6. Real-time models: Verifying Timed Automata

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https://fm-dcc.github.io/sv2526

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Real-Time & Embedded
Computing Systems

Behavioural Equivalences

Definition

A timed trace over a timed LTS is a (finite or infinite) sequence $\langle t_1, a_1 \rangle, \langle t_2, a_2 \rangle, \cdots$ in $\mathcal{R}_0^+ \times Act$ such that there exists a path

$$\langle \ell_0, \eta_0 \rangle \xrightarrow{d_1} \langle \ell_0, \eta_1 \rangle \xrightarrow{a_1} \langle \ell_1, \eta_2 \rangle \xrightarrow{d_2} \langle \ell_1, \eta_3 \rangle \xrightarrow{a_2} \cdots$$

such that

$$t_i = t_{i-1} + d_i$$

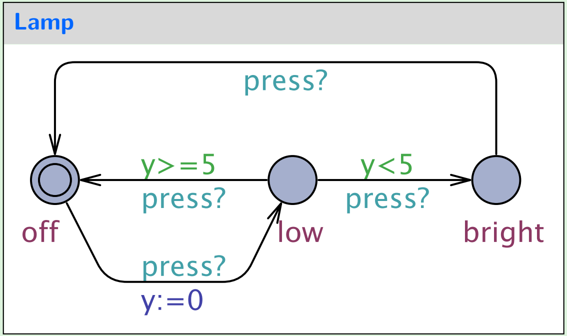
with $t_0 = 0$ and, for all clock x, $\eta_0\, x = 0$.

Intuitively, each t_i is an absolute time value acting as a time-stamp.

Warning

All results from now on are given over an arbitrary timed LTS; they naturally apply to $\mathcal{T}(ta)$ for any timed automata ta.

Ex. 6.1: Write 4 possible timed traces

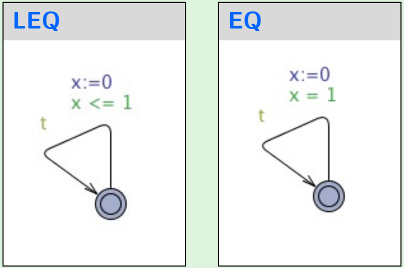


Given a timed trace tc, the corresponding untimed trace is $(\pi_2)^\omega\ tc$.

Definition

- two states s_1 and s_2 of a timed LTS are timed-language equivalent if the set of finite timed traces of s_1 and s_2 coincide;
- ... similar definition for untimed-language equivalent ...

Ex. 6.2: Why?



are not timed-language equivalent

Bisimulation

Timed bisimulation (between states of timed LTS)

A relation R is a timed simulation iff whenever $s_1 R s_2$, for any action a and delay d,

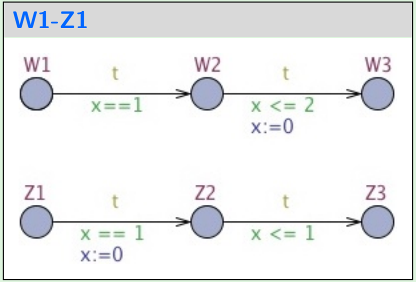
$$s_1 \xrightarrow{a} s_1' \;\Rightarrow\; \text{there is a transition} \quad s_2 \xrightarrow{a} s_2' \wedge s_1' R s_2'$$

$$s_1 \xrightarrow{d} s_1' \;\Rightarrow\; \text{there is a transition} \quad s_2 \xrightarrow{d} s_2' \wedge s_1' R s_2'$$

And a timed bisimulation if its converse is also a timed simulation.

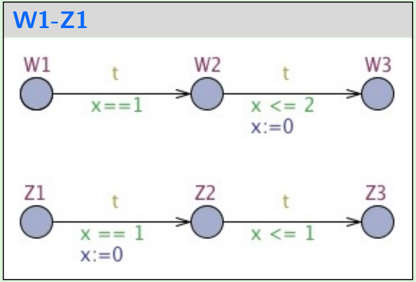
Bisimulation

Example



W1 bisimilar to Z1?

