Lab 5 – Study case: Modeling and Control of a Drone

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Contents

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The idea

In this lab, the model of a small drone will be presented as well as a simple control strategy.

1 Theoretical background

Modeling a drone system is relatively complex because it considers 6 degrees of freedom (3 position coordinates and 3 rotation coordinates). Moreover, these elements can (and are) viewed in two distinct coordinate systems:

- inertial coordinate system: this is the ”fixed” system, the one in which we measure the drone’s trajectory relative to the ground;

- body coordinate system: this is the ”mobile” system, centered on the drone’s center of mass.

We are interested in both coordinate systems because certain quantities are naturally measured in one of them while other quantities are measured in the other system.

If we need to use them simultaneously, we are interested in how we can reformulate the coordinates from one system to the other. In this respect, ”Euler angles” are required to show the inclinations of the mobile frame relative to the fixed one. These lead to the rotation matrix (by definition it is orthogonal, thus, ${}^B_I R = \left({}^I_B R\right)^{-1} = \left({}^I_B R\right)^\top$):

$$
{}^I_B R = \begin{bmatrix} c\theta c\psi & s\phi s\theta c\psi - c\phi s\psi & c\phi s\theta c\psi + s\phi s\psi \\ c\theta s\psi & s\phi s\theta s\psi + c\phi c\psi & c\phi s\theta s\psi - s\phi c\psi \\ -s\theta & s\phi c\theta & c\phi c\theta \end{bmatrix} \tag{1}
$$

The notations ”s” and ”c” are the shortened forms of ”sin” and ”cos”.

The subscripts ”B” and ”I” show that the matrix transforms the coordinates from the ”fixed” frame to the ”mobile” one. We will attach these subscripts to various quantities we use to indicate in which coordinate system they are measured. The structure of the drone is presented in Fig. 1. The same figure depicts the angular velocities and thrust forces generated by the 4 motors.

1.1 Inputs acting on the drone

We consider 4 motors oriented perpendicular to the drone plane. Their speeds, denoted by ω_i, define the control variables that affect the drone’s behavior:

- thrust force;

- torque moments about the rotation axes.

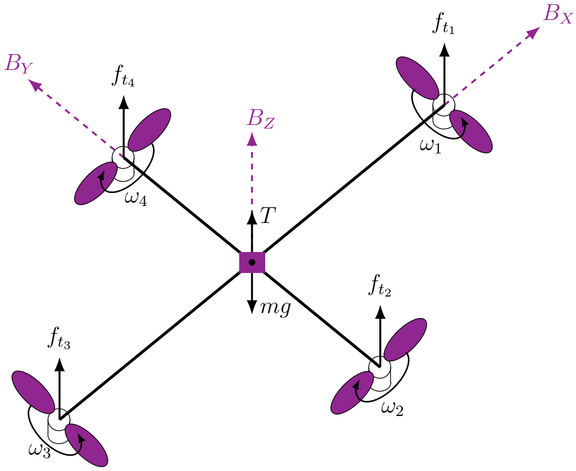

Figure 1: Drone structure

Thrust force

The total thrust force is given by:

$$
{}^{B}\vec{T} = \begin{bmatrix} 0 \\ 0 \\ T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ K_T \sum_{i=1}^{4} \omega_i^2 \end{bmatrix} \tag{2}
$$

Notice that the vector ${}^{B}\vec{T}$ has only one non-zero component: the motors are positioned perpendicular to the drone's plane, and therefore "push" the drone perpendicularly.

