Chap6-Week10

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Review Exercise: Find all eigenvalues and eigenvectors

$A = \begin{bmatrix} -1 & -2 \\ 6 & 6 \end{bmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -2 \\ 6 & 6-\lambda \end{vmatrix} = (\lambda-6)(\lambda+1)+12 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3) = 0$

$\lambda_1 = 2, \quad \lambda_2 = 3$

For $\lambda_1 = 2$: $\begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix} \vec{\xi}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \vec{\xi}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \alpha, \quad \alpha \neq 0$

For $\lambda_2 = 3$: $\begin{bmatrix} -4 & -2 \\ 6 & 3 \end{bmatrix} \vec{\xi}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \vec{\xi}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \alpha, \quad \alpha \neq 0$

$B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$\det(B - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4 = (\lambda - 1 - 2i)(\lambda - 1 + 2i) = 0$

$\lambda = 1 \pm 2i$

For $\lambda_1 = 1 + 2i$,

$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \vec{\xi}_1 = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} \alpha, \quad \alpha \neq 0, \ \alpha \in \mathbb{C}$

$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} i$

For $\lambda_2 = 1 - 2i$, $\quad \vec{\xi}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} i$

Chap6.1-6.3 Eigenvalues and Eigenvectors

W9
1. Be familiar with the eigenvalues and eigenvectors, master the skill in finding eigenpairs
   a. Characteristic equation
   b. Eigenspace, algebraic multiplicity of an eigenvalue
   c. Eigenvalues, determinant and trace

W10
2. Diagonalization
   a. With n distinct eigenvalues
   b. With <n distinct eigenvalues
3. Non-diagonalizable? (Generalized eigenvectors and Jordan form)

Why diagonalization?

If $A = PDP^{-1}$, then $A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$

$A^n = PD^nP^{-1}$

Computing $D^k = \begin{bmatrix} \lambda_1^k & & \\ & \lambda_2^k & \\ & & \ddots \ \lambda_n^k \end{bmatrix}$ is much easier than computing A^k.

△ Diagonalization

Definition 6.1.2

An $n \times n$ matrix A is said to be **diagonalizable** if there exists a nonsingular matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

The k th diagonal entry of D is an eigenvalue of A corresponding to the eigenvector, the k th column vector of P.

Theorem 6.1.8

$\Leftrightarrow$

An $n \times n$ matrix A is diagonalizable $\boxed{\text{if and only if}}$ A has n linearly independent eigenvectors.

Theorem 6.1.9

If $\lambda_1, \cdots, \lambda_k$ are distinct eigenvalues of an $n \times n$ matrix A corresponding to eigenvectors $v_1, \cdots, v_k$ then $\{v_1, \cdots, v_k\}$ is linearly independent. (If A has n distinct eigenvalues then A is diagonalizable.)

Proof by contradiction:

Assume that $\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_r\}$ are linearly independent, $r < k$, then

$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r + c_{r+1}\vec{v}_{r+1} = \vec{0} \quad ①$

for $c_{r+1} \neq 0$ and $c_1, \ldots, c_r$ are not all zeros, i.e.,

$\vec{v}_{r+1}$ is linearly dependent on $\{\vec{v}_1, \cdots, \vec{v}_r\}$.

$c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + \cdots + c_{r+1} A\vec{v}_{r+1} = \vec{0}$

$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_{r+1}\lambda_{r+1}\vec{v}_{r+1} = \vec{0} \quad ②$

② $- \lambda_{r+1} \times$ ①:

$c_1(\lambda_1 - \lambda_{r+1})\vec{v}_1 + c_2(\lambda_2 - \lambda_{r+1})\vec{v}_2 + \cdots + c_r(\lambda_r - \lambda_{r+1})\vec{v}_r = \vec{0}$

↳ some of $c_i(\lambda_i - \lambda_{r+1})$ are nonzero, which contradicts the linear independency of $\{\vec{v}_1, \cdots, \vec{v}_r\}$.

