

GEOMETRIC REALIZATION OF COVERING COMPLEXES

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ABSTRACT. We prove that the geometric realization of a covering complex is a covering space. Also, this holds true for the universal covering complex. Finally, we prove that the fundamental group of a complex coincides with the fundamental group of the geometric realization.

INTRODUCTION

The theory of abstract simplicial complexes is a useful tool in the calculation of fundamental groups. This fact appears explicitly in the paper [5] of A. Weil, For any connected abstract simplicial complex S its edge-path group $E(S)$ is naturally isomorphic to the fundamental group of its geometric realization $\pi_1(|S|)$. The edge-path group could be described explicitly by generators and relations. As reference for these facts, see the book of I. Singer and J. Thorpe [4].

1. PRELIMINARIES

We recall that, given a topological space X , a covering space on X is a continuous map $p: E \rightarrow X$, such that for every $x \in X$, there is an open neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets U_λ , $\lambda \in \Lambda$, and $p|_{U_\lambda}: U_\lambda \rightarrow U$ is a homeomorphism. We recommend the book of P. May [1] and the book of J. Rotman [3] as reference of covering spaces.

An abstract simplicial complex is a pair (S, \mathcal{K}) where S is a set and \mathcal{K} is a family of non-empty finite subsets of S such that, if $\sigma \subseteq \tau$ and $\tau \in \mathcal{K}$ then $\sigma \in \mathcal{K}$. We call complexes to the abstract simplicial complexes. If $\sigma \in \mathcal{K}$, then the dimension of σ is $|\sigma| - 1$, and we denote it by $\dim(\sigma)$. The elements of \mathcal{K} of dimension n are called n -simplices, and we denote the set of n -simplices by \mathcal{K}_n . The 0-simplices are called vertices. An edge e in (S, \mathcal{K}) is a pair of vertices (x, y) where $\{x, y\} \in \mathcal{K}$, x is the origin of the edge e and we denote it by $\text{orig}(e)$, and y is the end of the edge e and we denote it by $\text{end}(e)$. A path α in (S, \mathcal{K}) is a finite sequence of edges e_1, \dots, e_n such that $\text{end}(e_i) = \text{orig}(e_{i+1})$ with $i = 1, \dots, n - 1$. We define $\text{orig}(\alpha) = \text{orig}(e_1)$ and $\text{end}(\alpha) = \text{end}(e_n)$.

A morphism between complex (S_1, \mathcal{K}_1) and (S_2, \mathcal{K}_2) is a map $f: S_1 \rightarrow S_2$ such that $f(\sigma) \in \mathcal{K}_2$ for any $\sigma \in \mathcal{K}_1$. We call a morphism of complexes a simplicial map. We denote by \mathcal{C} to the category of complexes and simplicial maps.

We denote by \mathcal{T} to the category of topological spaces and continuous maps.

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2. GEOMETRIC REALIZATION OF COVERING COMPLEXES

Definition 2.1. The geometric realization of a complex (S, \mathcal{K}) is the set of all function $\phi: S \rightarrow [0, 1]$ such that:

- $\text{supp}(\phi) \in \mathcal{K}$
- $\sum_{s \in S} \phi(s) = 1$

We denote this set by $|S, \mathcal{K}|$. We may think $[0, 1]^S$ as the direct limit of $[0, 1]^A$ where A ranges over all finite subsets of S . So we give the $|S, \mathcal{K}|$ the subspace topology.

Definition 2.2. The geometric realization of an abstract simplicial complex (S, \mathcal{K}) is given by the following formula: first we give a total order to S . Then, for any simplex $\sigma = \{s_0 < s_1 < \dots < s_q\}$ we define $|\sigma| = \Delta^q$, the standard topological q -simplex and we associate to the vertex s_q the q -th vertex of Δ^q . If $\tau = \{s_0 < \dots < s_q\}$ is a simplex and $\sigma = \{s_{q_1} < \dots < s_{q_k}\} \subseteq \tau$, we define $i_\sigma^\tau: |\sigma| \rightarrow |\tau|$ to be the affine function such that maps the j -th vertex of $|\sigma|$ to the q_j -th vertex of $|\tau|$. Thus we take $|S|$ as the colimit over this system. If $f: (S_1, \mathcal{K}_1) \rightarrow (S_2, \mathcal{K}_2)$ it's a morphism of abstract simplicial complexes, then we can define $|f|: |S_1| \rightarrow |S_2|$ as the colimit of the affine functions $|f|_\sigma: |\sigma| \rightarrow |f(\sigma)|$. $|-|: \mathcal{C} \rightarrow \mathcal{T}$ it's a functor.

Definition 2.3. Let (S, \mathcal{K}) be a complex. We say that (S, \mathcal{K}) is connected if for any pair of vertices x, y of (S, \mathcal{K}) there is a path α such that $\text{orig}(\alpha) = x$ and $\text{end}(\alpha) = y$.

The following definition is due J. Rotman in [2].

Definition 2.4. Let (S, \mathcal{K}) be a complex. A covering of (S, \mathcal{K}) is a pair $((T, \mathcal{L}), p)$ where (T, \mathcal{L}) is a complex and $p: T \rightarrow S$ is a morphism of complexes such that:

- (T, \mathcal{L}) is a connected complex.
- For every $\sigma \in \mathcal{L}$, $p^{-1}(\sigma) = \bigcup_{i \in I} \sigma_i$ where the family $\{\sigma_i\}_{i \in I}$ is a pairwise disjoint family of simplices of \mathcal{K} such that $p|_{\sigma_i}: \sigma_i \rightarrow \sigma$ is bijective.

The map p is called projection and the simplices σ_i are called sheets over σ .

Proposition 2.1. Let (S, \mathcal{K}) be an abstract simplicial complex and $((T, \mathcal{L}), p)$ an abstract simplicial covering of (S, \mathcal{K}) . Then $(|T, \mathcal{L}|, |p|)$ is covering of $|S, \mathcal{K}|$.

Proof. □

3. GEOMETRIC REALIZATION OF THE UNIVERSAL COVERING COMPLEX

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