# 2. Algorithm Correctness

**José Proença** Pedro Ribeiro

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CISTER – U.Porto, Porto, Portugal

`https://fm-dcc.github.io/alg2425`

# Motivation

slides by Pedro Ribeiro, slides 1

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# Correctness and Loop Invariants

Pedro Ribeiro

DCC/FCUP

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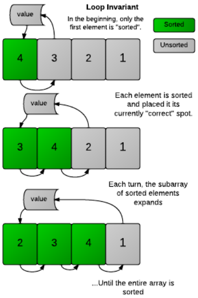
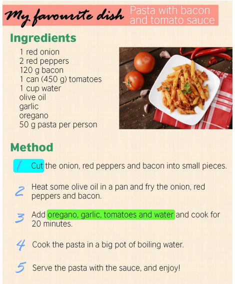
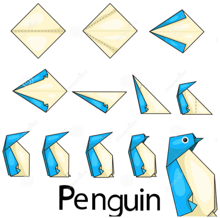


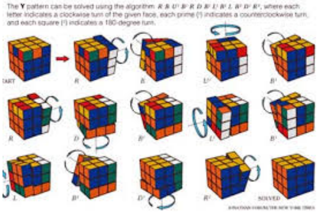
# On Algorithms

What are algorithms? A set of **instructions** to solve a **problem**.

- The problem is the **motivation** for the algorithm
- The instructions need to be **executable**
- Typically, there are **different algorithms** for the same problem [how to choose?]
- **Representation**: description of the instructions that is understandable for the intended audience



# On Algorithms

## "Computer" Science version

- An algorithm is a **method** for solving a (computational) problem
- Algorithms are the **ideas** behind the programs and are independent from the programming language, the machine, ...
- A **problem** is characterized by the description of its **input** and **output**

A classical example:

### Sorting Problem

**Input:** a sequence of $\langle a_1, a_2, \ldots, a_n \rangle$ of $n$ numbers
**Output:** a permutation of the numbers $\langle a_1^{'}, a_2^{'}, \ldots, a_n^{'} \rangle$ such that $a_1^{'} \leq a_2^{'} \leq \ldots \leq a_n^{'}$

### Example instance for the sorting problem

**Input:** 6 3 7 9 2 4
**Output:** 2 3 4 6 7 9

# On Algorithms

## What do we aim for?

- What **properties** do we want on an algorithm?

**Correction**

It has to solve correctly **all instances** of the problem

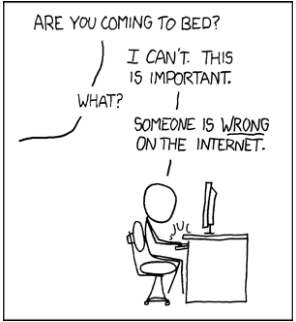
**Efficiency**

The performance (**time** and **memory**) has to be adequate

- This course is about **designing** correct and efficient algorithms and how to **prove** they meet the specifications

# About correction

- In this lecture we will (mostly) worry about **correction**
  - Given an algorithm, it is not often obvious or trivial to know if it is **correct**, and even less so to **prove** this.
  - By learning how to reason about correctness, we also gain **insight** into what really makes an algorithm work



# Specification

# When is an algorithm correct?

**Ex. 2.1:** What do these functions do?

```
int fa (int x, int y){
  // pre: True
  ...
  // pos: (m == x || m == y) &&
  //      (m >= x && m >= y)
  return m;
}
```

```
int fb (int x, int y){
  // pre: x >= 0 && y >= 0
  ...
  // pos: x % r == 0 && y % r == 0
  return r;
}
```

```
int fc (int x, int y){
  // pre: x > 0 && y > 0
  ...
  // pos: r % x == 0 && r % y == 0
  return r;
}
```

```
int fd (int a[], int N){
  // pre: N>0
  ...
  // pos:
  //  (forall_{0<=i<N} x<=a[i]) &&
  //  (exists_{0<=i<N} x==a[i])
  return x;
}
```

# When is an algorithm correct?

**Ex. 2.2: Formulate pre- and post-conditions:**

`int prod (int x, int y)` – product of two integers

`int gcd (int x, int y)` – greatest common divisor of 2 positive integers

`int sum (int v[], int N)` – sum of elements in an array

`int maxPOrd (int v[], int N)` – length of the longest sorted prefix of an array

`int isSorted (int v[], int N)` – tests if an array is sorted (growing)

