AFM Brief Solution to Assignment 3

1. A stochastic process is generated by the equation

$$dX_t = \mu dt + \sigma dW_t.$$

with the initial condition $X_0 = 1$. Which equation governs the process $Y(t, X_t) = (1+t)^2 e^{aX_t}$? Here a is a constant.

Solution: According to Ito's lemma, we have

$$dY_t = \left[\frac{\partial Y}{\partial t} + \mu \frac{\partial Y}{\partial X_t} + \frac{1}{2}\sigma^2 \frac{\partial^2 Y}{\partial X_t^2}\right] dt + \sigma \frac{\partial Y}{\partial X_t} dW_t.$$

As

$$\frac{\partial Y}{\partial t} = \frac{2Y}{1+t},\ \frac{\partial Y}{\partial X_t} = a(1+t)^2 e^{aX_t} = aY,\ \frac{\partial^2 Y}{\partial X_t^2} = a^2(1+t)^2 e^{aX_t} = a^2 Y,$$

we have

$$dY_t = \left[\frac{2}{1+t} + \mu a + \frac{1}{2}\sigma^2 a^2\right] Y_t dt + \sigma a Y_t dW_t.$$

with the initial condition $Y_0 = e^a$

2. Consider a stochastic process X_t governed by the equation

$$dX_t = X_t(\mu dt + \sigma dW_t),$$

with unknown constant μ and σ. It is known that the process $Y_t = e^{-3t}X_t^2$ is governed by the equation

$$dY_t = Y_t(dt + dW_t).$$

Determine the value of μ and σ.

Solution: According to Ito's lemma, we have

$$dY_t = \left[\frac{\partial Y}{\partial t} + \mu X_t \frac{\partial Y}{\partial X_t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 Y}{\partial X_t^2}\right] dt + \sigma X_t \frac{\partial Y}{\partial X_t} dW_t.$$

As

$$\frac{\partial Y}{\partial t} = -3e^{-3t}X_t^2,\ \frac{\partial Y}{\partial X_t} = 2e^{-3t}X_t,\ \frac{\partial^2 Y}{\partial X_t^2} = 2e^{-3t},$$

we have

$$\begin{aligned} dY_t &= \left[-3e^{-3t}X_t^2 + 2\mu e^{-3t}X_t^2 + \sigma^2 e^{-3t}X_t^2\right] dt + 2\sigma X_t^2 e^{-3t} dW_t \\ &= \left[-3 + 2\mu + \sigma^2\right] Y_t dt + 2\sigma Y_t dW_t \\ &= Y_t\left[(-3 + 2\mu + \sigma^2)dt + 2\sigma dW_t\right]. \end{aligned}$$

Hence we get

$$\left\{\begin{array}{l} -3 + 2\mu + \sigma^2 = 1 \\ 2\sigma = 1 \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{l} \mu = \frac{15}{8} \\ \sigma = \frac{1}{2} \end{array}\right.$$

3. Find $u(X,t)$ and $v(X,t)$ where

$$dX_t = udt + vdW_t$$

and

(a) $X_t = W_t^3$,
(b) $X_t = -7t^2 + W_t^3$,
(c) $X_t = g(t)e^{9W_t}$,

where f is a bounded, differentiable function.

Solution:

(a) Take $f(t, W_t) = W_t^4$. Then,

$$\frac{\partial f}{\partial t} = 0, \ \frac{\partial f}{\partial W_t} = 3W_t^2, \ \frac{\partial^2 f}{\partial W_t^2} = 6W_t.$$

As

$$df(t, W_t) = \left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W_t^2}\right] dt + \frac{\partial f}{\partial W_t} dW_t,$$

we have

$$dX_t = 3W_t dt + 3W_t^2 dW_t = 3X_t^{\frac{1}{3}} dt + 3X_t^{\frac{2}{3}} dW_t,$$

hence, $u(X,t) = 3X_t^{\frac{1}{3}}$ and $v(X,t) = 3X_t^{\frac{2}{3}}$.

(b) Take $f(t, W_t) = -7t^2 + W_t^3$. Then,

$$\frac{\partial f}{\partial t} = -14t, \ \frac{\partial f}{\partial W_t} = 3W_t^2, \ \frac{\partial^2 f}{\partial W_t^2} = 6W_t.$$

As

$$df(t, W_t) = \left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W_t^2}\right] dt + \frac{\partial f}{\partial W_t} dW_t,$$

we have

$$dX_t = (-14t+3W_t)dt+3W_t^2 dW_t = [-14t+3(X_t+7t^2)^{\frac{1}{3}}]dt+3(X_t+7t^2)^{\frac{2}{3}} dW_t,$$

hence, $u(X,t) = -14t + 3(X_t + 7t^2)^{\frac{1}{3}}$ and $v(X,t) = 3(X_t + 7t^2)^{\frac{2}{3}}$.

