Derivatives of the solution of the first order approximation of a DSGE model

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The first order approximation of a DSGE model is computed around the deterministic steady state. Computing the derivatives of this approximation with respect to the model parameters requires to compute

1. the derivatives of the steady state with respect to the parameters.
2. the derivatives of the coefficients of the linear approximation with respect to the parameters
3. the derivatives of solution the Unilateral Quadratic Matrix Equation (UQME) used to obtain the solution of a linear rational expectation model
4. the derivatives of the coeficients of response to shocks

We consider the general model

$$\mathbb{E}_t\left\{f(y_{t+1}, y_t, y_{t-1}, \epsilon_t; \theta)\right\} = 0$$

where

$$\begin{aligned} \mathbb{E}_t\left\{\epsilon_t\right\} &= 0 \\ \mathbb{E}_t\left\{\epsilon_t \epsilon_t'\right\} &= \Sigma_\epsilon \\ \mathbb{E}_t\left\{\epsilon_t \epsilon_\tau'\right\} &= 0 \quad t \neq \tau \end{aligned}$$

y_t is a vector of n endogenous variables and ϵ_t, a vector of m exogenous shocks.

1 Derivatives of the steady state

The steady state $\bar{y}$ satisfies

$$f(\bar{y}, \bar{y}, \bar{y}, 0; \theta) = 0$$

and

$$\frac{\partial \bar{y}}{\partial \theta} = -\left(\frac{\partial f}{\partial y_{t+1}} + \frac{\partial f}{\partial y_t} + \frac{\partial f}{\partial y_{t-1}}\right)^{-1} \frac{\partial f}{\partial \theta}$$

2 Derivatives of the coefficents of the linear approximation of the model

The first order approximation of the model is written

$$A\mathbb{E}_t\hat{y}_{t+1} + B\hat{y}_t + C\hat{y}_{t-1} + Du_t = 0$$

where $\hat{y}_t = y_t - \bar{y}$ and

$$\begin{aligned}
A &= \left.\frac{\partial f}{\partial y_{t+1}}\right|_{\bar{y}} \\
B &= \left.\frac{\partial f}{\partial y_t}\right|_{\bar{y}} \\
C &= \left.\frac{\partial f}{\partial y_{t-1}}\right|_{\bar{y}} \\
D &= \left.\frac{\partial f}{\partial u_t}\right|_{\bar{y}}
\end{aligned}$$

Then,

$$\begin{aligned}
A &= \frac{\partial^2 f}{\partial y_{t+1}\partial\theta} + \left(\frac{\partial^2 f}{\partial y_{t+1}\partial y_{t+1}} + \frac{\partial^2 f}{\partial y_t\partial y_{t+1}} + \frac{\partial^2 f}{\partial y_{t-1}\partial y_{t+1}}\right)\left(e_n \otimes \frac{\partial\bar{y}}{\partial\theta}\right) \\
B &= \frac{\partial^2 f}{\partial y_t\partial\theta} + \left(\frac{\partial^2 f}{\partial y_{t+1}\partial y_t} + \frac{\partial^2 f}{\partial y_t\partial y_t} + \frac{\partial^2 f}{\partial y_{t-1}\partial y_t}\right)\left(e_n \otimes \frac{\partial\bar{y}}{\partial\theta}\right) \\
C &= \frac{\partial^2 f}{\partial y_{t-1}\partial\theta} + \left(\frac{\partial^2 f}{\partial y_{t+1}\partial y_{t-1}} + \frac{\partial^2 f}{\partial y_t\partial y_{t-1}} + \frac{\partial^2 f}{\partial y_{t-1}\partial y_{t-1}}\right)\left((e_n \otimes \frac{\partial\bar{y}}{\partial\theta}\right) \\
D &= \frac{\partial^2 f}{\partial u_t\partial\theta} + \left(\frac{\partial^2 f}{\partial y_{t+1}\partial u_t} + \frac{\partial^2 f}{\partial y_t\partial u_t} + \frac{\partial^2 f}{\partial y_{t-1}\partial u_t}\right)\left(e_n \otimes \frac{\partial\bar{y}}{\partial\theta}\right)
\end{aligned}$$

where e_n is a 1-vector of size n.

3 Derivatives of the solution of the UQME

We assume that the solution of the first order approximation of the model take the form

$$\hat{y}_t = G\hat{y}_{t-1} + Hu_t$$

Matrix H must satisfies the UQME

$$AG^2 + BG + C = 0$$

4 Derivatives of the coefficients of response to the shocks

Given matrix G, matrix H must satisfy

$$AGH + BH + D = 0$$

and

$$H = -(AG + B)^{-1} D$$

Then

$$\frac{\partial H}{\partial \theta_i} = -(AG + B) \left(\frac{\partial A}{\partial \theta_i} GH + A \frac{\partial G}{\partial \theta_i} H + \frac{\partial B}{\partial \theta_i} H + D \right)$$