Seminar 9

1. a) Studiați continuitatea funcției f;

b) Determinați $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$;

c) Studiați diferențiabilitatea funcției f,

unde:

i) $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y)=\begin{cases} \frac{xy}{x^2+y^2}; & (x,y)\neq(0,0) \\ 0; & (x,y)=(0,0). \end{cases}$

Soluție. a) Vezi Seminar 5.

b) Fie $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$.

$$\frac{\partial f}{\partial x}(x,y)=\frac{(xy)'_x\,(x^2+y^2)-xy\,(x^2+y^2)'_x}{(x^2+y^2)^2}=$$

$$=\frac{y(x^2+y^2)-xy\cdot 2x}{(x^2+y^2)^2}.$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{(xy)'_y\,(x^2+y^2)-xy\,(x^2+y^2)'_y}{(x^2+y^2)^2}=$$

$$=\frac{x(x^2+y^2)-xy\cdot 2y}{(x^2+y^2)^2}.$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_1)-f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f((0,0)+t(1,0))-f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0.$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_2)-f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f((0,0)+t(0,1))-f(0,0)}{t} = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0.$$

c) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ continue pe $\mathbb{R}^2 \setminus \{(0,0)\}$
$\mathbb{R}^2 \setminus \{(0,0)\}$ deschisă
$\Bigg| \Rightarrow f$ diferențiabilă pe $\mathbb{R}^2 \setminus \{(0,0)\}$.

Studiem diferențiabilitatea lui f în $(0,0)$.

f nu e continuă în $(0,0)$ (vezi Seminar 5) $\Rightarrow$

$\Rightarrow f$ nu e diferențiabilă în $(0,0)$. $\square$

ii) $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y)=\begin{cases} \dfrac{x^5 \cdot y^2}{x^8+y^4} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0). \end{cases}$

__Soluție.__ a) f continuă pe $\mathbb{R}^2\setminus\{(0,0)\}$ (operații cu funcții elementare).

Studiem continuitatea lui f în $(0,0)$.

__Varianta 1__

Fie $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$.

$$|f(x,y)-f(0,0)| = \left|\frac{x^5y^2}{x^8+y^4}-0\right| = \frac{|x^5y^2|}{x^8+y^4} =$$

$$=|x|\,\frac{x^4y^2}{x^8+y^4}.$$

$$\frac{x^8+y^4}{2} \underset{\substack{\uparrow \\ \text{inegalitatea mediilor}}}{\geq} \sqrt{x^8y^4} = |x^4y^2| = x^4y^2 \Rightarrow \frac{x^4y^2}{x^8+y^4}\leq\frac{1}{2}.$$

$$|f(x,y)-f(0,0)| = |x|\underbrace{\frac{x^4y^2}{x^8+y^4}}_{\leq\frac{1}{2}} \leq \frac{1}{2}|x| \xrightarrow[(x,y)\to(0,0)]{} 0.$$

Deci f este continuă în $(0,0)$.

<u>Varianta 2</u>

Fie $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$.

$$|f(x,y)-f(0,0)| = \left|\frac{x^5y^2}{x^8+y^4}-0\right| = \frac{|x|^5\,|y|^2}{x^8+y^4} =$$

$$=\left(\frac{|x|^8}{x^8+y^4}\right)^{\frac{5}{8}}\cdot\left(\frac{|y|^4}{x^8+y^4}\right)^{\frac{2}{4}}\cdot\left(x^8+y^4\right)^{\frac{5}{8}+\frac{2}{4}-1} =$$

$$=\underbrace{\left(\frac{x^8}{x^8+y^4}\right)^{\frac{5}{8}}}_{\leq 1}\cdot\underbrace{\left(\frac{y^4}{x^8+y^4}\right)^{\frac{2}{4}}}_{\leq 1}\cdot\left(x^8+y^4\right)^{\frac{5+4-8}{8}} \leq$$

$$\leq\left(x^8+y^4\right)^{\frac{1}{8}}\xrightarrow[(x,y)\to(0,0)]{} 0.$$

Deci f este continuă în $(0,0)$.

b) Fie $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$.

