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On the classification of triangulated categories with finiteness conditions

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F. N. Enhanced finite triangulated categories . Journal of the Institute of Mathematics of Jussieu 21 (2022) p. 741–783 [10.1017/S1474748020000250](https://doi.org/10.1017/S1474748020000250)

2. Triangulated categories with a $d\mathbb{Z}$ -cluster tilting object $d \geq 1$

G. Jasso, F. N. The derived Auslander–Iyama correspondence . With an appendix by B. Keller. arXiv [10.48550/arXiv.2208.14413](https://arxiv.org/abs/2208.14413)

G. Jasso, F. N., B. Keller. The Donovan–Wemyss Conjecture via the Derived Auslander–Iyama correspondence . The Abel Symposium 2022. Springer [10.1007/978-3-031-57789-5_4](https://doi.org/10.1007/978-3-031-57789-5_4)

3. The Calabi–Yau property Work in progress with G. Jasso

4. Open questions

1. Additively finite triangulated categoires



Definitions

additively finite category $\mathcal{C} \xrightarrow{\sim} \text{proj } \Lambda$ f.d. projective (right) Λ -modules

basic additive generator $\mathcal{C} \xrightarrow{\sim} \Lambda$ Lambda algebra (f.d. basic)
 $\mathcal{C}(c,c)$

differential graded algebra $A \rightsquigarrow D(A)$ derived category of A -modules

(small) algebraic $\mathcal{T} \xrightarrow{\sim} \text{per } A$ thick subcategory generated by A

triangulated category $c \mapsto A$ enhancement

quasi-isomorphism $A \approx B \Rightarrow \text{per } A \simeq \text{per } B$ triangulated equivalence



$A \mapsto B$

Classify additively finite algebraic triangulated categories and their enhancements

Examples

1. $\underline{\text{mod}} \Gamma$ stable module category

Γ f.d. self-injective algebra of finite representation type

$\Lambda = \underline{\text{End}}(\Pi)$ stable Auslander algebra of Γ , $\Pi = \Pi_1 \oplus \dots \oplus \Pi_n$ indecomposable non-projectives

$A = \underline{\text{End}}(P)$ P complete resolution of Π

2. $C\Gamma R$ stable category of maximal Cohen-Macaulay (CM) modules

R $C\Gamma$ algebra of finite $C\Gamma$ representation type

3. $\text{proj } \Lambda$ deformed preprojective algebra of generalized Dynkin type ADEL

A  not born algebraic

Λ must be self-injective [Freyd'66]

Auslander-Reiten quiver of Γ [Xiao-Zhu'04] $k = \bar{k}$

Uniqueness of enhancements in the standard algebraic case [Amiot'07, Keller'18] $k = \bar{k}$

Theorem (triangulated Auslander correspondence) [Hanihara'20] k perfect TFAE

- [Amiot'07]
1. $\text{proj } \Lambda$ admits a triangulated category structure
 2. Λ is twisted 3-periodic: in $\underline{\text{Mod}}\Lambda^e$ $\Omega_{\Lambda^e}^3 \Lambda \xrightarrow{\cong} \Lambda_\sigma$ twisted Λ -bimodule $\sigma \in \text{Aut}(\Lambda)$

Theorem [Pi'22] Λ connected non-separable $\Lambda \setminus \mathbb{J}_\Lambda$ separable TFAE k arbitrary

1. Λ is twisted 3-periodic
2. $\exists!$ dga A such that $\text{proj } \Lambda \simeq \text{perf } A: \Lambda \mapsto A$

Hochschild cohomology

Amiot triangulated structure
induced by

$$\begin{array}{ccccc} \Lambda_\sigma & \xhookrightarrow{i} & P_3 & \rightarrow & P_2 \rightarrow P_1 \xrightarrow{P} \Lambda \\ & & \in \text{proj } \Lambda^e & & \end{array}$$

$$\in \text{Ext}_{\Lambda^e}^3(\Lambda, \Lambda_\sigma) = \text{HH}^3(\Lambda, \Lambda_\sigma)$$

