Seminar 2

① $\begin{cases} x+\alpha y+z=1 \\ \alpha x-y+z=1 \\ x+y-z=2 \end{cases}$ Discutie după $\alpha \in \mathbb{R}$.

$$A=\left(\begin{array}{ccc|c} 1 & \alpha & 1 & 1 \\ \alpha & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \end{array}\right)$$

$$\Delta=\det A=\begin{vmatrix} 1 & \alpha & 1 \\ \alpha & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}=\begin{vmatrix} 1 & \alpha & 1 \\ \alpha-1 & -(\alpha+1) & 0 \\ 2 & \alpha+1 & 0 \end{vmatrix}=$$

$$=1\cdot(-1)^{1+3}(\alpha+1)\begin{vmatrix} \alpha-1 & -1 \\ 2 & 1 \end{vmatrix}=(\alpha+1)(\alpha-1+2)=$$

$$=(\alpha+1)^2$$

Cazul 1) $\det A \neq 0$, $\alpha \in \mathbb{R}\setminus\{-1\}$

$\operatorname{rang} A=\operatorname{rang}\bar{A}=3$

SCD de tip Cramer

$$x=\frac{\Delta x}{\Delta} \qquad \Delta x=\begin{vmatrix} 1 & \alpha & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}=$$

$$y=\frac{\Delta y}{\Delta}$$

$$z=\frac{\Delta z}{\Delta} \qquad =\begin{vmatrix} 1 & \alpha & 1 \\ 0 & -(\alpha+1) & 0 \\ 3 & \alpha+1 & 0 \end{vmatrix}=3(\alpha+1)$$

$$x = \frac{3(\alpha+1)}{(\alpha+1)^2} = \frac{3}{\alpha+1}$$

$$\Delta y = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ \alpha-1 & 0 & 0 \\ 2 & 3 & 0 \end{vmatrix} = 3(\alpha-1)$$

$$y = \frac{3(\alpha-1)}{(\alpha+1)^2}$$

$$\Delta z = \begin{vmatrix} 1 & \alpha & 1 \\ \alpha & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & \alpha & 1 \\ \alpha-1 & -(\alpha+1) & 0 \\ -1 & 1-2\alpha & 0 \end{vmatrix} =$$

$$= (\alpha-1)(1-2\alpha) - (\alpha+1) = \alpha - 2\alpha^2 - 1 + 2\alpha - \alpha - 1 =$$

$$= -2\alpha^2 + 2\alpha - 2$$

$$z = \frac{-2(\alpha^2 - \alpha + 1)}{(\alpha+1)^2}$$

Cazul II $\Delta = 0$

$$\alpha = -1 \Rightarrow A = \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \end{array}\right)$$

$$\Delta_p = \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} = -2 \neq 0 \Rightarrow rg A = 2$$

$$\Delta c = \begin{vmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2 \cdot (-1)^{1+1} \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix}$$

$= 2 \cdot (-3) \neq 0$

$rang\,\bar{A} = 3 \Rightarrow$ Si.

(2) $\begin{cases} x+2y+3z=0 \\ 4x+5y+6z=0 \\ x+\alpha^2 z=0 \end{cases}$ $A=\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 1 & 0 & \alpha^2 & 0 \end{array}\right)$

$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & \alpha^2 \end{vmatrix} = 5\alpha^2+12-15-8\alpha^2=$

$= -3\alpha^2-3 = -3(\alpha^2+1) \neq 0 \ \forall \alpha \in \mathbb{R}$

SCD $rg\,A = rg\,\bar{A} = 3$ $(x,y,z)=(0,0,0)$

(3) ΔABC, $a, b, c \to$ lungimi laturi

$\begin{cases} ay + bx = c \\ cx + az = b \\ bz + cy = a \end{cases}$

$(\forall)$ $\Delta ABC \Rightarrow$ SCD și unică

(x_0, y_0, z_0) verifică $x_0, y_0, z_0 \in (-1, 1)^2$

$A = \left(\begin{array}{ccc|c} b & a & 0 & c \\ c & 0 & a & b \\ 0 & c & b & a \end{array}\right)$

