Chap6-Week11

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$6.1 + 6.3$ Diagonalizable Matrix $A_{n\times n} = PDP^{-1}$

Defective matrix * $B_{n\times n} = PJP^{-1}$

6.4 Orthogonally Diagonalizable Matrix — Hermitian Matrix ("Symmetric"); Unitary Matrix ("Orthogonal")

Sec 6.4 Hermitian Matrices

- Hermitian matrix
- Unitary matrix
- Properties

Complex : $\mathbb{C}$, $\mathbb{C}^n$, $\mathbb{C}^{n\times n}$ Real : $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$

$M \in \mathbb{C}^{n\times n}$ $A \in \mathbb{R}^{n\times n}$

△ Hermitian Matrix △ Symmetric Matrix

△ Unitary Matrix △ Orthogonal Matrix

The concept of orthogonality $\longleftrightarrow$ Complex Inner Products

Basic notations and concepts :

If $\alpha = a + ib \in \mathbb{C}$, then the length of α is

$$|\alpha| = \sqrt{\bar{\alpha}\alpha} = \sqrt{(a-ib)(a+ib)} = \sqrt{a^2+b^2}$$

If $\vec{z} = (z_1, z_2, \cdots, z_n)^T \in \mathbb{C}^n$, then the length of $\vec{z}$ is

$$\|\vec{z}\| = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{1/2} = (\bar{z}_1 z_1 + \cdots + \bar{z}_n z_n)^{1/2}$$
$$= (\bar{\vec{z}}^T \vec{z})^{1/2} = (\vec{z}^H \vec{z})^{1/2} = \sqrt{\vec{a}^T\vec{a} + \vec{b}^T\vec{b}}$$

Notation : $\vec{z}^H = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n) = (\bar{\vec{z}})^T$

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E.g.

$\alpha = 2 + 3i$, $|\alpha| = \sqrt{2^2+3^2} = \sqrt{13}$

$\vec{z} = (5+i, 1-3i)^T \in \mathbb{C}^2$,

$\bar{\vec{z}} = (5-i, 1+3i)^T$, a column vector

$\vec{z}^H = \bar{\vec{z}}^T = (5-i, 1+3i)$, a row vector

$$\|\vec{z}\|^2 = \vec{z}^H\vec{z} = (5-i, 1+3i)\begin{pmatrix}5+i\\1-3i\end{pmatrix}$$
$$= (5-i)(5+i) + (1+3i)(1-3i)$$
$$= (25+1) + (1+9) = 36$$

$$\vec{z} = \overset{\vec{a}}{\begin{pmatrix}5\\1\end{pmatrix}} + i\overset{\vec{b}}{\begin{pmatrix}1\\-3\end{pmatrix}}$$

$$\|\vec{z}\|^2 = \vec{a}^T\vec{a} + \vec{b}^T\vec{b} = (5,1)\begin{pmatrix}5\\1\end{pmatrix} + (1,-3)\begin{pmatrix}1\\-3\end{pmatrix} = 36$$

$\vec{z} = \vec{a} + i\vec{b}$

$\vec{z}^H = \vec{a}^T - i\vec{b}^T$

$$\|\vec{z}\| = \sqrt{\vec{z}^H\vec{z}} = \sqrt{(\vec{a}^T - i\vec{b}^T)(\vec{a}+i\vec{b})} = \sqrt{\vec{a}^T\vec{a} \underline{- i\vec{b}^T\vec{a} + i\vec{a}^T\vec{b}} + \vec{b}^T\vec{b}}$$
$$= \sqrt{\|\vec{a}\|^2 + \|\vec{b}\|^2} = \sqrt{\vec{a}^T\vec{a} + \vec{b}^T\vec{b}}$$

△ Complex Inner Products

Let V be a vector space over the complex numbers. An **inner product** on V is an operation that assigns to each pair of vectors $\mathbf{z}$ and $\mathbf{w}$ in V a complex number $\langle \mathbf{z}, \mathbf{w} \rangle$ satisfying the following conditions.

I. $\langle \mathbf{z}, \mathbf{z} \rangle \geq 0$, with equality if and only if $\mathbf{z} = \mathbf{0}$.
II. $\langle \mathbf{z}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{z} \rangle}$ for all $\mathbf{z}$ and $\mathbf{w}$ in V. Order matters!
III. $\langle \alpha\mathbf{z} + \beta\mathbf{w}, \mathbf{u} \rangle = \alpha\langle \mathbf{z}, \mathbf{u} \rangle + \beta\langle \mathbf{w}, \mathbf{u} \rangle$.

