# Basics about Sets

REPLAN team

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# Outline

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# Motivation

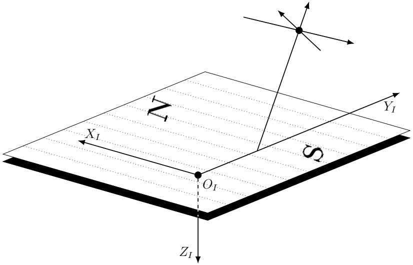
- Modeling a drone system is relatively complex because it considers 6 degrees of freedom:
  - 3 position coordinates;
  - 3 rotation coordinates.
- These elements can (and are) considered in two distinct coordinate systems:
  - inertial coordinate system: this is the "fixed" system, in which we measure the drone's trajectory relative to the ground;
  - body coordinate system: this is the "mobile" system, centered in the drone's center of mass.
  - We are interested in both coordinate systems because certain quantities are naturally measured in one of these systems, while other quantities, in the other system.

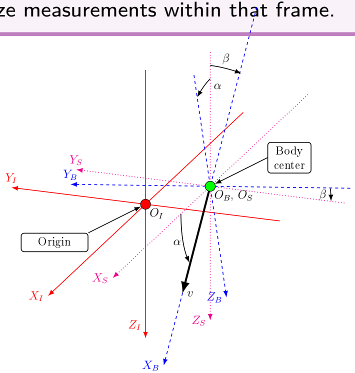
# Outline

# Inertial and Body frames

**Definition [physics]**

A frame of reference (or reference frame) consists of an abstract coordinate system and the set of physical reference points that uniquely fix (locate and orient) the coordinate system and standardize measurements within that frame.




- A fixed reference frame fixed to the environment, not to the moving subject, and used commonly in describing the motions of different body parts is the global frame.

# Inertial and Body frames

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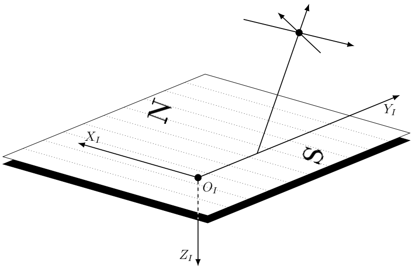
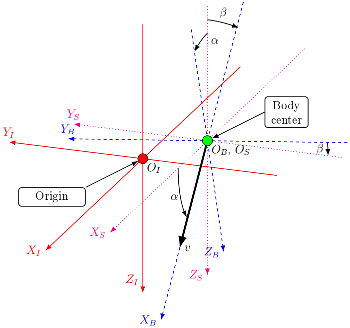



- A moving reference frame can translate and/or rotate. When a reference frame is either fixed or moving with a constant velocity, it is an inertial frame.

# About coordinate frames

**Info**

Posing the problem is half the solution!

Depending on problem specifics and / or requirements, we may consider[1] the:

- Earth-Centered Inertial (ECI) Frame
- Earth-Centered Earth-Fixed (ECEF) Frame
- North-East-Down (NED) Frame
- Body Frame

[1] D. H. Titterton (2004). "Strapdown inertial navigation technology". In: *The Institution of Engineering and Technology*, Chapter 3.3.

# Earth-Centered Inertial (ECI) Frame

**ECI frame**

Also called *i-frame*, is a fundamental reference frame used in celestial mechanics and satellite dynamics. It is an inertial frame of reference $\Rightarrow$ not subject to the rotational motion of the Earth.

- **The origin** of the frame is located at the center of the Earth.
- **The x-axis** points toward the vernal equinox, a fixed direction in space defined by the intersection of the Earth's equatorial plane and the ecliptic plane.
- **The z-axis** is aligned with the Earth's rotational axis, pointing toward the North Celestial Pole.
- **The y-axis** is perpendicular to both the $x$- and $z$-axes, completing the right-handed coordinate system.

# Earth-Centered Earth-Fixed (ECEF) Frame

**ECEF frame**

Also called the *e-frame*, rotates with the Earth, making it a fixed reference relative to the Earth's surface. It is the frame in which GNSS[a] receivers typically output their position and velocity computations.

