Risk Management in Finance - Market Risk III – Value at Risk and Expected Shortfall

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The Question Being Asked in VaR

"What loss level is such that we are X% confident it will not be exceeded in N business days?"

Loss Variable and Loss Ditribution

- We denote the *portfolio value* at time t as $V(t)$.
- The *time horizon* is $\triangle t$ (e.g. a day, ten days, a month, a year).
- The *profit* in the time interval from t to $t + \triangle t$ is

$$V(t + \triangle t) - V(t).$$

- The *loss variable*

$$L_{[t,t+\triangle t]} := -(V(t + \triangle t) - V(t))$$

is the negative profit. Its law is the *loss distribution*. Later, when the time horizon is clear according to context, we often write L instead of $L_{[t,t+\triangle t]}$.

Definition of Value at Risk

Suppse a loss variable L and a confidence level $\alpha \in (0, 1)$ is given. Let F_L be the distribution function of L, i.e.,

$$F_L(x) = P(L \leq x), \quad x \in \mathbb{R}.$$

The *Value-at-Risk* (VaR) for level α is defined as

$$\mathrm{VaR}_\alpha(L) = \inf\{x \in \mathbb{R} : F_L(x) \geq \alpha\} = \inf\{x \in \mathbb{R} : \mathrm{P}(L \leq x) \geq \alpha\}.$$

Remark: From the statistical point of view, $\mathrm{VaR}_\alpha(L)$ is just the α-quantile of the loss distribution. Recall that for a probability distribution with distribution function F, its α-quantile is defined as

$$q_\alpha(F) = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}.$$

Properties of Value at Risk

For any $a > 0$ and $b \in \mathbb{R}$,

$$\begin{aligned}
\mathrm{VaR}_\alpha(aL+b) &= \inf\{l \in \mathbb{R} \,:\, \mathrm{P}(aL+b \le l) \ge \alpha\} \\
&= \inf\{l \in \mathbb{R} \,:\, \mathrm{P}(L \le (l-b)/a) \ge \alpha\} \\
&= \inf\{al'+b \in \mathbb{R} \,:\, \mathrm{P}(L \le l') \ge \alpha\}, \quad \text{[let } l' = (l-b)/a] \\
&= a \inf\{l' \in \mathbb{R} \,:\, \mathrm{P}(L \le l') \ge \alpha\} + b \\
&= a\mathrm{VaR}_\alpha(L) + b.
\end{aligned}$$

Example: VaR for Normal Loss Distribution

- Suppose the loss distribution is normal, i.e., $L \sim \mathrm{N}(\mu, \sigma^2)$.
- This means that $L = \mu + \sigma L'$, where $L' \sim \mathrm{N}(0, 1)$.
- Note $\mathrm{VaR}_\alpha(L') = \Phi^{-1}(\alpha)$, where Φ is the distribution function of a standard normal random variable.
- We may compute the Value-at-Risk of L as

$$\mathrm{VaR}_\alpha(L) = \mu + \sigma\Phi^{-1}(\alpha).$$

Example 12.1 (page 257)

- The gain from a portfolio during six month is normally distributed with mean \$2 million and standard deviation \$10 million
- The 1% point of the distribution of gains is $2-2.33\times10$ or $-$ \$21.3 million
- The VaR for the portfolio with a six month time horizon and a 99% confidence level is \$21.3 million.

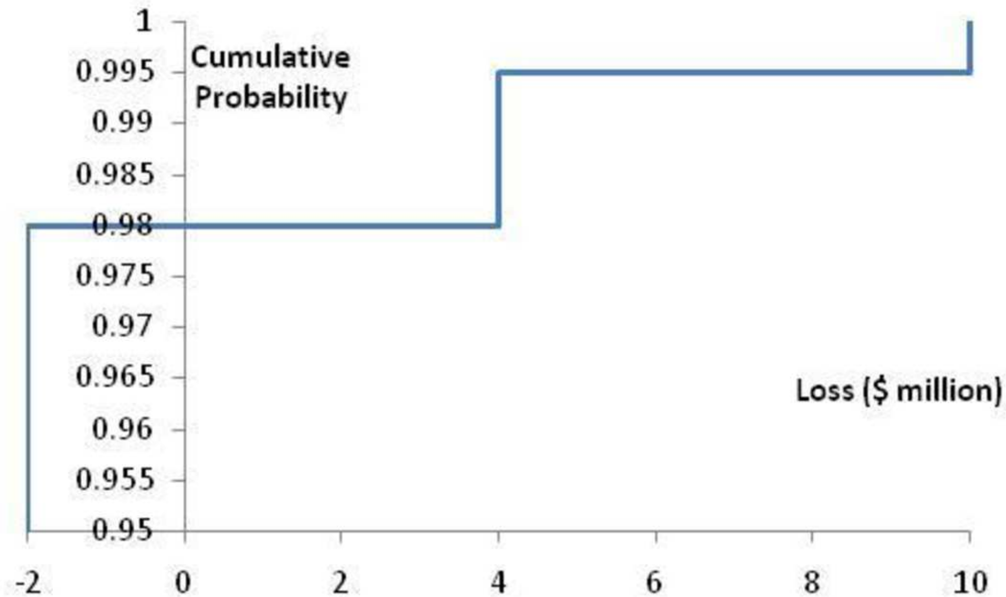
Example 12.2 (page 258)

