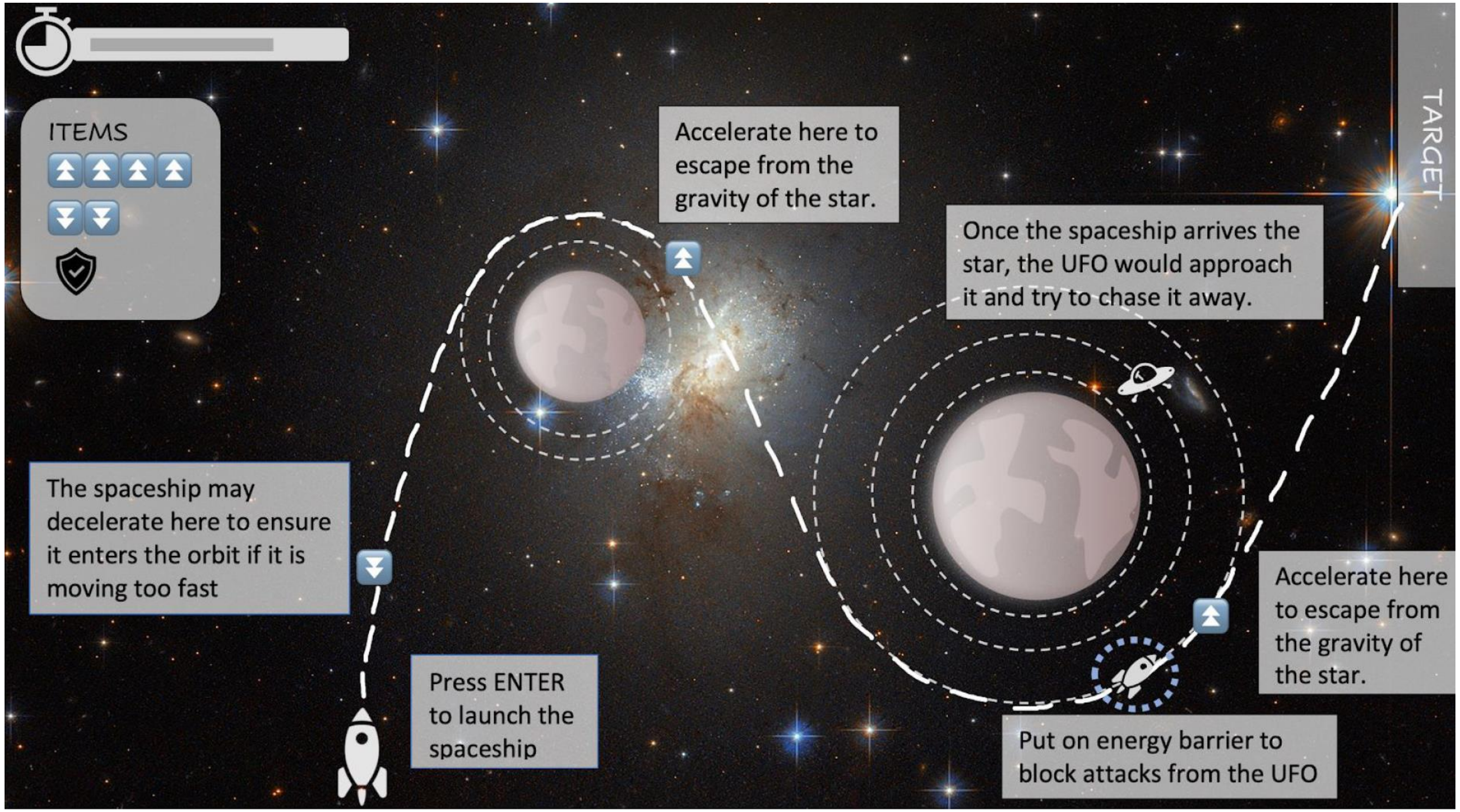
Physical Simulation



Overview

1. Equation of Motion

- Examples
- Ordinary Differentiable Equations (ODE)
- Solving ODEs

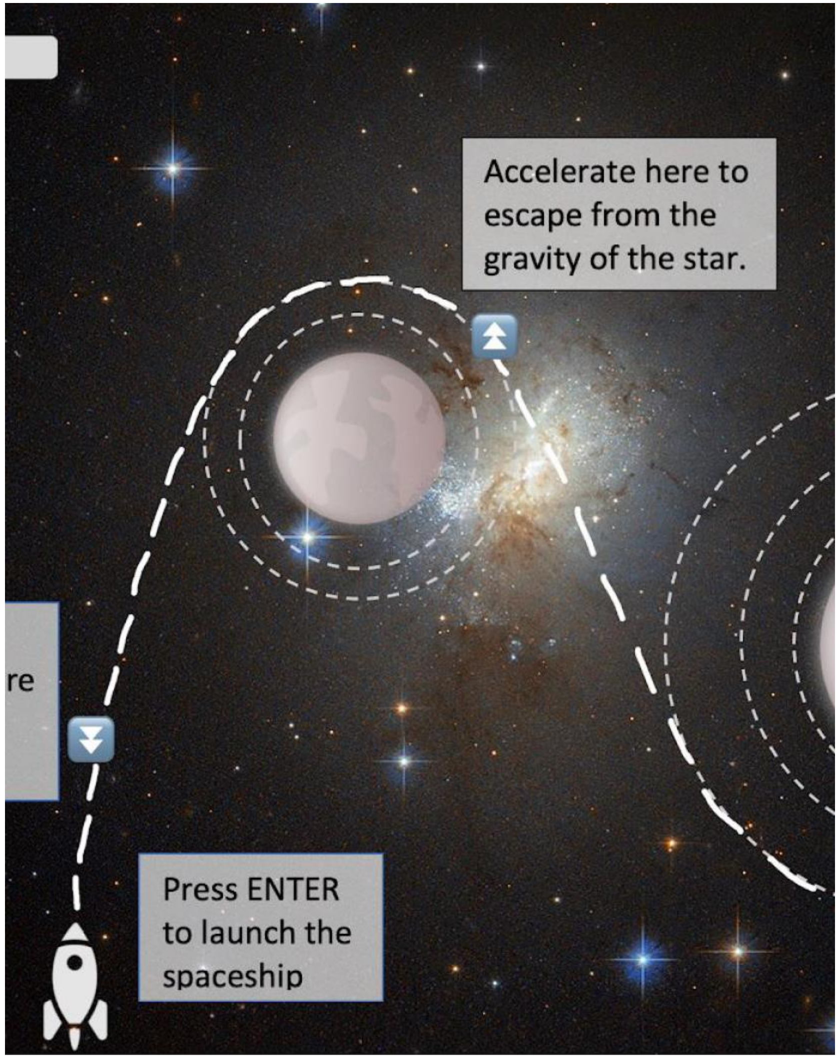
2. Collision and Reaction Forces

Physics

Learning goals:

- ***Connect your theoretical math knowledge to applications***

- ***Properly simulate object motion and their interaction in your game***



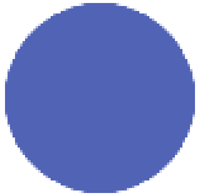
Basic Particle Simulation (first try)

How to compute the change in velocity?

$$\boldsymbol{d_t} = \boldsymbol{t_{i+1}} - \boldsymbol{t_i}$$

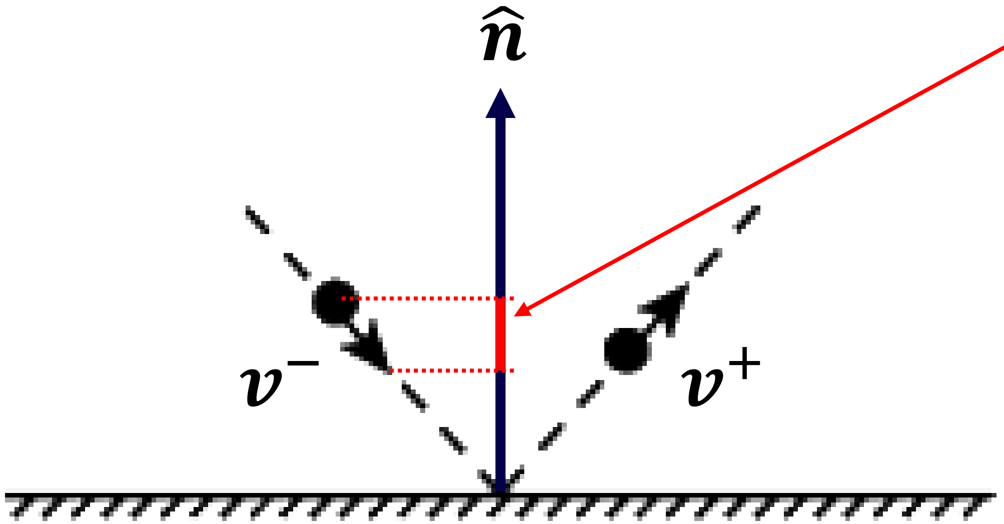
$$\vec{\boldsymbol{v}}_{i+1} = \vec{\boldsymbol{v}}_i + \boldsymbol{\Delta v}$$

$$\vec{\boldsymbol{p}}_{i+1} = \vec{\boldsymbol{p}}(\boldsymbol{t_i}) + \vec{\boldsymbol{v}}_i \boldsymbol{d_t}$$



Particle-Plane Collision

- ***In direction of normal***



Velocity along normal
(v projected on normal
by the dot product)

Apply change
along normal
(magnitude
times direction)

Frictionless

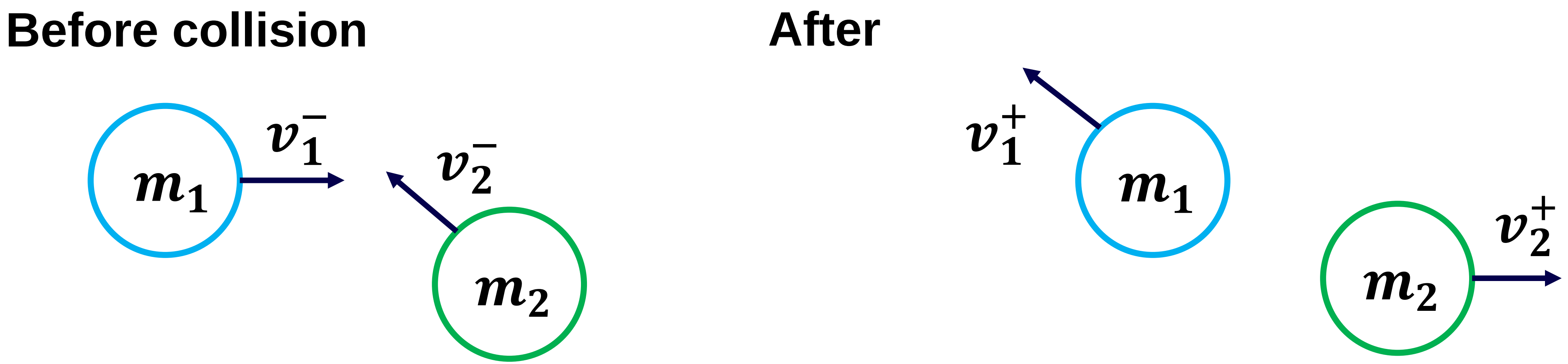
$$\Delta v = 2(v^- \circ \hat{n})\hat{n}$$

$$v^+ = v^- + \Delta v$$

Loss of energy

$$\Delta v = (1 + \epsilon)(v^- \circ \hat{n})\hat{n}$$

Particle-Particle Collisions (spherical objects)



Response:

$$v_1^+ = v_1^- - \frac{2m_2}{m_1+m_2}\frac{\langle v_1^- - v_2^-\rangle \cdot \langle p_1 - p_2\rangle}{\|p_1 - p_2\|^2}\langle p_1 - p_2\rangle$$

$$v_2^+ = v_2^- - \frac{2m_1}{m_1+m_2}\frac{\langle v_2^- - v_1^-\rangle \cdot \langle p_2 - p_1\rangle}{\|p_2 - p_1\|^2}\langle p_2 - p_1\rangle$$

- This is in terms of velocity
- next: derivation via impulse and forces

From Velocities (Δv) to Forces (F) and back

Force relates to mass and acceleration

$$\mathbf{F} = ma$$

A change in velocity related to acceleration over time

$$\Delta\mathbf{v} = \Delta t\, a$$

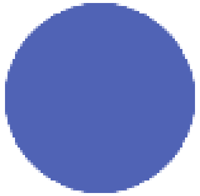
In terms of forces

$$\Delta\mathbf{v} = \Delta t\, \frac{F}{m}$$

Basic Particle Simulation (first try)

How to compute the change in velocity?

$$\boldsymbol{d_t = t_{i+1} - t_i}$$
$$\boldsymbol{\vec{v}_{i+1} = \vec{v}_i + \Delta v}$$
$$\boldsymbol{\vec{p}_{i+1} = \vec{p}(t_i) + \vec{v}_i d_t}$$



Forces are omnipresent

- **Gravity**

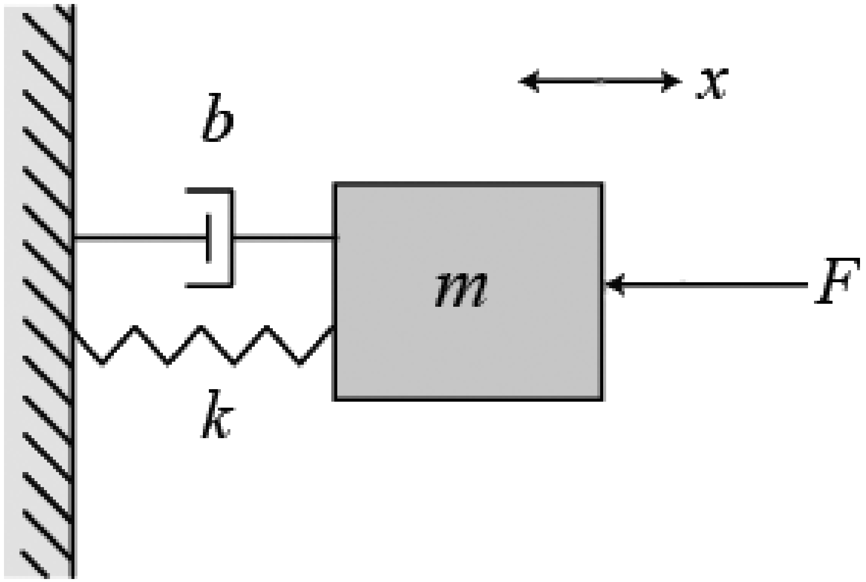
$$F = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

- **Viscous damping**

$$F = -bv$$

- **Spring & dampers**

$$F = -kx - bv$$



Gravity direction?

