Mixed integer in motion planning

REPLAN team

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Outline

Outline

Motivation

- Flexible mathematical model for the formulation of decision and control problems based on optimization
  - combinatorial allocation problem
  - multicast routing problem
- Flexible mathematical model for the formulation of collision avoidance problems involving the control of Multi-Agent Systems
  - path following with obstacle and collision avoidance
  - formation control with collision avoidance
- Fast off-the-shelf solvers available
  - CPLEX, Gurobi, Mosek, etc.
- Strong theoretical foundations
  - characterization of tractable special cases
  - NP-hard in general, but can also solve many large problems in practice

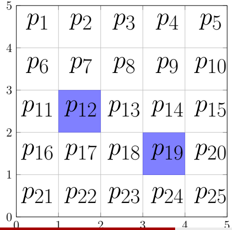
Mixed integer programs

Mixed Integer Programming (MIP) is a branch of mathematical optimization where:

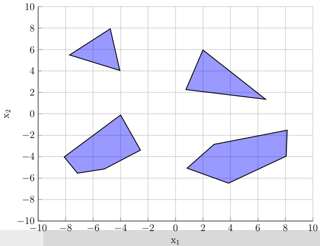
- (some) variables can take binary as well as integer values
- the goal is to find a solution that minimizes an objective function under a given set of constraints
- problems can easily grow to large sizes, execution time increases exponentially

MIP in motion planning:

- algebraic/combinatorial: involves logical decisions and/or selection from a priori known alternatives

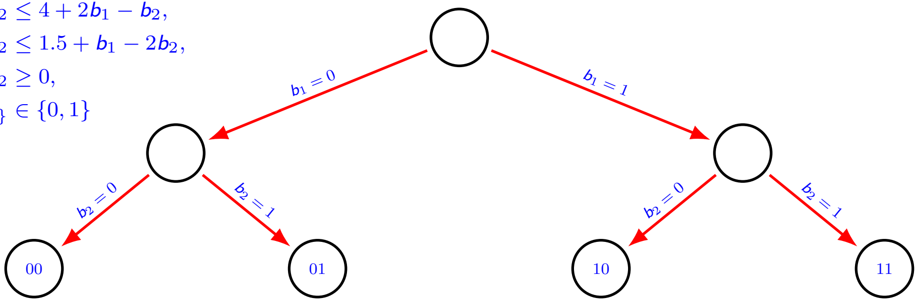


- geometrical: efficient mixed integer descriptions for non-convex regions



Mixed integer programs (example)

$$
\begin{aligned}
\min_{x,b} \quad & x_1 + 2x_2 - 2b_2 \\
\text{s.t.} \quad & 3x_1 + 0.5x_2 \leq 4 + 2b_1 - b_2, \\
& x_2 \leq 1.5 + b_1 - 2b_2, \\
& x_2 \geq 0, \\
& b_{\{1,2\}} \in \{0,1\}
\end{aligned}
$$



$$
\begin{aligned}
\min_{x,b} \quad & x_1 + 2x_2 \\
\text{s.t.} \quad & 3x_1 + 0.5x_2 \leq 4, \\
& x_2 \leq 1.5, \\
& x_2 \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
\min_{x,b} \quad & x_1 + 2x_2 - 2 \\
\text{s.t.} \quad & 3x_1 + 0.5x_2 \leq 3, \\
& x_2 \leq -0.5, \\
& x_2 \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
\min_{x,b} \quad & x_1 + 2x_2 \\
\text{s.t.} \quad & 3x_1 + 0.5x_2 \leq 6, \\
& x_2 \leq 2.5, \\
& x_2 \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
\min_{x,b} \quad & x_1 + 2x_2 - 2 \\
\text{s.t.} \quad & 3x_1 + 0.5x_2 \leq 5, \\
& x_2 \leq 0.5, \\
& x_2 \geq 0,
\end{aligned}
$$

Outline

Recap: Hyperplane and half-space

Let's introduce the basic notions of

- **hyperplane**: the set of form

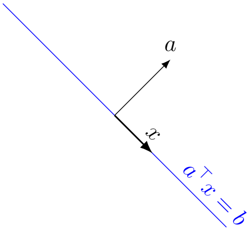
$$\left\{x \in \mathbb{R}^n : \ a^\top x = b\right\}$$

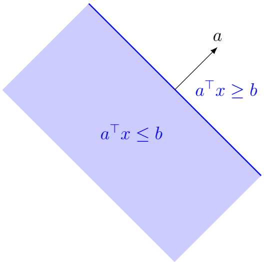
- **halfspace**: the set of form

$$\left\{x \in \mathbb{R}^n : \ a^\top x \leq b\right\}$$

for $a \neq 0$ and $(a, b) \in \mathbb{R}^n \times \mathbb{R}$

A pair (a, b) will determine three sets, the halfspaces $\mathcal{H}^-, \mathcal{H}^+$ and the hyperplane $\mathcal{H}^0$ which is their separating boundary.