[a] Global Navigation Satellite System

- **The origin** of the frame is located at the center of the Earth.
- **The x-axis** points toward the intersection of the Earth's equatorial plane and the Prime Meridian (Greenwich Meridian).
- **The z-axis** is aligned with the Earth's rotational axis, pointing toward the North Pole.
- **The y-axis** is perpendicular to both the $x$- and $z$-axes, completing the right-handed coordinate system and pointing approximately toward $90^\circ$ East longitude.

**Caution!**

The ECEF frame rotates with an angular velocity $\omega_{ie}$ relative to inertial space, where $\omega_{ie}$ represents the Earth's rotation rate.

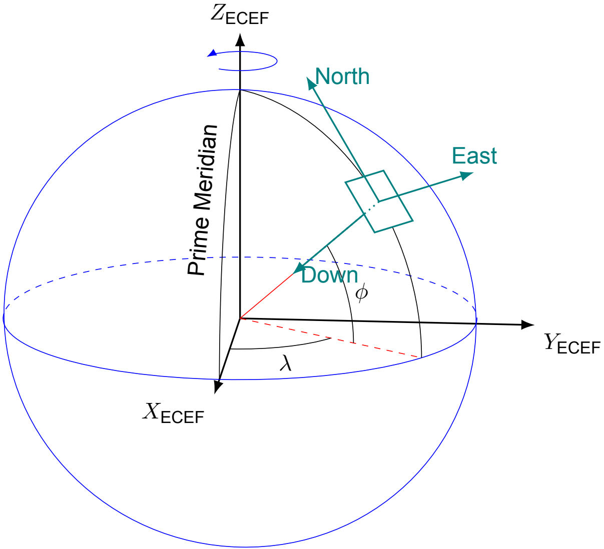
# North-East-Down (NED) Frame

**NED frame**

Also called the *n-frame* or The Navigation Frame is a local tangent plane frame commonly used in navigation. It provides a convenient reference for representing position, velocity, and orientation relative to the Earth's surface at a specific location.

- **The origin** of the frame is located at a specific point on the Earth's surface, typically at the location of interest.
- **The x-axis** (North axis) points toward the local true North, aligned tangentially to the Earth's surface.
- **The y-axis** (East axis) points tangentially to the Earth's surface and perpendicular to the x-axis, toward the local true East.
- **The z-axis** (Down axis) points downward, normal to the Earth's surface, completing the right-handed coordinate system.

# Illustration of the NED frame within the ECEF frame



# Body Frame

## Body Frame

Also called the *b-frame*, is a reference frame fixed to a vehicle or navigation platform.

- **The origin** is located at the center of the vehicle or navigation platform.
- **The x-axis** points forward along the vehicle's longitudinal axis, typically in the direction of travel.
- **The y-axis** points to the right of the vehicle, along the lateral axis.
- **The z-axis** points downward, completing the right-handed coordinate system and aligned with the vertical axis of the vehicle.

## Caution!

The *b-frame* is a non-inertial frame because it rotates with the turning body. It is important in navigation systems as it serves as the local frame for measurements from onboard sensors, such as accelerometers and gyroscopes.

# Frame conversions

Conversions between any of the presented navigation frames are possible[2]. The most popular ones are:

- Direct Cosine Matrix (DCM)
- quaternion-based
- Euler-angle rotation matrices

[2] P. D. Groves (2015). “Principles of GNSS, inertial, and multisensor integrated navigation systems, [Book review]”. In: *IEEE Aerospace and Electronic Systems Magazine* 30.2. Publisher: IEEE, pp. 26–27, Chapter 2.5.

# Direct Cosine Matrix (DCM) I

**DCM**

A mathematical representation of attitude that describes the orientation of one reference frame relative to another. It is denoted as $C_\alpha^\beta$, where $\alpha$ and $\beta$ indicate the two frames involved.

