Seminar 3

Spații vectoriale. SLi. SLD. SG
Baza

6) $(\mathbb{R}^3, +, \cdot)|_{\mathbb{R}}$, $S = \{(1, m, 1), (m, 1, 1), (1, 0, m)\} \subset \mathbb{R}^3$

$m \in \mathbb{R}$

1) $m = ?$ a.i. S este SLi
2) $m = ?$ a.i. S este SLD
3) Dacă $m = 2$, at. S este bază

Obs: $S = \{v_1, v_2, v_3\}$ SLi $\iff (\forall)\ a, b, c$ din $\mathbb{R}$ a.i. $av_1 + bv_2 + cv_3 = 0_{\mathbb{R}^2} \Rightarrow a = b = c = 0$

S este SLD $\iff \exists\ a, b, c \in \mathbb{R}^3$ nu toți nuli a.i. $av_1 + bv_2 + cv_3 = 0_{\mathbb{R}^3}$

S este SG $\iff \mathbb{R}^3 = \langle S \rangle \iff (\forall)\ x \in \mathbb{R}^3, \exists\ a, b, c \in \mathbb{R}$ a.i. $x = av_1 + bv_2 + cv_3$

$S \to$ bază $\iff S$ este SLi și S este SG

$\dim_{\mathbb{R}} \mathbb{R}^3 = 3$ $B_0 = \{(1, 0, 0) = e_1, e_2 = (0, 0, 0), e_3 = (0, 0, 1)\}$ bază canonică și card. $B_0 = 3$

Fie $a, b, c \in \mathbb{R}$ a.i. $a(1, m, 1) + b(m, 1, 1) + c(1, 0, m) = (0, 0, 0)$

$(a + bm + c, am + b, a + b + cm) = (0, 0, 0)$

$$\begin{cases} a + bm + c = 0 \\ am + b = 0 \\ a + b + cm = 0 \end{cases}$$

$$\overline{A} = \left(\begin{array}{ccc|c} 1 & m & 1 & 0 \\ m & 1 & 0 & 0 \\ 1 & 1 & m & 0 \end{array}\right)$$

$$\det A = \begin{vmatrix} 1 & m & 1 \\ m & 1 & 0 \\ 1 & 1 & m \end{vmatrix} = \begin{vmatrix} 1-m & m & 1 \\ m-1 & 1 & 0 \\ 0 & 1 & m \end{vmatrix} =$$

$$= (m-1) \begin{vmatrix} -1 & m & 1 \\ +1 & 1 & 0 \\ 0 & 1 & m \end{vmatrix} = (m-1) \begin{vmatrix} 0 & m & 1 \\ 0 & m+1 & 1 \\ 0 & 1 & m \end{vmatrix}$$

$$= -1(m-1) \begin{vmatrix} m+1 & 1 \\ 1 & m \end{vmatrix} = (1-m)(m(m+1)-1)$$

$$= (1-m)(m^2+m-1) \neq 0$$

$1-m \neq 0 \Rightarrow m \neq 1$

$m^2+m-1 \neq 0 \Rightarrow$

$\Delta = 1+4$

$$= (m-1) \begin{pmatrix} 0 & m+1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & m \end{pmatrix} = (m-1)\left(- \begin{vmatrix} m+1 & 1 \\ 1 & 0 \end{vmatrix} \right)$$

$$= -(m-1)[(m+1)\cdot m - 1] = -(m-1)(m^2+m-1)$$

a) S SLK $\Rightarrow$ sistemul are sol unică nula $\Leftrightarrow$

$\det A \neq 0 \Leftrightarrow m \in \mathbb{R} \setminus \{1, \frac{-1 \pm \sqrt{5}}{2}\}$

b) S SLD $\Leftrightarrow$ sistem are şi sol nenule $\Leftrightarrow$

$\det A = 0 \Leftrightarrow m \in \{1, \frac{-1 \pm \sqrt{5}}{2}\}$

c) S bază $m=2$, $S = \{(1,2,1)(2,1,1),(1,0,2)\}$

$\Rightarrow$ SL şi SG

Conform lui a) S SL

S SG $\Leftrightarrow \forall x \in \mathbb{R}^3 \; \exists a, b, c \in \mathbb{R}$ a.î.