Example



W1 bisimilar to Z1?

$$\langle\langle W1, \{x \mapsto 0\}\rangle, \langle Z1, \{x \mapsto 0\}\rangle\rangle \in R$$

where

$$\begin{array}{rcllll} R & = & \{\langle\langle W1, \{x \mapsto d\}\rangle & , \langle Z1, \{x \mapsto d\}\rangle\rangle & \mid d \in \mathcal{R}_0^+\} & \cup \\ & & \{\langle\langle W2, \{x \mapsto d+1\}\rangle & , \langle Z2, \{x \mapsto d\}\rangle\rangle & \mid d \in \mathcal{R}_0^+\} & \cup \\ & & \{\langle\langle W3, \{x \mapsto d\}\rangle & , \langle Z3, \{x \mapsto e\}\rangle\rangle & \mid d, e \in \mathcal{R}_0^+\} & \end{array}$$

Untimed bisimulation

A relation R is an untimed simulation iff whenever $s_1 R s_2$, for any action a and delay t,

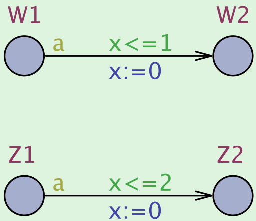
$$s_1 \xrightarrow{a} s_1' \;\Rightarrow\; \text{there is a transition} \quad s_2 \xrightarrow{a} s_2' \wedge s_1' R s_2'$$

$$s_1 \xrightarrow{d} s_1' \;\Rightarrow\; \text{there is a transition} \quad s_2 \xrightarrow{d'} s_2' \wedge s_1' R s_2'$$

And it is an untimed bisimulation if its converse is also an untimed simulation.

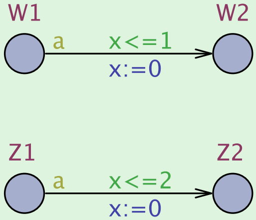
Alternatively, it can be defined over a modified LTS in which all delays are abstracted on a unique, special transition labelled by ϵ.

Ex. 6.3: W1 bisimilar to Z1?



Untimed Bisimulation

Ex. 6.3: W1 bisimilar to Z1?



$$\langle\langle W1, \{x \mapsto 0\}\rangle, \langle Z1, \{x \mapsto 0\}\rangle\rangle \in R$$

where

$$R \;=\; \{\langle\langle W1, \{x \mapsto d\}\rangle\ ,\langle Z1, \{x \mapsto d'\}\rangle\rangle \mid 0 \le d \le 1, 0 \le d' \le 2\} \;\cup$$
$$\quad \ldots$$

Behavioural Properties

The satisfaction problem

Given a timed automata *ta* and a property ϕ, show that

$$\mathcal{T}(ta) \models \phi$$

Properties: expression and satisfaction

The satisfaction problem

Given a timed automata *ta* and a property ϕ, show that

$$\mathcal{T}(ta) \models \phi$$

- in which logic language shall ϕ be specified?
- how is $\models$ defined?

Expressing properties: Uppaal

Uppaal variant of CTL

- state formulae: describes individual states in $\mathcal{T}(ta)$
- path formulae: describes properties of paths in $\mathcal{T}(ta)$

State formulae

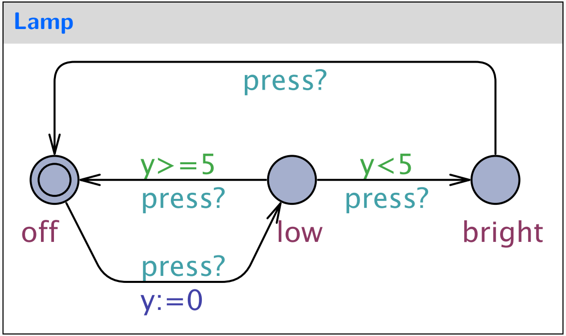
$$\Psi \ ::= ta.\ell \mid g_c \mid g_d \mid \texttt{deadlock} \mid \texttt{not}\ \Psi \mid \Psi\ \texttt{or}\ \Psi \mid \Psi\ \texttt{and}\ \Psi \mid \Psi\ \texttt{imply}\ \Psi$$

Any expression which can be evaluated to a boolean value for a state (typically involving the clock constraints used for guards and invariants and similar constraints over integer variables):

`x >= 8`, `i == 8 and x < 2`, ...

Additionally,

- $ta.\ell$ which tests current location: $(\ell, \eta) \models ta.\ell$
  provided (ℓ, η) is a state in $\mathcal{T}(ta)$
- `deadlock`: $(\ell, \eta) \models \forall_{d \in \mathcal{R}_0^+}$. there is no transition from $\langle \ell, \eta + d \rangle$