- each column of the DCM $\in \mathbb{R}3 \times 3$, represents a unit vector of the $\alpha$-frame's axes projected along the $\beta$-frame's axes:

$$C_\alpha^\beta = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \qquad \text{where:}$$

- $c_{ij}$ represents the cosine of the angle between the $i$-th axis of the $\alpha$-frame and the $j$-th axis of the $\beta$-frame.
- The columns of $C_\alpha^\beta$ correspond to the unit vectors of the $\alpha$-frame expressed in the $\beta$-frame.
- The rows of $C_\alpha^\beta$ correspond to the unit vectors of the $\beta$-frame expressed in the $\alpha$-frame.

# Direct Cosine Matrix (DCM) II

- Transformations between reference frames using the DCM:

**vector transformation**

$$r^{\beta} = C_{\alpha}^{\beta} \cdot r^{\alpha},$$

**attitude' rate of change**

$$\dot{C}_{\alpha}^{\beta} = C_{\alpha}^{\beta} \cdot \Omega_{\alpha\beta}^{\alpha},$$

where $\Omega_{\alpha\beta}^{\alpha}$ is the skew-symmetric matrix of the angular velocities $\omega_x$, $\omega_y$, and $\omega_z$ of the frame $\alpha$ w.r.t $\beta$, with:

$$\Omega_{\alpha\beta}^{\alpha} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}.$$

- The inverse transformation is equivalent to the transposition of the DCM matrix:

$$C_{\beta}^{\alpha} = \left( C_{\alpha}^{\beta} \right)^{\top},$$

where $C_{\beta}^{\alpha}$ is the dcm representing the transformation from the $\beta$-frame back to the $\alpha$-frame.

# Quaternions I

**Quaternions**

It is a four-parameter representation based on the idea that a transformation from one coordinate frame to another can be achieved through a single rotation about a vector $\mu$, defined with respect to the reference frame.

- Its elements are functions of $\mu$ and the magnitude of the rotation:

$$\mathbf{q} = \begin{bmatrix} a & b & c & d \end{bmatrix}^\top = \begin{bmatrix} \cos(\mu/2) & (\mu_x/|\mu|)\sin(|\mu|/2) & (\mu_y/|\mu|)\sin(|\mu|/2) & (\mu_z/|\mu|)\sin(|\mu|/2) \end{bmatrix}^\top,$$

  where $\mu_x$, $\mu_y$, and $\mu_z$ are the components of the angle vector $\mu$, and $|\mu|$ is the magnitude of $\mu$.
- It can also be expressed as a four-parameter complex number with a real component $a$ and three imaginary components $b$, $c$, and $d$:

$$\mathbf{q} = a + ib + jc + kd.$$

- The product of two quaternions, $\mathbf{q} = a + ib + jc + kd$ and $\mathbf{p} = e + if + jg + kh$, respects:

$$q \cdot p = ea - bf - cg - dh + (af + be + ch - dg)i + (ag + ce - bh + df)j + (ah + de + bg - cf)k$$

# Quaternions II

A vector $r^{\beta}$, defined in frame $\beta$, can be expressed in the reference frame $\alpha$ using a quaternion rotation:

- Define quaternion $r^{\beta^*}$ in function of vector $r^{\beta}$:

$$r^{\beta} = ix + jy + kz, \quad r^{\beta^*} = 0 + ix + jy + kz.$$

- The vector $r^{\alpha}$, expressed in frame $\alpha$, is given by:

$$r^{\alpha} = q\, r^{\beta^*}\, q^*,$$

  where $q$ is the quaternion representing the rotation from frame $\beta$ to frame $\alpha$, and $q^* = a - ib - jc - kd$ is the conjugate of $q$.

- The propagation of the quaternion $q$, is governed by:

$$\dot{q} = 0.5\, q \cdot \begin{bmatrix} 0 & \omega_x & \omega_y & \omega_z \end{bmatrix}^{\top},$$

  where $\omega_x$, $\omega_y$, and $\omega_z$ are the components of the angular velocity of the body frame $\beta$ w.r.t. the reference frame $\alpha$.

