SEMINAR 1
ŞIRURI DE NUMERE REALE

Limite de şiruri remarcabile

1) $\lim\limits_{n\to\infty} a^n = \begin{cases} 0, & a\in(-1,1) \\ 1, & a=1 \\ +\infty, & a>1 \\ \nexists, & a\le 1 \end{cases}$

2) $\lim \left(1+\frac{1}{n}\right)^n = e \in (2,3)$ - nr lui Euler

3) $\lim \left(1+\frac{1}{2}+\ldots+\frac{1}{n} - \ln n\right) = c \in (0,1)$ - constanta lui Euler

4) $\lim\limits_{n\to\infty} x_n = 0 \Rightarrow \lim\limits_{n\to\infty} \frac{a^{x_n}-1}{x_n} = \ln a,\ a>0$

$\lim \frac{(1+x_n)^r-1}{x_n} = r$

5) $\lim\limits_{n\to\infty} \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right) = +\infty\ (-\ln n+\ln n)$

Exerciţiul 1: Calculaţi următoarele limite de şiruri

a) $\lim\limits_{n\to\infty} \sqrt[p]{n}\cdot a^n,\ a>0,\ p\in\mathbb{N},\ p\ge 2$

b) $\lim\limits_{n\to\infty} \sqrt[n]{1+\frac{1}{2}+\ldots+\frac{1}{n}}$

c) $\lim\limits_{n\to\infty} \left(\sqrt{n^2+n+1} - \sqrt[3]{n^3+n+1}\right)$

d) $\lim\limits_{n\to\infty} \left(\frac{\sqrt[n]{a}+1}{2}\right)^n,\ a>0$

e) $\lim\limits_{n\to\infty} \frac{1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}}}{\sqrt{n}}$

Rezolvare a) $\lim_{n\to\infty} \sqrt[p]{n}\cdot a^n$

Caz I: $a=1 \Rightarrow \lim_{n\to\infty} \sqrt[p]{n}\cdot a^n = \lim \sqrt[p]{n} = +\infty$

Caz II: $a>1 \Rightarrow \lim_{n\to\infty} \sqrt[p]{n}\cdot a^n = +\infty$

Caz III: $a\in(0,1) \Rightarrow \lim_{n\to\infty} \sqrt[p]{n}\cdot a^n$ (caz de nedeterminare $\infty\cdot\infty$)

Se rezolvă cu criteriul raportului:

Fie $x_n = \sqrt[p]{n}\cdot a^n$

$x_n>0$, $(\forall)\, n\in\mathbb{N}^*$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\sqrt[p]{n+1}\cdot a^{n+1}}{\sqrt[p]{n}\cdot a^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{\frac{1}{p}}\cdot a = a < 1 \Rightarrow$$

$\Rightarrow$ dacă limita este 1 nu putem afla răspunsul

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$ – limita raportului

$$\begin{cases} L<1 \Rightarrow \lim_{n\to\infty} x_n = 0 \\ L>1 \Rightarrow \lim_{n\to\infty} x_n = +\infty \\ L=1 \Rightarrow \text{Criteriul este inutilizabil} \end{cases}$$

$\Rightarrow \lim_{n\to\infty} x_n = 0$

Rezolvare b) $\lim_{n\to\infty} \sqrt[n]{1+\frac{1}{2}+\ldots+\frac{1}{n}}$

Se aplică Criteriul Radicalului; dacă mai (∃) termeni se introduc sub radical.

Fie $x_n = 1+\frac{1}{2}+\ldots+\frac{1}{n}$

$x_n=0$, $(\forall)\, n\in\mathbb{N}^*$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{1+\frac{1}{2}+\ldots+\frac{1}{n}+\frac{1}{n+1}}{1+\frac{1}{2}+\ldots+\frac{1}{n}} = \lim_{n\to\infty} \left(1+\frac{\overset{\to 0}{\frac{1}{n+1}}}{\underset{\to+\infty}{1+\frac{1}{2}+\ldots+\frac{1}{n}}}\right) = 1$$

$\Rightarrow \exists \lim_{n\to\infty} \sqrt[n]{x_n} = 1$

Dacă (∃) $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$ atunci (∃) și $\lim_{n\to\infty} \sqrt[n]{x_n} = L$

Rezolvare c) $\lim\limits_{n\to\infty}\left(\sqrt{n^2+n+1}-\sqrt[3]{n^3+n+1}\right)$

$$\lim_{n\to\infty}\left(\sqrt{n^2+n+1}-\sqrt[3]{n^3+n+1}\right)=\lim_{n\to\infty}\left(n\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}-n\sqrt[3]{1+\frac{1}{n^3}+\frac{1}{n^2}}\right)$$

$$=\lim_{n\to\infty} n\left(\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}-\sqrt[3]{1+\frac{1}{n^2}+\frac{1}{n^3}}\right)$$

$$=\lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}-\sqrt{1+\frac{1}{n^2}+\frac{1}{n^3}}}{\frac{1}{n}}$$

(schimb. de var.)

$$-\text{pt. } L'H:\ \lim_{x\to\infty}\frac{\sqrt{1+x+x^2}-\sqrt[3]{1+x^2+x^3}}{x}=\lim_{x\to 0}\qquad \frac{1}{n}=x$$

$$=\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1-\left(\sqrt{1+x^2+x^3}-1\right)}{}$$

$$=\lim_{x\to 0}\left[\frac{(1+x+x^2)^{\frac{1}{2}}-1}{x}-\frac{(1+x+x^3)^{\frac{1}{3}}-1}{x}\right]$$

$x+1$ $\qquad$ x^2+x

$$=\lim_{x\to 0}\left[\underbrace{\frac{(1+x+x^2)^{\frac{1}{2}}-1}{x+x^2}}_{\downarrow\ \frac{1}{2}}\cdot\underbrace{\frac{x+x^2}{x}}_{\downarrow\ 1}-\underbrace{\frac{(1+x^2+x^3)^{\frac{1}{3}}}{x^2+x^3}}_{\downarrow\ \frac{1}{3}}\cdot\underbrace{\frac{x^2+x^3}{x}}_{\downarrow\ 0}\right]$$

$$=\frac{1}{2}\cdot 1-\frac{1}{3}\cdot 0=\frac{1}{2}$$

$$\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-\sqrt[3]{1+x^2+x^3}}{x}=\frac{1}{2}\xrightarrow{y_n=\frac{1}{n}}\lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}-\sqrt[3]{1+\frac{1}{n^2}+\frac{1}{n^3}}}{\frac{1}{n}}$$

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Rezolvare: d) $\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+1}{2}\right)^n,\ a>0$

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+1}{2}\right)^n=\lim_{n\to\infty}\left(1-1+\frac{\sqrt[n]{a}+1}{2}\right)^n=\lim_{n\to\infty}\left(1+\frac{\sqrt[n]{a}-1}{2}\right)^n$$

$$=\lim_{n\to\infty}\Bigg[\underbrace{\left(1+\frac{\sqrt[n]{a}-1}{2}\right)^{\frac{2}{\sqrt[n]{a}-1}}}_{\searrow e}\Bigg]^{\frac{\sqrt[n]{a}-1}{2}\cdot n}=e^{\lim_{n\to+\infty}\frac{n(\sqrt[n]{a}-1)}{2}}=e^{\frac{1}{2}\lim_{n\to\infty}\frac{a^{\frac{1}{n}}-1}{\frac{1}{n}}}$$

$$=e^{\frac{1}{2}\ln a}=e^{\ln(a^{\frac{1}{2}})}=a^{\frac{1}{2}}=\sqrt{a}$$

Rezolvare: e) $\lim_{n\to\infty}\frac{1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}}}{\sqrt{n}}$

$$\left.\begin{aligned}&1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}}>1+\frac{1}{2}+\ldots+\frac{1}{n},\ (\forall)n\geq 2\\&\lim_{n\to\infty}1+\frac{1}{2}+\ldots+\frac{1}{n}=+\infty\end{aligned}\right\}\text{; Aplicăm criteriul majorării:}\Rightarrow\lim_{n\to\infty}\left(1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}}\right)=+\infty$$

Notăm: $x_n=1+\frac{1}{\sqrt{2}}+\ldots+\sqrt{n}$

$y_n=\sqrt{n}$

$\lim_{n\to\infty}y_n=+\infty$

$y_{n+1}>y_n,\ (\forall)\ n\in\mathbb{N}$; $\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}=$

$$=\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n+1}}}{\sqrt{n+1}-\sqrt{n}}=\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n+1}}}{\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}}=\lim_{n\to\infty}\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}}$$

$$=\lim_{n\to\infty}\left(1+\sqrt{\frac{n}{n+1}}\right)=2$$

$\xRightarrow[\text{S-C}]{\text{lema}}\ \exists\ \lim_{n\to\infty}\frac{x_n}{y_n}=2$

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Temă: Se consideră $(x_n)_{n\in\mathbb{N}}$ un șir din $\mathbb{R}$ pentru care $\exists \lim\limits_{n\to\infty} [(n+1)x_{n+1} - x_n] = \ell \in \overline{\mathbb{R}}$. Demonstrați că $\exists \lim\limits_{n\to\infty} x_n = \ell$

IDEE DE REZOLVARE: $(n+1)x_{n+1} - x_n;\ n = \dfrac{a_{n+1} - a_n}{a_{n+1} - b_n}$

Exercițiul 2: Se consideră șirul $(x_n)_{n\in\mathbb{N}^*}$ definit prin relația de recurență $x_{n+1} = \dfrac{n \cdot x_n}{n + x_n^2}\ \forall\ n\in\mathbb{N}^*$ și $x_1 > 0$. Arătați că șirul este convergent și calculați $\lim\limits_{n\to\infty} x_n$.

Demonstrăm prin inducție că $x_n > 0\ (\forall) n\in\mathbb{N}^*$

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$x_n > 0\ (\forall) n\in\mathbb{N} \Rightarrow (x_n)_{n\in\mathbb{N}^*}$ mărginit inferior

$\dfrac{x_{n+1}}{x_n} = \dfrac{n\cdot \cancel{x_n}}{n + x_n^2}\cdot\dfrac{1}{\cancel{x_n}} = \dfrac{n}{n + x_n^2} < 1\ (\forall) n\in\mathbb{N} \Rightarrow$ șir str. $\searrow$ $\Rightarrow$ șirul (x_n) e convergent

$\exists \lim\limits_{n\to\infty} x_n = \ell \in \mathbb{R}$; $\underbrace{x_{n+1}}_{\to \ell} = \dfrac{n x_n}{n + x_n^2}$

$\ell \geq 0$

$x_{n+1} = \dfrac{n x_n}{n + x_n} = \dfrac{\cancel{n} x_n}{\cancel{n}\left(1 + \frac{x_n^2}{n}\right)} = \dfrac{x_n}{1 + \frac{x_n^2}{n}}$

$\ell = \dfrac{\ell}{1+0} \Rightarrow \ell = \ell$

Temă: Calculați $\lim\limits_{n\to\infty} [(n+1)x_{n+1} - n x_n] = \mathcal{L}$

$\Rightarrow \lim\limits_{n\to\infty} x_n = \mathcal{L} \Rightarrow \ell = \mathcal{L}$

SEMINAR 2
ȘIRURI DE NUMERE REALE

Exercițiul 1: Știind că $(\exists) \lim_{n\to\infty} \left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n\right)=c\in(0,1)$.
Calculați următoarele limite:

a) $\lim_{n\to\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)$

b) $\lim_{n\to\infty}\left(1+\frac{1}{2^\alpha}+\ldots+\frac{1}{n^\alpha}\right), \alpha\le 1$

Rezolvare: a) $\lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\right]=\lim_{n\to\infty}\left(\underbrace{1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n}_{c}\right)+\underbrace{\ln n}_{\infty}\Big]=\infty$

Rezolvare: b) $2^\alpha\le 2 \Rightarrow \frac{1}{2^\alpha}\ge\frac{1}{2}$

$3^\alpha\le 3\Rightarrow\frac{1}{3^\alpha}\ge\frac{1}{3}$

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$n^\alpha\le n\Rightarrow\frac{1}{n^\alpha}\ge\frac{1}{n}$

$$\left.\begin{array}{l}1+\frac{1}{2^\alpha}+\ldots+\frac{1}{n^\alpha}\ge 1+\frac{1}{2}+\ldots+\frac{1}{n}\ (\forall)\ n\ge 2\\ \lim_{n\to\infty} 1+\frac{1}{2}+\ldots+\frac{1}{n}=\infty\end{array}\right\}\Rightarrow(\exists)\lim_{n\to\infty}1+\frac{1}{2^\alpha}+\ldots+\frac{1}{n^\alpha}=\infty$$

Exercițiul 2: Se consideră șirul $(x_n)_{n\in\mathbb{N}}$ a.î. $(\exists)\lim_{n\to\infty}\sqrt{n}\,(x_{n+1}-x_n)=$
$l\in(0,\infty)$ și să se calculeze:

a) $\lim_{n\to\infty} x_n$

b) limita $\lim_{n\to\infty}\sqrt[n]{x_n}$

Rezolvare a) $(\forall)\ \varepsilon>0\ (\exists)\ n_\varepsilon \in \mathbb{N}$ a.î. $|\sqrt{n}(x_{n+1}-x_n)-l| < \varepsilon$, $(\forall)\ n \geq n_\varepsilon$

$$-\varepsilon < \sqrt{n}(x_{n+1}-x_n) - l < \varepsilon\ (\forall)\ n \geq n_\varepsilon$$

$$-\varepsilon + l < \sqrt{n}(x_{n+1}-x_n) < \varepsilon + l,\ (\forall)\ n \geq n_\varepsilon$$

Fie $\varepsilon = \frac{l}{2}$

$$\Rightarrow \underbrace{\frac{l}{2} < \sqrt{n}(x_{n+1}-x_n)} < \frac{3l}{2},\ (\forall)\ n \geq n_2$$

$$\frac{l}{2\sqrt{n}} < x_{n+1} - \cancel{x_n},\ (\forall)\ n \geq n_2$$

$$\frac{l}{2\sqrt{n-1}} < \cancel{x_n} - \cancel{x_{n-1}}$$

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$$\frac{l}{2\sqrt{n_2}} < \cancel{x_{n_2+1}} - x_{n_2}$$

$$\frac{l}{2}\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n-1}} + \dots + \frac{1}{\sqrt{n_2}}\right) < x_{n+1} - x_{n_2}\ (\forall)\ n \geq 2$$

$$x_{n+1} > \frac{l}{2}\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n-1}} + \dots + \frac{1}{\sqrt{n_2}}\right) + x_{n_2}\ (\forall)\ n \geq n_2$$

$$\lim_{n\to\infty}\left[\frac{l}{2}\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n-1}} + \dots + \frac{1}{\sqrt{n_2}}\right) + x_{n_2}\right] = \lim_{n\to\infty}\left[\underbrace{x_{n_2}}_{ct} + \underbrace{\frac{l}{2}}_{>0}\underbrace{\left(\frac{1}{n^{\frac{1}{2}}} + \dots + \frac{1}{(n-1)^{\frac{1}{2}}}\right)}_{\text{ex 1(b)}\ \infty} + x_{n_2}\right] =$$

$$(1),(2) \xRightarrow[\text{majorării}]{\text{crit.}} (\exists) \lim_{n\to\infty} x_n = \infty$$

soluția 2: $\lim_{n\to\infty} \sqrt{n}(x_{n+1}-x_n) = \lim_{n\to\infty} \frac{x_{n+1}-x_n}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{x_{n+1}-x_n}{1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}-(1+\frac{1}{\sqrt{2}}$

Rezolvare b) $\lim_{n\to\infty} \sqrt{n}(x_{n+1}-x_n) - l \in (0,\infty)$

$$\lim_{n\to\infty} x_n = \infty$$

Trebuie să calculăm $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$

$$\lim_{n\to\infty} x_{n+1} - x_n = \lim_{n\to\infty} \frac{(x_{n+1}-x_n)\sqrt{n}}{\underbrace{\sqrt{n}}_{\to\infty}} = 0$$

$$\lim_{n\to\infty}(x_{n+1}-x_n) = \lim_{n\to\infty} x_n\left(\frac{x_{n+1}}{x_n} - 1\right) = 0$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} - 1 = \lim_{n\to\infty} a_n\left(\frac{a_{n+1}}{a_n} - 1\right) = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \Rightarrow (\exists) \lim_{n\to\infty} \sqrt[n]{a_n} = 1$$

Exercițiul 3: Se consideră $(a_n)_{n\in\mathbb{N}^*}$ un șir def. prin relația de recurență $a_{n+1} = \frac{a_n \cdot n}{n + a_n^2}$, $(\forall) n \in \mathbb{N}^*$ și $a_1 > 0$. Arătați că șirul este convergent și calculați $\lim_{n\to\infty} a_n$. (șir desc)

Se dem. prin inducție că $a_n > 0$ $(\forall) n\in\mathbb{N}^* \Rightarrow$ șirul e mărginit inferior (1)

