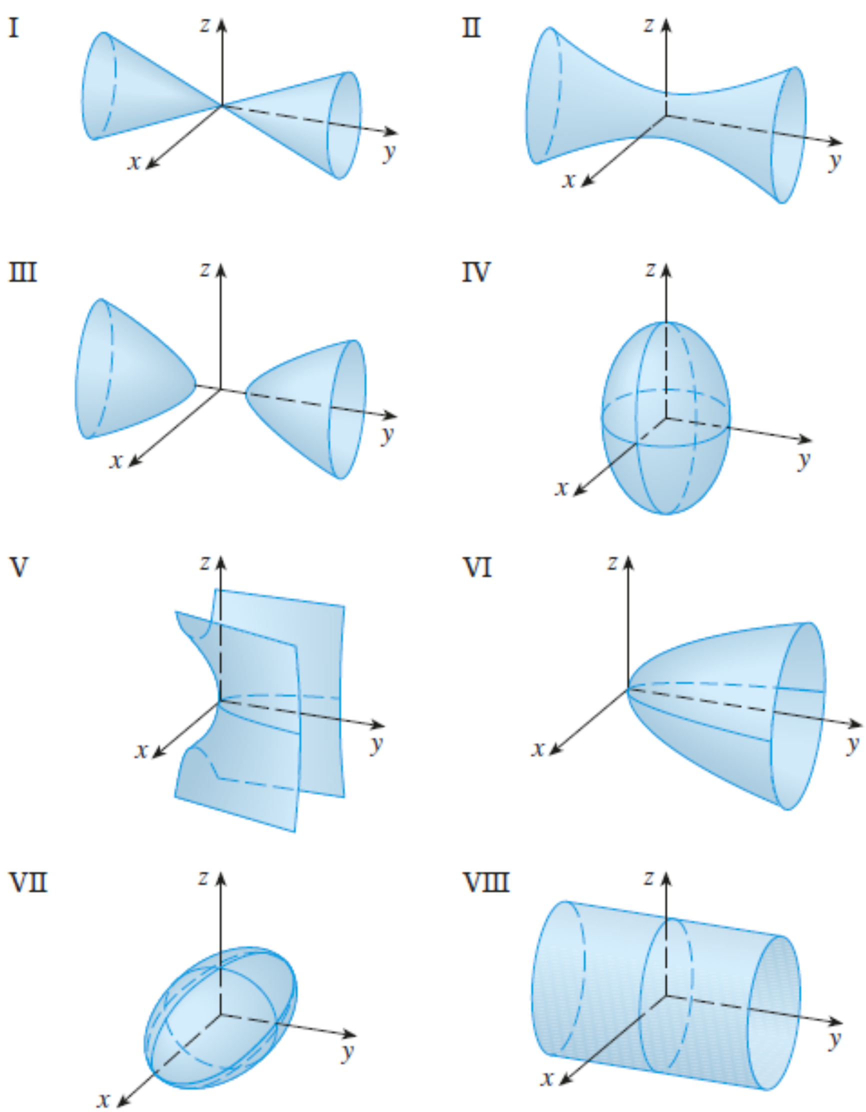
24. $-x^2 + y^2 - z^2 = 1$

25. $y = 2x^2 + z^2$

26. $y^2 = x^2 + 2z^2$

27. $x^2 + 2z^2 = 1$

28. $y = x^2 - z^2$



(21) This is the equation of an ellipsoid: $x^2 + 4y^2 + 9z^2 = x^2 + \frac{y^2}{(1/2)^2} + \frac{z^2}{(1/3)^2} = 1$, with x-intercepts ± 1, y-intercepts $\pm\frac{1}{2}$ and z-intercepts $\pm\frac{1}{3}$. So the major axis is the x-axis and the only possible graph is VII.

(22) This is the equation of an ellipsoid: $9x^2 + 4y^2 + z^2 = \frac{x^2}{(1/3)^2} + \frac{y^2}{(1/2)^2} + z^2 = 1$, with x-intercepts $\pm\frac{1}{3}$, y-intercepts $\pm\frac{1}{2}$ and z-intercepts ± 1. So the major axis is the z-axis and the only possible graph is IV.

(23) $x^2 - y^2 + z^2 = 1$ is the equation of a hyperboloid of one sheet, with $a = b = c = 1$. Since the coefficient of y^2 is negative, the axis of the hyperboloid is the y-axis. Hence, the correct graph is II.

(24) $-x^2 + y^2 - z^2 = 1$ is the equation of a hyperboloid of two sheets, with $a = b = c = 1$. This surface does not intersect the xz-plane at all, so the axis of the hyperboloid is the y-axis. Hence, the correct graph is III.

