Chapter 2 Determinants

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Section 2.2 Properties of Determinants

Type I Operation: Two rows of A are interchanged.

Proposition Let A and E be $n \times n$ matrices. If E is a Type I elementary matrix, then $\det(EA) = \det(E)\det(A)$ where $\det(E) = -1$.

Idea of Proof Use mathematical induction.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} = -a_{21}\begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix} + a_{22}\begin{vmatrix} a_{31} & a_{33} \\ a_{11} & a_{13} \end{vmatrix} - a_{23}\begin{vmatrix} a_{31} & a_{32} \\ a_{11} & a_{12} \end{vmatrix}$$

$$= -a_{21}\left(-\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}\right) + a_{22}\left(-\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}\right) - a_{23}\left(-\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}\right)$$

by induction assumption

Type II Operation: A row of A is multiplied by a nonzero constant α.

Proposition Let A be an $n \times n$ matrix. Let E denote the elementary matrix of Type II formed from I_n by multiplying the ith row by the nonzero constant α, then $\det(EA) = \det(E)\det(A)$ where $\det(E) = \alpha$.

Proof Expanding $\det(EA)$ by cofactors along the ith row given

$$\det(EA) = \alpha a_{i1}A_{i1} + \alpha a_{i2}A_{i2} + \cdots + \alpha a_{in}A_{in} = \alpha \det(A).$$

In particular, when $A = I$,

$$\det(E) = \det(EI) = \alpha \det(I) = \alpha$$

and hence,

$$\det(EA) = \alpha \det(A) = \det(E)\det(A).$$

Type III Operation: A multiple of one row is added to another row.

Proposition Let A and E be $n \times n$ matrices. If E is a Type III elementary matrix, then $\det(EA) = \det(E)\det(A)$ where $\det(E) = 1$.

Proof Let E be the elementary matrix of type III formed from I by adding c times the ith row to the jth row. Since E is triangular and its diagonal elements are all 1, it follows that $\det(E) = 1$. If $\det(EA)$ is expanded by cofactors along the jth row,

$$\begin{aligned}
\det(EA) =& (a_{j1} + ca_{i1})A_{j1} + (a_{j2} + ca_{i2})A_{j2} + \cdots + (a_{jn} + ca_{in})A_{jn} \\
=& (a_{j1}A_{j1} + \cdots + a_{jn}A_{jn}) + c(a_{i1}A_{j1} + \cdots + a_{in}A_{jn}) \\
=& \det(A) + c(0) \qquad\qquad \text{by the lemma on the next slide} \\
=& \det(A)
\end{aligned}$$

Thus,

$$\det(EA) = \det(A) = \det(E)\det(A).$$

Lemma Let A be an $n \times n$ matrix. If A_{jk} denotes the cofactor of a_{jk} for $k = 1, \cdots, n$, then

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A), & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \qquad (\sharp)$$

Proof If $i = j$, $(\sharp)$ is just the cofactor expansion of $\det(A)$ along the ith row of A.
When $i \neq j$, let A^* be the matrix obtained by replacing the jth row of A by the ith row of A. Since two rows of A^* are the same, its determinant is zero. It follows from the cofactor expansion of $\det(A^*)$ along the jth row that

$$\begin{aligned} 0 = \det(A^*) &= a_{i1}A^*_{j1} + a_{i2}A^*_{j2} + \cdots + a_{in}A^*_{jn} \\ &= a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} \end{aligned}$$

Summary If E is an elementary matrix, then $\det(EA) = \det(E)\det(A)$ where

$$\det(E) = \begin{cases} -1, & \text{if } E \text{ is of type I} \\ \alpha \neq 0, & \text{if } E \text{ is of type II} \\ 1, & \text{if } E \text{ is of type III} \end{cases}$$

Similar results hold for column operations:

Property If E is an elementary matrix, then $\det(AE) = \det(A)\det(E)$.

Proof If E is an elementary matrix, then E^T is also an elementary matrix. Then

$$\begin{aligned} \det(AE) &= \det\left((AE)^T\right) \\ &= \det\left(E^T A^T\right) \\ &= \det\left(E^T\right)\det\left(A^T\right) \\ &= \det(E)\det(A) \end{aligned}$$

Summary

The effects of row or column operations have on the value of the determinant:

1. Interchanging two rows or columns of a matrix changes the sign of the determinant.
2. Multiplying a single row or column of a matrix by a scalar has the effect of multiplying the value of the determinant by that scalar.
3. Adding a multiple of one row or column to another does not change the value of the determinant.

Example Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find $\det(A)$ and $\det(3A)$.

$$\det(3A) = \det\left(3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = \det\begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} = 3\det\begin{pmatrix} 1 & 2 \\ 9 & 12 \end{pmatrix}$$
$$= 3 \cdot 3\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 3^2(-2) = -18,$$

which is $3^2\det(A)$.

Example
$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 8 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -6 \\ 0 & -6 & -12 \end{vmatrix} \qquad (-4R_1 + R_2 \to R_2, -7R_1 + R_3 \to R_3)$$
$$= (-1)\begin{vmatrix} 1 & 2 & 3 \\ 0 & -6 & -12 \\ 0 & 0 & -6 \end{vmatrix} \qquad (R_2 \leftrightarrow R_3)$$
$$= -36$$

Example Re-visit

Example Compute $\det(C)$ for $C = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix}$.

Extra Exercises*

1. $$\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix}$$

2. $$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

<u>Answer:</u> 30 and -1. Simplify the problem by elementary row/col operations first: changing it into a triangular matrix.

