

GEOMETRIC REALIZATION OF COVERING COMPLEXES

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ABSTRACT. We prove that the geometric realization of a covering complex is a covering space. Also, this holds true for the universal covering complex.

INTRODUCTION

The theory of abstract simplicial complexes is a useful tool in the calculation of fundamental groups. This fact appears explicitly in the paper [7] of A. Weil, For any connected abstract simplicial complex S its edge-path group $E(S)$ is naturally isomorphic to the fundamental group of its geometric realization $\pi_1(|S|)$. The edge-path group could be described explicitly by generators and relations. As reference for these facts, see the book of I. Singer and J. Thorpe [5].

We base our definition of covering complex given by J. Rotman in [3], but there are other definitions covering complex as the one in [1]. The definition of J. Rotman

1. PRELIMINARIES

We recall that, given a topological space X , a covering space on X it's a continuous map $p: E \rightarrow X$, such that for every $x \in X$, there is an open neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets U_λ , $\lambda \in \Lambda$, and $p|_{U_\lambda}: U_\lambda \rightarrow U$ it's a homeomorphism. We recommend the book of P. May [2] and the book of J. Rotman [4] as reference of covering spaces.

An abstract simplicial complex is a pair (S, \mathcal{K}) where S is a set and \mathcal{K} is a family of non-empty finite subsets of S such that:

- $\bigcup \mathcal{K} = S$.
- If $\sigma \subseteq \tau$ and $\tau \in \mathcal{K}$ then $\sigma \in \mathcal{K}$.

We call complexes to the abstract simplicial complexes. If $\sigma \in \mathcal{K}$, then the dimension of σ is $|\sigma| - 1$, and we denote it by $\dim(\sigma)$. The elements of \mathcal{K} of dimension n are called n -simplices, and we denote the set of n -simplices by \mathcal{K}_n . The 0-simplices are called vertices. The dimension of (S, \mathcal{K}) is defined as the supremum of $\dim(\sigma)$ where σ ranges over \mathcal{K} , we denote it by $\dim(S, \mathcal{K})$. This dimension may be infinite. We call the complex (S, \mathcal{K}) finite, if its dimension is finite.

An edge e in (S, \mathcal{K}) is a pair of vertices (x, y) where $\{x, y\} \in \mathcal{K}$, x is the origin of the edge e and we denote it by $\text{orig}(e)$, and y is the end of the edge e and we denote by $\text{end}(e)$. A path α in (S, \mathcal{K}) is a finite sequence of edges e_1, \dots, e_n such that $\text{end}(e_i) = \text{orig}(e_{i+1})$ with $i = 1, \dots, n - 1$. We define $\text{orig}(\alpha) = \text{orig}(e_1)$ and $\text{end}(\alpha) = \text{end}(e_n)$.

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A morphism between complex (S_1, \mathcal{K}_1) and (S_2, \mathcal{K}_2) is a map $f: S_1 \rightarrow S_2$ such that $f(\sigma) \in \mathcal{K}_2$ for any $\sigma \in \mathcal{K}_1$. We call a morphism of complexes a simplicial map. We denote by \mathcal{C} to the category of complexes and simplicial maps.

As reference of complexes, we recommend [5] and [6].

We denote by \mathcal{T} to the category of topological spaces and continuous maps.

2. GEOMETRIC REALIZATION OF COVERING COMPLEXES

Definition 2.1. *The geometric realization of a complex (S, \mathcal{K}) is the set of all function $\phi: S \rightarrow [0, 1]$ such that:*

- $\text{supp}(\phi) \in \mathcal{K}$
- $\sum_{s \in S} \phi(s) = 1$

We denote this set by $|(S, \mathcal{K})|$. We may think $[0, 1]^S$ as the direct limit of $[0, 1]^A$ where A ranges over all finite subsets of S . So we give the $|(S, \mathcal{K})|$ the subspace topology.

