GEOMETRIC REALIZATION OF COVERING COMPLEXES

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Abstract. We prove that the geometric realization of a covering complex is a covering space. Also, this holds true for the universal covering complex.

Introduction

The theory of abstract simplicial complexes is a useful tool in the calculation of fundamental groups. This fact appears explicitly in the paper [7] of A. Weil, ... For any connected abstract simplicial complex S is edge-path group $E(S)$ is naturally isomorphic to the fundamental group of its geometric realization $\pi_1(|S|)$. The edge-path group could be described explicitly by generators and relations. As reference for these facts, see the book of I. Singer and J. Thorpe [5].

We base our definition of covering complex given by J. Rotman in [3], but there are other definitions covering complex as the one in [1]. The definition of J. Rotman

1. Preliminaries

We recall that, given a topological space X, a covering space on X it's a continuous map $p\colon E \to X$, such that for every $x \in X$, there is an open neighborhood U such that $p^{-1}(U)$ it is a disjoint union of open sets U_λ, $\lambda \in \Lambda$, and $p|_{U_\lambda}\colon U_\lambda \to U$ it's a homeomorphism. We recommend the book of P. May [2] and the book of J. Rotman [4] as reference of covering spaces.

An abstract simplicial complex is a pair $(S, \mathcal{K})$ where S is a set and $\mathcal{K}$ is a family of non-empty finite subsets of S such that:

- $\bigcup \mathcal{K} = S$.
- If $\sigma \subseteq \tau$ and $\tau \in \mathcal{K}$ then $\sigma \in \mathcal{K}$.

We call complexes to the abstract simplicial complexes. If $\sigma \in \mathcal{K}$, then the dimension of σ is $|\sigma| - 1$, and we denote it by $dim(\sigma)$. The elements of $\mathcal{K}$ of dimension n are called n-simplices, and we denote the set of n-simplices by $\mathcal{K}_n$. The 0-simplices are called vertices. The dimension of $(S, \mathcal{K})$ is defined as the supremum of $dim(\sigma)$ where σ ranges over $\mathcal{K}$, we denote it by $dim(S, \mathcal{K})$. This dimension may be infinite. We call the complex $(S, \mathcal{K})$ finite, if S is finite. In particular, a finite complex has finite dimension. A complex $(S, \mathcal{K})$ is called discrete, if $\mathcal{K} = \{\{s\} \mid s \in S\}$.

An edge e in $(S, \mathcal{K})$ is a pair of vertices (x, y) where $\{x, y\} \in \mathcal{K}$, x is the origin of the edge e and we denote it by $orig(e)$, and y is the end of the edge e and we denote by $end(e)$. A path α in $(S, \mathcal{K})$ is a finite sequence of edges $e_1, \ldots, e_n$ such that $end(e_i) = orig(e_{i+1})$ with $i = 1, \ldots, n-1$. We define $orig(\alpha) = orig(e_1)$ and $end(\alpha) = end(e_n)$.

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A morphism between complex $(S_1, \mathcal{K}_1)$ and $(S_2, \mathcal{K}_2)$ is a map $f\colon S_1 \longrightarrow S_2$ such that $f(\sigma) \in \mathcal{K}_2$ for any $\sigma \in \mathcal{K}_2$. We call a morphism of complexes a simplicial map. We denote by $\mathcal{C}$ to the category of complexes and simplicial maps.

As reference of complexes, we recommend [5] and [6].

We denote by $\mathcal{T}$ to the category of topological spaces and continuous maps.

2. Geometric Covering Complexes

Definition 2.1. *Let $(S, \mathcal{K})$ be a complex. We say that $(S, \mathcal{K})$ is connected if for any pair of vertices x, y of $(S, \mathcal{K})$ there is a path α such that $orig(\alpha) = x$ and $end(\alpha) = y$.*

The following definition is due J. Rotman in [3].

