1

## = Seminar 1 → Șiruri de $\mathbb{R}$ =

DEF// "A" $\subset \mathbb{N}$ → MULȚIME NUMĂRABILĂ i.e. (ADICĂ), $(\exists)$ o bij. ÎNTRE EA și $\mathbb{N}$. ȘIR NR $\mathbb{R}$ = $f: A \to \mathbb{R}$.

NOTAȚII: 1) $f(n) \overset{\text{not}}{=} x_n$, $(\forall)\, n \in A$

2) CONFORM DEF. MAI SUS și ① ⇒ ȘIRUL DE NR. $\mathbb{R}$, NOTAT $(x_n)_{n\in A}$ SAU $(x_n)_n$.

OBS// ÎN GENERAL, $A = \mathbb{N}$ sau $A = \mathbb{N}^*$, CAZURI ÎN CARE VOM SCRIE $(x_n)_{n\in\mathbb{N}}$, $(x_n)_{n\geq 0}$ sau $(x_n)_n$, RESPECTIV $(x_n)_{n\in\mathbb{N}^*}$, $(x_n)_{n\geq 1}$ sau $(x_n)_n$.

$(x_n)_n \subset \mathbb{R}$

DEF// • 1) $l \in \mathbb{R}$. SPUNEM CĂ $(x_n)_n$ ARE LIMITA "$l$" și SCRIEM $\lim\limits_{n\to\infty} x_n = l$ dacă $(\forall)\, \varepsilon > 0$, $(\exists)\, n_\varepsilon \in \mathbb{N}$ a.î. $(\forall)\, n \geq n_\varepsilon$, avem $|x_n - l| < \varepsilon$.

• 2) $(x_n)_n$ ARE LIMITA "$+\infty$" DACĂ $(\forall)\, \varepsilon > 0$, $(\exists)\, n_\varepsilon \in \mathbb{N}$ a.î. $(\forall)\, n \geq n_\varepsilon$, avem $x_n > \varepsilon$. $\left(\lim\limits_{n\to\infty} x_n = +\infty\right)$

• 3) $(x_n)_n$ ARE LIMITĂ "$-\infty$" DACĂ $(\forall)\, \varepsilon > 0$, $(\exists)\, n_\varepsilon \in \mathbb{N}$ a.î. $(\forall)\, n \geq n_\varepsilon$, avem $x_n < -\varepsilon$. $\left(\lim\limits_{n\to\infty} x_n = -\infty\right)$

Șiruri nr $\mathbb{R}$
- → cu limită
  - → finită (CONVERGENTE)
  - → infinită (DIVERGENTE)
- → fără limită (divergente)

Ex: 1) $x_n = \frac{1}{n}$, $(\forall)\, n \in \mathbb{N}^*$. ARĂTAȚI, FOLOSIND DEF, CĂ $l = 0$.

Sol: $\lim\limits_{n\to\infty} x_n = 0 \Leftrightarrow (\forall)\, \varepsilon > 0$, $(\exists)\, n_\varepsilon \in \mathbb{N}$ a.î. $(\forall)\, n \geq n_\varepsilon$, avem $|x_n - 0| < \varepsilon$

Fie $\varepsilon > 0$, căutăm $n_\varepsilon \in \mathbb{N}$ a.î. $(\forall)\, n \geq n_\varepsilon$, avem $|x_n - 0| < \varepsilon$

$|x_n - 0| < \varepsilon \Leftrightarrow \left|\frac{1}{n} - 0\right| < \varepsilon \Leftrightarrow \frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$.

Luăm $n_\varepsilon = \left[\frac{1}{\varepsilon} + 1\right]$ și OBȚINEM CONCLUZIA.

## • CRITERIUL RAPORTULUI PT. ȘIRURI CU TERMENI STRICT (+):

$(x_n)_n \subseteq (0;+\infty)$ a.î. $(\exists) \lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} \overset{not}{=} l \in [0;+\infty]$

1) $l < 1 \Rightarrow \lim\limits_{n\to\infty} x_n = 0$

2) $l > 1 \Rightarrow \lim = +\infty$

3) $l = 1 \Rightarrow$ NU ȘTIM (DECIDE)

2) $a > 0$ și $x_n = n \cdot a^n$, $(\forall) n \in \mathbb{N}^*$.

