Chapter 1 Matrices and System of Equations

Section 1.2 Row Echelon Form

Definition (Row Echelon Form) A matrix is said to be in *row echelon form* if

(i) the first nonzero entry in each nonzero row is 1;

$$\left(\begin{array}{cccccccccc|c} 1 & \star & \star & \star & & \star & \star & \star & \cdots & \star & \star \\ 0 & 0 & 1 & \star & & \star & \star & \star & \cdots & \star & \star \\ 0 & 0 & 0 & 0 & & 0 & 0 & 1 & \cdots & \star & \star \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & \cdots & 0 & \star \\ & & & & \cdots & & & & & & \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right)$$ where $\star$ could be any number

Definition (Row Echelon Form) A matrix is said to be in *row echelon form* if

(ii) if row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k;

$$\left(\begin{array}{cccccccccc|c}
1 & \star & \star & \star & & \star & \star & \star & \cdots & \star & \star \\
0 & 0 & 1 & \star & & \star & \star & \star & \cdots & \star & \star \\
0 & 0 & 0 & 0 & & 0 & 0 & 1 & \cdots & \star & \star \\
0 & 0 & 0 & 0 & & 0 & 0 & 0 & \cdots & 0 & \star \\
 & & & & \cdots & & & & & & \\
0 & 0 & 0 & 0 & & 0 & 0 & 0 & \cdots & 0 & 0
\end{array}\right) \quad \text{where } \star \text{ could be any number}$$

Definition (Row Echelon Form) A matrix is said to be in *row echelon form* if

(iii) if there are rows whose entries are all zero, they are below the rows having nonzero entries.

$$\left(\begin{array}{cccccccc|c} 1 & \star & \star & \star & \star & \star & \star & \cdots & \star & \star \\ 0 & 0 & 1 & \star & \star & \star & \star & \cdots & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \star \\ & & & & \cdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right)$$ where $\star$ could be any number

Definition (Row Echelon Form) A matrix is said to be in *row echelon form* if

(i) the first nonzero entry in each nonzero row is 1;

(ii) if row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k;

(iii) if there are rows whose entries are all zero, they are below the rows having nonzero entries.

$$\left(\begin{array}{cccccccccc|c} 1 & \star & \star & \star & & \star & \star & \star & \cdots & \star & \star \\ 0 & 0 & 1 & \star & & \star & \star & \star & \cdots & \star & \star \\ 0 & 0 & 0 & 0 & & 0 & 0 & 1 & \cdots & \star & \star \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & \cdots & 0 & \star \\ & & & & \cdots & & & & & & \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right)$$ where $\star$ could be any number

Example $\begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ are in row echelon form (ref).

$$\left(\begin{array}{ccccccccc|c} 1 & \star & \star & \star & \star & \star & \star & \cdots & \star & \star \\ 0 & 0 & 1 & \star & \star & \star & \star & \cdots & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \star \\ & & & & \cdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right)$$ where $\star$ could be any number

Extra Exercises

Determine the following matrices are in row echelon form or not:

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Extra Exercises

Determine the following matrices are in row echelon form or not:

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} Y & N & N \\ Y & N & N \end{pmatrix}$$

Gaussian Elimination The process of using elementary row operations I, II, and III to transform a linear system into one whose augmented matrix is in row echelon form is called *Gaussian Elimination*.

Example

Solve $\begin{cases} -x - y + 3z = 3 \\ x + z = 3 \\ 3x - y + 7z = 15 \end{cases}$ using Gaussian Elimination.

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Example

Solve $\left\{\begin{array}{rcrcrcr} -x & - & y & + & 3z & = & 3 \\ x & & & + & z & = & 3 \\ 3x & - & y & + & 7z & = & 15 \end{array}\right.$ using Gaussian Elimination.

