# 6. Data Structures

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`https://fm-dcc.github.io/alg2425`

- Algorithm Correctness
- Complexity: worst/best-case analysis
- Asymptotic analysis

- Recursive algorithms
- Average-case and randomized algorithms
- Dynamic programming
- Amortized analysis
- Data structures

- Sets and Sequences
- Buffers:
    - Stacks
    - Queues
    - Priority queues
- Dictionaries
    - Hashtables
    - Search trees

**We have seen that**

Different data structures are better at different operations

**We will see**

Useful data structures and associated operations (code)

**Examples**

Arrays can have operations to implement sets, multisets, trees, etc.

# Sets and Sequences

```
#define MAXS 100
typedef int SetInt [MAXS] ;
```

Given SetInt s:

$5 \in s \iff s[5]\text{!}=0$

```
#define MAXMS 100
typedef int MSetInt [MAXS] ;
```

Given MSetInt ms:

$\{4, 4\} \subseteq ms \iff ms[4] \geq 2$

```
void initSet      (SetInt);
int  searchSet    (SetInt, int);
int  addSet       (SetInt, int);
int  emptySet     (SetInt);
void unionSet     (SetInt, SetInt,
                   SetInt);
void intersectSet (SetInt, SetInt,
                   SetInt);
void differenceSet(SetInt, SetInt,
                   SetInt);
```

```
void initMSet      (MSetInt);
int  searchMSet    (MSetInt, int);
int  addMSet       (MSetInt, int);
int  emptyMSet     (MSetInt);
void unionMSet     (MSetInt, MSetInt,
                    MSetInt);
void intersectMSet (MSetInt, MSetInt,
                    MSetInt);
void differenceMSet(MSetInt, MSetInt,
                    MSetInt);
```

**Ex. 6.1:** What is the expected cost of each function? Could you implement them?

```
typedef struct list {
  int value ;
  struct list *next;
} *LInt;
```

```
LInt add (int x, LInt l) {
  LInt new =
    malloc(sizeof(struct list));
  if (new != NULL) {
    new->value=x;
    new->next=l ;
  }
  return new;
}
```

```
LInt dda (int x, LInt l) {
  LInt pt = l;
  while (pt != NULL) pt = pt->next;
  pt = malloc(sizeof(struct list));
  pt -> value = x;
  pt -> next = NULL ;
  return l ;
}
```

## Sequences – Recall linked lists (fixed)

```c
typedef struct list {
  int value ;
  struct list *next;
} *LInt;
```

```c
LInt add (int x, LInt l) {
  LInt new =
    malloc(sizeof(struct list));
  if (new != NULL) {
    new->value=x;
    new->next=l ;
  }
return new;
}
```

```c
LInt dda (int x, LInt l) {
  LInt pt = l, prev;
  while (pt != NULL) {
    prev = pt; pt = pt->next; }
  pt = malloc(sizeof(struct list));
  pt->value = x;
  pt->next = NULL ;
  if (l==NULL)  l = pt;
  else prev->next = pt;
  return l;
}
```

**Ex. 6.2:** What is the possible complexity of lookup, concat, reverse?

```
typedef struct list {
  int value ;
  struct list *next;
} *LInt ;
```

**Idea for reverseRec**

1. reverse the tail

2. add head to the end

```
reverseRec([])          = []
reverseRec([x1,x2,...]) = dda(x1,reverseRec([x2,...]))
```

**Sequences – reverse analysis**

```
LInt reverseRec (LInt l) {
  LInt r, pt;
  if (l==NULL || l->next==NULL)
      r=l;
  else {
    r = pt = reverseRec (l->next);
    while (pt->next != NULL)
      pt = pt->next;
    pt->next = l;
    l->next = NULL;
  }
  return r; }
```

```
LInt reverseLoop (LInt l) {
  LInt r, tmp;
  r = NULL;
  while (l != NULL) {
    tmp=l; l=l->next;
    tmp->next=r; r=tmp;
  }
  return r;
}
```

**Ex. 6.3:** What is the complexity of each reverse?

**Ex. 6.4:** What is the (informal) loop invariant in reverseLoop, assuming:
pre:l==$l_0$ and post:r==rev($l_0$)?

```
https://docs.scala-lang.org/
overviews/collections-2.13/
performance-characteristics.html
```

