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Design of LQI Control Systems with Stable Inner Loops

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I. Introduction

AS depicted in Fig. 1, LQI (Linear Quadratic with Integral) controllers are static feedback controllers based upon the augmented states (plant states plus integral of tracking errors) . They stabilize the augmented plants and minimize the quadratic cost functionals associated with step exogenous inputs (step reference inputs and step disturbances)[1]. In the case when not all states are measured, optimal state estimation (Kalman filtering) can be used to realize the LQI controllers.

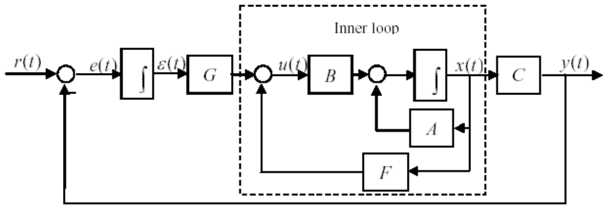

Fig.1 LQI control system

LQI control systems have been applied to a wide area, e.g. dynamic positioning of ocean floating platform[2], electro-dynamic suspension control[3], depth and heading control of unmanned underwater vehicle[4], and attitude/position control of unmanned air vehicle[5].

However, to the best of our knowledge, the problem of designing LQI control systems with stable inner loops remains unresolved. The main reason to stabilize the inner loops is that their instability results in the closed loops of poor dynamic performance. As stated in Ref. 6, for stable unit feedback systems, unstable open-loop poles impose fundamental constraints on achievable performance of closed loops. This is true with LQI control systems since the outer loops are based upon unit feedback and the closed-loop stability is guaranteed by LQI design. Moreover, their open-loop stability is equivalent to that of the inner loops. Therefore, in order to remove the fundamental constraints

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on the performance of the outer loops imposed by the open-loop instability, it is always desirable for LQI controllers to stabilize the inner loops.

The rest of this paper is organized as follows. The problem is formulated in Section II and solved in Section III. Then a case study is presented in Section IV to verify the proposed approach. And finally the paper is concluded in Section V.

II. Problem Formulation

A. A Brief Review of LQI Control

Consider a LTI (linear time-invariant) plant subject to step state and output disturbances

$$\dot{x}(t) = Ax(t) + Bu(t) + d_x \tag{1}$$

$$y(t) = Cx(t) + d_y \tag{2}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ and $y(t) \in \Re^m$ denote plant state, control and output, respectively. $d_x \in \Re^n$ and $d_y \in \Re^m$ are step disturbances with constant but unknown magnitudes, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ and $C \in \Re^{m\times n}$ are matrices of proper dimensions.

Define integral of tracking error under step reference $r(t) = r(t \geq 0)$ as

$$\varepsilon(t) = \int_0^t (r - y(\tau))d\tau \tag{3}$$

From Eq. (2) and (3), one can obtain

$$\dot{\varepsilon}(t) = r - d_y - Cx(t) \tag{4}$$

Combining Eq. (1) and Eq. (4) yields the augmented system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\varepsilon}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \varepsilon(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} d_x \\ r - d_y \end{bmatrix} \tag{5}$$

Or equivalently

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{\tilde{\varepsilon}}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \tilde{\varepsilon}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \tilde{u}(t) \tag{6}$$

where the tildes denote the biased variables which are obtained by subtracting their steady-state values from their temporal values, i.e.

$$\tilde{x}(t) = x(t) - x_\infty, \tilde{\varepsilon}(t) = \varepsilon(t) - \varepsilon_\infty, \tilde{u}(t) = u(t) - u_\infty \tag{7}$$

While ε_∞ can be arbitrary (it can be forced to zero by adding a feed-forward path in the LQI controller, see Ref. 7 for example), x_∞ and u_∞ are determined by Eq. (5) as follows

$$\begin{bmatrix} x_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} -d_x \\ r - d_y \end{bmatrix} \tag{8}$$

The LQI control problem is to find an optimal state feedback controller

$$u(t) = \begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} x(t) \\ \varepsilon(t) \end{bmatrix} \tag{9}$$

Or equivalently

$$\tilde{u}(t) = \begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \tilde{\varepsilon}(t) \end{bmatrix} \tag{10}$$

to stabilize the augmented plant (6) and minimize the quadratic cost functional

$$J = \int_0^\infty [\left(\tilde{x}^T(t) \quad \tilde{\varepsilon}^T(t)\right) Q \left(\tilde{x}^T(t) \quad \tilde{\varepsilon}^T(t)\right)^T + \tilde{u}^T(t) R \tilde{u}(t)] dt \tag{11}$$

with weighting matrices $Q = Q^T \geq 0$ and $R = R^T > 0$.

