GEOMETRIC REALIZATION OF COVERING COMPLEXES

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ABSTRACT. We prove that the geometric realization of a covering complex is a covering space. Also, this holds true for the universal covering complex. Finally, we prove that the fundamental group of a complex coincides with the fundamental group of the geometric realization.

INTRODUCTION

1. PRELIMINARIES

We recall that, given a topological space X, a covering space on X it's a continuous map $p\colon E \to X$, such that for every $x \in X$, there is an open neighborhood U such that $p^{-1}(U)$ it's a disjoint union of open sets U_λ, $\lambda \in \Lambda$, and $p|_{U_\lambda}\colon U_\lambda \to U$ it's a homeomorphism. We recommend the book of P. May [1] and the book of J. Rotman [3] as reference of covering spaces.

An abstract simplicial complex is a pair $(S, \mathcal{K})$ where S is a set and $\mathcal{K}$ is a family of non-empty finite subsets of S such that, if $\sigma \subseteq \tau$ and $\tau \in \mathcal{K}$ then $\sigma \in \mathcal{K}$. We call complexes to the abstract simplicial complexes. If $\sigma \in \mathcal{K}$, then the dimension of σ is $|\sigma| - 1$, and we denote it by $dim(\sigma)$. The elements of $\mathcal{K}$ of dimension n are called n-simplices, and we denote the set of n-simplices by $\mathcal{K}_n$. The 0-simplices are called vertices. An edge e in $(S, \mathcal{K})$ is a pair of vertices (x, y) where $\{x, y\} \in \mathcal{K}$, x is the origin of the edge e and we denote it by $orig(e)$, and y is the end of the edge e and we denote by $end(e)$. A path α in $(S, \mathcal{K})$ is a finite sequence of edges $e_1, \dots, e_n$ such that $end(e_i) = orig(e_{i+1})$ with $i = 1, \dots, n-1$. We define $orig(\alpha) = orig(e_1)$ and $end(\alpha) = end(e_n)$.

The geometric realization of a complex $(S, \mathcal{K})$ is the set of all function $\phi\colon S \longrightarrow [0, 1]$ such that:

- $supp(\phi) \in \mathcal{K}$
- $\sum_{s \in S} \phi(s) = 1$

We denote this set by $|(S, \mathcal{K})|$. We may think $[0, 1]^S$ as the direct limit of $[0, 1]^A$ where Aranges over all finite subsets of S. So we give the $|(S, \mathcal{K})|$ the subspace topology.

A morphism between complex $(S_1, \mathcal{K}_1)$ and $(S_2, \mathcal{K}_2)$ is a map $f\colon S_1 \longrightarrow S_2$ such that $f(\sigma) \in \mathcal{K}_2$ for any $\sigma \in \mathcal{K}_2$.

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2. Geometric Realization of Covering Complexes

Definition 2.1. *Let $(S, \mathcal{K})$ be a complex. We say that $(S, \mathcal{K})$ is connected if for any pair of vertices x, y of $(S, \mathcal{K})$ there is a path α such that $orig(\alpha) = x$ and $end(\alpha) = y$.*

The following definition is due J. Rotman in [2].

Definition 2.2. *Let $(S, \mathcal{K})$ be a complex. A covering of $(S, \mathcal{K})$ is a pair $((T, \mathcal{L}), p)$ where $(T, \mathcal{L})$ is a complex and $p\colon T \longrightarrow S$ is a morphism of complexes such that:*

- *$(T, \mathcal{L})$ is a connected complex.*
- *For every $\sigma \in \mathcal{L}$, $p^{-1}(\sigma) = \bigcup_{i \in I} \sigma_i$ where the family $\{\sigma_i\}_{i \in I}$ is a pairwise disjoint family of simplices of $\mathcal{K}$ such that $p|_{\sigma_i}\colon \sigma_i \longrightarrow \sigma$ is bijective.*

The map p is called projection and the simplices σ_i are called sheets over σ.

Proposition 2.1. *Let $(S, \mathcal{K})$ be an abstract simplicial complex and $((T, \mathcal{L}), p)$ an abstract simplicial covering of $(S, \mathcal{K})$. Then $(|(T, \mathcal{L})|, |p|)$ is covering of $|(S, \mathcal{K})|$.*

Proof. □

3. Geometric Realization of the Universal Covering Complex

4. Fundamental Group

References

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[2] Joseph Rotman. Covering complexes with applications to algebra. *The Rocky Mountain Journal of Mathematics*, 3(4):641–674, 1973.

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