

GEOMETRIC REALIZATION OF COVERING COMPLEXES

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ABSTRACT. We prove that the geometric realization of a covering complex is a covering space. Also, this holds true for the universal covering complex.

INTRODUCTION

The theory of abstract simplicial complexes is a useful tool in the calculation of fundamental groups. This fact appears explicitly in the paper [7] of A. Weil, For any connected abstract simplicial complex S its edge-path group $E(S)$ is naturally isomorphic to the fundamental group of its geometric realization $\pi_1(|S|)$. The edge-path group could be described explicitly by generators and relations. As reference for these facts, see the book of I. Singer and J. Thorpe [5].

We base our definition of covering complex given by J. Rotman in [3], but there are other definitions covering complex as the one in [1]. The definition of J. Rotman

1. PRELIMINARIES

We recall that, given a topological space X , a covering space on X is a continuous map $p: E \rightarrow X$, such that for every $x \in X$, there is an open neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets U_λ , $\lambda \in \Lambda$, and $p|_{U_\lambda}: U_\lambda \rightarrow U$ is a homeomorphism. We recommend the book of P. May [2] and the book of J. Rotman [4] as reference of covering spaces.

An abstract simplicial complex is a pair (S, \mathcal{K}) where S is a set and \mathcal{K} is a family of non-empty finite subsets of S such that:

- $\bigcup \mathcal{K} = S$.
- If $\sigma \subseteq \tau$ and $\tau \in \mathcal{K}$ then $\sigma \in \mathcal{K}$.

We call complexes to the abstract simplicial complexes. If $\sigma \in \mathcal{K}$, then the dimension of σ is $|\sigma| - 1$, and we denote it by $\dim(\sigma)$. The elements of \mathcal{K} of dimension n are called n -simplices, and we denote the set of n -simplices by \mathcal{K}_n . The 0-simplices are called vertices. The dimension of (S, \mathcal{K}) is defined as the supremum of $\dim(\sigma)$ where σ ranges over \mathcal{K} , we denote it by $\dim(S, \mathcal{K})$. This dimension may be infinite. We call the complex (S, \mathcal{K}) finite, if S is finite. In particular, a finite complex has finite dimension. A complex (S, \mathcal{K}) is called discrete, if $\mathcal{K} = \{\{s\} \mid s \in S\}$.

An edge e in (S, \mathcal{K}) is a pair of vertices (x, y) where $\{x, y\} \in \mathcal{K}$, x is the origin of the edge e and we denote it by $\text{orig}(e)$, and y is the end of the edge e and we denote by $\text{end}(e)$. A path α in (S, \mathcal{K}) is a finite sequence of edges e_1, \dots, e_n such that $\text{end}(e_i) = \text{orig}(e_{i+1})$ with $i = 1, \dots, n-1$. We define $\text{orig}(\alpha) = \text{orig}(e_1)$ and $\text{end}(\alpha) = \text{end}(e_n)$.

Date: October 28, 2019.

2000 Mathematics Subject Classification. Primary ****, ****; Secondary ****, ****.

Key words and phrases. Abstract Simplicial Complexes, Coverings, Fundamental Group.

A morphism between complex (S_1, \mathcal{K}_1) and (S_2, \mathcal{K}_2) is a map $f: S_1 \rightarrow S_2$ such that $f(\sigma) \in \mathcal{K}_2$ for any $\sigma \in \mathcal{K}_1$. We call a morphism of complexes a simplicial map. We denote by \mathcal{C} to the category of complexes and simplicial maps.

As reference of complexes, we recommend [5] and [6].

We denote by \mathcal{T} to the category of topological spaces and continuous maps.

2. GEOMETRIC COVERING COMPLEXES

Definition 2.1. Let (S, \mathcal{K}) be a complex. We say that (S, \mathcal{K}) is connected if for any pair of vertices x, y of (S, \mathcal{K}) there is a path α such that $\text{orig}(\alpha) = x$ and $\text{end}(\alpha) = y$.

The following definition is due J. Rotman in [3].

Definition 2.2. Let (S, \mathcal{K}) be a complex. A covering of (S, \mathcal{K}) is a pair $((T, \mathcal{L}), p)$ where (T, \mathcal{L}) is a complex and $p: T \rightarrow S$ is a simplicial map such that:

- (T, \mathcal{L}) is a connected complex.
- For every $\sigma \in \mathcal{K}$, $p^{-1}(\sigma) = \bigcup_{i \in I} \sigma_i$ where the family $\{\sigma_i\}_{i \in I}$ is a pairwise disjoint family of simplices of \mathcal{L} such that $p|_{\sigma_i}: \sigma_i \rightarrow \sigma$ is bijective.

The map p is called projection and the simplices σ_i are called sheets over σ .