For simplicity in notation, we define

$$\hat{A} := \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \hat{B} := \begin{bmatrix} B \\ 0 \end{bmatrix}, K := \begin{bmatrix} F & G \end{bmatrix} \tag{12}$$

It is well known that the LQI control problem is solvable if $(\hat{A}, \hat{B})$ is stabilizable and $(Q, \hat{A})$ has no unobservable modes on the imaginary axis. The optimal state feedback gain matrix is

$$K = -R^{-1}\hat{B}^T P \tag{13}$$

where $P = P^T \geq 0$ is solution to the algebraic Riccati equation (ARE)

$$P\hat{A} + \hat{A}^T P - P\hat{B}R^{-1}\hat{B}^T P + Q = 0 \tag{14}$$

The dynamics of the LQI control system can then be described by

$$\dot{\tilde{\eta}}(t) = \hat{A}_{cl}\tilde{\eta}(t) \tag{15}$$

where $\tilde{\eta}(t) = \begin{bmatrix}\tilde{x}^T(t) & \tilde{\varepsilon}^T(t)\end{bmatrix}^T$, and

$$\hat{A}_{cl} = \hat{A} + \hat{B}K = \hat{A} - \hat{B}R^{-1}\hat{B}P \tag{16}$$

B. The Constraints on the Performance of LQI Control Systems

Since the stability of the closed loop system (15) is guaranteed by the LQ design, the LQI control system can be interpreted as a stable unit feedback system with the open-loop system (cf. Fig. 1)

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\varepsilon}(t)\end{bmatrix} = \begin{bmatrix}A+BF & BG\\ 0 & 0\end{bmatrix}\begin{bmatrix}x(t)\\ \varepsilon(t)\end{bmatrix} + \begin{bmatrix}0\\ e(t)\end{bmatrix} + \begin{bmatrix}d_x\\ 0\end{bmatrix} \tag{17}$$

For such systems, unstable poles, non-minimum phase zeros and time delays in open loop plant impose fundamental constraints on achievable performance in control loops and also imply restrictions on the bandwidth of the control loop. More specifically, a stable unit feedback system which has real right-half-plane (RHP) open loop pole(s), must have overshoot in its step response and the amount of the overshoot is related to the rise time and the location of the unstable pole [6].

This suggests the open-loop system in a stable unit feedback system to have no RHP poles to achieve desirable dynamic performance. Since the open loop poles of the LQI control system is composed of nulls (with m multiplicity) and the poles of $(A+BF)$, the dynamic performance of LQI control system can be improved by making $(A+BF)$ stable, i.e. making the inner loop stable (cf. Fig. 1).

Thus our design problem to synthesize an LQI controller which also stabilizes the inner loop of the LQI control system.

III. Main Result

Since there is freedom in choosing the weighting matrices Q and R in LQI control system design, a natural idea is to utilize the freedom to achieve stable inner loops.

Here the design freedom in choosing the weighting matrix Q is utilized to achieve LQI controllers which also stabilize the inner loops. The main result is summarized in Theorem 1.

Theorem 1: For a LTI plant, suppose that $(\hat{A}, \hat{B})$ is stabilizable, $R = R^T > 0$ and $Q = Q^T > 0$ and $Q = diag(Q_{11}, Q_{22})$ with $Q_{11} \in \Re^{n\times n}$ and $Q_{22} \in \Re^{m\times m}$, then the LQI controller exists and the inner loop is stable.

