Optimal trajectories

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Outline

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The idea of dynamical model

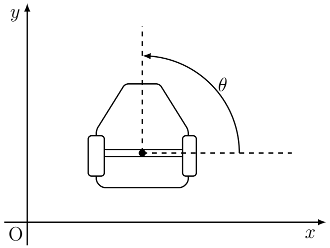
- We need to characterize the evolution of a system; we define

$$\dot{x}(t) = f(\underset{\text{states}}{x(t)},\quad \underset{\text{inputs}}{u(t)},\quad \underset{\text{parameters / disturbances}}{p})$$

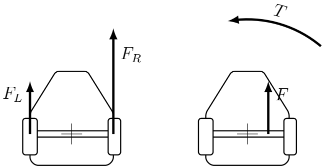
- we may define many types of systems:
  - the differential drive
  - the “simple” car (with many variations)
  - complex actuators and/or more complex environments (aerial, under water)

The unicycle – mathematical model

- The simplest mathematical model
- The two wheels function independently
- The movement is defined by position coordinates (x, y) in the horizontal plane and orientation with respect to the horizontal axis (θ)



$$\begin{cases} \dot{x} &= \frac{r}{2}(u_l + u_r)\cos\theta \\ \dot{y} &= \frac{r}{2}(u_l + u_r)\sin\theta \\ \dot{\theta} &= \frac{r}{L}(u_r - u_l) \end{cases} \Rightarrow \begin{cases} \dot{x} &= u_V\cos\theta \\ \dot{y} &= u_V\sin\theta \\ \dot{\theta} &= u_\theta \end{cases}$$



The car is controlled by inputs $u_V = r(u_l + u_r)/2$ and $u_\theta = r(u_r - u_l)/L$.

The simple car – mathematical model

- The "differential drive" mathematical model is not sufficiently realistic
- Closer to reality: "the simple car" which cannot sideslip[a]
- Changing orientation (θ) is done by changing the wheel angle through command action u_θ
- Changing the position (x, y) depends on orientation (θ) and commanded velocity (u_V)

$$\dot{x} = u_V \cos\theta$$
$$\dot{y} = u_V \sin\theta$$
$$\dot{\theta} = \frac{u_V}{L} \tan u_\theta$$

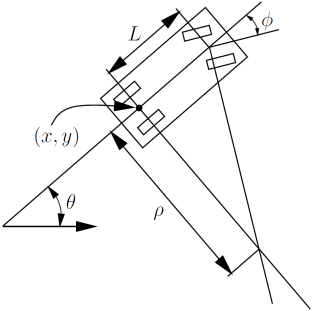


Fig. 13.1 from LaValle 2006

[a] S. LaValle (2006). *Planning Algorithms / Motion Planning*. Cambridge university press.

The simple car – limitations

- The car cannot turn “in place”; it has a maximal turn angle $\phi_{\max} < \pi/2$, or, equivalently stated, a minimum turn radius $\rho_{\min} = L/\tan\phi_{\max}$
- In fact, the turning radius should be a function of the current velocity!
- The velocity is constrained as well: $u_{min} \leq u_v \leq u_{max}$.
- the model is nonholonomic because there appear differential constraints which cannot be integrated away, in our case:

$$-\dot{x}\sin\phi + \dot{y}\cos\phi = 0.$$

  Equivalently stated, it is not possible to reformulate these constraints into a new form, where there are no derivatives.

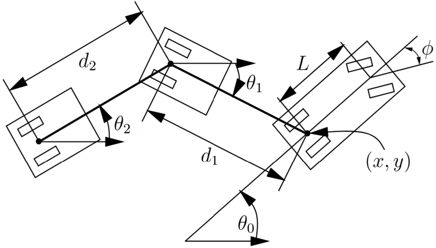
The simple car – variations

Depending on the nature of U which bounds $[u_v, u_\theta]$, we define:

- **tricycle**: $U = [-1, 1] \times [-\pi/2, \pi/2]$; it can (unrealistically) rotate in place if $u_\theta = \pm\pi/2$;
- **simple car**: $U = [-1, 1] \times (-\theta_{max}, \theta_{max})$; it has a minimum turning radius $\rho_{min} = L \tan /\theta_{max}$
- **Reeds-Shepp car**: $U = \{-1, 0, 1\} \times (-\theta_{max}, \theta_{max})$; it has the modes reverse, park or forward;
- **Dubins car**: $U = \{0, 1\} \times (-\theta_{max}, \theta_{max})$; it has the modes park or forward (NOT reverse).