Assuming a flat earth:

$$F = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

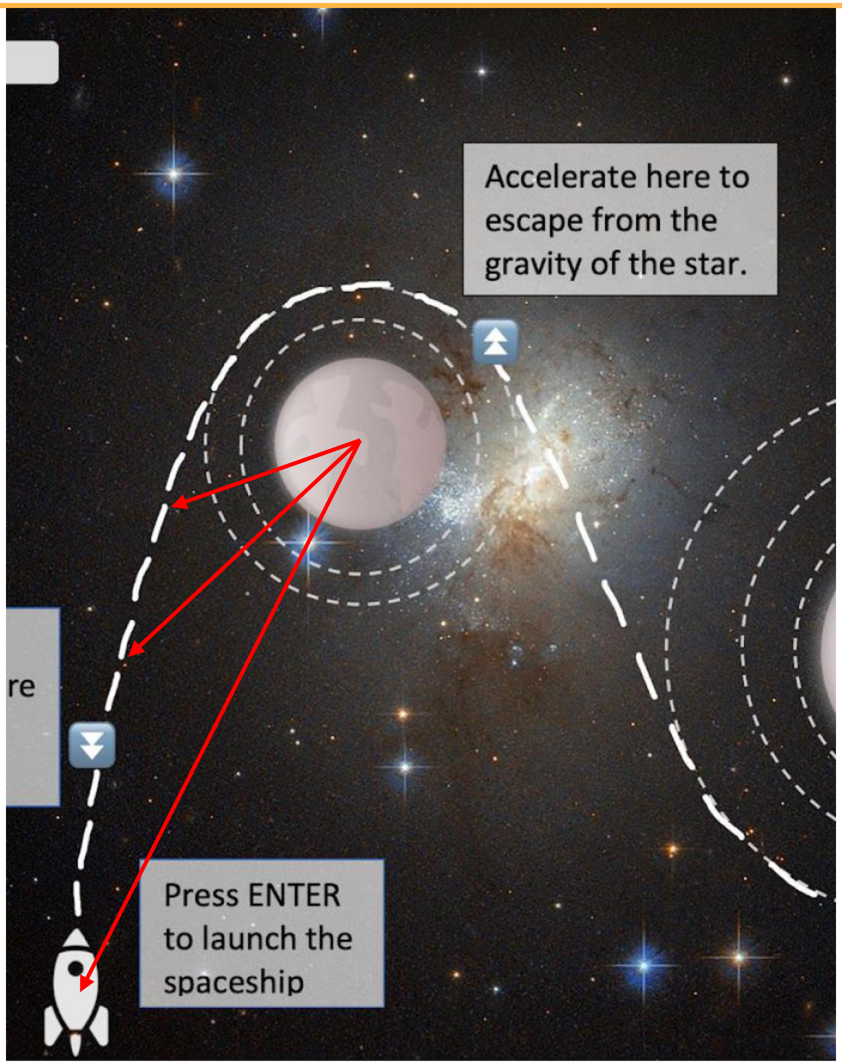
Assuming a spherical earth:

$$F = -mg \begin{bmatrix} a \\ b \end{bmatrix}$$

How to compute the vector (a,b) and g ?

Newton's law of universal gravitation

$$F = G\frac{m_1 m_2}{r^2}$$



Multiple forces?

Forces add up (and cancel):

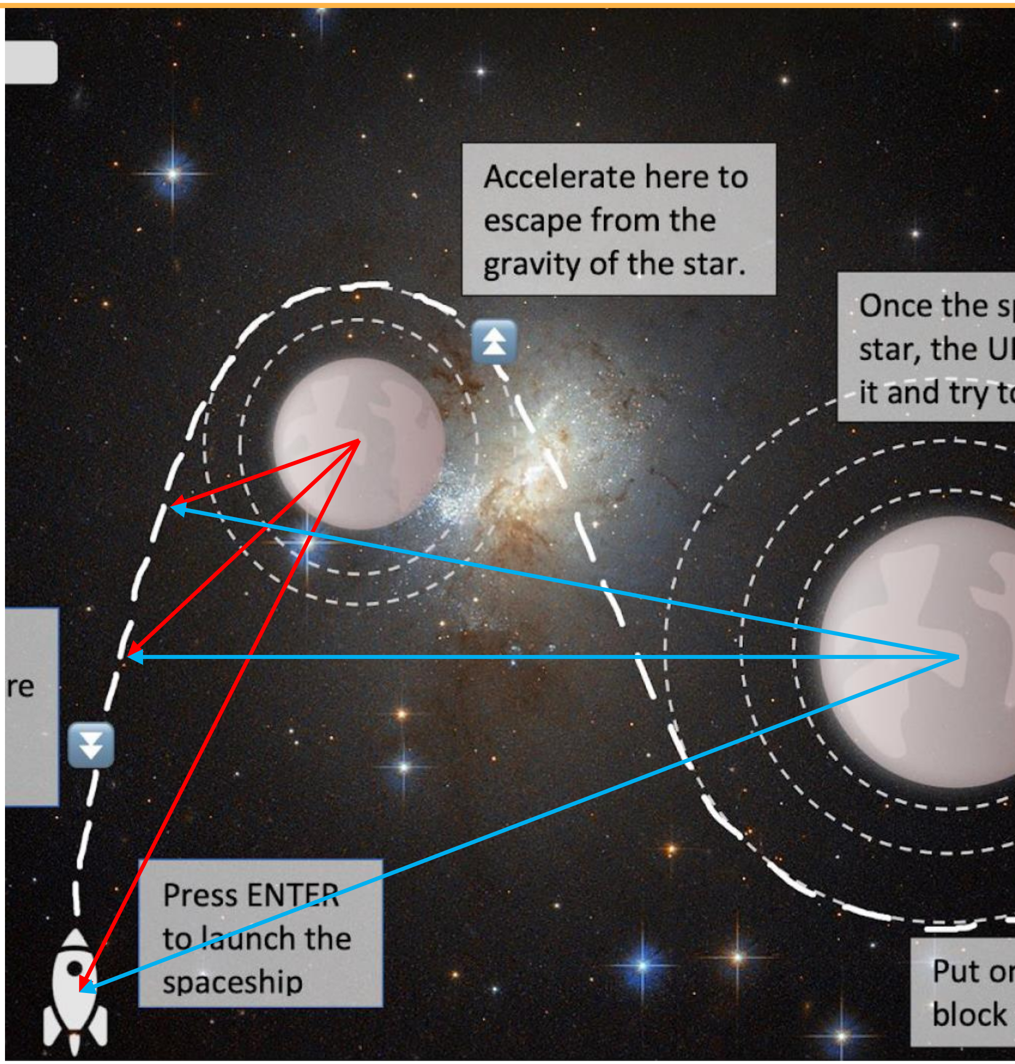
$$F = -m g_1 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} - m g_2 \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

- ***This holds for all types of forces!***

- ***Notation you might see:***

$$F = \sum_i F_i = \sum F_i = \sum F$$

$$\vec{F} = F$$



Your game idea does not need forces?

Are you sure?

- ***Particle effects***
- ***Fake forces***
- ***Proxy forces***
  - Simulate crowd behaviour

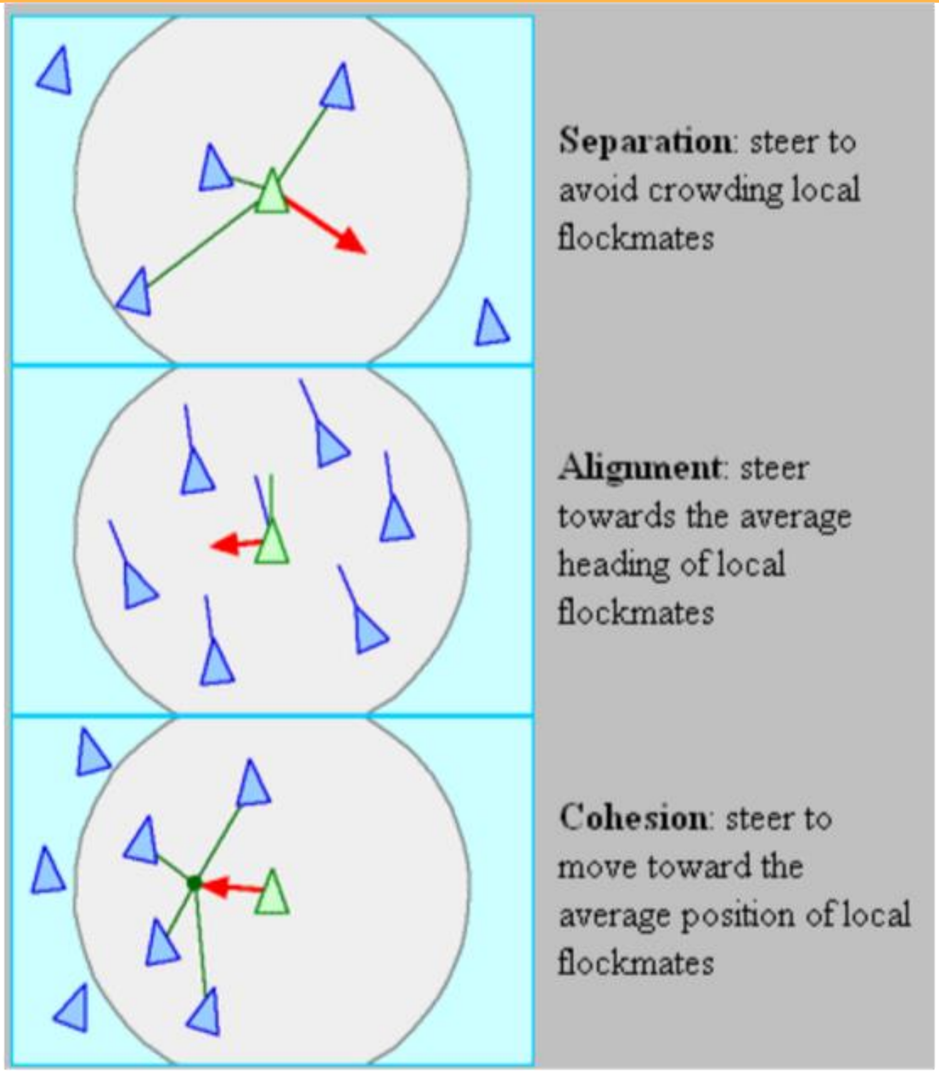
Take it as a chance to connect dry math with a practical application!