Def (6.1.7) Diagonalizable

$$A_{n \times n} = PDP^{-1}$$

where

$D = \begin{bmatrix} \ddots & & 0 \\ & \lambda_k & \\ 0 & & \ddots \end{bmatrix}_{n \times n}$ and $P = \begin{bmatrix} | & & | & & | \\ \cdots & & \vec{v}_k & & \cdots \\ | & & | & & | \end{bmatrix}_{n \times n}$

k^{th} diagonal entry $\leftarrow$ $k = 1, 2, \cdots, n$ $\rightarrow$ k^{th} column

$A = PDP^{-1} \overset{\text{nonsingular } P}{\Longleftrightarrow} AP = PD$

To verify $AP = PD$ for an invertible P.

$\underset{}{A} \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}}_{P} \overset{?}{=} \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}_{n \times n}}_{P} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}_{n \times n}}_{D}$

$\begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \end{bmatrix}_{n \times n}$

△ **Thm (6.1.8)**

$A_{n \times n}$ diagonalizable $\Longleftrightarrow$ A has n linearly independent eigenvectors

Thm (6.1.9)

Distinct eigenvalues $\Rightarrow$ Linearly independent eigenvectors.

Corollary $\Uparrow$ $\quad \Downarrow$ n linearly independent eigenvectors.

If $A_{n \times n}$ has n distinct eigenvalues, then A is diagonalizable.

Question True or False?

$\lambda_1 = 5, \ \vec{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}; \quad \lambda_2 = -2, \ \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

① ✓ $A = \begin{pmatrix} 3 & -2 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 5 & \\ & -2 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 4 & 2 \end{pmatrix}^{-1}$?

② ✓ $A = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} -2 & \\ & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}^{-1}$?

③ ✗ $A = \begin{pmatrix} 3 & -2 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 5 & \\ & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 4 & 2 \end{pmatrix}^{-1}$?

④ ✗ $A = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}^{-1}$?

Check $A = PDP^{-1}$ or $AP = PD$.

↳ Such diagonalization is NOT unique.

Recall Example:

$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

has $\lambda_1 = 5, \quad \lambda_2 = -2$

$\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Exercise Diagonalizable or not?

$A = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 7 & 9 \\ 0 & -2 & 3 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & & \\ * & 1 & \\ * & * & \ddots \ n \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

△ Diagonalizable Matrix with < n Distinct Eigenvalues

E.g. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \lambda_1 = \lambda_2 = 2$

For $\lambda = 2$, $\vec{0} = (A - 2I)\vec{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{u} \rightarrow E_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$

Let

$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow A = PDP^{-1}$

Since there are 2 linearly independent eigenvectors, A is diagonalizable.

Recall: algebraic multiplicity geometric multiplicity

E.g. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \det(\lambda I - A) = (\lambda - 2)^2 = 0 \rightarrow$ (corresponding eigenvectors) $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

almu $(\lambda = 2) = 2$ $\quad$ gemu $(\lambda = 2) = 2$

$B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \rightarrow \det(\lambda I - B) = (\lambda - 2)^2 = 0 \rightarrow (2I - B)\vec{v} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \vec{v} = \vec{0}$

almu $(\lambda = 2) = 2$ $\quad$ $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, gemu $(\lambda = 2) = 1$.

Remark:

The algebraic multiplicity and the geometric multiplicity of an eigenvalue are not always equal.

Thm

For each i, $almu(\lambda_i) = gemu(\lambda_i) \iff A_{n\times n}$ is diagonalizable.

$Tr(A) := \sum_{i=1}^{n} a_{ii}$

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $Tr(A) = 15$

△ Sum / Product of Eigenvalues

Thm

$\lambda = 0$, $\det(-A) = (-1)^n|A| = C_0$

Suppose $p(\lambda) = \det(\lambda I_n - A) = \lambda^n + C_{n-1}\lambda^{n-1} + \cdots + C_1\lambda + C_0$.

Then $C_{n-1} = -Tr(A)$ and $C_0 = (-1)^n \det(A)$

proof: Take $\lambda = 0$ in $p(\lambda)$

$\det(-A) = C_0 = (-1)^n \det(A)$.

As for $C_{n-1} = -Tr(A)$, use the determinant of $\lambda I_n - A$.

