GEOMETRIC REALIZATION OF COVERING COMPLEXES

FRANK MURPHY-HERNANDEZ AND LUIS MACÍAS-BARRALES

Abstract. We prove that the geometric realization of a covering complex is a covering space. Also, this holds true for the universal covering complex.

Introduction

The theory of abstract simplicial complexes is a useful tool in the calculation of fundamental groups. This fact appears explicitly in the paper [9] of A. Weil, ... For any connected abstract simplicial complex S is edge-path group $E(S)$ is naturally isomorphic to the fundamental group of its geometric realization $\pi_1(|S|)$. The edge-path group could be described explicitly by generators and relations. As reference for these facts, see the book of I. Singer and J. Thorpe [7].

We base our definition of covering complex given by J. Rotman in [4], but there are other definitions covering complex as the one in [1]. The definition of J. Rotman

1. Preliminaries

We recall that, given a topological space X, a covering space on X it's a continuous map $p\colon E \to X$, such that for every $x \in X$, there is an open neighborhood U such that $p^{-1}(U)$ it is a disjoint union of open sets U_λ, $\lambda \in \Lambda$, and $p|_{U_\lambda}\colon U_\lambda \to U$ it's a homeomorphism. We recommend the book of P. May [3] and the book of J. Rotman [6] as reference of covering spaces.

An abstract simplicial complex is a pair $(S, \mathcal{K})$ where S is a set and $\mathcal{K}$ is a family of non-empty finite subsets of S such that:

- $\bigcup \mathcal{K} = S$.
- If $\sigma \subseteq \tau$ and $\tau \in \mathcal{K}$ then $\sigma \in \mathcal{K}$.

We call complexes to the abstract simplicial complexes. If $\sigma \in \mathcal{K}$, then the dimension of σ is $|\sigma| - 1$, and we denote it by $dim(\sigma)$. The elements of $\mathcal{K}$ of dimension n are called n-simplices, and we denote the set of n-simplices by $\mathcal{K}_n$. We define the n-skeleton of $(S, \mathcal{K})$ as the complex $(S, \bigcup_{m=1}^{0} \mathcal{K}_m)$, and we denote it by $sk_n(S, \mathcal{K})$. The 0-simplices are called vertices. The dimension of $(S, \mathcal{K})$ is defined as the supremum of $dim(\sigma)$ where σ ranges over $\mathcal{K}$, we denote it by $dim(S, \mathcal{K})$. This dimension may be infinite. We call the complex $(S, \mathcal{K})$ finite, if S is finite. In particular, a finite complex has finite dimension. A complex $(S, \mathcal{K})$ is called indiscrete, if $\mathcal{K} = \{\{s\} \mid s \in S\}$.

An edge e in $(S, \mathcal{K})$ is a pair of vertices (x, y) where $\{x, y\} \in \mathcal{K}$, x is the origin of the edge e and we denote it by $orig(e)$, and y is the end of the edge e and we denote by $end(e)$. A path α in $(S, \mathcal{K})$ is a finite sequence of edges $e_1, \ldots, e_n$ such

Date: November 7, 2019.

2000 *Mathematics Subject Classification.* Primary ****, ****; Secondary ****, ****.

Key words and phrases. Abstract Simplicial Complexes, Coverings, Fundamental Group.

that $end(e_i) = orig(e_{i+1})$ with $i = 1, \dots, n-1$. We define $orig(\alpha) = orig(e_1)$ and $end(\alpha) = end(e_n)$.

A morphism between complex $(S_1, \mathcal{K}^1)$ and $(S_2, \mathcal{K}^2)$ is a map $f\colon S_1 \longrightarrow S_2$ such that $f(\sigma) \in \mathcal{K}^2$ for any $\sigma \in \mathcal{K}^1$. We call a morphism of complexes a simplicial map. We denote by $\mathcal{C}$ to the category of complexes and simplicial maps.

If X is a non empty set, we denote by $D(X)$ the complex given $(X, \mathcal{D}_X)$ where $\mathcal{D}_X$ is the set of all non empty finite subsets of X. .As reference of complexes, we recommend [7] and [8].

