PT

Solution to Assignment 6

1.

(a) We have

$$
\begin{aligned}
P\{X=1\} &= F(1) - \lim_{x\to 1^-} F(x) = 1/2 - 1/4 = 1/4,\\
P\{X=2\} &= F(2) - \lim_{x\to 2^-} F(x) = 11/12 - (1/2+1/4) = 1/6,\\
P\{X=3\} &= F(3) - \lim_{x\to 3^-} F(x) = 1 - 11/12 = 1/12.
\end{aligned}
$$

(b) It holds

$$
\begin{aligned}
P\left\{\frac{1}{2} < X < \frac{3}{2}\right\} &= P\left\{\frac{1}{2} < X \leq \frac{3}{2}\right\} - P\left(X = \frac{3}{2}\right)\\
&= F(3/2) - F(1/2) - \left(F(3/2) - \lim_{x\to \frac{3}{2}^-} F(x)\right)\\
&= \lim_{x\to \frac{3}{2}^-} F(x) - F(1/2)\\
&= \frac{5}{8} - \frac{1}{8} = \frac{1}{2}.
\end{aligned}
$$

2. Let X be your earnings.

$$
\begin{aligned}
P(X=10) &= \frac{4}{52},\\
P(X=5) &= \frac{4}{52},\\
P(X=1) &= \frac{22}{52}\cdot\frac{\binom{26}{2}}{\binom{51}{2}} = \frac{11}{102},\\
P(X=-1) &= 1 - \frac{2}{13} - \frac{11}{102},
\end{aligned}
$$

and so

$$EX = \frac{10}{13} + \frac{5}{13} + \frac{11}{102} - 1 + \frac{2}{13} + \frac{11}{102} = \frac{4}{13} + \frac{11}{51} > 0.$$

3. Note

$$E\left[c^X\right] = cp + c^{-1}(1-p).$$

Hence, $1 = E\left[c^X\right]$ if

$$cp + c^{-1}(1-p) = 1,$$

or, equivalently

$$pc^2 - c + 1 - p = 0,$$

or

$$(pc - 1 + p)(c - 1) = 0.$$

Thus, $c = (1-p)/p$.

4. Since $E[X] = np, \operatorname{Var}(X) = np(1-p)$, we are given that $np = 6, np(1-p) = 2.4$. Thus, $1 - p = 0.4$, or $p = 0.6, n = 10$. Hence,

$$P\{X = 5\} = \binom{10}{5}(0.6)^5(0.4)^5.$$

5. The number of outcomes is $\binom{11}{3}$. X can have values $-3, -2, -1, 0, 1, 2$, and 3 . Let us start with 0 . This can occur with one ball of each color or with 3 blue balls:

$$P(X = 0) = \frac{3 \cdot 3 \cdot 5 + \binom{5}{3}}{\binom{11}{3}} = \frac{55}{165}.$$

To get $X = 1$, we can have 2 red and 1 white, or 1 red and 2 blue:

$$P(X = 1) = P(X = -1) = \frac{\binom{3}{2}\binom{3}{1} + \binom{3}{1}\binom{5}{2}}{\binom{11}{3}} = \frac{39}{165}.$$

The probability that $X = -1$ is the same because of symmetry between the roles that the red and the white balls play. Next, to get $X = 2$ we must have 2 red balls and 1 blue:

$$P(X = -2) = P(X = 2) = \frac{\binom{3}{2}\binom{5}{1}}{\binom{11}{3}} = \frac{15}{165}.$$

Finally, a single outcome (3 red balls) produces $X = 3$:

$$P(X = -3) = P(X = 3) = \frac{1}{\binom{11}{3}} = \frac{1}{165}.$$

All the seven probabilities should add to 1 , which can be used either to check the computations or to compute the seventh probability (say, $P(X = 0)$) from the other six.

6. For each child, independently, the probability of the rr genotype is $\frac{1}{4}$. If X is the number of rr children, then X is Binomial $\left(n, \frac{1}{4}\right)$. Therefore,

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - \left(\frac{3}{4}\right)^n - n \cdot \frac{1}{4}\left(\frac{3}{4}\right)^{n-1}$$

7.

(a) N is Geometric $\left(\frac{5}{6}\right)$:

$$P(N = n) = \left(\frac{1}{6}\right)^{n-1} \cdot \frac{5}{6}$$

where $n = 1, 2, 3, \ldots$.

(b) $EN = \frac{6}{5}$.

(c) By symmetry, P (you win) $= \frac{1}{2}$.

(d) You get paid \$10 with probability $\frac{5}{12}$, \$1 with probability $\frac{1}{12}$, and 0 otherwise, so your expected winnings are $\frac{51}{12}$.