Example

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$\begin{aligned}
\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix} \quad -C_1+C_2 \to C_2, \ -C_1+C_3 \to C_3 \\
&= \begin{vmatrix} y-x & z-x \\ y^2-x^2 & z^2-x^2 \end{vmatrix} \quad \text{expand along the first row} \\
&= \begin{vmatrix} y-x & z-x \\ (y-x)(y+x) & (z-x)(z+x) \end{vmatrix} \\
&= (y-x)(z-x)\begin{vmatrix} 1 & 1 \\ y+x & z+x \end{vmatrix} \\
&= (y-x)(z-x)(z-y)
\end{aligned}$$

More Properties of Determinant

Theorem (Re-visit) Let A be an $n \times n$ matrix,

(i) If A has a row or column consisting entirely of zeros, then $\det(A) = 0$.

(ii) If A has two identical rows or two identical columns, then $\det(A) = 0$.

Exercises: Prove the theorem.

Tips: (i) Expand along the row or column that **contains the most zeros**.
(ii) Use Type I row operation.

Theorem (iii) A determinant can be expressed as the sum of two determinants by expressing every element in any row (or column) as the sum of two terms. For example,

$$\det\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$=\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}+\det\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Extra Exercises*

Factorize

1. $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$
2. $$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$
3. $$\begin{vmatrix} a+b & b+c & c+a \\ a^2+b^2 & b^2+c^2 & c^2+a^2 \\ a^3+b^3 & b^3+c^3 & c^3+a^3 \end{vmatrix}$$

Answer: (a) $(b-a)(c-a)(c-b)$. (b) $(a+b+c)(a^2+b^2+c^2-ac-ab-bc)$. (c) $2abc(b-a)(c-a)(c-b)$

Theorem A matrix A is singular if and only if $\det(A) = 0$.

Proof Let U be the reduced row echelon form of A. Then there exists a sequence of elementary matrices E_i's, such that $U = E_k E_{k-1} \cdots E_1 A$.
It follows that

$$\begin{aligned} \det(U) = \det(E_k E_{k-1} \cdots E_1 A) &= \det(E_k) \det(E_{k-1} \cdots E_1 A) \\ &\vdots \\ &= \det(E_k) \det(E_{k-1}) \cdots \det(E_1) \det(A) \end{aligned}$$

Since $\det(E_i) \neq 0$, then $\det(A) = 0$ if and only if $\det(U) = 0$.

- If A is singular, then the last row of U must be 0, and hence $\det(U) = 0$.
- If A is nonsingular, then $U = I$ and $\det(U) = \det(I) = 1$, which implies $\det(A) \neq 0$.

From now on, we have one more equivalent condition for nonsingularity.

Theorem (Equivalent Conditions for Nonsingularity)
Let A be an $n \times n$ matrix. The following are equivalent:
(a) A is nonsingular;
(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$;
(c) A is row equivalent to I. (I is the reduced row echelon form of A. A can be written as a product of elementary matrices.)
(d) The system $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b} \in \mathbf{R}^m$,
(e) $\det A \neq 0$.

Theorem If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

Proof when B is nonsingular If B is nonsingular, B can be written as a product of elementary matrices, i.e. $B = E_k E_{k-1} \cdots E_1$ where E_i are elementary. Thus,

$$\begin{aligned}
\det(AB) &= \det(AE_k E_{k-1} \cdots E_1) \\
&= \det(A)\det(E_k)\det(E_{k-1}) \cdots \det(E_1) \\
&= \det(A)\det(E_k E_{k-1} \cdots E_1) \\
&= \det(A)\det(B)
\end{aligned}$$

Proof when B is singular Exercise

Theorem Let $\left(\begin{array}{c|c} A_{k\times k} & B_{k\times l} \\ \hline C_{l\times k} & D_{l\times l}\end{array}\right)$ be a $(k+l)\times(k+l)$ block matrix. If $C_{l\times k} = O_{l\times k}$, then

$$\det\left(\begin{array}{c|c} A_{k\times k} & B_{k\times l} \\ \hline O_{l\times k} & D_{l\times l}\end{array}\right) = \det(A)\det(D).$$

Similarly, if $B_{k\times l} = O_{k\times l}$, then

$$\det\left(\begin{array}{c|c} A_{k\times k} & O_{k\times l} \\ \hline C_{l\times k} & D_{l\times l}\end{array}\right) = \det(A)\det(D).$$

Proof

$$\begin{aligned}
\det\left(\begin{array}{c|c} A_{k\times k} & B_{k\times l} \\ \hline O_{l\times k} & D_{l\times l}\end{array}\right) &= \det\left(\left[\begin{array}{c|c} I_{k\times k} & O_{k\times l} \\ \hline O_{l\times k} & D_{l\times l}\end{array}\right]\left[\begin{array}{c|c} A_{k\times k} & B_{k\times l} \\ \hline O_{l\times k} & I_{l\times l}\end{array}\right]\right) \\
&= \det\left(\begin{array}{c|c} I_{k\times k} & O_{k\times l} \\ \hline O_{l\times k} & D_{l\times l}\end{array}\right)\det\left(\begin{array}{c|c} A_{k\times k} & B_{k\times l} \\ \hline O_{l\times k} & I_{l\times l}\end{array}\right) \\
&= \det(A)\det(D)
\end{aligned}$$

The proof when $B_{k\times l} = O_{k\times l}$ is similar.