We denote by $\mathbb{P}(X)$ the power set of a set of X. If $f\colon X \longrightarrow Y$ is a function, then f induces a map $f_\circ\colon \mathbb{P}(X) \longrightarrow \mathbb{P}(Y)$ where $f_\circ(A)$ is the direct image of A under f for $A \in \mathbb{P}(X)$.

We denote by $\mathcal{T}$ to the category of topological spaces and continuous maps.

For S a G-set and $s \in S$, we denote by G_s the stabilizer of s and by $o(s)$ the orbit of s. The set of orbits of S is a partition of S, so the quotient set is denoted by S/G. We say that the action is free, for any $g \in G$ and $s \in S$ $gs = s$ implies $g = e$. Any G-set can be descomposed as the disjoint union of its orbits, so $S = \coprod_{o(s) \in S/G} o(s)$. As reference for G-sets, we have the books of M. Aschbacher [2], and of J. Rotman [5].

2. Covering Complexes

Definition 2.1. *Let $(S, \mathcal{K})$ be a complex. We say that $(S, \mathcal{K})$ is connected if for any pair of vertices x, y of $(S, \mathcal{K})$ there is a path α such that $orig(\alpha) = x$ and $end(\alpha) = y$.*

The following definition is due J. Rotman in [4].

Definition 2.2. *Let $(S, \mathcal{K})$ be a complex. A covering of $(S, \mathcal{K})$ is a pair $((T, \mathcal{L}), p)$ where $(T, \mathcal{L})$ is a complex and $p\colon T \longrightarrow S$ is a simplicial map such that:*

- *$(T, \mathcal{L})$ is a connected complex.*
- *For every $\sigma \in \mathcal{K}$, $p^{-1}(\sigma) = \bigcup_{i \in I} \sigma_i$ where the family $\{\sigma_i\}_{i \in I}$ is a pairwise disjoint family of simplices of $\mathcal{L}$ such that $p|_{\sigma_i} : \sigma_i \longrightarrow \sigma$ is bijective.*

The map p is called projection and the simplices σ_i are called sheets over σ.

We observe that p is surjective and $(S, \mathcal{K})$ is connected. For our geometric porpoises we need a stronger definition of covering complex.

Definition 2.3. *Let $(S, \mathcal{K})$ be a complex. A geometric covering of $(S, \mathcal{K})$ is a covering $((T, \mathcal{L}), p)$ such that for any simplex $\sigma \in \mathcal{L}_n$, $p(\sigma) \in \mathcal{K}_n$. In other words, p preserves the dimension of the simplices. We have that geometric coverings preserve the dimension of the complexes. By definition all geometric coverings are coverings.*

Example 2.1. *Let $(S, \mathcal{K})$ be the complex $D(\mathbb{Z}_2)$, and $(T, \mathcal{L})$ be the complex $D(\mathbb{Z}_4)$. We consider the canonical projection $p\colon \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$ which is a simplicial map. This simplicial map does not preserve the dimension of the simplices. We notice that $p^{-1}(\{[0]\}) = \{[0]\} \cup \{[2]\}$, $p^{-1}(\{[1]\}) = \{[1]\} \cup \{[3]\}$, and $p^{-1}(\{[0], [1]\}) = \{[0], [2]\} \cup \{[1], [3]\}$ and $D(\mathbb{Z}_4)$ is a connected complex, so p is a covering which is not a geometric covering.*

Proposition 2.1. *Let $(S, \mathcal{K})$ be a complex, and $((T, \mathcal{L}), p)$ a covering complex of $(S, \mathcal{K})$. If $\sigma \in \mathcal{K}$ is a maximal simplex, then p^{-1} is the disjoint union of maximal simplices.*

