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# On the classification of triangulated categories with finiteness conditions

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# Table of contents

1. Additively finite triangulated categories $d=1$

F.M. Enhanced finite triangulated categories. Journal of the Institute of Mathematics of Jussieu 21 (2022) p.741–783 10.1017/S1474748020000250

2. Triangulated categories with a $d\mathbb{Z}$-cluster tilting object $d \geq 1$

G. Jasso, F.M. The derived Auslander–Iyama correspondence. With an appendix by B. Keller. arXiv 10.48550/arXiv.2208.14413

G. Jasso, F.M, B. Keller. The Donovan–Wemyss Conjecture via the Derived Auslander–Iyama correspondence. The Abel Symposium 2022. Springer 10.1007/978-3-031-57789-5_4

3. The Calabi–Yau property Work in progress with G. Jasso

4. Open questions

# 1. Additively finite triangulated categoires

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## Definitions

additively finite category $\quad \mathcal{C} \xrightarrow{\sim} \operatorname{proj} \Lambda$ $\quad$ f.d. projective (right) $\Lambda$-modules

basic additive generator $\quad c \longmapsto \Lambda \cong \mathcal{C}(c,c)$ $\quad$ Lambda algebra (f.d. basic)

differential graded algebra $\quad A \rightsquigarrow D(A)$ $\quad$ derived category of $A$-modules

(small) algebraic $\quad \mathcal{T} \xrightarrow{\sim} \operatorname{per} A \subset D(A)$ $\quad$ thick subcategory generated by $A$

triangulated category $\quad c \longmapsto A$ $\quad$ enhancement

quasi-isomorphism $\quad A \simeq B \Rightarrow \operatorname{per} A \simeq \operatorname{per} B$ $\quad$ triangulated equivalence

 $\quad \cancel{\Leftarrow} \quad A \mapsto B$

**Classify additively finite algebraic triangulated categories and their enhancements**

# Examples

1. $\underline{\mathrm{mod}}\,\Gamma$ stable module category

$\Gamma$ f.d. self-injective algebra of finite representation type

$\Lambda = \underline{\mathrm{End}}(M)$ stable Auslander algebra of $\Gamma$, $M = M_1 \oplus \cdots \oplus M_n$ indecomposable non-projectives

$A = \mathrm{End}(P)$ $P$ complete resolution of $M$

2. $\underline{\mathrm{CM}}\,R$ stable category of maximal Cohen-Macaulay (CM) modules

$R$ CM algebra of finite CM representation type

3. $\mathrm{proj}\,\Lambda$ deformed preprojective algebra of generalized Dynkin type ADEL

$A$ ? not born algebraic

$\Lambda$ must be self-injective [Freyd'66]

Auslander-Reiten quiver of $\mathcal{T}$ [Xiao-Zhu'04] $k=\bar{k}$

Uniqueness of enhancements in the standard algebraic case [Amiot'07, Keller'18] $k=\bar{k}$

**Theorem (triangulated Auslander correspondence) [Hanihara'20]** $k$ perfect TFAE

[Amiot'07]

1. $\operatorname{proj}\Lambda$ admits a triangulated category structure
2. $\Lambda$ is twisted 3-periodic: in $\underline{\operatorname{mod}}\,\Lambda^e$ $\Omega^3_{\Lambda^e}\Lambda \cong {}_1\Lambda_\sigma$ twisted $\Lambda$-bimodule $\sigma \in \operatorname{Aut}(\Lambda)$

**Theorem [M'22]** $\Lambda$ connected non-separable $\Lambda/J_\Lambda$ separable TFAE $k$ arbitrary

1. $\Lambda$ is twisted 3-periodic
2. $\exists!$ dga $A$ such that $\operatorname{proj}\Lambda \simeq \operatorname{per}A : \Lambda \mapsto A$

Hochschild cohomology

Amiot triangulated structure induced by $\boxed{{}_1\Lambda_\sigma \overset{i}{\hookrightarrow} \underbrace{P_3 \to P_2 \to P_1}_{\in \operatorname{proj}\Lambda^e} \overset{p}{\twoheadrightarrow} \Lambda} \in \operatorname{Ext}^3_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma) = HH^3(\Lambda, {}_1\Lambda_\sigma)$

- shift $X[-1] = X \otimes_\Lambda {}_1\Lambda_\sigma$
- exact triangles $M \otimes_\Lambda (P_1 \otimes_\Lambda {}_1\Lambda_\sigma \to P_3 \to P_2 \to P_1)$ $M \in \operatorname{mod}\Lambda$

$P_1 \otimes_\Lambda {}_1\Lambda_\sigma \to {}_1\Lambda_\sigma \xrightarrow{i} P_3$

meta-triangle

## Algebraic $\Rightarrow$ Amiot

$\mathrm{proj}\,\Lambda \simeq \mathcal{T} \simeq \mathrm{per}\,A : \Lambda \mapsto c \mapsto A$ $\quad X[-1] = X \otimes_\Lambda {}_1\Lambda_\sigma \quad \sigma \in \mathrm{Aut}(\Lambda)$

graded Lambda algebra

$$H^*A = \bigoplus_{n\in\mathbb{Z}} \mathcal{T}(c, c[n]) = \bigoplus_{n\in\mathbb{Z}} {}_1\Lambda_\sigma^{\otimes_\Lambda(-n)} = \bigoplus_{n\in\mathbb{Z}} {}_1\Lambda_{\sigma^{-n}} = \Lambda(\sigma) = \frac{\Lambda\langle t^{\pm 1}\rangle}{(\lambda t - \sigma(\lambda) t)_{\lambda\in\Lambda}} \quad |t| = -1$$

minimal model $(\Lambda(\sigma), m_3, m_4, \dots, m_n, \dots)$ $A_\infty$-algebra [Kadeishvili'88]