$\lim\limits_{n\to\infty} x_n = ?$

Sol: $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim\limits_{n\to\infty} \frac{(n+1)a^{n+1}}{n \cdot a^n} = a$

Dacă
$a < 1 \Rightarrow l = 0$
$a = 1 \Rightarrow$ NU DECIDE
$a > 1 \Rightarrow l = +\infty$

$a = 1 \Rightarrow x_n = n \Rightarrow \lim\limits_{n\to\infty} x_n = +\infty$

## • CRITERIUL RADICALULUI .....:

$(x_n)_n \subseteq (0;+\infty)$ a.î. $(\exists) \lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} \overset{not}{=} l \in [0;+\infty] \Rightarrow$

$\Rightarrow (\exists) \lim\limits_{n\to\infty} \sqrt[n]{x_n}$ și $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = l$

3) a) $\lim\limits_{n\to\infty} \sqrt[n]{n} = ?$

$\sqrt[1]{a} = a$; $(\forall) a \in \mathbb{R}$.

Fie $x_n = n$, $(\forall) n \in \mathbb{N}^*$

$\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim\limits_{n\to\infty} \frac{n+1}{n} = 1$.

CONFORM CRITERIULUI RAD., $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = 1$.

b) $\lim\limits_{n\to\infty} \sqrt[n]{n!} = ?$

Fie $x_n = n!$, $(\forall) n \in \mathbb{N}^*$

$\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim\limits_{n\to\infty} \frac{(n+1)!}{n!} = \lim\limits_{n\to\infty} (n+1) = \infty \Rightarrow l = \infty$

• WEIESSTRASS: ȘIR MONOTON + MĂRGINIT $\Rightarrow$ CONVERGENT. (LIMITĂ FINITĂ)

OBS// RECIPROCA E FALSĂ.

4) $x_n = \frac{(-1)^n}{n}$, $(\forall) n \in \mathbb{N}^*$. DEM. CONVERGENȚA, DAR NU E MONOTON.

Sol: $-1 \leq (-1)^n \leq 1$, $(\forall) n \in \mathbb{N}^* \Leftrightarrow -\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$, $(\forall)$ $n \in \mathbb{N}^* \Rightarrow \lim\limits_{n\to\infty} x_n = 0 \Rightarrow$ CONVERGENT

$-\frac{1}{n} \to 0$, $\frac{(-1)^n}{n} \to 0$, $\frac{1}{n} \to 0$

$X_1 = -1$; $X_2 = \frac{1}{2}$; $X_3 = -\frac{1}{3}$ $\Rightarrow X_1 < X_2 > X_3 \Rightarrow$ NU E MONOTON

* ORICE ȘIR DE NR. $\mathbb{R}$ CONVERGENT ESTE MĂRGINIT.

2

# Seminar 2

1) $X_n = 1 + \frac{1}{2^2} + \dots + \frac{1}{n^2}$ ; $(\forall)\ n \in \mathbb{N}^*$. ARĂTAȚI CĂ $(x_n)_n$ E CONVERGENT.

ⓐ • MONOTONIE: Fie $n \in \mathbb{N}^*$

$$X_{n+1} - X_n = \cancel{1} + \cancel{\frac{1}{2^2}} + \dots + \cancel{\frac{1}{n^2}} + \frac{1}{(n+1)^2} - \cancel{1} - \cancel{\frac{1}{2}} - \dots - \cancel{\frac{1}{n^2}} = \frac{1}{(n+1)^2} > 0 \Rightarrow$$

$\Rightarrow (X_n)_n \nearrow$

ⓑ • MĂRGINIRE:

$$\frac{1}{2^2} < \frac{1}{1\cdot 2}$$

$$\frac{1}{3^2} < \frac{1}{2\cdot 3}$$

$$\vdots$$

$$\frac{1}{n^2} < \frac{1}{(n-1)n}$$

$(+)$

$$1 + \frac{1}{2^2} + \dots + \frac{1}{n^2} < 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{(n-1)n} \Leftrightarrow$$

$$\Leftrightarrow X_n < 1 + \left(1 - \cancel{\frac{1}{2}}\right) + \dots + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n}\right) \Leftrightarrow X_n < 2 - \frac{1}{n} \ ; (\forall)\ n \in \mathbb{N}^*$$

DECI $X_n < 2 \Rightarrow (X_n)_n$ MĂRGINIT SUPERIOR

ⓐ + ⓑ $\Rightarrow$ CONV.

