

2017-2018 学年第一学期《微积分 I-1》期末试卷参考答案 (A 卷)

一、1. 解: 令  $y' = 1 - \frac{2}{1+x} = \frac{x-1}{1+x} = 0$ , 求得驻点  $x=1$

当  $-1 < x < 1$  时,  $y' < 0$ ; 当  $x > 1$  时,  $y' > 0$ ,

因此函数  $y = x - 2\ln(1+x)$  在  $(-1, 1)$  上单调减少, 在  $(1, +\infty)$  上单调增加.

故函数  $y = x - 2\ln(1+x)$  在  $x=1$  处取得极小值, 极小值为  $1 - 2\ln 2$ .

$$2. \text{ 解: } \lim_{x \rightarrow 0} \frac{\int_0^{x^2} e^{-t^2} dt}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{e^{-x^4} \cdot 2x}{2 \sin x \cos x} = 1$$

$$3. \text{ 解一: } \int \frac{1-2x}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-x^2}} dx + \int \frac{-2x}{\sqrt{1-x^2}} dx \\ = \arcsin x + 2\sqrt{1-x^2} + C$$

$$\text{解二: 令 } x = \sin t, \text{ 则 } \int \frac{1-2x}{\sqrt{1-x^2}} dx = \int (1-2\sin t) dt \\ = t + 2\cos t + C = \arcsin x + 2\sqrt{1-x^2} + C$$

$$4. \text{ 解: } \int x \arctan x dx = \frac{1}{2} x^2 \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx \\ = \frac{1}{2} x^2 \arctan x - \frac{1}{2} \int (1 - \frac{1}{1+x^2}) dx \\ = \frac{1}{2} (x^2 + 1) \arctan x - \frac{x}{2} + C$$

$$5. \text{ 解: } \lim_{n \rightarrow \infty} \left( \frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+2n}} + \cdots + \frac{1}{\sqrt{n^2+n^2}} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left( \frac{1}{\sqrt{1+\frac{1}{n}}} + \frac{1}{\sqrt{1+\frac{2}{n}}} + \cdots + \frac{1}{\sqrt{1+\frac{n}{n}}} \right)$$

$$= \int_0^1 \frac{1}{\sqrt{1+x}} dx = 2\sqrt{1+x} \Big|_0^1 = 2\sqrt{2} - 2$$

$$6. \text{ 解: } \int e^{-x} f(x) dx = \int e^{-x} d(e^x \sin x) = e^{-x} (e^x \sin x) + \int \sin x dx \\ = \sin x - \cos x + C$$

$$7. \text{ 解: } s = 4 \int_0^{\frac{\pi}{2}} \sqrt{(-3a \cos^2 t \sin t)^2 + (3a \sin^2 t \cos t)^2} dt = 12a \int_0^{\frac{\pi}{2}} \sin t \cos t dt \\ = 6a \sin^2 t \Big|_0^{\frac{\pi}{2}} = 6a$$

$$\text{二、1. 解一: 令 } x = \sin t, \text{ 则 } \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{(1+x)\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin t} dt$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1 - \sin t}{\cos^2 t} dt = \left( \tan t + \frac{1}{\cos t} \right) \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 2.$$

解二：令  $x = \sin t$ ，则  $\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{(1+x)\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin t} dt$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{(\sin \frac{t}{2} + \cos \frac{t}{2})^2} dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2\cos^2(\frac{\pi}{4} - \frac{t}{2})} dt$$

$$= -\tan\left(\frac{\pi}{4} - \frac{t}{2}\right) \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \tan \frac{3\pi}{8} - \tan \frac{\pi}{8} = 2.$$

解三：令  $x = \sin t$ ，则  $\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{(1+x)\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin t} dt$

令  $\tan \frac{t}{2} = u$ ，则  $\int \frac{1}{1+\sin t} dt = 2 \int \frac{1}{(1+u)^2} du = -\frac{2}{1+u} + C = -\frac{2}{1+\tan \frac{t}{2}} + C$ ，

