Vectori proprii. Valori proprii. Diagonalizare

$f:\mathbb{R}^4\to\mathbb{R}^4$, $f(x)=(x_2-x_3+x_4, x_2-x_3+x_4, x_4, x_4)$

a) val propr

b) ssp propr.

c). un reper $\mathcal{R}$ în $\mathbb{R}^4$ a.î. $[f]_{\mathcal{R},\mathcal{R}}$ e diag

a) $[f]_{R_0R_0}=\begin{pmatrix}0&1&-1&1\\0&1&-1&1\\0&0&0&1\\0&0&0&1\end{pmatrix}$

$P(\lambda)=\det(A-\lambda I_4)=\begin{vmatrix}-\lambda&1&-1&1\\0&1-\lambda&-1&1\\0&0&-\lambda&1\\0&0&0&1-\lambda\end{vmatrix}=$

$=(-\lambda)^2(1-\lambda)^2=0$

$V_{\lambda_1}=\{x\in\mathbb{R}^4 \mid f(x)=\lambda_1 x\}=\ker f$

$f(x)=0$; $A\lambda=0_{4,1}$

$\dim_{\mathbb{R}} V_{\lambda_1}=4-rg(A)=4-2=2=m_{\lambda_1}$ $\begin{pmatrix}0&1&-1&1\\0&1&-1&1\\0&0&0&1\\0&0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=$

$=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$, $\Delta=\begin{vmatrix}-1&1\\0&1\end{vmatrix}\neq 0$

$V_{\lambda_1}=\{(x_1,x_2,x_2,0) \mid x_1,x_2\in\mathbb{R}\}$

$<\underbrace{\{(0,1,1,0),(1,0,0,0)\}}_{R_1 \text{ reper în } V_{\lambda_1}}>$

$V_{\lambda_2} = \{x \in \mathbb{R}^4 \mid f(x) = \lambda_2 x\}$

$f(x) \Leftrightarrow Ax - x\ (A - I_4)x = O_{4,1}$

$(A - I_4)X = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; $\begin{cases} x_2 - x_3 = x_1 - x_4 \\ -x_3 = -x_4 \\ x_3 = x_4 \\ x_2 = x_1 \end{cases}$

$\dim V_{\lambda_2} = 4 - rg(A - I_4) = 4 - 2 = 2$

$V_{\lambda_2} = \{(x_1, x_1, x_4, x_4) \mid x_1, x_4 \in \mathbb{R}\} = <\underbrace{\{(1,1,0,0),(0,0,1,1)\}}_{R_2 \text{ reper în } V_{\lambda_2}}>$

<u>Teoremă</u>: 1) $\lambda_1 = 0$, $\lambda_2 = 1 \in \mathbb{R}$

$m_{\lambda_1} = 2$; $m_{\lambda_2} = 2$

2 $\dim V_{\lambda_i} = m_{\lambda_i}$, $i = \overline{1,2}$

$\Rightarrow \exists R = R_1 \cup R_2$ a.î.

$[f]_{R,R} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\mathbb{R}^4 = V_{\lambda_1} \oplus V_{\lambda_2}$

$f \in End(\mathbb{R}^3)$ $R_0 = \{e_1, e_2, e_3\}$ reper canonic

$f(e_1) = e_2$

$f(e_2) = e_1 + e_2 + e_3$

$f(e_3) = e_2$

a) să se det. R reper în $\mathbb{R}^3$ a.î. $[f]_{R,R} =$ diag

b) $A = [f]_{R_0,R_0}$; $A^n = ?$

Seminar 7

① ex 8:

$f \in End(\mathbb{R}^3)$; $R_0 = \{e_1, e_2, e_3\}$ reper canonic

a) $\begin{cases} f(e_1) = e_2 \\ f(e_2) = e_1 + e_2 + e_3 \\ f(e_3) = e_2 \end{cases}$

Să se det R reper în $\mathbb{R}^3$ a.î. $[f]_{R,R} = diag$

b) $A = [f]_{R_0, R_0}$; $A^n = ?$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$f: \mathbb{R}^3 \to \mathbb{R}^3$; $f(x) = (x_2, x_1 + x_2 + x_3, x_2)$

$$P(\lambda) = \det(A - \lambda \cdot I_3) = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = \cancel{(-\lambda)^2(1-\lambda)}$$

