2022-23 First Semester
MATH1063 Linear Algebra II (1003)

Assignment 4 Suggested Solutions

1. **Proof:**

   $\langle A\mathbf{x}, \mathbf{y}\rangle = (\mathbf{x}^T A^T)\mathbf{y} = (\mathbf{x}^T)(A^T\mathbf{y}) = \langle \mathbf{x}, A^T\mathbf{y}\rangle$.
   Use the result of part(a), $\langle A^T A\mathbf{x}, \mathbf{x}\rangle = \langle A\mathbf{x}, (A^T)^T\mathbf{x}\rangle = \langle A\mathbf{x}, A\mathbf{x}\rangle = \|A\mathbf{x}\|^2$.

2. **Proof:**

   $A\mathbf{x} = \mathbf{b}$ is consistent $\Leftrightarrow$ $\mathbf{b} \in \mathrm{Col}(A) \xLeftrightarrow{\mathrm{Col}(A)\perp \mathrm{N}(A^T)}$ $\mathbf{b}$ is orthogonal to $\mathrm{N}(A^T)$.

3. **Solution:** Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, then $\mathrm{Col}(A) = S$. Notice that $\mathrm{rank}(A) = 2$, the projection of $\mathbf{v} = (2, 7, 10)^T$ onto $\mathrm{Col}(A)$ is

$$A(A^TA)^{-1}A^T\mathbf{v} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ 17 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 12 \\ 17 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 7 \\ 22 \\ 29 \end{bmatrix}.$$

4. **Solution:**

   (a) $A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad A^TA = \begin{bmatrix} 5 & -5 \\ -5 & 10 \end{bmatrix}, \quad A^T\mathbf{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$

$$\rightarrow \quad (A^TA)^{-1}A^T\mathbf{b} = \frac{1}{25} \begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

   (b) $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix}, A^TA = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}, A^T\mathbf{b} = \begin{bmatrix} 6 \\ 3 \\ 7 \end{bmatrix}.$

$$(A^TA)^{-1}A^T\mathbf{b} = \frac{1}{30} \begin{bmatrix} 11 & 1 & -3 \\ 1 & 11 & -3 \\ -3 & -3 & 9 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1.6 \\ 0.6 \\ 1.2 \end{bmatrix}.$$

5. **Solution:** The least-square solution $\hat{\mathbf{x}}$ relates to $\mathbf{b}$ in the way that $\hat{\mathbf{b}} = A\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{b}$ onto $\mathrm{Col}(A)$ and such $\hat{\mathbf{b}}$ is unique.

   Here $\mathrm{Col}(A) = \mathrm{span}\left\{(1, 2, -1)^T\right\}$, consequently, $\mathrm{rank}(A) \neq 2$, we can't use $\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$. Instead,

$$\hat{\mathbf{b}} = \frac{(3, 2, 1)(1, 2, -1)^T}{(1, 2, -1)(1, 2, -1)^T}(1, 2, -1)^T = (1, 2, -1)^T.$$

Solve $\hat{\mathbf{b}} = A\hat{\mathbf{x}}$ for $\hat{\mathbf{x}}$, and we have $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad c \in \mathbb{R}$. Infinitely many solutions.

6. (a) A is symmetric since

$$A^T = (\mathbf{x}\mathbf{y}^T + \mathbf{y}\mathbf{x}^T)^T = (\mathbf{x}\mathbf{y}^T)^T + (\mathbf{y}\mathbf{x}^T)^T = \mathbf{y}\mathbf{x}^T + \mathbf{x}\mathbf{y}^T = A.$$

Method 1: Prove part c first, then use the result to prove part b.

(c)

$$\begin{aligned} A &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \\ &= \left[\, x_1\mathbf{y} \,\middle|\, x_2\mathbf{y} \,\middle|\, \cdots \,\middle|\, x_n\mathbf{y} \,\right] + \left[\, y_1\mathbf{x} \,\middle|\, y_2\mathbf{x} \,\middle|\, \cdots \,\middle|\, y_n\mathbf{x} \,\right] \\ &= \left[\, x_1\mathbf{y} + y_1\mathbf{x} \,\middle|\, x_2\mathbf{y} + y_2\mathbf{x} \,\middle|\, \cdots \,\middle|\, x_n\mathbf{y} + y_n\mathbf{x} \,\right] \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{Col}(A) &= \mathrm{Span}\{x_i\mathbf{y} + y_i\mathbf{x}\}_{i=1}^n = \left\{ a_1(x_1\mathbf{y} + y_1\mathbf{x}) + \cdots + a_n(x_n\mathbf{y} + y_n\mathbf{x}) \,\middle|\, a_1, \cdots, a_n \in \mathbb{R}. \right\} \\ &= \left\{ \left(\sum_{i=1}^n a_i x_i\right)\mathbf{y} + \left(\sum_{i=1}^n a_i y_i\right)\mathbf{x} \,\middle|\, a_1, \cdots, a_n \in \mathbb{R}. \right\} \end{aligned}$$

which is in fact spanned by $\mathbf{x}$ and $\mathbf{y}$. Thus, $\mathrm{Col}(A) = S$. $\mathrm{rank}(A) = 2$ since $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

(b) By part c, $S = \mathrm{Col}(A)$, then $S^\perp = \mathrm{Col}(A)^\perp = \mathrm{N}(A^T) = \mathrm{N}(A)$ since $A = A^T$.

Method 2:

(b) To prove $\mathrm{N}(A) = S^\perp$, we need to show $\mathrm{N}(A) \subseteq S^\perp$ and $S^\perp \subseteq \mathrm{N}(A)$.
For any vector $\mathbf{z}$ in $\mathbf{R}^n$,

$$A\mathbf{z} = \mathbf{x}\mathbf{y}^T\mathbf{z} + \mathbf{y}\mathbf{x}^T\mathbf{z} = c_1\mathbf{x} + c_2\mathbf{y},$$

where $c_1 = \mathbf{y}^T\mathbf{z}, c_2 = \mathbf{x}^T\mathbf{z}$.
If $\mathbf{z}$ is in $\mathrm{N}(A)$, then

$$\mathbf{0} = A\mathbf{z} = c_1\mathbf{x} + c_2\mathbf{y}$$

since $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, we have $\mathbf{y}^T\mathbf{z} = c_1 = 0$ and $\mathbf{x}^T\mathbf{z} = c_2 = 0$. So $\mathbf{z}$ is orthogonal to both $\mathbf{x}$ and $\mathbf{y}$. Since $\mathbf{x}$ and $\mathbf{y}$ span S, it follows that $\mathbf{z} \in S^\perp$.
Conversely, if $\mathbf{z}$ is in $S^\perp$, then $\mathbf{z}$ is orthogonal to both $\mathbf{x}$ and $\mathbf{y}$. It follows that

$$A\mathbf{z} = c_1\mathbf{x} + c_2\mathbf{y} = \mathbf{0}$$

since $c_1 = \mathbf{y}^T\mathbf{z} = 0$ and $c_2 = \mathbf{x}^T\mathbf{z} = 0$, therefore $\mathbf{z} \in \mathrm{N}(A)$. Thus, $\mathrm{N}(A) = S^\perp$.

(c) Since $\dim S = 2$ and $N(A) = S^{\perp}$, $\dim S + \dim S^{\perp} = n$, we have

$$\dim N(A) = \dim S^{\perp} = n - 2.$$

It follows from the Rank-Nullity Theorem that the rank of A must be 2.