6. Probabilities: Markov chains and statistical model checking

José Proença

System Verification (CC4084) 2025/2026

CISTER – U.Porto, Porto, Portugal

https://fm-dcc.github.io/sv2526

U.PORTO
FACULDADE DE CIÊNCIAS
UNIVERSIDADE DO PORTO

CISTER - Research Centre in
Real-Time & Embedded
Computing Systems

Where we are

Syllabus

- Introduction to model-checking
- CCS: a simple language for concurrency
  - Syntax
  - Semantics
  - Equivalence
  - mCRL2: modelling
- Dynamic logic
  - Syntax
  - Semantics
  - Relation with equivalence
  - mCRL2: verification
- Timed Automata
  - Syntax
  - Semantics (composition, Zeno)
  - Equivalence
  - UPPAAL: modelling
- Temporal logics (LTL/CTL)
  - Syntax
  - Semantics
  - UPPAAL: verification
- Probabilistic and stochastic systems
  - Going probabilistic
  - UPPAAL: monte-carlo

Going probabilistic

Systems can get very complex

- E.g., 5 components, 3 possible traces each
- No communication (pure interleaving)
- Many permutations

Motivation

Systems can get very complex

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- Many permutations
- More components, more traces – untreatable

Motivation

Systems can get very complex

- E.g., 5 components, 3 possible traces each
- No communication (pure interleaving)
- Many permutations
- More components, more traces – untreatable

- Verifying deadlock freedom (and others) requires traversing all states
- Approximation:
  - traverse only part of the states
  - give more priority to some actions
  - return (statistically) likelihood of a given property

Recall: A taxonomy of transition systems

- $\alpha : S \to N \times S$ Moore machine
- $\alpha : S \to \texttt{Bool} \times S^N$ deterministic automata
- $\alpha : S \to \texttt{Bool} \times \mathrm{P}(S)^N$ non-deterministic automata (reactive)

- $\alpha : S \to \mathrm{P}(N \times S)$ non deterministic LTS (generative)
- $\alpha : S \to (S + 1)^N$ partial deterministic LTS
- $\alpha : S \to \mathrm{P}(S)$ unlabelled TS

- $\alpha : S \to \mathrm{D}(S)$ Markov chain

Bringing probabilities to transition systems

Markov chains

$$\alpha : S \to \mathrm{D}(S)$$

where $D(S)$ is the set of all discrete probability distributions on set S

A Markov chain goes from a state s to a state s' with probability p if

$$\alpha(s) = \mu \quad \text{with} \quad \mu(s') = p > 0$$

Recall discrete distributions

Recall

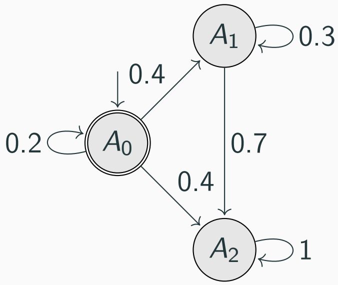
$\mu : S \to [0,1]$ is a discrete probability distribution if

- $\{s \in S \mid \mu(s) > 0\}$, is finite (called the support of μ), and
- $\sum_{s \in S} \mu(s) = 1$

Examples

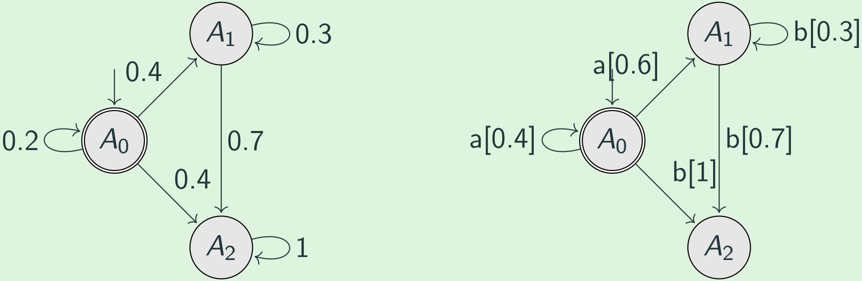
- Dirac distribution: $\mu_s^1 = \{s \to 1\}$
- Product distribution: $(\mu_1 \times \mu_2)\langle s,t\rangle = \mu_1(s) \times \mu 2(t)$