A car pulling k trailers

$$
\begin{aligned}
\dot{x} &= u_V \cos\theta_0 \\
\dot{y} &= u_V \sin\theta_0 \\
\dot{\theta}_0 &= \frac{u_V}{L} \tan u_\theta \\
\dot{\theta}_1 &= \frac{u_V}{d_1} \sin(\theta_0 - \theta_1) \\
&\vdots \\
\dot{\theta}_k &= \frac{u_V}{d_k} \left(\prod_{j=1}^{k-1} \cos(\theta_{j-1} - \theta_j) \right) \sin(\theta_{k-1} - \theta_k)
\end{aligned}
$$



Extending a model by adding integrators

- Instantaneous actions are not realistic $\Rightarrow$ smooth them by adding integrators
- An arbitrary number of integrators can be added.

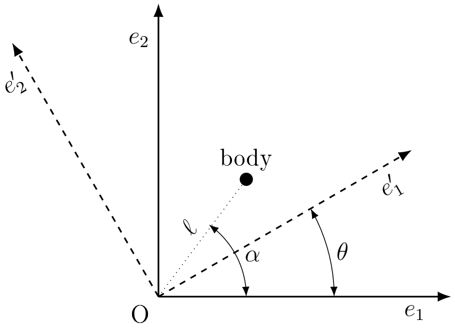
A better unicycle model

$$\dot{x} = v\cos\theta, \qquad \dot{v} = u_V$$
$$\dot{y} = v\sin\theta, \qquad \dot{\omega} = u_\omega$$
$$\dot{\theta} = \omega$$

A continuous-steering car

$$\dot{x} = \cos\theta, \qquad \dot{\phi} = \omega$$
$$\dot{y} = \sin\theta, \qquad \dot{\omega} = u_\omega$$
$$\dot{\theta} = \frac{\tan\phi}{L}$$

Intermezzo: Rotations in 2D



- Rotating is equivalent with multiplying with a matrix of form:

$$R(\theta) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

- The determinant is $det(R) = 1 \Rightarrow$ the multiplication does not change the moby's volume.
- In addition, the two column vectors of R are orthogonal:

$$\cos\phi \times (-\sin\phi) + \sin\phi \times \cos\phi = 0$$

Outline

Lagrangian mechanics

- Considering a cost denoted as $L(q, \dot{q}, t)$, $\tilde{q}$ denotes the trajectory which minimizes

$$\Phi(\tilde{q}) = \int_T L(q(t), \dot{q}(t), t) dt$$

  along a time interval; for example we may penalize the path length: $L(q, \dot{q}, t) = \sqrt{1 + \dot{q}^2}$.

- The standard Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

- Applying actions means that we add "generalized forces" in the right side

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = u.$$

- Gathering into $x = (q, \dot{q})$ allows to define an implicit (or, hopefully even explicit) dynamical equation.
- The general form ("torque-control" formulation) is

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u$$

Two-link manipulator example

- kinetic energy components

$$K_1(\dot q) = \frac{1}{2} m_1 \ell_1 \dot\theta_1^2 + \frac{1}{2} I_1 \dot\theta_1^2, \quad K_2(\dot q) = \frac{1}{2} p^\top p + \frac{1}{2} I_1 \left(\dot\theta_1 + \dot\theta_2\right)^2$$

where $p_1 = d_1 \cos\theta_1 + \ell_2 \cos\theta_2$, $p_2 = d_1 \sin\theta_1 + \ell_2 \sin\theta_2$.

