

4. Modal Logic & Verification

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<https://fm-dcc.github.io/sv2425>



CISTER - Research Centre in
Real-Time & Embedded
Computing Systems

- Introduction to model-checking
- CCS: a simple language for concurrency
 - Syntax
 - Semantics
 - Equivalence
 - mCRL2: modelling
- Dynamic logic
 - Syntax
 - Semantics
 - Relation with equivalence
 - mCRL2: verification
- Timed Automata
 - Syntax
 - Semantics (composition, Zeno)
 - Equivalence
 - UPPAAL: modelling
- Temporal logics (LTL/CTL)
 - Syntax
 - Semantics
 - UPPAAL: verification
- Probabilistic and stochastic systems
 - Going probabilistic
 - UPPAAL: monte-carlo

Recall: What's in a logic?

A language

i.e. a collection of well-formed expressions to which meaning can be assigned.

A semantics

describing how language expressions are interpreted as statements about something.

A deductive system

i.e. a collection of rules to derive in a purely syntactic way facts and relationships among semantic objects described in the language.

Note

- a purely syntactic approach (up to the 1940's; the **sacred form**)
- a model theoretic approach (A. Tarski legacy)

- sentences
- models & satisfaction: $\mathcal{M} \models \phi$
- validity: $\models \phi$ (ϕ is satisfied in every possible structure)
- logical consequence: $\Phi \models \phi$ (ϕ is satisfied in every model of Φ)
- theory: $Th \Phi$ (set of logical consequences of a set of sentences Φ)

Deductive systems \vdash

- sequents
 - Hilbert systems
 - natural deduction
 - tableaux systems
 - resolution
 - ...
-
- derivation and proof
 - deductive consequence: $\Phi \vdash \phi$
 - theorem: $\vdash \phi$

- A deductive system \vdash is **sound** wrt a semantics \models if for all sentences ϕ

$$\vdash \phi \implies \models \phi$$

(every theorem is valid)

- ... **complete** ...

$$\models \phi \implies \vdash \phi$$

(every valid sentence is a theorem)

For logics with **negation** and a **conjunction** operator

- A sentence ϕ is **refutable** if $\neg\phi$ is a theorem (i.e. $\vdash \neg\phi$)
- A set of sentences Φ is **refutable** if some finite conjunction of elements in Φ is refutable
- ϕ or Φ is **consistent** if it is not refutable.

$$\mathcal{M} \models \phi$$

- Propositional logic (logic of **uninterpreted assertions**; models are **truth assignments**)
- Equational logic (formalises **equational** reasoning; models are **algebras**)
- First-order logic (logic of **predicates** and **quantification** over structures; models are **relational structures**)
- **Modal logics**
- ...

Modal Logic

*Over the years modal logic has been applied in many different ways. It has been used as a tool for reasoning about **time**, **beliefs**, **computational systems**, **necessity** and **possibility**, and much else besides.*

*These applications, though diverse, have something important in common: the key ideas they employ (flows of time, relations between epistemic alternatives, transitions between computational states, networks of possible worlds) can all be represented as **simple graph-like structures**.*

Modal logics are

- **tools to talk about relational, or graph-like structures.**
- **fragments of classical ones**, with restricted forms of quantification ...
- ... which tend to be **decidable** and described in a pointfree notations.

Syntax

$$\phi ::= p \mid \text{true} \mid \text{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid \langle \alpha \rangle \phi \mid [\alpha] \phi$$

where $p \in \text{PROP}$ and $\alpha \in \text{ACT}$

Disjunction (\vee) and equivalence (\leftrightarrow) are defined by abbreviation.

The *signature* of the basic modal language is determined by sets:

- **PROP** of **propositional** symbols (typically assumed to be denumerably infinite) and
- **ACT** of **structured actions** (or programs), also called **modality** symbols.

Syntax

$$\phi ::= p \mid \text{true} \mid \text{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid \langle\alpha\rangle\phi \mid [\alpha]\phi$$

where $p \in \text{PROP}$ and $\alpha \in \text{ACT}$

Ex. 4.1: Interpreting formulas

- $\langle\text{drinkCoffee}\rangle \text{energetic}$: I will now **drink coffee** and will be in an **energetic** state
- $[\text{drink}] \neg \text{thirsty}$: If I **drink** anything now, I will not be in a **thirsty** state

Syntax

$$\phi ::= p \mid \text{true} \mid \text{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid \langle\alpha\rangle\phi \mid [\alpha]\phi$$

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- $[\text{drink}] \neg \text{thirsty}$: If I **drink** anything now, I will not be in a **thirsty** state
- $[\text{something}^*] [\text{pressCoffee}] \langle\text{getCoffee}\rangle \text{true}$:
If I do *something* any number of times, and then
I *press the coffee* button, then
I will *get my coffee* – and **that's it**.