Definition 2.2. *Let $(S, \mathcal{K})$ be a complex. A covering of $(S, \mathcal{K})$ is a pair $((T, \mathcal{L}), p)$ where $(T, \mathcal{L})$ is a complex and $p\colon T \longrightarrow S$ is a simplicial map such that:*

- *$(T, \mathcal{L})$ is a connected complex.*
- *For every $\sigma \in \mathcal{K}$, $p^{-1}(\sigma) = \bigcup_{i\in I} \sigma_i$ where the family $\{\sigma_i\}_{i\in I}$ is a pairwise disjoint family of simplices of $\mathcal{L}$ such that $p|_{\sigma_i}\colon \sigma_i \longrightarrow \sigma$ is bijective.*

The map p is called projection and the simplices σ_i are called sheets over σ.

We observe that p is surjective and $(S, \mathcal{K})$ is connected. For our geometric porpoises we need a stronger definition of covering complex.

Definition 2.3. *Let $(S, \mathcal{K})$ be a complex. A geometric covering of $(S, \mathcal{K})$ is a covering $((T, \mathcal{L}), p)$ such that for any simplex $\sigma \in \mathcal{L}_n$, $p(\sigma) \in \mathcal{K}_n$. In other words, p preserves the dimension of the simplices. We have that geometric coverings preserve the dimension of the complexes.*

Proposition 2.1. *Let $(S, \mathcal{K})$ be a complex, and $((T, \mathcal{L}), p)$ a covering complex of $(S, \mathcal{K})$. If $\sigma \in \mathcal{K}$ is a maximal simplex, then p^{-1} is the disjoint union of maximal simplices.*

Proof. Let $\sigma \in \mathcal{K}$ be a maximal simplex with $p^{-1}(\sigma) = \bigcup_{i\in I} \sigma_i$ where the family $\{\sigma_i\}_{i\in I}$ is a pairwise disjoint family of simplices of $\mathcal{K}$ such that $p|_{\sigma_i}\colon \sigma_i \longrightarrow \sigma$ is bijective. If σ_j is not maximal for some $j \in I$, then there is a simplex $\tau \in \mathcal{L}$ such that $\sigma_j \subset \tau$. As σ is maximal and $\sigma = p(\sigma_j) \subseteq p(\tau)$, we have that $p(\tau) = \sigma$. This contradicts the fact that p preserves the dimension. Therefore σ_i is a maximal simplex. □

Definition 2.4. *Let $(S, \mathcal{K})$ and $(T, \mathcal{L})$ be complexes and $f\colon (S, \mathcal{K}) \longrightarrow (T, \mathcal{L})$ be a simplicial map. We say that f reflects maximal simplices, if $\sigma \in \mathcal{K}$ is such that $f(\sigma)$ is a maximal simplex, then σ is a maximal simplex.*

Proposition 2.2. *Let $(S, \mathcal{K})$ be a finite dimensional complex, and $((T, \mathcal{L}), p)$ a covering complex of $(S, \mathcal{K})$. Then $((T, \mathcal{L}), p)$ is a geometric covering complex of $(S, \mathcal{K})$ if and only if p reflects maximal simplices.*

Proof. Conjetura □

3. Geometric Realization of Covering Complexes

Definition 3.1 (Geometric Realization)**.** *The geometric realization of a complex $(S, \mathcal{K})$ is the set of all function $\phi\colon S \longrightarrow [0, 1]$ such that:*