# Buffers (stacks and queues)

```
#define MAX 1000
typedef struct stack {
  int values [MAX];
  int sp;
} Stack;
```

```
typedef struct cell {
  int value;
  struct cell *next;
} Cell , *Stack;
```

```
typedef struct stack {
  int size;
  int *values;
  int sp;
} Stack;
```

```
#define MAX 1000
typedef struct stack {
  int values [MAX];
  int sp;
} Stack;
```

with static arrays

```
typedef struct cell {
  int value;
  struct cell *next;
} Cell , *Stack;
```

with linked lists

```
typedef struct stack {
  int size;
  int *values;
  int sp;
} Stack;
```

with dynamic arrays

**Ex. 6.5:** (Informally) what is the expected complexity of: push, pop, head?

```
void push (Stack *s , int x){
  if (s->sp == s->size)
    doubleArray (s);
  s->values[s->sp++] = x;
}

void doubleArray (Stack *s){
  s->size *= 2;
  s->values =
    realloc(s->values, s->size);
}
```

```
int pop (Stack *s){
  // reduces by half when only
  // 25% capacity is used
  ...
}

void halfArray (Stack *s){
  ...
}
```

**Ex. 6.6:** Implement the optimised pop function and discuss its complexity.

```c
#define MAX 1000
typedef struct queue
{
  int values [MAX];
  int start, size;
} Queue;
```

```c
typedef struct cell {
  int value ;
  struct cell *prox ;
} Cell ;

typedef struct queue {
  struct cell *start, *end;
} Queue;
```

```c
typedef struct queue
{
  int max;
  int *values;
  int start, size;
} Queue;
```

```
#define MAX 1000
typedef struct queue
{
  int values [MAX];
  int start, size;
} Queue;
```

```
typedef struct cell {
  int value ;
  struct cell *prox ;
} Cell ;

typedef struct queue {
  struct cell *start, *end;
} Queue;
```

```
typedef struct queue
{
  int max;
  int *values;
  int start, size;
} Queue;
```

with static arrays
(circular)

with linked lists

with dynamic arrays
(circular)

**Ex. 6.7:** (Informally) what is the complexity of: init, isEmpty, enqueue, dequeue?

- Binary tree

- Each node is smaller than any of its children

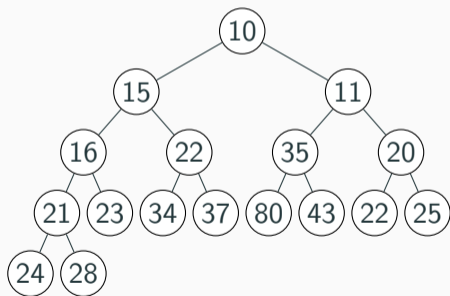- Implemented as an array

```
#define MAX 1000
typedef struct prQueue {
  int values [MAX];
  int size ;
} PriorityQ ;
```

**Tree example in the board**

```
    size=17   0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16
  values:  [10 15 11 16 22 35 20 21 23 34 37 80 43 22 25 24 28]
```
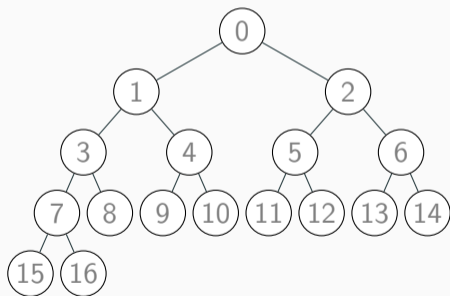
**Tree example**

```
size=17    0   1   2   3   4   5   6   7   8   9  10  11  12  13  14  15  16
values:   [10  15  11  16  22  35  20  21  23  34  37  80  43  22  25  24  28]
```

### Tree example

```
size=17    0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16
values:  [10 15 11 16 22 35 20 21 23 34 37 80 43 22 25 24 28]
```

**Ex. 6.8: Using the previous example, provide an expression to:**

1. calculate the index of the *left* tree given a position i
2. calculate the index of the *right* tree given a position i
3. calculate the index of the *parent* of a given a position i
4. calculate the index of the *first leaf*

**Ex. 6.9: Define** `bubbleUp(int i, int h[])`

Used to add elements. Fixes a min-heap by swapping the i-th element with the parent while needed.

**Ex. 6.10: Define** `bubbleDown(int i, int h[], int N)`

Used to remove elements. Fixes a min-heap by swapping the i-th element with one of the children while needed.

Buffers (stacks and queues)

### Ex. 6.11: Define the following operations:

- `void empty (PriorityQueue *q)` – initialises the queue;

- `int isEmpty (PriorityQueue *q)` – tests if q is empty;

- `int add (int x, PriorityQueue *q)` – adds a value x, returning 0 when the queue is full;

- `int remove (PriorityQueue *q, int *rem)` – removes the next element, and copies it to *rem*.