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n\cancel{a_n}}{n+a_n^2}}{\cancel{a_n}} = \frac{n}{n+a_n^2} < 1 \ (\forall) n\in\mathbb{N}^* \Rightarrow (a_n)_{n\in\mathbb{N}^*} \text{ desc } (2)$$

$$(1),(2) \xrightarrow{T.W.} (\exists) \lim_{n\to\infty} a_n = l \in \mathbb{R}$$

$$a_{n+1} = \frac{n a_n}{n + a_n^2} = \frac{\cancel{n} a_n}{\cancel{n}\left(1 + \frac{a_n^2}{n}\right)} = \frac{a_n}{1 + \frac{a_n^2}{n}} \ (\forall) n\in\mathbb{N}^*$$

$$l = \frac{l}{1+0} \Rightarrow l = l$$

$$\lim_{n\to\infty}\left[(n+1)a_{n+1} - n a_n\right] = \lim_{n\to\infty}\left[(n+1)\frac{n a_n}{n+a_n^2} - n a_n\right]$$

$$= \lim_{n\to\infty}\left[n a_n\left(\frac{n+1}{n+a_n^2} - 1\right)\right] = \lim_{n\to\infty}\left[n x_n\left(\frac{n+1-n-x_n^2}{n+x_n^2}\right)\right] = \lim_{n\to\infty} n x_n \frac{1-x_n^2}{n+a_n^2}$$

$$= \lim_{n\to\infty} \frac{\cancel{n} x_n(1-a_n^2)}{\cancel{n}\left(1+\frac{x_n^2}{n}\right)} = \frac{l(1-l^2)}{1+0} = l - l^3$$

$$\lim_{n\to\infty}\left[(n+1)a_{n+1} - n x_n\right] = l - l^3 \Rightarrow \lim_{n\to\infty} a_n = l - l^3$$

$$l = l - l^3 \Rightarrow l^3 = 0 \Rightarrow l = 0$$

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Exercițiul 4: $x_n = \frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \dots + \frac{1}{n\ln n}$; $n \in \mathbb{N}, n \geq 2$. Să se studieze convergența lui $(x_n)_{n\in\mathbb{N}}$

Rezolvare: monotonia:

$$x_{n+1} - x_n = -\frac{1}{(n+1)\ln(n+1)} > 0 \ (\forall) n \geq 2$$

$\Rightarrow (x_n)_{n\in\mathbb{N}}$ e st $\nearrow$

$f'(x) = \frac{1}{x\ln x} \Rightarrow f(x) = \int \frac{1}{x\ln x}dx = \ln(\ln x) + \mathcal{C}$

Se definește funcția $f:[k,k+1] \to \mathbb{R}$, $f(x) = \ln(\ln x)$

$$\left.\begin{array}{l} f \text{ continuă pe } [k,k+1] \\ f \text{ derivabilă pe } [k,k+1] \end{array}\right\} \Rightarrow (\exists) c \in [k,k+1] \text{ a.î. } f'(c) = \frac{f(k+1)-f(k)}{\cancel{k}+1-\cancel{k}}$$

$$\Rightarrow f'(c) = f(k+1) - f(k)$$

$$\ln(\ln(k+1)) - \ln(\ln k) = \frac{1}{c\ln c}$$

$$k < c < k+1$$

$$k\ln k < c\ln c < (k+1)\ln(k+1)$$

$$\frac{1}{(k+1)\ln(k+1)} < \ln(\ln(k+1)) - \ln(\ln k) < \frac{1}{k\ln k} \ (\forall) k > 2$$

$k=2$: $\cancel{\ln(\ln 3)} - \ln(\ln 2) < \frac{1}{2\ln 2}$

$k=3$: $\ln(\ln 4) - \cancel{\ln(\ln 3)} < \frac{1}{3\ln 3}$

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$k=n$: $\ln(\ln(n+1)) - \cancel{\ln(\ln n)} < \frac{1}{n\ln n}$ $\quad (+)$

$$\ln(\ln(n+1)) - \ln(\ln 2) < x_n, \ (\forall) n \geq 2$$

$$x_n > \ln(\ln(n+1)) - \ln(\ln 2), \ (\forall) n \geq 2$$

$$\lim_{n\to\infty} \ln(\ln(n+1)) - \ln(\ln 2) = \infty$$

$\Rightarrow (\exists) \lim_{n\to\infty} x_n = \infty$, deci $(x_n)_n$ e convergent

Temă: 1) Fie $\alpha > 1$ și $x_n = 1 + \frac{1}{2^\alpha} + \dots + \frac{1}{n^\alpha}$, $(\forall)\ n \in \mathbb{N}^*$

Dem. că șirul e convergent

Indicație: monotonia + (șir ↗)

$f'(x) = \frac{1}{x^\alpha} = x^{-\alpha} \Rightarrow f(x) = \int x^{-\alpha}\,dx = \frac{x^{1-\alpha}}{1-\alpha}$

Alegem $f : [k, k+1] \to \mathbb{R}$, $f(x) = \frac{x^{1-\alpha}}{1-\alpha}$ și apl. th. Lagrange + fol prima inegalitate.

SEMINAR 3
SERII DE NUMERE REALE

Exercițiul 1: Să se studieze natura următoarelor serii de numere reale:

a) $\sum_{n=0}^{\infty} \frac{n}{(2n+1)!!}$, unde $(2n+1)!! = 1\cdot 3\cdot 5\cdot\ldots\cdot(2n+1)$

b) $\sum_{n=0}^{\infty} \frac{n!\,a^n}{(1+a)(1+2a)\ldots(1+na)}$, $a\geq 0$

c) $\sum_{n=0}^{\infty} \frac{1}{n+a^n}$

d) $\sum_{n=1}^{\infty} \frac{\sin\frac{1}{n(n+1)}}{\cos\frac{1}{n}\cos\frac{1}{n+1}}$

e) $\sum_{n=1}^{\infty} \frac{n}{n+1}\cdot\frac{\sin\frac{1}{n(n+1)}}{\cos\frac{1}{n}\cos\frac{1}{n+1}}$

f) $\sum_{n=1}^{\infty} \frac{a^n\cdot n!}{n^n}$

Rezolvare a) $y_n = \frac{n}{(2n+1)!!}$, $n\in\mathbb{N}$

$y_n > 0\ (\forall)\, n\in\mathbb{N}^*$

Metoda 1: Se calc $\lim\limits_{n\to\infty} s_n$

Metoda 2: Aplicarea criteriilor de convergență

$s_n = \underset{0}{y_0} + y_1 + \ldots + y_n = y_1 + y_2 + \ldots + y_n$

$y_k = \frac{k}{(2k+1)!!} = \frac{k}{1\cdot 3\cdot\ldots\cdot(2k+1)} = \frac{1}{2}\cdot\frac{2k}{1\cdot 3\cdot(2k+1)} = \frac{1}{2}\cdot\frac{2k+1-1}{1\cdot 3\cdot\ldots\cdot(2k+1)}$

$= \frac{1}{2}\left(\frac{1}{1\cdot 3\cdot\ldots\cdot(2k-1)} - \frac{1}{1\cdot 3\cdot\ldots\cdot(2k+1)}\right)$, $k\in\{1,2,\ldots,n\}$

$$s_n = \frac{1}{2}\left(1 - \cancel{\frac{1}{1\cdot 3}} + \cancel{\frac{1}{1\cdot 3}} - \frac{1}{1\cdot 3\cdot 5} + \ldots - \frac{1}{1\cdot 3\cdot\ldots\cdot(2n+1)}\right)$$

$$s_n = \frac{1}{2}\left(1 - \frac{1}{1\cdot 3\cdot\ldots\cdot(2n+1)}\right)$$

$\lim\limits_{n\to\infty} s_n = \frac{1}{2}$. Seria este convergentă și are ~~limita~~ suma $\frac{1}{2}$

Rezolvare b) $x_n = \dfrac{n!\,a^n}{(1+a)(1+2a)\ldots(1+n\cdot a)}$, $a>0$, $n\in\mathbb{N}^*$

$x_n > 0$ $(\forall)$ $n\in\mathbb{N}^*$

$$s_n = x_1 + x_2 + \ldots + x_n = \frac{a}{1+a} + \frac{2!\,a^2}{(1+a)(1+2a)} + \ldots + \frac{n!\,a^n}{(1+a)(1+2a)\cdot\ldots\cdot(1+n\cdot a)}$$

$\lim\limits_{n\to\infty} s_n = ?$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\overset{(n+1)\ a}{\cancel{(n+1)!}\,\cancel{a^{n+1}}}}{(\cancel{1+a})(\cancel{1+2a})\cdot\ldots\cdot(\cancel{1+na})(1+(n+1)a)} \cdot \frac{\cancel{(1+a)}\cancel{(1+2a)}\cdot\ldots\cdot\cancel{(1+na)}}{\cancel{n!}\cdot\cancel{a^n}}$$

$$= \frac{(n+1)a}{1+(n+1)a} = 1$$

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{1+(n+1)a}{a(n+1)} - 1\right) = \lim_{n\to\infty} n\cdot\frac{1}{a(n+1)} = \frac{1}{a} = \ell$$

I $a\in(0,1)$, $\ell>1 \Rightarrow$ serie convergentă

II $a>1$, $\ell<1 \Rightarrow$ serie divergentă

III $a=1 \Rightarrow \ell = 1$

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=2}^{\infty} \frac{1}{n} \Rightarrow \text{serie divergentă}$$

Rezolvare c) $a_n = \frac{1}{n+a^n}$, $a>0$, $n \in \mathbb{N}$

$a_n > 0 \ (\forall)\ n \in \mathbb{N}$

$s_n = a_0 + a_1 + \ldots + a_n = \frac{1}{1+a} + \frac{1}{2+a^2} + \ldots + \frac{1}{n+a^n}$

$\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{n+a^n}{n+1+a^{n+1}} =$

I: $a > 1$

$$\lim_{n\to\infty} \frac{n+a^n}{n+1+a^{n+1}} = \lim_{n\to\infty} \frac{a^n\left(\overset{\to 0}{\frac{n}{a^n}}+1\right)}{a^{n+1}\left(\underset{\to 0}{\frac{n+1}{a^{n+1}}}+1\right)} = \frac{1}{a} < 1 \Rightarrow \text{serie converg.}$$

II: $a \in (0,1]$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+a^n}{n+1+a^{n+1}} = \lim_{n\to\infty} \frac{n\left(1+\frac{a^n}{n}\right)}{(n+1)\left(1+\frac{a^{n+1}}{n+1}\right)}$$

$a \leq 1 \Rightarrow a^n \leq 1 \,|+n \Rightarrow a^n + n \,|^{-1} \Rightarrow \frac{1}{a^n+n} \geq \frac{1}{1+a}$

$\Rightarrow a_n \geq \frac{1}{1+n} \ (\forall)\ n \in \mathbb{N}$

$\sum\limits_{n=0}^{+\infty} \frac{1}{n+1} = \sum\limits_{n+1=1}^{+\infty} \frac{1}{n+1} \Rightarrow$ serie convergentă

$\Rightarrow \sum\limits_{n=0}^{\infty} a_n$ e divergentă

Rezolvare d) $x_n = \dfrac{\sin\frac{1}{n+1}}{\cos\frac{1}{n} + \cos\frac{1}{n+1}}$, $n \in \mathbb{N}$

$\frac{1}{n(n+1)}, \frac{1}{n}, \frac{1}{n+1} \in \left[0, \frac{\pi}{2}\right)$, $n \in \mathbb{N}^* \Rightarrow x_n > 0$, $(\forall)\ n \in \mathbb{N}^*$

$$x_n = \frac{\sin\frac{1}{n(n+1)}}{\cos\frac{1}{n}\cos\frac{1}{n}} = \frac{\frac{\sin n(n+1)}{\frac{1}{n(n+1)}} + \frac{1}{n(n+1)}}{\cos\frac{1}{n}\cdot\cos\frac{1}{n+1}} = \frac{\frac{\sin\frac{1}{\sin(n+1)}}{\frac{1}{n(n+1)}}}{\cos\frac{1}{n}\cdot\cos\frac{1}{n+1}}$$

Alegem $y_n = \frac{1}{n(n+1)}$

$y_n > 0, (\forall)\ n \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\frac{\sin\frac{1}{n(n+1)}}{\frac{1}{n(n+1)}}}{\cos\frac{1}{n}\cdot\cos\frac{1}{n+1}} = 1 \in (0, +1)$$

$\Rightarrow \sum\limits_{n=1}^{\infty} x_n$ și $\sum\limits_{n=1}^{\infty} y_n$ au aceeași natură

$s_n = y_1+y_2+\dots+y_{n-1} = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n(n+1)} = 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\dots+\frac{1}{n}-\frac{1}{n+1}$

$= 1-\frac{1}{n+1} = \frac{n}{n+1} \Rightarrow s_n$ convergentă $\Rightarrow x_n$ convergent

Rezolvare e) $x_n = \frac{n}{n+1}\cdot\frac{\sin\frac{1}{(n+1)\cdot n}}{\cos\frac{1}{n}\cdot\cos\frac{1}{n+1}}$

Alegem $a_n = \frac{n}{n+1}$ și $b_n = \frac{\sin\frac{1}{n(n+1)}}{\cos\frac{1}{n}\cdot\cos\frac{1}{n+1}}$

$x_n = a_n\cdot b_n, \forall n\in\mathbb{N}^*$

$\lim\limits_{n\to\infty}\frac{x_n}{b_n} = \lim\limits_{n\to\infty} a_n = 1\in(0,+\infty) \Rightarrow \sum\limits_{n=1}^{\infty} x_n$ și $\sum\limits_{n=1}^{\infty} b_n$ au ac. nat.

Din punctul d) știm că $\sum\limits_{n=1}^{\infty} b_n$ e convergentă

$\Rightarrow \sum\limits_{n=1}^{\infty} x_n$ e convergentă

Rezolvare f) $x_n = \frac{a^n\cdot n!}{n^n}$, $a>0, a\neq e, n\in\mathbb{N}$

$x_n>0\ (\forall)\ n\in\mathbb{N}^*$

$\lim\limits_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim\limits_{n\to\infty}\frac{a^{n+1}\,(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{a^n\, n!} = \lim\limits_{n\to\infty}\frac{a\cdot n^n}{(n+1)^n}$

$= \lim\limits_{n\to\infty} a\left(\frac{n}{n+1}\right)^n = \lim\limits_{n\to\infty} a\left(1+\frac{n}{n+1}-1\right)^n = \lim\limits_{n\to\infty} a\left[\left(1+\frac{1}{n+1}\right)^{(n+1)}\right]^{\frac{-n}{n+1}}$

$= 1+e^{\lim\limits_{n\to\infty}\frac{n}{n+1}} = \frac{a}{e} = l$

I) $a\in(0,e) \Rightarrow l<1$ serie convergentă

II) $a>e \Rightarrow l>1$ serie divergentă

Temă: 1. Fie $(a_n)_{n\in\mathbb{N}}$ și $(b_n)_{n\in\mathbb{N}}$ două șiruri din $\mathbb{R}_+^*$

a) Dacă $\exists\, n_0\in\mathbb{N}$ pentru care $\frac{a_{n+1}}{a_n}\le\frac{b_{n+1}}{b_n}$, $\forall n\ge n_0$ și $\sum\limits_{n=0}^{\infty} b_n$ e convergentă, atunci $\sum\limits_{n=0}^{\infty} a_n$ este convergentă

b) Dacă $\exists\, n_0\in\mathbb{N}$ pt. care $\frac{a_{n+1}}{a_n}\le\frac{b_{n+1}}{b_n}$ și $\sum\limits_{n=0}^{\infty} a_n$ este divergentă, atunci $\sum\limits_{n=0}^{\infty} b_n$ este divergentă

1. Studiați natura seriei $\sum_{n=1}^{\infty} \frac{e^{n} \cdot n!}{n^n}$

$\boxed{\text{Ind}}$: $\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$ $(\forall)\, n \geq 2$

$a_{n+1} \leq a_n \cdot \frac{b_{n+1}}{b_n}$ $(\forall)\, n \geq n_0$

REMEMBER

! PENTRU EXAMEN -Exercițiul 1-

IMPORT

SEMINAR 4

SERII DE NUMERE REALE

TIPOLOGIA UNUI SPAȚIU METRIC

Exercițiul I: Fie $(a_n)_{n \in \mathbb{N}}$ un șir din $\mathbb{R}_+$. Să se studieze natura seriei $\sum_{n=1}^{\infty} \frac{a_n}{(1+a_1)(1+a_2)(1+a_3)\dots(1+a_n)}$

$$x_n = \frac{a_n}{(1+a_1)(1+a_2)\dots(a_n)}, (\forall)\, n \in \mathbb{N}^*, \; x_n \geq 0 \; (\forall)\, n \in \mathbb{N}^*$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{a_{n+1}}{(1+a_{n+1})(a_n)} = ?$$

$s_n = x_1 + x_n + \dots + x_n$

$$x_k = \frac{a_k}{(1+a_1)(1+a_2)\dots(1+a_k)} = \frac{1+a_k-1}{(1+a_1)(1+a_2)\dots(1+a_k)} = \frac{1}{(1+a_1)\dots(1+a_{k-1})} - \frac{1}{(1+a_1)\dots(1+a_k)}$$

$(\forall)\, k \geq 2$

$$s_n = \frac{a_1}{1+a_1} + \frac{1}{1+a_1} - \cancel{\frac{1}{(1+a_1)(1+a_2)}} + \cancel{\frac{1}{(1+a_1)(1+a_2)}} - \frac{1}{(1+a_1)(1+a_2)(1+a_3)} + \dots + \cancel{\frac{1}{(1+a_1)(1+a_2)}}$$

$$\frac{1}{(1+a_1)(1+a_2)\dots(1+a_n)} = \frac{a_1}{1+a_1} + \frac{1}{1+a_1} - \frac{1}{(1+a_1)(1+a_2)\dots(1+a_n)}$$

$$= 1 - \frac{1}{(1+a_1)(1+a_2)\dots(1+a_n)}$$

Seria $\sum_{n=0}^{\infty} a_n$ poate fi convergentă sau divergentă având suma $+\infty$

Cazul I: $\sum_{n=0}^{\infty} a_n$ e serie divergentă cu suma $+\infty$

$\Rightarrow \lim_{n\to\infty} a_0 + a_1 + \dots + a_n = \infty$

$(1+a_1)(1+a_2)\dots(1+a_n) \geq a_1 + a_2 + \dots + a_n$ $\Bigg|$ $\Rightarrow \lim_{n\to\infty} (1+a_1)\dots(1+a_n) = \infty$

$\Rightarrow (\exists) \lim_{n\to\infty} s_n = 1 \Rightarrow \sum_{n=1}^{\infty} x_n$ e convergentă

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Cazul II: seria $\sum_{n=0}^{\infty} a_n$ e convergentă

