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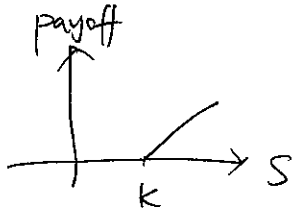
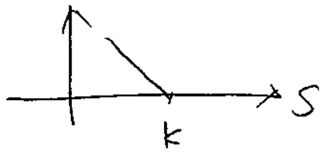
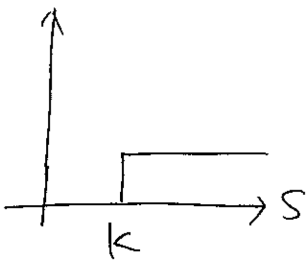
Vanilla option:

	payoff	
call	$\max(S-k,0)$	
put	$\max(k-S,0)$	

Exotic option (options which are not vanilla)

① Binary option

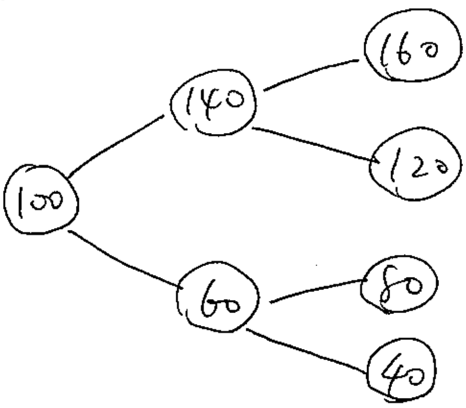
call



put

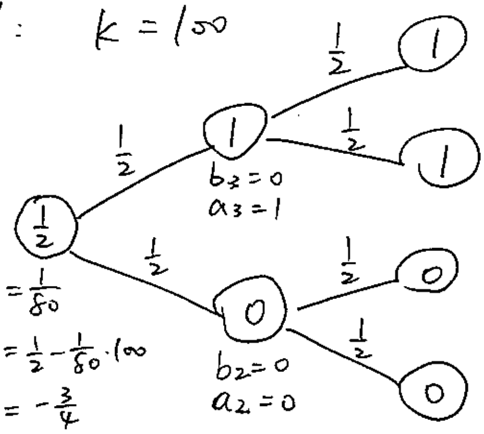
example: $r=0$

S:



$\Rightarrow q_i = \frac{1}{2}, \quad i=1,2\cdots 7$

V: $k=100$



$b_3 = 0$
$a_3 = 1$

$b_1 = \frac{1}{80}$
$a_1 = \frac{1}{2} - \frac{1}{80}\cdot 100$
$= -\frac{3}{4}$

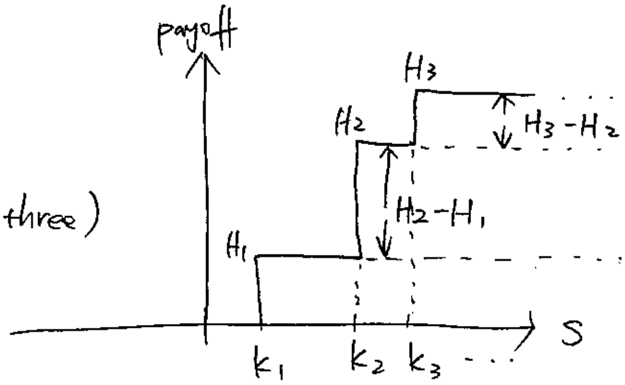
$b_2 = 0$
$a_2 = 0$

replicate strategy:

$$\begin{cases} b = \dfrac{V^+ - V^-}{S^+ - S^-} \\ a = V_0 - bS_0 \end{cases}$$

How does the replicate strategy work?

② Digital option



We can add several (here is three) binary options together to obtain a digital option.

However, this way need to calculate several times of binomial trees. Therefore, for binomial tree model, it is better to use the original payoff of a digital option to calculate the price of this option.

However, in continuous time, we have closed-form formula for a binary option. So we can easily add them together to get a digital option.

Let's try to find the closed-form formula of a binary option by simple comparison. We know the formula of European option, which is

$$E_q\{ e^{-r(T-t)} \cdot \max(S-k,0) \mid \mathcal{F}_t \} = S N(d_1) - k e^{-r(T-t)} N(d_2)$$

where $d_{1,2} = \dfrac{\ln\frac{S}{k} + (r \pm \frac{\sigma^2}{2})(T-t)}{\sqrt{(T-t)\sigma^2}}$. ①

$$\text{LHS} = \underset{S>k}{E_q}\{ e^{-r(T-t)} (S-k) \mid \mathcal{F}_t \} = \underset{S>k}{E_q}[e^{-r(T-t)} \cdot S \mid \mathcal{F}_t] - \underset{S>k}{E_q}[e^{-r(T-t)} k \mid \mathcal{F}_t]$$

$$= \underset{S>k}{E_q}[e^{-r(T-t)} \cdot S \mid \mathcal{F}_t] - k e^{-r(T-t)} \underset{S>k}{E_q}[1 \mid \mathcal{F}_t] \quad ②$$