(25) There are no real values of x and z that satisfy this equation, $y = 2x^2 + z^2$, for $y < 0$, so this surface does not extend to the left of the xz-plane. The surface intersects the plane $y = k > 0$ in an ellipse. Notice that y occurs to the first power whereas x and z occur to the second power. So the surface is an elliptic paraboloid with axis the y-axis. Its graph is VI.

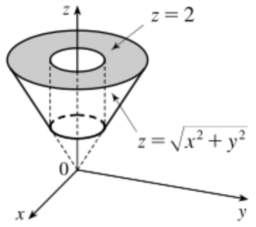
(26) $y^2 = x^2 + 2z^2$ is the equation of a cone with axis the y-axis. Its graph is I.

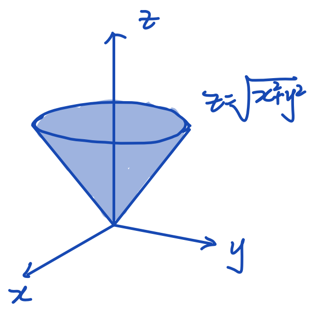
(27) $x^2 + 2z^2 = 1$ is the equation of a cylinder because the variable y is missing from the equation. The intersection of the surface and the xz-plane is an ellipse. Its graph is VIII.

(28) $y = x^2 - z^2$ is the equation of a hyperbolic paraboloid. The trace in the xy-plane is the parabola $y = x^2$. So the correct graph is V.

- Answer:21 VII, 22 IV, 23 II, 24 III, 25 VI, 26 I, 27 VIII, 28 V

2. Sketch the region bounded by the surface $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 = 1$ for $1 \le z \le 2$





$\lim_{t\to 0} \frac{t^2}{\sin^2 t} = \lim_{t=0} \frac{2t}{2\sin t\cos t} = \lim_{t\to 0} \frac{1}{\cos^2 t - \sin^2 t} = 1$

3. Find the limit of the vector function:

$$\lim_{t\to 0}\left(e^{-3t}\vec{i} + \frac{t^2}{\sin^2 t}\vec{j} + \cos 2t\,\vec{k}\right)$$

Solution: $\vec{i} + \vec{j} + \vec{k}$

4. Find the **unit tangent vector** $\vec{T}(t)$ for the given value t: $\vec{r}(t) = \cos t\,\vec{i} + 3t\,\vec{j} + 2\sin 2t\,\vec{k}$ at $t = 0$

Solution: $\vec{r}'(t) = -\sin t\,\vec{i} + 3\,\vec{j} + 4\cos 2t\,\vec{k}$ and $\vec{r}'(0) = 0\,\vec{i} + 3\,\vec{j} + 4\,\vec{k}$

$$\vec{T}(0) = \frac{\vec{r}'(0)}{|\vec{r}'(0)|} = \frac{1}{5}\left(3\vec{j} + 4\vec{k}\right)$$

5. Find the parametric equation for the **tanget line** to the curve with the given parametric equations

$$x = t\cos t, \qquad y = t, \qquad z = t\sin t$$

at the point $(-\pi, \pi, 0)$

Solution: *Step 1*:We differentiate each component of $\vec{r}(t)$

$$\vec{r}'(t) = (\cos t - t\sin t, 1, \sin t + t\cos t)$$

Step 2: at $(-\pi, \pi, 0)$, then $t = \pi$

$$\vec{r}'(\pi) = (-1, 1, -\pi)$$

Step 3: the parametric equation for the **tanget line**

$$x = -\pi - t, \qquad y = \pi + t, \qquad z = -\pi t$$

6. Evaluate the integral

$$\int_0^1 \left(\frac{1}{t+1}\vec{i} + \frac{1}{t^2+1}\vec{j} + \frac{t}{t^2+1}\vec{k}\right) dt$$

Solution:

$$\int_0^1 \left(\frac{1}{t+1}\vec{i} + \frac{1}{t^2+1}\vec{j} + \frac{t}{t^2+1}\vec{k}\right) dt = \ln(t+1)\,\vec{i} + \tan^{-1} t\,\vec{j} + \frac{1}{2}\ln\left(t^2+1\right)\vec{k}\,\Big|_0^1$$

$$= \ln 2\,\vec{i} + \frac{\pi}{4}\vec{j} + \frac{1}{2}\ln 2\,\vec{k}$$

7. If $\vec{r}(t) = (t^4, t, t^2)$, find $\vec{r}'(t)$, $\vec{T}(1)$, $\vec{r}''(t)$ and $\vec{r}'(t) \times \vec{r}''(t)$