Proof:

Since $Q > 0$, one has $rank(\lambda I - \hat{A}^T) = rank(Q) = n + m$ (λ is any eigenvalue of $\hat{A}$). Therefore $(Q, \hat{A})$ is observable. By taking into account the stabilizability of $(\hat{A}, \hat{B})$ and $R = R^T > 0$, one can find that all the existing conditions of LQI controllers are met [8].

Moreover, the observability of $(Q, \hat{A})$ guarantees the positive definiteness of the solution matrix P to ARE (14), i.e. $P > 0$ [8].

By partitioning P into

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \tag{18}$$

with $P_{11} \in R^{n\times n}$, $P_{22} \in R^{m\times m}$ and $P_{12} \in R^{n\times m}$, the following matrix inequalities can be obtained in terms of Schur complement lemma,

$$P_{22} > 0 \tag{19}$$

$$\overline{P}_{11} := P_{11} - P_{12}P_{22}^{-1}P_{12}^T > 0 \tag{20}$$

Substituting (12) and (18) into ARE (14), it can be expanded as

$$
\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}+\begin{bmatrix} A^T & -C^T \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}+\begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix}
-\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\begin{bmatrix} B \\ 0 \end{bmatrix}R^{-1}\begin{bmatrix} B \\ 0 \end{bmatrix}^T\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}=0 \tag{21}
$$

Pre-multiplying $\begin{bmatrix} I & -P_{12}P_{22}^{-1} \end{bmatrix}$ and aft-multiplying $\begin{bmatrix} I & -P_{12}P_{22}^{-1} \end{bmatrix}^T$ on both sides of Eq. (21) yields

$$
\overline{P}_{11}A + A^T\overline{P}_{11} - \overline{P}_{11}BR^{-1}B^T\overline{P}_{11} + Q_{11} + P_{12}P_{22}^{-1}Q_{22}P_{22}^{-1}P_{12}^T = 0 \tag{22}
$$

where Eq. (20) has been applied.

Similarly, pre-multiplying $\begin{bmatrix} 0 & I \end{bmatrix}$ and aft-multiplying $\begin{bmatrix} 0 & I \end{bmatrix}^T$ on both sides of Eq. (21) yields

$$
Q_{22} = P_{12}^T BR^{-1}B^T P_{12} \tag{23}
$$

Inserting Eq.(23) into Eq. (22) yields

$$
\overline{P}_{11}A + A^T\overline{P}_{11} - \overline{P}_{11}BR^{-1}B^T\overline{P}_{11} + Q_{11} + P_{12}P_{22}^{-1}P_{12}^T BR^{-1}B^T P_{12}P_{22}^{-1}P_{12}^T = 0 \tag{24}
$$

According to (20), it can be rewritten as

$$
\overline{P}_{11}A + A^T\overline{P}_{11} - \overline{P}_{11}BR^{-1}B^T\overline{P}_{11} + Q_{11} + (P_{11} - \overline{P}_{11})BR^{-1}B^T(P_{11} - \overline{P}_{11}) = 0 \tag{25}
$$

or

$$
\overline{P}_{11}(A - BR^{-1}B^T P_{11}) + (A - BR^{-1}B^T P_{11})^T\,\overline{P}_{11} = -Q_{11} - P_{11}BR^{-1}B^T P_{11} = 0 \tag{26}
$$

Due to positive definiteness of Q_{11} and P_{11}, one has

$$
\overline{P}_{11}(A - BR^{-1}B^T P_{11}) + (A - BR^{-1}B^T P_{11})^T\,\overline{P}_{11} < 0 \tag{27}
$$

Since $\overline{P}_{11}$ is positive definite, $(A - BR^{-1}B^T P_{11})$ is stable according to Lyapunov's stability theorem.

By recalling Eqs. (12) and (13), on has

$$
F = -R^{-1}B^T P_{11} \tag{28}
$$

Therefore $(A + BF)$, i.e. the inner loop, is stable and this concludes the proof.