Definition 2.2. *The geometric realization of an abstract simplicial complex (S, \mathcal{K}) is given by the following formula: first we give a total order to S . Then, for any simplex $\sigma = \{s_0 < s_1 < \dots < s_q\}$ we define $|\sigma| = \Delta^q$, the standar topological q -simplex and we asociate to the vertex s_q the q -th vertex of Δ^q . If $\tau = \{s_0 < \dots < s_q\}$ is a simplex and $\sigma = \{s_{q_1} < \dots < s_{q_k}\} \subseteq \tau$, we define $i_\sigma^\tau: |\sigma| \rightarrow |\tau|$ to be the affine function such that maps the j -th vertex of $|\sigma|$ to the q_j -th vertex of $|\tau|$. Thus we take $|S|$ as the colimit over this system. If $f: (S_1, \mathcal{K}_1) \rightarrow (S_2, \mathcal{K}_2)$ it's a morphism of abstract simplicial complexes, then we can define $|f|: |S_1| \rightarrow |S_2|$ as the colimit of the affine functions $|f|_\sigma: |\sigma| \rightarrow |f(\sigma)|$. $|-|: \mathcal{C} \rightarrow \mathcal{T}$ it's a functor.*

Definition 2.3. *Let (S, \mathcal{K}) be a complex. We say that (S, \mathcal{K}) is connected if for any pair of vertices x, y of (S, \mathcal{K}) there is a path α such that $\text{orig}(\alpha) = x$ and $\text{end}(\alpha) = y$.*

The following definition is due J. Rotman in [3].

Definition 2.4. *Let (S, \mathcal{K}) be a complex. A covering of (S, \mathcal{K}) is a pair $((T, \mathcal{L}), p)$ where (T, \mathcal{L}) is a complex and $p: T \rightarrow S$ is a morphism of complexes such that:*

- (T, \mathcal{L}) is a connected complex.
- For every $\sigma \in \mathcal{K}$, $p^{-1}(\sigma) = \bigcup_{i \in I} \sigma_i$ where the family $\{\sigma_i\}_{i \in I}$ is a pairwise disjoint family of simplices of \mathcal{L} such that $p|_{\sigma_i}: \sigma_i \rightarrow \sigma$ is bijective.

The map p is called projection and the simplices σ_i are called sheets over σ .

We observe that p is surjective and (S, \mathcal{K}) is connected. For our geometric porpoises we need a stronger definition of covering complex.

Definition 2.5. *Let (S, \mathcal{K}) be a complex. A geometric covering of (S, \mathcal{K}) is a covering $((T, \mathcal{L}), p)$ such that for any simplex $\sigma \in \mathcal{L}_n$, $p(\sigma) \in \mathcal{K}_n$. In other words, p preserves the dimension of the simplices.*

Proposition 2.1. *Let (S, \mathcal{K}) be a complex, and $((T, \mathcal{L}), p)$ a covering complex of (S, \mathcal{K}) . If $\sigma \in \mathcal{K}$ is a maximal simplex, then p^{-1} is the disjoint union of maximal simplices.*

Proof. Let $\sigma \in \mathcal{K}$ be a maximal simplex with $p^{-1}(\sigma) = \bigcup_{i \in I} \sigma_i$ where the family $\{\sigma_i\}_{i \in I}$ is a pairwise disjoint family of simplices of \mathcal{K} such that $p|_{\sigma_i}: \sigma_i \rightarrow \sigma$ is bijective. If σ_j is not maximal for some $j \in I$, then there is a simplex $\tau \in \mathcal{L}$ such

that $\sigma_j \subset \tau$. As σ is maximal and $\sigma = p(\sigma_j) \subseteq p(\tau)$, we have that $p(\tau) = \sigma$. This contradicts the fact that p preserves the dimension. Therefore σ_i is a maximal simplex. \square

Proposition 2.2. *Let (S, \mathcal{K}) be an abstract simplicial complex and $((T, \mathcal{L}), p)$ an abstract simplicial covering of (S, \mathcal{K}) . Then $(|(T, \mathcal{L})|, |p|)$ is covering of $|(S, \mathcal{K})|$.*

Proof. \square

3. GEOMETRIC REALIZATION OF THE UNIVERSAL COVERING COMPLEX

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