- shift $X[-1] = X \otimes_{\Lambda} \Lambda_\sigma$

- exact triangles $\Lambda \otimes_{\Lambda} (\Lambda \otimes_{\Lambda} \Lambda_\sigma \rightarrow P_3 \rightarrow P_2 \rightarrow P_1) \quad \Lambda \in \text{Mod } \Lambda$

$$\begin{array}{ccc} P \otimes_{\Lambda} \Lambda_\sigma & \xrightarrow{i} & \Lambda \\ \Lambda \otimes_{\Lambda} \Lambda_\sigma & \xrightarrow{i} & \Lambda_\sigma \end{array}$$

meta-triangle



Algebraic \Rightarrow Amiot

$$\text{proj } \Lambda \simeq T \simeq \text{per } A : \Lambda \hookrightarrow C \hookrightarrow A \quad X[-1] = X \otimes_{\Lambda} \Lambda_{\sigma} \quad \sigma \in \text{Aut}(\Lambda)$$

$$H^*A = \bigoplus_{n \in \mathbb{Z}} T(c, c[n]) = \bigoplus_{n \in \mathbb{Z}}, \Lambda_{\sigma}^{\otimes(1-n)} = \bigoplus_{n \in \mathbb{Z}}, \Lambda_{\sigma^{-n}} = \Lambda(\sigma) = \frac{\Lambda(t^{\pm 1})}{(\lambda t - \sigma(\lambda)t)_{\lambda \in \Lambda}} \quad |t| = -1$$

graded Lambda algebra

minimal model $(\Lambda(\sigma), m_3, m_4, \dots, m_n, \dots)$ A_{∞} -algebra [Kadeishvili '88]

$$m_n : \Lambda(\sigma) \otimes \cdots \otimes \Lambda(\sigma) \rightarrow \Lambda(\sigma) \quad |m_n| = 2-n \quad + \text{equations}$$

$j : \Lambda \hookrightarrow \Lambda(\sigma)$ inclusion of degree 0 part [Baues-Dreckmann '89, Benson-Krause-Schwede '04 ...]

$$j^* : HH^{3,-1}(\Lambda(\sigma), \Lambda(\sigma)) \longrightarrow HH^{3,-1}(\Lambda, \Lambda(\sigma)) = HH^3(\Lambda, \Lambda_{\sigma}) = \text{Ext}_{\Lambda}^3(\Lambda, \Lambda_{\sigma})$$

universal Massey product (UMP) $\{m_3\} \mapsto j^*\{m_3\}$ restricted universal Massey product (rUMP)

The algebraic triangulated structure on $\text{proj } \Lambda$ is Amiot's for the rUMP $j^*\{m_3\}$

$$j^*\{m_3\} \text{ represented by } \Lambda_{\sigma} \xrightarrow{i} P_3 \xrightarrow{P_3} P_2 \xrightarrow{P_2} P_1 \xrightarrow{P_1} \Lambda \Leftrightarrow j^*\{m_3\} \text{ unit in } \underline{HH}^{*,*}(\Lambda, \Lambda(\sigma))$$

$\epsilon \text{proj } \Lambda^e$

Hochschild-Tate cohomology

Example (cusp) $k = \mathbb{C}$

$$\mathcal{T} = \underline{\text{CM}}(R) \quad R = k[[x, y]]/(y^2 - x^3) \quad C = (x, y) \text{ additive generator}$$

$$\Lambda \equiv k[\varepsilon]/(\varepsilon^2) \quad \zeta(\varepsilon) = -\varepsilon \quad \Lambda(\sigma) = k<\varepsilon, t^{\pm 1}> / (\varepsilon^2, \varepsilon t + t\varepsilon) \quad |t| = -1$$

stable Auslander algebra graded Lambda algebra

$$A = \text{End} \left(\cdots \rightarrow R^2 \xrightarrow{\begin{pmatrix} y & x \\ x^2 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & -x \\ -x^2 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & x \\ x^2 & y \end{pmatrix}} R^2 \rightarrow \cdots \right) \text{ enhancement (2-periodic)}$$

$$\varepsilon: \begin{pmatrix} 0 & \\ 1 & y+x^2 \\ y+x^2 & 1 \end{pmatrix} \begin{pmatrix} 0 & \\ y & 1-y \\ 1-x^2 & x^2 \end{pmatrix} \begin{pmatrix} 0 & \\ 1 & y+x^2 \\ y+x^2 & 1 \end{pmatrix} \begin{pmatrix} 0 & \\ y & 1-y \\ 1-x^2 & x^2 \end{pmatrix}$$

