

# GEOMETRIC REALIZATION OF COVERING COMPLEXES

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ABSTRACT. We prove that the geometric realization of a covering complex is a covering space. Also, this holds true for the universal covering complex.

## INTRODUCTION

The theory of abstract simplicial complexes is a useful tool in the calculation of fundamental groups. This fact appears explicitly in the paper [7] of A. Weil, .... For any connected abstract simplicial complex  $S$  its edge-path group  $E(S)$  is naturally isomorphic to the fundamental group of its geometric realization  $\pi_1(|S|)$ . The edge-path group could be described explicitly by generators and relations. As reference for these facts, see the book of I. Singer and J. Thorpe [5].

We base our definition of covering complex given by J. Rotman in [3], but there are other definitions covering complex as the one in [1]. The definition of J. Rotman

## 1. PRELIMINARIES

We recall that, given a topological space  $X$ , a covering space on  $X$  is a continuous map  $p: E \rightarrow X$ , such that for every  $x \in X$ , there is an open neighborhood  $U$  such that  $p^{-1}(U)$  is a disjoint union of open sets  $U_\lambda$ ,  $\lambda \in \Lambda$ , and  $p|_{U_\lambda}: U_\lambda \rightarrow U$  is a homeomorphism. We recommend the book of P. May [2] and the book of J. Rotman [4] as reference of covering spaces.

An abstract simplicial complex is a pair  $(S, \mathcal{K})$  where  $S$  is a set and  $\mathcal{K}$  is a family of non-empty finite subsets of  $S$  such that:

- $\bigcup \mathcal{K} = S$ .
- If  $\sigma \subseteq \tau$  and  $\tau \in \mathcal{K}$  then  $\sigma \in \mathcal{K}$ .

We call complexes to the abstract simplicial complexes. If  $\sigma \in \mathcal{K}$ , then the dimension of  $\sigma$  is  $|\sigma| - 1$ , and we denote it by  $\dim(\sigma)$ . The elements of  $\mathcal{K}$  of dimension  $n$  are called  $n$ -simplices, and we denote the set of  $n$ -simplices by  $\mathcal{K}_n$ . We define the  $n$ -skeleton of  $(S, \mathcal{K})$  as the complex  $(S, \bigcup_{m=1}^n \mathcal{K}_m)$ , and we denote it by  $sk_n(S, \mathcal{K})$ . The 0-simplices are called vertices. The dimension of  $(S, \mathcal{K})$  is defined as the supremum of  $\dim(\sigma)$  where  $\sigma$  ranges over  $\mathcal{K}$ , we denote it by  $\dim(S, \mathcal{K})$ . This dimension may be infinite. We call the complex  $(S, \mathcal{K})$  finite, if  $S$  is finite. In particular, a finite complex has finite dimension. A complex  $(S, \mathcal{K})$  is called indiscrete, if  $\mathcal{K} = \{\{s\} \mid s \in S\}$ .

An edge  $e$  in  $(S, \mathcal{K})$  is a pair of vertices  $(x, y)$  where  $\{x, y\} \in \mathcal{K}$ ,  $x$  is the origin of the edge  $e$  and we denote it by  $orig(e)$ , and  $y$  is the end of the edge  $e$  and we denote it by  $end(e)$ . A path  $\alpha$  in  $(S, \mathcal{K})$  is a finite sequence of edges  $e_1, \dots, e_n$  such

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that  $\text{end}(e_i) = \text{orig}(e_{i+1})$  with  $i = 1, \dots, n-1$ . We define  $\text{orig}(\alpha) = \text{orig}(e_1)$  and  $\text{end}(\alpha) = \text{end}(e_n)$ .

A morphism between complex  $(S_1, \mathcal{K}^1)$  and  $(S_2, \mathcal{K}^2)$  is a map  $f: S_1 \rightarrow S_2$  such that  $f(\sigma) \in \mathcal{K}^2$  for any  $\sigma \in \mathcal{K}^1$ . We call a morphism of complexes a simplicial map. We denote by  $\mathcal{C}$  to the category of complexes and simplicial maps.