$\Rightarrow (\exists) \lim_{n\to\infty} a_1+a_2+\ldots+a_n = l \in \mathbb{R}$

$x_n = \dfrac{a_n}{(a_1+1)(a_2+2)\ldots(a_n+n)} \leq a_n \; (\forall) n \in \mathbb{N}^*$

$\sum_{n=0}^{\infty} a_n$ e serie convergentă

$\Rightarrow \sum_{n=1}^{\infty} x_n$ e serie convergentă

TIPOLOGIA UNUI SPAȚIU METRIC

(X,d) spațiu metric $\quad \tau_d = \{\phi\} \cup \{G \in X \mid G \neq \phi, (\forall) x \in G, \exists r>0$ a.î. $B(x,r) \subseteq G\}$

PROPRIETĂȚI:

1. $B(x,r) \in \tau_d \; (\forall) x \in X, (\forall) r>0$

   $B[x,r]$ mulțime închisă $(\forall) x \in X, (\forall) r>0$

2. $A \subseteq X$

   $x \in \mathring{A} \Leftrightarrow (\exists) r>0$ a.î. $B(x,r) \subseteq A$

   $x \in \overline{A} \Leftrightarrow (\exists) r>0, B(x,r) \cap A \neq \phi$

   $x \in \overline{A} \Leftrightarrow (\exists) (x_n)_{n\in\mathbb{N}}$ șir din A a.î. $\lim_{n\to\infty} x_n = x$

   $x \in A' \Leftrightarrow (\forall) r>0, B(x,r) \cap A \setminus \{x\} \neq \phi$

   $x_0 \in \overline{A} \Leftrightarrow (\exists)(x_n)_{n\in\mathbb{N}}$ șir din A a.î. $\lim_{n\to\infty} x_n = x_0$

   $x_0 \in A' \Leftrightarrow (\exists)(x_n)_{n\in\mathbb{N}}$ șir din $A \setminus \{x_0\}$ a.î. $\lim_{n\to\infty} x_n = x_0$

   $x \in \mathrm{Iz}\,A \Leftrightarrow \exists\, r>0$ a.î. $B(x_0,r) \cap A = \{x_0\}$

Exemplu: $(\mathbb{R},d)$ spațiu metric

$d(a,b) = |a-b|$ distanța uzuală a lui $\mathbb{R}$

$\tau_d \overset{not}{=} \tau_{\mathbb{R}}$ tipologia uzuală a lui $\mathbb{R}$

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$B(x,r) = (x-r, x+r)$

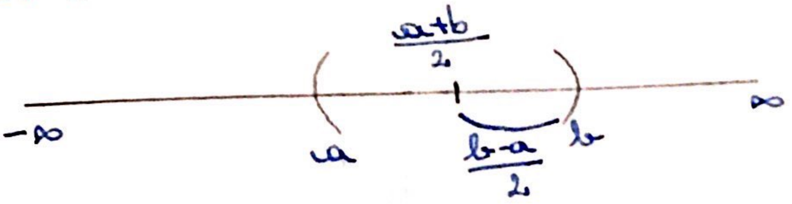
$B(x,r) = [x-r, x+r]$

Exercițiul 2: a) Fie $a < b \in \mathbb{R}$. Arătați că (a,b), $(a,-\infty)$, $(-\infty,b) \in \tau_{\mathbb{R}}$ și $[a,b]$, $[a,\infty)$, $(-\infty,b]$ sunt mulțimi închise în $\mathbb{R}$.

b) Fie $a < b$. Arătați că $[a,b)$, $(a,b]$, $\mathbb{Q}$ și $\mathbb{R}\setminus\mathbb{Q}$ nu sunt mulțimi deschise și mulțimi închise în $\mathbb{R}$.

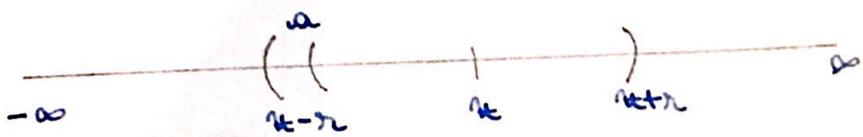
c) Arătați că $\mathbb{N}$, $\mathbb{Z}$ și orice mulțime finită a lui $\mathbb{R}$ sunt mulțimi închise în $\mathbb{R}$.

Rezolvare a) (a,b)



$B\left(\frac{a+b}{2}, \frac{b-a}{2}\right) \in \tau_{\mathbb{R}}$

(a,∞)



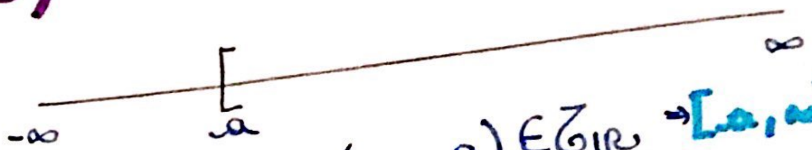
$(\forall)\ x \in (a,\infty)\ (\exists)\ r>0,\ r < x-a$ a.î. $B(x,r) \subseteq (a,\infty)$

$\Rightarrow (a,\infty) \in \tau_{\mathbb{R}}$

$(-\infty,b)$

$$\left.\begin{aligned} &(-\infty,b) = \bigcup_{k=1}^{\infty} (b-k, k) \\ &(b-k) \in \tau_{\mathbb{R}}\ (\forall)\ k \in \mathbb{N}^* \end{aligned}\right\} \Rightarrow (-\infty,b) \in \tau_{\mathbb{R}}$$

$[a,\infty)$

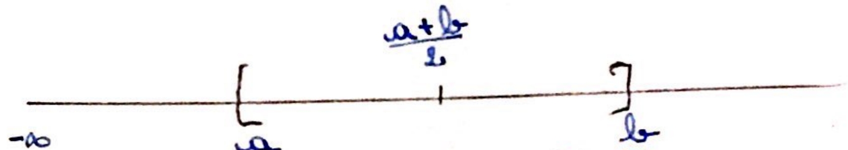


$C_{\mathbb{R}}[a,\infty) = \mathbb{R}\setminus[a,\infty) = (-\infty,a) \in \tau_{\mathbb{R}} \Rightarrow [a,\infty)$ mulț. închisă

$(-\infty, b]$

$C_{\mathbb{R}}(-\infty,b] = \mathbb{R}\setminus(-\infty,b] = (b,\infty) \in \tau_{\mathbb{R}} \Rightarrow (-\infty,b]$ mulț. închisă

$[a,b]$

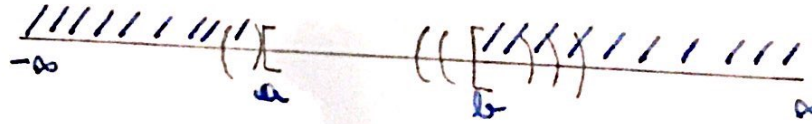


$[a,b] = B[\frac{a+b}{2}, \frac{a-b}{2}]$ mulțime închisă

Rezolvare b) $[a,b)$

$\exists)\ r>0,\ B(a,r) \not\subset [a,b) \Rightarrow [a,b) \in \tau_{\mathbb{R}}$

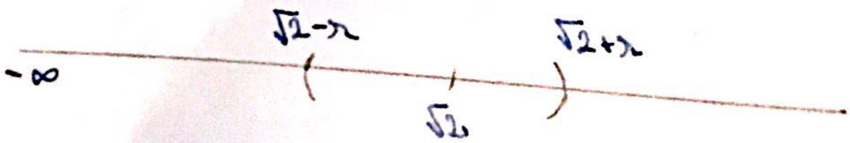
$C_{\mathbb{R}}[a,b) = \mathbb{R}\setminus[a,b) = (-\infty,a) \cup [b,\infty)$



$\forall)\ r>0,\ B(b,r) \not\subset C_{\mathbb{R}}[a,b) \Rightarrow C_{\mathbb{R}}[a,b) \notin \tau_{\mathbb{R}} \Rightarrow [a,b)$ nu e închisă

Temă: $(a,b]$, $\mathbb{Q}$

$\mathbb{R}\setminus\mathbb{Q}$



$\sqrt{2} \in \mathbb{R}\setminus\mathbb{Q}$

$(\forall)\ r>0\ (\exists)\ b \in \mathbb{Q},\ b \in (\sqrt{2}-r, \sqrt{2}+r)$

$(\forall)\ r>0,\ B(\sqrt{2}, r) \not\subseteq \mathbb{R}\setminus\mathbb{Q} \Rightarrow \mathbb{R}\setminus\mathbb{Q} \notin \tau_{\mathbb{R}}$

$C_{\mathbb{R}}(\mathbb{R}\setminus\mathbb{Q}) = \mathbb{Q}$

$(\forall)\ r>0\ (\exists)\ \beta \in \mathbb{R}\setminus\mathbb{Q}$

$\beta \in (1-r, 1+r)$

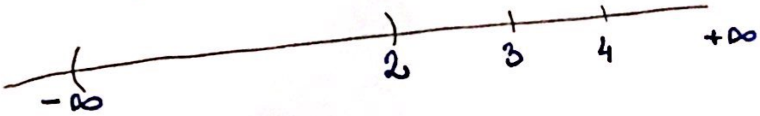
$(\forall)\ r>0,\ B(1, r) \not\subseteq \mathbb{Q} \Rightarrow \mathbb{Q} \in \tau_{\mathbb{R}} \Rightarrow$ **R\Q nu e închisă**

SEMINAR 5

Exercițiul 1: Fie $A = (-\infty, 2) \cup \{3,4\}$. Să se determine $\mathring{A}, \bar{A}, \mathrm{Fr}A$, (interior, aderența, frontieră)

$A', \mathrm{Iz}\,A$ (pct de acumul, pct izolate; derivat, continuabile)

Rezolvare: $\mathring{A} = ?$



$\mathring{A} \subseteq A$

$\left.\begin{array}{l}(-\infty, 2) \subseteq A \\ (-\infty, 2) \in \tau_{\mathbb{R}}\end{array}\right\} \Rightarrow (-\infty, 2) \subseteq \mathring{A}$

$(-\infty, 2) \subseteq \mathring{A} \subseteq A$

$(\nexists)\ r>0$ a.î. $B(3, r) \subseteq A \Rightarrow 3 \notin \mathring{A}$

$(\forall)\ r>0\ B(4, r) \subseteq A \Rightarrow 4 \notin A$

Rezolvare: $\bar{A} = ?$

$A \subseteq \bar{A}$

$A = (-\infty, 2] \cup \{3,4\}$

$\left.\begin{array}{l}(-\infty, 2] \text{ mulț. închisă} \\ \{3,4\} \text{ închisă}\end{array}\right\} \Rightarrow \bar{A} \subseteq (-\infty, 2] \cup \{3,4\}$ închisă

$A \subseteq \overline{A} \subseteq (-\infty,2)\cup\{3,4\}$

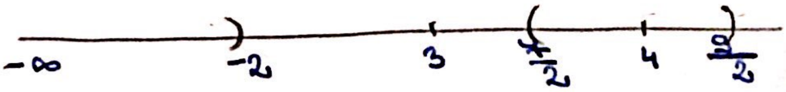
$(\forall)\ r>0,\ B(2,r)\cap A \neq \emptyset \Rightarrow 2\in\overline{A}$

$\overline{A} = (-\infty,2]\cup\{3,4\}$

$A\neq\overline{A} \Rightarrow A$ nu este închisă

$Fr\,A = \overline{A}\setminus \mathring{A} = \{2,3,4\}$

$A' \subseteq \overline{A} \Rightarrow A' \subseteq (-\infty,2)\cup\{3,4\}$



$B(4,\frac{1}{2})\cap(A\setminus\{4\})=\emptyset \Rightarrow 4\notin A'$

$B(3,\frac{1}{2})\cap(A/\{3\})=\emptyset \Rightarrow 3\notin A'$

$x_n = 2-\frac{1}{n},\ n\in\mathbb{N}^*$

$$\left.\begin{array}{l} \lim\limits_{n\to\infty} x_n = 2 \\ x_n\in A,\ (\forall)\, n\in\mathbb{N}^* \\ x_n\neq 2,\ (\forall)\, n\in\mathbb{N}^* \end{array}\right\} \Rightarrow 2\in A'$$

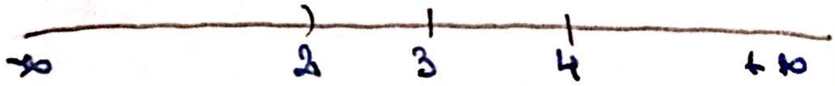
Fie $x\in(-\infty,2)$

$(\forall)\ r>0,\ B(x,r)\cap A\setminus\{x\}\neq\emptyset \Rightarrow x\in A'$

$A' = (-\infty,2]$

$f: D\subseteq\mathbb{R}\to\mathbb{R}$

$Izo\,A \subseteq A\setminus A' \Rightarrow Izo\,A = \{3,4\}$



$B(3,\frac{1}{2})\cap A = \{3\} \Rightarrow 3\in Izo\,A$

$B(4,\frac{1}{2})\cap A = \{4\} \Rightarrow 4\in Izo\,A$

$\Rightarrow Izo\,A = \{3,4\}$

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Observatie: Fie $f:(-\infty,2)\cup\{3,4\}\to\mathbb{R}$ o funcție arbitrară
f este continuă în $\{3,4\}$ ($\{3,4\}\in$ Izo A)
f nu este derivabilă în $\{3,4\}$ ($\{3,4\}\in A\setminus A'$)

	$\mathbb{Q}$	$\mathbb{R}\setminus\mathbb{Q}$	$\mathbb{N}$	$\mathbb{Z}$
$\mathring{A}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\bar{A}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{N}$	$\mathbb{Z}$
Fr A	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{N}$	$\mathbb{Z}$
A'	$\mathbb{R}$	$\mathbb{R}$	$\emptyset$	$\emptyset$
Izo A	$\emptyset$	$\emptyset$	$\mathbb{N}$	$\mathbb{Z}$

Observații: 1) Orice funcție $f:\mathbb{N}\to\mathbb{R}$ nu este derivabilă pe $\mathbb{N}$

2.) $\overline{\mathbb{Q}}=\overline{\mathbb{R}\setminus\mathbb{Q}}=\mathbb{R}$

$(\forall)\, x\in\mathbb{R}\ (\exists)\ (x_n)_{n\in\mathbb{N}}$ din $\mathbb{Q}$ a. î. $\lim\limits_{n\to\infty} x_n=x$

$(\forall)\, x\in\mathbb{R}\ (\exists)(y_n)_{n\in\mathbb{N}}$ din $\mathbb{R}\setminus\mathbb{Q}$ a. î. $\lim\limits_{n\to\infty} y_n=x$

Exercițiul 2: Fie $f:\mathbb{R}\to\mathbb{R}$ o funcție continuă a. î. $f(x+\frac{1}{n})=f(x)$ $(\forall)\ x\in\mathbb{R}$, $(\forall)\ n\in\mathbb{N}^*$. Ar că f este funcție constantă.

$f(\frac{1}{n})=f(0)$, $(\forall)\ n\in\mathbb{N}$

$f(\frac{2}{n})=f(\frac{1}{n}+\frac{1}{n})=f(\frac{1}{n})=f(0)$ $(\forall)\ n\in\mathbb{N}^*$

Dem. prin inducție că:

$f(\frac{k}{n})=f(0)$ $(\forall)\ k,n\in\mathbb{N}^*$ (după k) ①

$f(-\frac{2}{n})=f(-\frac{2}{n}+\frac{1}{n})=f(\frac{1}{n})=f(0)$

Dem prin inducție că:

$f(-\frac{k}{n})=f(0)$, $(\forall)\ k,n\in\mathbb{N}$ (după k) ②

①,② $\Rightarrow f(\frac{m}{n})=f(0)$, $(\forall)\ m\in\mathbb{Z}^*$, $n\in\mathbb{N}^*$

$\Rightarrow f(a)=f(0)$, $(\forall)\ a\in\mathbb{Q}$ ③

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Fie $x \in \mathbb{R} \Rightarrow (\exists)(y_n)_{n\in\mathbb{N}}$ din $\mathbb{Q}$ a.î. $\lim_{n\to\infty} y_n = x$

$$\left.\begin{array}{l} y_n \xrightarrow[n\to\infty]{} x \\ f \text{ continuă în } x \end{array}\right\} \Rightarrow f(y_n) \xrightarrow[n\to\infty]{} f(x)$$

$$\underbrace{\lim_{n\to\infty} f(y_n)}_{= f(0)} = f(x) \Rightarrow \lim_{n\to\infty} f(0) = f(x) \Rightarrow f(0) = f(x)\ (\forall)\ x \in \mathbb{R}$$

Exercițiul 3: Studiați continuitatea funcției $f: \mathbb{R}^2 \to \mathbb{N}$

$$f(x,y) = \begin{cases} \frac{xy^2}{x+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

f continuă pe $\mathbb{R}^2 \setminus \{(0,0)\}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\substack{x\to 0 \\ y\to 0}} \frac{xy^2}{x^2+y^2} = \lim_{n\to\infty}\left(\frac{1}{n}, \frac{1}{n}\right) = (0,0)$$

$$\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{2n} = 0$$

$$\lim_{n\to\infty}\left(\frac{1}{2n}, \frac{1}{3n}\right) = (0,0)$$

$$\lim_{n\to\infty} f\left(\frac{1}{2n}, \frac{1}{3n}\right) = \lim_{n\to\infty} \frac{\frac{1}{12n^3}}{\frac{13}{36n^2}} = \lim_{n\to\infty} \frac{36}{12\cdot 13\cdot n^2} = 0$$

Vom demonstra că $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Evaluăm $|f(x,y) - l|$

$$0 \leq |f(x,y) - 0| = \left|\frac{xy^2}{x^2+y^2}\right| = \frac{|x|y^2}{x^2+y^2} \leq \frac{|x|\cancel{y^2}}{\cancel{y^2}} = |x|$$

