ANNALS OF MATHEMATICS
Vol. 72, No. 2, September, 1960
Printed in Japan

ON DISCRETE SUBGROUPS OF LIE GROUPS

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(Received February 1, 1960)

1. Let G be a topological group and Γ an arbitrary group; one may think of Γ as being provided with the discrete topology. Consider the space $G^{(\Gamma)}$ of all mappings of Γ into G; this is the same as the product $\prod_{\gamma\in\Gamma}G_\gamma$, where G_γ is the same as G for every $\gamma\in\Gamma$, and will be provided with the usual product topology. The set $\mathfrak{R}=\mathfrak{R}(\Gamma, G)$ of all representations of Γ into G may be described as the subset of $G^{(\Gamma)}$, consisting of all the mappings r of Γ into G which satisfy $r(\gamma\gamma')=r(\gamma)r(\gamma')$ for every pair γ, γ' of elements of Γ; this is a closed subset of $G^{(\Gamma)}$ and will be provided with the topology induced on it by that of $G^{(\Gamma)}$; with that topology (the so-called "topology of pointwise convergence"), $\mathfrak{R}$ will be called the space of representations of Γ into G. If Γ is generated by a family of elements $(\gamma_\alpha)_{\alpha\in A}$, indexed by a set A, a representation r of Γ into G is uniquely determined by the elements $r(\gamma_\alpha)$, so that there is a one-to-one correspondence between $\mathfrak{R}$ and a certain subset of the set $G^{(A)}$ of all mappings of A into G. More precisely, Γ is then a homomorphic image of the free group Γ' with the generators $(\gamma'_\alpha)_{\alpha\in A}$; let φ be the homomorphism of Γ' onto Γ which maps γ'_α onto γ_α for every $\alpha\in A$; let Δ' be the kernel of φ; the elements of Δ', being elements of Γ', are "words" $w(\gamma')$ in the γ'_α, and we have then, for every such "word", $w(\gamma)=\varepsilon$, where ε is the neutral element of Γ; these are the "relations between the generators γ_α of Γ". The space $\mathfrak{R}'=\mathfrak{R}(\Gamma', G)$ is then in an obvious one-to-one correspondence with $G^{(A)}$, and it is a trivial matter to verify that this is a homeomorphism when $G^{(A)}$ is provided with the product topology in a manner similar to that described above. Then $\mathfrak{R}$ is in an obvious one-to-one correspondence with the closed subset $\mathfrak{R}_1$ ef $\mathfrak{R}'$, consisting of all the representations of Γ' into G which map Δ' into the neutral element e of G; and it is again a trivial matter to verify that this is a homeomorphism. We will identify $\mathfrak{R}'$ with $G^{(A)}$, and $\mathfrak{R}$ with $\mathfrak{R}_1$, by means of these correspondences. Let us assume further that $(w_\beta)_{\beta\in B}$ is a family of elements of Δ', such that Δ' is generated by the w_β and by their images under all inner automorphisms of Γ'; if we write each such element as a "word" $w_\beta(\gamma)$, we say then that the relations between the γ_α are "generated" by the "fundamental relations" $w_\beta(\gamma)=\varepsilon$; and $\mathfrak{R}$, as a subset of $\mathfrak{R}'=G^{(A)}$, is then the set of the elements (g_α) of $G^{(A)}$ which satisfy the relations $w_\beta(g)=e$.

Particular interest attaches to those representations r of Γ into G which are injective (or, as one also says, faithful) and such that $r(\Gamma)$ is a discrete subgroup of G with compact quotient space $G/r(\Gamma)$; of course G has then to be locally compact. Clearly a representation r has these properties if and only if there are a neighborhood U of e in G such that $r^{-1}(U) = \{\varepsilon\}$ and a compact subset K of G such that $G = K \cdot r(\Gamma)$. Let us denote by $\mathfrak{R}_0 = \mathfrak{R}_0(\Gamma, G)$ the set of all such representations, considered as a subset of $\mathfrak{R}$. It has been conjectured by A. Selberg that, if G is a semi-simple Lie group, $\mathfrak{R}_0$ is an open subset of $\mathfrak{R}$. This will now be proved for all Lie groups. More precisely, the following theorem will be proved:

Let G be a connected Lie group; let Γ be a discrete group, and r_0 an injective representation of Γ into G, such that $r_0(\Gamma)$ is discrete in G with compact quotient-space $G/r_0(\Gamma)$. Then there are a neighborhood U of e in G, a compact subset K of G, and a neighborhood $\mathfrak{U}$ of r_0 in the space $\mathfrak{R}$ of all representations of Γ in G, such that $r^{-1}(U) = \{\varepsilon\}$ and $G = K \cdot r(\Gamma)$ for all $r \in \mathfrak{U}$. Moreover, Γ has then a finite set of generators with a finite set of fundamental relations.

The last statement is included only for the sake of completeness, since it is an immediate consequence of the fact that the fundamental group of any compact manifold has the property in question (incidentally, a proof for this is included in what follows) and that the same is true of the fundamental group of any connected Lie group. In view of these facts, $\mathfrak{R}$ can be described as has been done above, using a finite set of generators (γ_α) and a finite set of fundamental relations (w_β); then the space denoted above by $\mathfrak{R}'$ is the product of finitely many Lie groups, isomorphic to G, and is therefore a connected real-analytic manifold; $\mathfrak{R}$ is the subset of $\mathfrak{R}'$ defined by the finitely many real-analytic relations $w_\alpha(g) = e$, and is therefore a real-analytic subset of $\mathfrak{R}'$. The theorem stated above implies, as we have said, that the subset of $\mathfrak{R}$ which has been denoted by $\mathfrak{R}_0$ is open on $\mathfrak{R}$; if G is the hyperbolic group (the quotient of $SL(2, \mathbf{R})$ by its center), it is known that $\mathfrak{R}_0$ is actually a manifold; the question naturally arises whether this is true for $\mathfrak{R}_0$, or rather for each connected component of $\mathfrak{R}_0$, whenever G is a Lie group.

2. As will be seen in no. 11, the general case of our theorem can be reduced to the case when G is simply connected. When G is such, $G/r_0(\Gamma)$ is a compact manifold V whose fundamental group is isomorphic to Γ. We first deal with some purely topological aspects of this situation.

In nos. 2–5, we will denote by V any connected manifold (it would be enough for our purposes to assume that V is connected and locally simply connected). Let $\tilde{V}$ be the universal covering of V, with the projection

p onto V; let Γ be the fundamental group of V, considered as a group of automorphisms of $\tilde{V}$, so that V can be identified with $\tilde{V}/\Gamma$. We write $x\gamma$ for the image of a point x of $\tilde{V}$ under an element γ of Γ, and ε for the neutral element of Γ.