If  $X$  is a non empty set, we denote by  $D(X)$  the complex given  $(X, \mathcal{D}_X)$  where  $\mathcal{D}_X$  is the set of all non empty finite subsets of  $X$ . As reference of complexes, we recommend [5] and [6].

We denote by  $\mathbb{P}(X)$  the power set of a set of  $X$ . If  $f: X \rightarrow Y$  is a function, then  $f$  induces a map  $f_\circ: \mathbb{P}(X) \rightarrow \mathbb{P}(Y)$  where  $f_\circ(A)$  is the direct image of  $A$  under  $f$  for  $A \in \mathbb{P}(X)$ .

We denote by  $\mathcal{T}$  to the category of topological spaces and continuous maps.

## 2. COVERING COMPLEXES

**Definition 2.1.** Let  $(S, \mathcal{K})$  be a complex. We say that  $(S, \mathcal{K})$  is connected if for any pair of vertices  $x, y$  of  $(S, \mathcal{K})$  there is a path  $\alpha$  such that  $\text{orig}(\alpha) = x$  and  $\text{end}(\alpha) = y$ .

The following definition is due J. Rotman in [3].

**Definition 2.2.** Let  $(S, \mathcal{K})$  be a complex. A covering of  $(S, \mathcal{K})$  is a pair  $((T, \mathcal{L}), p)$  where  $(T, \mathcal{L})$  is a complex and  $p: T \rightarrow S$  is a simplicial map such that:

- $(T, \mathcal{L})$  is a connected complex.
- For every  $\sigma \in \mathcal{K}$ ,  $p^{-1}(\sigma) = \bigcup_{i \in I} \sigma_i$  where the family  $\{\sigma_i\}_{i \in I}$  is a pairwise disjoint family of simplices of  $\mathcal{L}$  such that  $p|_{\sigma_i}: \sigma_i \rightarrow \sigma$  is bijective.

The map  $p$  is called projection and the simplices  $\sigma_i$  are called sheets over  $\sigma$ .

We observe that  $p$  is surjective and  $(S, \mathcal{K})$  is connected. For our geometric porpoises we need a stronger definition of covering complex.

**Definition 2.3.** Let  $(S, \mathcal{K})$  be a complex. A geometric covering of  $(S, \mathcal{K})$  is a covering  $((T, \mathcal{L}), p)$  such that for any simplex  $\sigma \in \mathcal{L}_n$ ,  $p(\sigma) \in \mathcal{K}_n$ . In other words,  $p$  preserves the dimension of the simplices. We have that geometric coverings preserve the dimension of the complexes. By definition all geometric coverings are coverings.

**Example 2.1.** Let  $(S, \mathcal{K})$  be the complex  $D(\mathbb{Z}_2)$ , and  $(T, \mathcal{L})$  be the complex  $D(\mathbb{Z}_4)$ . We consider the canonical projection  $p: \mathbb{Z}_4 \rightarrow \mathbb{Z}_2$  which is a simplicial map. This simplicial map does not preserve the dimension of the simplices. We notice that  $p^{-1}(\{[0]\}) = \{[0]\} \cup \{[2]\}$ ,  $p^{-1}(\{[1]\}) = \{[1]\} \cup \{[3]\}$ , and  $p^{-1}(\{[0], [1]\}) = \{[0], [2]\} \cup \{[1], [3]\}$  and  $D(\mathbb{Z}_4)$  is a connected complex, so  $p$  is a covering which is not a geometric covering.

**Proposition 2.1.** Let  $(S, \mathcal{K})$  be a complex, and  $((T, \mathcal{L}), p)$  a covering complex of  $(S, \mathcal{K})$ . If  $\sigma \in \mathcal{K}$  is a maximal simplex, then  $p^{-1}$  is the disjoint union of maximal simplices.