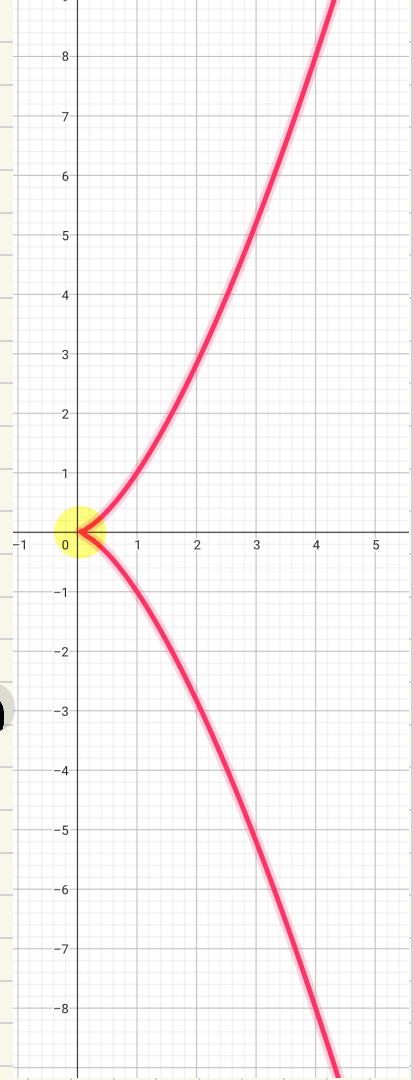
$$\underline{\text{HH}}^{*,*}(\Lambda, \Lambda(\sigma)) = k[u^{\pm 1}, \varepsilon t, t^{\pm 2}] \quad |u| = (1, -1) \quad |\varepsilon t| = (0, -1) \quad |t^2| = (0, -2)$$

rump $u^3 t^2$ Hochschild-Tate cohomology

minimal model

$(\Lambda(\sigma), m_3, 0 \dots)$ A_α -algebra $m_3(\dots, \dots) = 0$

$$m_3(\varepsilon, \varepsilon, \varepsilon) = t \quad m_3(\dots, \dots) = 0 \quad m_3 \text{ } k(t^{\pm 1})\text{-linear}$$



Examples (the role of the separability condition) [Jasso - 11.'23]

1. $J = \underline{\text{mod}}\ K[\epsilon]/(\epsilon^2)$

$C = K$ additive generator

graded Lambda algebra

Hochschild-Tate cohomology

$\Lambda = \underline{\text{End}}(k) = k \Rightarrow J = \text{mod } k \quad \sigma = \text{id}$

$\Lambda(\sigma) = k\langle t^{\pm 1} \rangle \quad |t| = -1$

$\underline{\text{HH}}^{\bullet, *}(k, \Lambda(\sigma)) = 0$

stable Auslander algebra

$A = \underline{\text{End}}(\dots \rightarrow k[\epsilon]/(\epsilon^2) \xrightarrow{\epsilon} k[\epsilon]/(\epsilon^2) \xrightarrow{\epsilon} k[\epsilon]/(\epsilon^2) \rightarrow \dots) \xleftarrow{\sim} k\langle t^{\pm 1} \rangle \quad d(t) = 0$

enhancement

formal

$$\dots \leftarrow k[\epsilon]/(\epsilon^2) \xleftarrow{\epsilon} k[\epsilon]/(\epsilon^2) \xleftarrow{\epsilon} k[\epsilon]/(\epsilon^2) \leftarrow \dots \longleftrightarrow t$$

$k = \ell[\sqrt{x}] \supset \ell = \mathbb{F}_2(x)$ new non-perfect ground field k/ℓ non-separable extension

2. $J = \underline{\text{mod}}\ \Gamma \quad \Gamma = \ell\langle \epsilon, y \rangle / (\epsilon^2, y^2 + \epsilon + x, y\epsilon + \epsilon y + \epsilon)$ $C = \Gamma / (\epsilon) = k$ additive generator
not a k -algebra

$\Lambda = \underline{\text{End}}(k) = k \Rightarrow J = \text{mod } k \quad \sigma = \text{id}$

stable Auslander algebra

$A = \underline{\text{End}}(\dots \rightarrow \Gamma \xrightarrow{\epsilon} \Gamma \xrightarrow{\epsilon} \Gamma \rightarrow \dots)$ neither

$A \neq k\langle t^{\pm 1} \rangle$ non-equivalent enhancements

$\Lambda(\sigma) = k\langle t^{\pm 1} \rangle$

graded Lambda algebra

$\underline{\text{HH}}^{\bullet, *}(k, \Lambda(\sigma)) = k[t^{\pm 1}, u^{\pm 1}]$

$|t| = -1$

$|t| = (0, -1)$

$|u| = (1, 0)$

cUMPS

$u^3 t$

0

$u: k \hookrightarrow k \otimes k \xrightarrow{\text{mult.}} k \in \text{Ext}_{k \otimes k}^1(k, k)$
 $t \mapsto \sqrt{x} \otimes 1 + 1 \otimes \sqrt{x}$

Algebraic \Leftarrow Amiot $\text{char } k \neq 2$

B graded algebra, $HH^{*,*}(B, B)$ commutative + shifted Lie algebra

(Gerstenhaber relation) $[x, y \cdot z] = [x, y] \cdot z + (-1)^{(|x|-1)|y|} y \cdot [x, z]$