$$\frac{\partial f}{\partial x}(x,y)=\frac{(x^5y^2)'_x\,(x^8+y^4)-x^5y^2(x^8+y^4)'_x}{(x^8+y^4)^2}=$$

$$=\frac{5x^4y^2(x^8+y^4)-x^5y^2\cdot 8x^7}{(x^8+y^4)^2}.$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{(x^5y^2)'_y\,(x^8+y^4)-x^5y^2(x^8+y^4)'_y}{(x^8+y^4)^2}=$$

$$=\frac{2x^5y\,(x^8+y^4)-x^5y^2\cdot 4y^3}{(x^8+y^4)^2}.$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{t\to 0}\frac{f((0,0)+te_1)-f(0,0)}{t}=$$

$$=\lim_{t\to 0}\frac{f((0,0)+t(1,0))-f(0,0)}{t}=\lim_{t\to 0}\frac{f((0,0)+(t,0))-f(0,0)}{t}=$$

$$=\lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t}=\lim_{t\to 0}\frac{0-0}{t}=0.$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{t\to 0}\frac{f((0,0)+te_2)-f(0,0)}{t}=$$

$$= \lim_{t\to 0} \frac{f((0,0)+t(0,1))-f(0,0)}{t} = \lim_{t\to 0} \frac{f((0,0)+(0,t))-f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0.$$

c) $\left.\begin{array}{l} \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ continue pe } \mathbb{R}^2\setminus\{(0,0)\} \\ \mathbb{R}^2\setminus\{(0,0)\} \text{ deschisă} \end{array}\right| \Rightarrow f$ este diferențiabilă pe $\mathbb{R}^2\setminus\{(0,0)\}$.

Studiem diferențiabilitatea lui f în $(0,0)$.

Dacă f ar fi diferențiabilă în $(0,0)$ atunci $\underbrace{df(0,0)}_{T} : \mathbb{R}^2 \to \mathbb{R}$, $df(0,0)(u,v) = \frac{\partial f}{\partial x}(0,0)\cdot u + \frac{\partial f}{\partial y}(0,0)\cdot v = 0\cdot u + 0\cdot v = 0.$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-df(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{x^5y^2}{x^8+y^4} - 0 - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^5y^2}{(x^8+y^4)\sqrt{x^2+y^2}}$$

Fie $(x_n, y_n) = \left(\frac{1}{\sqrt{n}}, \frac{1}{n}\right) \forall n \in \mathbb{N}^*$. Avem

$\lim\limits_{n\to\infty} (x_n, y_n) = (0,0)$ și $\lim\limits_{n\to\infty} \frac{x_n^5 y_n^2}{(x_n^8+y_n^4)\sqrt{x_n^2+y_n^2}} =$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^2\sqrt{n}} \cdot \frac{1}{n^2}}{\left(\frac{1}{n^4}+\frac{1}{n^4}\right)\sqrt{\frac{1}{n}+\frac{1}{n^2}}} =$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^4\sqrt{n}}}{\frac{2}{n^4}\cdot\frac{\sqrt{n+1}}{n}} = \lim_{n\to\infty} \frac{1}{\cancel{n^4}\sqrt{n}} \cdot \frac{\cancel{n^4}\cdot n}{2\sqrt{n+1}} =$$

$$= \lim_{n\to\infty} \frac{n}{2\sqrt{n}\sqrt{n+1}} = \frac{1}{2} \neq 0.$$

Deci $\lim\limits_{(x,y)\to(0,0)} \frac{x^5y^2}{(x^8+y^4)\sqrt{x^2+y^2}} \neq 0$, i.e. f nu

e diferențiabilă în $(0,0)$. $\square$

iii) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{y^3}{x^4+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0). \end{cases}$

<u>Soluție.</u> a) f continuă pe $\mathbb{R}^2 \setminus \{(0,0)\}$ (operații cu funcții elementare)

Studiem continuitatea lui f în $(0,0)$.

Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

$$|f(x,y) - f(0,0)| = \left| \frac{y^3}{x^4+y^2} - 0 \right| = \frac{|y|^3}{x^4+y^2} =$$

$$= |y| \underbrace{\frac{y^2}{x^4+y^2}}_{\leq 1} \leq |y| \xrightarrow[(x,y)\to(0,0)]{} 0 \Rightarrow$$

$\Rightarrow f$ e continuă în $(0,0)$.

b) Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

$$\frac{\partial f}{\partial x}(x,y) = \frac{(y^3)'_x (x^4+y^2) - y^3 (x^4+y^2)'_x}{(x^4+y^2)^2} =$$

$$= \frac{0(x^4+y^2) - y^3 \cdot 4x^3}{(x^4+y^2)^2}.$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{(y^3)'_y (x^4+y^2) - y^3 (x^4+y^2)'_y}{(x^4+y^2)^2} =$$

$$= \frac{3y^2(x^4+y^2) - y^3 \cdot 2y}{(x^4+y^2)^2}.$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_1) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f((0,0)+(t,0)) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0.$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_2) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f((0,0)+t(0,1)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f((0,0)+(0,t)) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^3}{0^4+t^2} - 0}{t} =$$

$$= \lim_{t\to 0} \frac{\cancel{t}^{\,1}}{\cancel{t}_{\,1}} = 1.$$

c) $\left.\begin{array}{l}\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ continue pe } \mathbb{R}^2\setminus\{(0,0)\} \\ \mathbb{R}^2\setminus\{(0,0)\} \text{ deschisă}\end{array}\right| \Rightarrow f$ dife-

rențiabilă pe $\mathbb{R}^2\setminus\{(0,0)\}$.

Studiem diferențiabilitatea lui f în $(0,0)$.

Dacă f ar fi diferențiabilă în $(0,0)$, atunci

$df(0,0) : \mathbb{R}^2 \to \mathbb{R}$, $df(0,0)(u,v) = \frac{\partial f}{\partial x}(0,0)\cdot u +$

$+ \frac{\partial f}{\partial y}(0,0)\cdot v = 0\cdot u + 1\cdot v = v.$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-df(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{y^3}{x^4+y^2} - 0 - \frac{x^4+y^2}{\cancel{x^4+y^2}}\,y}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\cancel{y^3} - x^4y - \cancel{y^3}}{(x^4+y^2)\sqrt{x^2+y^2}} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{-x^4 y}{(x^4+y^2)\sqrt{x^2+y^2}}.$$

Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

$$\left| -\frac{x^4 y}{(x^4+y^2)\sqrt{x^2+y^2}} - 0 \right| = \frac{x^4|y|}{(x^4+y^2)\sqrt{x^2+y^2}} =$$

$$= \frac{x^2|y|}{x^4+y^2} \cdot \frac{x^2}{\sqrt{x^2+y^2}} \leq \frac{1}{2} \frac{x^2}{\sqrt{x^2+y^2}} =$$

$\leq \frac{1}{2}$ (Explicație: $\frac{x^4+y^2}{2} \geq \sqrt{x^4y^2} = x^2|y|$)

inegalitatea mediilor

$$= \frac{1}{2} \cdot |x| \cdot \frac{|x|}{\sqrt{x^2+y^2}} \leq \frac{1}{2}|x| \xrightarrow[(x,y)\to(0,0)]{} 0.$$

≤ 1 (Explicație: $\sqrt{x^2+y^2} \geq \sqrt{x^2} = |x|$)

Deci $\lim_{(x,y)\to(0,0)} \frac{-x^4 y}{(x^4+y^2)\sqrt{x^2+y^2}} = 0$, i.e. f e diferențiabilă în $(0,0)$. $\square$

iv) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=\begin{cases}\dfrac{x^7y^8}{\sqrt{x^{28}+y^{28}}} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0).\end{cases}$

<u>Soluție</u>. a) f continuă pe $\mathbb{R}^2\setminus\{(0,0)\}$ (operații cu funcții elementare)

Studiem continuitatea lui f în $(0,0)$.