- All outcomes between a loss of $50 million and a gain of $50 million are equally likely for a one-year project
- The VaR for a one-year time horizon and a 99% confidence level is $49 million

Examples 12.3 and 12.4 (page 258)

- A one-year project has a 98% chance of leading to a gain of \$2 million, a 1.5% chance of a loss of \$4 million, and a 0.5% chance of a loss of \$10 million
- The VaR with a 99% confidence level is \$4 million
- What if the confidence level is 99.9%?
- What if it is 99.5%?

Cumulative Loss Distribution for Examples 12.3 and 12.4 (Figure 12.3, page 258)



VaR and Regulatory Capital

- Regulators have traditionally used VaR to calculate the capital they require banks to keep
- The market-risk capital has been based on a 10-day VaR estimated where the confidence level is 99%
- Credit risk and operational risk capital are based on a one-year 99.9% VaR

Advantages of VaR

- It captures an important aspect of risk in a single number
- It is easy to understand
- It asks the simple question: “How bad can things get?”

Expected Shortfall for Continuous Loss Distributions

Suppse a *continuous* loss *variable* L and a confidence level $\alpha \in (0,1)$ is given. Let f be the density function of L, i.e.,

$$P(L \leq x) = \int_{-\infty}^{x} f(z)dz, \quad x \in \mathbb{R}.$$

The *Expected Shortall* (ES) of L at confidence level α is defined as

$$\mathrm{ES}_\alpha(L) = E[L|L \geq \mathrm{VaR}_\alpha(L)] = \frac{\int_{\mathrm{VaR}_\alpha(L)}^{\infty} zf(z)dz}{1-\alpha}.$$

Remark: The definition of expected shortall for a general loss distribution is more tricky and will be given in a few moments.

Example: ES for Normal Loss Distribution

Suppose that $L \sim N(0,1)$. Let ϕ and Φ be the density and distribution function of L. Then

$$
\begin{aligned}
\mathrm{ES}_\alpha(L) &= \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^{\infty} l\phi(l)\mathrm{d}l \\
&= \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^{\infty} l\frac{1}{\sqrt{2\pi}}e^{-l^2/2}\mathrm{d}l \\
&= \frac{1}{1-\alpha}[-\frac{1}{\sqrt{2\pi}}e^{-l^2/2}]_{\Phi^{-1}(\alpha)}^{\infty} \\
&= \frac{1}{1-\alpha}[-\phi(l)]_{\Phi^{-1}(\alpha)}^{\infty} \\
&= \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}.
\end{aligned}
$$

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Example: ES for Normal Loss Distribution continued