- potential energy components

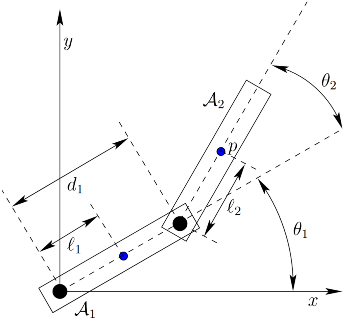
$$V_1(q) = m_1 g \ell_1 \sin\theta_1, \qquad V_2(q) = m_2 g (d_1 \sin\theta_1 + \ell_2 \sin\theta_2)$$

- the system's Lagrangian is

$$L(q, \dot q) = K_1(\dot\theta_1) + K_2(\dot\theta_1, \dot\theta_2) - V_1(\theta_1) - V_2(\theta_1, \theta_2)$$

- the result is (with some additional notation):

$$m_{11}\ddot\theta_1 + m_{12}\ddot\theta_2 - 2r\dot\theta_1\dot\theta_2 - r\dot\theta_2^2 + g_1(q) = u_1$$
$$m_{22}\ddot\theta_1 + m_{21}\ddot\theta_2 + r\dot\theta_1^2 = g_2(q) = u_2$$



- configuration $q = (\theta_1, \theta_2)$
- get the dynamical equation associated!

Hamiltonian mechanics

- define p, a generalized momentum vector, with the i-th component given by $p_i = \partial L / \partial \dot{q}_i$
- the Hamiltonian function (the total energy of a conservative system) is defined as

$$H(p, q) = p^\top \dot{q} - L(q, \dot{q}) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q, \dot{q})$$

- we get the dynamic equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = \frac{\partial H}{\partial q_i}$$

- harder to use to construct dynamics but useful for the Pontryagin minimum principle!

Pontryagin's minimum principle

- Provides necessary conditions for an optimal trajectory (the Hamilton-Jacobi-Bellman equation provides sufficient conditions)

- Define the adjoint variables and the Hamiltonian function

$$\lambda_i = \frac{\partial G^\star}{\partial x_i}, \qquad H(x, u, \lambda) = \ell(x, u) + \sum_{i=1}^{n} \lambda_i f_i(x, u)$$

  when we care only about minimizing the time, $\ell(x, u) = 0$.

- The optimal control action is the result of

$$u^\star(t) = \arg \min_{u \in U(x)} \{H(x(t), u(t), \lambda(t))\}$$

  and, furthermore, the adjoint dynamics are given by

$$\dot{\lambda} = -\frac{\partial G^\star}{\partial x_i} = g(x, \lambda, u^\star)$$

Optimal planning for the double integrator

- Consider $\ddot{q} = u$ with $u \in \mathcal{U} = [-1, 1]$;
- The goal is to go from some initial state x_i to a final state x_f in minimum time, while respecting the constraints.
- The dynamics and Hamiltonian are

$$\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & u \end{cases}, \qquad H(x, u, \lambda) = 1 + \lambda_1 x_2 + \lambda_2 u$$

- Applying Pontryagin's principle, we have that

$$u^\star(t) = \arg \min_{u \in [-1,1]} \{1 + \lambda_1(t) x_2(t) + \lambda_2(t) u(t)\} \qquad \Rightarrow \qquad u^\star(t) = \begin{cases} -\mathrm{sign}(\lambda_2(t)), & \forall \lambda_2(t) \neq 0 \\ [-1, 1], & \lambda_2(t) = 0 \end{cases}$$

- Solving the adjoint equations, gives

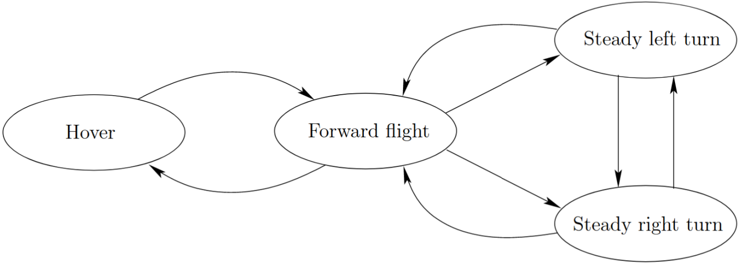
$$\begin{cases} \dot{\lambda}_1(t) & = 0 \\ \dot{\lambda}_2(t) & = -\lambda_1(t) \end{cases} \qquad \Rightarrow \qquad \begin{cases} \lambda_1(t) & = c_1, \\ \lambda_2(t) & = c_2 - c_1 t \end{cases}$$

with c_1, c_2 depending on the choices of the boundary conditions x_i, x_f

Outline

Why spend the effort in solving optimal control problems?

