2021-22 First Semester
MATH1083 Calculus II (1003)

Assignment 3

Due Date: 11:30am 7/Mar/2021(Tue).

- Write down your **Chinese name** and **student number**. Write neatly on **A4-sized** paper and **show your steps**.
- **Late submissions or answers without details will not be graded.**

1. Find the radius of convergence and interval of convergence of the power series.
[For these question, you will have to apply the Ratio Test to solve for x and then test the ***two endpoints****.]*

(a)

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} x^n$$

Solution: Let

$$a_n = \frac{(-1)^n 4^n}{\sqrt{n}} x^n$$

We use Ratio Test. If $x \neq 0$, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}4^{n+1}}{\sqrt{n+1}}x^{n+1}}{\frac{(-1)^n 4^n}{\sqrt{n}}x^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{4\sqrt{n}}{\sqrt{n+1}}\right||x| \\ &= 4|x|\end{aligned}$$

By Ratio Test, when $4|x| < 1$, that is when

$$|x| < \frac{1}{4}$$

the given series is convergent. **Thus the radius of convergence is** $R = 1/4$.
Then we test the series at the **two endpoints**:
When $x = 1/4$,$a_n = \frac{(-1)^n}{\sqrt{n}} 1^n$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

is **convergent** by **Alternating Convergence Theorem.**
When $x = -1/4$,$a_n = \frac{(-1)^n}{\sqrt{n}}(-1)^n = \frac{1}{\sqrt{n}}$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is **divergent** as it is a p-series with $p = 1/2 < 1$.Therefore the interval of convergence is $(-1/4, 1/4]$.

(b)

$$\sum_{n=1}^{\infty} n!\,(2x-1)^n$$

Solution: If $a_n = n!\,(2x-1)^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(2x-1)^{n+1}}{n!\,(2x-1)^n}\right| = \lim_{n\to\infty}|n+1||2x-1| = \infty$$

By Ratio Test, the series diverges when $x \neq 1/2$. Thus the given series convergest only when $x = 1/2$.
Therefore, the r**adius of convergence** $R = 0$ and the **interval of convergence** is $\{1/2\}$.
[Remark: If the interval of convergence is a single number, you need to put it inside a $\{\}$ which denotes an interval contains only one point.]

(c)

$$\sum_{n=1}^{\infty} \frac{(x-2)^n\, 2^n}{n^n}$$

Solution: If $a_n = \frac{(x-2)^n 2^n}{n^n}$, then

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}\, 2^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{(x-2)^n\, 2^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^{n+1}}\right| |x-2| \\
&= \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^n \frac{1}{(n+1)}\right| |x-2| \\
&= 0 < 1
\end{aligned}$$

for all x. Thus by the Ratio Test, the series converges for all values of x, the radius of convergence is $R = \infty$, and the interval of convergence is $\mathbb{R}$ or $(-\infty, \infty)$.
(Note: $\mathbb{R}$ denotes the real number set, similarly we have $\mathbb{C}$ denote all the complex number set, $\mathbb{N}$ denotes all the natural number. $\mathbb{N} \in \mathbb{R} \in \mathbb{C}$)

2. Find a power series representation for the function and determine the interval of convergence

$$\frac{1}{(1+x)^3}$$

[Hint: You will need to find the power series for $1/(1+x)^2$ first]

Solution: Since the power series for

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n\, x^n$$

with radius of convergence

$$R = 1$$

and using differentiation

$$-\left(\frac{1}{1+x}\right)' = \frac{1}{(1+x)^2}$$

So we can have the power series representation for

$$\begin{aligned}
\frac{1}{(1+x)^2} &= -\left(\sum_{n=0}^{\infty} (-1)^n\, x^n\right)' \\
&= \sum_{n=1}^{\infty} (-1)^{n+1}\, n x^{n-1}
\end{aligned}$$

with the **same radius of convergence**

$$R = 1$$

Since

$$-\frac{1}{2}\left(\frac{1}{(1+x)^2}\right)' = \frac{1}{(1+x)^3}$$

Therefore

$$\begin{aligned}
\frac{1}{(1+x)^3} &= -\frac{1}{2}\left(\sum_{n=1}^{\infty} (-1)^{n+1}\, n x^{n-1}\right)' \\
&= \frac{1}{2}\sum_{n=2}^{\infty} (-1)^n\, n\,(n-1)\, x^{n-2}
\end{aligned}$$

