Chapter 1 Matrices and System of Equations

Section 1.3 Matrix Arithmetic

Definition (Matrix, plural: matrices)
An array of mn numbers $a_{ij}, i = 1, \cdots, m; j = 1, \cdots, n$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called an $m \times n$ matrix, denoted by $A = (a_{ij})$, where A has m rows and n columns, and a_{ij} is called the (i, j)-entry of A in ith row and the jth column.

Definition (Row vector, column vector) A $1 \times n$ matrix is a *row vector*. An $m \times 1$ matrix a *column vector*.

Definition (Equal matrices) Two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be *equal* if $a_{ij} = b_{ij}$ for each i and j.

Remark Two equal matrices are of the same size.

Example $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 6 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Example $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & a \\ 4 & b & 6 \end{pmatrix}$ if and only if $a = 3$ and $b = 5$.

Definition (Augmented matrix) When an $m \times r$ matrix B attached to an $m \times n$ matrix A, the *augmented matrix* is an $m \times (n + r)$ matrix denoted by $(A|B)$, and

$$(A|B) = \left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & b_{11} & \cdots & b_{1r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{mn} & \cdots & a_{mn} & b_{m1} & \cdots & b_{mr} \end{array} \right).$$

Example If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}$ then $(A|A) = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \\ 5 & 6 & 5 & 6 \end{array} \right)$ and $(A|B) = \left(\begin{array}{cc|c} 1 & 2 & 10 \\ 3 & 4 & 20 \\ 5 & 6 & 30 \end{array} \right)$.

Definition (Square matrix) An $m \times n$ matrix A is called a *square matrix* of order n if $m = n$.

Example $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 0 & 1 \\ 3 & 5 & 7 \end{pmatrix}$ are square matrices.

$C = \begin{pmatrix} 1 & 2 \end{pmatrix}, D = \begin{pmatrix} 2 & 4 \\ 8 & 0 \\ 3 & 5 \end{pmatrix}$ are no square matrices.

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Definition (Diagonal / Off-diagonal) The entries $a_{11}, \cdots, a_{nn}$ of a square matrix are called *diagonal elements* while the entries $a_{ij}, i \neq j$ are called *off-diagonal elements*.

Example In the above example, the diagonal elements of A are $1, 4$. The diagonal elements of B are $2, 0, 7$.

Definition (Matrix Addition) If $A = (a_{ij})$ and $B = (b_{ij})$ are both $m \times n$ matrices, then *sum* $A + B$ is the $m \times n$ matrix whose (i, j) entry is $a_{ij} + b_{ij}$, i.e. $A + B = (a_{ij} + b_{ij})$.

Example Let

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 2 & 3 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & -4 \\ 8 & -1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 \\ 9 & -1/2 \end{pmatrix}$$

Then

$$A + B = \begin{pmatrix} 1+1 & (-1)+0 & 4+(-4) \\ 2+8 & 3+(-1) & 0+1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 10 & 2 & 1 \end{pmatrix}$$

Since the sizes don't match, none of these is defined: $A + C$, $C + A$, $B + C$, $C + B$.

Definition (Scalar Multiplication (Scalar product)) If $A = (a_{ij})$ is an $m \times n$ matrix and α is a scalar, then *scalar multiple* αA *of* A *by* α is the $m \times n$ matrix whose (i, j) entry is αa_{ij}.

Example $5 \begin{pmatrix} 3 & -1 \\ -2 & 5 \\ 0 & -9 \end{pmatrix} = \begin{pmatrix} 15 & -5 \\ -10 & 25 \\ 0 & -45 \end{pmatrix}.$

Definition (Matrix Multiplication) If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times r$ matrix, then the *product* $AB = C = (c_{ij})$ is the $m \times r$ matrix whose entries are defined by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}.$$

$$AB = \begin{pmatrix} & \vdots & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ & \vdots & & \end{pmatrix} \begin{pmatrix} & b_{1j} & \\ \dots & b_{2j} & \dots \\ & \vdots & \\ & b_{nj} & \end{pmatrix} = \begin{pmatrix} & \vdots & \\ \dots & c_{ij} & \dots \\ & \vdots & \end{pmatrix}.$$

Example Let $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 0 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

$$AB = \begin{pmatrix} 1(-2) + 2(3) & 1(0) + 2(1) & 1(1) + 2(1) \\ 0(-2) + (-1)(3) & 0(0) + (-1)(1) & 0(1) + (-1)(1) \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & -1 & -1 \end{pmatrix}.$$

Generally $AB \neq BA$, even the matrix multiplication is valid.

Example Compute $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

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Solution

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} (1)(0)+(0)(3) & (1)(0)+(0)(4) \\ (0)(0)+(0)(3) & (0)(0)+(0)(4) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (0)(1)+(0)(0) & (0)(0)+(0)(0) \\ (3)(1)+(4)(0) & (3)(0)+(4)(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}.$$

If $AB = AC$ for a nonzero matrix A, then it is not true in general that $B = C$.

For example, if we consider

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 5 & 6 \\ 0 & 0 \end{pmatrix},$$

then

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = AC$$

However $B \neq C$.

Definition (Transpose) The transpose of an $m \times n$ matrix $A = (a_{ij})$ is the $n \times m$ matrix $B = (b_{ij})$ defined by $b_{ji} = a_{ij}$, for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. The transpose of A is denoted by A^T.

In some books, the transpose of A is denoted by A^t or A'.

Example

$$\begin{pmatrix} 1 & 5 & 3 \\ 7 & 1 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 7 \\ 5 & 1 \\ 3 & 6 \end{pmatrix} \text{ and } \begin{pmatrix} -5 & -2 \\ 3 & 0 \end{pmatrix}^T = \begin{pmatrix} -5 & 3 \\ -2 & 0 \end{pmatrix}.$$

Example Let $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 0 & 1 \\ 3 & 1 & 1 \end{pmatrix}$. Then

$$B^T A^T = \begin{pmatrix} -2 & 3 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1(-2) + 2(3) & 0(-2) + (-1)(3) \\ 1(0) + 2(1) & 0(0) + (-1)(1) \\ 1(1) + 2(1) & 0(1) + (-1)(1) \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 2 & -1 \\ 3 & -1 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1(-2) + 2(3) & 1(0) + 2(1) & 1(1) + 2(1) \\ 0(-2) + (-1)(3) & 0(0) + (-1)(1) & 0(1) + (-1)(1) \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 2 & 3 \\ -3 & -1 & -1 \end{pmatrix}.$$

Theorem

Each of the following statements is valid for any scalars α and for any matrices A, B for which the indicated operations are defined.

$$\begin{aligned} (A^T)^T &= A \\ (\alpha A)^T &= \alpha A^T \\ (A+B)^T &= A^T + B^T \\ (AB)^T &= B^T A^T \end{aligned}$$

Definition (Symmetric) A square matrix $A = (a_{ij})$ is called a symmetric matrix if $A = A^T$.

Example

$\begin{pmatrix} 1 & 5 & 3 \\ 5 & 2 & 6 \\ 3 & 6 & 0 \end{pmatrix}$ is symmetric. $\begin{pmatrix} 1 & 5 & 3 \\ 5 & 2 & 6 \\ 0 & 6 & 0 \end{pmatrix}$ is not symmetric.

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Example Let A be a matrix and let $B = AA^T$. Show that B is symmetric.

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Solution [Idea: We have to show $B^T = B$.]

$$\begin{aligned} B^T &= (AA^T)^T \\ &= (A^T)^T A^T \\ &= AA^T \\ &= B \end{aligned}$$

Since $B^T = B$, B is symmetric.