Chapter 1 Infinite Sequence and Series

Definition 1.1.1 (Sequence)

A sequence $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, \cdots, a_n, \cdots\}.$$

Each number in the sequence is called a **term** of the sequence. We call a_n the **general term** of the sequence. The subscript n that appears in a_n is called an **index**, and it indicates the order of terms in the sequence. The choice of a starting index is arbitrary, but sequences usually begin with $n = 0$ or $n = 1$.

A sequence may be generated by a **recurrence relation** of the form $a_{n+1} = f(a_n)$, for $n = 1, 2, \cdots,$ where a_1 is given. A sequence may also be defined with an **explicit formula** of the form $a_n = f(n)$, for $n = 1, 2, \cdots$.

Harmonic sequence: $\left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$. The general term is $a_n = \frac{1}{n}$.

Arithmetic sequence: $\{a, a+d, a+2d, \cdots\}$ where a is the first term and d is the common difference. The general term is $a_n = a + (n-1)d$. The recurrence relation is $a_{n+1} = a_n + d,\ a_1 = a$. $\{1, 1, 1, \cdots\}$ and $\{1, 2, 3, \cdots\}$ are examples of arithmetic sequence.

Geometric sequence: $\{a, ar, ar^2, \cdots\}$ where a is the first term and r is the common ratio. The general term is $a_n = ar^{n-1}$. The recurrence relation is $a_{n+1} = ra_n$. $\{1, 2, 4, \cdots\}$ is an example of geometric sequence.

Fibonacci sequence: $\{1, 1, 2, 3, 5, 8, 13, \cdots\}$. The recurrence relation is $a_{n+2} = a_{n+1} + a_n$, where $a_2 = a_1 = 1$.

Example 1.1.2

Use the recurrence relation for $\{a_n\}$ to write the first four terms of the sequence

$$a_1 = 1,\ a_{n+1} = \frac{1+4a_n+\sqrt{1+24a_n}}{16}.$$

Solution

$a_1 = 1$ (given)

$$a_2 = \frac{1+4+\sqrt{1+24}}{16} = \frac{5+\sqrt{25}}{16} = \frac{5}{8} = 0.625$$

$$a_3 = \frac{1+4\left(\frac{5}{8}\right)+\sqrt{1+24\left(\frac{5}{8}\right)}}{16} = \frac{1+\frac{5}{2}+\sqrt{1+15}}{16} = \frac{15}{32} = 0.46875$$

$$a_4 = \frac{1+4\left(\frac{15}{32}\right)+\sqrt{1+24\left(\frac{15}{32}\right)}}{16} = \frac{1+\frac{15}{8}+\sqrt{1+\frac{45}{4}}}{16} = \frac{23/8+7/2}{16} = \frac{51}{128} = 0.3984375.$$

Definition 1.1.3 (Limit of a Sequence)

If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases, then we say $\lim_{n\to\infty} a_n = L$ exists, and the sequence **converges** to L. If the terms of the sequence do not approach a single number as n increases, the sequence has no limit, and the sequence **diverges**.

Here is the formal definition:

For every $\varepsilon > 0$, there exists a positive integer N such that if $n > N$ then $|a_n - L| < \varepsilon$.

Example 1.1.4

Determine the following sequences converge or diverge. If converge, find their limits.

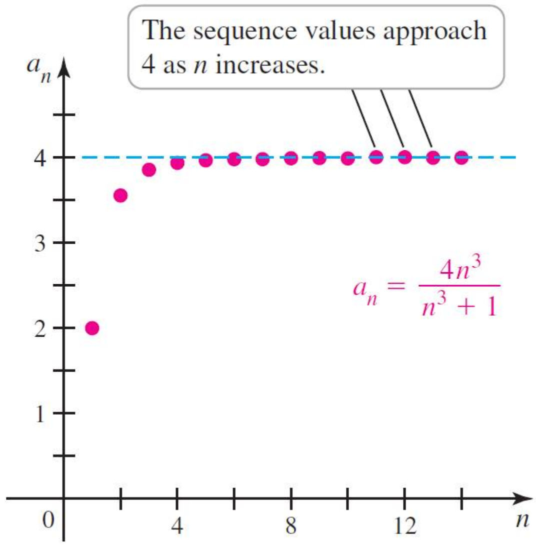
(a) $\dfrac{4n^3}{n^3+1}$

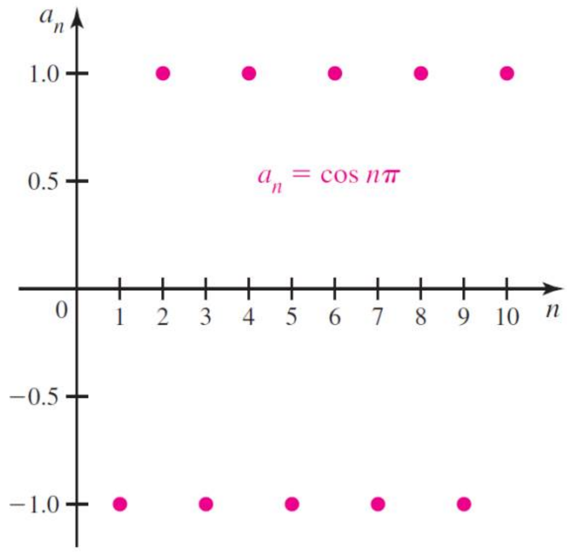
(b) $\cos n\pi$

Solution

(a) $\lim_{n\to\infty}\frac{4n^3}{n^3+1}=\lim_{n\to\infty}\frac{4}{1+n^{-3}}=4.$ The sequence converges to 4.

(b) $\{\cos n\pi\}=\{-1,1,-1,1,\cdots\}.$ The sequence diverges.





Theorem 1.1.5

If $\lim_{x\to\infty} f(x)=L$ and $f(n)=a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L.$

Theorem 1.1.6

Assume that the sequences $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then

(a) $\lim_{n\to\infty}(a_n \pm b_n)=A\pm B$

(b) $\lim_{n\to\infty} ca_n = cA,$ where c is a real number

(c) $\lim_{n\to\infty} a_n b_n = AB$

(d) $\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{A}{B},$ if $B\neq 0.$

(e) If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

(f) $\lim\limits_{n\to\infty} \sqrt[p]{a_n} = \sqrt[p]{\lim\limits_{n\to\infty} a_n}$ if $p > 0$ and all $a_n \geq 0$.

(g) If $\{a_n\}$ is bounded above, that is $a_n \leq M$ for all n, then $A \leq M$. If $\{a_n\}$ is bounded below, that is $N \leq a_n$ for all n, then $N \leq A$.

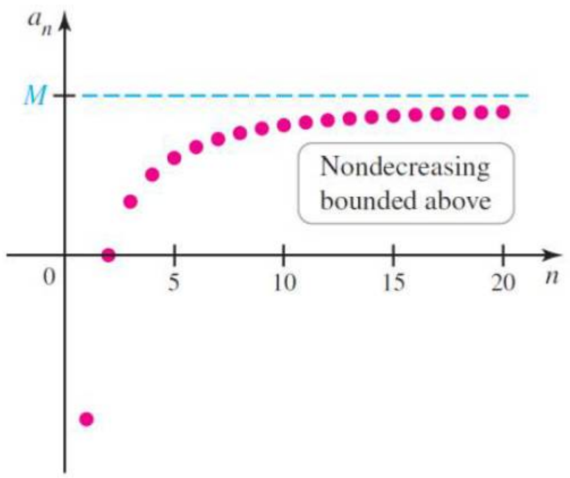
(h) If $a_n \leq b_n$ for all n, then $A \leq B$.

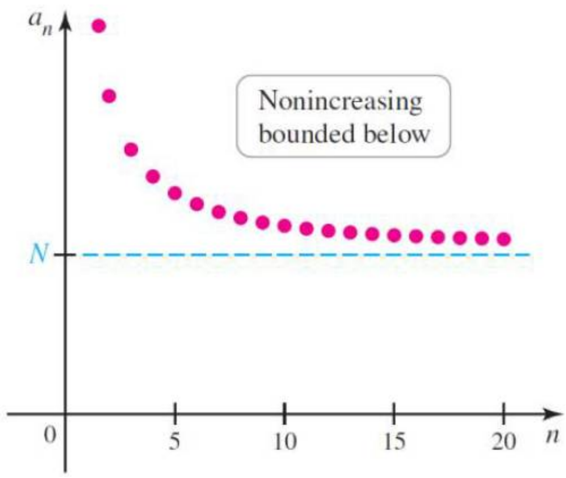
Definition 1.1.7

A sequence is said to be **monotonic increasing** if $a_n \leq a_{n+1}$ for $n = 1, 2, 3, \cdots$. A sequence is said to be **monotonic decreasing** if $a_n \geq a_{n+1}$ for $n = 1, 2, 3, \cdots$. A sequence that is either monotonic increasing or monotonic decreasing is said to be **monotonic**.

Theorem 1.1.8 (Bounded Monotonic Sequence Theorem)

A bounded above (below) monotonic increasing (decreasing) sequence converges.





Theorem 1.1.9

If $\lim\limits_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim\limits_{n\to\infty} f(a_n) = f(L)$.

Example 1.1.10

Consider Example 1.2. Show that limit of $\{a_n\}$ exists and find its limit.

Solution

Clearly, $a_1 \geq \frac{1}{3}$. Suppose $a_n \geq \frac{1}{3}$ for a positive integer n. Then

$$a_{n+1} = \frac{1+4a_n+\sqrt{1+24a_n}}{16} \geq \frac{1+4/3+\sqrt{1+8}}{16} = \frac{1}{3}.$$

By induction, $\{a_n\}$ is bounded below by $\frac{1}{3}$.

$$\begin{aligned} a_n - a_{n+1} &= a_n - \frac{1+4a_n+\sqrt{1+24a_n}}{16} \\ &= \frac{12a_n - 1 - \sqrt{1+24a_n}}{16} \\ &= \frac{(24a_n+1) - 2\sqrt{1+24a_n} - 3}{32} \\ &= \frac{(\sqrt{1+24a_n} - 3)(\sqrt{1+24a_n}+1)}{32} \geq 0 \end{aligned}$$

$\{a_n\}$ is monotonic decreasing and bounded below by $\frac{1}{3}$. By Theorem 1.1.8, limit of $\{a_n\}$ exists. Let $L = \lim_{n\to\infty} a_n$.

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1+4a_n+\sqrt{1+24a_n}}{16} = \frac{1+4\lim_{n\to\infty} a_n + \sqrt{1+24\lim_{n\to\infty} a_n}}{16} = \frac{1+4L+\sqrt{1+24L}}{16}$$

$$16L = 1+4L+\sqrt{1+24L}$$

$$12L - 1 = \sqrt{1+24L}$$

$$144L^2 - 24L + 1 = 1 + 24L$$

$$144L^2 = 48L$$

$$L = \frac{1}{3} \quad \text{or} \quad 0 \text{ (rejected)}$$

Example 1.1.11 (Exponential constant)

Prove that $a_n = \left(1+\frac{1}{n}\right)^n$ is an increasing sequence and bounded above by 3.

Solution

By binomial expansion,

$$
\begin{aligned}
a_n=\left(1+\frac{1}{n}\right)^n &= \binom{n}{0}+\binom{n}{1}\frac{1}{n}+\binom{n}{2}\frac{1}{n^2}+\cdots+\binom{n}{k}\frac{1}{n^k}+\cdots+\binom{n}{n}\frac{1}{n^n}\\
&=1+\frac{1}{1!}\cdot\frac{n}{n}+\frac{1}{2!}\cdot\frac{n(n-1)}{n^2}+\cdots+\frac{1}{k!}\cdot\frac{n(n-1)\cdots(n-k+1)}{n^k}+\cdots+\frac{1}{n!}\cdot\frac{n!}{n^n}\\
&=1+\frac{1}{1!}+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)+\cdots+\frac{1}{n!}\cdot\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)\\
&<1+\frac{1}{1!}+\frac{1}{2!}\cdots+\frac{1}{k!}+\cdots+\frac{1}{n!}\\
&<1+\frac{1}{1!}+\frac{1}{1\cdot 2}\cdots+\frac{1}{k(k-1)}+\cdots+\frac{1}{n(n-1)}\\
&=1+1+\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{k-1}-\frac{1}{k}\right)+\left(\frac{1}{k}-\frac{1}{k+1}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right)\\
&=3-\frac{1}{n}\\
&<3
\end{aligned}
$$

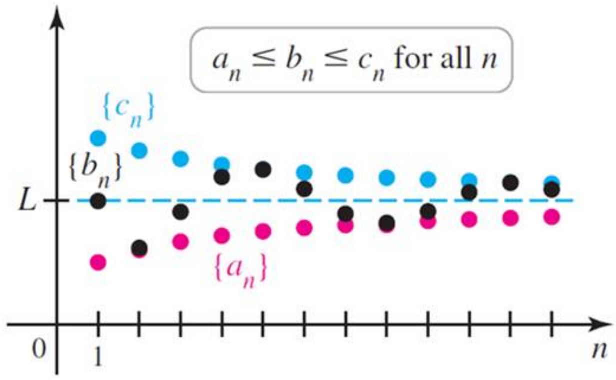
$$
\begin{aligned}
a_{n+1}&=1+\frac{1}{1!}+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\cdots+\frac{1}{k!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right)+\cdots+\\
&\quad\frac{1}{n!}\cdot\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)+\frac{1}{(n+1)!}\cdot\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)\\
&>1+\frac{1}{1!}+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\cdots+\frac{1}{k!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right)+\cdots+\frac{1}{n!}\cdot\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)\\
&>1+\frac{1}{1!}+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)+\cdots+\frac{1}{n!}\cdot\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)=a_n
\end{aligned}
$$

By Bounded Monotonic Sequence Theorem, $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim\limits_{n\to\infty}a_n$ exists. Such limit is called exponential constant e, i.e.,

$$
\begin{aligned}
e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n &=1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{k!}+\cdots\\
&\approx 2.71828182845904523536028747135266249775724709369995\cdots
\end{aligned}
$$

Theorem 1.1.12 (Squeeze Theorem)

Let $\{a_n\},\{b_n\}$, and $\{c_n\}$ be sequences with $a_n\le b_n\le c_n$ for all n. If $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}c_n=L$, then $\lim_{n\to\infty} b_n = L.$



Example 1.1.13

Find the limit of the following sequences.

(a) $\sqrt[n]{3^n+5^n}$

(b) $\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\cdots+\dfrac{1}{\sqrt{n^2+n}}$

(c) $(\sin n)\sin\left(\dfrac{1}{n}\right)$

Solution

(a) Since $5=\sqrt[n]{5^n}\le\sqrt[n]{3^n+5^n}\le\sqrt[n]{5^n+5^n}=5\sqrt[n]{2}\to 5,$ by Squeeze Theorem $\lim_{n\to\infty}\sqrt[n]{3^n+5^n}=5.$

(b) Since $1\leftarrow\dfrac{1}{\sqrt{1+n^{-1}}}=\dfrac{n}{\sqrt{n^2+n}}\le\dfrac{1}{\sqrt{n^2+1}}+\cdots+\dfrac{1}{\sqrt{n^2+n}}\le\dfrac{n}{\sqrt{n^2+1}}=\dfrac{1}{\sqrt{1+n^{-2}}}\to 1,$ by Squeeze Theorem $\lim_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\cdots+\dfrac{1}{\sqrt{n^2+n}}\right)=1.$

(c) Since $0\le\left|(\sin n)\sin\left(\dfrac{1}{n}\right)\right|=|\sin n|\left|\sin\left(\dfrac{1}{n}\right)\right|\le\left|\sin\left(\dfrac{1}{n}\right)\right|\to 0,$ by Squeeze Theorem $\lim_{n\to\infty}\left|(\sin n)\sin\left(\dfrac{1}{n}\right)\right|=0.$ By Theorem 1.1.6(e), $\lim_{n\to\infty}(\sin n)\sin\left(\dfrac{1}{n}\right)=0.$

Definition 1.2.1 (Infinite Series)

Given a set of numbers $\{a_1, a_2, \cdots, a_n, \cdots\}$, the sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

is called an **infinite series**. Its sequence of partial sums $\{S_n\}$ has the terms

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ &\vdots \\ S_n &= \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n, \quad \text{for } n = 1, 2, 3, \cdots \end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series **converges** to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums diverges, the infinite series also **diverges**.

Whether a series converges does not depend on a finite number of terms added to or removed from the series. Specifically, if M is a positive integer, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ both converge or both diverge. However, the value of a convergent series does change of nonzero terms are added or deleted.

Example 1.2.2 (Telescoping Series)

Evaluate the following series.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} + \sqrt{k+1}}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

Solution

(a) $\dfrac{1}{\sqrt{k}+\sqrt{k+1}}=\dfrac{1}{\sqrt{k}+\sqrt{k+1}}\cdot\dfrac{-\sqrt{k}+\sqrt{k+1}}{-\sqrt{k}+\sqrt{k+1}}=\dfrac{-\sqrt{k}+\sqrt{k+1}}{-k+k+1}=-\sqrt{k}+\sqrt{k+1}.$

Incorrect method:

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}+\sqrt{k+1}}&=\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots\\&=\left(-\sqrt{1}+\sqrt{2}\right)+\left(-\sqrt{2}+\sqrt{3}\right)+\left(-\sqrt{3}+\sqrt{4}\right)+\cdots\\&=-1\end{aligned}$$

Misconception: It seems that every term is cancelled except -1. Indeed, if we consider the partial sum S_n, there is a missing term $\sqrt{n+1}$.

Correct method:

$$S_n=\sum_{k=1}^{n}\frac{1}{\sqrt{k}+\sqrt{k+1}}=\left(-\sqrt{1}+\sqrt{2}\right)+\left(-\sqrt{2}+\sqrt{3}\right)+\cdots+\left(-\sqrt{n}+\sqrt{n+1}\right)=\sqrt{n+1}-1$$
$$\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}+\sqrt{k+1}}=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\sqrt{n+1}-1=\infty\quad\text{(diverges)}$$

(b) $$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k(k+1)}&=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)}\\&=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)\\&=\lim_{n\to\infty}\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\\&=\lim_{n\to\infty}\left(\frac{1}{1}-\frac{1}{n+1}\right)\\&=1\end{aligned}$$

Theorem 1.2.3 (Geometric Series)

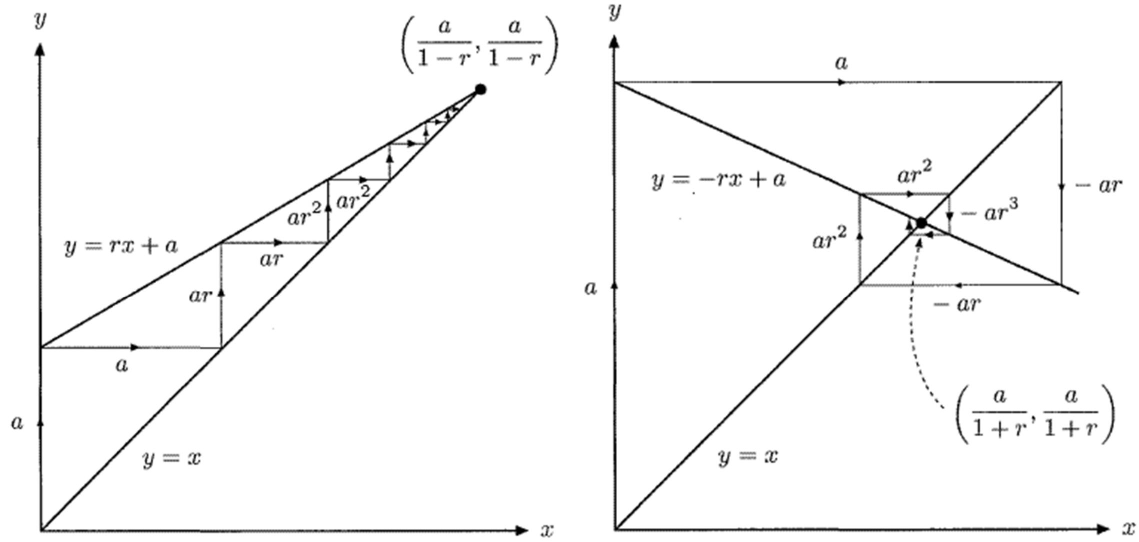
Let $a\neq 0$ and r be real numbers. If $|r|<1$, then $\sum_{k=0}^{\infty}ar^k=\dfrac{a}{1-r}$. If $|r|\geq 1$, then the series diverges.

Solution

$$\begin{array}{rl} S_n = & a + ar + ar^2 + \cdots + ar^{n-1} \\ -)\ rS_n = & \quad\ ar + ar^2 + \cdots + ar^{n-1} + ar^n \\ \hline S_n - rS_n = & a \qquad\qquad\qquad\qquad\qquad\ - ar^n \end{array}$$

We have $S_n = \begin{cases} a\dfrac{1-r^n}{1-r} & r \neq 1 \\ na & r = 1. \end{cases}$ Since $\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \leq -1 \text{ or } r > 1, \end{cases}$

$$\sum_{k=0}^{\infty} ar^k = \begin{cases} \dfrac{a}{1-r} & |r| < 1 \\ \text{does not exist} & \text{otherwise.} \end{cases}$$



$a + ar + ar^2 + \cdots = \frac{a}{1-r}, \quad 0 < r < 1.$ $a - ar + ar^2 - \cdots = \frac{a}{1+r}, \quad 0 < r < 1.$

Theorem 1.2.4 (Divergence Test)

If $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Equivalently, if $\lim_{k\to\infty} a_k \neq 0$ or does not exist, then the series diverges.

Proof

Let $S = \sum_{k=1}^{\infty} a_k$ and $S_n = \sum_{k=1}^{n} a_k$. Then $\lim_{k\to\infty} a_k = \lim_{k\to\infty}(S_k - S_{k-1}) = S - S = 0$.

Example 1.2.5

$\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+1}$ diverges as $\lim_{n\to\infty}(-1)^n \frac{n}{n+1}$ does not exist. Theorem 1.2.4 cannot be used to determine convergence. $\lim_{n\to\infty}\frac{1}{n}=0$ does not mean that $\sum_{n=1}^{\infty}\frac{1}{n}$ converges. Indeed, it diverges.

$$\begin{aligned}
&1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots \\
&\geq 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots \\
&=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots \\
&=\infty \quad \text{(diverges)}
\end{aligned}$$

Theorem 1.2.6

1. Suppose $\sum a_k$ converges to A and let c be a real number. The series $\sum ca_k$ converges and $\sum ca_k = c\sum a_k = cA$.

2. Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum (a_k \pm b_k)$ converges and $\sum (a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$.

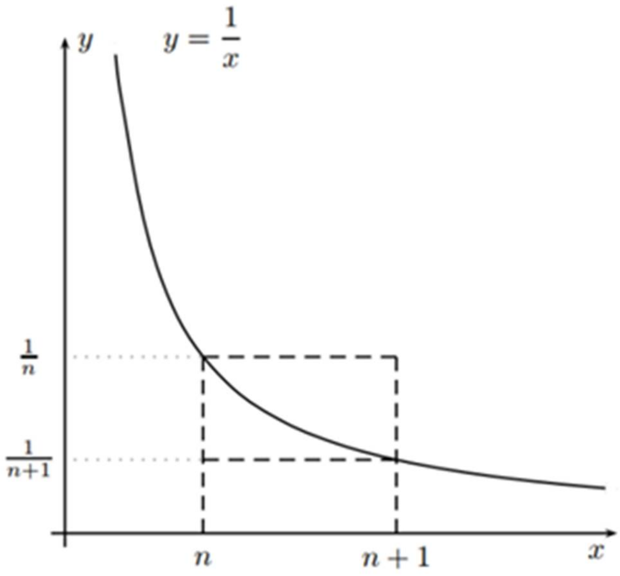
Example 1.2.7

(a) Prove that $\frac{1}{n+1} < \ln(n+1)-\ln n < \frac{1}{n}$.

(b) Let $a_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln(n+1)$, show that the sequence $\{a_n\}$ is increasing and bounded above by 1.

(c) Hence find the value of $\lim_{n\to\infty}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}\right)$.

Solution



(a) From the above picture, $\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\,dx = \ln(n+1) - \ln n < \frac{1}{n}$.

(b) $$\begin{aligned} a_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1) \\ &< 1 + (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln n - \ln(n-1)) - \ln(n+1) \\ &= 1 + \ln n - \ln(n+1) < 1 \end{aligned}$$

$$a_n - a_{n-1} = \frac{1}{n} - \ln(n+1) + \ln n > 0$$

By Theorem 1.1.8, the sequence $\{a_n\}$ converges to a number γ, known as Euler constant.

$$\gamma = \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1)\right) \approx 0.5772156649$$

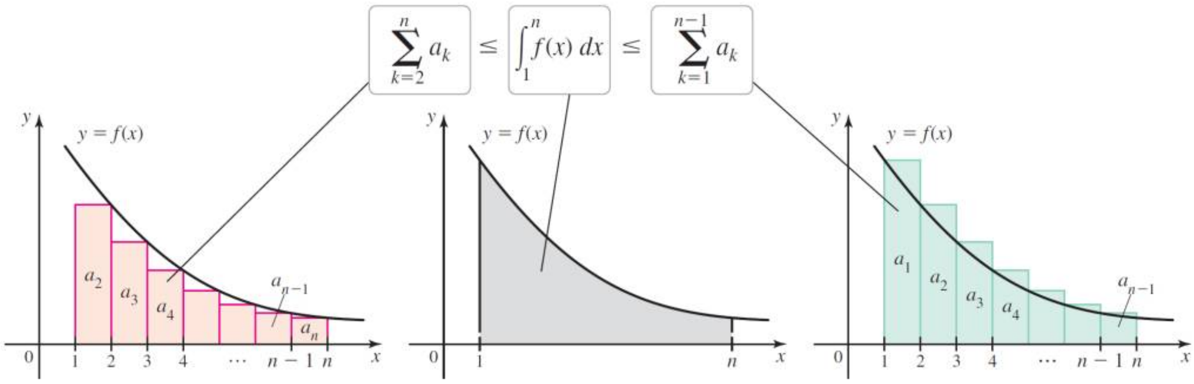
(c) $$\begin{aligned} &\lim_{n\to\infty}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}\right) \\ &= \lim_{n\to\infty}\left[\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)\right] \\ &= \lim_{n\to\infty}\left[\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\right] \\ &= \lim_{n\to\infty}\left[\left(a_{2n} + \ln(2n+1)\right) - \left(a_n + \ln(n+1)\right)\right] \\ &= \gamma - \gamma + \lim_{n\to\infty}\ln\left(\frac{2n+1}{n+1}\right) \\ &= \ln 2. \end{aligned}$$

Theorem 1.3.1 (Integral Test)

Suppose f is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for k = 1, 2, 3,…. Then

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \int_1^{\infty} f(x)dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is not, in general, equal to the value of the series.



When we use the integral test, it is not necessary to start the series or the integral at n = 1. For instance, in testing the series

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} \quad \text{we use} \quad \int_2^{\infty} \frac{1}{\ln x} dx.$$

	Sequences / Series	**Functions**
Independent variable	n	x
Dependent variable	a_n	$f(x)$
Domain	Integers e.g., $n = 0, 1, 2, 3, \ldots$	Real numbers e.g., $\{x: x \geq 0\}$
Accumulation	Sums	Integrals
Accumulation over a finite interval	$\sum_{k=0}^{n} a_k$	$\int_0^n f(x)\,dx$
Accumulation over an infinite interval	$\sum_{k=0}^{\infty} a_k$	$\int_0^{\infty} f(x)\,dx$

Example 1.3.2

Use the integral test to determine whether the following series converge.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

(b) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

(c) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Solution

$\frac{d}{dx}\left(\frac{x}{x^2+1}\right) = \frac{(x^2+1)-2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} \le 0$ on $[1,\infty)$. Therefore $\frac{x}{x^2+1}$ is continuous, positive, and decreasing on $[1,\infty)$. Furthermore $\frac{1}{x^2+1}$ and $\frac{1}{x \ln x}$ are continuous, positive, and decreasing on $[1,\infty)$ and $[2,\infty)$ respectively.

(a) $\int_1^{\infty} \frac{1}{x^2+1} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1} dx = \lim_{t\to\infty} \tan^{-1} x \Big|_1^t = \lim_{t\to\infty} (\tan^{-1} t - \tan^{-1} 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

By Integral Test, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent.

(b) $\int_1^{\infty} \frac{x}{x^2+1} dx = \lim_{t\to\infty} \int_1^t \frac{x}{x^2+1} dx = \frac{1}{2} \lim_{t\to\infty} \ln(x^2+1)\Big|_1^t = \frac{1}{2} \lim_{t\to\infty} (t^2+1) - \frac{1}{2}\ln 2 = \infty$

By Integral Test, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is divergent.

(c) $\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x \ln x} dx = \lim_{t\to\infty} \int_1^t \frac{1}{\ln x} d \ln x = \lim_{t\to\infty} \ln(\ln x)\Big|_2^t = \lim_{t\to\infty} (\ln(\ln t) - \ln(\ln 2)) = \infty$

By Integral Test, the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is divergent.

Theorem 1.3.3 (Convergence of the p-Series)

The p-seires $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges, for $p > 1$, and diverges $p \le 1$.

The above theorem follows from Integral test and Example 5.50 (p-test) in Calculus I.

Theorem 1.3.4

Let f be a continuous, positive, decreasing function, for $x \ge 1$, and let $a_k = f(k)$, for $k = 1$, 2, 3,…. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series and let $S_n = \sum_{k=1}^{n} a_k$ be the sum of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies

$$R_n \le \int_n^\infty f(x)dx.$$

Furthermore, the exact value of the series is bounded as follows:

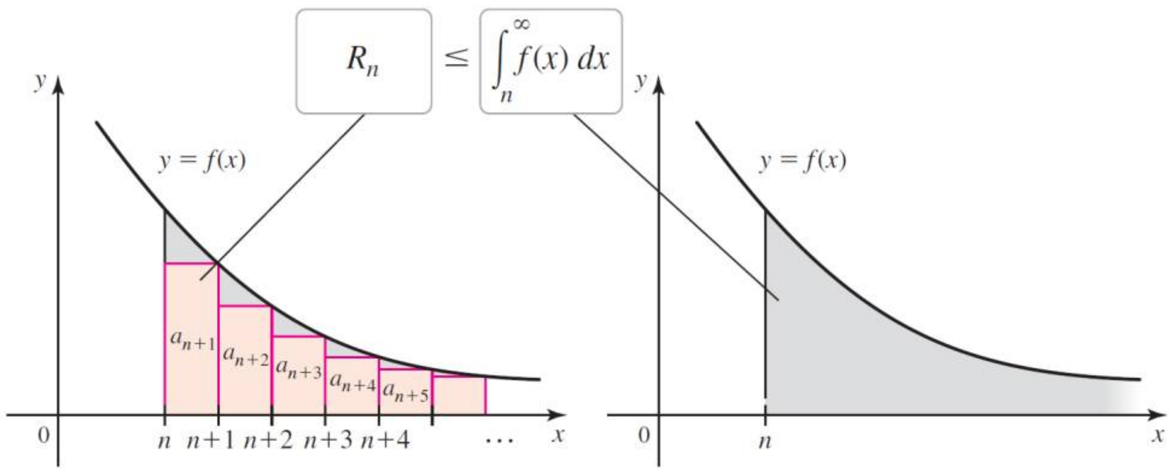
$$S_n + \int_{n+1}^\infty f(x)dx \le \sum_{k=1}^{\infty} a_k \le S_n + \int_n^\infty f(x)dx.$$

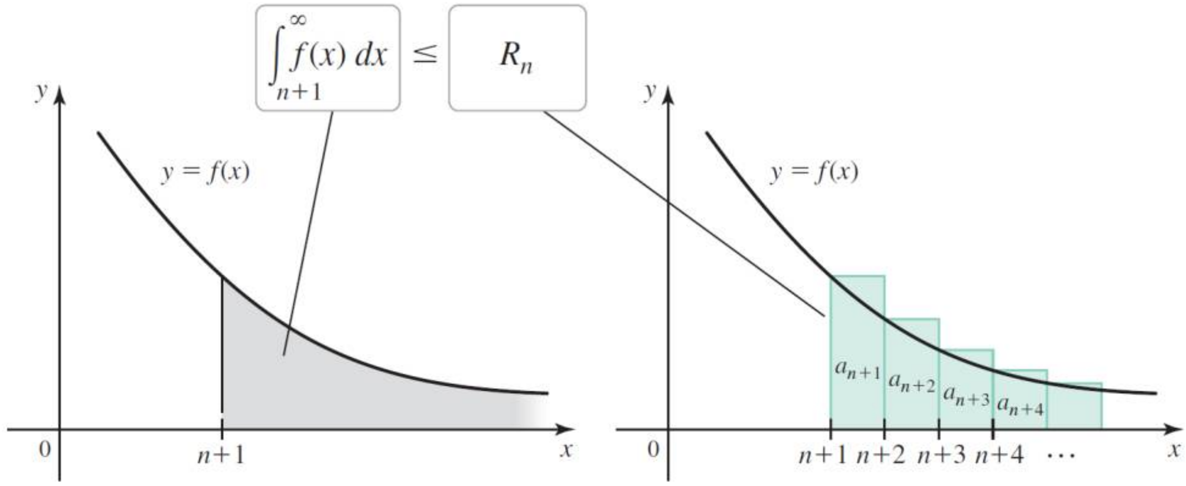
Proof

From the graph below, we have $R_n \le \int_n^\infty f(x)dx.$

Furthermore,

$$\int_{n+1}^\infty f(x)dx \le R_n \le \int_n^\infty f(x)dx$$
$$\int_{n+1}^\infty f(x)dx \le S - S_n \le \int_n^\infty f(x)dx$$
$$S_n + \int_{n+1}^\infty f(x)dx \le S \le S_n + \int_n^\infty f(x)dx$$





Example 1.3.5

(a) How many terms of the convergent p-series $S = \sum_{k=1}^{\infty} \frac{1}{k^2}$ must be summed to obtain an approximation that is within 10^{-3} of the exact value of the series?

(b) Find the approximation to the series using 50 terms of the series.

Solution

(a) $R_n \leq \int_n^{\infty} f(x)dx = \int_n^{\infty} \frac{1}{x^2} dx = \frac{1}{n}$. To ensure $R_n \leq 10^{-3}$, we must choose n so that $1/n \leq 10^{-3}$, which implies that $n \geq 1000$.

(b) The series is bounded as follows:

$$S_{50} + \int_{51}^{\infty} \frac{dx}{x^2} \leq S \leq S_{50} + \int_{50}^{\infty} \frac{dx}{x^2}$$
$$S_{50} + \frac{1}{51} \leq S \leq S_{50} + \frac{1}{50}.$$

Since $S_{50} \approx 1.625133$, we have $1.644741 < S < 1.645133$. Taking the average of these 2 bounds as our approximation of S, we find that $S \approx 1.644937$. (The actual value of S is $\frac{\pi^2}{6} \approx 1.644934067$.)

Definition 1.4.1

An **alternating series** is a series whose terms are alternately positive and negative. That is,

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + a_3 - a_4 + \cdots, \quad a_k > 0$$

Consider the series

$$S = 1-1+1-1+1-1+\cdots$$

Student A claims that

$$S = (1-1)+(1-1)+(1-1)+\cdots = 0.$$

Student B claims that

$$S = 1+(-1+1)+(-1+1)+(-1+1)+\cdots = 1.$$

Student C claims that

$$\begin{aligned} S &= 1-(1-1+1-1+1-1+\cdots) = 1-S \\ 2S &= 1 \\ S &= \frac{1}{2}. \end{aligned}$$

Q. Are they correct?

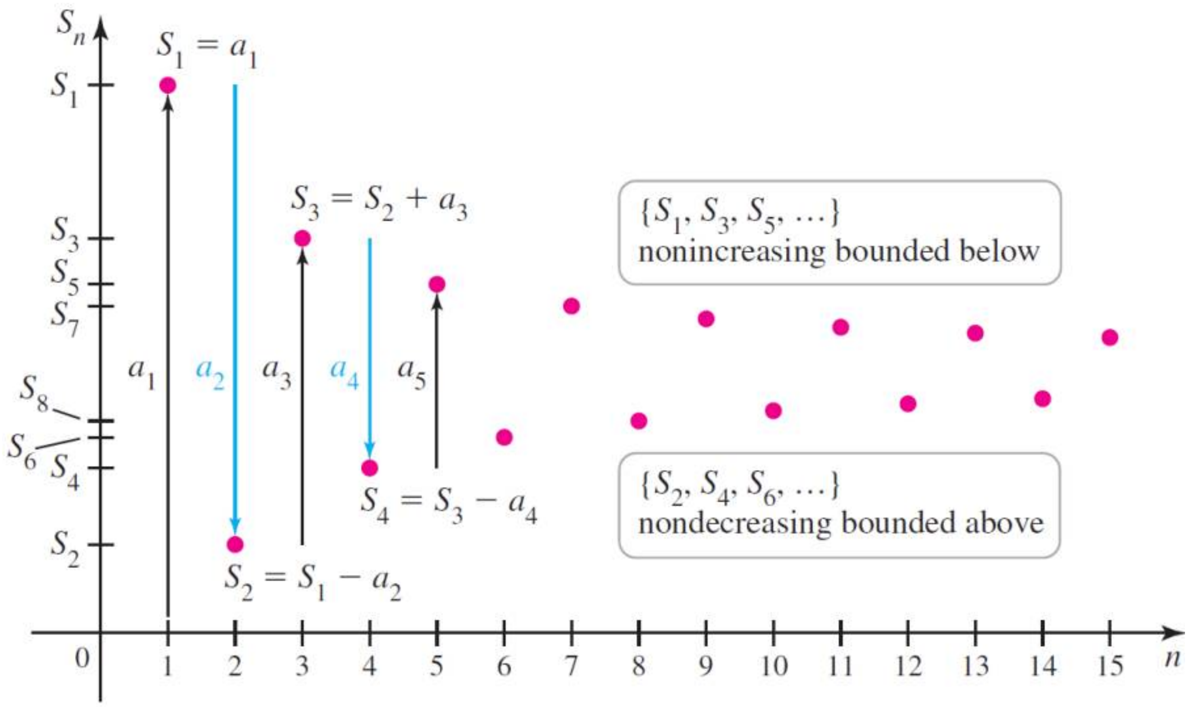
A. None of them are correct. Consider the partial sum:

$$S_k = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

The series S diverges.

Theorem 1.4.2 (Alternating Series Test)

If the alternating series $\sum(-1)^{k+1}a_k$, satisfies $a_k \geq a_{k+1} > 0$ for all k and $\lim_{k\to\infty} a_k = 0$ then the series $\sum(-1)^{k+1}a_k$, is convergent.



Proof

From the above figure, $\{S_1, S_3, S_5, \cdots\}$ form a non-increasing sequence that is bounded below by S_2. On the other hand, $\{S_2, S_4, S_6, \cdots\}$ form a non-decreasing sequence that is bounded above by S_1. By Bounded Monotonic Sequence Theorem, $\lim_{k\to\infty} S_{2k-1} (= L)$ and $\lim_{k\to\infty} S_{2k} (= L')$ exist. Since

$$\underbrace{\lim_{k\to\infty} S_{2k}}_{L} = \underbrace{\lim_{k\to\infty} S_{2k-1}}_{L'} - \underbrace{\lim_{k\to\infty} a_{2k}}_{0},$$

$L = L'$. Hence $\lim_{k\to\infty} S_k = L$.

Example 1.4.3

Use Alternating Series Test to determine whether the following series converge.

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

(b) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

(c) $1-\frac{2}{3}+\frac{3}{5}-\frac{4}{7}+\cdots$

Solution

(a) Since $\frac{1}{k}\geq\frac{1}{k+1}>0$ for all k and $\lim_{k\to\infty}\frac{1}{k}=0$, $1-\frac{2}{3}+\frac{3}{5}-\frac{4}{7}+\cdots$ converges by Alternating Series Test. Its value is ln2 (Example 1.2.7).

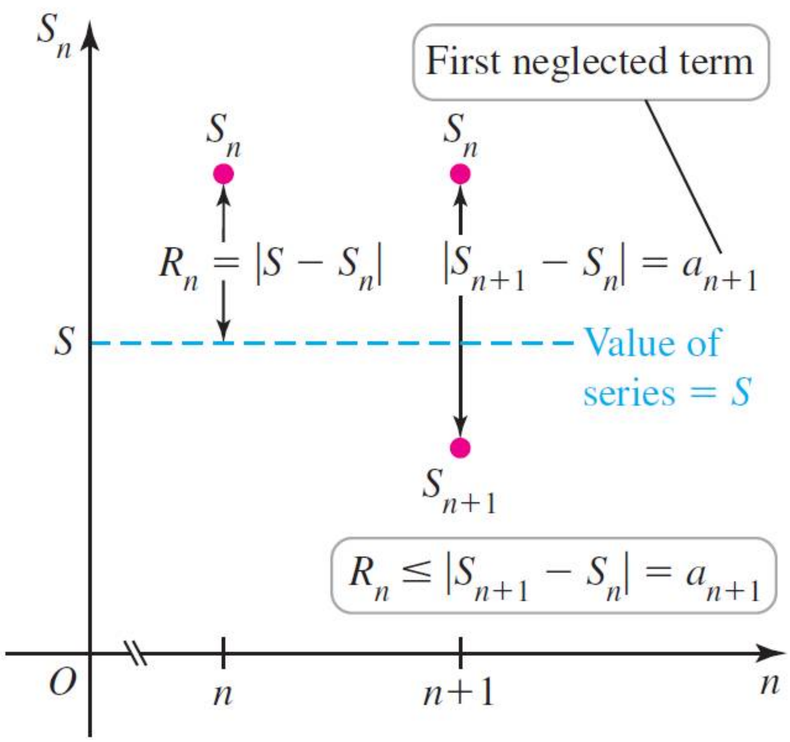
(b) Since $\frac{1}{k^2}\geq\frac{1}{(k+1)^2}>0$ for all k and $\lim_{k\to\infty}\frac{1}{k^2}=0$, $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$ converges by Alternating Series Test.

(c) Since $\lim_{k\to\infty}\frac{k}{2k-1}=\frac{1}{2}\neq 0$, $1-\frac{2}{3}+\frac{3}{5}-\frac{4}{7}+\cdots$ diverges by Divergence Test.

Theorem 1.4.4

Suppose the alternating series $\sum(-1)^{k+1}a_k$, satisfies $a_k\geq a_{k+1}>0$ for all k and $\lim_{k\to\infty}a_k=0$.

Let $R_n=|S-S_n|$ be the remainder. Then $R_n\leq a_{n+1}$.



Example 1.4.5

(a) How many terms of the series $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ are required to approximate the value of the series with a remainder less than 10^{-6}?

(b) If $n = 9$ terms of the series $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$ are summed, what is the maximum error committed in approximating the value of the series (which is $1 - e^{-1}$)?

Solution

(a) The series is expressed as the sum of the first n terms plus the remainder:

$$\ln 2 = \underbrace{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n}}_{S_n = \text{ the sum of the first } n \text{ terms}} + \underbrace{\frac{(-1)^{n+2}}{n+1} + \cdots}_{R_n = |S - S_n|}$$

$$R_n = |S - S_n| \le a_{n+1} = \frac{1}{n+1}.$$

To ensure that the remainder is less than 10^{-6}, we require that

$$a_{n+1} = \frac{1}{n+1} < 10^{-6}, \quad \text{or} \quad n+1 > 10^{6}.$$

Therefore, it takes 1 million terms of the series to approximate $\ln 2$ with a remainder less than 10^{-6}.

(b) The series is expressed as the sum of the first 9 terms plus the remainder:

$$1 - e^{-1} = \underbrace{1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{1}{9!}}_{S_9 = \text{ the sum of the first 9 terms}} \underbrace{- \frac{1}{10!} - \cdots}_{R_9 = |S - S_9|}$$

$$R_9 = |S - S_9| \le a_{10} = \frac{1}{10!} \approx 2.8 \times 10^{-7}.$$

$$\begin{aligned} \text{Actual Error} &= \left| \left(1 - e^{-1}\right) - \left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{1}{9!}\right) \right| \\ &\approx |0.632120559 - 0.632120811| \\ &= 2.5 \times 10^{-7}. \end{aligned}$$

The actual error satisfies the given inequality.

Theorem 1.5.1 (Comparison Test)

Let $\sum a_k$ and $\sum b_k$ be series with positive terms with $0 < a_k \le b_k$ for all k. If $\sum b_k$ converges, then $\sum a_k$ converges. In other words, if $\sum a_k$ diverges, then $\sum b_k$ diverges.

Proof

Let $s_n = \sum_{k=1}^{n} a_k$. If $\sum b_k$ converges, then $s_n = \sum_{k=1}^{n} a_k \le \sum_{k=1}^{n} b_k \le \sum_{k=1}^{\infty} b_k$ and $s_{n+1} = s_n + a_n \ge s_n$.

By Theorem 1.1.8 (Bounded Monotonic Sequence Theorem), $\sum a_k$ converges.

In using the Comparison Test we must, of course, have some known series for the purpose of comparison. Most of the time we use p-series $\sum \frac{1}{n^p}$.

Example 1.5.2

Determine whether the following series converge.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

(b) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$

Solution

(a) Clearly $0 < \frac{1}{\sqrt{n^3+1}} \le \frac{1}{n^{\frac{3}{2}}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges by p-test. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges by Comparison Test.

(b) Let us compare $(\ln n)^2$ with n.

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2\ln x}{x} = \lim_{x\to\infty} \frac{2}{x} < 1 \quad \Rightarrow (\ln n)^2 < n \quad \Rightarrow \frac{1}{n} < \frac{1}{(\ln n)^2} \quad \text{for large } n.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test, $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ diverges by Comparison Test.

Theorem 1.6.1 (Limit Comparison Test)

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let

$$\lim_{k\to\infty}\frac{a_k}{b_k}=L.$$

(a) If $0<L<\infty$, then $\sum a_k$ and $\sum b_k$ either both converge and both diverge.

(b) If $L=0$ and $\sum b_k$ converges, then $\sum a_k$ converges.

(c) If $L=\infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Proof (outline)

If $0<L<\infty$, $a_k \approx Lb_k$ for large values k. Then

$$a_k+a_{k+1}+a_{k+2}+\cdots \approx L\left(b_k+b_{k+1}+b_{k+2}+\cdots\right)$$

Thus, $\sum a_k$ and $\sum b_k$ either both converge or both diverge.

If $L=0$, b_k is large relative to a_k. If $\sum b_k$ converges, then $\sum a_k$ converges.

If $L=\infty$, b_k is small relative to a_k. If $\sum b_k$ diverges, then $\sum a_k$ diverges.

Remark:

(i) $L=0$ and $\sum b_k$ diverges does not imply $\sum a_k$ diverges. For instances, consider $a_k=\frac{1}{k^2}$ and $b_k=\frac{1}{k}$. Then $\lim_{k\to\infty}\frac{a_k}{b_k}=\lim_{k\to\infty}\frac{k}{k^2}=0$. $\sum b_k$ diverges but $\sum a_k$ converges by p-test.

(ii) $L=\infty$ and $\sum b_k$ converges does not imply $\sum a_k$ converges. For instances, consider $a_k=\frac{1}{k}$ and $b_k=\frac{1}{k^2}$. Then $\lim_{k\to\infty}\frac{a_k}{b_k}=\lim_{k\to\infty}\frac{k^2}{k}=0$. $\sum b_k$ converges but $\sum a_k$ diverges by p-test.

Theorem 1.6.2 (Growth Rates of Sequences)

The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\} << \{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$:

$$\{(\ln n)^q\} << \{n^p\} << \{n^p (\ln n)^r\} << \{n^{p+s}\} << \{a^n\} << \{n!\} << \{n^n\}.$$

The ordering applies for positive real number p, q, r, s, and $a > 1$.

Example 1.6.3

Use the Limit Comparison Test to determine whether the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2n+3}{4n^4+5n^2+6}$

(b) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Solution

(a) Let $a_n = \dfrac{n^2+2n+3}{4n^4+5n^2+6}$ and $b_n = \dfrac{1}{n^2}$. Since $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4+2n^3+3n^2}{4n^4+5n^2+6} = \frac{1}{4}$ and $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2+2n+3}{4n^4+5n^2+6}$ converges by Limit Comparison Test.

(b) Let $a_n = \sin\left(\dfrac{1}{n}\right)$ and $b_n = \dfrac{1}{n}$. Since $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$ and $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges by Limit Comparison Test.

(c) Let $a_n = \dfrac{1}{n^{1+\frac{1}{n}}}$ and $b_n = \dfrac{1}{n}$. Then $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \lim\limits_{n\to\infty}\dfrac{n^{-1-\frac{1}{n}}}{n^{-1}} = \lim\limits_{n\to\infty} n^{-\frac{1}{n}}$.

Let $f(x) = x^{-\frac{1}{x}}$. We have

$$\begin{aligned}\ln\left(\lim_{x\to\infty} f(x)\right) &= \lim_{x\to\infty}\left(\ln f(x)\right)\\ &= \lim_{x\to\infty}\frac{-\ln x}{x} \quad \text{by l'Hospital's rule}\\ &= \lim_{n\to\infty}\frac{-1}{x} = 0.\end{aligned}$$

Therefore $\lim\limits_{x\to\infty} x^{-\frac{1}{x}} = \lim\limits_{x\to\infty} f(x) = e^0 = 1$. Since $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$ diverges by the p-test, it follows that $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{1+\frac{1}{n}}}$ also diverges by limit comparison test.

(d) Let $a_n = \dfrac{1}{\sqrt{n}\ln n}$ and $b_n = \dfrac{1}{n}$. Then $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \lim\limits_{n\to\infty}\dfrac{\sqrt{n}}{\ln n}$. Let $f(x) = \dfrac{\sqrt{x}}{\ln x}$. We have

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty}\frac{\sqrt{x}}{\ln x}.\\ &= \lim_{x\to\infty}\frac{0.5x^{-0.5}}{x^{-1}} \quad \text{by l'Hospital's rule}\\ &= \lim_{n\to\infty} 0.5x^{0.5} = \infty.\end{aligned}$$

Since $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$ diverges by the p-test, it follows that $\sum\limits_{n=2}^{\infty}\dfrac{1}{\sqrt{n}\ln n}$ also diverges by limit comparison test.