Notes

- if there is only one modality in the signature (i.e., ACT is a singleton), write simply $\Diamond\phi$ and $\Box\phi$
- the language has some redundancy: in particular modal connectives are **dual** (as quantifiers are in first-order logic): $[\alpha]\phi$ is equivalent to $\neg\langle\alpha\rangle\neg\phi$

Example

Models as LTSs over Act.

$ACT = Act$ (sets of actions)

$\langle a \rangle \phi$ can be read as “it **must** observe a , and ϕ must hold after that.”

$[a]\phi$ can be read as “**if** it observes a , then ϕ must hold after that.”

$\mathcal{M}, s \models \phi$ – what does it mean?

Model definition

A **model** for the language is a pair $\mathcal{M} = \langle \mathcal{L}, V \rangle$, where

- $\mathcal{L} = \langle S, \text{ACT}, \longrightarrow \rangle$ is an **LTS**:
 - S is a non-empty set of states (or points)
 - ACT are the labels consisting of (structured) action symbols (or modality symbols)
 - $\longrightarrow \subseteq S \times \text{ACT} \times S$ is the transition relation
- $V : \text{PROP} \longrightarrow \mathcal{P}(S)$ is a **valuation**.

When $\text{ACT} = 1$

- $\Diamond\phi$ and $\Box\phi$ instead of $\langle \cdot \rangle \phi$ and $[\cdot] \phi$
- $\mathcal{L} = \langle S, \longrightarrow \rangle$ instead of $\mathcal{L} = \langle S, \text{ACT}, \longrightarrow \rangle$
- $\longrightarrow \subseteq S \times S$ instead of $\longrightarrow \subseteq S \times \text{ACT} \times S$

Satisfaction: for a model \mathcal{M} and a state s

$\mathcal{M}, s \models \text{true}$

$\mathcal{M}, s \not\models \text{false}$

$\mathcal{M}, s \models p$

iff $s \in V(p)$

$\mathcal{M}, s \models \neg\phi$

iff $\mathcal{M}, s \not\models \phi$

$\mathcal{M}, s \models \phi_1 \wedge \phi_2$

iff $\mathcal{M}, s \models \phi_1$ and $\mathcal{M}, s \models \phi_2$

$\mathcal{M}, s \models \phi_1 \rightarrow \phi_2$

iff $\mathcal{M}, s \not\models \phi_1$ or $\mathcal{M}, s \models \phi_2$

$\mathcal{M}, s \models \langle \alpha \rangle \phi$

iff there exists $v \in S$ st $s \xrightarrow{\alpha} v$ and $\mathcal{M}, v \models \phi$

$\mathcal{M}, s \models [\alpha] \phi$

iff for all $v \in S$ st $s \xrightarrow{\alpha} v$ and $\mathcal{M}, v \models \phi$

Satisfaction

A formula ϕ is

- **satisfiable in a model** \mathcal{M} if it is satisfied at some point of \mathcal{M}
- **globally satisfied** in \mathcal{M} ($\mathcal{M} \models \phi$) if it is satisfied at all points in \mathcal{M}
- **valid** ($\models \phi$) if it is globally satisfied in all models
- **a semantic consequence** of a set of formulas Γ ($\Gamma \models \phi$) if for all models \mathcal{M} and all points s , if $\mathcal{M}, s \models \Gamma$ then $\mathcal{M}, s \models \phi$

Process logic (**Hennessy-Milner logic**)

- $\text{PROP} = \emptyset$ (hence $V = \emptyset$)
- $S = \mathcal{P}$ is a set states in a labelled transition system, typically process terms
- structured actions are built by the grammar $K := a \in \text{Act} \mid K + K$
- the underlying LTS is given by $\mathcal{L} = \langle \mathcal{P}, \text{Act}, \{ \langle p, a, p' \rangle \mid a \in \text{Act} \} \rangle$