Solution: $\vec{r}'(t) = (4t^3, 1, 2t)$, $\vec{r}''(t) = (12t^2, 0, 2)$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{(4t^3, 1, 2t)}{\sqrt{16t^6 + 4t^2 + 1}}$$

so

$$\vec{T}(1) = \frac{(4, 1, 2)}{\sqrt{21}}$$

$\vec{r}'(t) \times \vec{r}''(t) = (2, 16t^3, -12t^2)$

8. If $\overrightarrow{u}(t) = (\sin t, \cos t, t)$ and $\overrightarrow{v}(t) = (t, \cos t, \sin t)$ use chain rule to find

$$\frac{d}{dt}\left[\overrightarrow{u}(t) \cdot \overrightarrow{v}(t)\right]$$

Solution: since $\overrightarrow{u}'(t) = (\cos t, -\sin t, 1)$ and $\overrightarrow{v}'(t) = (1, -\sin t, \cos t)$

$$\begin{aligned}\frac{d}{dt}\left[\overrightarrow{u}(t) \cdot \overrightarrow{v}(t)\right] &= \frac{d\overrightarrow{u}(t)}{dt} \cdot \overrightarrow{v}(t) + \overrightarrow{u}(t) \cdot \frac{d\overrightarrow{v}(t)}{dt} \\ &= (\cos t, -\sin t, 1) \cdot (t, \cos t, \sin t) + (\sin t, \cos t, t) \cdot (1, -\sin t, \cos t) \\ &= 2t\cos t - 2\sin t \cos t + 2\sin t\end{aligned}$$

9. Find the length of the curve

$$\overrightarrow{r}(t) = \cos t\,\overrightarrow{i} + \sin t\,\overrightarrow{j} + \ln\cos t\,\overrightarrow{k} \qquad 0 \le t \le \frac{\pi}{4}$$

Step 1:$\overrightarrow{r}'(t) = -\sin t\,\overrightarrow{i} + \cos t\,\overrightarrow{j} + \frac{\sin t}{\cos t}\overrightarrow{k}$, then we can apply the formula
Step 2:be careful with the domain for t

$$\begin{aligned}L &= \int_0^{\pi/4} \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt \\ &= \int_0^{\pi/4} \sqrt{[-\sin t]^2 + [\cos t]^2 + \left[\frac{\sin t}{\cos t}\right]^2} dt \\ &= \int_0^{\pi/4} \frac{1}{\cos t} dt \\ &= [\ln|\sec x + \tan x|]_0^{\pi/4} = \ln(\sqrt{2}+1)\end{aligned}$$

$\int \sec x\, dx = \ln(\sec x + \tan x)$

10. a) Find the arc length function for the curve measured from the point P in the direction of increasing t and then b) reparametrize the curve with respect to arc length starting from P. c) Find the point **4 units** along the curve (in the direction of increasing t) from P.

$$\overrightarrow{r}(t) = (5-t)\,\overrightarrow{i} + (4t-3)\,\overrightarrow{j} + 3t\,\overrightarrow{k} \qquad P(4,1,3)$$

Solution. *Step 1*: The **initial point** $P(4,1,3)$ corresponds to the parameter value $t = 1$ because we need to let $5 - t = 4$, $4t - 3 = 1$ and $3t = 3$. Then we have $r'(t) = -\vec{i} + 4\vec{j} + 3\vec{k}$

$$\frac{ds}{dt} = \left|\overrightarrow{r'}(t)\right| = \sqrt{1^2 + 4^2 + 3^2} = \sqrt{26}$$

so

$$s = s(t) = \int_1^t \left|\overrightarrow{r'}(u)\right| du = \int_1^t \sqrt{26}du = \sqrt{26}\,(t-1)$$

therefore

$$t = \frac{s}{\sqrt{26}} + 1$$

b) and the reparametrization is

$$\overrightarrow{r}(t(s)) = \left(4 - \frac{s}{\sqrt{26}}\right)\overrightarrow{i} + \left(4\frac{s}{\sqrt{26}} + 1\right)\overrightarrow{j} + 3\left(\frac{s}{\sqrt{26}} + 1\right)\overrightarrow{k}$$

c) when $s = 4$, the point is

$$\overrightarrow{r}(t(4)) = \left(4 - \frac{4}{\sqrt{26}}\right)\overrightarrow{i} + \left(\frac{16}{\sqrt{26}} + 1\right)\overrightarrow{j} + \left(\frac{12}{\sqrt{26}} + 3\right)\overrightarrow{k}$$

11. 1) Find the unit tangent and unit normal vectors $\overrightarrow{T}(t)$ and $\overrightarrow{N}(t)$. 2) Find the curvature.

