Tutoring Analiză Matematică - Zama Debu
SiSC Tutoring ♥

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Sub I

1. $f_n(x) = \frac{\sin(nx)}{n^3+1}$ - să se stud. conv. simplă și uniformă a acestui șir.

- Conv. simplă -> calculăm $\lim\limits_{0 \le n\to\infty} |f_n(x)| = \lim \frac{|\sin(nx)|}{n^3+1} \le$

! $|\sin x| \le 1, \forall x \in \mathbb{R}$

$\le \lim\limits_{n\to\infty} \frac{1}{n^3+1} = 0$

$\Rightarrow \lim\limits_{n\to\infty} |f(x)| = 0 \Rightarrow \lim\limits_{n\to\infty} f_n(x) = 0 \Rightarrow (f_n)_{n\ge1}$ - $\forall x \in \mathbb{R}$

- converge simplu pe $\mathbb{R}$ la funcția limită $f:\mathbb{R}\to\mathbb{R}$, $f(x)=0$. $\Rightarrow f_n \xrightarrow[\mathbb{R}]{CS} f$ (converge simplu)

- Conv. uniformă : $|f_n(x) - f(x)| =$

$= \left|\frac{\sin(nx)}{n^3+1}\right| \le \frac{1}{n^3+1} = a_n, \forall n \ge 1, \forall x \in \mathbb{R}$ (1)

$\lim\limits_{n\to\infty} a_n = 0$ (2)

Din (1) și (2) $\xrightarrow[\text{majorării}]{\text{Crit.}}$ $f_n \xrightarrow[\mathbb{R}]{u} f$ -> converge uniform

2. $\sum\limits_{n=1}^{\infty} \frac{2^n}{3^n} \cdot (\underbrace{4x-1}_{y})^n$, $x \in \mathbb{R}$

- Notăm $a_n = \frac{2^n}{3^n}$, $y = 4x-1 \Rightarrow$

$\Rightarrow$ seria se scrie de forma $\sum\limits_{n=1}^{\infty} a_n \cdot y^n$.

- $\omega = \lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \frac{2}{3} \Rightarrow R = \frac{1}{\omega} = \frac{3}{2}$

- Cf. T. Abel $\Rightarrow \begin{cases} \text{pt. } y \in \left(-\frac{3}{2}; \frac{3}{2}\right) \Rightarrow \text{Serie conv. abs.} \\ \text{pt. } y \in \left(-\infty; -\frac{3}{2}\right) \cup \left(\frac{3}{2}; \infty\right) \Rightarrow \text{Serie div.} \end{cases}$

$\Rightarrow$ Serie conv.

Pt. $y = \frac{3}{2} \Rightarrow \sum\limits_{n=1}^{\infty} \frac{2^n}{3^n} \cdot \left(\frac{3}{2}\right)^n = \sum\limits_{n=1}^{\infty} \underbrace{1}_{b_n}$

$\lim\limits_{n\to\infty} b_n = 1 \ne 0 \xrightarrow[\text{suf. de div.}]{\text{Crit.}}$ S. Divergentă

*(Crit. necesar de convergență)

Pt. $y = -\frac{3}{2} \Rightarrow \sum\limits_{n=1}^{\infty} \frac{2^n}{3^n}\left(-\frac{3}{2}\right)^n = \sum\limits_{n=1}^{\infty} \underbrace{(-1)^n}_{c_n}$

$\left.\begin{matrix} c_{2n} \to 1 \\ c_{2n+1} \to -1 \end{matrix}\right\} \Rightarrow \nexists \lim c_n \Rightarrow c_n \not\to 0 \Rightarrow$

$\xRightarrow{CSD}$ SD.

Seria de puteri e conv. pt. $y \in \left(-\frac{3}{2}; \frac{3}{2}\right) \Leftrightarrow$

$$\Leftrightarrow -\frac{3}{2} < 4x-1 < \frac{3}{2} \quad \Big| +1$$

$$-\frac{1}{2} < 4x < \frac{5}{2} \quad \Big| \cdot \frac{1}{4}$$

$$-\frac{1}{8} < x < \frac{5}{8} \Rightarrow C_x = \left(-\frac{1}{8}; \frac{5}{8}\right) \to \text{mulțimea de convergență}$$

* Dacă $w = 0 \Rightarrow R = \infty$

• Seria geometrică: $\boxed{\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}}, \forall q \in (-1;1)$

$$\sum_{n=1}^{\infty} \left(\frac{2(4x-1)}{3}\right)^n = \sum_{n=1}^{\infty} \left(\frac{8x-2}{3}\right)^n =$$

$$= \frac{1}{1-\frac{8x-2}{3}} - 1$$

$$\Rightarrow S_{(x)} = \sum_{n=1}^{\infty} \frac{2^n}{3^n}(4x-1)^n = \frac{8x-2}{5-8x}, \forall x \in \left(-\frac{1}{8}; \frac{5}{8}\right)$$

!