Fie $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$.

$$|f(x,y)-f(0,0)|=\left|\frac{x^7y^8}{\sqrt{x^{28}+y^{28}}}-0\right| = |y|\frac{|x^7y^7|}{\sqrt{x^{28}+y^{28}}}.$$

$$\frac{x^{28}+y^{28}}{2}\underset{\uparrow}{\geq}\sqrt{x^{28}y^{28}}=|x^{14}y^{14}|=x^{14}y^{14}\Rightarrow$$

inegalitatea mediilor

$$\Rightarrow \frac{\sqrt{x^{28}+y^{28}}}{\sqrt{2}}\geq\sqrt{x^{14}y^{14}}=|x^7y^7|\Rightarrow$$

$$\Rightarrow \frac{|x^7y^7|}{\sqrt{x^{28}+y^{28}}}\leq\frac{1}{\sqrt{2}}.$$

$$|f(x,y)-f(0,0)|=|y|\frac{|x^7y^7|}{\sqrt{x^{28}+y^{28}}}\leq\frac{1}{\sqrt{2}}|y|\xrightarrow[(x,y)\to(0,0)]{}0\Rightarrow$$

$\Rightarrow f$ e continuă în $(0,0)$.

b) Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

$$\frac{\partial f}{\partial x}(x,y) = \frac{(x^7 y^8)'_x \sqrt{x^{28}+y^{28}} - (x^7 y^8)\left(\sqrt{x^{28}+y^{28}}\right)'_x}{x^{28}+y^{28}} =$$

$$= \frac{7x^6 y^8 \sqrt{x^{28}+y^{28}} - x^7 y^8 \cdot \frac{1}{2\sqrt{x^{28}+y^{28}}} \cdot 28x^{27}}{x^{28}+y^{28}}.$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{(x^7 y^8)'_y \sqrt{x^{28}+y^{28}} - (x^7 y^8)\left(\sqrt{x^{28}+y^{28}}\right)'_y}{x^{28}+y^{28}} =$$

$$= \frac{8x^7 y^7 \sqrt{x^{28}+y^{28}} - (x^7 y^8) \cdot \frac{1}{2\sqrt{x^{28}+y^{28}}} \cdot 28y^{27}}{x^{28}+y^{28}}.$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t \to 0} \frac{f((0,0)+te_1) - f(0,0)}{t} =$$

$$= \lim_{t \to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t \to 0} \frac{f((0,0)+(t,0)) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0.$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_2)-f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f((0,0)+t(0,1))-f(0,0)}{t} = \lim_{t\to 0} \frac{f((0,0)+(0,t))-f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0.$$

c) $\left.\begin{array}{l} \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ continue pe } \mathbb{R}^2 \setminus \{(0,0)\} \\ \mathbb{R}^2 \setminus \{(0,0)\} \text{ deschisă} \end{array}\right| \Rightarrow f$ dife-

rențiabilă pe $\mathbb{R}^2 \setminus \{(0,0)\}$.

Studiem diferențiabilitatea lui f în $(0,0)$.

Dacă f ar fi diferențiabilă în $(0,0)$, atunci

$df(0,0): \mathbb{R}^2 \to \mathbb{R}$, $df(0,0)(u,v) = \frac{\partial f}{\partial x}(0,0) \cdot u +$

$+ \frac{\partial f}{\partial y}(0,0) \cdot v = 0 \cdot u + 0 \cdot v = 0.$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-df(0,0)\left((x,y)-(0,0)\right)}{\|(x,y)-(0,0)\|} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{x^7y^8}{\sqrt{x^{28}+y^{28}}}-0-0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^7y^8}{\sqrt{x^{28}+y^{28}}\sqrt{x^2+y^2}}.$$

Fie $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \ \forall n \in \mathbb{N}^*$.

Avem $\lim_{n\to\infty}(x_n, y_n) = (0,0)$ și

$$\lim_{n\to\infty} \frac{x_n^7 y_n^8}{\sqrt{x_n^{28}+y_n^{28}}\sqrt{x_n^2+y_n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n^{15}}}{\sqrt{\frac{2}{n^{28}}}\sqrt{\frac{2}{n^2}}} =$$

$$= \lim_{n\to\infty} \frac{1}{\cancel{n^{15}}} \cdot \frac{\cancel{n^{15}}}{2} = \frac{1}{2} \neq 0.$$

Deci $\lim_{(x,y)\to(0,0)} \frac{x^7y^8}{\sqrt{x^{28}+y^{28}}\sqrt{x^2+y^2}} \neq 0$, i.e.

f nu e diferențiabilă în $(0,0)$. $\square$

2. Fie $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y)=\begin{cases} xy\sin\frac{1}{x^2+y^2} & ;(x,y)\neq(0,0) \\ 0 & ;(x,y)=(0,0). \end{cases}$

a) Studiați continuitatea funcției f;

b) determinați $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ și arătați că acestea nu sunt continue în $(0,0)$;

c) Arătați că f este diferențiabilă pe $\mathbb{R}^2$.