$m_n : \Lambda(\sigma) \otimes \overset{n}{\cdots} \otimes \Lambda(\sigma) \longrightarrow \Lambda(\sigma)$ $\quad |m_n| = 2-n$ $\quad$ + equations

$j : \Lambda \hookrightarrow \Lambda(\sigma)$ inclusion of degree 0 part [Baues-Dreckmann'89, Benson-Krause-Schwede'04 ...]

$$j^* : HH^{3,-1}(\Lambda(\sigma), \Lambda(\sigma)) \longrightarrow HH^{3,-1}(\Lambda, \Lambda(\sigma)) = HH^3(\Lambda, {}_1\Lambda_\sigma) = \mathrm{Ext}^3_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma)$$

universal Massey product (UMP) $\{m_3\} \longmapsto j^*\{m_3\}$ restricted universal Massey product (rUMP)

The algebraic triangulated structure on $\mathrm{proj}\,\Lambda$ is Amiot's for the rUMP $j^*\{m_3\}$

$j^*\{m_3\}$ represented by ${}_1\Lambda_\sigma \overset{i}{\hookrightarrow} \underbrace{P_3 \to P_2 \to P_1}_{\in\,\mathrm{proj}\,\Lambda^e} \overset{p}{\twoheadrightarrow} \Lambda \Leftrightarrow j^*\{m_3\}$ unit in $\underline{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma))$

Hochschild-Tate cohomology

# Example (cusp) $k=\mathbb{C}$

$\mathcal{T} = \underline{CM}(R)$ $\quad R = k[[x,y]]/(y^2-x^3)$ $\quad C=(x,y)$ additive generator

$\Lambda \cong k[\varepsilon]/(\varepsilon^2)$ $\quad \sigma(\varepsilon) = -\varepsilon$ $\quad \Lambda(\sigma) = k\langle \varepsilon, t^{\pm 1}\rangle/(\varepsilon^2, \varepsilon t + t\varepsilon)$ $\quad |t|=-1$

stable Auslander algebra

graded Lambda algebra

$$A = \mathcal{E}nd\left(\cdots \to R^2 \xrightarrow{\begin{pmatrix} y & x \\ x^2 & y\end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & -x \\ -x^2 & y\end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & x \\ x^2 & y\end{pmatrix}} R^2 \to \cdots\right)$$

enhancement (2-periodic)

$$\varepsilon: \quad \begin{pmatrix} 1 & y+x^2 \\ y+x^2 & 1\end{pmatrix} \quad \begin{pmatrix} y & 1-y \\ 1-x^2 & x^2\end{pmatrix} \quad \begin{pmatrix} 1 & y+x^2 \\ y+x^2 & 1\end{pmatrix} \quad \begin{pmatrix} y & 1-y \\ 1-x^2 & x^2\end{pmatrix}$$

$\underline{HH}^{\bullet,*}(\Lambda,\Lambda(\sigma)) = k[u^{\pm 1}, \varepsilon t, t^{\pm 2}]$ $\quad |u|=(1,-1)$ $\quad |\varepsilon t|=(0,-1)$ $\quad |t^2|=(0,-2)$

Hochschild–Tate cohomology

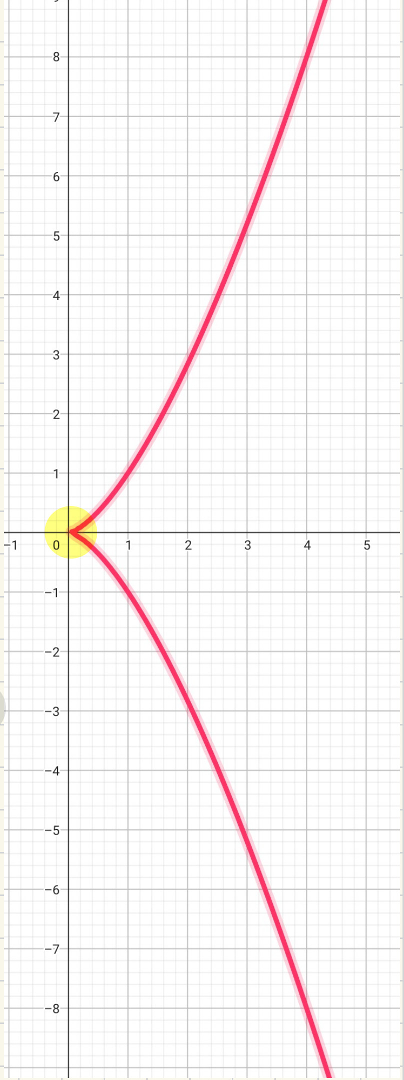
rUMP $\quad u^3t^{-2}$

minimal model

$u: {}_1\Lambda_\sigma \hookrightarrow \Lambda\otimes\Lambda \xrightarrow{\text{mult.}} \Lambda \in \mathrm{Ext}^1_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma)$