Metoda 2 pt. mulțimea de convergență C_x (dar nu merge mereu):

S.p. e o serie geom. cu rația $q = \frac{8x-2}{3}$ și este conv. $\Leftrightarrow q \in (-1;1) \Leftrightarrow -1 < \frac{8x-2}{3} < 1 \Big| \cdot 3 \Rightarrow$

$$\Rightarrow -3 < 8x-2 < 3 \Big| +2$$

$$-1 < 8x < 5 \Big| \cdot \frac{1}{8} \Rightarrow -\frac{1}{8} < x < \frac{5}{8}$$

$\Rightarrow S_x$ este conv. $\Leftrightarrow x \in \left(-\frac{1}{8}; \frac{5}{8}\right) = C_x$

Sub II

1. $f(x,y) = y^2 - 2y + x^2, \quad y^2 - x^2 = 1$

$$\underbrace{y^2 - x^2 - 1 = 0}_{g(x,y)}$$

• $L(x,y,\lambda) = f(x,y) + \lambda \cdot g(x,y)$

$$= y^2 - 2y + x^2 + \lambda(y^2 - x^2 - 1)$$

• $L'_x() = 2x - 2\lambda x = 0 \quad (1)$

$L'_y() = 2y - 2 + 2\lambda y = 0 \quad (2)$

$L'_\lambda() = y^2 - x^2 - 1 = 0 \quad (3)$

$(1) \Rightarrow 2x(1-\lambda) = 0 \Rightarrow$ a) $x = 0$; b) $\lambda = 1$

a) $x = 0 \overset{(3)}{\Rightarrow} y^2 - 1 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

Pt. $y = 1 \overset{(2)}{\Rightarrow} 2 - 2 + 2\lambda = 0 \Rightarrow \boxed{\lambda = 0}$

Pt. $y = -1 \overset{(2)}{\Rightarrow} -2 - 2 - 2\lambda = 0 \Rightarrow -4 - 2\lambda = 0 \Rightarrow \boxed{\lambda = -2}$

$A(0;1)$; $\lambda = 0$
$B(0;-1)$; $\lambda = -2$

b) $\lambda = 1 \overset{(2)}{\Rightarrow} 2y - 2 + 2y = 0 \Rightarrow 4y = 2 \Rightarrow \boxed{y = \frac{1}{2}} \overset{(3)}{\Rightarrow}$

$\Rightarrow \frac{1}{4} - x^2 - 1 = 0 \Rightarrow x^2 = -\frac{3}{4} \Rightarrow x_{1,2} \notin \mathbb{R}$

• Natura pc. staționare (cu mat. hessiană)

$L''_{x^2}(x,y) = [L'_x(x,y)]'_x = (2x - 2\lambda x)'_x =$
$= 2 - 2\lambda$

$L''_{xy}(\,) = [L'_x(\,)]'_y = 0$

$L''_{y^2}(\,) = [L'_y(\,)]'_y = 2 + 2\lambda$

$$\Rightarrow H(x,y) = \begin{pmatrix} L''_{x^2}(\,) & L''_{xy}(\,) \\ L''_{yx}(\,) & L''_{y^2}(\,) \end{pmatrix}$$

!

$$\Rightarrow H(x,y) = \begin{pmatrix} 2-2\lambda & 0 \\ 0 & 2+2\lambda \end{pmatrix}$$

$$H(A) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \left.\begin{matrix} \Delta_1 = 2 > 0 \\ \Delta_2 = 4 > 0 \end{matrix}\right\} \Rightarrow$$

$\Rightarrow A(0;1)$ pct. de minim local condiționat

$$H(B) = \begin{pmatrix} 6 & 0 \\ 0 & -2 \end{pmatrix}$$

$\left.\begin{matrix} \Delta_1 = 6 > 0 \\ \Delta_2 = -12 < 0 \end{matrix}\right\} \Rightarrow$ pt. a stabili natura punctului staționar scriem dif. de ordin 2 a lui L în B și diferențiem legătura

* Nu diferențiem când toți sunt cu plus sau când semnele alternează începând cu „<"

$d^2L(B) = L''_{x^2}(B)\,dx^2 + 2L''_{xy}(B)\,dx\,dy + L''_{y^2}(B)\,dy^2$
$= 6dx^2 - 2dy^2$

(Pt. A):
$d^2L(A) = 2dx^2 + 2dy^2 > 0 \Rightarrow d^2L(A)$ poz. def. $\Rightarrow$
$\Rightarrow A(0;1)$ pc. de minim local condiționat