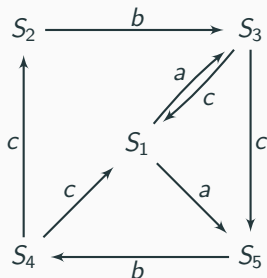
Satisfaction is abbreviated as

$$\begin{array}{ll} p \models \langle K \rangle \phi & \text{iff } \exists_{q \in \{p' \mid p \xrightarrow{a} p' \wedge a \in K\}} . q \models \phi \\ p \models [K] \phi & \text{iff } \forall_{q \in \{p' \mid p \xrightarrow{a} p' \wedge a \in K\}} . q \models \phi \end{array}$$

Process Logic Syntax (Hennessy-Milner Logic)

$$\phi ::= \text{true} \mid \text{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid \langle K \rangle \phi \mid [K] \phi$$

where $K := a \in \text{Act} \mid K + K$



Ex. 4.2: Prove:

1. $S_1 \models [a + b + c] (\langle b + c \rangle \text{true})$
2. $S_2 \models [a] (\langle b \rangle \text{true} \wedge \langle c \rangle \text{true})$
3. $S_1 \not\models [a] (\langle b \rangle \text{true} \wedge \langle c \rangle \text{true})$
4. $S_2 \models [b][c] (\langle a \rangle \text{true} \vee \langle b \rangle \text{true})$
5. $S_1 \models [b][c] (\langle a \rangle \text{true} \vee \langle b \rangle \text{true})$
6. $S_1 \models [a + b] \langle b + c \rangle (\langle a \rangle \text{true})$

$(P, <)$ a strict partial order with infimum 0

i.e., $P = \{0, a, b, c, \dots\}$,

$a \rightarrow b$ means $a < b$,

$a < b$ and $b < c$ implies $a < c$

$0 < x$, for any $x \neq 0$

there are no loops

some elements may not be comparable

- $P, x \models \Box \text{false}$ if x is a maximal element of P
- $P, 0 \models \Diamond \Box \text{false}$ iff ...
- $P, 0 \models \Box \Diamond \Box \text{false}$ iff ...

Temporal logic

- $\langle T, < \rangle$ where T is a set of time points (instants, execution states , ...) and $<$ is the **earlier than** relation on T .
- Thus, $\Box\varphi$ (respectively, $\Diamond\varphi$) means that φ holds in all (respectively, some) time points.

Epistemic logic (J. Hintikka, 1962)

- W is a set of agents
- $\alpha \models [K_i] \phi$ means that agent i always knows that ϕ is true.
- $\alpha \models \langle K_i \rangle \phi$ means that agent i can reach a state where he knows ϕ .
- $\alpha \models (\neg[K_i] \phi) \wedge (\neg[K_i] \neg\phi)$ means that agent i does not know whether ϕ is true or not.

Many variations exist, modelling knowledge and believes, knowledge of who knows what, distributed knowledge, etc.

Deontic logic (G.H. von Wright, 1951)

- Obligations and permissions: **must** and **can** do.
- $\alpha \models \Box \phi$ means ϕ is obligatory.
- $\alpha \models \Diamond \phi$ means ϕ is a possibility.

Each logic accepts a different set of *principles* or *rules* (with variations), that makes their interpretation different.

Ex. 4.3: Express the properties in Process Logic

- inevitability of a :
- progress (can always act):
- deadlock or termination (is stuck):

Ex. 4.4: What does this mean?

1. $\langle - \rangle$ false
2. $[-]$ true

“ $-$ ” stands for $\sum_{a \in \text{Act}} a$, and “ $-x$ ” abbreviates $\sum_{a \notin \text{Act}} a$

Recall syntax

$$\begin{aligned}
 \phi &::= \text{true} \\
 &| \text{false} \\
 &| \neg \phi \\
 &| \phi_1 \wedge \phi_2 \\
 &| \phi_1 \rightarrow \phi_2 \\
 &| \langle K \rangle \phi \\
 &| [K] \phi
 \end{aligned}$$

where $K := a \mid K + K$

Ex. 4.3: Express the properties in Process Logic

- inevitability of a : $\langle - \rangle \text{true} \wedge [-a] \text{false}$
- progress (can always act):
- deadlock or termination (is stuck):

Ex. 4.4: What does this mean?