$\det A = \begin{vmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{vmatrix} = -abc - abc = -2abc$

$\neq 0 \Rightarrow$ SCD

$$\Delta x = \begin{vmatrix} c & a & 0 \\ b & 0 & a \\ a & c & b \end{vmatrix} = a^3 - ac^2 - ab^2$$

$$= a(a^2 - c^2 - b^2)$$

$$= -a(b^2 + c^2 - a^2)$$

$$x = \frac{-a(b^2 + c^2 - a^2)}{-2abc} = \frac{(b^2 + c^2 - a^2)}{2bc}$$

T. cos $a^2 = b^2 + c^2 - 2bc\cos A$

$\Rightarrow x = \cos A$

Analog $y = \cos B$, $z = \cos C$

$A, B, C \in (0, \pi) \Rightarrow x, y, z \in (-1, 1)$

5) $\begin{cases} x + y + z = 0 \\ (b+c)x + (a+c)y + (a+b)z = 0 \\ bcx + acy + abz = 0 \end{cases}$

$a, b, c \in \mathbb{R}$, distincte.

$$A = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ b+c & a+c & a+b & 0 \\ bc & ac & ab & 0 \end{array}\right)$$

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ b+c & a+c & a+b \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ b+c & a-b & a-c \\ bc & c(a-b) & b(a-c) \end{vmatrix}$$

$$= 1\cdot(-1)^{1+1}\cdot(a-b)(a-c)\begin{vmatrix}1 & 1\\ c & b\end{vmatrix} =$$

$$=(a-b)(a-c)(b-c)\neq 0$$

a, b, c distincte

rang A = rang $\overline{A}$ ⇒ SCD. sistem omogen ⇒ soluție unică $(0,0,0)$

6) $$\begin{cases} x+2y = m+1\\ 2x+3y = m-1\\ mx+y = 3\end{cases}$$

$m=?$ a.i. sistem incompatibil

$$A=\left(\begin{array}{cc|c} \boxed{\begin{matrix}1 & 2\\ 2 & 3\end{matrix}} & & m+1\\ & & m-1\\ m & 1 & 3\end{array}\right)$$

$$\Delta_p = \begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix} = 3-4\neq 0$$

$$\Delta_c = \begin{vmatrix}1 & 2 & m+1\\ 2 & 3 & m-1\\ m & 1 & 3\end{vmatrix} = \begin{vmatrix}m+4 & 2 & m+1\\ m+4 & 3 & m-1\\ m+4 & 1 & 3\end{vmatrix} =$$

$$=(m+4)\begin{vmatrix}1 & 2 & m+1\\ 1 & 3 & m-1\\ 1 & 1 & 3\end{vmatrix} = (m+4)\begin{vmatrix}0 & 1 & m-2\\ 0 & 2 & m-4\\ 1 & 1 & 3\end{vmatrix}$$

$$=(m+4)\begin{vmatrix}1 & m-2\\ 2 & m-4\end{vmatrix} = (m+4)(m-\cancel{4}-2m+\cancel{4}) = (m+4)\cdot(-m)\neq 0$$

Răspuns $m \in \mathbb{R} \setminus \{-4, 0\}$.

10). Fie $A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 4 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

a) Să se scrie A în forma eșalon (resp. eșalon redusă)

b) $\operatorname{rg} A = ?$

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & 4 & 1 \\ 1 & 1 & 0 \end{pmatrix} \underset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & 4 & 1 \\ 3 & 1 & 2 \end{pmatrix} \underset{L_3 - 3L_1}{\sim}$$

$$\begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{4} & 1 \\ 0 & -2 & 2 \end{pmatrix} \underset{L_3 + \frac{L_2}{2}}{\sim} \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{4} & 1 \\ 0 & 0 & \boxed{\frac{5}{2}} \end{pmatrix} \sim$$

forma eșalon

$$\sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix} \underset{L_2 - \frac{L_3}{4}}{\sim} \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

forma eșalon redusă

$\operatorname{rg} A = 3$

II) b) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

A^{-1} ? Gauss-Jordan.