# Dictionaries

Dictionary: maps keys to values (hashset: just the keys)

(Keys are unique)

**Idea - using a hash function**

- *Magic* function `hash` converts a key into an index (number).
- This index points to the position of an array where the value *should* be found.
- Usually the size of the array is less than the set of possible keys, i.e., `hash` is not injective.
- If 2 keys have the same `hash` value, there is a colision that must be mitigated (alternative solutions exist).

**Hashtables: Closed and Open Addressing (also hashsets))**

**Closed Addressing (or chaining)**
- Table = *array of linked lists*
- Find value of key `k`:
  - go to index `hash(k)`
  - traverse list until `k`

**Open Addressing**
- Table = *just an array*
- Find value of key `k`:
  - go to index `hash(k)`
  - *"jump"* until `k`

**Some concerns**
- Use dynamic arrays (grow when the load factor (#keys/HSIZE) gets high)
  - Need to *rehash*
- Smart *jumps* (`probe` function to know where to jump)
- Need to *garbage collect* in open addressing
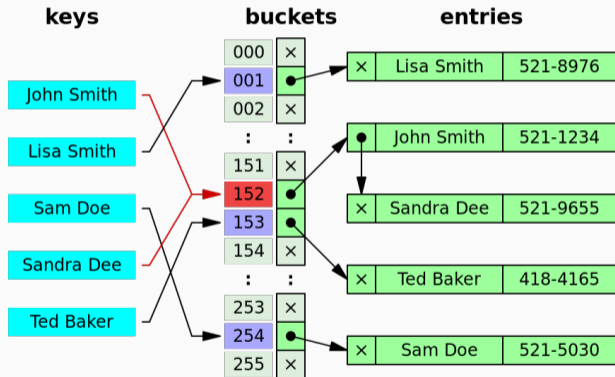
(from Wikipedia)

- int hash(int k, int size);

- void initTab(HTChain h);

- int lookup(HTChain h, int k, int *i);

- int update(HTChain h, int k, int i);

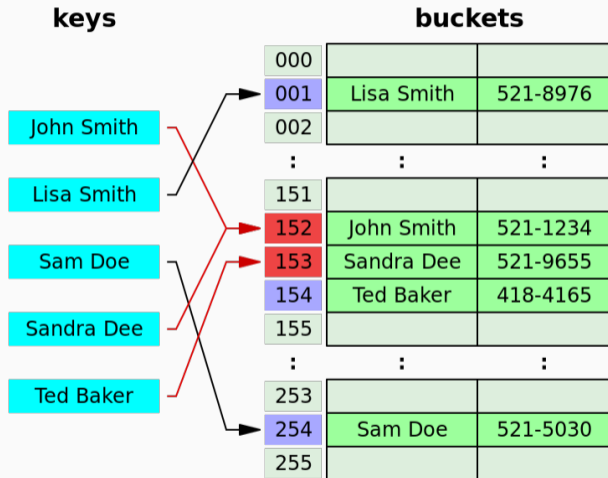- int remove(HTChain h, int k);

```c
#define HSIZE 1000

typedef struct bucket {
  int key;
  int info;
  struct bucket *next;
} *Bucket;

typedef Bucket
   HTChain[HSIZE];
```

**Ex. 6.12:** Implement hash and lookup

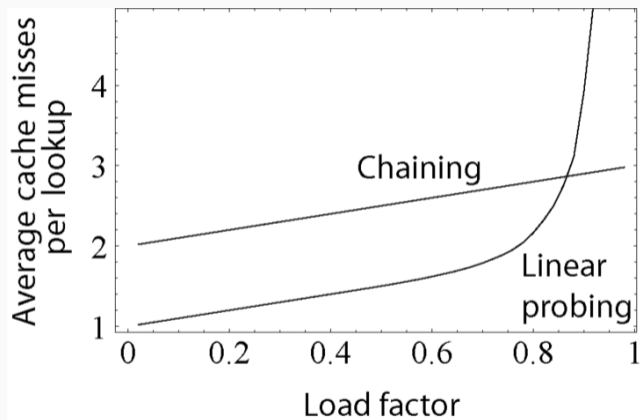**Ex. 6.13:** (Informally) what is the expected complexity of each function?