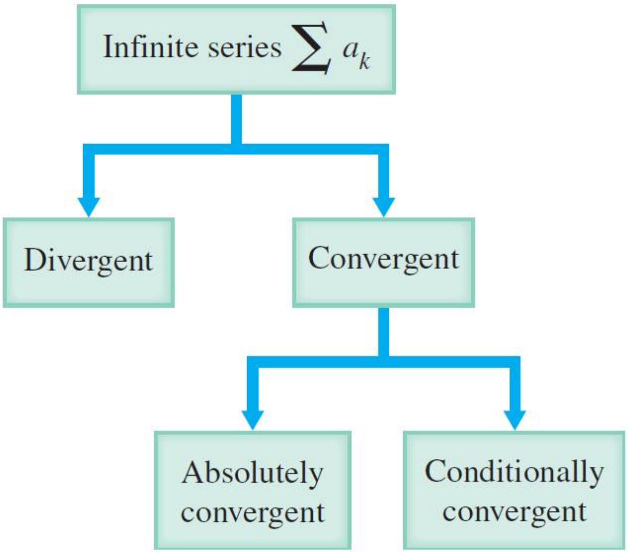
(e) Let $a_n = \dfrac{\ln n}{n^2}$ and $b_n = \dfrac{1}{n\sqrt{n}}$. Then $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \lim\limits_{n\to\infty}\dfrac{\ln n}{\sqrt{n}}$. Let $g(x) = \dfrac{\ln x}{\sqrt{x}}$. We have

$$\lim_{x\to\infty} g(x) \overset{\text{(d)}}{=} \lim_{x\to\infty}\frac{1}{f(x)} = 0.$$

Since $\sum\limits_{n=1}^{\infty}\dfrac{1}{n\sqrt{n}}$ converges by the p-test, it follows that $\sum\limits_{n=1}^{\infty}\dfrac{\ln n}{n^2}$ also converges by limit comparison test.

Definition 1.7.1

Assume the infinite series $\sum a_k$ converges. The series $\sum a_k$ **converges absolutely** if the series $\sum |a_k|$ converges. Otherwise, the series $\sum a_k$ **converges conditionally**.



Example 1.7.2

Determine whether the following series diverge, converge absolutely, or converge conditionally.

(a) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

(b) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$

(c) $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{1}{3n}+\cdots$

Solution

(a) By Example 1.4.3 and p-test, $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ converges conditionally.

(b) By Example 1.4.3 and p-test, $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$ converges absolutely.

(c) $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{1}{3n}=1+\frac{1}{2}+\cdots+\frac{1}{3n}-2\left(\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{3n}\right)$

$$\geq\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\frac{2}{3}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$$=\frac{1}{3}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\to\infty$$

$1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{1}{3n}+\cdots$ diverges by comparison test

Theorem 1.7.3

If $\sum|a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). If $\sum a_k$ diverges, then $\sum|a_k|$ diverges

Proof

Since $-|a_k|\leq a_k\leq|a_k|$, it follows that $0\leq|a_k|+a_k\leq 2|a_k|$. Convergence of $\sum|a_k|$ implies convergence of $\sum 2|a_k|$. By Comparison Test, $\sum\left(|a_k|+a_k\right)$ converges. Result follows from

$$\sum a_k=\sum\left(|a_k|+a_k-|a_k|\right)=\underbrace{\sum\left(|a_k|+a_k\right)}_{\text{converges}}-\underbrace{\sum|a_k|}_{\text{converges}}.$$

Example 1.7.4

Determine whether the series $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ is convergent or divergent.

Solution

This series has both positive and negative terms, but it is not alternating. Since $\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right|\leq\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right|$ converges by comparison test. That means $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ converge absolutely. Hence $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ is convergent.

If we rearrange the order of the terms in a finite sum, then of course the value of the sum remains unchanged. But this is not always the case for an infinite series. By a **rearrangement** of an infinite series $\sum a_k$ we mean a series obtained by simply changing the order of the terms. For instance, a rearrangement of $\sum a_k$ could start as follows:

$$a_1 + a_3 + a_2 + a_4 + a_5 + a_7 + a_6 + a_8 + \cdots$$

However, any conditionally convergent series can be rearranged to give a different sum.

To illustrate this fact let's consider the alternating harmonic series

$$\begin{aligned}
\ln 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \cdots \\
&= \underbrace{1 - \frac{1}{2}}_{\frac{1}{2}} - \frac{1}{4} + \underbrace{\frac{1}{3} - \frac{1}{6}}_{\frac{1}{6}} - \frac{1}{8} + \underbrace{\frac{1}{5} - \frac{1}{10}}_{\frac{1}{10}} - \frac{1}{12} + \cdots \qquad (+--+--+--\cdots) \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right) \\
&= \frac{1}{2}\ln 2
\end{aligned}$$

Theorem 1.7.5

If $\sum a_k$ is an absolutely convergent series with sum s, then any rearrangement of $\sum a_k$ has the same sum s.

Theorem 1.7.6

If $\sum a_k$ is a conditionally convergent series and r is any real number, then there is a rearrangement of $\sum a_k$ that has a sum equal to r.

Theorem 1.8.1 (Ratio Test)

Let $\sum a_k$ be an infinite series with nonzero terms and let $r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$.

1. If $0 \le r < 1$, the series converges.

2. If $r > 1$ (including $r = \infty$), the series diverges.

3. If $r = 1$, the test is inconclusive.

Proof (outline)

Suppose the limit r exists. As N gets large and $\left|\frac{a_{k+1}}{a_k}\right|$ approaches r for $k > N$, we have $|a_{k+1}| \approx r|a_k|$. The tail of the series,

$$\begin{aligned}|a_N| + |a_{N+1}| + |a_{N+2}| + \cdots &\approx |a_N| + r|a_N| + r^2|a_N| + \cdots \\ &= |a_N|\left(1 + r + r^2 + \cdots\right)\end{aligned}$$

behaves like a geometric series with ratio r, which is convergence for $0 \le r < 1$. The series $\sum a_k$ converges absolutely and therefore converges.

If $r > 1$, then, as N gets large and $|a_{k+1}| \approx r|a_k| \ge |a_k| \ge \cdots \ge |a_N| > 0$ for $k > N$. Therefore, $\lim_{k\to\infty} a_k \ne 0$. By Theorem 1.2.4, the series $\sum a_k$ diverges.

Example 1.8.2

Consider $\sum_{k=1}^{\infty}\frac{1}{k^2}$. Clearly $\left(\frac{1}{(k+1)^2}\right)\Big/\left(\frac{1}{k^2}\right) = \frac{k^2}{(k+1)^2} \to 1$. $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by p-test.

Consider $\sum_{k=1}^{\infty}\frac{1}{k}$. Clearly $\left(\frac{1}{k+1}\right)\Big/\left(\frac{1}{k}\right) = \frac{k}{k+1} \to 1$. $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges by p-test.

That is why Ratio test is inconclusive if $r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1$.

Example 1.8.3

Use Ratio test to determine whether the following series converge.

(a) $\sum_{k=1}^{\infty} \frac{2^k}{k!}$

(b) $\sum_{k=1}^{\infty} \frac{(-2)^k}{k^2}$

(c) $\sum_{k=1}^{\infty} \frac{k!}{k^k}$

(d) $\sum_{k=1}^{\infty} \sin\left(\frac{\pi}{2^k}\right)$

Solution

(a) $\frac{a_{k+1}}{a_k} = \left(\frac{2^{k+1}}{(k+1)!}\right) \bigg/ \left(\frac{2^k}{k!}\right) = \frac{2}{k} \to 0 < 1$

By Ratio test, $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ converges.

(b) $\left|\frac{a_{k+1}}{a_k}\right| = \left(\frac{2^{k+1}}{(k+1)^2}\right) \bigg/ \left(\frac{2^k}{k^2}\right) = \frac{2k^2}{(k+1)^2} \to 2 > 1$

By Ratio test, $\sum_{k=1}^{\infty} \frac{(-2)^k}{k^2}$ diverges.

(c) $\frac{a_{k+1}}{a_k} = \left(\frac{(k+1)!}{(k+1)^{k+1}}\right) \bigg/ \left(\frac{k!}{k^k}\right) = \frac{k^k}{(k+1)^k} = \left(1+\frac{1}{k}\right)^{-k} \to e^{-1} < 1$

By Ratio test, $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ converges.

(d) $\frac{a_{k+1}}{a_k} = \sin\left(\frac{\pi}{2^{k+1}}\right) \bigg/ \sin\left(\frac{\pi}{2^k}\right) = \frac{\sin \frac{\pi}{2^{k+1}}}{2\sin \frac{\pi}{2^{k+1}} \cos \frac{\pi}{2^{k+1}}} = \frac{1}{2\cos \frac{\pi}{2^{k+1}}} \to \frac{1}{2} < 1$

By Ratio test, $\sum_{k=1}^{\infty} \sin\left(\frac{\pi}{2^k}\right)$ converges.

Theorem 1.9.1 (Root Test)

Let $\sum a_k$ be an infinite series and let $L = \lim_{k\to\infty} \sqrt[k]{|a_k|}$.

1. If $0 \le L < 1$, the series converges.
2. If $L > 1$ (including $L = \infty$), the series diverges.
3. If $L = 1$, the test is inconclusive.

Example 1.8.2

$\lim_{x\to\infty} \ln x^{1/x} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0 \quad \Rightarrow \quad \lim_{x\to\infty} x^{1/x} = 1$. Therefore $\lim_{k\to\infty} \frac{1}{\sqrt[k]{k}} = \lim_{k\to\infty} \frac{1}{\sqrt[k]{k^2}} = 1$.

However $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges by p-test.

That is why Root test is inconclusive if $L = \lim_{k\to\infty} \sqrt[k]{|a_k|} = 1$.

Example 1.9.2

Use Ratio test to determine whether the following series converge.

(a) $\sum_{k=1}^{\infty} \left(\frac{2k+1}{4k+1}\right)^k$

(b) $\sum_{k=1}^{\infty} \frac{1}{2^k(2+\sin k)}$

Solution

(a) Since $\sqrt[k]{|a_k|} = \frac{2k+1}{4k+1} \to \frac{1}{2} < 1$, $\sum_{k=1}^{\infty} \left(\frac{2k+1}{4k+1}\right)^k$ converges by Root test.

(b) $1 \le 2 + \sin k \le 3 \quad \Rightarrow \quad 2^k \le 2^k(2+\sin k) \le 3 \cdot 2^k \quad \Rightarrow \quad 2 \le \sqrt[k]{2^k(2+\sin k)} \le 2\sqrt[k]{3} \to 2$

Since $\sqrt[k]{|a_k|} = \frac{1}{\sqrt[k]{2^k(2+\sin k)}} \to \frac{1}{2} < 1$, $\sum_{k=1}^{\infty} \frac{1}{2^k(2+\sin k)}$ converges by Root test.

Guidelines for testing a series of $\sum a_k$ for convergence:

1. Begin with the Divergence Test. If you show that $\lim_{k\to\infty} a_k \neq 0$, then the series diverges and your work is finished.

2. Is the series a special series?

- Geometric series: $\sum ar^k$ converges for $|r|<1$ and diverges for $|r|\geq 1$ $(a\neq 0)$.
- p-series: $\sum \frac{1}{k^p}$ converges for $p>1$, and diverges for $p\leq 1$.
- Exponential series: $\sum \frac{a^k}{k!}$ always converges.
- Check also for a telescoping series.

3. If the general k-th term a_k of the series involves $k!$ or a^k, where a is a constant, the Ratio Test is advisable. Series with $f(k)^k$ may yield to the Root Test.

4. If every term of series $\sum a_k$ is positive then try 4(i) and 4(ii)

(i) If the general k-th term a_k of the series looks like a continuous, positive, decreasing function $f(k)$ you can integrate, then try the integral test.

(ii) If the general k-th term a_k of the series is a rational function of k (or a root of a rational function), use the Comparison Test or Limit Comparison Test. Use the families of series $\sum b_k$ given in Step 2 as comparison series. The order of growth rates of sequences is useful for evaluating $\lim_{k\to\infty}\frac{a_k}{b_k}$.

5. If not every term of series $\sum a_k$ is positive then try to show absolutely convergence by 3, 4(i) or 4(ii). If this is the case, $\sum a_k$ converges.

6. It the series is an alternating series $\sum (-1)^{k+1} a_k$, try alternating series test if possible.

We now expand our discussion of series to the case where the terms of the series are functions of the variable x. The primary reason for studying series is that we can use them to represent functions. This opens up numerous possibilities for us, from approximating the values of transcendental functions to calculating derivatives and integrals of such functions, to studying differential equations. As well, defining functions as convergent series produces an explosion of new functions available to us, including many important functions, such as the Bessel functions $J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

A power series may converge for some values of x and diverge for other values of x. Consider the series

$$\sum_{k=0}^{\infty}(x-2)^k = 1+(x-2)+(x-2)^2+\cdots.$$

Notice that for each fixed x, this is a geometric series with $r=(x-2)$, which will converge whenever $|r|=|x-2|<1$ and diverge whenever $|r|=|x-2|\geq 1$. Further, for each x in (1, 3), the series converges to $\frac{1}{1-(x-2)}=\frac{1}{3-x}$.

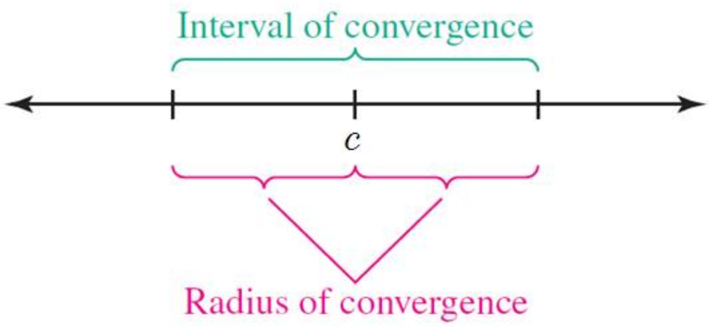
Definition 1.10.1 (Power Series)

A **power series** has the general form

$$f(x)=\sum_{k=0}^{\infty}a_k(x-c)^k = a_0+a_1(x-c)+a_2(x-c)^2+\cdots,$$

where c and a_k are real numbers, and x is a variable. The a_k 's are the **coefficients** of the power series and c is the **center** of the power series. The set of values of x for which the series converges is its **interval of convergence**. The **radius of convergence** of the power series, denoted R, is the distance from the center of the series to the boundary of the interval of convergence.

Notice that f looks like a polynomial. The only difference is that f has infinitely many terms.



Theorem 1.10.2 (Convergence of Power Series)

A power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ centered at c converges in one of three ways:

(a) The series converges absolutely for all x, in which case the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

(b) There is a real number $R > 0$ such that the series converges absolutely for $|x-c| < R$, and diverges for $|x-c| > R$, in which case the radius of convergence is R.

(c) The series converges only at c, in which case the radius of convergence is $R = 0$.

Finding interior of interval of convergence:

Consider the absolute sequence of $\sum_{n=0}^{\infty} a_n(x-c)^n$, that is, $\sum_{n=0}^{\infty} |a_n||x-c|^n$.

Ratio Test: The series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges if

$$r = \lim_{n\to\infty} \frac{|a_{n+1}||x-c|^{n+1}}{|a_n||x-c|^n} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x-c| < 1, \quad \text{i.e.,} \quad |x-c| < \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|.$$

Root Test: The series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges if

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n||x-c|^n} = \lim_{n\to\infty} \sqrt[n]{|a_n|}\, |x-c| < 1, \quad \text{i.e.,} \quad |x-c| < \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Use Ratio Test or Root Test to determine the radius of convergence R (including 0 and ∞):

$$R = \lim_{n\to\infty} \frac{1}{\left|\frac{a_{n+1}}{a_n}\right|} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \qquad \text{or} \qquad R = \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}}.$$

Then the series $\sum_{n=0}^{\infty} a_n (x-c)^n$ converges on the interval $(c-R, c+R)$.

Testing endpoints:

(i) If $\lim_{n\to\infty} a_n R^n \neq 0$, then $\sum_{k=0}^{\infty} a_k (x-c)^k$ diverges at two endpoints $c-R$ and $c+R$ by Divergence Test.

If (i) fails, i.e., $\lim_{n\to\infty} a_n R^n = 0$, it does not mean the series converges at both endpoints $c-R$ and $c+R$.

Assume all $a_n > 0$. Then proceed to (ii) and (iii).

(ii) Use Alternating Series Test to determine whether $\sum_{k=0}^{\infty} a_k (x-c)^k$ converges at $c-R$ or not.

(iii) Use Integral Test, Comparison Test or Limit Comparison Test… to determine whether $\sum_{k=0}^{\infty} a_k (x-c)^k$ converges at $c+R$ or not.

Thus there are four possibilities for the interval of convergence:

$$(c-R, c+R), \quad [c-R, c+R), \quad (c-R, c+R], \quad [c-R, c+R]$$

Example 1.10.3

Find the radius and interval of convergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$

(b) $\sum_{n=0}^{\infty} \sqrt{n}(x-2)^n$

(c) $\sum_{n=1}^{\infty}\frac{x^n}{n^n}$

(d) $\sum_{n=0}^{\infty}n!(-x)^n$

Solution

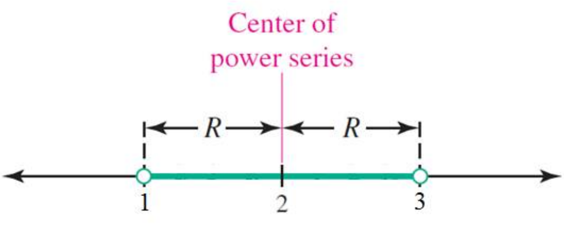
(a) By Ratio Test, we have $r=\lim_{n\to\infty}\left(\frac{|x|^{n+1}}{n+1}\right)\Big/\left(\frac{|x|^n}{n}\right)=\lim_{n\to\infty}\frac{n|x|}{n+1}=|x|$. $R=\lim_{n\to\infty}\frac{n+1}{n}=1$.

Thus the series $\sum_{n=1}^{\infty}\frac{x^n}{n}$ converges absolutely if $|x|<1$, i.e., $-1<x<1$. For $x=1$, $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (harmonic series). For $x=-1$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges by Alternating Series Test. The radius of convergence is 1 and the interval of convergence is $-1\le x<1$.

(b) By Ratio Test, we have

$$r=\lim_{n\to\infty}\left(\sqrt{n+1}\,|x-2|^{n+1}\right)\Big/\left(\sqrt{n}\,|x-2|^n\right)=\lim_{n\to\infty}\frac{\sqrt{n+1}\,|x-2|}{\sqrt{n}}=|x-2|.\ R=1.$$

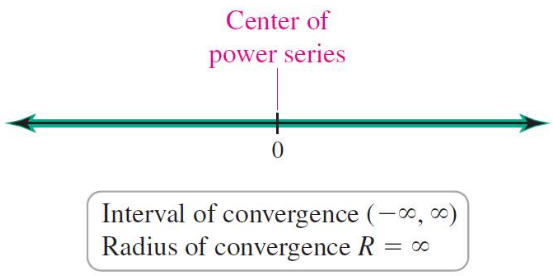
Thus the series $\sum_{n=0}^{\infty}\sqrt{n}(x-2)^n$ converges absolutely if $|x-2|<1$, i.e., $1<x<3$. Since $\lim_{n\to\infty}\sqrt{n}=\infty$, $\sum_{n=0}^{\infty}\sqrt{n}$ and $\sum_{n=0}^{\infty}(-1)^n\sqrt{n}$ diverge by divergence test. The radius of convergence is 1 and the interval of convergence is $1<x<3$.



Interval of convergence (1, 3)
Radius of convergence $R=1$

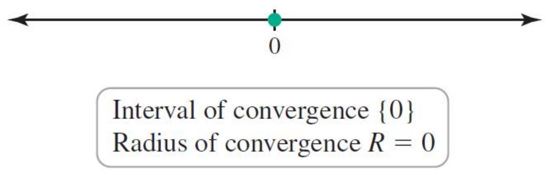
(c) By Root Test, we have $\rho=\lim_{n\to\infty}\sqrt[n]{\frac{|x|^n}{n^n}}=\lim_{n\to\infty}\frac{|x|}{n}=0$. $R=\lim_{n\to\infty}n=\infty$.

Thus the series $\sum_{n=1}^{\infty}\frac{x^n}{n^n}$ converges for any x. The radius of convergence is ∞ and the interval of convergence is $-\infty<x<\infty$.



(d) By Ratio Test, we have $r = \lim_{n\to\infty} \dfrac{(n+1)!|x|^{n+1}}{n!|x|^n} = \lim_{n\to\infty}(n+1)|x|$. $R = \lim_{n\to\infty} \dfrac{1}{n+1} = \infty$.

Thus the series $\sum_{n=0}^{\infty} n!(-x)^n$ converges only at $x = 0$. The radius of convergence is 0 and the interval of convergence is $\{0\}$.



Theorem 1.10.4

If the power series $\sum_{k=0}^{\infty} a_k (x-c)^k$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

is differentiable (and therefore continuous) on the interval $(c-R, c+R)$ and

(i) $f'(x) = \dfrac{d}{dx}a_0 + a_1\dfrac{d}{dx}(x-c) + a_2\dfrac{d}{dx}(x-c)^2 + \cdots = a_1 + 2a_2(x-c) + \cdots$

(ii) $\int f(x)dx = \text{constant} + a_0\int 1dx + a_1\int (x-c)dx + a_2\int (x-c)^2 dx + \cdots$

$$= \text{constant} + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots$$

or $\int_c^x f(t)dt = a_0(x-c) + a_1\dfrac{(x-c)^2}{2} + a_2\dfrac{(x-c)^3}{3} + \cdots$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Suppose that the power series $\sum_{k=0}^{\infty} a_k (x-c)^k$ has radius of convergence $R > 0$. As we've observed, this means that the series converges absolutely to some function f on the interval $(c-R, c+R)$. We have

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + \cdots,$$

for each $x \in (c-R, c+R)$. Differentiating term-by-term, we get that

$$f'(x) = \sum_{k=1}^{\infty} a_k k(x-c)^{k-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots,$$

again, for each $x \in (c-R, c+R)$. Likewise, we get

$$f''(x) = \sum_{k=1}^{\infty} a_k k(k-1)(x-c)^{k-1} = 2\cdot 1a_2 + 3\cdot 2a_3(x-c) + 4\cdot 3a_4(x-c)^2 + \cdots$$

and $f'''(x) = \sum_{k=1}^{\infty} a_k k(k-1)(k-2)(x-c)^{k-1} = 3\cdot 2\cdot 1a_3 + 4\cdot 3\cdot 2a_4(x-c) + \cdots$

and so on (all valid for $x \in (c-R, c+R)$). Notice that if we substitute $x = c$ in each of the above derivatives, all the terms of the series drop out, except the first one. We get

$$f(c) = a_0, \quad f'(c) = a_1, \quad f''(c) = 2!a_2, \quad f'''(c) = 3!a_3$$

and so on. Observe that in general, we have $f^{(k)}(c) = k!a_k$. That is

$$a_k = \frac{f^{(k)}(c)}{k!}, \quad \text{for } k = 0, 1, 2, \cdots$$

We have $f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$ for $x \in (c-R, c+R)$ if f is a series.

Definition 1.11.1 (Taylor / Maclaurin Series for a Function)

Suppose the function f has derivatives of all orders on an interval containing the point a. The **Taylor series for f centered at a** is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

A Taylor series centered at 0 is called a **Maclaurin series**.

There are two important questions we need to answer.

(i) Does a series constructed in this way converge and, if so, what is its radius of convergence?

(ii) If the series converges, it converges to a function. Does it converge to f? No. You may consider Question 1.12.10 in P. 50. But we may assume all series in table in P.51 valid.

Example 1.11.2

Find the Maclaurin series for the following functions. Give the interval of convergence.

(a) e^x

(b) $\sin x$

(c) $\cos x$

Solution

(a) If $f(x)=e^x$, then $f^{(n)}(x)=e^x$, so $f^{(n)}(0)=e^0=1$ for all n. Therefore the Taylor series for f at 0 (that is, the Maclaurin series) is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

To find the radius of convergence we let $a_n=\dfrac{x^n}{n!}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|=\frac{|x|}{n+1}\to 0<1$$

so, by the Ratio Test, the series converges for all x and the radius of convergence is $R=\infty$ and the interval of convergence is $-\infty<x<\infty$.

(b) Let $f(x)=\sin x$. First, we compute some derivatives and their value at $x=0$. We have

$f(x)=\sin x$	$f'(x)=\cos x$	$f''(x)=-\sin x$	$f'''(x)=-\cos x$	$f^{(4)}(x)=\sin x$
$f(0)=0$	$f'(0)=1$	$f''(0)=0$	$f'''(0)=-1$	$f^{(4)}(0)=0$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

To find the radius of convergence we let $a_n = (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{2n+1}}{(2n+1)!} \cdot \frac{(2n-1)!}{x^{2n-1}} \right| = \frac{|x^2|}{(2n+1)2n} \to 0 < 1$$

so, by the Ratio Test, the series converges for all x and the radius of convergence is $R = \infty$ and the interval of convergence is $-\infty < x < \infty$.

(c) Let $f(x) = \cos x$. First, we compute some derivatives and their value at $x = 0$. We have

$f(x) = \cos x$	$f'(x) = -\sin x$	$f''(x) = -\cos x$	$f'''(x) = \sin x$	$f^{(4)}(x) = \cos x$
$f(0) = 1$	$f'(0) = 0$	$f''(0) = -1$	$f'''(0) = 0$	$f^{(4)}(0) = 1$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

To find the radius of convergence we let $a_n = (-1)^n \frac{x^{2n}}{(2n)!}$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = \frac{|x^2|}{(2n+2)(2n+1)} \to 0 < 1$$

so, by the Ratio Test, the series converges for all x and the radius of convergence is $R = \infty$ and the interval of convergence is $-\infty < x < \infty$.

Example 1.11.3

Verify the following by Maclaurin series:

(a) $\frac{d}{dx} \sin x = \cos x$

(b) $\dfrac{d}{dx}\cos x = -\sin x$

(c) $\dfrac{d}{dx}e^x = e^x$

Solution

(a) The Maclaurin series for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

and it converges for $-\infty < x < \infty$. By Theorem 1.10.4, the differentiated series also converges for $-\infty < x < \infty$ and it converges to $(\sin x)'$. On differentiating, we have

$$\frac{d}{dx}\sin x = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \cos x.$$

(b) The Maclaurin series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

and it converges for $-\infty < x < \infty$. By Theorem 1.10.4, the differentiated series also converges for $-\infty < x < \infty$ and it converges to $(\cos x)'$. On differentiating, we have

$$\frac{d}{dx}\cos x = \frac{d}{dx}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) = -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = -\sin x.$$

(c) The Maclaurin series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

and it converges for $-\infty < x < \infty$. By Theorem 1.10.4, the differentiated series also converges for $-\infty < x < \infty$ and it converges to $\left(e^x\right)'$. On differentiating, we have

$$\frac{d}{dx}e^x = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x.$$

Example 1.11.4

Find Maclaurin for the following functions and give their intervals of convergence.

(a) $\tan^{-1} x$

(b) $\ln(1-x)$

(c) $\ln(1+x)$

Solution

(a) Since $\dfrac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots$ for $-1 < t < 1$,

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x 1 - t^2 + t^4 - t^6 + \cdots dt = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

for $-1 < x < 1$ by Theorem 1.10.4. Both $1 - \dfrac{1}{3} + \dfrac{1}{5} - \cdots$ and $(-1) - \dfrac{(-1)^3}{3} + \dfrac{(-1)^5}{5} - \cdots$ $= -\left(1 - \dfrac{1}{3} + \dfrac{1}{5} - \cdots\right)$ converge by alternating series test. Therefore the interval of convergence is $[-1, 1]$. In particular,

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

(b) Since $\dfrac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$ for $-1 < t < 1$,

$$\ln(1-x) = \int_0^x \frac{-1}{1-t} dt = -\int_0^x 1 + t + t^2 + \cdots dt = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

for $-1 < x < 1$ by Theorem 1.10.4. $-\left(1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots\right)$ diverges and $-(-1) - \dfrac{(-1)^2}{2} - \dfrac{(-1)^3}{3} - \cdots$ $= 1 - \dfrac{1}{2} + \dfrac{1}{3} - \cdots$ converges by alternating series test. Therefore the interval of convergence is $[-1, 1)$. In particular,

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

(c) Replace x by $-x$ in (b), we have $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$. The interval of convergence is $(-1, 1]$.

If p is a positive integer, then $(1+x)^p$ is a polynomial of degree p. In fact,

$$(1+x)^p = \binom{p}{0} + \binom{p}{1}x + \binom{p}{2}x^2 + \cdots + \binom{p}{p}x^p,$$

where the binomial coefficients $\binom{p}{k} = \dfrac{p!}{k!(p-k)!}$.

Our goal is to extend this idea to the functions $f(x) = (1+x)^p$, where p is a real number other than a nonnegative integer. The result is a Taylor series called the **binomial series**.

Definition 1.11.5 (Binomial Coefficients)

For real numbers p and integers $k \geq 1$,

$$\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \quad \binom{p}{0} = 1.$$

If $p = -n$ is a negative integer where n is a positive integer, then

$$\begin{aligned}\binom{-n}{k} &= \frac{(-n)(-n-1)(-n-2)\cdots(-n-k+1)}{k!} \\ &= (-1)^k \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!} \\ &= (-1)^k \binom{n+k-1}{k}.\end{aligned}$$

Theorem 1.11.6 (Binomial Series)

For real numbers $p \neq 0$, the Taylor series for $f(x) = (1+x)^p$ centered at 0 is the **binomial series**

$$\begin{aligned}\sum_{k=0}^{\infty} \binom{p}{k} x^k &= \sum_{k=0}^{\infty} \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!} x^k \\ &= 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots\end{aligned}$$

The series converges for $|x|<1$ (and possibly at the endpoints or even larger, depending on p).

If p is a nonnegative integer, the series terminates and results in a polynomial of degree p.

Proof

We only consider neither positive integer nor zero p.

$$\begin{aligned} f(x) &= (1+x)^p & \Rightarrow && f(0) &= 1 \\ f'(x) &= p(1+x)^{p-1} & \Rightarrow && f'(0) &= p \\ f''(x) &= p(p-1)(1+x)^{p-2} & \Rightarrow && f''(0) &= p(p-1) \\ f'''(x) &= p(p-1)(p-2)(1+x)^{p-3} & \Rightarrow && f'''(0) &= p(p-1)(p-2) \end{aligned}$$

In general, we have

$$f^{(k)}(0) = p(p-1)(p-2)\cdots(p-k+1).$$

Therefore,

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!} = \binom{p}{k}, \quad \text{for} \quad k = 0,1,2,\cdots.$$

Hence, we have $(1+x)^p = \binom{p}{0} + \binom{p}{1}x + \binom{p}{2}x^2 + \binom{p}{3}x^3 + \cdots$.

Holding p and x fixed,

$$\begin{aligned} r &= \lim_{k\to\infty}\left|\frac{x^{k+1}p(p-1)(p-2)\cdots(p-k)/(k+1)!}{x^k p(p-1)(p-2)\cdots(p-k+1)/k!}\right| \\ &= |x|\lim_{k\to\infty}\left|\frac{p-k}{k+1}\right| \\ &= |x|. \end{aligned}$$

Thus the series $\sum_{k=0}^{\infty}\binom{p}{k}x^k$ converges absolutely if $|x|<1$, i.e., $-1<x<1$.

In particular, we have the following when $-1<x<1$.

$$(1+x)^{-n} = \sum_{k=0}^{\infty}\binom{-n}{k}x^k = \sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}x^k$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

$$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \cdots$$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots$$

$$(1+x)^{-\frac{1}{3}} = 1 - \frac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 - \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \cdots$$

$$(1+x)^{-\frac{2}{3}} = 1 - \frac{2}{3}x + \frac{2\cdot 5}{3\cdot 6}x^2 - \frac{2\cdot 5\cdot 8}{3\cdot 6\cdot 9}x^3 + \cdots$$

Definition 1.12.1(Taylor Polynomials)

Let f be a function with n-th order derivative defined at a. The ***n*-th order Taylor polynomial** for f with its center at a, denoted p_n, has the property that it matches f in value, slope, concavity and all derivatives up to the n-th derivative at a; that is

$$p_n(a)=f(a),\quad p_n'(a)=f'(a),\quad p_n''(a)=f''(a),\quad p_n^{(n)}(a)=f^{(n)}(a).$$

The n-th order Taylor polynomial centered at a is

$$p_n(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

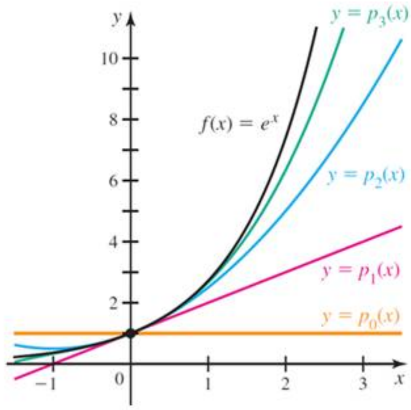
The **remainder** in using p_n to approximate f at the point x is

$$R_n(x)=f(x)-p_n(x).$$

Example 1.12.2

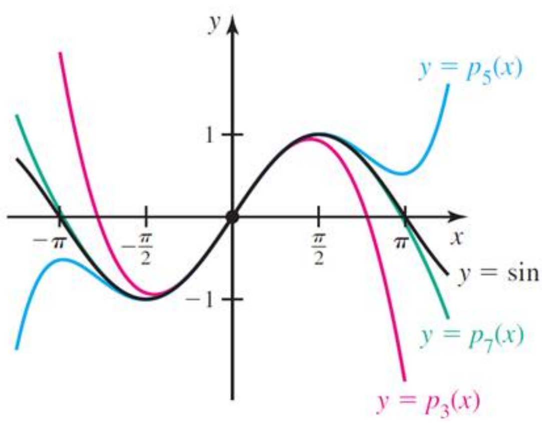
Let $f(x)=e^x$.

$$p_0(x)=1$$
$$p_1(x)=1+x$$
$$p_2(x)=1+x+\frac{x^2}{2!}$$
$$p_3(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$$



Let $f(x)=\sin x$.

$$p_1(x)=x$$
$$p_3(x)=x-\frac{x^3}{3!}$$
$$p_5(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}$$
$$p_7(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}$$



Theorem 1.12.3 (Taylor's Theorem)

Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a. For all x in I,

$$f(x) = p_n(x) + R_n(x),$$

where p_n is the nth-order Taylor polynomial for f centered at a, and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1},$$

for some point z between x and a. Furthermore, suppose there exists a number M such that $\left|f^{(n+1)}(z)\right| \le M,$ for all z between a and x inclusive. The remainder in the n-th order Taylor polynomial for f centered at a satisfies

$$|R_n(x)| = |f(x) - p_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Example 1.12.4

Use the Taylor polynomials of order $n = 0, 1, 2$ and 3 for $f(x) = e^x$ centered at 0 to approximate $e^{0.1}$ and $e^{-0.25}$. Find the absolute errors, $|f(x) - p_n(x)|$, in the approximations.

Solution

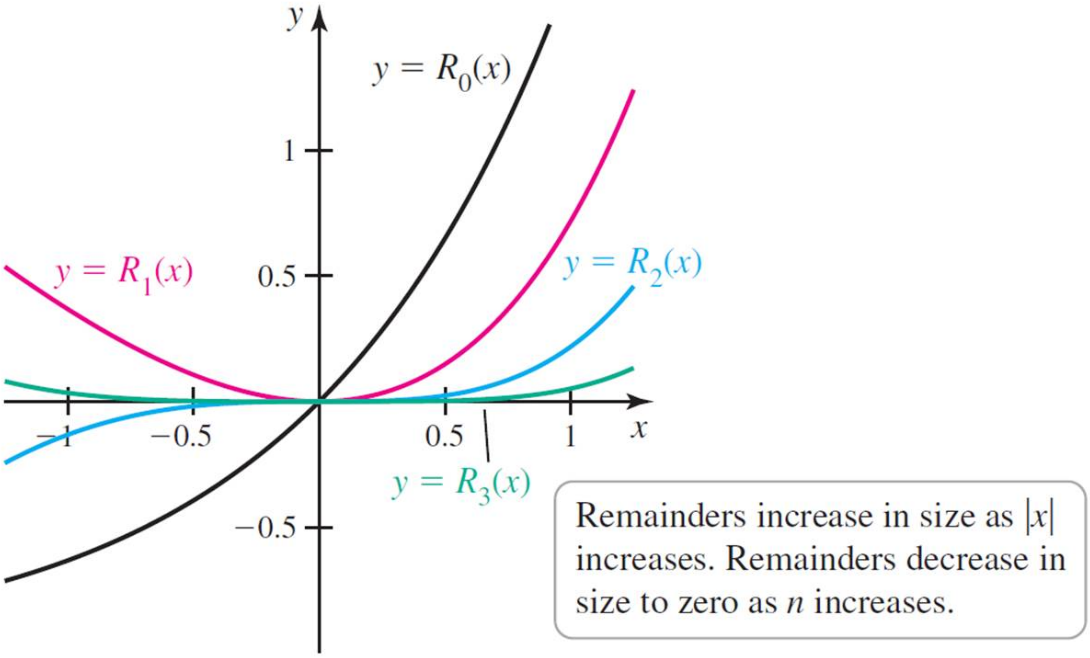
$$p_0(x) = 1, \quad p_1(x) = 1 + x, \quad p_2(x) = 1 + x + \frac{x^2}{2!}, \quad p_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}, \quad R_n(x) = \frac{e^z}{(n+1)!}x^{n+1}$$

The results are shown in the following table:

n	Approximations $p_n(0.1)$	Absolute Error $\lvert e^{0.1} - p_n(0.1)\rvert$	Approximations $p_n(-0.25)$	Absolute Error $\lvert e^{-0.25} - p_n(-0.25)\rvert$
0	1	1.05×10^{-1}	1	2.21×10^{-1}
1	1.1	5.17×10^{-3}	0.75	2.88×10^{-2}
2	1.105	1.71×10^{-4}	0.78125	2.45×10^{-3}
3	1.105167	4.25×10^{-6}	0.778646	1.55×10^{-4}

Reasonable approximations based on these calculations are $e^{0.1} \approx 1.105$ and $e^{-0.25} \approx 0.78.$

The actual value of $e^{0.1}$ and $e^{-0.25}$ are 1.10517… and 0.7788… respectively.



Example 1.12.5

(a) Find the Taylor polynomial $p_3(x)$ of order 3 for $f(x)=\sqrt{1+x}$ with center 0

(b) Show that $|R_3(x)| \le \frac{1}{4!}\cdot\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)|x^4|$ for $x > 0$.

(c) Approximate $\sqrt{18}=4\sqrt{1+x}$ by $p_3(x)$ for a suitable x.

Solution

(a) Note that $\binom{\frac{1}{2}}{0}=1,\ \binom{\frac{1}{2}}{1}=\frac{\frac{1}{2}}{1!}=\frac{1}{2},\ \binom{\frac{1}{2}}{2}=\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}=-\frac{1}{8},\ \binom{\frac{1}{2}}{3}=\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}=\frac{1}{16}.$

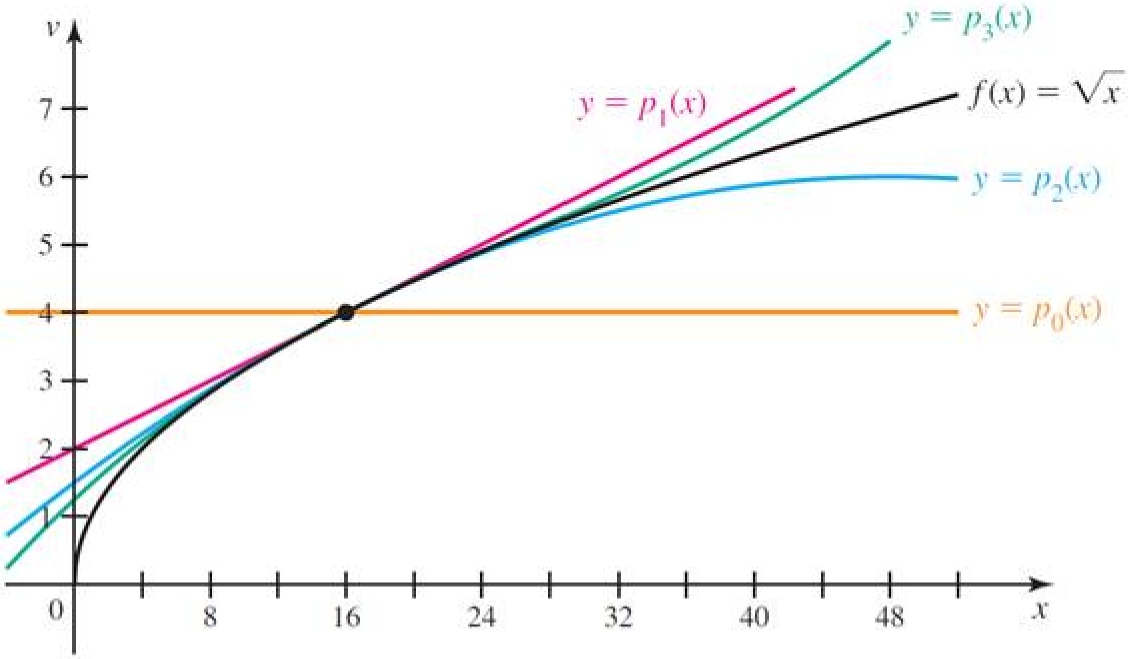
$$p_3(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$$

(b) $f^{(4)}(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(1+x)^{-\frac{7}{2}},\quad \left|f^{(4)}(x)\right| \le \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)$

$$|R_3(x)|=\left|\frac{f^{(4)}(z)}{4!}x^4\right| \le \frac{1}{4!}\cdot\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)|x^4|$$

(c) $\sqrt{18} = 4\sqrt{1+\frac{1}{8}} \approx 4\left(1+\frac{1}{2}\cdot\frac{1}{8}-\frac{1}{8}\cdot\frac{1}{8^2}+\frac{1}{16}\cdot\frac{1}{8^3}\right) = 4.24267578125$

The actual value of $\sqrt{18}$ is 4.2426406871192851…



Theorem 1.12.6

The n-th order Taylor polynomial for $f(x) = \ln(1-x)$ centered at 0 (Example 1.11.4) is

$$p_n(x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n}.$$

(a) How many terms of the Taylor polynomial are needed to approximate values of $f(x) = \ln(1-x)$ with an error less than 10^{-3} on the interval $[-\frac{1}{2}, \frac{1}{2}]$?

(b) What is the maximum error in approximating $\ln(1-x)$ by $p_3(x)$ for values of x in the interval $[-\frac{1}{2}, \frac{1}{2}]$?

Solution

(a) For any positive integer n, the remainder is $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$. Differentiating f several times reveals that

$$f^{(n+1)}(x) = -\frac{n!}{(1-x)^{n+1}}.$$

On the interval $[-\frac{1}{2},\frac{1}{2}]$, the maximum magnitude of this derivative occurs at $x=\frac{1}{2}$ and is $n!/\left(\frac{1}{2}\right)^{n+1}=n!2^{n+1}$. Therefore, a bound on the remainder is

$$|R_n(x)| \le \frac{n!2^{n+1}}{(n+1)!}|x|^{n+1} \le \frac{1}{n+1}, \quad \because |x| \le \frac{1}{2}.$$

To ensure that the error is less than 10^{-3} on the entire interval $[-\frac{1}{2},\frac{1}{2}]$, n must satisfy $|R_n(x)| \le \dfrac{1}{n+1} < 10^{-3}$ or $n > 999$.

(b) maximum error is $|R_3(x)| \le \dfrac{1}{3+1} = 0.25.$

Example 1.12.7

Approximate the value of the integral $\int_0^1 e^{-x^2}\,dx$ with an error no greater than 5×10^{-4}.

Solution

Consider the Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots,$$

which converges for $-\infty < x < \infty$. We replace x by $-x^2$ to obtain

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots,$$

which converges for $-\infty < x < \infty$.

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &= \left(x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)n!} + \cdots \right)\Bigg|_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \cdots + \frac{(-1)^n}{(2n+1)n!} + \cdots.\end{aligned}$$

The remainder in truncating series is less than the first neglected term, which is $\dfrac{(-1)^{n+1}}{(2n+3)(n+1)!}$ (Theorem 1.4.4).

When $n = 5$, we have $\dfrac{1}{13\cdot 6!} \approx 1.07\times 10^{-4}$. The sum of the terms of the series up to $n = 5$ gives the approximation

$$\int_0^1 e^{-x^2}\,dx \approx 1-\frac{1}{3}+\frac{1}{5\cdot 2!}-\frac{1}{7\cdot 3!}+\frac{1}{9\cdot 4!}-\frac{1}{11\cdot 5!} \approx 0.746729196.$$

The actual value of $\int_0^1 e^{-x^2}\,dx \approx 0.746824$.

Theorem 1.12.8

Suppose that f has derivatives of all orders in the interval $(c-R, c+R)$, for some $R > 0$. $\lim_{n\to\infty} R_n(x) = 0$ for all x in $(c-R, c+R)$ if and only if

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k,$$

for all x in $(c-R, c+R)$.

Proof

$$\lim_{n\to\infty} R_n(x) = 0 \text{ in } (c-R, c+R) \quad \Leftrightarrow \quad f(x) = \lim_{n\to\infty} p_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \text{ in } (c-R, c+R)$$

Example 1.12.9

Show that the Taylor series for $f(x) = e^x$ expanded about $x = 0$ converges to e^x.

Solution

In Example 1.12.4, the remainder term of e^x is

$$R_n(x) = \frac{e^z}{(n+1)!}x^{n+1},$$

where z is somewhere between x and 0 (and depends also on the value of n). We first find a bound on the size of e^z. Notice that if $x > 0$, then $0 < z < x$ and so, $e^z < e^x$. If $x \le 0$, then $x \le z \le 0$, so that $e^z \le e^0 = 1$.

Let $M = \max\{e^x, 1\}$. Then, for any x and any n, we have $e^z \le M$. The error estimate is

$$|R_n(x)| = \frac{e^z}{(n+1)!}|x|^{n+1} \le M\frac{|x|^{n+1}}{(n+1)!} = M\frac{|x|}{1}\cdot\frac{|x|}{2}\cdots\frac{|x|}{n+1}.$$

For any given x, $\frac{|x|^{n+1}}{(n+1)!} = \frac{|x|}{1}\cdot\frac{|x|}{2}\cdots\frac{|x|}{n+1} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} R_n(x) = 0$ for all x.

From Theorem 1.12.8, we conclude that the Taylor series converges to e^x for all x. That is,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

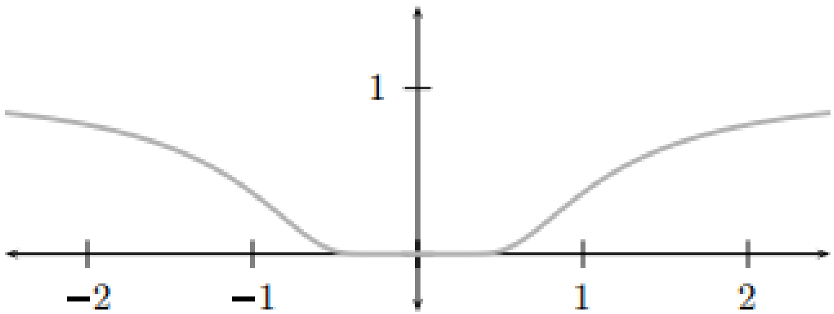
Q 1.12.10. Is every differentiable function equal to its Taylor series? i.e.,

$$f(x) \overset{???}{=} f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

A. No. A counterexample is provided by the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \ne 0, \\ 0 & \text{if } x = 0, \end{cases}$$

whose graph is shown below.



It can be shown that $f^{(n)}(0) = 0$ for any positive integers n. Its Taylor series is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = 0.$$

But $f(x)$ is not a zero function. Since $R_n(x) = f(x)$ does not converge to 0 for $x \ne 0$, $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$ does not converge to f by Theorem 1.12.8.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \quad \text{for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \le 1$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \quad \text{for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \binom{p}{0} = 1$$

Example 1.12.11

Evaluate $\lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x^4}$

Solution

You may use L'Hôpital's Rule to solve the problem. But you need to use many times! Using the Maclaurin series for $\cos x$, we have

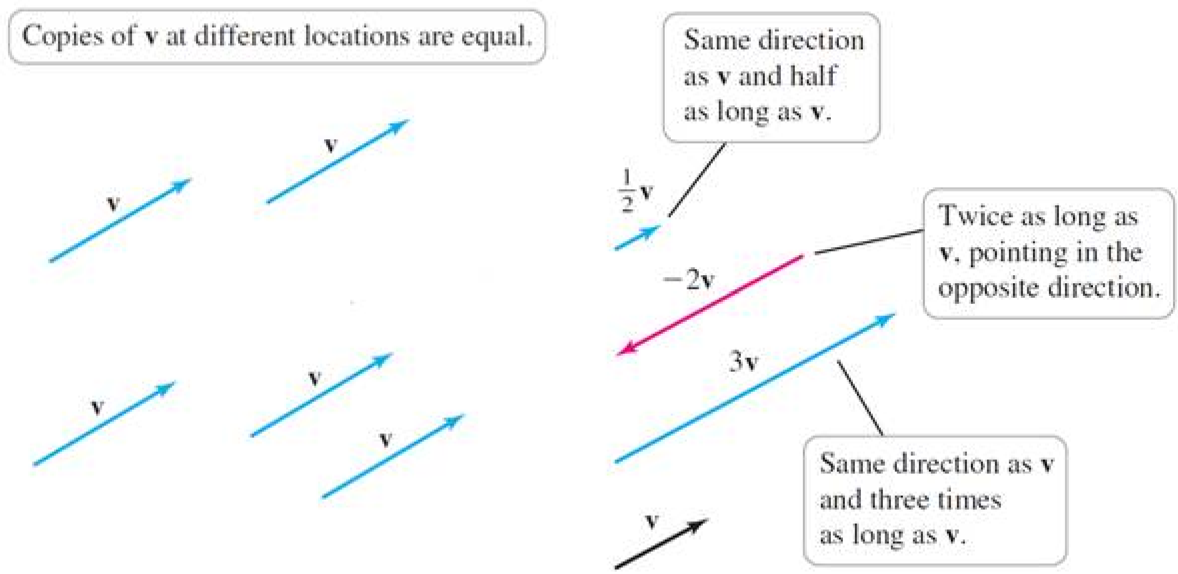
$$\begin{aligned}
\lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x^4} &= \lim_{x \to 0} \frac{x^2 + 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - 2}{x^4} \\
&= \lim_{x \to 0} \frac{2\left(\frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}{x^4} \\
&= \lim_{x \to 0} \left(\frac{2}{4!} - \frac{2}{6!} x^2 + \cdots\right) \\
&= \frac{1}{12}
\end{aligned}$$

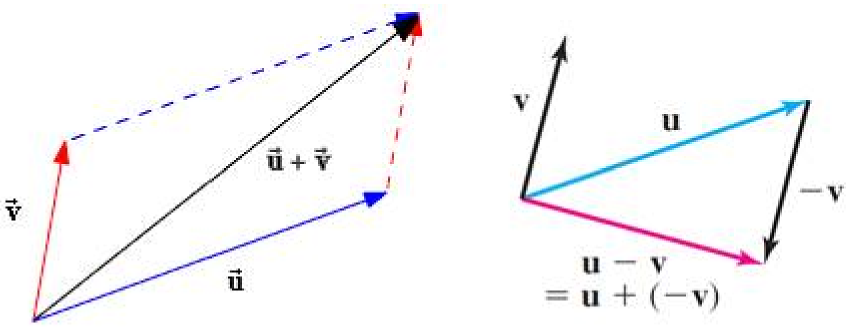
Chapter 2 Vectors and the Geometry of Space

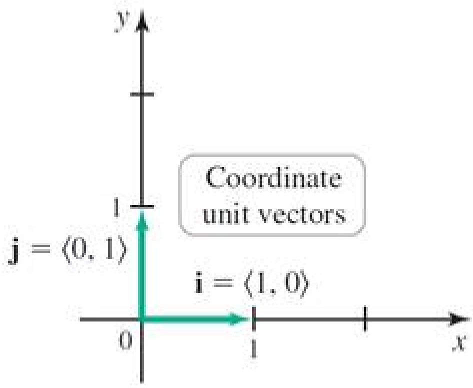
Vectors are quantities that have both **length** (or **magnitude**) and **direction**. Two vectors are **equal** if they have the same magnitude and direction. Quantities having magnitude but no direction are called **scalars**. One exception is the **zero vector**, denote **0**: It has length 0 and no direction.