Massey algebra (B, m) B graded alg. $m \in HH^{3,-1}(B, B)$ UMP $\frac{1}{2}[m, m] = 0$

Example $(H^*A, \pm m_3)$ the Massey algebra of A dga

Theorem (lifting units) [Pi'22] $\eta \in HH^{3,-1}(\Lambda, \Lambda(\sigma))$ unit in $\underline{HH}^{*,*}(\Lambda, \Lambda(\sigma))$

$\Rightarrow \exists ! m \in HH^{3,-1}(\Lambda(\sigma), \Lambda(\sigma))$ with

- $j^* m = \eta$ rUMP
- $\frac{1}{2}[m, m] = 0$

$(\underline{\text{mod}}_P \Lambda^e, \otimes)$

Example $(\Lambda(\sigma), m)$ the Massey algebra of the Amiot triangulated structure

on $\text{proj } \Lambda$ defined by $\eta: \Lambda_\sigma \xhookrightarrow{i} P_3 \rightarrow P_2 \rightarrow P_1 \xrightarrow{P} \Lambda \in \text{Ext}_{\Lambda^e}^3(\Lambda, \Lambda_\sigma) = HH^{3,-1}(\Lambda, \Lambda(\sigma))$
 $\in \text{proj } \Lambda^e$

$(HH^{*,*}(B, B), d = [m, -])$ Hochschild-Nassey complex of (B, m) $|d| = (2, -1)$

$HH^{*,*}(B, m)$ Hochschild-Nassey cohomology

Theorem [T'22] $HH^{p+2, -p}(B, m) = 0 \quad p > 1 \Rightarrow \exists! \text{ A dga } (H^*A, \{m_3\}) \cong (B, m)$

Proposition [T'20] $\cdots \rightarrow HH^{p-1, q}(\Lambda, \Lambda(\sigma)) \rightarrow HH^{p, q}(\Lambda(\sigma), \Lambda(\sigma)) \rightarrow HH^{p, q}(\Lambda, \Lambda(\sigma)) \xrightarrow{\text{id-conj. by } \sigma} HH^{p, q}(\Lambda, \Lambda(\sigma)) \rightarrow \cdots$

Theorem [T'22] $(\Lambda(\sigma), m)$ Nassey algebra and j^*m Hochschild-Tate unit

$$\Rightarrow HH^{p+2, q}(\Lambda(\sigma), m) = 0 \quad p > 1, q \in \mathbb{Z}$$

Idea of proof

Euler derivation $\delta \in HH^{1,0}(B, B)$ $\delta(b) = q \cdot b, b \in B^q \quad [\delta, x] = q \cdot x, x \in HH^{p,q}(B, B)$

$(HH^{*,*}(\Lambda(\sigma), \Lambda(\sigma)), d = [m, -]) \curvearrowleft m.$

$(HH^{*,*}(\Lambda(\sigma), \Lambda(\sigma)), d = [m, -]) \curvearrowleft \delta.$

endomorphism of complexes

null-homotopy for m .

of bidegree $(3, -1)$

$$m \cdot x = [m, \delta] \cdot x = [m, \delta \cdot x] + \delta \cdot [m, x]$$

Gerstenhaber
relation

bijective on bidegrees (p, q) for $p > 1$

$$x \in HH^{*,*}(\Lambda(\sigma), \Lambda(\sigma))$$

Theorem (derived Auslander correspondence) [Pi'22] K perfect

We have bijections between equivalence classes of:

A

1. A dga with additively finite per $A \ni A$ basic additive generator

↓

per A

quasi-isomorphism

2. \mathcal{T} additively finite algebraic triangulated category

↓

Amiot

↑ triangulated equivalence

↓

c additive generator

3. $(\Lambda, , \Lambda_\sigma)$ with

a) Λ f.d. basic self-injective algebra

b) $, \Lambda_\sigma$ twisted Λ -bimodule such that $\Sigma^3_{\Lambda^e} \Lambda \cong , \Lambda_\sigma$ in $\underline{\text{Mod}} \Lambda^e$

isomorphisms of (algebra, bimodule in $\underline{\text{Mod}} \Lambda^e$)

$(H^0 A, H^{-1} A)$

$(\mathcal{T}(c, c), \mathcal{T}(c, cc^{-1}))$

Corollary k arbitrary Λ separable we have bijections between

1. $\{ \text{A dga with } \text{per } \Lambda \cong \text{proj } \Lambda : \Lambda \mapsto \Lambda \} / \text{quasi-isomorphism}$

$\Lambda(\sigma)$ $d=0$ formal

2. $\{ \text{algebraic triangulated category } \mathcal{T} \cong \text{proj } \Lambda \} / \text{equivalence}$

$\mathcal{T} = \text{proj } \Lambda \quad X[-1] = X \underset{\Lambda}{\otimes} \Lambda \sigma \quad \text{exact triangles} \equiv \text{split triangles}$

3. $\{ \text{conjugacy classes in } \text{Out}(\Lambda) \}$

$\sigma \in \text{Out}(\Lambda)$

Example $\Lambda = k^n \quad \mathcal{T} = (\text{mod } k)^n \quad \text{Out}(\Lambda) = S_n$

{conjugacy classes in S_n } \cong {integer partitions of n }

the trivial partition corresponds to the only indecomposable triangulated structure, with $\sum (x_1 \dots x_n) = (x_2 \dots x_n x_1)$. The rest decompose as the partition.

