Derivative of unilateral quadratic matrix equation

Michel Juillard

August 18, 2023

Given equation

$$AX^2 + BX + C = 0,$$

compute $\frac{\partial X}{\partial z}$ given $\frac{\partial A}{\partial z}$, $\frac{\partial B}{\partial z}$, $\frac{\partial C}{\partial z}$.

Derivative of an implicit function:

$$\frac{\partial A}{\partial z}X^2 + A\frac{\partial X}{\partial z}X + AX\frac{\partial X}{\partial z} + B\frac{\partial X}{\partial z} + \frac{\partial B}{\partial z}X + \frac{\partial C}{\partial z} = 0$$

Can be reorganized as

$$A\frac{\partial X}{\partial z}X + (AX + B)\frac{\partial X}{\partial z} + \frac{\partial A}{\partial z}X^2 + \frac{\partial B}{\partial z}X + \frac{\partial C}{\partial z} = 0$$

or

$$D\frac{\partial X}{\partial z}E + F\frac{\partial X}{\partial z} + G = 0$$

This can be solved with GeneralizedSylvesterSolver.jl. Function `generalized_sylvester_solver!(a,b,c,d,1,ws)` solves

$$ax + bxc = d$$

where

$$\begin{aligned} a &= AX + B \\ b &= A \\ c &= X \\ d &= \frac{\partial A}{\partial z}X^2 + \frac{\partial B}{\partial z}X + \frac{\partial C}{\partial z} \end{aligned}$$

and `ws = GeneralizedSylvesterWs(n, n, n, 1)`.

The solution for the unknown matrix $x = \frac{\partial X}{\partial z}$ is returned into `d`. `generalized_sylvester_solver` transforms the problem as

$$x + a^{-1}bxc = a^{-1}d,$$

vectorizes it:

$$(I + c' \otimes (a^{-1}b)x = vec(a^{-1}d)$$

and solves for x with a special algorithm that never forms the Kronecker product explicitly.