Proof. Let $\sigma \in \mathcal{K}$ be a maximal simplex with $p^{-1}(\sigma) = \bigcup_{i\in I} \sigma_i$ where the family $\{\sigma_i\}_{i\in I}$ is a pairwise disjoint family of simplices of $\mathcal{K}$ such that $p|_{\sigma_i} \colon \sigma_i \longrightarrow \sigma$ is bijective. If σ_j is not maximal for some $j \in I$, then there is a simplex $\tau \in \mathcal{L}$ such that $\sigma_j \subset \tau$. As σ is maximal and $\sigma = p(\sigma_j) \subseteq p(\tau)$, we have that $p(\tau) = \sigma$. This contradicts the fact that p preserves the dimension. Therefore σ_i is a maximal simplex. □

Definition 2.4. *Let $(S,\mathcal{K})$ and $(T,\mathcal{L})$ be complexes and $f \colon (S,\mathcal{K}) \longrightarrow (T,\mathcal{L})$ be a simplicial map. We say that f reflects maximal simplices, if $\sigma \in \mathcal{K}$ is such that $f(\sigma)$ is a maximal simplex, then σ is a maximal simplex.*

Proposition 2.2. *Let $(S,\mathcal{K})$ be a finite dimensional complex, and $((T,\mathcal{L}),p)$ a covering complex of $(S,\mathcal{K})$. Then $((T,\mathcal{L}),p)$ is a geometric covering complex of $(S,\mathcal{K})$ if and only if p reflects maximal simplices.*

Proof. Conjetura □

Let $(S,\mathcal{K})$ and $(T,\mathcal{L})$ be complexes and $f \colon S \longrightarrow T$ be a function. We notice that f is a simplicial map if $f_\circ(\mathcal{K}) \subseteq \mathcal{L}$. So we denote the induced map by $f_* \colon \mathcal{K} \longrightarrow \mathcal{L}$.

Definition 2.5. *Let $(S,\mathcal{K})$ be a complex. A combinatoric covering of $(S,\mathcal{K})$ is a covering $((T,\mathcal{L}),p)$ such that $p_*^{-1}(\sigma)$ is a family of pairwise disjoint simplices.*

Example 2.2. *Let $(S,\mathcal{K})$ be the complex $D(\mathbb{Z}_2)$, and $(T,\mathcal{L})$ be the complex $sk_1(D(\mathbb{Z}_4))$. We consider the canonical projection $p \colon \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$ which is a simplicial map. This map is a geometric covering complex, but is not a combinatoric complex as $p_*^{-1}(\{[0]\}) = \{\{[0]\},\{[2]\},\{[0],[2]\}\}$ is not pairwise disjoint.*

Example 2.3. *Let $(S,\mathcal{K})$ be the complex $sk_1(D(\mathbb{Z}_3))$, and $(T,\mathcal{L})$ be the complex given by $T = \mathbb{Z}$ and $\mathcal{L} = \{\{n, n+1\} \mid n \in \mathbb{Z}\} \cup \{\{m\} \mid m \in \mathbb{Z}\}$. We consider the canonical projection $p \colon \mathbb{Z} \longrightarrow \mathbb{Z}_3$ which is a simplicial map. Moreover, $((T,\mathcal{L}),p)$ is a combinatoric covering complex.*

Proposition 2.3. *Let $(S,\mathcal{K})$ be a complex, and $((T,\mathcal{L}),p)$ a combinatoric covering complex of $(S,\mathcal{K})$. Then $((T,\mathcal{L}),p)$ is a geometric covering complex.*

Proof. Let $\sigma \in \mathcal{L}_n$. As p is a covering there is a family $\{\sigma_i\}_{i\in I}$ of pairwise disjoint simplices such that $p^{-1}(p(\sigma)) = \bigcup_{i\in I} \sigma_i$ and $|\sigma_i| = |p(\sigma)|$ for all $i \in I$. We observe that $\{\sigma_i\}_{i\in I}$ and $p_*^{-1}(\sigma)$ are partitions of $p^{-1}(p(\sigma))$. As we have that $\{\sigma_i\}_{i\in I} \subseteq p_*^{-1}(\sigma)$, then $\{\sigma_i\}_{i\in I} = p_*^{-1}(\sigma)$. So $\sigma = \sigma_j$ for some $j \in I$, and $|\sigma| = |p(\sigma)|$. Therefore $((T,\mathcal{L}),p)$ a geometric covering complex. □