Assume that $(\tilde{U}_i)_{i\in I}$ is a family of connected open subsets of $\tilde{V}$, indexed by a finite set I, with the following properties:

(a) *the sets* $\tilde{U}_i\gamma$, *for* $i \in I$, $\gamma \in \Gamma$, *are a covering of* $\tilde{V}$ (i.e., their union is $\tilde{V}$);

(b) *for every pair* (i, j) *there is at most one element* γ *of* Γ *such that* $\tilde{U}_i\gamma$ *meets* $\tilde{U}_j$.

For each i, put $U_i = p(\tilde{U}_i)$; applying (b) to $i = j$, we see that $\tilde{U}_i\gamma$ cannot meet $\tilde{U}_i$ unless $\gamma = \varepsilon$; this means that p induces on $\tilde{U}_i$ a bijective mapping, and therefore a homeomorphism, of $\tilde{U}_i$ onto U_i.

By (a), the sets U_i are a covering of V; call N the nerve of that covering; this is the set of all subsets J of I, such that $\bigcap_{j\in J} U_j$ is not empty. In particular, we have $\{i, j\} \in N$ if and only if $p(U_i)$ meets $p(U_j)$, i.e., if and only if there is γ in Γ such that $\tilde{U}_i\gamma$ meets $\tilde{U}_j$; by (b), when that is so, γ is uniquely determined by that condition; this element γ will be denoted by γ_{ij}. It is clear that $\gamma_{ii} = \varepsilon$ for all i in I, and that $\gamma_{ij}\gamma_{ji} = \varepsilon$ for all $\{i, j\} \in N$.

A subset J of I is in N if and only if, for some j in J, there are a point x in $\tilde{U}_j$ and elements γ_h of Γ such that x is in $\tilde{U}_h\gamma_h$ for all h in J, and then the same is true for all choices of j in J. But then we must have $\gamma_h = \gamma_{hj}$ for all h in J. Therefore J is in N if and only if, for some j in J, all pairs $\{j, h\}$ with h in J are in N, and the sets $\tilde{U}_h\gamma_{hj}$, for $h \in J$, have a non-empty intersection; and then the same is true for all j in J. In particular, take $J = \{i, j, k\}$; this is in N if and only if $\{i, j\}$ and $\{i, k\}$ are in N and there are points u_i in $\tilde{U}_i$, u_j in $\tilde{U}_j$, u_k in $\tilde{U}_k$, such that $u_i = u_j\gamma_{ji} = u_k\gamma_{ki}$; but then u_j is in $\tilde{U}_j$ and in $\tilde{U}_k\gamma_{ki}\gamma_{ij}$, so that we have $\gamma_{kj} = \gamma_{ki}\gamma_{ij}$.

As V is connected, the nerve N is connected; this means that I cannot be written as the union of two disjoint non-empty subsets I', I'', such that no pair $\{i', i''\}$, with i' in I' and i'' in I'', belongs to N. In fact, if this were not so, the unions of the $U_{i'}$ and of the $U_{i''}$ would be disjoint non-empty open subsets of V. We select some element i_0 of I as "origin", and, for each i in I, we select a "chain" C_i, i.e., a sequence $i_0, i_1, \cdots, i_m = i$, with the origin i_0 and the end-point $i_m = i$, such that, for $1 \leqq \mu \leqq m$, $\{i_{\mu-1}, i_\mu\}$ is in N. We put $\delta_i = \gamma_{i_0 i_1}\gamma_{i_1 i_2}\cdots\gamma_{i_{m-1}i_m}$ for the chain C_i. We take for C_{i_0} the chain i_0, i_0, so that $\delta_{i_0} = \varepsilon$. Now, for every $\{i, j\}$ in N, put $\sigma_{ij} = \delta_i\gamma_{ij}\delta_j^{-1}$. These elements satisfy the relations $\sigma_{ik} = \sigma_{ij}\sigma_{jk}$ whenever $\{i, j, k\}$ is in N (which implies, for $i = j = k$, that $\sigma_{ii} = \varepsilon$ for all i, and, for $i = k$, that $\sigma_{ij}\sigma_{ji} = \varepsilon$ for all $\{i, j\}$ in N), and, for every chain $C_i = (i_0, \cdots, i_m)$, the relation $\sigma_{i_0 i_1} \cdots \sigma_{i_{m-1} i_m} = \varepsilon$. It will now be shown that *the* σ_{ij} *generate* Γ, *and that the relations we have just written generate all the relations between them.*

3. As to the first assertion, we use the fact that the nerve $\tilde{N}$ of the covering of $\tilde{V}$ by the $\tilde{U}_i\gamma$ must be connected. Now a pair $\{(i, \gamma), (j, \gamma')\}$ is in $\tilde{N}$ if and only if $\tilde{U}_i\gamma$ meets $\tilde{U}_j\gamma'$, i.e., if and only if $\{i, j\}$ is in N and $\gamma = \gamma_{ij}\gamma'$. Let γ be any element of Γ; there must be a chain of $\tilde{N}$, with the origin (i_0, γ) and the end-point (i_0, ε); if this chain consists of the elements (i_μ, γ_μ), with $0 \leqq \mu \leqq m$, we must therefore have $i_0 = i_m$, $\gamma_0 = \gamma$, $\gamma_m = \varepsilon$, and, for $1 \leqq \mu \leqq m$, $\{i_{\mu-1}, i_\mu\} \in N$ and $\gamma_{\mu-1} = \gamma_{i_{\mu-1} i_\mu}\gamma_\mu$. This gives:

$$\gamma_0 = \gamma_{i_0 i_1}\gamma_{i_1 i_2} \cdots \gamma_{i_{m-1} i_m}\gamma_m ,$$

which, together with the relations $i_m = i_0$, $\gamma_0 = \gamma$, $\gamma_m = \varepsilon$, $\delta_{i_0} = \varepsilon$, and with the definition of the σ_{ij}, implies that we have

