# Seminarul 5

## Subspații vectoriale. Aplicații liniare

2) $(\mathbb{R}^3, +, \cdot)|\mathbb{R}$, $S = \{(1,2,3), (-1,1,5)\}$

$S' = \{(1,5,11), (2,1,-2), (3,6,9)\}$

a) $\langle S \rangle = \langle S' \rangle = V'$

b) Să se descrie $V'$ printr-un sistem de ec. liniare

c) Să se det $V''$ a.î. $\mathbb{R}^3 = V' \oplus V''$

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a) $\text{rang}\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 5 \end{pmatrix} = 2 = \max \overset{SLi}{\Longrightarrow} S \text{ SLi} \Rightarrow \dim \langle S \rangle = 2$

$\text{rang}\begin{pmatrix} 1 & 2 & 3 \\ 5 & 1 & 6 \\ 11 & -2 & 9 \end{pmatrix} < 3$, $\text{rang} = 2 \Rightarrow S'' = \langle \{(1,5,11),$

$, (2,1,-2)\} \rangle$ SLi maximal în $S'$

$\dim \langle S' \rangle = \dim \langle S'' \rangle = 2$

$$(2,1,-2) = (1,2,3) - (-1,-1,5)$$

$$(1,5,11) = \underset{a}{2}(1,2,3) + \underset{b}{1}(-1,1,5)$$

$$\left.\begin{array}{l} \langle S'' \rangle \subset \langle S' \rangle \\ \dim \langle S'' \rangle = \dim \langle S' \rangle \end{array}\right| \Rightarrow \langle S' \rangle = \langle S'' \rangle$$

b) $R' = \{(1,2,3), (-1,1,5)\}$ reper in $V'$

c) $\operatorname{rang}\begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 3 & 5 & 1 \end{pmatrix} = 3 \Rightarrow V'' = \langle \{e_3\} \rangle$

3) $(\mathbb{R}^3, +, \cdot)|_{\mathbb{R}}$, $V' = \{(x,y,z) \in \mathbb{R}^3 \mid \begin{cases} x-y+2z=0 \\ 2x+y+z=0 \end{cases}\}$

Să se descompună $x = (-1,3,4)$ în raport cu $\mathbb{R}^3 = V' \oplus V''$

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \end{pmatrix} \qquad V' = S(A)$$

$$\dim V' = 3 - \operatorname{rg} A = 3 - 2 = 1$$

$$\begin{cases} x - y = -2z \\ 2x + y = -z \end{cases} \; (+)$$

$$3x = -3z \Rightarrow x = -z \Rightarrow y = z$$

$V' = \{(-z, z, z) \mid z \in \mathbb{R}\}$

$V' = \langle\{(-1, 1, 1)\}\rangle = R'$

$rg\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = 3 = \text{maxim} \Rightarrow V'' = \langle\{e_1, e_2\}\rangle$

$(-1, 3, 4) = a(-1, 1, 1) + b(1, 0, 0) + c(0, 1, 0)$

$= (-a+b, a+c, a)$

$\begin{cases} -a+b = -1 \\ a+c = 3 \\ \boxed{a = 4} \end{cases} \Rightarrow \boxed{c = -1}$ $\quad -4 + b = -1$ $\quad \boxed{b = 3}$

$V' = 4(-1, 1, 1)$

$V'' = 3(1, 0, 0) + -1(0, 1, 0) = (3, -1, 0)$

## Aplicații liniare

2) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3,$

$2x_1 + 5x_2 + 3x_3, -3x_1 - 7x_2 - 4x_3)$

a) $f$ liniară

b) $\operatorname{Ker} f = ?$ Precizați un reper în $\operatorname{Ker} f$

c) $\operatorname{Im} f = ?$ — $\text{ll}$ — $\operatorname{Im} f$

d) $[f]_{R_0, R_0} = A = ?$ $R_0$ = reperul canonic în $\mathbb{R}^3$

a) $f(ax+by) = f(ax_1+by_1, ax_2+by_2, ax_3+by_3)$

$= (ax_1+by_1+2(ax_2+by_2)+ax_3+by_3,\ 2(ax_1+by_1)+$

$5(ax_2+by_2)+3(ax_3+by_3),\ -3(ax_1+by_1)-7(ax_2+by_2)$

$-4(ax_3+by_3))$

$= a f(x) + b f(y)$

b) $\operatorname{Ker} f = \{x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3}\} = \begin{cases} x_1+2x_2+x_3=0 \\ 2x_1+5x_2+3x_3=0 \\ -3x_1-7x_2-4x_3=0 \end{cases}$