A triple $\{P\}S\{Q\}$ is a valid Hoare triple when

if $[P$ *holds*$]$ and $[S$ *is executed*$]$ then $[Q$ *holds*$]$

**Ex. 2.3: Find initial states that show these are not valid (and fix pre-cond.)**

1. `{True} r=x+y; {r≥x}`
2. `{True} x=x+y; y=x-y; x=x-y; {x==y}`
3. `{True} x=x+y; y=x-y; x=x-y; {x≠y}`
4. `{True} if(x>y) r=x-y; else r=y-x; {r>0}`
5. `{True} while (x>0) {y=y+1; x=x-1;} {y>x}`

# Partial correctness

# Using rules for Hoare triples

$$\frac{P \Rightarrow Q[x \backslash E]}{\{P\}\ x := E\ \{Q\}} \qquad \frac{P \Rightarrow I \qquad \{I \land c\}\ S\ \{I\} \qquad (I \land \neg c) \Rightarrow Q}{\{P\}\ \texttt{while}\ c\ S\ \{Q\}}$$

1. **Initialisation:** $P \Rightarrow I$ *($P$ is the precondition right before the cycle)*
   Before the cycle the invariant holds.
2. **Maintenance:** $\{I \land c\}\ S\ \{I\}$ *(or $I \land c \Rightarrow I'$, where $I'$ is the invariant after $S$)*
   Assuming the invariant holds before an iteration; it must be valid after it.
3. **Termination/Usefulness:** $(I \land \neg c) \Rightarrow Q$ *(simplify $I \land c$ until obtain $Q$)*
   After the cycle the post-condition holds.

slides by Pedro Ribeiro, slides 1

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# Loops

- We will tackle one of the most fundamental (and most used) algorithmic patterns: a **loop** (e.g. `for` or `while` instructions)

**Example loop: summing integers from 1 to $n$**

$sum = 0$
$i = 1$
**while** $(i \leq n)$ {
  $sum = sum + i$
  $i = i + 1$
}

- We will talk about how to prove that a **loop** is correct
- We will show how this is also useful for **designing** new algorithms

# Loop Invariants

**Definition of Loop Invariant**

A **condition** that is necessarily true immediately before (and immediately after) each iteration of a loop

Note that this says nothing about its truth or falsity part way through an iteration.

*Instructions are for computers, invariants are for humans*

- The loop program statements are **"operational"**, they are **"how to do"** instructions
- Invariants are **"assertional"**, capturing **"what it means"** descriptions

# Anatomy of a loop

Consider a simple loop: **while (B)** { **S** }

- **Q**: precondition (assumptions at the beginning)
- **B**: the stop condition (defining when the loop end)
- **S**: the body of the loop (a set of statements)
- **R**: postcondition (what we want to be true at the end)

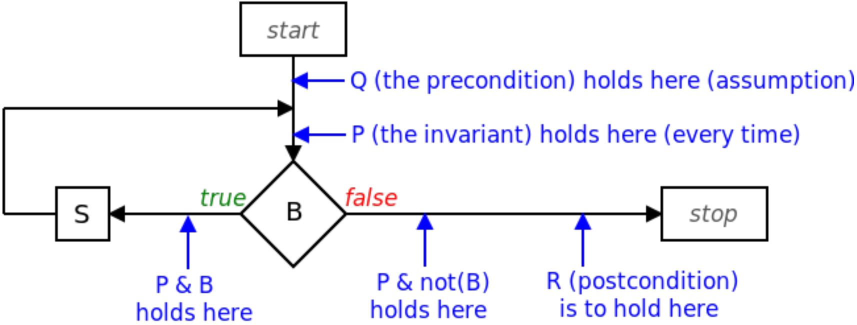
**Example loop: summing integers from 1 to $n$**

$sum = 0$
$i = 1$
**while** $(i \leq n)$ {
  $sum = sum + i$
  $i = i + 1$
}

- **Q**: $sum = 0$ and $i = 1$
- **B**: $i \leq N$
- **S**: $sum = sum + i$ followed by $i = i + 1$
- **R**: $sum = \sum_{i=1}^{n} i$

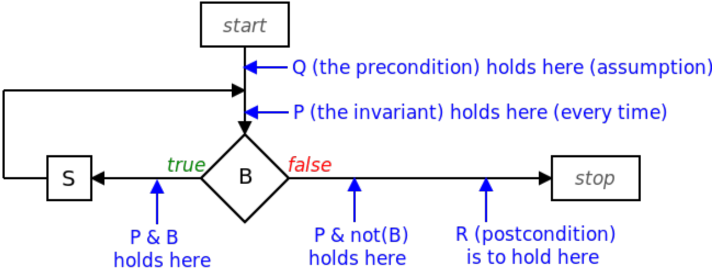
# The invariant?