Compare ② with the right hand side of ① :

$$\underset{S>k}{E_q}[1 \mid \mathcal{F}_t] = N(d_2) = N\left(\frac{\ln\frac{S}{k} + (r - \frac{\sigma^2}{2})(T-t)}{\sqrt{(T-t)\sigma^2}} \right) \quad ③$$

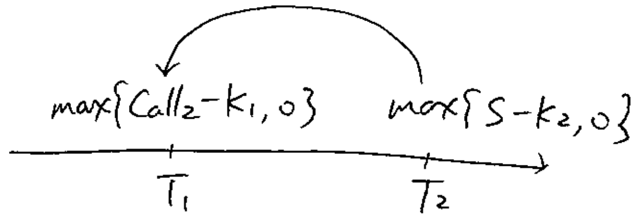
Binary option : $F(S) = \begin{cases} 1 & S \geq k \\ 0 & S < k \end{cases}$

$$V(t,S) = E_q[e^{-r(T-t)} F(S) \mid \mathcal{F}_t] = E_q[e^{-r(T-t)} \cdot (1 \mid \mathcal{F}_t, S \geq k)]$$

$$\overset{③}{=} e^{-r(T-t)} \cdot N\left(\frac{\ln\frac{S}{k} + (r - \frac{\sigma^2}{2})(T-t)}{\sqrt{(T-t)\sigma^2}} \right).$$

③ Compound options:

call-on-call:



let $C(T_1, T_2, K, S)$ / $P(T_1, T_2, k, S)$ be the price of a call/put option at T_1, with maturity T_2, strike price k and stock price, at T_1, S.

call-on-call: $\max\{C(T_1, T_2, k_2, S) - K_1, 0\}$

is a call option with underlying asset being a call option.

call-on-put: $\max\{P(T_1, T_2, k_2, S) - k_1, 0\}$

is a call option with underlying asset being a put option.

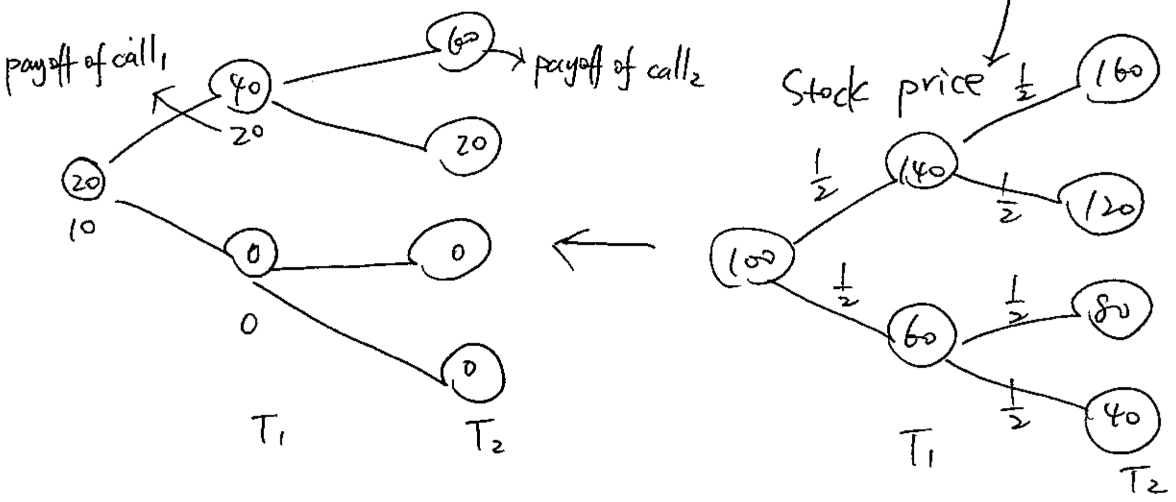
put-on-call: $\max\{k_1 - C(T_1, T_2, k_2, S), 0\}$

is a put option with underlying asset being a call option.

put-on-put: $\max\{k_1 - P(T_1, T_2, k_2, S), 0\}$

is a put option with underlying asset being a put option.

Example: $k_1 = 20$, $k_1 = 100$ for a call-on-call.



Let's look at something interesting. Based on the example above

	S	$C(t, T_2, 100, S)$	call-on-call
price at t	100	20	10
shares per 100	1	5	10
if $S=200$ at T_2, the profit	100%	400%	900%
if $S=100$ at T_2, the profit	0%	-100%	-100%
if $S=110$ at T_2, the profit	10%	-50%	0%
if $S=80$ at T_2, the profit	-20%	-100%	-100%

▲ Options are more risky, but have higher return than stock.