$(A \mid I_3) \sim (C \mid B) = (I_3 \mid A^{-1})$

$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1 + 1 + 1 - 1 = 2 \neq 0$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{L_2-L_1}{\sim}$$

$$\left(\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & \boxed{-2} & 0 & -1 & 1 & 0 \\ 0 & 0 & \boxed{-1} & -1 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & +\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 & -1 \end{array}\right) \underset{\sim}{L_1 - L_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \underset{\sim}{L_1 - L_2}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

$$\underbrace{}_{I_3} \qquad \underbrace{}_{A^{-1}}$$

2) $\begin{cases} 3x_1 + x_2 - 3x_3 = 4 \\ x_1 + 3x_2 - 2x_3 = -5 \\ 2x_1 + 2x_2 + 5x_3 = 7 \end{cases}$

$$\bar{A} = \left(\begin{array}{ccc|c} 3 & 1 & -3 & 4 \\ 1 & 3 & -2 & -5 \\ 2 & 2 & 5 & 7 \end{array}\right) \underset{L_1 \leftrightarrow L_2}{\sim}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & -5 \\ 3 & 1 & -3 & 4 \\ 2 & 2 & 5 & 7 \end{array}\right) \underset{\substack{L_2 - 3L_1 \\ L_3 - 2L_1}}{\sim}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & -5 \\ 0 & -8 & 3 & 19 \\ 0 & -4 & 9 & 17 \end{array}\right) \underset{\substack{L_2/-8 \\ L_3 + 4L_2}}{\sim} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -5 \\ 0 & 1 & -\frac{3}{8} & -\frac{19}{8} \\ 0 & 0 & \frac{15}{2} & \frac{15}{2} \end{array}\right)$$

$$\underset{L_3 / \frac{15}{2}}{\sim} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -5 \\ 0 & 1 & -\frac{3}{8} & -\frac{19}{8} \\ 0 & 0 & 1 & 1 \end{array}\right) \underset{\substack{L_2 + \frac{3}{8} \cdot L_3 \\ L_1 + 2 \cdot L_3}}{\sim}$$

$$= \left(\begin{array}{ccc|c} 1 & 3 & 0 & -5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \underset{L_1-3L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\begin{cases} x = 3 \\ y = -2 \\ z = 1 \end{cases}$$

ex suplimentar

$$\begin{cases} 3x_1 + 2x_2 + 5x_3 + 4x_4 = -1 \\ 2x_1 + x_2 + 3x_3 + 3x_4 = 0 \\ x_1 + 2x_2 + 3x_3 = -3 \end{cases}$$

Alg. Gauss Jordan

$$\overline{A} = \left(\begin{array}{cccc|c} 3 & 2 & 5 & 4 & -1 \\ 2 & 1 & 3 & 3 & 0 \\ 1 & 2 & 3 & 0 & -3 \end{array}\right) \underset{L_3 \leftrightarrow L_1}{\sim}$$

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 0 & -3 \\ 2 & 1 & 3 & 3 & 0 \\ 3 & 2 & 5 & 4 & -1 \end{array}\right) \underset{\substack{L_2-2L_1 \\ L_3-3L_1}}{\sim}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -3 \\ 0 & -3 & -3 & 3 & 6 \\ 0 & -4 & -4 & 4 & 8 \end{array}\right) \underset{\substack{L_2/-3 \\ L_3/-4}}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -3 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -1 & -2 \end{array}\right) \underset{L_3 - L_2}{\sim}$$

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 0 & -3 \\ 0 & \boxed{1} & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \underset{L_1 - 2L_2}{\sim} \left(\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 2 & 1 \\ 0 & \boxed{1} & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow x_3 = \alpha$

$x_4 = \beta$ $\Rightarrow x_1 = 1 - \alpha - 2\beta$

$x_2 = -2 - \alpha + \beta$

$\alpha, \beta \in \mathbb{R}$.

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