We observe that p is surjective and (S, \mathcal{K}) is connected. For our geometric porpoises we need a stronger definition of covering complex.

Definition 2.3. Let (S, \mathcal{K}) be a complex. A geometric covering of (S, \mathcal{K}) is a covering $((T, \mathcal{L}), p)$ such that for any simplex $\sigma \in \mathcal{K}_n$, $p(\sigma) \in \mathcal{K}_n$. In other words, p preserves the dimension of the simplices. We have that geometric coverings preserve the dimension of the complexes.

Proposition 2.1. Let (S, \mathcal{K}) be a complex, and $((T, \mathcal{L}), p)$ a covering complex of (S, \mathcal{K}) . If $\sigma \in \mathcal{K}$ is a maximal simplex, then p^{-1} is the disjoint union of maximal simplices.

Proof. Let $\sigma \in \mathcal{K}$ be a maximal simplex with $p^{-1}(\sigma) = \bigcup_{i \in I} \sigma_i$ where the family $\{\sigma_i\}_{i \in I}$ is a pairwise disjoint family of simplices of \mathcal{K} such that $p|_{\sigma_i}: \sigma_i \rightarrow \sigma$ is bijective. If σ_j is not maximal for some $j \in I$, then there is a simplex $\tau \in \mathcal{L}$ such that $\sigma_j \subset \tau$. As σ is maximal and $\sigma = p(\sigma_j) \subseteq p(\tau)$, we have that $p(\tau) = \sigma$. This contradicts the fact that p preserves the dimension. Therefore σ_i is a maximal simplex. \square

Definition 2.4. Let (S, \mathcal{K}) and (T, \mathcal{L}) be complexes and $f: (S, \mathcal{K}) \rightarrow (T, \mathcal{L})$ be a simplicial map. We say that f reflects maximal simplices, if $\sigma \in \mathcal{K}$ is such that $f(\sigma)$ is a maximal simplex, then σ is a maximal simplex.

Proposition 2.2. Let (S, \mathcal{K}) be a finite dimensional complex, and $((T, \mathcal{L}), p)$ a covering complex of (S, \mathcal{K}) . Then $((T, \mathcal{L}), p)$ is a geometric covering complex of (S, \mathcal{K}) if and only if p reflects maximal simplices.

Proof. Conjetura \square

3. GEOMETRIC REALIZATION OF COVERING COMPLEXES

Definition 3.1. The geometric realization of a complex (S, \mathcal{K}) is the set of all function $\phi: S \rightarrow [0, 1]$ such that:

- $\text{supp}(\phi) \in \mathcal{K}$
- $\sum_{s \in S} \phi(s) = 1$

We denote this set by $|(S, \mathcal{K})|$. We can give $|(S, \mathcal{K})|$ a metric topology given by:

$$d(\phi, \psi) = \sqrt{\sum_{s \in S} (\phi(s) - \psi(s))^2}$$

for $\phi, \psi \in |(S, \mathcal{K})|$. When we endowed $|(S, \mathcal{K})|$ with the metric topology we denote it by $|(S, \mathcal{K})|_d$. There is a second topology for $|(S, \mathcal{K})|$ called the coherent topology. For each simplex $\sigma \in \mathcal{K}$, we define its geometric realization $|\sigma|$ as the set of functions $\phi \in |(S, \mathcal{K})|$ with $\text{supp}(\phi) \subseteq \sigma$. We give to $|\sigma|$ the subspace topology inherited as subset of $|(S, \mathcal{K})|$. If we consider the inclusion $i_\sigma: |\sigma| \rightarrow |(S, \mathcal{K})|$, then coherent topology on $|(S, \mathcal{K})|$ is the largest topology which makes all the inclusions continuous. Usually, $|(S, \mathcal{K})|$ is considered with the coherent topology. We may characterize $|(S, \mathcal{K})|$ as the colimit in \mathcal{T} of the geometric realization of its simplices. So a function $f: |(S, \mathcal{K})| \rightarrow X$ is continuous if and only its restrictions to $|\sigma|$ is continuous for all $\sigma \in \mathcal{K}$. Especially, the identity $|(S, \mathcal{K})| \rightarrow |(S, \mathcal{K})|_d$ is continuous, so the coherent topology contains the metric topology. In particular, if (S, \mathcal{K}) is a finite complex, then both topologies coincide.

Proposition 3.1. *Let (S, \mathcal{K}) be a complex and $((T, \mathcal{L}), p)$ a finite geometric covering of (S, \mathcal{K}) . Then $|(T, \mathcal{L}), p|$ is covering of $|(S, \mathcal{K})|$.*

Proof.

□

4. GEOMETRIC REALIZATION OF THE UNIVERSAL COVERING COMPLEX

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