# SPA

## Solution to Assignment 2

1.

   (a) We only show that $\mathcal{F}_1$ is a $\sigma$-field, since the result can be obtained in the same way for $\mathcal{F}_2$. Set $A = \{1\}$. Then $\{2,3\}$ is $A^c$ and it is obvious that $\mathcal{F}_1$ satisfies the first two conditions in the definition of a $\sigma-$field. As for $\cup_{i=1}^{\infty} A_i$, $A_i \in \mathcal{F}_1$, this union can only be $A$, $A^c$, $\emptyset$, $\Omega$, all of which belong to $\mathcal{F}_1$, verifying that $\mathcal{F}_1$ is a $\sigma-$field. Indeed, there are only the following 5 cases:

      i. $A_i = \Omega$ for some $i$. Then $\cup_{i=1}^{\infty} A_i = \Omega$;
      ii. $A_i = A$ and $A_j = A^c$ for some $i, j$. Then $\cup_{i=1}^{\infty} A_i = \Omega$;
      iii. (i) and (ii) don't hold and $A_i = A$ for some $i$. Then $\cup_{i=1}^{\infty} A_i = A$;
      iv. (i) and (ii) don't hold and $A_i = A^c$ for some $i$. Then $\cup_{i=1}^{\infty} A_i = A^c$;
      v. (i),(ii),(iii) and (iv) don't hold. In this case $A_i = \emptyset$ for all $i$. So $\cup_{i=1}^{\infty} A_i = \emptyset$.

   (b) $\{1,2\} \cap \{2,3\} = \{2\} \notin \Omega$, so $\mathcal{F}_1 \cap \mathcal{F}_2$ is not a $\sigma-$field.

2. We set $A = \{1,2,3\}$, $B = \{3,4,5\}$. So $A^c = \{4,5\}$ and $B^c = \{1,2\}$ and $A \cap B = \{3\}$. So $\sigma(\mathcal{U}) = \big\{\{1,2,3\}, \{3,4,5\}, \{4,5\}, \{1,2\}, \{3\}, \{1,2,4,5\}, \Omega, \emptyset\big\}$.

3.

   (a) For $a \in \mathbb{R}$ and $n \in \mathbb{N}$, we know that $(a, a+\frac{1}{n}] \in \mathcal{B}(\mathbb{R})$. So $\cap_{n=1}^{\infty}(a, a+\frac{1}{n}] = (a, a] = \emptyset$ is also in $\mathcal{B}(\mathbb{R})$.

   (b) For $c \in \mathbb{R}$, since $\{c\} = \cap_{n=1}^{\infty}(c - \frac{1}{n}, c]$, it follows that $\{c\}$ is also in $\mathcal{B}(\mathbb{R})$.

   (c) It holds that

$$
\begin{aligned}
(a,b) &= (a,b] \setminus \{b\} \in \mathcal{B}(\mathbb{R}) \\
[a,b] &= (a,b] \cup \{a\} \in \mathcal{B}(\mathbb{R}) \\
[a,b) &= \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b) \in \mathcal{B}(\mathbb{R}).
\end{aligned}
$$

4.

(a) Substitutubg $Y_{t-1} = \theta_0 + Y_{t-2} + e_{t-1}$ into $Y_t = \theta_0 + Y_{t-1} + e_t$ and repeating until we get $e_1$, we obtain $Y_t = t\theta_0 + e_t + e_{t-1} + ... + e_1$.

(b) It holds

$$E[Y_t] = E[t\theta_0 + e_t + e_{t-1} + ... + e_1] = t\theta_0.$$

(c) Suppose $k \geq 0$. The autocovariance function for $(Y_t)$ is

$$\begin{aligned}
Cov(Y_t, Y_{t-k}) &= Cov(t\theta_0 + e_t + e_{t-1} + \cdots + e_1, (t-k)\theta_0 + e_{t-k} + e_{t-k-1} + \cdots + e_1) \\
&= Cov(e_t + e_{t-1} + \cdots + e_1, e_{t-k} + e_{t-k-1} + \cdots + e_1) \\
&= Var(e_{t-k} + e_{t-k-1} + \cdots + e_1) \\
&= (t-k)\sigma_e^2.
\end{aligned}$$

Equivalently, we can write $Cov(Y_t, Y_s) = \min\{s, t\}, \quad s, t \geq 0.$

5. It holds

$$E[X_n] = E[\xi \cos(\lambda n) + \eta \sin(\lambda n] = 0,$$

$$\begin{aligned}
&Cov(X_n, X_m) = E[X_n X_m] = E[(\xi \cos(\lambda n) + \eta \sin(\lambda n))(\xi \cos(\lambda m) + \eta \sin(\lambda m))] \\
=&E[\xi^2 \cos(\lambda n)\cos(\lambda m) + \eta^2 \sin(\lambda n)\sin(\lambda m)] \\
=&\sigma^2(\cos(\lambda n)\cos(\lambda m) + \sin(\lambda n)\sin(\lambda m)) \\
=&\sigma^2 \cos(\lambda(n-m)).
\end{aligned}$$

6. Consider $t_1 < t_2 < \cdots < t_n$. Set

$$A = \begin{bmatrix} \cos(\theta t_1) & \sin(\theta t_1) \\ \cos(\theta t_2) & \sin(\theta t_2) \\ ... & ... \\ \cos(\theta t_n) & \sin(\theta t_n) \end{bmatrix}.$$

Then

$$(X_{t_1}, X_{t_2}, ..., X_{t_n})^\top = A \times \begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

Sine $(\xi, \eta)$ is a two-dimensional Gaussian vector, $(X_{t_1}, X_{t_2}, ..., X_{t_n})^\top$ is also a Gaussian vector with expectation $\mathbf{0}$ and covariance matrix $\Sigma$, where

$$
\begin{aligned}
\Sigma &= AA^\top \\
&= \begin{bmatrix} \cos(\theta t_1) & \sin(\theta t_1) \\ \cos(\theta t_2) & \sin(\theta t_2) \\ ... & ... \\ \cos(\theta t_n) & \sin(\theta t_n) \end{bmatrix} \times \begin{bmatrix} \cos(\theta t_1) & \cos(\theta t_2) & \cdots \\ \sin(\theta t_1) & \sin(\theta t_2) & \cdots \end{bmatrix} \\
&= \begin{bmatrix} \cos^2(\theta t_1) + \sin^2(\theta t_1) & \cos(\theta t_1)\cos(\theta t_2) + \sin(\theta t_1)\sin(\theta t_2) & \cdots \\ \cos(\theta t_2)\cos(\theta t_1) + \sin(\theta t_2)\sin(\theta t_1) & \cos^2(\theta t_2) + \sin^2(\theta t_2) & \cdots \\ \cdots & \cdots & \cdots \\ \cos(\theta t_n)\cos(\theta t_1) + \sin(\theta t_n)\sin(\theta t_1) & \cdots & \cos^2(\theta t_n) + \sin^2(\theta t_n) \end{bmatrix} \\
&= \begin{bmatrix} 1 & \cos(\theta(t_1 - t_2)) & \cdots \\ \cos(\theta(t_2 - t_1)) & 1 & \cdots \\ \cdots & \cdots & \cdots \\ \cos(\theta(t_n - t_1)) & \cdots & 1 \end{bmatrix}.
\end{aligned}
$$

So the finite-dimensional distributions of $\{X_t : t \in \mathbb{R}\}$ are all Gaussian. This indicates that $\{X_t : t \in \mathbb{R}\}$ is a Gaussian process.