$$\gamma = \sigma_{i_0 i_1}\sigma_{i_1 i_2} \cdots \sigma_{i_{m-1} i_0} .$$

Now we prove our assertion about the relations between the σ_{ij}. Let Γ^* be a group generated by elements σ^*_{ij}, with the fundamental relations enumerated at the end of no. 2; the relations $F(\sigma^*_{ij}) = \sigma_{ij}$ determine a homomorphism of Γ^* onto Γ, and we have to show that this is an isomorphism. Let W be the union of the disjoint open sets $\tilde{U}_i \times \{(i, \gamma^*)\}$ in the product $\tilde{V} \times I \times \Gamma^*$, where I and Γ^* are provided with the discrete topology. In W, we introduce an equivalence relation R as follows. Two points (u_i, i, γ^*), (u_j, j, γ'^*), with $u_i \in \tilde{U}_i$, $u_j \in \tilde{U}_j$, will be called equivalent under R if and only if $p(u_i) = p(u_j)$ and $\gamma^* = \sigma^*_{ij}\gamma'^*$. This is reflexive, because $\sigma^*_{ii} = \varepsilon^*$ for all i (where ε^* is the neutral element of Γ^*), and symmetric, because $\sigma^*_{ij}\sigma^*_{ji} = \varepsilon^*$ for all $\{i, j\}$ in N; in order to prove transitivity, let (u_i, i, γ^*) be equivalent to (u_j, j, γ'^*), and the latter point to (u_k, k, γ''^*); then $p(u_i) = p(u_j) = p(u_k)$, so that $\{i, j, k\}$ is in N; also, we have $\gamma^* = \sigma^*_{ij}\gamma'^*$, $\gamma'^* = \sigma^*_{jk}\gamma''^*$, which, in view of the relations between the σ^*_{ij}, implies $\gamma^* = \sigma^*_{ik}\gamma''^*$; this proves that (u_i, i, γ^*) and (u_k, k, γ''^*) are equivalent. One sees at once that the equivalence relation R is open; therefore $V^* = W/R$ is a manifold.

We can define a mapping of W into $\tilde{V}$, by putting $f(u_i, i, \gamma^*) = u_i\delta_i^{-1}F(\gamma^*)$; if (u_i, i, γ^*) and (u_j, j, γ'^*) are equivalent under R, we have $p(u_i) = p(u_j)$, which implies $u_j = u_i\gamma_{ij}$ and therefore $u_j\delta_j^{-1} = u_i\delta_i^{-1}\sigma_{ij}$, and $\gamma^* = \sigma^*_{ij}\gamma'^*$; therefore those two points have the same image in $\tilde{V}$ under the mapping f, so that f determines a mapping φ of $V^* = W/R$ into $\tilde{V}$. The inverse image of $\tilde{U}_i\gamma$ by f consists of the points (u_j, j, γ^*) of W such that $u_j\delta_j^{-1}F(\gamma^*) = u_i\gamma$ for some $u_i \in \tilde{U}_i$; when that is so, we have $p(u_i) = p(u_j)$ and $\{i, j\} \in N$, so that (u_j, j, γ^*) is equivalent under R to $(u_i, i, \sigma_{ij}^*\gamma^*)$. Therefore the inverse image of $\tilde{U}_i\gamma$ by φ consists of the union of the images in V^* of the subsets $\tilde{U}_i \times \{(i, \gamma^*)\}$ of W for which $F(\gamma^*) = \delta_i\gamma$; as those images are disjoint open subsets of V^*, and as φ induces on each one of them a homeomorphism onto $\tilde{U}_i\gamma$, this proves that V^*, with the mapping φ onto $\tilde{V}$, is a covering of $\tilde{V}$; moreover, we see that φ is bijective if and only if F is so. As $\tilde{V}$ is simply connected, this implies that F is bijective if and only if V^* is connected.

4. In order to prove that V^* is connected, it will be enough, since the $\tilde{U}_i$ are connected, to show that the nerve of the covering of V^* by the images of the sets $\tilde{U}_i \times \{(i, \gamma^*)\}$ is connected. This will be done by constructing a "chain" of sets $\tilde{U}_{i_\nu} \times \{(i_\nu, \gamma_\nu^*)\}$, with $1 \leqq \nu \leqq n$, beginning with a given set $\tilde{U}_i \times \{(i, \gamma^*)\}$ and ending up with $\tilde{U}_{i_0} \times \{(i_0, \varepsilon^*)\}$, such that the images in V^* of any two consecutive sets of the chain have a common point. The latter condition will be fulfilled if and only if, for every ν, $\{i_{\nu-1}, i_\nu\} \in N$ and $\gamma_{\nu-1}^* = \sigma_{i_{\nu-1}i_\nu}{}^*\gamma_\nu^*$; when that is so, we have

$$\gamma^* = \gamma_1^* = \sigma_{i_1i_2}^*\sigma_{i_2i_3}^* \cdots \sigma_{i_{n-1}i_n}^*$$

with $i_1 = i$, and $i_n = i_0$. Conversely, if we can write γ^* in that form, we only have to denote by γ_ν^* the product of the last $n - \nu$ factors in the right-hand side in order to have a chain fulfilling the required conditions.

Now, since Γ^* is generated by the σ_{ij}^*, we can write for γ^* an expression consisting of factors σ_{jh}^* and $(\sigma_{jh}^*)^{-1}$; using the defining relations for Γ^*, we can replace each factor $(\sigma_{jh}^*)^{-1}$ by σ_{hj}^*, so that γ^* now appears as a product of factors σ_{jh}^*. For each factor σ_{jh}^* in that expression, we use the relations corresponding to the chains $C_j = (j_0, j_1, \cdots, j_p)$, $C_h = (h_0, h_1, \cdots, h_q)$, with $j_0 = h_0 = i_0$, $j_p = j$, $h_q = h$, in order to rewrite it as

$$\sigma_{jh}^* = \sigma_{i_0j_1}^*\sigma_{j_1j_2}^* \cdots \sigma_{j_{p-1}j}^*\sigma_{jh}^*\sigma_{hh_{q-1}}^* \cdots \sigma_{h_2h_1}^*\sigma_{h_1i_0}^* ;$$

substituting for the σ_{jh}^* the expressions in the right-hand sides, we get for γ^* an expression which is of the desired form except that it begins with i_0 instead of i; multiplying this to the left with the inverse of the relation corresponding to the chain C_i, we get what we want.

5. We add some remarks to the facts proved in nos. 2–4. Firstly, a covering such as (U_i), used above, can be characterized as follows, in terms of V alone. Let us say that a connected open subset U of V is *homotopically flat on* V if every closed path contained in U is homotopic to 0 on V; this will be so if and only if the inverse image $p^{-1}(U)$ of U in $\tilde{V}$ for p consists of disjoint connected components, each of which is mapped bijectively onto U by p. Now let $(U_i)_{i\in I}$ be a finite covering of V by *connected open sets* U_i, such that, *whenever* $U_i \cap U_j$ *is not empty*, $U_i \cup U_j$ *is homotopically flat on* V; and take for $\tilde{U}_i$ any connected component of $p^{-1}(U_i)$; these will satisfy conditions (a) and (b). The description given above of the fundamental group Γ of V by means of the generators σ_{ij} and of the relations written in no. 2 depends only upon the nerve N of the covering (U_i); moreover, after the group Γ has been so constructed, no. 4 proves that $\tilde{V}$ itself is isomorphic to the quotient of the union of the open subsets $U_i \times \{(i, \gamma)\}$ of $V \times I \times \Gamma$ by the equivalence relation between points (u_i, i, γ), (u_j, j, γ') of that union which is given by $u_i = u_j$, $\gamma = \sigma_{ij}\gamma'$. It would make no difference in this construction if we substituted the γ_{ij} for the σ_{ij}, since this merely amounts to transforming the equivalence relation by the mapping $(u, i, \gamma) \to (u, i, \delta_i\gamma)$ of the union of the sets $U_i \times \{(i, \gamma)\}$ onto itself.