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$$0 \le |f(x,y) - \ell| \le g(x,y)$$

$$0 \xrightarrow{(x,y)\to(0,0)} 0 \xleftarrow{(x,y)\to(0,0)} g(x,y)$$

$$0 \le |f(x,y) - 0| \le |x| \quad (\forall)\,(x,y) \in \mathbb{R} \setminus \{(0,0)\}$$

$$0 \xrightarrow{(x,y)\to(0,0)} 0$$

$\left.\begin{array}{l} f(x,y) = \ell = 0 \\ f(0,0) = 0 \end{array}\right\} \Rightarrow f$ continuă în $(0,0)$

TEMĂ: Stud. cont. fct. $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{x^2+y^4}, & (x,y) \ne (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^4+y^4}$$

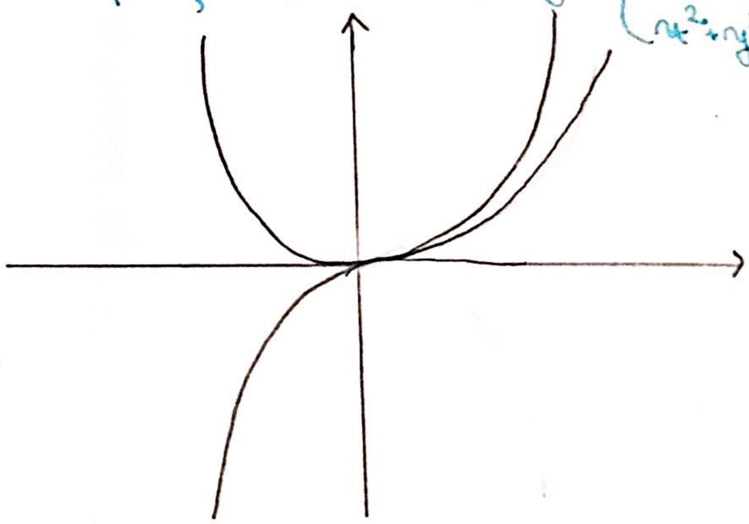
$f\left(\frac{1}{n}, 0\right)$ sau $f\left(\frac{1}{n}, \frac{1}{n^2}\right)$

SEMINAR 6

FUNCȚII CONTINUE. SERII DE FUNCȚII

Exercițiul 1: Studiați continuitatea funcției $f:\mathbb{R}^2\to\mathbb{R}, f(x,y)=\begin{cases}\frac{\sin(x^4+y^4)}{x^2+y^2}, & (x,y)\ne \\ x^2+y^2; & (x,y)=(0,\end{cases}$

f continuă pe $\mathbb{R}^2\setminus\{0,0\}$



$\left(\frac{1}{n},\frac{1}{n^2}\right)\to(0,0)$

$\left(\frac{1}{n},\frac{1}{n^2}\right)\to(0,0)$

$$\lim_{n\to\infty} g\left(\frac{1}{n},\frac{1}{n^2}\right)=\lim_{n\to\infty}\frac{\frac{1}{n^4}+\frac{1}{n^8}}{\frac{1}{n^2}+\frac{1}{n^4}}$$

$$=\lim_{n\to\infty}\frac{n^4+1}{n^4}\cdot\frac{n^4}{n^2+1}=0$$

$$\lim_{n\to\infty} g\left(\frac{1}{n},\frac{1}{n^3}\right)=\lim_{n\to\infty}\frac{\frac{1}{n^4}+\frac{1}{n^{12}}}{\frac{1}{n^2}+\frac{1}{n^8}}=\lim_{n\to\infty}\frac{n^2+1}{\cancel{n^{12}}\,n^6}\cdot\frac{\cancel{n^6}}{n^4+1}=0$$

Dem. că $(\exists)\lim\limits_{(x,y)\to(0,0)} g(x,y)=0=\ell$

$$0\le|g(x,y)-\ell|=\left|\frac{x^4+y^4}{x^2+y^2}-0\right|=\frac{x^4+y^4}{x^2+y^2}\le\frac{x^4+y^4+2x^2y^2}{x^2+y^2}\le\frac{(x^2+y^2)^2}{x^2+y^2}=$$

$$0\le|g(x,y)-\ell|\le x^2+y^2,\ (\forall)(x,y)\in\mathbb{R}^2\setminus\{0,0\}$$

0

$\Rightarrow(\exists)\lim\limits_{(y,x)\to(0,0)} f(x,y)=0$

$\Rightarrow f$ este continuă în $(0,0)$

Exercitiul 2: Fie $f:\mathbb{R}\to\mathbb{R}$ o funcție continuă pe $\mathbb{R}$, periodică cu $T>0$. Să se arate că $(\exists)\ c\in\mathbb{R}$ a.î. $f(c+\frac{T}{2})=f(c)$

REMEMBER
LA ACEST TIP DE PROBL. CONSTR. O FCT. AUXILIARĂ g

Fie $g:\mathbb{R}\to\mathbb{R}.\ g(x)=f(x)-f(x+\frac{T}{2})$

f continuă pe $\mathbb{R}$

$g(0)=f(0)-f(\frac{T}{2})$

$g(T)=f(T)-f(T+\frac{T}{2})$

$g(-\frac{T}{2})=f(-\frac{T}{2})-f(0)$

$g(\frac{T}{2})=f(\frac{T}{2})-f(T)=f(\frac{T}{2})-f(0)$

$g(0)\cdot g(\frac{T}{2})=-(f(\frac{T}{2})-f(0))^2\leq 0\Rightarrow(\exists)\ c\in[0,\frac{T}{2}]$ a. î. $g(c)=0$

$\Rightarrow f(c)-f(c+\frac{T}{2})=0\Rightarrow f(c)=f(c+\frac{T}{2})$

Exercitiul 3: Demonstrați că NU există funcții continue $f:\mathbb{R}\to\mathbb{R}$ astfel încât $f(f(x))=-x,\ (\forall)\ x\in\mathbb{R}$

REZOLVARE CU METODA REDUCERII LA ABSURD

Presupunem că $(\exists)\ f:\mathbb{R}\to\mathbb{R}$, funcție continuă astfel încât $f(f(x))=-x,\ (\forall)\ x\in\mathbb{R}$

Fie $x_1,x_2\in\mathbb{R}$ a. î. $f(x_1)=f(x_2)$

$f(x_1)=f(x_2)\ |\ f\Rightarrow f(f(x_1))=f(f(x_2))\Rightarrow -x_1=-x_2\Rightarrow x_1=x_2\Rightarrow f$ este SURJECTIVĂ

$\Rightarrow f$ este STRICT MONOTONĂ
- I. f strict crescătoare
- II. f strict descrescătoare

Cazul I: f strict crescătoare:

$a<b\Rightarrow f(a)<f(b)\ |\ f\Rightarrow f(f(a))<f(f(b))\Rightarrow -a<-b\Rightarrow a>b$ ✗

Cazul II: f strict descrescătoare:

$a<b\Rightarrow f(a)>f(b)\ |\ f\Rightarrow f(f(a))<f(f(b))\Rightarrow -a<b\Rightarrow a>b$ ✗

$\Rightarrow$ Presupunerea făcută este falsă.

Exercițiul 4: Să se arate că seria de funcții $\sum_{n=1}^{+\infty} \frac{x^n}{n}$ este absolut convergentă pe $(-1,1)$, dar nu este uniform convergentă pe $(-1,1)$

$f_n:(-1,1)\to\mathbb{R},\ f_n(x)=\frac{x^n}{n},\ (\forall)x\in(-1,1),\ (\forall)n\in\mathbb{N}^*$

Fie $x\in(-1,1)$

Demonstrăm că seria de nr. reale $\sum_{n=1}^{+\infty}\frac{x^n}{n}$ este ABSOLUT CONVERGENTĂ:

$$\sum_{n=1}^{+\infty}\left|\frac{x^n}{n}\right| = \sum_{n=1}^{+\infty}\underbrace{\frac{|x^n|}{n}}_{a_n}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{|x^{n+1}|}{n+1}\cdot\frac{n}{|x^n|}=\lim_{n\to\infty}\frac{|x|n}{n+1}=|x|<1$$

$\Rightarrow$ Seria de numere reale $\sum_{n=1}^{+\infty}\left|\frac{x^n}{n}\right|$ este CONVERGENTĂ $(\forall)x\in(-1,1)$

$\Rightarrow$ Seria de numere reale $\sum_{n=1}^{+\infty}\frac{x^n}{n}$ este ABSOLUT CONVERGENTĂ $(\forall)x\in(-1,1)$

$\Rightarrow$ Seria de funcții $\sum_{n=1}^{+\infty}\frac{x^n}{n}$ este ABSOLUT CONVERGENTĂ $(\forall)x\in(-1,1)$

Presupunem că seria de funcții $\sum_{n=1}^{\infty}$ este UNIFORM CONVERGENTĂ pe $(-1,1)$ $\Leftrightarrow (\forall)\varepsilon>0,\ (\exists)n_\varepsilon\in\mathbb{N}$ a.î.:

$$|f_{n+1}(x)+f_{n+2}(x)+\dots+f_{n+m}(x)|<\varepsilon,\ (\forall)x\in(-1,1),\ (\forall)n\geq n_\varepsilon,\ (\forall)m\in\mathbb{N}$$

$$m=n\Rightarrow\left|\frac{x^{n+1}}{n+1}+\frac{x^{n+2}}{n+2}+\dots+\frac{x^{2n}}{2n}\right|<\varepsilon,\ (\forall)x\in(-1,1),\ (\forall)n\geq n_\varepsilon$$

$$\Rightarrow\frac{x^{n+1}}{n+1}+\frac{x^{n+2}}{n+2}+\dots+\frac{x^{2n}}{2n}<\varepsilon,\ (\forall)x\in(0,1),\ (\forall)n\geq n_\varepsilon \quad ①$$

$$\frac{x^{n+1}}{n+1}+\frac{x^{n+2}}{n+2}+\dots+\frac{x^{2n}}{2n}\geq\frac{x^{2n}}{n+1}+\frac{x^{2n}}{n+2}+\dots+\frac{x^{2n}}{2n}=x^{2n}\left(\frac{1}{n+1}+\dots+\frac{1}{2n}\right)$$

$$\geq x^{2n}\left(\frac{1}{2n}+\frac{1}{2n}+\dots+\frac{1}{2n}\right)=\frac{x^{2n}}{2}\ (\forall)x\in[0,1)\ (\forall)n\in\mathbb{N}^* \quad ②$$

$$\xRightarrow{①\,\text{și}\,②}\frac{x^{2n}}{2}<\varepsilon\ (\forall)x\in[0,1),\ (\forall)n\geq n_\varepsilon$$

$$x^{2n}<2\varepsilon,\ (\forall)x\in\cancel{[0,1)},\ (\forall)n\geq n_\varepsilon\ \Big|\sup_{x\in[0,1)}\Rightarrow$$

$$\Rightarrow\sup_{x\in[0,1)}x^{2n}\leq 2\varepsilon,\ (\forall)n\geq n_\varepsilon\Rightarrow 1\leq 2\varepsilon,\ (\forall)n\geq n_\varepsilon,\ (\forall)\varepsilon>0$$

$Bn\,\underline{\ \ }\ \epsilon = \frac{1}{3} \Rightarrow 1 \le \frac{2}{3}x$

$\Rightarrow$ Seria de funcții NU este UNIFORM CONVERGENTĂ pe $(-1,1)$

Exercițiul 5: Să se demonstreze că seria de funcții $\sum_{n=1}^{+\infty} \frac{\sin(nx)}{n(n+1)}$ este absolut și uniform convergentă pe $\mathbb{R}$.

$f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{\sin(nx)}{n(n+1)}$, $(\forall)\, n \in \mathbb{N}^*$

$|f_n| = \left|\frac{\sin(nx)}{n(n+1)}\right| = \frac{|\sin(nx)|}{n(n+1)} \le \frac{1}{n(n+1)} \le \frac{1}{n^2}$, $(\forall)\, x \in \mathbb{R}$, $(\forall)\, n \in \mathbb{N}^*$ ②

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ – serie convergentă ①

①și② $\Rightarrow$ Seria de funcții $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n(n+1)}$ absolut și uniform converg pe R

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– SĂPTĂMÂNA 8 –

SEMINAR 7
FUNCȚII REALE DERIVABILE

Exercițiul 1: Să se determine punctele de extrem local ale funcției: $f(x) = \begin{cases} x^3, & x \le 0 \\ x \cdot e^{-2x}, & x > 0 \end{cases}$

f continuă pe $\mathbb{R}^*$

$\left.\begin{array}{l} l_d(0) = \lim\limits_{\substack{x\to 0\\ x>0}} f(x) = \lim\limits_{\substack{x\to 0\\ x>0}} x \cdot e^{-2x} = 0 \\ l_s(0) = \lim\limits_{\substack{x\to 0\\ x<0}} f(x) = \lim\limits_{\substack{x\to 0\\ x<0}} x^3 = 0 \end{array}\right\} \Rightarrow f$ continuă pe $\mathbb{R}$ în 0

$f(0) = 0$

f derivabilă pe $\mathbb{R}^*$

$\left.\begin{array}{l} \lim\limits_{\substack{x\to 0\\ x<0}} f'(x) = \lim\limits_{\substack{x\to 0\\ x<0}} 3x^2 = 0 \Rightarrow (\exists)\, f'_s(0) = 0 \\ \lim\limits_{\substack{x\to 0\\ x>0}} f'(x) = \lim\limits_{\substack{x\to 0\\ x>0}} \left[e^{-2x} + x(e^{-2x})(-2)\right] = 1 \Rightarrow (\exists)\, f'_d(0) = 1 \end{array}\right\} \Rightarrow f$ nu e derivabilă în 0

$$f'(x)=\begin{cases}3x^2, & x<0 \\ e^{-2x}(1-2x), & x>0\end{cases}$$

$$f'(x)=0 \Rightarrow \begin{cases}3x^2=0,\ x<0 \Rightarrow x\in\emptyset \\ e^{-2x}(1-2x)=0,\ x>0 \Rightarrow x=\frac{1}{2}\end{cases}$$

x	$-\infty$		0		$\frac{1}{2}$		∞
f'	+	+ + +	|	+ + + +	0	- - - - - - -	
f		$\nearrow$	0	$\nearrow$	$\frac{1}{2e}$	$\searrow$	

$x_0=\frac{1}{2}$ punct de maxim local

$f(x)\leq f(\frac{1}{2})$ $(\forall)$ $x\in\mathbb{R} \Rightarrow x_0=\frac{1}{2}$ punct de maxim global

Exercițiul 2: Determinați parametrul real $a>0$ care verifică inegalitatea $3^x+a^x\geq 6^x+4^x$ $(\forall)$ $x\in\mathbb{R}$

Fie $f:\mathbb{R}\to\mathbb{R}$, $f(x)=3^x+a^x-6^x-4^x$

f derivabilă pe $\mathbb{R}$

$\left.\begin{array}{l} f(x)\geq 0\ (\forall)\, x\in\mathbb{R} \\ f(0)=0 \end{array}\right\} \Rightarrow f(x)\geq f(0),\ (\forall)\, x\in\mathbb{R} \Rightarrow x_0=0$ punct de minim global

f derivabilă în x_0

$x_0\in\mathring{D} \Leftrightarrow 0\in\mathbb{R}$ (A) $\xRightarrow{\text{Th. Fermat}}$

$\xRightarrow{\text{Th. Fermat}} f'(0)=0$

$$f'(x)=3^x\ln 3+a^x\ln a-6^x\ln 6-4^x\ln 4$$

$$=\ln\frac{3a}{24}=\ln\frac{a}{8}$$

$$\ln\frac{a}{8}=0 \Rightarrow \frac{a}{8}=1 \Rightarrow a=8$$

Exercițiul 3: a) Demonstrați inegalitatea $\ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ $(\forall)\ x \in (0,\infty)$

b) Calculați $\lim\limits_{\substack{x \to 0 \\ x > 0}} \dfrac{\ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3}}{x^4}$

Rezolvare a) Pas 1: Identificăm în jurul cărui punct x_0 e scris mereu $x_0 = 0$

Pas 2: Dacă e cazul, modificăm D în funcție de x_0

Pas 3: Alegem funcția

$x_0 = 0$, $[0, \infty)$

$f : [0,\infty) \to \mathbb{R}$, $f(x) = \ln(1+x)$

f funcție de clasă C^∞ pe $[0,\infty) \Rightarrow f$ este derivabilă de 4 ori pe $[0,\infty)$

$f'(x) = \frac{1}{1+x}$

$f''(x) = \frac{-1}{(1+x)^2}$

$f'''(x) = \frac{2\cancel{(x+1)}}{(x+1)^{\cancel{4}}} = \frac{2}{(x+1)^3}$

$f''''(x) = \frac{-2\cdot 3\cancel{(x+1)^2}}{(x+1)^{\cancel{6}}} = \frac{-6}{(x+1)^4}$

$(\forall)\ x \in [0, +\infty)$, $x \neq 0$, $(\exists)\ c \in [0,+\infty)$, $c \in (0, x)$ a.î.

$$f(x) = \underbrace{f(0) + \frac{f'(0)}{1!}\cdot x + \frac{f''(0)}{2!}\cdot x^2 + \frac{f'''(0)}{3!}\cdot x^3}_{\text{Polinomul lui Taylor}} + \underbrace{\frac{f''''(c)}{4!}\, x^4}_{\text{Restul}}$$

$f(x) = \ln(x+1) =$

$$\ln(1+x) = x + \frac{(-x^2)}{2} + \frac{x^3}{3} - \frac{1}{4(x+1)^4}\cdot x^4$$

$\frac{-1}{4(x+1)}\cdot x^4 < 0 \Rightarrow \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ $(\forall)\ x \in (0,\infty)$

$\circledast$ de la a)

Rezolvare b) $\lim\limits_{\substack{x \to 0 \\ x>0}} \dfrac{\ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3}}{x^4} =$

$$= \lim\limits_{\substack{x \to 0 \\ x>0 \\ c\to 0 \\ c>0}} \dfrac{\frac{-1}{4(c+1)^4} \cdot x^4}{x^4} = -\frac{1}{4}$$