④ Barrier option

A barrier option is a type of option whose payoff depends on whether or not the underlying asset has reached or exceeded a predetermined price.

Down-and-out $\longrightarrow$ It can expire worthless if the underlying exceeds a certain price.

Down-and-in

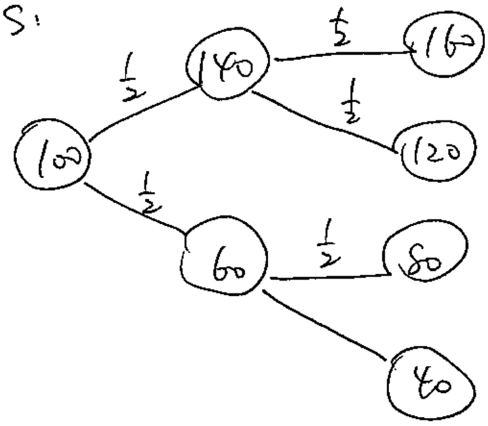
Up-and-out

Up-and-in $\longrightarrow$ It has no value until the underlying reaches a certain price.

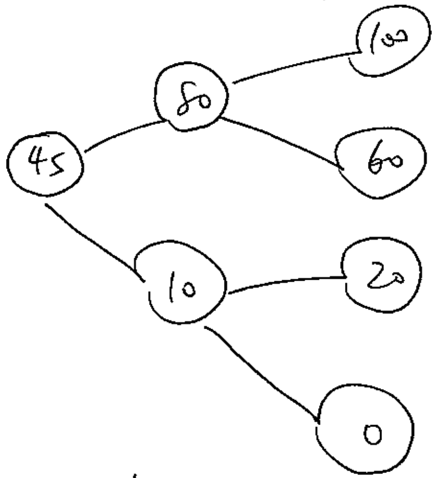
$$V_{out} + V_{in} = V_{no\text{-}barrier}$$

Example: Call with $K = 60$, $B = 70$

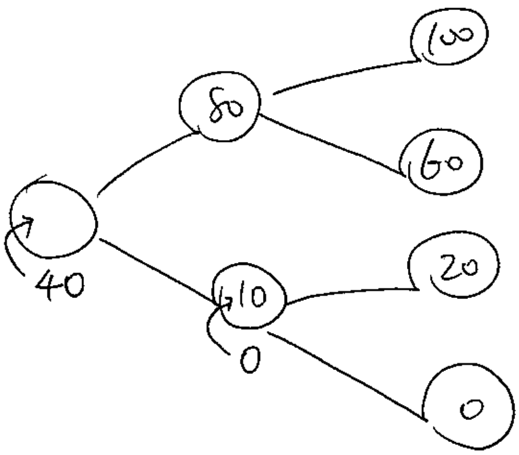
S:



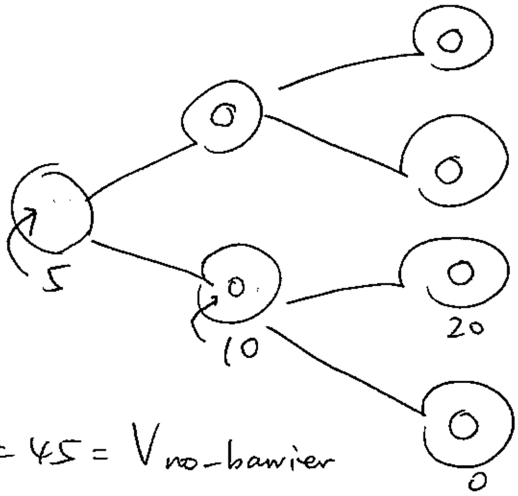
No barrier
Vanilla call



down-and-out



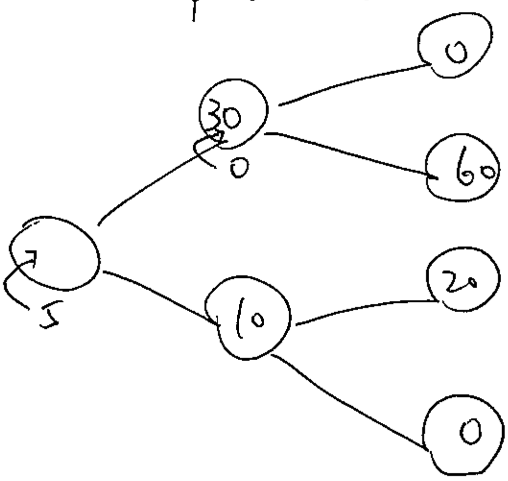
down-and-in



$\therefore \quad V_{out} + V_{in} = 40 + 5 = 45 = V_{no\text{-}barrier}$

Reset $B = 130$

up-and-out



up-and-in