Dif. legătura $\boxed{dg(B) = 0} \Leftrightarrow g'_x(B)\,dx + g'_y(B)\,dy =$
$= 0$ (4)

$g(x,y) = y^2 - x^2 - 1 \Rightarrow g'_x(x,y) = -2x \Rightarrow$

$\left.\begin{matrix} \Rightarrow g'_x(B) = 0 \\ g'_y(x,y) = 2y \Rightarrow g'_y(B) = -2 \end{matrix}\right\} \overset{(4)\text{ devine}}{\Longleftrightarrow}$

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$\Leftrightarrow -2dy = 0 \Rightarrow dy = 0$
$dx \in \mathbb{R}$

$\Rightarrow d^2L(B) = 6dx^2 > 0 \Rightarrow B(0;-1)$ - punct de minim local condiționat

2. (Metoda celor mai mici pătrate)

aceeași pas
toți din 2 în 2

x_i	1	2	3	4
y_i	5	8	9	10

$\Leftrightarrow \Rightarrow$

x_i	-3	-1	1	3
y_i	5	8	9	10

$\downarrow$
5
$f(5)$

Cu 5 valori:	-2 -1 0 1 2
	y rămâne mereu la fel!

Rez.: $f(x) = ax+b$

x_i — y_i (valoarea observată în practică)
— $f(x_i)$ (valoarea estimată)

$S(a,b) =$ suma erorilor la pătrat
$= \sum_{i=1}^{4} [f(x_i) - y_i]^2$

$(u^2)' = 2 \cdot u \cdot u'$

Determinăm $a, b \in \mathbb{R}$ a.î. S minim.

$S'_a() = \sum_{i=1}^{4} 2(ax_i + b - y_i) \cdot x_i = 0 \mid :2$

$S'_b() = \sum_{i=1}^{4} 2(ax_i + b - y_i) \cdot 1 = 0 \mid :2$

$\Rightarrow \begin{cases} a\sum x_i^2 + b\sum x_i - \sum x_i y_i = 0 \\ a\sum x_i + 4b - \sum y_i = 0 \end{cases}$ → se pot scrie direct.

x_i	y_i (val. obs.)	x_i^2	$x_i \cdot y_i$	$f(x_i)$ (val estimate)	$[f(x_i) - y_i]^2$
-3	5	9	-15	5,6	0,36
-1	8	1	-8	7,2	0,64
1	9	1	9	8,8	0,04
3	10	9	30	10,4	0,16
$\sum_{i=1}^{4}$: 0	32	20	16		1,2

SUMA PĂT. ERORILOR

$\Rightarrow \begin{cases} 20a - 16 = 0 \\ 4b - 32 = 0 \end{cases} \Rightarrow a = \frac{16}{20} = \frac{4}{5} = 0{,}8$

$b = \frac{32}{4} = 8$

$\Rightarrow f(x) = 0{,}8x + 8$

$\Rightarrow$ Prognoza pe anul următor este:

$f(5) = 0{,}8 \cdot 5 + 8 = 12$

* Dacă se cere suma pătratelor erorilor:

$f(-3) = 0{,}8 \cdot (-3) + 8 = -2{,}4 + 8 = 5{,}6$

Sub. III Ec. diferențială:

$$x^3 y' = 3x^2 y + x^4 e^{-\frac{3y}{x^3}}, \quad x<0 \quad (1)$$

→ schimbarea de variabilă $z = \frac{y}{x^3}$

S.V $\quad z = \frac{y}{x^3} \Rightarrow y = x^3 \cdot z$

$$\left.\begin{array}{l} y = x^3 z \\ y' = 3x^2 z + x^3 z' \end{array}\right\} \overset{(1)}{\Longrightarrow}$$

$$\Rightarrow x^3(3x^2 z + x^3 z') = 3x^2 \cdot x^3 z + x^4 e^{-\frac{3x^3 z}{x^3}} = \cancel{3x^5 z} +$$

$$+ x^6 z' = \cancel{3x^5 z} + x^4 e^{-3z} \quad \Big| \cdot \frac{1}{x^6}, \quad x<0$$

$z' = \frac{1}{x^2} \cdot e^{-3z}$ → ec. dif. cu var. separabile

$z' = f(x) \cdot g(z)$

$$\frac{dz}{dx} = \frac{1}{x^2} \cdot e^{-3z} \Big| \cdot e^{3z} \cdot dx = e^{3z} dz = \frac{1}{x^2} dx$$

$$\int e^{3z} dz = \int \frac{1}{x^2} dx$$

$$\int \frac{1}{x^2} dx = \int x^{-2} dx = -\frac{1}{x} + c.$$

$$\frac{1}{3} e^{3z} = -\frac{1}{x} + c; \quad c \in \mathbb{R}$$

$$\boxed{\begin{array}{l} (e^{ax})' = a \cdot e^{ax} \\ \int e^{ax} = \frac{1}{a} e^{ax} \end{array}}$$

!