Exercițiul 4: Fie $f:(0,-\infty) \to \mathbb{R}$ o funcție derivabilă pentru care $(\exists) \lim\limits_{x\to\infty} (f(x) + f'(x)) = l \in \overline{\mathbb{R}}$. Calculați $\lim\limits_{x\to+\infty} f(x)$ $(= l)$

Ideea rezolvării: Determinarea a două funcții derivabile $g,h:(0,+\infty)$.

a.î. $\begin{cases} f(x) + f'(x) = \dfrac{g'(x)}{h'(x)} & ① \\ f(x) = \dfrac{g(x)}{h(x)} & ② \end{cases}$

cercare elem

$$\frac{f(x)+f'(x)}{1} = \frac{g'(x)}{h'(x)} \Rightarrow h'(x) = 1 \Rightarrow h(x) = x \xrightarrow{\text{în } ②} f(x) = \frac{g(x)}{h(x)} \Leftrightarrow f(x) = \frac{g(x)}{x}$$

Fie $g:(0,+\infty) \to \mathbb{R}$, $g(x) = e^x f(x)$

$h:(0,+\infty) \to \mathbb{R}$, $h(x) = e^x$

g, h derivabilă pe $(0,+\infty)$

$\lim\limits_{x\to\infty} h(x) = \lim\limits_{x\to\infty} e^x = \infty$

!!!! Ipoteza inițială trebuie verificată

$$\lim\limits_{x\to\infty} \frac{g'(x)}{h'(x)} = \lim\limits_{x\to\infty} \frac{e^x(f(x)+f'(x))}{e^x} = l \in \overline{\mathbb{R}}$$

$\xRightarrow{L'H} (\exists) \lim\limits_{x\to\infty} \dfrac{g'(x)}{h'(x)}$

$= \lim\limits_{x\to\infty} \dfrac{g(x)}{h(x)} = l$

$\Rightarrow \lim\limits_{x\to\infty} \dfrac{e^x f(x)}{e^x} = l$

Exercițiul 5: Fie $f:\mathbb{R} \to [0,+\infty)$ o funcție derivabilă cu următoarele proprietăți:

a) $0 \le f'(x) \le f(x)$, $(\forall)\, x \in \mathbb{R}$

b) $(\exists)\, a \in \mathbb{R}$ a.î. $f(a) = 0$,

demonstrați că f este funcție nulă ($f(x) = 0$, $(\forall)\, x \in \mathbb{R}$)

$f'(x) \ge 0, (\forall)\, x \in \mathbb{R} \Rightarrow f$ este crescătoare pe $\mathbb{R}$

$x \le a \Rightarrow f(x) \le f(a) \Rightarrow \left.\begin{array}{l} f(x) \le 0 \\ f(x) \ge 0 \end{array}\right| \Rightarrow f(x) = 0$

x arbitrar ales

$f(x) = 0, (\forall)\, x \in (-\infty, a]$

Fie $y > a$

$f'(x) \le f(x) \Leftrightarrow f'(x) - f(x) \le 0 \,\Big|\, \cdot e^{-x} \Rightarrow$ ~~$f'(x) \le f(x)$~~ $e^{-x}(f'(x) - f(x)) \le 0$

$g(x) = e^{-x} f(x) \mid g'(x) = -e^{-x} f(x) + f'(x) \cdot e^{-x} = e^{-x}(f'(x) - f(x))$

Fie $g:\mathbb{R} \to \mathbb{R}$, $g(x) = e^{-x} f(x)$

g derivabilă

$g'(x) \le 0\ (\forall)\, x \in \mathbb{R}$

$\Rightarrow g$ e descrescătoare pe $\mathbb{R}$

$y > a \mid g \Rightarrow g(y) \le g(a) \Leftrightarrow e^{-y} f(y) \le e^{-a} f(a)$

$\Rightarrow e^{-y} f(y) \le 0$

$\Rightarrow \left.\begin{array}{l} f(y) \le 0 \\ f(y) \ge 0 \end{array}\right| \Rightarrow f(y) = 0$

$\left.\begin{array}{l} f(y) = 0\ (\forall)\, y \in (a, +\infty) \\ f(x) = 0\ (\forall)\, x \in (-\infty, a] \end{array}\right\} \Rightarrow f(x) = 0, (\forall)\, x \in \mathbb{R}$

Temă: a) Fie $f:(0,+\infty) \to \mathbb{R}$ o funcție derivabilă pentru care $(\exists) \lim_{x \to +\infty} (f(x) + x \cdot f'(x)) = l \in \overline{\mathbb{R}}$. Calculați $\lim_{x \to +\infty} f(x)$

b) Presupunem că $(\exists) \lim_{x \to \infty} f'(x) = l \in \mathbb{R}^*$. Calculați $\lim_{x \to \infty} f(x)$

SEMINAR 8

26 NOIEMBRIE 2019
– SĂPTĂMÂNA 9 –

– FUNCȚII DIFERENȚIABILE –

Exercițiul 1: Studiați diferențiabilitatea următoarelor funcții

$$\boxed{\begin{array}{l} f: D \subset X \to Y \\ df(x_0): X \to Y \end{array}}$$

a) $f: [-1,1] \to \mathbb{R}^2$
$f(x) = (\arcsin x, \sqrt{1-x^2})$

b) $f: (0,\infty) \times (0,+\infty) \to \mathbb{R}$
$f: (x,y) = xy + \frac{2}{x} + \frac{1}{y}$

c) $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0; & (x,y) = (0,0) \end{cases}$$

Rezolvare a) $f_1: [-1,1] \to \mathbb{R}$
$f_1(x) = \arcsin x$
$f_2: [-1,1] \to \mathbb{R}$
$f_2(x) = \sqrt{1-x}$

$\left.\begin{array}{l} f_1 \text{ derivabilă pe } (-1,1) \\ f_2 \text{ derivabilă pe } (-1,1) \end{array}\right\} \Rightarrow f \text{ derivabilă pe } (-1,1) \Rightarrow f \text{ e diferențiabilă pe } (-1,1)$

$$f'(x) = ((\arcsin(x))', (\sqrt{1-x^2})') = \left(\frac{1}{\sqrt{1-x^2}}, \frac{-x}{\sqrt{1-x^2}}\right), \forall x \in (-1,1)$$

$df(0): \mathbb{R} \to \mathbb{R}^2$

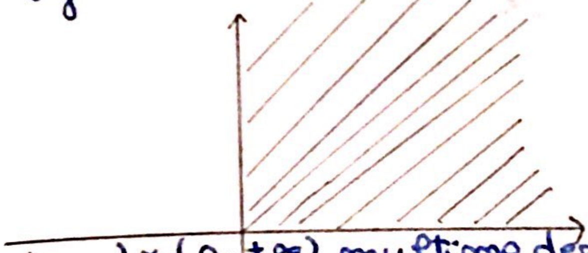
$$df(0)(x) = x \cdot f'(0) = x \cdot (1,0)$$
$$= (x,0)\ \forall x \in \mathbb{R}.$$

Rezolvare b) $f:(0,\infty)\times(0,+\infty)\to\mathbb{R}$,
$f:(x,y)=xy+\frac{2}{x}+\frac{1}{y}$

f continuă pe $(0,\infty)\times(0,\infty)$

$\frac{\partial f}{\partial x}(x,y)=\left(xy+\frac{2}{x}+\frac{1}{y}\right)'_x=y-\frac{2}{x^2}$, $(\forall)y,x\in(0,\infty)\times(0,\infty)$

$\frac{\partial f}{\partial y}(x,y)=\left(xy+\frac{2}{x}+\frac{1}{y}\right)'_y=x-\frac{2}{y^2}$, $(\forall)x,y\in(0,\infty)\times(0,\infty)$



$(0,+\infty)\times(0,+\infty)$ mulțime deschisă

$\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ funcții continue pe $(0,\infty)\times(0,\infty)$

$df(1,1):\mathbb{R}^2\to\mathbb{R}$

$df(1,1)(x,y)=x\cdot\frac{\partial f}{\partial x}(1,1)+y\frac{\partial f}{\partial y}=x\left(1-\frac{2}{1^2}\right)+y\left(1-\frac{1}{1^2}\right)=-x$, $(\forall)(x,y)\in\mathbb{R}^2$

Rezolvare c) $f:\mathbb{R}^2\to\mathbb{R}$
$$f(x,y)=\begin{cases}\frac{xy^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$$\frac{\partial f}{\partial x}(x,y)=\left(\frac{xy^2}{x^2+y^2}\right)'=\frac{y^2(x+y^2)-xy^2\cdot 2x}{(x^2+y^2)^2}=\frac{y^2x^2+y^4-2x^2y^2}{(x^2+y^2)^2}\ (\forall)(x,y)\in\mathbb{R}\setminus\{(0,0)\}$$

$$\frac{\partial f}{\partial y}(x,y)=\left(\frac{xy^2}{x^2+y^2}\right)'=\frac{2xy(x^2+y^2)-xy^2\cdot 2y}{(x^2+y^2)^2}$$

$$=\frac{2x^3y+2xy^3-2xy^3}{(x^2+y^2)^2}=\frac{2x^3y}{(x^2+y^2)^2},\ (\forall)(x,y)\in\mathbb{R}\setminus\{(0,$$

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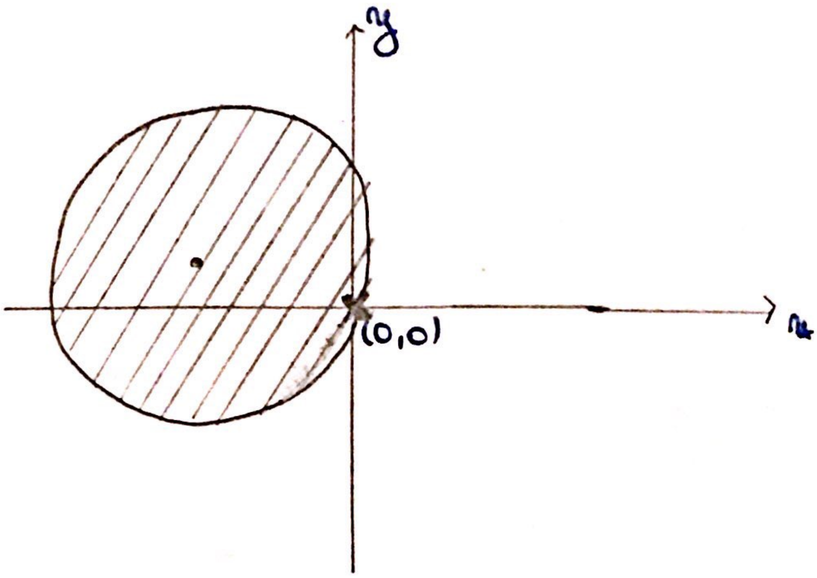


f este diferențiabilă pe $\mathbb{R}^2 \setminus \{(0,0)\}$

În (0,0) aplicăm definiția

$$\lim_{t\to 0}\frac{f(x_0+t\cdot e_1)-f(x_0)}{t} = \lim_{t\to 0}\frac{f((0,0)+t(1,0))-f(0,0)}{t}$$

$$= \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t}$$

$$= \lim_{t\to 0}\frac{\frac{t\cdot 0^2}{t^2+0^2}+0}{t} = \lim_{t\to 0}\frac{\frac{0}{t^2}+0}{t} =$$

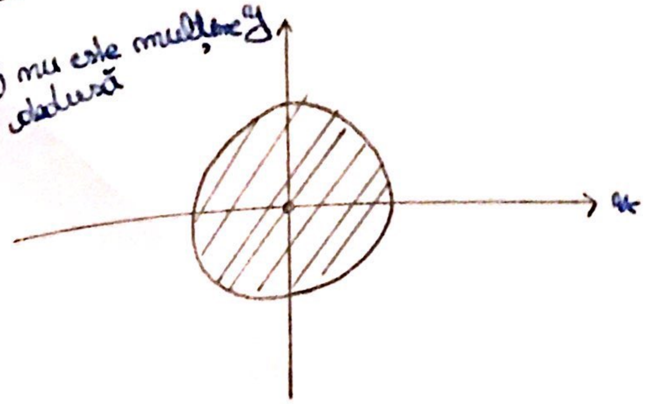
$$= \lim_{t\to 0}\frac{0}{t} = 0 \in \mathbb{R} \Rightarrow (\exists)\frac{\partial f}{\partial x}=0$$

$$\lim_{t\to 0}\frac{f(x_0+t\cdot e_2)-f(x_0)}{t} = \lim_{t\to 0}\frac{f((0,0)+t(0,1))-f(0,0)}{t}$$

$$= \lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t}$$

$$= \lim_{t\to 0}\frac{\frac{0\cdot t^2}{0^2+t^2}+0}{t} = 0 \in \mathbb{R} \Rightarrow (\exists)\frac{\partial f}{\partial y}=0$$

$\{(0,0)\}$ nu este mulțime închisă

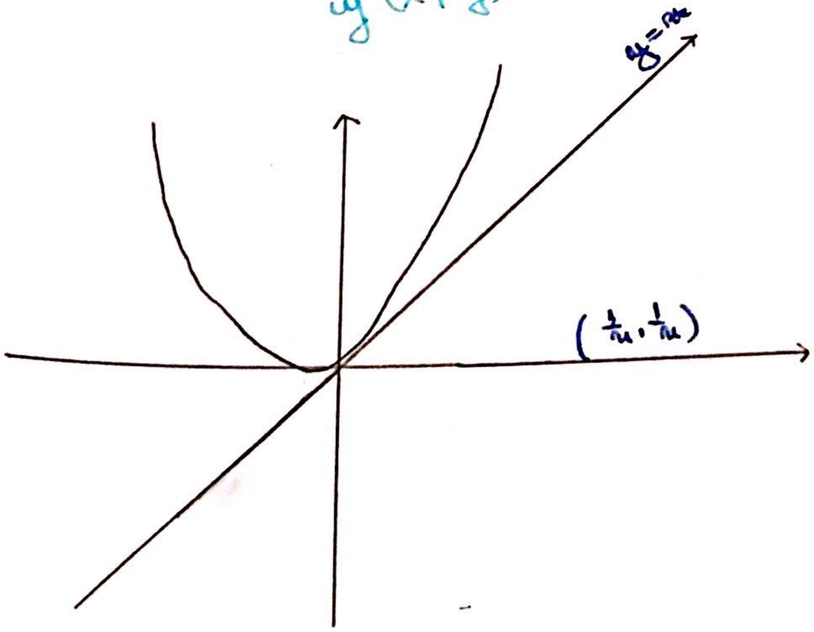


$T:\mathbb{R}^2 \to \mathbb{R}$

$T(x,y) = x\dfrac{\partial f}{\partial y}(0,0) + y\cdot\dfrac{\partial f}{\partial y}(0,0) = x\cdot 0 + y\cdot 0 = 0\ (\forall)(x,y)\in\mathbb{R}^2$

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y)-f(0,0)-T((x,y)-(0,0))|}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{\left|\frac{xy^2}{x^2+y^2}-0-0\right|}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \underbrace{\frac{|x|\cdot y^2}{(x^2+y^2)\cdot\sqrt{x^2+y^2}}}_{g(x,y)}$$



- domeniul de definiție $g(x,y)$

$$\lim_{n\to\infty} g\left(\frac{1}{n},\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^3}}{\frac{2}{n^2}\cdot\frac{\sqrt{2}}{n}} = \frac{1}{2\sqrt{2}}$$

$$\lim_{n\to\infty} g\left(\frac{1}{n},\frac{1}{n^2}\right) = \lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{1}{n^4}\right)\sqrt{\frac{1}{n^2}} \lim_{n\to\infty} \frac{\frac{1}{n^4}}{\frac{n^2+1}{n^2}} = \lim_{n\to\infty} \frac{n}{} \ ?$$

$\Rightarrow (\nexists)$

?

Exercițiul 2: Să se determine punctele critice ale funcției $f:(0,\infty)\times(0,\infty)\to\mathbb{R}$, $f(x,y) = xy + \frac{2}{x} + \frac{1}{y}$

Definiție: Fie $f: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ și $x_0 \in D \cap D'$. Elementul x_0 este PUNCT CRITIC al funcției f dacă f este diferențiabilă în x_0 și $df(x_0)=0$.

Se studiază diferențiabilitatea funcției f.

Conform exercițiului 1.b) f diferențiabilă pe $(0,+\infty)\times(0,+\infty)$

$$df(x,y)=0,\ (x,y)\in(0,\infty)\times(0,\infty) \Leftrightarrow \begin{cases} \frac{\partial f}{\partial x}(x,y)=0 \\ \frac{\partial f}{\partial y}(x,y)=0 \end{cases} \Leftrightarrow \begin{cases} y-\frac{2}{x^2}=0 \\ x-\frac{1}{y}=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y=\frac{2}{x^2} \\ \frac{1}{y^2}=x \end{cases} \Leftrightarrow \frac{1}{\frac{4}{x^4}} = x \Leftrightarrow \frac{x^4}{4} = x \Leftrightarrow x^4 - 4x = 0 \Leftrightarrow x(x^3-4)=0$$

$$\Rightarrow \begin{array}{l} x_1 = 0 \notin (0,\infty) \\ x_2 = \sqrt[3]{4} \in (0,+\infty) \end{array} \Rightarrow y = \frac{2}{\sqrt[3]{16}} = \frac{\cancel{2}}{\cancel{2}\sqrt{2}} = \frac{1}{\sqrt{2}} \in (0,\infty)$$

Punct critic $\left(\sqrt[3]{4}, \frac{1}{\sqrt{2}}\right)$

2 DECEMBRIE 2019
- SĂPTĂMÂNA 10 -

SEMINAR 9

- PUNCT DE EXTREM LOCAL. TEOREMA FCȚ. IMPLICITE -

Exercițiul 1: Determinați punctele de extrem local ale funcției $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 - 2x^2y + 3$.

PASUL 1: Se studiază continuitatea funcției și se identifică pct. de discontinuitate.