2. Triangulated categoires with a periodic cluster tilting object



Definition $d \geq 1$

[Iyama-Yoshino'08]

\mathcal{T} idempotent complete, hom-finite triangulated category

\mathcal{C} functorially finite full subcategory is $d\mathbb{Z}$ -cluster tilting if

1. $\mathcal{C} = \{x \in \mathcal{T} \mid \mathcal{T}(x, \mathcal{C}[n]) = 0 \quad \forall 0 < n < d\}$

$$= \{x \in \mathcal{T} \mid \mathcal{T}(\mathcal{C}, x[n]) = 0 \quad \forall 0 < n < d\}$$

2. $\mathcal{C} = \mathcal{C}[d]$

$d=1 \quad \mathcal{C} \subset \mathcal{T}$ $1\mathbb{Z}$ -cluster tilting $\Leftrightarrow \mathcal{C} = \mathcal{T}$

If additively finite $\mathcal{C} \xrightarrow{\sim} \text{proj } \Lambda$ f.d. basic self-injective algebra

basic $d\mathbb{Z}$ -cluster tilting $\mathcal{C} \xrightarrow{\sim} \Lambda$ Lambda algebra

Classify hom-finite algebraic triangulated categories

with a $d\mathbb{Z}$ -cluster tilting object and their enhancements

Examples (Amiot-Guo-Keller cluster categories) [Keller'08, Amiot'09, Guo'11]

$\mathcal{T} = \mathcal{C}(\Gamma) = \text{perf } \Gamma / \{ \text{I\kern-0.21emn with } \sum_{n \in \mathbb{Z}} \dim H^n \text{I\kern-0.21emn} \} \ni \Gamma$ $d\mathbb{Z}$ -cluster tilting for

1. $\Gamma = \text{I\kern-0.21emn}_{d+1}(A)$ derived $(d+1)$ -preprojective algebra of

A d -representation finite algebra ($\text{gldim } A = d$)

$\Lambda = \text{I\kern-0.21emn}_{d+1}(A)$ $(d+1)$ -preprojective algebra [Iyama-Oppermann'13]

2. $\Gamma = \Gamma(Q, W)$ completed Ginzburg dga of a quiver Q with potential W ($d=2$) provided ... [Keller-Yang'11, Herschend-Iyama'11]

$\Lambda = J(Q, W)$ the completed Jacobian algebra is f.d. self-injective

Theorem [Wemyss'18] $k=\mathbb{C}$ R isolated compound Du Val singularity (cDV)

with crepant resolutions

$\{ \text{resolutions of } \text{spec } R \} \cong \{ 2\mathbb{Z}\text{-cluster tilting objects in } \underline{\mathcal{C}(\Gamma)}(R) \}$

In this case Lambda algebras are called contraction algebras (connected)

[Jasso - N'23] κ arbitrary

Theorem \mathcal{T} hom-finite algebraic triangulated category

\Downarrow

c basic $d\mathbb{Z}$ -cluster tilting object

Λ Lambda algebra of c

If Λ is connected non-separable and $\Lambda/\mathbb{J}_\Lambda$ is separable TFAE

1. Λ is twisted $(d+2)$ -periodic, i.e. $\Sigma_{\Lambda^e}^{d+2} \cong \Lambda_\sigma$ in $\underline{\text{mod}}\Lambda^e$ for some $\sigma \in \text{Aut}(\Lambda)$
2. $\exists!$ dga A such that $\mathcal{T} \simeq \text{per } A : c \mapsto A$

Conjecture [Donovan-Wemyss'16] R_1, R_2 isolated cDV singularities

with crepant resolutions and contraction algebras Λ_1, Λ_2

isomorphism $R_1 \cong R_2 \Leftrightarrow D(\Lambda_1) \simeq D(\Lambda_2)$ triangulated equivalence