**DEF** Fie $X \neq \emptyset$. $d : X \times X \to [0;\infty)$ S.N. "METRIC/DISTANȚĂ" PE $X$, DACĂ:

1) $d(x,y) = 0 \Leftrightarrow x = y$, $(\forall)\ x,y \in X$
2) $d(x,y) = d(y,x)$ " — " — "
3) $d(x,y) + d(y,z) \geq d(x,z)$ " — " — " $\to$ INEGALITATEA TRIUNGHI

**DEF** $X \neq \emptyset$ și $d : X \times X \to [0;\infty)$. $(X,d)$ = "SPAȚIU METRIC".

**DEF** $(X,d) \to$ SPAȚIU M.; $(X_n)_n \subseteq X$ și $x \in X$. $(X_n)_n$ ARE $\lim = x$ ÎN RAPORT CU METRICA $d$ (SAU CONVERGE) ȘI NOT. $\lim\limits_{n\to\infty} x_n \overset{d}{=} x$ (SAU) $x_n \xrightarrow[n]{d} x$, DACĂ $\lim\limits_{n\to\infty} d(X_n, x) = 0$ (lim. ȘIR $\mathbb{R}$).

OBS. "ÎN RAPORT CU METRICA $d$" = "ÎN SPAȚIUL METRIC $(X,d)$".

2) $X \neq \emptyset$; $d: X \times X \to \mathbb{R}$; $d(x,y) = \begin{cases} 0; & x=y \\ 1; & x \neq y \end{cases}$

$\overline{(X,d) \to SP.M.}$

a) $d(x,y) \geq 0$; $(\forall)\ x,y \in X$ (EVIDENT) $\Rightarrow d: X \times X \to \mathbb{R}_+$

b) $d(x,y) = 0$; $(\forall)\ x,y \in X$ " — " — "

c) $d(x,y) = d(y,x)$; $(\forall)\ x,y \in X$ " — " — "

d) $d(x,z) \leq d(x,y) + d(y+z)$ ✓

$d(x,z) \in \{0,1\}$ și $d(x,y) + d(y,z) \in \{0,1,2\}$

$\begin{cases} \bullet \text{ DACĂ } d(x,z) = 0 \Rightarrow \text{EVIDENT} \\ \bullet \text{ " — " } d(x,z) = 1 \ (*) \end{cases}$

$\begin{cases} \bullet \text{ DACĂ SUMA} \in \{1,2\} \Rightarrow \text{EVIDENT} \\ \bullet \text{ DACĂ SUMA} = 0 \Rightarrow x = y = z, \text{ DAR } (*) \Rightarrow x \neq z \end{cases}$

$\not{\Delta}$

$a + b + c + d \Rightarrow (X,d) \to S.M.$

examen 3) $d_1: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $d_1((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$

$((x_n, y_n))_n \subseteq \mathbb{R}^2$ și $(x,y) \in \mathbb{R}^2$.

a) $(\mathbb{R}^2, d_1) \to S.M.$

b) ARĂTAȚI CĂ AVEM ECHIVALENȚA: $(x_n, y_n) \xrightarrow[n]{d_1} (x,y) \Leftrightarrow x_n \xrightarrow[n]{} x$ și $y_n \xrightarrow[n]{} y$.

a) NOT. $a = (x_1, x_2)$; $b = (y_1, y_2)$; $c = (z_1, z_2)$

$\bullet\ d_1(a,b) = |x_1 - y_1| + |x_2 - y_2| \geq 0$, $(\forall)\ a,b \in \mathbb{R}^2$

$\bullet\ d_1(a,b) \Leftrightarrow |x_1 - y_1| + |x_2 - y_2| = 0 \Leftrightarrow x_1 = y_1$ și $x_2 = y_2 \Leftrightarrow$

$\Leftrightarrow a = b$

$\bullet\ d_1(a,b) = |x_1 - y_1| + |x_2 - y_2|$

$d_1(b,a) = |y_1 - x_1| + |y_2 - x_2| = |-(x_1 - y_1)| + |-(x_2 - y_2)| =$

$= d_1(a,b)$

- $d_1(a,c) \leq d_1(a,b) + d_1(b,c)$ ✓

$d_1(a,c) = |x_1 - z_1| + |x_2 - z_2|$

$d_1(a,b) = |x_1 - y_1| + |x_2 - y_2|$

$d_1(b,c) = |y_1 - z_1| + |y_2 - z_2|$

$$\left.\begin{array}{l} |x_1 - z_1| = |x_1 + y_1 - y_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1| \\ |x_2 - z_2| = |x_2 + y_2 - y_2 - z_2| \leq |x_2 - y_2| + |y_2 - z_2| \end{array}\right\} \Rightarrow$$

$\Rightarrow |x_1 - z_1| + |x_2 - z_2| \leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2|$

b) Not. $a_n = (x_n, y_n)$ și $a = (x, y)$

"$\Rightarrow$": ȘTIM CĂ $a_n \xrightarrow[n]{d_1} a$

$a_n \xrightarrow[n]{d_1} a \overset{def.}{\Longleftrightarrow} d_1(a_n, a) \xrightarrow[n]{} 0 \Leftrightarrow |x_n - x| + |y_n - y| \xrightarrow[n]{} 0$