Example



$$\alpha : S \rightarrow (\mathrm{D}(S) + 1)^N$$

Ex. 6.1: Formalise the system below on the right as a function

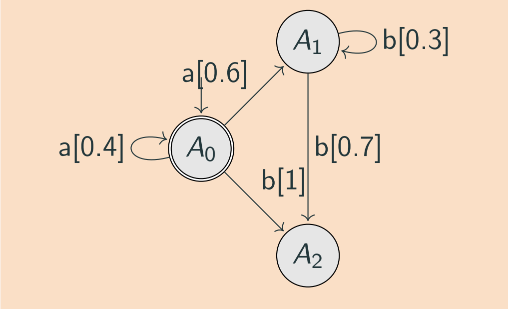


Notions of bisimulation arise naturally.

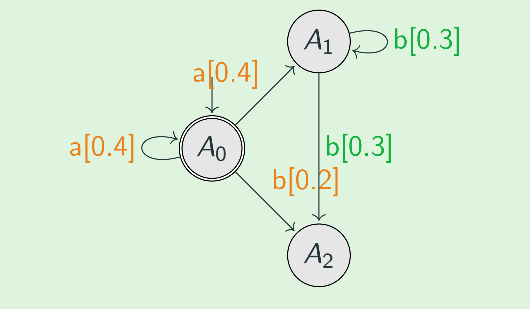
Generative PTS

$$\alpha : S \to \mathrm{D}((S \times N) + 1)$$

Before (reactive)



Ex. 6.2: Now (generative) – formalise it



Generative Timed PTS (as in Uppaal)

$$\alpha : S \to \mathrm{D}(S \times N) + 1$$

Generative Timed PTS (as in Uppaal)

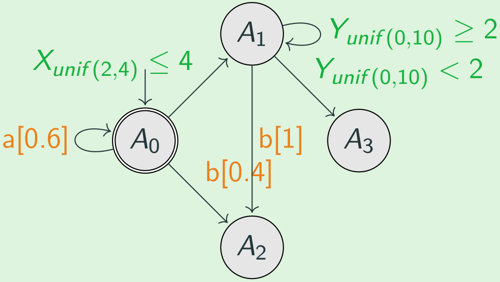
$$\alpha : S \rightarrow \mathrm{D}(S \times N) + 1$$

$$\alpha : S \rightarrow (\mathrm{D}(S \times N) + 1) \times \mathrm{D}_{cont}(\mathcal{R}_0^+ \times S)$$

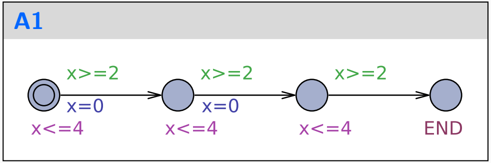
Notes

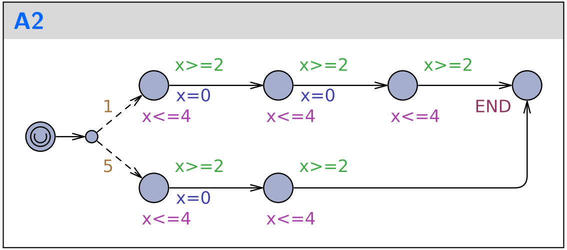
- Continuous time: continuous distribution
- Probabilities both at
  - discrete transitions and
  - continuous delays

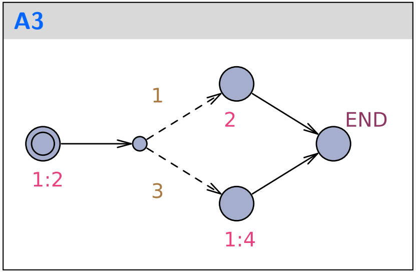
Ex. 6.3: Now (Timed PTS)