$(\Lambda(\sigma), m_3, 0 \ldots)$ $\quad A_\infty$-algebra $\quad 1 \mapsto \varepsilon\otimes 1 - 1\otimes\varepsilon$

$m_3(\varepsilon,\varepsilon,\varepsilon) = t$ $\quad m_3(\ldots 1 \ldots) = 0$ $\quad m_3$ $k\langle t^{\pm 1}\rangle$-linear



# Examples (the role of the separability condition) [Jasso–M.'23]

**1.** $\mathcal{J} = \underline{\mathrm{mod}}\, k[\varepsilon]/(\varepsilon^2)$ $\quad c = k$ additive generator $\quad$ graded Lambda algebra $\quad$ Hochschild–Tate cohomology

$\Lambda = \underline{\mathrm{End}}(k) = k \Rightarrow \mathcal{J} = \mathrm{mod}\, k \quad \sigma = \mathrm{id} \quad \Lambda(\sigma) = k\langle t^{\pm 1}\rangle \; |t| = -1 \quad \underline{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma)) = 0$

stable Auslander algebra

$A = \mathcal{E}nd(\cdots \to k[\varepsilon]/(\varepsilon^2) \xrightarrow{\varepsilon} k[\varepsilon]/(\varepsilon^2) \xrightarrow{\varepsilon} k[\varepsilon]/(\varepsilon^2) \to \cdots) \xleftarrow{\sim} k\langle t^{\pm 1}\rangle \quad d(t) = 0$

enhancement $\qquad$ formal

$\cdots \leftarrow k[\varepsilon]/(\varepsilon^2) \xleftarrow{-1} k[\varepsilon]/(\varepsilon^2) \xleftarrow{1} k[\varepsilon]/(\varepsilon^2) \leftarrow \cdots \quad \longleftarrow\!\!| \; t$

$k = \ell[\sqrt{x}] \supset \ell = \mathbb{F}_2(x)$ new non-perfect ground field $\quad k/\ell$ non-separable extension

**2.** $\mathcal{J} = \underline{\mathrm{mod}}\, \Gamma \quad \Gamma = \ell\langle \varepsilon, y\rangle/(\varepsilon^2, y^2 + \varepsilon + x, y\varepsilon + \varepsilon y + \varepsilon)$ $\quad c = \Gamma/(\varepsilon) = k$ additive generator

not a $k$-algebra

$\underline{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma)) = k[t^{\pm 1}, u^{\pm 1}]$

$\Lambda = \underline{\mathrm{End}}(k) = k \Rightarrow \mathcal{J} = \mathrm{mod}\, k \quad \sigma = \mathrm{id} \quad \Lambda(\sigma) = k\langle t^{\pm 1}\rangle$

stable Auslander algebra $\quad$ graded Lambda algebra $\quad |t| = -1 \quad |t| = (0,-1) \quad |u| = (1,0)$

$A = \mathcal{E}nd(\cdots \to \Gamma \xrightarrow{\varepsilon} \Gamma \xrightarrow{\varepsilon} \Gamma \to \cdots)$ $\quad$ neither $\qquad$ rUMPs $\quad u^3 t \quad 0$

$A \not\simeq k\langle t^{\pm 1}\rangle$ non-equivalent enhancements

$u\colon \; k \hookrightarrow k \otimes_\ell k \xrightarrow{\text{mult.}} k \in \mathrm{Ext}^1_{k \otimes_\ell k}(k, k)$

$1 \mapsto \sqrt{x} \otimes 1 + 1 \otimes \sqrt{x}$

## Algebraic ⇐ Amiot    char $k \neq 2$

$B$ graded algebra, $HH^{\bullet,*}(B,B)$ commutative + shifted Lie algebra

Gerstenhaber relation $[x, y\cdot z] = [x,y]\cdot z + (-1)^{(|x|-1)|y|}\, y\cdot[x,z]$

Massey algebra $(B,m)$ $B$ graded alg. $m \in HH^{3,-1}(B,B)$ UMP $\frac{1}{2}[m,m]=0$

Example $(H^*A, \{m_3\})$ the Massey algebra of $A$ dga

Theorem (lifting units) [M'22] $\eta \in HH^{3,-1}(\Lambda, \Lambda(\sigma))$ unit in $\underline{HH}^{\bullet,*}(\Lambda,\Lambda(\sigma))$

$\Rightarrow \exists!\, m \in HH^{3,-1}(\Lambda(\sigma), \Lambda(\sigma))$ with

- $j^* m = \eta$    rUMP    $(\underline{\mathrm{mod}}_p \Lambda^e, \otimes_\Lambda)$
- $\frac{1}{2}[m,m]=0$