The torque moments acting on the drone

The speed of each motor contributes a torque relative to each axis of the coordinate frame. For example, assuming constant rotation directions for each of the motors, we define the total torque about the ${}^{B}z$ axis:

$$
\tau_\psi = b(-\omega_1^2 + \omega_2^2 - \omega_3^2 + \omega_4^2) \tag{3}
$$

The torques corresponding to the ${}^{B}x$ and ${}^{B}y$ axes are derived from standard mechanical equations. The torque about the ${}^{B}x$ axis is obtained by decelerating motor 2 and accelerating motor 4. Similarly, the torque about the ${}^{B}y$ axis is obtained by decelerating motor 1 and accelerating motor 3.

$$\tau_\phi = LK_T(-\omega_2^2 + \omega_4^2), \tag{4a}$$

$$\tau_\theta = LK_T(-\omega_1^2 + \omega_3^2). \tag{4b}$$

In other words, the angular moments defined w.r.t. the mobile frame are:

$$ {}^{B}\tau = \begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} LK_T(-\omega_2^2 + \omega_4^2) \\ LK_T(-\omega_1^2 + \omega_3^2) \\ b(-\omega_1^2 + \omega_2^2 - \omega_3^2 + \omega_4^2) \end{bmatrix} \tag{5}$$

1.2 Mathematical model of the drone system

When modeling a drone from a mechanical point of view, it is assumed that it is a rigid body that can be described dynamically by using the Newton–Euler equations. In the inertial reference system, the centripetal force is zero, so only the weight and the thrust force contribute to defining the drone's acceleration:

$$ m \begin{bmatrix} {}^{I}\ddot{x} \\ {}^{I}\ddot{y} \\ {}^{I}\ddot{z} \end{bmatrix} = m \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} + {}^{I}_{B}R \cdot {}^{B}\vec{T}. \tag{6}$$

Although we wrote the translational part in the fixed frame, it is more natural to write the rotational part of the dynamics in the mobile frame. Expressed in vector form, the Newton–Euler rotation equation is:

$$ {}^{B}I\, {}^{B}\dot{\vec{\Omega}} + {}^{B}\vec{\Omega} \times \left({}^{B}I\, {}^{B}\vec{\Omega} \right) = {}^{B}\tau, \tag{7}$$

where ${}^{B}I$ — the inertia matrix — is (in a simplified manner) a diagonal matrix:

$$ {}^{B}I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \tag{8}$$

Above, we have defined the two subsystems of the drone's dynamics (translation and rotation). In order to write the entire dynamic model, we need to transform the angular accelerations from the mobile frame to the fixed one.

In the **Body frame**, it can be expressed as:

$$ {}^{B}\vec{\Omega} = {}^{B}_{I}R\, {}^{I}\vec{\Omega} = \begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & s\phi c\theta \\ 0 & -s\phi & c\phi c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = W\dot{\eta}\,, \tag{9}$$

where the angles are written as $\eta = \begin{bmatrix} \phi & \theta & \psi \end{bmatrix}^\top$.

Finally, the dynamic equations describing the drone's rotational and translational motions can be written in matrix form as follows:

$$\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix} = \begin{bmatrix}0\\0\\-g\end{bmatrix} + \frac{1}{m}\begin{bmatrix}c\phi s\theta c\psi + s\phi s\psi\\ c\phi s\theta s\psi - s\phi c\psi\\ c\phi c\theta\end{bmatrix} T, \tag{10a}$$

$$\begin{bmatrix}\dot{\Omega}_x\\ \dot{\Omega}_y\\ \dot{\Omega}_z\end{bmatrix} = \begin{bmatrix}(I_{yy}-I_{zz})I_{xx}^{-1}\Omega_y\Omega_z\\ (I_{zz}-I_{xx})I_{yy}^{-1}\Omega_z\Omega_x\\ (I_{xx}-I_{yy})I_{zz}^{-1}\Omega_x\Omega_y\end{bmatrix} + \begin{bmatrix}I_{xx}^{-1}\tau_\phi\\ I_{yy}^{-1}\tau_\theta\\ I_{zz}^{-1}\tau_\psi\end{bmatrix} \tag{10b}$$