Probabilities in Uppaal


A1
x>=2
x=0
x<=4
x>=2
x=0
x<=4
x>=2
x<=4
END


A2
1
5
x>=2
x=0
x<=4
x>=2
x=0
x<=4
x>=2
x<=4
END
x>=2
x=0
x<=4
x>=2
x<=4


A3
1
3
1:2
2
1:4
END

Stochastic Timed automata Definition

$$\langle \mathbf{L}, \mathbf{L_0}, \mathbf{Act}, \mathbf{C}, \mathbf{Tr}, \mathbf{Inv} \rangle$$

where

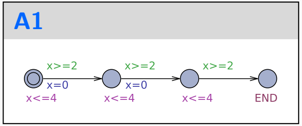
- L is a set of locations, and $L_0 \subseteq L$ the set of initial locations
- Act is a set of actions and C a set of clocks
- $Tr \subseteq L \times (\mathcal{C}(C) \cup \mathbb{N}) \times Act \times \mathcal{P}(C) \times L$ is the transition relation

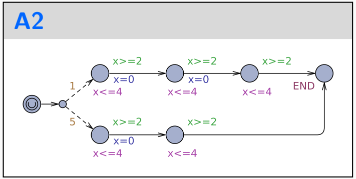
  $$\ell_1 \xrightarrow{g,a,U} \ell_2 \qquad\qquad \text{or} \qquad\qquad \ell_1 \xrightarrow{w,a,U} \ell_2$$

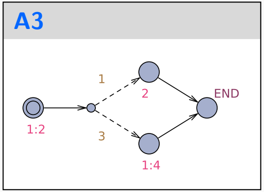
  denotes a transition from location ℓ_1 to ℓ_2, labelled by a, enabled if guard g is valid, which, when performed, resets the set U of clocks, with a probability given by the weight w
- $Inv : L \longrightarrow \mathcal{C}(C) + \mathbb{Q}$ is the assignment of invariants or rates (of an exponential distribution) to locations

where $\mathcal{C}(C)$ denotes the set of clock constraints over a set C of clock variables

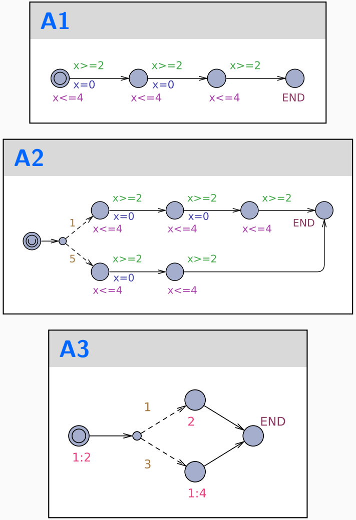
Again A1,A2,A3: Timed PTS





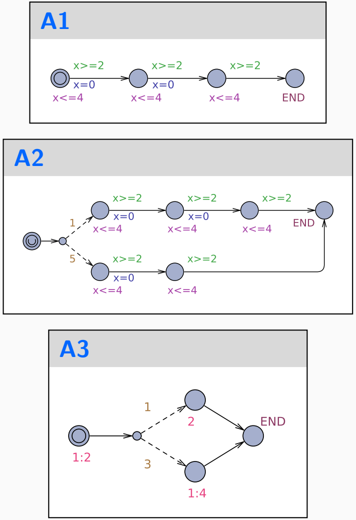


- Probability of $\langle A1_0, \overline{0}\rangle \xrightarrow{0.5} \langle A1_0, \overline{0.5}\rangle$?



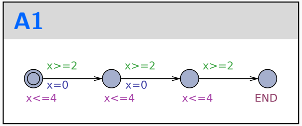
- Probability of $\langle A1_0, \overline{0}\rangle \xrightarrow{0.5} \langle A1_0, \overline{0.5}\rangle$?
- Probability of reaching $A1_1$ within 1?

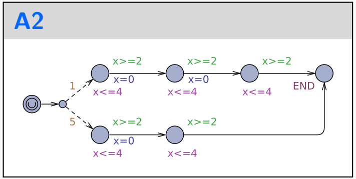
Again A1,A2,A3: Timed PTS

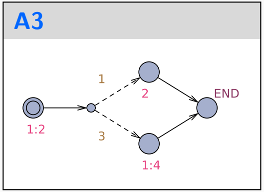


- Probability of $\langle A1_0, \overline{0}\rangle \xrightarrow{0.5} \langle A1_0, \overline{0.5}\rangle$?
- Probability of reaching $A1_1$ within 1?
- Probability of reaching $A1_1$ within 5?