1. $\langle - \rangle \text{false}$
2. $[-] \text{true}$

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Recall syntax

$$\begin{aligned} \phi &::= \text{true} \\ &| \text{false} \\ &| \neg \phi \\ &| \phi_1 \wedge \phi_2 \\ &| \phi_1 \rightarrow \phi_2 \\ &| \langle K \rangle \phi \\ &| [K] \phi \end{aligned}$$

where $K := a \mid K + K$

Ex. 4.5: Coffee-machine

1. The user can have tea or coffee.
2. The user can have tea but not coffee.
3. The user can have tea after having 2 consecutive coffees.

Ex. 4.6: *a*'s and *b*'s

1. It is possible to do *a* after 3 *b*'s, but not more than 1 *a*.
2. It must be possible to do *a* after [doing *a* and then *b*].
3. After doing *a* and then *b*, it is not possible to do *a*.

Ex. 4.7: Taxi network

- $\phi_0 =$ In a taxi network, a car can *collect* a passenger or be *allocated* by the Central to a pending service
- $\phi_1 =$ This applies only to cars already *on-service*
- $\phi_2 =$ If a car is *allocated* to a service, it must first *collect* the passenger and then *plan* the route
- $\phi_3 =$ On detecting an *emergency* the taxi becomes inactive
- $\phi_4 =$ A car *on-service* is not inactive

Process Logic with regular expressions

$$\phi ::= \text{true} \mid \text{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid \langle \alpha \rangle \phi \mid [\alpha] \phi$$

where $\alpha \in ACT$ are structured actions over a set Act :

$$\alpha := a \in Act \mid \alpha; \alpha \mid \alpha + \alpha \mid \alpha^*$$

More expressive than Process Logic. Used by mCRL2.

Process Logic with regular expressions

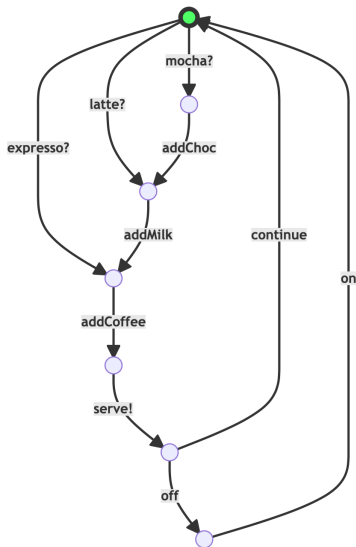
$$\phi ::= \text{true} \mid \text{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid \langle \alpha \rangle \phi \mid [\alpha] \phi$$

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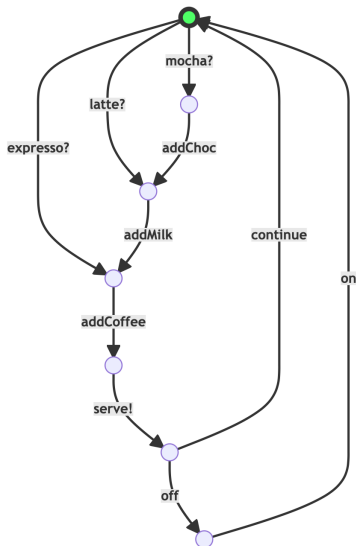
Examples

- “ $\langle a.b.c \rangle \text{true}$ ” means “ $\langle a \rangle \langle b \rangle \langle c \rangle \text{true}$ ”
- “ $[a.b.c] \text{false}$ ” means “ $[a][b][c] \text{false}$ ”
- “ $\langle a^*.b \rangle \text{true}$ ” means that b can be taken after some number of a 's.
- “ $\langle -^*.a \rangle \text{true}$ ” means that a can eventually be taken
- “ $[-^*] \langle a + b \rangle \text{true}$ ” means it is always possible to do a or b



Ex. 4.8: What does this mean?