Fie $\varepsilon > 0$. $|x_n - x| + |y_n - y| \xrightarrow[n]{} 0 \Rightarrow (\exists)\ n_\varepsilon \in \mathbb{N}$, a.î. $(\forall)$ $n \geq n_\varepsilon$ avem $|x_n - x| + |y_n - y| - 0 < \varepsilon$

Fie $n \geq n_\varepsilon$. $|x_n - x| \leq |x_n - x| + |y_n - y| < \varepsilon$

$|y_n - y| \leq |x_n - x| + |y_n - y| < \varepsilon$ $\square$

"$\Leftarrow$": ȘTIM $x_n \xrightarrow[n]{} x$ și $y_n \xrightarrow[n]{} y$

Fie $\varepsilon > 0$, $(\exists)\ n_\varepsilon^1 \in \mathbb{N}$ a.î. $(\forall)$ $n \geq n_\varepsilon^1$ avem $|x_n - x| < \frac{\varepsilon}{2}$

$y_n \xrightarrow[n]{} y \Rightarrow (\exists)\ n_\varepsilon^2 \in \mathbb{N}$ a.î. $(\forall)$ $n \geq n_\varepsilon^2$, $|y_n - y| < \frac{\varepsilon}{2}$

Fie $n_\varepsilon = \max(n_\varepsilon^1, n_\varepsilon^2)$; $n > n_\varepsilon$

$\underbrace{|x_n - x| + |y_n - y|}_{d_1(a_n, a)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\square$

= LIM. SUP. + INF. ȘIR $\mathbb{R}$ =

$(x_n)_n \subseteq \mathbb{R}$

DEF: $x \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. $x$ = "PCT. LIM. $(x_n)_n$" DACĂ $\exists$ subșir $(x_{n_k})_k \subseteq (x_n)_n$ a.î. $\lim_{k \to \infty} x_{n_k} = x$.

NOT: $\mathcal{L}((x_n)_n) \neq \emptyset = \{x \in \overline{\mathbb{R}} \mid x = \text{PCT. LIMITĂ } (x_n)_n\}$

PROP: $(\exists)$ UN CEL MAI "$>$" PCT. LIM. (FINIT SAU NU) și UN CEL MAI "$<$" PCT. LIM. (...) A LUI $(x_n)_n$.

DEF 1) CEL MAI ">" PCT. LIM. $(x_n)_n$ S.N. "LIM. SUPERIOARĂ" ($\overline{\lim}\, x_n$).

2) " — " "<" " ——— " " ——— " INFERIOARĂ" ($\underline{\lim}\, x_n$).

PROP 1) $\underline{\lim}\, x_n \leq \overline{\lim}\, x_n$

2) $(x_n)_n$ ARE $\lim \Leftrightarrow \underline{\lim} = \overline{\lim} = \lim\limits_{n\to\infty} x_n$.

examen

4) $\underline{\lim}\, x_n$; $\overline{\lim}\, x_n = ?$

a) $x_n = \dfrac{1+(-1)^n}{2} + (-1)^n \dfrac{n}{2n+1}$; $(\forall)\, n \in \mathbb{N}$

$$x_{2n} = \frac{1+(-1)^{2n}}{2} + (-1)^{2n}\frac{2n}{4n+1} = 1 + \frac{2n}{4n+1} \xrightarrow{n} \frac{3}{2}$$

$$x_{2n+1} = \frac{1+(-1)^{2n+1}}{2} + (-1)^{2n+1}\frac{2n+1}{4n+3} = 0 - \frac{2n+1}{4n+3} \xrightarrow{n} -\frac{1}{2}$$

$\mathbb{N} = 2\mathbb{N} \cup (2\mathbb{N}+1) \Rightarrow \mathcal{L}((x_n)_n) = \left\{-\frac{1}{2}; \frac{3}{2}\right\} \Rightarrow$