(from Wikipedia)

- int hash(int k, int size);

- void initTab(HashTable h);

- void lookup(HashTable h, int k, int *i);

- void update(HashTable h, int k, int i);

- void remove(HashTable h, int k);

- int find_probe (HashTable h, int k)
    - linear vs. quadratic probing (why quadratic?)

```
#define HSIZE 1000
#define STATUSFREE 0
#define STATUSUSED 1

typedef struct bucket {
  int status ;
  int key;
  int info;
} Bucket ;

typedef Bucket
  HashTable [HSIZE];
```

**Ex. 6.14:** Define a linear probing function and update.

(from Wikipedia)

# Removing with Open Addressing

- int hash(int k, int size);

- void initTab(HashTable h);

- void lookup(HashTable h, int k, int *i);

- void update(HashTable h, int k, int i);

- int find_probe (HashTable h, int k);

- void remove(HashTable h, int k);

```
#define HSIZE 1000
#define STATUSFREE 0
#define STATUSUSED 1
#define STATUSDEL 2

typedef struct bucket {
  int status ;
  int key;
  int info;
} Bucket ;

typedef Bucket
  HashTable [HSIZE];
```

**Ex. 6.15:** How would you implement update?
How would you implement a *garbageCollect* that removes deleted cells?
What is their complexity?

# Dictionaries with trees – not for evaluation

**We will see:**

- Height- and weight-balanced tree
- Self-balancing binary search tree
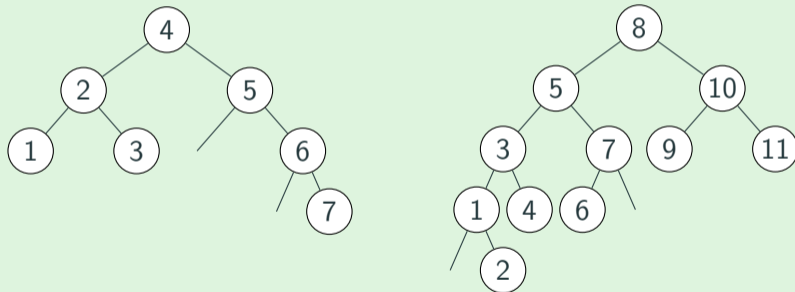    - AVL tree
    - Red-black tree

**Height-balanced**

- more used
- AVL: left-height = right-height $\pm$ 1
- Red-black: similar wrt *black*
- height = log $n$

**Weight-balanced**

- less used
- leafs-left/right $\geq \alpha \times$ leafs, $0 < \alpha < 1$
- better for lookup intensive systems

- By Adelson-Velsky and Landis
- Oldest self-balancing binary search tree data structure to be invented ('62)
- Binary (left-right) search (sorted) tree
- Labels in the nodes
- At every node, the height of left and right trees differ at most by 1
- Insertions and removals preserve this

| Function | Amortized | Worst Case | Amortized (RB) | Worst case (RB) |
|----------|-----------|------------|----------------|-----------------|
| Search | $\Theta(\log n)$ | $\mathcal{O}(\log n)$ | $\mathcal{O}(\log n)$ | $\mathcal{O}(\log n)$ |
| Insert | $\Theta(\log n)$ | $\mathcal{O}(\log n)$ | $\mathcal{O}(1)$ | $\mathcal{O}(\log n)$ |
| Delete | $\Theta(\log n)$ | $\mathcal{O}(\log n)$ | $\mathcal{O}(1)$ | $\mathcal{O}(\log n)$ |

## Ex. 6.16: Are these balanced?

# Update in an AVL tree

**See animation**

```
https://en.wikipedia.org/wiki/AVL_tree#/media/File:
AVL_Tree_Example.gif
```

4 rotations: left, right, right-left, right-right

```c
typedef struct avl {
  int bal;
  int key, info;
  struct avl *left , *right ;
} *AVL;
#define LEFT  -1
#define RIGHT 1
#define BAL   0

// returns 0 if key already existed
int updateAVL (AVL *a, int k, int i);
```

**Ex. 6.17:** How would you implement an **update without balancing**?

**Ex. 6.18:** How would you implement **AVL rotateRight(AVL a)**?

# Full code: updateAVL

```
int updateAVL(AVL *a,
              int k, int i){
  int g, u;
  *a =
    updateAVLRec(*a,k,i,&g,&u);
  return u;
}
```

```
AVL updateAVLRec(AVL a , int k ,
                 int i, int *g , int *u){
 if (a == NULL) {          // insert k->i here
   a = malloc (sizeof (struct avl ));
   a->key=k; a->info=i ; a->bal=BAL;
   a->left=a->right=NULL; *g=1; *u=0;
 } else if (a->key==k) {      // update k->i
   a->info=i; *g=0; *u=1;
 } else if (a->key > k) {      // update left
   a->left = updateAVLRec(a->left,k,i,g,u);
   if (*g == 1) switch (a->bal){ // balance
     case LEFT: a= fixLeft(a); *g=0; break;
     case RIGHT:a->bal=BAL;    *g=0; break;
     case BAL:  a->bal=LEFT;          break;
   }
 } else{ // a->key < k          update right
   // left <--> right
 }
 return  a ;
}
```

