CS146 - Homework

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1.5

```
public static int NumOfOnes(int n)
{
 if(n == 0) //Base Case
    return 0;

 if(n%2 == 0) //n is an even number
    return NumOfOnes(n/2);
 else
    return 1 + NumOfOnes(n/2);
}
```

1.7a.

To Prove: $logx < x$ for all $x > 0$
Consider the function $g(x) = x - log(x)$ for any real value of $x > 0$
$g(x) = x - \frac{ln(x)}{ln(2)}$ re-written using change of base
$g'(x) = 1 - \frac{1}{ln(2)*x}$ by taking the first derivative
Critical point is at $x = \frac{1}{ln(2)}$. The function $g(x)$ has a negative slope before this point and a positive slope after this point. Thus $x = \frac{1}{ln(2)}$ gives an absolute minimum for the function $g(x)$ on $x > 0$. Since $g(\frac{1}{ln(2)}) > 0$ the value of $g(x)$ before and after $x = \frac{1}{ln(2)}$ must be positive.
Therefore, $log(x)$ must be less than x for $x - log(x)$ to be > 0 which is what was to be proved.

1.7b.

To Prove: $log(A^B) = (B)log(A)$ Note: Using base 2 for logarithm
Let $logA = C$ equation $1^\star$
Which is $2^C = A$ by definition of logarithm
Raise each side to the B power for some integer B. $(2^C)^B = A^B$
Re-write L.H.S using rule of exponent multiplication. $2^{CB} = A^B$
Take the log of both sides. $log2^{CB} = logA^B$
Using logarithm rule change the L.H.S. $CB = logA^B$
Substitute the value of C from equation $1^\star$ on the L.H.S $logAB = logA^B$
Re-arrange the product on the L.H.S $(B)logA = logA^B$
Which is what was to be proved.

1.8a.

$$S_\infty = \sum_{i=0}^{\infty} \frac{1}{4^i} = \frac{1}{4^0} + \frac{1}{4^1} + \frac{1}{4^2} + ... + \frac{1}{4^\infty}$$

$$S_\infty = \sum_{i=0}^{\infty} \frac{1}{4^i} = 1 + \frac{1}{4} + \frac{1}{16} + ... + \frac{1}{4^\infty}$$

Formula for the sum of a convergent geometric series: $\frac{first\ term}{1-common\ ratio}$ or $\frac{a_1}{1-r}$

$$S_\infty = \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

1.8b.

$$S_\infty = \sum_{i=0}^{\infty} \frac{i}{4^i}$$

$S_1 = 0 + \frac{1}{4} = \frac{1}{4}$

$S_5 = 0 + \frac{1}{4} + \frac{1}{8} + \frac{3}{64} + \frac{1}{64} + \frac{5}{1024} = 0.44238$

$S_8 = 0 + \frac{1}{4} + \frac{1}{8} + \frac{3}{64} + \frac{1}{64} + \frac{5}{1024} + \frac{6}{4096} + \frac{7}{16384} + \frac{1}{8192} = 0.44439$

By considering what number the partial sums are approaching:

$$S_\infty = \sum_{i=0}^{\infty} \frac{i}{4^i} = 0.444444444444 = \frac{4}{9}$$

1.9

Estimate $\sum_{i=\frac{n}{2}}^{n} \frac{1}{i}$

$S_1 = \frac{1}{\frac{1}{2}} = 2$

$S_2 = \frac{1}{\frac{2}{2}} + \frac{1}{\frac{2}{2}} = 1 + 1 = 2$

$S_3 = \frac{1}{\frac{3}{2}} + \frac{1}{\frac{3}{2}} + \frac{1}{\frac{3}{2}} = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = 2$

$S_4 = \frac{1}{\frac{4}{2}} + \frac{1}{\frac{4}{2}} + \frac{1}{\frac{4}{2}} + \frac{1}{\frac{4}{2}} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2$

$S_n = 2$

1.11a.

$\sum_{i=1}^{n-2} F_i = F_n - 2$ where F_i is defined as Fibonacci number.

Proof will be by induction.

Base Case: $n = 3$ $\quad \sum_{i=1}^{3-2} F_1 = 1 = F_3 - 2 = 3 - 2$ base case is good

Assume true for all integers from base case up to k. $\sum_{i=1}^{k-2} F_i = F_k - 2$ (Inductive Hypothesis)

Prove also true for $k+1$ $\quad \sum_{i=1}^{(k+1)-2} F_i = F_{k+1} - 2$

By definition of Fibonacci number is $F_{k+1} = F_k + F_{k-1}$

Change R.H.S. to $= F_k + F_{k-1} - 2$

Rearrange R.H.S $= F_k - 2 + F_{k-1}$

On R.H.S. substitute for $F_k - 2$ from the inductive hypothesis $= \sum_{i=1}^{k-2} F_i + F_{k-1}$

$= \sum_{i=1}^{k-2} F_i + \sum_{i=1}^{k+1} F_{k-1} == \sum_{i=1}^{(k+1)-2} F_i$ which is the same as the L.H.S. End Proof.

1.11b.

Prove $F_n < \phi^n$ with $\phi = \frac{1+\sqrt{5}}{2}$

In mathematics $\dfrac{1+\sqrt{5}}{2}$ is defined as the golden ratio.

The Fibonacci numbers are related to the golden ratio and its conjugate by the equation $F_i = \dfrac{\phi^i - \phi^i}{\sqrt{5}}$ (this is noted in the book Introduction To Algorithms, by Cormen, pg.59)

Using this fact, the inequality can be re-written as $\dfrac{\phi^i - \phi^i}{\sqrt{5}} < \phi^i$

Or more simply $\dfrac{\phi^i}{\sqrt{5}} < \phi^i$

Which makes it obvious that the L.H.S. is less than the right since it is being divided by $\sqrt{5}$