Recap: Polyhedral sets

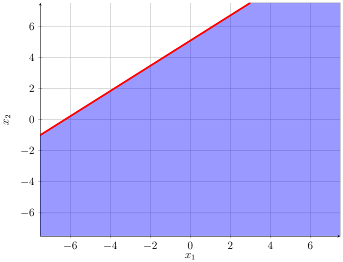
Dual representation:

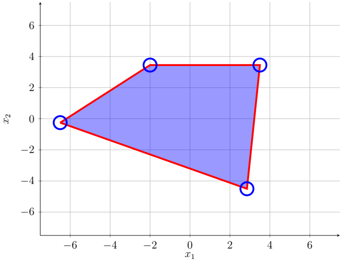
- half-space representation

$$X = \{x \in \mathbb{R}^d : F_i^\top x \leq \theta_i, i = 1 \dots n_h\},$$

- Vertex representation

$$X = \left\{x \in \mathbb{R}^d : x = \sum_{j=1}^{n_v} \alpha_j v_j,\ \sum_{j=1}^{n_v} \alpha_j = 1,\ \alpha_j \geq 0\right\}.$$





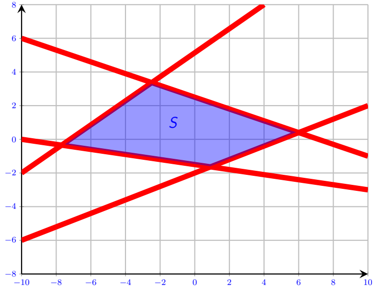
Defining characteristics:

- can approximate arbitrarily-well convex sets
- robust to small and medium-sized problem sizes
- can be embedded in large-scale LP/QP optimization problems

MIP for a polyhedral obstacle

Consider a bounded polyhedral set

$$S = \left\{x \in \mathbb{R}^n : h_i x \leq k_i,\ \ i = 1 : N\right\}$$



Any of the regions $\mathcal{R}^-(\mathcal{H}_i)$ of $\mathcal{C}(S)$ can be obtained by a suitable choice

$$\mathcal{R}^-(\mathcal{H}_i) \longleftrightarrow (\alpha_1, \ldots, \alpha_N)^i \triangleq (1, \ldots, 1, \underbrace{0}_{i}, 1, \ldots, 1)$$

MIP for a polyhedral obstacle

Consider a bounded polyhedral set

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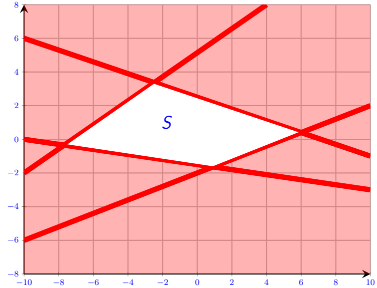
Consider the complement of S

$$\mathcal{C}(S) \triangleq cl(\mathbb{R}^n \setminus S) = \bigcup_i \mathcal{R}^-(\mathcal{H}_i), \quad i = 1 : N$$



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$$\mathcal{C}(S) \triangleq cl(\mathbb{R}^n \setminus S) = \bigcup_i \mathcal{R}^-(\mathcal{H}_i), \quad i = 1 : N$$

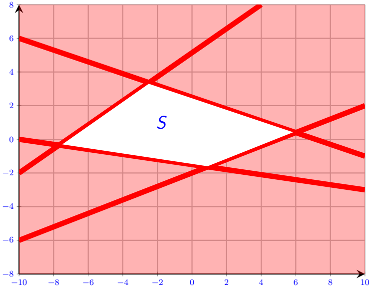
Define $\mathcal{C}(S)$ in a linear representation

$$-h_i x \leq -k_i + M\alpha_i, \quad i = 1 : \; N$$

$$\sum_{i=1}^{i=N} \alpha_i \leq N - 1$$

with $(\alpha_1, \ldots, \alpha_N) \in \{0, 1\}^{\;N}$

Any of the regions $\mathcal{R}^-(\mathcal{H}_i)$ of $\mathcal{C}(S)$ can be obtained by a suitable choice



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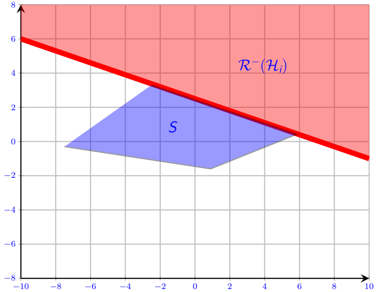
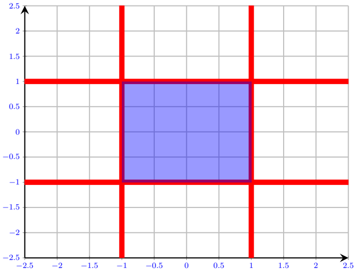


Illustrative example

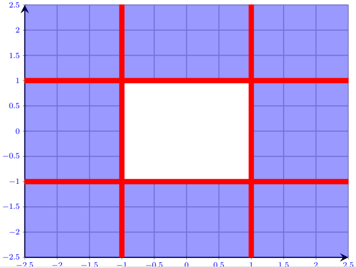
Consider a polytope $P \subset \mathbb{R}^2$ given by

$$\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



and its complement $\mathcal{C}(P)$ by

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} x \leq \begin{bmatrix} -1 + M\alpha_1 \\ -1 + M\alpha_2 \\ -1 + M\alpha_3 \\ -1 + M\alpha_4 \end{bmatrix}$$



in the classical mixed-integer formulation.