# Full code: updateAVL – fix-left

```
AVL fixLeft(AVL a) {
  AVL b, c;
  b=a->left ;
  if (b->bal==LEFT){
    a->bal = b->bal = BAL;
    a=rotateRight(a);
  } else {
    c = b->right ;
    switch (c->bal) {
      case LEFT:   a->bal=RIGHT; b->bal=BAL;  break;
      case RIGHT:  a->bal=BAL;   b->bal=LEFT; break;
      case BAL:    a->bal=BAL;   b->bal=BAL;
    }
    c->bal = BAL;
    a->left = rotateLeft(b);
    a = rotateRight(a);
  }
  return a;
}
```
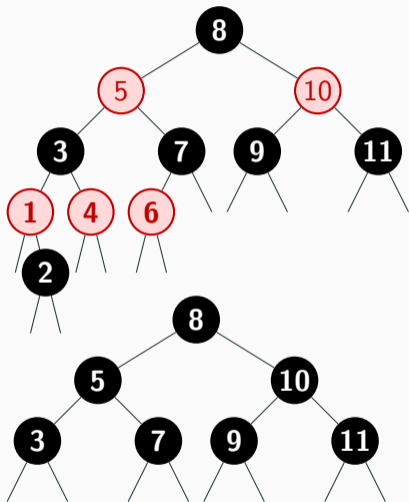
- **1.** Nodes are black or red
- **2.** Empty nodes count as black
- **3.** Red nodes have only black children
- **4.** All down paths from a root have equal black-height

- The root is black.
- Only 1 on the left is a RB tree
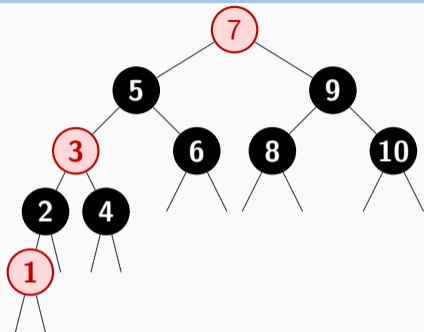
- 6 cases for insertion (with nesting)
- 6 cases for deletion (with nesting)

**Properties**

- height is $\mathcal{O}(\log n)$.
- no path from the root to a leaf is more than twice as long as a path to another leaf

| Function | Amortized (AVL) | Worst Case (AVL) | *Amortized* | *Worst case* |
|----------|-----------------|------------------|-------------|--------------|
| Search | $\Theta(\log n)$ | $\mathcal{O}(\log n)$ | $\mathcal{O}(\log n)$ | $\mathcal{O}(\log n)$ |
| Insert | $\Theta(\log n)$ | $\mathcal{O}(\log n)$ | $\mathcal{O}(1)$ | $\mathcal{O}(\log n)$ |
| Delete | $\Theta(\log n)$ | $\mathcal{O}(\log n)$ | $\mathcal{O}(1)$ | $\mathcal{O}(\log n)$ |

# Red-Black trees height is in $\mathcal{O}(\log n)$

- **1.** Nodes are black or red
- **2.** Empty nodes count as black
- **3.** Red nodes have only black children
- **4.** All down paths from a root have equal black-height

**Lemma: size of a subtree** − $size(x) \geq 2^{bh(x)} - 1$

- $bh(x)$ is the black-height of a node $x$
- base case: $2^{bh(\text{NULL})} - 1 = 2^0 - 1 = size(\text{NULL})$
- inductive case: For each child $c$ of $x$:
  $$bh(c) = bh(x) \text{ or } bh(c) = bh(x) - 1.$$
  Then $size(x) \geq 2 \times (2^{bh(x)-1} - 1) + 1$
  $$= 2^{bh(x)-1+1} - 2 + 1 = 2^{bh(x)-1}$$

**Theorem: Height of a RB tree is $\mathcal{O}(\log n)$**

- Let $h$ be the height of a RB tree $x$
- For any trace $x, \ldots, leaf$, more than half are black
- $\Rightarrow bh(h) \geq h/2$
- $\Rightarrow size(x) \geq 2^{h/2} - 1 \Leftrightarrow h \leq 2\log(n+1) \in \mathcal{O}(\log n)$