(înlocuim z-ul)

$$\frac{1}{3} e^{\frac{3y}{x^3}} = -\frac{1}{x} + c; \quad c \in \mathbb{R}$$

↳ forma implicită!

* Pt. forma explicită (opțional - doar dacă se cere)

$$\frac{1}{3} e^{\frac{3y}{x^3}} = -\frac{1}{x} + c; \quad c \in \mathbb{R} \quad \Big| \cdot 3$$

$$e^{\frac{3y}{x^3}} = -\frac{3}{x} + c, \quad c \in \mathbb{R} \text{ a.î. } -\frac{3}{x} + c > 0.$$

$$\ln e^{\frac{3y}{x^3}} = \ln \frac{cx-3}{x} \Leftrightarrow \frac{3y}{x^3} = \ln \frac{cx-3}{x} \quad \Big| \cdot \frac{x^3}{3}$$

$$y = \frac{x^3}{3} \ln \frac{cx-3}{x}.$$

$x<0 \Rightarrow cx-3<0 \Leftrightarrow cx<3$

$$y = \frac{x^3}{3} \ln \frac{cx-3}{x}, \quad x<0, \quad c \in \mathbb{R} \text{ a.î. } cx-3<0$$

Dacă $c>0 \Rightarrow cx-3<0 \Big| \cdot \frac{1}{c} > 0$

$$x < \frac{3}{c}$$

$c<0 \Rightarrow cx-3<0 \Big| \cdot \frac{1}{c} < 0$

$$x > \frac{3}{c} \Leftrightarrow x \in \left(\frac{3}{c}; 0\right)$$

$$y = \frac{x^3}{3} \ln \frac{cx-3}{x}, \quad c>0, \; x<0$$

sau

$y = \frac{x^3}{3}\ln\frac{cx-3}{x}$, $c<0$, $x \in \left(\frac{3}{c};0\right)$

Sub. IV Calc.:

1. $\int_{-2}^{\infty} e^{-x^2-4x+1}\,dx$

$$\int_0^{\infty} e^{-x^2} = \frac{\sqrt{\pi}}{2}$$

$$-x^2-4x+1 = -(x^2+4x-1) = -(x^2+4x+4-4-1) =$$
$$= -[(x+2)^2-5] = 5-(x+2)^2$$

$$I = \int_{-2}^{\infty} e^{5-(x+2)^2}\,dx = e^5\int_{-2}^{\infty} e^{-(x+2)^2}\,dx =$$

Fol. sch. de var. $\rightarrow x+2=y \Rightarrow x = y-2$

$(x)'dx = (y-2)'dy$

$dx = dy$

x	-2	∞
y	0	∞

$$\begin{cases} x=-2 \Rightarrow y=0 \\ x\to\infty \Rightarrow y\to\infty \end{cases}$$

$$I = e^5 \boxed{\int_0^{\infty} e^{-y^2}\,dy} = e^5\cdot\frac{\sqrt{\pi}}{2}$$

↳ Euler

2. $D = \{(x,y)\in\mathbb{R}^2 \mid \underbrace{-y \le x}_{1} \underbrace{\le y}_{2}, \underbrace{x^2+y^2\le 9}_{3}\}$

Să se calc. $\iint_D dx\,dy$.

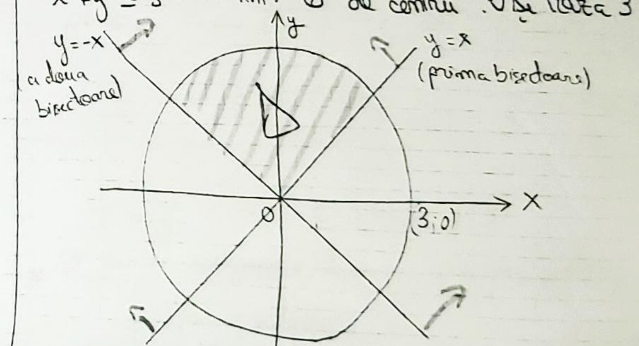
$-y = x \Leftrightarrow y = -x$

$x = y \Leftrightarrow y = x$

$x^2+y^2 = 9$ $\quad$ ($(x-a)^2+(y-b)^2 = r^2 \rightarrow \mathcal{C}$ (cerc) de centru (a,b) și rază r)

$\rightarrow$ centru $(0;0)$ și raza 3.