故 原式  $= -2\left(\frac{1}{1+\tan \frac{\pi}{8}} - \frac{1}{1-\tan \frac{\pi}{8}}\right) = \frac{4 \tan \frac{\pi}{8}}{1 - \tan^2 \frac{\pi}{8}} = 2$ 。

2. 解：令  $t = x^2$ ，则  $\int_0^{+\infty} x^3 e^{-x^2} dx = \frac{1}{2} \int_0^{+\infty} t e^{-t} dt = \frac{1}{2} (-te^{-t}) \Big|_0^{+\infty} + \int_0^{+\infty} e^{-t} dt$

$$= \frac{1}{2} (-e^{-t}) \Big|_0^{+\infty} = \frac{1}{2}.$$

3. 解一：  $\frac{3\sin x + 4\cos x}{2\sin x + \cos x} = 2 + \frac{-\sin x + 2\cos x}{2\sin x + \cos x}$ ，

则  $\int \frac{3\sin x + 4\cos x}{2\sin x + \cos x} dx = \int \left(2 + \frac{2\cos x - \sin x}{2\sin x + \cos x}\right) dx = 2x + \ln|2\sin x + \cos x| + C$ 。

解二：令  $\tan \frac{x}{2} = t$ ，则

$$\int \frac{3\sin x + 4\cos x}{2\sin x + \cos x} dx = 4 \int \frac{2+3t-2t^2}{(1+t^2)(1+4t-t^2)} dt$$

$$= \int \left( \frac{-2t+4}{1+t^2} + \frac{4-2t}{1+4t-t^2} \right) dt \quad (\text{或} = \int \left( \frac{-2t+4}{1+t^2} + \frac{1}{\sqrt{5}-2+t} - \frac{1}{\sqrt{5}+2-t} \right) dt)$$

$$= 4 \arctan t + \ln \left| \frac{1+4t-t^2}{1+t^2} \right| + C = 2x + \ln|2\sin x + \cos x| + C.$$

解三：令  $\tan x = t$ ，则

$$\begin{aligned}
\int \frac{3 \sin x + 4 \cos x}{2 \sin x + \cos x} dx &= \int \frac{3 \tan x + 4}{2 \tan x + 1} dx = \int \frac{3t + 4}{(2t + 1)(1 + t^2)} dt \\
&= \int \left( \frac{2}{2t + 1} + \frac{-t + 2}{1 + t^2} \right) dt \\
&= \ln |2t + 1| - \frac{1}{2} \ln(1 + t^2) + 2 \arctan t + C \\
&= 2 \arctan t + \ln \left| \frac{2t + 1}{1 + t^2} \right| + C \\
&= 2x + \ln |2 \sin x + \cos x| + C
\end{aligned}$$

4. 解: 记  $A = \int_0^{\frac{\pi}{2}} f(x) \sin x dx$ , 则

$$A = \int_0^{\frac{\pi}{2}} f(x) \sin x dx = \int_0^{\frac{\pi}{2}} \sin^4 x dx + 2A \cdot \int_0^{\frac{\pi}{2}} \sin x dx,$$

$$\text{故 } A = \int_0^{\frac{\pi}{2}} f(x) \sin x dx = -\int_0^{\frac{\pi}{2}} \sin^4 x dx = -\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = -\frac{3}{16} \pi.$$

$$\text{其中 } \int_0^{\frac{\pi}{2}} \sin^4 x dx = \frac{1}{4} \int_0^{\frac{\pi}{2}} (1 - \cos 2x)^2 dx = \frac{1}{4} \int_0^{\frac{\pi}{2}} (1 - 2 \cos 2x + \frac{1 + \cos 4x}{2}) dx = \frac{3}{16} \pi,$$