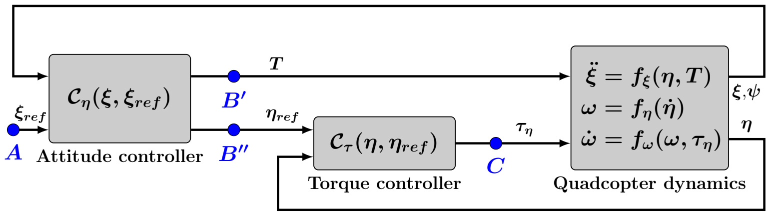

Figure 2: Quadcopter control scheme

1.3 High-level control strategies

By imposing a value for the angle ψ and a certain trajectory generated offline, one can calculate the reference angles η_{ref} and the thrust force T based on the matrix equation defining the translational motion:

$$\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}+g\end{bmatrix} = \frac{T}{m}\begin{bmatrix}c\phi s\theta c\psi + s\phi s\psi\\ c\phi s\theta s\psi - s\phi c\psi\\ c\phi c\theta\end{bmatrix} \tag{11}$$

First, the thrust force T is calculated as follows:

$$\ddot{x}^2 + \ddot{y}^2 + (\ddot{z}+g)^2 = \frac{T^2}{m^2}[c\phi^2 s\theta^2(c\psi^2 + s\psi^2) + s\phi^2(s\psi^2 + c\psi^2) + c\theta^2 c\phi^2] \tag{12a}$$

$$\ddot{x}^2 + \ddot{y}^2 + (\ddot{z}+g)^2 = \frac{T^2}{m^2}[c\phi^2(s\theta^2 + c\theta^2) + s\phi^2] \tag{12b}$$

$$\ddot{x}^2 + \ddot{y}^2 + (\ddot{z}+g)^2 = \frac{T^2}{m^2}[c\phi^2 + s\phi^2]\ddot{x}^2 + \ddot{y}^2 + (\ddot{z}+g)^2 = \frac{T^2}{m^2} \tag{12c}$$

Therefore, the thrust force can be rewritten as:

$$T = m\sqrt{\ddot{x}^2 + \ddot{y}^2 + (\ddot{z}+g)^2} \tag{13}$$

Usually, the angle ψ is free and is introduced as a reference. In other words:

$$\psi = \psi_{ref} \tag{14}$$

The value for the angle θ can be calculated using:

$$\ddot{x}c\psi + \ddot{y}s\psi = \frac{T}{m}[(c\phi s\theta c\psi + s\phi s\psi)c\psi + (c\phi s\theta s\psi - s\phi c\psi)s\psi] \tag{15a}$$

$$\ddot{x}c\psi + \ddot{y}s\psi = \frac{T}{m}[c\phi s\theta(c\psi^2 + s\psi^2)] \tag{15b}$$

$$\ddot{x}c\psi + \ddot{y}s\psi = \frac{T}{m}c\phi s\theta \tag{15c}$$

Thus, by taking the result from (15c) and dividing by $\ddot{z} + g$, we obtain:

$$\frac{\ddot{x}c\psi + \ddot{y}s\psi}{\ddot{z} + g} = \frac{\frac{T}{m}c\phi s\theta}{\frac{T}{m}c\phi c\theta} = \tan\theta \tag{16}$$

Hence, θ can be calculated from (16) by applying:

$$\theta = \arctan\left(\frac{\ddot{x}c\psi + \ddot{y}s\psi}{\ddot{z} + g}\right) \tag{17}$$

The value for the angle ϕ can be calculated using:

$$\ddot{x}s\psi - \ddot{y}c\psi = \frac{T}{m}[(c\phi s\theta c\psi + s\phi s\psi)s\psi - (c\phi s\theta s\psi - s\phi c\psi)c\psi] \tag{18a}$$

$$\ddot{x}s\psi - \ddot{y}c\psi = \frac{T}{m}[s\phi(s\psi^2 + c\psi^2)] \tag{18b}$$

$$\ddot{x}s\psi - \ddot{y}c\psi = \frac{T}{m}s\phi \tag{18c}$$

Taking the result from (18c) and multiplying by $\frac{m}{T}$, we get:

$$(\ddot{x}s\psi - \ddot{y}c\psi)\frac{m}{T} = s\phi \tag{19}$$

By substituting (13) for T in (19), we obtain:

$$s\phi = \frac{\ddot{x}s\psi - \ddot{y}c\psi}{\sqrt{\ddot{x}^2 + \ddot{y}^2 + (\ddot{z} + g)^2}} \tag{20}$$