$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix} \Rightarrow \det A = 0$

$\operatorname{Ker} f = S(A) \Rightarrow \dim \operatorname{Ker} f = 3 - \operatorname{rang} A = 1.$

$\begin{cases} x_1+2x_2 = -x_3 \quad |(-2) \\ 2x_1+5x_2 = -3x_3 \end{cases} \Rightarrow \begin{matrix} x_2 = -x_3 \\ x_1 = x_3 \end{matrix}$

$\operatorname{Ker} f = \{(x_3, -x_3, x_3) \mid x_3 \in \mathbb{R}\} = \langle \underbrace{\{(1,-1,1)\}}_{R'} \rangle$

c) Metoda I

$\operatorname{Im} f = \{y \in \mathbb{R}^3 \mid \exists\, x \in \mathbb{R}^3 \text{ a.î. } f(x) = y\}$

$$\begin{cases} x_1+2x_2+x_3=y_1 \\ 2x_1+5x_2+3x_3=y_2 \\ -3x_1-7x_2-4x_3=y_3 \end{cases}$$

$$A=\left(\begin{matrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ -3 & -7 & -4 \end{matrix}\right)\left|\begin{matrix} y_1 \\ y_2 \\ y_3. \end{matrix}\right. \quad SC$$

$$\Delta_C=0 \Rightarrow \begin{vmatrix} 1 & 2 & y_1 \\ 2 & 5 & y_2 \\ -3 & -7 & y_3 \end{vmatrix}=0 \Leftrightarrow \begin{vmatrix} 0 & 0 & y_1+y_2+y_3 \\ 2 & 5 & y_2 \\ -3 & -7 & y_3 \end{vmatrix}=0 \Rightarrow$$

$$\Rightarrow (y_1+y_2+y_3)=0 \Rightarrow \operatorname{Im} f=\{y\in\mathbb{R}^3 \mid y_1+y_2+y_3=0\}=$$

$$=\{(-y_2-y_3,y_2,y_3) \mid y_2,y_3\in\mathbb{R}\}.=\langle\underbrace{\{(-1,1,0),(-1,0,1)\}}_{R''}\rangle$$

<u>Teorema dimensiunii</u>

$$\underset{\overset{\parallel}{3}}{\dim\mathbb{R}^3}=\underset{\overset{\parallel}{1}}{\dim \operatorname{Ker} f}+\dim\operatorname{Im} f$$

$$3=1+\dim\operatorname{Im} f \Rightarrow \dim\operatorname{Im} f=2.$$

$R''$ reper pt $\operatorname{Im} f$

Metoda 2

$R' = \{(1,-1,1)\}$ reper în Ker $f$

extindem la un reper în $\mathbb{R}^3$

$$rg\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 3 \Rightarrow \{f(e_2), f(e_3)\} \text{ reper în Im} f$$

$f(e_2) = (2, 5, -7)$

$f(e_3) = (1, 3, -4)$

d) $R_0 \xrightarrow{A} R_0$

$$f(e_i) = \sum_{j=1}^{3} a_{ji} e_j, \quad i = \overline{1,3}$$

$f(e_1) = (1, 2, -3) = e_1 + 2e_2 - 3e_3$

$f(e_2) = (2, 5, -7) = 2e_1 + 5e_2 - 7e_3$

$f(e_3) = (1, 3, -4) = e_1 + 3e_2 - 4e_3$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix}$$

OBS: $f(x) = y \Leftrightarrow Y = AX$

$$\begin{pmatrix} x_1+2x_2+x_3 \\ 2x_1+5x_2+3x_3 \\ -3x_1-7x_2-4x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

3) $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f(x) = (3x_1 - 2x_2, 2x_1 - x_2, -x_1 + x_2)$

a) $f$ liniară

b) $f$ inj

c) $\operatorname{Im} f = ?$

d) $[f]_{R_0, R_0'} = A = ?$ $R_0, R_0'$ reper canonic în $\mathbb{R}^2$, respectiv $\mathbb{R}^3$

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a) $f(e_1) = f(1,0) = (3, 2, -1) = 3e_1' + 2e_2' - 1e_3'$

$f(e_2) = f(0,1) = (-2, -1, 1) = -2e_1' - 1e_2' + e_3'$

$$A = \begin{pmatrix} 3 & -2 \\ 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$f(x) = y \iff Y = AX$

b) $f$ inj $\iff \operatorname{Ker} f = \{0_{\mathbb{R}^2}\}$.

