Caculus II Math 1038 (1002&1003)

Monica CHEN

Chapter Sequences and Series

1. Tests for sequence convergence
   (a) divergent:
      i. find two subsequences with different limits p2(Dr Wong's note page 2)
   (b) convergent:
      i. find the limit. p4
      ii. squeeze theorem, p6,7
2. Series
   (a) Geometric series
   (b) telescoping series
3. **Tests for series convergence**
   (a) divergence test $\lim_{n\to\infty} a_n \neq 0$
   (b) integral test p13,14
   (c) alternating test: p17 $\sum(-1)^k a_k$ satisfy $a_k \geq a_{k+1} > 0$ (non-increasing) and $\lim_{k\to\infty} a_k = 0$, then it is convergent.
   (d) comparison test: p21-24 direct comparison and limit comparison
   (e) ratio test p28
   (f) root test p30

   page 31 Guidelines for testing series convergence
4. **Taylor's series (all)**
   (a) power series
   (b) Taylor/ Maclaurin series for simple functions: p37-41, **p51** e^x, $\sin x$, $\cos x$, $\ln x$, $\frac{1}{1-x}$, $\frac{1}{1+x}$, $\sqrt{1+x}$,...
   (c) Taylor's Theorem: p44- derivation of the coefficients of the first n terms of a Taylor polynomial
   (d) Remainder R_n and error estimate
   (e) applications: p51
      i. approximation of a function using the first n terms
      ii. evaluate a limit without using L'Hospital rule!

Chapter Vector and geometry of space (all)

1. Vector operations, notations
2. Dot product, angles, projection...
3. Cross product, areas...
4. vector and parametric equations
   (a) equation of a plane
   (b) distances
   (c) angle between two planes

Chapter Partial differentiation

1. functions of two variables
   (a) definition of limit
   (b) limit does NOT exist: find two paths which gives two different limits
   (c) limit exists:
      i. at a point in the domain $(a, b) \in D$: direct substitution
      ii. NOT in the domain $(a, b) \notin D$: prove by Squeeze Theorem, or by change of coordinate, or by definition
   (d) continuity
   Definition: f is continuous on D if f is continuous at every point (a, b) in D.

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

2. partial derivatives
   (a) differentiability:
   (b) **important relationships**: i $\to$ ii $\to$ iii $\to$ iv
      i. $f_x(a, b)$ and $f_y(a, b)$ are **continuous**
      ii. f **is differentiable at** (a, b)
      iii. f is continuous at (a, b)
      iv. $\lim_{(x,y)\to(a,b)} f(x, y) = L$ limit exist.
   iv $\not\to$ iii $\not\to$ ii $\not\to$ i.
   (c) to prove f is differentiable
      i. To show $f_x(a, b)$ and $f_y(a, b)$ exist AND are **continuous** at (a, b), .
   (d) to prove f is NOT differentiable:
      i. To show f is not continuous, or
      ii. by definition: ϵ_1 and $\epsilon_2 \not\to 0$.
   (e) tangent planes:
      i. equation of tangent plane through a point $P(x_0, y_0, z_0)$: $z = f(x, y)$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

      ii. vector equation of a normal line

$$(x, y, z) = (x_0, y_0, z_0) + t \langle f_x, f_y, -1\rangle$$

   (f) differentiation rules: product, quotient, chain rules
   (g) **chain rule** $z = f(x(s, t), y(s, t))$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial x} \cdot \frac{dy}{dt}$$
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial z}{\partial x} \cdot \frac{dy}{ds}$$

   (h) implicit differentiation
   (i) directional derivatives

$$D_{\overrightarrow{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = \langle f_x(x, y), f_y(x, y)\rangle \cdot \langle a, b\rangle = \nabla f \cdot \overrightarrow{u}$$

   (j) gradient vector
   (k) maximum rate of change

$$\max D_{\overrightarrow{u}} f(x, y) = |\nabla f|$$

(l) **optimization problems with two independent variables x and y**

i. critical point (a, b):
$$f_x(a, b) = f_y(a, b) = 0$$

ii. local maximum/minimum values: Second Derivative Test

iii. absolute extremum values: compare all the critical points and boundary points

iv. **method of Langrange Multiplier**

Chapter Multiple integrals

1. double integral over rectangles

   (a) Fubini's Theorem

2. double integrals over general regions

   (a) sketch the regions

      i. type I

      ii. type II

   (b) change integration order

   (c) polar coordinates

      i. area element: $dsdr = rd\theta dr$

   (d) surface area

3. *triple integrals

   (a) general region (x, y, z)