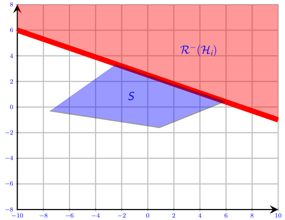
Logarithmic representation

For each region $\mathcal{R}^-(\mathcal{H}_i)$ a unique combination of binary variables $\lambda^i \in \{0,1\}^{\lceil log_2 N \rceil}$ is associated. Then, the affine functions $\alpha_i : \{0,1\}^{\lceil log_2 N \rceil} \to \{0\} \cup [1, \infty)$ are constructed:

$$\alpha_i(\lambda) = \sum_{k=0}^{\lceil log_2 N \rceil} \left(\lambda_k^i + (1 - 2\lambda_k^i) \cdot \lambda_k\right).$$

λ_k denotes the kth component of λ and λ_k^i its value for the tuple associated to region $\mathcal{R}^-(\mathcal{H}_i)$:

$$\alpha_i(\lambda) = \begin{cases} 0, & \text{only if } \lambda = \lambda^i \\ \geq 1, & \text{for any } \lambda \neq \lambda^i \end{cases}$$



which leads to the compact formulation

$$\begin{aligned} -h_i x &\leq -k_i + M\alpha_i(\lambda), \quad i = 1 : N, \\ 0 &\leq \beta_l(\lambda). \end{aligned}$$

Illustrative example

Consider a polytope $P \subset \mathbb{R}^2$ given by

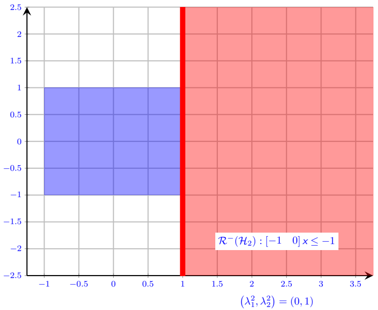
$$\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

and its complement $\mathcal{C}(P)$ by

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} x \leq \begin{bmatrix} -1 + M(\quad \lambda_1 + \lambda_2) \\ -1 + M(1 - \lambda_1 + \lambda_2) \\ -1 + M(1 + \lambda_1 - \lambda_2) \\ -1 + M(2 - \lambda_1 - \lambda_2) \end{bmatrix}$$

in the reduced MI formulation.

In the reduced representation only $N_0 = \lceil \log_2 4 \rceil = 2$ binary variables are needed.



For region $\mathcal{R}^-(\mathcal{H}_2)$ associate tuple $(\lambda_1^2, \lambda_2^2) = (0, 1)$ which leads to the mapping

$$\alpha_2 = 1 + \lambda_1 - \lambda_2$$

Interdicted tuples

In the mixed-integer representation we interdict tuples which describe the obstacle:

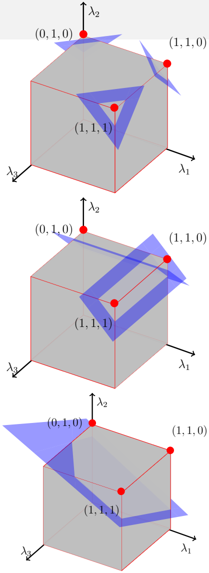
- in the classical formulation we force that at least one constraint is active:

$$\sum_{i=1}^{i=N} \alpha_i \leq N - 1$$

- in the logarithmic formulation
  - multiple constraints to interdict tuples[a]

$$0 < \beta_l(\lambda)$$

  - if the allocated tuples are ordered a single constraint suffices[b]



[a] F. Stoican, I. Prodan, and S. Olaru (2011). "Enhancements on the hyperplane arrangements in mixed integer techniques". In: *2011 50th IEEE Conference on Decision and Control and European Control Conference*. IEEE, pp. 3986–3991.

[b] R. J. Afonso and R. K. Galvão (2013). "Comments on Enhancements on the Hyperplanes Arrangements in Mixed-Integer Programming Techniques". In: *Journal of Optimization Theory and Applications*, pp. 1–8.

Outline

Min/max and scalar PWA modeling

Mixed-integer (MI) is useful whenever non-smooth functions have to be described:

- select the minimum from a list:

$$\underline{t} = \min_i x_i$$

MI form:

$$x_i - M(1 - z_i) \leq \underline{t} \leq x_i,$$
$$z_1 + \cdots + z_n = 1,$$

- select the maximum from a list:

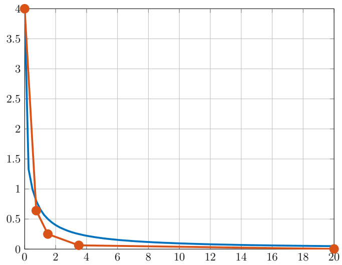
$$\bar{t} = \max_i x_i$$

MI form:

$$x_i \leq \bar{t} \leq x_i + M(1 - z_i),$$
$$z_1 + \cdots + z_n = 1,$$

- epigraf of a scalar PWA function defined by $(x_i, f(x_i))$:

$$x = \sum \alpha_i x_i,\ \sum \alpha_i f(x_i) \leq t$$
$$\alpha_i \leq z_i + z_{i-1},\ \alpha_i \geq 0, \sum \alpha_i = 1$$
$$\sum z_i = 1$$