Example $(\Lambda(\sigma), m)$ the Massey algebra of the Amiot triangulated structure

on $\mathrm{proj}\,\Lambda$ defined by $\eta: {}_1\Lambda_\sigma \overset{i}{\hookrightarrow} \underbrace{P_3 \to P_2 \to P_1}_{\in \mathrm{proj}\,\Lambda^e} \overset{p}{\twoheadrightarrow} \Lambda \in \mathrm{Ext}^3_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma) = HH^{3,-1}(\Lambda, \Lambda(\sigma))$

$(HH^{\bullet,*}(B,B), d=[m,-])$ Hochschild-Massey complex of $(B,m)$ $|d|=(2,-1)$

$HH^{\bullet,*}(B,m)$ Hochschild-Massey cohomology

**Theorem** [M'22] $HH^{p+2,-p}(B,m)=0$ $p>1$ $\Rightarrow \exists!$ A dga $(H^*A,\{m_3\}) \cong (B,m)$

**Proposition** [M'20] $\cdots \to HH^{p-1,q}(\Lambda,\Lambda(\sigma)) \to HH^{p,q}(\Lambda(\sigma),\Lambda(\sigma)) \to HH^{p,q}(\Lambda,\Lambda(\sigma)) \xrightarrow{\text{id}-\text{conj. by }\sigma} HH^{p,q}(\Lambda,\Lambda(\sigma)) \to \cdots$

**Theorem** [M'22] $(\Lambda(\sigma),m)$ Massey algebra and $j^*m$ Hochschild-Tate unit

$\Rightarrow HH^{p+2,q}(\Lambda(\sigma),m)=0$ $p>1$ $q\in\mathbb{Z}$

Idea of proof

Euler derivation $\delta \in HH^{1,0}(B,B)$ $\delta(b)=q\cdot b$, $b\in B^q$ $[\delta,x]=q\cdot x$, $x\in HH^{p,q}(B,B)$

$(HH^{\bullet,*}(\Lambda(\sigma),\Lambda(\sigma)), d=[m,-])$ ↻ $m\cdot$

endomorphism of complexes

of bidegree $(3,-1)$

bijective on bidegrees $(p,q)$ for $p>1$

$(HH^{\bullet,*}(\Lambda(\sigma),\Lambda(\sigma)), d=[m,-])$ ↻ $\delta\cdot$

null-homotopy for $m\cdot$

$m\cdot x=[m,\delta]\cdot x=[m,\delta\cdot x]+\delta\cdot[m,x]$

Gerstenhaber relation $x\in HH^{\bullet,*}(\Lambda(\sigma),\Lambda(\sigma))$

$\square$

## Theorem (derived Auslander correspondence) [Π'22] $k$ perfect

We have bijections between equivalence classes of:

$A$ $\longmapsto$ $\operatorname{per} A$ $\longmapsto$ $(H^0A, H^{-1}A)$ (Amiot)

1. $A$ dga with additively finite $\operatorname{per} A \ni A$ basic additive generator

quasi-isomorphism

2. $\mathcal{T}$ additively finite algebraic triangulated category

triangulated equivalence

$\mathcal{T}$ $\longmapsto$ $(\mathcal{T}(c,c), \mathcal{T}(c,c[-1]))$, $c$ additive generator

3. $(\Lambda, {}_1\Lambda_\sigma)$ with
   a) $\Lambda$ f.d. basic self-injective algebra
   b) ${}_1\Lambda_\sigma$ twisted $\Lambda$-bimodule such that $\Omega^3_{\Lambda^e}\Lambda \cong {}_1\Lambda_\sigma$ in $\underline{\operatorname{mod}}\,\Lambda^e$

isomorphisms of (algebra, bimodule in $\underline{\operatorname{mod}}\,\Lambda^e$)

**Corollary** k arbitrary $\Lambda$ separable we have bijections between

1. $\{A$ dga with $\operatorname{per} A \simeq \operatorname{proj} \Lambda : A \mapsto \Lambda\}/$ quasi-isomorphism

   $\Lambda(\sigma)$ $d=0$ formal

2. $\{$algebraic triangulated category $\mathcal{T} \simeq \operatorname{proj} \Lambda\}/$ equivalence

   $\mathcal{T} = \operatorname{proj} \Lambda$ $X[-1] = X \otimes_{\Lambda} {}_1\Lambda_\sigma$ exact triangles $\equiv$ split triangles

3. $\{$conjugacy classes in $\operatorname{Out}(\Lambda)\}$

   $\sigma \in \operatorname{Out}(\Lambda)$

**Example** $\Lambda = k^n$ $\mathcal{T} = (\operatorname{mod} k)^n$ $\operatorname{Out}(\Lambda) = S_n$

$\{$conjugacy classes in $S_n\} \cong \{$integer partitions of $n\}$

the trivial partition corresponds to the only indecomposable triangulated structure, with $\Sigma(X_1 \dots X_n) = (X_2 \dots X_n X_1)$. The rest decompose as the partition.