Hence if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of A, then

$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$
$= \lambda^n + (-\Sigma\lambda_i)\lambda^{n-1} + \cdots + (-1)^n \lambda_1 \cdots \lambda_n$

Comparing to Thm 6.1.6 we have

$\sum_{i=1}^{n} \lambda_i = Tr(A)$, $\prod_{i=1}^{n} \lambda_i = \det(A)$.

Exercise

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. $\begin{cases} \lambda_1 + \lambda_2 = 2+2 \\ \lambda_1\lambda_2 = 4-1 \end{cases} \to \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 3 \end{cases}$

$B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$. $\begin{cases} \lambda_1 = 2 \\ 2 + \lambda_2 + \lambda_3 = 8 \\ 2\lambda_2\lambda_3 = 2\cdot(3^2 - 1^2) = 16 \end{cases} \to \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 2 \\ \lambda_3 = 4 \end{cases}$

$\begin{cases} \lambda_2 + \lambda_3 = 6 \\ \lambda_2\lambda_3 = 8 \end{cases}$

E.g. $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

Eigenvalues: $\det(A - \lambda I_n) = (2-\lambda)[(3-\lambda)^2 - 1] = (2-\lambda)(3-\lambda-1)(3-\lambda+1) = 0$.

$\lambda_1 = \lambda_2 = 2$, $\lambda_3 = 4$ $almu(2) = 2$, $almu(4) = 1$

Eigenvectors:

For $\lambda_1 = 2$, $E_2 = N(A - 2I_3) = span\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$, $gemu(2) = 2$

$\vec{0} = (A - 2I_3)\vec{v} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}\vec{v} \to \begin{cases} -x_1 + x_2 + x_3 = 0 \\ x_2 \in \mathbb{R} \\ x_3 \in \mathbb{R} \end{cases} \to \vec{v} = \begin{pmatrix} x_2 + x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}x_2 + \begin{pmatrix}1\\0\\1\end{pmatrix}x_3$

For $\lambda_3 = 4$, $E_4 = N(A - 4I_3) = span\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$, $gemu(4) = 1$

$\vec{0} = (A - 4I)\vec{v} = \begin{pmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}\vec{v} \to \begin{cases} x_1 = 0 \\ -x_2 + x_3 = 0 \\ x_3 \in \mathbb{R} \end{cases} \to \vec{v} = \begin{pmatrix} 0 \\ x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}x_3$

Diagonalize A:

Let $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ ($\vec{v}_1$ $\vec{v}_2$ $\vec{v}_3$) $\vec{v}_1, \vec{v}_2 \in E_2$, linearly independent

or, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ ($\vec{v}_1$ $2\vec{v}_3$ $\vec{v}_1 + \vec{v}_2$) as long as P has n linearly independent cols.

$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ -1 & 3-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{vmatrix}$ Type Scaling

$= (2-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ -1 & 3-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{vmatrix}$ type III

$= (2-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix}$

$= (2-\lambda)\cdot 1 \cdot \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$

$= (2-\lambda)[(3-\lambda)^2 - 1] = 0$

$p(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \lambda - a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{n1} & & a_{nn} & \lambda - a_{nn} \end{vmatrix}$

$= \lambda^n + C_{n-1}\lambda^{n-1} + \cdots + C_0$

$C_0 = p(0) = \det(-A) = (-1)^n \det(A)$

$\lambda_1\lambda_2\cdots\lambda_n = \det(A)$

$p(\lambda) = (\lambda - \lambda_1)\cdots(\lambda - \lambda_n)$
$= \lambda^n + (-\Sigma\lambda_i)\lambda^{n-1} + \cdots + (-1)^n\lambda_1\lambda_2\cdots\lambda_n$

Conclusion on diagonalizable matrices:

$A_{n\times n}$ is diagonalizable if A has n distinct eigenvalues (See A)

$A_{n\times n}$ is diagonalizable $\iff$ A has n l.i. eigenvectors.
$almu(\lambda_i) = gemu(\lambda_i)$ for each i (See B)

△ Defective Matrix ☆☆

Def A square matrix is said to be defective if it is not diagonalizable.

The eigenspaces does not provide enough linearly independent eigenvectors. $\longrightarrow$ Enlarge E_λ to K_λ for more.

