HW7-sol

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Q1. Solution: (a) Notice that A is an upper triangular matrix, the eigenvalues of A are

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 1$$

For $\lambda_1 = 0$, the eigenspace E_0 is

$$E_0 = N(A) = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}.$$

$$\begin{pmatrix}0&0&1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}$$

For $\lambda_2 = \lambda_3 = 1$,

$$(A-I)\vec{x}_2 = \begin{pmatrix}-1&0&1\\0&0&0\\0&0&0\end{pmatrix}\vec{x}_2 = \vec{0} \quad \Rightarrow \quad \vec{x}_2 = \begin{pmatrix}a\\b\\a\end{pmatrix}, \quad a,b \in \mathbb{R}.$$

Thus, $E_1 = \text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$.

Hence A is diagonalizable since A has 3 linearly independent eigenvectors.

$$A = \underbrace{\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}0&&\\&1&\\&&1\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}}_{P^{-1}}$$

(b) A is not orthogonally diagonalizable since A is not symmetric by Q1.

Q2. Notice that the 2nd and 3rd rows of A are scalar multiples of the 1st row, then

$$\text{rank}(A) = 1 \quad \text{and} \quad \dim(N(A)) = 2$$

Hence $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = Tr(A) - \lambda_1 - \lambda_2 = (1+4+4) - 0 = 9$.

For $\lambda_1 = \lambda_2 = 0$, $A\vec{x}_1 = \vec{0} \Rightarrow \vec{x}_1 = \begin{pmatrix}2b-2c\\b\\c\end{pmatrix}$, $b, c \in \mathbb{R}$.

$$E_0 = N(A) = \text{span}\left\{\begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}\right\}.$$

← orthogonal? No.

For $\lambda_3 = 9$,

$$(A-9I)\vec{x}_3 = \begin{pmatrix}-8&-2&2\\-2&-5&-4\\2&-4&-5\end{pmatrix}\vec{x}_3 = \vec{0} \longrightarrow \left(\begin{array}{ccc|c}2&5&4&0\\0&1&1&0\\0&0&0&0\end{array}\right)$$

$$E_9 = \text{span}\left\{\begin{pmatrix}1\\-2\\2\end{pmatrix}\right\}. \quad \text{Take } \vec{q}_{p3} = \frac{1}{3}\begin{pmatrix}1\\-2\\2\end{pmatrix} \in E_9$$

Apply Gram-Schmidt process on $\left\{\underbrace{\begin{pmatrix}2\\1\\0\end{pmatrix}}_{\vec{x}_1}, \underbrace{\begin{pmatrix}-2\\0\\1\end{pmatrix}}_{\vec{x}_2}\right\}$ to obtain an orthonormal set.

Let $\vec{v}_1 = \vec{x}_1 = \begin{pmatrix}2\\1\\0\end{pmatrix}$ then

$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}\vec{x}_2 = \begin{pmatrix}-2\\0\\1\end{pmatrix} - \frac{\langle \vec{x}_2, \vec{v}_1\rangle}{\langle \vec{v}_1, \vec{v}_1\rangle}\vec{v}_1 = \begin{pmatrix}-2\\0\\1\end{pmatrix} - \frac{-4}{5}\begin{pmatrix}2\\1\\0\end{pmatrix} = \frac{1}{5}\begin{pmatrix}-2\\4\\5\end{pmatrix}$$

Take $\vec{q}_{p1} = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\\0\end{pmatrix}$, $\vec{q}_{p2} = \frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \frac{1}{\sqrt{45}}\begin{pmatrix}-2\\4\\5\end{pmatrix} = \frac{1}{3\sqrt{5}}\begin{pmatrix}-2\\4\\5\end{pmatrix}$

Hence

$$A = QDQ^T = \begin{bmatrix}2/\sqrt{5}&-2/3\sqrt{5}&1/3\\1/\sqrt{5}&4/3\sqrt{5}&-2/3\\0&\sqrt{5}/3&2/3\end{bmatrix}\begin{bmatrix}0&&\\&0&\\&&9\end{bmatrix}\begin{bmatrix}2/\sqrt{5}&1/\sqrt{5}&0\\-2/3\sqrt{5}&4/3\sqrt{5}&5/3\\1/3&-2/3&2/3\end{bmatrix}$$

Q3. proof: Since A is Hermitian, then $A^H = A$. For $\vec{x} \in \mathbb{C}^n$,

$$\bar{c} = \overline{\vec{x}^H A \vec{x}} = (\vec{x}^H A \vec{x})^H = \vec{x}^H A^H \vec{x} = \vec{x}^H A \vec{x} = c.$$

Thus, c is a real number.