# Euler Angles I

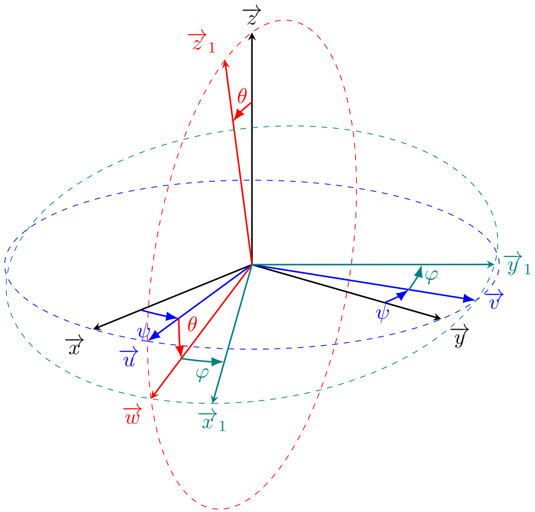
Euler angles are a method of representing the orientation of one coordinate frame relative to another using three successive rotations about different axes. These angles consist of:

- A rotation through angle $\psi$ about the reference $z$-axis (yaw),
- A rotation through angle $\theta$ about the new $y$-axis (pitch),
- A rotation through angle $\varphi$ about the new $x$-axis (roll).

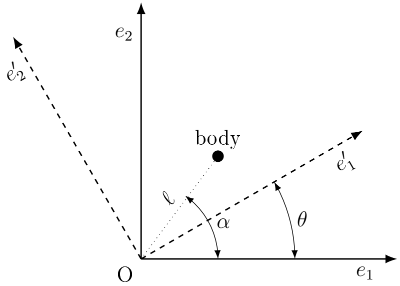
Several elements of interest:

- The order of rotations is not unique and thus the convention considered has to be known.
- Are very intuitive, making them popular for understanding and visualizing attitude.
- Suffer from a significant limitation known as **gimbal lock**, a singularity that occurs when the pitch angle ($\theta$) approaches $\pm 90^\circ$, causing a loss of one degree of rotational freedom.

# Euler Angles II



# Rotations in 2D




- Rotating is equivalent with multiplying with a matrix of form:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- The determinant is $\det(R) = 1 \Rightarrow$ the multiplication does not change the moby's volume.
- In addition, the two column vectors of $R$ are orthogonal:

$$\cos\theta \times (-\sin\theta) + \sin\theta \times \cos\theta = 0$$

# Rotations in 3D

- To switch from one coordinate system to the other, we need to express the rotation of the mobile frame w.r.t. the fixed frame:

$$
{}^{I}_{B}R = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}}_{R_x(\theta)} \underbrace{\begin{bmatrix} c\psi & 0 & s\psi \\ 0 & 1 & 0 \\ -s\psi & 0 & c\psi \end{bmatrix}}_{R_y(\psi)} \underbrace{\begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_z(\phi)} = \begin{bmatrix} c\theta c\psi & s\phi s\theta c\psi - c\phi s\psi & c\phi s\theta c\psi + s\phi s\psi \\ c\theta s\psi & s\phi s\theta s\psi + c\phi c\psi & c\phi s\theta s\psi - s\phi c\psi \\ -s\theta & s\phi c\theta & c\phi c\theta \end{bmatrix}
$$

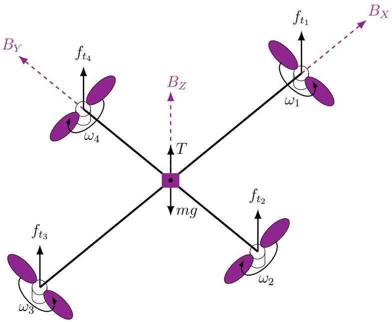
- The notations "$s\alpha$" and "$c\alpha$" are the shorthand forms for "$\sin\alpha$" and "$\cos\alpha$".