Vertex-based modeling for an arbitrary PWA

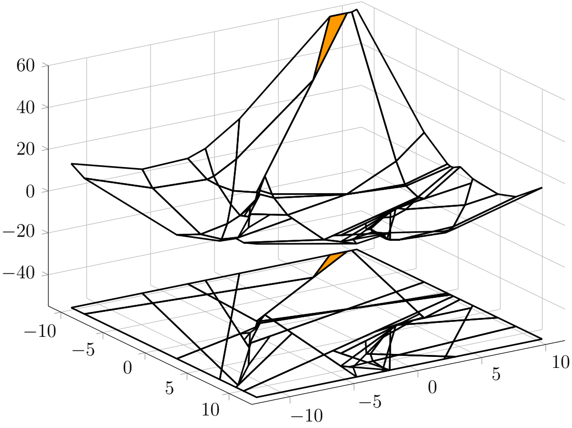
Consider the PWA function $f(x) : \mathbb{R}^n \mapsto R$ with support over the polyhedral partition $\bigcup_{i=1\dots n} R_i$ where the vertices of all regions R_i are stored in $\mathbb{V} = \{v_j\}_{j=1\dots m}$.

We give $f(x) \leq t$, the epigraph of $f(x)$, as:

$$x = \sum_{j=1}^{m} \alpha_j v_j, \qquad \sum_{j=1}^{m} \alpha_j f(v_j) \leq t,$$

$$\alpha_j \geq 0,\ \forall j = 1 \dots m, \qquad \sum_{j=1}^{m} \alpha_j = 1,$$

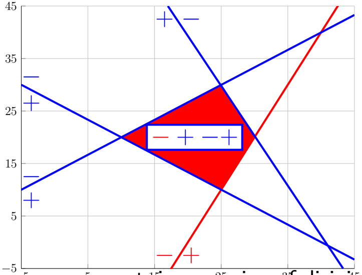
$$\alpha_j \leq \sum_{i:\ v_j \in R_i} z_i, \qquad \sum_{j=1}^{n} z_j = 1,$$



- generic formulation (holds for any PWA)
- but, the number of binary variables depends on the number of regions

About hyperplane arrangements...

We briefly recapitulate the *hyperplane arrangement (HA)* notion[1]:



- A hyperplane in $\mathbb{R}^n$ is the set

$$H_k = \{x \in \mathbb{R}^n | a_k^\top x = b_k\},$$

- Each hyperplane "cuts" the space into two disjoint (up to their boundary) half-spaces

$$H_k^\pm = \{x \in \mathbb{R}^n | \pm a_k^\top x \leq \pm b_k\}.$$

- With a collection of N hyperplanes $\mathbb{H} := \{H_k\}_{k=1\dots N}$, we arrive at a hyperplane arrangement, i.e., a union of disjoint cells $\mathcal{A}(\sigma)$ which cover the entire space:

$$\mathbb{R}^n = \mathcal{A}(\mathbb{H}) := \bigcup_{\sigma\in\Sigma} \mathcal{A}(\sigma) = \bigcup_{\sigma\in\Sigma} \left(\bigcap_{k=1\dots N} H_k^{\sigma_k} \right),$$

- Sign tuple $\sigma = (\sigma_1, \dots, \sigma_N) \in \Sigma \subset \{-,+\}^N$ denotes on which side cell $\mathcal{A}(\sigma)$ lies w.r.t. each of the hyperplanes H_k:

$$\sigma_k = `\pm' \text{ implies } \mathcal{A}(\sigma) \subseteq H_k^\pm$$

[1] G. M. Ziegler (2012). *Lectures on polytopes*. Vol. 152. Springer Science & Business Media.

… and their application in motion planning

We may characterize elements of interest strictly in terms of sign tuples σ:

- partition the list of sign tuples Σ into *allowed* – Σ° and *interdicted* – $\Sigma^\bullet$, thus characterizing the union of obstacles $\mathbb{P}$ and the feasible space $\mathbb{R}^n \setminus \mathbb{P}$:

$$\mathbb{P} = \bigcup_{\sigma^\bullet \in \Sigma^\bullet} \mathcal{A}(\sigma^\bullet), \quad \mathbb{R}^n \setminus \mathbb{P} = \bigcup_{\sigma^\circ \in \Sigma^\circ} \mathcal{A}(\sigma^\circ).$$

- An useful notion is the *merged cell*, characterized by a sign tuple $\sigma^\star \in \{-, +, \star\}^N$:

$$\mathcal{A}(\sigma^\star) = \bigcap_{\sigma_k^\star \neq \star} H_k^{\sigma_k^\star} = \bigcup_{\sigma} \mathcal{A}(\sigma), \text{ where } \begin{cases} \sigma_k = \sigma_k^\star, & \sigma_k^\star \neq \star \\ \sigma_k \in \{-, +\}, & \sigma_k^\star = \star \end{cases}.$$

Many interesting properties:

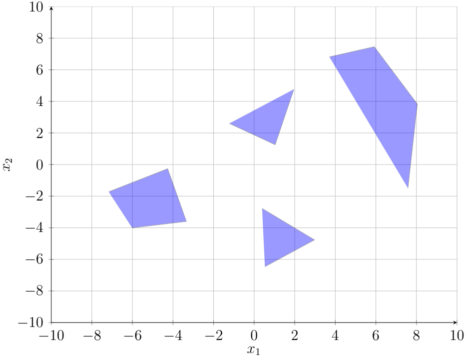
- the formulations naturally lead to mixed-integer representations
- provide a scaffolding over which to analyze the behavior of related functions
- the union of any two full-face neighboring regions is a convex set: exploited complexity reduction in PWA descriptions[2] and motion planning[3]

[2] T. Geyer, F. D. Torrisi, and M. Morari (2008). "Optimal complexity reduction of polyhedral piecewise affine systems". In: *Automatica* 44.7, pp. 1728–1740.
[3] D. Ioan, I. Prodan, S. Olaru, F. Stoican, and S.-I. Niculescu (2020). "Mixed-integer programming in motion planning". In: *Annual Reviews in Control*.