$x = (x_1, x_2, x_3) = a(1,2,1) + b(2,1,1) + c(1,0,2) = (a+2b+c, 2a+b, a+b+2c)$

$$\begin{cases} a+2b+c = x_1 \\ 2a+b = x_2 \\ a+b+2c = x_3 \end{cases}$$

$$A = \left(\begin{array}{ccc|c} 1 & 2 & 1 & x_1 \\ 2 & 1 & 0 & x_2 \\ 1 & 1 & 0 & x_3 \end{array}\right)$$

$|A| \neq 0 \Rightarrow \forall x$ are sol unică a, b, c.

$(\mathbb{R}^3, +, \cdot)|_{\mathbb{R}}$

$B_0 = \{e_1, e_2, e_3\}$ bază $\Rightarrow$ SG $\Big| \xrightarrow[\text{schimbului}]{Th}$ S este SG

S este SL

OBS: $(V, +, \cdot) | K$ spațiu vectorial n-dimensional.

$S = \{x_1, x_2, \ldots\ldots x_n\}$ urm af sunt adv

a) S este bază

b) S SLi

c) S SG.

OBS: n este nr. maxim de vect care formează SLi

n este nr minim de vec care formează SG

b) $S' = \{(1, a_1, a_1^2), (1, a_2, a_2^2), (1, a_3, a_3^2)\} \subset \mathbb{R}^3$

$a_1, a_2, a_3 \in \mathbb{R}$.

Ce relație verifică a_1, a_2, a_3 a.î. S' este bază

card $S' = 3 = \dim_{\mathbb{R}} \mathbb{R}^3 \Rightarrow S'$ bază $\Leftrightarrow S'$ este SLi

$$A = \left(\begin{array}{ccc} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{array}\right) \left| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right.$$

det Vandermond

$\det A = (a_3 - a_2)(a_3 - a_1)(a_2 - a_1) \neq 0 \Leftrightarrow$

$\Leftrightarrow a_1, a_2, a_3$ sunt distincte 2 câte 2.

8) $(\mathbb{R}_2[X] = \{P \in \mathbb{R}[X] \mid \text{grad } P \le 2\}, +, \cdot)/_{\mathbb{R}}$

spațiu vect polinoamelor grad ≤ 2.

a) $f = 2x^2 - 3x + 1$

$B_1 = \{f, f', f''\}$ bază generalizare.

$B_1 = \{2x^2 - 3x + 1, 4x - 3, 4\} \rightarrow B_1' = \{(1, -3, 2),$

$(-3, 4, 0), (4, 0, 0)\}$

$P = a_0 + a_1 x + a_2 x^2 \rightarrow (a_0, a_1, a_2)$

$B_0 = \{1, x, x^2\}$ bază canonica $\Rightarrow B_0' = \{(1, 0, 0),$

$(0, 1, 0), (0, 0, 1)\}$.

OBS: $f = \tilde{f} \Leftarrow$ fn poli descrescat

card $B_1' =$ card $B_0' = 3 \Rightarrow (B_1'$ este bază $\Leftrightarrow$

$\Leftrightarrow B_1'$ este SLi

$$\begin{vmatrix} 1 & -3 & 4 \\ -3 & 4 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 4 \begin{vmatrix} -3 & 4 \\ 2 & 0 \end{vmatrix} = -4 \cdot 4 \cdot 2 = -32 \neq 0 \Rightarrow$$

$\Rightarrow B_0'$ SLi $\Rightarrow$ baza

b) $B_2 = \{1, x - 1, (x-1)^2\}$ baza generalizare

$\boxed{M_1}$ $B_2' = \{(1, 0, 0), (-1, 1, 0), (1, -2, 1)\}$.