One may also note the following consequence of these results. By a *cochain of* I *in a group* G, let us understand a mapping $(i, j) \to x_{ij}$ into G of the set of those pairs (i, j) for which $\{i, j\}$ is in N; this will be called a *cocycle* if $x_{ik} = x_{ij}x_{jk}$ whenever $\{i, j, k\}$ is in N; two cocycles (x_{ij}), (y_{ij}) will be called *equivalent* if there is a mapping $i \to z_i$ of I into G such that $y_{ij} = z_i x_{ij} z_j^{-1}$ for every $\{i, j\} \in N$. Then the classes of equivalent cocycles of I in G are in a one-to-one correspondence with the classes of representations of Γ into G; as usual, two such representations are called equivalent if they can be derived from one another by an inner automorphism of G.

6. Now we go back to the Lie group G. Any quotient of G by a discrete subgroup may be considered as a Clifford-Klein form of G. In order to give a precise content to this concept, we introduce the notion of G-structure on a manifold. Let n be the dimension of G; let $\omega_1, \cdots, \omega_n$ be a basis for the right-invariant differential forms of degree 1 on G. These satisfy the Maurer-Cartan equations:

$$R_i(\omega) = d\omega_i - \sum_{j<k} c_{ijk}\omega_j\omega_k = 0 \ ,$$

where the c_{ijk} are the constants of structure for G. By a G-manifold, we shall understand an analytic variety V of dimension n, provided with n differential forms η_i, satisfying the equations $R_i(\eta) = 0$ and linearly independent at every point of V; the η_i are called the structural forms of

V. If V and V' are two G-manifolds, a mapping φ of V into V' will be called a G-mapping if it is analytic and if the inverse images by φ of the structural forms for V' are those for V; if φ is bijective, then its inverse is also a G-mapping, and φ is called a G-isomorphism. Every G-mapping of V into V' is a "local isomorphism" in the sense that each point of V has a neighborhood which is mapped G-isomorphically by φ onto its image in V'. It follows from Frobenius's theorem on completely integrable systems that, if V and V' are G-manifolds, a a point on V and a' a point on V', there is a G-isomorphism of a neighborhood of a on V onto a neighborhood of a' on V' which maps a onto a'; by the same theorem, any two such isomorphisms must coincide in some neighborhood of a on V; from this it follows that, if V is connected, two G-mappings of V into V' which coincide at one point must coincide everywhere.

The group G itself has a natural G-structure, determined by the forms ω_i; the automorphisms for that structure are the right-translations. Therefore the G-structure of G can be transported to the quotient G/Γ of G by any discrete subgroup Γ (this being understood as the space of right cosets $x\Gamma$). Every point of a G-manifold has a neighborhood which is isomorphic to a neighborhood of e (the neutral element of G) on G. A G-manifold V is called *complete* if there is an open neighborhood U of e in G with the following property: to every point a of V, there is a neighborhood W of a and a G-isomorphism of U onto W, mapping e onto a. The group G itself, and every group locally isomorphic to G, are complete G-manifolds; every compact G-manifold is complete.

If V is a connected and simply connected G-manifold, and V' a complete G-manifold, there is one and only one G-mapping of V into V' which maps a given point a of V onto a given point a' of V'; if at the same time V is complete, then, for such a mapping, V becomes a covering manifold of V' and therefore its universal covering. If we take for G the simply connected Lie group with the given structure, then this shows that every complete, connected and simply connected G-manifold is isomorphic to G. If now V is any complete connected G-manifold, its universal covering may be identified with G; if Γ is the fundamental group of V, it operates on G by G-automorphisms, i.e., by right-translations; this means that Γ can be identified with a discrete subgroup of G, and V with G/Γ. This applies in particular to every compact and connected G-manifold.

Let U be a connected open subset of the simply connected group G; let V be the quotient G/Γ of G by a discrete subgroup Γ; let U' be an open subset of V, and assume that there is a G-isomorphism φ of U onto U'. Let p be the canonical mapping of G onto $V = G/\Gamma$; take a point a in U;

as p is surjective, there is a point b of G such that $p(b) = \varphi(a)$. Then φ, and the mapping of U into V defined by $u \to p(ua^{-1}b)$, are G-mappings of U into V which coincide at a; as U is connected, they coincide everywhere on U. In other words, p induces on $Ua^{-1}b$ a G-isomorphism of that set onto U'; this means that U' is homotopically flat on V. In particular, the results of nos. 2–5 can be applied to any finite covering of a compact connected G-manifold V by connected open subsets U_i, with the property that, whenever U_i meets U_j, $U_i \cup U_j$ is G-isomorphic to an open subset of the simply connected group G. This idea will now be carried out more in detail.

7. As explained above, we assume, until further notice, that the Lie group G is connected and simply connected, and we proceed to prove for that case the theorem stated in no. 1. To simplify notations, we identify Γ with $r_0(\Gamma)$ by means of r_0; we put $V = G/\Gamma$; then G can be identified with the universal covering $\tilde{V}$ of V, and Γ with its fundamental group. In view of our assumptions and of the definitions in no. 6, V is a compact G-manifold.

As Γ is discrete, there is a neighborhood U_0 of e in G, containing no element of Γ except e. As G/Γ is compact, there is a compact subset K of G such that $G = K\Gamma$. We can choose an open neighborhood U of e in G, such that $s^{-1}u_1^{-1}u_2u_3^{-1}u_4s$ is in U_0 whenever s is in K and the u_i are in U; choosing once for all local coordinates $x_1, \cdots, x_n$ in a neighborhocd of e in G, we take for U a ball $\sum_i x_i^2 < \rho^2$ of sufficiently small radius ρ. Then, if s, s' are two elements of K, there is at most one $\gamma \in \Gamma$ such that $Us\gamma$ meets Us'; when that is so, we denote that element by $\gamma(s, s')$; we also put $\tau(s, s') = s\gamma(s, s')s'^{-1}$, so that $U\tau(s, s')$ meets U; we have $\gamma(s, s')\gamma(s', s) = e$, and a similar relation for τ. In particular, for every s in K, we have $\gamma(s, s) = e$, $\tau(s, s) = e$.