Solution

$$\left(\begin{array}{ccc|c} -1 & -1 & 3 & 3 \\ 1 & 0 & 1 & 3 \\ 3 & -1 & 7 & 15 \end{array}\right) \xrightarrow[3R_1+R_3\to R_3]{R_1+R_2\to R_2} \left(\begin{array}{ccc|c} -1 & -1 & 3 & 3 \\ 0 & -1 & 4 & 6 \\ 0 & -4 & 16 & 24 \end{array}\right)$$

$$\xrightarrow{-4R_2+R_3\to R_3} \left(\begin{array}{ccc|c} -1 & -1 & 3 & 3 \\ 0 & -1 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[-R_1\to R_1]{-R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -3 \\ 0 & 1 & -4 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & -3 \\ 0 & 1 & -4 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left\{\begin{array}{cccccr} x & +y & -3z & = & -3 & (1) \\ & y & -4z & = & -6 & (2) \\ & & 0 & = & 0 & (3) \end{array}\right.$$

(3) is always true.
From (2), $y = -6 + 4z$.
Substitute $y = -6 + 4z$ into (1), $x + (-6 + 4z) - 3z = -3$ and so $x = 3 - z$.
So, the solution are of the form $(3 - \alpha, -6 + 4\alpha, \alpha)$, where α is any real number.

For example, when $\alpha = 1$, $(3 - \alpha, -6 + 4\alpha, \alpha) = (2, -2, 1)$ and $(2, -2, 1)$ is a solution of the system. When $\alpha = 2$, $(3 - \alpha, -6 + 4\alpha, \alpha) = (1, 2, 2)$ and $(1, 2, 2)$ is a solution of the system.

Definition (Leading variable) In each row of a row echelon form, the first variable with a nonzero coefficient is the rows *leading variable*.

Or, in a row echelon form, the variable corresponding to *leading 1's* is the *leading variable*.

In the above example, x and y are leading variables.

Definition (Free variable) In a row echelon form, the variables that are not leading are *free variables*.

In the above example, z is the free variable.

Example

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 1 & 8 \\ 2 & 3 & 2 & 13 \end{array}\right) \xrightarrow[-2R_1+R_3\to R_3]{-R_1+R_2\to R_2} \xrightarrow{-R_2+R_3\to R_3} \xrightarrow{-R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Last equation gives $0x_1 + 0x_2 + 0x_3 = 1$. Since it is always false that $0 = 1$, this system is inconsistent and has no solution.

Rule If the row echelon form of the augmented matrix contains a row of the form $\left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right)$, then the system is inconsistent and has no solution.

Reduced Row Echelon Form A matrix is said to be in *reduced row echelon form* if
(i) the matrix is in row echelon form, and
(ii) the first nonzero entry in each row is the only nonzero entry in its column.

Example The following matrices are in reduced row echelon form.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example The blue entries make the row echelon forms not reduced.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Gauss-Jordan reduction The process of using elementary row operations to transform a matrix into reduced row echelon form is called *Gauss-Jordan reduction*.

Example Solve $\left\{\begin{array}{rcrcrcrcr} x_1 & - & x_2 & & & - & 2x_4 & = & 2 \\ x_1 & + & x_2 & + & 3x_3 & + & x_4 & = & 1 \\ & - & x_2 & + & x_3 & - & x_4 & = & 0 \end{array}\right.$ using Gauss-Jordan reduction.

Solution We first reduce the matrix to its row echelon form.

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 1 & 1 & 3 & 1 & 1 \\ 0 & -1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{-1R_1+R_2\to R_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 2 & 3 & 3 & 1 \\ 0 & -1 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{(1/2)R_2+R_3\to R_3} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 2 & 3 & 3 & 1 \\ 0 & 0 & \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right) \xrightarrow[(2/5)R_3\to R_3]{(1/2)R_2\to R_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 1 & \frac{3}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 1 & 3/2 & 3/2 & -1/2 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right)$$

(Eliminate x_3 in Row 1 and Row 2 by Row 3)

$$\xrightarrow{-(3/2)R_3+R_2\to R_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 1 & 0 & 6/5 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right)$$

(Eliminate x_2 in Row 1 by Row 2)

$$\xrightarrow{R_2+R_1\to R_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -4/5 & 9/5 \\ 0 & 1 & 0 & 6/5 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right)$$

x_4 is the free variable. The solutions set are of the form $(9/5 + (4/5)\alpha, -1/5 - (6/5)\alpha, -1/5 - (1/5)\alpha, \alpha)$, where α is a real number.