Again A1,A2,A3: Timed PTS

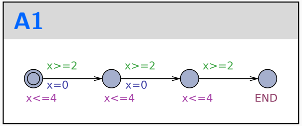


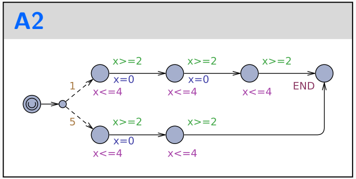


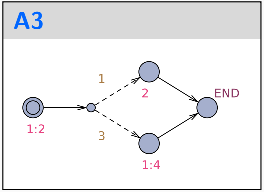


- Probability of $\langle A1_0, \overline{0}\rangle \xrightarrow{0.5} \langle A1_0, \overline{0.5}\rangle$?
- Probability of reaching $A1_1$ within 1?
- Probability of reaching $A1_1$ within 5?
- Probability of reaching $A2_1$ (above) within 5?

Again A1,A2,A3: Timed PTS

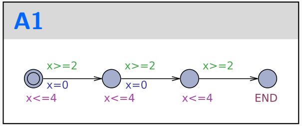


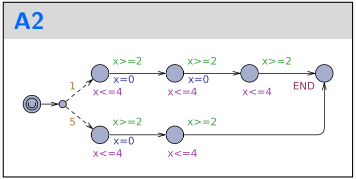


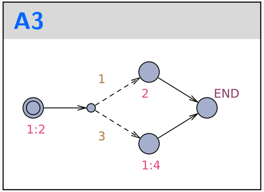


- Probability of $\langle A1_0, \overline{0} \rangle \xrightarrow{0.5} \langle A1_0, \overline{0.5} \rangle$?
- Probability of reaching $A1_1$ within 1?
- Probability of reaching $A1_1$ within 5?
- Probability of reaching $A2_1$ (above) within 5?
- Expected time to reach $A1_1$?

Again A1,A2,A3: Timed PTS

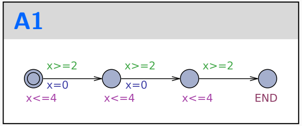


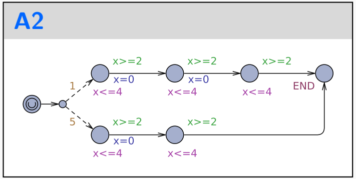


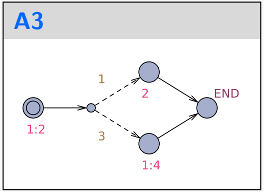


- Probability of $\langle A1_0, \overline{0}\rangle \xrightarrow{0.5} \langle A1_0, \overline{0.5}\rangle$?
- Probability of reaching $A1_1$ within 1?
- Probability of reaching $A1_1$ within 5?
- Probability of reaching $A2_1$ (above) within 5?
- Expected time to reach $A1_1$?
- Expected time to reach $A3_1$ or $A3_2$?

Again A1,A2,A3: Timed PTS



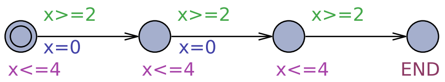




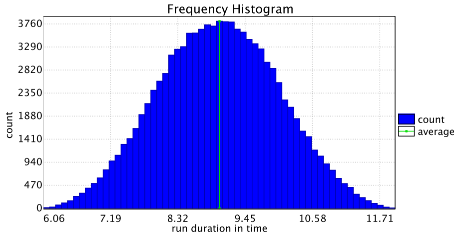
- Probability of $\langle A1_0, \overline{0} \rangle \xrightarrow{0.5} \langle A1_0, \overline{0.5} \rangle$?
- Probability of reaching $A1_1$ within 1?
- Probability of reaching $A1_1$ within 5?
- Probability of reaching $A2_1$ (above) within 5?
- Expected time to reach $A1_1$?
- Expected time to reach $A3_1$ or $A3_2$?
- Expected time to reach $A1_{END}$?
- Expected time to reach $A2_{END}$?
- Expected time to reach $A3_{END}$?

A1: When does it end?