Non-connected and non-convex regions

Consider the complement $\mathcal{C}(\mathbb{S}) = cl(\mathbb{R}^n \setminus \mathbb{S})$ of a union of polyhedral sets $\mathbb{S} = \bigcup_l S_l$

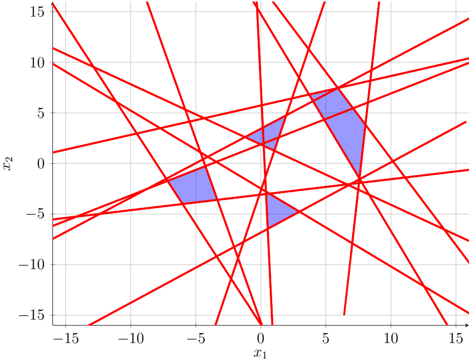
$$\mathcal{A}(\mathbb{H}) = \bigcup_{l=1,\dots,\gamma(N)} \underbrace{\left(\bigcap_{i=1}^{N} R^{\sigma_l(i)}(\mathcal{H}_i) \right)}_{A_l}$$



Non-connected and non-convex regions

Consider the complement $\mathcal{C}(\mathbb{S}) = cl(\mathbb{R}^n \setminus \mathbb{S})$ of a union of polyhedral sets $\mathbb{S} = \bigcup_l S_l$

$$
\begin{aligned}
&\vdots\\
A_l &\begin{cases} \sigma_l(1)h_1 x & \leq \sigma_l(1)k_1 + M\alpha_l(\lambda) \\ & \vdots \\ \sigma_l(N)h_N x & \leq \sigma_l(N)k_N + M\alpha_l(\lambda) \end{cases}\\
&\vdots\\
&0 \leq \beta_l(\lambda)
\end{aligned}
$$



Using the hyperplanes $\mathcal{H}_i$ we partition the space into disjoint cells A_l and we associate a linear combination of binary variables $\alpha_l(\lambda)$ to each cell.

Non-connected and non-convex regions

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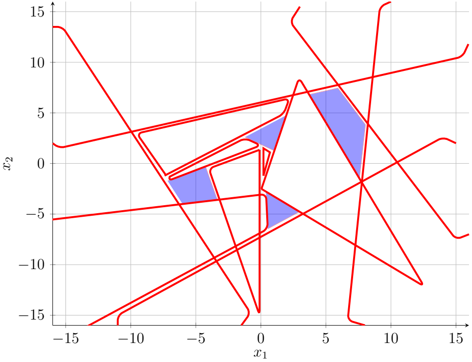
$$
\begin{array}{c}
\vdots \\
A_l \begin{cases} \sigma_l(1) h_1 x & \leq \sigma_l(1) k_1 + M\alpha_l(\lambda) \\ & \vdots \\ \sigma_l(N) h_N x & \leq \sigma_l(N) k_N + M\alpha_l(\lambda) \end{cases} \\
\vdots \\
0 \leq \beta_l(\lambda)
\end{array}
$$



The number of cells can be reduced through merging procedures.

Outline

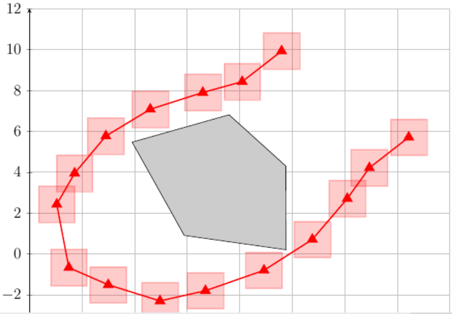
Obstacle and collision avoidance example

1. for any obstacle S_l and any agent characterized by its dynamical state $x_i(k)$ and the associated safety region S_i^a, the collision avoidance conditions are:

$$(\{x_i(k)\} \oplus S_i^a) \cap S_l = \emptyset, \quad \forall i = 1 \dots N_a, \ \forall l = 1 \dots N_o.$$

2. for any two agents characterized by their dynamical states $x_i(k)$, $x_j(k)$ and their associated safety regions S_i^a, S_j^a, the collision avoidance conditions are:

$$(\{x_i(k)\} \oplus S_i^a) \cap \left(\{x_j(k)\} \oplus S_j^a\right) = \emptyset, \quad \forall i, j = 1 \dots N_a, \ i \neq j.$$