*Proof.* Let  $\sigma \in \mathcal{K}$  be a maximal simplex with  $p^{-1}(\sigma) = \bigcup_{i \in I} \sigma_i$  where the family  $\{\sigma_i\}_{i \in I}$  is a pairwise disjoint family of simplices of  $\mathcal{K}$  such that  $p|_{\sigma_i}: \sigma_i \rightarrow \sigma$  is bijective. If  $\sigma_j$  is not maximal for some  $j \in I$ , then there is a simplex  $\tau \in \mathcal{L}$  such that  $\sigma_j \subset \tau$ . As  $\sigma$  is maximal and  $\sigma = p(\sigma_j) \subseteq p(\tau)$ , we have that  $p(\tau) = \sigma$ . This contradicts the fact that  $p$  preserves the dimension. Therefore  $\sigma_i$  is a maximal simplex.  $\square$

**Definition 2.4.** Let  $(S, \mathcal{K})$  and  $(T, \mathcal{L})$  be complexes and  $f: (S, \mathcal{K}) \rightarrow (T, \mathcal{L})$  be a simplicial map. We say that  $f$  reflects maximal simplices, if  $\sigma \in \mathcal{K}$  is such that  $f(\sigma)$  is a maximal simplex, then  $\sigma$  is a maximal simplex.

**Proposition 2.2.** Let  $(S, \mathcal{K})$  be a finite dimensional complex, and  $((T, \mathcal{L}), p)$  a covering complex of  $(S, \mathcal{K})$ . Then  $((T, \mathcal{L}), p)$  is a geometric covering complex of  $(S, \mathcal{K})$  if and only if  $p$  reflects maximal simplices.

*Proof.* Conjetura □

Let  $(S, \mathcal{K})$  and  $(T, \mathcal{L})$  be complexes and  $f: S \rightarrow T$  be a function. We notice that  $f$  is a simplicial map if  $f_*(\mathcal{K}) \subseteq \mathcal{L}$ . So we denote the induced map by  $f_*: \mathcal{K} \rightarrow \mathcal{L}$ .

**Definition 2.5.** Let  $(S, \mathcal{K})$  be a complex. A combinatoric covering of  $(S, \mathcal{K})$  is a covering  $((T, \mathcal{L}), p)$  such that  $p_*^{-1}(\sigma)$  is a family of pairwise disjoint simplices.

**Example 2.2.** Let  $(S, \mathcal{K})$  be the complex  $D(\mathbb{Z}_2)$ , and  $(T, \mathcal{L})$  be the complex  $sk_1(D(\mathbb{Z}_4))$ . We consider the canonical projection  $p: \mathbb{Z}_4 \rightarrow \mathbb{Z}_2$  which is a simplicial map. This map is a geometric covering complex, but is not a combinatoric complex as  $p_*^{-1}(\{[0]\}) = \{\{[0]\}, \{[2]\}, \{[0], [2]\}\}$  is not pairwise disjoint.

**Example 2.3.** Let  $(S, \mathcal{K})$  be the complex  $sk_1(D(\mathbb{Z}_3))$ , and  $(T, \mathcal{L})$  be the complex given by  $T = \mathbb{Z}$  and  $\mathcal{L} = \{\{n, n+1\} \mid n \in \mathbb{Z}\} \cup \{\{m\} \mid m \in \mathbb{Z}\}$ . We consider the canonical projection  $p: \mathbb{Z} \rightarrow \mathbb{Z}_3$  which is a simplicial map. Moreover,  $((T, \mathcal{L}), p)$  is a combinatoric covering complex.

**Proposition 2.3.** Let  $(S, \mathcal{K})$  be a complex, and  $((T, \mathcal{L}), p)$  a combinatoric covering complex of  $(S, \mathcal{K})$ . Then  $((T, \mathcal{L}), p)$  is a geometric covering complex.