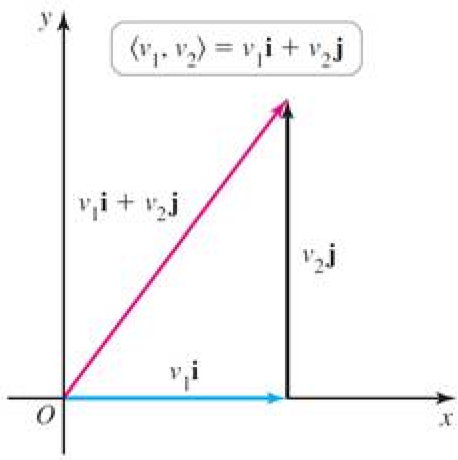
Definition 2.1.1

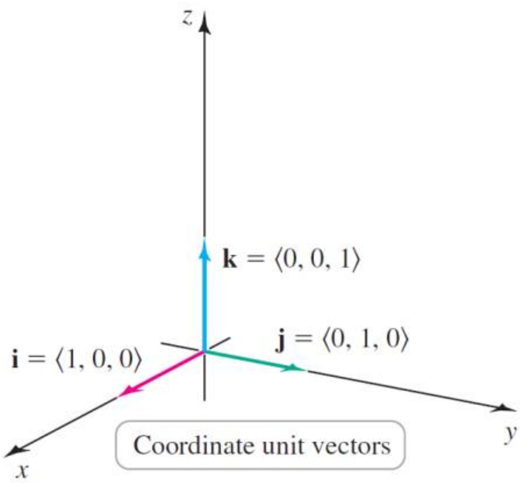
Given a scalar c and a vector $\boldsymbol{v}$, the **scalar multiple** of $c\boldsymbol{v}$ is a vector whose magnitude is $|c|$ multiplied by the magnitude of $\boldsymbol{v}$. If $c > 0$, then $c\boldsymbol{v}$ has the same direction as $\boldsymbol{v}$. If $c < 0$, then $c\boldsymbol{v}$ and $\boldsymbol{v}$ point in opposite directions. Two vectors are **parallel** if they are scalar multiples of each other.

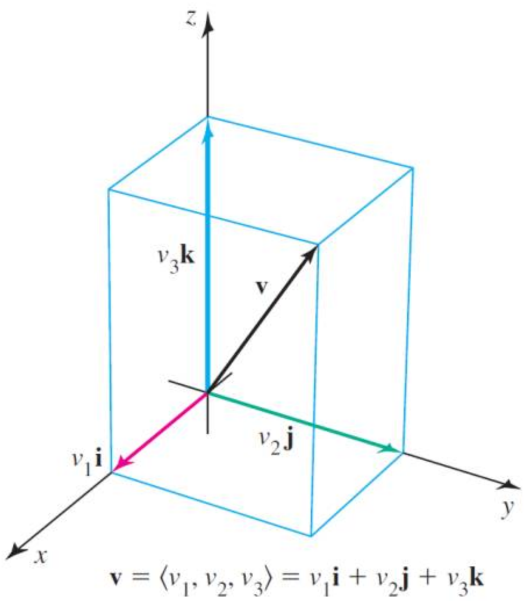












Definition 2.1.2

Let c be a scalar, $\boldsymbol{u}=(u_1,u_2)$ and $\boldsymbol{v}=(v_1,v_2)$ be two vectors in $\boldsymbol{R}^2$. Then

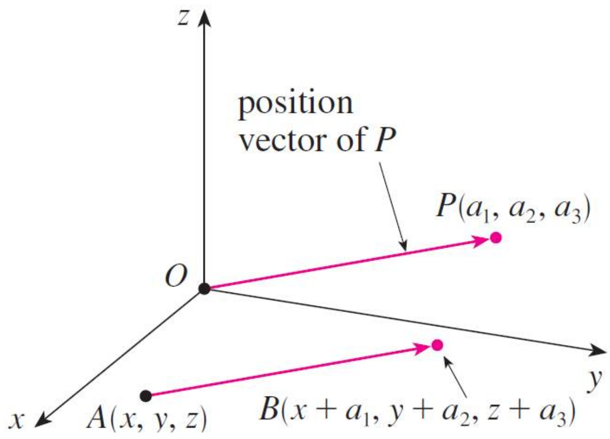
$$\boldsymbol{u}+\boldsymbol{v}=(u_1+v_1,u_2+v_2),\quad \boldsymbol{u}-\boldsymbol{v}=(u_1-v_1,u_2-v_2),\quad c\boldsymbol{u}=(cu_1,cu_2).$$

Definition 2.1.3

Let c be a scalar, $\boldsymbol{u}=(u_1,u_2,u_3)$ and $\boldsymbol{v}=(v_1,v_2,v_3)$ be two vectors in $\boldsymbol{R}^3$. Then

$$\boldsymbol{u}+\boldsymbol{v}=(u_1+v_1,u_2+v_2,u_3+v_3),\quad \boldsymbol{u}-\boldsymbol{v}=(u_1-v_1,u_2-v_2,u_3-v_3),\quad c\boldsymbol{u}=(cu_1,cu_2,cu_3).$$

The vector $\overrightarrow{OP}$ from the origin to the point is called the **position vector** of the point P. The vector $\overrightarrow{AB}$ is $\overrightarrow{OB} - \overrightarrow{OA}$.



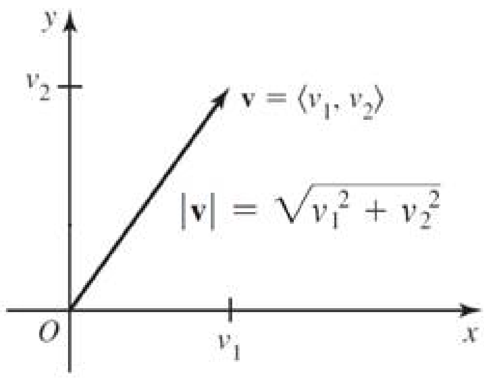
The **magnitude** or **length** of the vector $\boldsymbol{v}$ is the length of any of its representations and is denoted by the symbol $|\boldsymbol{v}|$. By Pythagorean Theorem we obtain the following formulas:

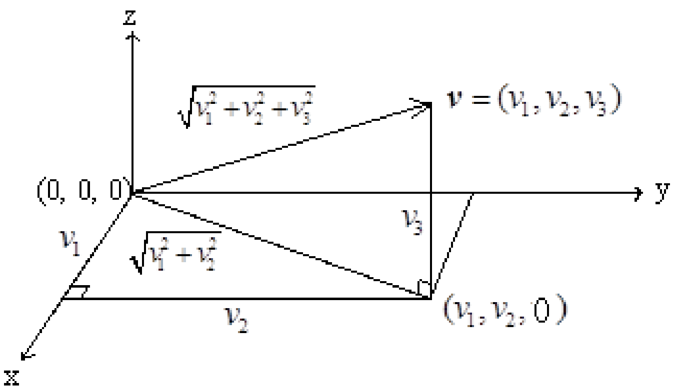
Definition 2.1.4

The length of the two-dimensional vector $\boldsymbol{v} = (v_1, v_2) \in \boldsymbol{R}^2$ is $|\boldsymbol{v}| = |(v_1, v_2)| = \sqrt{v_1^2 + v_2^2}$.

The length of the three-dimensional vector $\boldsymbol{v} = (v_1, v_2, v_3) \in \boldsymbol{R}^3$ is $|\boldsymbol{v}| = |(v_1, v_2, v_3)| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

The distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined to be $|\boldsymbol{u} - \boldsymbol{v}|$.





For any non-zero $\boldsymbol{v} = (v_1, v_2, v_3)$, a unit vector in the same direction as $\boldsymbol{v}$ is given by $\frac{1}{|\boldsymbol{v}|}\boldsymbol{v}$.

Definition 2.2.1

Let $\boldsymbol{u}=(u_1,u_2,u_3)^T$ and $\boldsymbol{v}=(v_1,v_2,v_3)^T$ be vectors in $\boldsymbol{R}^3$. The **dot product** of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by

$$\boldsymbol{u}\cdot\boldsymbol{v}=u_1v_1+u_2v_2+u_3v_3.$$

The dot product of two-dimensional vectors is defined in a similar fashion: $\boldsymbol{u}\cdot\boldsymbol{v}=u_1v_1+u_2v_2$.

Theorem 2.2.2

Suppose $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in $\boldsymbol{R}^3$ (or $\boldsymbol{R}^2$) and let c be a scalar in $\boldsymbol{R}$. Then

(i) $\boldsymbol{u}\cdot\boldsymbol{v}=\boldsymbol{v}\cdot\boldsymbol{u}$

(ii) $c(\boldsymbol{u}\cdot\boldsymbol{v})=(c\boldsymbol{u})\cdot\boldsymbol{v}=\boldsymbol{u}\cdot(c\boldsymbol{v})$

(iii) $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w})=\boldsymbol{u}\cdot\boldsymbol{v}+\boldsymbol{u}\cdot\boldsymbol{w}$

(iv) $\boldsymbol{v}\cdot\boldsymbol{v}=|\boldsymbol{v}|^2$ " = " holds if and only if $\boldsymbol{v}=\boldsymbol{0}$

(v) $\|c\boldsymbol{v}\|=|c||\boldsymbol{v}|$ for every vector $\boldsymbol{v}$ in $\boldsymbol{R}^3$ (or $\boldsymbol{R}^2$) and every scalar c in $\boldsymbol{R}$.

The proofs are easy.

Theorem 2.2.3

Let $\boldsymbol{u}=(u_1,u_2,u_3)$ and $\boldsymbol{v}=(v_1,v_2,v_3)$ in $\boldsymbol{R}^3$. Suppose θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. Then $\boldsymbol{u}\cdot\boldsymbol{v}=|\boldsymbol{u}||\boldsymbol{v}|\cos\theta$ where $0\le\theta\le\pi$. The same result also valid for $\boldsymbol{R}^2$.

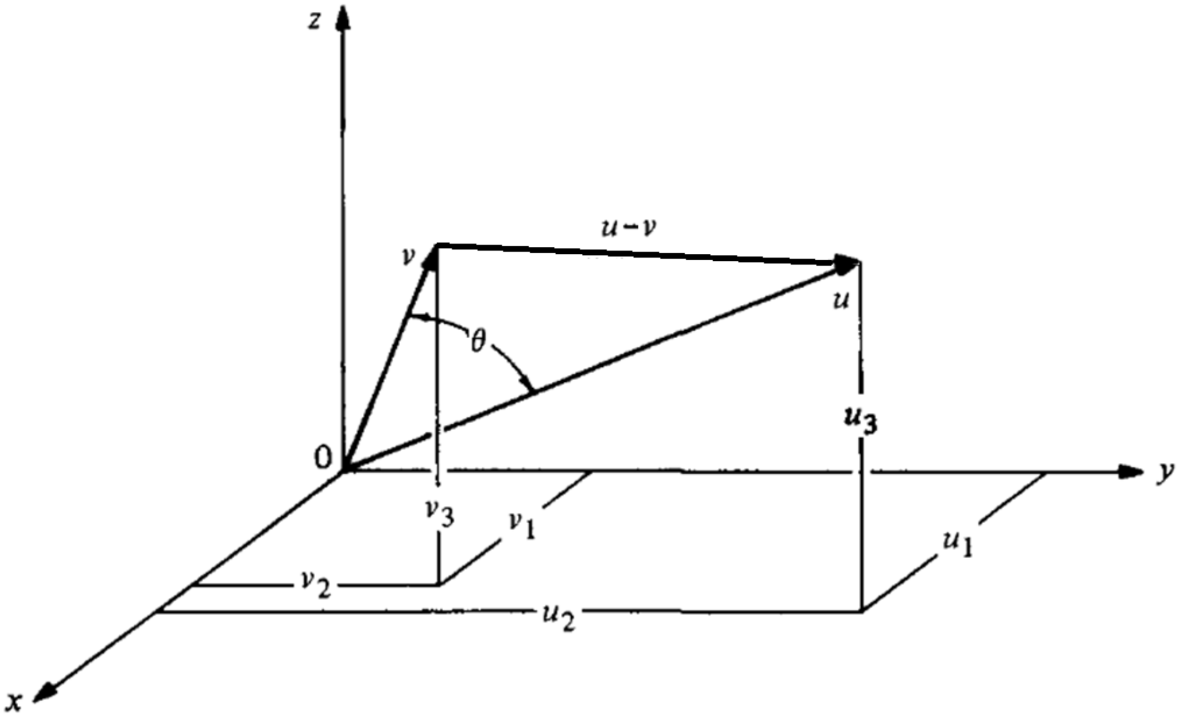
Proof

Consider a triangle formed by $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{u}-\boldsymbol{v}$. By the Law of Cosines, we have

$$|\boldsymbol{u}-\boldsymbol{v}|^2=|\boldsymbol{u}|^2+|\boldsymbol{v}|^2-2|\boldsymbol{u}||\boldsymbol{v}|\cos\theta.$$

$$\begin{aligned}|\boldsymbol{u}||\boldsymbol{v}|\cos\theta&=\frac{1}{2}\left(|\boldsymbol{u}|^2+|\boldsymbol{v}|^2-|\boldsymbol{u}-\boldsymbol{v}|^2\right)\\&=\frac{1}{2}\left(\boldsymbol{u}\cdot\boldsymbol{u}+\boldsymbol{v}\cdot\boldsymbol{v}-(\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v})\right)\end{aligned}$$

$$= \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}+\boldsymbol{v}\cdot\boldsymbol{v}-(\boldsymbol{u}\cdot\boldsymbol{u}-2\boldsymbol{u}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{v}))$$
$$= \boldsymbol{u}\cdot\boldsymbol{v}$$



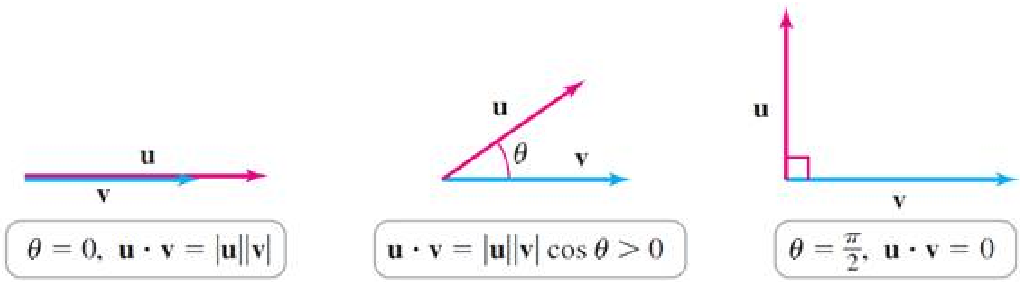
Corollary 2.2.4

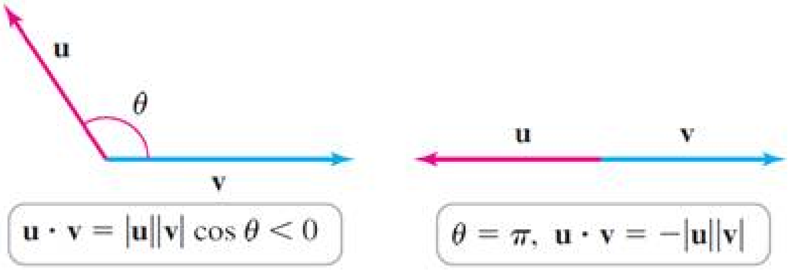
Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in $\boldsymbol{R}^3$ (or $\boldsymbol{R}^2$). Then

$$|\boldsymbol{u}\cdot\boldsymbol{v}| \le |\boldsymbol{u}||\boldsymbol{v}|$$

$\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel $(\theta = 0$ or $\theta = \pi)$ if and only if $|\boldsymbol{u}\cdot\boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}|$.

$\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular $(\theta = \pi/2)$ or either one of the vectors is the zero vector if and only if $\boldsymbol{u}\cdot\boldsymbol{v} = 0$.



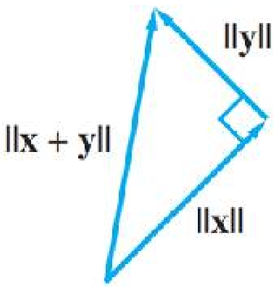


Definition 2.2.5

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{R}^n$ are said to be **orthogonal** (to each other) if $\boldsymbol{u}\cdot\boldsymbol{v}=0$.

Theorem 2.2.6

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if and only if $\|\boldsymbol{u}+\boldsymbol{v}\|^2=\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2$.



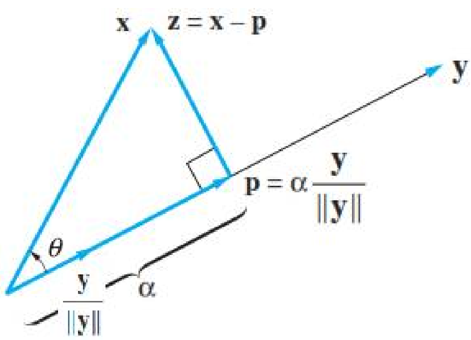
$$\|\mathbf{x}+\mathbf{y}\|^2=(\mathbf{x}+\mathbf{y})^T(\mathbf{x}+\mathbf{y})=\|\mathbf{x}\|^2+2\mathbf{x}^T\mathbf{y}+\|\mathbf{y}\|^2$$

$$\|\mathbf{x}+\mathbf{y}\|^2=\|\mathbf{x}\|^2+\|\mathbf{y}\|^2 \quad\Longrightarrow\quad \mathbf{x}^T\mathbf{y}=0.$$

Definition 2.2.7

Scalar projection of $\mathbf{x}$ onto $\mathbf{y}$: $\alpha=\dfrac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{y}\|}$

Vector projection of $\mathbf{x}$ onto $\mathbf{y}$: $\mathbf{p}=\alpha\dfrac{1}{\|\mathbf{y}\|}\mathbf{y}=\dfrac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y}$



$$\begin{aligned}\alpha &= \|\mathbf{x}\|\cos\theta\\ &= \frac{\|\mathbf{x}\|\ \|\mathbf{y}\|\cos\theta}{\|\mathbf{y}\|}\\ &= \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{y}\|}\end{aligned}$$

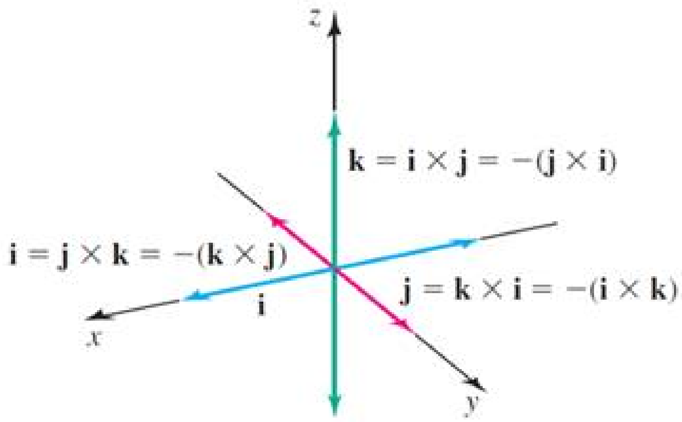
Definition 2.3.1 (Cross Product)

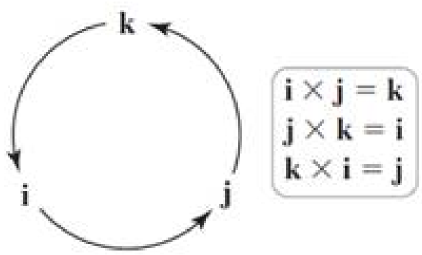
Let $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ be unit vectors in $\mathbb{R}^3$ on positive x, y and z-axis respectively. Let $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$ and $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$. The **cross product** $\boldsymbol{u}\times\boldsymbol{v}$ of $\boldsymbol{u}$ and $\boldsymbol{v}$ is the vector

$$\begin{aligned}\boldsymbol{u}\times\boldsymbol{v} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (u_2v_3 - u_3v_2)\boldsymbol{i} + (u_3v_1 - u_1v_3)\boldsymbol{j} + (u_1v_2 - u_2v_1)\boldsymbol{k} \\ &= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)\end{aligned}$$

Example 2.3.2

$$\begin{aligned}&\boldsymbol{i}\times\boldsymbol{j} = -(\boldsymbol{j}\times\boldsymbol{i}) = \boldsymbol{k}, \quad \boldsymbol{j}\times\boldsymbol{k} = -(\boldsymbol{k}\times\boldsymbol{j}) = \boldsymbol{i} \\ &\boldsymbol{k}\times\boldsymbol{i} = -(\boldsymbol{i}\times\boldsymbol{k}) = \boldsymbol{j}, \quad \boldsymbol{i}\times\boldsymbol{i} = \boldsymbol{j}\times\boldsymbol{j} = \boldsymbol{k}\times\boldsymbol{k} = \boldsymbol{0}\end{aligned}$$





Example 2.3.3

Let $\boldsymbol{u}$ = (1, 2, 3) and $\boldsymbol{v}$ = (4, 5, 6).

$$\begin{aligned}\boldsymbol{u}\times\boldsymbol{v} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} \\ &= (2\cdot 6 - 3\cdot 5)\boldsymbol{i} + (3\cdot 4 - 1\cdot 6)\boldsymbol{j} + (1\cdot 5 - 2\cdot 4)\boldsymbol{k} \\ &= (-3, 6, -3)\end{aligned}$$

Theorem 2.3.4

Let $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ be nonzero vectors in $\mathbb{R}^3$, and let a and b be scalars.

(i) $\boldsymbol{u}\times\boldsymbol{v} = -(\boldsymbol{v}\times\boldsymbol{u})$

(ii) $(a\boldsymbol{u})\times(b\boldsymbol{v}) = ab(\boldsymbol{u}\times\boldsymbol{v})$

(iii) $\boldsymbol{u}\times(\boldsymbol{v}+\boldsymbol{w}) = (\boldsymbol{u}\times\boldsymbol{v})+(\boldsymbol{u}\times\boldsymbol{w})$

The proofs are easy.

Q? Is $(\boldsymbol{u}\times\boldsymbol{v})\times\boldsymbol{w} = \boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w})$ for every vector $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$?

A. No. $(\boldsymbol{i}\times\boldsymbol{i})\times\boldsymbol{j} = \boldsymbol{0}$, $\boldsymbol{i}\times(\boldsymbol{i}\times\boldsymbol{j}) = \boldsymbol{i}\times\boldsymbol{k} = -\boldsymbol{j}$.

Indeed, $\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w}) = (\boldsymbol{u}\cdot\boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{w}$

Theorem 2.3.5

Let $\boldsymbol{u} = u_1\boldsymbol{i}+u_2\boldsymbol{j}+u_3\boldsymbol{k}$, $\boldsymbol{v} = v_1\boldsymbol{i}+v_2\boldsymbol{j}+v_3\boldsymbol{k}$ and $\boldsymbol{w} = w_1\boldsymbol{i}+w_2\boldsymbol{j}+w_3\boldsymbol{k}$. Then

$$\begin{aligned}(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{w} &= (u_2v_3-u_3v_2,\, u_3v_1-u_1v_3,\, u_1v_2-u_2v_1)\cdot(w_1,w_2,w_3)\\ &= (u_2v_3-u_3v_2)w_1+(u_3v_1-u_1v_3)w_2+(u_1v_2-u_2v_1)w_3\\ &= \begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}\end{aligned}$$

Clearly, $\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) = (\boldsymbol{v}\times\boldsymbol{w})\cdot\boldsymbol{u} = \begin{vmatrix} v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\\ u_1 & u_2 & u_3\end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix} = (\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{w}$

Suppose $\boldsymbol{u}\times\boldsymbol{v}\neq\boldsymbol{0}$. Clearly, $(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{u} = \begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ u_1 & u_2 & u_3\end{vmatrix} = 0$ and $(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{v} = \begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ v_1 & v_2 & v_3\end{vmatrix} = 0.$

That means $\boldsymbol{u}\times\boldsymbol{v}$ is perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

Furthermore, we also have

$$(\boldsymbol{u}\times\boldsymbol{v})\cdot(\boldsymbol{u}\times\boldsymbol{v}) = |\boldsymbol{u}\times\boldsymbol{v}|^2 > 0.$$

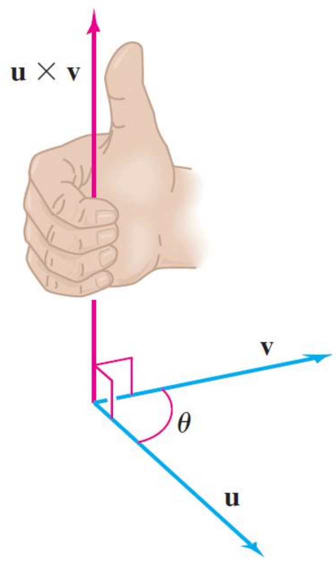
On the other hand, $(\boldsymbol{i}\times\boldsymbol{j})\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\boldsymbol{k} = 1 > 0.$ $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}\times\boldsymbol{v}$ has the same orientation as $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

The direction of $\boldsymbol{u}\times\boldsymbol{v}$ is given by the right hand rule:

When you put the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ tail to tail and let the fingers of you right hand curl from $\boldsymbol{u}$ to $\boldsymbol{v}$, the direction of $\boldsymbol{u}\times\boldsymbol{v}$ is the direction of your thumb, perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$ (see the below figure). However switching the order of $\boldsymbol{u}$ and $\boldsymbol{v}$ gives the opposite direction but in the same magnitude, that means

$$\boldsymbol{v}\times\boldsymbol{u} = -(\boldsymbol{u}\times\boldsymbol{v}).$$

When $\boldsymbol{u}\times\boldsymbol{v} = \boldsymbol{0}$, the direction of $\boldsymbol{u}\times\boldsymbol{v}$ is undefined.



Theorem 2.3.6

Given two nonzero vectors $\boldsymbol{u} = (u_1, u_2, u_3)$, and $\boldsymbol{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$, the magnitude of $\boldsymbol{u}\times\boldsymbol{v}$ is

$$|\boldsymbol{u}\times\boldsymbol{v}|=|\boldsymbol{u}||\boldsymbol{v}|\sin\theta,$$

where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ with $0\le\theta\le\pi$.

Solution

$$\begin{aligned}
|\boldsymbol{u}\times\boldsymbol{v}|^2 &= (u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2\\
&=(u_2v_3)^2-2(u_2v_3)(u_3v_2)+(u_3v_2)^2\\
&\quad+(u_3v_1)^2-2(u_3v_1)(u_1v_3)+(u_1v_3)^2\\
&\quad+(u_1v_2)^2-2(u_1v_2)(u_2v_1)+(u_2v_1)^2\\
&=(u_1v_1)^2+(u_1v_2)^2+(u_1v_3)^2+(u_2v_1)^2+(u_2v_2)^2+(u_2v_3)^2+(u_3v_1)^2+(u_3v_2)^2+(u_3v_3)^2\\
&\quad-\left[(u_1v_1)^2+(u_2v_2)^2+(u_3v_3)^2+2(u_1v_1)(u_2v_2)+2(u_2v_2)(u_3v_3)+2(u_1v_1)(u_3v_3)\right]\\
&=(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2\\
&=|\boldsymbol{u}|^2|\boldsymbol{v}|^2-(\boldsymbol{u}\cdot\boldsymbol{v})^2\\
&=|\boldsymbol{u}|^2|\boldsymbol{v}|^2-|\boldsymbol{u}|^2|\boldsymbol{v}|^2\cos^2\theta\\
&=|\boldsymbol{u}|^2|\boldsymbol{v}|^2\sin^2\theta
\end{aligned}$$

$|\boldsymbol{u}\times\boldsymbol{v}|=|\boldsymbol{u}||\boldsymbol{v}|\sin\theta$ where $0\le\theta\le\pi$.

Theorem 2.3.7

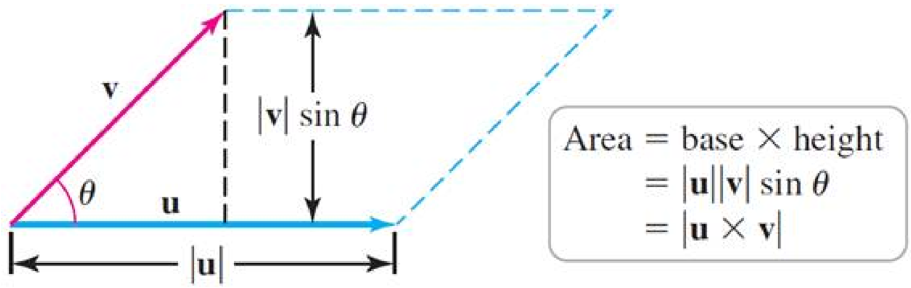
Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in $\mathbb{R}^3$. Suppose $\boldsymbol{u}\times\boldsymbol{v}=\boldsymbol{0}$ if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel.

Corollary 2.3.8

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two nonzero vectors in $\mathbb{R}^3$, If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two sides of a parallelogram, then the area of the parallelogram is

$$|\boldsymbol{u}\times\boldsymbol{v}|=|\boldsymbol{u}||\boldsymbol{v}|\sin\theta,$$

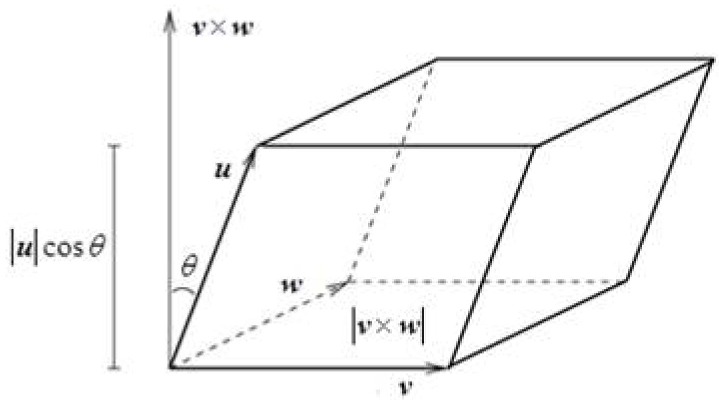
where $0\le\theta\le\pi$ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.



Theorem 2.3.9

The volume of the parallelepiped generated by the vectors generated by $\boldsymbol{u}=(u_1,u_2,u_3)$, $\boldsymbol{v}=(v_1,v_2,v_3)$ and $\boldsymbol{w}=(w_1,w_2,w_3)$ is the base area times its height, the absolute value of

$$|\boldsymbol{v}\times\boldsymbol{w}||\boldsymbol{u}|\cos\theta=\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w})=\begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}.$$



Example 2.3.10

Find the volume of triangular pyramid formed by (0, 0, 0), (1, 2, 3), (4, 5, 6) and (2, 1, 2).

Solution

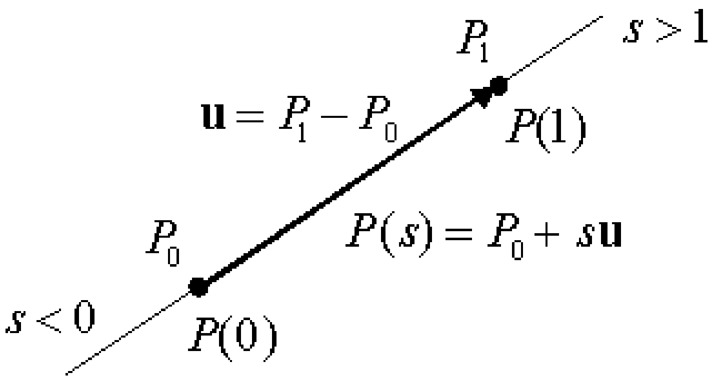
By Example 2.3.3, $\begin{vmatrix}2 & 1 & 2\\ 1 & 2 & 3\\ 4 & 5 & 6\end{vmatrix}=(-3,6,-3)\cdot(2,1,2)=-6$

volume $=\frac{1}{3}$(triangular base area)(height) $=\frac{1}{6}$(volume of the parallelepiped) $=\left|\frac{-6}{6}\right|=1$

In any dimension, the **vector equation** of a line L defined by two points P_0 and P_1 can be represented as:

$$P(s) = P_0 + s(P_1 - P_0) = P_0 + s\boldsymbol{u},$$

where the parameter s is a real number and $\boldsymbol{u} = P_1 - P_0$ is a direction vector. $P(s) = sP_1 + (1-s)P_0$ is a point on the line segment P_0P_1 when $0 \le s \le 1$. Further, if $s < 0$ then $P(s)$ is outside the segment on the P_0 side, and if $s > 1$ then $P(s)$ is outside the segment on the P_1 side.



Let us work out for the case in $\boldsymbol{R}^3$. Let $\boldsymbol{u} = (a, b, c)$ and $P_0 = (x_0, y_0, z_0)$. Then the **parametric equation** of L is

$$(x, y, z) = P(s) = P_0 + s\boldsymbol{u} = (x_0 + sa, y_0 + sb, z_0 + sc).$$

Another way of describing a line L is $(P(s) - P_0) // \boldsymbol{u}$ or $(P(s) - P_0) \times \boldsymbol{u} = (0,0,0)$.

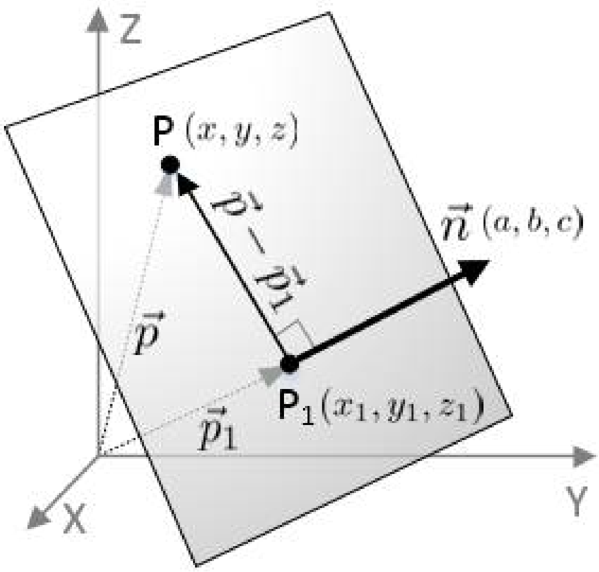
If none of a, b, or c is 0, we can solve each of these equations for s:

$$s = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

If one of a, b, or c is 0, we can still eliminate s. For instance, if $a = 0$, we could write the equations of L as

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This means that L lies in the vertical plane $x = x_0$.



Choose two distinct points $P=(x,y,z)$ and $P_1=(x_1,y_1,z_1)$ on the plane in $\mathbb{R}^3$. Any vector in the plane is perpendicular to vector (a, b, c). (a, b, c) is the normal vector of the plane.

An **equation of the plane** through point $P_1=(x_1,y_1,z_1)$ with normal vector $\boldsymbol{n}$ = (a, b, c) is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=(a,b,c)\cdot(x-x_1,y-y_1,z-z_1)=0$$

or $ax + by + cz + d = 0$ where $d=-(ax_1+by_1+cz_1)$.

Example 2.4.1

Find the equation of the plane that passes through the points

$$P_1=(1,1,2),\quad P_2=(2,3,3),\quad P_3=(3,-3,3).$$

Solution

Let $\boldsymbol{x}=\overrightarrow{P_1P_2}=(1,2,1)$ and $\boldsymbol{y}=\overrightarrow{P_1P_3}=(2,-4,1)$. The normal vector $\boldsymbol{n}$ must be orthogonal to both $\boldsymbol{x}$ and $\boldsymbol{y}$. If we set

$$\boldsymbol{n}=\boldsymbol{x}\times\boldsymbol{y}=(6,1,8),$$

then $\boldsymbol{n}$ will be a normal vector to the plane that passes through the given points. We can then use any one of the points to determine the equation of the plane. Using the point P_1, we see that the equation of the plane is

$$6(x-1)+(y-1)-8(z-2)=0.$$

Choose 3 distinct points P_1, P_2 and P_3 on a plane such that $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ are not parallel. The **parametric equation** of the plane passing through P_1, P_2 and P_3 is

$$(x, y, z) = P_1 + s\overrightarrow{P_1P_2} + t\overrightarrow{P_1P_3}.$$

The parametric equation of the plane in Example 2.4.1 is

$$(x, y, z) = (1, 1, 2) + s(1, 2, 1) + t(2, -4, 1) = (1 + s + 2t, 1 + 2s - 4t, 2 + s + t).$$

Example 2.4.2

(a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

(b) Find the parametric equation for the line of intersection L of these two planes.

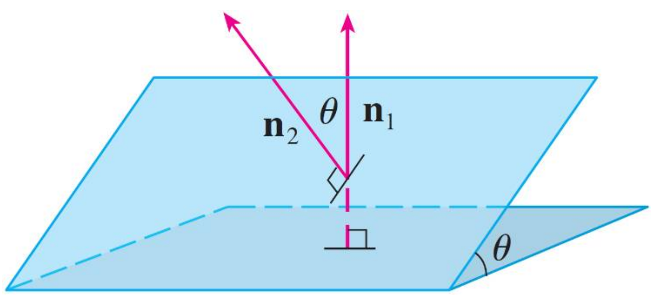
Solution

(a) The normal vectors of these planes are

$$\boldsymbol{n}_1 = (1, 1, 1) \text{ and } \boldsymbol{n}_2 = (1, -2, 3)$$

and so, if θ is the angle between the planes, then

$$\cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|} = \frac{1(1) + 1(-2) + 1(3)}{\sqrt{1+1+1} \cdot \sqrt{1+4+9}} = \frac{2}{\sqrt{42}}$$
$$\theta = \cos\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ.$$



(b) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -2 & 3 & 1 \end{array}\right) \xrightarrow{R_2 - R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & 2 & 0 \end{array}\right)$

That means $x+y+z=1$ and $-3y+2z=0$.

Let $z = 3s$. Then $y = 2s$ and $x=1-5s$. The parametric equation of L is

$$(x,y,z)=(1,0,0)+s(-5,2,3).$$

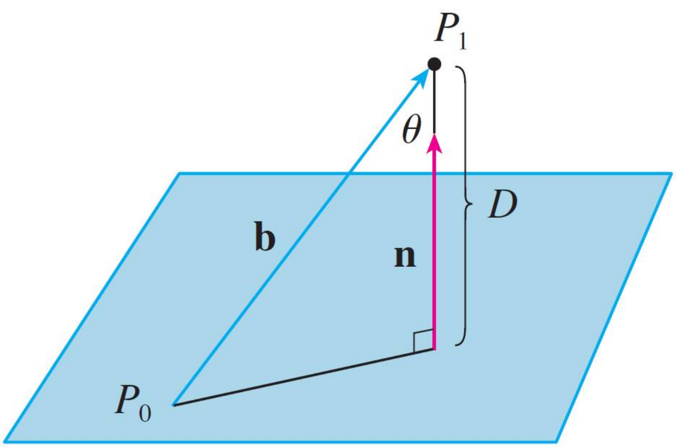
Example 2.4.3

Find a formula for the distance D from a point $P_1=(x_1,y_1,z_1)$ to the plane $ax + by + cz + d = 0$

Solution

Let $P_0=(x_0,y_0,z_0)$ be any point in the given plane. Then

$$\overrightarrow{P_0P_1}=(x_1-x_0,y_1-y_0,z_1-z_0)$$



You can see that the distance D from P_1 to the plane is equal to the absolute value of the scalar projection of $\overrightarrow{P_0P_1}$ onto the normal vector $\boldsymbol{n} = (a, b, c)$. Thus

$$\begin{aligned} D &= \frac{\left|\boldsymbol{n}\cdot\overrightarrow{P_0P_1}\right|}{|\boldsymbol{n}|} \\ &= \frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|ax_1+by_1+cz_1-ax_0-by_0-cz_0|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} \end{aligned}$$

Example 2.4.4

Find the distance between the parallel planes $10x+2y-2z=5$ and $5x+y-z=1$.

Solution

First we note that the planes are parallel because their normal vectors $(10,2,-2)$ and $(5,1,-1)$ are parallel. To find the distance D between the planes, we choose any point on one plane and calculate its distance to the other plane. In particular, if we put $y = z = 0$ in the equation of the first plane, we get $10x = 5$ and so (1/2, 0, 0) is a point in this plane. By Example 2.4.3, the distance between (1/2, 0, 0) and the plane $5x+y-z=1$ is

$$D=\frac{\left|5\cdot\frac{1}{2}+1\cdot 0-1\cdot 0-1\right|}{\sqrt{5^2+1^2+(-1)^2}}=\frac{\frac{3}{2}}{3\sqrt{3}}=\frac{\sqrt{3}}{6}$$

Nonparallel, nonintersecting lines are called **skew lines**.

Example 2.4.5 (distance between 2 skew lines)

Show that two lines $P(s)=(-3,-8,7)+(0,-1,1)s$ and $Q(t)=(6,3,0)+(-4,-3,0)t$ never meet. Find the distance between these 2 skew lines.

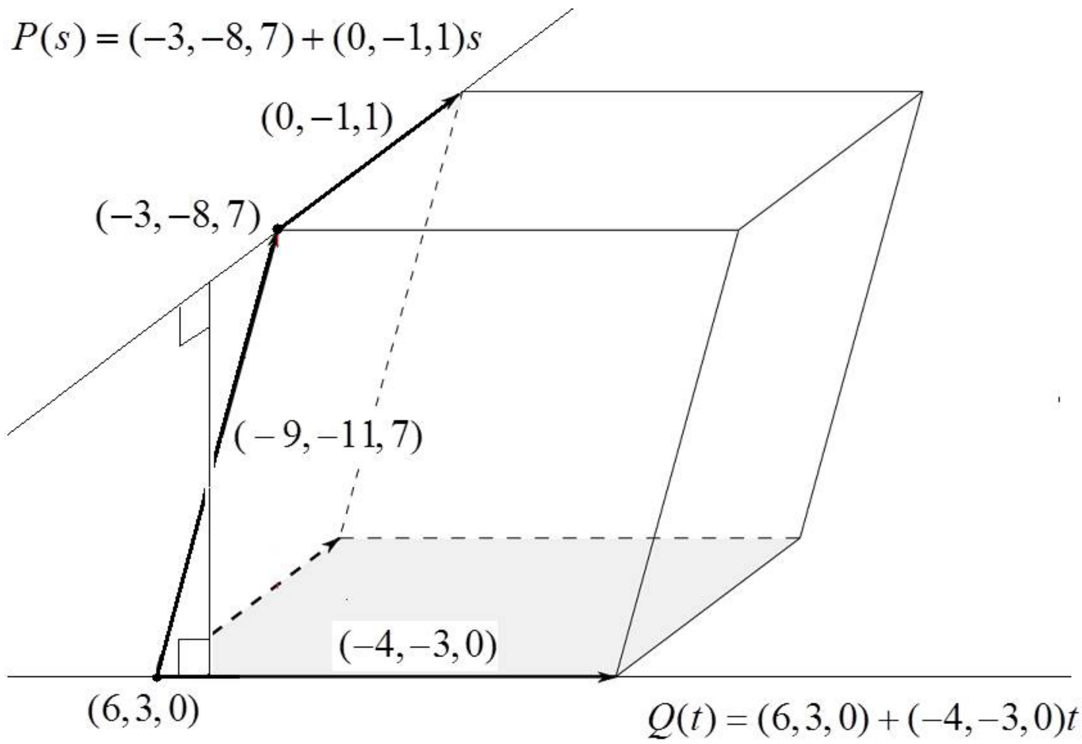
Solution

Suppose $(-3,-8,7)+(0,-1,1)s=(6,3,0)+(-4,-3,0)t$ for some s and t. That means

$$(-9,-11,7)+(0,-1,1)s-(-4,-3,0)t=(0,0,0) \qquad \text{or} \qquad \begin{pmatrix}-4 & 0 & -9\\ -3 & -1 & -11\\ 0 & 1 & 7\end{pmatrix}\begin{pmatrix}-t\\ s\\ 1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

However, $\det\begin{pmatrix}-4 & 0 & -9\\ -3 & -1 & -11\\ 0 & 1 & 7\end{pmatrix}=11\neq 0$. Hence $\begin{pmatrix}-t\\ s\\ 1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ impossible.

Distance between these 2 skew lines is height of parallelepiped.



Method 1:

The common normal vector must be orthogonal to both $(0,-1,1)$ and $(-4,-3,0)$ is

$$(0,-1,1)\times(-4,-3,0)=(3,-4,-4)$$

The equation of the plane passing through (6, 3, 0) with normal $(3,-4,-4)$ is

$$3(x-6)-4(y-3)-4z=0$$

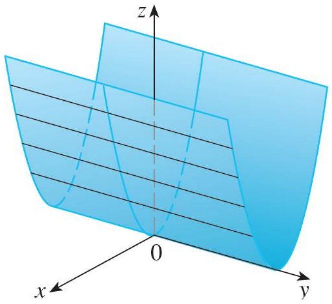
$$\begin{aligned}D&=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}\\&=\frac{|3(-3-6)-4(-8-3)-4(7)|}{\sqrt{3^2+(-4)^2+(-4)^2}}\\&=\frac{11}{\sqrt{41}}\end{aligned}$$

Method 2:

$$d=\frac{\text{volume of parallelopiped}}{\text{base area of parallelogram}}=\frac{\left|\det\begin{pmatrix}-4&-3&0\\0&-1&1\\-9&-11&7\end{pmatrix}\right|}{|(-4,-3,0)\times(0,-1,1)|}=\frac{11}{|(-3,4,4)|}=\frac{11}{\sqrt{41}}$$

In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces** (or cross-sections) of the surface.

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.



The surface $z = x^2$ is a parabolic cylinder.

A **quadric surface** is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

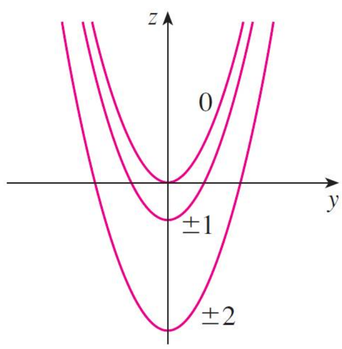
where $A, B, C, \ldots, J$ are constants, but by translation and rotation it can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

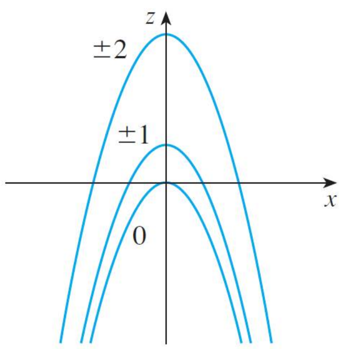
Example

Sketch the surface $z = y^2 - x^2$.

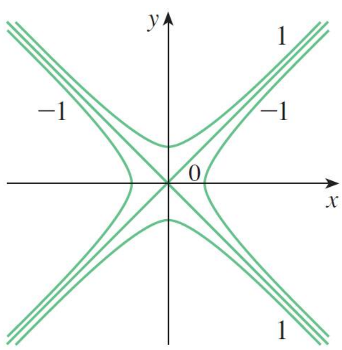
Solution



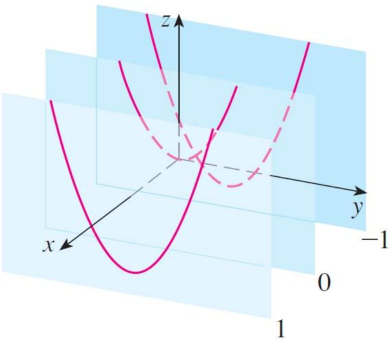
Traces in $x = k$ are $z = y^2 - k^2$.



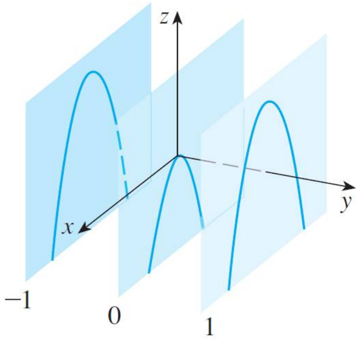
Traces in $y = k$ are $z = -x^2 + k^2$.



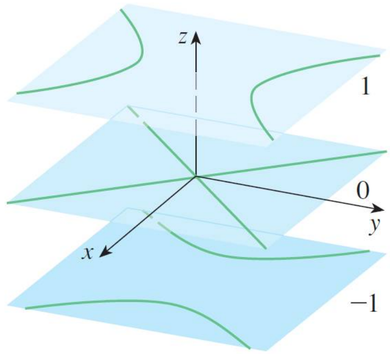
Traces in $z = k$ are $y^2 - x^2 = k$.



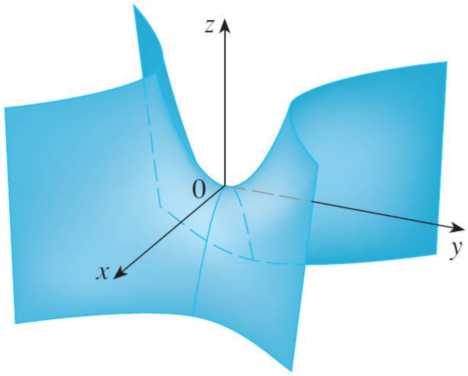
Traces in $x = k$

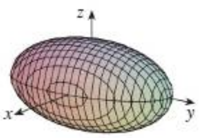
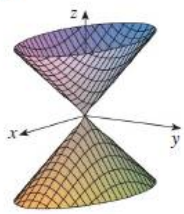
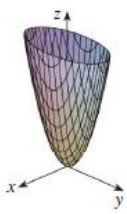
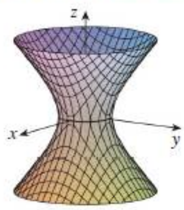
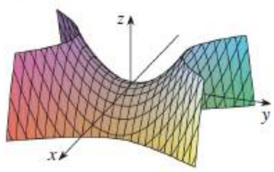
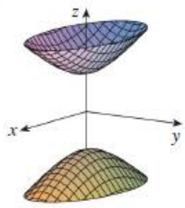


Traces in $y = k$



Traces in $z = k$

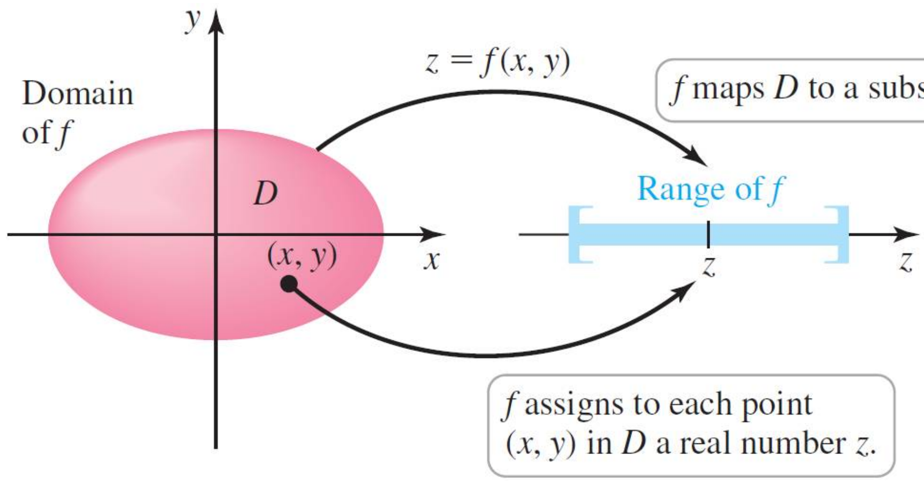


Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Chapter 3 Functions of Several Variables

Definition 3.1.1

A **function** f **of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. We sometimes write $f: D \subset \boldsymbol{R}^2 \to \boldsymbol{R}$. The set D is called the **domain** of f.