(c) Take $f(t, W_t) = g(t)e^{9W_t}$. Then,

$$\frac{\partial f}{\partial t} = g'(t)e^{9W_t}, \ \frac{\partial f}{\partial W_t} = 9g(t)e^{9W_t}, \ \frac{\partial^2 f}{\partial W_t^2} = 81g(t)e^{9W_t}.$$

As

$$df(t, W_t) = \left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W_t^2}\right] dt + \frac{\partial f}{\partial W_t} dW_t,$$

we have

$$dX_t = (g'(t)+\frac{81}{2}g(t))e^{9W_t}dt+9g(t)e^{9W_t}dW_t = \left[\frac{g'(t)}{g(t)} + \frac{81}{2}\right] X_t dt+9X_t dW_t,$$

hence, $u(X,t) = (\frac{g'(t)}{g(t)} + \frac{81}{2})X_t$ and $v(X,t) = 9X_t$.

4. S_t is generated by the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad \text{with } S_0 = a,$$

where μ, σ and a are constants. Evaluate the probability of $\lambda K \leq S_T \leq 2\lambda K$, where $T > 0, \lambda > 0$, and K is a positive constant.
Solution:
By solving the stochastic differential equation, we can get that

$$S_t = ae^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

Thus, we have

$$\begin{aligned} Prob(\lambda K \leq S_T \leq 2\lambda K) &= Prob\left(ae^{(\mu-\frac{1}{2}\sigma^2)T+\sigma W_T} \leq 2\lambda K\right) \\ &\quad - Prob\left(ae^{(\mu-\frac{1}{2}\sigma^2)T+\sigma W_T} \leq \lambda K\right) \\ &= Prob\left(\frac{W_T}{\sqrt{T}} \leq \frac{\ln\left(\frac{2\lambda K}{a}\right) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &\quad - Prob\left(\frac{W_T}{\sqrt{T}} \leq \frac{\ln\left(\frac{\lambda K}{a}\right) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right). \end{aligned}$$

As W_T follows the normal distribution with mean 0 and variance T, $\frac{W_T}{\sqrt{T}}$ follows the standard normal distribution, we get that

$$\begin{aligned} Prob(\lambda K \leq S_T \leq 2\lambda K) &= N\left(\frac{\ln\left(\frac{2\lambda K}{a}\right) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &\quad - N\left(\frac{\ln\left(\frac{\lambda K}{a}\right) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \end{aligned}$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution.

5. A stochastic process is generated by the equation

$$dX_t = \mu dt + \sigma X_t dW_t$$

with the initial condition of $X_0 = 1$. Which equation governs the process $Y(t, X_t) = e^{(1+t)X_t + t^2}$?
Solution: According to Ito's lemma, we have

$$dY_t = \left[\frac{\partial Y}{\partial t} + \mu \frac{\partial Y}{\partial X_t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 Y}{\partial X_t^2}\right] dt + \sigma X_t \frac{\partial Y}{\partial X_t} dW_t.$$

Since

$$\frac{\partial Y}{\partial t} = (X_t + 2t)Y, \ \frac{\partial Y}{\partial X_t} = (1+t)Y, \ \frac{\partial^2 Y}{\partial X_t^2} = (1+t)^2 Y, X_t = \frac{ln(Y_t) - t^2}{1+t}$$

we have

$$dY_t = \left[(X_t + 2t) + \mu(1+t) + \frac{1}{2}\sigma^2 X_t^2 (1+t)^2\right] Y_t dt + \sigma X_t (1+t) Y_t dW_t.$$

with the initial condition of $Y_0 = e$.

6. Consider a stochastic process X_t governed by the equation

$$dX_t = X_t(a(t)dt + b(t)dW_t),$$

where $a(t)$ and $b(t)$ are unknown functions of t. It is known that the process $Y_t = f(t)X_t^2$ is governed by the equation

$$dY_t = Y_t t(dt + 9t^2 dW_t).$$

Determine the functions $a(t)$ and $b(t)$.
Solution: According to Ito's lemma, we have

$$dY_t = \left[\frac{\partial Y}{\partial t} + a(t)X_t \frac{\partial Y}{\partial X_t} + \frac{1}{2} b(t)^2 X_t^2 \frac{\partial^2 Y}{\partial X_t^2}\right] dt + b(t) X_t \frac{\partial Y}{\partial X_t} dW_t.$$

As

$$\frac{\partial Y}{\partial t} = f'(t) X_t^2, \ \frac{\partial Y}{\partial X_t} = 2f(t) X_t, \ \frac{\partial^2 Y}{\partial X_t^2} = 2f(t),$$

we have

$$\begin{aligned} dY_t &= \left[f'(t) + f(t)(2a(t) + b(t)^2)\right] X_t^2 dt + 2f(t)b(t)X_t^2 dW_t \\ &= Y_t \left[\left(\frac{f'(t)}{f(t)} + 2a(t) + b(t)^2\right)dt + 2b(t)dW_t\right]. \end{aligned}$$

Hence we get

$$\left\{ \begin{array}{l} 2b(t) = 9t^3 \\ \frac{f'(t)}{f(t)} + 2a(t) + b(t)^2 = t \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} b(t) = \frac{9}{2}t^3 \\ a(t) = \frac{1}{2}(t - \frac{81}{4}t^6 - \frac{dlnf(t)}{dt}) \end{array} \right.$$