- *supp*(ϕ) $\in \mathcal{K}$
- $\sum_{s\in S}\phi(s)=1$

We denote this set by $|(S,\mathcal{K})|$. We can give $|(S,\mathcal{K})|$ a metric topology given by:

$$d(\phi,\psi)=\sqrt{\sum_{s\in S}(\phi(s)-\psi(s))^2}$$

for $\phi,\psi\in|(S,\mathcal{K})|$. When we endowed $|(S,\mathcal{K})|$ with the metric topology we denote it by $|(S,\mathcal{K})|_d$. There is a second topology for $|(S,\mathcal{K})|$ called the coherent topology. Foer each simplex $\sigma\in\mathcal{K}$, we define its geometric realization $|\sigma|$ as the set of functions $\phi\in|(S,\mathcal{K})|$ with supp(ϕ) $\subseteq\sigma$. We give to $|\sigma|$ the subspace topology inherited as subset of $|(S,\mathcal{K})|$. If we consider the inclusion $i_\sigma\colon|\sigma|\longrightarrow|(S,\mathcal{K})|$, then coherent topology on $|(S,\mathcal{K})|$ is the largest topology which makes all the inclusions continuous. Usually, $|(S,\mathcal{K})|$ is considered with the coherent topology. We may characterize $|(S,\mathcal{K})|$ as the colimit in $\mathcal{T}$ of the geometric realization of its simplices. So a function $f\colon|(S,\mathcal{K})|\longrightarrow X$ is continuous if and only its restrictions to $|\sigma|$ is continuous for all $\sigma\in\mathcal{K}$. Especially, the identity $|(S,\mathcal{K})|\longrightarrow|(S,\mathcal{K})|_d$ is continuous, so the coherent topology contains the metric topology. In particular, if $(S,\mathcal{K})$ is a finite complex, then both topologies coincide.

Definition 3.2. *Let $(S,\mathcal{K})$ be a complex and $s\in S$. We define $\phi_s\colon S\longrightarrow[0,1]$ as $\phi_s(t)=\delta_{st}$ for any $t\in S$ where δ is the Kronecker's delta.*

Definition 3.3. *If $f\colon(S,\mathcal{K})\longrightarrow(T,\mathcal{L})$ is a simplicial map, then it induces a continuous function $|f|\colon|(S,\mathcal{K})|\longrightarrow|(T,\mathcal{L})|$. If $\phi\in|(S,\mathcal{K})|$ with $\sigma=supp(\phi)$ then $\phi=\sum_{s\in\sigma}\phi(s)\phi_s$. So we define $|f|(\phi):=\sum_{s\in\sigma}\phi(s)\phi_{f(s)}$. In this way, the geometric realization is a functor from $\mathcal{C}$ to $\mathcal{T}$.*

Proposition 3.1. *Let $(S,\mathcal{K})$ be a complex and $((T,\mathcal{L}),p)$ a finite geometric covering of $(S,\mathcal{K})$. Then $|(T,\mathcal{L}),p)|$ is covering of $|(S,\mathcal{K})|$.*

Proof. Let $\phi\in|(S,\mathcal{K})|$ with $supp(\phi)=\sigma\in\mathcal{K}$. Then $p^{-1}(\sigma)=\bigcup_{i\in I}\sigma_i$ where the family $\{\sigma_i\}_{i\in I}$ is a pairwise disjoint family of simplices of $\mathcal{L}$ such that $p|_{\sigma_i}\colon\sigma_i\longrightarrow\sigma$ is bijective. We define $R=\min\{\frac{d(\phi,\phi_s}{2}\mid s\in\sigma\}$. So $\mathbb{B}_R(\phi)$ is an open neighborhood of ϕ. We affirm that $|p|^{-1}(\mathbb{B}_R(\phi))=\bigcup_{i\in I}\mathbb{B}_R(\phi_i)$ □

Proposition 3.2. *Let $(S,\mathcal{K})$ be a complex and $((T,\mathcal{L}),p)$ a finite dimensional geometric covering of $(S,\mathcal{K})$. Then $|(T,\mathcal{L}),p)|$ is covering of $|(S,\mathcal{K})|$.*

Proof. □

Proposition 3.3. *Let $(S,\mathcal{K})$ be a complex and $((T,\mathcal{L}),p)$ a geometric covering of $(S,\mathcal{K})$. Then $|(T,\mathcal{L}),p)|$ is covering of $|(S,\mathcal{K})|$.*

Proof. □

4. Geometric Realization of the Universal Covering Complex

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