PASUL 2: Se studiază diferențiabilitatea funcției și se identifică punctele în care f nu este diferențiabilă.

$$\frac{\partial f}{\partial x}(x,y) = (x^3 - 2x^2y + xy^2 + 3)'_x = 3x^2 - 4xy + y^2, (\forall)(x,y) \in \mathbb{R}^2$$

$$\frac{\partial f}{\partial y}(x,y) = (x^3 - 2x^2y + xy^2 + 3)'_x = -2x^2 + 2xy, (\forall)(x,y) \in \mathbb{R}^2$$

$\mathbb{R}^2$ mulțime deschisă

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ fcț. continue $\Rightarrow$ f este diferențiabilă pe $\mathbb{R}$.

PASUL 3: $\mathbb{R}^2$ $\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases}$

PASUL 4: Se studiază diferențiabilitatea de ord 2 și să se identifice punctele în care f nu este diferențiabilă de două ori.

$$\to \frac{\partial^2 f}{\partial^2 x}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) = (3x^2 - 4xy + y^2)'_x = 6x - 4y \ (\forall)(x,y) \in \mathbb{R}^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y) = (3x^2 - 4xy + y^2)' = -4x + 2y, (\forall)(x,y) \in \mathbb{R}^2$$

$$\frac{\partial^2 f}{\partial^2 y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y) = (-2x^2 + 2yx)'_y = 2x, (\forall)(x,y) \in \mathbb{R}^2$$

$\mathbb{R}^2$ mulțime deschisă

$\frac{\partial^2 f}{\partial^2 x}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial^2 y}$ funcții continue

$\Rightarrow f$ e diferențiabilă de două ori pe $\mathbb{R}^2$

PASUL 5: (Aplicarea crit. în pct. critice în care f este diferențiabilă de două ori și identif. pct. critice în care nu se pronunță)

$$H_f(\alpha,\alpha) = \begin{pmatrix} \frac{\partial^2 f}{\partial^2 x}(\alpha,\alpha) & \frac{\partial^2 f}{\partial x \partial y}(\alpha,\alpha) \\ \frac{\partial^2 f}{\partial y \partial x}(\alpha,\alpha) & \frac{\partial^2 f}{\partial^2 y}(\alpha,\alpha) \end{pmatrix} = \begin{pmatrix} 2\alpha & -2\alpha \\ -2\alpha & 2\alpha \end{pmatrix}$$

$\Delta_1 = 2\alpha$

$\Delta_2 = \begin{vmatrix} 2x & -2\alpha \\ -2x & 2\alpha \end{vmatrix} = 4\alpha^2 - \alpha^2 = 0$

I $\alpha < 0 \Rightarrow \Delta_1 < 0,\ \Delta_2 = 0 \Rightarrow$ nu se pronunță

II $\alpha > 0 \Rightarrow \Delta_1 > 0,\ \Delta_2 = 0 \Rightarrow$ —— " ——

III $\alpha = 0 \Rightarrow \Delta_1 = 0,\ \Delta_2 = 0 \Rightarrow$ —— " ——

PASUL 6: Se aplică def. pct de extrem local în urm. categorii de pct.:

a) Puncte de discontinuitate

b) Puncte în care fct. f nu este diferențiabilă

c) Puncte în care f nu ~~există~~ este dif de două ori

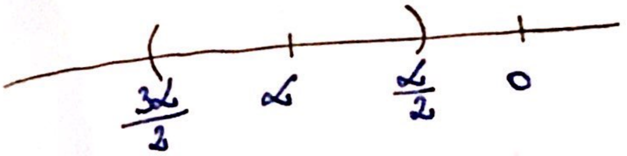
d) Pct. critice în care crit nu se pronunță

I: $\alpha<0$:

$f(x,y)-f(\alpha,\alpha)=x^3-2x^2y+y^2x+3-(\alpha^3-2\alpha^3+\alpha^3+3)$

$=x^3-2x^2y+xy^2$

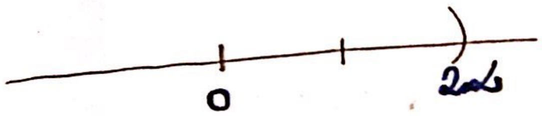
$=x(x-y)^2\le 0 \quad (\forall)(x,y)\in\left(\frac{3\alpha}{2},\frac{\alpha}{2}\right)\times\left(\frac{3\alpha}{2},\frac{\alpha}{2}\right)$



$\Rightarrow$ (α,α) punct de maxim local

II: $\alpha>0$:

$f(x,y)-f(\alpha,\alpha)=x(x-y)^2\ge 0 \quad (\forall)(x,y)\in(0,2\alpha)\times(0,2\alpha)$



$\Rightarrow$ (α,α) punct de minim

III: $\alpha=0$:

$f(x,y)-f(\alpha,\alpha)=x(x-y)^2$

$f\left(\frac{1}{n},\frac{1}{n^2}\right)-f(0,0)=\frac{1}{n}\left(\frac{1}{n}-\frac{1}{n^2}\right)^2>0 \qquad n\in\mathbb{N}^*$

$f\left(-\frac{1}{n},\frac{1}{n^2}\right)-f(0,0)=-\frac{1}{n}\left(-\frac{1}{n}-\frac{1}{n^2}\right)^2<0$

$\Rightarrow$ $(0,0)$ nu este punct de extrem

$E=\{(\alpha,\alpha)\mid \alpha\in\mathbb{R}^*\}$

Exercițiul 2: Să se demonstreze că ecuația $2x^2+2y^2+z^2-8xz-z+8=0$ are soluții definite implicit sub forma $z=f(x,y)$ pe o vecinătate a pct $(2,0,1)$. Calc $\frac{\partial z}{\partial x}(2,0)$ și $\frac{\partial z}{\partial y}(2,0)$

Fie $f:\mathbb{R}^3\to\mathbb{R}$, $f(x,y,z)=2x^2+2y^2+z^2-8xz-z+8$

$\mathbb{R}^3$ mulțime deschisă

$$\frac{\partial f}{\partial x}(x,y,z) = 4x - 8z \quad (\forall)\ (x,y,z) \in \mathbb{R}^3$$

$$\frac{\partial f}{\partial y}(x,y,z) = 4y \quad (\forall) \quad (x,y,z) \in \mathbb{R}^3$$

$$\frac{\partial f}{\partial z}(x,y,z) = 2z - 8x - 1 \quad (\forall)(x,y,z) \in \mathbb{R}^3$$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ fct. continuă, $f \in C^1(\mathbb{R}^3)$

$f(2,0,1) = 8 + 0 \cancel{+1} - 16 \cancel{-1} + 8 = 0$

$\frac{\partial f}{\partial z}(2,0,1) = 2 - 16 - 1 = -15 \neq 0$

$\underset{x_0}{(2,0)}\ \underset{y_0}{(1)}$ $\Big|$ $\Rightarrow (\exists)\ r_1, r_2 > 0$ a. î. $B((2,0), r_1) \times B(1, r_2) \subseteq \mathbb{R}^3$

$(\exists!) \varphi: B((2,0), r_1) \to B(1, r_2)$ fct. de clasă C^1

care următoarele proprietăți:

1) $\varphi((2,0)) = 1$

2) $f(x, y, \varphi(x,y)) = 0\ (\forall)(x,y) \in B((2,0), r_1)$

$f(x, \varphi(x)) = 0$

necunoscută secundară

necunoscută principală

Ec. $f(x,y,z) = 0$ care sol. def. implicit sub forma $z = \varphi(x,y)$

$(\forall)\ z = \varphi(x,y)\ (\forall)\ (x,y) \in B((2,0), r_1)$

$$\frac{\partial z}{\partial x}(2,0) \overset{not}{=} \frac{\partial \varphi}{\partial x}(2,0) = -\frac{\frac{\partial f}{\partial x}(2,0,1)}{\frac{\partial f}{\partial z}(2,0,1)} = \frac{0}{-15} = 0$$

$$\frac{\partial z}{\partial y}(2,0) \overset{not}{=} \frac{\partial \varphi}{\partial y}(2,0) = \frac{\frac{\partial f}{\partial y}(2,0,1)}{\frac{\partial f}{\partial z}(2,0,1)} = \frac{0}{-15} = 0$$

SEMINAR 10

- FUNCȚII INTEGRABILE RIEMANN -

Exemplul 1: Să se studieze integrabilitatea Riemann a funcției

$f:[0,1] \to \mathbb{R}$, $f(x) = \begin{cases} \frac{e^{2x}}{e^x + e^{1-x}}, & x \in (0,1) \\ 10, & x = 0 \\ -1, & x = 1 \end{cases}$

f e continuă pe $(0,1) \Rightarrow D_f \subseteq \{(0,1)\} \Rightarrow D_f$ mulțimea finită sau vidă

$\Rightarrow D_f$ mulțime neglijabilă Lebesgue

Se observă că $f(x) \geq -1$ $(\forall)\, x \in [0,1]$

$\frac{e^{2x}}{e^x + e^{1-x}} \leq \frac{e^{2x}}{e^x} = e^x \leq e$ $(\forall)\, x \in [0,1]$

$\Rightarrow f(x) \leq 10$ $(\forall)\, x \in [0,1]$

$\Rightarrow f$ este mărginită ②

①, ② $\Rightarrow f \in \mathcal{R}([0,1])$

Fie $g:[0,1] \to \mathbb{R}$, $g(x) = \frac{e^{2x}}{e^x + e^{1-x}}$ $(\forall)\, x \in [0,1]$

$g \in \mathcal{R}([0,1])$

$\{x \in [0,1] \mid f(x) \neq g(x)\} = \{0,1\}$ mulțime finită $\Rightarrow \int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx = \int_0^1 \frac{e^{2x}}{e^x + e^{1-x}}\,dx$

$u = e^x$

$dx = \frac{1}{u}\,du$

$$\int_0^1 \frac{e^{2x}}{e^x + e^{1-x}}\,dx = \int_1^e \frac{u^2}{u + \frac{e}{u}} \frac{du}{u} = \int_1^e \frac{u^2}{u^2+e}\,du = \int_1^e \frac{u^2+e-e}{u^2+e}\,du$$

$$= \int_1^e 1\,du = \int_1^e 1\,du - \int_1^e \frac{e}{u^2+e}\,du = \left(u - \cancel{e}^{\sqrt{e}} \cdot \frac{1}{\cancel{\sqrt{e}}} \cdot \operatorname{arctg} \frac{u}{\sqrt{e}}\right)\Bigg|_1^e$$

$$= e - \sqrt{e}\operatorname{arctg}\sqrt{e} - \left(1 - \sqrt{e}\operatorname{arctg}\frac{1}{\sqrt{e}}\right) = e - 1 + \sqrt{e}\left(\operatorname{arctg}\frac{1}{\sqrt{e}} - \operatorname{arctg}\sqrt{e}\right).$$

Exercițiul 2: Să se calculeze urm. limite:

a) $\lim\limits_{n \to \infty} \int_0^1 \frac{x^{2n+3}}{4+x^{3n}}\,dx$

b) $\lim\limits_{\substack{x \to 0 \\ x>0}} \int_x^{2x} \frac{\operatorname{arctg} t}{t}\,dt$

c) $\lim\limits_{\substack{x \to 0 \\ x \to 0}} \int_x^{2x} \frac{\operatorname{arctg} t}{t^2}\,dt$

Rezolvare 2. a) $\lim\limits_{n \to \infty} \int_0^1 \frac{x^{2n+3}}{4+x^{3n}}\,dx$

$f_n : [0,1] \to \mathbb{R},\ f_n(x) = \frac{x^{2n+3}}{4+x^{3n}}\ (\forall)\, n \in \mathbb{N}^*$

fie $x \in [0,1]$

Calculați $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{x^{2n+3}}{4+x^{3n}} = \begin{cases} \frac{3}{4}, & x \in [0,1) \\ \frac{4}{5}, & x = 1 \end{cases}$

$A = [0,1]$

$f : [0,1] \to \mathbb{R},\ f(x) = \begin{cases} \frac{3}{4}, & x \in [0,1) \\ \frac{4}{5}, & x = 1 \end{cases}$

$f_n \xrightarrow[{[0,1]}]{s} f.$

$\left.\begin{array}{l} f \text{ nu este continuă în } x = 1 \\ f_n \text{ cont în } x = 1\ (\forall)\, n \in \mathbb{N}^* \end{array}\right\} \Rightarrow f_n \cancel{\xrightarrow[{[0,1]}]{u}} f$

$|f_n(x)| = \left|\frac{x^{2n+3}}{4+x^{3n}}\right| = \frac{x^{2n+3}}{4+x^{3n}} \le \frac{4}{4+x^{3n}} \le 1,\ (\forall)\, x \in [0,1]\ (\forall)\, n \in \mathbb{N}$

$f_n \xrightarrow[{[0,1]}]{\Delta} f$

$|f_n(x)| \le 1\ (\forall)\, x \in [0,1],\ (\forall)\, n \in \mathbb{N}^*$

$f_n \in \mathcal{R}([0,1])\ (\forall)\, n \in \mathbb{N}^*$

$f \in \mathcal{R}([0,1])$

$\Rightarrow \lim\limits_{n\to\infty} \int_0^1 \frac{x^{2n+3}}{4+x^{3n}}\,dx = \int_0^1 \lim\limits_{n\to\infty} \frac{x^{2n+3}}{4+x^{3n}} = \frac{3}{4} x \Big|_0^1 = \frac{3}{4}$

Rezolvare 2b) $\lim\limits_{x \to 0} \int_x^{2x} \frac{\operatorname{arctg} t}{t}\, dt$

Fie $f:(0,\infty) \to \mathbb{R}$, $f(t) = \frac{\operatorname{arctg}(t)}{t}$

f e continuă pe $(0,\infty)$

Fie $x>0 \Rightarrow f|_{[x,2x]} \in \mathcal{R}([x,2x])$

Aplicăm teorema de medie pentru $f|_{[x,2x]} \Rightarrow (\exists)\, c \in (x,2x)$ a.î.

$\int_x^{2x} f(t)\, dt = f(c)(b-a) = f(c)\cdot x = \frac{\operatorname{arctg} c}{c} \cdot x.$

$$\lim_{\substack{x\to 0\\ x>0}} \int_x^{2x} \frac{\operatorname{arctg} t}{t}\, dt = \lim_{\substack{x\to 0\\ c\to 0}} \underbrace{\frac{\operatorname{arctg} c}{c}}_{\downarrow\, 1} \cdot \underbrace{x}_{\downarrow\, 0} = 0$$

$$x \le c \le 2x$$
$$\searrow \quad \downarrow \quad \swarrow$$
$$0$$

Rezolvare 2c) $\lim\limits_{\substack{t\to 0\\ x\to 0}} \int_x^{2x} \frac{\operatorname{arctg} t}{t^2}\, dt$

Fie $f,g:(0,\infty) \to \mathbb{R}$, $f(t) = \frac{\operatorname{arctg} t}{t}$, $g(t) = \frac{1}{t}$

f, g cont pe $(0,\infty)$

Fie $x>0$

$f|_{[x,2x]}, g|_{[x,2x]} \in \mathcal{R}([x,2x])$

$g(t) \ge 0$ $(\forall)\, t \in [x,2x]$

f are prop lui Darboux

$\Rightarrow (\exists)\, c \in (x,2x)$ a.î. $\int_x^{2x} f(t)\cdot g(t)\, dt = \frac{\operatorname{arctg} c}{c} \cdot \int_x^{2x} \frac{1}{t}\, dt = \frac{\operatorname{arctg} c}{c} \cdot \ln t \Big|_x^{2x}$

$= \frac{\operatorname{arctg} c}{c} \cdot \ln 2$

$$\lim_{x\to 0} \int_x^{2x} \frac{\operatorname{arctg} t}{t^2}\, dt = \lim_{\substack{x\to 0\\ c\to 0}} \frac{\operatorname{arctg} c}{c} \cdot \ln 2 = \ln 2$$

Exercițiul 3 : Fie $f:[1,2] \to \mathbb{R}$ o funcție continuă astfel că $\int_1^2 f(t)dt = 0$
$(\exists)\, c \in (1,2)$ a. î. $\int_1^c f(x)\,dx = c \cdot f(c) \iff \frac{1}{c}\int_1^c f(x)\,dx = f(c)$

$g(x) = \int_1^x f(t)\,dt$

$\left.\begin{array}{l} g(1)=0 \\ g(2)=0 \end{array}\right| \Rightarrow (\exists)\, c \in (1,2)$ a. î. $g'(c)=0$

$g'(x) = f(x)\ (\forall)\, x \in (1,2) \Rightarrow (\exists)\, c \in (1,2)$ a. î. $f(c)=0$

fie $g:[1,2] \to \mathbb{R},\ g(x) = \frac{1}{x}\int_1^x f(t)\,dt$

$g(1)=0$
$g(2) = \frac{1}{2}\int_1^2 f(t)\,dt = 0$

$g'(x) = -\frac{1}{x^2}\int_1^x f(t)\,dt + \frac{1}{x}\left(\int_1^x f(t)\,dt\right)'$

$= \frac{-1}{x^2}\int_1^x f(t)\,dt + \frac{1}{x} f(x)\ (\forall)\, x \in [1,2]$

Aplicăm teorema lui Rolle fct. g
$(\exists)\, c \in (1,2)$ a. î. ș $g'(c)=0$
$\Rightarrow (\exists)\, c \in (1,2)$ a. î. $\frac{-1}{c^2} \cdot \int_1^c f(t)\,dt + \frac{1}{c} f(c) = 0$

$\frac{f(c)}{c} = \frac{1}{c^2}\int_1^c f(t)\,dt$
$\Rightarrow c \cdot f(c) = \int_1^c f(t)\,dt$

abcd

SEMINAR 11

16 DECEMBRIE 2019 – SĂPTĂMÂNA 12 –

– INTEGRALE IMPROPRII –

Exercițiul 1: Studiați natura integralelor improprii

a) $\int_{0+0}^{1} \frac{1}{x^2}$ și $\int_{0+0}^{1} \frac{tg\,x}{x^3}$

Fie $f, g: (0,1] \to \mathbb{R}$, $f(x) = \frac{1}{x}$, $g(x) = \frac{tg\,x}{x^3}$

f, g continue pe $(0,1] \Rightarrow f, g \in R_{loc}((0,1])$

$f(x) > 0, (\forall) x \in (0,1]$

$g(x) > 0, (\forall) x \in (0,1]$

$\int_{0+0}^{1} \frac{1}{x^2} dx = -\frac{1}{x}\Big|_{0+0}^{1} = -\frac{1}{1} - \lim_{\substack{x\to 0\\ x>0}} \left(-\frac{1}{x}\right) = +\infty$

$\int_{0+0}^{1} f(x)\,dx = +\infty \Rightarrow \int_{0+0}^{1} f(x)\,dx$ divergentă.