Call p, as in no. 2, the canonical mapping of G onto $V = G/\Gamma$. For $s \in K$, $Us\gamma$ cannot meet Us unless $\gamma = e$; this means that p induces on Us a homeomorphism (more precisely, a G-isomorphism) of Us onto $p(Us)$. Call $N(K)$ the nerve of the covering of V by the sets $p(Us)$. A pair $\{s, s'\}$ is in $N(K)$ if and only if there is $\gamma \in \Gamma$ such that $Us\gamma$ meets Us', in which case we have $\gamma = \gamma(s, s')$; a finite subset S of K is in $N(K)$ if and only if, for some s_0 in S, every pair $\{s_0, s\}$, with $s \in S$, is in $N(K)$ and there are elements u_s of U such that $u_{s_0} = u_s\tau(s, s_0)$ for every $s \in S$; then the same is true for every choice of s_0 in S. Applying this to a set $S = \{s, s', s''\}$, we see (just as in no. 2) that, if this set is in $N(K)$, we have

$$\gamma(s, s'') = \gamma(s, s')\gamma(s', s'') ,$$

and a similar relation with τ instead of γ.

8. As K is compact, we can choose a finite subset S of K such that K is contained in the union of the sets Us for $s \in S$; then the sets Us, for $s \in S$, have the properties (a) and (b) stated for the sets U_i in no. 2. The nerve of the covering of V by the sets $p(Us)$ is the intersection $N(S)$ of $N(K)$ with the set of all subsets of S. For each T in $N(S)$, we can choose, once and for all, elements $u_T(t)$ of U such that all the elements $u_T(t)t$, for $t \in T$, have the same image in V; this means that we have $u_T(t) = u_T(t')\tau(t', t)$ for all t, t' in T.

Call φ the mapping of $U \times S$ into V defined by $\varphi(u, s) = p(us)$; as this is surjective, and as V is compact, there is a compact subset X of $U \times S$ such that $\varphi(X) = V$; for each $s \in S$, call X_s the compact subset of U such that $X_s \times \{s\} = X \cap (U \times \{s\})$. Let U' be the ball $\sum_i x_i^2 < \rho'^2$, where ρ' is taken $<\rho$ and so close to ρ that U' contains all the sets X_s and all the points $u_T(t)$ for $T \in N(S), t \in T$. Then $U' \times S$ contains X, so that $\varphi(U' \times S) = V$, which means that the sets $p(U's)$, for $s \in S$, are still a covering of V; therefore the sets $U's$, for $s \in S$, still have the properties (a), (b) of no. 2. For $T \in N(S)$ and $t \in T$, the point $u_T(t)t$ is in $U't$; this shows that T is in the nerve of the covering of V by the sets $\varphi(U's)$; as this nerve is obviously contained in $N(S)$, it is still $N(S)$.

We can now define G and V by means of the non-connected manifolds $U' \times S \times \Gamma$, $U' \times S$, with suitable identifications. Call f the mapping of $U' \times S \times \Gamma$ into G defined by $f(u, s, \gamma) = us\gamma$; one sees at once that two points (u, s, γ), (u', s', γ') have the same image by f in G if and only if we have $\{s, s'\} \in N(S)$, $\gamma = \gamma(s, s')\gamma'$, $u' = u\tau(s, s')$; if we write R for this relation, it follows from this that R is an equivalence relation and that G can be identified with $(U' \times S \times \Gamma)/R$; as R is compatible with the G-structure of $U' \times S \times \Gamma$, G can be identified with $(U' \times S \times \Gamma)/R$, not only as a topological space, but even as a G-manifold. Similarly, V, as a G-manifold, can be identified with $(U' \times S)/R'$, where R' is the equivalence relation between pairs (u, s), (u', s') of $U' \times S$ defined by $\{s, s'\} \in N(S)$, $u' = u\tau(s, s')$.

Furthermore, we can use the sets Us to define a set of generators for Γ in the manner explained in no. 2. In order to do this, we have to choose an element s_0 of S, and, for each $s \in S$, a chain C_s as described in no. 2; that being done, we define elements $\delta(s)$, $\sigma(s, s')$ of Γ, in the manner explained there; the $\sigma(s, s')$ are then generators for Γ, and the relations are those stated in no. 2.

9. We now modify as follows the equivalence relations R, R' defined above. Once and for all, we choose a neighborhood Ω of e in G, such that $U'\Omega \subset U$, $u_T(t)\Omega^{-1} \subset U'$ for every $T \in N(S)$ and every $t \in T$, and that the closure of $X_s\Omega^{-1}$, for every $s \in S$, is a compact subset Y_s of U'. Now take a representation $\gamma \to \bar{\gamma}$ of Γ into G; this will be given by assigning the images $\bar{\sigma}(s, s')$ of the generators $\sigma(s, s')$ of Γ in that representation, which can be taken arbitrarily, subject to the relations between the $\sigma(s, s')$. We put

$$\bar{\tau}(s, s') = s\delta(s)^{-1}\bar{\sigma}(s, s')\delta(s')s'^{-1}$$
$$\omega(s, s') = \bar{\tau}(s, s')\tau(s, s')^{-1} .$$

As a consequence of the relations between the $\bar{\sigma}(s, s')$, we have $\bar{\tau}(s, s)=e$ for all $s \in S$, $\bar{\tau}(s, s')\bar{\tau}(s', s)=e$ for all $\{s, s'\} \in N(S)$, $\bar{\tau}(s, s'')=\bar{\tau}(s, s')\bar{\tau}(s', s'')$ for all $\{s, s', s''\} \in N(S)$. Moreover, if the representation $\gamma \to \bar{\gamma}$ is close enough to the identity mapping of Γ onto itself, the elements $\omega(s, s')$ will be in Ω for all $\{s, s'\} \in N(S)$. It will now be shown that, when this is so, the $\bar{\sigma}(s, s')$ generate a discrete subgroup $\bar{\Gamma}$ of G, isomorphic to Γ, such that $G/\bar{\Gamma}$ is compact.