A1



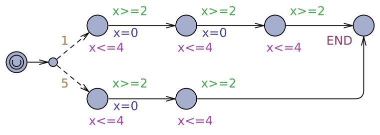
A1's histogram



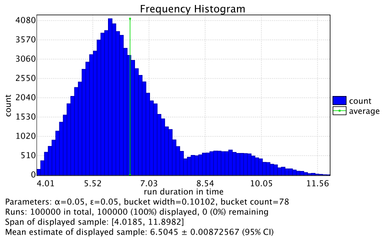
Parameters: α=0.05, ε=0.05, bucket width=0.10002, bucket count=59
Runs: 102000 in total, 102000 (100%) displayed, 0 (0%) remaining
Span of displayed sample: [6.06618, 11.9675]
Mean estimate of displayed sample: 9.00732 ± 0.00614869 (95% CI)

- Run 102000 times
- Histogram: how many times it took [9..9.1] seconds?
- ...

A2



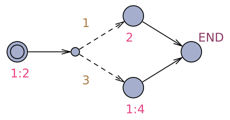
A2's histogram



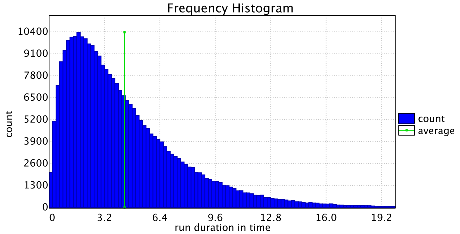
- Run 100000 times
- Histogram: how many times it took [9..9.1] seconds?
- ...

A3: When does it end?

A3



A3's histogram



Parameters: α=0.05, ε=0.05, bucket width=0.19991, bucket count=100
Runs: 300000 in total, 299181 (99.727%) displayed, 819 (0.273%) remaining
Span of displayed sample: [0.0045298, 19.9958]
Mean estimate of displayed sample: 4.31514 ± 0.0118734 (95% CI)

- Run 300000 times
- Histogram: how many times it took [9..9.1] seconds?
- ...

Probabilistic queries in Uppaal

Probabilistic queries

- `Pr[c<=10; 100]([] safe)` – runs 100 stochastic simulations and estimates the probability of `safe` remaining true within 10 cost units, based on 100 runs.
- `Pr[<=10](<> good)` – runs a number of stochastic simulations and estimates the probability of `good` eventually becoming true within 10 time units. The number of runs is decided based on the probability interval precision ($\pm\varepsilon$) and confidence level (level of significance α).
- `Pr[<=10](<> good) >= 0.5` – checks if the probability of reaching good within 10 time units is greater than 50% (less runs than calculating the probability, using "Walds's algorithm")
- `E[<=10; 100](max: cost)` runs 100 stochastic simulations and estimates the maximal value of cost expression over 10 time units of stochastic simulation.

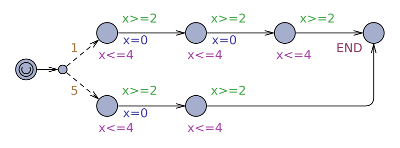
More at `https://docs.uppaal.org/language-reference/query-syntax/statistical_queries/`

Running a single simulation

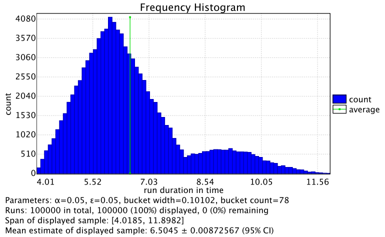
- `simulate[<=10] { x, y }` creates one stochastic simulation run of up to 10 time units in length and plot the values of `x` and `y` expressions over time (after checking, right-click the query and choose a plot).
- Variations: `[c<=10] / [#<=10]` – based on clock c or based on the number of transitions.

Replicate the histograms

A2



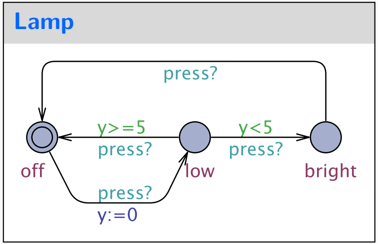
A2's histogram



Ex. 6.4: Replicate the visualisation

Ex. 6.5: Replicate the visualisation also for A1 and A3

Exercise: create a stochastic simulation of the lamp



Ex. 6.6: Adapt the model to make it stochastic

Ex. 6.7: Adapt requirements to make them probabilistic

1. The lamp can become bright;
2. The lamp will eventually become bright;
3. The lamp can never be on for more than 3600s;
4. It is possible to never turn on the lamp;
5. Whenever the light is bright, the clock y is non-zero;
6. Whenever the light is bright, it will eventually become off.