Proof assembled by Keller

$i=1,2$ $\begin{cases} \Lambda_i \text{ contraction algebra of } c_i \in \underline{\text{C}\Gamma}(R_i) \text{ 2\mathbb{Z}-cluster tilting object} \\ A_i \text{ dga with } \text{per } A_i \simeq \underline{\text{C}\Gamma}(R_i): A_i \mapsto c_i \Rightarrow H^0 A_i = \Lambda_i \text{ connected} \end{cases}$

\Rightarrow [Wemyss'18, Dugas'15]

$\Leftarrow D(\Lambda_1) \simeq D(\Lambda_2) \Rightarrow$ we can assume $\Lambda_1 \cong \Lambda_2$ changing c_2 [August '20]

$\Rightarrow A_1 \simeq A_2$ [Jasso-M.'23] $\Rightarrow R_1 \cong R_2$ [Hua-Keller'18]

The only separable case is $\Lambda = \mathbb{C}$



Theorem (derived Auslander-Iyama correspondence) [Jasso - 17'23] K perfect

We have bijections between equivalence classes of:

A 1. A dga with hom-finite $\text{per } A \ni A$ basic $d\mathbb{U}$ -cluster tilting

\downarrow
 $\text{(per } A, A)$

quasi-isomorphism

2. (\mathcal{T}, c) with

no
Amiot

a) \mathcal{T} hom-finite algebraic triangulated category

b) $c \in \mathcal{T}$ basic $d\mathbb{U}$ -cluster tilting

triangulated equivalence preserving the object

(\mathcal{T}, c)

$(H^0 A, H^{-d} A)$ 3. $(\Lambda, \Lambda_\sigma)$ with

a) Λ f.d. basic self-injective algebra

b) Λ_σ twisted Λ -bimodule such that $\Sigma_{\Lambda^e}^{d+2} \Lambda \cong \Lambda_\sigma$ in $\underline{\text{Mod}} \Lambda^e$

isomorphisms of (algebra, bimodule in $\underline{\text{Mod}} \Lambda^e$)

$(\mathcal{T}(c, c), \mathcal{T}(c, cc-d))$

Definition [Geiss - Keller - Opperman '13]

A $(d+2)$ -angulated category consists of

1. \mathcal{C} additive category

2. $\Sigma : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ self-equivalence d -shift

3. $X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_{d+2} \rightarrow \Sigma X_1$, exact $(d+2)$ -angles

+ axioms like

forall object $X \xrightarrow{1} X \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow \Sigma X$ exact

forall map $Y \xrightarrow{f} Z \rightarrow X_3 \rightarrow \dots \rightarrow X_{d+2} \rightarrow \Sigma Y$ exact

$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \dots \rightarrow X_{d+2} \rightarrow \Sigma X_1$, exact

forall square $\begin{matrix} \downarrow g & \downarrow & \dots & \downarrow & \downarrow \Sigma g \\ Y_1 \rightarrow Y_2 \rightarrow Y_3 \rightarrow \dots \rightarrow Y_{d+2} \rightarrow \Sigma Y_1 \end{matrix}$ exact

$Y_1 \rightarrow Y_2 \rightarrow Y_3 \rightarrow \dots \rightarrow Y_{d+2} \rightarrow \Sigma Y_1$ exact

$d=1$ 3-angulated \equiv triangulated

Theorem T idempotent complete hom-finite triangulated category TFAE

1. $\mathcal{C} \subset \mathcal{T}$ $d\mathbb{Z}$ -cluster tilting
2.
 - a) $\mathcal{T} = \text{thick } (\mathcal{C})$ and $\mathcal{C}[d] = \mathcal{C}$
 - b) $\mathcal{T}(\mathcal{C}, \mathcal{C}[n]) \quad \forall 0 < n < d$ d -rigidity
 - c) \mathcal{C} $(d+2)$ -angulated with $\Sigma = [d]$ and exact $(d+2)$ -angles



1. \Rightarrow 2. [Geiss - Keller - Opperman '13]

1. \Leftarrow 2. [Jasso - M. '23]

These $(d+2)$ -angulated categories are called normal

Theorem [Lin'19] Amiot - Lin $(d+2)$ -angulated structure on $\text{proj } \Lambda$ induced by

$$\boxed{\begin{array}{c} \Lambda_\sigma \xrightarrow{i} P_{d+2} \rightarrow \cdots \rightarrow P_2 \rightarrow P_1 \xrightarrow{P} \Lambda \\ \in \text{proj } \Lambda^e \\ \in \text{Ext}_{\Lambda^e}^{d+2}(\Lambda_1, \Lambda_\sigma) = \text{HH}^{d+2}(\Lambda_1, \Lambda_\sigma) \end{array}}$$