In fact, call $\bar{R}$ the relation between elements (u, s, γ), (u', s', γ') of $U' \times S \times \Gamma$, given by $\{s, s'\} \in N(S)$, $\gamma=\gamma(s, s')\gamma'$, $u'=u\bar{\tau}(s, s')$; call $\bar{R}'$ the relation between elements (u, s), (u', s') of $U' \times S$, given by $\{s, s'\} \in N(S)$, $u' = u\bar{\tau}(s, s')$. We first prove that these are equivalence relations. It is clear that they are reflexive and symmetric. Assume that $\bar{R}'$ holds for (u, s), (u'', s'') and also for (u', s'), (u'', s''); this can be written as $\{s, s''\} \in N(S)$, $\{s', s''\} \in N(S)$, and

$$u'' = \bar{u}\tau(s, s'') = \bar{u}'\tau(s', s'') ,$$

with $\bar{u} = u\omega(s, s')$, $\bar{u}' = u'\omega(s, s')$, so that $\bar{u}, \bar{u}$ are in $U'\Omega$, hence in U. Therefore those relations imply that $\{s, s', s''\}$ is in $N(S)$; in view of the relations between the $\bar{\tau}(s, s')$, it follows at once from this that $\bar{R}'$ holds for (u', s'), (u'', s''). The same proof shows that $\bar{R}$ is also an equivalence relation. As the equivalence relations $\bar{R}$, $\bar{R}'$ are open and are compatible with the G-structures of $U' \times S \times \Gamma$ and of $U' \times S$, the quotients $\bar{G} = (U' \times S \times \Gamma)/\bar{R}$, $\bar{V}=(U' \times S)/\bar{R}'$ are G-manifolds. We denote by $\bar{f}$ and by $\bar{\varphi}$ the canonical mappings of $U' \times S \times \Gamma$ onto $\bar{G}$ and of $U' \times S$ onto $\bar{V}$, respectively.

We now prove that $\bar{V}$ is compact, by showing that it is the union of the images under φ of the compact subsets $Y_s \times \{s\}$ of $U' \times S$. In fact, let (u, s) be any point of $U' \times S$; as V is the union of the sets $p(X_s s)$, there is an element s' of S and a point x of $X_{s'}$ such that (x, s') is equivalent to (u, s) for R', which means that $\{s, s'\}$ is in $N(S)$ and that $u = x\tau(s', s)$. Then, if we put $y = x\omega(s', s)^{-1}$, y is in $X_{s'}\Omega^{-1}$, hence in $Y_{s'}$, and we have $u = y\tau(s', s)$, which shows that (u, s) is equivalent to (y, s') for $\bar{R}'$. This proves our assertion.

The sets $\bar{\varphi}(U' \times \{s\})$, for $s \in S$, are an open covering of $\bar{V}$. We now prove that the nerve of this covering is still $N(S)$. In fact, assume first that T is in that nerve; then every pair $\{t, t'\}$ of elements of T must be in $N(S)$, and there are elements u_t of U' for every $t \in T$, such that $u_{t'} = u_t\bar{\tau}(t, t')$ for all t, t' in T. Taking t_0 in T, and putting $\bar{u}_t = u_t\omega(t, t_0)$ for all $t \in T$, we get $u_{t_0} = \bar{u}_t\tau(t, t_0)$ for all $t \in T$; as the $\bar{u}_t$ are in $U'\Omega$, hence in U, this implies that T is in $N(S)$. Conversely, let T be any set in $N(S)$; taking t_0 in T, put $\bar{u}_t = u_T(t)\omega(t, t_0)^{-1}$; then we have, by the definition of the elements $u_T(t)$, $\bar{u}_{t_0} = \bar{u}_t\bar{\tau}(t, t_0)$ for all $t \in T$; as the $\bar{u}_t$ are in U', this shows that T is in the nerve of the covering of $\bar{V}$ by the sets $\bar{\varphi}(U' \times \{s\})$.

10. As $\bar{V}$ is compact, it is a complete G-manifold. It has the covering by the sets $\bar{\varphi}(U' \times \{s\})$, with the nerve $N(S)$. If $\{s, s'\}$ is in $N(S)$, the union of the sets $\bar{\varphi}(U' \times \{s\})$, $\bar{\varphi}(U' \times \{s'\})$ is the quotient of the union of the two sets $U' \times \{s\}$, $U' \times \{s'\}$ by the equivalence relation induced on that union by $\bar{R}'$; it is at once seen that this quotient is isomorphic to $U' \cup U'\bar{\tau}(s, s')$; as explained in no. 6, this implies that it is homotopically flat on $\bar{V}$. By the results in no. 5, this shows that the fundamental group of $\bar{V}$ is isomorphic to Γ, and furthermore that the universal covering of $\bar{V}$ is the manifold constructed as explained in no. 5; but it is at once seen that this is nothing else than $\bar{G}$. We have thus proved that $\bar{G}$ is connected and simply connected, as well as complete. As such, it is isomorphic to G and may be identified with it; as the isomorphism between $\bar{G}$ and G is determined only up to right-translation, we may assume the identification to be made in such a way that the point $\bar{f}(e, s_0, e)$ of $\bar{G}$ is identified with the point s_0 of G.

As two G-isomorphisms of U' into G can differ only by a right-translation, the sets $\bar{f}\big(U' \times \{(s, \gamma)\}\big)$, in the identification we have just made of $\bar{G}$ with G, must appear as right-translates of U', i.e., as sets of the form $U'\rho(s, \gamma)$. We will now determine the factors $\rho(s, \gamma)$, or at any rate the factors $\rho(s_0, \gamma)$ and $\rho(s, e)$, since that will be enough for our purpose. Since we have $\bar{f}(e, s_0, e) = s_0$, we have $\rho(s_0, e) = s_0$.

Let $\{s, s'\}$ be in $N(S)$; then $\bar{\varphi}(U' \times \{s\})$, $\bar{\varphi}(U' \times \{s'\})$ have a point in common, which is the image by $\bar{\varphi}$ of a point (u, s) of $U' \times \{s\}$ and of a point (u', s') of $U' \times \{s'\}$, so that we have $u = u'\bar{\tau}(s', s)$. Take any γ in Γ, and put $\gamma' = \gamma(s', s)\gamma$; then the points (u, s, γ), (u', s', γ') have the same image by $\bar{f}$. On the other hand, by the definition of $\rho(s, \gamma)$, we have

$$\bar{f}(u, s, \gamma) = u\rho(s, \gamma), \qquad \bar{f}(u', s', \gamma') = u'\rho(s', \gamma') .$$

