

Lab Course

Scientific Computing

Worksheet 1

distributed: 15.10.2014

due to: 27.10.2014, 3:00 pm, submission on Moodle

personal presentation: 28.10.2014 (exact slots will be announced)

We examine the following ordinary differential equation describing the dynamics of the population of a certain species:

$$\dot{p} = \left(1 - \frac{p}{10}\right) \cdot p \quad (1)$$

with initial condition

$$p(0) = 1. \quad (2)$$

The analytical solution is given by

$$p(t) = \frac{10}{1 + 9e^{-t}}.$$

We use this rather simple equation with a known exact solution to examine the properties of different numerical methods.

- a) Use `matlab` to plot the function $p(t)$ in a graph.
- b) Consider a general initial value problem

$$\dot{y} = f(y), \quad y(0) = y_0.$$

Implement the following explicit numerical methods with variable stepsize δt and end time t_{end}

- 1) explicit Euler method,

- 2) method of Heun,
- 3) Runge-Kutta method (fourth order)

as a `matlab` function depending on the right hand side $f(y)$, the initial value y_0 , the stepsize δt and the end time t_{end} . The output of the function shall be a vector containing all computed approximate values for y .

- c) For each of the three methods implemented, compute approximate solutions for equation (1) with initial conditions (2), end time $t_{end} = 5$, and with time steps $\delta t = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$. For each case, compute the approximation error

$$E = \sqrt{\frac{\delta t}{5} \sum_k (p_k - p_{k,exact})^2},$$

where p_k denotes the approximation, $p_{exact,k}$ the exact solution at $t = \delta t \cdot k$.

Plot your solutions in one graph per method (together with the given solution from a)) and write down the errors in the tabulars below.

- d) For each of the three methods, determine the factor by which the error is reduced if the step size δt is halved. Write down the results in the tabular below.
- e) In general, we do not know the exact solution of an equation we have to solve numerically (otherwise, we would not have to use a numerical method, in fact ;)). To anyhow guess the accuracy of a method, we can use the difference between our best approximation (the one with the smallest time step δt) and the other approximations:

$$\tilde{E} = \sqrt{\frac{\delta t}{5} \sum_k (p_k - p_{k,best})^2},$$

where p_k denotes the approximation with time step δt , $p_{best,k}$ the best approximation at $t = \delta t \cdot k$.

Compute \tilde{E} for all time steps and methods used, write down the results in the tabulars below and compare them to the exact error.

explicit Euler method ($q = 1$)				
δt	1	1/2	1/4	1/8
error	0.78	0.38	0.19	0.09
error red.		2.05	2.00	2.11
error app.	0.094	0.021	0.004	0

method of Heun ($q = 2$)				
δt	1	1/2	1/4	1/8
error	0.275	0.072	0.019	0.005
error red.		3.819	3.789	3.800
error app.	0.14	0.019	0.002	0

Runge-Kutta method ($q = 4$)				
δt	1	1/2	1/4	1/8
error	0.007894	0.000522	0.000035	0.000002
error red.		15.123	14.914	17.5
error app.	0.004281	0.000159	0.000006	0

Questions:

- 1) By which factor is the error reduced for each halving of δt if you apply a
 - first order ($O(\delta t)$),
 - second order ($O(\delta t^2)$),
 - third order ($O(\delta t^3)$),
 - fourth order ($O(\delta t^4)$)method.
- 2) For which integer q can you conclude that the error of the
 - a) explicit Euler method,
 - b) method of Heun,
 - c) Runge-Kutta method (fourth order)behaves like $O(\delta t^q)$?
- 3) Is a higher order method always more accurate than a lower order method (for the same stepsize δt)?
- 4) Assume you have to compute the solution up to a certain prescribed accuracy limit and that you see that you can do with less time steps if you use the Runge-Kutta-method than if you use Euler or the method of Heun. Can you conclude in this case that the Runge-Kutta method is the most efficient one of the three alternatives?