(a) $\overrightarrow{r}(t) = \left\langle t^2, \sin t - t\cos t, \cos t + t \sin t \right\rangle, t > 0$

(b) $\overrightarrow{r}(t) = \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle, t > 0$

Solution: (a)$\overrightarrow{r}'(t) = \langle 2t, t\sin t, t\cos t\rangle$ $|\overrightarrow{r}'(t)| = \sqrt{5}t$, the **unit tangent vector**

$$\overrightarrow{T}(t) = \frac{\overrightarrow{r}'(t)}{|\overrightarrow{r}'(t)|} = \frac{1}{\sqrt{5}t}\langle 2t, t\sin t, t\cos t\rangle = \frac{1}{\sqrt{5}}\langle 2, \sin t, \cos t\rangle$$

so $\overrightarrow{T}'(t) = \frac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle$ and $\left|\overrightarrow{T}'(t)\right| = \frac{1}{\sqrt{5}}$, the principal **unit normal vector**:

$$\overrightarrow{N}(t) = \frac{\overrightarrow{T}'(t)}{\left|\overrightarrow{T}'(t)\right|} = \langle 0, \cos t, -\sin t\rangle$$

The **curvature**

$$\kappa = \left|\frac{\overrightarrow{T}'(t)}{\overrightarrow{r}'(t)}\right| = \left|\frac{\frac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle}{\sqrt{5}t}\right| = \frac{1}{5t}$$

(b) $\overrightarrow{r}'(t) = \left\langle \sqrt{2}, e^t, -e^{-t} \right\rangle, t > 0$ $|\overrightarrow{r}'(t)| = \sqrt{e^{2t} + 2 + e^{-2t}} = e^t + e^{-t}$the **unit tangent vector**

$$\overrightarrow{T}(t) = \frac{\overrightarrow{r}'(t)}{|\overrightarrow{r}'(t)|} = \frac{1}{e^t + e^{-t}}\left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle$$

so $\overrightarrow{T}'(t) = \frac{1}{(e^t+e^{-t})^2}\left\langle \sqrt{2}\left(e^t - e^{-t}\right), 2, -2\right\rangle$ and $\left|\overrightarrow{T}'(t)\right| = \frac{\sqrt{2}}{(e^t+e^{-t})}$, the principal **unit normal vector**:

$$\overrightarrow{N}(t) = \frac{\overrightarrow{T}'(t)}{\left|\overrightarrow{T}'(t)\right|} = \frac{1}{(e^t + e^{-t})}\left\langle \left(e^t - e^{-t}\right), \sqrt{2}, -\sqrt{2}\right\rangle$$

The **curvature**

$$\kappa = \left|\frac{\overrightarrow{T}'(t)}{\overrightarrow{r}'(t)}\right| = \left|\frac{\frac{1}{(e^t+e^{-t})^2}\left\langle \sqrt{2}\left(e^t - e^{-t}\right), 2, -2\right\rangle}{e^t + e^{-t}}\right| = \frac{\sqrt{2}}{\left(e^t + e^{-t}\right)^2}$$

12. Find the curvature

$$\overrightarrow{r}(t) = \sqrt{6}t^2\overrightarrow{i} + 2t\overrightarrow{j} + 2t^3\overrightarrow{k}$$

Use theorem 10 $\kappa = \frac{|\overrightarrow{r}'(t)\times\overrightarrow{r}''(t)|}{|\overrightarrow{r}'(t)|^3}$.

Solution: $\overrightarrow{r}'(t) = 2\sqrt{6}t\overrightarrow{i} + 2\overrightarrow{j} + 6t^2\overrightarrow{k}$, and $\overrightarrow{r}''(t) = 2\sqrt{6}\overrightarrow{i} + 0\overrightarrow{j} + 12t\overrightarrow{k}$

$$\begin{aligned}
\kappa &= \frac{|\overrightarrow{r}'(t)\times\overrightarrow{r}''(t)|}{|\overrightarrow{r}'(t)|^3} \\
&= \frac{\left|\left(2\sqrt{6}t\overrightarrow{i} + 2\overrightarrow{j} + 6t^2\overrightarrow{k}\right)\times\left(2\sqrt{6}\overrightarrow{i} + 0\overrightarrow{j} + 12t\overrightarrow{k}\right)\right|}{\left|2\sqrt{6}t\overrightarrow{i} + 2\overrightarrow{j} + 6t^2\overrightarrow{k}\right|^3} \\
&= \frac{\sqrt{6}}{2\left(3t^2+1\right)^2}
\end{aligned}$$