- The indices "B" and "I" indicate that the matrix transforms coordinates from the "fixed" frame to the "mobile" frame.

- The rotation matrix is by definition orthogonal, therefore, ${}^{B}_{I}R = \left({}^{I}_{B}R\right)^{-1} = \left({}^{I}_{B}R\right)^{\top}$.

# Outline

# Mechanical structure




- The angular velocities and thrust forces developed by the four motors are observable.
- We consider four motors oriented perpendicularly to the plane of the drone.
- the rotations, $\omega_i$, define the control magnitudes that affect the drone's behavior:
  - thrust force: $T$;
  - torque moments w.r.t. the axes of rotation: $\tau_\theta, \tau_\phi, \tau_\psi$ .

# Thrust force & torques - block **(a)**

- The total thrust force is given by:

$$
{}^{B}\vec{T} = \begin{bmatrix} 0 \\ 0 \\ \tau_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \beta \sum_{i=1}^{4} \omega_i^2 \end{bmatrix}
$$

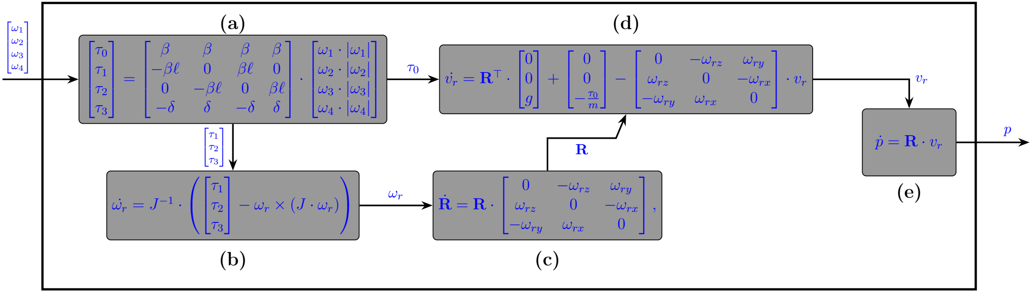
- The vector ${}^{B}\vec{T}$ has a single non-zero component: the motors are positioned perpendicular to the plane of the drone, thus "pushing" the drone vertically.
- The rotation speed of each motor contributes a torque moment relative to each of the axes of the coordinate frame.
- Assuming constant rotation directions for each of the motors, we obtain:

$$
\begin{aligned}
\tau_1 &= \tau_\phi = \beta\ell(-\omega_2^2 + \omega_4^2), \\
\tau_2 &= \tau_\theta = \beta\ell(-\omega_1^2 + \omega_3^2) \\
\tau_3 &= \tau_\psi = \delta(-\omega_1^2 + \omega_2^2 - \omega_3^2 + \omega_4^2).
\end{aligned}
$$

- In other words, the angular moments defined in relation to the mobile frame are:

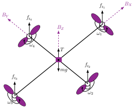
$$
{}^{B}\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} \ell\beta(-\omega_2^2 + \omega_4^2) \\ \ell\beta(-\omega_1^2 + \omega_3^2) \\ \delta(-\omega_1^2 + \omega_2^2 - \omega_3^2 + \omega_4^2) \end{bmatrix}
$$

# Quadcopter dynamics - mathematical model 1/5

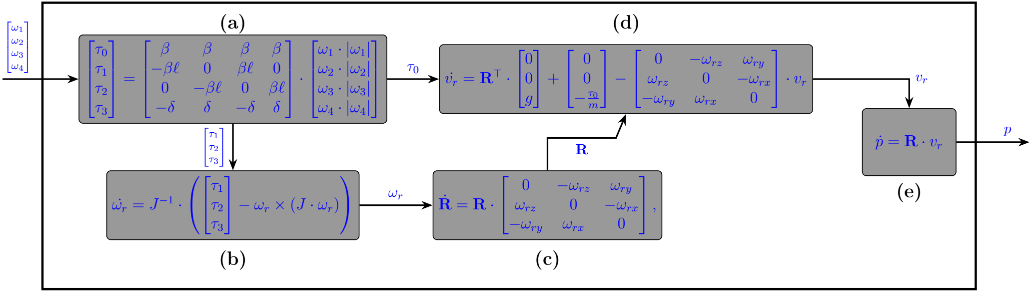



Block **(a)**

- $\beta$ - thrust factor
- $\delta$ - drag factor
- $\ell$ - the distance between any rotor and the center of the robot