$\Rightarrow \left.\begin{array}{l} \underline{\lim}\, x_n = -\frac{1}{2} \\ \overline{\lim}\, x_n = \frac{3}{2} \end{array}\right\} \Rightarrow$ SUNT DIFERITE $\Rightarrow \nexists \lim\limits_{n\to\infty} x_n$

b) $x_n = 1 + 2\cdot(-1)^{n+1} + 3\cdot(-1)^{\frac{n(n+1)}{2}}$

c) $x_n = \sin \dfrac{n\pi}{3}$

d) $x_n = \cos \dfrac{n\pi}{3}$

e) $x_n = \dfrac{n\cdot \cos \frac{n\pi}{2}}{n^2+1}$

f) $x_n = \left(1-\dfrac{1}{n}\right)^n \left[\dfrac{1}{2} + (-1)^n\right] + \cos \dfrac{n\pi}{2}$

g) $x_n = \dfrac{2+(-1)^n}{1+n^{(-1)^n}} + \sin \dfrac{n\pi}{2}$

**3**

# = Seminar 3 =

→ examen

1) $\underline{\lim}$ și $\overline{\lim}\; x_n$, PRECIZÂND DACĂ $(\exists)\ \lim\limits_{n\to\infty} x_n = ?$

a) $x_n = 1+2\cdot(-1)^{n+1}+3\cdot(-1)^{\frac{n(n+1)}{2}}, \ (\forall)\ n\in\mathbb{N}$

$x_{2n} = 1+2\cdot(-1)^{2n+1}+3\cdot(-1)^{n(2n+1)} =$

$= 1+2\cdot(-1)^{2n+1}+3\cdot(-1)^{n(2n+1)}$

$$\begin{cases} x_{2\cdot(2n)} = 1-2+3\cdot(-1)^{2n(4n+1)} = -1+3 = 2 \xrightarrow[n]{} 2 \\ x_{2\cdot(2n+1)} = 1-2+3\cdot(-1)^{2(2n+1)(2(2n+1)+1)} = -1-3 = -4 \xrightarrow[n]{} -4 \end{cases}$$

$$\begin{cases} x_{4n+1} = 1+2(-1)^{4n+2}+3\cdot(-1)^{\frac{(4n+1)(4n+2)}{2}} = \\ \quad = 1+2-3 = 0 \to 0 \\ x_{4n+3} = 1+2\cdot(-1)^{4n+3}+3\cdot(-1)^{\frac{(4n+3)(4n+4)}{2}} = 1+2+3 = 6 \xrightarrow[n]{} 6 \end{cases}$$

$\mathbb{N} = 4\cdot\mathbb{N} \cup 4\cdot(\mathbb{N}+1) \cup 4\cdot(\mathbb{N}+2) \cup 4\cdot(\mathbb{N}+3) \Rightarrow$

$\Rightarrow L((x_n)_n) = \{2,-4,0,6\}$

$\underline{\lim}\; x_n = -4$; $\overline{\lim} = 6$ $\Rightarrow$ DIF $\Rightarrow (\nexists) \lim$

b) $x_n = \sin\frac{n\cdot\pi}{3}$; $(\forall)\ n\in\mathbb{N}$

$$\begin{cases} \sin n\pi = 0 \\ \cos n\cdot\pi = (-1)^n \end{cases} ; \ (\forall)\ n\in\mathbb{N}$$

$x_{6n} = \sin\frac{{}^{2}\cancel{6}n\cdot\pi}{\cancel{3}} = 0 \xrightarrow[n]{} 0$

$x_{6n+1} = \sin\frac{(6n+1)\pi}{3} = \sin\left(2n\pi+\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt3}{2} \xrightarrow[n]{} \frac{\sqrt3}{2}$

$x_{6n+2} = \sin\left(2n\pi+\frac{2\pi}{3}\right) = \sin\frac{2\pi}{3} = 2\cdot\sin\frac{\pi}{3}\cdot\cos\frac{\pi}{3} = \frac{\sqrt3}{2} \to \frac{\sqrt3}{2}$

$x_{6n+3} = \sin(2n+1)\pi = 0 \to 0$

$x_{6n+4} = \sin\left(2n\pi+\frac{4\pi}{3}\right) = \sin\frac{4\pi}{3} = \sin\left(\pi+\frac{\pi}{3}\right) =$

$= \sin\pi\cdot\cos\frac{\pi}{3}+\cos\pi\cdot\sin\frac{\pi}{3} = -\sin\frac{\pi}{3} \to -\frac{\sqrt3}{2}$

$x_{6n+5} = \sin\left(2n\pi+\frac{5\pi}{3}\right) = \sin\frac{5\pi}{3} = \sin\left(2\pi-\frac{\pi}{3}\right) = \sin\left(-\frac{\pi}{3}\right) =$

$= -\sin\frac{\pi}{3} \to -\frac{\sqrt{3}}{2}$

$\mathbb{N} = 6\mathbb{N} \cup (6\mathbb{N}+1) \cup \ldots \cup (6n+5) \Rightarrow \mathcal{L}((x_n)_n) = \{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\} \Rightarrow$