- The effort is carried offline, during runtime, it is just a manner of tracking the trajectory (a simple feedback mechanism)
- It allows to do implement a local planning method (LPM) where the trajectory is decomposed in predefined and parametrizable motion primitives
- at each new transition, from the available list, a feasible motion primitive is selected



The minimum-length trajectory for the Dubins car

- We consider a simplified Dubins car model:

$$\begin{cases} \dot{x} = & \cos\theta \\ \dot{y} = & \sin\theta \\ \dot{\theta} = & u \end{cases}$$

- In the simplest case, we are interested in planning a trajectory between two arbitrary configurations (configuration = a combination of position and vehicle orientation)

$$(x(t_i), y(t_i), \theta(t_i)) \quad \longrightarrow \quad (x(t_f), y(t_f), \theta(t_f))$$

- The cost to optimize is

$$L(\tilde{q}, \tilde{u}) = \int_{t_i}^{t_f} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt \qquad \Leftarrow \qquad \text{path length}$$

for $u \in U = [-\tan\phi_{max}, \tan\phi_{max}]$.

The minimum-length trajectory for the Dubins car (II)

- It can be shown that any trajectory of minimum length follows a combination of steps of form:
  - rotation at max turn angle in one direction (clockwise or counter-clockwise)
  - movement in a straight line
  - rotation at max turn angle again (clockwise or counter-clockwise)
- For any pair of configurations there are only 6 optimal possibilities to plan a Dubins trajectory:

$$\{LRL, RLR, LSL, LSR, RSL, RSR\}$$

  where, $L -$ left, $R -$ right și $S -$ straight.

- If we let C to cover both L/R, we have only two base words

$$\{CCC, CSC\}$$

Dubins trajectories – illustrations

- To each component we associated a scalar (arc angle or segment length):

$$\{L_\alpha R_\beta L_\gamma, R_\alpha L_\beta R_\gamma, L_\alpha S_d L_\gamma, L_\alpha S_d R_\gamma, R_\alpha S_d L_\gamma, R_\alpha S_d R_\gamma\},$$

where $\alpha, \gamma \in [0, 2\pi)$, $\beta \in (\pi, 2\pi)$ și $d \geq 0$.

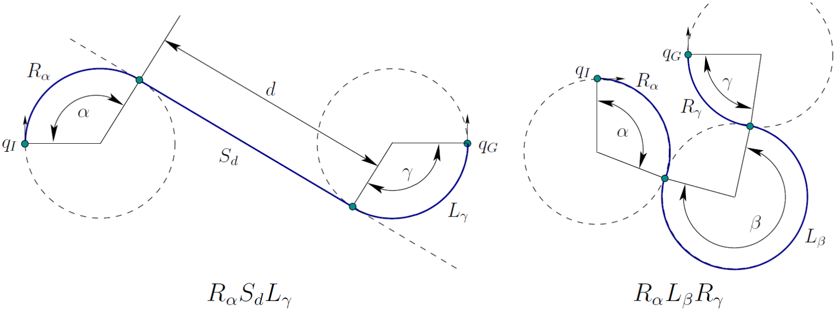


Fig. 15.4 from LaValle 2006

- Out of all these possibilities we have to select the one which is the shortest.

Calculating the optimal trajectory

We still need to calculate the trajectory:

- which of the 6 words is optimal (corresponds to the shortest path)?
- what are then, the values α, β, γ, d which characterize a particular word?

