Non-unital Gelfand duality

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Abstract

We give a proof of a version of non-unital Gelfand duality: The category of commutative C^*-algebras with *non-degenerate* $*$-homomorphisms is contravariantly equivalent to the category of locally compact Hausdorff spaces with *proper* continuous maps. We also derive some basic consequences from this equivalence of categories.

Preliminary remark: The content of this note should be essentially well-known, but the author doesn't know a proper reference and has only found less detailed versions in the literature. The note is an adaptation of a German version from 2007 which is available at `http://www.matheplanet.com/matheplanet/nuke/html/article.php?sid=1111`.

Let us introduce the two categories with which we will work.

A $*$-homomorphism of commutative C^*-algebras $f : A \to B$ is called *non-degenerate* if the linear span of $\{f(a) \cdot b : a \in A, b \in B\}$ is dense in B. Clearly non-degenerate $*$-homomorphisms compose. We will denote by $\mathsf{CommC^*Alg}$ the category of commutative C^*-algebras with non-degenerate $*$-homomorphisms. Notice that the trivial C^*-algebra $\{0\}$ is a terminal object in $\mathsf{CommC^*Alg}$, but it is not initial because of our choice of morphisms; in fact, there is no initial object at all.

We will also denote by $\mathsf{LoCompHaus}$ the category of locally compact Hausdorff spaces with proper continuous maps; recall that a continuous map is called *proper* if preimages of compact subsets are compact. This category has $\emptyset$ as an initial object, but no terminal object.

Theorem 1 (Gelfand duality)**.** *There is an equivalence of categories*

$$\mathsf{CommC^*Alg}^{\mathrm{op}} \simeq \mathsf{LoCompHaus}\,.$$

Explicitly, a locally compact Hausdorff space X corresponds to the commutative C^*-algebra of continuous functions on X which vanish at infinity.

Proof. First, we recall the unital case: Let $\mathsf{CommC^*Alg}_1$ denote the category of unital commutative C^*-algebras with unital $*$-homomorphisms. Let $\mathsf{CompHaus}$ denote the category of compact Hausdorff spaces with continuous maps. There is a functor

$$C : \mathsf{CompHaus} \to \mathsf{CommC^*Alg}_1^{\mathrm{op}}$$

which maps a compact Hausdorff space X to the C^*-algebra $C(X)$ of continuous functions on X (everything is $\mathbb{C}$-valued here) and a continuous map $f : X \to Y$ to the pullback $*$-homomorphism $f^* : C(Y) \to C(X)$, $h \mapsto h \circ f$. Conversely, we define a functor

$$\Phi_1 : \mathsf{CommC^*Alg}_1^{\mathrm{op}} \to \mathsf{CompHaus}$$

as follows: A commutative unital C^*-algebra A is mapped to its *unital Gelfand spectrum* $\Phi_1(A)$ whose underlying set consists of the unital $*$-homomorphisms $A \to \mathbb{C}$ and which is equipped with the subspace topology inherited from the product space $\prod_{a \in A} \mathbb{C}$ (also known as the weak-$*$-topology). The Banach-Alaoglu theorem implies that, in fact, $\Phi_1(A)$ is compact. A unital $*$-homomorphism $f : A \to B$ is mapped to the obvious continuous map $f^* : \Phi_1(B) \to \Phi_1(A)$. Now, if X is a compact Hausdorff space, there is a natural continuous map

$$\alpha : X \to \Phi_1(C(X)),\ x \mapsto (f \mapsto f(x)).$$

Urysohn's Lemma implies that α is injective and it is a standard exercise that α is surjective using the compactness of X. It follows that α is a homeomorphism. Hence, $\mathrm{id}_{\mathsf{CompHaus}} \cong \Phi_1 \circ C$. Conversely, if A is a commutative unital C^*-algebra, there is a natural unital $*$-homomorphism

$$\beta : A \to C(\Phi_1(A)),\ a \mapsto (\phi \mapsto \phi(a)).$$

The unital Gelfand representation theorem asserts that β is an isomorphism. Hence, $\mathrm{id}_{\mathsf{CommC^*Alg}_1^{\mathrm{op}}} \cong C \circ \Phi_1$. This shows that C and Φ_1 induce an (adjoint) equivalence of categories