$$\text{因此, } f(x) = \sin^3 x - \frac{3}{8} \pi.$$

$$\begin{aligned}
5. \text{ 解: } \int_{-1}^1 (1 + x + \sqrt{1 - x^2})^2 dx &= \int_{-1}^1 (1 + x^2 + 1 - x^2 + 2x + 2\sqrt{1 - x^2} + 2x\sqrt{1 - x^2}) dx \\
&= \int_{-1}^1 (2 + 2x + 2\sqrt{1 - x^2} + 2x\sqrt{1 - x^2}) dx \\
&= 2 \int_{-1}^1 dx + 2 \int_{-1}^1 \sqrt{1 - x^2} dx \\
&= 4 + \pi.
\end{aligned}$$

$$\text{三、解: } A = \int_0^2 (x + 2 - x^2) dx = \left( \frac{1}{2} x^2 + 2x - \frac{1}{3} x^3 \right) \Big|_0^2 = 2 + 4 - \frac{8}{3} = \frac{10}{3};$$

$$B = \pi \int_0^2 [(x + 2)^2 - x^4] dx = \frac{\pi}{3} (x + 2)^3 \Big|_0^2 - \frac{\pi}{5} x^5 \Big|_0^2 = \frac{\pi}{3} (64 - 8) - \frac{32}{5} \pi = \frac{184}{15} \pi.$$

$$\text{四、证明一: 作辅助函数 } F(t) = \int_a^t x f(x) dx - \frac{a+t}{2} \int_a^t f(x) dx, t \in [a, b].$$

$$\text{因为 } F'(t) = t f(t) - \frac{a+t}{2} f(t) - \frac{1}{2} \int_a^t f(x) dx = \frac{1}{2} \int_a^t [f(t) - f(x)] dx$$

因为  $f(x)$  为单调增加函数, 故当  $t > a$  时,  $F'(t) > 0$ , 即  $F(x)$  为  $[a, b]$  上单调增加函数。

$$\text{所以, } F(b) = \int_a^b x f(x) dx - \frac{a+b}{2} \int_a^b f(x) dx > F(a) = 0, \text{ 即 } \int_a^b x f(x) dx > \frac{a+b}{2} \int_a^b f(x) dx.$$

证明二：因为  $f(x)$  为单调增加函数，故

$$\begin{aligned} & \int_a^b xf(x)dx - \frac{a+b}{2} \int_a^b f(x)dx = \int_a^b (x - \frac{a+b}{2})f(x)dx \\ &= f(a) \int_a^\eta (x - \frac{a+b}{2})dx + f(b) \int_\eta^b (x - \frac{a+b}{2})dx \quad (\text{由积分第二中值定理, } a < \eta < b) \\ &= f(a) [\frac{1}{2}(\eta - \frac{a+b}{2})^2 - \frac{(b-a)^2}{8}] + f(b) [\frac{(b-a)^2}{8} - \frac{1}{2}(\eta - \frac{a+b}{2})^2] \\ &= \frac{1}{2} [f(b) - f(a)](\eta - a)(\eta - b) > 0. \end{aligned}$$

五、解：(1) 令  $t = a + b - x$ ，则  $\int_a^b f(x)dx = -\int_b^a f(a + b - t)dt = \int_a^b f(a + b - x)dx$ 。

$$(2) \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x}{x(\pi - 2x)} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2(\frac{\pi}{2} - x)}{(\frac{\pi}{2} - x)[\pi - 2(\frac{\pi}{2} - x)]} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2 x}{(\pi - 2x)x} dx,$$

故 
$$\begin{aligned} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x}{x(\pi - 2x)} dx &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x}{x(\pi - 2x)} dx + \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2 x}{(\pi - 2x)x} dx = \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{x(\pi - 2x)} dx \\ &= \frac{1}{\pi} \left[ \frac{1}{2} \ln x - \frac{1}{2} \ln(\pi - 2x) \right] \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left( \frac{1}{2x} + \frac{1}{\pi - 2x} \right) dx \\ &= \frac{1}{\pi} \left[ \frac{1}{2} \ln 2 + \frac{1}{2} \ln 2 \right] = \frac{\ln 2}{\pi}. \end{aligned}$$