Proxy Forces (= fake forces)

- **Behavior forces: [“Boids”, Craig Reynolds, SIGGRAPH 1987]**
- **flocking birds, schooling fish, etc.**

- **Attract to goal location (like gravity)**
  - ***E.g., waypoint determined by shortest path search***
- **Repulsion if close**
- **Align orientation to neighbors**
- **Center to neighbors**

- **Forces add up!**



Simulation Basics

Simulation loop…

1. ***Equations of Motion***
   - sum forces & torques
   - solve for accelerations: $\vec{\boldsymbol{F}} = \boldsymbol{ma}$
2. ***Numerical integration***
   - update positions, velocities
3. ***Collision detection***
4. ***Collision resolution***

What we did so far: Forward Euler

Forces only $\vec{F} = ma$

$$d_t = t_{i+1} - t_i$$

$$\vec{v}_{i+1} = \vec{v}_i + (\vec{F}(t_i)/m)d_t$$

$$\vec{p}_{i+1} = \vec{p}(t_i) + \vec{v}_{i+1}d_t$$

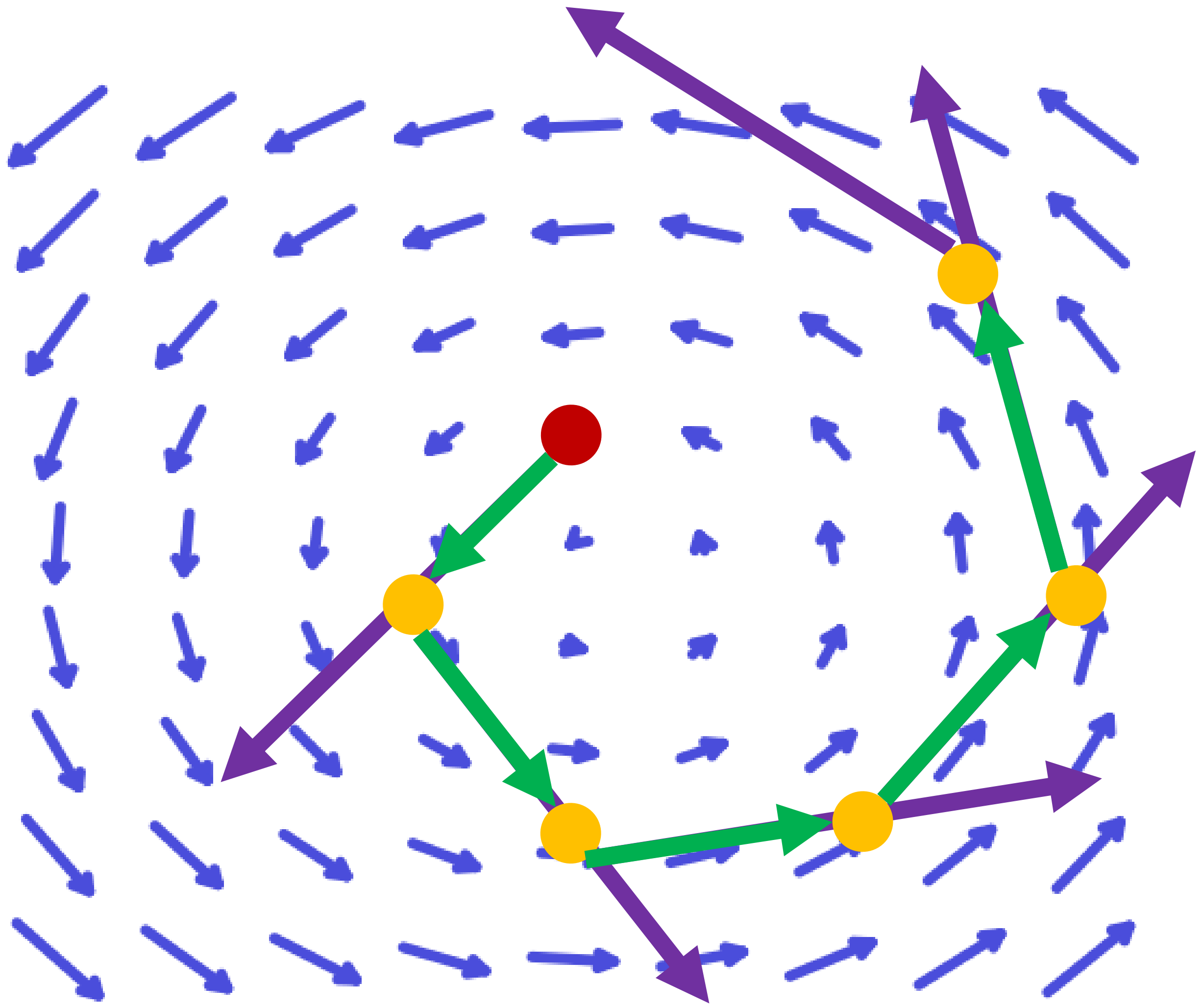
acceleration $= \frac{\partial v}{\partial t}$

get values at time t_{i+1} from values at time t_i

Issues? Alternatives?

How can we discretize this?

Issue: extrapolation

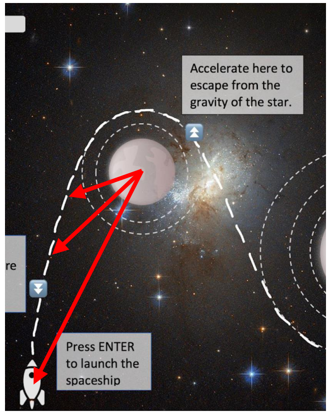


Which forces depend on t?

- **Gravity**

$$F = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

$$F = -mg \begin{bmatrix} a \\ b \end{bmatrix}$$

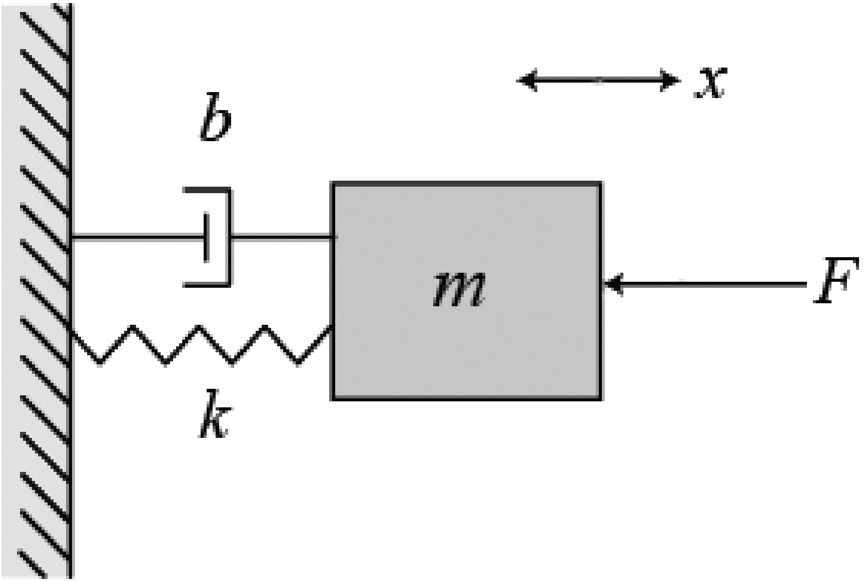


- **Viscous damping**

$$F = -bv$$

- **Spring & dampers**

$$F = -kx - bv$$



Basic Particle Simulation: Small Problem…

$$d_t = t_{i+1} - t_i$$

$$\vec{v}_{i+1} = \vec{v}_i + (\vec{F}(t_{???})/m)d_t$$

$$\vec{p}_{i+1} = \vec{p}(t_i) + \vec{v}_{i+1}d_t$$

Equations of motion describe state (equilibrium)

- **Involves quantities and their derivatives**
  - **-> we need to solve differential equations**

Lets start from scratch

Given:

$$\vec{F} = m\,\frac{\partial^2 x}{\partial t^2}$$

Wait!
There is no position x in this equation?!
Only contains acceleration a!

How to solve such differential equation?

Desired: the position x at time t

$$x$$

Newtonian Physics as First-Order Diff. Eq. (DE)

Second-order DE

$$\vec{F} = m \frac{\partial^2 x}{\partial t^2}$$

acceleration

$= \frac{\partial v}{\partial t}$

Now we have an x!

First-order DE

$$\frac{\partial}{\partial t}\begin{bmatrix}\vec{x} \\ \vec{v}\end{bmatrix} = \begin{bmatrix}\vec{v} \\ \vec{F}/m\end{bmatrix}$$

velocity

$= \frac{\partial x}{\partial t}$

Higher-order DEs can be turned into a first-order DE with additional variables and equations!

Newtonian Physics as First-Order DE

- **Motion of one particle**

Second-order DE

$$\vec{F} = m\,\frac{\partial^2 x}{\partial t^2}$$

First-order DE

$$\frac{\partial}{\partial t}\begin{bmatrix}\vec{x}\\ \vec{v}\end{bmatrix} = \begin{bmatrix}\vec{v}\\ \Sigma\vec{F}/m\end{bmatrix}$$

- **Motion of many particles**

$$\frac{\partial}{\partial t}\begin{bmatrix}\overrightarrow{x_1}\\ \overrightarrow{v_1}\\ \overrightarrow{x_2}\\ \overrightarrow{v_2}\\ \vdots\\ \overrightarrow{x_n}\\ \overrightarrow{v_n}\end{bmatrix} = \begin{bmatrix}\overrightarrow{v_1}\\ \overrightarrow{F_1}/m_1\\ \overrightarrow{v_2}\\ \overrightarrow{F_2}/m_2\\ \vdots\\ \overrightarrow{v_n}\\ \overrightarrow{F_n}/m_n\end{bmatrix}$$

Overview

Different DE solvers

- ***Forward Euler***
  (take current accel. to update vel., current vel. to update pos.)

- ***Midpoint Method & Trapezoid Method***
  (mix current and approximations of future vel. & acc. Estimates)

- ***Backwards Euler***
  (solve for future pos., vel., and accel. jointly)
  - *May require an iterative solver*

Recap: Forward Euler

Forces only $\vec{F} = \boldsymbol{ma}$

acceleration $= \frac{\partial v}{\partial t}$

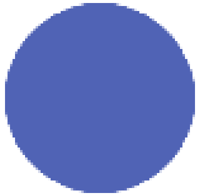
$$\boldsymbol{d_t = t_{i+1} - t_i}$$

$$\boldsymbol{\vec{v}_{i+1} = \vec{v}_i + (\vec{F}(t_i)/m)d_t}$$

$$\boldsymbol{\vec{p}_{i+1} = \vec{p}(t_i) + \vec{v}_{i+1}d_t}$$

get values at time t_{i+1} from values at time t_i

Issues? Alternatives?



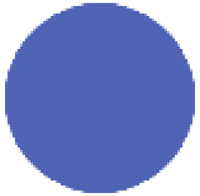
Idea: Backwards Euler

- **Viscous damping**
  $F = -bv$
- **Spring & dampers**
  $F = -kx - bv$

$$d_t = t_{i+1} - t_i$$
$$\vec{v}_{i+1} = \vec{v}_i + (\vec{F}(t_{i+1})/m)d_t$$
$$\vec{p}_{i+1} = \vec{p}(t_i) + \vec{v}_{i+1}d_t$$

get values at time t_{i+1} from states at time t_i and forces at t_{i+1}

Issues?



Differential Equations

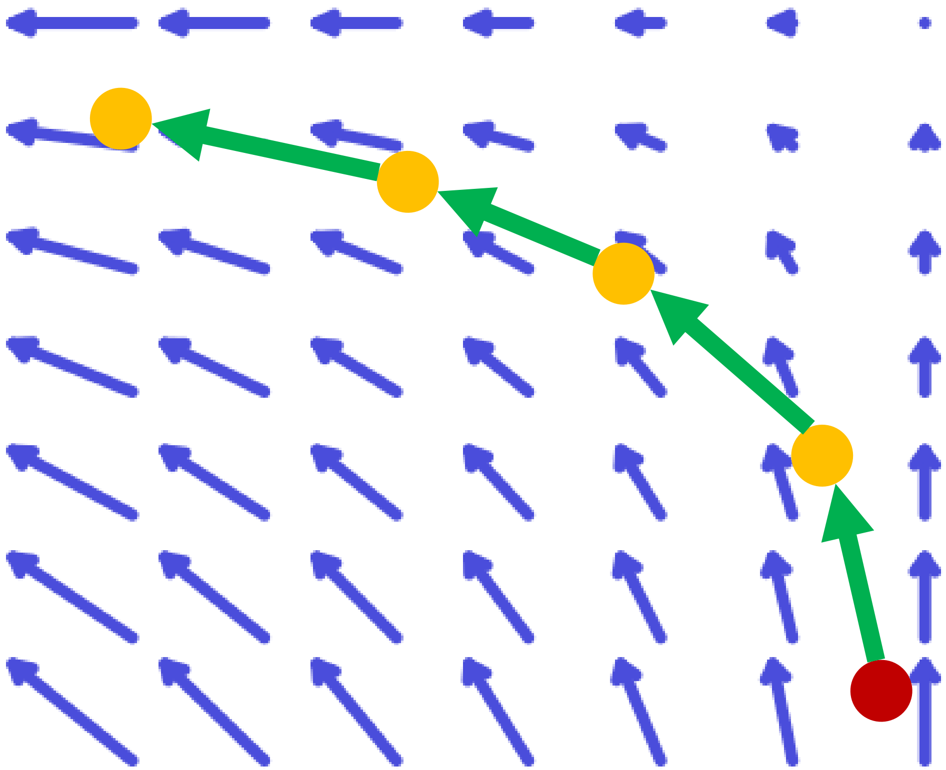
$$\frac{\partial}{\partial t}\vec{X}(t) = f(\vec{X}(t), t)$$