Se obs. că prin metoda clasice NU știm să calculăm.

$\int_{0+0}^{1} \frac{tg\,x}{x^3} dx$. O vom compara folosind f. fol. crit. cu ineg.

$f(x) - g(x) = \frac{1}{x^2} - \frac{tg\,x}{x^3} = \frac{x - tg\,x}{x^3} > 0$

$h'(x) = 1 - (1 + tg^2 x) = -tg^2 x < 0\ (\forall) x \in (0,1] \Rightarrow h$ e s. ↘ pe $(0,1]$

$\Rightarrow h(x) < \lim_{x\to 0} h(x)\ (\forall) x \in (0,1] \Rightarrow h(x) < 0\ (\forall) x \in (0,1] \Rightarrow f(x) < g(x)\ (\forall) x \in (0,1]$ (1)

$\int_{0+0}^{1} f(x)\,dx$ e diverg. (2)

(1), (2) $\Rightarrow \int_{0+0}^{1} g(x)$ e diverg

b) Fie $f: [1,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{\sqrt[7]{x^5} + \sqrt[11]{x^4}}$

f continuă pe $[1,\infty) \Rightarrow f \in R_{loc}([1,\infty))$

$f(x) > 0, (\forall) x \in [1,\infty)$

$f(x) = \frac{1}{x^{\frac{5}{7}} + x^{\frac{4}{11}}}$; $\frac{5}{7} > \frac{4}{11}$ $\left(\frac{55}{77}\right) > \left(\frac{28}{77}\right)$ $\Rightarrow$ deci $g: [1,\infty) \to \mathbb{R}$, $g(x) = \frac{1}{x^{\frac{5}{7}}}$ (F.C.F)

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{\frac{5}{7}}}{x^{\frac{5}{7}} + x^{\frac{4}{11}}} = 1 \in (0,\infty)$

$\lim_{x\to\infty} \frac{f(x)}{g(x)} \in (0,\infty) \Rightarrow \int_1^{\infty} f(x)$ și $\int_1^{\infty} g(x)\,dx$ au aceeași natură

$\int_1^{\infty} g(x)\,dx = \int_1^{\infty} \frac{1}{x^{\frac{5}{7}}} dx = \frac{x^{-\frac{5}{7}+1}}{-\frac{5}{7}+1}\Big|_1^{\infty} = \frac{7}{2} x^{\frac{2}{7}}\Big|_1^{\infty} = \lim_{x\to\infty} \frac{7}{2} x^{\frac{2}{7}} - \frac{7}{2} = \infty$

$\Rightarrow \int_1^{\infty} g(x)$ diverg $\Rightarrow \int_1^{\infty} f(x)\,dx$ diverg.

Temă: Stab. natura urm. integr. improprii:

$\int_{0+0} \frac{}{\sqrt{x^5} + \sqrt[4]{x^4}}$

Exercițiul 2: Calculați următoarele integrale improprii:

a) $\int_0^{\frac{\pi}{2}} \sqrt{\sin^5 x \cos^3 x}\,dx = \int_0^{\frac{\pi}{2}} \sin^{\frac{5}{2}} x \cos^{\frac{3}{2}} x\,dx = I$

$\beta(p,q) = 2\int_{0+0}^{\frac{\pi}{2}-0} \sin^{2p-1} x \cdot \cos^{2q-1} x\,dx$

$2p-1 = \frac{5}{2} \iff p = \frac{7}{4}$

$2q-1 = \frac{3}{2} \iff q = \frac{5}{4}$

$I = \frac{1}{2}\beta\left(\frac{7}{4},\frac{5}{4}\right) = \frac{1}{2}\frac{\Gamma(\frac{7}{4})\Gamma(\frac{5}{4})}{\Gamma(3)} = \frac{\Gamma(\frac{7}{4})\Gamma(\frac{5}{4})}{4}$

$\Gamma(\frac{7}{4}) = \Gamma(\frac{3}{4}+1) = \frac{3}{4}\Gamma(\frac{3}{4})$

$I = \frac{\frac{3}{4}\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{4} = \frac{\frac{3}{16}\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{4} = \frac{\frac{3}{16}\Gamma(1-\frac{1}{4})\Gamma(\frac{1}{4})}{4}$

$= \frac{\frac{3}{16}\frac{\pi}{\sin\frac{1}{4}\pi}}{4} = \frac{3\pi}{\cancel{16}_{8}}\frac{\cancel{2}}{\sqrt{2}} = \frac{3\pi}{8\sqrt{2}} = \frac{3\pi}{32\sqrt{2}}$

b) $\int_{0+0}^{+\infty} \frac{1}{(1+x^2)^4}\,dx = I$

$\beta(p,q) = \int_{0+0}^{\infty} \frac{x^{p-1}}{(1+x)^{p+q}}\,dx$

Notăm: $t = x^2 \Rightarrow x = \sqrt{t}$

$dx = \frac{1}{2\sqrt{t}}\,dt$

$x \to 0,\ x>0 \Rightarrow t \to 0,\ t>0$

$x \to +\infty \Rightarrow t \to +\infty$

$I = \int_{0+0}^{\infty} \frac{1}{(1+t)^4}\cdot\frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\int_{0+0}^{\infty}\frac{t^{-\frac{1}{2}}}{(1+t)^4}\,dt$

$\begin{cases} p-1 = -\frac{1}{2} \to p = \frac{1}{2} \\ p+q = 4 \to q = \frac{7}{2} \end{cases}$

$I = \frac{1}{2}\beta\left(\frac{1}{2},\frac{7}{2}\right) = \frac{1}{2}\frac{\Gamma(\frac{1}{2})\cdot\Gamma(\frac{7}{2})}{\Gamma(4)} = \frac{1}{2}\cdot\frac{\sqrt{\pi}\cdot\Gamma(\frac{7}{2})}{6}$

$\Gamma(\frac{7}{2}) = \Gamma(\frac{5}{2}+1) = \frac{5}{2}\Gamma(\frac{5}{2}) = \frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma(\frac{3}{2}) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(\frac{1}{2}) = \frac{15\sqrt{\pi}}{8}$

$= \sqrt{\pi}\cdot\frac{15}{8}\sqrt{\pi}$ —

c) $\int_{0+0}^{1-0} \frac{x^3}{\sqrt{1-x^2}}\,dx = I$

$$\beta(p,q) = \int_{0+0}^{1-0} x^{p-1}\cdot(1-x)^{q-1}\,dx$$

$x^2 = t \Rightarrow x = \sqrt{t}$

$dx = \frac{1}{2\sqrt{t}} = dt$

$\left.\begin{matrix} x \to 0 \\ x > 0 \end{matrix}\right\} \Rightarrow \begin{matrix} t \to 0 \\ t > 0 \end{matrix}$

$\left.\begin{matrix} x \to 1 \\ x < 1 \end{matrix}\right\} \Rightarrow \begin{matrix} t \to 1 \\ t > 0 \end{matrix}$

$$\Rightarrow I = \int_{0+0}^{1-0} \frac{t\sqrt{t}}{\sqrt{1-t}}\cdot\frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\int_{0+0}^{1-0}\frac{t}{\sqrt{1-t}}\,dt$$

$$= \frac{1}{2}\int_{0+0}^{1-0}\frac{t}{\sqrt{1-t}}\,dt = \frac{1}{2}\int_{0+0}^{1-0} t(1-t)^{-\frac{1}{2}}\,dt$$

$\begin{cases} p-1 = 1 \Rightarrow p = 2 \\ q-1 = -\frac{1}{2} \Rightarrow q = \frac{1}{2} \end{cases}$

$$\Rightarrow I = \frac{1}{2}\beta\left(2,\frac{1}{2}\right) = \frac{1}{2}\cdot\frac{\Gamma(2)\,\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{2})} = \frac{1}{2}\cdot\frac{1\cdot\sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})} = \frac{2}{3}$$

Exercițiul 3: Calculați $\int_0^{1-0} \sqrt[6]{\frac{x}{1-x}}\,dx$

$$\beta(p,q) = \int_{0+0}^{1-0} x^{p-1}(1-x)^{q-1}\,dx$$

$$I = \int_0^{1-0} x^{\frac{1}{6}}(1-x)^{-\frac{1}{6}}\,dx$$

$$\beta(p,q) = \int_{0+0}^{1-0} x^{p-1}(1-x)^{q-1}$$

$$I = \int_0^{1-0} x^{\frac{1}{6}}(1-x)^{-\frac{1}{6}}\,dx$$

$\begin{cases} p-1 = \frac{1}{6} \Rightarrow p = \frac{7}{6} \\ q-1 = -\frac{1}{6} \Rightarrow q = \frac{5}{6} \end{cases}$ $\Rightarrow I = \frac{1}{2}\beta\left(\frac{7}{6},\frac{5}{6}\right) = \frac{1}{2}\frac{\Gamma(\frac{7}{6})\cdot\Gamma(\frac{5}{6})}{\Gamma(2)} = \Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{5}{6}\right)$

$$\Rightarrow I = \Gamma\left(1+\frac{1}{6}\right)\cdot\Gamma\left(\frac{5}{6}\right) = \frac{1}{6}\cdot\Gamma\left(\frac{1}{6}\right)\cdot\Gamma\left(\frac{5}{6}\right) = \frac{1}{6}\frac{\pi}{\sin\frac{\pi}{6}} = \frac{1}{6}\cdot\frac{\pi}{\frac{1}{2}} = \frac{1}{6}\cdot 2\pi = \frac{\pi}{3}$$

SEMINAR 12
– INTEGRALA MULTIPLĂ –

6 IANUARIE 2020
– SĂPTĂMÂNA 13 –

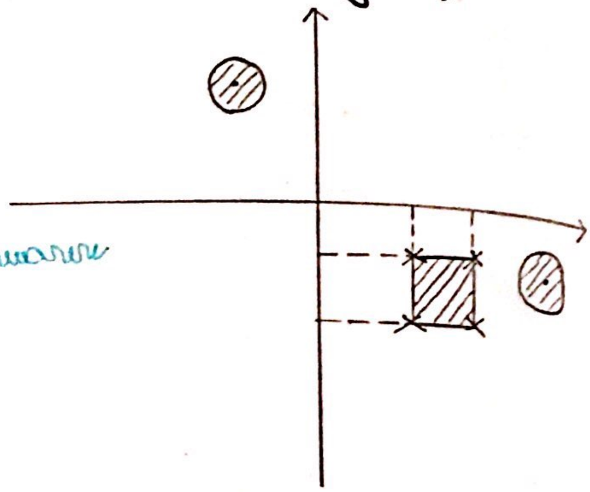
Exercițiul 1: Fie $\Delta = [1,2]\times[-2,-1] \subseteq \mathbb{R}^2$

a) Arătați că $\Delta \in \mathcal{J}(\mathbb{R}^2)$ și calc. $\mu(\Delta)$

b) Calc. $\iint_\Delta \frac{x^2}{y^3}\,dx\,dy$

Rezolvare a) Δ interval închis 2-dimensional $\Rightarrow \Delta \in \mathcal{J}(\mathbb{R}^2)$

$$\left.\begin{array}{l} \mu(\Delta) = (2-1)(-1+2) = 1 \\ f:\Delta \to \mathbb{R} \\ f(x,y) = \frac{x^2}{y^3} \\ f \text{ cont pe } \Delta \\ \Delta \in \mathcal{J}(\mathbb{R}^2) \\ \text{din desen obs. că } \Delta \text{ mulț. închisă} \end{array}\right\} \Rightarrow f \text{ este integr. Riemann pe } \Delta$$



Rezolvare b) $\iint_\Delta \frac{x^2}{y^3}\,dx\,dy = \int_{-2}^{-1}\left(\int_1^2 \frac{x^2}{y^3}dx\right)dy = \int_{-2}^{-1}\left(\frac{1}{y^3}\cdot\frac{x^3}{3}\Big|_2^1\right)dy$

$= \int_{-2}^{-1}\frac{7}{3y^3}dy = \frac{7}{3}\int_2^{-1} y^{-3}dy = \frac{7}{3}\cdot\frac{y^{-2}}{-2}\Big|_{-2}^{-1} = \frac{-7}{6}\cdot\frac{1}{y^2}\Big|_{-2}^{-1} = \frac{-7}{6}\left(\frac{4}{1}-\frac{1}{4}\right)$

$= -\frac{7}{\cancel{6}_2}\cdot\frac{\cancel{3}}{4} = -\frac{7}{8}$

Exercițiul 2: Fie $\Delta = \{(x,y)\in\mathbb{R}^2 \,/\, y \geq x^2,\ y \leq 2x+3\}$

a) Arătați că $\Delta \in \mathcal{J}(\mathbb{R}^2)$ și calc. $\mu(\Delta)$

b) ~~Calc~~ $\iint_\Delta x^2 y\,dx\,dy$

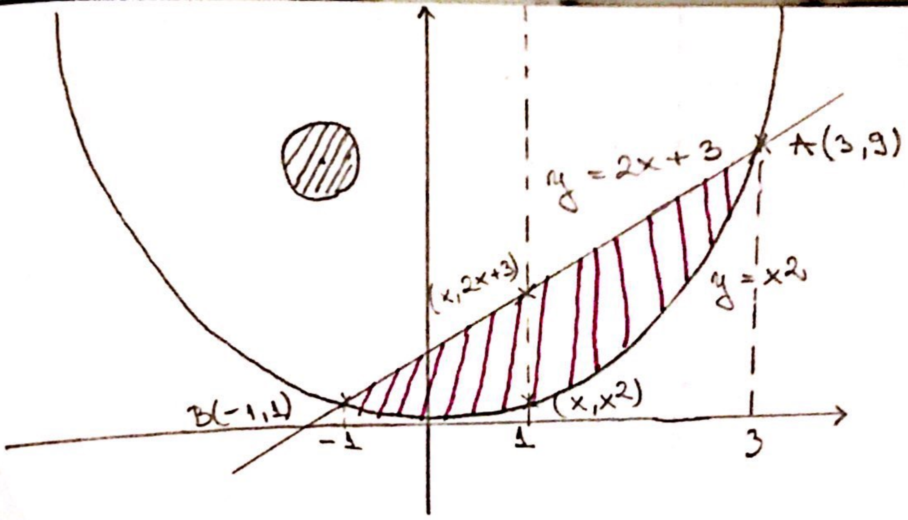
$\begin{cases} y = x^2 \\ y = 2x+3 \end{cases}$ $\quad x^2 = 2x+3$

$x^2 - 2x - 3 = 0$

$\Delta = 4+12 = 16$

$x_{1,2} = \frac{2\pm 4}{2} \Rightarrow 3,\ -1$

$A(3,9),\ B(-1,1)$



$$\Delta: \begin{cases} -1 \le x \le 3 \\ \underbrace{x^2}_{f_1(x)} \le y \le \underbrace{2x+3}_{f_2(x)} \end{cases} \Rightarrow \Delta \text{ mulțime simplă în raport cu axa } Oy$$

$$\Downarrow$$

$$\Delta \in \mathcal{J}(\mathbb{R}^2)$$

$$\mu(\Delta) = \iint_\Delta 1 \, dx\, dy = \int_{-1}^{3} \left(\int_{x^2}^{2x+3} 1 \, dy \right) dx = \int_{-1}^{3} \left(y \Big|_{x^2}^{2x+3} \right) dx = \int_{-1}^{3} (2x+3-x^2)\, dx = \frac{32}{3}$$

$$\Downarrow$$

$$A(\Gamma_{f,g}) = \int_a^b |f(x) - g(x)|\, dx$$

Rezolvare b) $f: \Delta \to \mathbb{R}$, $f(x,y) = x^2 y$

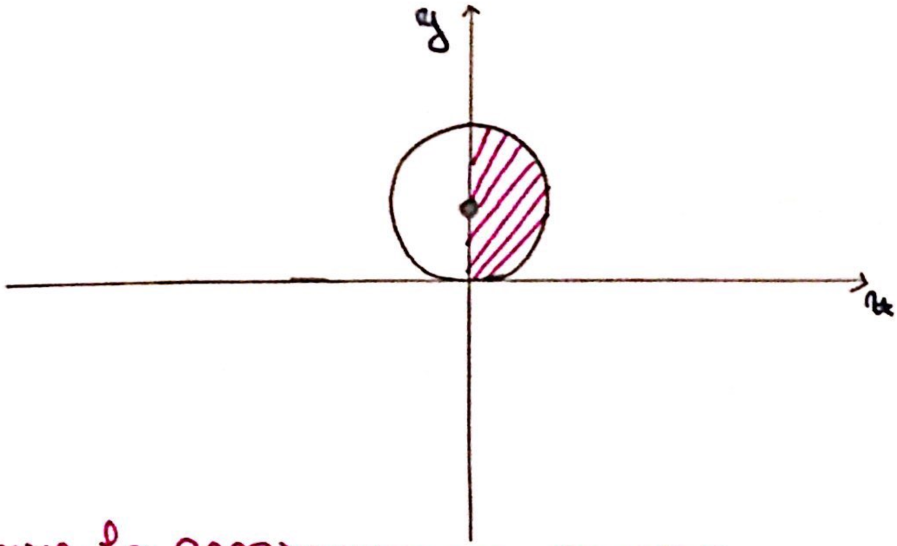
$$\left.\begin{array}{l} f \text{ continuă pe } \Delta \\ \Delta \in \mathcal{J}(\mathbb{R}^2) \\ \Delta \text{ mulțime închisă} \end{array}\right\} \Rightarrow f \text{ integrabilă Riemann pe } \Delta$$

$$\iint_\Delta x^2 y \, dx dy = \int_{-1}^{3} \left(\int_{x^2}^{2x+3} x^2 y \, dy \right) dx = \int_{-1}^{3} \left(x^2 \cdot \frac{y^2}{2}\right)\Big|_{x^2}^{2x+3} dx = \int_{-1}^{3} \left(x^2 \cdot \frac{(2x+3)^2}{2} - \frac{x^6}{2} \right) dx$$

$$= \int_{-1}^{3} \frac{x^2(4x^2+12x+9) - x^6}{2}\, dx$$

$$= \int_{-1}^{3} \frac{4x^4 + 12x^3 + 9x^2 - x^6}{2}\, dx = \left(\frac{4x^5}{10} + \frac{12x^4}{8} + \frac{9x^3}{6} - \frac{x^7}{14} \right)\Big|_{-1}^{3} = \ldots$$