Ex. 6.4: Write a state formula

1. The lamp is `low`
2. Not `off` and $y > 25$
3. If it is `low` or `bright`, then $y \leq 3600$

Path formulae

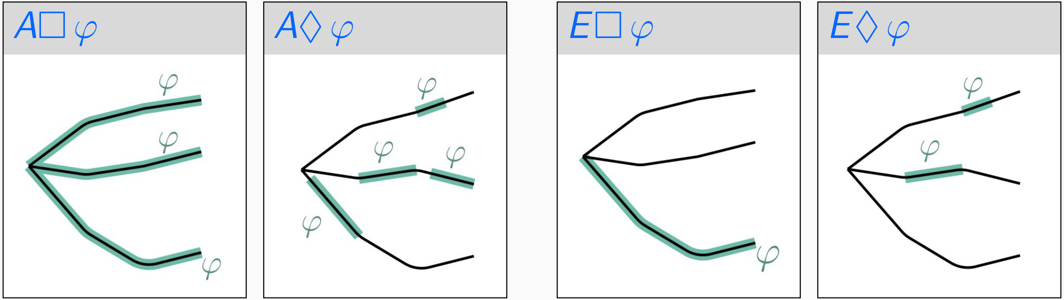
$$\Pi \ ::= A\square\,\Psi \mid A\lozenge\,\Psi \mid E\square\,\Psi \mid E\lozenge\,\Psi \mid \Phi \leadsto \Psi$$

where

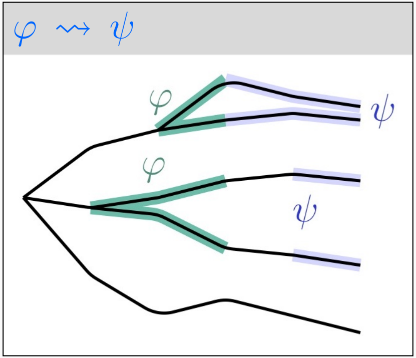
- A, E quantify (universally and existentially, resp.) over paths
- $\square$, $\lozenge$ quantify (universally and existentially, resp.) over states in a path

also notice that

$$\Phi \leadsto \Psi \ \stackrel{\text{abv}}{=}\ A\square\,(\Phi \Rightarrow A\lozenge\,\Psi)$$


$A\Box\,\varphi$
φ
φ
φ
$A\Diamond\,\varphi$
φ
φ
φ
φ
$E\Box\,\varphi$
φ
$E\Diamond\,\varphi$
φ
φ

Expressing properties: Uppaal



Example

If a message is sent, it will eventually be received – `send(m)` $\rightsquigarrow$ `received(m)`

Reachability properties

$E\Diamond\,\phi$

Is there a path starting at the initial state, such that a state formula ϕ is eventually satisfied?

- Often used to perform sanity checks on a model:
  - is it possible for a sender to send a message?
  - can a message possibly be received?
  - ...
- Do not by themselves guarantee the correctness of the protocol (i.e. that any message is eventually delivered), but they validate the basic behavior of the model.

Safety properties

$A\Box\,\phi$ **and** $E\Box\,\phi$

Something bad will never happen
or something bad will possibly never happen

Examples

- In a nuclear power plant the temperature of the core is always (invariantly) under a certain threshold.
- In a game a safe state is one in which we can still win, ie, will possibly not loose.

In Uppaal these properties are formulated positively: something good is invariantly true.

Liveness properties

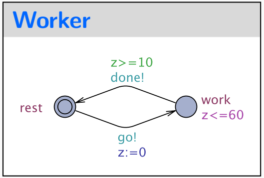
$A\Diamond\,\phi$ **and** $\phi \rightsquigarrow \psi$

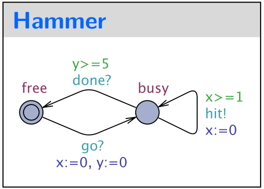
Something good will *eventually happen*
or if *something* happens, then *something else* will eventually happen!

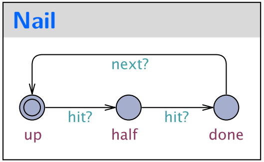
Examples

- When pressing the on button, then eventually the television should turn on.
- In a communication protocol, any message that has been sent should eventually be received.

Exercise: worker, hammer, nail - revisited





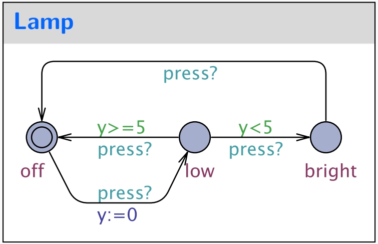


Ex. 6.5: Write properties and explain them

1. Using $E\Diamond$
2. Using $E\Box$
3. Using $A\Diamond$
4. Using $A\Box$
5. Using $\leadsto$

(Practice in UPPAAL)