<u>Soluție.</u> a) f continuă pe $\mathbb{R}^2 \setminus \{(0,0)\}$ (operații cu funcții elementare).

Studiem continuitatea lui f în $(0,0)$.

Fie $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$.

$$|f(x,y)-f(0,0)| = \left|xy\sin\frac{1}{x^2+y^2}-0\right| =$$

$$= |xy|\underbrace{\left|\sin\frac{1}{x^2+y^2}\right|}_{\leq 1} \leq |xy| \xrightarrow[(x,y)\to(0,0)]{} 0 \Rightarrow f \text{ e}$$

continuă în $(0,0)$.

b) Fie $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$.

$$\frac{\partial f}{\partial x}(x,y) = y \sin\frac{1}{x^2+y^2} + xy\left(\cos\frac{1}{x^2+y^2}\right)\cdot\left(\frac{-2x}{(x^2+y^2)^2}\right) =$$

$$= y \sin\frac{1}{x^2+y^2} - \frac{2x^2y}{(x^2+y^2)^2}\cos\frac{1}{x^2+y^2}.$$

$$\frac{\partial f}{\partial y}(x,y) = x \sin\frac{1}{x^2+y^2} + xy\left(\cos\frac{1}{x^2+y^2}\right)\cdot\left(\frac{-2y}{(x^2+y^2)^2}\right) =$$

$$= x \sin\frac{1}{x^2+y^2} - \frac{2xy^2}{(x^2+y^2)^2}\cos\frac{1}{x^2+y^2}.$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0}\frac{f((0,0)+te_1)-f(0,0)}{t} =$$

$$= \lim_{t\to 0}\frac{f((0,0)+t(1,0))-f(0,0)}{t} = \lim_{t\to 0}\frac{f((0,0)+(t,0))-f(0,0)}{t} =$$

$$= \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{0-0}{t} = 0.$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0}\frac{f((0,0)+te_2)-f(0,0)}{t} =$$

$$= \lim_{t\to 0}\frac{f((0,0)+t(0,1))-f(0,0)}{t} = \lim_{t\to 0}\frac{f((0,0)+(0,t))-f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0.$$

Avem
$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} y \sin\frac{1}{x^2+y^2} - \frac{2x^2y}{(x^2+y^2)^2}\cos\frac{1}{x^2+y^2} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0). \end{cases}$$

Fie $(x_n, y_n) = \left(\frac{1}{2\sqrt{n\pi}}, \frac{1}{2\sqrt{n\pi}}\right)$ $\forall\, n\in\mathbb{N}^*$.

Avem $\lim\limits_{n\to\infty}(x_n, y_n) = (0,0)$ și $\lim\limits_{n\to\infty}\frac{\partial f}{\partial x}(x_n, y_n) =$

$$= \lim_{n\to\infty}\left(\frac{1}{2\sqrt{n\pi}}\sin 2n\pi - \frac{\cancel{2}\cdot\frac{1}{4n\pi}\cdot\frac{1}{2\sqrt{n\pi}}}{\left(\frac{\cancel{2}}{\underset{2}{\cancel{4}}n\pi}\right)^2}\cos 2n\pi\right) =$$

$$= \lim_{n\to\infty}\left(0 - \frac{1}{4n\pi\sqrt{n\pi}}\cdot\frac{4n^2\pi^2}{1}\right) = -\infty \neq 0 =$$

$$= \frac{\partial f}{\partial x}(0,0) \Rightarrow \frac{\partial f}{\partial x} \text{ nu e continuă în } (0,0).$$

Analog se arată că $\frac{\partial f}{\partial y}$ nu e continuă în $(0,0)$ (cu aceleași șiruri $(x_n)_n$ și $(y_n)_n$).

c) $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ continue pe $\mathbb{R}^2 \setminus \{(0,0)\}$

$\mathbb{R}^2 \setminus \{(0,0)\}$ deschisă

$\Rightarrow$ f diferențiabilă pe $\mathbb{R}^2 \setminus \{(0,0)\}$.