$\Rightarrow \underline{\lim} = -\frac{\sqrt{3}}{2}$; $\overline{\lim}\, x_n = \frac{\sqrt{3}}{2} \Rightarrow$ DIF $\Rightarrow (\nexists) \lim$

c) $x_n = (1-\frac{1}{n})^n [\frac{1}{2} + (-1)^n] + \cos\frac{n\pi}{2}$, $(\forall) n \in \mathbb{N}^*$

$x_{4n} = (1-\frac{1}{4n})^{4n} \cdot [\frac{1}{2} + (-1)^{4n}] + \cos\frac{\cancel{2}4n\pi}{\cancel{2}} =$

$= (1-\frac{1}{4n})^{4n} \cdot \frac{3}{2} + 1 \to \frac{3}{2e} + 1$

$x_{4n+1} = (1-\frac{1}{4n+1})^{4n+1} \cdot [\frac{1}{2} + (-1)^{4n+1}] + \underbrace{\cos\frac{(4n+1)\pi}{2}}_{\cos(2n\pi + \frac{\pi}{2})} \to -\frac{1}{2e} + \cos\frac{\pi}{2} = -\frac{1}{2e}$

$x_{4n+2} = \ldots = \ldots + \cos(2n\pi + \pi) \to \frac{3}{2e} - 1$

$x_{4n+3} = \ldots = \ldots \xrightarrow[n]{} -\frac{1}{2e}$

$\mathbb{N}^* = (4\mathbb{N}^*+2) \cup (4\mathbb{N}^*+1) \cup 4\mathbb{N}^* \cup (4\mathbb{N}^*+3) \cup \{1,2,3\}$

$(1-\frac{1}{n})^n \to \frac{1}{e}$; $(1+\frac{1}{n})^n \to e$

$\mathcal{L}((x_n)_n) = \{\frac{3}{2e}+1; -\frac{1}{2e}; \frac{3}{2e}-1\} \Rightarrow \begin{cases} \underline{\lim} = \frac{3}{2e} - 1 \\ \overline{\lim} = \frac{3}{2e} + 1 \end{cases} \Rightarrow (\nexists) \lim$

COMPARĂM

$-\frac{1}{2e} \boxed{?} \frac{3}{2e} - 1$

$-\frac{1}{2e} \boxed{?} \frac{3-2e}{2e}$

## = SERII DE NR R =

1) SUME SERII = ?

a) $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = ?$ $\Rightarrow$ Fie $x_n = \frac{n}{(n+1)!}$, $(\forall) n \in \mathbb{N}^*$

$y_n = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} \frac{k}{(k+1)!} = \sum_{k=1}^{n} \left(\frac{k+1}{(k+1)!} - \frac{1}{(k+1)!}\right) = \sum_{k=1}^{n} \left(\frac{1}{k!} - \frac{1}{(k+1)!}\right) =$

$= 1 - \cancel{\frac{1}{1\cdot 2}} + \cancel{\frac{1}{1\cdot 2}} - \cancel{\frac{1}{3!}} + \ldots + \cancel{\frac{1}{n!}} - \frac{1}{(n+1)!} = 1 - \frac{1}{(n+1)!}$, $(\forall) n \in \mathbb{N}^*$

$\lim_{n\to\infty} y_n = 1 \Rightarrow \sum_{n\geq 1}^{\infty} x_n = 1$

5) Fie $a>0$ și $f:(0,a]\to\mathbb{R}_+^*$, a.î. $(\exists)\ \lim\limits_{x\to 0}\frac{f(x)}{x} \overset{not.}{=} l\neq 0$

ARĂTAȚI CĂ $\sum\limits_{n=1}^{\infty}\sqrt[n]{n!\cdot f(a)\cdot f(\frac{a}{2})\cdot\ldots\cdot f(\frac{a}{n})}$ DIV.

Fie $x_n=\sqrt[n]{n!\cdot f(a)\cdot\ldots\cdot f(\frac{a}{n})}$, $(\forall)\ n\in\mathbb{N}^*$

VOM ARĂTA CĂ $\lim\limits_{n\to\infty} x_n\neq 0$. PT. ASTA, VOM FOLOSI CR. RADICALULUI PT. ȘIRURI cu TERMENI STRICT POZITIVI:

$$\lim_{n\to\infty}\frac{\cancel{(n+1)!}\cdot \cancel{f(a)}\cdot\ldots\cdot \cancel{f(\frac{a}{n})}\cdot f(\frac{a}{n+1})}{\cancel{n!}\cdot\ldots\cdot \cancel{f(\frac{a}{n})}}=\lim_{n\to\infty}(n+1)\cdot f\left(\frac{a}{n+1}\right)=$$

$$=\lim_{n\to\infty}\frac{\cancel{(n+1)}\cdot f\left(\frac{a}{n+1}\right)}{\frac{a}{n+1}}\cdot\frac{a}{\cancel{n+1}}=a\cdot l$$

$\to l$ (IPOTEZĂ)

DECI $\lim\limits_{n\to\infty}x_n=a\cdot l\neq 0$. PRIN URMARE, SERIA E DIVERGENTĂ.