Given that $\vec{X}_0 = \vec{X}(t_0)$

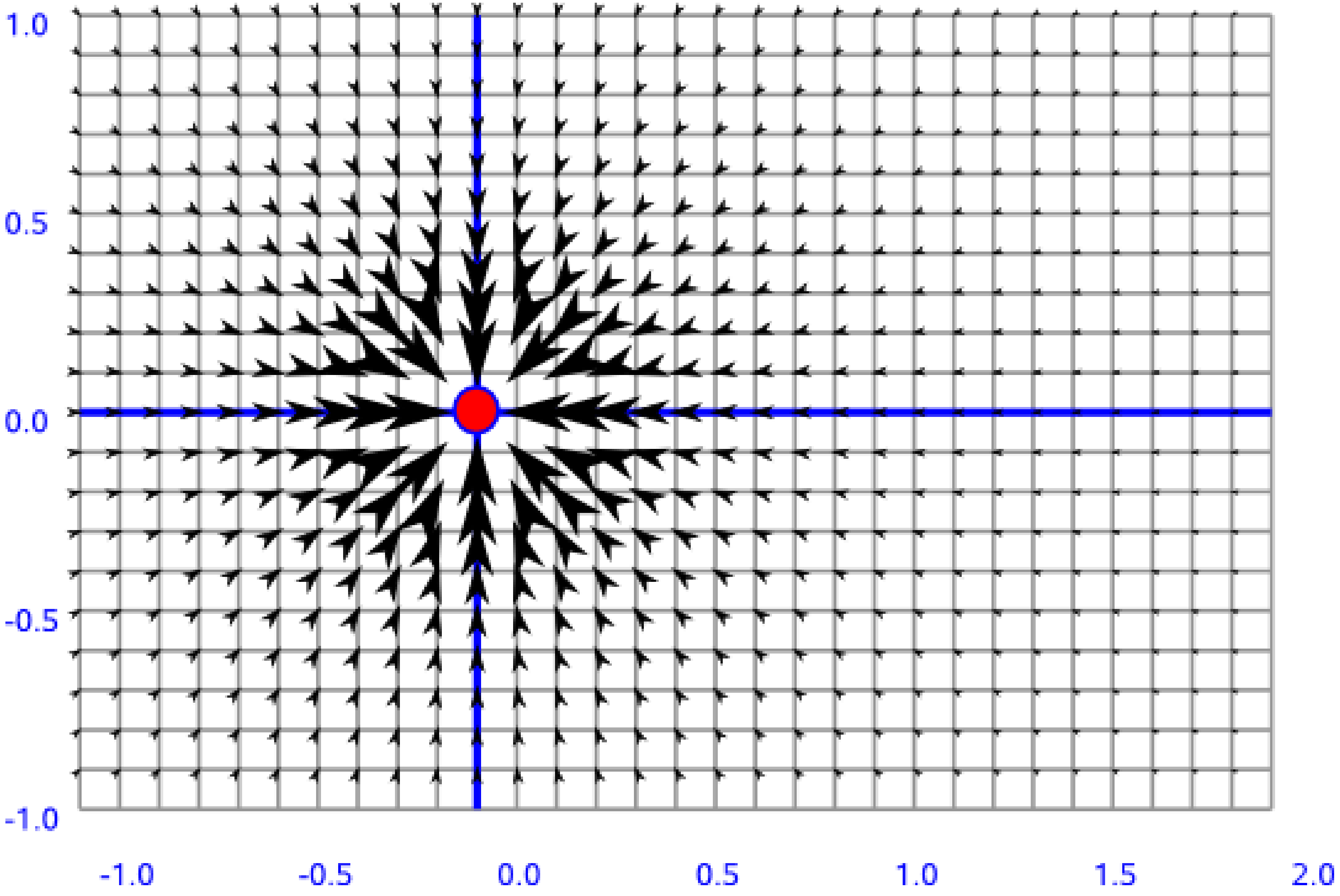
Compute $\vec{X}(t)$ **for** $\mathrm{t} > t_0$

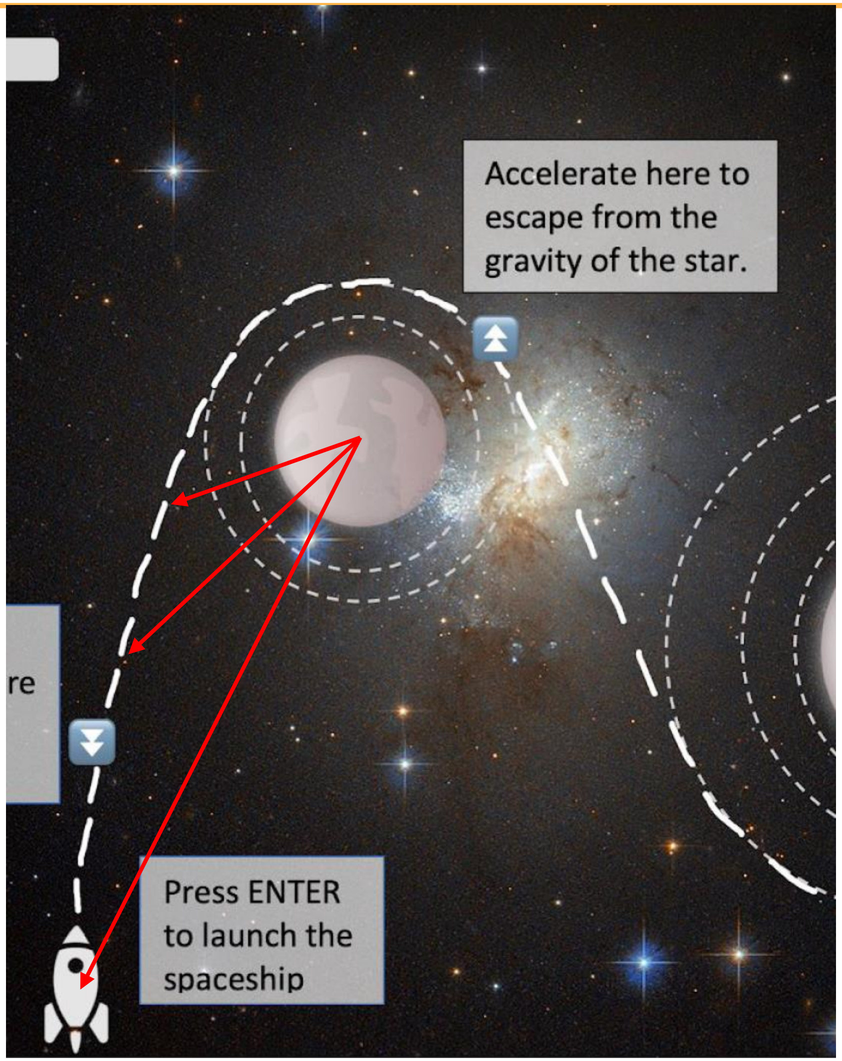
$$\Delta\vec{X}(t) = f(\vec{X}(t), t)\Delta t$$

- **Simulation:**
  - ***path through state-space***
  - ***driven by vector field***



Gravitational field





https://www.euclideanspace.com/maths/geometry/space/fields/index.htm

DE Numerical Integration: Explicit (Forward) Euler

$$\frac{\partial}{\partial t}\vec{X}(t) = f(\vec{X}(t), t)$$

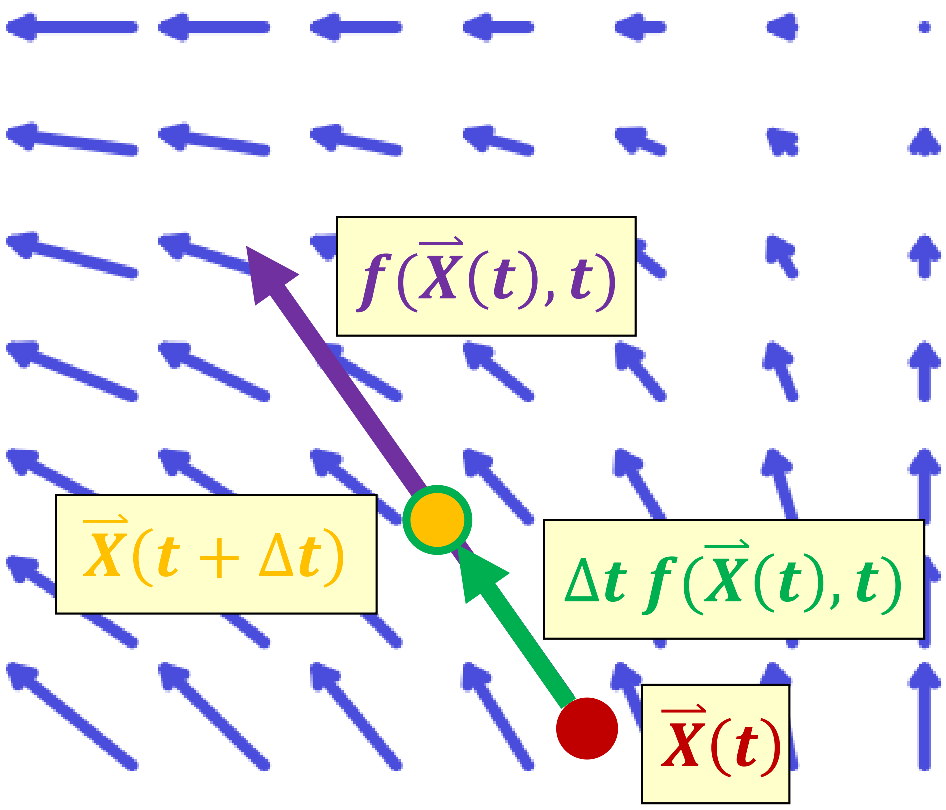
Given that $\vec{X}_0 = \vec{X}(t_0)$

Compute $\vec{X}(t)$ **for** $\mathrm{t} > \mathrm{t}_0$

$$\Delta t = \mathrm{t}_i - \mathrm{t}_{i-1}$$

$$\Delta\vec{X}(\mathrm{t}_{i-1}) = \Delta t\, f(\vec{X}(\mathrm{t}_{i-1}), \mathrm{t}_{i-1})$$

$$\vec{X}_i = \vec{X}_{i-1} + \Delta t\, f(\vec{X}_{i-1}, \mathrm{t}_{i-1})$$

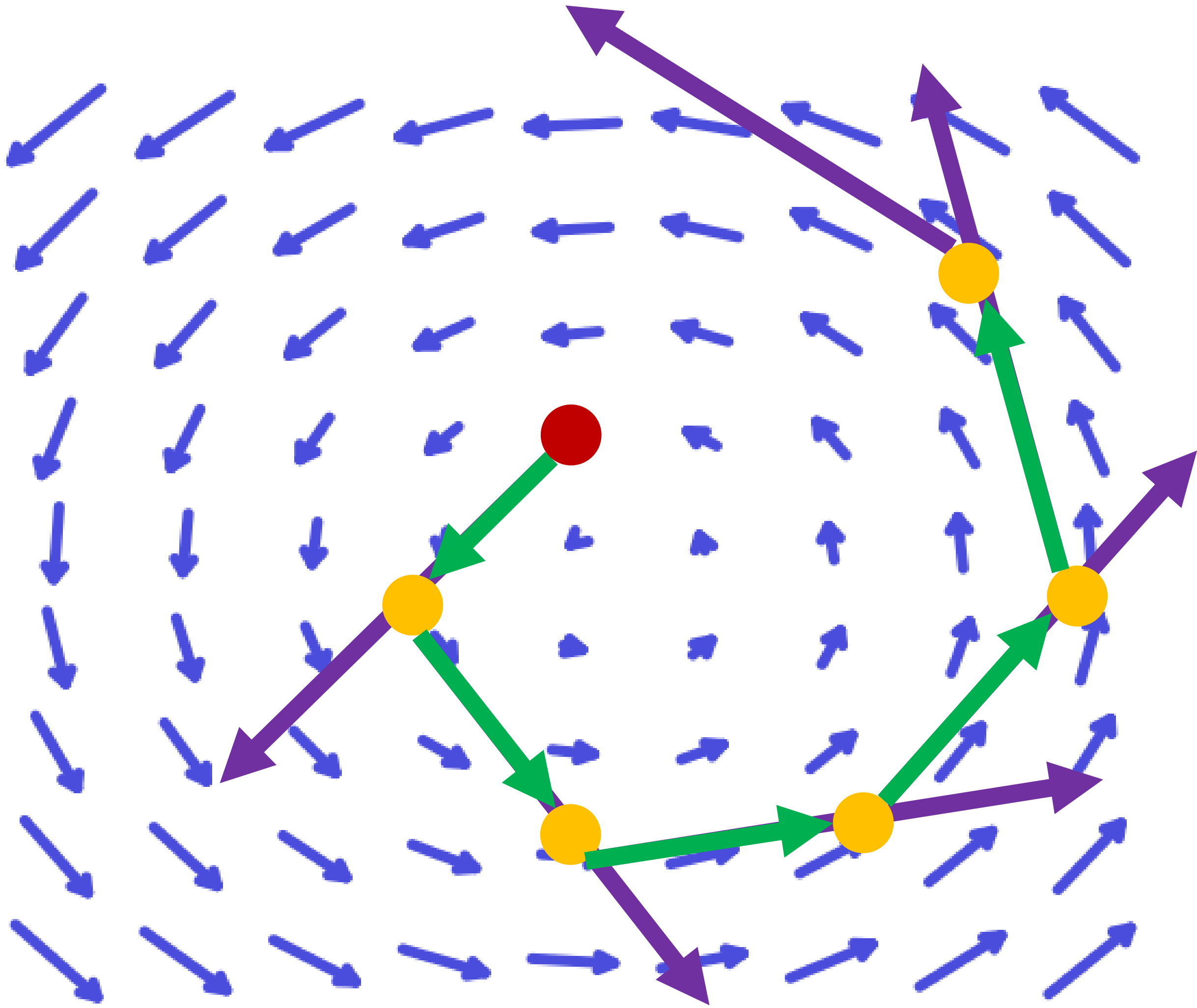


Explicit Euler Problems

- Solution spirals out
  - *Even with small time steps*
  - *Although smaller time steps are still better*

