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Grothendieck Topologies for the Simplex Category

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Abstract Toda subcategorías reflectiva de una categoría de pregavillas es una categoría de gavillas.

En conjuntos simpliciales los complejos de Kan, son una subcategorías reflectiva???

Keywords Simplicial Set · Sheaf · Grothendieck Topology

1 Introduction

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2 Preliminaries

We denote the simplex category as Δ. It is the category of non-empty finite ordinals and monotone maps. For a natural number n, we put $[n]$ as $\{0, \dots n\}$. In this manner $[n]$ is the ordinal $n+1$.

We denote the category of sets and functions as $\mathcal{S}$. A simplicial set K is contravariant functor from the simplex category Δ into $\mathcal{S}$. For reference of simplicial sets, we recommend [1] and [4].

For a category $\mathcal{A}$ and an object A in $\mathcal{A}$, a sieve S over A is a family of morphisms in $\mathcal{A}$ with codomain A such that $fg \in S$, if $f \in S$ and fg is defined. We denote the family of sieves over A as $\Omega(A)$. We have that $\Omega(A)$ is ordered by the inclusion. Moreover, the intersection of sieves is a sieve. So for a family of morphisms $\mathcal{X}$ with codomain A, we have the sieve $c(\mathcal{X})$ generated by $\mathcal{X}$. In particular, for a morphism f in $\mathcal{A}$, we have that

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$c(f) = \{fg \in \mathcal{A} \mid cod(g) = dom(f)\}$. We observe that $c(\mathcal{X}) = \bigcup_{f\in\mathcal{X}} c(f)$. Thus for $\mathcal{X}$ and $\mathcal{Y}$ families of morphisms with codomain A, if $\mathcal{X} \subseteq \mathcal{Y}$ then $c(\mathcal{X}) \subseteq c(\mathcal{Y})$.

The references that we recommend for topos theory are [2] and [3].

3 Sieves in the simplex category

Proposition 1 *Let $f\colon [m] \longrightarrow [n]$ be a morphism in Δ. Then*

$$c(f) = \{g\colon [k] \longrightarrow [n] \in \Delta \mid im(g) \subseteq im(f)\}$$

Proof Let $g \in c(f)$. Then there is $h \in \mathcal{A}$ such that $g = fh$. It follows that $im(g) \subseteq im(f)$.

Let $g\colon [k] \longrightarrow [n]$ with $im(g) \subseteq im(f)$. We build a function $h\colon [k] \longrightarrow [m]$ given by $h(x) = \min\{y \in [n] \mid f(y) = g(x)\}$. As $im(g) \subseteq im(f)$, the set $\{y \in [n] \mid f(y) = g(x)\}$ is not empty for any $x \in [k]$. By construction h satifies that $g = fh$. We define $A_x = \{y \in [n] \mid g(x) \leq f(y)\}$ for $x \in [k]$. So $h(x) = \min A_x$. If $x \leq x'$ in $[k]$, then $A_{x'} \subseteq A_x$. Thus $h(x) = \min A_x \leq A_{x'} = h(x')$. Therefore h is monotone and $g = fh$.

Proposition 2 *Let S be a sieve over $[n] \in \Delta$. Then there is a minimal family $\mathcal{X}$ contained in S such $c(\mathcal{X}) = S$. Moreover, $\mathcal{X}$ is finite.*

Proof

Proposition 3 *Let $f\colon [m] \longrightarrow [n]$ and $\alpha\colon [k] \longrightarrow [n]$ be two morphisms in Δ. Then*

$$\alpha^*(c(f)) = \{g\colon [l] \longrightarrow [m] \in \Delta \mid im(g) \subseteq \alpha^{-1}(im(f))\}.$$

Proof

4 Grothendieck Topologies in the simplex category

5 Simplicial Sets as Sheaves

References

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