Exercițiul 3: Calculați $\iint_D \left(1+\sqrt{x^2+y^2}\right) dx\,dy$, unde
$D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2-y\leq 0,\ x\geq 0\}$



Trecerea la COORDONATELE POLARE

$$\begin{cases} x=R\cos\alpha \\ y=R\sin\alpha \\ R\geq 0 \end{cases}$$
$\alpha\in(0,2\pi)$

$D\longrightarrow D'$
$dx\,dy \longrightarrow ?\ dR\,d\alpha$

$x^2+y^2-y+\frac{1}{4}=\frac{1}{4}$
$x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$

$D:\begin{cases} x^2+y^2-y\leq 0 \\ x\geq 0 \end{cases} \longrightarrow D':\begin{cases} R^2\cos^2\alpha+R^2\sin^2\alpha-R\sin\alpha\leq 0 \\ R\cos\alpha\geq 0 \end{cases}$

$R\geq 0$
$\alpha\in[0,2\pi]$

$\Rightarrow D':\{R^2\leq R\sin\alpha \Rightarrow D':\begin{cases} R\leq\sin\alpha \\ \cos\alpha\geq 0 \end{cases}$
$\alpha\in[0,2\pi]$

$\Rightarrow D':\begin{cases} 0\leq\sin\alpha \\ \cos\alpha\geq 0 \\ R\geq 0 \\ R\leq\sin\alpha \end{cases} \Rightarrow \begin{cases} 0\leq\alpha\leq\frac{\pi}{2} \\ \underset{f_1(\alpha)}{0}\leq R\leq \underset{f_2(\alpha)}{\sin\alpha} \end{cases}$

$$D' = \begin{cases} 0 \le \alpha \le \frac{\pi}{2} \\ 0 \le R \le \sin\alpha \end{cases} \Rightarrow D' \text{ este simplă în raport cu } OR \quad (1)$$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
$= 1 - \sin^2\alpha$
$= 2\cos^2\alpha - 1$

$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$

$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(R,\alpha)}\right| dR\,d\alpha \qquad \frac{\partial(x,y)}{\partial(R,\alpha)} = \begin{vmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial \alpha} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial \alpha} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & -R\sin\alpha \\ \sin\alpha & R\cos\alpha \end{vmatrix} = R\cos^2\alpha + R\sin^2\alpha = R$$

$dx\,dy = |R|\,dR\,d\alpha \quad (2)$

$$\iint (1+\sqrt{x^2+y^2})\,dx\,dy = \iint_D 1+\sqrt{R^2\cos^2\alpha + R^2\sin^2\alpha} = \iint_D 1+|R|\,|R|\,dR\,d\alpha$$

$$\overset{(1),(2)}{\Longrightarrow} \iint_D = \int_0^{\frac{\pi}{2}}\left(\left(\frac{R^2}{2}+\frac{R^3}{3}\right)\Big|_0^{\sin\alpha}\right)d\alpha = \int_0^{\frac{\pi}{2}}\left(\frac{\sin^2\alpha}{2}+\frac{\sin^3\alpha}{3}\right)d\alpha = \int_0^{\frac{\pi}{2}} \sin^2 d\alpha = \frac{\pi}{8}+\frac{2}{9}$$

$$\int_0^{\frac{\pi}{2}}\sin^2\alpha\,d\alpha = \int_0^{\frac{\pi}{2}}\frac{1-\cos 2\alpha}{2}\,d\alpha = \left(\frac{\alpha}{2}-\frac{\sin 2\alpha}{4}\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}}\sin^3\alpha\,d\alpha = \int_0^{\frac{\pi}{2}}\frac{3\sin\alpha - \sin 3\alpha}{4}\,d\alpha = \left(-\frac{3\cos\alpha}{4}+\frac{\cos 3\alpha}{12}\right)\Big|_0^{\frac{\pi}{2}} = \frac{3}{4}-\frac{1}{12}=\frac{2}{3}$$

SEMINAR 13

- INTEGRALA CURBILINIE -

Exercițiul 1: Calculați următoarele integrale curbilinii

a) $\int_d x\,dl$, unde $d(t) = (1-t, \sqrt{4-t^2})\ \forall t \in [0,1]$

Rezolvare:

$d:[0,1] \to \mathbb{R}^2$, $d(t) = (1-t, \sqrt{4-t^2})$

d drum de clasă C^1 în $\mathbb{R}^2$ $d_1(t)$ $d_2(t)$

$f:\mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = x$ cont. pe $\mathbb{R}^2$

$\operatorname{Im} d \subseteq \mathbb{R}^2$

$$\int_d x\,dl = \int_0^1 (1-t)\,l'(t)\,dt = \int_0^1 (1-t)\sqrt{1+\frac{t^2}{4-t^2}}$$

$$= \int_0^1 (1-t)\sqrt{1+\frac{t^2}{4-t^2}}\,dt = \int_0^1 (1-t)\sqrt{\frac{4-t^2+t^2}{4-t^2}}\,dt$$

$$= \int_0^1 (1-t)\frac{2}{\sqrt{4-t^2}} = 2\int_0^1 (1-t)\cdot\frac{1}{\sqrt{4-t^2}}\,dt$$

$$= 2\int_0^1 \frac{1}{\sqrt{4-t^2}}\,dt - 2\int_0^1 \frac{t}{\sqrt{4-t^2}}\,dt$$

$$= 2\cdot\frac{1}{2}\arcsin\frac{t}{2}\Big|_0^1 + 2\sqrt{4-t^2}\Big|_0^1$$

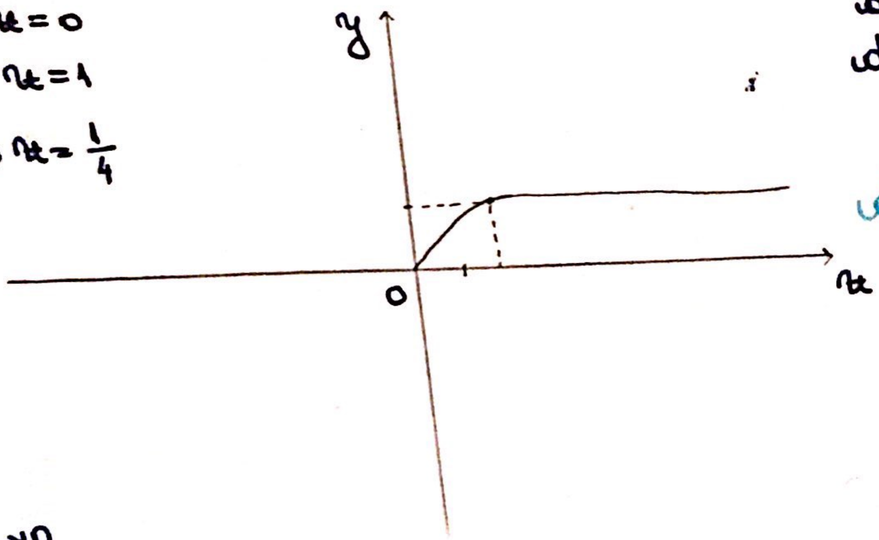
$$= \arcsin\frac{1}{2} + 2\sqrt{3} - 4$$

b) $\int_c y^2\,dl$ unde $c: x = \frac{y^4}{4}$, $y \in [0,2]$

$y = 0 \Rightarrow x = 0$

$y = 2 \Rightarrow x = 1$

$y = 1 \Rightarrow x = \frac{1}{4}$



$d : [0,2] \to \mathbb{R}^2$

$d(t) = \Big(\underbrace{\frac{t^4}{4}}_{d_1(t)}, \underbrace{t}_{d_2(t)}\Big)$

d drum de clasă C^1 în $\mathbb{R}^2$

$F : \mathbb{R}^2 \to \mathbb{R}$

$F(x,y) = y^2$ continuă pe $\mathbb{R}^2$

$\operatorname{Im} d = C \subseteq \mathbb{R}^2$

$d : [0,2] \to \mathbb{R}^2$

$d(t) = \Big(\underbrace{\frac{t^4}{4}}_{d_1(t)}, \underbrace{t}_{d_2(t)}\Big)$

$$\int_c y^2\,dl = \int_d y^2\,dl = \int_0^2 (d_2(t))^2 \cdot l'(t)\,dt$$

$$= \int_0^2 t^2 \sqrt{(d_1'(t))^2 + (d_2'(t))^2}\,dt$$

$$= \int_0^2 t^2 \sqrt{t^6 + t}\,dt$$

$$= \frac{1}{3}\int_0^2 (t^3)' \sqrt{(t^3)^2 + 1}\,dt$$

$$= \frac{1}{3}\int_0^8 \sqrt{x^2+1}$$

$$= \frac{1}{3}\int_0^8 \frac{x^2+1}{\sqrt{x^2+1}}\,dx$$

$$= \frac{1}{3}\int_0^8 \frac{x^2+1}{\sqrt{x^2+1}}\,dx$$

$$= \frac{1}{3}\int_0^8 \frac{x^2}{\sqrt{x^2+1}}\,dx + \frac{1}{3}\int_0^8 \frac{1}{\sqrt{x^2+1}}\,dx$$

$$= \frac{1}{3} \ln\left|\frac{x+\sqrt{x^2+1}}{x}\right| + c \Big|_0^8 + \frac{1}{3}\int_0^8 x \cdot \frac{x}{\sqrt{x^2+1}}\,dx$$

$$= \alpha + \frac{1}{3}\left(x \cdot \sqrt{\frac{x^2+1}{x^2}}\right)\Big|_0^8 - \frac{1}{3}\int_0^8 \sqrt{x^2+1}\,dx$$

$$= \alpha + \beta - \frac{1}{3}\int_0^8 \sqrt{x^2+1}\,dx \quad = \frac{\alpha+\beta}{2}$$

$t^3 = x$

$t = 0 \Rightarrow x = 0$

$t = 2 \Rightarrow x = 8$

$3t^2\,dt = dx$

Se propune schimbarea de variabilă $tg^2\alpha + 1 = \frac{1}{\cos^2\alpha}$

Exercițiul 2: Calculați următoarele integrale curbilinii:

a) $\int_d \sqrt{yz}\,dx + \sqrt{zx} + \sqrt{xy}\,dz$, unde $d(t) = (t, t^2, t^3)$ $(\forall)\ t \in [0,1]$

b) $\int_c xy\,dx + (x^2-y^2)\,dy$, unde C este segmentul

Rezolvarea a)

$\int_d \omega$, ω = formă diferențială

$\omega(x,y,z) = \sqrt{yz}\,dx + \sqrt{zx}\,dy + \sqrt{xy}\,dz$

$d : [0,1] \to \mathbb{R}^2$

$d(t) = (t, t^2, t^3)$

$d_1(t)\ d_2(t)\ d_3(t)$

d este drum de clasă C^1 în $\mathbb{R}^3$

$P_1, P_2, P_3 \to \mathbb{R}$

$$\left.\begin{aligned} P_1(x,y,z) &= \sqrt{yz} \\ P_2(x,y,z) &= \sqrt{xz} \\ P_3(x,y,z) &= \sqrt{xy} \end{aligned}\right\} \text{funcții continue pe } \mathbb{R}_+^3$$

$\operatorname{Imd} \subseteq \mathbb{R}_+^3$

$d_1(t)\ d_2(t)\ d_3'(t)$

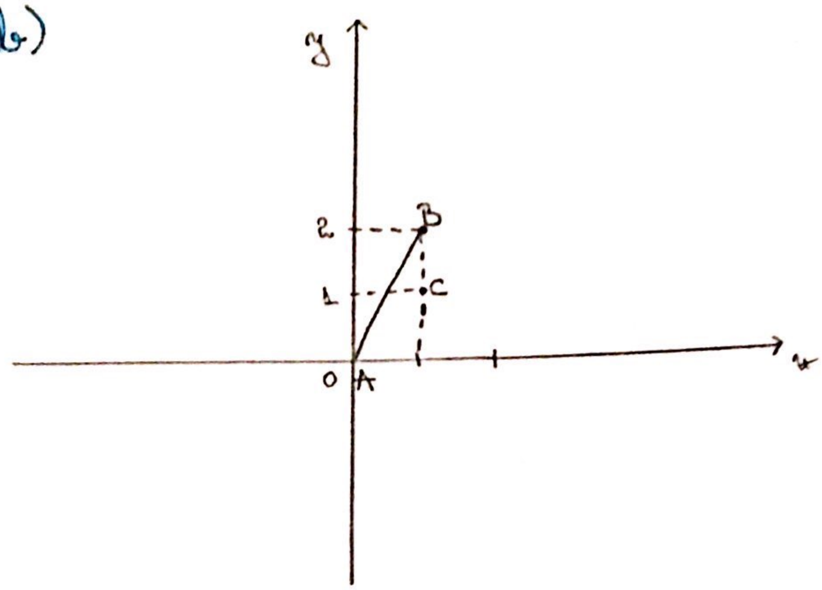
$$\int_d \sqrt{yz}\,dx + \sqrt{zx}\,dy + \sqrt{xy}\,dz = \int_0^1 \sqrt{t^2 \cdot t^3} \cdot t'\,dt + \sqrt{t^3 \cdot t}\,(t^2)'\,dt + \sqrt{t \cdot t^2} \cdot (t^3)'\,dt$$

$d_2(t)\ d_3(t)\ d_1'(t)\,dt\ d_3(t)\ d_1(t)\ d_2'(t)\,dt$

$$= \int_0^1 t^{\frac{5}{2}}\,dt + 2\int_0^1 t^3\,dt + 3\int_0^1 t^{\frac{3}{2}} \cdot t^2\,dt = \frac{2}{7} \cdot t^{\frac{7}{2}}\Big|_0^1 + 2 \cdot \frac{t^4}{4}\Big|_0^1 + 3\,\frac{2}{3} \cdot t^{\frac{3}{2}}\Big|_0^1$$

$$= \frac{2}{7} + \frac{1}{2} + \frac{2}{5}$$

b)



$\int_C xy\,dx + (x^2 - y^2)\,dy$

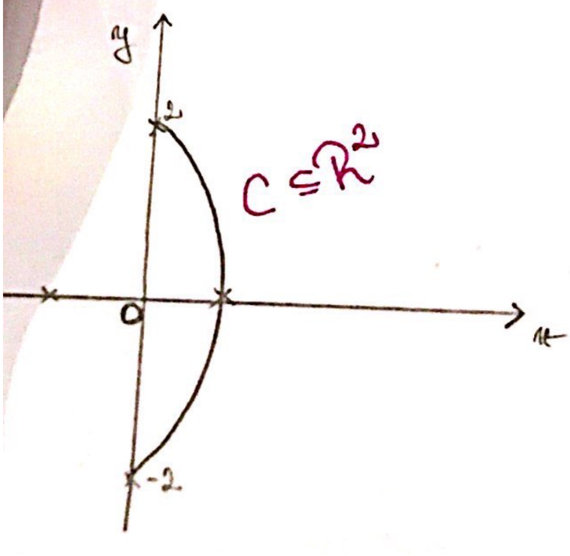
$C = [AB]$

$d : [0,1] \longrightarrow \mathbb{R}^2, d(t) = (1-t)\cdot(x_A, y_A) + t\cdot($

$d(t) = (1-t)\cdot(0,0) + t(1;2) = t(1,2) = t\cdot(1,2) = (\underbrace{t}_{d_1}, \underbrace{2t}_{d_2})$

d - drum de clasă C^1 în $\mathbb{R}^2$

Exercițiul 3: Calculați $\int_C \sqrt{1-x^2}\,dx + y\,dy$, unde $C: x^2 + \frac{y^2}{4} = 1, x \geq 0$



$x = 1 \Rightarrow y = 0$

$x = -1 \Rightarrow y = 0$

$y = 2 \Rightarrow x = 0$

$y = -2 \Rightarrow x =$

$d : [0, \pi] \longrightarrow \mathbb{R}^2$

$d(t) = (\underbrace{\sin t}_{d_1(t)}, 2\underbrace{\cos t}_{d_2(t)})$

$x \geq 0 \Rightarrow \sin t \geq 0 \Rightarrow t \in [0, \pi]$

↳ sin pozitiv în cadranele I și II

$P_1, P_2 : \quad \longrightarrow \mathbb{R}^2$

$P_1(x, y) = \sqrt{1-x}$ ⟵ $x \in [-1, 1]$, $y \in \mathbb{R}$ $\Rightarrow [-1, 1]$ produs cartezian cu $\mathbb{R}$ } funcții continue

$P_2(x, y) = y$

Avem de justificat că imaginea lui D este inclusă în $[-1, 1] \times \mathbb{R}$

$\operatorname{Im} d = C' \subseteq [-1, 1] \times \mathbb{R}$