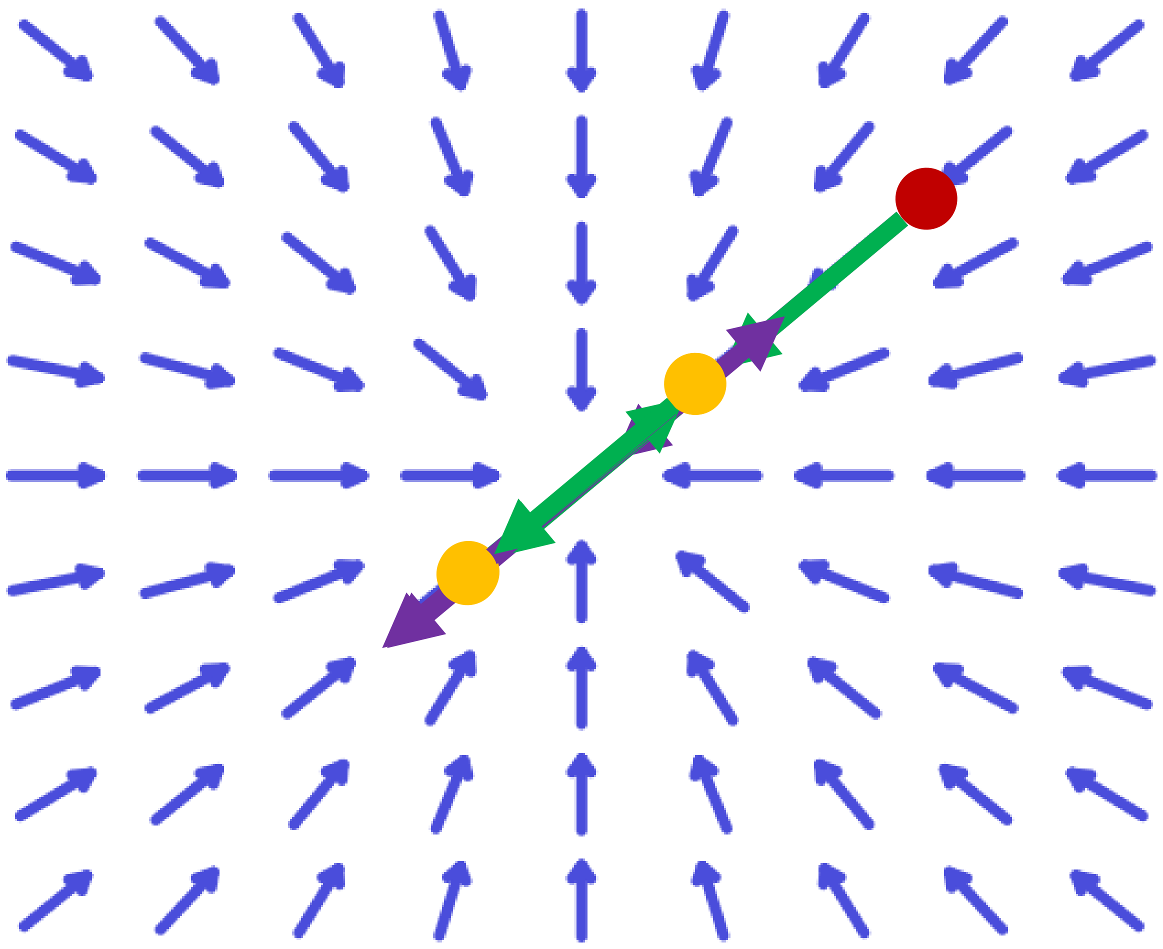
Definition: Explicit

- *Closed-form/analytic solution*
- no iterative solve required



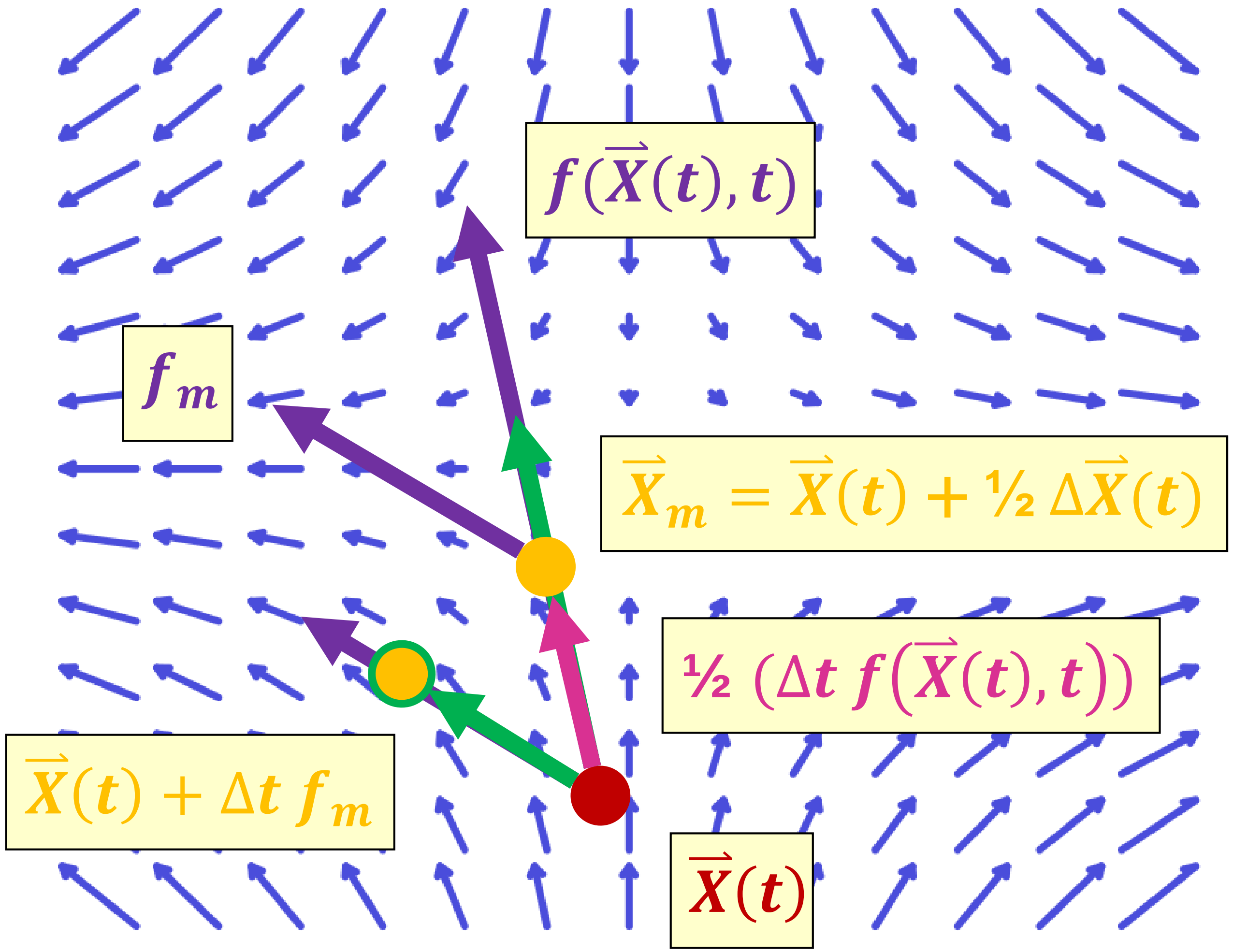
Explicit Euler Problems

- Can lead to **instabilities**



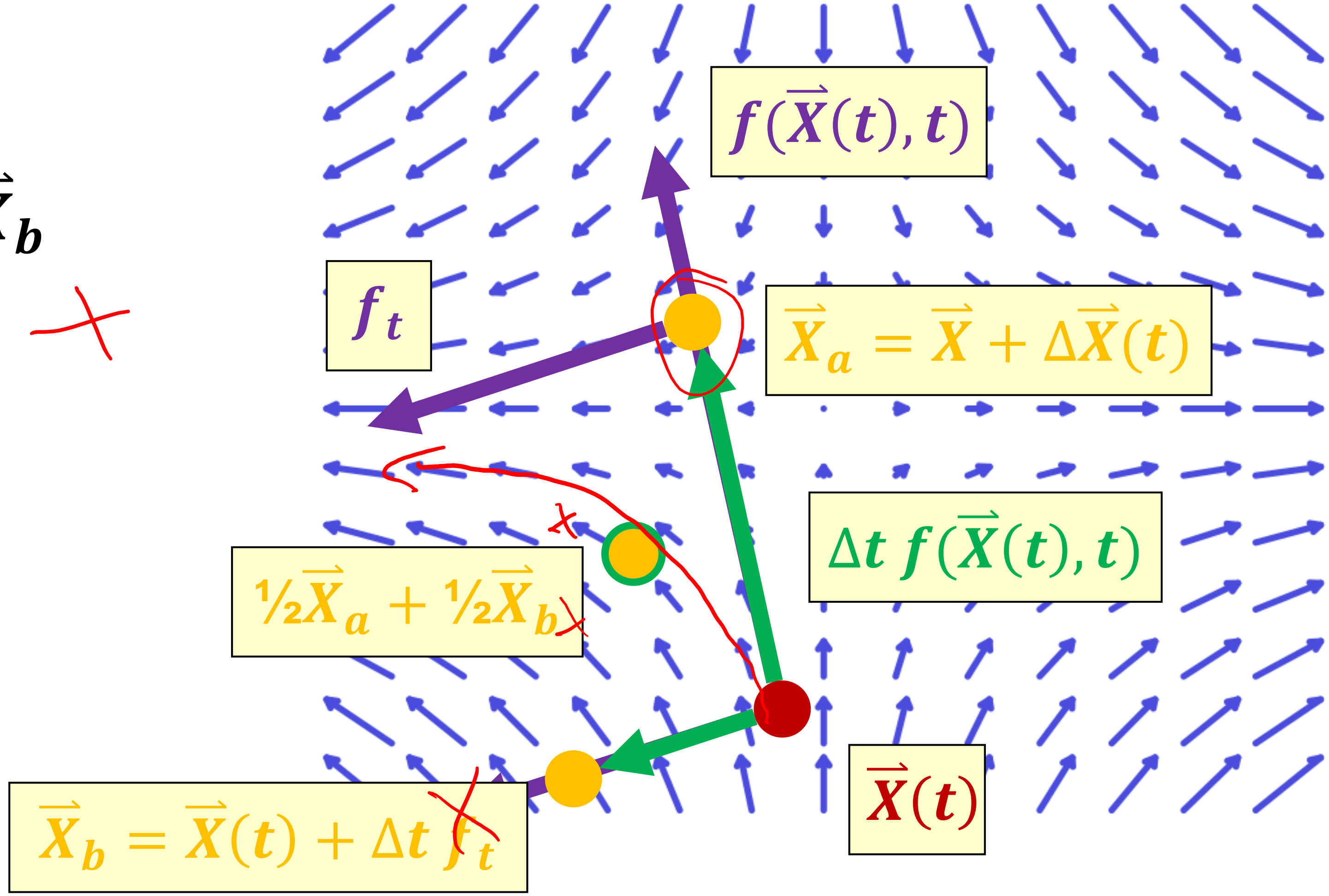
Midpoint Method

1. ½ Euler step
2. evaluate f_m at $\vec{X}_m$
3. full step using f_m



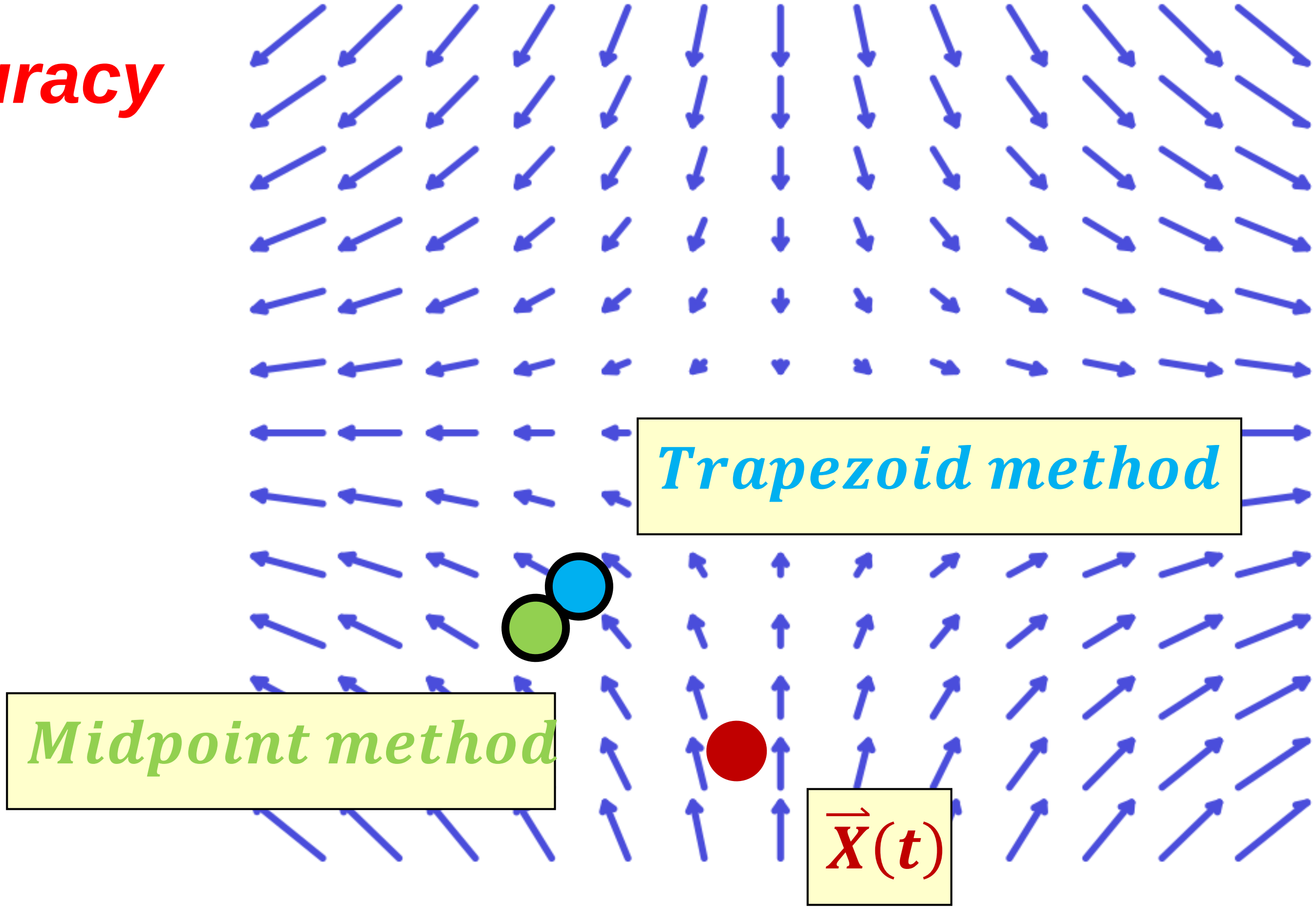
Trapezoid Method

1. full Euler step get $\vec{X}_a$
2. evaluate f_t at $\vec{X}_a$
3. full step using f_t get $\vec{X}_b$
4. average $\vec{X}_a$ and $\vec{X}_b$

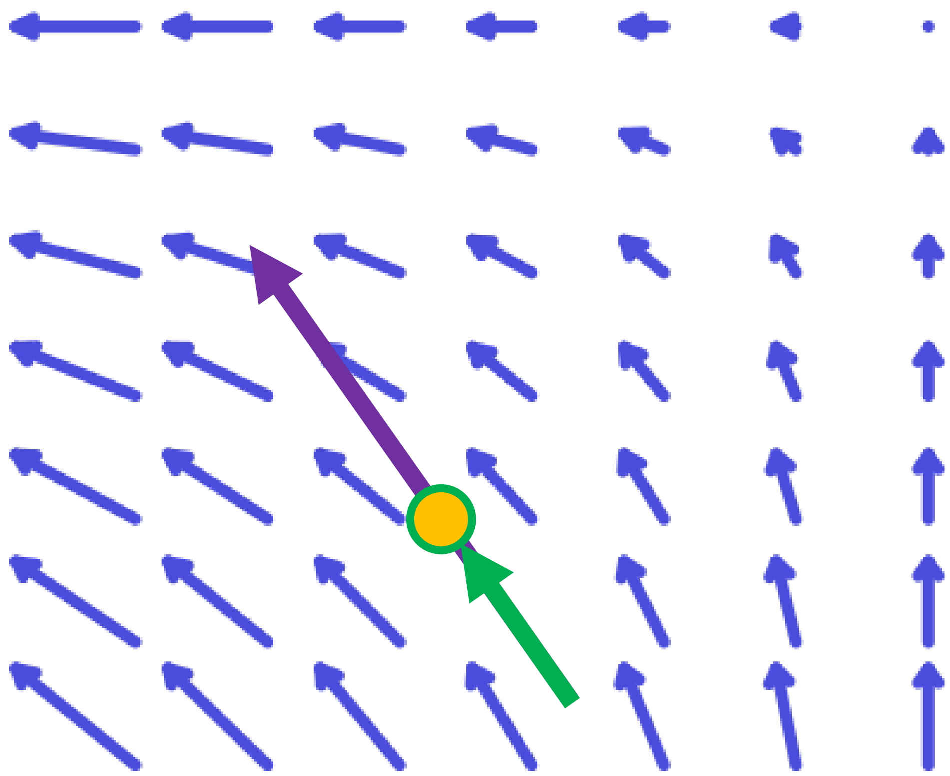


Midpoint & Trapezoid Method

- **Not exactly the same**
  - *But same order of accuracy*

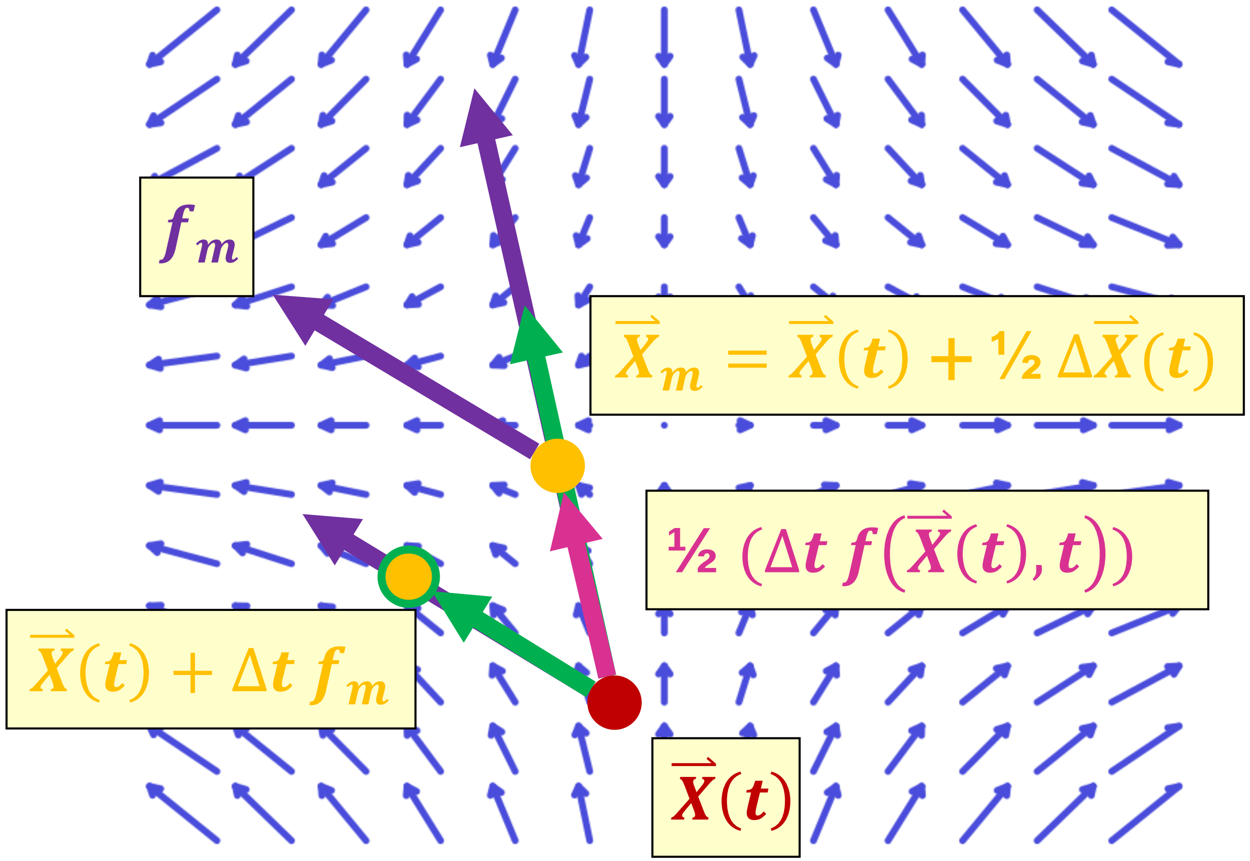


Explicit Euler: Code



```
void takeStep(ParticleSystem* ps, float h)
{
        velocities = ps->getStateVelocities()
        positions = ps->getStatePositions()
        forces = ps->getForces(positions, velocities)
        masses = ps->getMasses()
        accelerations = forces / masses
        newPositions = positions + h*velocities
        newVelocities = velocities  + h*accelerations
        ps->setStatePositions(newPositions)
        ps->setStateVelocities(newVelocities)
}
```

Midpoint Method: Code

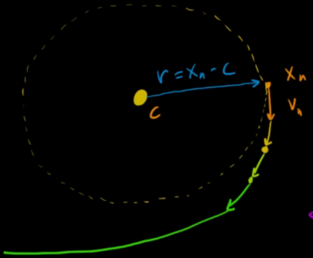



```
void takeStep(ParticleSystem* ps, float h)
{
        velocities = ps->getStateVelocities()
        positions = ps->getStatePositions()
        forces = ps->getForces(positions, velocities)
        masses = ps->getMasses()
        accelerations = forces / masses
        midPositions = positions + 0.5*h*velocities
        midVelocities = velocities  + 0.5*h*accelerations
        midForces = ps->getForces(midPositions, midVelocities)
        midAccelerations = midForces / masses
