GEOMETRIC REALIZATION OF COVERING COMPLEXES

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Abstract. We prove that the geometric realization of a covering complex is a covering space. Also, this holds true for the universal covering complex. Finally, we prove that the fundamental group of a complex coincides with the fundamental group of the geometric realization.

Introduction

The theory of abstract simplicial complexes is a useful tool in the calculation of fundamental groups. This fact appears explicitly in the paper [5] of A. Weil, ... For any connected abstract simplicial complex S is edge-path group $E(S)$ is naturally isomorphic to the fundamental group of its geometric realization $\pi_1(|S|)$. The edge-path group could be described explicitly by generators and relations. As reference for these facts, see the book of I. Singer and J. Thorpe [4].

1. Preliminaries

We recall that, given a topological space X, a covering space on X it's a continuous map $p\colon E \to X$, such that for every $x \in X$, there is an open neighborhood U such that $p^{-1}(U)$ it is a disjoint union of open sets U_λ, $\lambda \in \Lambda$, and $p|_{U_\lambda}\colon U_\lambda \to U$ it's a homeomorphism. We recommend the book of P. May [1] and the book of J. Rotman [3] as reference of covering spaces.

An abstract simplicial complex is a pair $(S, \mathcal{K})$ where S is a set and $\mathcal{K}$ is a family of non-empty finite subsets of S such that:

- $\bigcup \mathcal{K} = S$.
- If $\sigma \subseteq \tau$ and $\tau \in \mathcal{K}$ then $\sigma \in \mathcal{K}$.

We call complexes to the abstract simplicial complexes. If $\sigma \in \mathcal{K}$, then the dimension of σ is $|\sigma| - 1$, and we denote it by $dim(\sigma)$. The elements of $\mathcal{K}$ of dimension n are called n-simplices, and we denote the set of n-simplices by $\mathcal{K}_n$. The 0-simplices are called vertices. The dimension of $(S, \mathcal{K})$ is defined as the supremum of $dim(\sigma)$ where σ ranges over $\mathcal{K}$, we denote it by $dim(S, \mathcal{K})$. This dimension may be infinite. We call the complex $(S, \mathcal{K})$ finite, if its dimension is finite.

An edge e in $(S, \mathcal{K})$ is a pair of vertices (x, y) where $\{x, y\} \in \mathcal{K}$, x is the origin of the edge e and we denote it by $orig(e)$, and y is the end of the edge e and we denote by $end(e)$. A path α in $(S, \mathcal{K})$ is a finite sequence of edges $e_1, \dots, e_n$ such that $end(e_i) = orig(e_{i+1})$ with $i = 1, \dots, n-1$. We define $orig(\alpha) = orig(e_1)$ and $end(\alpha) = end(e_n)$.

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A morphism between complex $(S_1, \mathcal{K}_1)$ and $(S_2, \mathcal{K}_2)$ is a map $f\colon S_1 \longrightarrow S_2$ such that $f(\sigma) \in \mathcal{K}_2$ for any $\sigma \in \mathcal{K}_2$. We call a morphism of complexes a simplicial map. We denote by $\mathcal{C}$ to the category of complexes and simplicial maps.

We denote by $\mathcal{T}$ to the category of topological spaces and continuous maps.

2. Geometric Realization of Covering Complexes

Definition 2.1. *The geometric realization of a complex $(S, \mathcal{K})$ is the set of all function $\phi\colon S \longrightarrow [0,1]$ such that:*

- *$supp(\phi) \in \mathcal{K}$*
- *$\sum_{s\in S} \phi(s) = 1$*

We denote this set by $|(S,\mathcal{K})|$. We may think $[0,1]^S$ as the direct limit of $[0,1]^A$ where Aranges over all finite subsets of S. So we give the $|(S,\mathcal{K})|$ the subspace topology.

Definition 2.2. *The geometric realization of an abstract simplical complex $(S, \mathcal{K})$ is given by the following formula: first we give a total order to S. Then, for any simplex $\sigma = \{s_0 < s_1 < \ldots < s_q\}$ we define $|\sigma| = \Delta^q$, the standar topological q-simplex and we asociate to the vertex s_q the q-th vertex of Δ^q. If $\tau = \{s_0 < \ldots < s_q\}$ is a simplex and $\sigma = \{s_{q_1} < \ldots < s_{q_k}\} \subseteq \tau$, we define $i^\tau_\sigma\colon |\sigma| \to |\tau|$ to be the affine function such that maps the j-th vertex of $|\sigma|$ to the q_j-th vertex of $|\tau|$. Thus we take $|S|$ as the colimit over this system. If $f\colon (S_1, \mathcal{K}_1) \to (S_2, \mathcal{K}_2)$ it's a morphism of abstract simplicial complexes, then we can define $|f|\colon |S_1| \to |S_2|$ as the colimit of the affine functions $|f|_\sigma|\colon |\sigma| \to |f(\sigma)|$. $|-|\colon \mathcal{C} \to \mathcal{T}$ it's a functor.*

Definition 2.3. *Let $(S, \mathcal{K})$ be a complex. We say that $(S, \mathcal{K})$ is connected if for any pair of vertices x, y of $(S, \mathcal{K})$ there is a path α such that $orig(\alpha) = x$ and $end(\alpha) = y$.*

The following definition is due J. Rotman in [2].

Definition 2.4. *Let $(S, \mathcal{K})$ be a complex. A covering of $(S, \mathcal{K})$ is a pair $((T, \mathcal{L}), p)$ where $(T, \mathcal{L})$ is a complex and $p\colon T \longrightarrow S$ is a morphism of complexes such that:*

- *$(T, \mathcal{L})$ is a connected complex.*
- *For every $\sigma \in \mathcal{L}$, $p^{-1}(\sigma) = \bigcup_{i\in I} \sigma_i$ where the family $\{\sigma_i\}_{i\in I}$ is a pairwise disjoint family of simplices of $\mathcal{K}$ such that $p|_{\sigma_i}\colon \sigma_i \longrightarrow \sigma$ is bijective.*

The map p is called projection and the simplices σ_i are called sheets over σ.

We observe that p is surjective and $(S, \mathcal{K})$ is connected.

Proposition 2.1. *Let $(S, \mathcal{K})$ be a complex, and $((T, \mathcal{L}), p)$ a covering complex of $(S, \mathcal{K})$. If $\sigma \in \mathcal{K}$ is a maximal simplex, then p^{-1} is the disjoint union of maximal simplices.*

Proof. Let $\sigma \in \mathcal{K}$ be a maximal simplex with $p^{-1}(\sigma) = \bigcup_{i\in I} \sigma_i$ where the family $\{\sigma_i\}_{i\in I}$ is a pairwise disjoint family of simplices of $\mathcal{K}$ such that $p|_{\sigma_i}\colon \sigma_i \longrightarrow \sigma$ is bijective. If σ_j is not maximal for some $j \in I$, then there is a simplex $\tau \in \mathcal{L}$ such that $\sigma_j \subset \tau$. As σ is maximal and $\sigma = p(\sigma_j) \subseteq p(\tau)$, we have that $p(\tau) = \sigma$. .... □

Proposition 2.2. *Let $(S, \mathcal{K})$ be an abstract simplicial complex and $((T, \mathcal{L}), p)$ an abstract simplicial covering of $(S, \mathcal{K})$. Then $(|(T, \mathcal{L})|, |p|)$ is covering of $|(S, \mathcal{K})|$.*

Proof. □

3. Geometric Realization of the Universal Covering Complex

References

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