# Quadcopter dynamics - mathematical model 2/5

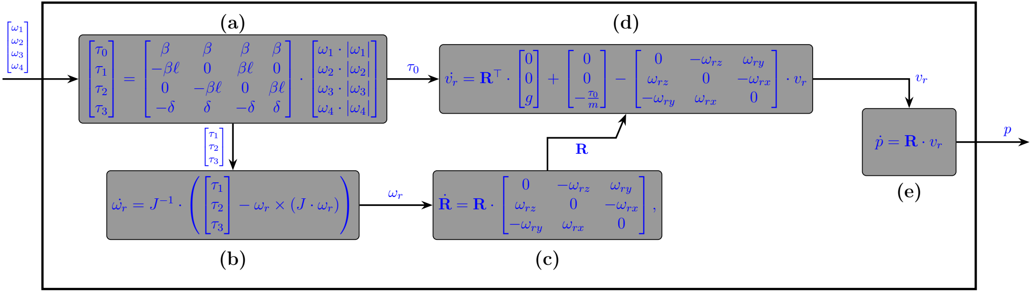



Block **(b)**

- $J$ - the inertia matrix
- $\omega_r$ - angular velocity

$$J = \int_V \rho(x,y,z) \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} dx\, dy\, dz,$$

# Quadcopter dynamics - mathematical model 3/5

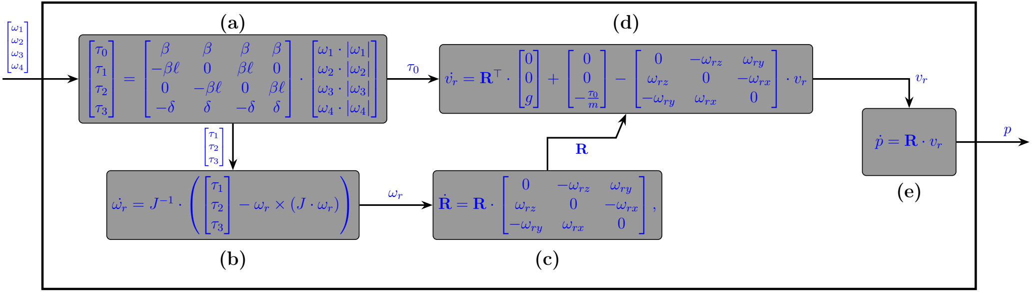



Block **(c)**

- $\mathbf{R}$ - rotation/orientation matrix
- $\dot{\mathbf{R}}$ - the derivative of $\mathbf{R}$
- $\mathbf{R} = R_x(\theta) R_y(\psi) R_z(\phi)$

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\0 & c\theta & -s\theta\\0 & s\theta & c\theta\end{bmatrix}}_{R_x(\theta)} \underbrace{\begin{bmatrix}c\psi & 0 & s\psi\\0 & 1 & 0\\-s\psi & 0 & c\psi\end{bmatrix}}_{R_y(\psi)} \underbrace{\begin{bmatrix}c\phi & -s\phi & 0\\s\phi & c\phi & 0\\0 & 0 & 1\end{bmatrix}}_{R_z(\phi)}$$