- **P**: an invariant (condition that holds at the start of each iteration)




- To be **useful**, the invariant $P$ that we seek should be such that: $P \wedge not(B) \rightarrow R$
  - For the example sum loop, it could be: $sum = \sum_{i=1}^{i-1} i$

# How to show that an invariant is really one?




- First, show that $Q \rightarrow P$
  *(truth precondition Q guarantees truth of invariant P)*
  - For the example sum loop: sum=0 which is $= \sum_{i=1}^{0} i$
- If $P \wedge B$, then after executing $S$, then $P$ holds after executing $S$
  *(the statements S of the loop guarantee that P is respected)*
  - For the example sum loop: $\sum_{i=1}^{i-1} + i = \sum_{i=1}^{i}$

# How to show that an invariant is really one?

**Initialization**

The invariant is true prior to the first iteration of the loop

**Maintenance**

If it is true before an iteration of the loop, it remains true before the next iteration

**Termination**

When the loop terminates, the invariant gives us a useful property that helps show that the algorithm is correct

# Exercises

```c
int mult1 (int x, int y){
  // pre: x>=0
  int a, b, r;
  a=x; b=y; r=0;
  while (a!=0){
    r = r+b;
    a = a-1;
  }
  // pos: r == x * y
  return r;
}
```

```c
int mult2 (int x, int y){
  // pre: x>=0
  int a, b, r;
  a=x; b=y; r=0;
  while (a!=0) {
    if (a%2 == 1) r = r+b;
    a=a/2;
    b=b*2;
  // pos: r == x * y
  return r;
}
```

**Ex. 2.4: Check if *Initialization* and *Maintenance* holds for these formulae**

| | | |
|---|---|---|
| $r == a * b$ | $r \geq 0$ | $b == 0$ |
| $a \geq 0$ | $a == x$ | $a * b == x * y$ |
| $b \geq 0$ | $a \neq x$ | $a * b + r == x * y$ |

# Exercises

```c
int mult1 (int x, int y){
  // pre: x>=0
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```c
int mult2 (int x, int y){
  // pre: x>=0
  int a, b, r;
  a=x; b=y; r=0;
  while (a!=0) {
    if (a%2 == 1) r = r+b;
    a=a/2;
    b=b*2;
  // pos: r == x * y
  return r;
}
```

**Ex. 2.5:** Find loop invariants to prove partial correctness

# Some intuition – mult1(4,5)

```
1  int mult1 (int x, int y){
2    // pre: x>=0
3    int a, b, r;
4    a=x; b=y; r=0;
5    while (a>0){
6      r = r+b;
7      a = a-1;
8    }
9    // pos: r == x * y
10   return r;
11 }
```

| line | x | y | a | b | r |
|---|---|---|---|---|---|
| 4 | 4 | 5 | 4 | 5 | 0 |
| 6 | 4 | 5 | 4 | 5 | 5 |
| 7 | 4 | 5 | 3 | 5 | 5 |
| 6 | 4 | 5 | 3 | 5 | 10 |
| 7 | 4 | 5 | 2 | 5 | 10 |
| 6 | 4 | 5 | 2 | 5 | 15 |
| 7 | 4 | 5 | 1 | 5 | 15 |
| 6 | 4 | 5 | 1 | 5 | 20 |
| 7 | 4 | 5 | 0 | 5 | 20 |
| 10 | 4 | 5 | 0 | 5 | 20 |

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| line | x | y | a | b | r |
|---|---|---|---|---|---|
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| 6 | 4 | 5 | 3 | 5 | 10 |
| 7 | 4 | 5 | 2 | 5 | 10 |
| 6 | 4 | 5 | 2 | 5 | 15 |
| 7 | 4 | 5 | 1 | 5 | 15 |
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- x and y never change
- r grows proportionally as a shrinks
- guess:
  $I \triangleq \mathtt{a*y} + \mathtt{r} = \mathtt{x*y}$