△ Generalized Eigenspace K_λ and Jordan Canonical Form

$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

upper bidiagonal matrix

Linear Algebra II by Chiu Fai WONG

Theorem 6.1.29 $m_i = almu(\lambda_i)$, $\forall i$

Let $A \in M_{n\times n}(C)$ be an $n \times n$ matrix in C and $\lambda_1, \lambda_2, \cdots, \lambda_k$ be distinct eigenvalues of A with corresponding multiplicities $m_1, m_2, \cdots m_k$. For $1 \le i \le k$, let β_i be an ordered basis for K_{λ_i}. Then

(a) $\beta_i \cap \beta_j = \varnothing$ for $i \ne j$. distinct eigenvalues $\Rightarrow$ linearly independent generalized eigenvector

(b) $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis for C^n.

(c) $K_{\lambda_i} = N\left((A - \lambda I_n)^{m_i}\right)$ and $\dim\left(K_{\lambda_i}\right) = m_i$ for all i. $\dim(K_{\lambda_i}) = almu(\lambda_i)$

Theorem 6.1.30

Let $A \in M_{n\times n}(C)$ and $v \in C^n$ be a generalized eigenvector of A corresponding to the eigenvalue λ. Suppose that p is the smallest positive integer such that $(A - \lambda I_n)^p(v) = \mathbf{0}$. Let

$\gamma = \left\{(A - \lambda I_n)^{p-1}(v), \cdots, (A - \lambda I_n)(v), v\right\}$. Then $[A]_\gamma = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}$. — elementary Jordan block

The above matrix is called is called Jordan block corresponding to eigenvalue λ

Hence $A = P \begin{pmatrix} J_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \end{pmatrix} P^{-1}$ where — Jordan matrix

$J_1 = \begin{pmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1 \end{pmatrix}$, $J_2 = \begin{pmatrix} \lambda_2 & 1 & & \\ & \lambda_2 & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_2 \end{pmatrix}$

△ Elementary Jordan Block

$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$

△ Jordan Form Matrix

$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}$

E.g. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ $\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ $\begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$ $\begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 0 & b \end{bmatrix}$

△ Check point for Jordan canonical form

Which of the following matrices are of Jordan canonical form?

A. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ B. $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ C. $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ D. $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

E. $\begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ F. $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ G. $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ H. $\begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

$$\text{Hence } A = P\begin{pmatrix} \boxed{\begin{matrix}\lambda_1 & \ddots & \\ & \ddots & 1 \\ & & \lambda_1\end{matrix}}_{J_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boxed{\begin{matrix}\lambda_2 & 1 & & \\ & \lambda_2 & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_2\end{matrix}}_{J_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \boxed{\begin{matrix}\lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k\end{matrix}}_{J_k} \end{pmatrix} P^{-1} \text{ where}$$

Jordan form ...

J_i contains eigenvalues of A on the diagonal

$$P = \big((A-\lambda_1 I_n)^{p_1-1}(\boldsymbol{v}_1), \cdots, (A-\lambda_1 I_n)(\boldsymbol{v}_1), \boldsymbol{v}_1, \cdots, (A-\lambda_k I_n)^{p_k-1}(\boldsymbol{v}_k), \cdots, (A-\lambda_k I_n)(\boldsymbol{v}_k), \boldsymbol{v}_k\big).$$

linearly independent generalized eigenvectors of A.

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E. $\begin{bmatrix} \frac12 & 1 & 0 \\ 0 & \frac12 & 1 \\ 0 & 0 & \frac12 \end{bmatrix}$ F. $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ G. $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ H. $\begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

I. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ J. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ K. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

Brain Storm :

1. Suppose a 3x3 matrix has $\lambda_{1,2,3} = a$, find all possible Jordan forms to this matrix.
2. What if it is a 4x4 matrix with $\lambda_{1,2,3,4} = a$?