## 2. Triangulated categoires with a periodic cluster tilting object

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**Definition** $d \geq 1$ [Iyama–Yoshino '08]

$\mathcal{T}$ idempotent complete, hom-finite triangulated category

$\cup$

$\mathcal{C}$ functorially finite full subcategory is $d\mathbb{Z}$-cluster tilting if

1. $\mathcal{C} = \{x \in \mathcal{T} \mid \mathcal{T}(x, \mathcal{C}[n]) = 0 \quad \forall\, 0 < n < d\}$

$\quad = \{x \in \mathcal{T} \mid \mathcal{T}(\mathcal{C}, x[n]) = 0 \quad \forall\, 0 < n < d\}$

2. $\mathcal{C} = \mathcal{C}[d]$

$d = 1$ $\quad \mathcal{C} \subset \mathcal{T}$ $1\mathbb{Z}$-cluster tilting $\Leftrightarrow$ $\mathcal{C} = \mathcal{T}$

If additively finite $\quad \mathcal{C} \xrightarrow{\sim} \operatorname{proj} \Lambda$ f.d. basic self-injective algebra

basic $d\mathbb{Z}$-cluster tilting $\quad c \longmapsto \Lambda \qquad$ Lambda algebra

Classify hom-finite algebraic triangulated categories with a $d\mathbb{Z}$-cluster tilting object and their enhancements

## Examples (Amiot–Guo–Keller cluster categories) [Keller'08, Amiot'09, Guo'11]

$\mathcal{T} = \mathcal{C}(\Gamma) = \operatorname{per}\Gamma / \{M \text{ with } \sum_{n\in\mathbb{Z}} \dim H^n M\} \ni \Gamma$ $d\mathbb{Z}$-cluster tilting for

1. $\Gamma = \Pi_{d+1}(A)$ derived $(d+1)$-preprojective algebra of
   $A$ $d$-representation finite algebra ($\operatorname{gl\,dim} A = d$)

   $\Lambda = \Pi_{d+1}(A)$ $(d+1)$-preprojective algebra [Iyama–Oppermann'13]

2. $\Gamma = \Gamma(Q, W)$ completed Ginzburg dga of a quiver $Q$ with
   potential $W$ ($d=2$) provided … [Keller–Yang'11, Herschend–Iyama'11]

   $\Lambda = J(Q, W)$ the completed Jacobian algebra is f.d. self-injective

**Theorem** [Wemyss'18] $k = \mathbb{C}$ $R$ isolated compound Du Val singularity (cDV) with crepant resolutions

$\{\text{resolutions of } \operatorname{spec} R\} \cong \{2\mathbb{Z}\text{-cluster tilting objects in } \underline{\mathrm{CM}}(R)\}$

In this case Lambda algebras are called contraction algebras (connected)

[Jasso-M'23] $k$ arbitrary

**Theorem** $\mathcal{T}$ hom-finite algebraic triangulated category

$\mathcal{T} \ni c$ basic $d\mathbb{Z}$-cluster tilting object

$\Lambda$ Lambda algebra of $c$

If $\Lambda$ is connected non-separable and $\Lambda / J_\Lambda$ is separable TFAE

1. $\Lambda$ is twisted $(d+2)$-periodic, i.e. $\Omega^{d+2}_{\Lambda^e} \Lambda \cong {}_1\Lambda_\sigma$ in $\underline{\mathrm{mod}}\,\Lambda^e$ for some $\sigma \in \mathrm{Aut}(\Lambda)$

2. $\exists !$ dga $A$ such that $\mathcal{T} \simeq \mathrm{per}\, A : c \mapsto A$

**Conjecture** [Donovan–Wemyss'16] $R_1, R_2$ isolated cDV singularities with crepant resolutions and contraction algebras $\Lambda_1, \Lambda_2$

isomorphism $R_1 \cong R_2 \Leftrightarrow D(\Lambda_1) \simeq D(\Lambda_2)$ triangulated equivalence

**Proof** assembled by Keller

$i=1,2$ $\begin{cases} \Lambda_i \text{ contraction algebra of } c_i \in \underline{\mathrm{CM}}(R_i) \quad 2\mathbb{Z}\text{-cluster tilting object} \\ A_i \text{ dga with } \operatorname{per} A_i \simeq \underline{\mathrm{CM}}(R_i) : A_i \mapsto c_i \end{cases}$ $\Rightarrow$ $\boxed{H^0 A_i = \Lambda_i \text{ connected}}$

$\Rightarrow$ [Wemyss'18, Dugas'15]

$\Leftarrow$ $D(\Lambda_1) \simeq D(\Lambda_2) \Rightarrow$ we can assume $\Lambda_1 \cong \Lambda_2$ changing $c_2$ [August'20]

$\Rightarrow A_1 \simeq A_2$ [Jasso–M.'23] $\Rightarrow R_1 \cong R_2$ [Hua–Keller'18]

The only separable case is $\Lambda = \mathbb{C}$ $\square$

**Theorem (derived Auslander–Iyama correspondence)** [Jasso–M '23] $\mathbb{k}$ perfect

We have bijections between equivalence classes of:

$A$

$\downarrow$

$(\operatorname{per} A, A)$

$\downarrow$

$(H^0 A, H^{-d} A)$

no Amiot $\uparrow$

1. $A$ dga with hom-finite $\operatorname{per} A \ni A$ basic $d\mathbb{Z}$-cluster tilting

quasi-isomorphism

2. $(\mathcal{T}, c)$ with
   a) $\mathcal{T}$ hom-finite algebraic triangulated category
   b) $c \in \mathcal{T}$ basic $d\mathbb{Z}$-cluster tilting

triangulated equivalence preserving the object

3. $(\Lambda, {}_1\Lambda_\sigma)$ with
   a) $\Lambda$ f.d. basic self-injective algebra
   b) ${}_1\Lambda_\sigma$ twisted $\Lambda$-bimodule such that $\Omega^{d+2}_{\Lambda^e}\Lambda \cong {}_1\Lambda_\sigma$ in $\underline{\operatorname{mod}}\,\Lambda^e$

isomorphisms of (algebra, bimodule in $\operatorname{mod} \Lambda^e$)