$x^2+y^2 \le 9 \rightarrow$ int. $\mathcal{C}$ de centru O și raza 3



Pt. pc. $(3;0) \Rightarrow 0 \le 3$ (A) $\Rightarrow$ pc. $(3,0) \in$ semiplanului $-y \le x$

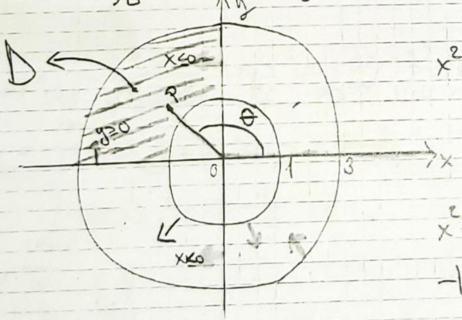
Pt. $(3,0)$ în $x \le y \Rightarrow 3 \le 0$ (F) $\Rightarrow$ pc. $(3;0) \notin$ semiplanului $x \le y$.

Met. 1 $\boxed{\iint_D dx\,dy = \text{Aria}(D)} = \frac{1}{4} \cdot \pi r^2 = \frac{1}{4} \cdot \pi \cdot 9 = \frac{9\pi}{4}$.

Metoda 2 (cu coordonate polare) exterior de cerc

Exercițiul 1: $D = \{(x,y) \in \mathbb{R}^2 \mid 1 \le x^2+y^2 \le 9,\ x \le 0,\ y \ge 0\}$. $\iint_D (x^2+y^2)^2\,dx\,dy$

interior de cerc.



$x^2+y^2=1$

$=9=3^2$

$x^2+y^2 \ge 1$

$-\|- \ge 9$

Calculăm integrala: (Trecerea la coord. polare)

Sch. var.: $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \Rightarrow I = \iint_{D^*} f(\rho\cos\theta, \rho\sin\theta) \cdot \left|\frac{D(x,y)}{D(\rho,\theta)}\right| d\rho\, d\theta$

$$= \iint_{D^*} f(\rho\cos\theta, \rho\sin\theta) \cdot \underbrace{\left|\frac{D(x,y)}{D(\rho,\theta)}\right|}_{=\rho} d\rho\, d\theta$$

$$D^* = \left\{(\rho;\theta) \in \mathbb{R}^2 \mid 1 \le \rho \le 3,\ \frac{\pi}{2} \le \theta \le \pi\right\}$$

$$I = \int_1^3 \left(\int_{\frac{\pi}{2}}^{\pi} \underbrace{(\rho^2\cos^2\theta + \rho^2\sin^2\theta)}_{\rho^2\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}}{}^2 \cdot \rho\, d\rho\, d\theta\right.$$

$$I = \int_1^3 \left(\int_{\frac{\pi}{2}}^{\pi} \rho^5\, d\theta\right) d\rho = \int_1^3 \rho^5 \cdot \theta \Big|_{\frac{\pi}{2}}^{\pi} d\rho =$$

$$= \int_1^3 \rho^5 \left(\pi - \frac{\pi}{2}\right) d\rho = \frac{\pi}{2} \cdot \frac{\rho^6}{6}\Big|_1^3 = \frac{\pi}{12}(3^6 - 1) =$$

$$= \frac{\pi}{\cancel{12}} \cdot \cancel{728} = \frac{182\pi}{3}$$

Exercițiul 2: $\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n+1}}{n}}_{a_n} (\underbrace{3x+1}_{y})^n;\ x \in \mathbb{R}$

a) int. de conv.

b) mult. de conv.

c) suma seriei

• $a_n = \frac{(-1)^{n+1}}{n}$; $y = 3x+1 \Rightarrow \sum_{n=1}^{\infty} a_n y^n$.

a_{n+1} $\frac{1}{a_n}$

• $\omega = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \left| \frac{(-1)^{n+2}}{n+2} \cdot \frac{n}{(-1)^{n+1}} \right| =$

$= \lim \frac{n}{n+1} = 1$

• $R = \frac{1}{\omega} \Rightarrow R = 1$.

$\Rightarrow$ int. de conv. pt. seria in y este

$J_y = (-1;1)$

$-1 < y < 1 \Leftrightarrow -1 < 3x+1 < 1 \mid -1$

$-2 < 3x < 0 \mid \cdot \frac{1}{3}$

$-\frac{2}{3} < x < 0 \Rightarrow J_x = \left(-\frac{2}{3}; 0\right)$

b) • Cf. T. Abel $\Rightarrow \begin{cases} \text{pt. } y \in (-1;1) \Rightarrow \text{seria abs. conv.} \\ \Rightarrow \text{seria este conv.} \\ \text{pt. } y \in (-\infty; -1) \cup (1; \infty) \Rightarrow SD. \end{cases}$

Pt. $y = -1 \Rightarrow \sum \frac{(-1)^{n+1}}{n} \cdot (-1)^n = \sum \frac{(-1)^{2n+1}}{n} =$

$= -1 \sum_{n=1}^{\infty} \frac{1}{n}$ (serie armonică) este divergentă.

