ASP

Solution to Assignment 6

1. The chain is irreducible and aperiodic. Moreover, (a) $\pi = \left[\frac{10}{21}, \frac{5}{21}, \frac{6}{21}\right]$ and (b) the limit is $\pi_0 = \frac{10}{21}$.

2. Consider S_n mod 13. This is a Markov chain with states $0, 1 \ldots, 12$ and transition matrix is
$$\begin{bmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \ldots & 0 \\ 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \ldots & 0 \\ & & & & & \vdots & & & & \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \ldots & 0 \end{bmatrix}.$$
This is a doubly stochastic matrix with $\pi_i = \frac{1}{13}$, for all i. So the answer is $\frac{1}{13}$.

3. As the chain is irreducible and aperiodic, P_{ij}^n converges to $\pi_j, j = 0, 1, 2, 3, 4$, where π is given by $\pi = \left[\frac{12}{37}, \frac{6}{37}, \frac{4}{37}, \frac{3}{37}, \frac{12}{37}\right]$.

4. Consider a chain with two states 1 and 2. Let
$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
Then the chain has two equivalent classes $\{1\}$ and $\{2\}$, and the chain is not irreducible. Note that
$$\pi P = \pi$$
for any row vector π, so any distribution is a stationary distribution for this chain.

5.

   (a) Let $\{T_n, n = 1, 2, \ldots\}$ be the sequence of interarrival times. Then $T_n, n = 1, 2, \ldots,$ are independent identically distributed exponential random variables having mean $1/\lambda$. Then
$$\begin{aligned} E(S_4) &= E(T_1 + T_2 + T_3 + T_4) \\ &= 4E(T_1) = 4/\lambda; \end{aligned}$$

(b) Noting that $N(4)-N(2)$ and $N(1)$ are independent and $N(4)-N(2)\sim N(2)$, we have

$$\begin{aligned} E(N(4)-N(2)\,|\,N(1)=3) &= E(N(4)-N(2)) \\ &= E(N(2)) \\ &= 2\lambda. \end{aligned}$$

6.

(a) It holds

$$\begin{aligned} P(N(1)\le 2) &= P(N(1)=0)+P(N(1)=1)+P(N(1)=2) \\ &= e^{-2}(\frac{2^0}{0!}+\frac{2^1}{1!}+\frac{2^2}{2!}) = 5e^{-2}; \end{aligned}$$

(b) Note that $N(2)-N(1)$ and $N(1)$ are independent and $N(2)-N(1)\sim N(1)$, we have

$$\begin{aligned} P(N(1)=1,\, N(2)=3) &= P(N(1)=1)P(N(2)=3\,|\,N(1)=1) \\ &= P(N(1)=1)P(N(2)-N(1)=3-1\,|\,N(1)=1) \\ &= P(N(1)=1)P(N(2)-N(1)=2) \\ &= e^{-2}\left(\frac{2^1}{1!}\right)e^{-2}\left(\frac{2^2}{2!}\right) = 4e^{-4}; \end{aligned}$$

7.

(a) It holds

$$\begin{aligned} P(N(3)=6\,|\,N(1)=2) &= P(N(3)-N(1)=6-2\,|\,N(1)=2) \\ &= P(N(3)-N(1)=4) \\ &= e^{-4}\frac{4^4}{4!} = \frac{32}{3}e^{-4}; \end{aligned}$$

(b) It holds

$$\begin{aligned} E(N(1)N(2)) &= E(N(1)^2)+E(N(1)(N(2)-N(1)) \\ &= \lambda^2+\lambda+E(N(1)E(N(2)-N(1)) \\ &= \lambda^2+\lambda+\lambda\cdot\lambda = 2\lambda^2+\lambda = 10. \end{aligned}$$