$\mathcal{T}$

$\downarrow$

$(\mathcal{T}(c,c), \mathcal{T}(c,c[-d]))$

**Definition** [Geiss - Keller - Opperman '13]

A $(d+2)$ angulated category consists of

1. $\mathcal{C}$ additive category
2. $\Sigma : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ self-equivalence $d$-shift
3. $X_1 \to X_2 \to \cdots \to X_{d+2} \to \Sigma X_1$ exact $(d+2)$-angles

\+ axioms like

$\forall$ object $X \xrightarrow{1} X \to 0 \to \cdots \to 0 \to \Sigma X$ exact

$\forall$ map $Y \xrightarrow{f} Z \to X_3 \to \cdots \to X_{d+2} \to \Sigma Y$ exact

$\forall$ square
$$\begin{array}{ccccccccccc} X_1 & \to & X_2 & \to & X_3 & \to & \cdots & \to & X_{d+2} & \to & \Sigma X_1 \\ \downarrow g & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \Sigma g \\ Y_1 & \to & Y_2 & \to & Y_3 & \to & \cdots & \to & Y_{d+2} & \to & \Sigma Y_1 \end{array}$$
with top row exact and bottom row exact

$d = 1$ 3-angulated $\equiv$ triangulated

**Theorem** $\mathcal{T}$ idempotent complete hom-finite triangulated category TFAE

1. $\mathcal{C} \subset \mathcal{T}$ $d\mathbb{Z}$-cluster tilting
2. a) $\mathcal{T} = \operatorname{thick}(\mathcal{C})$ and $\mathcal{C}[d] = \mathcal{C}$

   b) $\mathcal{T}(\mathcal{C}, \mathcal{C}[n])\ \forall\, 0<n<d$ $d$-rigidity

   c) $\mathcal{C}$ $(d+2)$-angulated with $\Sigma = [d]$ and exact $(d+2)$-angles

$$\begin{tikzcd}
 & X_2 \arrow[rr] \arrow[dr] \arrow[d, phantom, "\text{exact}"] & & X_3 \arrow[dr] \arrow[d, phantom, "\text{exact}"] & \cdots & & & X_{d+1} \arrow[dr] \arrow[d, phantom, "\text{exact}"] & \\
X_1 \arrow[ur] & & X_{2.5} \arrow[ll, "+1"] \arrow[ur] \arrow[u, phantom, "\#"] & & X_{3.5} \arrow[ll, "+1"] & \cdots \arrow[l] & X_{d.5} \arrow[l] \arrow[ur] & & X_{d+2} \arrow[ll, "+1"]
\end{tikzcd}$$

$d$ exact triangles in $\mathcal{T}$

$\in \mathcal{C}$

1. $\Rightarrow$ 2. [Geiss - Keller - Opperman '13]

1. $\Leftarrow$ 2. [Jasso - M. '23]

These $(d+2)$-angulated categories are called normal

**Theorem** [Lin'19] Amiot-Lin $(d+2)$-angulated structure on proj $\Lambda$ induced by

$$\boxed{{}_1\Lambda_\sigma \xrightarrow{i} \underbrace{P_{d+2} \to \cdots \to P_2 \to P_1}_{\in \operatorname{proj}\Lambda^e} \xrightarrow{p} \Lambda} \in \operatorname{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma) = HH^{d+2}(\Lambda, {}_1\Lambda_\sigma)$$


- d-shift $\Sigma^{-1}X = X \otimes_\Lambda {}_1\Lambda_\sigma$
- exact $(d+2)$-angles $M \otimes_\Lambda (P_1 \otimes_\Lambda {}_1\Lambda_\sigma \to P_{d+2} \to \cdots \to P_2 \to P_1)$ $M \in \operatorname{mod}\Lambda$

  $P \otimes_\Lambda {}_1\Lambda_\sigma \to {}_1\Lambda_\sigma \xrightarrow{i}$

  meta-$(d+2)$-angle

**Example** [Erdmann-Skowroński'08, Dugas'10]

$\Lambda$ self-injective algebra of finite representation type

Algebraic normal
$\Rightarrow$ Amiot–Lin

enhancement

$\mathcal{T} \simeq \operatorname{per} A \ : \ c \mapsto A$
$\cup$
$d\mathbb{Z}$-cluster tilting $\longrightarrow$ $\mathcal{C} \simeq \operatorname{proj} \Lambda \ : \ c \mapsto \Lambda$