$\Rightarrow \sum -\frac{1}{n}$ este tot divergentă

Pt. $y = 1 \Rightarrow \sum \frac{(-1)^{n+1}}{n} = \sum (-1)^{n+1} \cdot \frac{1}{n} \rightarrow$

$\rightarrow$ serie alternată

Not. $u_n = \frac{1}{n}$

$\lim u_n = \lim \frac{1}{n} = 0$.

$\frac{u_{n+1}}{u_n} = \frac{n}{n+1} < 1 \Rightarrow (u_n)_{n\geq 1}$ șir strict descrescător

$\xRightarrow{\text{Crit. Leibniz}}$ Serie convergentă

$\Rightarrow$ Seria de puteri este convergentă pentru

$y \in (-1; 1]$ $\Leftrightarrow$

$\Leftrightarrow -1 < 3x+1 \leq 1 \Leftrightarrow x \in \left(-\frac{2}{3}; 0\right] = C_x$

Mulțimea de convergență este $\left(-\frac{2}{3}; 0\right] = C_x$.

c) Notăm $S(x) = \sum \frac{(-1)^{n+1}}{n} (3x+1)^n$,

$x \in \left(-\frac{2}{3}; 0\right]$

$$S(x) = (-1)\sum_{n=1}^{\infty} \frac{(-3x-1)^n}{n} \quad ; \quad x \in \left(-\frac{2}{3}; 0\right]$$

Fol. T. de derivare termen cu termen a seriilor de puteri pe mulț. de conv. rezultă:

$$S'(x) = \left[(-1)\sum \frac{(-3x-1)^n}{n}\right]' = (-1)\sum\left[\frac{(-3x-1)^n}{n}\right]'$$

$$= (-1)\sum \frac{\cancel{n}(-3x-1)^{n-1}\cdot(-3)}{\cancel{n}} = 3\sum_{n=1}^{\infty}(-3x-1)^{n-1} =$$

$$= 3\left[\underbrace{1 + (-3x-1) + (-3x-1)^2 + \dots}_{1+q+q^2+\dots = \frac{1}{1-q},\ q\in(-1;1)}\right] = 3\cdot\frac{1}{1-(-3x-1)} =$$

$$= \frac{3}{3x+2}$$

$$\Rightarrow S(x) = \int \frac{3}{3x+2}\,dx = \ln|3x+2| + C, \quad C \in \mathbb{R}$$

$$|3x+2| = \begin{cases} 3x+2, & 3x+2>0 \Leftrightarrow x > -\frac{2}{3} \\ -3x-2, & x \le -\frac{2}{3} \end{cases}$$

$$\int \frac{1}{x}\,dx = \ln x$$

$$\int \frac{1}{u}\cdot u' = \ln|u|$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(3x+1)^n = \ln(3x+2) + C$$

$$x = -\frac{1}{3} \Rightarrow 0 = \ln 1 + C \Rightarrow C = 0.$$

$$\Rightarrow \boxed{S(x) = \ln(3x+2)}$$

Exercițiul 3: $\sum_{n=1}^{\infty} n\left(\frac{2x+5}{3x+1}\right)^n\cdot(-1)^{n-1}$, $x \in \mathbb{R}\setminus\left\{-\frac{1}{3}\right\}$

a) mulț. de conv.

b) suma.

a) $a_n = n\cdot(-1)^{n-1}$, $y = \frac{2x+5}{3x+1}$

$$\omega = \lim\left|\frac{a_{n+1}}{a_n}\right| = 1 \Rightarrow R = 1$$

T. Abel $\Rightarrow \begin{cases} y \in (-1;1) \text{ Ser. a. conv.} \Rightarrow \text{Ser. conv} \\ y \in (-\infty;-1)\cup(1;\infty) \Rightarrow \text{S. div} \end{cases}$

Pt. $y = -1 \Rightarrow \sum_{n=1}^{\infty} \underbrace{(-1)\cdot n}_{b_n}$

$$\lim b_n = -\infty \neq 0 \xRightarrow{CSD} \text{Serie div.}$$

Pt. $y = -1 \Rightarrow \underbrace{\sum n(-1)^{n-1}}_{c_n}$ (serie alternată)

$$\lim c_{2n} = -\infty \neq 0 \Rightarrow c_{2n} \not\to 0 \Rightarrow c_n \not\to 0$$