$$\mathsf{CompHaus} \simeq \mathsf{CommC^*Alg}_1^{\mathrm{op}}.$$

Now let us derive the non-unital case. There is a functor

$$C_0 : \mathsf{LoCompHaus} \to \mathsf{CommC^*Alg}^{\mathrm{op}}$$

which maps a locally compact Hausdorff space X to the commutative C^*-algebra $C_0(X)$ of continuous functions $f : X \to \mathbb{C}$ which *vanish at infinity*, i.e. for every $\varepsilon > 0$ there is a compact subset $K \subseteq X$ such that $f(X \setminus K) \subseteq \overline{B}_\varepsilon(0)$. If $f : X \to Y$ is a proper continuous map, observe that $f^* : C_0(Y) \to C_0(X)$, $h \mapsto h \circ f$ is a well-defined $*$-homomorphism. It is non-degenerate: Let $b \in C_0(X)$ and $\varepsilon > 0$. Choose some compact subset $K \subseteq X$ with $b(X \setminus K) \subseteq \overline{B}_\varepsilon(0)$. Since $f(K) \subseteq Y$ is compact, we find a compactly supported function $a : Y \to [0,1]$ which equals 1 on $f(K)$. Then we have $\|f^*(a) \cdot b - b\| \leq 2\varepsilon$. Thus, $\{f^*(a) \cdot b : a \in C_0(Y),\, b \in C_0(X)\}$ is dense in $C_0(X)$, as desired.

In the other direction, we define the functor

$$\Phi : \mathsf{CommC^*Alg}^{\mathrm{op}} \to \mathsf{LoCompHaus}$$

as follows: A commutative C^*-algebra A is mapped to its *Gelfand spectrum* $\Phi(A)$ whose underlying set consists of the $*$-homomorphisms $A \to \mathbb{C}$ which are surjective (equivalently, non-zero) and which is equipped with the subspace topology inherited from $\prod_{a \in A} \mathbb{C}$. It follows immediately that $\Phi(A)$ is locally compact Hausdorff. We map a non-degenerate $*$-homomorphism $f : A \to B$ to the obvious continuous map $f^* : \Phi(B) \to \Phi(A)$, which we claim to be proper: As we will see later, f induces a unital $*$-homomorphism $f^+ : A^+ \to B^+$, which then induces a continuous map $\Phi_1(f^+) : \Phi_1(B^+) \to \Phi_1(A^+)$, which identifies with $(f^*)^+ : \Phi(B)^+ \to \Phi(A)^+$, and continuity at ∞ means that f^* is proper.

The next step is to construct functors

$$\begin{aligned} + \ &: \ \mathsf{LoCompHaus} \to \mathsf{CompHaus} \\ + \ &: \ \mathsf{CommC^*Alg} \to \mathsf{CommC^*Alg}_1 \end{aligned}$$

such that the squares of functors

$$\begin{tikzcd} \mathsf{LoCompHaus} \arrow[r, "+"] \arrow[d, "C_0"'] & \mathsf{CompHaus} \arrow[d, "C"] \\ \mathsf{CommC^*Alg}^{\mathrm{op}} \arrow[r, "+"] & \mathsf{CommC^*Alg}_1^{\mathrm{op}} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{LoCompHaus} \arrow[r, "+"] & \mathsf{CompHaus} \\ \mathsf{CommC^*Alg}^{\mathrm{op}} \arrow[r, "+"] \arrow[u, "\Phi"] & \mathsf{CommC^*Alg}_1^{\mathrm{op}} \arrow[u, "\Phi_1"'] \end{tikzcd}$$

commute up to isomorphism. To achieve this, we map a locally compact Hausdorff space X to its *Alexandrov compactification* X^+, whose points are those of X and some new point denoted by ∞. The topology is made in such a way that X becomes an open subspace of X^+ and the open neighborhoods of ∞ are of the form $X \setminus K \cup \{\infty\}$ for compact subsets $K \subseteq X$. A proper continuous map $f : X \to Y$ extends to a continuous map $f^+ : X^+ \to Y^+$ by mapping $\infty \mapsto \infty$; notice that continuity at ∞ is precisely what properness for f means.