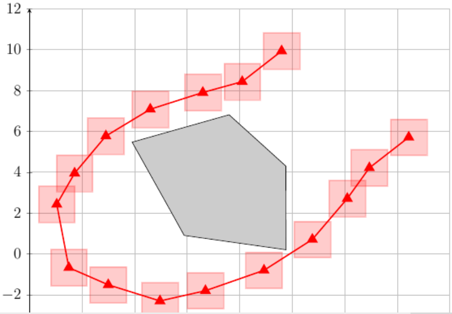
Obstacle and collision avoidance example

1. for any obstacle S_l and any agent characterized by its dynamical state $x_i(k)$ and the associated safety region S_i^a, the collision avoidance conditions are:

$$x_i(k) \notin (\{-S_i^a\} \oplus S_l), \quad \forall i = 1 \dots N_a, \ \forall l = 1 \dots N_o,$$

2. for any two agents characterized by their dynamical states $x_i(k)$, $x_j(k)$ and their associated safety regions S_i^a, S_j^a, the collision avoidance conditions are:

$$x_i(k) - x_j(k) \notin \left(\{-S_i^a\} \oplus S_j^a\right), \quad \forall i, j = 1 \dots N_a, \ i \neq j.$$



Implementation for the motion planning problem

$$
\begin{aligned}
\min_{u_k \dots u_{k+N-1}} \; & \sum_{i=1}^{N} \left(x_{k+i} - \bar{x}\right)^\top \left(x_{k+i} - \bar{x}\right) + \sum_{i=0}^{N-1} u_{k+i}^\top u_{k+i} \\
\text{s.t.} \quad & x_{k+i+1} = A x_{k+i} + B u_{k+i}, \\
& |u_{k+i}| \leq \bar{u}, \\
& |x_{k+i+1}| \leq \bar{x}, \\
& x_{k+i+1} \notin P, \qquad \forall i = 1 : N.
\end{aligned}
$$

- we define the cost to be minimized (effort along the path): $u_{k+i}^\top u_{k+i}$, distance to target $\left(x_{k+i} - \bar{x}\right)^\top \left(x_{k+i} - \bar{x}\right)$, etc.);
- we force constraint validation (on input: $|u_{k+i}| \leq \bar{u}$, on state: $|x_{k+i+1}| \leq \bar{x}$, on obstacle avoidance: $x_{k+i+1} \notin P$);
- we apply the constraints and cost over a finite prediction horizon (of length N) and from the sequence of obtained inputs $\{u_k, \dots, u_{n+N-1}\}$ we apply the first, u_k;
- we increment the index $k \mapsto k+1$ and repeat the previous steps.

Obstacle avoidance problems

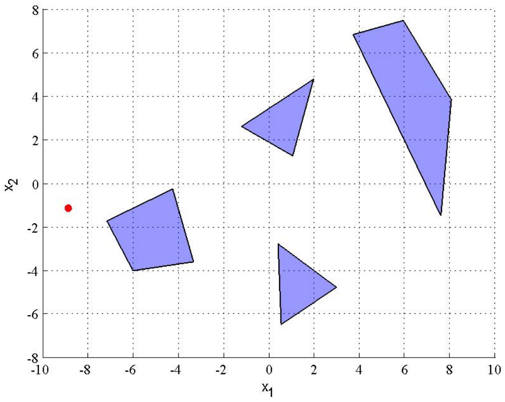
Consider a dynamical agent characterized by the LTI dynamics:

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k),$$

with $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ the agent state, input and output, respectively.

Collision avoidance condition:
For any obstacle S_l and an agent characterized by its dynamical state $x(k)$ we have:

$$\{x(k)\} \cap S_l = \emptyset, \quad \forall l = 1 \dots N_o.$$



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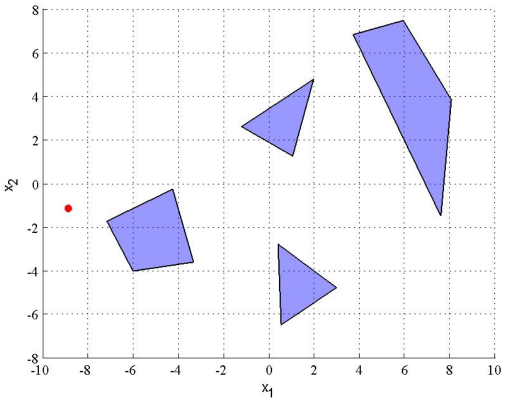
MIP representation of the feasible space:

- 14 hyperplanes
- 106 regions obtained with hyperplane arrangements
- 10 cells describing the interdicted regions
- 96 cells describing the feasible region

- $N_0 = 12$ the number of the binary variables



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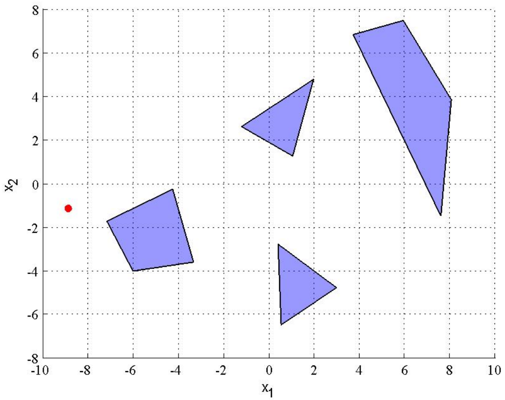
Solve the MIQP optimization problem over a finite prediction horizon:

$$\begin{aligned} u^* = \arg \min_{u(k),\dots u(k+N_p-1)} & \sum_{i=0}^{N_p-1} \|x(k+i+1)\|_Q + \|u(k+i)\|_R, \\ \text{s.t. } & x(k+i+1) = Ax(k+i) + Bu(k+i), \\ & y(k+i) \in \mathcal{Y},\ u(k+i) \in \mathcal{U}, \\ & x(k+i+1) \notin \mathbb{S}, \quad l = 1 \dots N_o. \end{aligned}$$