E.g. Define a function : $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$

$$\langle \vec{z}, \vec{w} \rangle := \vec{w}^H \vec{z}$$

① Show that $\langle \cdot, \cdot \rangle$ is a complex inner product. — simple exercises

② Compute $\langle \vec{z}, \vec{w} \rangle$ and $\langle \vec{w}, \vec{z} \rangle$ for

$$\vec{z} = \begin{pmatrix}5-i\\1+i\end{pmatrix}, \quad \vec{w} = \begin{pmatrix}1+2i\\i\end{pmatrix}, \quad \bar{\vec{w}} = \begin{pmatrix}1-2i\\-i\end{pmatrix}$$

$$\langle \vec{z}, \vec{w} \rangle = \vec{w}^H\vec{z} = (1-2i, -i)\begin{pmatrix}5-i\\1+i\end{pmatrix} = (1-2i)(5-i) - i(1+i) = 5-2-11i-i+1$$
$$= 4-12i$$

$$\langle \vec{w}, \vec{z} \rangle = \vec{z}^H\vec{w} = \left(\begin{pmatrix}5\\1\end{pmatrix} - \begin{pmatrix}-1\\1\end{pmatrix}i\right)^T\left[\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix}i\right] = (5\ \ 1)\begin{pmatrix}1\\0\end{pmatrix} + (5, 1)\begin{bmatrix}2\\1\end{bmatrix}i - (-1\ \ 1)\begin{bmatrix}1\\0\end{bmatrix}i + (-1\ \ 1)\begin{pmatrix}2\\1\end{pmatrix}$$
$$= 4+12i$$

so that $\langle \vec{z}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{z} \rangle}$

△ Hermian Matrix

Let $M = (m_{ij})$ be an $m\times n$ with $m_{ij} = a_{ij} + ib_{ij}$ for $i = 1, \cdots, m$; $j = 1, \cdots, n$

Then we denote M as

$$M = A + iB$$

where A and B are $m\times n$ real-valued matrices.

We define the conjugate of M as $\bar{M}$

$$\bar{M} = A - iB$$

by taking the conjugate of each entries in M.

Then

$$M^H = (\bar{M})^T = (A - iB)^T = A^T - iB^T$$

E.g. $M = \begin{bmatrix}3 & 2-i\\2+i & 4\end{bmatrix}$

$$M^H = (\bar{M})^T = \begin{bmatrix}3 & 2+i\\2-i & 4\end{bmatrix}^T = \begin{bmatrix}3 & 2-i\\2+i & 4\end{bmatrix}$$

Since $M = M^H$, M is a Hermitian matrix.

Questions : Is a real-valued symmetric matrix Hermitian?

Yes!

__Properties__: If A and B are elements of $\mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{n\times r}$, then

I. $(A^H)^H = (\underbrace{A_{Re}^T - iA_{Im}^T}_{A^H})^H = (A_{Re}^T + iA_{Im}^T)^T = A_{Re} + iA_{Im} = A$

II. $(\alpha A + \beta B)^H = \bar{\alpha}A^H + \bar{\beta}B^H$, $\forall \alpha, \beta \in \mathbb{C}$

III. $(AC)^H = C^H A^H$

__Def__. (Hermitian Matrix)

$$A = A^H$$

Remark : A real Hermitian matrix is a symmetric matrix.

__Thm__ (i) The eigenvalues of a Hermitian matrix are **all real**.

real?

__E.g__ $A = \begin{bmatrix}2 & 1-i\\1+i & 1\end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix}2-\lambda & 1-i\\1+i & 1-\lambda\end{vmatrix} = (2-\lambda)(1-\lambda) - (1+i)(1-i)$$
$$= (\lambda^2 - 3\lambda + 2) - (1^2 - i^2) = \lambda(\lambda-3)$$

$\lambda_1 = 0$, $\lambda_2 = 3$

$$\det(\lambda I - A) = \begin{vmatrix}\lambda-2 & -(1-i)\\-(1+i) & \lambda-2\end{vmatrix} = \lambda(\lambda-3) = 0$$

For $\lambda_1 = 0$,

$$(A - \lambda_1 I)\vec{v}_1 = \begin{bmatrix}2 & 1-i\end{bmatrix}\begin{bmatrix}a\end{bmatrix} = \begin{bmatrix}0\end{bmatrix}$$

Remark: A real Hermitian matrix is a symmetric matrix.