Thus, the angle ϕ can be calculated from (20) by applying:

$$\phi = \arcsin\left(\frac{\ddot{x}s\psi - \ddot{y}c\psi}{\sqrt{\ddot{x}^2 + \ddot{y}^2 + (\ddot{z} + g)^2}}\right) \tag{21}$$

Finally, the calculation formulas for the reference angles and the thrust force are:

$$T = m\sqrt{\ddot{x}^2 + \ddot{y}^2 + (\ddot{z} + g)^2} \tag{22}$$

$$\phi_{ref} = \arcsin\left(\frac{\ddot{x}s\psi - \ddot{y}c\psi}{\sqrt{\ddot{x}^2 + \ddot{y}^2 + (\ddot{z} + g)^2}}\right) \tag{23}$$

$$\theta_{ref} = \arctan\left(\frac{\ddot{x}c\psi + \ddot{y}s\psi}{\ddot{z} + g}\right) \tag{24}$$

$$\psi_{ref} = \psi \tag{25}$$

By means of the formulas (22), relationships are highlighted between the accelerations $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ and the angle ψ on one hand, and the thrust force T and the angles ϕ and θ on the other hand. Thus, if one wishes to influence the accelerations, and therefore implicitly the position of the drone, T, ϕ, and θ must be chosen appropriately.

Under these conditions, a few reference quantities $\ddot{x}_{ref}$, $\ddot{y}_{ref}$, and $\ddot{z}_{ref}$ are defined so that in the end we obtain:

$$\ddot{x} = \ddot{x}_{ref} \tag{26}$$

$$\ddot{y} = \ddot{y}_{ref} \tag{27}$$

$$\ddot{z} = \ddot{z}_{ref} \tag{28}$$

We have already discussed about Bezier curves in a prior lab. We now implement a B-spline construction. These are obtained by a recurrence relation of the form:

$$B_{\ell,d,\zeta}(\tau) = \frac{\tau - \tau_\ell}{\tau_{\ell+d} - \tau_\ell} B_{\ell,d-1,\zeta}(\tau) + \frac{\tau_{\ell+d+1} - \tau}{\tau_{\ell+d+1} - \tau_{\ell+1}} B_{\ell+1,d-1,\zeta}(\tau), \tag{29a}$$

$$B_{\ell,0,\zeta}(\tau) = \begin{cases} 1, & \tau \in [\tau_\ell, \tau_{\ell+1}), \\ 0, & \text{otherwise.} \end{cases}, \quad \ell = 1 \dots n. \tag{29b}$$

The parameters d and $\zeta = \{\underbrace{\tau_1, \dots \tau_1}_{d+1}, \tau_2, \dots, \tau_{n-1}, \underbrace{\tau_n, \dots \tau_n}_{d+1}\}$ define the order of the B-spline function and its "knot vector". Then, the B-spline curve is obtained simply by multiplying the basis functions with control points P_ℓ:

$$\begin{bmatrix} x_{ref}(\tau) \\ y_{ref}(\tau) \\ z_{ref}(\tau) \end{bmatrix} = \sum_{\ell=1}^{n} P_\ell B_{\ell,d,\zeta}(\tau). \tag{30}$$

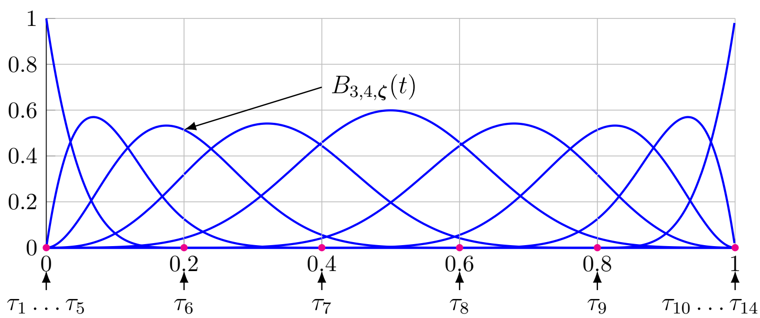

Figure 3: B-spline basis functions for $d = 4$

2 Implementation

3 Proposed exercises