On the algebraic side, we map a commutative C^*-algebra A to its *unitalization* A^+, whose elements are of the form (a, λ) with $a \in A$, $\lambda \in \mathbb{C}$. The vector space structure is the evident one. The multiplication is defined by $(a, \lambda) \cdot (b, \mu) = (ab + a\mu + \lambda b, \lambda\mu)$ with unit $1 := (0, 1)$. The involution is defined by $(a, \lambda)^* := (a^*, \overline{\lambda})$. The norm $\|(a, \lambda)\|' = \max(\|a\|, \|\lambda\|)$ makes A^* an involutive Banach algebra, but the C^*-condition is not satisfied. Instead, define $\|(a, \lambda)\|$ as the operator norm (with respect to $\|-\|'$) of the linear operator $A \to A$ which maps x to $xa + x\lambda$. A non-degenerate $*$-homomorphism $f : A \to B$ is mapped to the evident unital $*$-homomorphism $f^+ : A^+ \to B^+$, $(a, \lambda) \mapsto (f(a), \lambda)$.

For a locally compact Hausdorff space X we have a natural isomorphism of unital C^*-algebras

$$\gamma : C_0(X)^+ \xrightarrow{\cong} C(X^+)$$

which is given by mapping $(f, \lambda) \in C_0(X)^+$ to the continuous function $X^+ \to \mathbb{C}$ which extends $\lambda + f : X \to \mathbb{C}$ by $\infty \mapsto \lambda$. Notice that continuity at ∞ precisely means that f vanishes at infinity. Recall that naturality means that for every proper continuous map $X \to Y$ the induced diagram

$$\begin{tikzcd} C_0(X)^+ \arrow[r] & C(X^+) \\ C_0(Y)^+ \arrow[r] \arrow[u] & C(Y^+) \arrow[u] \end{tikzcd}$$

commutes. For a commutative C^*-algebra A we have a natural isomorphism of compact Hausdorff spaces

$$\delta : \Phi_1(A^+) \xrightarrow{\cong} \Phi(A)^+$$

which is given by mapping $\phi : A^+ \to \mathbb{C}$ to $\phi|_A : A \to \mathbb{C}$ if $\phi|_A$ is non-zero and otherwise (i.e. when ϕ is the projection $(a, \lambda) \mapsto \lambda$) to ∞. Naturality means that for every non-degenerate $*$-homomophism $A \to B$ the induced diagram

$$\begin{tikzcd} \Phi_1(A^+) \arrow[r] & \Phi(A)^+ \\ \Phi_1(B^+) \arrow[r] \arrow[u] & \Phi(B)^+ \arrow[u] \end{tikzcd}$$

commutes.

Now we can conclude from the unital Gelfand duality that for locally compact Hausdorff spaces X we have natural isomorphisms

$$X \xrightarrow{\cong} X^+ \setminus \{\infty\} \xrightarrow{\alpha} \Phi_1(C(X^+)) \setminus \{\alpha(\infty)\} \xrightarrow{\gamma^*} \Phi_1(C_0(X)^+) \setminus \{\gamma^*(\alpha(\infty))\}$$
$$\xrightarrow{\delta} \Phi(C_0(X))^+ \setminus \{\delta(\gamma^*(\alpha(\infty)))\} \overset{!}{=} \Phi(C_0(X))^+ \setminus \{\infty\} \xrightarrow{\cong} \Phi(C_0(X)).$$

An inspection shows that the composition $X \xrightarrow{\cong} \Phi(C_0(X))$ maps a point $x \in X$ to the surjective $*$-homomorphism $C_0(X) \to \mathbb{C}$ which evaluates at x. On the other hand, for commutative C^*-algebras A we get

$$A \cong \ker(A^+ \xrightarrow{pr_1} \mathbb{C}) \cong \ker(C(\Phi_1(A^+)) \xrightarrow{pr_1 \circ \beta^{-1}} \mathbb{C})$$

$$\cong \ker(C_0(\Phi(A))^+ \xrightarrow{pr_1 \circ \beta^{-1} \circ \delta \circ \gamma} \mathbb{C}) \overset{!}{=} \ker(C_0(\Phi(A))^+ \xrightarrow{pr_1} \mathbb{C}) \cong C_0(\Phi(A)).$$

An inspection shows that the composition $A \xrightarrow{\cong} C_0(\Phi(A))$ maps an element $a \in A$ to the continuous function $\Phi(A) \to \mathbb{C}$ which evaluates at a.