For a linear system of m equations and n unknowns, the number of solutions can be summarized as follows.

	$m > n$	$m = n$	$m < n$
no solution	most likely	possible	little chance
unique solution	little chance	most likely	impossible
infinity many solutions	little chance	possible	most likely

Definition (Overdetermined System) A system of m linear equations in n unknowns is *overdetermined* if there are more equations than unknowns ($m > n$).

Definition (Underdetermined System) A system of m linear equations in n unknowns is *underdetermined* if there are fewer equations than unknowns ($m < n$).

Definition (Homogeneous Systems) A system of linear equations is said to be *homogeneous* if the constants on the righthand side are all zero.

Example

This system

$$\begin{cases} x - y \qquad - 2w = 0 \\ x + y + 3z + w = 0 \\ \quad - y + z - w = 0 \end{cases}$$

is homogeneous.

Property of homogeneous systems Any homogeneous system is consistent because $(0, 0, \cdots, 0)$ is a solution of it. This solution $(0, 0, \cdots, 0)$ is said to be trivial.

Existence and Uniqueness

1. Is the system consistent? (i.e. Does a solution **exist**?)
2. If a solution exists, is it **unique**?
3. What is the solution?

No Solution

If the **row echelon form of the augmented matrix** contains a row of the form

$$\left(\begin{array}{ccc|c} 0 & \cdots & 0 & k \end{array}\right)$$

for $\boxed{k \neq 0}$, the system is **inconsistent**. Otherwise, it is consistent.

Example Consider

$$\begin{cases} x_1 & +2x_2 & -2x_3 & = 5 \\ & x_2 & -x_3 & = 2 \\ & x_2 & -x_3 & = 5 \end{cases}$$

Then its augmented matrix

$$(A|\mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 2 & -2 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

$0x_3 = -3$ is never true! The original system is inconsistent!

Unique

If the system is consistent and the nonzero rows of the **row echelon form** of the matrix form a strictly triangular system

$$\left(\begin{array}{cccc|c} \boxed{1} & * & \cdots & * & * \\ 0 & \boxed{1} & \ddots & \vdots & * \\ \vdots & \ddots & \ddots & * & \vdots \\ 0 & \cdots & 0 & \boxed{1} & * \end{array}\right)$$

the system will have a **unique solution** (**Consistent + No free variables**).

Infinitely Many

The system has **infinitely many solutions**:

$$\left(\begin{array}{cccc|c} 1 & * & \cdots & * & * \\ 0 & 1 & \ddots & \vdots & * \\ \vdots & \ddots & \ddots & * & \vdots \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right)$$

when the last row of the **row echelon form** must be all zeros (**Consistent + At least 1 free variable**).

Extra exercises

(a). Determine if the following matrices are in ref? rref? or neither?

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right), \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 4 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right), \left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 4 \\ 0 & 1 & 3 & 0 & 1 \end{array}\right), \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{array}\right)$$

Extra exercises

(b). Overdetermined? Underdetermined? Neither?

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right), \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 4 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right), \left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 4 \\ 0 & 1 & 3 & 0 & 1 \end{array}\right), \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{array}\right)$$

Extra exercises

(c). Based on the following augmented matrices, determine if the corresponding linear systems are consistent or not.

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right), \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 4 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right), \left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 4 \\ 0 & 1 & 3 & 0 & 1 \end{array}\right), \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{array}\right)$$

Extra exercises

(d). If consistent, how many solutions?

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right), \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 4 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right), \left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 4 \\ 0 & 1 & 3 & 0 & 1 \end{array}\right), \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{array}\right)$$