# Some intuition – mult1(4,5)

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- x and y never change
- r grows proportionally as a shrinks
- guess:
  $I \triangleq \mathtt{a*y} + \mathtt{r} = \mathtt{x*y}$
- Need to show:

$$\mathtt{x>=0} \Rightarrow I'$$

$$I \wedge \mathtt{a>0} \Rightarrow I'$$

$$I \wedge \neg(\mathtt{a>0}) \Rightarrow \mathtt{r} = \mathtt{x*y}$$

# Some intuition – mult1(4,5)

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- x and y never change
- r grows proportionally as a shrinks
- guess: $I \triangleq \texttt{a*y} + \texttt{r} = \texttt{x*y}$
- Need to show:

$$\texttt{x>=0} \Rightarrow I'$$

$$I \land \texttt{a>0} \Rightarrow I'$$

$$I \land \neg(\texttt{a>0}) \Rightarrow \texttt{r} = \texttt{x*y}$$

- (Not all works – enrich invariant!)

```c
int serie(int n){
  // pre: n>=0
  int r=0, i=1;
  // inv: ??
  while (i!=n+1) {
    r = r+i; i = i+1;
  }
  // pos: r == n * (n+1) / 2;
  return r;
}
```

```c
int mod(int x, int y) {
  // pre: x>=0 && y>0
  int r = x;
  while (y <= r) {
    r = r-y;
  }
  // pos: 0 <= r < y && exists_{q}
       x == q*y + r
  return r;
}
```

**Ex. 2.5:** Find loop invariants

# Even more exercises (@home)

```
int minInd (int v[], int N) {
  // pre: N>0
  int i = 1, r = 0;
  // inv: ???
  while (i<N) {
    if (v[i] < v[r]) r = i;
    i = i+1; }
  // pos: 0 <= r < N && forall_{0 <= k < N} v[r] <= v[k]
return r; }

int minimum (int v[], int N) {
  // pre: N>0
  int i = 1, r = v[0];
  // inv: ???
  while (i!=N) {
     if (v[i] < r) r = v[i];
     i=i+1; }
  // pos: (forall_{0 <= k < N} r <= v[k]) &&
  //      (exists_{0 <= p < N} r == v[p])
  return r;
}

int sum (int v[], int N) {
  // pre: N>0
  int i = 0, r = 0;
  // inv: ???
  while (i!=N) {
    r = r + v[i]; i=i+1;
  }
  // pos: r == sum_{0 <= k < N} v[k]
  return r;
}
```

```
int sqr1 (int x) {
  // pre: x>=0
  int a = x, b = x, r = 0;
  // inv: ??
  while (a!=0) {
    if (a%2 != 0) r = r + b;
    a=a/2; b=b*2;
  }
  // pos: r == x^2
  return r;
  }

int sqr2 (int x){
  // pre: x>=0
  int r = 0, i = 0, p = 1;
  // inv: ??
  while (i<x) {
    i = i+1; r = r+p; p = p+2;
  }
  // pos: r == x^2
  return r;
}

int ssearch (int x, int a[], int N){
  // pre: N>0 &&
  //      forall_{0 < k < N-1} a[k-1]<=a[k]
  int p = -1, i = 0;
  // inv: ??
  while (p == -1 && i < N && x >= a[i]) {
    if (a[i] == x) p = i;
    i = i+1;
  }
  // pos: (p == -1 && forall_{0 <= k < N} a[k] != x) ||
  //      ( (0 <= p < N ) && x == a[p])
  return p;
}
```

# Complete correctness

# Partial/Complete correctness

Given $\{P\}\ S\ \{Q\}$

**Partial correctness**

if [$P$ *holds*] and [$S$ *is executed*] then [$Q$ *holds*]

**Complete correctness**

if [$P$ *holds*] and [$S$ *is executed*] then [$Q$ *holds*] AND $S$ terminates

# Partial/Complete correctness

Given $\{P\}\ S\ \{Q\}$

**Partial correctness**

if [$P$ *holds*] and [$S$ *is executed*] then [$Q$ *holds*]

**Complete correctness**

if [$P$ *holds*] and [$S$ *is executed*] then [$Q$ *holds*] AND $S$ terminates

**Enough to show the existence of a loop variant**

# Loop variant

Technique that measures the distance between the current state and the final state.

**A loop variant $V$ is an integer expression s.t.**

- is positive in the beginning of each round ($c \wedge I \Rightarrow V > 0$)
- decreases in every round ($c \wedge I \Rightarrow V > V'$)

```
r=x;
q=0;
while (y <= r) {
  r = r-y;
  q = q+1;
}
```

- $V = r - y$ is not a good variant
- ...

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- $V = r - y$ is not a good variant
- $V = r - y + 1$ is a good variant

$y \leq r \Rightarrow V > 0$ at each round

$V > V'$ after each round

# Exercises

```
int sum(int v[], int N) {
  int i = 0, r = 0;
  while (i!=N) {
    // variant: ???
    r = r + v[i];
    i=i+1;
  }
  return r;
}
```

**Ex. 2.6:** Find variant above

**Ex. 2.7:** Find variants of the loops in previous exercises (when searching for invariants)