Exercise: write formulas

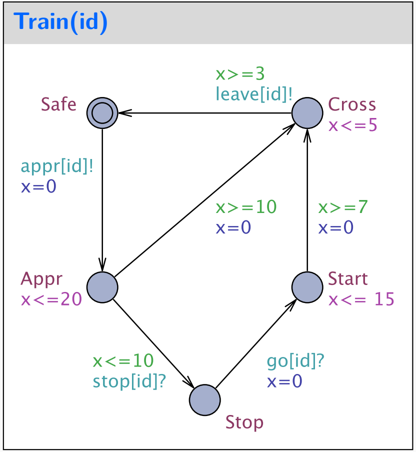


Ex. 6.6: Write formulas, and say which ones are true

1. The lamp can become bright;
2. The lamp will eventually become bright;
3. The lamp can never be on for more than 3600s;
4. It is possible to never turn on the lamp;
5. Whenever the light is bright, the clock y is non-zero;
6. Whenever the light is bright, it will eventually become off.

Examples: proving mutual exclusion

The train gate example (1/2)

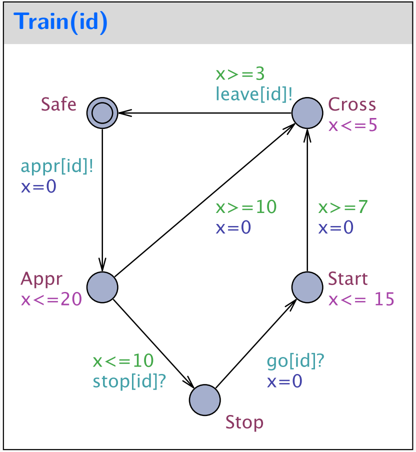


(Train 0 can reach the cross)

(Train 0 can be crossing bridge while Train 1 is waiting to cross)

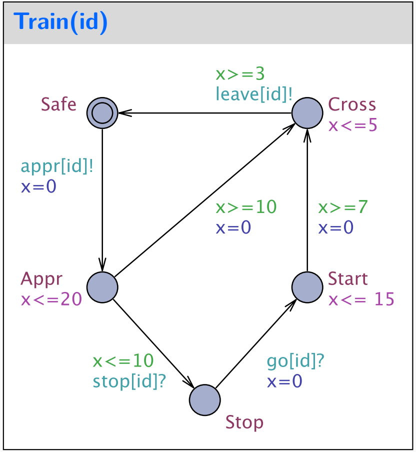
(Train 0 can cross bridge while the other trains are waiting to cross)

The train gate example (1/2)



- `E<> Train(0).Cross`
  (Train 0 can reach the cross)
- `E<> Train(0).Cross and Train(1).Stop`
  (Train 0 can be crossing bridge while Train 1 is waiting to cross)
- ```
  E<> Train(0).Cross and
          (forall (i:id-t)
                i != 0 imply Train(i).Stop)
  ```
  (Train 0 can cross bridge while the other trains are waiting to cross)
  ```

The train gate example (2/2)



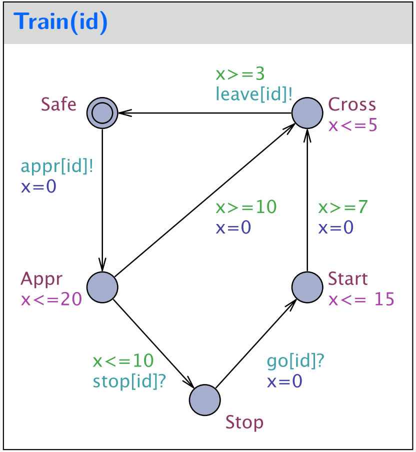
There can never be N elements in the queue

There is never more than one train crossing the bridge

Whenever a train approaches the bridge, it will eventually cross

The system is deadlock-free

The train gate example (2/2)



- `A[] Gate.list[N] == 0`
  There can never be N elements in the queue
- `A[] forall (i:id-t) forall (j:id-t) Train(i).Cross && Train(j).Cross imply i == j`
  There is never more than one train crossing the bridge
- `Train(1).Appr -> Train(1).Cross`
  Whenever a train approaches the bridge, it will eventually cross
- `A[] not deadlock`
  The system is deadlock-free

Properties

- mutual exclusion: no two processes are in their critical sections at the same time
- deadlock freedom: if some process is trying to access its critical section, then eventually some process (not necessarily the same) will be in its critical section; similarly for exiting the critical section

Mutual exclusion

The Problem

- Dijkstra's original asynchronous algorithm (1965) requires, for n processes to be controlled, $\mathcal{O}(n)$ read-write registers and $\mathcal{O}(n)$ operations.
- This result is a theoretical limit (proved by Lynch and Shavit in 1992) which compromises scalability.

Mutual exclusion

The Problem

- Dijkstra's original asynchronous algorithm (1965) requires, for n processes to be controlled, $\mathcal{O}(n)$ read-write registers and $\mathcal{O}(n)$ operations.
- This result is a theoretical limit (proved by Lynch and Shavit in 1992) which compromises scalability.

but it can be overcome by introducing specific timing constraints

Two *timed* algorithms:

- Fisher's protocol (included in the UPPAAL distribution)
- Lamport's protocol

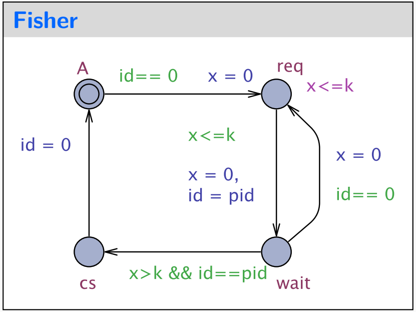
The algorithm

repeat
    repeat
        await $id = 0$
        $id := i$
        delay(k)
    until $id = i$
    (critical section)
    $id := 0$
forever

Comments

- One shared read/write register (the variable *id*)
- Behaviour depends crucially on the value for k — the time delay
- Constant k should be larger than the longest time that a process may take to perform a step while trying to get access to its critical section
- This choice guarantees that whenever process i finds $id = i$ on testing the loop guard it can enter safely ist critical section: all other processes are out of the loop or with their index in *id* overwritten by i.

Fisher's algorithm in Uppaal



- Each process uses a local clock x to guarantee that the upper bound between between its successive steps, while trying to access the critical section, is k (cf. invariant in state *req*).
- Invariant in state *req* establishes k as such an upper bound
- Guard in transition from *wait* to *cs* ensures the correct delay before entering the critical section

Fisher's algorithm in Uppaal

Properties