Example 3.1.2

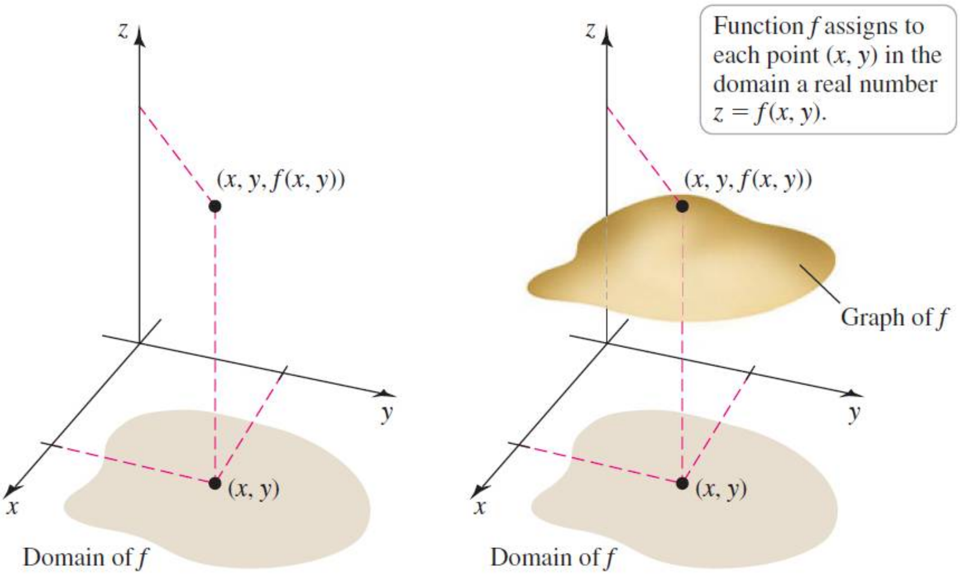
Clearly, domain of $f(x,y)=\sqrt{1-x^2-y^2}$ is $\{(x,y)\in \boldsymbol{R}^2 \mid x^2+y^2\le 1\}$ and domain of $g(x,y)=\dfrac{1}{x+y}$ is $\{(x,y)\in \boldsymbol{R}^2 \mid x+y\ne 0\}$.

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are **independent variables** and z is the **dependent variable**. [Compare this with the notation $y = f(x)$ for functions of a single variable.]

Likewise, a **function of three variables** is a rule that assigns a real number $f(x, y, z)$ to each ordered triple of real numbers (x, y, z) in the domain $D \subset \boldsymbol{R}^3$ of the function. We sometimes write $f: D \subset \boldsymbol{R}^3 \to \boldsymbol{R}$ to indicate that f maps points in three dimensions to real numbers.

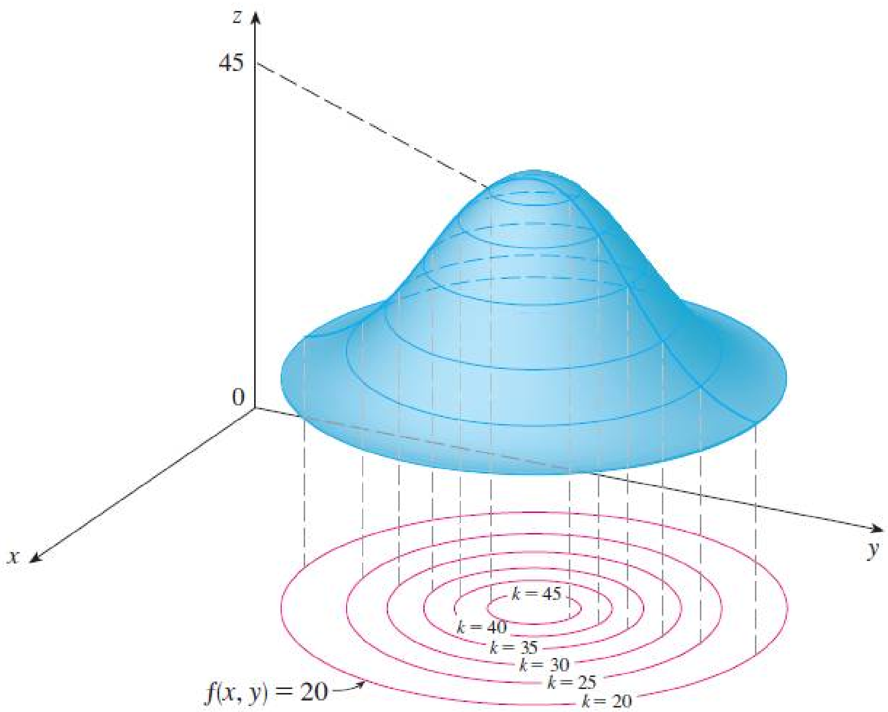
Definition 3.1.3

If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\boldsymbol{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.



Definition 3.1.4

The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the image of f, that is, $\{f(x, y) \mid (x, y) \in D\}$).



Definition 3.2.1

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of** $f(x, y)$ **as** (x, y) **approaches** (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

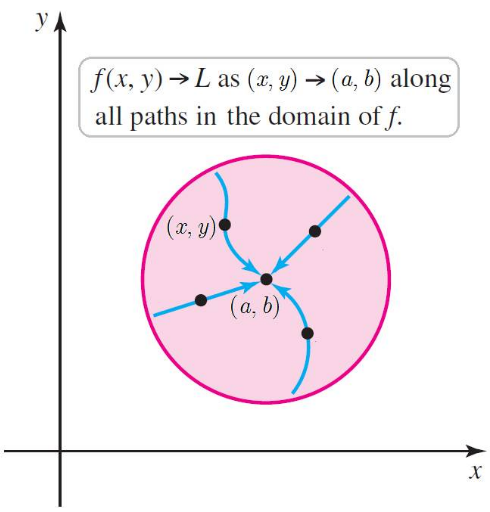
if for every $\varepsilon > 0$, there is a corresponding $\delta > 0$ such that

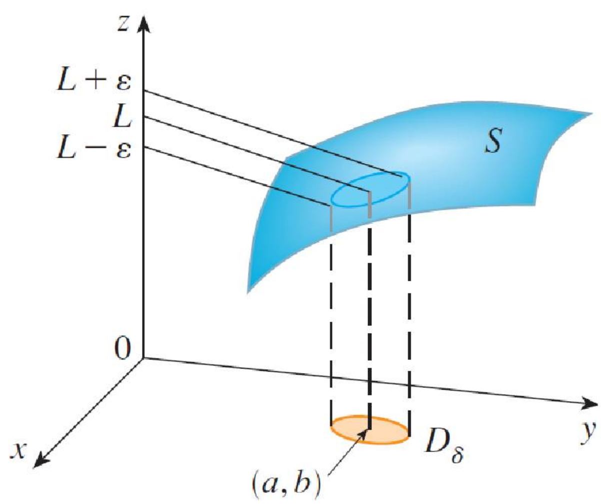
$$\text{if } (x,y) \in D \text{ and } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{then } |f(x,y) - L| < \varepsilon.$$

Definition 3.2.1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). The definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the direction of approach. Therefore, the limit exists only if $f(x, y)$ approaches L as (x, y) approaches (a, b) *along all possible paths* (not necessary straight path) in the domain of f.

Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b)

Unlike the case for functions of a single variable where there are just two paths approaching a given point (corresponding to left- and right-hand limits), in two dimensions there are infinitely many paths (and you obviously can't check each one individually).





If $f(x,y)\to L_1$ as $(x,y)\to(a,b)$ along a path C_1 and $f(x,y)\to L_2$ as $(x,y)\to(a,b)$ along a path C_2, where $L_1\neq L_2$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Keep in mind that $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ is neither $\lim\limits_{y\to b}\left(\lim\limits_{x\to a} f(x,y)\right)$ nor $\lim\limits_{x\to a}\left(\lim\limits_{y\to b} f(x,y)\right)$ even both limits exist and equal.

Example 3.2.2

Consider $f(x,y)=\dfrac{xy}{x^2+y^2}$.

If (x, y) approaches (0, 0) along $y = mx$, then $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=mx}} \dfrac{xy}{x^2+y^2}=\lim\limits_{x\to 0}\dfrac{mx^2}{x^2+m^2x^2}=\dfrac{m}{1+m^2}$.

Say for instance, $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=0}} \dfrac{xy}{x^2+y^2}=0$ and $\lim\limits_{\substack{(x,y)\to(0,0)\\ x=y}} \dfrac{xy}{x^2+y^2}=\lim\limits_{x\to 0}\dfrac{x^2}{x^2+x^2}=\dfrac{1}{2}$.

Since we have obtained different limits along different paths, the given limit does not exist.

Theorem 3.2.3

Suppose that $|f(x,y)-L|\leq g(x,y)$ for all (x, y) in the interior of some circle centered at (a, b), except possibly at (a, b). If $\lim\limits_{(x,y)\to(a,b)} g(x,y)=0$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)=L$.

Example 3.2.4

Consider $f(x,y)=\dfrac{x^2y}{x^2+y^2}$.

$\left|\dfrac{x^2y}{x^2+y^2}\right|=\left|\dfrac{x^2}{x^2+y^2}\right||y|\leq\left|\dfrac{x^2+y^2}{x^2+y^2}\right||y|=|y|$ and $\lim\limits_{(x,y)\to(0,0)}|y|=0$. Therefore $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y}{x^2+y^2}=0$.

Theorem 3.2.5 (Limit Laws for Functions of Two Variables)

Let L and M be real numbers and suppose that $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = M$.

Assume c is a constant, and m and n are integers.

(a) $\lim_{(x,y)\to(a,b)} \left(f(x,y) \pm g(x,y) \right) = L \pm M$

(b) $\lim_{(x,y)\to(a,b)} \left(cf(x,y) \right) = cL$

(c) $\lim_{(x,y)\to(a,b)} \left(f(x,y) g(x,y) \right) = LM$

(d) $\lim_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$, provided $M \neq 0$.

(e) $\lim_{(x,y)\to(a,b)} \left(f(x,y)^n \right) = L^n$

(f) If m and n have no common factors and $n \neq 0$, then $\lim_{(x,y)\to(a,b)} \left(f(x,y)^{m/n} \right) = L^{m/n}$, where we assume $L > 0$ if n is even.

Definition 3.2.6

A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

Theorem 3.2.7

Suppose that $f(x, y)$ is continuous at (a, b) and $g(x)$ is continuous at the point $f(a, b)$. Then

$$g \circ f(x,y) = g(f(x,y))$$

is continuous at (a, b).

Polynomials, trigonometric functions, inverse trigonometric functions, rational functions, algebraic functions, exponential functions and logarithmic functions are continuous at all points of their domains, and limits of these functions may be evaluated by direct substitution at all points of their domains.

Example 3.2.8

Determine the points at which the following function is continuous.

$$f(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^4} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

Solution

The function $\dfrac{xy^2}{x^2+y^4}$ is a rational function, so it is continuous at all points in $\boldsymbol{R}^2$ except (0, 0).

It is easy to verify that $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=mx}} \dfrac{xy^2}{x^2+y^4}=0=f(0,0).$

However, consider parabolic paths of the form $x=my^2$, where m is a nonzero constant.

$$\lim_{\substack{(x,y)\to(0,0)\\ x=my^2}} \frac{xy^2}{x^2+y^4}=\lim_{y\to 0}\frac{(my^2)y^2}{(my^2)^2+y^4}=\lim_{y\to 0}\frac{my^4}{(1+m^2)y^4}=\frac{m}{1+m^2}.$$

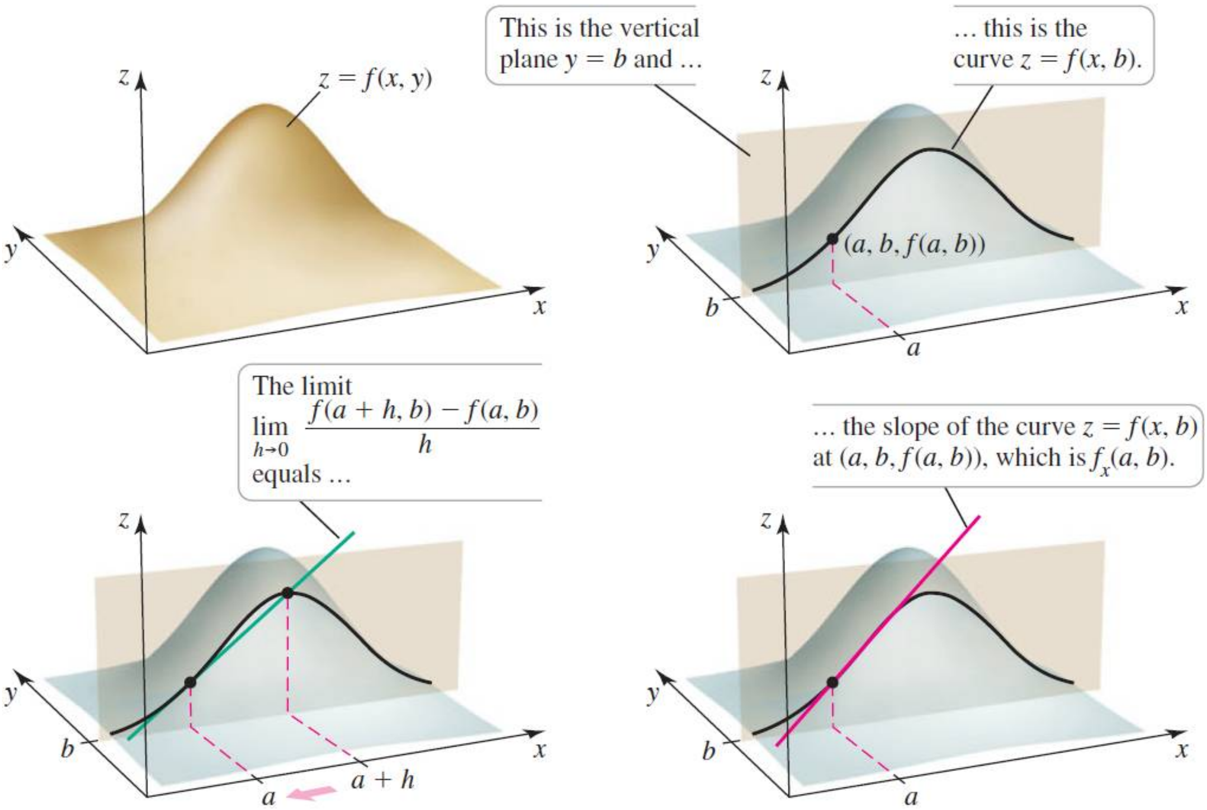
Because function values approach two different numbers along two different paths, the limit at (0, 0) does not exist, and f is not continuous at (0, 0).

The work we have done with limits and continuity of functions of two variables extends to functions of three or more variables. Specifically, the limit laws of Theorem 3.2.5 apply to functions of the form $w=f(x, y, z)$. Compositions of continuous functions (Theorem 3.2.7) of the form $f(g(x, y, z))$ are also continuous.

Suppose we move along the surface $z = f(x, y)$, starting at the point $(a, b, f(a, b))$ in such a way that $y = b$ is fixed and only x varies. The resulting path is a curve (a trace) on the surface that varies in the x-direction. This curve is the intersection of the surface with the vertical plane $y = b$; it is described by $z = f(x, b)$, which is a function of the single variable x. We know how to compute the slope of this curve: It is the ordinary derivative of $f(x, b)$ with respect to x. This derivative is called the *partial derivative of f with respect to x*, denoted $\partial f / \partial x$ or f_x. When evaluated at (a, b) its value is defined by the limit

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h},$$

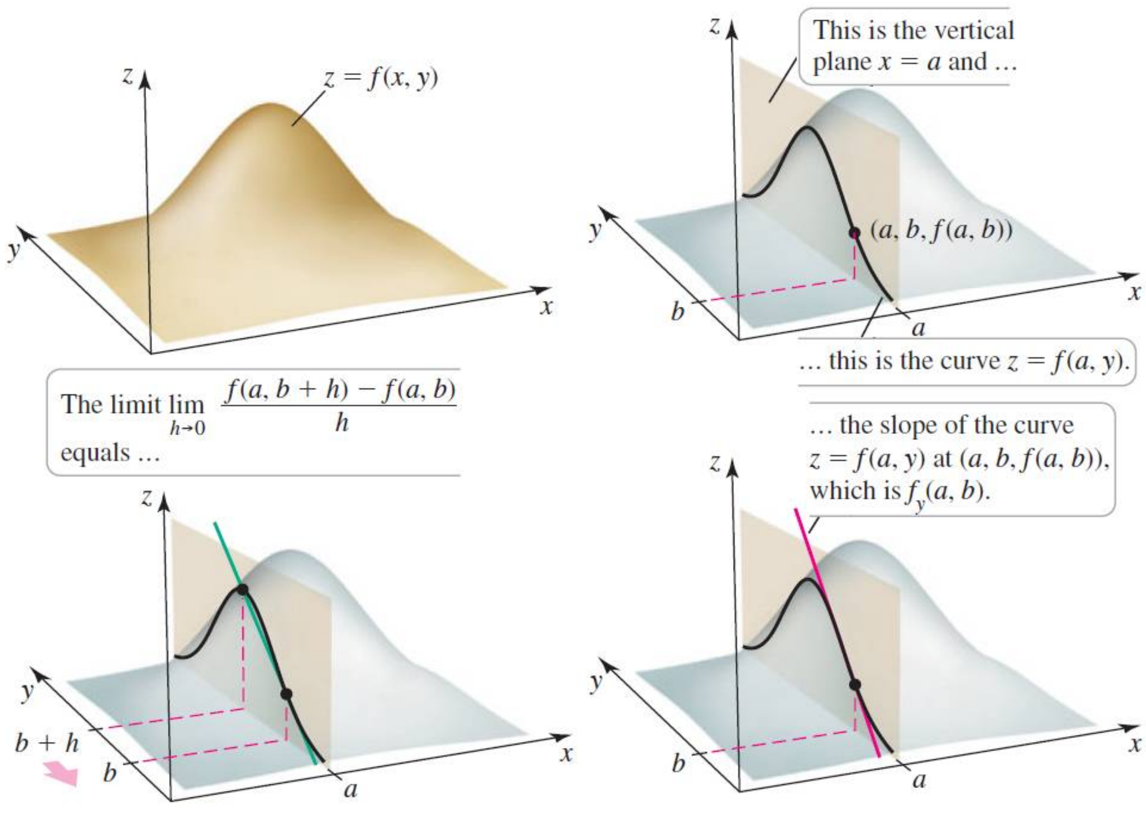
provided this limit exists. Notice that the y-coordinate is fixed at $y = b$ in this limit. If we replace (a, b) by the variable point (x, y), then f_x becomes a function of x and y.



In a similar way, we can move along the surface $z = f(x, y)$ from the point $(a, b, f(a, b))$ in such a way that $x = a$ is fixed and only y varies. Now, the result is a trace described by $z = f(a, y)$, which is the intersection of the surface and the plane $x = a$. The slope of this curve at (a, b) is given by the ordinary derivative of $f(a, y)$ with respect to y. This derivative is called the *partial* *derivative of f with respect to y*, denoted $\partial f/\partial y$ or f_y. When evaluated at (a, b), it is defined by the limit

$$f_y(a,b)=\lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h},$$

provided this limit exists. If we replace (a, b) by the variable point (x, y), then f_y becomes a function of x and y.



Definition 3.3.1

The **partial derivative of** f **with respect to** x **at the point** (a, b) is

$$f_x(a,b)=\lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}.$$

The **partial derivative of** f **with respect to** y **at the point** (a, b) is

$$f_y(a,b)=\lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h},$$

provided these limits exist. We write

$$f_x(x,y)=f_x=\frac{\partial f}{\partial x}=f_1=D_1f=D_xf \quad \text{and} \quad f_y(x,y)=f_y=\frac{\partial f}{\partial y}=f_2=D_2f=D_yf.$$

Example 3.3.2

Let $f(x,y)=x^3+x^2y^3-2y^2$. Calculate $f_x(2,1)$ and $f_y(2,1)$.

Solution

Holding y constant and differentiating with respect to x, we get

$$f_x(x,y)=3x^2+2xy^3 \quad \text{and} \quad f_x(2,1)=3\cdot 2^2+2\cdot 2\cdot 1^3=16.$$

Holding x constant and differentiating with respect to y, we get

$$f_y(x,y)=3x^2y^2-4y \quad \text{and} \quad f_y(2,1)=3\cdot 2^2\cdot 1^2-4\cdot 1=8$$

Partial derivatives can also be defined for functions of three or more variables. In general, if u is a function of n variables, $u=f(x_1,x_2,\cdots,x_n)$, its partial derivative with respect to the i-th variable x_i is

$$\frac{\partial u}{\partial x_i}=\lim_{h\to 0}\frac{f(x_1,\cdots,x_i+h,\cdots x_n)-f(x_1,\cdots,x_i,\cdots,x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i}=\frac{\partial f}{\partial x_i}=f_{x_i}=f_i=D_if.$$

Just as we have higher-order derivatives of functions of one variable, we also have higher order partial derivatives for two variables. If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x,(f_x)_y,(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f. If $z=f(x,y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$ first differentiate with respect to x and then with respect to y

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$ first differentiate with respect to y and then with respect to x

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Example 3.3.3

Find the second partial derivatives of $f(x,y) = x^3 + x^2y^3 - 2y^2$.

Solution

In Example 3.3.2 we found that

$$f_x(x,y) = 3x^2 + 2xy^3 \qquad \text{and} \qquad f_y(x,y) = 3x^2y^2 - 4y$$

Therefore

$$f_{xx}(x,y) = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x + 2y^3, \qquad f_{xy}(x,y) = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2,$$

$$f_{yx}(x,y) = \frac{\partial}{\partial x}(3x^2y^2 - 4y) = 6xy^2, \qquad f_{yy}(x,y) = \frac{\partial}{\partial y}(3x^2y^2 - 4y) = 6x^2y - 4$$

Example 3.3.4

Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$ if $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Solution

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{(x^2+y^2)(3x^2y - y^3) - 2x^2y(x^2-y^2)}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{k\to 0}\frac{\frac{\partial f}{\partial x}(0,k)-\frac{\partial f}{\partial x}(0,0)}{k} = \lim_{k\to 0}\frac{1}{k}\cdot\frac{-k^5}{k^4} = -1$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{0-0}{k} = 0$$

$$\frac{\partial f}{\partial y}(x,y) = -\frac{xy^4+4x^3y^2-x^5}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h\to 0}\frac{\frac{\partial f}{\partial y}(h,0)-\frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h\to 0}\frac{1}{h}\cdot\frac{h^5}{h^4} = 1$$

Theorem 3.3.5 (Clairaut)

Assume that f is defined on an open set D of $\boldsymbol{R}^2$, and f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

The reason why Theorem 3.3.5 fails in Example 3.3.4 is the following:

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}\right) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} = \frac{\partial}{\partial x}\left(-\frac{xy^4+4x^3y^2-x^5}{(x^2+y^2)^2}\right) = f_{yx}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}}\frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} = \lim_{y\to 0}\frac{-y^6}{y^6} = -1, \quad \lim_{\substack{(x,y)\to(0,0)\\ y=0}}\frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} = \lim_{x\to 0}\frac{x^6}{x^6} = 1$$

Both $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ and $\lim_{(x,y)\to(0,0)} f_{yx}(x,y)$ are not exist. Hence both f_{xy} and f_{yx} are not continuous at (0, 0).

Example 3.3.6

$\dfrac{\partial^2}{\partial x \partial y}\left(\dfrac{e^{xy}}{x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{xe^{xy}}{x}\right) = \dfrac{\partial e^{xy}}{\partial x} = ye^{xy}$ but $\dfrac{\partial^2}{\partial y \partial x}\left(\dfrac{e^{xy}}{x}\right)$ is more complicated.

We can, of course, define third-, fourth- or even higher-order partial derivatives. Theorem 3.3.5 can be extended to show that as long as the partial derivatives are all continuous in an open set, the order of differentiation doesn't matter.

Recall that a function f of one variable is differentiable at $x = a$ provided the limit

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

exists. We define the quantity

$$\varepsilon = \underbrace{\frac{f(a+\Delta x) - f(a)}{\Delta x}}_{\text{slope of secant}} - \underbrace{f'(a)}_{\substack{\text{slope of} \\ \text{tangent}}},$$

where ε is viewed as a function of Δx and $\lim_{\Delta x \to 0} \varepsilon = 0$. Rearranging terms,

$$\Delta y = f(a+\Delta x) - f(a) = f'(a)\Delta x + \varepsilon \Delta x.$$

The analogous requirement with several variables is the definition of differentiability for functions or two (or more) variables.

Definition 3.4.1

The function $z = f(x, y)$ is **differentiable at** (a, b) provided $f_x(a,b)$ and $f_y(a,b)$ exist and the change $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$ equals

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where for fixed a and b, ε_1 and ε_2 are functions that depend only on Δx and Δy, with $(\varepsilon_1, \varepsilon_2) \to (0,0)$ as $(\Delta x, \Delta y) \to (0,0)$. A function is **differentiable** on an open set D if it is differentiable at every point of D.

Theorem 3.4.2

Suppose the function f has partial derivatives f_x and f_y defined on an open set containing (a, b), with f_x and f_y continuous at (a, b). Then f is differentiable at (a, b).

Proof

$$
\begin{aligned}
\Delta z &= f(a+\Delta x, b+\Delta y) - f(a,b) \\
&= [f(a+\Delta x, b+\Delta y) - f(a, b+\Delta y)] + [f(a, b+\Delta y) - f(a,b)] \\
&= f_x(u, b+\Delta y)[(a+\Delta x) - a] + f_y(a,v)[(b+\Delta y) - b] \quad \text{by Mean Value Theorem} \\
&= f_x(u, b+\Delta y)\Delta x + f_y(a,v)\Delta y
\end{aligned}
$$

Here, u is some value between a and $a+\Delta x$, and v is some value between b and $b+\Delta y$.

$$
\begin{aligned}
\Delta z &= f_x(u, b+\Delta y)\Delta x + f_y(a,v)\Delta y \\
&= f_x(a,b)\Delta x + [f_x(u, b+\Delta y) - f_x(a,b)]\Delta x + f_y(a,b)\Delta y + [f_y(a,v) - f_y(a,b)]\Delta y \\
&= f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y
\end{aligned}
$$

where $\varepsilon_1 = f_x(u, b+\Delta y) - f_x(a,b)$ and $\varepsilon_2 = f_y(a,v) - f_y(a,b)$. Finally, observe that if f_x and f_y are both continuous in some open region containing (a, b), then $(\varepsilon_1, \varepsilon_2) \to (0,0)$ as $(\Delta x, \Delta y) \to (0,0)$.

Theorem 3.4.3

If a function f is differentiable at (a, b), then it is continuous at (a, b).

Proof

By the definition of differentiability,

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $(\varepsilon_1, \varepsilon_2) \to (0,0)$ as $(\Delta x, \Delta y) \to (0,0)$. That means

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \Delta z = 0.$$

Also, because $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$, it follows that

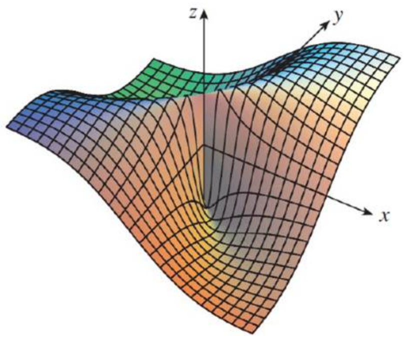
$$\lim_{(\Delta x, \Delta y) \to (0,0)} f(a+\Delta x, b+\Delta y) = f(a,b),$$

which implies continuity of f at (a, b).

Example 3.4.4

Discuss the differentiability and continuity of the function

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

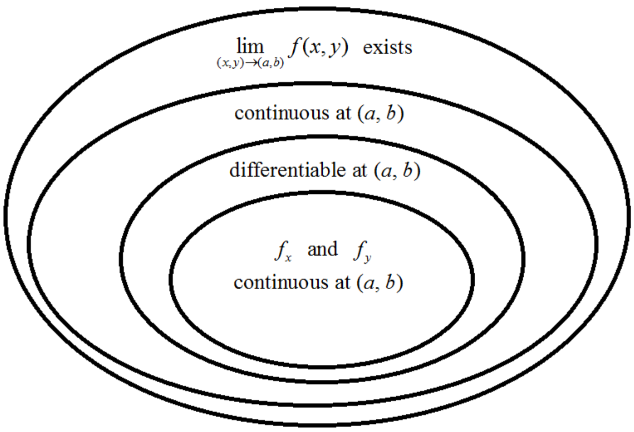


Solution

In Example 3.2.2, $\lim_{(x,y)\to(0,0)}\dfrac{xy}{x^2+y^2}$ does not exist. So f is not continuous at (0, 0). By Theorem 3.4.3, it follows that f is not differentiable at (0, 0). The above figure shows the discontinuity of f at the origin. Furthermore

$$f_x(0,0)=\lim_{h\to0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{0-0}{h}=0$$
$$f_y(0,0)=\lim_{h\to0}\frac{f(0,0+h)-f(0,0)}{h}=\lim_{h\to0}\frac{0-0}{h}=0$$

Despite the fact that f is not differentiable at (0, 0), its first partial derivatives exist at (0, 0). Existence of first partial derivatives at a point is not enough to ensure differentiability at that point.



Example 3.4.5

Let $f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$ Determine whether f is differentiable at (0,0) or not.

Solution

In Example 3.2.4, f is continuous at (0, 0). We cannot use the trick "not continuous at (0, 0) implies not differentiable at (0, 0)". On the other hand,

$$f_x(x,y)=\frac{2xy^3}{(x^2+y^2)^2} \quad \text{and} \quad f_y(x,y)=\frac{x^2(x^2-y^2)}{(x^2+y^2)^2}$$

are not continuous at (0, 0). Theorem 3.4.2 does not work. We need to use Definition 3.4.1.

Clearly, $f_x(0,0)=f_y(0,0)=0$. Suppose it is differentiable at (0, 0). We have

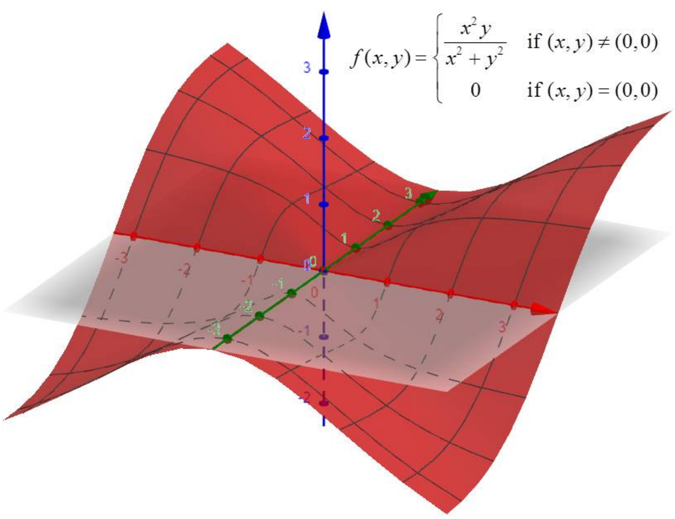
$$\begin{aligned}\Delta z &= f_x(0,0)h+f_y(0,0)k+\varepsilon_1h+\varepsilon_2k\\ f(h,k)-f(0,0)&=\varepsilon_1h+\varepsilon_2k\\ \frac{h^2k}{h^2+k^2}&=\varepsilon_1h+\varepsilon_2k\end{aligned}$$

where $(\varepsilon_1,\varepsilon_2)\to(0,0)$ as $(h,k)\to(0,0)$. Consider $(h,k)\to(0,0)$ along direction $h=k$,

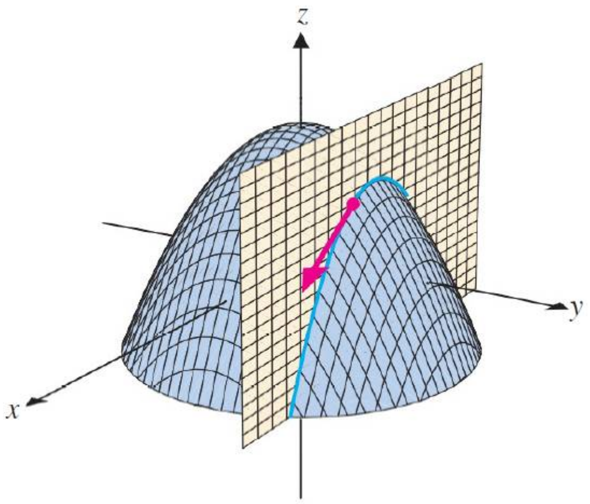
$$\frac{h^2k}{h^2+k^2}=\varepsilon_1h+\varepsilon_2k \Rightarrow \frac{h^3}{2h^2}=(\varepsilon_1+\varepsilon_2)h$$

$$\Rightarrow \varepsilon_1+\varepsilon_2=\frac{h^3}{2h^3}=\frac{1}{2}\not\longrightarrow 0 \quad \text{as} \quad h\to 0$$

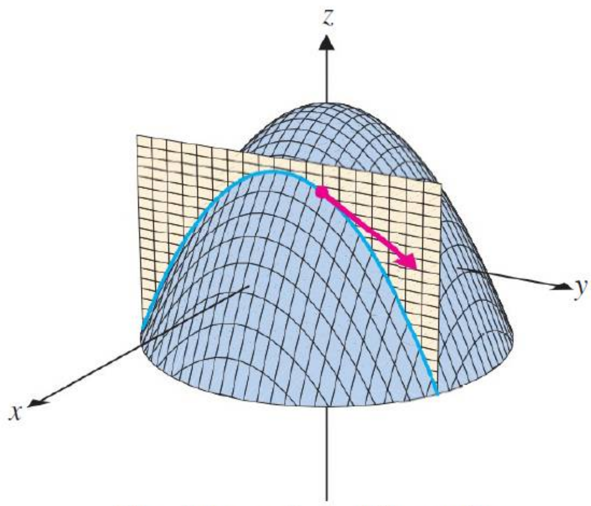
That is $(\varepsilon_1,\varepsilon_2)\not\longrightarrow(0,0)$. f is not differentiable at $(0,0)$.



Suppose f is differentiable at (a, b). Imagine intersecting the surface $z = f(x, y)$ with the plane $y = b$, as shown in bottom left figure. A curve in the plane $y = b$ whose slope at $x = a$ is given by $f_x(a,b)$. Along the tangent line at $x = a$, a change of 1 unit in x corresponds to a change of $f_x(a,b)$ in z. A vector with the same direction as the tangent line is $\langle 1, 0, f_x(a,b)\rangle$. Similarly, intersecting the surface $z = f(x, y)$ with the plane $x = a$, as shown in bottom right, we get a curve lying in the plane $x = a$, whose slope at $y = b$ is given by $f_y(a,b)$. A vector with the same direction as the tangent line at $y = b$ is then $\langle 0, 1, f_y(a,b)\rangle$.



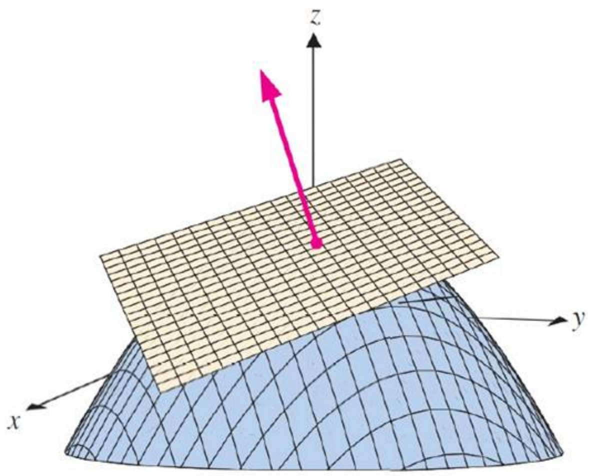
The intersection of the surface $z = f(x, y)$ with the plane $y = b$



The intersection of the surface $z = f(x, y)$ with the plane $x = a$

A vector normal to the plane is then given by the cross product:

$$\langle 0, 1, f_y(a,b)\rangle \times \langle 1, 0, f_x(a,b)\rangle = \langle f_x(a,b), f_y(a,b), -1\rangle.$$



Tangent plane and normal vector

Theorem 3.5.1

Let f be differentiable at the point (a, b). An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

The normal line of the tangent plane at $(a, b, f(a, b))$ is given by

$$(x, y, z) = (a, b, f(a,b)) + t\left\langle f_x(a,b),\ f_y(a,b), -1\right\rangle.$$

Example 3.5.2

Find equations of the tangent plane and the normal line to $z = 6 - x^2 - y^2$ at the point (1, 2, 1).

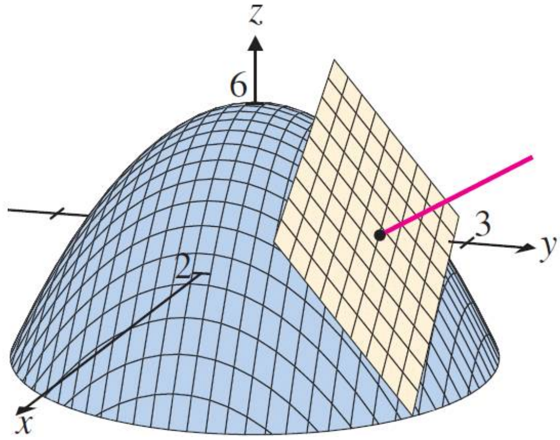
Solution

For $f(x, y) = 6 - x^2 - y^2$, we have $f_x = -2x$ and $f_y = -2y$. This gives us $f_x(1, 2) = -2$ and $f_y(1, 2) = -4$. f_x and f_y are continuous everywhere. By Theorem 3.4.2, f is differentiable everywhere. A normal vector is then $\langle -2, -4, -1\rangle$ and from Theorem 3.5.1, an equation of the tangent plane is

$$z = 1 - 2(x-1) - 4(y-2).$$

Furthermore, equations of the normal line are

$$(x, y, z) = (1, 2, 1) + t\langle -2, -4, -1\rangle.$$



In the one-variable case, if f is differentiable at a, the linear approximation to the curve $y = f(x)$ at the point $(a, f(a))$ is

$$L(x) = f(a) + f'(a)(x - a).$$

The two-variable case is analogous.

Definition 3.5.3 (Linear Approximation)

Let f be differentiable at (a, b). The linear approximation to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the tangent plane at that point, given by the equation

$$L(x, y) = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b).$$

Example 3.5.4

Let $f(x, y) = \dfrac{5}{x^2 + y^2}$.

(a) Find the linear approximation to the function at the point $(-1, 2, 1)$.

(b) Use the linear approximation to estimate the value of $f(-1.05, 2.1)$.

Solution

(a) The partial derivatives of f are

$$f_x = -\frac{10x}{(x^2 + y^2)^2} \quad \text{and} \quad f_x = -\frac{10y}{(x^2 + y^2)^2}.$$

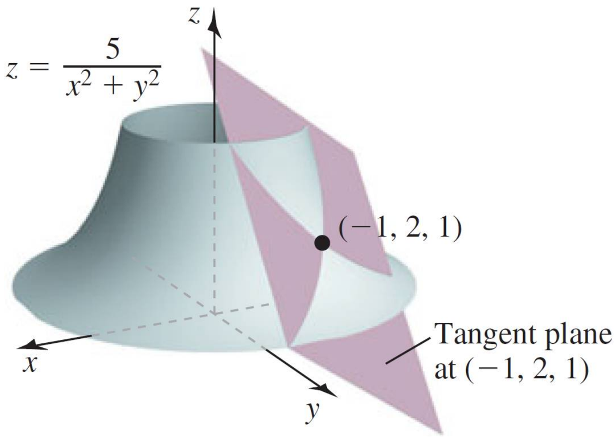
f_x and f_y are continuous everywhere except (0, 0). By Theorem 3.4.2, f is differentiable everywhere except $(0, 0)$. Evaluated at $(-1, 2)$, we have $f_x(-1, 2) = \frac{2}{5}$ and $f_y(-1, 2) = -\frac{4}{5}$. Therefore, the linear approximation to the function at $(-1, 2, 1)$ is

$$\begin{aligned} L(x, y) &= f(-1,2) + f_x(-1,2)(x + 1) + f_y(-1,2)(y - 2) \\ &= 1 + \tfrac{2}{5}(x + 1) - \tfrac{4}{5}(y - 2) \end{aligned}$$

(b) The value of the function at the point $(-1.05, 2.1)$ is approximated by the value of the linear approximation at that point, which is

$$L(-1.05, 2.1) = 1 + \tfrac{2}{5}(-0.05) - \tfrac{4}{5}(0.1) = 0.9.$$

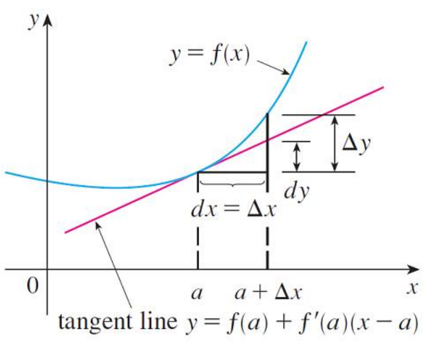
The actual value of $f(-1.05, 2.1) \approx 0.907$.

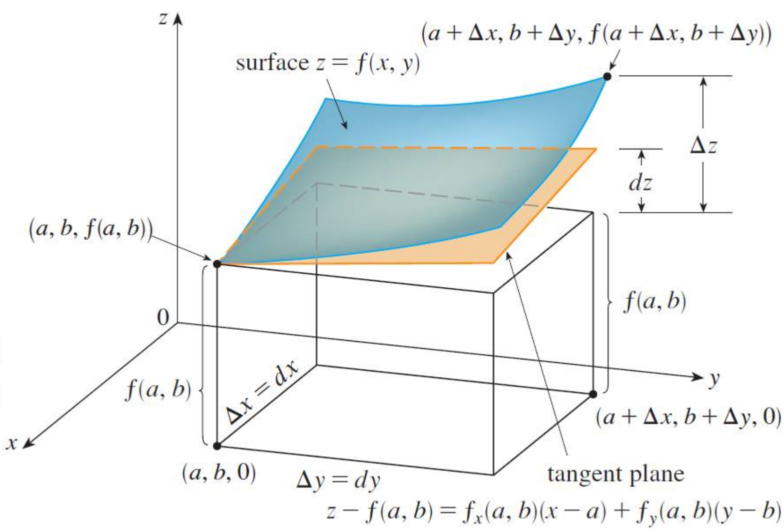


For a differentiable function of one variable, $y = f(x)$, we define the differential dx to be an independent variable; that is, dx can be given the value of any real number. The differential of y is then defined as

$$dy = f'(x)dx.$$

The below figure on the left shows the relationship between the increment Δy and the differential dy: Δy represents the change in height of the curve $y = f(x)$ and dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.





For functions of the form $z = f(x, y)$, we start with the linear approximation to the surface

$$f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

The exact change in the function between the points (a, b) and (x, y) is

$$\Delta z = f(x,y) - f(a,b).$$

Replacing $f(x, y)$ by its linear approximation, the change Δz is approximated by

$$\Delta z \approx \overbrace{L(x,y) - f(a,b)}^{dz} = \underbrace{f_x(a,b)\overbrace{(x-a)}^{dx}}_{\substack{\text{change in } z \text{ due} \\ \text{to change in } x}} + \underbrace{f_y(a,b)\overbrace{(y-b)}^{dy}}_{\substack{\text{change in } z \text{ due} \\ \text{to change in } y}}.$$

Definition 3.5.5

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz, also called the **total differential**, is defined by

$$dz = f_x(a,b)dx + f_y(a,b)dy.$$

Example 3.5.6

Let $z = f(x,y) = \dfrac{5}{x^2 + y^2}$. Approximate the change in z when the independent variables change from $(-1,2)$ to $(-0.93,1.94)$.

Solution

If the independent variables change from $(-1,2)$ to $(-0.93,1.94)$, then $dx = 0.07$ (an increase) and $dy = -0.06$ (a decrease). Using the values of the partial derivatives evaluated in Example 3.5.4, the corresponding change in z is approximately

$$dz = f_x(-1,2)dx + f_y(-1,2)dy = 0.4(0.07) + (-0.8)(-0.06) = 0.076.$$

The actual change is $\Delta z = f(-0.93,1.94) - f(-1,2) \approx 0.08$.

Recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

We now extend the chain rule to functions of several variables. This takes several slightly different forms, depending on the number of independent variables, but each is a variation of the already familiar chain rule for functions of a single variable.

Theorem 3.6.1 (Chain Rule)

If $z = f(x(t), y(t))$, where $x(t)$ and $y(t)$ are differentiable and $f(x, y)$ is a differentiable function of x and y, then

$$\frac{dz}{dt} = \frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

With $z = f(x(t), y(t))$, the dependent variable is z and the sole independent variable is t. x and y are **intermediate variables**.

Proof

Since $f(x, y)$ is differentiable, from Definition 3.4.1 we have

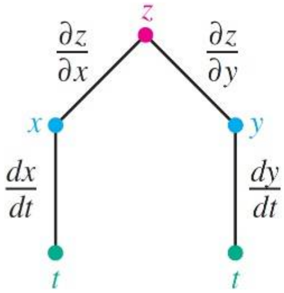
$$\Delta z = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $(\varepsilon_1, \varepsilon_2) \to (0,0)$ as $(\Delta x, \Delta y) \to (0,0)$. Dividing both sides of this equation by Δt, we have

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x} \cdot \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \cdot \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}.$$

Since $x(t)$ and $y(t)$ are differentiable, $(\Delta x, \Delta y) \to (0,0)$ as $\Delta t \to 0$. Hence $\lim_{\Delta t \to 0} \varepsilon_1 = \lim_{\Delta t \to 0} \varepsilon_2 = 0$.

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x} \cdot \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \cdot \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} + \left(\lim_{\Delta t \to 0} \varepsilon_1 \right) \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \left(\lim_{\Delta t \to 0} \varepsilon_2 \right) \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\begin{aligned}\frac{d^2z}{dt^2} &= \frac{\partial f}{\partial x}\cdot\frac{d^2x}{dt^2} + \left(\frac{\partial^2 f}{\partial x^2}\cdot\frac{dx}{dt} + \frac{\partial^2 f}{\partial y\partial x}\cdot\frac{dy}{dt}\right)\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{d^2y}{dt^2} + \left(\frac{\partial^2 f}{\partial x\partial y}\cdot\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\cdot\frac{dy}{dt}\right)\cdot\frac{dy}{dt} \\ &= \frac{\partial f}{\partial x}\cdot\frac{d^2x}{dt^2} + \frac{\partial f}{\partial y}\cdot\frac{d^2y}{dt^2} + \frac{\partial^2 f}{\partial x^2}\cdot\left(\frac{dx}{dt}\right)^2 + 2\frac{\partial^2 f}{\partial x\partial y}\cdot\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial^2 f}{\partial y^2}\cdot\left(\frac{dy}{dt}\right)^2 \quad \text{if } f_{xy} = f_{yx}\end{aligned}$$

Example 3.6.2

If $z = x^4 + y^4$, where $x = \sin t$ and $y = \cos t$, find dz/dt when $t = \pi/6$.

Solution

Clearly, $f_x(x,y) = 4x^3$ and $f_y(x,y) = 4y^3$. Both first partial derivatives are continuous everywhere. z is differentiable everywhere by Theorem 3.4.2.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = 4x^3\cos t + 4y^3(-\sin t).$$

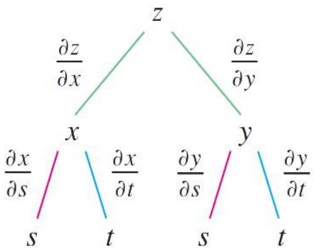
At $t = \pi/6 : x = 1/2,\ y = \sqrt{3}/2.$ $f_x(1/2, \sqrt{3}/2) = \frac{1}{2}.$ $f_y(1/2, \sqrt{3}/2) = \frac{3\sqrt{3}}{2}$

$$\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{6}} = \frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{3\sqrt{3}}{2}\cdot\frac{1}{2} = -\frac{\sqrt{3}}{2}.$$

Theorem 3.6.3

Suppose that $z = f(x, y)$, where f is a differentiable function of x and y and where $x = x(s, t)$ and $y = y(s, t)$ both have first-order partial derivatives. Then we have the chain rules:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \quad\text{and}\quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$



$$\boxed{\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \end{aligned}}$$

$$\frac{\partial^2 z}{\partial s^2} = \frac{\partial f}{\partial x}\cdot\frac{\partial^2 x}{\partial s^2} + \frac{\partial f}{\partial y}\cdot\frac{\partial^2 y}{\partial s^2} + \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial s}\right)^2 + 2\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial s}\frac{\partial y}{\partial s} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial s}\right)^2 \quad \text{if } f_{xy} = f_{yx}$$

$$\frac{\partial^2 z}{\partial s\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial^2 x}{\partial s\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial^2 y}{\partial s\partial t} + \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial t}\frac{\partial x}{\partial s}\right) + \frac{\partial^2 f}{\partial y\partial x}\left(\frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right) + \frac{\partial^2 f}{\partial x\partial y}\left(\frac{\partial y}{\partial t}\frac{\partial x}{\partial s}\right) + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial t}\frac{\partial y}{\partial s}\right) = \frac{\partial^2 z}{\partial t\partial s}$$

if $x_{st} = x_{ts}$, $y_{st} = y_{ts}$ and $f_{xy} = f_{yx}$

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial f}{\partial x}\cdot\frac{\partial^2 x}{\partial t^2} + \frac{\partial f}{\partial y}\cdot\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 \quad \text{if } f_{xy} = f_{yx}$$

Example 3.6.4

For a differentiable function $f(x, y)$ with continuous second partial derivatives, $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$f_r = f_x\cos\theta + f_y\sin\theta \quad \text{and} \quad f_\theta = -f_x r\sin\theta + f_y r\cos\theta$$

Solution

Clearly, $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$. From Theorem 3.6.3, we have

$$f_r = \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta,$$
$$f_\theta = \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -f_x r\sin\theta + f_y r\cos\theta$$

Theorem 3.6.5

Suppose that u is a differentiable function of the n variables $x_1, x_2, \cdots, x_n$ and each x_i is a differentiable function of the m variables $t_1, t_2, \cdots, t_m$. Then u is a function of $t_1, t_2, \cdots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f. If F is differentiable and $\partial F/\partial y \neq 0$, we obtain

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0.$$

Since $\frac{dx}{dx}=1$, we have $\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\frac{F_x}{F_y}$.

