Technische Universität München
Institut für Informatik
Kaveh Rahnema
Emily Mo-Hellenbrand
Benjamin Uekermann

WT 2014/2015

Lab Course
Scientific Computing

Worksheet 1

distributed: 15.10.2014
due to: 27.10.2014, 3:00 pm, submission on Moodle
personal presentation: 28.10.2014 (exact slots will be announced)

We examine the following ordinary differential equation describing the dynamics of the population of a certain species:

$$\dot{p} = \left(1 - \frac{p}{10}\right) \cdot p \tag{1}$$

with initial condition

$$p(0) = 1. \tag{2}$$

The analytical solution is given by

$$p(t) = \frac{10}{1 + 9e^{-t}}.$$

We use this rather simple equation with a known exact solution to examine the properties of different numerical methods.

a) Use `matlab` to plot the function $p(t)$ in a graph.

b) Consider a general initial value problem

$$\dot{y} = f(y), \quad y(0) = y_0.$$

Implement the following explicit numerical methods with variable stepsize δt and end time t_{end}

1) explicit Euler method,

2) method of Heun,

3) Runge-Kutta method (fourth order)

as a `matlab` function depending on the right hand side $f(y)$, the initial value y_0, the stepsize δt and the end time t_{end}. The output of the function shall be a vector containing all computed approximate values for y.

c) For each of the three methods implemented, compute approximate solutions for equation (1) with initial conditions (2), end time $t_{end} = 5$, and with time steps $\delta t = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$. For each case, compute the approximation error

$$E = \sqrt{\frac{\delta t}{5} \sum_k \left(p_k - p_{k,exact}\right)^2},$$

where p_k denotes the approximation, $p_{exact,k}$ the exact solution at $t = \delta t \cdot k$.

Plot your solutions in one graph per method (together with the given solution from **a)**) and write down the errors in the tabulars below.

d) For each of the three methods, determine the factor by which the error is reduced if the step size δt is halved. Write down the results in the tabular below.

e) In general, we do not know the exact solution of an equation we have to solve numerically (otherwise, we would not have to use a numerical method, in fact ;)). To anyhow guess the accuracy of a method, we can use the difference between our best approximation (the one with the smallest time step δt) and the other approximations:

$$\tilde{E} = \sqrt{\frac{\delta t}{5} \sum_k \left(p_k - p_{k,best}\right)^2},$$

where p_k denotes the approximation with time step δt, $p_{best,k}$ the best approximation at $t = \delta t \cdot k$.

Compute $\tilde{E}$ for all time steps and methods used, write down the results in the tabulars below and compare them to the exact error.

explicit Euler method ($q = 1$)				
δt	1	1/2	1/4	1/8
error	0.78	0.38	0.19	0.09
error red.		2.05	2.00	2.11
error app.	0.687	0.289	0.095	0

method of Heun ($q = 2$)				
δt	1	1/2	1/4	1/8
error	0.275	0.072	0.019	0.005
error red.		3.819	3.789	3.800
error app.	0.27	0.067	0.014	0

Runge-Kutta method ($q = 4$)				
δt	1	1/2	1/4	1/8
error	0.007894	0.000522	0.000035	0.000002
error red.		15.123	14.914	17.5
error app.	0.007892	0.00052	0.000032	0

Questions:

1) By which factor is the error reduced for each halfing of δt if you apply a

   - first order ($O(\delta t)$),
   - second order ($O\left(\delta t^2\right)$),
   - third order ($O\left(\delta t^3\right)$),
   - fourth order ($O\left(\delta t^4\right)$)

   method.

2) For which integer q can you conclude that the error of the

   a) explicit Euler method,
   b) method of Heun,
   c) Runge-Kutta method (fourth order)

   behaves like $O\left(\delta t^q\right)$?

3) Is a higher order method always more accurate than a lower order method (for the same stepsize δt)?

4) Assume you have to compute the solution up to a certain prescribed accuracy limit and that you see that you can do with less time steps if you use the Runge-Kutta-method than if you use Euler or the method of Heun. Can you conclude in this case that the Runge-Kutta method is the most efficient one of the three alternatives?