2021-22 First Semester
MATH1083 Calculus II (1002)

Assignment 1

Due Date: 21/Feb/2021(Tue), before 11:30am.

- Write down your **Chinese name** and **student number**. Write neatly on **A4-sized** paper and **show your steps**.
- **Late submissions or answers without details will not be graded.**

1. What is the difference between a sequence and a series?

   Answer: A sequence is an ordered list of numbers whereas a series is the sum of a list of numbers.

2. Find the value of c such that

$$\sum_{n=0}^{\infty} e^{nc} = 10$$

   Solution:

$$\sum_{n=0}^{\infty} e^{nc} = \lim_{n\to\infty} \frac{1-e^{cn}}{1-e^{c}} = \frac{1}{1-e^{c}} = 10$$

   We can solve for c

$$c = \ln 0.9$$

3. Determine whether the following sequence converges or diverges. If it converges, find the limit.

   prove sequence converges: $\lim_{n\to\infty} a_n$ exists.

   (a)

$$a_n = (-1)^n \frac{e^n}{n!}$$

   (b)

$$b_n = \underline{n^{-\frac{1}{n}}}$$ ∞^0形式取ln

   Solution:(a)

$$\lim_{n\to\infty} a_n = 0$$

   (b) we can let $c_n = \ln(b_n) = -\frac{\ln n}{n}$, and $\lim_{n\to\infty} c_n = 0$,so

$$\lim_{n\to\infty} b_n = 1$$

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4. Show the sequence $\{a_n\}$ given by

$$a_1 = \sqrt{2}, \qquad a_{n+1} = \sqrt{2+a_n}$$

   is monotonic and bounded. Apply Monotonic Sequence Theorem to show

$$\lim_{n\to\infty} a_n$$

   exists, and find the limit.

   Solution: First we prove this sequence is bounded by 1 from below and 2 from above by mathematical induction:

$$1 < a_1 < 2,$$

   if $a_n < 2$, then

$$1 < a_{n+1} = \sqrt{2+a_n} < \sqrt{2+2} = 2,$$

   Second we prove that it is monotonically increasing: **[There are many different ways to prove this]**.

Since $1 < a_n < 2$, we can have $a_n - 2 < 0$ and $a_n + 1 > 0$, so $(a_n - 2)(a_n + 1) < 0$, expand the left, we can have

$$a_n^2 - a_n - 2 < 0$$

so

$$a_n^2 < a_n + 2,$$

$$a_n < \sqrt{2 + a_n} = a_{n+1}.$$

Then applying the Bounded Monotonic Sequence Theorem, we have the sequence $\{a_n\}$ converges.

5. Determine whether the following series converges or diverges.

$$\underline{\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}} \quad \infty^1 \xrightarrow{\text{change}} \infty^0$$

Solution: Use Limit Comparison Test and results from Q3b, let

$$a_n = \frac{1}{n^{1+1/n}}, \qquad b_n = \frac{1}{n}$$

ef w

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$$

therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$

diverges as $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

6. Show that if $a_n > 0$, and $\sum a_n$ is convergent, then

$$\sum_{n=1}^{\infty} \ln(1 + a_n)$$

is convergent. → 记 $\lim\limits_{n\to\infty} \frac{a_n}{\ln(1+a_n)}$

Proof: Since $\sum a_n$ is convergent, so we have

$$\lim_{n\to\infty} a_n = 0$$

Let

$$f(x) = \frac{x}{\ln(1+x)} \qquad \lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{x\to 0} x$$

and applying L'Hopital's rule, we have

$$\lim_{x\to 0} \frac{x}{\ln(1+x)} = \lim_{x\to 0} \frac{1}{\frac{1}{1+x}} = 1$$

Using **limit comparison test**

$$\lim_{n\to\infty} \frac{a_n}{\ln(1+a_n)} = 1$$

since $\sum a_n$ is convergent, it follows $\sum \ln(1 + a_n)$ converges.

4. Show the sequence $\{a_n\}$ given by

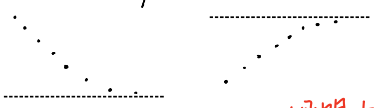
$$a_1 = \sqrt{2}, \qquad a_{n+1} = \sqrt{2 + a_n}$$

is monotonic and bounded. Apply Monotonic Sequence Theorem to show

$$\lim_{n\to\infty} a_n$$

exists, and find the limit.

Monotonic Sequence Theorem



证明 bounded

monotonic increasing，证明 bounded above

monotonic decreasing，证明 bounded below

Clearly $a_1 < 2$. Suppose $a_n < 2$ for a positive integer n. then $a_{n+1} = \sqrt{2+a_n} < \sqrt{2+2} = 2$

Thus, by induction, $\{a_n\}$ is bounded above by 2.

或 $\frac{a_{n+1}}{a_n}$

$$a_{n+1} - a_n = \sqrt{2+a_n} - a_n = \frac{(\sqrt{2+a_n} - a_n)(\sqrt{2+a_n} + a_n)}{\sqrt{2+a_n} + a_n} = \frac{2 + a_n - a_n^2}{\sqrt{2+a_n} + a_n} = \frac{-(a_n-2)(a_n+1)}{\sqrt{2+a_n} + a_n}$$

来证明 monotonic increasing / decreasing

Clearly $a_n > 0$ for positive integer n. $\frac{-(a_n-2)(a_n+1)}{\sqrt{2+a_n} + a_n} > 0$, $a_{n+1} > a_n$.

$\{a_n\}$ is monotonic increasing and bounded above by 2. By monotonic sequence theorem, limit of $\{a_n\}$ exists. let $L = \lim_{n\to\infty} a_n$. 必须写这步才得出 limit 值。

$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n} = \sqrt{2 + \lim_{n\to\infty} a_n} = \sqrt{2+L}$

$L^2 - (2+L) = 0 \quad (L-2)(L+1) = 0 \quad L = 2$ or -1 (reject).

Thus, the limit of $\{a_n\}$ is 2.