$\Rightarrow \cancel{\lambda_1 = 0 \; ; \; m_{\lambda_1} = 2}$

$\cancel{\lambda_2 = 1 \; ; \; m_{\lambda_2} = 1}$

$\cancel{V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_1 \cdot x = 0\}}$

$$= \lambda \cdot \begin{vmatrix} -\lambda & 1 & \\ 0 & 1-\lambda & 1 \\ \lambda & 1 & -\lambda \end{vmatrix} = \lambda \cdot \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} =$$

$c_1 + c_3$

$$= \lambda \cdot \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 2 & -\lambda \end{vmatrix} = -\lambda \cdot (\lambda^2 - \lambda - 2) =$$

$L_3 + L_1$

$$= -\lambda(\lambda+1)(\lambda-2)$$

4'

$\lambda_1 = 0$; $m_{\lambda_1} = 1$

$\lambda_2 = -1$; $m_{\lambda_2} = 1$

$\lambda_3 = 2$; $m_{\lambda_3} = 1$

$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid f(x) = 0 \} = \ker f$

$f(x) = 0 \iff A X = 0_{3,1}$; $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\dim_{\mathbb{R}} V_{\lambda_1} = 3 - rg A = 3 - 2 = 1 = m_{\lambda_1}$

sistem: $\begin{cases} x_2 = 0 \\ x_1 + x_2 = -x_3 \end{cases} \Rightarrow \cancel{x_1 = x_3}\ x_1 = -x_3$

$V_{\lambda_1} = \{ (x_3, 0, -x_3) \mid x_3 \in \mathbb{R} \} = \langle \underbrace{\{(1, 0, -1)\}}_{R_1 \text{ reper pt. } V_{\lambda_1}} \rangle$

$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid f(x) = \lambda_2 \cdot x \}$

$f(x) = -x \iff A X = -X$

$A X + X = 0$

$(A + I_3) X = 0 \iff \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} X = 0$

$\dim_{\mathbb{R}} V_{\lambda_2} = 3 - rg(A+I_3) = 3-2=1; = m_{\lambda_2}$

sistemul: $\begin{cases} x_1+x_2=0 \\ x_1+2x_2=-x_3 \end{cases} \Rightarrow x_1=-x_2 \Big\} \Rightarrow x_2=-x_3$

$\Rightarrow V_{\lambda_2} = \{(x_3, -x_3, x_3) \mid x_3 \in \mathbb{R}\} =$

$= \langle \underbrace{\{(1,-1,1)\}}_{B_2 \text{ reper în } V_{\lambda_2}} \rangle$

$V_{\lambda_3} = \{X \in \mathbb{R}^3 \mid f(X) = \lambda_3 \cdot X\}$

$f(X) = 2X \Rightarrow AX = 2X$

$\Rightarrow AX - 2X = 0$

$(A - 2I_3)X = 0$

$A - 2I_3 = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$

$\dim V_{\lambda_3} = 3-2 = 1 = m_{\lambda_3}$

sist: $\begin{cases} -2x_1+x_2=0 \Rightarrow x_2=2x_1 \\ x_1-x_2=-x_3 \end{cases} \Rightarrow x_2=2x_3$

(+)

$-x_1 = -x_3 \Rightarrow x_1 = x_3$

$\Rightarrow V_{\lambda_3} = \{(x_3, 2x_3, x_3) \mid x_3 \in \mathbb{R}\}$

$= \langle \underbrace{\{(1,2,1)\}}_{R_3 \text{ reper în } V_{\lambda_3}} \rangle$

din Th $\Rightarrow [f]$

$R = R_1 \cup R_2 \cup R_3$ și $[f]_{R,R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = A'$

$R = \{(-1,0,1), (1,-1,1), (1,2,1)\}$

$R_0 \xrightarrow{C} R$, $C = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

$A' = C^{-1} \cdot A \cdot C$

$A = C A' C^{-1}$

$A^n = C A' \cancel{C^{-1}} \cdot \cancel{C} A' \cancel{C^{-1}} \cdot \cancel{C} A' \cancel{C^{-1}} \cdots \cancel{C} A' C^{-1} = C \cdot A'^n \cdot C^{-1}$