Example 3.6.6

Find y' if $x^3+y^3=6xy$.

Solution

$F(x,y)=x^3+y^3-6xy$. $\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{3x^2-6y}{3y^2-6x}=-\frac{x^2-2y}{y^2-2x}$

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable and $\partial F/\partial z \neq 0$, then

$$\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}=0.$$

Since $\frac{\partial x}{\partial x}=1$ and $\frac{\partial y}{\partial x}=0$, we have $\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}=-\frac{F_x}{F_z}$. Similarly, $\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}=-\frac{F_y}{F_z}$.

Example 3.6.7

Find $\partial z/\partial x$ and $\partial z/\partial y$ if $x^3+y^3+z^3+6xyz=1$.

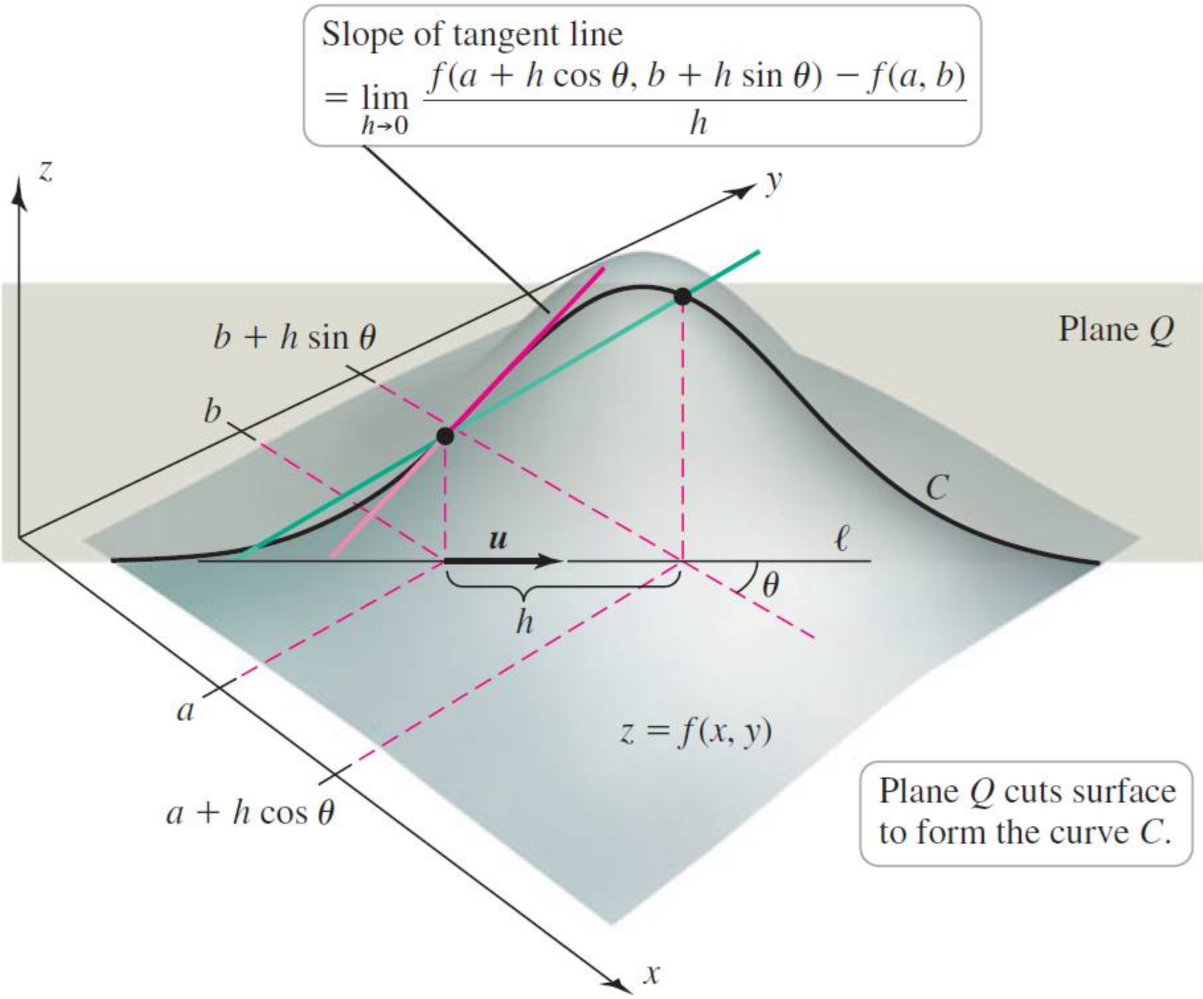
Solution

$F(x,y,z)=x^3+y^3+z^3+6xyz-1$.

$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{3x^2+6yz}{3z^2+6xy}=-\frac{x^2+2yz}{z^2+2xy}$, $\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{3y^2+6xz}{3z^2+6xy}=-\frac{y^2+2xz}{z^2+2xy}$

Let $(a, b, f(a, b))$ be a point on the surface $z = f(x, y)$ and let $\boldsymbol{u}$ be a unit vector in the xy-plane. Our aim is to find the rate of change of f in the direction $\boldsymbol{u}$ at (a, b). In general, this rate of change is neither $f_x(a,b)$ nor $f_y(a,b)$ (unless $\boldsymbol{u} = \langle 1, 0\rangle$ or $\boldsymbol{u} = \langle 0, 1\rangle$), but it turns out to be a combination of $f_x(a,b)$ and $f_y(a,b)$.

The below figure shows the unit vector $\boldsymbol{u}$ at an angle θ to the positive x-axis; its components are $\boldsymbol{u} = \langle u_1, u_2\rangle = \langle \cos\theta, \sin\theta\rangle$. The derivative we seek must be computed along the line l in the xy-plane through (a, b) in the direction of $\boldsymbol{u}$. A neighboring point P, which is h units from (a, b) along l, has coordinates $P(a + h\cos\theta, b + h\sin\theta)$.



Now imagine the plane Q perpendicular to the xy-plane, containing l. This plane cuts the surface $z = f(x, y)$ in a curve C. Consider two points on C corresponding to (a, b) and P; they have z-coordinates $f(a, b)$ and $f(a + h\cos\theta, b + h\sin\theta)$. The slope of the secant line between these points is

$$\frac{f(a+h\cos\theta, b+h\sin\theta)-f(a,b)}{h}.$$

The derivative of f in the direction of $\boldsymbol{u}$ is obtained by letting $h \to 0$; when the limit exists, it is called the *directional derivative of f at (a, b) in the direction of $\boldsymbol{u}$*. It gives the slope of the line tangent to the curve C in the plane Q.

Definition 3.7.1 (Directional Derivative)

Let f be differentiable at (a, b) and let $\boldsymbol{u} = \langle \cos\theta, \sin\theta \rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a, b) in the direction of $\boldsymbol{u}$** is

$$D_{\boldsymbol{u}} f(a,b) = \lim_{h \to 0} \frac{f(a+h\cos\theta, b+h\sin\theta)-f(a,b)}{h},$$

provided the limit exists.

Theorem 3.7.2

Let f be differentiable at (a, b) and let $\boldsymbol{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The directional derivative of f at (a, b) in the direction of $\boldsymbol{u}$ is

$$D_{\boldsymbol{u}} f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2 = \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle$$

Proof

Let $g(h) = f(a+hu_1, b+hu_2)$. Then, $g(0) = f(a, b)$ and so, from Definition 3.7.1, we have

$$D_{\boldsymbol{u}} f(a,b) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2)-f(a,b)}{h} = \lim_{h \to 0} \frac{g(h)-g(0)}{h} = g'(0).$$

If we define $x = a + hu_1$ and $y = b + hu_2$, we have $g(h) = f(x, y)$. From the chain rule, we have

$$g'(h) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dh} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dh} = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2.$$

Finally, taking $h = 0$ gives us

$$D_{\boldsymbol{u}} f(a,b) = g'(0) = f_x(a,b)u_1 + f_y(a,b)u_2 = \left(f_x(a,b), f_y(a,b) \right) \cdot (u_1, u_2)$$

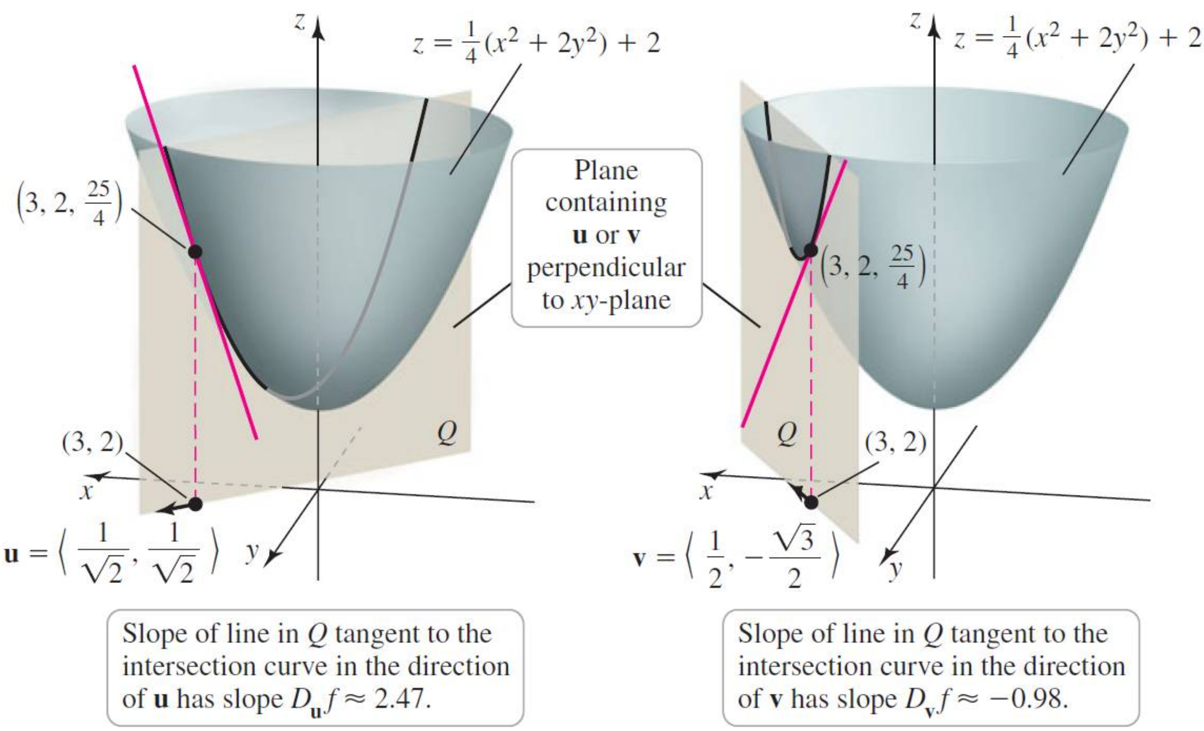
Example 3.7.3

Consider the paraboloid $z = f(x, y) = \frac{1}{4}(x^2 + 2y^2) + 2$. Find the directional derivative of f at (3, 2) in the directions of $\boldsymbol{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ and $\boldsymbol{u} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$.

Solution

We see that $f_x = x/2$ and $f_y = y$; evaluated at (3, 2), we have $f_x(3,2) = 3/2$ and $f_y(3,2) = 2$. The directional derivatives in the directions $\boldsymbol{u}$ and $\boldsymbol{v}$ are

$$D_{\boldsymbol{u}} f(3,2) = \left\langle f_x(3,2), f_y(3,2) \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{3}{2} \cdot \frac{1}{\sqrt{2}} + 2 \cdot \frac{1}{\sqrt{2}} = \frac{7}{2\sqrt{2}} \approx 2.47$$
$$D_{\boldsymbol{v}} f(3,2) = \left\langle f_x(3,2), f_y(3,2) \right\rangle \cdot \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle = \frac{3}{2} \cdot \frac{1}{2} + 2 \cdot \left(-\frac{\sqrt{3}}{2}\right) = \frac{3}{4} - \sqrt{3} \approx -0.98$$



Definition 3.7.4 (Two Dimensional Gradient)

Let f be differentiable at the point (x, y). The **gradient** of f at (x, y) is the vector-valued function

$$\nabla f(x, y) = \left\langle f_x(x, y), f_y(x, y) \right\rangle = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

The directional derivative of f at (a, b) in the direction of the unit vector $\boldsymbol{u}$ can be written

$$D_{\boldsymbol{u}} f(a,b) = \nabla f(a,b) \cdot \boldsymbol{u}$$

The gradient of f at (3, 2) in Example 3.7.3 is $\langle 3/2, 2 \rangle$.

Since a directional derivative gives the rate of change of a function in a given direction, it's reasonable to ask in what direction a function has its maximum or minimum rate of change. Using properties of the dot product, we have

$$D_{\boldsymbol{u}} f(a,b) = \nabla f(a,b) \cdot \boldsymbol{u} = \|\nabla f(a,b)\| \|\boldsymbol{u}\| \cos\theta = \|\nabla f(a,b)\| \cos\theta,$$

where θ is the angle between the gradient vector at (a, b) and the direction unit vector $\boldsymbol{u}$.

Notice that $\|\nabla f(a,b)\| \cos\theta$ has its maximum value when $\theta = 0$, so that $\cos\theta = 1$. The directional derivative is then $\|\nabla f(a,b)\|$. This occurs when $\nabla f(a, b)$ and $\boldsymbol{u}$ are in the *same* direction, so that $\boldsymbol{u} = \dfrac{\nabla f(a,b)}{\|\nabla f(a,b)\|}$. Similarly, the minimum value of the directional derivative occurs when $\theta = \pi$, so that $\cos\theta = -1$. In this case, $\nabla f(a, b)$ and $\boldsymbol{u}$ have *opposite* directions, so that $\boldsymbol{u} = -\dfrac{\nabla f(a,b)}{\|\nabla f(a,b)\|}$. Finally, observe that when $\theta = \frac{\pi}{2}$, $\boldsymbol{u}$ is perpendicular to $\nabla f(a, b)$ and the directional derivative in this direction is zero. We summarize these observations in Theorem 3.7.5.

Theorem 3.7.5

Suppose that f is a differentiable function of x and y at the point (a, b). Then

(i) the maximum rate of change of f at (a, b) is $\|\nabla f(a,b)\|$, occurring in the direction of the gradient;

(ii) the minimum rate of change of f at (a, b) is $-\|\nabla f(a,b)\|$, occurring in the direction opposite the gradient;

(iii) the rate of change of f at (a, b) is 0 in the directions orthogonal to $\nabla f(a, b)$.

Example 3.7.6

Consider the bowl-shaped paraboloid $z = f(x, y) = 4 + x^2 + 3y^2$.

(a) If you are located on the paraboloid at the point $(2, -\frac{1}{2}, \frac{35}{4})$, in which direction should you move in order to maximize rate of increase (steepest ascent) and maximize rate of decrease (steepest descent) on the surface? What are the rate of changes?

(b) At the point (3, 1, 16), in what direction(s) is there no change in the function values?

Solution

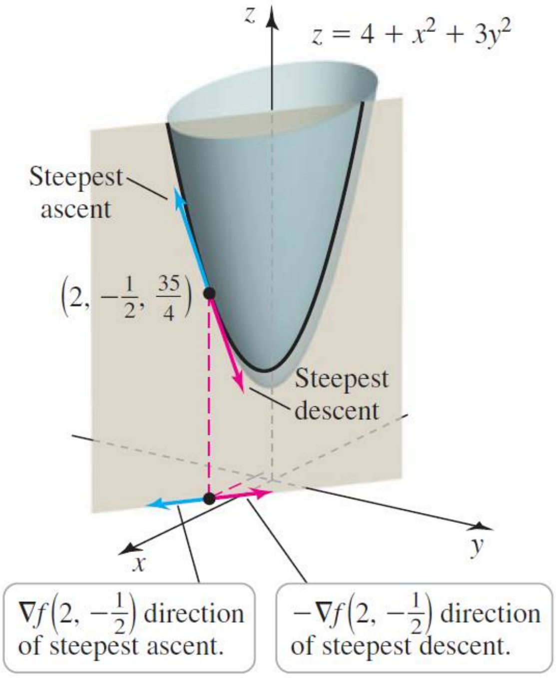
(a) At the point $(2, -\frac{1}{2})$, the value of the gradient is

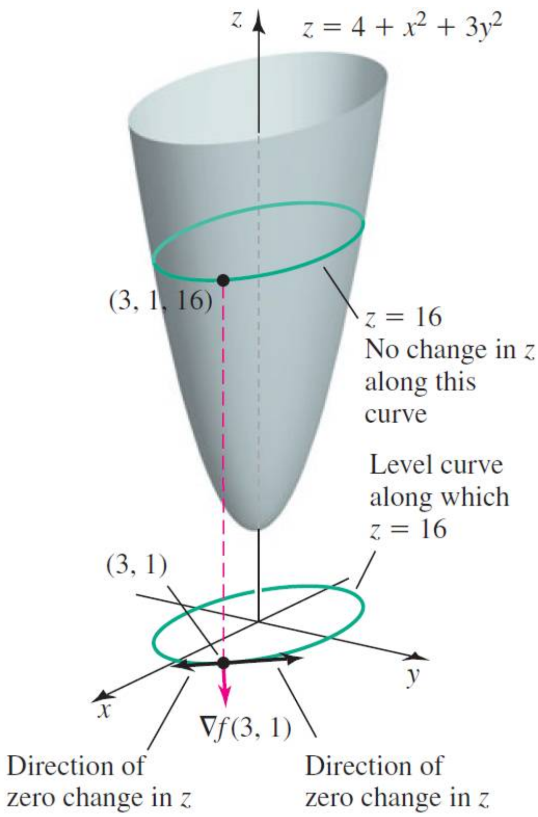
$$\nabla f(x, y) = \langle 2x, 6y \rangle\big|_{(2,-\frac{1}{2})} = \langle 4, -3 \rangle.$$

Therefore, the direction of maximize rate of increase in the xy-plane is in the direction of the gradient vector $\langle 4, -3 \rangle$, (or $\boldsymbol{u} = \frac{1}{5}\langle 4, -3 \rangle$, as a unit vector). The rate of change is $|\nabla f(2, -\frac{1}{2})| = |\langle 4, -3 \rangle| = 5$.

The direction of maximize rate of decrease is the direction of $-\nabla f(2, -\frac{1}{2}) = \langle -4, 3 \rangle$ (or $\boldsymbol{u} = \frac{1}{5}\langle -4, 3 \rangle$, as a unit vector). The rate of change is $-|\nabla f(2, -\frac{1}{2})| = -5$.

(b) At the point (3, 1), the value of the gradient is $\nabla f(3, 1) = \langle 6, 6 \rangle$. The function has zero change if we move in either of the two directions orthogonal to $\langle 6, 6 \rangle$; these two directions are parallel to $\langle 6, -6 \rangle$. In terms of unit vectors, the directions of no change are $\boldsymbol{u} = \frac{1}{\sqrt{2}}\langle -1, 1 \rangle$ and $\boldsymbol{u} = \frac{1}{\sqrt{2}}\langle 1, -1 \rangle$.





Theorem 3.7.7

Given a function f differentiable at (a, b), the line tangent to the level curve of f at (a, b) is orthogonal to the gradient $\nabla f(a, b)$, provided $\nabla f(a,b) \neq \mathbf{0}$.

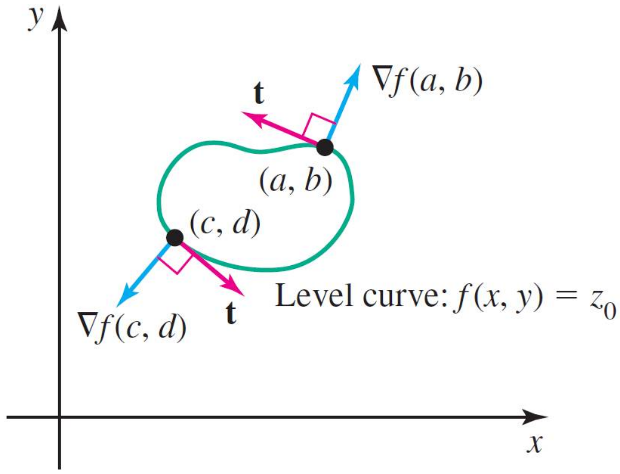
Proof

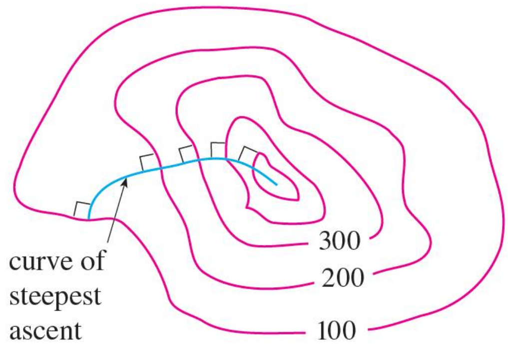
In P.94, the slope of the line tangent to the level curve $f(x,y) = z_0$, where z_0 is a constant, is $y'(x) = -f_x / f_y$.

It follows that any vector that points in the direction of the tangent line at the point (a, b) is a scalar multiple of the vector $\boldsymbol{t} = \langle -f_y(a,b), f_x(a,b) \rangle$. At that same point, the gradient points in the direction $\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$. The dot product of $\boldsymbol{t}$ and $\nabla f(a, b)$ is

$$\boldsymbol{t} \cdot \nabla f(a,b) = \langle -f_y(a,b), f_x(a,b) \rangle \cdot \langle f_x(a,b), f_y(a,b) \rangle = -f_y f_x + f_x f_y \big|_{(a,b)} = 0.$$

which implies that $\boldsymbol{t}$ and $\nabla f(a, b)$ are orthogonal.





The line tangent to the curve at (a, b) is orthogonal to $\nabla f(a, b)$. Therefore, if (x, y) is a point on the tangent line, then $\nabla f(a,b)\cdot\langle x-a, y-b\rangle = 0$, which, when simplified, gives an equation of the line tangent to the curve $f(x,y) = z_0$:

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) = 0.$$

The **path of steepest ascent** / **descent** is a curve that remains perpendicular to each level curve through which it passes.

Example 3.7.8

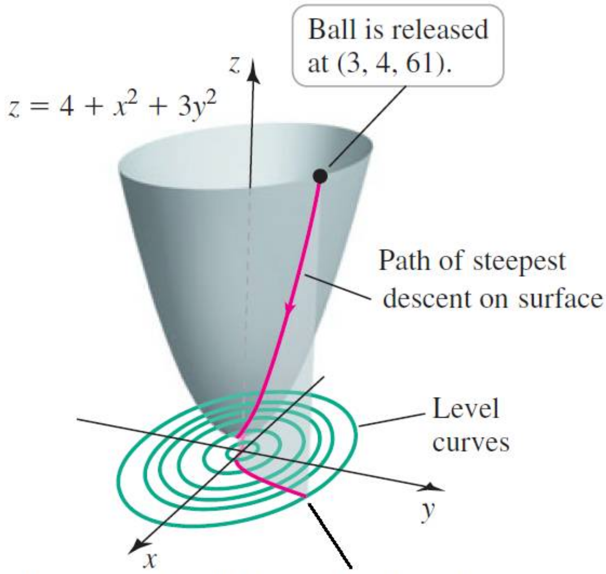
Consider the paraboloid $z = f(x,y) = 4 + x^2 + 3y^2$. Beginning at the point (3, 4, 61) on the surface, find the path in the *xy*-plane that points in the direction of steepest descent on the surface.

Solution

Imagine releasing a ball at (3, 4, 61) and assume that it rolls in the direction of steepest descent at all points. The projection of this path in the *xy*-plane points in the direction of $-\nabla f(x,y) = \langle -2x, -6y\rangle$, which means that at the point (x, y) the line tangent to the path has slope $y'(x) = (-6y)/(-2x) = 3y/x$. Therefore, the path in the *xy*-plane satisfies $y'(x) = 3y/x$ and passes through the initial point (3, 4). The solution to this differential equation is

$$\frac{dy}{y}=\frac{3dx}{x} \quad \Rightarrow \quad \int\frac{dy}{y}=\int\frac{3dx}{x} \quad \Rightarrow \quad \ln y = 3\ln x + C \quad \Rightarrow \quad y=e^C x^3$$

At (3, 4), we have $4=e^C 3^3$. The projection of the path of steepest descent in the *xy*-plane is the curve $y=\frac{4}{27}x^3$. The descent ends at (0, 0), which corresponds to the vertex of the paraboloid. At all points of the descent, the curve in the *xy*-plane is orthogonal to the level curves of the surface.



Projection of path of steepest descent on *xy*-plane, $y=\dfrac{4x^3}{27}$, is orthogonal to level curves.

Theorem 3.7.9

Suppose $\boldsymbol{u}=\langle u_1,u_2\rangle$ is a unit vector and f has continuous second partial derivatives. Then

$$\begin{aligned}
D_{\boldsymbol{u}}^2 f(x,y) &= D_{\boldsymbol{u}}\left(D_{\boldsymbol{u}} f(x,y)\right)\\
&= D_{\boldsymbol{u}}\left(f_x(x,y)u_1 + f_y(x,y)u_2\right)\\
&= D_{\boldsymbol{u}}\left(f_x(x,y)\right)u_1 + D_{\boldsymbol{u}}\left(f_y(x,y)\right)u_2\\
&= \left(f_{xx}(x,y)u_1 + f_{xy}(x,y)u_2\right)u_1 + \left(f_{yx}(x,y)u_1 + f_{yy}(x,y)u_2\right)u_2\\
&= f_{xx}(x,y)u_1^2 + 2f_{xy}(x,y)u_1u_2 + f_{yy}(x,y)u_2^2 \quad \text{(Theorem 3.3.5)}
\end{aligned}$$

Definition 3.7.10

The **directional derivative** of $f(x, y, z)$ at the point (a, b, c) and in the direction of the unit vector $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ is given by

$$D_{\boldsymbol{u}} f(a,b,c) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a,b,c)}{h},$$

provided the limit exists. The **gradient** of $f(x, y, z)$ is the vector-valued function

$$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

provided all the partial derivatives are defined.

Theorem 3.7.11

If f is a differentiable function of x, y and z and $\boldsymbol{u}$ is any unit vector, then

$$D_{\boldsymbol{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \boldsymbol{u}.$$

Theorem 3.7.12

Suppose that $f(x, y, z)$ has continuous partial derivatives at $P = (a, b, c)$ and $\nabla f(a,b,c) \neq \mathbf{0}$. Then, $\nabla f(a, b, c)$ is a normal vector to the tangent plane to the level surface $f(x,y,z) = k$ at the point (a, b, c). Further, the equation of the tangent plane is

$$f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0.$$

Solution

Since $\nabla f(a,b,c) \neq 0$, we may assume $f_x(a,b,c) \neq 0$. Suppose $f(a,b,c) = k$. Consider the intersection of the level surface $f(x,y,z) = k$ and the plane $z = c$. The equation of cross section is $f(x,y,c) = k$. In the proof of Theorem 3.7.7, the tangent vector at (a, b, c) of $f(x,y,c) = k$ is $\langle -f_y(a,b,c), f_x(a,b,c), 0 \rangle$.

Similarly, the tangent vector at (a, b, c) of $f(x,b,z) = k$ is $\langle -f_z(a,b,c), 0, f_x(a,b,c) \rangle$. Therefore, the normal vector of the tangent plane of $f(x,y,z) = k$ at (a, b, c) is

$$\langle -f_y(P), f_x(P), 0 \rangle \times \langle -f_z(P), 0, f_x(P) \rangle = f_x(P) \langle f_x(P), f_y(P), f_z(P) \rangle = f_x(P) \nabla f(P)$$

Since we assume $f_x(a,b,c) \neq 0$, we may take $\nabla f(a,b,c)$ as another normal vector. Hence the equation of the tangent plane is

$$f_x(P)(x-a)+f_y(P)(y-b)+f_z(P)(z-c)=\langle x-a, y-b, z-c\rangle \cdot \langle f_x(P), f_y(P), f_z(P)\rangle = 0.$$

We refer to the line through (a, b, c) in the direction of $\nabla f(a, b, c)$ as the **normal line** to the surface at the point (a, b, c). Observe that this has parametric equations

$$(x, y, z) = (a,b,c) + t\langle f_x(a,b,c), f_y(a,b,c), f_z(a,b,c)\rangle.$$

Example 3.7.13

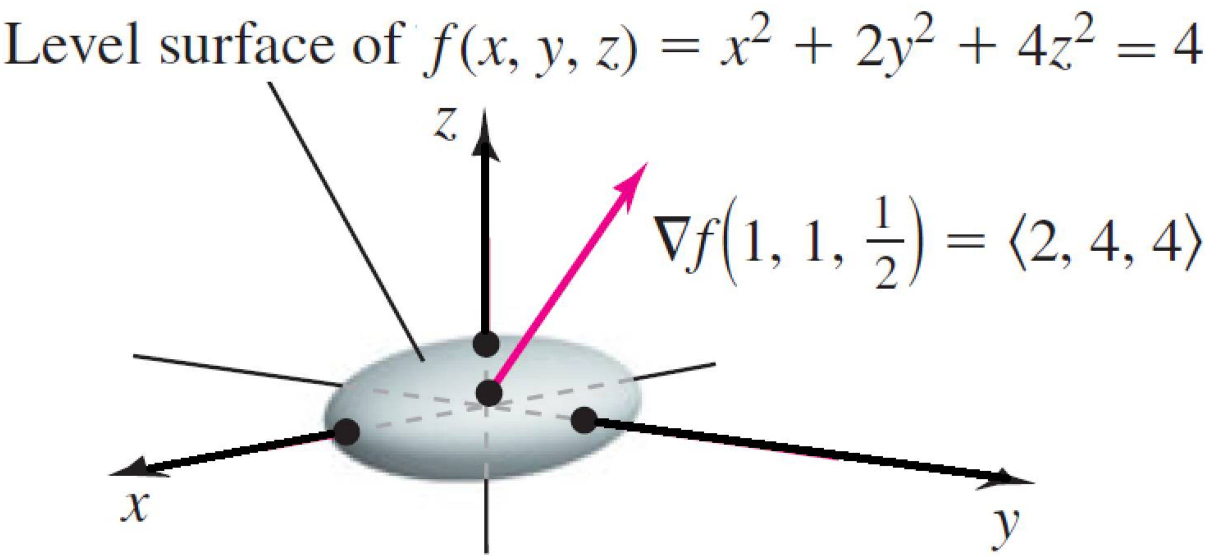
Find equations of the tangent plane and the normal line to $f(x, y, z) = x^2 + 2y^2 + 4z^2 = 4$ at the point $(1, 1, \frac{1}{2})$.

Solution

We have $\nabla f = (2x, 4y, 8z)$ and $\nabla f(1, 1, \frac{1}{2}) = \langle 2, 4, 4\rangle$. The equation of the tangent plane is

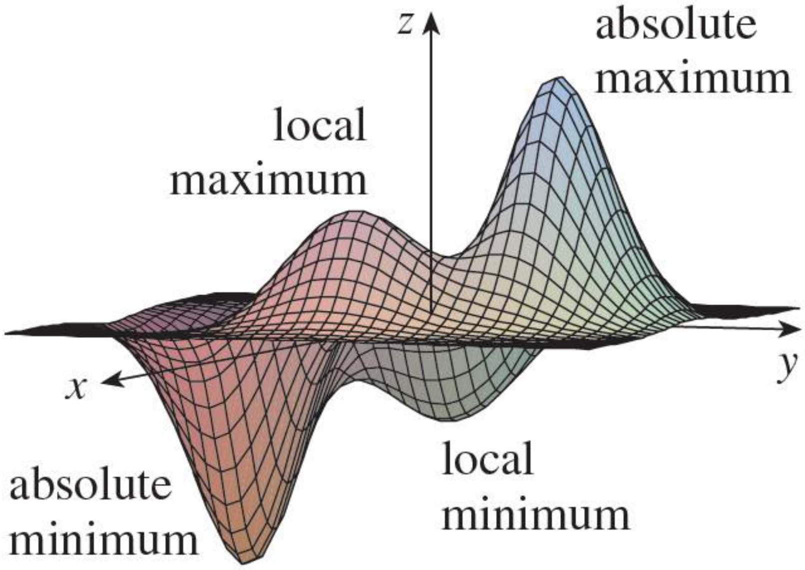
$$2(x-1)+4(y-1)+4(z-\tfrac{1}{2})=0.$$

The normal line has parametric equations $(x, y, z) = (1, 1, \frac{1}{2}) + t\langle 2, 4, 4\rangle$.



Definition 3.8.1 (Local Extreme Values)

A function f has a **local maximum value** at (a, b) if there is an open disk D centered at (a, b), for which $f(a, b) \geq f(x, y)$ for all $(x, y) \in D$. Similarly, A function f has a **local minimum value** at (a, b) if there is an open disk D centered at (a, b), for which $f(a, b) \leq f(x, y)$ for all $(x, y) \in D$. In either case, $f(a, b)$ is called a **local extreme value** or **local extremum** of f.



Definition 3.8.2

The point (a, b) is a **critical point** of the function $f(x, y)$ if (a, b) is in the domain of f and either $f_x(a,b) = f_y(a,b) = 0$ or one or both of f_x and f_y do not exist at (a, b).

Theorem 3.8.3

If $f(x, y)$ has a local extremum at (a, b), then (a, b) must be a critical point of f.

Solution

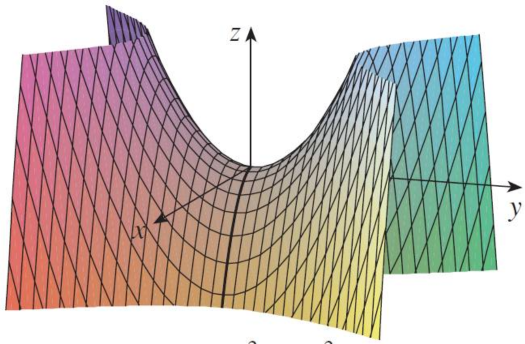
Suppose that $f(x, y)$ has a local extremum at (a, b). $g(x) = f(x, b)$ has a local extremum at $x = a$. Then either $f_x(a,b) = g'(a) = 0$ or does not exist. Similarly, $h(y) = f(a, y)$ has a local extremum at $y = b$. Then either $f_y(a,b) = h'(b) = 0$ or does not exist. In conclusion, (a,b) must be a critical point of f.

Critical points do not necessarily correspond to local maxima or minima.

Definition 3.8.4 (Saddle Point)

A function f has a **saddle point** at a critical point (a, b) if, in every open disk centered at (a,b), there are points (x, y) for which $f(x, y) > f(a, b)$ and points for which $f(x, y) < f(a, b)$.

Consider the hyperbolic paraboloid $f(x,y) = y^2 - x^2$. We can easily check that $f_x(0,0) = f_y(0,0) = 0$. (0, 0) is a critical of f. The graph of f has a horizontal tangent plane (z = 0) at the origin. You can see that f (0, 0) = 0 is a maximum in the direction of the x-axis but a minimum in the direction of the y-axis. (0, 0) is a saddle point.



$z = y^2 - x^2$

Theorem 3.8.5 (Second Derivative Test)

Suppose that the second partial derivatives of f are continuous throughout an open disk centered at the point (a, b), where $f_x(a,b) = f_y(a,b) = 0$. Let

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2.$$

(1) If $D(a, b) > 0$ and $f_{xx}(a,b) > 0$, then f has a local minimum value at (a, b).

(2) If $D(a, b) > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum value at (a, b).

(3) If $D(a, b) < 0$, then f has a saddle point at (a, b).

(4) If $D(a, b) = 0$, then the test is inconclusive.

Proof

Consider the black curve in P.95, the intersection of the graph $z = f(x, y)$ and the vertical plane Q passing through (a, b) along a unit vector $\boldsymbol{u} = \langle u_1, u_2 \rangle$.

Suppose $f_x(a,b) = f_y(a,b) = 0$. The first directional derivative of f at (a, b) along a unit vector $\boldsymbol{u}$ is $D_{\boldsymbol{u}} f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2 = 0$.

In Theorem 3.7.9, the second directional derivative of f at (a, b) along $\boldsymbol{u}$ is

$$D_{\boldsymbol{u}}^2 f(a,b) = f_{xx}(a,b)u_1^2 + 2f_{xy}(a,b)u_1u_2 + f_{yy}(a,b)u_2^2.$$

Recall that a quadratic polynomial

$$as^2 + bst + ct^2 \begin{cases} > 0 \text{ for all } s, t \text{ if } a > 0 \text{ and } b^2 - 4ac < 0 \\ < 0 \text{ for all } s, t \text{ if } a < 0 \text{ and } b^2 - 4ac < 0 \\ \boxed{\begin{array}{c} > 0 \text{ for some } (s, t) \text{ and } < 0 \text{ for another} \\ (s, t) \text{ for any open disk center at } (0, 0) \end{array}} \text{ if } b^2 - 4ac > 0 \end{cases}$$

If $D(a, b) < 0$ (Case (3)), then $D_{\boldsymbol{u}}^2 f(a,b) > 0$ for some (x, y) and $D_{\boldsymbol{u}}^2 f(x,y) < 0$ for another (x, y) for any open disk center at (a, b). That means f has a saddle point at (a, b).

Suppose $D(a, b) > 0$. Either $f_{xx}(a,b)$ or $f_{yy}(a,b)$ is non-zero. Assume $f_{xx}(a,b) \neq 0$. Interchange x and y if not.

Case (1): $D(a, b) > 0$ and $f_{xx}(a,b) > 0$.

$D_{\boldsymbol{u}}^2 f(a,b) > 0$ for any direction $\boldsymbol{u}$. That means all curves of cross section are concave up. f has a local minimum value at (a, b).

Case (2): $D(a, b) > 0$ and $f_{xx}(a,b) < 0$.

$D_{\boldsymbol{u}}^2 f(a,b) > 0$ for any direction $\boldsymbol{u}$. That means all curves of cross section are concave down. f has a local maximum value at (a, b).

Example 3.8.6

Find the local maximum and minimum values and saddle points of $f(x,y)=x^4+y^4-4xy+1$.

Solution

We first locate the critical points: $f_x=4x^3-4y$ and $f_y=4y^3-4x$.

Setting these partial derivatives equal to 0, we obtain the equations $y=x^3$ and $x=y^3$.

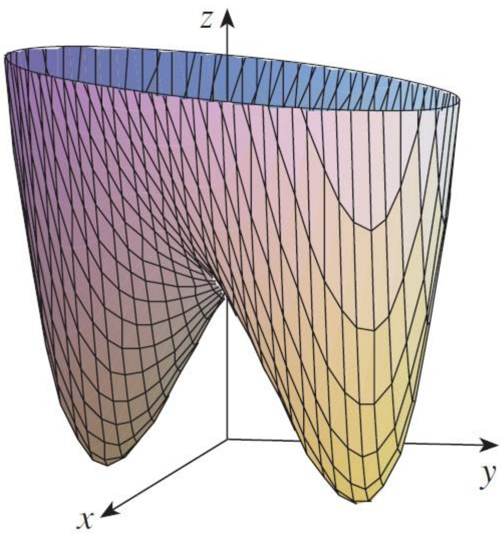
Solving these equations we obtain $x=0,\pm1$. The three critical points are (0, 0), (1, 1), and $(-1,-1)$.

Next we calculate the second partial derivatives and $D(x, y)$:

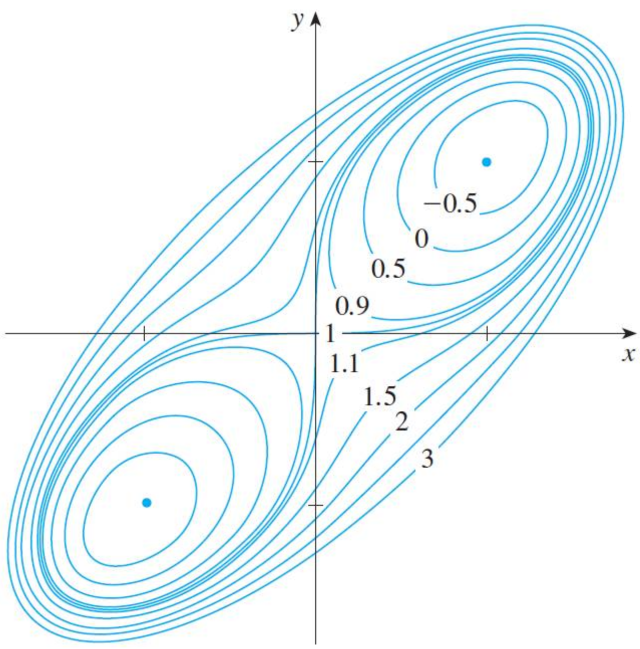
$$f_{xx}=12x^2,\quad f_{xy}=-4,\quad f_{yy}=12y^2,\quad D(x,y)=f_{xx}f_{yy}-f_{xy}^2=144x^2y^2-16.$$

Since $D(0,0)=-16<0$, it follows from case (3) of the Second Derivatives Test that the origin is a saddle point.

Since $D(1,1)=128>0$ and $f_{xx}(1,1)=12>0$, we see from case (1) of the test that $f(1,1)=-1$ is a local minimum. Similarly, we have $D(-1,-1)=128>0$ and $f_{xx}(-1,-1)=12>0$, so $f(-1,-1)=-1$ is also a local minimum.



$z=x^4+y^4-4xy+1$



Example 3.8.7

Apply the Second Derivative Test to function $f(x,y)=2-xy^2$ and interpret the results.

Solution

The critical points of f satisfy the conditions

$$f_x(x,y)=-y^2=0 \quad \text{and} \quad f_y(x,y)=-2xy=0.$$

The solutions of these equations have the form (a, 0), where a is a real number. It is easy to check that the second partial derivatives evaluated at (a, 0) are

$$f_{xx}(a,0)=f_{xy}(a,0)=0 \quad \text{and} \quad f_{yy}(a,0)=-2a.$$

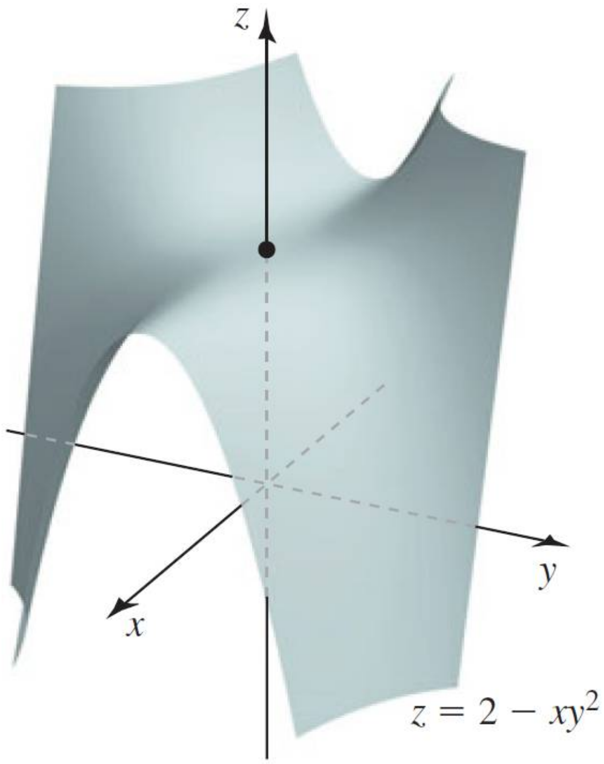
Therefore, the discriminant is $D(a, 0) = 0$.

Indeed, (0, 0) is a saddle because $f(\varepsilon,\varepsilon)<f(0,0)=2<f(-\varepsilon,\varepsilon)$ where $\varepsilon>0$.

(a, 0) is a local maximum for $a > 0$ because $f(a,\pm\varepsilon)=2-a\varepsilon^2<f(a,0)=2$ where $\varepsilon>0$.

(a, 0) is a local minimum for $a < 0$ because $f(a,\pm\varepsilon)=2-a\varepsilon^2>f(a,0)=2$ where $\varepsilon>0$.

That is why the Second Derivative test is inconclusive if $D(a, b) = 0$.



Example 3.8.8

Suppose there are m pairs of distinct datum $(x_1, y_1), (x_2, y_2), \cdots, (x_m, y_m)$. Find the best fitted line $y = \beta_0 + \beta_1 x$ with least squared error $\sum_{j=1}^{m} (\beta_0 + \beta_1 x_j - y_j)^2$.

Solution

Let $E(\beta_0, \beta_1) = \sum_{j=1}^{m} (\beta_0 + \beta_1 x_j - y_j)^2$ be the squared error function of β_0 and β_1. The critical point of E satisfy

$$E_{\beta_0} = \frac{\partial}{\partial \beta_0} \sum_{j=1}^{m} (\beta_0 + \beta_1 x_j - y_j)^2 = 2\sum_{j=1}^{m} (\beta_0 + \beta_1 x_j - y_j) = 2\left(m\beta_0 + \beta_1 \sum_{j=1}^{m} x_j - \sum_{j=1}^{m} y_j \right) = 0$$

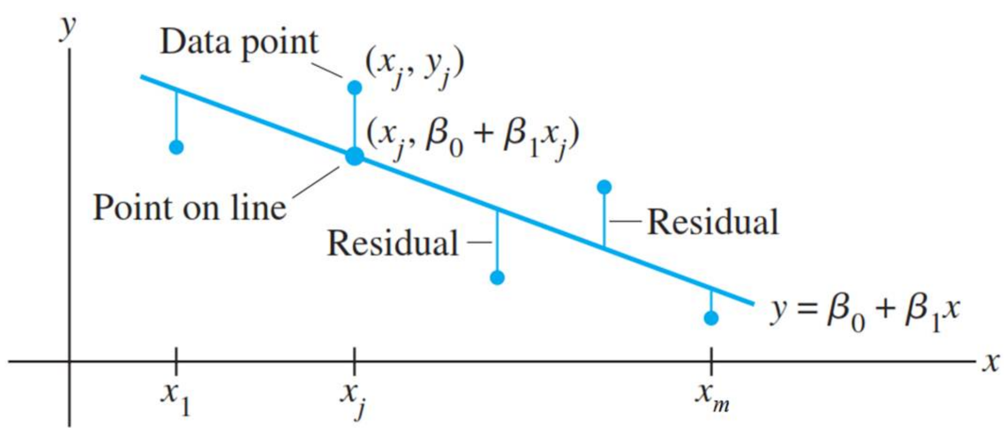
$$E_{\beta_1} = \frac{\partial}{\partial \beta_1} \sum_{j=1}^{m} (\beta_0 + \beta_1 x_j - y_j)^2 = 2\sum_{j=1}^{m} x_j(\beta_0 + \beta_1 x_j - y_j) = 2\left(\beta_0 \sum_{j=1}^{m} x_j + \beta_1 \sum_{j=1}^{m} x_j^2 - \sum_{j=1}^{m} x_j y_j \right) = 0$$

$$\begin{pmatrix} m & \sum_{j=1}^{m} x_j \\ \sum_{j=1}^{m} x_j & \sum_{j=1}^{m} x_j^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{m} y_j \\ \sum_{j=1}^{m} x_j y_j \end{pmatrix}$$

The critical point is $\beta_0 = \frac{1}{m}\sum_{j=1}^{m} y_j - \frac{\beta_1}{m}\sum_{j=1}^{m} x_j$, $\beta_1 = \dfrac{m\sum_{j=1}^{m} x_j y_j - \sum_{j=1}^{m} x_j \sum_{j=1}^{m} y_j}{m\sum_{j=1}^{m} x_j^2 - \left(\sum_{j=1}^{m} x_j\right)^2}$

$$E_{\beta_0\beta_0} = 2m,\ E_{\beta_0\beta_1} = 2\sum_{j=1}^{m} x_j,\ E_{\beta_1\beta_1} = 2\sum_{j=1}^{m} x_j^2,\ D(\beta_0, \beta_1) = 4m\sum_{j=1}^{m} x_j^2 - 4\left(\sum_{j=1}^{m} x_j\right)^2 = 4m^2 Var[X] > 0$$

By second derivative test, the square error is minimum at (β_0, β_1).



Fitting a line to experimental data.

Definition 3.8.9 (Absolute Maximum, Minimum Values)

We call $f(a, b)$ the **absolute maximum** of f on the region R if $f(a, b) \geq f(x, y)$ for all $(x, y) \in R$. Similarly, $f(a, b)$ is called the **absolute minimum** of f on R if $f(a, b) \leq f(x, y)$ for all $(x, y) \in R$. In either case, $f(a, b)$ is called an **absolute extremum** of f.

Theorem 3.8.10 (Extreme Value Theorem)

Suppose that $f(x, y)$ is continuous on the closed and bounded region $R \subset \boldsymbol{R}^2$. Then f has both an absolute maximum and an absolute minimum on R. Further, an absolute extremum may only occur at a critical point in R or at a point on the boundary of R.

Let f be continuous on a closed bounded set $R \subset \boldsymbol{R}^2$. To find the absolute maximum and minimum values of f on R:

(1) Determine the values of f at all critical points in R.

(2) Find the maximum and minimum values of f on the boundary of R.

(3) The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on R, and the least function value found in Steps 1 and 2 is the absolute minimum value of f on R.

Example 3.8.11

Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 - 2x + 2y + 5$ on the set $R = \{(x, y) : x^2 + y^2 \leq 4\}$ (the closed disk centered at (0, 0) with radius 2).

Solution

The critical points of f satisfy the equations

$$f_x(x, y) = 2x - 2 = 0 \qquad \text{and} \qquad f_y(x, y) = 2y + 2 = 0,$$

which have the solution $x = 1$ and $y = -1$. The value of the function at this point is $f(1, -1) = 3$.