card $B_2' = 3 = \dim \mathbb{R}_2[X] \Rightarrow B_2'$ bază $\Leftrightarrow B_2'$ SLi

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow$$

$\Rightarrow B_2'$ SLi

Deci B_2' bază

$\boxed{M_2}$: Dezvoltare în serie Taylor în jurul lui x_0

$$f(x) = f(x_0) + \frac{f'(x_0)(x - x_0)}{1!} + \frac{f''(x_0)(x - x_0)^2}{2!} + \dots + \frac{f^{(k)}(x_0)(x - x_0)^k}{k!} + \dots +$$

B_2 SG $\Leftrightarrow$ ~~$\mathbb{R}_2[x] =$~~ $\mathbb{R}_2[x] = \langle B_2 \rangle$

$(\forall)\, p = a_0 + a_1 x + a_2 x^2 \in \mathbb{R}_2[x]$, $\exists\, a, b, c \in \mathbb{R}$

a.î. $p = a \cdot 1 + b(x-1) + c(x-1)^2$

$$p(x) = p(1) + \frac{p'(1)(x-1)}{1!} + \frac{p''(1)(x-1)^2}{2!}$$

$a = p(1) = a_0 + a_1 + a_2$ $\qquad p'(x) = a_1 + 2a_2 x$

$b = \frac{p'(1)}{1} = 2a_2 + a_1$ $\qquad p''(x) = 2a_2$

$c = \frac{p''(1)}{2!} = \frac{2a_2}{2} = a_2$.

B_2 SG $\overset{OBS}{\Longrightarrow}$ B_2 bază

$B = \{(1,1,1,1),(0,3,1,1),(1,0,0,1),(x,1,1,1)\}$ baza în $\mathbb{R}^4$

card $B = 4 = \dim_{\mathbb{R}} \mathbb{R}^4 \Rightarrow B$ bază $\Leftrightarrow$ ...

$|A| = \ldots$

$\Rightarrow c(2-x) = 0 \Leftrightarrow x = \ldots$

11) $(\mathcal{C}(\mathbb{R}), +, \cdot)|_{\mathbb{R}}$

a) $S = \{f_1, f_2, f_3\}$

$f_1(x) = 1$, $f_2(x) = \sin x$, $f_3(x) = \cos x$

S este SLi

Fie $a, b, c \in \mathbb{R}$ a.i. $a f_1 + b \cdot f_2 + c f_3 = 0$

$a + b \sin x + c \cos x = 0$, $\forall x \in \mathbb{R}$

$x = 0$ $\quad a + 0 + c = 0$

$x = \frac{\pi}{2}$ $\quad a + b + 0 = 0$

$x = \pi$ $\quad a + 0 + (-1) \cdot c = 0$

$$\bar{A} = \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right)$$

$$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = (-1)^{2+2} \cdot 1 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} =$$

$= -1 - 1 = -2 \neq 0 \Rightarrow S = SLi$

b) $S' = \{g_1, g_2, g_3\}$, $g_1(x) = 1$, $g_2(x) = \cos x$,

$g_3(x) = \sin^2 \frac{x}{2}$

S' este SLD

$1 - \cos x = 2\sin^2 \frac{x}{2} \Rightarrow 1 - \cos x - 2\sin^2 \frac{x}{2} = 0$

$\underset{a}{\underset{\shortparallel}{1}} \cdot 1 + \underset{b}{\underset{\shortparallel}{(-1)}} \cos x + \underset{c}{\underset{\shortparallel}{(-2)}} \cdot \sin^2 \frac{x}{2} = 0$

etc SLD

c) $S'' = \{ h_1, h_2, h_3 \}$, $h_1(x) = e^x$, $h_2(x) = e^{-x}$,

$h_3(x) = ch(x) = \frac{e^x + e^{-x}}{2}$

$h_1(x) + h_2(x) - 2h_3(x) = 0$

$\begin{cases} a = 1 \\ b = 1 \\ c = -2 \end{cases}$