ASP Additional Examples on Martingales

1. Suppose that $\{N_t, t \geq 0\}$ is a Poisson process with parameter $\lambda = 1$. Find $E(N_1 \mid N_2)$ and $E(N_2 \mid N_1)$.

2. Let $Y_1, Y_2, \ldots$ be nonnegative i.i.d. random variables with $EY_m = 1$ and $P(Y_m = 1) < 1$. Show that $X_n = \prod_{m \leq n} Y_m$ defines a martingale with respect to $X_1, X_2, \ldots$

3. (**Lognormal stock prices**) Consider $X_i = e^{\eta_i}$, where $\eta_i \sim N(\mu, \sigma^2)$ and (η_i) being i.i.d. For what values of μ and σ is $M_n = M_0 \cdot X_1 \cdots X_n$ a martingale?

4. Let $X_1, X_2, X_3, \ldots$ be independent identically distributed random variables. Let $m(t) = \mathbb{E}\left(e^{tX_1}\right)$ be the moment generating function of X_1 (and hence of each X_i). Fix t and assume $m(t) < \infty$. Let $S_0 = 0$ and for $n > 0$,

$$S_n = X_1 + \cdots + X_n$$

Let $M_n = m(t)^{-n} e^{tS_n}$. Show that M_n is a martingale with respect to $X_1, X_2, \ldots$

5. Let $X_1, X_2, \ldots$ be independent with $\mathbb{P}(X_i = -1) = q, \mathbb{P}(X = 1) = p$, where $p + q = 1$. Let $S_0 = j$, where $j \in \mathbb{N}$ is a constant, and $S_n = S_0 + X_1 + \cdots + X_n$. Suppose $p \neq q$. Show that $M_n = (q/p)^{S_n}$ is a martingale with respect to $X_1, X_2, \ldots$

Solutions:

1.

(a) By "taking out what is known" and the "role of independence",

$$E[N_2 \mid N_1] = E[(N_2 - N_1) + N_1 \mid N_1] = E[N_2 - N_1] + N_1 = N_1 + 1.$$

(b) $E[N_1 \mid N_2 = k] = E[N_1 \mid N_1 + (N_2 - N_1) = k]$. In assignment 1, we know that

$$P(X_1 = j \mid X_1 + X_2 = k) = \binom{k}{j} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^j \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{k-j},$$

where X_1 and X_2 are independent Poisson random variables with parameters λ_1 and λ_2, respectively; that is, conditioning on $X_1 + X_2 = k$, X_1 has a binomial distribution with parameters k and $\lambda_1/(\lambda_1 + \lambda_2)$. Therefore, conditioning on $N_2 = k$, N_1 has a binomial distribution with parameters k and $1/2$. So

$$E[N_1 \mid N_2 = k] = \sum_{j=0}^{k} j \binom{k}{j} \cdot \left(\frac{1}{2}\right)^j \left(\frac{1}{2}\right)^{k-j} = \frac{k}{2},$$

and thus $E[N_1 \mid N_2] = \frac{N_2}{2}$.

2. Let $\mathcal{F}_n = \sigma(X_1, X_2, \cdots X_n)$. Obviously, $(X_n)_{n\geq 1}$ is adapted with respect to $(\mathcal{F}_n)_{n\geq 1}$, and $E[|X_n|] = E[X_n] = E\left[\prod_{m=1}^{n} Y_m\right] = \prod_{m=1}^{n} E[Y_m] = 1 < \infty$. Moreover, by "taking out what is known" and the "role of independence",

$$E[X_{n+1} \mid \mathcal{F}_n] = E\left[\left(\prod_{m=1}^{n} Y_m\right) \cdot Y_{n+1} \mid \mathcal{F}_n\right]$$
$$= \left(\prod_{m=1}^{n} Y_m\right) \cdot E[Y_{n+1} \mid \mathcal{F}_n] = \left(\prod_{m=1}^{n} Y_m\right) E[Y_{n+1}] = X_n.$$

So $(X_n)_{n\geq 1}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 1}$.