Studiem diferențiabilitatea lui f în (0,0).

Dacă f ar fi diferențiabilă în (0,0), atunci

$$df(0,0): \mathbb{R}^2 \to \mathbb{R}, \quad df(0,0)(u,v) = \frac{\partial f}{\partial x}(0,0)\cdot u + \frac{\partial f}{\partial y}(0,0)\cdot v = 0\cdot u + 0\cdot v = 0.$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-df(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{xy\sin\frac{1}{x^2+y^2} - 0 - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}\sin\frac{1}{x^2+y^2}$$

Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\sin\frac{1}{x^2+y^2} - 0\right| = \frac{|xy|}{\sqrt{x^2+y^2}}\underbrace{\left|\sin\frac{1}{x^2+y^2}\right|}_{\leq 1} \leq \frac{|xy|}{\sqrt{x^2+y^2}} =$$

$$= |x| \frac{|y|}{\sqrt{x^2+y^2}} \le |x| \xrightarrow[(x,y)\to(0,0)]{} 0.$$

≤ 1 (Explicație: $\sqrt{x^2+y^2} \ge \sqrt{y^2} = |y|$)

Deci $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} \sin\frac{1}{x^2+y^2} = 0$, i.e. f este diferențiabilă în $(0,0)$.

Așadar f este diferențiabilă pe $\mathbb{R}^2$. $\square$

3. Fie $\varphi : \mathbb{R}^2 \to \mathbb{R}$ o funcție diferențiabilă și $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = \varphi(xy, x^2+y^2-z^2)$. Arătați că $xz \frac{\partial f}{\partial x}(x,y,z) - yz\frac{\partial f}{\partial y}(x,y,z) + (x^2-y^2)\frac{\partial f}{\partial z}(x,y,z) = 0 \ \forall (x,y,z) \in \mathbb{R}^3$.

Soluție.

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "g"] \arrow[rr, bend right, "f=\varphi\circ g"'] & \mathbb{R}^2 \arrow[r, "\varphi"] & \mathbb{R} \end{tikzcd}$$

Fie $g : \mathbb{R}^3 \to \mathbb{R}^2$, $g(x,y,z) = (xy, x^2+y^2-z^2)$.

Fie $u, v : \mathbb{R}^3 \to \mathbb{R}$, $u(x,y,z) = xy$, $v(x,y,z) = x^2+y^2-z^2$.

Avem $f = \varphi \circ g$.

$$\frac{\partial g}{\partial x}(x,y,z) = \left(\frac{\partial u}{\partial x}(x,y,z),\ \frac{\partial v}{\partial x}(x,y,z)\right) =$$

$$= (y, 2x) \quad \forall (x,y,z) \in \mathbb{R}^3.$$

$$\frac{\partial g}{\partial y}(x,y,z) = \left(\frac{\partial u}{\partial y}(x,y,z),\ \frac{\partial v}{\partial y}(x,y,z)\right) =$$

$$= (x, 2y) \quad \forall (x,y,z) \in \mathbb{R}^3.$$

$$\frac{\partial g}{\partial z}(x,y,z) = \left(\frac{\partial u}{\partial z}(x,y,z),\ \frac{\partial v}{\partial z}(x,y,z)\right) =$$

$$= (0, -2z) \quad \forall (x,y,z) \in \mathbb{R}^3.$$

$\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}$ continue pe $\mathbb{R}^3$

$\mathbb{R}^3$ deschisă

$\Rightarrow$ g diferențiabilă pe $\mathbb{R}^3$.

φ diferențiabilă pe $\mathbb{R}^2$

g diferențiabilă pe $\mathbb{R}^3$

$\Rightarrow$ $f = \varphi \circ g$ diferențiabilă pe $\mathbb{R}^3$.