SERII CU TERMENI POZITIVI:

1) STUDIAȚI CONV. $\sum\limits_{n=1}^{\infty}\frac{1\cdot 4\cdot 7\cdot\ldots\cdot(3n-2)}{3\cdot 6\cdot 9\cdot\ldots\cdot(3n)}\cdot x^n$, $x>0$

# = Seminar 4 =

4

1) CONV = ?

a) $\sum_{n=1}^{\infty} \frac{1\cdot 4\cdot \ldots \cdot (3n-2)}{3\cdot 6\cdot 9\cdot \ldots \cdot 3n}\cdot x^n$ ; $x>0$ $\begin{cases} \text{CONV}, x<1 \\ \text{DIV}, x\geq 1 \end{cases}$

SOL: Fie $x_n = \frac{1\cdot 4\cdot \ldots \cdot (3n-2)}{3\cdot 6\cdot \ldots \cdot 3n}$, $(\forall) n\in\mathbb{N}^*$

CR. RAPORTULUI PT. SERII

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{\cancel{1\cdot 4\cdot \ldots \cdot (3n-2)}\cdot (3n+1)}{\cancel{3\cdot 6\cdot \ldots}\cdot (3n+3)}\cdot \cancel{x^{n+1}} \cdot \frac{\cancel{3\cdot 6\cdot \ldots \cdot 3n}}{\cancel{1\cdot 4\cdot \ldots \cdot (3n-2)}}\cdot \cancel{(x^n)^{-1}} =$$

$$= \lim_{n\to\infty} \frac{3n+1}{3n+3}\cdot x = x$$

I) $x<1 \Rightarrow$ CONV.

II) $x>1 \Rightarrow$ DIV.

III) $x=1 \Rightarrow$ CR. NU DECIDE, DAR, ÎN ACEST CAZ, SERIA DEVINE:

$$\sum_{n=1}^{\infty} \frac{1\cdot 4\cdot \ldots \cdot (3n-2)}{3\cdot 6\cdot \ldots \cdot 3n}$$

APLICĂM CR. RAABE-DUHAMEL PT.

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{3n+3}{3n+1} - 1\right) = \lim_{n\to\infty} \frac{2n}{3n+3} = \frac{2}{3} < 1 \Rightarrow \text{DIV.}$$

b) $\sum_{n=1}^{\infty} \left(\frac{an^2+4n+2}{bn^2+3n+1}\right)^n$ ; $a,b>0$

Fie $x_n = \left(\frac{an^2+4n+2}{bn^2+3n+1}\right)^n$

CR. RADICALULUI

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{an^2+4n+2}{bn^2+3n+1} = \frac{a}{b}$$

I) $\frac{a}{b}<1$ (i.e. $a<b$) $\Rightarrow$ CONV.

II) $\frac{a}{b}>1$ (i.e. $a>b$) $\Rightarrow$ DIV.

III) $\frac{a}{b}=1$ (i.e. $a=b$) $\Rightarrow$ CR. NU DECIDE, DAR, SERIA DEVINE:

$$\sum_{n=1}^{\infty} \left(\frac{an^2+4n+2}{an^2+3n+1}\right)^n$$

$$\lim_{n\to\infty}\left(\frac{an^2+4n+2}{an^2+3n+1}\right)^n = \lim_{n\to\infty}\left(1+\frac{an^2+4n+2-an^2-3n+1}{an^2+3n+1}\right)^n =$$

$$= \lim_{n\to\infty}\Bigg[\underbrace{\left(1+\frac{n+1}{an^2+3n+1}\right)^{\frac{an^2+3n+1}{n+1}}}_{e}\Bigg]^{\frac{n^2+n}{an^2+3n+1}} = e^{\lim\limits_{n\to\infty}\frac{n^2+n}{an^2+3n+1}} = e^{\frac{1}{a}} \neq 0$$

$\Rightarrow$ SERIA DIV.