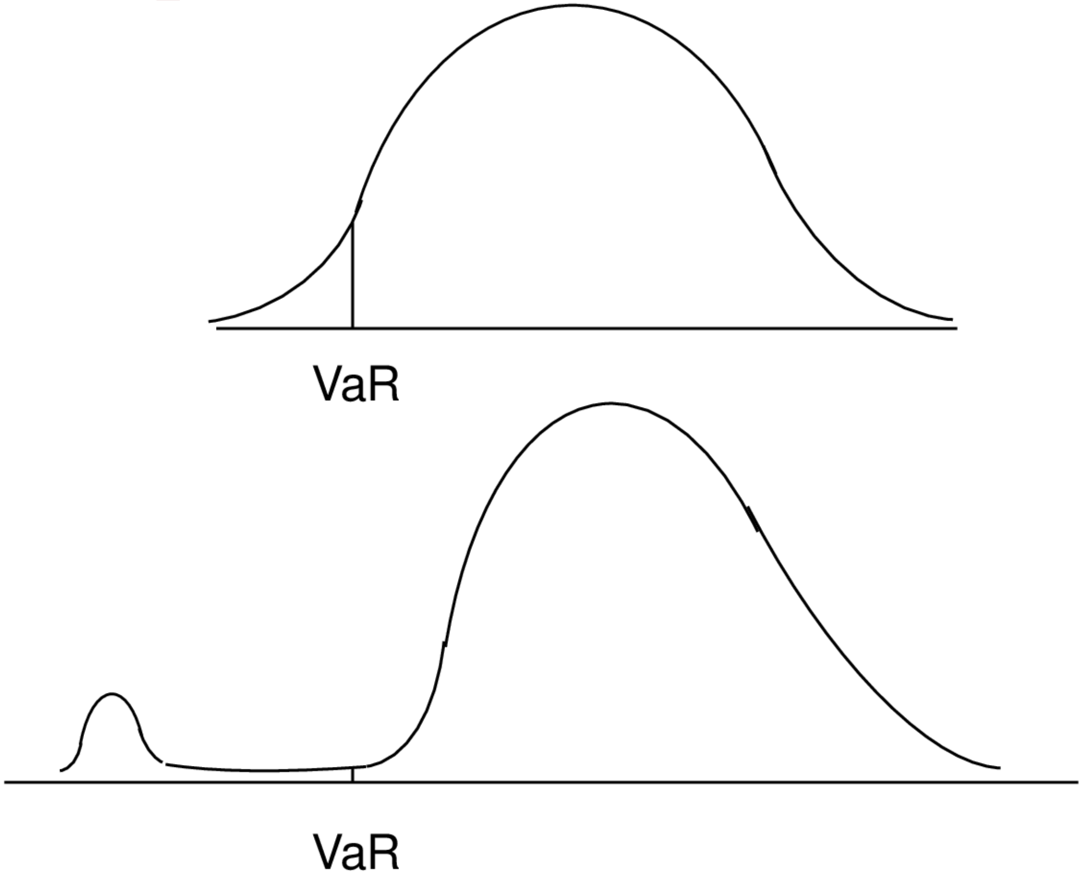
Suppose that $L' \sim \mathrm{N}(\mu, \sigma^2)$. Then

$$\begin{aligned}
\mathrm{ES}_\alpha(L') &= \mathrm{E}(L' | L' \geq \mathrm{VaR}_\alpha(L')) \\
&= \mathrm{E}(\mu + \sigma L | L \geq \mathrm{VaR}_\alpha(L)) \\
&= \mu + \sigma \mathrm{ES}_\alpha(L) \\
&= \mu + \sigma \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}.
\end{aligned}$$

VaR vs. Expected Shortfall

- VaR is the loss level that will not be exceeded with a specified probability
- Expected shortfall (ES) is the expected loss given that the loss is greater than the VaR level (also called C-VaR and Tail Loss)
- Regulators have indicated that they plan to move from using VaR to using ES for determining market risk capital
- Two portfolios with the same VaR can have very different expected shortfalls

Distributions with the Same VaR but Different Expected Shortfalls



Expected Shortfall for General Loss Distributions

Suppse a loss *variable* L and a confidence level $\alpha \in (0, 1)$ is given. Let F be the distribution function of L, i.e.,

$$F(x) = P(L \leq x), \quad x \in \mathbb{R}.$$

Let

$$\lambda := \frac{F(\mathrm{VaR}_\alpha(L)) - \alpha}{1 - \alpha}.$$

The *Expected Shortall* (ES) of L at confidence level α is defined as

$$\mathrm{ES}_\alpha(L) = \lambda \cdot \mathrm{VaR}_\alpha(L) + (1 - \lambda) \cdot E[L | L > \mathrm{VaR}_\alpha(L)]).$$

Coherent Risk Measures

So far we have introduced VaR and ES, both of which map any loss variable (distribution) to a real number. In general, we may consider the following.

Definition A mapping $L \mapsto \varrho(L)$ assiging a number to any loss variable L is called *coherent risk measure* if the following axioms hold:

1. *translation invariance*: $\varrho(L+b) = \varrho(L) + b$ for any loss L and any real number b;
2. *positive homogeneity*: $\varrho(aL) = a\varrho(L)$ for any number $a \geq 0$;
3. *monotonicity*: $\varrho(L) \leq \varrho(\overline{L})$ if $L \leq \overline{L}$;
4. *convexity*: $\varrho(L+\overline{L}) \leq \varrho(L) + \varrho(\overline{L})$ for any losses $L, \overline{L}$.

VaR vs Expected Shortfall

- VaR satisfies the first three conditions but not the fourth one
- ES satisfies all four conditions.