- $\langle -^*; \text{serve!} \rangle$ true
- $[-^*; (\text{addChoc} + \text{addMilk}); \text{serve!}]$ false
- $[-^*; \text{addCoffee}] \langle \text{serve!} \rangle$ true
- $\langle - \rangle$ true
- $[-^*] \langle - \rangle$ true
- $[-^*.a] \langle b \rangle$ true
- $[-^*.send] \langle (-send)^*.recv \rangle$ true



Ex. 4.9: Express using logic

1. The user **can only have coffee** after the **coffee button** is pressed.
2. The user **must have coffee** after the **coffee button** is pressed.
3. It is always possible to **turn off** the coffee machine.
4. It is always possible to reach a state where the coffee machine can be **turned off**.
5. It is never possible to **add chocolate** right after pressing the *latte button*.

mCRL2 Tools

Slides 3:

<https://fm-dcc.github.io/sv2425/slides/3-mcrl2.pdf>

Bisimulation and modal equivalence

Definition

Given two models $\mathcal{M} = \langle \mathcal{L}, V \rangle$ and $\mathcal{M}' = \langle \mathcal{L}', V' \rangle$, a **bisimulation of \mathcal{L} and \mathcal{L}'** is also a **bisimulation of \mathcal{M} and \mathcal{M}'** if,

whenever $s R s'$, then $V(s) = V'(s')$

Lemma (invariance: bisimulation implies modal equivalence)

Given two models \mathcal{M} and \mathcal{M}' , and a bisimulation R between their states:

if two states s, s' are related by R (i.e. sRs'),

then s, s' satisfy the same basic modal formulas.

(i.e., for all ϕ : $\mathcal{M}, s \models \phi \Leftrightarrow \mathcal{M}', s' \models \phi$)

Hence

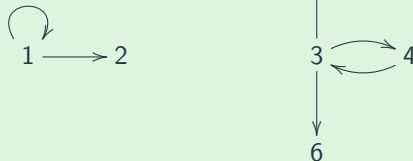
Given 2 models \mathcal{M} and \mathcal{M}' , if you can find ϕ such that

$$\mathcal{M} \models \phi \text{ and } \mathcal{M}' \not\models \phi$$

(or vice-versa) then they are NOT bisimilar.

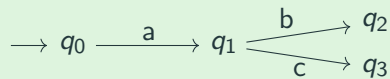
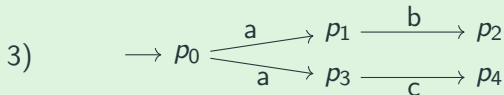
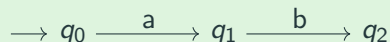
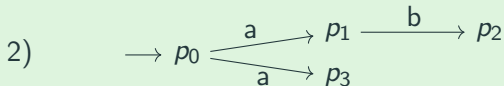
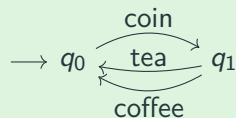
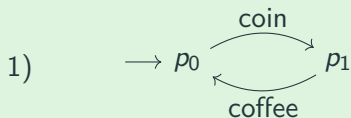
Ex. 4.10: Bisimilarity and modal equivalence

Consider the following transition systems:



Give a modal formula that can be satisfied at point 1 but not at 3.

Ex. 4.11: Find distinguishing modal formula



Richer modal logics

can be obtained in different ways, e.g.

- axiomatic extensions
- introducing more complex satisfaction relations
- support novel semantic capabilities
- ...

Examples

- richer temporal logics
- hybrid logic
- modal μ -calculus

Until and Since

$\mathcal{M}, s \models \phi \mathcal{U} \psi$ iff there **exists** r st $s \leq r$ and $\mathcal{M}, r \models \psi$, and
for all t st $s \leq t < r$, one has $\mathcal{M}, t \models \phi$

$\mathcal{M}, s \models \phi \mathcal{S} \psi$ iff there **exists** r st $r \leq s$ and $\mathcal{M}, r \models \psi$, and
for all t st $r < t \leq s$, one has $\mathcal{M}, t \models \phi$

- Defined for temporal frames $\langle T, < \rangle$ (transitive, asymmetric).
- note the $\exists \forall$ qualification pattern: these operators are neither diamonds nor boxes.
- More general definition for other frames – it becomes more expressive than modal logics.

