2023-24 First Semester

MATH2023 Ordinary and Partial Differential Equations (1002)

Assignment 3 Suggested Solution

1. (a) **Characteristic Eqn**: $r^2 + 3r = 0 \quad \to \quad r_1 = 0,\ r_2 = -3$

   General solution:
$$y = C_1 + C_2 e^{-3t}, \quad C_1, C_2 \in \mathbb{R}.$$

   Set in ICs:
$$\begin{cases} -2 = y(0): & C_1 + C_2 = -2, \\ 3 = y'(0): & -3C_2 = 3. \end{cases} \to \begin{cases} C_1 = -1 \\ C_2 = -1 \end{cases}.$$

   Solution to IVP:
$$y = -1 - e^{-3t}$$

   (b) **Characteristic Eqn**: $r^2 + 2r - 3 = 0 \quad \to \quad r_1 = 1, r_2 = -3$

   General solution:
$$y = C_1 e^t + C_2 e^{-3t}, \quad C_1, C_2 \in \mathbb{R}.$$

   Solution to IVP:
$$y = \frac{1}{4}e^t - \frac{1}{4}e^{-3t}$$

   (c) **Characteristic Eqn**: $r^2 - 10r + 24 = 0 \quad \to \quad r_1 = 4, r_2 = 6$

   General solution:
$$y = C_1 e^{4t} + C_2 e^{6t}, \quad C_1, C_2 \in \mathbb{R}.$$

   Solution to IVP:
$$y = (3\alpha - \frac{1}{2}\beta)e^{4t} + (-2\alpha + \frac{1}{2}\beta)e^{6t}$$

2. (a) Substitute $y_1(t) = 1$ into the equation,
$$yy'' + (y')^2 = 0 = RHS$$

   Substitute $y_2(t) = t^{1/2}$ into the equation and
$$\begin{aligned} yy'' + (y')^2 &= t^{1/2}\left(-\frac{1}{4}t^{-3/2}\right) + \left(\frac{1}{2}t^{-1/2}\right)^2 \\ &= 0 \end{aligned}$$

   Thus, $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are both solutions to the differential equation.

   (b) Substitute $y(t) = c_1 + c_2 t^{1/2}$ into the equation and
$$\begin{aligned} yy'' + (y')^2 &= \left(c_1 + c_2 t^{1/2}\right)\left(-\frac{1}{4}c_2 t^{-3/2}\right) + \left(\frac{1}{2}c_2 t^{-1/2}\right)^2 \\ &= -\frac{1}{4}c_1 c_2 t^{-3/2} \\ &\neq 0 \\ &\neq RHS \end{aligned}$$

Hence, in general, $y = c_1 + c_2 t^{1/2}$ is not a solution of this equation.

(c) The principle of superposition is not contradicted, since the equation is not linear.

3. (a) Let $v = y'$, then $\frac{\mathrm{d}v}{\mathrm{d}t} = y''$, the given DE becomes

$$v' + tv^2 = 0 \quad \to \quad \int \frac{1}{v^2}\mathrm{d}v = \int -t\mathrm{d}t \quad \to v(t) = \frac{2}{c+t^2}, \ v \neq 0, \ c \in \mathbb{R}.$$

Then $y' = v = 2/(c+t^2)$, integrate once w.r.t. t, we have

$$y(t) = \begin{cases} \frac{2}{k}\arctan(t/k) + c_1, & \text{when } c = k^2 > 0 \\ -\frac{2}{t} + c_1, & \text{when } c = 0 \\ \frac{1}{k}\ln\left|\frac{t-k}{t+k}\right| + c_1, & \text{when } c = -k^2 < 0 \\ c_2, & \text{for } v \equiv 0 \end{cases}, \qquad c_1, c_2 \in \mathbb{R}, k > 0.$$

(b) Let $v = y'$, then $\frac{\mathrm{d}v}{\mathrm{d}t} = y''$,

$$v' + v = e^{-t}$$

Solve the first order linear DE, we have

$$v(t) = e^{-t}(t + C), \qquad C \in \mathbb{R}$$

Substitute back $y' = v$ and integrate w.r.t. t,

$$y(t) = \int te^{-t}\mathrm{d}t + C\int e^{-t}\mathrm{d}t + C_1 = -te^{-t} + \int e^{-t}\mathrm{d}t + C\int e^{-t}\mathrm{d}t + C_1$$

$$y(t) = -(t + C + 1)e^{-t} + C_1, \qquad C_1, C \in \mathbb{R}$$

4. Proof: Assume the equation is exact then

$$P(x)y'' + Q(x)y' + R(x)y = [P(x)y']' + [f(x)y]' = 0,$$

where $f(x)$ is to be determined. Expansion yields

$$\begin{aligned} & P'(x)y' + P(x)y'' + f'(x)y + f(x)y' &= 0 \\ \to \quad & P(x)y'' + (P'(x) + f(x))y' + f'(x)y &= 0 \end{aligned}$$

i.e., $\begin{cases} P'(x) + f(x) = Q(x) & (1) \\ f'(x) = R(x) & (2) \end{cases}$

Take derivative on (1) and substitute $f'(x) = Q' - P''$ into (2), we have

$$P''(x) + R(x) - Q'(x) = 0$$

(a) We have $P''(x) - Q'(x) + R(x) = 2 - 1 - 1 = 0. \quad \rightarrow \quad$ It is exact.
Hence $f(x) = Q(x) - P'(x) = x - 2x = -x$ and

$$[x^2y']' - [xy]' = 0$$

Integrate on both side:

$$\begin{aligned} x^2y' - xy &= C, \quad C \in \mathbb{R}, \\ \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} &= \frac{C}{x^2}, \end{aligned}$$

Take $\mu(x) = \exp\left(\int -\frac{1}{x}\mathrm{d}x\right) = \frac{1}{x}$, then

$$\begin{aligned} \left[\frac{1}{x}y\right]' &= \frac{C}{x^3} \\ y &= \frac{C_1}{x} + C_2x, \qquad C_{1,2} \in \mathbb{R}. \end{aligned}$$

(b) $P''(x) - Q'(x) + R(x) = 0 - 4x + x = -3x \neq 0 \quad \rightarrow \quad$ Not exact.

5. Standard Form:

$$y'' + \frac{1}{x}y' + \frac{x^2 - v^2}{x^2}y = 0, \quad x \neq 0.$$

By Abel's theorem,

$$W(y_1, y_2) = Ce^{-\int \frac{1}{x}\mathrm{d}x} = C/x$$

Since $W(y_1, y_2)(1) = C/1 = 1 \rightarrow \quad C = 1$. Hence the Wronskian is

$$W(y_1, y_2) = 1/x, \quad x \neq 0.$$