        newPositions = positions + h*midVelocities
        newVelocities = velocities  + h*midAccelerations
        ps->setStatePositions(newPositions)
        ps->setStateVelocities(newVelocities)
}
```

NUMERICAL INTEGRATION

$$h = \Delta t$$

$$f(x,t) = \cancel{gt + v_0}$$



$$x_{n+1} = x_n + h v_n$$

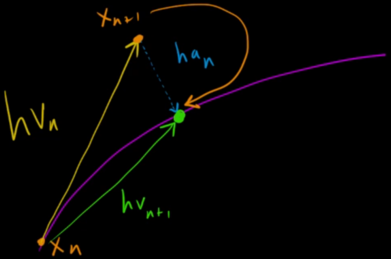
$$v_{n+1} = v_n + h a_n$$

$$a(x_n) = G \frac{m_{sun}}{|r|^3} \vec{r}$$

SEMI-IMPLICIT EULER

$$v_{n+1} = v_n + h a_n$$

$$x_{n+1} = x_n + h v_{n+1}$$



Implicit (Backward) Euler:

- **Use forces at destination**

Solve system of equations

$$\frac{\partial}{\partial t}\begin{bmatrix}\vec{x}\\ \vec{v}\end{bmatrix} = \begin{bmatrix}\vec{v}\\ \Sigma\vec{F}/m\end{bmatrix}$$

$$\boxed{\begin{aligned}\boldsymbol{x_{n+1}} &= \boldsymbol{x_n} + \boldsymbol{h}\,\boldsymbol{v_{n+1}}\\ \boldsymbol{v_{n+1}} &= \boldsymbol{v_n} + \boldsymbol{h}\left(\frac{\boldsymbol{F_{n+1}}}{\boldsymbol{m}}\right)\end{aligned}}$$

- **Types of forces:**
  - ***Gravity***

    $$F = \begin{bmatrix}0\\ -mg\end{bmatrix}$$

  - ***Viscous damping***

    $$F = -bv$$

  - ***Spring & dampers***

    $$F = -kx - bv$$

Implicit (Backward) Euler:

- Use forces at destination + derivative at the destination

Solve system of equations

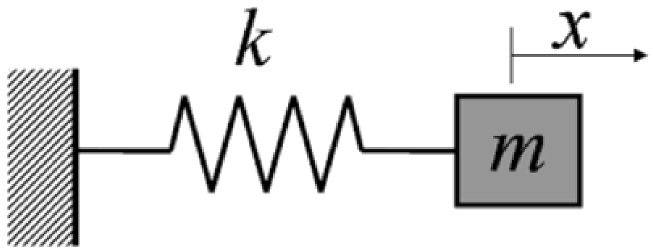
$$\frac{\partial}{\partial t}\begin{bmatrix}\vec{x}\\ \vec{v}\end{bmatrix} = \begin{bmatrix}\vec{v}\\ \Sigma\vec{F}/m\end{bmatrix}$$

$$\boxed{\begin{aligned} \boldsymbol{x_{n+1}} &= \boldsymbol{x_n} + \boldsymbol{h}\,\boldsymbol{v_{n+1}} \\ \boldsymbol{v_{n+1}} &= \boldsymbol{v_n} + \boldsymbol{h}\left(\frac{\boldsymbol{F_{n+1}}}{\boldsymbol{m}}\right) \end{aligned}}$$

Key idea: use velocity estimated at next step instead of current !