Thm (i) The eigenvalues of a Hermitian matrix are **all real**.

proof: $\vec{x}^H A\vec{x} = \vec{x}^H\lambda\vec{x} = \lambda\|\vec{x}\|^2 \rightarrow \lambda = \frac{\vec{x}^H A\vec{x}}{\|\vec{x}\|^2}$ real? real

Consider $\overline{\vec{x}^H A\vec{x}} = \vec{x}^H A^H(\vec{x}^H)^H = \vec{x}^H A^H\vec{x} = \vec{x}^H A\vec{x}$

Hence, $\vec{x}^H A\vec{x}$ is also a real number and $\lambda = \frac{\vec{x}^H A\vec{x}}{\|\vec{x}\|^2}$ is real.

Method 2: $\vec{x}^H A\vec{x} = \vec{x}^H A^H\vec{x} = (A\vec{x})^H\vec{x} = (\lambda\vec{x})^H\vec{x} = \bar{\lambda}\vec{x}^H\vec{x}$

$\rightarrow \lambda\|\vec{x}\|^2 = \bar{\lambda}\|\vec{x}\|^2$, $\|\vec{x}\|^2>0$, $\|\vec{x}\|^2\in\mathbb{R}$

i.e. $\lambda = \bar{\lambda}$

All eigenvalues of a Hermitian matrix are real.

(ii) Furthermore, eigenvectors belonging to distinct eigenvalues are **orthogonal**.

proof: Consider $\lambda_1\neq\lambda_2$ with eigenvectors $\vec{x}_1$ and $\vec{x}_2$ of A, respectively

$$\vec{x}_2^H A\vec{x}_1 = \vec{x}_2^H\lambda_1\vec{x}_1 = \lambda_1\vec{x}_2^H\vec{x}_1 = \lambda_1\langle\vec{x}_1,\vec{x}_2\rangle$$

$$\parallel$$

$$\vec{x}_2^H A^H\vec{x}_1 = (A\vec{x}_2)^H\vec{x}_1 = (\lambda_2\vec{x}_2)^H\vec{x}_1 = \bar{\lambda}_2\vec{x}_2^H\vec{x}_1$$

Since eigenvalues of A are all real, so $\bar{\lambda}_2 = \lambda_2$ and

$$(\lambda_1-\lambda_2)\vec{x}_2^H\vec{x}_1 = 0 \rightarrow \langle\vec{x}_1,\vec{x}_2\rangle = 0$$

$$\vec{x}_1\perp\vec{x}_2$$

For $\lambda_1=0$,

$$(A-\lambda_1 I)\vec{v}_1 = \begin{bmatrix}2 & 1-i\\ 1+i & 1\end{bmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\rightarrow \begin{cases}2a+(1-i)b=0\\ \underline{(1+i)a+b=0}\end{cases} \rightarrow \begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}1\\ -1-i\end{pmatrix}\alpha,\ \alpha\neq0.$$

For $\lambda_2=3$

$$(A-\lambda_2 I)\vec{v}_2 = \begin{bmatrix}-1 & 1-i\\ 1+i & -2\end{bmatrix}\vec{v}_2 = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\rightarrow \vec{v}_2 = \begin{bmatrix}1-i\\ 1\end{bmatrix}\beta,\ \beta\neq0.$$

Notice that $\langle\vec{v}_1,\vec{v}_2\rangle = \vec{v}_2^H\vec{v}_1 = (1+i\ \ 1)\begin{pmatrix}1\\ -1-i\end{pmatrix} = 0$

△ Unitary Matrix $\longleftrightarrow$ orthogonal matrix

Def (Unitary) An $n\times n$ matrix is **unitary** if its columns form an **orthonormal basis of $\mathbb{C}^n$**.

Thm $U_{n\times n}$ is unitary $\Leftrightarrow$ $U^H U = I$.

Remark: A real unitary matrix is an orthogonal matrix.

Corollary If the eigenvalues of a Hermitian matrix A are distinct, then there exists a unitary matrix U that diagonalizes A.

i.e. $A = UDU^{-1} = UDU^H$

E.g. $A = \begin{bmatrix}0 & 2 & -1\\ 2 & 3 & -2\\ -1 & -2 & 0\end{bmatrix}$

proof: (in the textbook).

Thm (Spectral Thm)

If A is Hermitian, then there exists a unitary matrix U that diagonalizes A, that is,

$$A = UDU^H.$$

proof till next lecture.