c) $\sum\limits_{n=2}^{\infty}\frac{1}{n\cdot\ln n}$

Fie $x_n = \frac{1}{n\cdot\ln n}$; $(\forall)\, n\geq 2$, și $(x_n)_n \searrow$

CR. CONDENSĂRII

$$\sum_{n=2}^{\infty}\frac{1}{n\cdot\ln n} \sim \sum_{n=2}^{\infty} 2^n\cdot x_{2^n}$$

$$\sum_{n=2}^{\infty} 2^n\cdot x_{2^n} = \sum_{n=2}^{\infty}\cancel{2^n}\cdot\frac{1}{\cancel{2^n}\cdot\ln 2^n} = \sum_{n=2}^{\infty}\frac{1}{n\cdot\ln 2}$$

STUDIEM CONV.:

$$\sum_{n=2}^{\infty}\frac{1}{n\cdot\ln 2} =$$

Fie $a_n = \frac{1}{n\ln 2}$; $b_n = \frac{1}{n}$, $(\forall)\, n\geq 2$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{\cancel{n}\cdot\ln 2}\cdot\cancel{n} = \frac{1}{\ln 2}\in(0,\infty)$$

(CR. COMPARAȚIEI CU LIMITĂ)

$$\sum_{n=2}^{\infty} a_n \sim \sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty}\frac{1}{n} \to \text{DIV (SERIA ARMONICĂ GENERALIZATĂ)}$$

d) $\sum\limits_{n=1}^{\infty}\frac{\sin\frac{1}{n^2}}{\cos\frac{1}{n}\cdot\cos\frac{1}{n+1}} =$

$\frac{1}{n^2}; \frac{1}{n}; \frac{1}{n+1}\in\left(0;\frac{\pi}{2}\right)$, $(\forall)\, n\in\mathbb{N}^*$

$\sin x > 0$, $\cos x > 0$ / $(\forall)\, x\in\left(0,\frac{\pi}{2}\right)$

$\Rightarrow$ SERIE TERMENI + (CR. COMP. CU lim)

Fie $a_n = \frac{\sin\frac{1}{n^2}}{\cos\frac{1}{n}\cdot\cos\frac{1}{n+1}}$; $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}} \cdot \frac{1}{\cos\frac{1}{n}\cdot\cos\frac{1}{n+1}} = 1 \in (0,\infty)$$

DECI $\sum_{n=1}^{\infty} a_n \sim \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ CONV (SERIE ARMONICĂ GENERALIZATĂ CU $\alpha = 2$)

e) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt[n]{n}}$ ; $x>0$

Fie $x_n = \frac{x^n}{\sqrt[n]{n}}$, $(\forall)\, n \in \mathbb{N}^*$

CR. RAPORTULUI

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{x \cdot \sqrt[n]{n}}{\sqrt[n+1]{n+1}} = x$$

$\lim_{n\to\infty} \sqrt[n]{n} = 1$ (SEMINAR 1)

I) $x<1 \Rightarrow$ CONV.

II) $x>1 \Rightarrow$ DIV

III) $x = 1 \Rightarrow$ CR. NU DECIDE, DAR SERIA $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$

$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 \neq 0 \Rightarrow$ DIV

f) $\sum_{n=1}^{\infty} \underbrace{\frac{n!\, x^n}{(\alpha+1)\cdot\ldots\cdot(\alpha+n)}}_{x_n}$, $\alpha>0$, $x>0$ (CR. RAPORT)

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)x}{\alpha+n+1} = x$$

I) $x<1 \Rightarrow$ CONV

II) $x>1 \Rightarrow$ DIV

III) $x=1 \Rightarrow$ CR. NU DECIDE; SERIA DEVINE $\sum_{n=1}^{\infty} \frac{n!}{(\alpha+1)\ldots(\alpha+n)}$

(CR RAABE - DUHAMEL)

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{\alpha+n+1}{n+1} - 1\right) = \alpha > 0$$

I) $\alpha<1 \Rightarrow$ DIV

II) $\alpha>1 \Rightarrow$ CONV

III) $\alpha = 1 \Rightarrow$ CR. NU DECIDE. SERIA: $\sum_{n=1}^{\infty} \frac{n!}{2\cdot3\cdot4\cdot\ldots\cdot(n+1)} =$

$= \sum_{n=1}^{\infty} \frac{1}{n+1} \sim \sum_{n=1}^{\infty} \frac{1}{n}$ DIV.

(SERIE ARM. $\alpha=1$)

9) $\sum_{n=2}^{\infty} \frac{1}{\ln n} =$

Fie $a_n = \frac{1}{\ln n}$ și $b_n = \frac{1}{n}$; $(\forall)\ n \geq 2$

(CR. COMP. CU LIMITĂ)

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{x}{\ln x} \overset{l'H}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{x}} = \infty$

$\sum b_n$ DIV. $\Rightarrow$ $\sum a_n$ DIV