or

$$\frac{1}{(1+x)^3} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n (n+2)(n+1)x^n$$

with the same radius of convergence

$$R = 1$$

Then we test the endpoints:
When $x = 1$, $a_n = (-1)^n (n+2)(n+1)$, the series is

$$\sum_{n=1}^{\infty}(-1)^n (n+2)(n+1)$$

is **divergent.**
When $x = -1$,$a_n = (n+2)(n+1)$, the series is

$$\sum_{n=1}^{\infty}(n+2)(n+1) = \infty$$

is divergent. Then the interval of convergence is $(-1, 1)$.

3. Find the Taylor series for $f(x)$ centered at the given value of a, and find the associated **radius of convergence.**

(a) $f(x) = e^{2x}, \qquad a = 3$

Solution: $f(x) = e^{2x}$, $f(a) = e^6$
$f'(x) = 2e^{2x}$, $f'(a) = 2e^6$
$f''(x) = 2^2e^{2x}$, $f''(a) = 4e^6$
$f'''(x) = 2^3e^{2x}$, $f'''(a) = 8e^6$
and therefore they Taylor series at $a = 3$ is

$$\begin{aligned} & e^6 + 2e^6(x-3) + \frac{1}{2!}2^2e^6(x-3)^2 + \frac{1}{3!}2^3e^6(x-3)^3 + \cdots \\ =& e^6\sum_{n=0}^{\infty}\frac{2^n}{n!}(x-3)^n \end{aligned}$$

By the **Ratio Test**,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right||x-3| = \lim_{n\to\infty}\left|\frac{2}{n+1}\right||x-3| = 0$$

,therefore series converges for all values of x and the **radius of convergence** is $R = \infty$.

4. Use binomial series to expand the given function as a power series and find the radius of convergence

$$(2-x)^{3/4}$$

Solution: Using the binomial series with $k = 3/4$, and then we have

$$\begin{aligned}(2-x)^{3/4} &= 2^{3/4}\left(1-\frac{x}{2}\right)^{3/4} \\ &= 2^{3/4}\sum_{n=0}^{\infty}\binom{\frac{3}{4}}{n}\left(-\frac{x}{2}\right)^n \\ &= 2^{3/4}\left[1 - \frac{3}{4}\left(\frac{x}{2}\right) + \frac{\frac{3}{4}\cdot\left(-\frac{1}{4}\right)}{2!}\left(\frac{x}{2}\right)^2 - \frac{\frac{3}{4}\cdot\left(-\frac{1}{4}\right)\cdot\left(-\frac{5}{4}\right)}{3!}\left(\frac{x}{2}\right)^3 + \cdots + (-1)^n\frac{\frac{3}{4}\cdot\left(-\frac{1}{4}\right)\cdots\cdot\left(\frac{3}{4}-n+1\right)}{n!}\left(\frac{x}{2}\right)^n + \cdots\right.\end{aligned}$$

[Please pay attention to how to express a infinite series (in red)].
Then we use the Ratio Test to find the radius of convergence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)!\left(\frac{3}{4}-n\right)}{n+1}\right|\left|\frac{x}{2}\right| = \left|\frac{x}{2}\right| < 1$$

therefore $|x| < 2$ and $R = 2$ is the radius of convergence.

3. (b) $f(x)=\frac{1}{x^2}$, $a=1$

Solution: $f(x)=\frac{1}{x^2}$ $f(a)=1$

$f'(x)=-\frac{2}{x^3}$ $f(a)=-2$

$f''(x)=\frac{6}{x^4}$ $f(a)=6$

$f^{(3)}(x)=-\frac{24}{x^5}$ $f(a)=-24$

$$f(x)=1+(-2)(x-1)+\frac{6}{2!}(x-1)^2-\frac{24}{3!}(x-1)^3+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)!}{n!}(x-1)^n=\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n$$