We have proven $\mathrm{id}_{\mathsf{LoCompHaus}} \cong \Phi \circ C_0$ and $\mathrm{id}_{\mathsf{CommC^*Alg}^{\mathrm{op}}} \cong C_0 \circ \Phi$, so that Φ and C_0 induce the desired equivalence (which is, in fact, part of an adjoint equivalence). $\square$

Remark 2. There is the following alternative and more general version of non-unital Gelfand duality: Let $\mathsf{CommC^*Alg}'$ denote the category of commutative C^*-algebras with all $*$-homomorphisms. This category is equivalent to the slice category $\mathsf{CommC^*Alg}_1 / \mathbb{C}$ via the unitalization resp. by taking the kernel in the other direction. The equivalence $\mathsf{CommC^*Alg} \simeq \mathsf{CompHaus}^{\mathrm{op}}$ maps $\mathbb{C} \mapsto \{\star\}$ and induces an equivalence

$$\mathsf{CommC^*Alg}_1 / \mathbb{C} \simeq \mathsf{CompHaus}^{\mathrm{op}} / \{\star\} \cong (\{\star\} / \mathsf{CompHaus})^{\mathrm{op}} \cong \mathsf{CompHaus}_*^{\mathrm{op}},$$

where $\mathsf{CompHaus}_*$ is the category of pointed compact Hausdorff spaces equipped with pointed continuous maps. This, in turn, is equivalent to the category $\mathsf{LoCompHaus}'$ of locally compact Hausdorff spaces where a morphism $X \to Y$ is a continuous map $X^+ \to Y^+$ between the Alexandrov compactifications which maps $\infty \mapsto \infty$; this may also be described as a continuous map $X \to Y^+$ such that for all compact subsets $K \subseteq Y$ the preimage in X is compact. Thus, we have an equivalence

$$\mathsf{CommC^*Alg}'^{\mathrm{op}} \simeq \mathsf{LoCompHaus}'.$$

From this equivalence we may also derive again the equivalence between the subcategories $\mathsf{CommC^*Alg}^{\mathrm{op}} \simeq \mathsf{LoCompHaus}$ once we observe that a morphism in $\mathsf{LoCompHaus}'$ lies in $\mathsf{LoCompHaus}$ if and only if its image in $\mathsf{CommC^*Alg}'$ lies in $\mathsf{CommC^*Alg}$. There is even a third variant of non-unital Gelfand duality, where we have locally compact Hausdorff spaces with arbitrary continuous maps on the geometric side and commutative C^*-algebras with suitably defined non-degenerate $*$-homomorphisms into multiplier algebras on the algebraic side; see for instance *Functoriality of Rieffel's Generalised Fixed-Point Algebras for Proper Actions* by van Huef-Raeburn-Williams.

Every equivalence of categories preserves limits, colimits, monomorphisms and epimorphisms. Let us apply this to Gelfand duality:

Corollary 3. *Let $f : X \to Y$ be a proper continuous map of locally compact Hausdorff spaces and $f^* : C_0(Y) \to C_0(X)$ be the associated $*$-homomorphism. The following are equivalent:*

1. *f is a monomorphism in* $\mathsf{LoCompHaus}$
2. *f is injective*
3. *f^* is an epimorphism in* $\mathsf{CommC^*Alg}$
4. *f^* is surjective*

Proof. 1. $\Leftrightarrow$ 3. follows formally from Gelfand duality. 1. $\Rightarrow$ 2. The forgetful functor $\mathsf{LoCompHaus} \to \mathsf{Set}$ is represented by $\{\star\}$ and hence preserves monomorphisms. 2. $\Rightarrow$ 1. and 4. $\Rightarrow$ 3. follow from the fact that faithful functors reflect mono- and epimorphisms. 2. $\Rightarrow$ 4. Let f be injective. Since f is a proper map into a locally compact Hausdorff space, f is a closed map. It follows that f factors as

$$X \xrightarrow{p} Z \xrightarrow{i} Y,$$

where i is the inclusion of a closed subspace (the image of f) and p is an isomorphism. Therefore, it suffices to prove that the map $i^* : C_0(Y) \to C_0(Z)$ is surjective. It is enough to prove that the map $(i^*)^+ : C_0(Y)^+ \to C_0(Z)^+$ is surjective. This map identifies with $(i^+)^* : C(Y^+) \to C_0(Z^+)$. The claim follows from Tietze's extension theorem applied to the compact Hausdorff space Y^+ and its closed subspace $i^+ : Z^+ \to Y^+$. □