$\xRightarrow{CSD}$ Serie divergentă

$\Rightarrow$ Seria de puteri e conv. pt. $y \in (-1;1) \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \frac{2x+5}{3x+1} > -1 \Leftrightarrow \frac{5x+6}{3x+1} > 0 \\ \frac{2x+5}{3x+1} < 1 \Leftrightarrow \frac{4-x}{3x+1} < 0 \end{cases}$$

$$\left.\begin{array}{l} x \in (-\infty; -\frac{6}{5}) \cup (-\frac{1}{3}; \infty) \\ x \in (-\infty; -\frac{1}{3}) \cup (4; \infty) \end{array}\right\} \cap \Longrightarrow$$

$\Longrightarrow x \in (-\infty; -\frac{6}{5}) \cup (4; \infty) = C_x$ - mult. de convergență.

b) Suma:

Notăm: $S(x) = \sum n \left(\frac{2x+5}{3x+1}\right)^n \cdot (-1)^{n-1}$, $x \in C_x$

$S(x) = (-1) \cdot \sum n \left(\frac{-2x-5}{3x+1}\right)^n$; $x \in C_x$.

Notăm: $\frac{-2x-5}{3x+1} = y \Rightarrow T(y) = \sum_{n=1}^{\infty} n \cdot y^n$

(la final înmulțim cu -1)

Cons. $\sum_{n=1}^{\infty} y^n = \frac{1}{1-y} - 1 = \frac{y}{1-y}$, $\forall y \in (-1;1)$

Fol. T. de deriv. term. cu term. a ser. de puteri $\Rightarrow$

$\Rightarrow \left(\sum y^n\right)' = \sum (y^n)' \Leftrightarrow \left(\frac{y}{1-y}\right)' = \sum n \cdot y^{n-1} \,\Big|\cdot y$

$\Rightarrow \sum n y^n = y \cdot \frac{1-\cancel{y}+\cancel{y}}{(1-y)^2} \Rightarrow \sum n y^n =$

$= \frac{y}{(1-y)^2}$, $\forall y \in (-1;1)$ (1)

Pt. $y = \frac{-2x-5}{3x+1}$ in (1) $\Rightarrow \sum n \left(\frac{-2x-5}{3x+1}\right)^n =$

$= \frac{\frac{-2x-5}{3x+1}}{\left(1+\frac{2x+5}{3x+1}\right)^2} = \frac{(-2x-5)(3x+1)}{(5x+6)^2} \,\Big|\cdot(-1) \Rightarrow$

$\Rightarrow S(x) = \frac{(2x+5)(3x+1)}{(5x+6)^2}$, $\forall x \in C_x$.

Exercițiu 4: $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{(n+1)!} \cdot (4x-1)^n =$

$= \sum \frac{(1-4x)^n}{n!}$

$a^n \cdot b^n = (a \cdot b)^n$

!

$\sum_{n=0}^{\infty} \frac{a^n}{n!}$ conv. $\forall a \in \mathbb{R}$.

Și are suma $\boxed{\sum \frac{a^n}{n!} = e^a}$, $\forall a \in \mathbb{R}$ (1)

a) Int. de conv.

- $a_n = \frac{1}{n!}$; $y = 1-4x \Rightarrow \sum a_n \cdot y^n$.
- $\omega = \lim \left|\frac{a_{n+1}}{a_n}\right| = 0 \Rightarrow R = \infty$

$\Rightarrow J_y = (-\infty; \infty) = (-R; R)$

$J_x = (-\infty; \infty)$

b) Suma pt. $x = \frac{1}{8}$.

$$x = \frac{1}{8} \Rightarrow \sum_{n=1}^{\infty} = \frac{\left(1-\frac{1}{2}\right)^n}{n!} = \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n!} = e^{\frac{1}{2}} - 1 =$$

$= \sqrt{e} - 1$

Exercițiu 5: $f_n : (-1; 1] \to \mathbb{R}$, $f_n(x) = \frac{n^2 \cdot x^n}{2n^2+1}$, $n \geq 1$

$(f_n)_{n\geq 1} \cancel{\xrightarrow[(-1;1]]{u.c.}}$ (să arătăm că nu e unif. convergent)

Studiem convergența simplă:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \underbrace{\frac{n^2}{2n^2+1}}_{\to \frac{1}{2}} \cdot x^n =$$

$$= \frac{1}{2} \cdot \lim_{n\to\infty} x^n = \begin{cases} \frac{1}{2} \cdot 0 = 0, & x \in (-1; 1). \\ \frac{1}{2}, & x = 1 \end{cases} \Rightarrow$$