Idea[1]:

- do coordinate changes to simplify the calculations;
- for a particular word, the segment and/or arc length are the unique solution of a system of three equations
- provide conditions involving initial and final orientation for which a given word is the shortest

[1] A. M. Shkel and V. Lumelsky (2001). “Classification of the Dubins set”. In: *Robotics and Autonomous Systems* 34.4. Publisher: Elsevier, pp. 179–202.

Moving from one configuration to the next

- transformation after each operation:

$$L_\nu(x, y, \phi) = (x + \sin(\phi + \nu) - \sin\phi, y - \cos(\phi + \nu) + \cos\phi, \phi + \nu)$$
$$R_\nu(x, y, \phi) = (x - \sin(\phi - \nu) + \sin\phi, y + \cos(\phi - \nu) - \cos\phi, \phi - \nu)$$
$$S_\nu(x, y, \phi) = (x + \nu\cos\phi, y + \nu\sin\phi, \phi)$$

- after the change of coordinates, we have

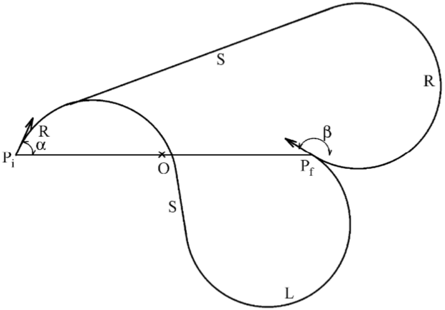
$$(0, 0, \alpha) \quad \rightarrow \quad (d, 0, \beta)$$

- the length of the path is given as $t + p + q$, where (taking LRL as example)

$$L_q(R_p(L_t(0, 0, \alpha))) = (d, 0, \beta)$$

- consider only the "long path case":

$$d > \sqrt{4 - (|\cos\alpha + \cos\beta|)^2 + |\sin\alpha| + |\sin\beta|}$$



Solution for the LSL case

A system of three scalar equations:

$$
\begin{aligned}
p\cos(\alpha+t)-\sin\alpha+\sin\beta &= d\\
p\sin(\alpha+t)+\cos\alpha-\cos\beta &= 0\\
\alpha+t+q &= \beta[mod\,2\pi]
\end{aligned}
$$

whose solution is given as:

$$
\begin{aligned}
t &= -\alpha+\arctan\frac{\cos\beta-\cos\alpha}{d+\sin\alpha-\sin\beta}[mod\,2\pi]\\
p &= \sqrt{2+d^2-2\cos(\alpha-\beta)+2d(\sin\alpha-\sin\beta)}\\
q &= \beta-\arctan\frac{\cos\beta-\cos\alpha}{d+\sin\alpha-\sin\beta}[mod\,2\pi]
\end{aligned}
$$

- Then, the length of the path is

$$\mathcal{L}_{\mathsf{LSL}} = -\alpha+\beta+p_{\mathsf{LSL}}$$

- The existence condition is that $p \geq 0$.
- There are 4×4 cases, for each combination of quadrant in which the pair α, β may stay
- There are also symmetries which characterize trajectories

$$C_1SC_2[\alpha,\beta](t,p,q) \approx \overline{C}_1\overline{S}C_2[-\alpha,-\beta](t,p,q) \approx \overline{C}_2\overline{SC}_1[\beta,\alpha](q,p,t) \approx C_2SC_1[-\beta,-\alpha](q,p,t)$$