Theorem 1 suggests a simple design approach to guarantee the stability of the inner loop of a LQI control system: to set the state weighting matrix Q to be diagonal block matrices, i.e. to make the quadratic cost functional have the following decoupled form:

$$J = \int_0^{\infty} [\tilde{x}^T(t) Q_{11} \tilde{x}(t) Q + \tilde{\varepsilon}^T(t) Q_{22} \tilde{\varepsilon}(t) + \tilde{u}^T(t) R \tilde{u}(t)]\, dt \tag{29}$$

IV. A Case Study

To verify the effectiveness of the approach proposed in Section III, a case study is presented in this section.

For a given SISO plant with state space realization

$$A = \begin{bmatrix} 0.5413 & 0.0434 & 0.7566 \\ 0.6938 & 0.0901 & 0.6315 \\ 0.0787 & 0.1426 & 0.6173 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$$

the design approach will be utilized to synthesize a LQI control system with stable inner loop.

It can readily be checked that $(\hat{A}, \hat{B})$ is stabilizable (cf. Eq. (12)). Hence the LQI control problem is solvable if the weighting matrices are chosen that $R > 0$ (R is a scalar here), $Q = Q^T \geq 0$ and $(Q, \hat{A})$ has no unobservable modes on the imaginary axis.

For simplicity, the control weighting matrix is selected as $R = 1$. As for the state weighting matrix Q, two different choices are made here for comparison between the proposed approach and the conventional LQI design.

A. Conventional LQI Design

The conventional LQI design does not necessarily generate a control system with a stable inner loop since the state weighting matrix Q does not have to be a form of diagonal block matrix.

Here we choose Q as a non-diagonal block matrix

$$Q = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \end{array}\right]$$

It can also be checked that the existing conditions of LQI controllers are fully met, and the resulting optimal state feedback gain matrices can be obtained by applying the LQI control design: $F = [-3.5495 \quad -4.7734 \quad -19.9028]$ and $G = -1$.

Therefore the eigenvalues of $(A + BF)$, i.e. the inner loop are 0.0495 and $-1.3257 \pm 1.1668i$.

Since the inner loop has a RHP pole, it imposes fundamental constraints on the dynamic performance of the LQI control system. This is can be confirmed by the step response ($r = 1, d_x = d_y = 0$) of the LQI control system (dotted line in Fig .2) which suffers an overshoot over 50% and a long settling time.

B. Proposed LQI Design

As suggested by Theorem 1, we choose the state weighing matrix Q to be a diagonal block matrix

$$Q = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Note that all the off-diagonal elements of Q_{11} are simply set to zero here although they do not have to be set to zero..

Simple calculations reveal that the LQI controller exists. And the optimal state feedback gain matrices are: $F = [-3.9669 \quad -5.7594 \quad -28.2727]$ and $G = -1$.

The eigenvalues of $(A + BF)$ are -0.0703 and $-1.3240 \pm 1.3170i$, therefore the inner loop is stable. And

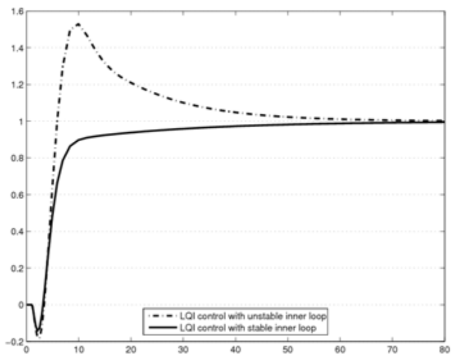

Fig. 2 Step responses of LQI control systems

the constraints on the dynamic performance of LQI control system, which are imposed by the RHP poles of the inner loop (if exist), vanish.

The step response of the LQI control system is plotted in Fig. 2 (solid line), from where one can conclude that the proposed design approach is superior to the conventional design due to significant improvement on the dynamic performance of LQI control system.

V. Conclusions

LQI control systems can be regarded as stable unit feedback systems whose achievable dynamic performance is fundamentally constrained by the unstable poles of the open loops. A simple design approach for synthesis of LQI controllers, which also stabilizes the inner loops, is proposed in this paper. The only difference between the proposed LQI design approach and the conventional LQI design approach lies in the selection of the state weighting matrix, i.e. to make the state weighting matrix have a form of diagonal matrix.

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