We now determine the maximum and minimum values of f on the boundary of R, which is a circle of radius 2 described by the parametric equations

$$x = 2\cos\theta, \quad y = 2\sin\theta \qquad \text{for} \qquad 0 \le \theta \le 2\pi.$$

Substituting x and y in terms of θ into the function f, we obtain a new function $g(\theta)$ gives the values of f on the boundary of R:

$$\begin{aligned} g(\theta) &= (2\cos\theta)^2 + (2\sin\theta)^2 - 2(2\cos\theta) + 2(2\sin\theta) + 5 \\ &= 4(\cos^2\theta + \sin^2\theta) - 4\cos\theta + 4\sin\theta + 5 \\ &= -4\cos\theta + 4\sin\theta + 9 \end{aligned}$$

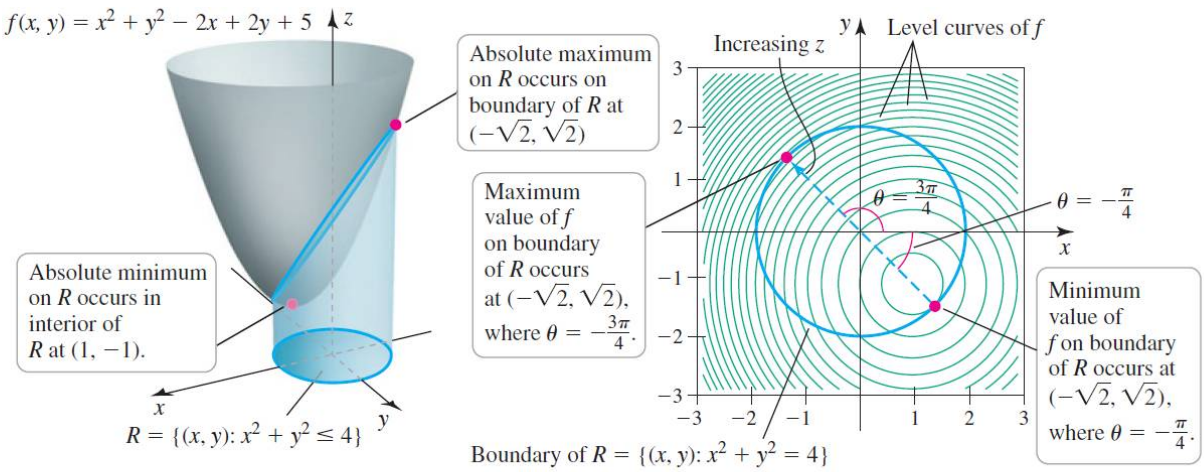
Finding the maximum and minimum boundary values is now a one-variable problem. The critical points of g satisfy

$$g'(\theta) = 4\sin\theta + 4\cos\theta = 0,$$

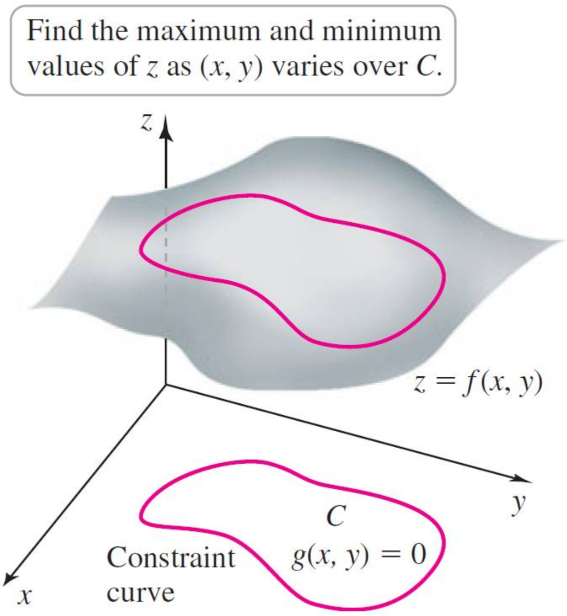
or $\tan\theta = -1$. Therefore, g has critical points $\theta = 3\pi/4$ and $\theta = 7\pi/4$, which correspond to the points $\left(-\sqrt{2}, \sqrt{2}\right)$ and $\left(\sqrt{2}, -\sqrt{2}\right)$. The function values at these points are $f\left(-\sqrt{2}, \sqrt{2}\right) = 9 + 4\sqrt{2}$ and $f\left(\sqrt{2}, -\sqrt{2}\right) = 9 - 4\sqrt{2}$. Furthermore, the value of g at 0 and 2π is $g(0) = g(2\pi) = f(2,0) = 5$. We have four function values to consider:

$$f(1,-1) = 3, \quad f\left(\sqrt{2}, -\sqrt{2}\right) = 9 - 4\sqrt{2} \approx 3.3, \quad f\left(-\sqrt{2}, \sqrt{2}\right) = 9 + 4\sqrt{2} \approx 14.7, \quad f(2,0) = 5.$$

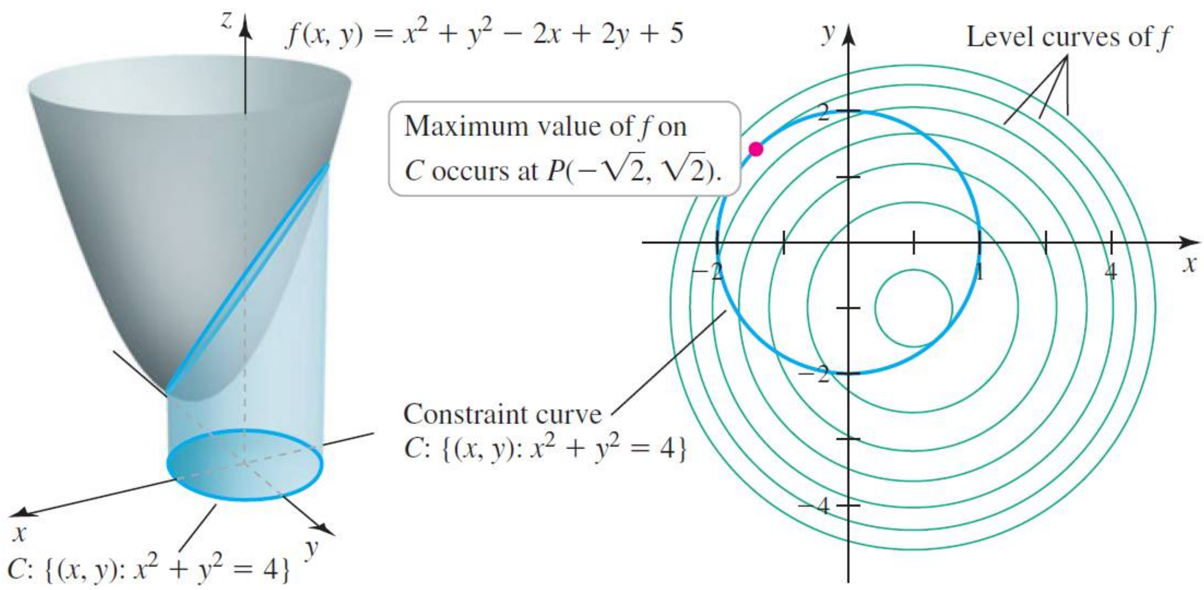
The greatest value, $f\left(-\sqrt{2}, \sqrt{2}\right) \approx 14.7$, is the absolute maximum value, and it occurs at a boundary point. The least value, $f(1,-1) = 3$, is the absolute minimum value, and it occurs at an interior point



We start with a typical constrained optimization problem with two independent variables and give its method of solution; a generalization to more variables then follows. We seek maximum and/or minimum values of a differentiable function f (the **objective function**) with the restriction that x and y must lie on a **constraint** curve C in the xy-plane given by $g(x, y) = 0$.



The problem and a method of solution are easy to visualize if we refer to Example 3.8.11. Part of that problem was to find the maximum value of $f(x,y) = x^2 + y^2 - 2x + 2y + 5$ on the circle $C = \{(x,y): x^2 + y^2 = 4\}$. We see the level curves of f and the point $P\left(-\sqrt{2}, \sqrt{2}\right)$ on C at which f has a maximum value. Imagine moving along C toward P; as we approach P, the values of f increase and reach a maximum value at P. Moving past P, the values of f decrease.



The value of f is maximum when level curve of f and constraint curve just touch each other, that is, when they have a common tangent line. (Otherwise, the value of $f(x, y)$ could be increased further.) This means that the normal lines at the point (a, b) where they touch are identical. So the gradient vectors are parallel; that is, $\nabla f(a,b) = \lambda \nabla g(a,b)$ for some scalar λ.

Theorem 3.9.1

Let f be a differentiable function in a region of $\boldsymbol{R}^2$ that contains the smooth curve C given by $g(x, y) = 0$. Assume that f has a local extreme value (relative to values of f on C) at a point $P(a, b)$ on C. Then $\nabla f(a, b)$ is orthogonal to the line tangent to C at P. Suppose $\nabla g(a,b) \neq \mathbf{0}$. It follows that there is a real number λ (called a **Lagrange multiplier**) such that $\nabla f(a,b) = \lambda \nabla g(a,b)$.

Proof

Because C is smooth it can be expressed parametrically in the form $C: \boldsymbol{r}(t) = \langle x(t), y(t) \rangle$, where x and y are differentiable functions on an interval in t that contains t_0 with $P(a,b) = (x(t_0), y(t_0))$. As we vary t and follow C, the rate of change of f is given by the Chain Rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = \nabla f(x,y) \cdot \boldsymbol{r}'(t).$$

At the point $(x(t_0), y(t_0)) = (a,b)$ at which f has a local maximum or minimum value, we have $f'(t_0) = 0$, which implies that $\nabla f(a,b) \cdot \boldsymbol{r}'(t_0) = 0$. Because $\boldsymbol{r}'(t)$ is tangent to C, the gradient $\nabla f(a, b)$ is orthogonal to the line tangent to C at P.

To prove the second assertion, note that the constraint curve C given by $g(x, y) = 0$ is also a level curve of the surface $z = g(x, y)$. Recall that gradients are orthogonal to level curves. Therefore, at the point $P(a, b)$, $\nabla g(a, b)$ is orthogonal to C at (a, b). Because both $\nabla f(a, b)$ and $\nabla g(a, b)$ are orthogonal to C, the two gradients are parallel, so there is a real number λ such that $\nabla f(a,b) = \lambda \nabla g(a,b)$.

Let the objective function f and the constraint function g be differentiable on a region of $\boldsymbol{R}^2$ with $\nabla g(a,b) \neq \mathbf{0}$ on the curve $g(x, y) = 0$. To locate the maximum and minimum values of f subject to the constraint $g(x, y) = 0$, carry out the following steps.

(1) Find the values of x, y, and λ (if they exist) that satisfy the equations

$$\nabla f(x,y) = \lambda \nabla g(x,y) \qquad \text{and} \qquad g(x,y) = 0.$$

(2) Among the values (x, y) found in Step 1, select the largest and smallest corresponding function values, which are the maximum and minimum values of f subject to the constraint.

Example 3.9.2

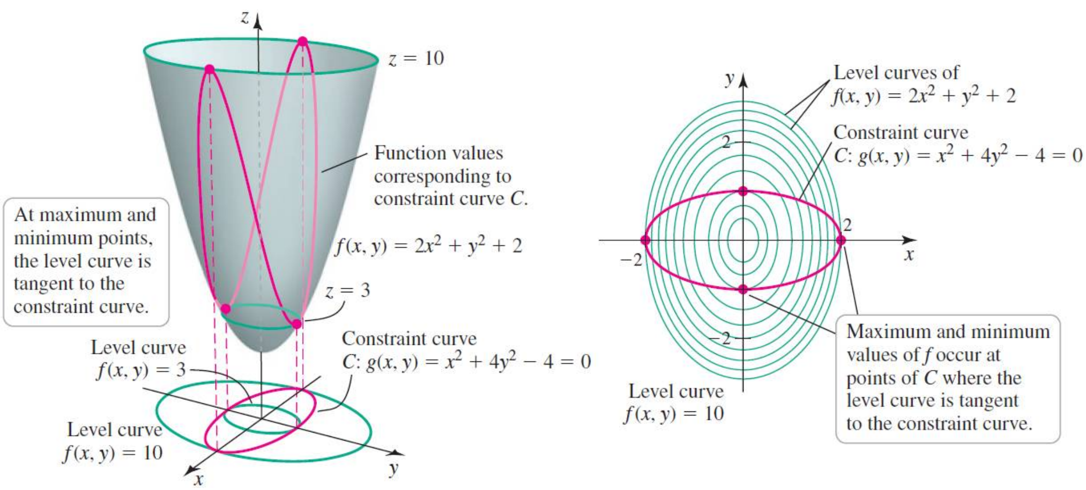
Find the maximum and minimum values of the objective function $f(x,y) = 2x^2 + y^2 + 2$, where x and y lie on the ellipse C given by $g(x,y) = x^2 + 4y^2 - 4 = 0$.

Solution

Noting that $\nabla f(x,y) = \langle 4x, 2y \rangle$ and $\nabla g(x,y) = \langle 2x, 8y \rangle$, the equations that result from $\nabla f = \lambda \nabla g$ and the constraint are

$$f_{2,2} = \frac{m_{2,2} - 1}{m_{2,2}} \begin{cases} \underbrace{\langle 4x, 2y \rangle = \lambda \langle 2x, 8y \rangle}_{\nabla f = \lambda \nabla g} \\ \underbrace{x^2 + 4y^2 - 4 = 0}_{g(x,y)=0} \end{cases} \Rightarrow \begin{cases} x(2-\lambda) = 0 & (1) \\ x(1-4\lambda) = 0 & (2) \\ x^2 + 4y^2 - 4 = 0 & (3) \end{cases}$$

The solutions of equation (1) are $x = 0$ or $\lambda = 2$. If $x = 0$, then equation (3) implies that $y = \pm 1$ and (2) implies that $\lambda = \frac{1}{4}$. On the other hand, if $\lambda = 2$, then equation (2) implies that $y = 0$; from (3), we get $x = \pm 2$. Therefore, the candidates for locations of extreme values are $(0, \pm 1)$, with $f(0, \pm 1) = 3$, and $(\pm 2, 0)$, with $f(\pm 2, 0) = 10$. We see that the maximum value of f on C is 10, which occurs at (2, 0) and $(-2, 0)$; the minimum value of f on C is 3, which occurs at (0, 1) and $(0, -1)$.



This kind of argument also applies to the problem of finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$. Thus the point (x, y, z) is restricted to lie on the level surface S with equation $g(x, y, z) = 0$. Instead of the level curves, we consider the level surfaces $f(x, y, z) = c$ and argue that if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x, y, z) = c$ is tangent to the level surface $g(x, y, z) = 0$ and so the corresponding gradient vectors are parallel. That means $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.

Example 3.9.3

Find the minimum distance between the point (3, 4, 0) and the surface of the cone $z^2 = x^2 + y^2$.

Solution

Since (3, 4, 0) is in the xy-plane, we anticipate two solutions, one for each sheet of the cone. The distance between (3, 4, 0) and any point (x, y, z) on the cone is

$$d(x, y, z) = \sqrt{(x-3)^2 + (y-4)^2 + z^2}.$$

It is easier to work with the *square* of the distance because if a point minimizes $d(x, y, z)^2$, it also minimizes $d(x, y, z)$. Therefore, we define

$$f(x, y, z) = d(x, y, z)^2 = (x-3)^2 + (y-4)^2 + z^2.$$

Now we proceed with Lagrange multipliers; the conditions are

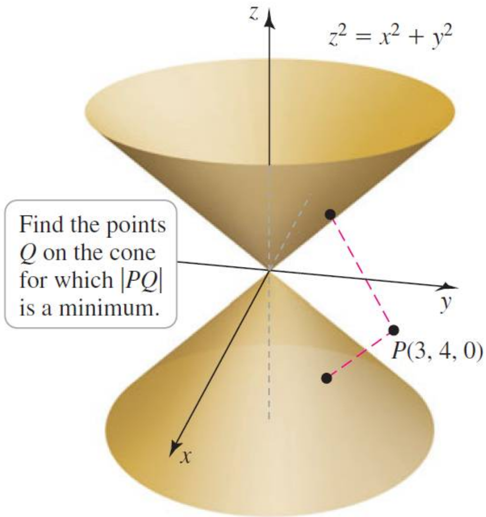
$$\begin{cases} f_x(x, y, z) = \lambda g_x(x, y, z), \text{ or } 2(x-3) = \lambda(-2x), \text{ or } x(1+\lambda) = 3, & (4) \\ f_y(x, y, z) = \lambda g_y(x, y, z), \text{ or } 2(y-4) = \lambda(-2y), \text{ or } y(1+\lambda) = 4, & (5) \\ f_z(x, y, z) = \lambda g_z(x, y, z), \text{ or } 2z = \lambda(2z), \text{ or } z = \lambda z, \text{ and} & (6) \\ g(x, y, z) = z^2 - x^2 - y^2 = 0. & (7) \end{cases}$$

The solutions of equation (6) (the simplest of the four equations) are either $z = 0$, or $\lambda = 1$ and $z \neq 0$. In the first case, if $z = 0$, then by equation (7), $x = y = 0$; however, $x = 0$ and $y = 0$ do not satisfy (4) and (5). So no solution results from this case.

On the other hand if $\lambda = 1$, then by (4) and (5), we find that $x = \frac{3}{2}$ and $y = 2$. Using (7), the corresponding values of z are $\pm\frac{5}{2}$. Therefore, the two solutions and the values of f are

$$x = \tfrac{3}{2}, \quad y = 2, \quad z = \tfrac{5}{2} \quad \text{with } f(\tfrac{3}{2}, 2, \tfrac{5}{2}) = \tfrac{25}{2}, \text{ and}$$
$$x = \tfrac{3}{2}, \quad y = 2, \quad z = -\tfrac{5}{2} \quad \text{with } f(\tfrac{3}{2}, 2, -\tfrac{5}{2}) = \tfrac{25}{2}.$$

You can check that moving away from $(\frac{3}{2}, 2, \pm\frac{5}{2})$ in any direction on the cone has the effect of increasing the values of f. Therefore, the points correspond to *local* minima of f. The minimum distance is $\sqrt{\frac{25}{2}} = 5/\sqrt{2}$.

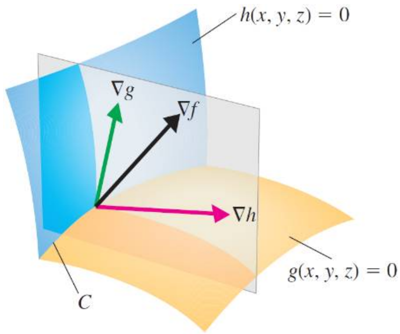


Suppose now that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints of the form $g(x, y, z) = 0$ and $h(x, y, z) = 0$. Geometrically, this means that we are looking for the extreme values of f when (x, y, z) is restricted to lie on the curve of intersection C of the level surfaces $g(x, y, z) = 0$ and $h(x, y, z) = 0$. Suppose f has such an extreme value at a point $P(a, b, c)$. We know from the beginning of this section that ∇f is orthogonal to C at P. But we also know that ∇g is orthogonal to $g(x, y, z) = 0$ and ∇h is orthogonal to $h(x, y, z) = 0$, so ∇g and ∇h are both orthogonal to C. This means that the gradient vector $\nabla f(a, b, c)$ is in the plane determined by $\nabla g(a, b, c)$ and $\nabla h(a, b, c)$. (We assume that these gradient vectors are not zero and not parallel.) So there are numbers λ and μ (called Lagrange multipliers) such that

$$\nabla f(a,b,c) = \lambda \nabla g(a,b,c) + \mu \nabla h(a,b,c).$$

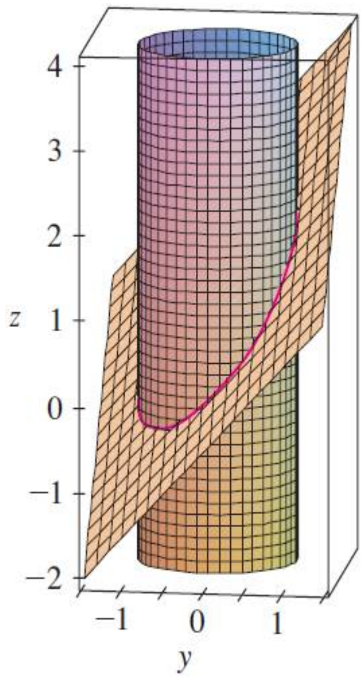
The method of Lagrange multipliers for the case of two constraints then consists of finding the point (x, y, z) and the Lagrange multipliers λ and μ (for a total of five unknowns) satisfying the five equations defined by:

$$\begin{cases} f_x(x,y,z) = \lambda g_x(x,y,z) + \mu h_x(x,y,z) \\ f_y(x,y,z) = \lambda g_y(x,y,z) + \mu h_y(x,y,z) \\ f_z(x,y,z) = \lambda g_z(x,y,z) + \mu h_z(x,y,z) \\ g(x,y,z) = 0 \\ h(x,y,z) = 0 \end{cases}$$



Example 3.9.4

Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.



Solution

We maximize the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z - 1$ and $h(x, y, z) = x^2 + y^2 - 1$. The Lagrange condition is $\nabla f = \lambda\nabla g + \mu\nabla h$, so we solve the equations

$$\begin{cases} 1 = \lambda + 2x\mu & (8) \\ 2 = -\lambda + 2y\mu & (9) \\ 3 = \lambda & (10) \\ x - y + z = 1 & (11) \\ x^2 + y^2 = 1 & (12) \end{cases}$$

Putting $\lambda = 3$ [from (10)] in (8), we get $2x\mu = -2$, so $x = -1/\mu$. Similarly, (9) gives $y = 5/(2\mu)$. Substitution in (12) then gives

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

and so $\mu^2 = \frac{29}{4}$, $\mu = \pm\sqrt{29}/2$. Then $x = \mp 2/\sqrt{29}$, $y = \pm 5/\sqrt{29}$, and, from (11), $z = 1 - x + y$ $= 1 \pm 7/\sqrt{29}$. The corresponding values of f are

$$\mp\frac{2}{\sqrt{29}} + 2\left(\pm\frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}.$$

Therefore the maximum value of f on the given curve is $3 + \sqrt{29}$.

Chapter 4 Multiple Integrals

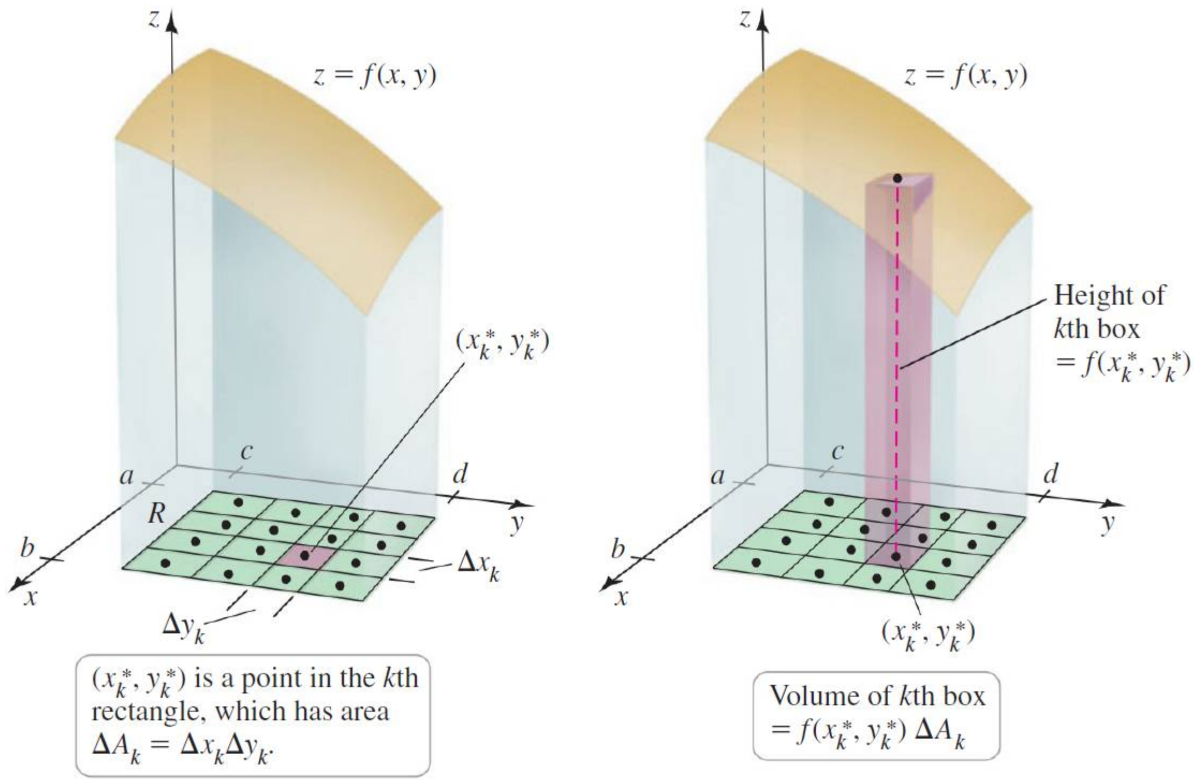
We assume that $z = f(x, y)$ is a nonnegative function defined on a *rectangular* region $R = \{(x, y) : a \le x \le b, c \le y \le d\}$. A **partition** of R is formed by dividing R into n rectangular subregions using lines parallel to the x- and y-axes (not necessarily uniformly spaced). The subregions may be numbered in any systematic way; for example, left to right, and then bottom to top. The side lengths of the k-th rectangle are denoted Δx_k and Δy_k, so the area of the k-th subregion is $\Delta A_k = \Delta x_k \Delta y_k$. We also let (x_k^*, y_k^*) be any point in the k-th subregion, for $1 \le k \le n$.

To approximate the volume of the solid bounded by the surface $z = f(x, y)$ and the region R, we construct boxes on each of the n subregions; each box has a height of $f(x_k^*, y_k^*)$ and a base with area ΔA_k, for $1 \le k \le n$. Therefore, the volume of the k-th box is

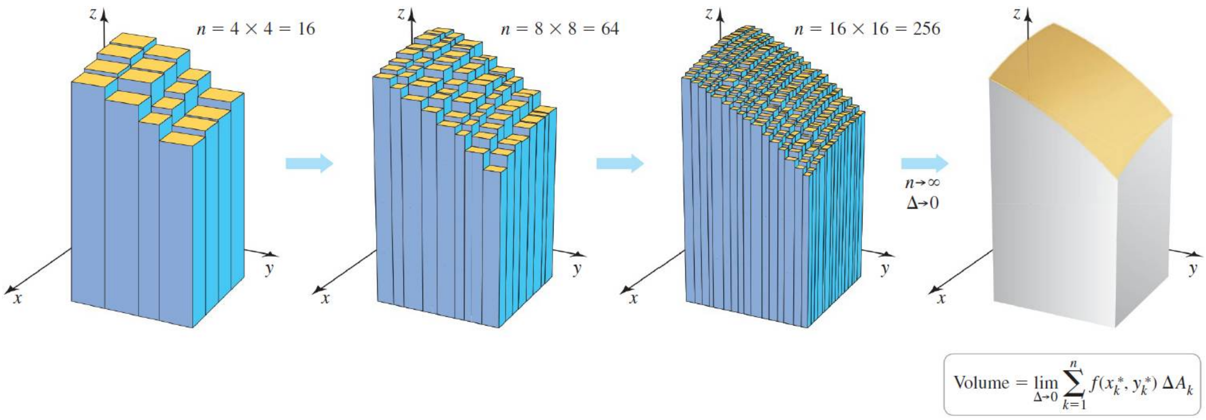
$$f(x_k^*, y_k^*)\Delta A_k = f(x_k^*, y_k^*)\Delta x_k \Delta y_k.$$

The sum of the volumes of the n boxes gives an approximation to the volume of the solid:

$$V = \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k.$$



We now let Δ be the maximum length of the diagonals of the rectangular subregions in the partition. As $\Delta \to 0$, the areas of *all* the subregions approach zero $(\Delta A_k \to 0)$ and the number of subregions increases $(n \to \infty)$. If the approximations given by these Riemann sums have a limit as $\Delta \to 0$, then we define the volume of the solid to be that limit.



Definition 4.1.1

For any function f defined on the rectangle $R = \{(x, y) \mid a \le x \le b \text{ and } c \le y \le d\}$, we define the **double integral** of f over R by

$$\iint_R f(x, y)dA = \lim_{\Delta \to 0} \sum_{i=1}^{n} f(u_i, v_i)\Delta A_i,$$

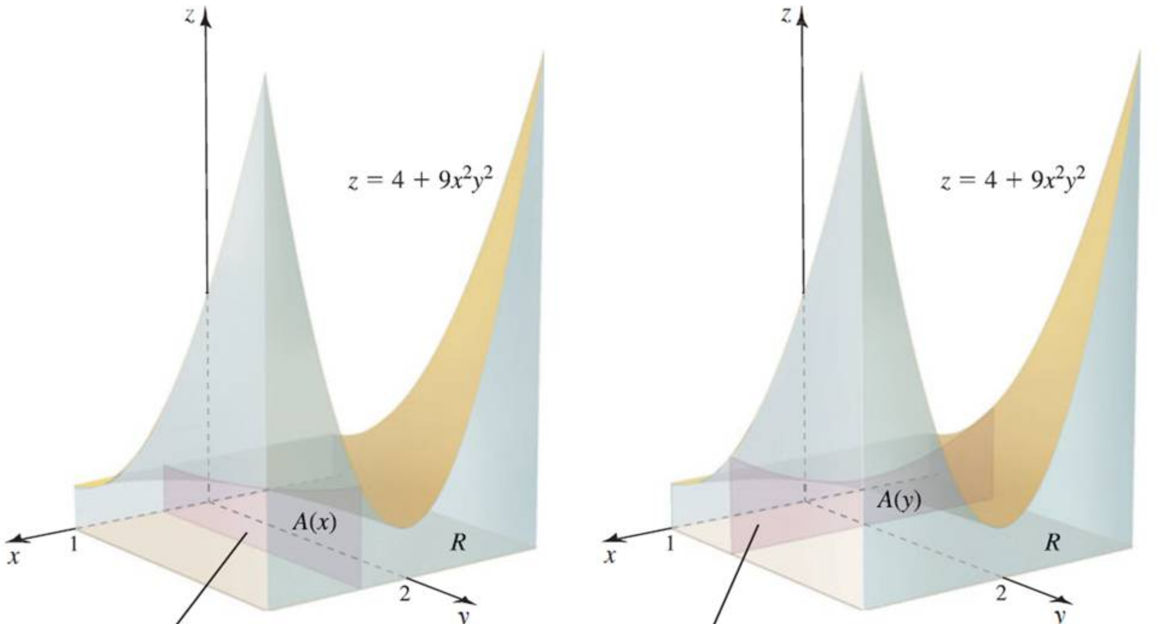
provided the limit exists and is the same for every choice of the evaluation points (u_i, v_i) in R_i, for $i = 1, 2, \cdots, n$. When this happens, we say that f is **integrable** over R.

Suppose we wish to compute the volume of the solid region bounded by the surface $z = f(x, y) = 4 + 9x^2y^2$ over the rectangular region $R = \{(x, y) : -1 \le x \le 1, 0 \le y \le 2\}$. By definition, the volume is given by the double integral

$$V = \iint_R f(x, y)dA = \iint_R (4 + 9x^2y^2)dA.$$

According to the General Slicing Method, we can compute this volume by taking slices through the solid parallel to the y-axis and perpendicular to the xy-plane. The slice at the point x has a cross-sectional area denoted $A(x)$. In general, as x varies, the area $A(x)$ also changes, so we integrate these cross-sectional areas from $x = -1$ to $x = 1$ to obtain the volume

$$V = \int_0^1 A(x)dx.$$



$A(x) = \int_0^2 (4 + 9x^2y^2)dy$ | $V = \int_{-1}^{1}\int_0^2 (4 + 9x^2y^2)\,dy\,dx$ | $A(y) = \int_{-1}^{1} (4 + 9x^2y^2)dx$ | $V = \int_0^2\int_{-1}^{1} (4 + 9x^2y^2)\,dx\,dy$

The important observation is that for a fixed value of x, $A(x)$ is the area of the plane region under the curve $z = 4 + 9x^2y^2$. This area is computed by integrating f with respect to y from $y = 0$ to $y = 2$, holding x fixed; that is,

$$A(x) = \int_0^2 (4 + 9x^2y^2)dy,$$

where $-1 \le x \le 1$, and x is treated as a constant in the integration. Substituting for $A(x)$, we have

$$V = \int_{-1}^{1} A(x)dx = \int_{-1}^{1} \underbrace{\left[\int_0^2 (4 + 9x^2y^2)dy\right]}_{A(x)} dx.$$

Similarly, $A(y) = \int_{-1}^{1} (4 + 9x^2y^2)dx$ is the area of the plane region under the curve $z = 4 + 9x^2y^2$ where $0 \le y \le 2$, and y is treated as a constant in the integration. We have

$$V = \int_0^2 A(y)dy = \int_0^2 \underbrace{\left[\int_{-1}^{1} (4 + 9x^2y^2)dx\right]}_{A(y)} dy.$$

In general, we have the following theorem.

Theorem 4.1.2 (Fubini)

Suppose that f is integrable over the rectangle $R = \{(x, y) \mid a \le x \le b \text{ and } c \le y \le d\}$. Then we can write the double integral of f over R as either of the iterated integrals:

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy.$$

Example 4.1.3

Find the volume of the solid bounded by the surface $z = 4 + 9x^2y^2$ over the region $R = \{(x,y) : -1 \le x \le 1, 0 \le y \le 2\}$. Use both possible orders of integration.

Solution

If we first integrate with respect to x, the area of a cross section of the solid for a fixed value of y is given by $A(y)$ (figure on the left). The volume of the region is

$$\begin{aligned}\iint_R (4+9x^2y^2)dA &= \int_0^2 \underbrace{\int_{-1}^1 (4+9x^2y^2)dx}_{A(y)}dy \\ &= \int_0^2 \left[4x + 3x^3y^2\right]_{-1}^1 dy \\ &= \int_0^2 (8+6y^2)dy \\ &= \left[8y + 2y^3\right]_0^2 \\ &= 32.\end{aligned}$$

Alternatively, if we integrate first with respect to y, the area of a cross section of the solid for a fixed value of x is given by $A(x)$ (figure on the right). The volume of the region is

$$\begin{aligned}\iint_R (4+9x^2y^2)dA &= \int_{-1}^1 \underbrace{\int_0^2 (4+9x^2y^2)dy}_{A(x)}dx \\ &= \int_{-1}^1 \left[4y + 3x^2y^3\right]_0^2 dx \\ &= \int_{-1}^1 (8+24x^2)dx\end{aligned}$$

$$= \left[8x + 8x^3 \right]_{-1}^{1}$$
$$= 32.$$

Example 4.1.4

Evaluate $\iint_R y\sin(xy)dA,$ where $R = [1,2]\times[0,\pi].$

Solution

If we first integrate with respect to x, we get

$$\begin{aligned}
\iint_R y\sin(xy)dA &= \int_0^{\pi}\int_1^2 y\sin(xy)dxdy \\
&= \int_0^{\pi}\left[-\cos(xy)\right]_{x=1}^{x=2} dy \\
&= \int_0^{\pi} -\cos(2y)+\cos y dy \\
&= \left[-\tfrac{\sin 2y}{2}+\sin y\right]_0^{\pi} \\
&= 0
\end{aligned}$$

If we reverse the order of integration and first integrate with respect to y, we get

$$\iint_R y\sin(xy)dA = \int_1^2\int_0^{\pi} y\sin(xy)dydx$$

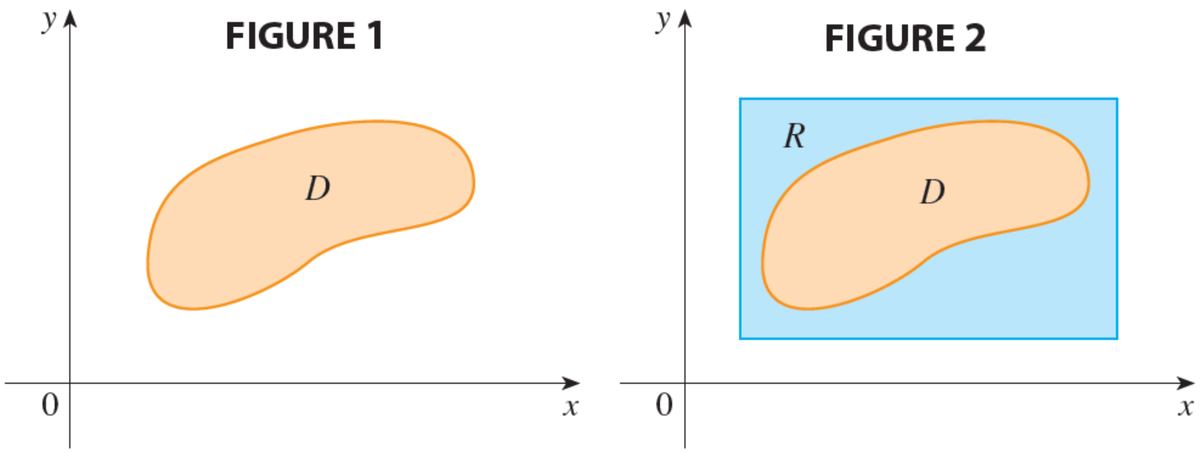
but this order of integration is much more difficult than the method given in this example because it involves integration by parts twice.

In the special case where $f(x, y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a particularly simple form. To be specific, suppose that $f(x, y) = g(x)\, h(y)$ and $R = [a,b]\times[c,d]$. Then Fubini's Theorem gives

$$\begin{aligned}
\iint_R f(x,y)dA &= \int_c^d\int_a^b g(x)h(y)dxdy \\
&= \int_c^d\left(\int_a^b g(x)h(y)dx\right)dy \\
&= \int_c^d h(y)\left(\int_a^b g(x)dx\right)dy \quad [h(y) \text{ is a constant in the inner integral}] \\
&= \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)
\end{aligned}$$

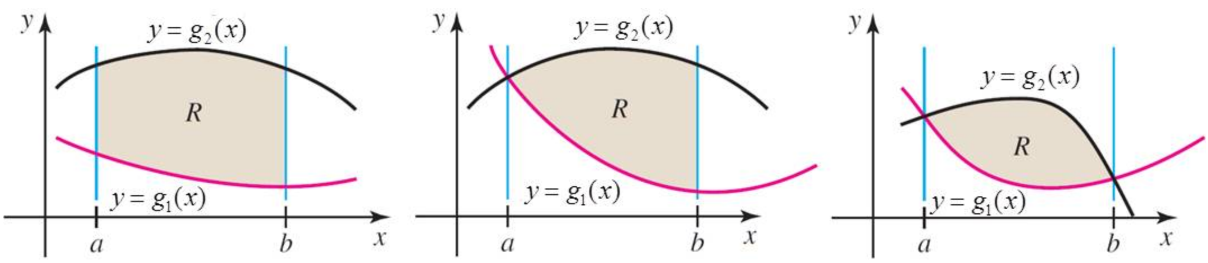
For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function *f* not just over rectangles but also over regions *D* of more general shape, such as the one illustrated in Figure 1. We suppose that *D* is a bounded region, which means that *D* can be enclosed in a rectangular region *R* as in Figure 2. Then we define a new function *F* with domain *R* by

$$F(x,y)=\begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$



If *F* is integrable over *R*, then we define the **double integral of** f **over** *D* by

$$\iint_D f(x,y)dA=\iint_R F(x,y)dA.$$



Theorem 4.2.1

Suppose that f is continuous on the region R defined by $R=\{(x,y)\,|\,a\le x\le b$ and $g_1(x)\le y\le g_2(x)\}$, for continuous functions g_1 and g_2, where $g_1(x)\le g_2(x)$, for all *x* in $[a, b]$. Then,

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

With integrals over nonrectangular regions, the order of integration cannot be simply switched; that is,

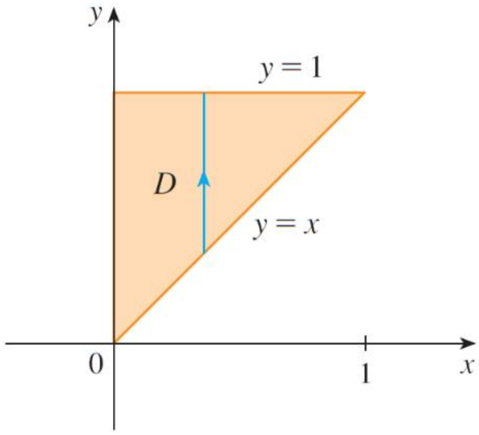
$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx \neq \int_{g_1(x)}^{g_2(x)} \int_a^b f(x,y)dxdy.$$

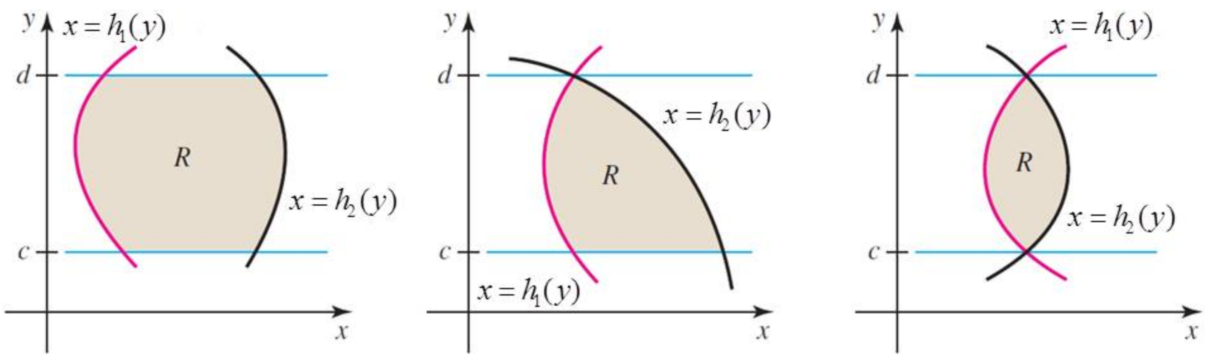
Example 4.2.2

Evaluate $\int_0^1 \int_x^1 2xydydx.$

Solution

$$\int_0^1 \int_x^1 2xydydx = \int_0^1 xy^2\Big|_x^1 dx = \int_0^1 (x - x^3)dx = \frac{x}{2} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}.$$



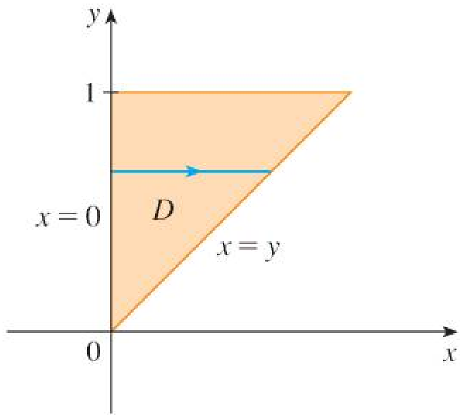


Theorem 4.2.3

Suppose that f is continuous on the region R defined by $R = \{(x,y) \mid c \le y \le d$ and $h_1(y) \le x \le h_2(y)\}$, for continuous functions h_1 and h_2, where $h_1(y) \le h_2(y)$, for all y in $[a, b]$. Then,

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy.$$

Example 4.2.4



Evaluate $\int_0^1 \int_0^y 2xy\,dx\,dy$.

Solution

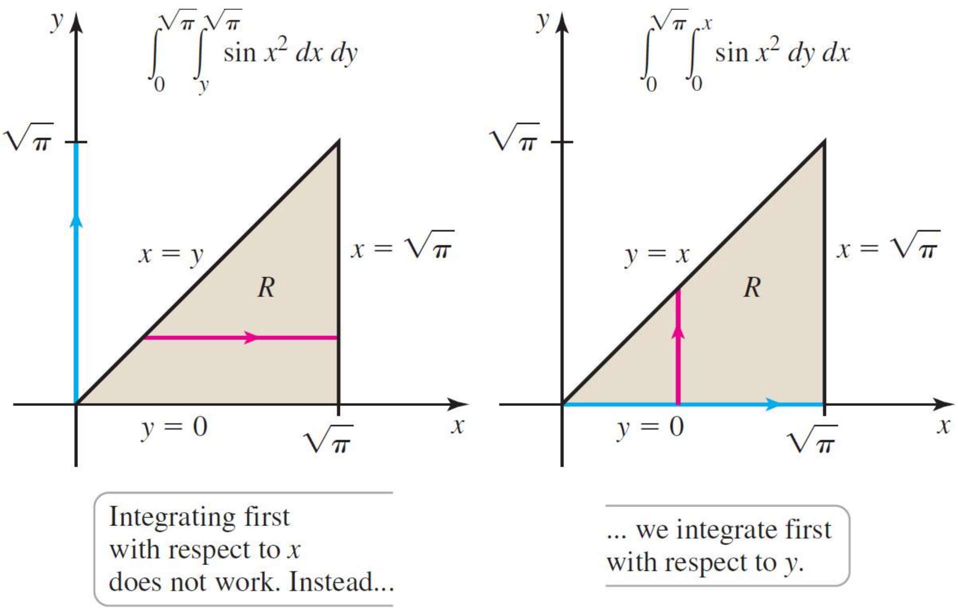
$$\int_0^1 \int_0^y 2xy\,dx\,dy = \int_0^1 x^2 y \Big|_0^y dy = \int_0^1 y^3 dy = \frac{y^4}{4}\Big|_0^1 = \frac{1}{4}$$

Example 4.2.5

Evaluate $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin\left(x^2\right) dx\,dy$.

Solution

$$\begin{aligned}
\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin\left(x^2\right) dx\,dy &= \int_0^{\sqrt{\pi}} \int_0^{x} \sin\left(x^2\right) dy\,dx \\
&= \int_0^{\sqrt{\pi}} y \sin\left(x^2\right)\Big|_0^x dx \\
&= \int_0^{\sqrt{\pi}} x \sin\left(x^2\right) dx \\
&= -\frac{1}{2}\cos\left(x^2\right)\Big|_0^{\sqrt{\pi}} \\
&= 1
\end{aligned}$$



For the general regions, we have the following properties:

(i) $\iint_D [af(x,y)+bg(x,y)]\,dA = a\iint_D f(x,y)\,dA + b\iint_D g(x,y)\,dA$ where a and b are constants.

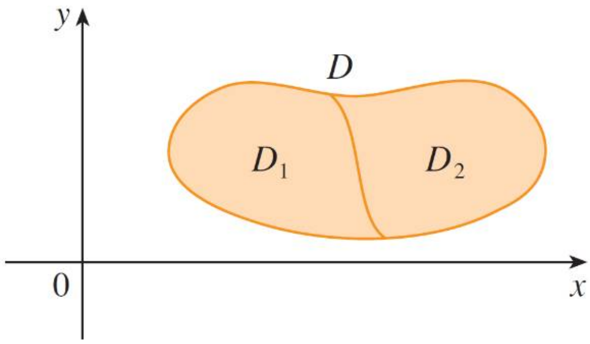
(ii) If $f(x,y) \ge g(x,y)$ for all (x, y) in D, then

$$\iint_D f(x,y)\,dA \ge \iint_D g(x,y)\,dA$$

The next property of double integrals is similar to the property of single integrals given by the equation $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

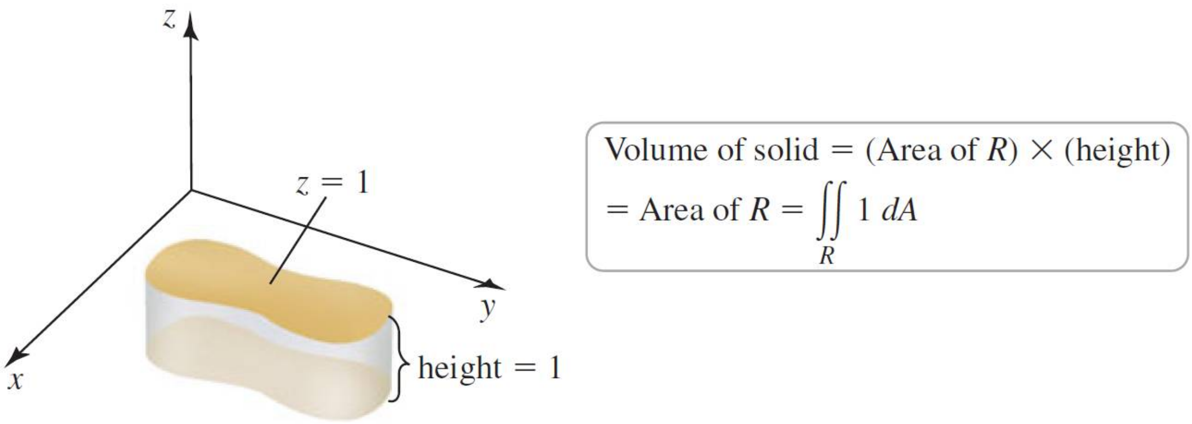
(iii) If $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap except perhaps on their boundaries, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

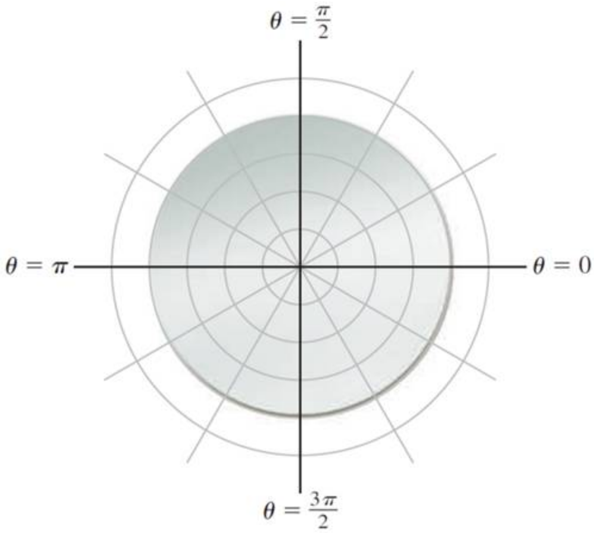


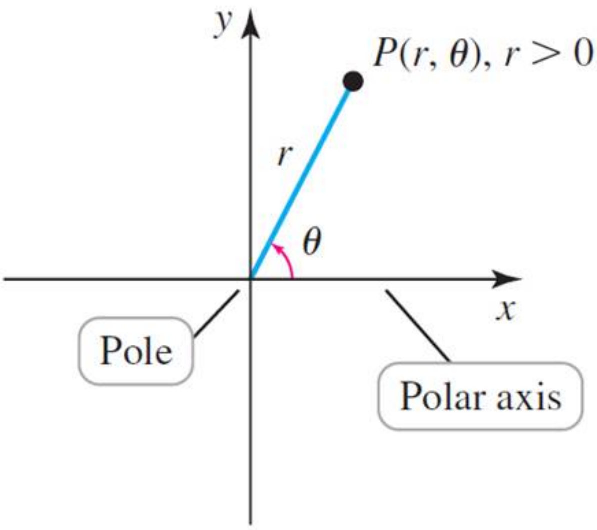
(iv) If we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D:

$$\iint_D 1\,dA = A(D).$$

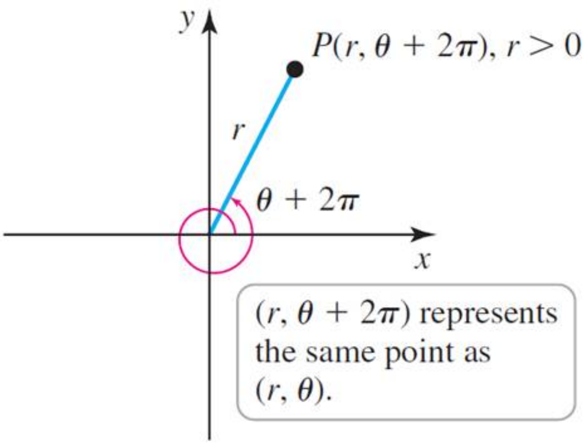


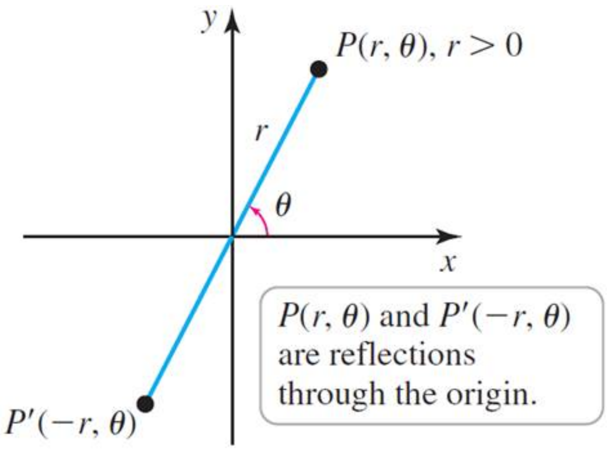
The **polar coordinate system** is a two-dimensional coordinate system in which each point P on a plane is determined by a distance r from a fixed O that is called the **pole** (or origin) and an angle θ from positive x-axis (**polar axis**). The point P is represented by the ordered pair (r,θ) and r,θ are called **polar coordinates**.