Corollary 4. *Let X be a locally compact Hausdorff space. There is an inclusion-reversing bijection between the C^*-ideals of $C_0(X)$ and the closed subsets of X. Here, a closed subset $Z \subseteq X$ corresponds to the C^*-ideal $I(Z) = \{f \in C_0(X) : f|_Z = 0\}$ of $C_0(X)$, which satisfies $C_0(X)/I(Z) \cong C_0(Z)$.*

Proof. The partial order of C^*-ideals of $C_0(X)$ is isomorphic to the category of surjective $*$-homomorphisms $C_0(X) \to A$, where A is some commutative C^*-algebra (as a full subcategory of the slice category $C_0(X)/\mathsf{CommC^*Alg}$). By Gelfand duality and our previous result, this category is contravariantly equivalent to the category of injective proper continuous maps $i : Y \to X$. This category is equivalent to the partial order of closed subsets of X. Explicitly, a closed subset $Z \subseteq X$ gets mapped to the inclusion $i : Z \hookrightarrow X$, which then gets mapped to the surjection $i^* : C_0(X) \to C_0(Z)$, which then gets mapped to the ideal $\ker(i^*) = I(Z)$. □

Corollary 5. *Let X be a locally compact Hausdorff space. There is a bijection between the points of X and the maximal C^*-ideals of $C_0(X)$, given by mapping $x \in X$ to the maximal C^*-ideal $\{f \in C_0(X) : f(x) = 0\}$.*

Proof. This follows formally from the previous result by looking at maximal proper elements of the involved partial orders (resp. their dual). Of course, we have already used this result more or less in our proof of Gelfand duality anyway. □

Corollary 6. *Let $f : X \to Y$ be a proper continuous map of locally compact Hausdorff spaces and $f^* : C_0(Y) \to C_0(X)$ be the associated $*$-homomorphism. The following are equivalent:*

1. *f is an epimorphism in* $\mathsf{LoCompHaus}$
2. *f is surjective*
3. *f^* is a monomorphism in* $\mathsf{CommC^*Alg}$
4. *f^* is injective*

Proof. 1. $\Leftrightarrow$ 3. follows formally from Gelfand duality. 2. $\Rightarrow$ 4. is clear. 4. $\Rightarrow$ 2. Since f is proper, the image $f(X)$ is a closed subset of Y. If f is not surjective, we find some $y \in Y \setminus f(X)$. By Urysohn's Lemma there is some $g \in C(Y^+)$ with $g|_{f(X)^+} \equiv 0$ and $g(y) = 1$. Because of $g(\infty) = 0$, g restricts to some $h \in C_0(Y)$ with $h|_{f(X)} \equiv 0$ and $h(y) = 1$. Thus, f^* is not injective. 2. $\Rightarrow$ 1. is clear. 1. $\Rightarrow$ 2. uses the theory of perfect mappings (see for instance Engelking's *General topology*, section 3.7); I am obliged to Niels Diepeveen for sharing this argument. We consider the pushout of the inclusion $f(X) \hookrightarrow Y$ of itsself in Top, i.e. the quotient space $Q = (Y \times \{0,1\})/\sim$ with $(y,0) \sim (y,1)$ if $y \in f(X)$, equipped with the natural projection $p : Y \times \{0,1\} \to Q$. The fibers of p are finite, hence compact, and p is closed, as can be easily checked using the fact that $f(X) \subseteq Y$ is closed. This means that p is perfect. Since $Y \times \{0,1\}$ is locally compact and Hausdorff, it then follows that Q is locally compact and Hausdorff, too. Perfect maps between locally compact Hausdorff maps are precisely the proper continuous maps. In particular, p is proper. Composing p with the two natural embeddings $Y \hookrightarrow Y \times \{0,1\}$

(which are clearly proper) produces two proper continuous maps $g_1, g_2 : Y \to Q$ which satisfy $g_1 \circ f = g_2 \circ f$ by construction. Since f is an epimorphism, it follows that $g_1 = g_2$. But this means that f is surjective. $\square$