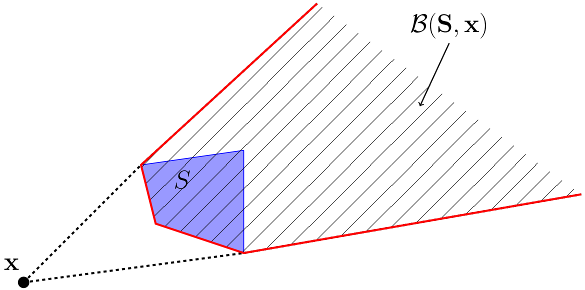
Outline

Shadow region description

We can define the "shadow" region $\mathcal{B}(S, x)$ as the collection of all the points from $\mathbb{R}^n$ which are "in the shadow" from the point of view of x:

$$\mathcal{B}(S, x) = \{y : [x, y] \cap S \neq \emptyset\}$$

- S is the obstacle
- x is the sensor/agent



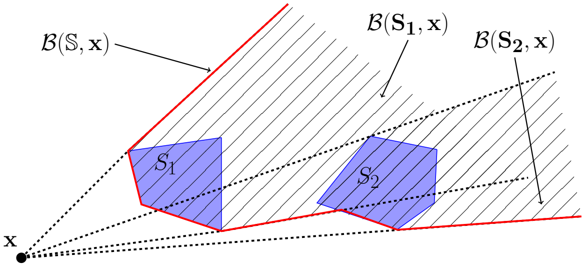
If the segment $[x, y]$ intersects S it means that point y is "hidden" by obstacle S and therefore is not "visible" from the point of view of x.

Shadow region description

We can define the "shadow" region $\mathcal{B}(\mathbb{S}, x)$ as the collection of all the points from $\mathbb{R}^n$ which are "in the shadow" from the point of view of x:

$$\mathcal{B}(\mathbb{S}, x) = \{y \in \mathbb{R}^n : [x, y] \cap \mathbb{S} \neq \emptyset\} = \bigcup_{l=1}^{N_o} \mathcal{B}(S_l, x)$$

- $\mathbb{S} \triangleq \bigcup_{l=1}^{N_o} S_l$ is the collection of obstacles
- x is the sensor/agent



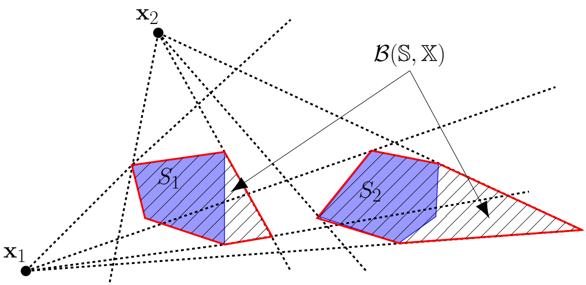
If the segment $[x, y]$ intersects S it means that point y is "hidden" by obstacle $S \in \mathbb{S}$ and therefore is not "visible" from the point of view of x.

Shadow region description

We can define the "shadow" region $\mathcal{B}(\mathbb{S}, \mathbb{X})$ as the collection of all the points from $\mathbb{R}^n$ which are "in the shadow" from the point of view of $\mathbb{X}$:

$$\begin{aligned}\mathcal{B}(\mathbb{S}, \mathbb{X}) &= \bigcap_{k=1}^{N_a} \mathcal{B}(\mathbb{S}, x_k) = \bigcap_{k=1}^{N_a} \left[\left(\bigcup_{l=1}^{N_o} \mathcal{B}(S_l, x_k)\right)\right] \\ &= \bigcap_{k=1}^{N_a} \left(\bigcup_{l=1}^{N_o} \mathcal{B}(S_l, x_k)\right)\end{aligned}$$

- $\mathbb{S} \triangleq \bigcup_{l=1}^{N_o} S_l$ is the collection of obstacles
- $\mathbb{X} \triangleq \{x_1, \ldots, x_{N_a}\}$ is the collection of sensors/agents



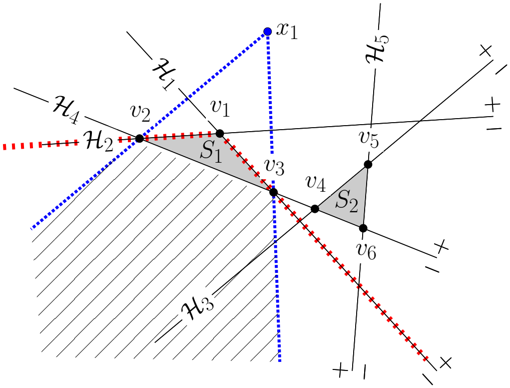
If the segment $[x, y]$ intersects S it means that point y is "hidden" by obstacle $S \in \mathbb{S}$ and therefore is not "visible" from the point of view of $x \in \mathbb{X}$.