```
% P(1) requests access => it will eventually wait
P(1).req --> P(1).wait
% the algorithm is deadlock-free
A[] not deadlock
% mutual exclusion invariant
A[] forall (i:int[1,6]) forall (j:int[1,6])
   P(i).cs && P(j).cs imply i == j
```

- The algorithm is deadlock-free
- It ensures mutual exclusion if the correct timing constraints.
- ... but it is critically sensible to small violations of such constraints: for example, replacing $x > k$ by $x \geq k$ in the transition leading to cs compromises both mutual exclusion and liveness.

Lamport's algorithm

The algorithm

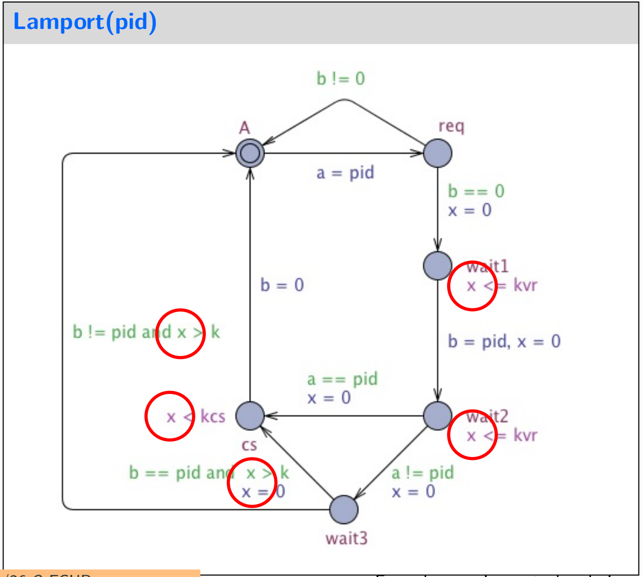
$$
\begin{aligned}
\text{start}: \quad & a := i \\
& \text{if } b \neq 0 \text{ then goto start} \\
& b := i \\
& \text{if } a \neq i \text{ then delay}(k) \\
& \qquad\qquad \text{else if } b \neq i \text{ then goto start} \\
& \textit{(critical section)} \\
& b := 0
\end{aligned}
$$

Lamport's algorithm

Comments

- Two shared read/write registers (variables a and b)
- Avoids forced waiting when no other processes are requiring access to their critical sections

Lamport's algorithm in Uppaal



Model time constants:

- k — time delay
- kvr — max bound for register access
- kcs — max bound for permanence in critical section

Typically $\boxed{k \;\geq\; kvr + kcs}$

Experiments

	k	kvr	kcs	verified?
Mutual Exclusion	4	1	1	Yes
Mutual Exclusion	2	1	1	Yes
Mutual Exclusion	1	1	1	No
No deadlock	4	1	1	Yes
No deadlock	2	1	1	Yes
No deadlock	1	1	1	Yes