Decision table for finding the shortest path

	1	2	3	4
1	RSL	if $S_{12} < 0$ then RSR if $S_{12} > 0$ then RSL	if $S_{13} < 0$ then RSR if $S_{13} > 0$ then LSR	if $S^1_{14} > 0$ then LSR if $S^2_{14} > 0$ then RSL, else RSR
2	if $S_{21} < 0$ then LSL if $S_{21} > 0$ then RSL	if $S^1_{22} < 0$ then LSL if $S^1_{22} > 0$ then RSL if $S^2_{22} < 0$ then RSR if $S^2_{22} > 0$ then RSL	RSR	if $S_{24} < 0$ then RSR if $S_{24} > 0$ then RSL
3	if $S_{31} < 0$ then LSL if $S_{31} > 0$ then LSR	LSL	if $S^1_{33} < 0$ then RSR if $S^1_{33} > 0$ then LSR if $S^2_{33} < 0$ then LSL if $S^2_{33} > 0$ then LSR	if $S_{34} < 0$ then RSR if $S_{34} > 0$ then LSR
4	if $S^1_{41} > 0$ then RSL if $S^2_{41} > 0$ then LSR, else LSL	if $S_{42} < 0$ then LSL if $S_{42} > 0$ then RSL	if $S_{43} < 0$ then LSL if $S_{43} > 0$ then LSR	LSR

Passing through waypoints

The multi-point Markov-Dubins problem:

- consider initial, intermediary and final waypoints

$$WP_i = (x_i, y_i, \phi_i)$$

- define the full configuration for WP_0 (initial) and WP_N (final) waypoints, but only the position components for WP_i, $\forall 1 \leq i \leq N-1$

- check for each type of primitive between consecutive waypoints
- find best intermediary orientations to reduce the total path length
- at each segment we need to change again the coordinates

Surprisingly difficult to solve! (dynamic programming, non-linear mixed-integer programming, etc.)

Outline

Reeds-Shepp trajectories

- For the Reeds-Shepp car, in contrast to the Dubins car, we allow backwards movement. The simplified model is:

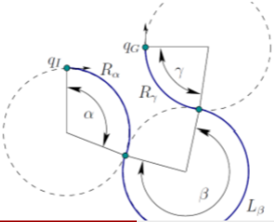
$$\begin{cases} \dot{x} &= u_1 \cos\theta \\ \dot{y} &= u_1 \sin\theta \\ \dot{\theta} &= u_1 u_2 \end{cases}, \text{ where } u_1 \in \{-1, 1\},\ u_2 \in [-\tan\phi_{\max}, \tan\phi_{\max}].$$

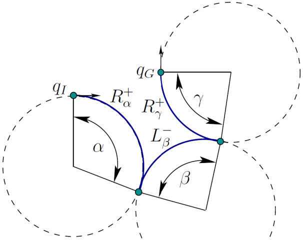
- We now have 48 possible primitives which are composed from up to 5 distinct movements ("|" denotes a change in gear, e.g., from reverse to forward):

$$\{C|C|C,\ CC|C,\ C|CC,\ CSC,\ CC_\beta|C_\beta C,\ C|C_\beta C_\beta|C,\ C|C_{\pi/2}SC,\ CSC_{\pi/2}|C,\ C|C_{\pi/2}SC_{\pi/2}|C\}$$

  $\pi/2$ and β appear to denote some particular linking with the previous primitive.

- An illustrated trajectory (compared with the 'only forward' case of the Dubins car):





Variations for the differential drive

- The problem of achieving a configuration in a minimum time can be posed for various models and with constraints imposed on them

- For the differential robot (whose two wheels that can move independently), if we penalize the effort of rotation in cost

$$L(\tilde{q}, \tilde{u}) = \int_{t_i}^{t_f} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} + |\dot{\theta}(t)| dt,$$

  we obtain the Balkcom-Mason trajectories, composed of at most 5 components that take values from a set of 4 distinct movements $\Rightarrow$ 40 distinct trajectories

- another idea is to penalize the total amount of wheel rotation:

Symbol	Left wheel	Right wheel
forward	1	1
reverse	-1	-1
turn left	-1	1
turn right	1	-1

$$\{\curvearrowright,\ \Downarrow,\ \Downarrow\curvearrowright,\ \curvearrowright\Downarrow\curvearrowright,\ \Uparrow\curvearrowleft_{\pi}\Downarrow,\ \curvearrowleft\Downarrow\curvearrowright,\ \Downarrow\curvearrowright\curvearrowright,\ \curvearrowleft\Downarrow\curvearrowright\Uparrow,\ \Uparrow\curvearrowleft\Downarrow\curvearrowright\Uparrow\}$$