Lambda algebra

$X[-d] = X \otimes_\Lambda {}_1\Lambda_\sigma$ $\quad$ $X \in \operatorname{proj} \Lambda$
$\sigma \in \operatorname{Aut}(\Lambda)$

d-sparse graded Lambda algebra

$$H^*A = \bigoplus_{n\in\mathbb{Z}} \mathcal{T}(c, c[n]) = \bigoplus_{dn \in d\mathbb{Z}} {}_1\Lambda_\sigma^{\otimes_\Lambda(-n)} = \bigoplus_{dn\in d\mathbb{Z}} {}_1\Lambda_{\sigma^{-n}} = \Lambda(\sigma, d) = \frac{\Lambda\langle t^{\pm 1}\rangle}{(\lambda t - \sigma(\lambda) t)_{\lambda\in\Lambda}} \quad |t| = -d$$

minimal model $(\Lambda(\sigma,d), m_{d+2}, m_{2d+2}, \dots, m_{dn+2}, \dots)$ d-sparse $A_\infty$-algebra

$j : \Lambda \hookrightarrow \Lambda(\sigma, d)$ inclusion of degree 0 part

$$j^* : HH^{d+2,-d}(\Lambda(\sigma,d), \Lambda(\sigma,d)) \longrightarrow HH^{d+2,-d}(\Lambda, \Lambda(\sigma,d)) = HH^{d+2}(\Lambda, {}_1\Lambda_\sigma) = \operatorname{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma)$$

$(d+2)$-UMP $\quad \{m_{d+2}\} \longmapsto j^*\{m_{d+2}\} \quad (d+2)$-rUMP

The normal $(d+2)$-angulated structure on $\operatorname{proj}\Lambda$ is Amiot's for the rUMP $j^*\{m_{d+2}\}$

$j^*\{m_{d+2}\}$ represented by ${}_1\Lambda_\sigma \overset{i}{\hookrightarrow} \underbrace{P_{d+2} \to \cdots \to P_2 \to P_1}_{\in \operatorname{proj}\Lambda^e} \overset{p}{\twoheadrightarrow} \Lambda \iff j^*\{m_{d+2}\}$ unit in $\underline{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma,d))$

Example (Reid's pagoda) $k=\mathbb{C}$ $d=2$

$\mathcal{J} = \underline{CM}(R)$ $\quad R = k[[u,v,x,y]]/(uv+x^2-xy^3)$

$c=(u,x)$ additive generator $\quad$ cDV singularity

$\Lambda \cong k[\varepsilon]/(\varepsilon^3)$ stable Auslander algebra

$\sigma = \mathrm{id}$

$\Lambda(\sigma,d) = \Lambda[t^{\pm1}]$ $\quad |t| = -2$ graded Auslander algebra

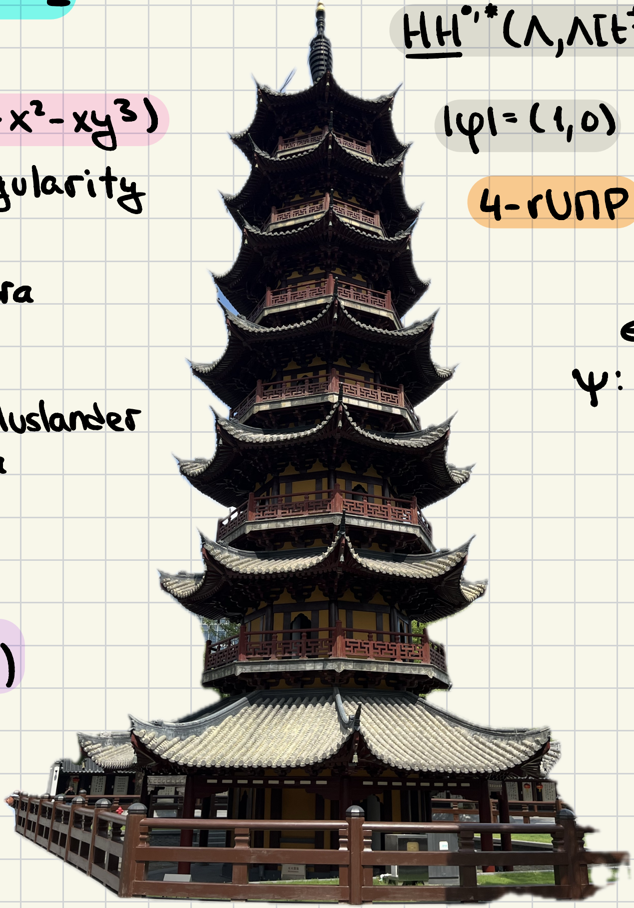
$S = k[[x,y]]/(x^2-xy^3)$

$$A = \mathcal{E}nd(\cdots \to S \xrightarrow{x} S \xrightarrow{x-y^3} S \xrightarrow{x} S \to \cdots)$$

$\varepsilon$: $\circlearrowleft y \quad \circlearrowleft y \quad \circlearrowleft y \quad \circlearrowleft y$

$$\varepsilon^3 = 0: \quad \cdots \to S \xleftarrow{1} S \xleftarrow{-1} S \xleftarrow{1} S \to \cdots$$

enhancement by Knörrer periodicity



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$$\underline{HH}^{\bullet,*}(\Lambda,\Lambda[t^{\pm1}]) = k[\varphi,\psi^{\pm1},\varepsilon,t^{\pm1}]/(\varepsilon^2)$$