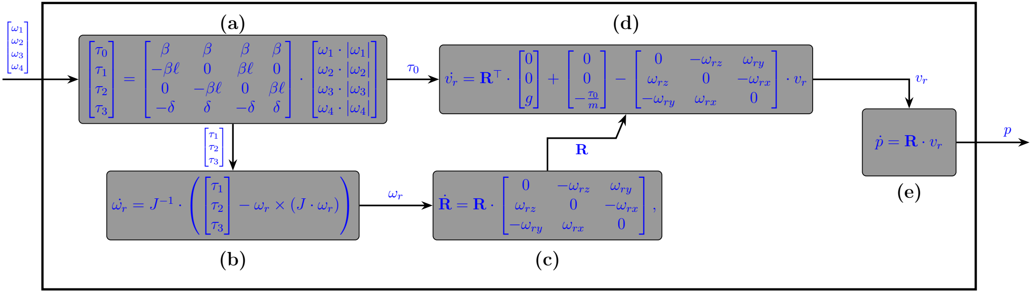
# Quadcopter dynamics - mathematical model 4/5




Block **(d)**

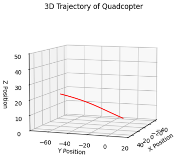
- linear velocity equation within *body frame*
- $v_r$ - linear velocity of the quadcopter
- $\omega_r$ - angular velocity of the quadcopter - body frame
- $m$ - mass
- $g$ - gravitational acceleration

# Quadcopter dynamics - mathematical model 5/5

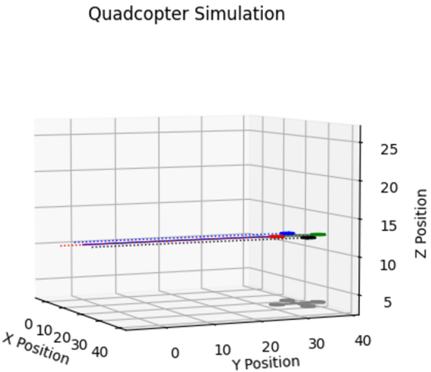


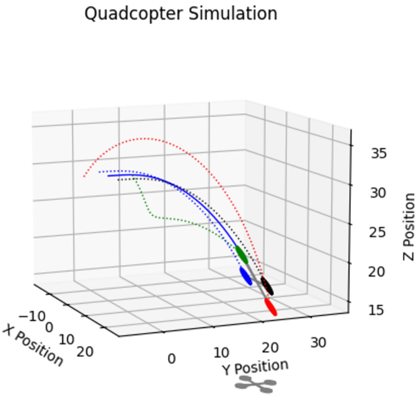

Block **(e)**

- $p$ - position (within global frame )
- $p \in \mathbb{R}^3$, $p = \begin{bmatrix} x_I & y_I & z_I \end{bmatrix}^\top$



# Quadcopter dynamics - simulation(`roblib.py`)



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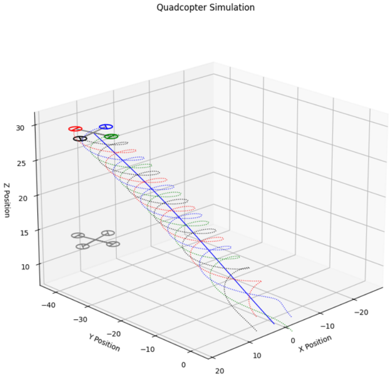
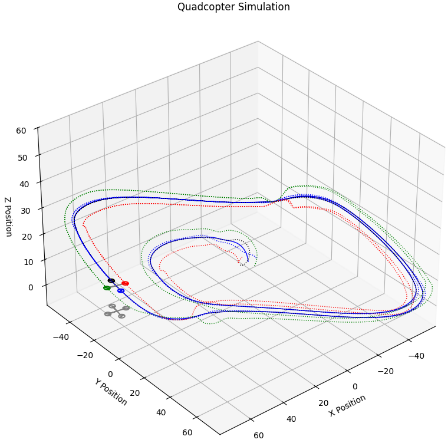
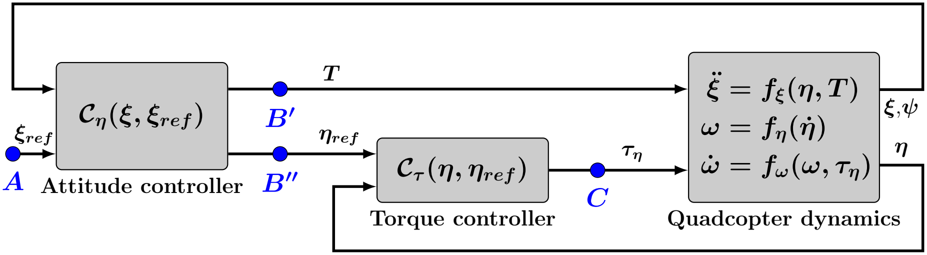