Corollary 7. *For finite families of locally compact Hausdorff spaces $(X_i)_{i\in I}$ we have an isomorphism of C^*-algebras*

$$C_0(\coprod_{i\in I} X_i) \cong \prod_{i\in I} C_0(X_i).$$

Proof. The isomorphism can be checked directly, but we would like to illustrate how it follows from Gelfand duality. One checks that the disjoint union $\coprod_{i\in I} X_i$ equipped with the inclusion maps $X_i \hookrightarrow \coprod_{i\in I} X_i$ is a coproduct of $(X_i)_{i\in I}$ in $\mathsf{LoCompHaus}$; the finiteness of I is essential here because of our choice of morphisms. Likewise, one checks that the product of C^*-algebras (equipped with the maximum norm) is in fact the product in $\mathsf{CommC^*Alg}$. The claim follows since $C_0 : \mathsf{LoCompHaus} \to \mathsf{CommC^*Alg}^{\mathrm{op}}$ preserves coproducts (which actually can already be derived from the adjunction to Φ, which is completely formal). $\square$

Corollary 8. *Let X, Y be two compact Hausdorff spaces. Then, there is an isomorphism of C^*-algebras*

$$C(X) \otimes_{\max} C(Y) \cong C(X \times Y).$$

Proof. It is known that the maximal tensor product $A \otimes_{\max} B$ of two arbitrary unital C^*-algebras is the universal unital C^*-algebra equipped with two unital $*$-homomorphisms $A \to A \otimes_{\max} B \leftarrow B$ which commute elementwise in the obvious sense. If A, B are commutative, $A \otimes_{\max} B$ is commutative, too, and it follows that it is the coproduct of A and B in $\mathsf{CommC^*Alg}_1$. Since $C : \mathsf{CompHaus} \to \mathsf{CommC^*Alg}_1^{\mathrm{op}}$ preserves products (as does every equivalence of categories), the claim follows. $\square$

Remark 9. If X, Y are just locally compact Hausdorff space, we also have

$$C_0(X) \otimes_{\max} C_0(Y) \cong C_0(X \times Y).$$

A reduction to the compact case is possible, using the following result: If A, B are two C^*-algebras, the canonical map $(A \otimes_{\max} B)^+ \to A^+ \otimes_{\max} B^+$ is the equalizer of the two evident maps $A^+ \otimes_{\max} B^+ \rightrightarrows A^+ \times B^+$. Details omitted.

Corollary 10. *Let X be a locally compact Hausdorff space. Then, there are isomorphisms of C^*-algebras*

$$C_b(X) \cong C(\beta(X)) \cong M(C_0(X)),$$

where β denotes the Stone-Čech-compactification and M the multiplier algebra.

Proof. We will use that the functor $\beta : \mathsf{Top} \to \mathsf{CompHaus}$ is left adjoint to the inclusion functor. Since any bounded continuous function $X \to \mathbb{C}$ has image in some compact subset $K \subseteq \mathbb{C}$, we get a bijection of underlying sets (the colimit taken over the partial order of compact subsets $K \subseteq \mathbb{C}$)

$$C_b(X) \cong \varinjlim_K \mathrm{Hom}_{\mathsf{Top}}(X, K) \cong \varinjlim_K \mathrm{Hom}_{\mathsf{CompHaus}}(\beta(X), K) \cong C(\beta(X)),$$

which is easily seen to be a $*$-homomorphism, thus an $*$-isomorphism.

Now let us check that the inclusion $C_0(X) \hookrightarrow C_b(X)$ satisfies the universal property of the multiplier algebra, i.e. that this is an C^*-ideal and that for every other C^*-ideal

$C_0(A) \hookrightarrow A$ there is a unique unital $*$-homomorphism $A \to C_b(X)$ such that

$$\begin{tikzcd} A \arrow[rr, dashed] & & C_b(X) \\ & C_0(X) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes. An easy computation shows that $C_0(X) \hookrightarrow C_b(X)$ is a C^*-ideal. Using $C_b(X) \cong C(\beta(X))$, that ideal corresponds to the closed subset $\beta(X) \setminus X$ of $\beta(X)$. Now, if $C_0(X) \hookrightarrow C(Y)$ is a C^*-ideal, where Y is compact Hausdorff, X corresponds to some open subset of Y. The universal property of $\beta(X)$ yields a unique continuous map $\beta(X) \to Y$ which extends $X \to Y$. By Gelfand duality, this corresponds to a unique unital $*$-homomorphism $C(Y) \to C(\beta(X))$ such that the diagram with $C_0(X)$ commutes. Thus, we are done when A is commutative. Sorry, we won't discuss the general case. $\square$