*Proof.* Let  $\sigma \in \mathcal{L}_n$ . As  $p$  is a covering there is a family  $\{\sigma_i\}_{i \in I}$  of pairwise disjoint simplices such that  $p^{-1}(p(\sigma)) = \bigcup_{i \in I} \sigma_i$  and  $|\sigma_i| = |p(\sigma)|$  for all  $i \in I$ . We observe that  $\{\sigma_i\}_{i \in I}$  and  $p_*^{-1}(\sigma)$  are partitions of  $p^{-1}(p(\sigma))$ . As we have that  $\{\sigma_i\}_{i \in I} \subseteq p_*^{-1}(\sigma)$ , then  $\{\sigma_i\}_{i \in I} = p_*^{-1}(\sigma)$ . So  $\sigma = \sigma_j$  for some  $j \in I$ , and  $|\sigma| = |p(\sigma)|$ . Therefore  $((T, \mathcal{L}), p)$  a geometric covering complex. □

### 3. G-COVERING COMPLEXES

**Definition 3.1.** Let  $G$  be a group and  $(S, \mathcal{K})$  be a complex. We say that  $(S, \mathcal{K})$  is a  $G$ -complex, if  $S$  is a  $G$ -set and  $g\sigma \in \mathcal{K}$  for all  $\sigma \in \mathcal{K}$ . We notice that in particular  $\mathcal{K}$  is a  $G$ -set.

**Definition 3.2.** Let  $G$  be a group and  $(S, \mathcal{K})$  be a  $G$ -complex. We define  $\mathcal{K}^G := \{\pi(\sigma) \subseteq S/G \mid \sigma \in \mathcal{K}\}$  where  $\pi: S \rightarrow S/G$  is the canonical projection.

**Proposition 3.1.** Let  $G$  be a group and  $(S, \mathcal{K})$  be a  $G$ -complex. Then  $(S/G, \mathcal{K}^G)$  is a complex. Moreover,  $\pi$  is a simplicial map.

*Proof.* □

**Definition 3.3.** Let  $G$  be a group and  $(S, \mathcal{K})$  be a  $G$ -complex. We say that the action is discontinuous, if  $g\sigma \cap \sigma = \emptyset$  for any  $\sigma \in \mathcal{K}$ .

**Proposition 3.2.** Let  $G$  be a group and  $(S, \mathcal{K})$  be a  $G$ -complex with discontinuous action. Then  $((S/G, \mathcal{K}^G), \pi)$  is covering complex.

*Proof.* □

**Example 3.1.**

## 4. GEOMETRIC REALIZATION OF COVERING COMPLEXES

**Definition 4.1** (Geometric Realization). *The geometric realization of a complex  $(S, \mathcal{K})$  is the set of all function  $\phi: S \rightarrow [0, 1]$  such that:*

- $\text{supp}(\phi) \in \mathcal{K}$
- $\sum_{s \in S} \phi(s) = 1$

We denote this set by  $|(S, \mathcal{K})|$ . We can give  $|(S, \mathcal{K})|$  a metric topology given by:

$$d(\phi, \psi) = \sqrt{\sum_{s \in S} (\phi(s) - \psi(s))^2}$$

for  $\phi, \psi \in |(S, \mathcal{K})|$ . When we endowed  $|(S, \mathcal{K})|$  with the metric topology we denote it by  $|(S, \mathcal{K})|_d$ . There is a second topology for  $|(S, \mathcal{K})|$  called the coherent topology. For each simplex  $\sigma \in \mathcal{K}$ , we define its geometric realization  $|\sigma|$  as the set of functions  $\phi \in |(S, \mathcal{K})|$  with  $\text{supp}(\phi) \subseteq \sigma$ . We give to  $|\sigma|$  the subspace topology inherited as subset of  $|(S, \mathcal{K})|$ . If we consider the inclusion  $i_\sigma: |\sigma| \rightarrow |(S, \mathcal{K})|$ , then coherent topology on  $|(S, \mathcal{K})|$  is the largest topology which makes all the inclusions continuous. Usually,  $|(S, \mathcal{K})|$  is considered with the coherent topology. We may characterize  $|(S, \mathcal{K})|$  as the colimit in  $\mathcal{T}$  of the geometric realization of its simplices. So a function  $f: |(S, \mathcal{K})| \rightarrow X$  is continuous if and only its restrictions to  $|\sigma|$  is continuous for all  $\sigma \in \mathcal{K}$ . Especially, the identity  $|(S, \mathcal{K})| \rightarrow |(S, \mathcal{K})|_d$  is continuous, so the coherent topology contains the metric topology. In particular, if  $(S, \mathcal{K})$  is a finite complex, then both topologies coincide.

We observe that if  $\sigma, \tau \in \mathcal{K}$  are disjoint, then  $|\sigma|$  and  $|\tau|$  are disjoint.

**Definition 4.2.** Let  $(S, \mathcal{K})$  be a complex and  $s \in S$ . We define  $\phi_s: S \rightarrow [0, 1]$  as  $\phi_s(t) = \delta_{st}$  for any  $t \in S$  where  $\delta$  is the Kronecker's delta.

**Definition 4.3.** If  $f: (S, \mathcal{K}) \rightarrow (T, \mathcal{L})$  is a simplicial map, then it induces a continuous function  $|f|: |(S, \mathcal{K})| \rightarrow |(T, \mathcal{L})|$ . If  $\phi \in |(S, \mathcal{K})|$  with  $\sigma = \text{supp}(\phi)$  then  $\phi = \sum_{s \in \sigma} \phi(s) \phi_s$ . So we define  $|f|(\phi) := \sum_{s \in \sigma} \phi(s) \phi_{f(s)}$ . In this way, the geometric realization is a functor from  $\mathcal{C}$  to  $\mathcal{T}$ .

**Proposition 4.1.** Let  $(S, \mathcal{K})$  be a complex and  $((T, \mathcal{L}), p)$  a finite geometric covering of  $(S, \mathcal{K})$ . Then  $|(T, \mathcal{L}), p|$  is covering of  $|(S, \mathcal{K})|$ .

*Proof.* Let  $\phi \in |(S, \mathcal{K})|$  with  $\text{supp}(\phi) = \sigma \in \mathcal{K}$ . Then  $p^{-1}(\sigma) = \bigcup_{i \in I} \sigma_i$  where the family  $\{\sigma_i\}_{i \in I}$  is a pairwise disjoint family of simplices of  $\mathcal{L}$  such that  $p|_{\sigma_i}: \sigma_i \rightarrow \sigma$  is bijective. We define  $R = \min\{\frac{d(\phi, \phi_s)}{2} \mid s \in \sigma\}$ . So  $\mathbb{B}_R(\phi)$  is an open neighborhood of  $\phi$ . We affirm that  $|p|^{-1}(\mathbb{B}_R(\phi)) = \bigcup_{i \in I} \mathbb{B}_R(\phi_i)$   $\square$

**Proposition 4.2.** Let  $(S, \mathcal{K})$  be a complex and  $((T, \mathcal{L}), p)$  a finite dimensional geometric covering of  $(S, \mathcal{K})$ . Then  $|(T, \mathcal{L}), p|$  is covering of  $|(S, \mathcal{K})|$ .

*Proof.*  $\square$

**Proposition 4.3.** Let  $(S, \mathcal{K})$  be a complex and  $((T, \mathcal{L}), p)$  a geometric covering of  $(S, \mathcal{K})$ . Then  $|(T, \mathcal{L}), p|$  is covering of  $|(S, \mathcal{K})|$ .

*Proof.*  $\square$

## 5. GEOMETRIC REALIZATION OF THE UNIVERSAL COVERING COMPLEX

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