A Short, Friendly Glance at the Artificial Compressibility Method

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(Dated: September 17, 2015)

The Navier-Stokes (NS) equations are solved numerically in a two-step procedure - computing pressure at the current timestep from the existing velocity field, and then solution of the pressure Poisson equation at the next timestep. In the 1960s, the latter step was considered computationally expensive - this fact was the inspiration behind exploring artificial compressibility.

We pause on the Lattice Boltzmann Method (LBM) front to delve into a different approach to solving the NS equations, i.e. the artificial compressibility method (ACM). We then show the striking similarity between ACM and LBM, and compare their merits and de-merits.

I. NS PROBLEMS

The dimensionless NS equations governing the time-varying motion of an incompressible viscous fluid are the momentum equation,

$$\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{U} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2 \mathbf{U} \tag{1}$$

and the continuity equation,

$$\nabla \cdot \mathbf{U} = 0 \tag{2}$$

Two immediate issues are seen with these equations - the non-linear coupling of the velocities, and the absence of pressure in the continuity equation (2). The first issue establishes the need for a numerical method. The second requires for a slight re-writing of the equations. A simple explicit differenciation in time of equation (1) leads to the following time-discrete formulation:

$$\mathbf{U}^{(n+1)} = \underbrace{\mathbf{U}^{(n)} + dt\left[\frac{1}{Re}\Delta\mathbf{U}^{(n)} - (\mathbf{U}^{(n)}\cdot\nabla)\mathbf{U}^{(n)}\right]}_{\mathbf{Z}^{(n)}} - dt\left(\nabla p^{(n+1)}\right) \tag{3}$$

Note the implicit use of pressure - at any given time n, the implicit pressure $p^{(n+1)}$ is not known. We substitute the value of $\mathbf{U}^{(n+1)}$ from equation (3) in the continuity equation (2), and rearrange a little to obtain:

$$\nabla^2 p^{(n+1)} = \frac{1}{dt}\nabla \cdot \mathbf{Z}^{(n)} \tag{4}$$

Equation (4) is known as the pressure-Poisson equation (PPE). It is an elliptical equation in $p^{(n+1)}$, whose solution is used to update velocities using equation (3). The PPE needs to be solved at every time-step to ensure mass conservation. The problem is that this computation can be quite expensive, as we known from our courses in numerical programming.

II. ARTIFICIAL COMPRESSIBILITY

In 1967, Chorin [1] introduced the ACM as a means to speed up convergence of stationary problems of an incompressible viscous fluid. The new set of equations comprises of the momentum equation, which remains the same as equation (1), and the modified continuity equation,

$$\partial_t \rho^* + \nabla \cdot \mathbf{U} = 0 \tag{5}$$

Here, ρ^* represents the artificial density, and it is related to the artificial pressure p^* and the artificial compressibility δ^* as,

$$p^* = \frac{\rho^*}{\delta^*} \tag{6}$$

The benifit of such a representation is seen when discretizing the two equations in time:

$$\mathbf{U}^{(n+1)} = \mathbf{U}^{(n)} + dt\left[\frac{1}{Re}\Delta\mathbf{U}^{(n)} - (\mathbf{U}^{(n)}\cdot\nabla)\mathbf{U}^{(n)}\right] - dt\left(\nabla p^{(n+1)}\right) \tag{7}$$

$$p^{*(n+1)} = p^{*(n)} - \frac{1}{\delta^*}\nabla \cdot \mathbf{U}^{(n)} \tag{8}$$

Since the pressure is now present (and in a linear form) in the continuity equation (5), it can be calculated without having to solve a large elliptic problem. As a result, the computational speed is greatly increased. One can use their favourite space discretization technique to resolve the spatial derivatives. Note that the artificial density, pressure and compressibility (ρ^*, p^*, δ^*) are not the same as their real counterparts (ρ, p, δ), at least for an incompressible viscous fluid model.

As far as modeling a fluid as incompressible and viscous, this temporal dependence on time is non-physical. However, if the solution of equations (1) and (5) converges with time to a fixed value (i.e. if the solution is *stationary*), the dependence on δ^* vanishes, and the equations represent exactly the incompressible viscous model.

Some important facts about the ACM are listed below:

1. Stability of the ACM is governed by the space- and time-discretization, as well as the choice of δ^* [1]. Several modifications to the exact form of the continuity equation (5) have been proposed in literature that improve the stability of the method (see for example [2] and [3]).

2. Choice of grid is another major factor that determines the correctness of the solutions allowed by the ACM. Staggered grids are preferred, but solutions on a co-located grid can be computed with additional correction terms [3].

3. In case of time-dependent (*transient*) flows The ACM is physically realistic if a *weakly-compressible fluid model is employed* (i.e. model with constant sound speed) [4].

III. LBM-ACM: A COMPARISON

As we have seen in the talks so far, the lattice Boltzmann equation (derived from the Bhatnagar-Gross-Krook approximation of the collision operator) is written as,

$$f_i\left(\mathbf{x}+\mathbf{e}_i\Delta t, t+\Delta t\right) = f_i\left(\mathbf{x}, t\right) + \frac{1}{\tau_f}\left(f_i^{eq} - f_i\right) \tag{9}$$

where the terms have their usual meaning [5]. On conducting Chapman-Enskog expansion of the second order, this leads to the macroscopic equations,

$$\partial_t \mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} = -\nabla p + \frac{1}{\mathrm{Re}}\left[\nabla^2\mathbf{U} + \nabla\left(\nabla\cdot\mathbf{U}\right)\right] \tag{10}$$

and

$$\mathrm{M}^2\partial_t p + \nabla\cdot\mathbf{U} = 0 \tag{11}$$

The Mach number $\mathbf{M}$ relates to the relaxation time τ_f through the relation $\mathbf{M}^2 = \mathrm{Re}\left(\tau_f - 0.5\Delta t\right)$.

One can see a strong resemblance of equations (10) and (11) to the ACM equations (1) and (5). The only difference is the term $\nabla\left(\nabla\cdot\mathbf{U}\right)$, which disappears as steady state is achieved. Again, equations (10) and (11) are *not* the incompressible NS equations except for stationary flows.

A. ACM Notes

- Faster convergence than traditional methods of solving the NS equations
- Asymptotic convergence to correct solution for *stationary* incompressible flows
- For *weakly compressible* flows, transient results are also correct

B. LBM Notes

- Very similar computational and numerical behavior to ACM
- **More accurate pressure results** compared to ACM [4]

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[3] Ohwada,T., Asinari,P., Yabusaki,D. (2011). Artificial compressibility method and lattice Boltzmann method: Similarities and differences. *Comp. Math. App.*. 61:3461-3474.

[4] He,X., Doolen,G.D., Clark,T. (2002). Comparison of the Lattice Boltzmann Method and the Artificial Compressibility Method for Navier-Stokes Equations. *J. Comput. Phy.*. 179:439-451.

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