- d -shift $\Sigma^{-1}X = X \otimes_{\Lambda_1} \Lambda_\sigma$
- exact $(d+2)$ -angles $\Pi \otimes_{\Lambda_1} (P_1 \otimes_{\Lambda_1} \Lambda_\sigma \rightarrow P_{d+2} \rightarrow \cdots \rightarrow P_2 \rightarrow P_1) \quad \Pi \in \text{mod } \Lambda$
 $P_1 \otimes_{\Lambda_1} \Lambda_\sigma \rightarrow \Lambda_1 \rightarrow \Lambda_\sigma$ meta- $(d+2)$ -angle



Example [Erdmann - Skowroński '08, Dugas '10]

Λ self-injective algebra of finite representation type

Algebraic normal

\Rightarrow Amiot-Lin

$$\begin{aligned} \mathcal{T} &\simeq \text{per } A : C \mapsto A \\ \cup \\ \mathcal{C} &\simeq \text{proj } \Lambda : C \mapsto \Lambda \end{aligned}$$

enhancement
Lambda algebra

dZ-cluster tilting

$$X[-d] = X \otimes_{\Lambda} \Lambda_{\sigma}$$

$$X \in \text{proj } \Lambda$$

 $\sigma \in \text{Aut}(\Lambda)$

$$H^*A = \bigoplus_{n \in \mathbb{Z}} \mathcal{T}(C, C[n]) = \bigoplus_{dn \in d\mathbb{Z}}, \Lambda_{\sigma}^{\otimes(-n)} = \bigoplus_{dn \in d\mathbb{Z}}, \Lambda_{\sigma^{-n}} = \Lambda(\sigma, d) = \frac{\Lambda(t^{\pm 1})}{(\lambda t - t \sigma(\lambda))}_{\lambda \in \Lambda} \quad |t| = -d$$

minimal model $(\Lambda(\sigma, d), m_{d+2}, m_{2d+2}, \dots, m_{dn+2}, \dots)$ d-sparse A_{∞} -algebra

$j : \Lambda \hookrightarrow \Lambda(\sigma, d)$ inclusion of degree 0 part

$$j^* : HH^{d+2, -d}(\Lambda(\sigma, d), \Lambda(\sigma, d)) \longrightarrow HH^{d+2, -d}(\Lambda, \Lambda(\sigma, d)) = HH^{d+2}(\Lambda, \Lambda_{\sigma}) = \text{Ext}_{\Lambda^e}^{d+2}(\Lambda, \Lambda_{\sigma})$$

(d+2)-UMP $\{m_{d+2}\} \mapsto j^* \{m_{d+2}\}$ (d+2)-rUMP

The normal (d+2)-angulated structure on $\text{proj } \Lambda$ is Amiot's for the rUMP $j^* \{m_{d+2}\}$

$$j^* \{m_{d+2}\} \text{ represented by } \Lambda_{\sigma} \xleftarrow{i} P_{d+2} \rightarrow \dots \rightarrow P_2 \rightarrow P_1 \xrightarrow{P} \Lambda \Leftrightarrow j^* \{m_{d+2}\} \text{ unit in } \underline{HH}^{*, *}(\Lambda, \Lambda(\sigma, d))$$

 $\in \text{proj } \Lambda^e$

Example (Reid's pagoda) $K = \mathbb{C}$ $d = 2$

$$J = \underline{\mathcal{C}\Gamma}(R) \quad R = K[[u, v, x, y]]/(uv + x^2 - xy^3)$$

$C = (u, x)$ basic $\mathbb{Z}\mathbb{Z}$ -c.t. cDV singularity

$\Lambda \cong K[\epsilon]/(\epsilon^3)$ stable Auslander algebra

$\sigma = \text{id}$

$\Lambda(\sigma, d) = \Lambda[t^{\pm 1}]$ $|t| = -2$ graded Auslander algebra

$$S = K[[x, y]]/(x^2 - xy^3)$$

$$A = \text{End}(\dots \rightarrow S \xrightarrow{x} S \xrightarrow{x-y^3} S \xrightarrow{x} S \rightarrow \dots)$$

$$\epsilon: \begin{matrix} \circ \\ y \end{matrix} \quad \begin{matrix} \circ \\ y \end{matrix} \quad \begin{matrix} \circ \\ y \end{matrix} \quad \begin{matrix} \circ \\ y \end{matrix}$$

$$\epsilon^3 = 0: \dots \rightarrow S \overset{-1}{\leftarrow} S \overset{-1}{\leftarrow} S \overset{-1}{\leftarrow} S \rightarrow \dots$$

enhancement by Knörrer periodicity



Longhua Shanghai

$$\underline{\mathcal{HH}}^{*,*}(\Lambda, \Lambda[t^{\pm 1}]) = K[\varphi, \varphi^{\pm 1}, \epsilon, t^{\pm 1}] / (\epsilon^2)$$