1 block $\begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$

2 blocks $\left(\begin{array}{cc|c} a & 1 & 0 \\ 0 & a & 0 \\ \hline 0 & 0 & a \end{array}\right)$ $\begin{pmatrix} a & & \\ & a & 1 \\ & & a \end{pmatrix}$

3 blocks $\begin{pmatrix} a & & \\ & a & \\ & & a \end{pmatrix}$

Diagonalization & Eigenpairs

- Diagonalizable
  - Def $A_{n\times n} = PDP^{-1}$, where D .... and P ...
  - Thm $A_{n\times n}$ is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.
  - Case I: $A_{n\times n}$ has n distinct eigenvalues.
  - Case II: $A_{n\times n}$ has k (< n) distinct eigenvalues. $\text{almu}(\lambda_i) = \text{gemu}(\lambda_i)$, $\forall i$.
- Non-diagonalizable: $A_{n\times n}$ has k (< n) distinct eigenvalues. $\text{almu}(\lambda_i) > \text{gemu}(\lambda_i)$, for some i.
  - $A = CJC^{-1}$, where J is the Jordan canonical form and C contains generalized eigenvectors.

defective matrices

Lack of linearly independent (l.i.) eigenvectors!

Enlarge the eigenspaces to make a basis!

generalized eigenvectors

△ What is a "generalized eigenvector"?

Def Generalized Eigenvector

A nonzero vector $\vec{v}$ which satisfies $(A-\lambda I_n)^p \vec{v} = \vec{0}$ for some positive integer p is called a generalized eigenvector of A corresponding to λ.

△ Generalized Eigenspace

Let A be an nxn matrix and suppose λ is an eigenvalue of A with $\text{almu}(\lambda) = m$.
Then there exists some $p \le m$ s.t.

$$\dim(N(A-\lambda I_n)^p) = m$$

and we call $K_\lambda = N(A-\lambda I_n)^p$ as the generalized eigenspace of λ.

> Exercise:
> Use mathematical induction to show that
> $N(A-\lambda I)^k \subseteq N(A-\lambda I)^{k+1}$, for positive integer k.

λ is an eigenvalue of A

Eigenvector	nonzero vectors	Generalized Eigenvector
$(A-\lambda I)\vec{v} = \vec{0}$		$(A-\lambda I)^p \vec{v} = \vec{0}$ for some $p \in \mathbb{Z}^+$

$p = 1$

Eigenspace	Generalized Eigenspace
$E_\lambda = N(A-\lambda I_n)$	$K_\lambda = N(A-\lambda I)^p$
$\dim(E_\lambda) = \text{gemu}(\lambda)$	$\dim(K_\lambda) = \text{almu}(\lambda)$

Remark: Recall Thm $\text{gemu}(\lambda_i) \le \text{almu}(\lambda_i)$, $\forall i$.

E.g. $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ with $\lambda_{1,2,3} = 3$

For $\lambda = 3$,

$$E_3 = N(A-3I) = N\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

A is defective since $\text{almu}(3) = 3 > \text{gemu}(3) = 2$.

To find a generalized eigenvector $\vec{y}$:

Consider

$$N(A-3I)^2 = N\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{R}^3.$$

Take $\vec{y} \in N(A-3I)^2 \setminus N(A-3I)$.

Then $\vec{y} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ is a generalized eigenvector of A belonging to $\lambda = 3$.

Let $\vec{v} = (A-3I)\vec{y} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, then

$\{(A-3I)\vec{y}, \vec{y}\}$ is a chain of generalized eigenvectors.

Notice that $\vec{v} = (A-3I)\vec{y} \in N(A-3I)$ is an eigenvector w.r.t. $\lambda = 3$.

To form matrix J and C:

$$J = \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 3 & 0 \\ \hline 0 & 0 & 3 \end{array}\right] \quad \text{or} \quad J = \left[\begin{array}{c|cc} 3 & 0 & 0 \\ \hline 0 & 3 & 1 \\ 0 & 0 & 3 \end{array}\right]$$

\# of linearly independent eigenvectors
=
\# of Jordan blocks

$$C = [(A-3I)\vec{y} \mid \vec{y} \mid \vec{x}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad C = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & -1 \\ 1 & 0 & 3 \end{bmatrix}$$

$(A-3I)\vec{y}_2$ $\quad \vec{y}_2$

Use MATLAB to check $A = CJC^{-1}$.