$\Rightarrow f_n \xrightarrow[(-1;1]]{\text{conv. simplu}} f$

P.p. $f_n \xrightarrow[(-1;1]]{c.u} f$

Cum f_n cont. pe $(-1; 1]$, $\forall n \geq 1$ $\Big\}$ $\xrightarrow{\text{transfer. cont.}}$

f cont. pe $(-1; 1]$ (2) contradicție

$\lim\limits_{\substack{x\to 1 \\ x<1}} f(x) = 0 \neq f(1) = \frac{1}{2} \Rightarrow f$ nu e cont. în $x = 1$. (3)

$$f(x) = \begin{cases} 0; & x \in (-1; 1) \\ \frac{1}{2}; & x = 1 \end{cases}$$

Din (2) și (3) $\Rightarrow$ CONTRADICȚIE $\Rightarrow$ propunerea este falsă $\Rightarrow f_n \cancel{\xrightarrow[(-1;1]]{c.u}} f$.

Exercițiu 6: $\sum_{n=1}^{\infty} \underbrace{\frac{2^n + (-1)^n}{3^n + (-1)^n}}_{a_n}$. Natura seriei

$a_n > 0$, $\forall n \geq 1 \Rightarrow$ folosim CRIT. RĂDĂCINII

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n\left[1 + \underbrace{\left(-\frac{1}{2}\right)^n}_{\to 0}\right]}{3^n\left[1 + \underbrace{\left(-\frac{1}{3}\right)^n}_{\to 0}\right]}} = \frac{2}{3} < 1$$

$\overset{\text{Crit.}}{\underset{\text{rad.}}{\Longrightarrow}}$ seria este CONVERGENTĂ

Exercițiu 7: $\int_0^\infty \frac{x^4}{(1+x^6)^1} \longrightarrow$ (adăugăm noi la puterea 1)

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$
$$\beta(a,b) = \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}}\,dx$$

Sch. var: $x^6 = y$

$x = y^{\frac{1}{6}}$ (derivăm)

$dx = \frac{1}{6}y^{\frac{1}{6}-1}\,dy$

x	0	∞
y	0	∞

$$J = \int_0^\infty \frac{\left(y^{\frac{1}{6}}\right)^4}{(1+y)^1} \cdot \frac{1}{6}y^{-\frac{5}{6}}\,dy =$$

$$= \frac{1}{6}\int_0^\infty \frac{y^{\frac{2}{3}} \cdot y^{-\frac{5}{6}}}{(1+y)^1}\,dy = \frac{1}{6}\int_0^\infty \frac{y^{-\frac{1}{6}}}{(1+y)^1}\,dy$$

$a - 1 = -\frac{1}{6} \Rightarrow a = -\frac{1}{6} + 1 = \frac{5}{6} \Rightarrow \boxed{a = \frac{5}{6}}$

$a + b = 1 \Rightarrow \frac{5}{6} + b = 1 \Rightarrow \boxed{b = \frac{1}{6}}$

$$\Rightarrow J = \frac{1}{6} \cdot \beta\left(\frac{5}{6}, \frac{1}{6}\right)$$

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Dacă $\left.\begin{array}{l} a+b=1 \\ a,b>0 \end{array}\right\} \Rightarrow \beta(a,b) = \frac{\pi}{\sin(a\cdot\pi)}$

$\beta(a,b) = \beta(b,a)$, $\forall\, a,b>0$

$$\Rightarrow J = \frac{1}{6} \cdot \beta\left(\frac{1}{6}; \frac{5}{6}\right) = \frac{1}{6} \cdot \frac{\pi}{\sin\left(\frac{1}{6}\cdot\pi\right)} =$$

$\boxed{\frac{1}{6} + \frac{5}{6} = 1}$

$$= \frac{1}{6} \cdot \frac{\pi}{\frac{1}{2}} = \frac{1}{6} \cdot 2\pi = \frac{\pi}{3}$$

Exercițiu 8: Limite iterate și lim. globale.

$f(x,y) = \frac{x+y}{x-y}$, $f: \mathbb{R}^2 - \{(x,y) \in \mathbb{R}^2 \mid y = x\}$

Limite iterate

$$l_{x,y}(0;0) = \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{x+y}{x-y}\right)$$

$$= \lim_{x\to 0}\frac{x}{x} = \lim_{x\to 0} 1 = 1$$

$\lim_{y,x} = -1$

$l_{x,y}(0;0) = 1 \neq l_{y,x}(0;0) = -1 \Rightarrow$

$\Rightarrow \nexists \lim_{(x,y)\to(0;0)} f(x,y)$.

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Presupunem $\exists \lim_{(x,y)\to(0;0)} f(x,y) = l \Rightarrow l_{x,y}(0;0) =$

$= l_{y,x}(0,0) = l$ CONTRADICȚIE $\Rightarrow$ presupunerea e falsă $\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$.