SPA Assignment 3 Solution

1. We let X_n be the number of papers in the n-th evening. If we know X_n, then the behavior of X_{n+1} only depends on X_n and not on $X_0, X_1, \ldots, X_{n-1}$. So (X_n) is a Markov chain. Since Mr. Smith will take all the paper to the bin if there are at least five papers, X_n can only take 0,1,2,3,4. Thus the state space $S = \{0, 1, 2, 3, 4\}$. The transition matrix is

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

2.

(a) Because whether or not it rains today depends on previous weather conditions through the last two days but not yesterday only.

(b) $\boldsymbol{P} = \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix}.$

(c) It holds that

$$\begin{aligned} P_{30}^2 + P_{31}^2 &= P_{31}P_{10} + P_{33}P_{11} + P_{33}P_{31} \\ &= (.2)(.5) + (.8)(0) + (.2)(0) + (.8)(.2) \\ &= .26. \end{aligned}$$

3. Consider $\boldsymbol{P} = \begin{pmatrix} 0 & 0.8 & 0.2 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0 & 0.5 \end{pmatrix}$. Its characteristic polynomial is

$$\det(\lambda I_3 - \boldsymbol{P}) = \det\begin{pmatrix} \lambda & -0.8 & -0.2 \\ -0.5 & \lambda - 0.1 & -0.4 \\ -0.5 & 0 & \lambda - 0.5 \end{pmatrix} = \lambda^3 - 0.6\lambda^2 - 0.45\lambda + 0.05.$$

It can be factorized as

$$\begin{aligned} \lambda^3 - 0.6\lambda^2 - 0.45\lambda + 0.05 &= (\lambda - 1)(\lambda^2 + 0.4\lambda + 0.05) \qquad \text{(1 is an eigenvalue of } \boldsymbol{P}\text{)} \\ &= (\lambda - 1)\left(\lambda + \frac{1}{2}\right)\left(\lambda - \frac{1}{10}\right) \end{aligned}$$

$$\lambda = 1, N\begin{pmatrix} 1 & -0.8 & -0.2 \\ -0.5 & 0.9 & -0.4 \\ -0.5 & 0 & 0.5 \end{pmatrix} = span\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

$$\lambda = -\frac{1}{2}, N\begin{pmatrix} -0.5 & -0.8 & -0.2 \\ -0.5 & -0.6 & -0.4 \\ -0.5 & 0 & -1 \end{pmatrix} = span\left\{\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}\right\}$$

$$\lambda = \frac{1}{10}, N\begin{pmatrix} 0.1 & -0.8 & -0.2 \\ -0.5 & 0 & -0.4 \\ -0.5 & 0 & -0.4 \end{pmatrix} = span\left\{\begin{pmatrix} 16 \\ 7 \\ -20 \end{pmatrix}\right\}$$

$$\begin{pmatrix} 0 & 0.8 & 0.2 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0 & 0.5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 16 \\ 1 & -1 & 7 \\ 1 & -1 & -20 \end{pmatrix}\begin{pmatrix} 1 & & \\ & -0.5 & \\ & & 0.1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 16 \\ 1 & -1 & 7 \\ 1 & -1 & -20 \end{pmatrix}^{-1}.$$

$$\begin{aligned} \boldsymbol{P}^n &= \begin{pmatrix} 0 & 0.8 & 0.2 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0 & 0.5 \end{pmatrix}^n \\ &= \begin{pmatrix} 1 & 2 & 16 \\ 1 & -1 & 7 \\ 1 & -1 & -20 \end{pmatrix}\begin{pmatrix} 1 & & \\ & -0.5 & \\ & & 0.1 \end{pmatrix}^n\begin{pmatrix} 1 & 2 & 16 \\ 1 & -1 & 7 \\ 1 & -1 & -20 \end{pmatrix}^{-1} \\ &= \frac{1}{27}\begin{pmatrix} 1 & 2 & 16 \\ 1 & -1 & 7 \\ 1 & -1 & -20 \end{pmatrix}\begin{pmatrix} 1 & & \\ & (-0.5)^n & \\ & & 0.1^n \end{pmatrix}\begin{pmatrix} 9 & 8 & 10 \\ 9 & -12 & 3 \\ 0 & 1 & -1 \end{pmatrix} \end{aligned}$$

4.

Consider $\boldsymbol{P} = \begin{pmatrix} & \frac{1}{2} & & \frac{1}{2} \\ \frac{1}{3} & & \frac{1}{3} & \frac{1}{3} \\ & \frac{1}{2} & & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \end{pmatrix}$. Its characteristic polynomial is

$$\det(\lambda I_4 - \boldsymbol{P}) = \det\left(\lambda I_2 - \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}\right)\det\left(\lambda I_2 - \begin{pmatrix} 0 & 1 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}\right) = \lambda(\lambda + \tfrac{1}{3})(\lambda - 1)(\lambda + \tfrac{2}{3}).$$

$$\text{eigenvalue } \lambda_1 = 1, \text{ eigenvector } v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{eigenvalue } \lambda_2 = 0, \text{ eigenvector } v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\text{eigenvalue } \lambda_3 = -\tfrac{1}{3}, \text{ eigenvector } v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{eigenvalue } \lambda_4 = -\tfrac{2}{3}, \text{ eigenvector } v_4 = \begin{pmatrix} 3 \\ -2 \\ 3 \\ -2 \end{pmatrix}$$

$$\boldsymbol{P} = \begin{pmatrix} & \frac{1}{2} & & \frac{1}{2} \\ \frac{1}{3} & & \frac{1}{3} & \frac{1}{3} \\ & \frac{1}{2} & & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & -2 \\ 1 & -1 & 0 & 3 \\ 1 & 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & -\frac{1}{3} & \\ & & & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & -2 \\ 1 & -1 & 0 & 3 \\ 1 & 0 & -1 & -2 \end{pmatrix}^{-1}.$$

$$\boldsymbol{P}^n = \begin{pmatrix} & \frac{1}{2} & & \frac{1}{2} \\ \frac{1}{3} & & \frac{1}{3} & \frac{1}{3} \\ & \frac{1}{2} & & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \end{pmatrix}^n$$

$$= \begin{pmatrix} 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & -2 \\ 1 & -1 & 0 & 3 \\ 1 & 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & -\frac{1}{3} & \\ & & & -\frac{2}{3} \end{pmatrix}^n \begin{pmatrix} 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & -2 \\ 1 & -1 & 0 & 3 \\ 1 & 0 & -1 & -2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & -2 \\ 1 & -1 & 0 & 3 \\ 1 & 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & (-\frac{1}{3})^n & \\ & & & (-\frac{2}{3})^n \end{pmatrix} \begin{pmatrix} 0.2 & 0.3 & 0.2 & 0.3 \\ 0.5 & 0 & -0.5 & 0 \\ 0 & 0.5 & 0 & -0.5 \\ 0.1 & -0.1 & 0.1 & -0.1 \end{pmatrix}$$