$$|\varphi| = (1, 0) \quad |\psi| = (2, 0) \quad |\epsilon| = (0, 0)$$

$$4\text{-runP} \quad \psi^2 t \quad |t| = (0, -2)$$

$$\in \text{Ext}_{\Lambda}^2(\Lambda, \Lambda)$$

$$\psi: \Lambda \hookrightarrow \Lambda \otimes \Lambda \xrightarrow{\text{mult.}} \Lambda \otimes \Lambda \xrightarrow{\epsilon} \Lambda$$

$$1 \mapsto \epsilon \otimes 1 - 1 \otimes \epsilon$$

$$1 \mapsto \epsilon^2 \otimes 1 + \epsilon \otimes \epsilon + 1 \otimes \epsilon^2$$

minimal model

$$(\Lambda[t^{\pm 1}], m_4, 0 \dots)$$

2-sparse A_∞ -algebra

computed in joint discussions with Booth and Wemyss

Corollary k arbitrary Λ separable we have bijections between

1. { A dga with $H^0\Lambda \cong \Lambda$ hom-finite per Λ
 $d\mathbb{Z}$ -cluster tilting $\Lambda \in \text{per } \Lambda$ } / quasi-isomorphism
 $\Lambda(\mathfrak{s}, d) \quad \mathfrak{d} = 0$ formal
2. { hom-finite algebraic triangulated category $T \ni c$ $d\mathbb{Z}$ -cluster tilting object with $T(c, c) \cong \Lambda$ } / triangulated equivalence preserving the object
 $T = (\text{proj } \Lambda)^d \ni c = (\Lambda \circ \dots \circ)$
 $\Sigma^{-1}(x_1 \dots x_d) = (x_2 \dots x_d, x_1 \otimes_{\Lambda} \Lambda)$ exact triangles \equiv split triangles
3. { conjugacy classes in $\text{Out}(\Lambda)$ } independent of d
 $\sigma \in \text{Out}(\Lambda)$

3. The Calabi-Yau property



B graded algebra

DB graded dual B-bimodule $(DB)^i = \text{Hom}_k(B^{-i}, k)$

B is (bimodule right) n-Calabi-Yau (n-CY) if $B[n] \cong DB$ as B-bimodules

B n-CY \Rightarrow Batalin-Vilkovisky operator $\Delta: HH^{p,q}(B, B) \rightarrow HH^{p-1, q}(B, B)$

$$\Delta(x \cdot y) = \Delta(x) \cdot y + (-1)^{|x|} x \cdot \Delta(y) + [x, y], \quad \Delta^2 = 0.$$

The n-CY definition extends to graded categories and to dgas.

A triangulated category T is n-CY if the associated graded category T^\bullet , with the same objects and $T^\bullet(x, y) = T(x, y[i])$, is.

A n-CY dga \Rightarrow per A = T n-CY $\Rightarrow H^*A$, $T^\bullet(x, x)$, $x \in T$, n-CY.

Examples (Amiot-Guo-Keller cluster categories)

$\mathcal{T} = \mathcal{C}(\Gamma)$ is d -CY for

1. $\Gamma = \prod_{d+1}(A)$ derived $(d+1)$ -preprojective algebra of
A d -representation finite algebra

2. $\Gamma = \Gamma(Q, W)$ completed Ginzburg dga of a quiver Q with
potential W ($d=2$) with $J(Q, W)$ f.d. self-injective

[Jasso-N.]

Theorem A dga with hom-finite $\text{per } A \ni A$ basic $d\mathbb{Z}$ -cluster tilting

A d -CY dga $\Leftrightarrow H^* A \cong \Lambda(s, d)$ d -CY graded algebra and $\Delta(\{M_{d+2}\}) = 0$

Proof Homotopy theory of operads, their algebras and their bimodules

This equation holds in all examples that we have been able to compute

4. Open problems



Λ f.d. algebra which is

- basic
- connected
- non-separable
- twisted $(d+2)$ -periodic

Find explicitly:

1. A dga with hom-finite $\text{per } A \ni A$ $d\mathbb{Z}$ -cluster tilting and $H^0 A = \Lambda$
2. $(\Lambda(s,d), m_{d+2}, m_{2d+2}, \dots, m_{nd+2}, \dots)$ minimal model of A (A_∞ -algebra)
3. Down-to-earth description of $T = \text{per } A$
4. d -CY structure on A or $(\Lambda(s,d), \{m_{nd+2}\}_{n \geq 1})$ when $\Lambda(s,d)$ is d -CY
and $\Delta(\{m_{d+2}\}) = 0$
5. Does this equation always hold? Proof or counterexample



**THANKS
A LOT**