We extend the meaning of polar coordinates (r,θ) to the case in which r is negative by agreeing that the points $(-r,\theta)$ and (r,θ) lie in the same line through O and at the same distance $|r|$ from O, but on opposite sides of O. If $r>0$, the point (r,θ) lies in the same quadrant as θ; if $r<0$, it lies in the quadrant on the opposite side of the pole.





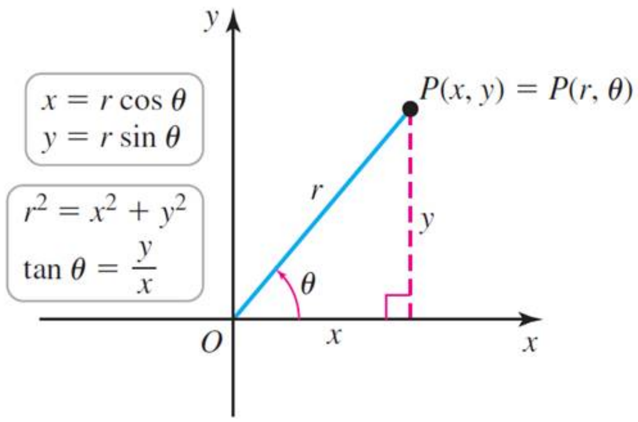
PROCEDURE **Converting Coordinates**

A point with polar coordinates (r, θ) has xy-coordinates (x, y), where

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

A point with xy-coordinates (x, y) has polar coordinates (r, θ), where

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \tan\theta = \frac{y}{x}.$$



Example 4.3.1

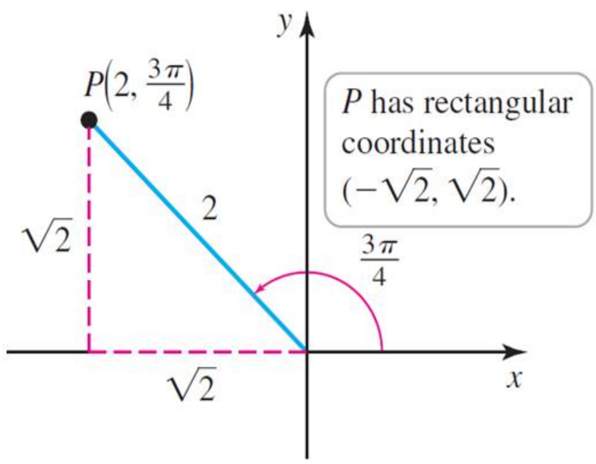
(a) Convert the point $P\left(2, \dfrac{3\pi}{4}\right)$ from polar to xy-coordinates.

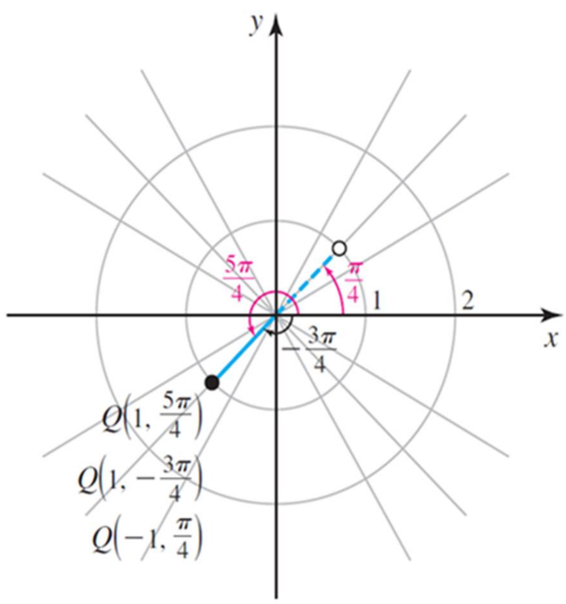
(b) Represent the point Q with xy-coordinates $\left(-\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$ in terms of polar coordinates.

Solution

(a) The x-y coordinate of P is $\left(2\cos\frac{3\pi}{4}, 2\sin\frac{3\pi}{4}\right) = \left(-\sqrt{2}, \sqrt{2}\right)$

(b) $\left(1, \dfrac{5\pi}{4}\right), \left(1, -\dfrac{3\pi}{4}\right), \left(-1, \dfrac{\pi}{4}\right), \cdots$





Example 4.3.2

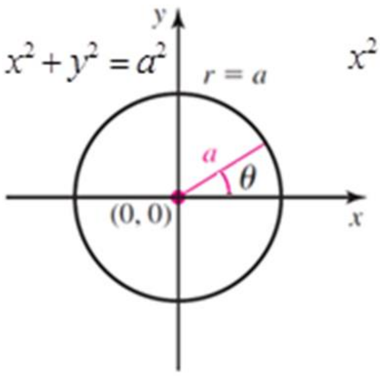
The equation $r = a$ describes a circle of radius $|a|$ centered at (0, 0).

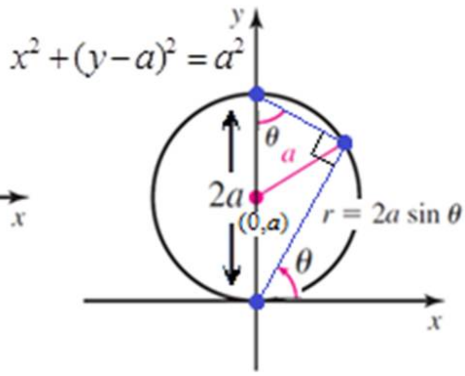
The equation $r = 2a \sin \theta$ describes a circle of radius $|a|$ centered at $(0, a)$.

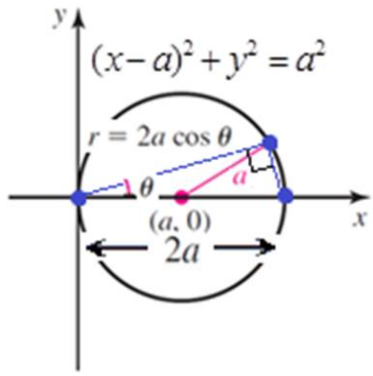
$$\begin{aligned} x^2 + y^2 &= r^2 \\ &= 2a(2a\sin\theta)\sin\theta && \because r = 2a\sin\theta \\ &= 2ar\sin\theta && \because y = r\sin\theta \\ &= 2ay \end{aligned}$$

$$\Rightarrow x^2 + y^2 - 2ay = 0 \quad \Rightarrow x^2 + y^2 - 2ay + a^2 = a^2 \quad \Rightarrow x^2 + (y-a)^2 = a^2$$

Similarly, the equation $r = 2a \cos \theta$ describes a circle of radius $|a|$ centered at $(a, 0)$.





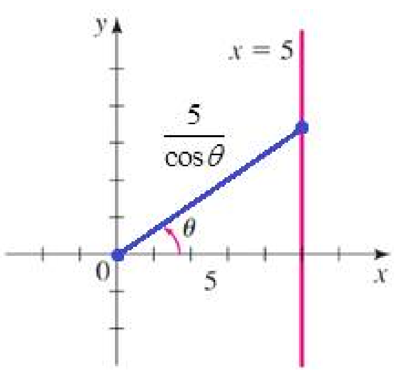


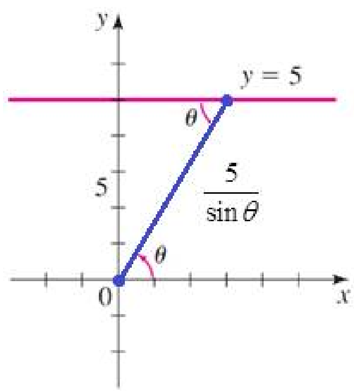
The polar equation $\theta = \alpha$ is a straight line passing through origin with slope $\tan\alpha$.

The polar equation $r = a\sec\theta$ describes a vertical straight $x = a$ line passing through $(a, 0)$.

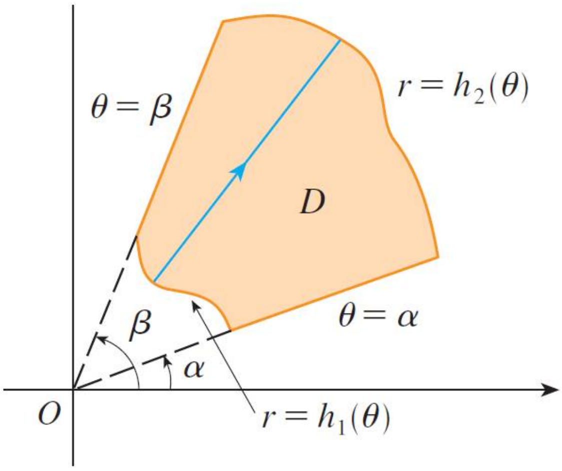
The polar equation $r = a\csc\theta$ describes a vertical straight line $y = a$ passing through $(0, a)$.

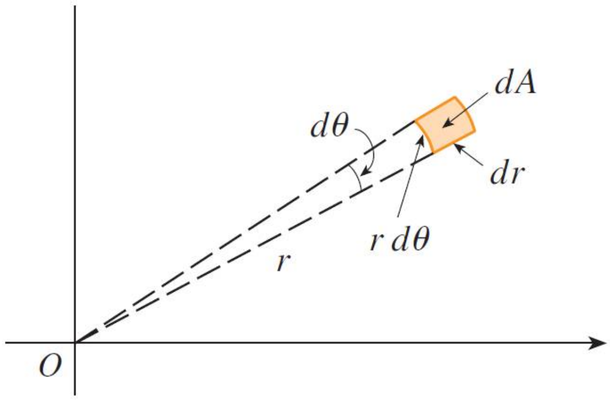
$$x = r\cos\theta = \frac{5}{\cos\theta}\cos\theta = 5 \qquad\qquad y = r\sin\theta = \frac{5}{\sin\theta}\sin\theta = 5$$





In some cases, the description of R in terms of rectangular coordinates is rather complicated, but R is easily described using polar coordinates. The "infinitesimal" polar rectangle can be thought of as an ordinary rectangle with dimensions $r\,d\theta$ and dr and therefore has "area" $dA = r\,dr\,d\theta$.





Theorem 4.3.3

Suppose that $f(r,\theta)$ is continuous on the region $R=\{(r,\theta)\,|\,\alpha\le\theta\le\beta \text{ and } g_1(\theta)\le r\le g_2(\theta)\}$, where $0\le g_1(\theta)\le r\le g_2(\theta)$ for all θ in $[\alpha, \beta]$. Then,

$$\iint_R f(x,y)dA=\int_\alpha^\beta\int_{g_2(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

<u>Be careful not to forget the additional factor r on the right side.</u>

Example 4.3.4

Find the volume of the solid bounded by the paraboloid $z=f(x,y)=9-x^2-y^2$ and the xy-plane.

Solution

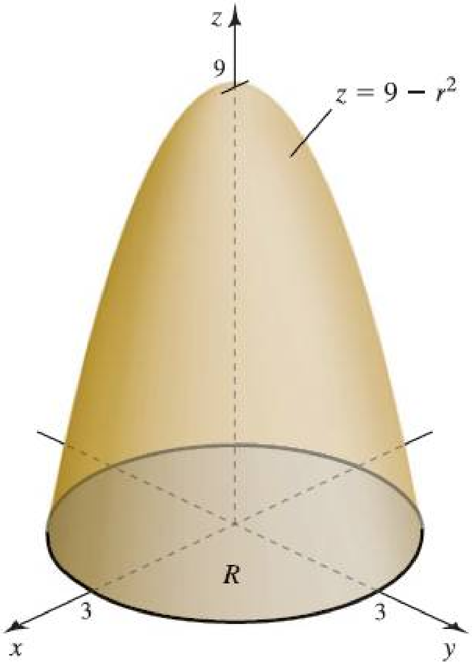
The paraboloid intersects the xy-plane ($z = 0$) when $z=9-x^2-y^2=9-r^2=0$, or $r = 3$. Therefore, the intersection curve is the circle of radius 3 centered at the origin. The resulting region of integration is the disk $R=\{(r,\theta)\,|\,0\le r\le 3, 0\le\theta\le 2\pi\}$. Integrating over R in polar coordinates, the volume is

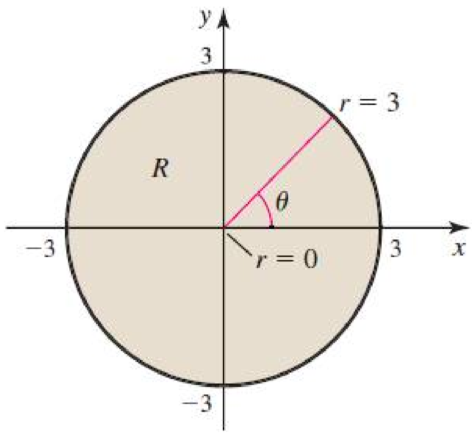
$$V=\iint_R (9-x^2-y^2)dA=\int_0^{2\pi}\int_0^3 (9-r^2)rdrd\theta=\int_0^3 (9r-r^3)dr\int_0^{2\pi} d\theta=2\pi\left[\frac{9r^2}{2}-\frac{r^4}{4}\right]_0^3=\frac{81}{2}\pi$$

If we had used rectangular coordinates instead of polar coordinates, then we would have obtained

$$V=\iint_R (9-x^2-y^2)dA=\int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (9-x^2-y^2)dydx$$

which is not easy to evaluate.





Example 4.3.5

Find the volume of the solid that lies under the paraboloid $z=x^2+y^2$, above the *xy*-plane, and inside the cylinder $x^2+y^2=2x$.

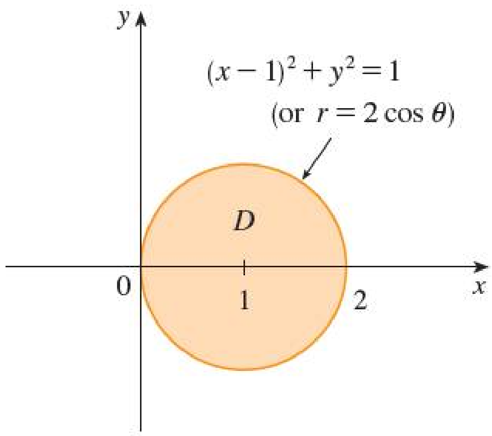
Solution

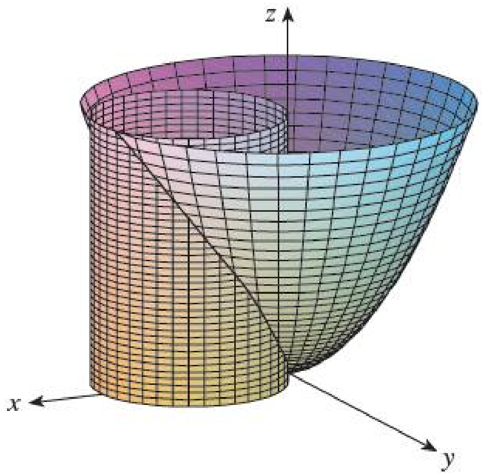
The solid lies above the disk D whose boundary circle has equation $x^2+y^2=2x$ or, after completing the square, $(x-1)^2+y^2=1$.

The polar equation of the boundary circle becomes $r = 2\cos\theta$. Thus the disk D is given by

$$D=\{(r,\theta)\,|\,-\pi/2\le\theta\le\pi/2,\ 0\le r\le 2\cos\theta\}.$$

$$V = \iint_D (x^2+y^2)\,dA$$
$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2 \cdot r\,dr\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$
$$= 4\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta \quad \left[\cos^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1+2\cos 2\theta+\cos^2 2\theta}{4} = \frac{1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)}{4}\right]$$
$$= \int_{-\pi/2}^{\pi/2} 1+2\cos 2\theta+\tfrac{1}{2}(1+\cos 4\theta)\,d\theta$$
$$= \left[\tfrac{3}{2}\theta+\sin 2\theta+\tfrac{1}{8}\sin 4\theta\right]_{-\pi/2}^{\pi/2}$$
$$= \frac{3\pi}{2}$$





Example 4.3.6

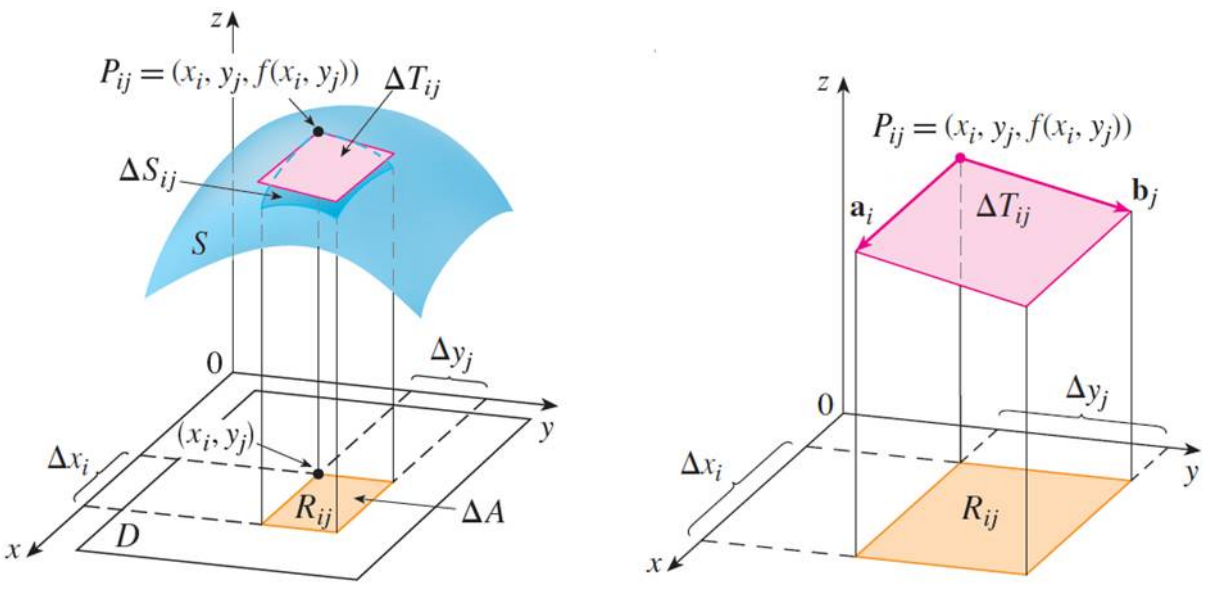
Calculate $\int_0^\infty e^{-x^2}\,dx.$

Solution

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \iint_{x,y>0} e^{-(x^2+y^2)}\,dA$$
$$\int_0^\infty\int_0^\infty e^{-x^2}\cdot e^{-y^2}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$$
$$\int_0^\infty e^{-x^2}dx\cdot\int_0^\infty e^{-y^2}dy = \int_0^{\pi/2}\left[-e^{-r^2}\right]_0^\infty d\theta = \frac{\pi}{2}$$
$$\int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-y^2}dy = \sqrt{\frac{\pi}{2}}$$

Let S be a surface with equation $z = f(x, y)$, where f has continuous partial derivatives. For simplicity in deriving the surface area formula, we assume that $f(x, y) \geq 0$ and the domain D of f is a rectangle. We divide D into small rectangles R_{ij} with area $\Delta A = \Delta x \Delta y$. Let (x_i, y_j) be the corner of R_{ij} closest to the origin and $P_{ij}(x_i, y_j, f(x_i, y_j))$ be the point on S directly above it. The tangent plane to S at P_{ij} is an approximation to S near P_{ij}. So the area ΔT_{ij} of the part of this tangent plane (a parallelogram) that lies directly above R_{ij} is an approximation to the area ΔS_{ij} of the part of S that lies directly above R_{ij}. Thus the sum $\sum\sum \Delta T_{ij}$ is an approximation to the total area of S, and this approximation appears to improve as the number of rectangles increases. Therefore we define the **surface area** of S to be

$$A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$



Let Δx_i and Δy_j be the length of edges of rectangle R_{ij} which are parallel to x-axis and y-axis respectively. Let the vectors $\mathbf{a}_i$ and $\mathbf{b}_j$ form two adjacent sides of the parallelogram (the above figure). The equation of tangent plane of $z = f(x, y)$ at (x_i, y_j) is given by

$$z - f(x_i, y_j) = f_x(x_i, y_j)(x - x_i) + f_y(x_i, y_j)(y - y_j).$$

Therefore $\mathbf{a}_i = \langle \Delta x_i, \underbrace{0}_{\substack{\text{no change in } y \\ \text{on tangent plane}}}, \underbrace{f_x(x_i, y_j)\Delta x_i}_{\substack{\text{change in } z \text{ on} \\ \text{tangent plane}}} \rangle$ and $\mathbf{b}_j = \langle \underbrace{0}_{\substack{\text{no change in } x \\ \text{on tangent plane}}}, \Delta y_j, \underbrace{f_y(x_i, y_j)\Delta y_j}_{\substack{\text{change in } z \text{ on} \\ \text{tangent plane}}} \rangle$.

$$\mathbf{a}_i \times \mathbf{b}_j = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x_i & 0 & f_x(x_i, y_j)\Delta x_i \\ 0 & \Delta y_j & f_y(x_i, y_j)\Delta y_j \end{vmatrix} = \left\langle -f_x(x_i, y_j)\Delta x_i \Delta y_j, -f_y(x_i, y_j)\Delta x_i \Delta y_j, \Delta x_i \Delta y_j \right\rangle$$

This gives us $\Delta T_{ij} = \|\mathbf{a}_i \times \mathbf{b}_j\| = \sqrt{f_x(x_i, y_j)^2 + f_y(x_i, y_j)^2 + 1}\,\overbrace{\Delta x_i \Delta y_j}^{\Delta A}$. Thus

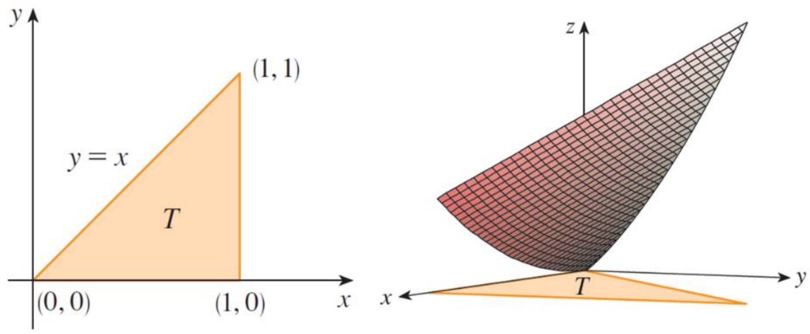
$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij} = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{f_x(x_i, y_j)^2 + f_y(x_i, y_j)^2 + 1}\,\Delta A = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\,dA$$

Example 4.4.1

Find the surface area of the part of the surface $z = f(x, y) = x^2 + 2y$ that lies above the triangular region $T = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le x\}$.

Solution

$$\begin{aligned} A &= \iint_T \sqrt{1 + f_x^2 + f_y^2}\,dA \\ &= \int_0^1 \int_0^x \sqrt{1 + 2^2 + (2x)^2}\,dy\,dx \\ &= \int_0^1 x\sqrt{5 + 4x^2}\,dx \\ &= \tfrac{1}{8} \cdot \tfrac{2}{3}(5 + 4x^2)^{3/2}\Big|_0^1 \\ &= \tfrac{1}{12}(27 - 5\sqrt{5}) \end{aligned}$$



Example 4.4.2

Find the surface area of the part of the paraboloid $z = f(x, y) = 9 - x^2 - y^2$ that lies above the xy-plane (Example 4.3.4).

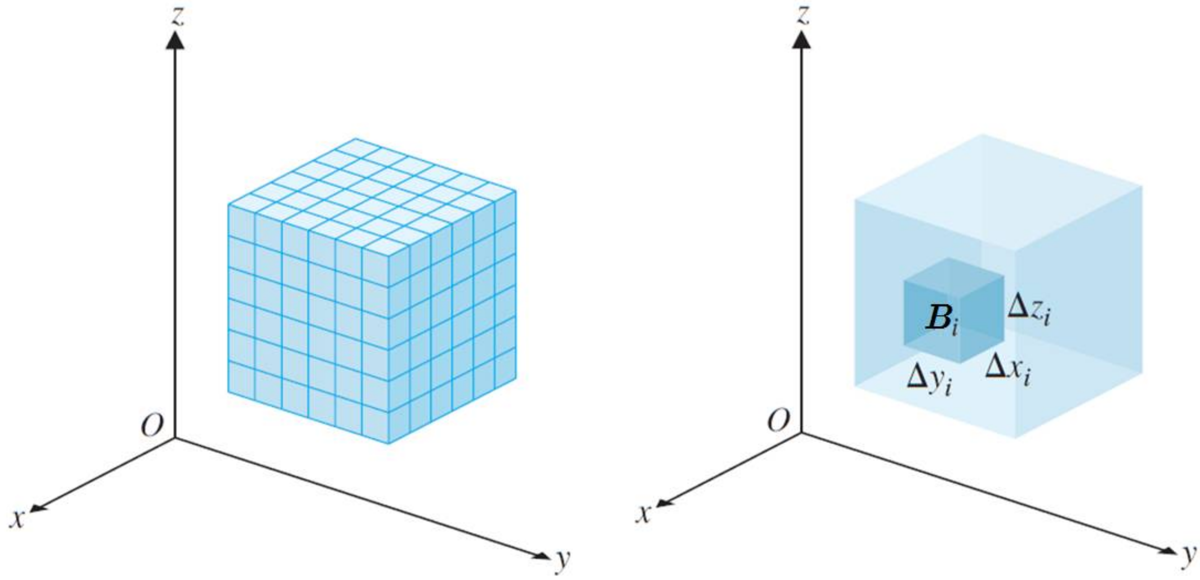
Solution

$$A = \iint_R \sqrt{1 + f_x^2 + f_y^2}\,dA = \iint_R \sqrt{1 + 4x^2 + 4y^2}\,dA = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\,r\,dr\,d\theta = 2\pi \cdot \tfrac{2}{3}(1 + 4r^2)^{3/2}\Big|_0^3 \approx 938.55$$

Consider a function $w = f(x, y, z)$ that is defined on a closed and bounded region D of $\boldsymbol{R}^3$. The graph of f is the set of points $(x, y, z, f(x, y, z))$, where (x, y, z) is in D, for which there is no complete three-dimensional representation. Since the graph of $w = f(x, y, z)$ lies in a four dimensional space, we can't even visualize such graph. Despite the difficulties in representing f in $\boldsymbol{R}^3$, we may still define the integral of f over D. Let's first deal with the simplest case where f is defined on a rectangular box:

$$B = \{(x, y, z) \mid a \le x \le b, c \le y \le d, r \le z \le s\}.$$

We begin by partitioning the region B by slicing it by planes parallel to the *xy*-plane, planes parallel to the *xz*-plane and planes parallel to the *yz*-plane. Notice that this divides B into a number of smaller boxes. Number the smaller boxes in any order: $B_1, B_2, \cdots, B_n$. For each box B_i $(i = 1, 2, \cdots, n)$, call the x, y and z dimensions of the box $\Delta x_i, \Delta y_i$ and Δz_i, respectively.



The volume of the box B_i is then $\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$. As we did in both one and two dimensions, we pick any point (u_i, v_i, w_i) in the box B_i and form the Riemann sum

$$\sum_{i=1}^{n} f(u_i, v_i, w_i) \Delta V_i.$$

In this three-dimensional case, we define the norm of the partition $\|P\|$ to be the longest diagonal of any of the boxes B_i, $i = 1, 2, \cdots, n$. The smaller is $\|P\|$, the more is the accuracy of the approximation. We can now define the triple integral of $f(x, y, z)$ over B.

Definition 4.5.1

For any function $f(x, y, z)$ defined on the rectangular box B, we define the **triple integral** of f over B by

$$\iiint_B f(x,y,z)dV = \lim_{\|P\|\to 0}\sum_{i=1}^{n} f(u_i, v_i, w_i)\Delta V_i,$$

provided the limit exists and is the same for every choice of evaluation points (u_i, v_i, w_i) in B_i, for $i = 1, 2, \cdots, n.$ When this happens, we say that f is **integrable** over B.

Theorem 4.5.2 (Fubini)

If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x,y,z)dV = \int_a^b \int_c^d \int_r^s f(x,y,z)dzdydx$$

When B is a rectangular box, the order of the triple integral is irrelevant.

Example 4.5.3

Evaluate the triple integral $\iiint_B xyz^2 dV,$ where B is the rectangular box given by $B = [0, 1] \times [-1, 2] \times [0, 3]$.

Solution

We could use any of the six possible orders of integration.

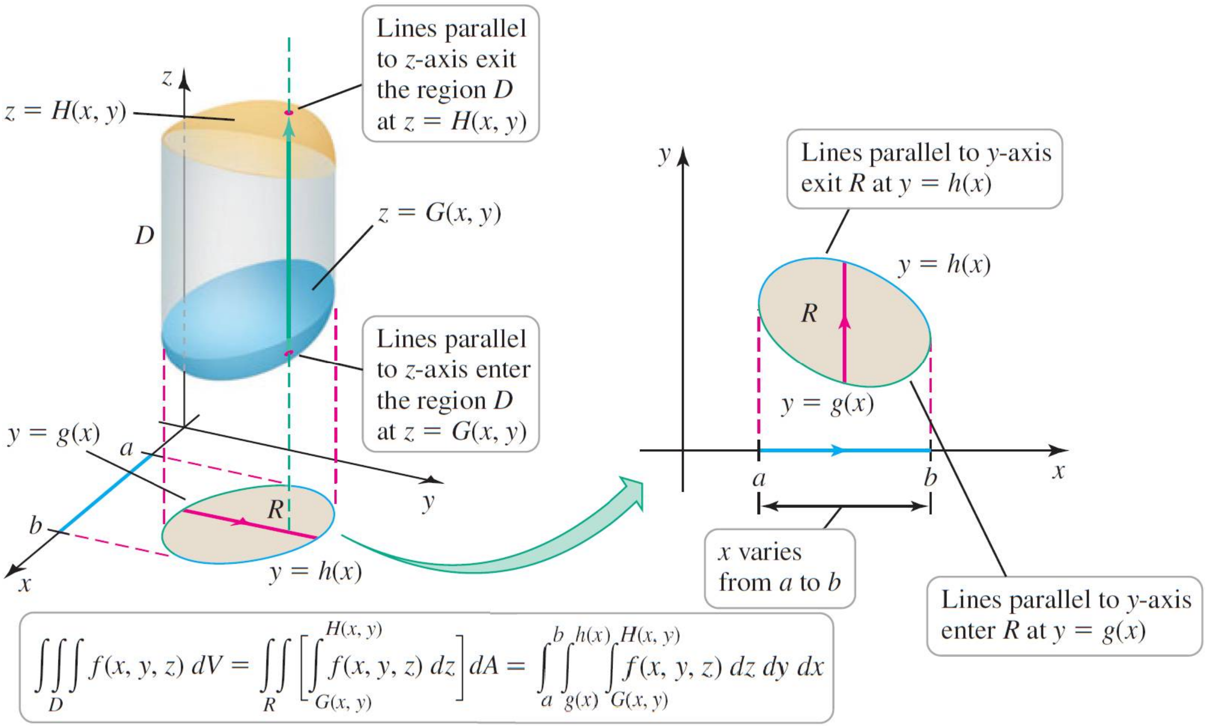
$$\begin{aligned}\iiint_B xyz^2 dV &= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 dxdydz \\ &= \int_0^1 xdx \int_{-1}^2 ydy \int_0^3 z^2 dz \\ &= \left[\frac{x^2}{2}\right]_0^1 \cdot \left[\frac{y^2}{2}\right]_{-1}^2 \cdot \left[\frac{z^3}{3}\right]_0^3 \\ &= \frac{1}{2}\cdot\frac{2^2-1}{2}\cdot\frac{3^3}{3} \\ &= \frac{27}{4}\end{aligned}$$

Now we define the **triple integral over a general bounded region *R*** in three-dimensional space (a solid) by much the same procedure that we used for double integrals. We enclose *R* in a box $B = \{(x, y, z) \mid a \le x \le b, c \le y \le d, r \le z \le s\}$. Then we define *F* so that it agrees with *f* on *R* but is 0 for points in *B* that are outside *R*. By definition,

$$\iiint_R f(x,y,z)dV = \iiint_B F(x,y,z)dV.$$

Suppose a region *D* in $\boldsymbol{R}^3$ is bounded above by a surface $z = H(x, y)$ and below by a surface $z = G(x, y)$. These two surfaces determine the limits of integration in the *z*-direction.

Once we know the upper and lower boundaries of *D*, the next step is to project the region *D* onto the *xy*-plane to form a region that we call *R*. You can think of *R* as the shadow of *D* in the *xy*-plane. Assume *R* is bounded above and below by the curves $y = h(x)$ and $y = g(x)$, respectively, and bounded on the right and left by the lines $x = a$ and $x = b$, respectively. The remaining integration over *R* is carried out as a double integral.



To integrate over all points of *D* we

- first integrate with respect to *z* from $z = G(x, y)$ to $z = H(x, y)$,
- then integrate with respect to *y* from $y = g(x)$ to $y = h(x)$, and

- finally integrate with respect to x from $x = a$ to $x = b$.

Theorem 4.5.4 (Triple Integrals)

Let f be continuous over the region

$$D = \{(x,y,z) \mid a \le x \le b,\ g(x) \le y \le h(x),\ G(x,y) \le z \le H(x,y)\}$$

where g, h, G, and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral

$$\iiint_B f(x,y,z)dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,y,z)dzdydx.$$

In general, the order of integral follows the below table:

Inner integral	Variable is bounded by functions of 2 other variables
Middle integral	Variable is bounded by functions of 1 variable which is not the variable of the inner integral
Outer integral	Variable is bounded by 2 constants

Example 4.5.5

Find the volume of the solid bounded by the graphs of $z = 4 - y^2$, $x + z = 4$, $x = 0$ and $z = 0$.

Solution

$R = \{(x,y,z) \mid 0 \le x \le 4 - z,\ 0 \le z \le 4 - y^2,\ -2 \le y \le 2\}$.

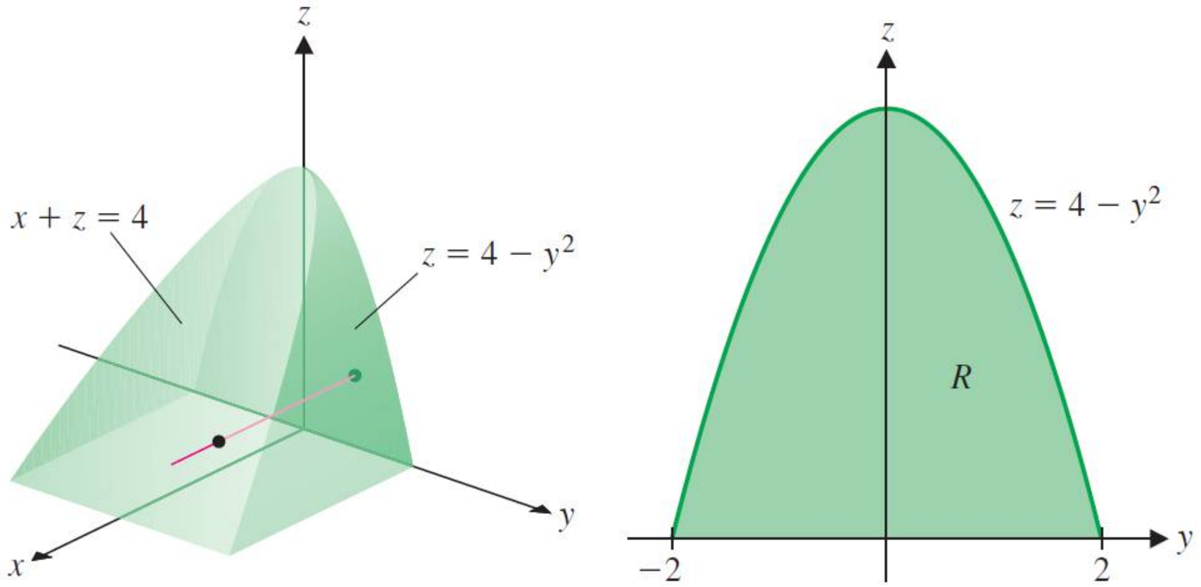
$$\begin{aligned}\iiint_R 1dV &= \int_{-2}^{2}\int_0^{4-y^2}\int_0^{4-z} 1dxdzdy \\ &= \int_{-2}^{2}\int_0^{4-y^2} (4-z)dzdy \\ &= \int_{-2}^{2}\left[4z - \frac{z^2}{2}\right]_0^{4-y^2} dy \\ &= \int_{-2}^{2}\left[4(4-y^2) - \frac{(4-y^2)^2}{2}\right]dy\end{aligned}$$

$$= 2\int_0^2 \left[4(4-y^2) - \frac{(4-y^2)^2}{2} \right] dy$$

$$= 2\int_0^2 \left[8 - \frac{y^4}{2} \right] dy$$

$$= 2\left[8y - \frac{y^5}{10} \right]_0^2$$

$$= \frac{128}{5}$$



Example 4.5.6

Calculate $\iiint\limits_R x^4 \cos(xz)dV$ where $R = \{(x,y,z) \mid 0 \le y \le 1, 0 \le z \le y, \sqrt{y} \le x \le 1\}$.

Solution

If we integrate first with respect to x, then z, and then y, then

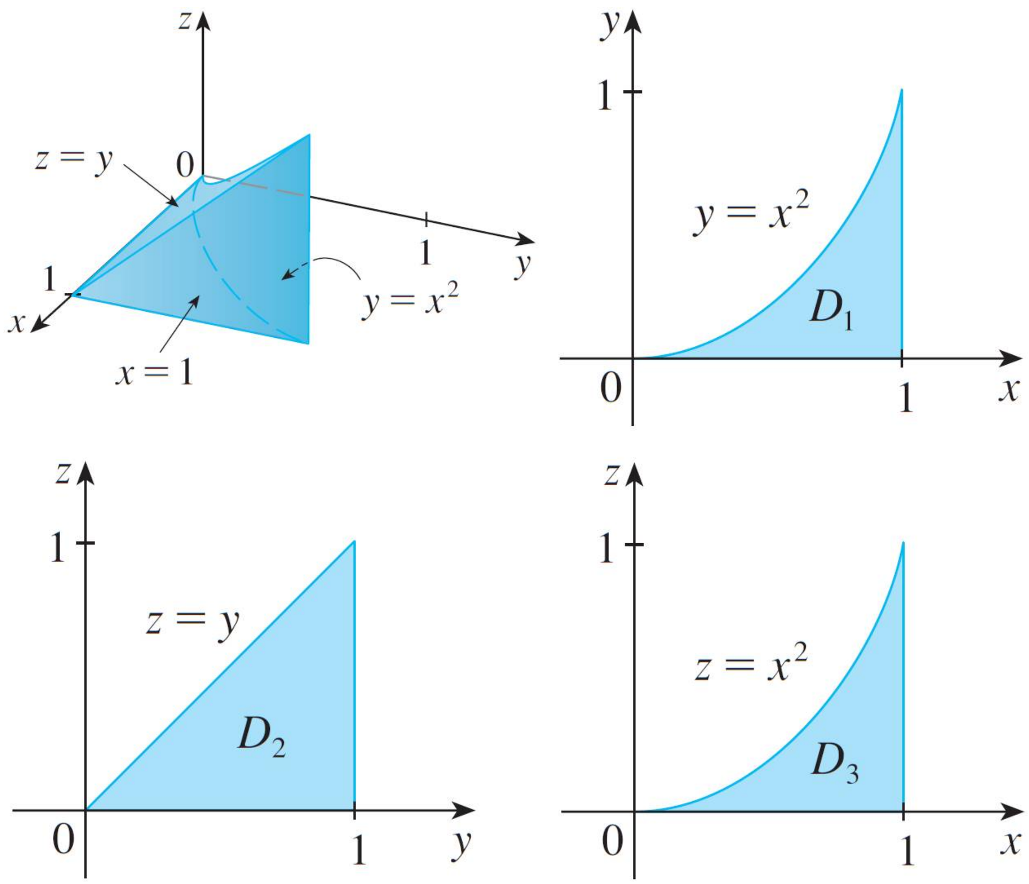
$$\iiint\limits_R x^4 \cos(xz)dV = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 x^4 \cos(xz)dxdzdy$$

is difficult. Indeed, $0 \le z \le y \le x^2 \le 1$ gives $R = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x^2, 0 \le z \le y\}$.

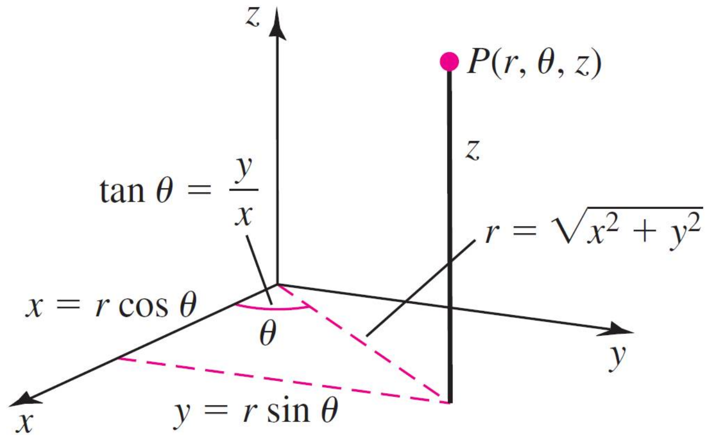
$$\iiint\limits_R x^4 \cos(xz)dV = \int_0^1 \int_0^{x^2} \int_0^y x^4 \cos(xz)dzdydx$$

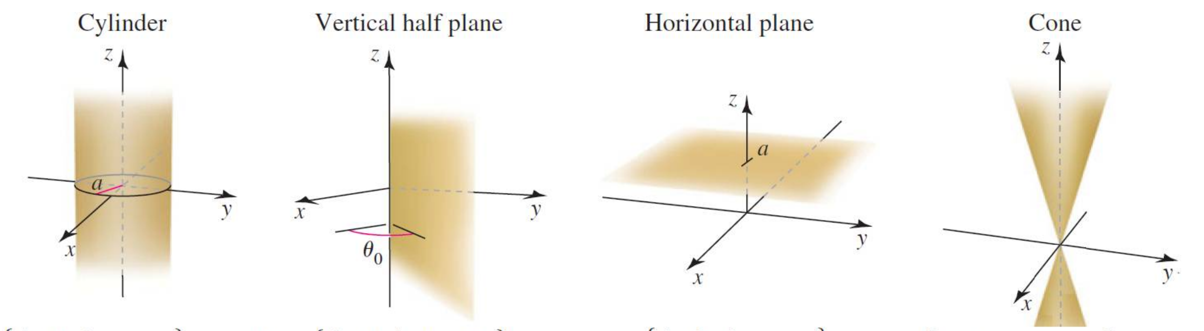
$$
\begin{aligned}
&= \int_0^1 \int_0^{x^2} \int_{z=0}^{z=y} x^3 \cos(xz) d(xz) dy dx \\
&= \int_0^1 \int_0^{x^2} \left[x^3 \sin(xz) \right]_{z=0}^{z=y} dy dx \\
&= \int_0^1 \int_{y=0}^{y=x^2} x^2 \sin(xy) d(xy) dx \\
&= \int_0^1 \left[-x^2 \cos(xy) \right]_{y=0}^{y=x^2} dx \\
&= \int_0^1 x^2 \left(1 - \cos(x^3) \right) dx \\
&= \frac{1}{3} \int_0^1 \left(1 - \cos(x^3) \right) dx^3 \\
&= \frac{1}{3} \left[x^3 - \sin(x^3) \right]_0^1 \\
&= \frac{1 - \sin 1}{3}
\end{aligned}
$$

Other order of the triple integrals such as $\int_0^1 \int_{\sqrt{y}}^1 \int_0^y x^4 \cos(xz) dz dx dy$, $\int_0^1 \int_z^1 \int_{\sqrt{y}}^1 x^4 \cos(xz) dx dy dz$, $\int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} x^4 \cos(xz) dy dx dz$ and $\int_0^1 \int_0^{x^2} \int_z^{x^2} x^4 \cos(xz) dy dz dx$ also give you the same answer but the computation is complicated.



When we extend polar coordinates from $\boldsymbol{R}^2$ to $\boldsymbol{R}^3$, the result is **cylindrical coordinates**. In this coordinate system, a point P in $\boldsymbol{R}^3$ has coordinates (r,θ,z), where r is the distance between P and the z-axis and θ is the usual polar angle measured counterclockwise from the positive x-axis. As in Cartesian coordinates, the z-coordinate is the signed vertical distance between P and the xy-plane. Any point in $\boldsymbol{R}^3$ can be represented by cylindrical coordinates using the intervals $0 \le r < \infty, 0 \le \theta \le 2\pi,$ and $-\infty < z < \infty.$

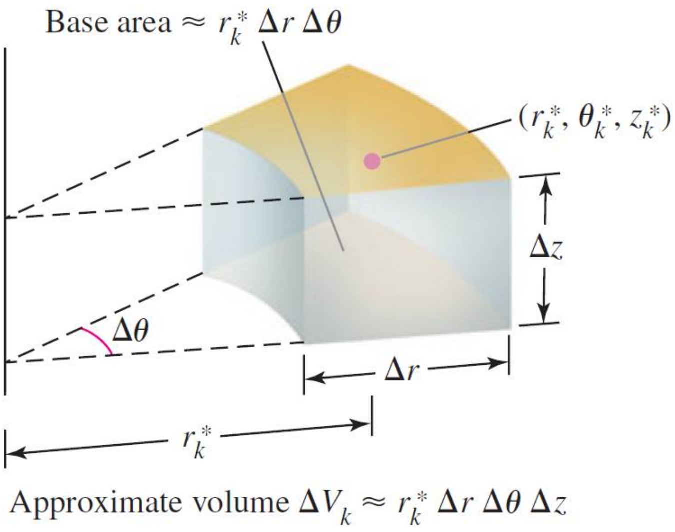




$\{(r,\theta,z): r = a\}, a > 0$ $\quad\{(r,\theta,z): \theta = \theta_0\}$ $\quad\{(r,\theta,z): z = a\}$ $\quad\{(r,\theta,z): z = ar\}, a \neq 0$

Among the uses of cylindrical coordinates is the evaluation of triple integrals. We begin with a region D in $\boldsymbol{R}^3$ and partition it into cylindrical wedges formed by changes of $\Delta r, \Delta\theta,$ and Δz in the coordinate directions. Those wedges that lie entirely within D are labeled from $k =$ 1 to $k = n$ in some convenient order. We let $(r_k^*, \theta_k^*, z_k^*)$ be an arbitrary point in the kth wedge.

As shown in below figure, the base of the kth wedge is a polar rectangle with an approximate area of $r_k^*\Delta r\Delta\theta.$ The height of the wedge is $\Delta z.$ Multiplying these dimensions together, the approximate volume of the wedge is $\Delta V_k = r_k^*\Delta r\Delta\theta\Delta z,$ for $k = 1,\cdots,n.$



We now assume that f is continuous on D and form a Riemann sum over the region by adding function values multiplied by the corresponding approximate volumes:

$$\sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*)\Delta V_k = \sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*) r_k^* \Delta r \Delta\theta \Delta z.$$

Let Δ be the maximum value of $\Delta r, \Delta\theta,$ and $\Delta z,$ for $k = 1, \cdots, n.$ As $n \to \infty$ and $\Delta \to 0,$ the Riemann sums approach a limit called the **triple integral of f over D in cylindrical coordinates**:

$$\iiint_D f(x, y, z)\,dV = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*) r_k^* \Delta r \Delta\theta \Delta z.$$

Suppose D is a region in $\boldsymbol{R}^3$ consisting of points between the surfaces $z = G(x, y)$ and $z = H(x, y),$ where x and y belong to a region R in the xy-plane and $G(x, y) \le H(x, y)$ on R. Assuming f is continuous on D, the triple integral of f over D may be expressed as

$$\iiint_D f(x, y, z)\,dV = \iint_R \left(\int_{G(x,y)}^{H(x,y)} f(x, y, z)\,dz \right) dA.$$

Suppose the region R is described in polar coordinates by

$$R = \{(r, \theta) : g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\},$$

where $0 \le g(\theta) \le h(\theta),$ for all θ in $[\alpha, \beta]$. The effect is a change of variables from rectangular to cylindrical coordinates. Letting $x = r\cos\theta$ and $y = r\sin\theta$, we have the following result:

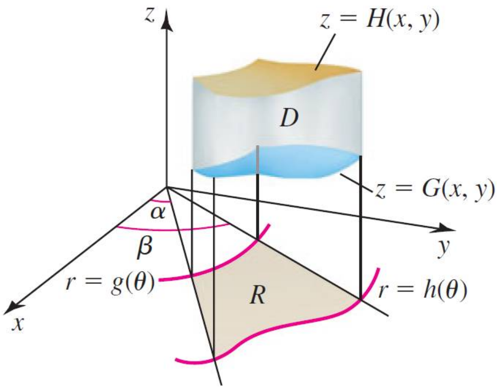
Theorem 4.6.1

Let f be continuous over the region

$$D = \{(r, \theta, z) : g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta, G(x, y) \le z \le H(x, y)\}.$$

Then f is integrable over D and the triple integral of f over D in cylindrical coordinates is

$$\iiint_D f(x, y, z)dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} rf(r\cos\theta, r\sin\theta, z)dzdrd\theta.$$



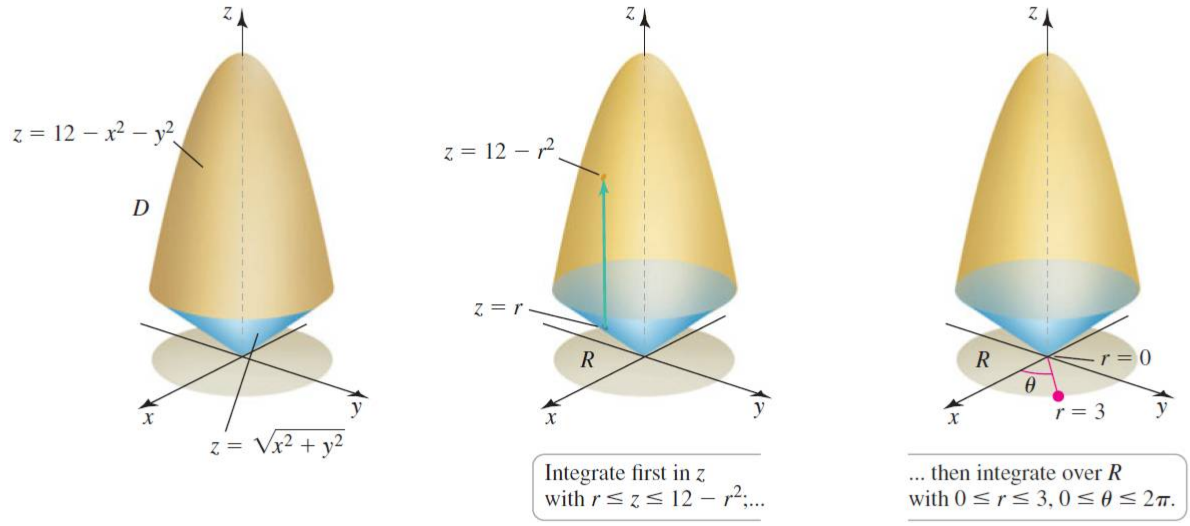
Example 4.6.2

Find the volume of the solid D between the cone $z = \sqrt{x^2 + y^2}$ and the inverted paraboloid $z = 12 - x^2 - y^2$.