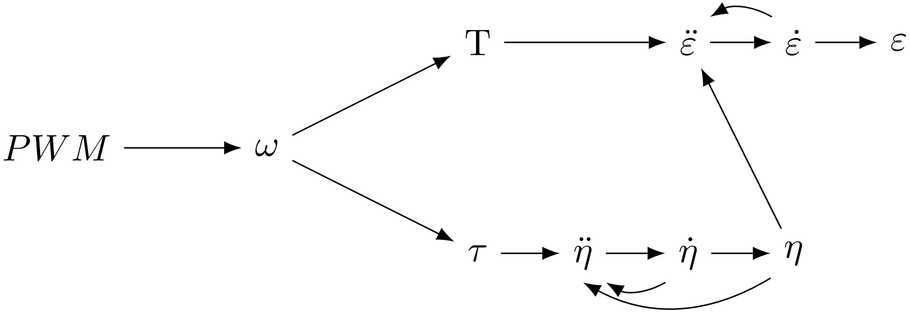
# Outline

# Typical control scheme




- cascaded control loop: high level (translation) and low level (rotation);
- sometimes only intermediary points are given, not even the reference trajectory;
- internally, torque moments and thrust force are converted into motor speeds (another control block)

# From propeller velocity to the drone's position




- PWM $=$ Pulse Width Modulation;
- The commands sent to the motors (PWM) propagate through the drone's dynamics to eventually lead to the desired position $\epsilon$;
- Note: the torque moments are proportional to $\epsilon^{(4)}$.

# Flat representation – I

- By imposing a value for $\psi$ and an offline-generated trajectory, the reference angles, $\eta_{ref}$, and the thrust force, $T$, can be calculated based on the translation equation:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} + g \end{bmatrix} = \frac{T}{m} \begin{bmatrix} c\phi s\theta c\psi + s\phi s\psi \\ c\phi s\theta s\psi - s\phi c\psi \\ c\phi c\theta \end{bmatrix}$$

- We express all quantities in flat representation, where we denote

$$z_1 = x, \ , z_2 = y, \ z_3 = z, \text{ and } z_4 = \psi.$$

# Flat representation – II

- We can represent the four control magnitudes through the relations

$$
\begin{aligned}
T &= m\sqrt{\ddot{z}_1^2 + \ddot{z}_2^2 + (\ddot{z}_3 + g)^2},\\
\phi_{ref} &= \arcsin\left(\frac{\ddot{z}_1 \sin z_4 - \ddot{z}_2 \cos z_4}{\sqrt{\ddot{z}_1^2 + \ddot{z}_2^2 + (\ddot{z}_3 + g)^2}}\right),\\
\theta_{ref} &= \arctan\left(\frac{\ddot{z}_1 \cos z_4 + \ddot{z}_2 \sin z_4}{\ddot{z}_3 + g}\right),\\
\psi_{ref} &= z_4.
\end{aligned}
$$

- Assuming that the reference angles are tracked ($\phi_{ref} \leftarrow \phi$ and $\theta_{ref} \leftarrow \theta$), it can be shown that the drone's accelerations follow the reference profiles:

$$\ddot{x} \leftarrow \ddot{z}_1,\ \ddot{y} \leftarrow \ddot{z}_2,\ \ddot{z} \leftarrow \ddot{z}_3.$$

- By correctly choosing the reference accelerations, a desired trajectory profile ($x \leftarrow z_1, y \leftarrow z_2, z \leftarrow z_3$) is obtained.