This gives

$$\rho(s', \gamma(s', s)\gamma) = \bar{\tau}(s', s)\rho(s, \gamma) .$$

If we call $\bar{\delta}(s)$, $\bar{\gamma}(s', s)$ the images of $\delta(s)$, $\gamma(s', s)$ in the given representation $\gamma \rightarrow \bar{\gamma}$ of Γ, the definition of $\bar{\tau}(s', s)$ can be written

$$\bar{\tau}(s', s) = \lambda(s')\bar{\gamma}(s', s)\lambda(s)^{-1}$$

with $\lambda(s)$ defined by

$$\lambda(s) = s\delta(s)^{-1}\bar{\delta}(s) ;$$

we note that, by the definition of the elements $\delta(s)$, we have $\delta(s_0) = e$, hence $\bar{\delta}(s_0) = e$ and $\lambda(s_0) = s_0$. Now, putting $\rho'(s, \gamma) = \lambda(s)^{-1}\rho(s, \gamma)$, we can write as follows the relations obtained above:

$$\rho'(s', \gamma(s', s)\gamma) = \bar{\gamma}(s', s)\rho'(s, \gamma) .$$

Let $s_0, s_1, \cdots, s_m$ be such that $\{s_{\mu-1}, s_\mu\} \in N(S)$ for $1 \leqq \mu \leqq m$, and put

$$\gamma = \gamma(s_0, s_1)\gamma(s_1, s_2) \cdots \gamma(s_{m-1}, s_m) .$$

By induction on m, we deduce from the above formula that we have

$$\rho'(s_0, \gamma) = \bar{\gamma}\rho'(s_m, e) .$$

It was proved in no. 3 that one can find a sequence $s_0, s_1, \cdots, s_m$ with the property stated above, and such that $s_m = s_0$ and that γ is any given element of Γ. As we have $\rho'(s_0, e) = e$, this shows that we have, for every γ in Γ, $\rho'(s_0, \gamma) = \bar{\gamma}$, and therefore

$$\rho(s_0, \gamma) = s_0\bar{\gamma} .$$

On the other hand, $\bar{V}$ is the quotient of $\bar{G}$, i.e., of G, by the fundamental group Γ of $\bar{V}$, considered as operating on $\bar{G}$; and the construction of the universal covering in no. 5 shows that an element γ of that fundamental group operates on $\bar{G}$ by mapping the class of (u, s, γ'), for the relation $\bar{R}$, onto the class of $(u, s, \gamma'\gamma)$; in particular, it maps $\bar{f}(e, s_0, e) = s_0$ onto $\bar{f}(e, s_0, \gamma) = \rho(s_0, \gamma) = s_0\bar{\gamma}$, i.e., it operates on it by right-multiplication by $\bar{\gamma}$. But the operation on $\bar{G}$ of any element of the fundamental group of $\bar{V}$ is an automorphism of the G-structure and is therefore a right-translation; this shows that it is the right-translation $\bar{\gamma}$, and that $\bar{V}$ is the same as $G/\bar{\Gamma}$, where $\bar{\Gamma}$ is the image of Γ by the representation $\gamma \rightarrow \bar{\gamma}$.

We now observe that $\bar{f}(U' \times \{(s_0, e)\}) = U's_0$ is mapped isomorphically onto its image $\bar{\varphi}(U' \times \{s_0\})$ by the projection of $\bar{G}$ onto $\bar{V}$. In view of what we have just proved, this means that we cannot have $u's_0 = u''s_0\bar{\gamma}$ unless $\gamma = e$; therefore we cannot have $\bar{\gamma} \in s_0^{-1}U's_0$ unless $\gamma = e$.

Furthermore, applying the relations found above to the chain $C_s = (s_0, s_1, \cdots, s_m)$ with $s_m = s$, we get

$$\rho'(s_0, \delta(s)) = \bar{\delta}(s)\rho'(s, e)$$

which, in view of what we have already proved, gives $\rho'(s, e) = e$ and $\rho(s, e) = \lambda(s)$. Now $\bar{V}$ is the image of the union in G of the sets

$$\bar{f}(U' \times \{(s, e)\}) = U'\rho(s, e) = U's\delta(s)^{-1}\bar{\delta}(s) ,$$

which are respectively contained in the sets Us provided the $\bar{\delta}(s)$ are close enough to the $\delta(s)$. If we call K' the closure of the union of the sets Us for $s \in S$, this means that we have $G = K'\bar{\Gamma}$, provided the representation $\gamma \rightarrow \bar{\gamma}$ is taken close enough to the identity mapping of Γ onto itself. This completes the proof of the theorem stated in no. 1, for the case when G is simply connected.

11. Let us still denote by G the connected and simply connected Lie group with the given structure, and let G' be a connected Lie group, locally isomorphic to G. We can then identify G' with a quotient G/Z, where Z is a discrete subgroup of the center of G, and the fundamental group of G' is isomorphic to Z. As it is known that G' is homeomorphic to the product of a maximal compact subgroup G'' of G' with an open ball, Z is also the fundamental group of a compact group G'' and is therefore finitely generated. Call p the canonical mapping of G onto $G'=G/Z$.

Let Γ' be a discrete subgroup of G' with compact quotient-space G'/Γ'; put $\Gamma = p^{-1}(\Gamma')$. Then Γ is a discrete subgroup of G, containing Z; we can identify Γ' with Γ/Z and G/Γ with G'/Γ', so that G/Γ is compact; we may therefore apply to G and Γ the theorem in no. 1. Let (γ_i) be a set of generators for Γ, such that there is a finite set $R_\lambda(\gamma) = e$ of fundamental relations between them; these may be chosen for instance as described above. Let (ζ_j) be a finite set of generators for Z; each of these can be expressed in terms of the γ_i; for each, take one such expression, say $\zeta_j = F_j(\gamma)$. Then Γ' is generated by the $\gamma'_i = p(\gamma_i)$ and the relations $R_\lambda(\gamma') = e'$, $F_j(\gamma') = e'$ are clearly a fundamental set of relations for the γ'_i, if e' denotes the neutral element of G'.

Let r_0, r'_0 be the identity mappings of Γ and of Γ', respectively, onto themselves. Let U be an open neighborhood of e in G, such that p induces on U a homeomorphism of U onto $U' = p(U)$; let φ be the homeomorphism of U' onto U, inverse to p. Let r' be a representation of Γ' into G'; assume, first of all, that, for every i, $r'(\gamma_i')$ is in $U'\gamma_i'$, and put

$$\bar{\gamma}_i = \varphi\big(r'(\gamma_i')\gamma_i'^{-1}\big)\gamma_i \ .$$

Then we have $p(\bar{\gamma}_i) = r'(\gamma_i')$. As the $r'(\gamma_i')$ satisfy the fundamental relations written above for Γ', the elements $R_\lambda(\bar{\gamma})$, $F_j(\bar{\gamma})$ are all in Z; they depend continuously upon the γ_i', hence upon r', and coincide respectively with e and with the ζ_j if $r' = r_0'$; if we take r' so close to r_0' that $R_\lambda(\bar{\gamma})$ is in U for every λ and $F_j(\bar{\gamma})$ in $U\zeta_j$ for every j, then, since $U \cap Z = \{e\}$, we must have $R_\lambda(\bar{\gamma}) = e$, $F_j(\bar{\gamma}) = \zeta_j$ for all λ and j. This means that there is a representation r of Γ into G, such that $r(\gamma_i) = \bar{\gamma}_i$ for every i, and that this induces on Z the identity mapping of Z onto itself. Moreover, it is clear that r depends continuously upon r'.