Temporal logics - rewrite using \mathcal{U}

- $\Diamond\psi =$
- $\Box\psi =$

Temporal logics - rewrite using \mathcal{U}

- $\Diamond\psi = tt\mathcal{U}\psi$
- $\Box\psi =$

Temporal logics - rewrite using \mathcal{U}

- $\Diamond\psi = tt\mathcal{U}\psi$
- $\Box\psi = \neg(\Diamond\neg\psi) = \neg(tt\mathcal{U}\neg\psi)$

$$\phi := \text{true} \mid p \mid \phi_1 \wedge \phi_2 \mid \neg \phi \mid \bigcirc \phi \mid \phi_1 \mathcal{U} \phi_2$$

mutual exclusion	$\Box(\neg c_1 \vee \neg c_2)$
liveness	$\Box \Diamond c_1 \wedge \Box \Diamond c_2$
starvation freedom	$(\Box \Diamond w_1 \rightarrow \Box \Diamond c_1) \wedge (\Box \Diamond w_2 \rightarrow \Box \Diamond c_2)$
progress	$\Box(w_1 \rightarrow \Diamond c_1)$
weak fairness	$\Diamond \Box w_1 \rightarrow \Box \Diamond c_1$
eventually forever	$\Diamond \Box w_1$

- First temporal logic to reason about reactive systems [Pnueli, 1977]
- Formulas are interpreted over **execution paths**
- Express **linear-time properties**

state formulas to express properties of a state:

$$\Phi := \text{true} \mid \Phi \wedge \Phi \mid \neg \Phi \mid \exists \psi \mid \forall \psi$$

path formulas to express properties of a path:

$$\psi := \bigcirc \Phi \mid \Phi \mathcal{U} \Psi$$

mutual exclusion	$\forall \square (\neg c_1 \vee \neg c_2)$
liveness	$\forall \square \forall \Diamond c_1 \wedge \forall \square \forall \Diamond c_2$
order	$\forall \square (c_1 \vee \forall \bigcirc c_2)$

- Branching time structure encode transitive, irreflexive but not necessarily linear flows of time
- flows are **trees**: past linear; branching future

Motivation

Add the possibility of **naming** points and reason about their **identity**

Compare:

$$\Diamond(r \wedge p) \wedge \Diamond(r \wedge q) \rightarrow \Diamond(p \wedge q)$$

with

$$\Diamond(i \wedge p) \wedge \Diamond(i \wedge q) \rightarrow \Diamond(p \wedge q)$$

for $i \in \text{NOM}$ (a **nominal**)

Syntax

$$\phi ::= \dots \mid p \mid \langle \alpha \rangle \phi \mid [\alpha] \phi \mid i \mid @_i \phi$$

where $p \in \text{PROP}$ and $\alpha \in \text{ACT}$ and $i \in \text{NOM}$

Nominals i

- Are special propositional symbols that hold exactly on one state (the state they **name**)
- In a model the **valuation** V is extended from

$$V : \text{PROP} \longrightarrow \mathcal{P}(S)$$

to

$$V : \text{PROP} \longrightarrow \mathcal{P}(S) \quad \text{and} \quad V : \text{NOM} \longrightarrow S$$

where NOM is the set of nominals in the model

- Satisfaction:

$$\mathcal{M}, s \models i \quad \text{iff } s = V(i)$$

The $@_i$ operator $\mathcal{M}, s \models \text{true}$ $\mathcal{M}, s \not\models \text{false}$ $\mathcal{M}, s \models p$ iff $s \in V(p)$ $\mathcal{M}, s \models \neg \phi$ iff $\mathcal{M}, s \not\models \phi$ $\mathcal{M}, s \models \phi_1 \wedge \phi_2$ iff $\mathcal{M}, s \models \phi_1$ and $\mathcal{M}, s \models \phi_2$ $\mathcal{M}, s \models \phi_1 \rightarrow \phi_2$ iff $\mathcal{M}, s \not\models \phi_1$ or $\mathcal{M}, s \models \phi_2$ $\mathcal{M}, s \models \langle \alpha \rangle \phi$ iff **there exists** $v \in S$ st $s \xrightarrow{\alpha} v$ and $\mathcal{M}, v \models \phi$ $\mathcal{M}, s \models [\alpha] \phi$ iff **for all** $v \in S$ st $s \xrightarrow{\alpha} v$ and $\mathcal{M}, v \models \phi$ $\mathcal{M}, s \models @_i \phi$ iff $\mathcal{M}, u \models \phi$ and $u = V(i)$ $[u \text{ is the state denoted by } i]$

Summing up

- basic hybrid logic is a simple notation for capturing the **bisimulation-invariant fragment of first-order logic with constants and equality**, i.e., a mechanism for equality reasoning in propositional modal logic.
- comes **cheap**: up to a polynomial, the complexity of the resulting decision problem is no worse than for the basic modal language