Example 12.5 and 12.7

- Each of two independent projects has a probability 0.98 of a loss of $1 million and 0.02 probability of a loss of $10 million
- What is the 97.5% VaR for each project?
- What is the 97.5% expected shortfall for each project?
- What is the 97.5% VaR for the portfolio?
- What is the 97.5% expected shortfall for the portfolio?

Examples 12.6 and 12.8

- A bank has two $10 million one-year loans. Possible outcomes are as follows

Outcome	Probability
Neither Loan Defaults	97.5%
Loan 1 defaults, loan 2 does not default	1.25%
Loan 2 defaults, loan 1 does not default	1.25%
Both loans default	0.00%

- If a default occurs, losses between 0% and 100% are equally likely. If a loan does not default, a profit of 0.2 million is made.
- What is the 99% VaR and expected shortfall of each project
- What is the 99% VaR and expected shortfall for the portfolio

Spectral Risk Measures

1. A spectral risk measure assigns weights to quantiles of the loss distribution
2. VaR assigns all weight to Xth percentile of the loss distribution
3. Expected shortfall assigns equal weight to all percentiles greater than the Xth percentile
4. For a coherent risk measure weights must be a non-decreasing function of the percentiles

Normal Distribution Assumption

- When losses (gains) are normally distributed with mean μ and standard deviation σ

$$\text{VaR} = \mu + \sigma N^{-1}(X)$$

$$\text{ES} = \mu + \sigma \frac{e^{-Y^2/2}}{\sqrt{2\pi}(1-X)}$$

Changing the Time Horizon

- If losses in successive days are independent, normally distributed, and have a mean of zero

$$T\text{-day VaR} = 1\text{-day VaR} \times \sqrt{\mathrm{T}}$$

$$T\text{-day ES} = 1\text{-day ES} \times \sqrt{T}$$

Extension

- If there is autocorrelation ρ between the losses (gains) on successive days, we replace $\sqrt{T}$ by

$$\sqrt{T+2(T-1)\rho+2(T-2)\rho^2+2(T-3)\rho^3+\ldots+2\rho^{T-1}}$$

in these equations

Ratio of *T*-day VaR to 1-day VaR (Table 12.1, page 266)

	T=1	T=2	T=5	T=10	T=50	T=250
ρ=0	1.0	1.41	2.24	3.16	7.07	15.81
ρ=0.05	1.0	1.45	2.33	3.31	7.43	16.62
ρ=0.1	1.0	1.48	2.42	3.46	7.80	17.47
ρ=0.2	1.0	1.55	2.62	3.79	8.62	19.35

Choice of VaR Parameters

- Time horizon should depend on how quickly portfolio can be unwound. Regulators are planning to move toward a system where ES is used and the time horizon depends on liquidity. (See *Fundamental Review of the Trading Book*)
- Confidence level depends on objectives. Regulators use 99% for market risk and 99.9% for credit/operational risk.
- A bank wanting to maintain a AA credit rating might use confidence levels as high as 99.97% for internal calculations.

Aggregating VaRs

An approximate approach that seems to works well is

$$\text{VaR}_{\text{total}} = \sqrt{\sum_i \sum_j \text{VaR}_i \text{VaR}_j \rho_{ij}}$$

where VaR_i is the VaR for the ith segment, $\text{VaR}_{\text{total}}$ is the total VaR, and ρ_{ij} is the coefficient of correlation between losses from the ith and jth segments

VaR Measures for a Portfolio where an amount x_i is invested in the ith component of the portfolio (page 268-270)

- Marginal VaR: $$\frac{\partial \text{VaR}}{\partial x_i}$$

- Incremental VaR: Incremental effect of the *i*th component on VaR

- Component VaR: $$x_i \frac{\partial \text{VaR}}{\partial x_i}$$

Properties of Component VaR

- The component VaR is approximately the same as the incremental VaR
- The total VaR is the sum of the component VaR's (Euler's theorem)
- The component VaR therefore provides a sensible way of allocating VaR to different activities

Back-testing (page 270-273)

- Back-testing a VaR calculation methodology involves looking at how often exceptions (loss > VaR) occur
- Alternatives: a) compare VaR with actual change in portfolio value and b) compare VaR with change in portfolio value assuming no change in portfolio composition
- Suppose that the theoretical probability of an exception is p (=1−X). The probability of m or more exceptions in n days is

$$\sum_{k=m}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

Bunching

- Bunching occurs when exceptions are not evenly spread throughout the back testing period
- Statistical tests for bunching have been developed by Christoffersen (See page 200)