3. Similar to the last problem, we let $\mathcal{F}_n = \sigma(M_0, M_1, \cdots, M_n)$. If $(M_n)_{n\geq 0}$ is a martingale, it is to satisfy

$$E[M_{n+1} \mid \mathcal{F}_n] = M_n, \quad n \geq 0,$$

where

$$E[M_{n+1} \mid \mathcal{F}_n] = E\left[M_0 \cdot X_{n+1} \cdot \left(\prod_{m=1}^{n} X_m\right) \mid \mathcal{F}_n\right]$$
$$= \left(\prod_{m=1}^{n} X_m\right) \cdot E[X_{n+1}] = M_n \cdot E[X_{n+1}].$$

So $E[X_{n+1}]$ needs to be 1, that is, $E[e^{\eta_i}] = 1$. Since $E[e^{\eta_i}] = e^{\mu + \frac{\sigma^2}{2}}$, the parameters μ, σ must be such that

$$\mu + \frac{\sigma^2}{2} = 0.$$

4. Let $\mathcal{F}_n = \sigma(X_1, X_2, \cdots, X_n)$. Then

$$M_n = \frac{e^{tS_n}}{m(t)^n} = \frac{e^{t(X_1 + \cdots + X_n)}}{m(t)^n},$$

thus $(M_n)_{n\geq 1}$ is adapted with respect to $(\mathcal{F}_n)_{n\geq 1}$. Moreover,

$$E[|M_n|] = E[M_n] = E\left[\frac{e^{t(X_1+\cdots+X_n)}}{m(t)^n}\right] = \frac{1}{m(t)^n} E\left[e^{tX_1} \cdots e^{tX_n}\right] = \frac{1}{m(t)^n}\left(E\left[e^{tX_1}\right]\right)^n < \infty.$$

Also,

$$E[M_{n+1} \mid \mathcal{F}_n] = E\left[\frac{e^{tS_{n+1}}}{m(t)^{n+1}} \mid \mathcal{F}_n\right] = \frac{1}{m(t)^{n+1}} E\left[e^{tS_n} \cdot e^{tX_{n+1}} \mid \mathcal{F}_n\right] = \frac{e^{tS_n}}{m(t)^{n+1}} \cdot E\left[e^{tX_{n+1}} \mid \mathcal{F}_n\right].$$

Since X_{n+1} is independent of $X_1, X_2, \cdots, X_n$, $e^{tX_{n+1}}$ is also independent of $X_1, X_2, \cdots, X_n$. So

$$E[M_{n+1} \mid \mathcal{F}_n] = \frac{e^{tS_n}}{m(t)^{n+1}} \cdot E\left[e^{tX_{n+1}}\right] = \frac{e^{tS_n}}{m(t)^{n+1}} \cdot m(t) = \frac{e^{tS_n}}{m(t)^n} = M_n.$$

Therefore, $(M_n)_{n\geq 1}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 1}$.

5. Let $\mathcal{F}_n = \sigma\left(X_1, X_2, \cdots, X_n\right)$. Then

$$M_n = \left(\frac{q}{p}\right)^{S_n} = \left(\frac{q}{p}\right)^{j+X_1+\cdots+X_n},$$

thus $(M_n)_{n\geq 1}$ is adapted with respect to $(\mathcal{F}_n)_{n\geq 1}$. Moreover,

$$\begin{aligned} E\left[|M_n|\right] = E\left[M_n\right] &= E\left[\left(\frac{q}{p}\right)^{j+X_1+\cdots+X_n}\right] \\ &= \left(\frac{q}{p}\right)^{j} E\left[\left(\frac{q}{p}\right)^{X_1}\cdots\left(\frac{q}{p}\right)^{X_n}\right] = \left(\frac{q}{p}\right)^{j}\left(E\left[\left(\frac{q}{p}\right)^{X_1}\right]\right)^{n} < \infty. \end{aligned}$$

And

$$\begin{aligned} E\left[M_{n+1} \mid \mathcal{F}_n\right] &= E\left[\left(\frac{q}{p}\right)^{S_{n+1}} \mid \mathcal{F}_n\right] = \left(\frac{q}{p}\right)^{S_n} \cdot E\left[\left(\frac{q}{p}\right)^{X_{n+1}} \mid \mathcal{F}_n\right] \\ = \left(\frac{q}{p}\right)^{S_n} \cdot E\left[\left(\frac{q}{p}\right)^{X_{n+1}}\right] &= \left(\frac{q}{p}\right)^{S_n} \cdot \left[\left(\frac{q}{p}\right)^{-1} \cdot q + \left(\frac{q}{p}\right)^{1} \cdot p\right] \\ = \left(\frac{q}{p}\right)^{S_n} &= M_n. \end{aligned}$$

So $(M_n)_{n\geq 1}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 1}$.