Solution

The equation of the cone becomes $z = r > 0$, and the equation of the paraboloid becomes $z = 12 - r^2$. The inner integral in z runs from the cone $z = r$ (the lower surface) to the paraboloid $z = 12 - r^2$ (the upper surface). We project D onto the xy-plane to produce the region R, whose boundary is determined by the intersection of the two surfaces. Equating the z-coordinates in the equations of the two surfaces, we have $12 - r^2 = r$, or $(r - 3)(r + 4) = 0$. Because $r > 0$, the relevant root is $r = 3$. Therefore, the projection of D on the xy-plane is $R = \{(r, \theta) : 0 \le r \le 3, 0 \le \theta \le 2\pi\}$, which is a disk of radius 3 centered at (0, 0). The volume of the region is

$$\iiint_D dV = \int_0^{2\pi} \int_0^3 \int_r^{12-r^2} rdzdrd\theta = \int_0^{2\pi} \int_0^3 (12 - r^2 - r)rdrd\theta = \left[6r^2 - \frac{r^4}{4} - \frac{r^3}{3}\right]_0^3 \cdot \int_0^{2\pi} d\theta = \frac{99\pi}{2}$$



Example 4.6.3

Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)dzdydx.$

Solution

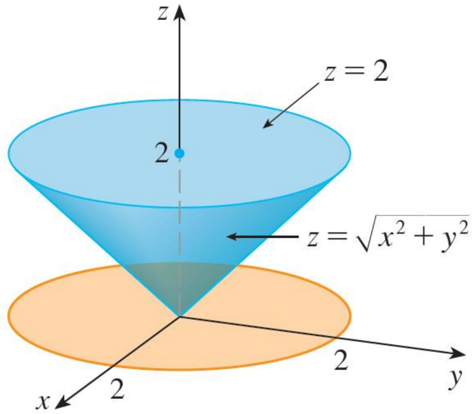
This is a triple integral over the solid region

$$D=\{(x,y,z):-2\le x\le 2,-\sqrt{4-x^2}\le y\le\sqrt{4-x^2},\sqrt{x^2+y^2}\le z\le 2\}$$

and the projection of E onto the xy-plane is the disk $x^2+y^2\le 4$. The lower surface of D is the cone $z=\sqrt{x^2+y^2}$ and its upper surface is the plane $z = 2$. This region has a much simpler description in cylindrical coordinates:

$$D=\{(r,\theta,z):0\le r\le 2,0\le\theta\le 2\pi,r\le z\le 2\}.$$

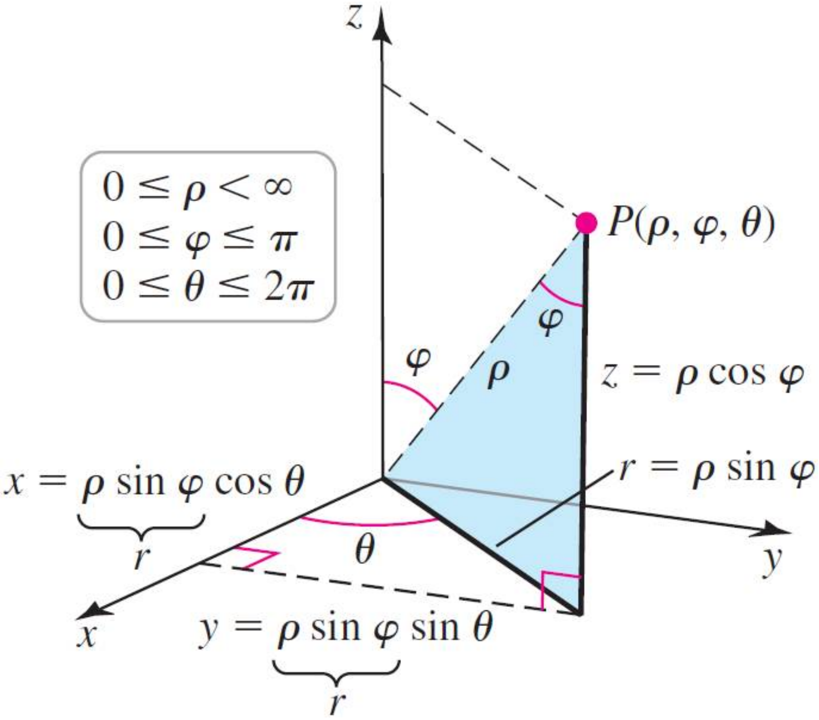
Therefore we have



$$\begin{aligned}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)dzdydx&=\iiint_D f(x,y,z)dV\\&=\int_0^{2\pi}\int_0^2\int_r^2 r^2 rdzdrd\theta\\&=\int_0^{2\pi}1d\theta\int_0^2 r^3(2-r)dr\\&=2\pi\left[\tfrac{1}{2}r^4-\tfrac{1}{5}r^5\right]_0^2\\&=\tfrac{16}{5}\pi\end{aligned}$$

In **spherical coordinates**, a point P in $\boldsymbol{R}^3$ is represented by three coordinates (ρ, φ, θ).

- ρ is the distance from the origin to P.
- φ is the angle between the positive z-axis and the line OP.
- θ is the same angle as in cylindrical coordinates; it measures rotation about the z-axis relative to the positive x-axis.

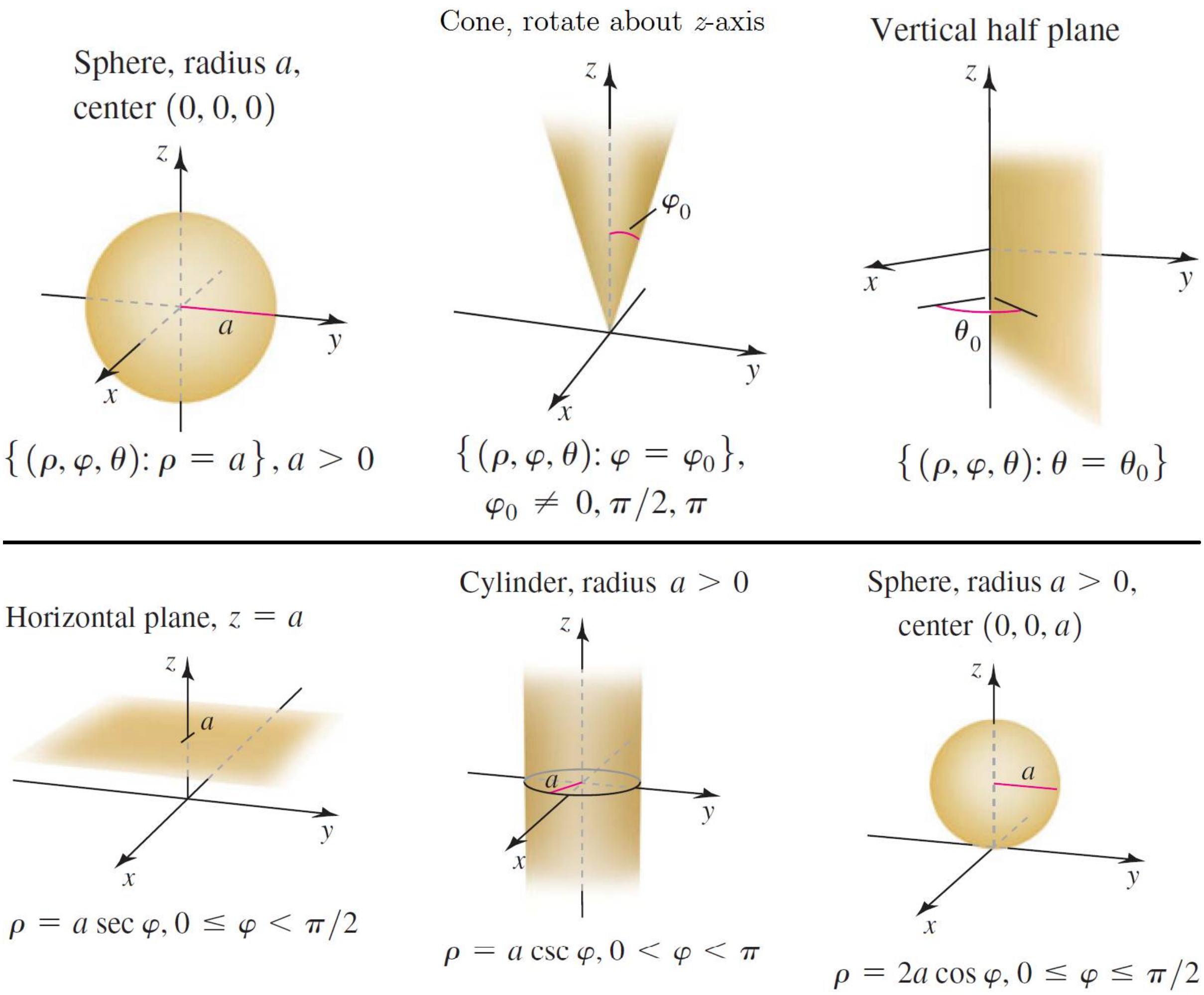


All points in $\boldsymbol{R}^3$ can be represented by spherical coordinates using the intervals $0 \leq \rho < \infty, 0 \leq \varphi \leq \pi,$ and $0 \leq \theta \leq 2\pi$.

The above figure allows us to find the relationships among rectangular and spherical coordinates. Given the spherical coordinates (ρ, φ, θ) of a point P, the distance from P to the z-axis is $r = \rho \sin \varphi$. We also see from the above figure that $x = r\cos\theta = \rho\sin\varphi\cos\theta$, $y = r\sin\theta = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$.

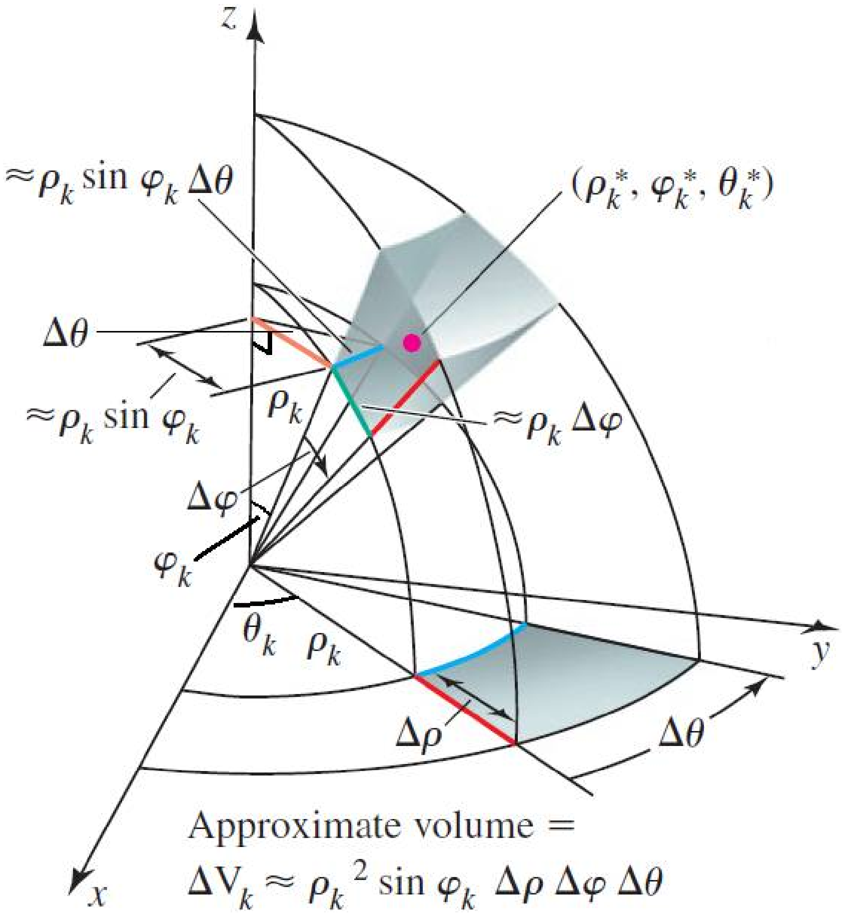
Transformations Between Spherical and Rectangular Coordinates

Rectangular $\rightarrow$ Spherical	**Spherical $\rightarrow$ Rectangular**
$\rho^2 = x^2 + y^2 + z^2$	$x = \rho \sin \varphi \cos \theta$
Use trigonometry to find	$y = \rho \sin \varphi \sin \theta$
φ and θ	$z = \rho \cos \varphi$



We now investigate triple integrals in spherical coordinates over a region D in $\boldsymbol{R}^3$. The region D is partitioned into "spherical boxes" that are formed by changes of $\Delta\rho$, $\Delta\varphi$, and $\Delta\theta$ in the coordinate directions. Those boxes that lie entirely within D are labeled from $k = 1$ to $k = n$. We let $(\rho_k, \varphi_k, \theta_k)$ be an arbitrary point in the k-th box.

To approximate the volume of a typical box, note that the length of the box in the ρ-direction is $\Delta\rho$ (the below figure). The approximate length of the k-th box in the θ-direction is the length of an arc of a circle of radius $\rho_k \sin\varphi_k$ subtended by an angle $\Delta\theta$; this length is $\rho_k \sin\varphi_k \Delta\theta$. The approximate length of the box in the φ-direction is the length of an arc of radius ρ_k subtended by an angle $\Delta\varphi$; this length is $\rho_k \Delta\varphi$. Multiplying these dimensions together, the approximate volume of the k-th spherical box is $\Delta V_k = \rho_k^2 \sin\varphi_k \Delta\rho\Delta\varphi\Delta\theta$, for $k = 1, \cdots, n$.



We now assume that f is continuous on D and form a Riemann sum over the region by adding function values multiplied by the corresponding approximate volumes:

$$\sum_{k=1}^{n} f(\rho_k^*, \varphi_k^*, \theta_k^*)\Delta V_k = \sum_{k=1}^{n} f(\rho_k^*, \varphi_k^*, \theta_k^*)\rho_k^2 \sin\varphi_k \Delta\rho\Delta\varphi\Delta\theta.$$

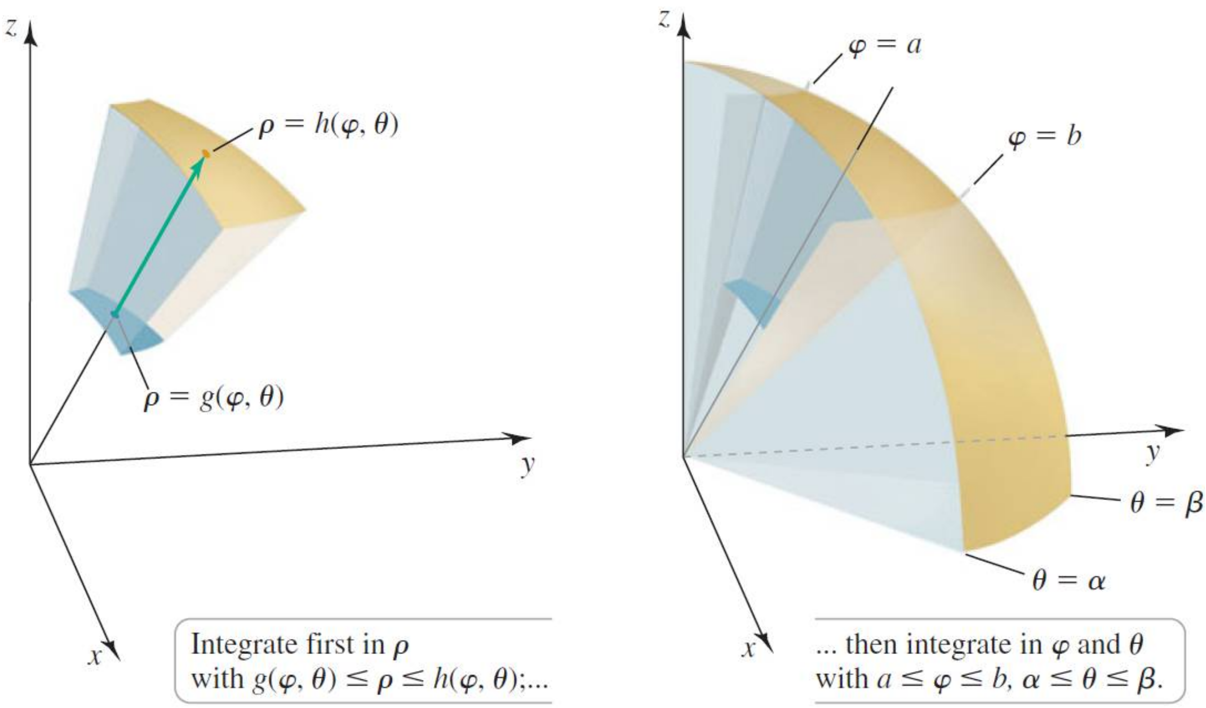
We let Δ denote the maximum value of $\Delta\rho, \Delta\varphi,$ and $\Delta\theta.$ As $n \to \infty$ and $\Delta \to 0,$ the Riemann sums approach a limit called the **triple integral of f over D in spherical coordinates**:

$$\iiint_D f(\rho,\varphi,\theta)dV = \lim_{\Delta\to 0}\sum_{k=1}^{n} f(\rho_k^*, \varphi_k^*, \theta_k^*)\rho_k^2 \sin\varphi_k \Delta\rho\Delta\varphi\Delta\theta.$$

We consider the region of integration has the form

$$D = \{(\rho,\varphi,\theta) : g(\varphi,\theta) \le \rho \le h(\varphi,\theta),\, a \le \varphi \le b,\, \alpha \le \theta \le \beta\}.$$

In other words, D is bounded in the ρ-direction by two surfaces given by g and h. In the angular directions, the region lies between two cones $(a \le \varphi \le b)$ and two half planes $(\alpha \le \theta \le \beta)$.



To integrate over D we

- first integrate with respect to ρ from $\rho = g(\varphi,\theta)$ to $\rho = h(\varphi,\theta)$,
- then integrate with respect to φ from $\varphi = a$ to $\varphi = b$, and
- finally integrate with respect to θ from $\theta = \alpha$ to $\theta = \beta$.

Notice that the middle and outer integrals, with respect to θ and φ, may be done in either order.

Theorem 4.7.1

Let f be continuous over the region

$$D = \{(\rho,\varphi,\theta) : g(\varphi,\theta) \le \rho \le h(\varphi,\theta), a \le \varphi \le b, \alpha \le \theta \le \beta\}.$$

Then f is integrable over D, and the triple integral of f over D in spherical coordinates is

$$\iiint_D f(x,y,z)dV = \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2 \sin\varphi d\rho d\varphi d\theta$$

Example 4.7.2

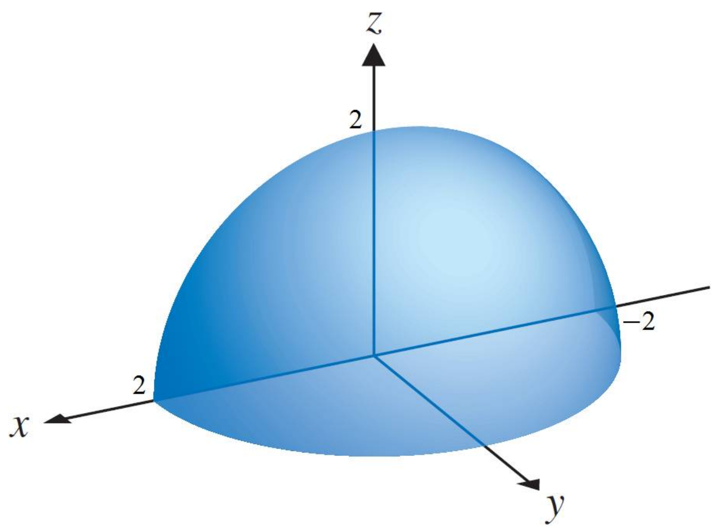
Evaluate the triple iterated integral $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}}(x^2+y^2+z^2)dzdydx$

Solution

The solid over which we are integrating is a portion of a sphere, as follows.

Notice that for each x in the interval $[-2, 2]$ indicated by the outermost limits of integration, y varies from 0 (corresponding to the x-axis) to $y=\sqrt{4-x^2}$ (the top semicircle of radius 2 centered at the origin). Finally, z varies from 0 (corresponding to the xy-plane) up to $z=\sqrt{4-x^2-y^2}$ (the top hemisphere of radius 2 centered at the origin). The solid Q over which we are integrating is then the half of the hemisphere that lies above the first and second quadrants of the xy-plane. In spherical coordinates, this portion of the sphere is obtained if we let ρ range from 0 up to 2, φ range from 0 up to $\pi/2$ and θ range from 0 to π. The integral then becomes

$$\begin{aligned}\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}}(x^2+y^2+z^2)dzdydx &= \int_{0}^{\pi}\int_{0}^{\pi/2}\int_{0}^{2}\rho^2(\rho^2\sin\varphi)d\rho d\varphi d\theta \\ &= \int_{0}^{2}\rho^4 d\rho\int_{0}^{\pi/2}\sin\varphi d\varphi\int_{0}^{\pi}1d\theta \\ &= \left[\frac{\rho^5}{5}\right]_0^2\cdot[-\cos\varphi]_0^{\pi/2}\cdot\pi \\ &= \frac{32}{5}\pi\end{aligned}$$



Example 4.7.3

Find the volume lying inside the sphere $x^2+y^2+z^2=2z$ and inside the cone $x^2+y^2=z^2$.

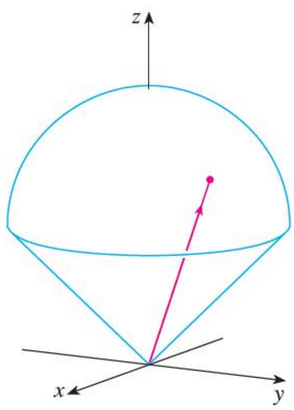
Solution

Notice that by completing the square in the equation of the sphere, we get

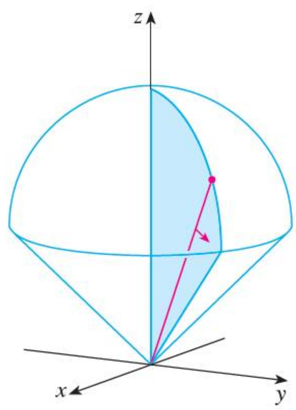
$$x^2+y^2+(z-1)^2=x^2+y^2+(z^2-2z+1)=1$$

the sphere of radius 1, centered at the point (0, 0, 1).

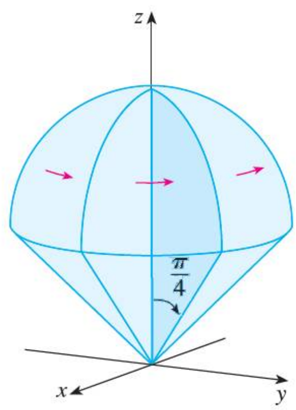
$$\begin{aligned}
&x^2+y^2+z^2=2z \quad\Rightarrow\quad \rho^2=2\rho\cos\varphi \quad\Rightarrow\quad \rho=0 \text{ or } 2\cos\varphi\\
&x^2+y^2=z^2 \quad\Rightarrow\quad (\rho\sin\varphi\cos\theta)^2+(\rho\sin\varphi\sin\theta)^2=(\rho\cos\varphi)^2\\
&\qquad\qquad\qquad\Rightarrow\quad \sin^2\varphi=\cos^2\varphi\\
&\qquad\qquad\qquad\Rightarrow\quad \varphi=\pi/4 \text{ or } -\pi/4 \text{ (rejected)}\\
&Q=\{(\rho,\varphi,\theta):0\le\rho\le 2\cos\varphi,\ 0\le\varphi\le\pi/4,\ 0\le\theta\le 2\pi\}\\
&V=\iiint_Q 1dV\\
&\ \ =\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\varphi}\rho^2\sin\varphi d\rho d\varphi d\theta\\
&\ \ =\int_0^{2\pi}1d\theta\int_0^{\pi/4}\left[\frac{\rho^3}{3}\right]_0^{2\cos\varphi}\sin\varphi d\varphi\\
&\ \ =\frac{16\pi}{3}\int_0^{\pi/4}\cos^3\varphi\sin\varphi d\varphi\\
&\ \ =-\frac{16\pi}{3}\left[\frac{\cos^4\varphi}{4}\right]_0^{\pi/4}\\
&\ \ =-\frac{4\pi}{3}\left(\frac{1}{4}-1\right)\\
&\ \ =\pi
\end{aligned}$$



ρ varies from 0 to cos ϕ while ϕ and θ are constant.



ϕ varies from 0 to $\pi/4$ while θ is constant.



θ varies from 0 to 2π.

In one-variable calculus we often use a change of variable (a substitution) to simplify an integral. We can write the Substitution Rule as

$$\int_a^b f(x)dx = \int_c^d f(g(u))g'(u)du = \int_c^d f(x(u))\frac{dx}{du}du.$$

A change of variables can also be useful in double and triple integrals.

We consider a change of variables that is given by a **transformation** T from the uv-plane to the xy-plane:

$$T(u, v) = (x, y)$$

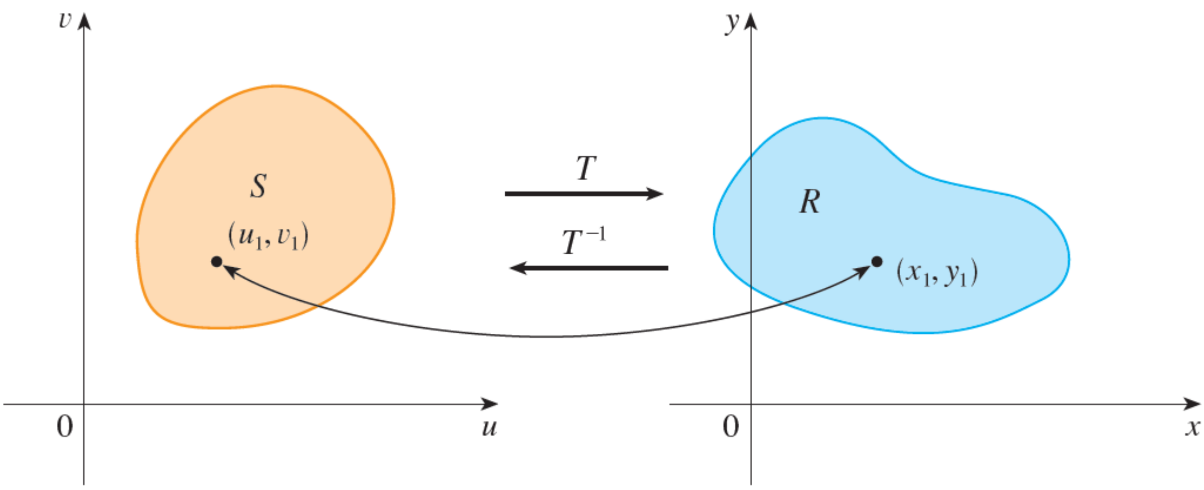
where x and y are related to u and v by the equations

$$x = g(u, v) \qquad \text{and} \qquad y = h(u, v)$$

or, as we sometimes write, $x = x(u, v)$ and $y = y(u, v)$.

We usually assume that T is a C^1 **transformation**, which means that g and h have continuous first-order partial derivatives.

A transformation T is really just a function whose domain and range are both subsets of $\boldsymbol{R}^2$. If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (x_1, y_1) If no two points have the same image, T is called **one-to-one**. ($T(P) = T(Q)$ only when $P = Q$, where P and Q are points in domain). The below figure shows the effect of a transformation T on a region S in the uv-plane. T transforms S into a region R in the xy-plane called the **image of** S, consisting of the images of all points in S.



Example 4.8.1

A transformation is defined by the equations

$$x = u^2 - v^2 \quad \text{and} \quad y = 2uv$$

Find the image of the square $S = \{(u, v) \mid 0 \le u, v \le 1\}$.

Solution

The transformation maps the boundary of S into the boundary of the image. So we begin by finding the images of the sides of S. The first side, S_1, is given by $v = 0$ $(0 \le u \le 1)$. From the given equations we have $x = u^2$, $y = 0$, and $0 \le x \le 1$. Thus S_1 is mapped into the line segment from (0, 0) to (1, 0) in the xy-plane. The second side, S_2, is $u = 1$ $(0 \le v \le 1)$ and, putting $u = 1$ in the given equations, we get

$$x = 1 - v^2, \quad y = 2v$$

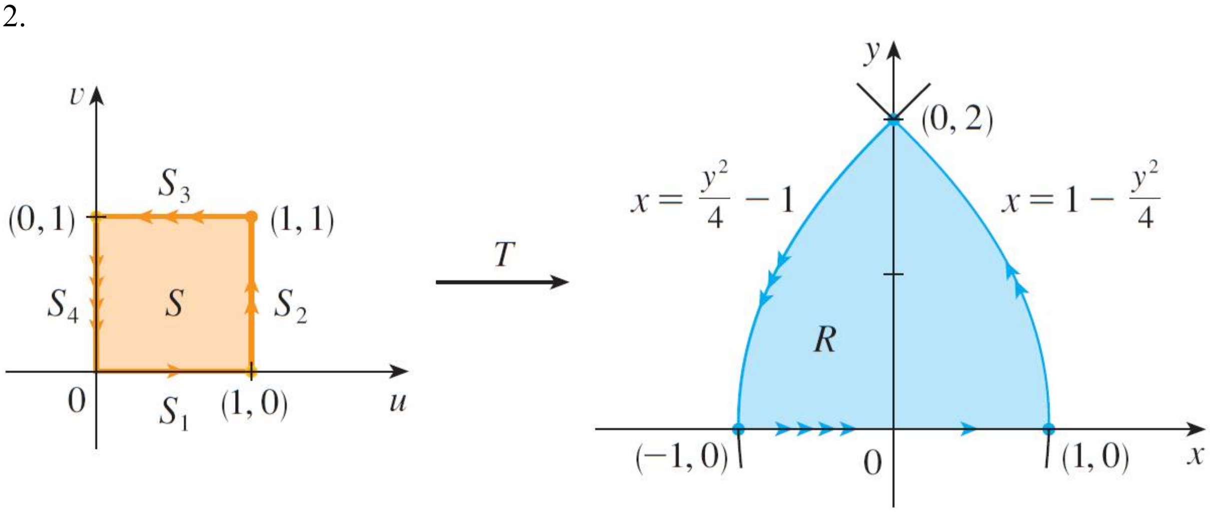
Eliminating v, we obtain

$$x = 1 - \frac{y^2}{4}, \quad 0 \le x \le 1 \tag{1}$$

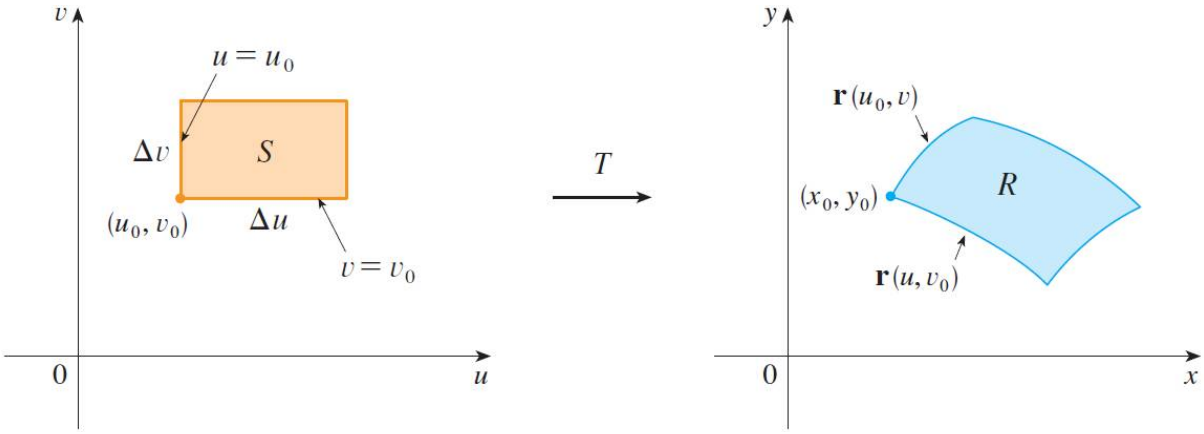
which is part of a parabola. Similarly, S_3 is given by $v = 1$ $(0 \le u \le 1)$, whose image is the parabolic arc

$$x = \frac{y^2}{4} - 1, \quad -1 \le x \le 0 \tag{2}$$

Finally, S_4 is given by $u = 1$ $(0 \le v \le 1)$ whose image is $x = -v^2$, $y = 0$, that is, $-1 \le x \le 0$. (Notice that as we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.) The image of S is the region R (shown in the below figure) bounded by the x-axis and the parabolas given by Equations 1 and 2.



We start with a small rectangle S in the uv-plane whose lower left corner is the point (u_0, v_0) and whose dimensions are Δu and Δv.



The image of S is a region R in the xy-plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$. The vector

$$\boldsymbol{r}(u,v) = g(u,v)\mathbf{i} + h(u,v)\mathbf{j}$$

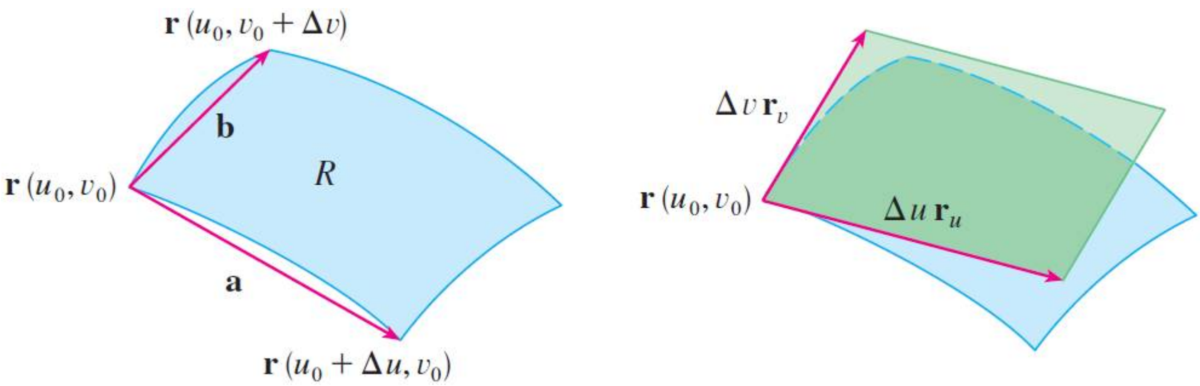
is the position vector of the image of the point (u, v). The equation of the lower side of S is $v = v_0$, whose image curve is given by the vector function $\boldsymbol{r}(u, v_0)$. The tangent vector at (x_0, y_0) to this image curve is

$$\boldsymbol{r}_u(u,v) = g_u(u_0,v_0)\mathbf{i} + h_u(u_0,v_0)\mathbf{j} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}$$

Similarly, the tangent vector at (x_0, y_0) to the image curve of the left side of S (namely, $u = u_0$) is

$$\boldsymbol{r}_v(u,v) = g_v(u_0,v_0)\mathbf{i} + h_v(u_0,v_0)\mathbf{j} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

On the other hand, $\boldsymbol{r}_u = \lim_{\Delta u \to 0} \frac{\boldsymbol{r}(u_0 + \Delta u, v_0) - \boldsymbol{r}(u_0, v_0)}{\Delta u}$ and $\boldsymbol{r}_v = \lim_{\Delta v \to 0} \frac{\boldsymbol{r}(u_0, v_0 + \Delta v) - \boldsymbol{r}(u_0, v_0)}{\Delta v}$.



We can approximate the image region $R = T(S)$ by a parallelogram determined by the secant vectors

$$\boldsymbol{r}(u_0+\Delta u, v_0)-\boldsymbol{r}(u_0,v_0)\approx \Delta u\cdot \boldsymbol{r}_u \qquad \text{and} \qquad \boldsymbol{r}(u_0, v_0+\Delta v)-\boldsymbol{r}(u_0,v_0)\approx \Delta v\cdot \boldsymbol{r}_v$$

This means that we can approximate R by a parallelogram determined by the vectors $\Delta u\cdot \boldsymbol{r}_u$ and $\Delta v\cdot \boldsymbol{r}_v$. Therefore we can approximate the area of R by the area of this parallelogram, which, from Corollary 2.3.8, is

$$|(\Delta u\cdot \boldsymbol{r}_u)\times(\Delta v\cdot \boldsymbol{r}_v)| = |\boldsymbol{r}_u\times \boldsymbol{r}_v|\Delta u\Delta v = \left|\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{pmatrix}\right|\Delta u\Delta v = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}\right|\Delta u\Delta v \qquad (3)$$

Definition 4.8.2 (Jacobian Determinant of a Transformation of Two Variables)

Given a transformation T: $x = g(u, v)$, $y = h(u, v)$, where g and h are differentiable on a region of the uv-plane, the **Jacobian determinant** (or **Jacobian**) of T is

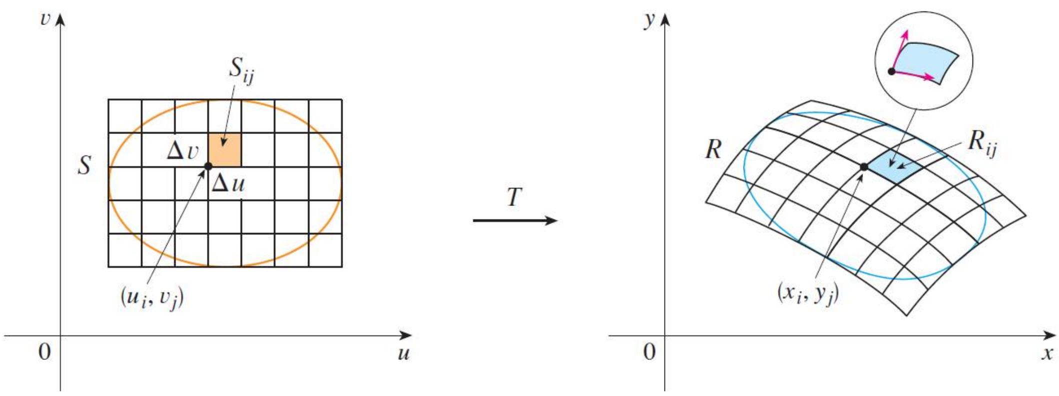
$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

With this notation we can use equation 3 to give an approximation to the area ΔA of R:

$$\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\Delta v \qquad (4)$$

where the Jacobian is evaluated at (u_0, v_0).

Next we divide a region S in the uv-plane into rectangles S_{ij} and call their images in the xy-plane R_{ij}.



Applying the approximation (4) to each R_{ij}, we approximate the double integral of f over R as follows:

$$\iint_R f(x,y)dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,y_j)\Delta A \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_i,v_j),h(u_i,v_j))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\Delta v$$

where the Jacobian is evaluated at (u_i,v_j). Notice that this double sum is a Riemann sum for the integral

$$\iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv.$$

Theorem 4.8.3

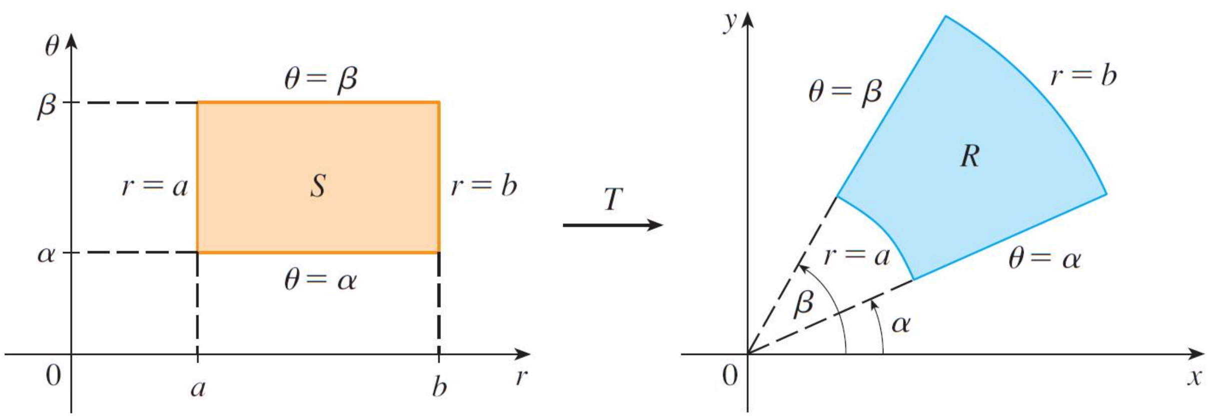
Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv.$$

Example 4.8.4

Use Theorem 4.8.3 to derive the evaluation formula for polar coordinates ($r > 0$):

$$\iint_R f(x,y)dA = \iint_S f(r\cos\theta, r\sin\theta)rdrd\theta.$$



Solution

Here the transformation T from the $r\theta$-plane to the xy-plane is given by

$$x = g(r,\theta) = r\cos\theta \qquad \text{and} \qquad y = h(r,\theta) = r\sin\theta.$$

T maps an ordinary rectangle in the $r\theta$-plane to a polar rectangle in the xy-plane. The Jacobian of T is

$$J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r > 0$$

Thus Theorem 4.8.3 gives

$$\iint_R f(x,y)dA = \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r dr d\theta$$

which is the same as equation in Theorem 4.3.3 for $g_1(\theta) = a, g_2(\theta) = b.$

Example 4.8.5

Use the change of variables $x = u^2 - v^2, y = 2uv$ to evaluate the integral $\iint_R ydA,$ where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x,\ y \geq 0.$

Solution

In Example 4.8.1 we discovered that $T(S) = R$, where S is the square $[0, 1] \times [0, 1]$. S is a much simpler region than R. First we need to compute the Jacobian:

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0.$$

Therefore, by Theorem 4.8.3,

$$\begin{aligned} \iint_R ydA &= \iint_S 2uv\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \\ &= \int_0^1 \int_0^1 2uv(4u^2 + 4v^2) dudv \end{aligned}$$

$$
\begin{aligned}
&= 8\int_0^1\int_0^1 (u^3v+uv^3)\,dudv \\
&= 8\int_0^1 \left[\tfrac{1}{4}u^4v+\tfrac{1}{2}u^2v^3\right]_{u=0}^{u=1} dv \\
&= \int_0^1 (2v+4v^3)\,dv \\
&= \left[v^2+v^4\right]_0^1 \\
&= 2
\end{aligned}
$$

Example 4.8.6

Let R be the region in the first quadrant bounded by the parabolas $x=y^2$, $x=y^2-4$, $x=9-y^2$, and $x=16-y^2$. Evaluate $\iint_R y^2\,dA$.

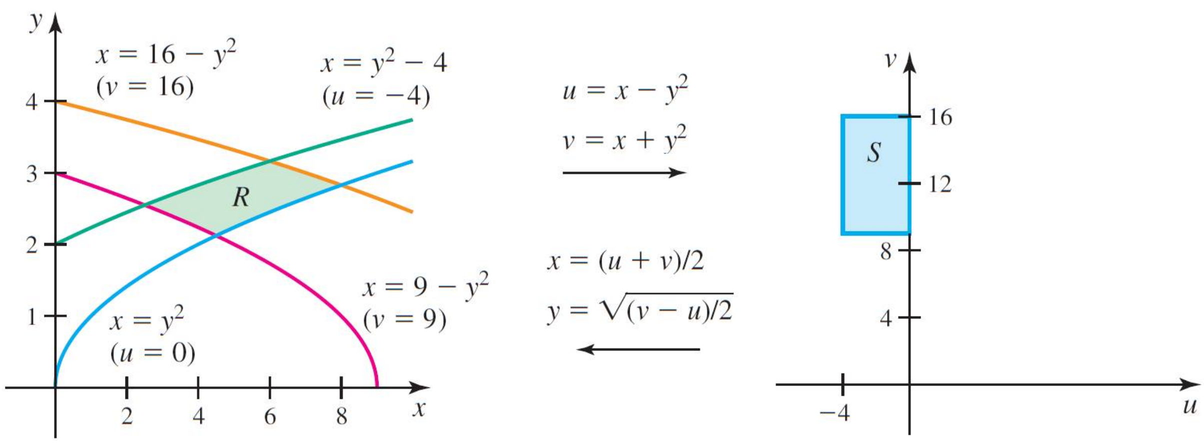
Solution

When the region R is bounded by two pairs of "parallel" curves in the families $g(x,y)=a_1, g(x,y)=a_2$ and $h(x,y)=b_1, h(x,y)=b_2$. In this case the new region of integration is a rectangle $S=\{(u,v): a_1\le u\le a_2, b_1\le v\le b_2\}$, where $u=g(x,y)$ and $v=h(x,y)$. Therefore, the new variables are

$$u=x-y^2,\quad v=x+y^2.$$

The boundary curves of S are $u=-4$, $u=0$, $v=9$, and $v=16$. Therefore, the new region is $S=\{(u,v): -4\le u\le 0, 9\le v\le 16\}$. To compute the Jacobian, we must find the transformation T by writing x and y in terms of u and v. Solving for x and y, and observing that $y\ge 0$ for all points in R, we find that

$$T:\quad x=\frac{u+v}{2},\quad y=\sqrt{\frac{v-u}{2}}.$$



The points of S satisfy $v > u$, so $\sqrt{v-u}$ is defined. Now the Jacobian may be computed:

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2\sqrt{2(v-u)}} & \frac{1}{2\sqrt{2(v-u)}}\end{vmatrix}=\frac{1}{2\sqrt{2(v-u)}}.$$

The change of variables proceeds as follows:

$$\begin{aligned}\iint_R y^2\,dA&=\int_9^{16}\int_{-4}^{0}\underbrace{\frac{v-u}{2}}_{y^2}\cdot\underbrace{\frac{1}{2\sqrt{2(v-u)}}}_{|J(u,v)|}dudv\\
&=\frac{1}{4\sqrt{2}}\int_9^{16}\int_{-4}^{0}\sqrt{v-u}\,dudv\\
&=\frac{1}{4\sqrt{2}}\cdot\frac{2}{3}\int_9^{16}-\left[(v-u)^{3/2}\right]_{-4}^{0}dv\\
&=\frac{1}{6\sqrt{2}}\int_9^{16}(v+4)^{3/2}-v^{3/2}dv\\
&=\frac{1}{6\sqrt{2}}\cdot\frac{2}{5}\left[(v+4)^{5/2}-v^{5/2}\right]_9^{16}\\
&=\frac{\sqrt{2}}{30}\left(32\cdot 5^{5/2}-13^{5/2}-781\right)\\
&\approx 18.79\end{aligned}$$

There is a similar change of variables formula for triple integrals. Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations

$$x = g(u, v, w), \qquad y = h(u, v, w), \qquad z = k(u, v, w)$$

The **Jacobian** of T is the following 3×3 determinant:

$$J(u,v,w)=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$$

Theorem 4.8.7

Suppose that the region S in uvw-space is mapped onto the region R in xyz-space by the one-to-one transformation T defined by $x = g(u, v, w)$, $y = h(u, v, w)$ and $z = l(u, v, w)$, where g, h and l have continuous first partial derivatives in S. If f is continuous in R and the Jacobian

$$\iiint_R f(x,y,z)dV = \iiint_S f(g(u,v,w),h(u,v,w),l(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dudvdw.$$

Example 4.8.8

Use Theorem 4.8.7 to derive the formula for triple integration in spherical coordinates.

Solution

Here the change of variables is given by

$$x = \rho\sin\varphi\cos\theta, \quad y = \rho\sin\varphi\sin\theta, \quad z = \rho\cos\varphi.$$

We compute the Jacobian as follows:

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} &= \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix} \\
&= \cos\varphi\begin{vmatrix} -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \end{vmatrix} - \rho\sin\varphi\begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix} \\
&= \cos\varphi\left(-\rho^2\sin\varphi\cos\varphi\sin^2\theta - \rho^2\sin\varphi\cos\varphi\cos^2\theta\right) \\
&\quad -\rho\sin\varphi\left(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta\right) \\
&= -\rho^2\sin\varphi\cos^2\varphi - \rho^2\sin\varphi\sin^2\varphi \\
&= -\rho^2\sin\varphi
\end{aligned}$$

Since $0 \le \varphi \le \pi$, we have $\sin\varphi \ge 0$. Therefore

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = \left|-\rho^2\sin\varphi\right| = \rho^2\sin\varphi$$

and Theorem 4.8.7 gives

$$\iiint_R f(x,y,z)dV = \iiint_S f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi d\rho d\theta d\varphi.$$

Theorem 4.8.9

Use a change of variables to evaluate $\iiint_D xz dV$, where D is a parallelepiped bounded by the planes $y = x$, $y = x + 2$, $z = x$, $z = x + 3$, $z = 0$, and $z = 4$.

Solution

The key is to note that D is bounded by three pairs of parallel planes.

- $y - x = 0$ and $y - x = 2$
- $z - x = 0$ and $z - x = 3$
- $z = 0$ and $z = 4$

These combinations of variables suggest the new variables

$$u = y - x, \qquad v = z - x, \qquad \text{and} \qquad w = z.$$

With this choice, the new region of integration (the right above figure) is the rectangular box

$$S = \{(u, v, w) : 0 \le u \le 2, 0 \le v \le 3, 0 \le w \le 4\}.$$

To compute the Jacobian, we must express x, y, and z in terms of u, v, and w. A few steps of algebra lead to the transformation

$$T: \quad x = w - v, \quad y = u - v + w, \quad \text{and} \quad z = w.$$

The resulting Jacobian is

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} = \det\begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 1.$$

Noting that the integrand is $xz = (w - v)w = w^2 - vw$, the integral may now be evaluated:

$$\begin{aligned}
\iiint_D xz\,dV &= \iiint_S (w^2 - vw)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw \\
&= \int_0^4\int_0^3\int_0^2 (w^2 - vw)dudvdw \\
&= 2\int_0^4\int_0^3 (w^2 - vw)dvdw \\
&= 2\int_0^4 \left[vw^2 - \frac{v^2 w}{2}\right]_0^3 dw \\
&= 2\int_0^4 \left(3w^2 - \frac{9w}{2}\right)dw \\
&= 2\left[w^3 - \frac{9w^2}{2}\right]_0^4 \\
&= 56
\end{aligned}$$