We have thus defined a continuous mapping $r' \to r$ into $\mathfrak{R}(\Gamma, G)$ of a neighborhood of r_0' in $\mathfrak{R}(\Gamma', G')$, such that, for every r', r induces the identity mapping on Z and that $p\big(r(\gamma)\big) = r'\big(p(\gamma)\big)$ for every $\gamma \in \Gamma$. It is now a trivial matter to verify that, since the theorem in no. 1 holds for G, Γ and r_0, it holds also for G', Γ' and r_0'.

12. Now, going back to the notations of no. 1, we consider again a connected Lie group G (which we do not assume to be simply connected) and a discrete group Γ with a finite set $(\gamma_1, \cdots, \gamma_N)$ of generators and a finite set $R_\lambda(\gamma) = \varepsilon$ of fundamental relations between them. As explained there, we can identify the space $\mathfrak{R} = \mathfrak{R}(\Gamma, G)$ of representations of Γ into G with the real-analytic subset of $G^{(N)}$ defined by the relations $R_\lambda(g_1, \cdots, g_N) = e$; and the set $\mathfrak{R}_0$ of those injective representations $r \in \mathfrak{R}$ for which $r(\Gamma)$ is discrete with compact quotient-space is an open subset of $\mathfrak{R}$.

Let X be a topological space or a differentiable or real-analytic manifold; let f be a continuous resp. differentiable resp. real-analytic mapping of X into $G^{(N)}$, such that $f(X) \subset \mathfrak{R}_0$; for each $x \in X$, write r_x instead of $f(x)$; r_x is then, for every $x \in X$, a representation of Γ into G, with the properties specified above. We make Γ operate on the space $X \times G$ by putting, for every $x \in X$, $g \in G$:

$$(x, g)\gamma = \big(x, g \cdot r_x(\gamma)\big) \ .$$

It is clear that Γ operates on $X \times G$ continuously resp. differentiably resp. real-analytically. Take any $x \in X$, $g \in G$; since in any case the mapping $x \to r_x$ is continuous, the theorem in no. 1 shows that there is a neighborhood Y of x in X, and a neighborhood U of e in G, such that, for every $y \in Y$, $r_y(\gamma) \in U$ implies $\gamma = \varepsilon$; if now U' is a neighborhood of e in G, such that $U'^{-1}U' \subset U$, one sees at once that the neighborhood $Y \times gU'$ of (x, g) in $X \times G$ has no point in common with its image under any element γ of G, other than ε. This means that we can define the quotient $S = (X \times G)/\Gamma$ as a topological space, or as a differentiable or real-analytic manifold, as the case may be; $X \times G$ is then a covering of S (its universal covering, if both X and G are simply connected). As the operations of Γ on $X \times G$ are compatible with the projection from $X \times G$ to X, that projection determines a mapping p of S onto X, which is continuous, resp. differentiable, resp. real-analytic; for every $x \in X$, $p^{-1}(x)$ is no other than the compact manifold $G/r_x(\Gamma)$. By the theorem in no. 1, for every $x \in X$, there is a neighborhood Y of x and a compact subset K of G such that $p^{-1}(Y)$ is the image of $Y \times K$ in the canonical mapping of $X \times G$ onto S; this implies at once that, to every compact subset K_1 of X, there is a compact subset K_2 of G such that $p^{-1}(K_1)$ is the image of $K_1 \times K_2$, hence compact; in other words, p is a proper mapping of S onto X. All this applies to the case $X = \mathfrak{R}_0$, f being the identity mapping of $\mathfrak{R}_0$ onto itself. Also, p has locally, in a sufficiently small neighborhood of each point of S, the same local properties as the projection from $X \times G$ to X; in particular, if X is a differentiable manifold, p is a differentiable mapping which has everywhere maximal rank (i.e., the Jacobian matrix of p has everywhere a rank equal to the dimension of X).

We now apply this to the case when X is an open interval, say $-1 < x < 1$, with its natural analytic structure. By means of a locally finite covering of S, we can put on S a differentiable ds^2; using Grauert's theorem,[1] we can even do this real-analytically. This ds^2 can be mapped back into $X \times G$; this defines on $X \times G$ an analytic ds^2, invariant under Γ. The elementary theory of differential equations shows now that the orthogonal trajectories of the fibres $\{x\} \times G$ in $X \times G$, i.e., the curves which (for the given ds^2) are at every point orthogonal to the fibre through that point, make up a family of curves which can be written as $g = F(x, g_0)$, this being the equation of the orthogonal trajectory through the point $(0, g_0)$; and F is an analytic function of (x, g_0). If, for each x, we put $F_x(g_0) = F(x, g_0)$, F_x is then an analytic homeomorphism of G onto itself. As the differential equations of the orthogonal trajectories are invariant under Γ, we have, for every $\gamma \in \Gamma$:

$$F_x\big(g_0 \cdot r_0(\gamma)\big) = F_x(g_0) \cdot r_x(\gamma) \,.$$

[1] H. Grauert, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. of Math., 68 (1958), 460–471. It will be shown elsewhere that the same could also be done without making use of Grauert's theorem.

Dividing by the equivalence relations determined in G by the operations of $r_0(\Gamma)$ and of $r_x(\Gamma)$, respectively, we see that F_x determines an analytic homeomorphism of $G/r_0(\Gamma)$ onto $G/r_x(\Gamma)$.

Now it is known[2] that, on any connected real-analytic set, any two points can be joined by a succession of analytic arcs. Therefore:

With the notations of no. 1, *let* r, r' *be two points in the same connected component of* $\mathfrak{R}_0$. *Then there is an analytic homeomorphism* F *of* G *onto itself, such that*

$$F\big(g \cdot r(\gamma)\big) = F(g) \cdot r'(\gamma)$$

for every $g \in G$ *and every* $\gamma \in \Gamma$; *and this determines an analytic homeomorphism of* $G/r(\Gamma)$ *onto* $G/r'(\Gamma)$.

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[2] A. H. Wallace, *Sheets of real analytic varieties*, Canad. J. Math., 12 (1960), 51–67 (see lemma 5.2(4), p. 66). The same result is already proved in H. Whitney et F. Bruhat, *Quelques propriétés fondamentales des ensembles analytiques-réels*, Comment. Math. Helv., 33 (1959), 132–160 (see Prop. 2, p. 141), but the formulation, as given there, is slightly weaker than what we need.