$|\varphi| = (1,0)$ $\quad |\psi| = (2,0)$ $\quad |\varepsilon| = (0,0)$

4-rU∩P $\quad \psi^2 t$ $\quad |t| = (0,-2)$

$\in \mathrm{Ext}^2_{\Lambda^e}(\Lambda,\Lambda)$

$$\psi: \Lambda \hookrightarrow \Lambda\otimes\Lambda \to \Lambda\otimes\Lambda \xrightarrow{\text{mult.}} \Lambda$$

$1 \mapsto \varepsilon\otimes 1 - 1\otimes\varepsilon$

$1 \mapsto \varepsilon^2\otimes 1 + \varepsilon\otimes\varepsilon + 1\otimes\varepsilon^2$

minimal model

$(\Lambda[t^{\pm1}], m_4, 0 \ldots)$

2-sparse $A_\infty$-algebra

computed in joint discussions with Booth and Wemyss

**Corollary** $k$ arbitrary $\Lambda$ separable we have bijections between

1. $\{A$ dga with $H^0 A \cong \Lambda$ hom-finite per $A$ $d\mathbb{Z}$-cluster tilting $A \in \operatorname{per} A\}$ / quasi-isomorphism

   $\Lambda(\sigma, d)$ $\partial = 0$ formal

2. $\{$ hom-finite algebraic triangulated category $\mathcal{T} \ni c$ $d\mathbb{Z}$-cluster tilting object with $\mathcal{T}(c,c) \cong \Lambda\}$ / triangulated equivalence preserving the object

   $\mathcal{T} = (\operatorname{proj} \Lambda)^d \ni c = (\Lambda\, 0 \dots 0)$

   $\Sigma^{-1}(X_1 \dots X_d) = (X_2 \dots X_d\, X_1 \otimes_{\Lambda} {}_1\Lambda_\sigma)$ exact triangles $\equiv$ split triangles

3. $\{$conjugacy classes in $\operatorname{Out}(\Lambda)\}$ independent of $d$

   $\sigma \in \operatorname{Out}(\Lambda)$

# 3. The Calabi-Yau property

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$B$ graded algebra

$DB$ graded dual $B$-bimodule $(DB)^i = \mathrm{Hom}_k(B^{-i}, k)$

$B$ is (bimodule right) $n$-Calabi-Yau ($n$-CY) if $B[n] \cong DB$ as $B$-bimodules

$B$ $n$-CY $\Rightarrow$ Batalin-Vilkovisky operator $\Delta : HH^{p,q}(B,B) \longrightarrow HH^{p-1,q}(B,B)$

$$\Delta(x \cdot y) = \Delta(x) \cdot y + (-1)^{|x|} x \cdot \Delta(y) + [x,y], \quad \Delta^2 = 0.$$

The $n$-CY definition extends to graded categories and to dgas.

A triangulated category $\mathcal{T}$ is $n$-CY if the associated graded category $\mathcal{T}^\bullet$, with the same objects and $\mathcal{T}^i(x,y) = \mathcal{T}(x, y[i])$, is.

A $n$-CY dga $\Rightarrow$ per $A = \mathcal{T}$ $n$-CY $\Rightarrow$ $H^*A$, $\mathcal{T}^\bullet(x,x)$, $x \in \mathcal{T}$, $n$-CY.

## Examples (Amiot–Guo–Keller cluster categories)

$\mathcal{T} = \mathcal{C}(\Gamma)$ is $d$-CY for

1. $\Gamma = \Pi_{d+1}(A)$ derived $(d+1)$-preprojective algebra of $A$ $d$-representation finite algebra
2. $\Gamma = \Gamma(Q, W)$ completed Ginzburg dga of a quiver $Q$ with potential $W$ $(d=2)$ with $J(Q,W)$ f.d. self-injective

[Jasso–M.]

**Theorem** A dga with hom-finite $\operatorname{per} A \ni A$ basic $d\mathbb{Z}$-cluster tilting $A$ $d$-CY dga $\iff$ $H^*A \cong \Lambda(\sigma, d)$ $d$-CY graded algebra and $\boxed{\Delta(\{m_{d+2}\}) = 0}$

Proof Homotopy theory of operads, their algebras and their bimodules

This equation holds in all examples that we have been able to compute

# 4. Open problems

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$\Lambda$ f.d. algebra which is
- basic
- connected
- non-separable
- twisted $(d+2)$-periodic

**Find explicitly:**

1. A dga with hom-finite $\operatorname{per} A \ni A$ $d\mathbb{Z}$-cluster tilting and $H^0A = \Lambda$
2. $(\Lambda(\sigma,d), \boxed{m_{d+2}}, m_{2d+2}, \dots, m_{nd+2}, \dots)$ minimal model of $A$ ($A_\infty$-algebra)
3. Down-to-earth description of $\mathcal{T} = \operatorname{per} A$
4. $d$-CY structure on $A$ or $(\Lambda(\sigma,d), \{m_{nd+2}\}_{n\geq 1})$ when $\Lambda(\sigma,d)$ is $d$-CY and $\boxed{\Delta(\{m_{d+2}\}) = 0}$
5. Does this equation always hold? Proof or counterexample

THANKS
A LOT