$(A')^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 2^n \end{pmatrix}$

$\det C = \begin{vmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} \underset{L_3 + L_1}{=} \begin{vmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 2 & 2 \end{vmatrix} = (-1) \begin{vmatrix} -1 & 2 \\ 2 & 2 \end{vmatrix} = 6$

$C^t = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$; $C^* = \begin{pmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

$$A^n = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{pmatrix} =$$

$= \ldots$

(2) Ex n: $f \in End(\mathbb{R}^3)$;

$\lambda_1 = 3$; $\lambda_2 = -2$; $\lambda_3 = 1$; valorile proprii și

$v_1 = (-3, 2, 1)$; $v_2 = (-2, 1, 0)$; $v_3 = (-6, 3, 1)$

$[f]_{R_0, R_0}$

$R_0 = \{v_1, v_2, v_3\}$; reper în $\mathbb{R}^3$ a.î.

$$[f]_{R, R} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} := A'$$

$R_0 \xrightarrow{C} R$; $C = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$

$A' = C^{-1} A C$

$A = C A' C^{-1}$

$\det C = 1$

$$C^t = \begin{pmatrix} -3 & 2 & 1 \\ -2 & 1 & 0 \\ -6 & 3 & 1 \end{pmatrix}; \quad C^* = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & -3 \\ -1 & -2 & 1 \end{pmatrix} = C^{-1}$$

$$A = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & -3 \\ -1 & -2 & 1 \end{pmatrix}$$

7. ex6

$f: \mathbb{R}^4 \to \mathbb{R}^4$,

$$[f]_{R_0, R_0} = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix};$$

a) val pr. vect pr.;

b) $U = \langle e_1 + 2e_2,\ e_2 + e_3 + 2e_4 \rangle$ subspațiu invariant al lui f (i.e. $f(U) \subset U$).

a) $$P(\lambda) = \det(A - \lambda I_4) = \begin{vmatrix} 1-\lambda & 0 & 2 & -1 \\ 0 & 1-\lambda & 4 & -2 \\ 2 & -1 & -\lambda & 1 \\ 2 & -1 & -1 & 2-\lambda \end{vmatrix} L_4 - L_3$$

$$= \begin{vmatrix} 1-\lambda & 0 & 2 & -1 \\ 0 & 1-\lambda & 4 & -2 \\ 2 & -1 & -\lambda & 1 \\ 0 & 0 & \lambda-1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 1 & -1 \\ 0 & 1-\lambda & 2 & -2 \\ 2 & -1 & 1-\lambda & 1 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda) \cdot \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 2 \\ 2 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)\left((1-\lambda)^3 - 2\cancel{(1-\lambda)} + 2\cancel{(1-\lambda)}\right)$$

$= (1-\lambda)^4$; $\lambda_1 = 1$; $m_{\lambda_1} = 4$;

$V_{\lambda_1} = \{ x \in \mathbb{R}^4 \mid f(x) = x \}$

$AX = X$; $AX - X = 0$

$(A - I_4) \cdot X = 0$;

$$\det(A - \lambda_1) = \begin{vmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 4 & -2 \\ 2 & -1 & -1 & 1 \\ 2 & -1 & -1 & 1 \end{vmatrix}$$

sist: $\begin{cases} 4x_3 - 2x_4 = 0 \\ -x_3 + x_4 = 0 \Rightarrow x_4 = x_3 \\ \quad -2x_1 + x_2 \end{cases} \Rightarrow \begin{matrix} x_3 = 0 \\ x_4 = 0 \end{matrix}$

$\Leftrightarrow \begin{cases} 2x_3 - x_4 = 0 \\ -x_3 + x_4 = -2x_1 + x_2 \end{cases}$

$x_3 = -2x_1 + x_2$

$x_4 = 2x_3 = -4x_1 + 2x_2$

$V_{\lambda_1} = \{(x_1, x_2, -2x_1 + x_2, -4x_1 + 2x_2) \mid x_1, x_2 \in \mathbb{R}\} =$

$\Rightarrow 2 \{(1, 0, -2, -4), (0, 1, 1, 2)\}>$

din $V_{\lambda_1} = 4 - 2 = 2 \Rightarrow$ A nu se poate diagonaliza

se obs că $V \cong V_{\lambda_1}$ ssp. invariant