$x \in \operatorname{Ker} f \Rightarrow f(x) = 0_{\mathbb{R}^3}$

$$\begin{cases} 3x_1 - 2x_2 = 0 \\ 2x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases} \qquad A = \begin{pmatrix} 3 & 2 \\ 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$rg A = 2 = max \Rightarrow SCD \Rightarrow (x_1, x_2) = (0,0)$.

T. dimensiunii

$$\dim R^2 = \underbrace{\dim Ker f}_{\overset{\parallel}{0}} + \dim Im f \Rightarrow \dim Im f = 2$$

$y \in Im f \Rightarrow \exists x \in R^2 \mid f(x) = y.$

$$\begin{cases} 3x_1 - 2x_2 = y_1 \\ 2x_1 - x_2 = y_2 \\ -x_1 + x_2 = y_3 \end{cases} \qquad A = \left(\begin{array}{cc|c} 3 & 2 & y_1 \\ 2 & -1 & y_2 \\ -1 & 1 & y_3 \end{array}\right)$$

$$\Delta_c = 0 \Rightarrow \begin{vmatrix} 3 & -2 & y_1 \\ 2 & -1 & y_2 \\ -1 & 1 & y_3 \end{vmatrix} = 0 \Leftrightarrow 1 \cdot y_1 - 1 y_2 + 1 \cdot y_3 = 0$$

$\Rightarrow Im f = \{ y \in R^3 \mid y_1 - y_2 + y_3 = 0 \}$

$$= \{ (\overset{(y_1, y_1, 0) + (0, y_3, y_3)}{y_1, y_1 + y_3, y_3}) \mid y_1, y_3 \in R \} =$$

$$= \langle \{ (1,1,0), (0,1,1) \} \rangle$$

11) $f : R_2[X] \to R_1[X]$ liniară

$f(X+2) = X+1$, $f(-X^2+3) = 2X+3$

$f(2X+5) = -X+1$

Determinați $f$.

$$\begin{cases} 2f(1)+f(x)=x+1 \quad /\cdot(-2) \\ 3f(1)-f(x^2)=2x+3 \\ 5f(1)+2f(x)=-x+1 \end{cases} \Rightarrow \begin{cases} -4f(1)-2f(x)=-2x-2 \\ 5f(1)+2f(x)=-x+1 \quad (+) \\ \hline 3f(1)-f(x^2)=2x+3 \end{cases}$$

$$\begin{cases} f(1)=-3x-1 \\ 3f(1)-f(x^2)=2x+3 \Rightarrow f(x^2)=-11x-6 \\ 2f(1)+f(x)=x+1 \Rightarrow f(x)=7x+3 \end{cases}$$

$f(a_0+a_1x+a_2x^2)=a_0f(1)+a_1f(x)+a_2f(x^2)=$

$=a_0(-3x-1)+a_1(7x+3)+a_2(-11x-6)=$

$=-a_0+3a_1-6a_2+(-3a_0-7a_1-11a_2)x$

$[f]_{R_0,R_0'}=A=?$ unde $R_0=\{1,x,x^2\}$ reper canonic din $R_2[X]$

$R_0'=\{1,x\}$ reper canonic din $R_1[X]$

$$A=\begin{pmatrix} -1 & 3 & -6 \\ -3 & 7 & -11 \end{pmatrix}$$

7) $f \in End(V)$ a.î. $f^2 = 0$

Să se arate că $g = id_V + f \in Aut(V)$

$A^2 = O_n \Rightarrow I_n + A$ inversabilă

$I_n^2 = I_n^2 - A^2 = (I_n - A)(I_n + A)$

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