Example: Spring Force

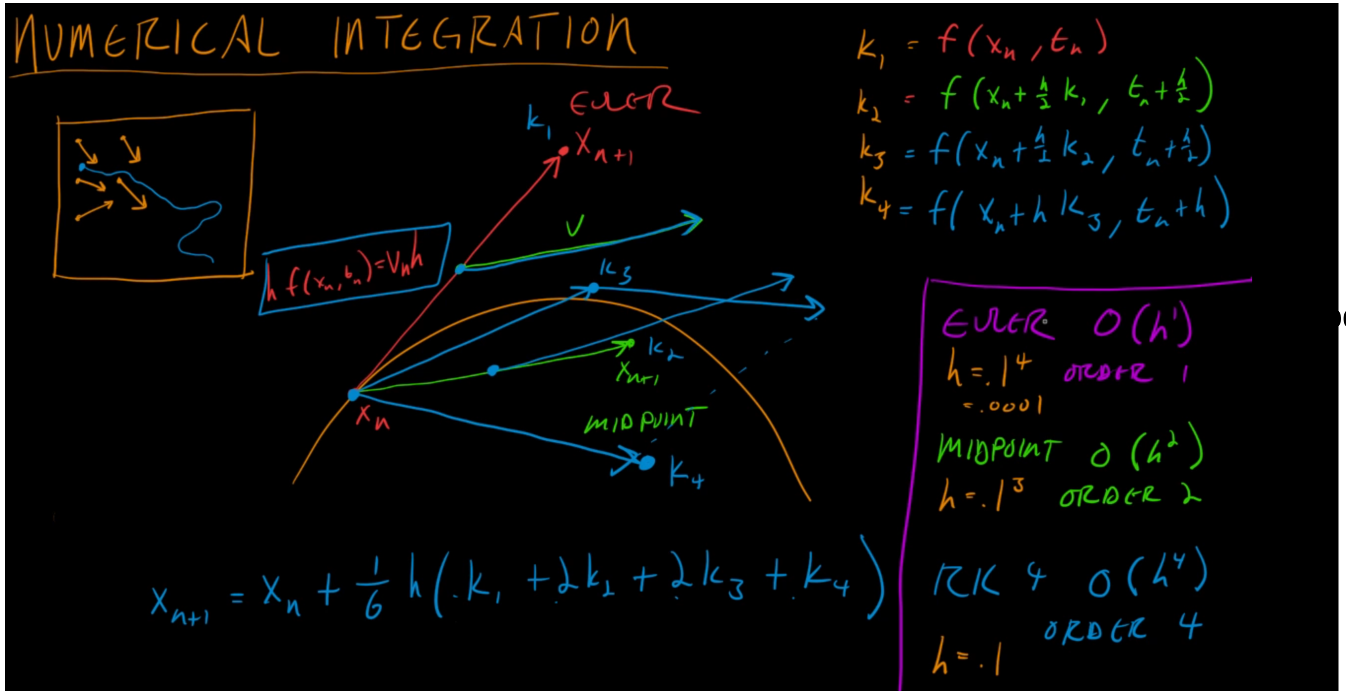
$$F = -kx$$



$$\boxed{\begin{aligned} \boldsymbol{x_{n+1}} &= \boldsymbol{x_n} + \boldsymbol{h}\,\boldsymbol{v_{n+1}} \\ \boldsymbol{v_{n+1}} &= \boldsymbol{v_n} + \boldsymbol{h}\left(\frac{-\boldsymbol{k}\,\boldsymbol{x_{n+1}}}{\boldsymbol{m}}\right) \end{aligned}}$$

Analytic or iterative solve?

Rung-Kutta Order 4



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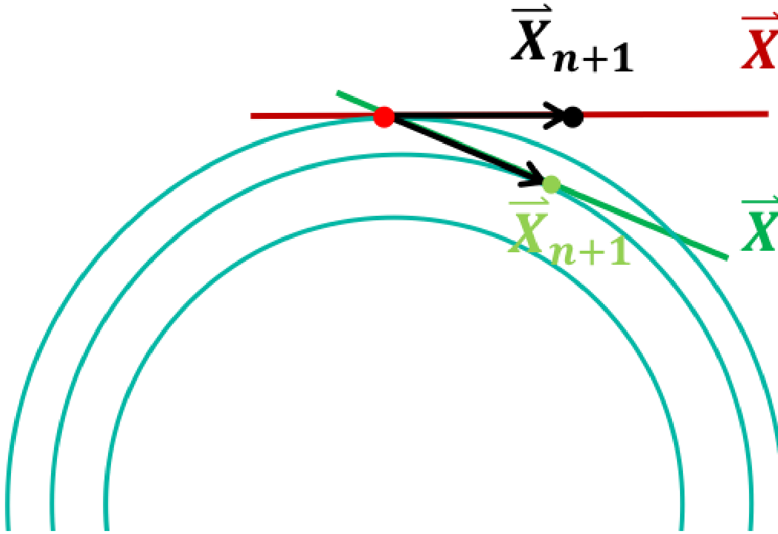
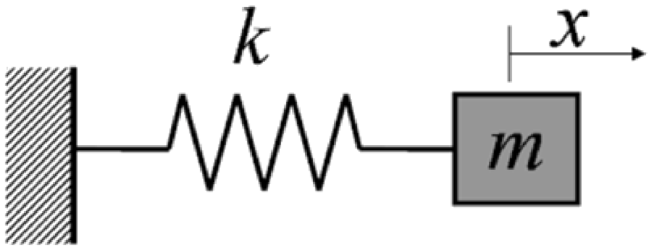
Forward vs Backward



$$\vec{X}_{n+1} = \vec{X}_n + \Delta t\, f(\vec{X}_n)$$

$$\vec{X}_{n+1} = \vec{X}_n + \Delta t\, f(\vec{X}_{n+1})$$



Forward Euler

$$x_{n+1} = x_n + h\, v_n$$
$$v_{n+1} = v_n + h\left(\frac{-k\, x_n}{m}\right)$$

Backward Euler

$$x_{n+1} = x_n + h\, v_{n+1}$$
$$v_{n+1} = v_n + h\left(\frac{-k\, x_{n+1}}{m}\right)$$

Could one apply the Trapezoid Method?

Particles:
Newtonian Physics as First-Order DE

- **Motion of many particles?**

$$\frac{\partial}{\partial t}\begin{bmatrix}\overrightarrow{x_1}\\ \overrightarrow{v_1}\\ \overrightarrow{x_2}\\ \overrightarrow{v_2}\\ \vdots\\ \overrightarrow{x_n}\\ \overrightarrow{v_n}\end{bmatrix}=\begin{bmatrix}\overrightarrow{v_1}\\ \overrightarrow{F_1}/m_1\\ \overrightarrow{v_2}\\ \overrightarrow{F_2}/m_2\\ \vdots\\ \overrightarrow{v_n}\\ \overrightarrow{F_n}/m_n\end{bmatrix}$$

- **Interaction of particles?**

Multiple-particle collision

- ***naïve implementation is likely unstable***
  - *Objects pushing inside each other*

- ***Further reading:***
- https://box2d.org/publications/
  - *In particular https://box2d.org/files/ErinCatto_ModelingAndSolvingConstraints_GDC2009.pdf*

Simulation Basics

Simulation loop…

1. ***Equations of Motion***
2. ***Numerical integration***
3. ***Collision detection***
4. ***Collision resolution***

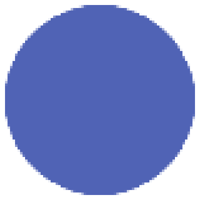
Collisions

- Collision detection
  - *Broad phase: AABBs, bounding spheres*
  - *Narrow phase: detailed checks*
- Collision response
  - *Collision impulses*
  - *Constraint forces: resting, sliding, hinges, ….*

Basic Particle Simulation (first try)

Forces only $\vec{F} = ma$

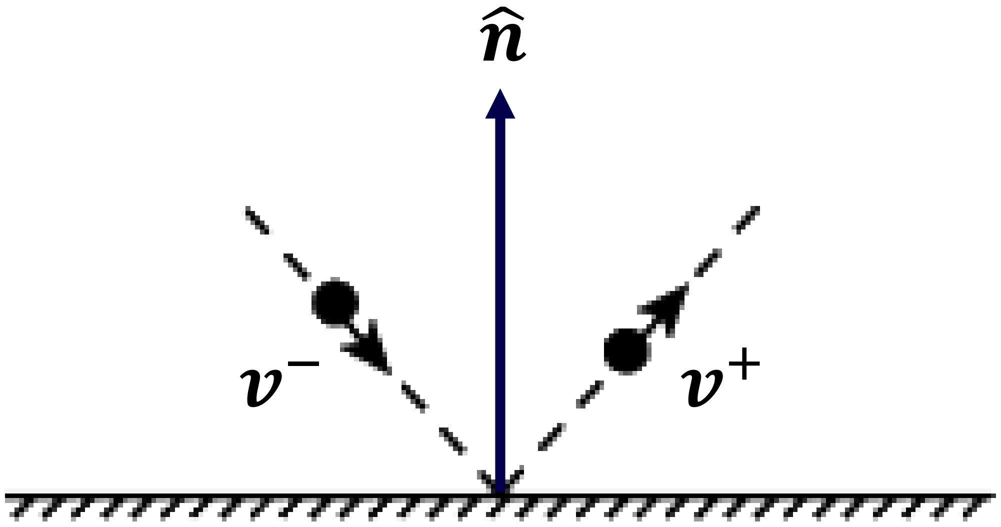
$$d_t = t_{i+1} - t_i$$
$$\vec{v}_{i+1} = \vec{v}(t_i) + (\vec{F}(t_i)/m)d_t$$
$$\vec{p}_{i+1} = \vec{p}(t_i) + \vec{v}(t_{i+1})d_t$$



Particle-Plane Collisions

- ***Apply an 'impulse' of magnitude j***
  - Inversely proportional to mass of particle
- ***In direction of normal***

Impulse in physics: Integral of F over time
In games: an instantaneous step change (not physically possible), i.e., the force applied over one time step of the simulation



$$j = (1+\epsilon)(v^- \cdot \hat{n})m$$

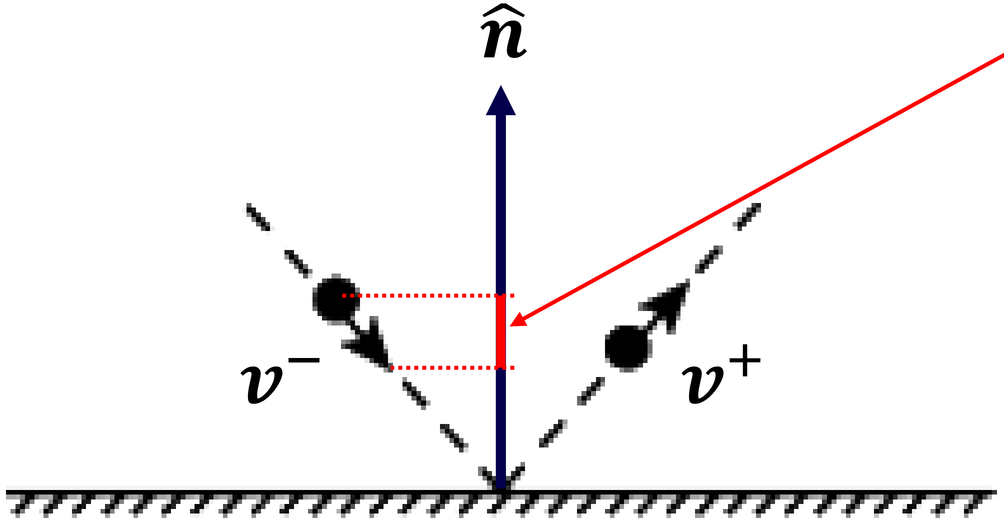
$$\vec{j} = j\,\hat{n}$$

$$v^+ = \frac{\vec{j}}{m} + v^-$$

What is the effect of ϵ ?

Recap: Particle-Plane Collisions (in terms of vel.)

- ***Change in direction of normal***



Velocity along normal
(v projected on normal
by the dot product)

Apply change
along normal
(magnitude
times direction)

Frictionless

$$\Delta v = 2(\overline{v^- \cdot \hat{n}})\overline{\hat{n}}$$

$$v^+ = v^- + \Delta v$$

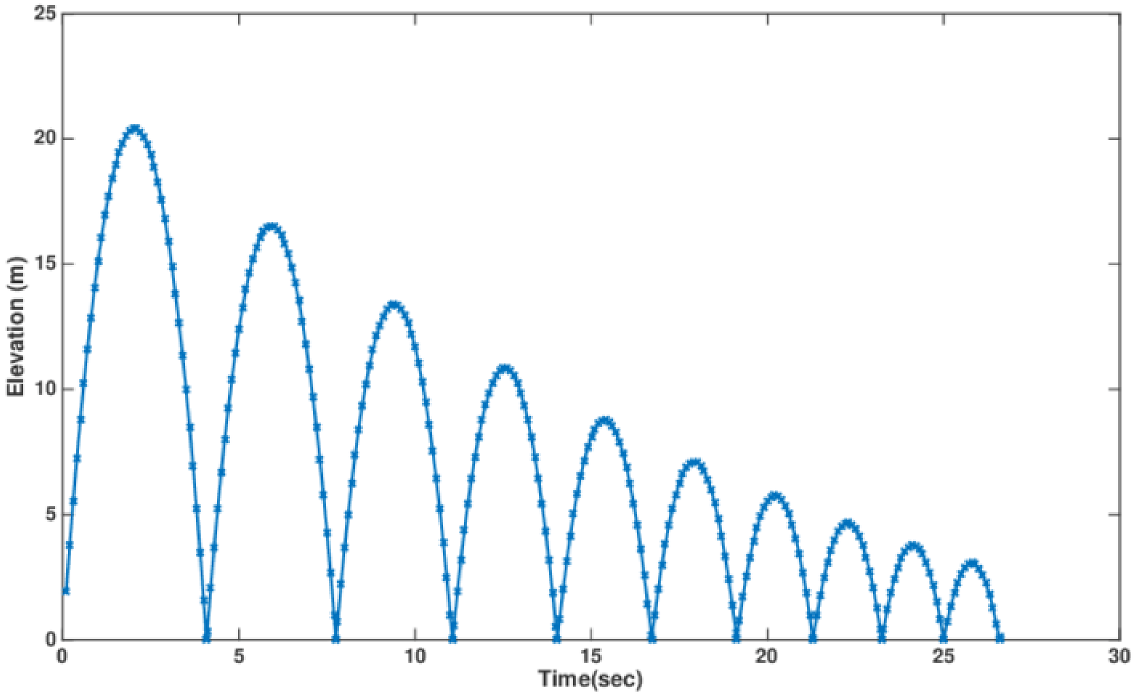
Loss of energy

$$\Delta v = (1+\epsilon)(v^- \cdot \hat{n})\hat{n}$$

Why use ‘Impulse’?