Illustrative example (I)

Recall that $\mathcal{B}(\sigma_1, x_1) = \mathrm{Cone}(x_1, S_1) \cap \mathcal{H}_1^- \cap \mathcal{H}_2^-$ and $\mathcal{B}(\sigma_1, \sigma) = \mathcal{H}_1^- \cap \mathcal{H}_2^-$.

$x^+ \in \mathcal{B}(\sigma_1, x_1)$

$$
\begin{aligned}
x^+ &= x + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3, \\
&\ldots\ldots \\
&\beta_3 \leq M\,(5 - \sigma(1) - \sigma(2) - \sigma(3) - \sigma(4) - \sigma(5))\,, \\
&\ldots\ldots \\
&\beta_1 \geq 0,\ \beta_2 \geq 0,\ \beta_3 \geq 0, \\
-h_1 x^+ &\leq -k_1 + M(1 - \sigma(1)), \\
-h_2 x^+ &\leq -k_2 + M(1 - \sigma(2)), \\
h_3 x^+ &\leq \quad k_3 + M\sigma(3), \\
h_4 x^+ &\leq \quad k_4 + M\sigma(4), \\
h_5 x^+ &\leq \quad k_5 + M\sigma(5).
\end{aligned}
$$



Illustrative example (I)

We have that $x^+ \in \mathcal{B}(S_1 \cup S_2, \sigma) \Rightarrow x^+ \in \mathcal{B}(\sigma_1, \sigma) \cup \mathcal{B}(\sigma_2, \sigma)$ which means that

$$x^+ \in \mathcal{B}(S_1 \cup S_2, \sigma) \Leftrightarrow \begin{cases} \sigma^+(1) + \sigma^+(2) & = 0 \\ \text{OR} & \\ \sigma^+(2) + \sigma^+(3) & = 0. \end{cases}$$

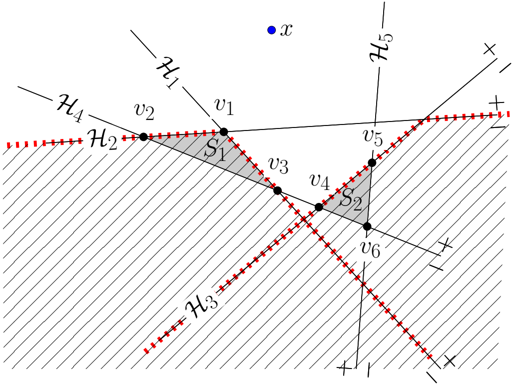
Then,

$$x^+ \notin \mathcal{B}(S_1 \cup S_2, \sigma) \Leftrightarrow \begin{cases} \sigma^+(1) + \sigma^+(2) & > 0 \\ \text{AND} & \\ \sigma^+(2) + \sigma^+(3) & > 0. \end{cases}$$

that is, the future position cannot be in

- region $\mathcal{H}_1^+ \cap \mathcal{H}_2^+$ AND
- region $\mathcal{H}_2^+ \cap \mathcal{H}_3^+$

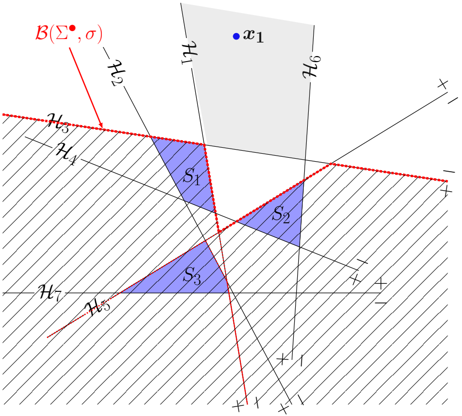
$x^+ \notin \mathcal{B}(\sigma_1, \sigma) \cup \mathcal{B}(\sigma_2, \sigma)$



Illustrative example (a multi-obstacle environment)

The over-approximated shadow region $\mathcal{B}(\Sigma^{\bullet}, \sigma)$ has the mixed-integer representation:

$$\begin{aligned}
|1-\sigma^{+}(1)| + |1-\sigma^{+}(3)| &\leq N(1-\alpha^{1}),\\
|1-\sigma^{+}(3)| + |\sigma^{+}(5)| &\leq N(1-\alpha^{2}),\\
|1-\sigma^{+}(1)| + |1-\sigma^{+}(2)| + |1-\sigma^{+}(3)| &\\
+ |1-\sigma^{+}(4)| + |\sigma^{+}(5)| &\leq N(1-\alpha^{3}),\\
\alpha^{1} + \alpha^{2} + \alpha^{3} &\geq 1.
\end{aligned}$$



Illustrative example (a multi-obstacle environment)

The over-approximated visible region $\overline{\mathcal{B}(\Sigma^{\bullet}, \sigma)}$, has the mixed-integer representation:

$$
\begin{aligned}
|1-\sigma^{+}(1)| + |1-\sigma^{+}(3)| &> 0, \\
|1-\sigma^{+}(3)| + |\sigma^{+}(5)| &> 0, \\
|1-\sigma^{+}(1)| + |1-\sigma^{+}(2)| + |1-\sigma^{+}(3)| & \\
+ |1-\sigma^{+}(4)| + |\sigma^{+}(5)| &> 0.
\end{aligned}
$$