   (b) cylindrical coordinates (r, θ, z): volume element $\Delta V = r\Delta r\Delta\theta\Delta z$

   (c) spherical coordinates (ρ, ϕ, θ): volume element $\Delta V = \rho^2 \sin\phi\Delta\rho\Delta\theta\Delta\phi$

4. Change of variables

   (a) Transformation from uv-plane to xy-plane
   $$T(u, v) = (x, y)$$
   where $x = g(u, v)$ or $x = x(u, v)$ and $y = h(u, v)$ or $y(u, v)$, and $(u, v) \in S \subset \mathbb{R}^2$ and $(x, y) \in R \subset \mathbb{R}^2$

   (b) inverse transformation T^{-1} from xy-plane to uv-plane
   $$T^{-1}(x, y) = (u, v)$$

   (c) Jacobian determinant of a transformation of two variables
   $$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$
   remark: Jacobian can be positive and negative.

   (d) areas element ΔA in S (image) in xy-plane
   $$\Delta A \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u\Delta v$$
   remark: areas are non-negative, so we use the **absolute value** of Jacobian.

   (e) double integral of f over R and double integral of f over S
   $$\iint_R f(x, y)dA = \iint_S f(g(u, v), h(u, v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv$$

5. **Change of variables**

(a) Problem-solving strategy

i. sketch the region R in the xy-plane and write the **equations** of the curves of the boundaries if they are not given.

ii. choose $u(x,y)$, $v(x,y)$, and the transformation T ($x = g(u,v)$, $y = h(u,v)$) if they are not given, depending on the region R and integrand $f(x,y)$. e.g. parallel lines/curves.

iii. determine the **new limit (upper and lower bounds of integrals for u, v)** of the integration in uv-plane

iv. find the Jacobian $J(u,v)$

v. replace the variables in the integrand $f(x,y) \to f(x(u,v), y(u,v))$

vi. replace $dydx$ or $dxdy$ by $|J(u,v)|\, dudv$

$$\iint_R f(x,y)dA = \iint_S f(g(u,v), h(u,v))|J(u,v)|dudv$$

(b) understand all the examples and do lot of exercises

6. About exam:

(a) time: 9:30-11:30 am 3 June 2023. Two hours only!! (Do not stuck at one particular question, move on and come back later)

(b) closed book: pen and photo ID ONLY!! no calculator, electronic devices, notes, books, etc.

(c) final score: quiz (5% each, 10% total), assigment (1% each, 15% total), mid-term (15%), final (60%)

i. 40% of the final score has been finished, you can compute your scores

$$S_1 = \sum_{i=1}^{10} Ass_i \cdot 1\% + Q_1 \cdot 5\% + Q_2 \cdot 5\% + Mid \cdot 15\%$$

all the scores are out of 100.

ii. the final score out of 100 will be determined by yourself $S_2 = Final \cdot 60\%$

iii. scenarios A: submitted all the homework and get an average of 80%, $Q_1 = 40$, $Mid = 80$,$Q_2 = 90$

$$S_1 = 80 \cdot 15\% + 40 \cdot 5\% + 80 \cdot 15\% + 90 \cdot 5\% = 30.5$$

if $Final = 80$, oveall

$$S = S_1 + 80 \cdot 60\% = 78.5$$

iv. scenarios B: submitted all the homework and get an average of 90%, $Q_1 = 60$, $Mid = 90$,$Q_2 = 100$

$$S_1 = 90 \cdot 15\% + 60 \cdot 5\% + 90 \cdot 15\% + 100 \cdot 5\% = 35$$

if $Final = 70$,

$$S = S_1 + 70 \cdot 60\% = 78$$

if $Final = 90$,

$$S = S_1 + 90 \cdot 60\% = 89$$

if $Final = 50$,

$$S = S_1 + 50 \cdot 60\% = 65$$

(d) sleep well, work hard, eat less.