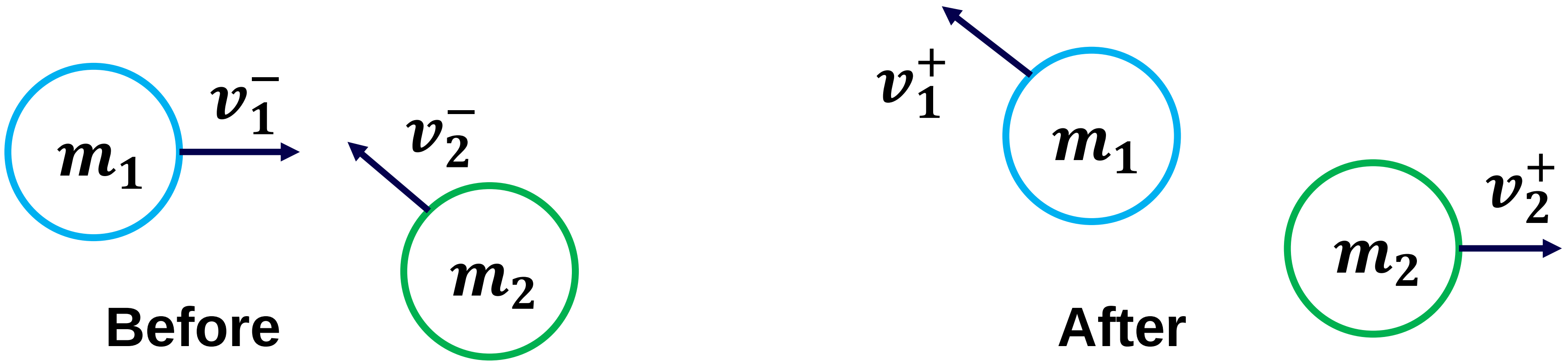
- ***Integrates with the physics solver***

- ***How to integrate damping?***



Particle-Particle Collisions (radius=0)

- **Particle-particle frictionless elastic impulse response**



- **Momentum is preserved**

$$m_1 v_1^- + m_2 v_2^- = m_1 v_1^+ + m_2 v_2^+$$

- **Kinetic energy is preserved**

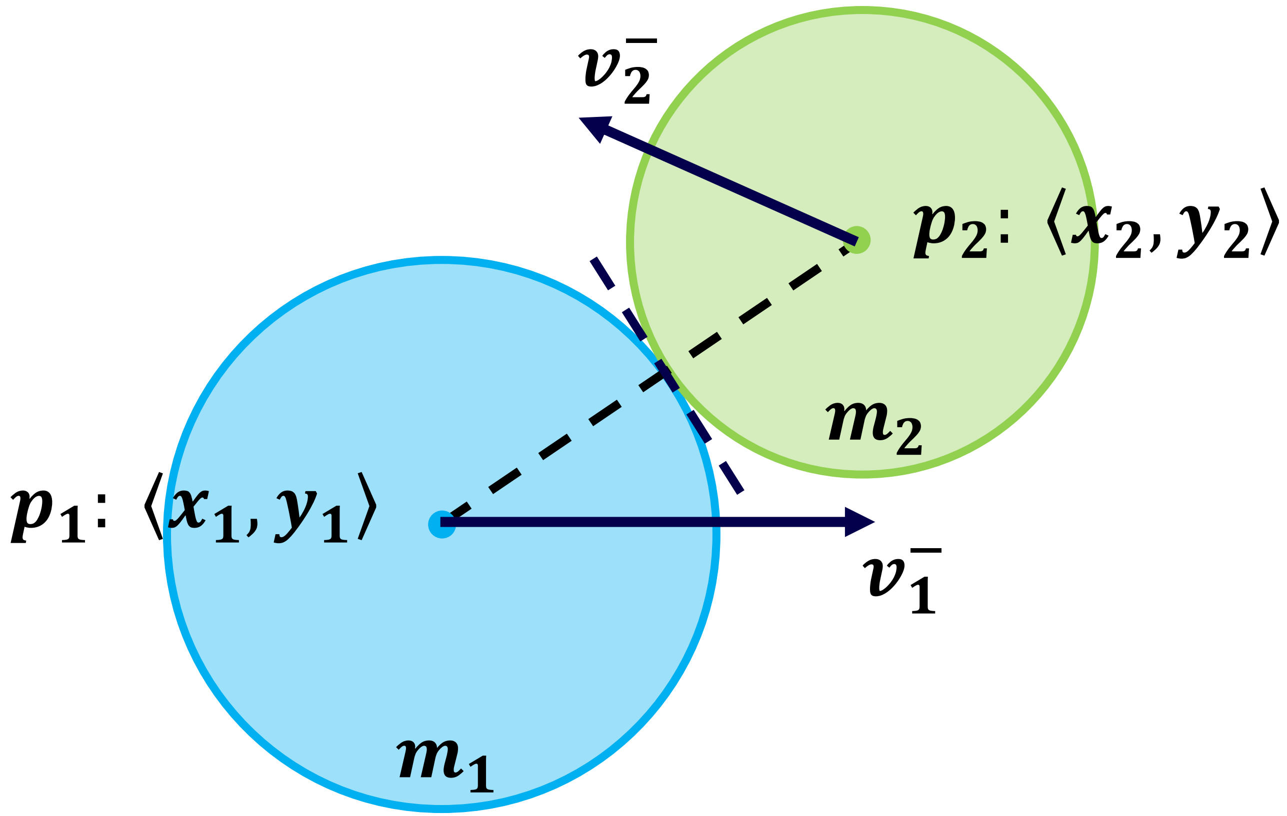
$$½\, m_1 {v_1^-}^2 + ½\, m_2 {v_2^-}^2 = ½\, m_1 {v_1^+}^2 + ½\, m_2 {v_2^+}^2$$

- **Velocity is preserved in tangential direction**

$$t \circ v_1^- = t \circ v_1^+ , \qquad t \circ v_2^- = t \circ v_2^+$$

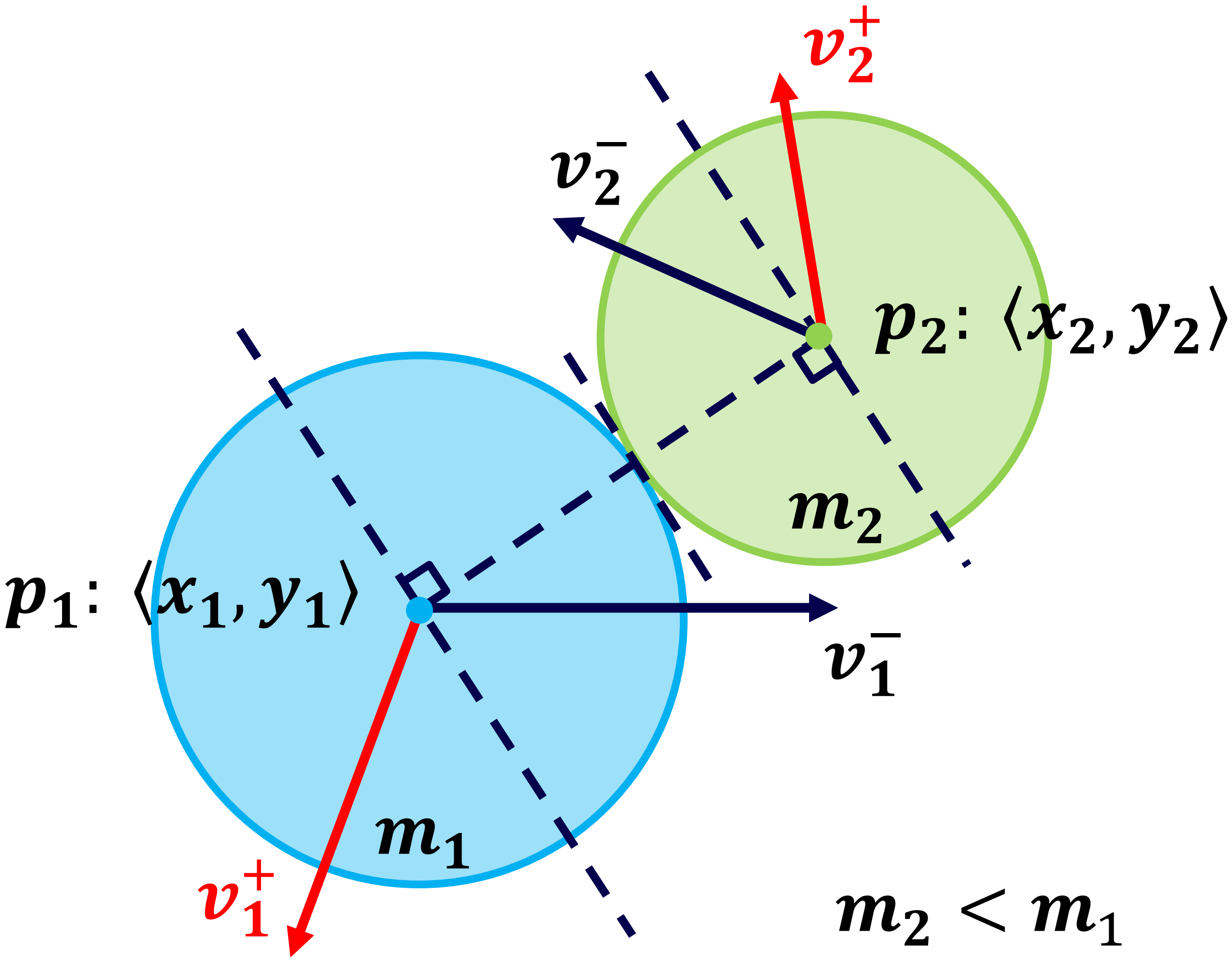
Particle-Particle Collisions (radius >0)

- **What we know…**
  - ***Particle centers***
  - ***Initial velocities***
  - ***Particle Masses***
- **What we can calculate…**
  - ***Contact normal***
  - ***Contact tangent***



Particle-Particle Collisions (radius >0)

- Impulse **direction** reflected across **tangent**
- Impulse **magnitude** proportional to **mass of other particle**



Particle-Particle Collisions (radius >0)

- **More formally…**

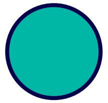
$$\boldsymbol{v}_1^+ = \boldsymbol{v}_1^- - \frac{2\boldsymbol{m}_2}{\boldsymbol{m}_1 + \boldsymbol{m}_2} \frac{\langle \boldsymbol{v}_1^- - \boldsymbol{v}_2^- \rangle \cdot \langle \boldsymbol{p}_1 - \boldsymbol{p}_2 \rangle}{\|\boldsymbol{p}_1 - \boldsymbol{p}_2\|^2} \langle \boldsymbol{p}_1 - \boldsymbol{p}_2 \rangle$$

$$\boldsymbol{v}_2^+ = \boldsymbol{v}_2^- - \frac{2\boldsymbol{m}_1}{\boldsymbol{m}_1 + \boldsymbol{m}_2} \frac{\langle \boldsymbol{v}_2^- - \boldsymbol{v}_1^- \rangle \cdot \langle \boldsymbol{p}_2 - \boldsymbol{p}_1 \rangle}{\|\boldsymbol{p}_2 - \boldsymbol{p}_1\|^2} \langle \boldsymbol{p}_2 - \boldsymbol{p}_1 \rangle$$

- This is in terms of velocity, what would the corresponding impulse be?

Rigid Body Dynamics
(rotational motion of objects?)

- **From particles to rigid bodies…**



Particle

Rigid body

$$state = \begin{cases} \vec{x}\ position \\ \vec{v}\ velocity \end{cases}$$

$\mathbb{R}^4$ in 2D
$\mathbb{R}^6$ in 3D

$$state = \begin{cases} \vec{x}\ position \\ \vec{v}\ velocity \\ R\ rotation\ matrix\ 3x3 \\ \vec{w}\ angular\ velocity \end{cases}$$

$\mathbb{R}^{12}$ in 3D

DEMOS

1. Code on Files for particles and integration methods.
Objectives: take a look at the code and forces field, comparison between methods

2. Unreal Particles level: ContentExamples, level Particles_intro.
Objectives: understand how they are simulated, efforts needed for programmers, curves, etc.

3. Unreal Physics tutorial

Objectives:

- Applyimpulse on drop objects, check PlayerCharacter blueprint

- radial force

- understanding Skeletal Mesh bones and physical asset
- constraints
- thruster (X axis neg force constant applied)
- angular motors on constraints
- show angular, linear limits for constraints