By the ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)(n+2)}{n+1}\right||x-1|$$

$$=|x-1|<1,$$

therefore the radius of convergence is $R=1$

☆ binomial series

$$(1+x)^p=\sum_{k=0}^{\infty}\binom{p}{k}x^k$$

$$=\sum_{k=0}^{\infty}\frac{p(p-1)\cdots(p-k+1)}{k!}x^k$$

radius $=1$

if $p=-n$

$$\binom{-n}{k}=\frac{(-n)(-n-1)(-n-2)\cdots(-n-k+1)}{k!}$$

$$=(-1)^k\frac{n(n+1)(n+2)\cdots(n+k-1)}{k!}$$

$$=(-1)^k\binom{n+k-1}{k}$$

☆ Taylor polynomials

n-th order Taylor polynomial centered at a

$$P_n(x)=f(a)+\frac{f'(a)(x-a)}{1!}+\frac{f''(a)(x-a)^2}{2!}+\cdots+\frac{f^{(n)}(a)(x-a)^n}{n!}$$

remainder $R_n(x)=f(x)-P_n(x)$

$$=\frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

$|f^{(n+1)}(z)|\le M$, z is between x and a.

$$|R_n(x)|=|f(x)-P_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1}$$

5. Approximate function $f(x)$ by a Taylor's polynomial with degree n at center a, and **use Taylor's Inequality** to estimate the accuracy of the approximation when $f(x) \approx T_n(x)$ when x lies in the given interval.

(a) $f(x) = 1/x$, $a = 1$, $n = 2$, $0.7 \le x \le 1.3$
Solution: First let us compute the first 4 coefficients of Taylor series:

$$f(x) = \frac{1}{x}, \qquad f(1) = 1$$
$$f'(x) = -\frac{1}{x^2}, \qquad f'(1) = -1$$
$$f''(x) = \frac{2}{x^3}, \qquad f''(1) = 2$$
$$f'''(x) = -\frac{6}{x^4}, \qquad f'''(1) = -6$$

Thus the **second-degree Taylor polynomials** is

$$\begin{aligned} T_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &= 1 - (x-1) + (x-1)^2 \end{aligned}$$

for $0.7 \le x \le 1.3$, we have $|x-1| < 0.3$. Then we use Taylor's Inequality, and for $z \in (0.7, 1.3)$, $\max|f'''(z)| = \ 6/0.7^4$ (*it attains the maximum at* $x = 0.7$):

$$|R_2(x)| = \left|\frac{f'''(z)(x-1)^3}{3!}\right| \le \ \frac{1}{0.7^4} \times 0.3^3 = 0.11245$$

[*Remark: If the question does not ask for Taylor's Inequality, you can also use use the* ***Alternating Series Estimation Theorem*** *(Easier!) for this alternating. The error is at most* $|a_3|$ *which is*

$$|R_2(x)| \le |a_3| = \left|\frac{f'''(1)(x-1)^3}{3!}\right| = \left|(x-1)^3\right| \le 0.3^3 = 0.027$$

]

(b) $f(x) = x \ln x$, $a = 1$, $n = 3$, $0.5 \le x \le 1.5$
Solution: First let us compute the first 5 coefficients of Taylor series:

$$f(x) = x\ln x, \qquad f(1) = 0$$
$$f'(x) = \ln x + 1, \qquad f'(1) = 1$$
$$f''(x) = \frac{1}{x}, \qquad f''(1) = 1$$
$$f'''(x) = -\frac{1}{x^2}, \qquad f'''(1) = -1$$
$$f''''(x) = \frac{2}{x^3}, \qquad f''''(1) = 2$$

Thus the **third-degree Taylor polynomials** is

$$\begin{aligned} T_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{3}(x-1)^3 \end{aligned}$$

for $0.5 \le x \le 1.5$, we have $|x-1| < 0.5$. Then we use Taylor's Inequality, and for $z \in (0.5, 1.5)$,

$$R_3(x) \le \max|f''''(z)| = \frac{2}{0.5^3} = M$$

(*it attains the maximum at* $x = 0.5$), and

$$\max\left|(x-1)^4\right| = 0.5^4$$

$$|R_3(x)| = \left|\frac{f''''(z)(x-1)^4}{4!}\right| = \frac{M}{4!}\left|(x-1)^4\right| \le \frac{\frac{2}{0.5^3}}{4!} \times 0.5^4 = 0.04167$$