Q4. proof: $B = \alpha I - SAS^{-1} \longrightarrow BS = \alpha S - SA$

Multiply both sides by $\vec{x}$ on the right,

$$BS\vec{x} = \alpha S\vec{x} - SA\vec{x} = \alpha S\vec{x} - S(\lambda\vec{x})$$
$$= \alpha S\vec{x} - \lambda S\vec{x}$$

i.e. $B(S\vec{x}) = (\alpha - \lambda)(S\vec{x})$

Since S is nonsingular, $\vec{x} \neq \vec{0}$ is an eigenvector of A, then $\vec{y} = S\vec{x} \neq \vec{0}$

Thus $\vec{y}$ is an eigenvector of B corresponding to an eigenvalue $(\alpha - \lambda)$.

Q5. proof: (i) For $k=1$, $\forall\ \vec{x} \in N(A-\lambda I)$,

$$(A-\lambda I)^2\vec{x} = (A-\lambda I)(A-\lambda I)\vec{x} = (A-\lambda I)\vec{0} = \vec{0}$$

so $\vec{x} \in N(A-\lambda I)^2$. $N(A-\lambda I) \subseteq N(A-\lambda I)^2$.

The statement holds for $k=1$.

(ii) For $k > 1$, $k \in \mathbb{Z}^+$, $\forall\ \vec{y} \in N(A-\lambda I)^k$,

$$(A-\lambda I)^{k+1}\vec{y} = (A-\lambda I)(A-\lambda I)^k\vec{y} = (A-\lambda I)\vec{0} = \vec{0}$$

so $N(A-\lambda I)^k \subseteq N(A-\lambda I)^{k+1}$ for any positive integer k.

Q6. proof: For $A \in \mathbb{R}^{n\times n}$, if A is orthogonally diagonalizable, then

$$A = QDQ^T$$

where Q is an orthogonal matrix and D is a diagonal matrix

$$A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = QD^TQ^T = QDQ^T = A. \quad \square$$

Q7. proof: If $A_{n\times n}$ is skew-symmetric and λ is an eigenvalue of A, then

$$A\vec{x} = \lambda\vec{x} \quad \text{for some nonzero } \vec{x} \in \mathbb{C}^n.$$

then

$$\vec{x}^H A\vec{x} = \lambda\vec{x}^H\vec{x} \quad \rightarrow \quad \lambda = \frac{(\vec{x}^H A\vec{x})}{(\vec{x}^H\vec{x})}$$

→ purely imaginary

→ $\vec{x}^H\vec{x} > 0$ real

Consider $\overline{\vec{x}^H A\vec{x}}$. Let $\vec{x}^H A\vec{x} = a + ib$ for some $a, b \in \mathbb{R}$, then

$$a - ib = \overline{\vec{x}^H A\vec{x}} = (\vec{x}^H A\vec{x})^H = \vec{x}^H A^H\vec{x} = -\vec{x}^H A\vec{x} = -(a+ib)$$

and we have $2a - ib + ib = 0 \rightarrow a = 0$.

So $\vec{x}^H A\vec{x}$ is purely imaginary and since $\vec{x}^H\vec{x}$ is always real,

$$\lambda = \frac{\vec{x}^H A\vec{x}}{\vec{x}^H\vec{x}} \quad \text{is purely imaginary.}$$

proof: Let $A_{n\times n}$ be Hermitian, then A is diagonalizable with

$$A = UDU^H$$

where U is an unitary matrix and D is diagonal.

Thus,

$$A = [u_1\ u_2\ \cdots\ u_n]\begin{bmatrix}\lambda_1&&&\\&\lambda_2&&\\&&\ddots&\\&&&\lambda_n\end{bmatrix}\begin{bmatrix}u_1^H\\u_2^H\\\vdots\\u_n^H\end{bmatrix}$$

$$= [\lambda_1 u_1\ \ \lambda_2 u_2\ \cdots\ \lambda_n u_n]\begin{bmatrix}u_1^H\\u_2^H\\\vdots\\u_n^H\end{bmatrix}$$

$$= \lambda_1 u_1 u_1^H + \lambda_2 u_2 u_2^H + \cdots + \lambda_n u_n u_n^H$$