3. G-Covering Complexes

Definition 3.1. *Let G be a group and $(S,\mathcal{K})$ be a complex. We say that $(S,\mathcal{K})$ is a G-complex, if S is a G-set and $g\sigma \in \mathcal{K}$ for all $\sigma \in \mathcal{K}$. We notice that in particular $\mathcal{K}$ is a G-set.*

Definition 3.2. *Let G be a group and $(S,\mathcal{K})$ be a G-complex. We define $\mathcal{K}^G := \{\pi(\sigma) \subseteq S/G \mid \sigma \in \mathcal{K}\}$ where $\pi \colon S \longrightarrow S/G$ is the canonical projection.*

Proposition 3.1. *Let G be a group and $(S,\mathcal{K})$ be a G-complex. Then $(S/G, \mathcal{K}^G)$ is a complex. Moreover, π is a simplicial map.*

Proof. Let $o(x) \in S/G$. Then $\pi(\{x\}) = \{xG\}$. As $\{x\} \in \mathcal{K}$, we have that $\{xG\} \in \mathcal{K}^G$. □

Definition 3.3. *Let G be a group and $(S,\mathcal{K})$ be a G-complex. We say that the action is wandering, if for any $g \in G$ $g\sigma \cap \sigma \neq \emptyset$ implies that $g = e$.*

Example 3.1. *Let G be a group and S be a G-set. If the action over S is free, then the indiscrete complex over S is G complex with wandering action.*

Proposition 3.2. *Let G be a group and $(S,\mathcal{K})$ be a G-complex with discontinuous action. Then $((S/G,\mathcal{K}^G),\pi)$ is covering complex.*

Proof. □

Definition 3.4. *Let G be a group, $(S,\mathcal{K})$ be a G-complex, and $\sigma \in \mathcal{K}$. We say that σ is wandering, if for any $x, y \in \sigma$ $o(x) = o(y)$ implies that $x = y$.*

Proposition 3.3. *Let G be a group, $(S,\mathcal{K})$ be a G-complex, and $\sigma \in \mathcal{K}$. Then σ is wandering if and only if $|\sigma \cap o(x)| = 1$ for any $x \in \sigma$.*

Proof. □

Definition 3.5. *Let G be a group and S be a G-set. We define $\mathcal{K}[G]$ as the set of all elements of $\mathcal{D}_S$ which are wandering.*

Proposition 3.4. *Let G be a group and S be a G-set. Then $(S,\mathcal{K}[G])$ is a complex with discontinuous action.*

Proof. □

Proposition 3.5. *Let G be a group and $(S,\mathcal{K})$ be a G-complex with discontinuous action. Then $\mathcal{K} \subseteq \mathcal{K}[G]$.*

Proof. □

Example 3.2.

4. Geometric Realization of Covering Complexes

Definition 4.1 (Geometric Realization)**.** *The geometric realization of a complex $(S,\mathcal{K})$ is the set of all function $\phi\colon S \longrightarrow [0,1]$ such that:*

- *$supp(\phi) \in \mathcal{K}$*
- *$\sum_{s\in S}\phi(s) = 1$*

We denote this set by $|(S,\mathcal{K})|$. We can give $|(S,\mathcal{K})|$ a metric topology given by:

$$d(\phi,\psi) = \sqrt{\sum_{s\in S}(\phi(s)-\psi(s))^2}$$

for $\phi,\psi \in |(S,\mathcal{K})|$. When we endowed $|(S,\mathcal{K})|$ with the metric topology we denote it by $|(S,\mathcal{K})|_d$. There is a second topology for $|(S,\mathcal{K})|$ called the coherent topology. Foer each simplex $\sigma \in \mathcal{K}$, we define its geometric realization $|\sigma|$ as the set of functions $\phi \in |(S,\mathcal{K})|$ with $supp(\phi) \subseteq \sigma$. We give to $|\sigma|$ the subspace topology inherited as subset of $|(S,\mathcal{K})|$. If we consider the inclusion $i_\sigma\colon |\sigma| \longrightarrow |(S,\mathcal{K})|$, then coherent topology on $|(S,\mathcal{K})|$ is the largest topology which makes all the inclusions continuous. Usually, $|(S,\mathcal{K})|$ is considered with the coherent topology. We may characterize $|(S,\mathcal{K})|$ as the colimit in $\mathcal{T}$ of the geometric realization of its simplices. So a function $f\colon |(S,\mathcal{K})| \longrightarrow X$ is continuous if and only its restrictions to $|\sigma|$ is continuous for all $\sigma \in \mathcal{K}$. Especially, the identity $|(S,\mathcal{K})| \longrightarrow |(S,\mathcal{K})|_d$ is continuous, so the coherent topology contains the metric topology. In particular, if $(S,\mathcal{K})$ is a finite complex, then both topologies coincide.

We observe that if $\sigma, \tau \in \mathcal{K}$ are disjoint, then $|\sigma|$ and $|\tau|$ are disjoint.

Definition 4.2. *Let $(S, \mathcal{K})$ be a complex and $s \in S$. We define $\phi_s : S \longrightarrow [0,1]$ as $\phi_s(t) = \delta_{st}$ for any $t \in S$ where δ is the Kronecker's delta.*

Definition 4.3. *If $f : (S, \mathcal{K}) \longrightarrow (T, \mathcal{L})$ is a simplicial map, then it induces a continuous function $|f| : |(S, \mathcal{K})| \longrightarrow |(T, \mathcal{L})|$. If $\phi \in |(S, \mathcal{K})|$ with $\sigma = supp(\phi)$ then $\phi = \sum_{s \in \sigma} \phi(s)\phi_s$. So we define $|f|(\phi) := \sum_{s\in\sigma} \phi(s)\phi_{f(s)}$. In this way, the geometric realization is a functor from $\mathcal{C}$ to $\mathcal{T}$.*

Proposition 4.1. *Let $(S, \mathcal{K})$ be a complex and $((T, \mathcal{L}), p)$ a finite geometric covering of $(S, \mathcal{K})$. Then $|(T, \mathcal{L}), p)|$ is covering of $|(S, \mathcal{K})|$.*

Proof. Let $\phi \in |(S, \mathcal{K})|$ with $supp(\phi) = \sigma \in \mathcal{K}$. Then $p^{-1}(\sigma) = \bigcup_{i \in I} \sigma_i$ where the family $\{\sigma_i\}_{i\in I}$ is a pairwise disjoint family of simplices of $\mathcal{L}$ such that $p|_{\sigma_i} : \sigma_i \longrightarrow \sigma$ is bijective. We define $R = \min\{\frac{d(\phi,\phi_s}{2} \mid s \in \sigma\}$. So $\mathbb{B}_R(\phi)$ is an open neighborhood of ϕ. We affirm that $|p|^{-1}(\mathbb{B}_R(\phi)) = \bigcup_{i\in I} \mathbb{B}_R(\phi_i)$ □

Proposition 4.2. *Let $(S, \mathcal{K})$ be a complex and $((T, \mathcal{L}), p)$ a finite dimensional geometric covering of $(S, \mathcal{K})$. Then $|(T, \mathcal{L}), p)|$ is covering of $|(S, \mathcal{K})|$.*

Proof. □

Proposition 4.3. *Let $(S, \mathcal{K})$ be a complex and $((T, \mathcal{L}), p)$ a geometric covering of $(S, \mathcal{K})$. Then $|(T, \mathcal{L}), p)|$ is covering of $|(S, \mathcal{K})|$.*

Proof. □

5. Geometric Realization of the Universal Covering Complex

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Facultad de Ciencias, UNAM, Mexico City
Email address: murphy@ciencias.unam.mx

Instituto de Matemáticas, UNAM, Mexico City