PT

Solution to Assignment 11

1. Let $W_i, i = 1, 2$, denote the i^{th} outcome.

$$\begin{aligned}
\operatorname{Cov}(X, Y) &= \operatorname{Cov}(W_1 + W_2, W_1 - W_2) \\
&= \operatorname{Cov}(W_1, W_1) - \operatorname{Cov}(W_2, W_2) \\
&= \operatorname{Var}(W_1) - \operatorname{Var}(W_2) = 0
\end{aligned}$$

2. $f_Y(y) = e^{-y} \int \frac{1}{y} e^{-x/y} dx = e^{-y}$. In addition, the conditional distribution of X given that $Y = y$ is exponential with mean y. Hence,

$$E[Y] = 1, E[X] = E[E[X \mid Y]] = E[Y] = 1$$

Since, $E[XY] = E[E[XY \mid Y]] = E[YE[X \mid Y]] = E\left[Y^2\right] = 2$ (since Y is exponential with mean 1, it follows that $E\left[Y^2\right] = 2\big)$. Hence, $\operatorname{Cov}(X, Y) = 2 - 1 = 1$.

3. We can use $\operatorname{Cov}(X, Y) = EXY - EXEY$. We have $EX = \frac{3}{2}$ and

$$\begin{aligned}
EY &= E[E[Y \mid X]] && \text{(law of total expectations)} \\
&= E\left[\frac{1}{X}\right] && (\text{ since } Y \mid X \sim \text{Exponential}(X)) \\
&= \int_1^2 \frac{1}{x} dx \\
&= \ln 2
\end{aligned}$$

We also have

$$\begin{aligned}
EXY &= E[E[XY \mid X]] && \text{(law of total expectations)} \\
EXY &= E[XE[Y \mid X]] && (\text{ since } E[X \mid X = x] = x) \\
&= E\left[X \frac{1}{X}\right] && (\text{ since } Y \mid X \sim \text{Exponential}(X)) \\
&= 1
\end{aligned}$$

Thus,

$$\operatorname{Cov}(X, Y) = E[XY] - (EX)(EY) = 1 - \frac{3}{2} \ln 2.$$

4. It holds

$$\begin{aligned}
&\operatorname{Cov}(Y_n, Y_n) = \operatorname{Var}(Y_n) = 3\sigma^2 \\
&\operatorname{Cov}(Y_n, Y_{n+1}) = \operatorname{Cov}(X_n + X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2} + X_{n+3}) \\
&= \operatorname{Cov}(X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2}) = \operatorname{Var}(X_{n+1} + X_{n+2}) = 2\sigma^2 \\
&\operatorname{Cov}(Y_n, Y_{n+2}) = \operatorname{Cov}(X_{n+2}, X_{n+2}) = \sigma^2 \\
&\operatorname{Cov}(Y_n, Y_{n+j}) = 0 \text{ when } j \geq 3
\end{aligned}$$

5.

$$\operatorname{Cov}(X, Y) = b \operatorname{Var}(X), \operatorname{Var}(Y) = b^2 \operatorname{Var}(X)$$

$$\rho(X, Y) = \frac{b \operatorname{Var}(X)}{\sqrt{b^2} \operatorname{Var}(X)} = \frac{b}{|b|}$$

6.

(a) $\rho_{X_1+X_2,X_2+X_3} = \frac{\operatorname{Cov}(X_1+X_2,X_2+X_3)}{\sqrt{\operatorname{Var}(X_1+X_2)}\sqrt{\operatorname{Var}(X_2+X_3)}} = \frac{\operatorname{Cov}(X_2,X_2)}{\sqrt{\operatorname{Var}(X_1)+\operatorname{Var}(X_2)}\sqrt{\operatorname{Var}(X_2)+\operatorname{Var}(X_3)}} = \frac{1}{2}$

(b) Since $\operatorname{Cov}(X_1 + X_2, X_3 + X_4) = 0$, we have $\rho_{X_1+X_2,X_3+X_4} = 0$.

7. We have

$$\begin{aligned}
&E[X] = 0.9, \quad E[Y] = 0.2, \quad \operatorname{Var}[X] = 0.09, \quad \operatorname{Var}[Y] = 0.16, \quad E[XY] = p_{X,Y}(1,1) \\
&\operatorname{Cov}(X, Y) = E[XY] - E[X]E[Y] = p_{X,Y}(1,1) - 0.18 = \rho_{X,Y}\sigma_X\sigma_Y = -0.5(0.3)(0.4) = -0.06 \\
&p_{X,Y}(1,1) = 0.12 \\
&p_{X,Y}(1,0) + p_{X,Y}(1,1) = p_X(1) = 0.9 \\
&p_{X,Y}(1,0) = 0.78 \\
&p_{X,Y}(0,1) + p_{X,Y}(1,1) = p_Y(1) = 0.2 \\
&p_{X,Y}(0,1) = 0.08 \\
&p_{X,Y}(0,0) + p_{X,Y}(0,1) = p_X(0) = 0.1 \\
&p_{X,Y}(0,0) = 0.02
\end{aligned}$$

8. We have $f_{X,Y}(x, y) = e^{-x}$ for $x > 0$ and $0 < y < 1$. In order to find the joint density of U and V, we need to first express X and Y in terms of U and V, i.e., find the inverse function of the transformation. Note that all variables considered here: X, Y, U, V, are positive with probability one. We have

$$UV = XY \cdot \frac{X}{Y} = X^2$$

so that $X = \sqrt{UV}$. Furthermore, $\frac{U}{V} = \frac{XY}{XY^{-1}} = Y^2$. This gives us $Y = \sqrt{UV^{-1}}$. Therefore, the inverse transformation, written in terms of regular variables, is

$$(x, y) = \left(\sqrt{uv}, \sqrt{uv^{-1}}\right).$$

This results in

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\left[\begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array}\right] = \det\left[\begin{array}{cc} \frac{1}{2\sqrt{uv}} \cdot v & \frac{1}{2\sqrt{uv}} \cdot u \\ \frac{1}{2\sqrt{uv^{-1}}} \cdot v^{-1} & -\frac{1}{2\sqrt{uv^{-1}}} \cdot uv^{-2} \end{array}\right] = -\frac{1}{2v}$$

Therefore

$$f_{U,V}(u,v) = f_{X,Y}(x(u,v), y(u,v)) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{e^{-\sqrt{uv}}}{2v}$$

for $\sqrt{uv} > 0$ and $0 < \sqrt{uv^{-1}} < 1$, i.e., $u > 0, v > 0$, and $u < v$.