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "g=(u,v)"] \arrow[rr, bend right, "f=\varphi\circ g"'] & \mathbb{R}^2 \arrow[r, "\varphi"] & \mathbb{R} \end{tikzcd}$$

(x,y,z) $\qquad$ (u,v)

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{\partial(\varphi\circ g)}{\partial x}(x,y,z) = \frac{\partial \varphi}{\partial u}(g(x,y,z))\cdot$$

$$\cdot\frac{\partial u}{\partial x}(x,y,z) + \frac{\partial \varphi}{\partial v}(g(x,y,z))\cdot\frac{\partial v}{\partial x}(x,y,z) =$$

$$= \frac{\partial \varphi}{\partial u}(xy,\ x^2+y^2-z^2)\cdot y + \frac{\partial \varphi}{\partial v}(xy,\ x^2+y^2-z^2)\cdot 2x,$$

$$\forall (x,y,z)\in\mathbb{R}^3.$$

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial(\varphi\circ g)}{\partial y}(x,y,z) = \frac{\partial \varphi}{\partial u}(g(x,y,z))\cdot$$

$$\cdot\frac{\partial u}{\partial y}(x,y,z) + \frac{\partial \varphi}{\partial v}(g(x,y,z))\cdot\frac{\partial v}{\partial y}(x,y,z) =$$

$$= \frac{\partial \varphi}{\partial u}(xy,\ x^2+y^2-z^2)\cdot x + \frac{\partial \varphi}{\partial v}(xy,\ x^2+y^2-z^2)\cdot 2y$$

$$\forall (x,y,z)\in\mathbb{R}^3.$$

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{\partial(\varphi\circ g)}{\partial z}(x,y,z) = \frac{\partial \varphi}{\partial u}(g(x,y,z))\cdot$$

$$\cdot\frac{\partial u}{\partial z}(x,y,z) + \frac{\partial \varphi}{\partial v}(g(x,y,z))\cdot\frac{\partial v}{\partial z}(x,y,z) =$$

$$= \frac{\partial \varphi}{\partial u}(xy,\ x^2+y^2-z^2)\cdot 0 + \frac{\partial \varphi}{\partial v}(xy,\ x^2+y^2-z^2)\cdot(-2z)$$

$$\forall (x,y,z)\in\mathbb{R}^3.$$

$$xz\frac{\partial f}{\partial x}(x,y,z)-yz\frac{\partial f}{\partial y}(x,y,z)+(x^2-y^2)\frac{\partial f}{\partial z}(x,y,z)=$$

$$=xz\left(\frac{\partial \varphi}{\partial u}(xy,x^2+y^2-z^2)\cdot y+\frac{\partial \varphi}{\partial v}(xy,x^2+y^2-z^2)\cdot 2x\right)-$$

$$-yz\left(\frac{\partial \varphi}{\partial u}(xy,x^2+y^2-z^2)\cdot x+\frac{\partial \varphi}{\partial v}(xy,x^2+y^2-z^2)\cdot 2y\right)+$$

$$+(x^2-y^2)\left(\frac{\partial \varphi}{\partial u}(xy,x^2+y^2-z^2)\cdot 0+\frac{\partial \varphi}{\partial v}(xy,x^2+y^2-z^2)\cdot(-2z)\right)=$$

$$=\cancel{xyz\frac{\partial \varphi}{\partial u}(xy,x^2+y^2-z^2)}+\cancel{2x^2z\frac{\partial \varphi}{\partial v}(xy,x^2+y^2-z^2)}-$$

$$-\cancel{xyz\frac{\partial \varphi}{\partial u}(xy,x^2+y^2-z^2)}-\cancel{2y^2z\frac{\partial \varphi}{\partial v}(xy,x^2+y^2-z^2)}-$$

$$-\cancel{2x^2z\frac{\partial \varphi}{\partial v}(xy,x^2+y^2-z^2)}+\cancel{2y^2z\frac{\partial \varphi}{\partial v}(xy,x^2+y^2-z^2)}=$$

$$=0. \quad \square$$