Lawvere Metric Spaces and Quantales

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Lawvere, F.W.: Metric spaces, generalized logic, and closed categories. Rendiconti del seminario matematico e fisico di Milano **43**, 135–166 (1973), **reprints in TAC, No. 1, 1-37 (2002)**

Dagnino, F., Farjudian, A., Moggi, E.: Robustness in metric spaces over continuous quantales and the Hausdorff-Smyth monad, **in ICTAC (2023)**

Qualitative vs Quantitative

- equal/different (objects) vs how much different
- near/distant (points) vs how much distant
- faster/slower (program) vs how much faster

$$\begin{tikzcd} \text{quantitative} \arrow[d, "\textit{abstraction}" description] \\ \text{qualitative} \end{tikzcd} \qquad \begin{tikzcd} \text{metric space} \arrow[d, "\text{open ball topology}"] \\ \text{topological space} \arrow[d, "\text{specialization preorder}"] \\ \text{preorder} \end{tikzcd}$$

Question: What quantities should one use?

Summary

Goal

Present some mathematical tools for quantitative analyses

Some Uses

- general framework to define the notion of robustness [DFM2023]
- measure program differences (Ugo Dal Lago)
- measure incompleteness of abstract interpretations (Roberto Giacobazzi)

1. metric space (X, d), topological space (X, τ), open ball topology τ_d
2. categories and categories enriched over an ordered monoid (more generally over a monoidal category)
3. the ordered monoid $\mathbb{R}_+$ and Lawvere metric spaces
4. quantales: definition, examples, uses

Bottom-up Approach

From concrete examples to more abstract/general mathematical notions

D (X, d) metric space $\stackrel{\triangle}{\iff}$ X set and $d\colon X^2 \to [0, \infty)$ metric, i.e.

1. $d(x, y) + d(y, z) \geq d(x, z)$ triangular inequality
2. $0 \geq d(x, x)$ identity or equivalently
   $0 = d(x, x)$
3. $d(x, y) = d(y, x)$ **symmetry**
4. $0 \geq d(x, y) \wedge 0 \geq d(y, x) \implies x = y$ **separation** or equivalently
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D open ball topology $\tau_d \subseteq \mathrm{P}(X)$ for metric space (X, d) generated by **open balls** $B(x, \delta) \stackrel{\triangle}{=} \{y | d(x, y) < \delta\}$ with $x \in X$ and $\delta > 0$

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P τ_d is T_2, i.e., $x \neq y \iff \exists O_x, O_y \in \tau_d . x \in O_x \wedge y \in O_y \wedge O_x \# O_y$

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D specialization preorder $x \leq_\tau y \overset{\triangle}{\iff} \forall O \in \tau . x \in O \implies y \in O$ for topological space (X, τ). The preorder $\leq_{\tau_d}$ is equality on X.

Categories and V-enriched Categories [Kelly1982]

Top-down Approach by Lawvere

Derive mathematical notions as instances of more abstract notions

- get definitions/theorems by *auto-pilot*!
- does one get the *same outcome* of the bottom-up approach?

A (locally small) category C consists of

- a class C of *objects*
- a *hom-set* $C(a, b)$ of *arrows* for each $a, b \in C$
- an arrow $\mathrm{id}_a \in C(a, a)$ for each $a \in C$
- a map $\circ_{a,b,c}\colon C(a, b) \times C(b, c) \to C(a, c)$ for each $a, b, c \in C$

such that $h \circ (g \circ f) = (h \circ g) \circ f$ and $\mathrm{id} \circ f = f = f \circ \mathrm{id}$ when $f \in C(a, b)$, $g \in C(b, c)$, $h \in C(c, d)$, we write $g \circ f$ for $\circ(f, g)$.

V-enrichment

Given a *monoidal category* $(V, \otimes, \mathrm{u}, \ldots)$, a V-enriched category C is a category where the set $C(a, b)$ is replaced by an object in V.

Categories and V-enriched Categories [Kelly1982]

Preorders are categories whose hom-sets have at most one element.

V-enrichment

Given an **ordered monoid** $(V, \sqsubseteq, \otimes, \mathrm{u})$, i.e., $\otimes\colon V^2 \to V$ monotonic and $\mathrm{u} \in V$ such that $x \otimes (y \otimes z) = (x \otimes y) \otimes z$ and $\mathrm{u} \otimes x = x = x \otimes \mathrm{u}$

a (small) V-enriched category C (V-**category** for short) consists of

- a set C of *objects*
- an object $C(a, b)$ in V of *arrows* for each $a, b \in C$, such that
- $\mathrm{u} \sqsubseteq C(a, a)$ for each $a \in C$ and
- $C(a, b) \otimes C(b, c) \sqsubseteq C(a, c)$ for each $a, b, c \in C$.

When V is an preorder, the underlying category C_0 is a preorder, since $C_0(a, b) = V(\mathrm{u}, C(a, b))$ has at most one element.

Lawvere Metric Spaces [Lawvere1973]

The category $\mathcal{M}et$ of Lawvere metric spaces is the *2-category* of $\mathbb{R}_+$-categories, where $\mathbb{R}_+$ is the ordered monoid $([0,\infty], \geq, +, 0)$, i.e.

obj (X, d) with X set and $d\colon X^2 \to [0,\infty]$ such that
$0 \geq d(x,x)$ and $d(x,y) + d(y,z) \geq d(x,y)$

arr $f\colon (X,d) \to (X',d')$ $\mathbb{R}_+$-functor (aka **short map**), i.e.,
$f\colon X \to X'$ such that $d(x,y) \geq d'(fx, fy)$

nat $f \to f'\colon (X,d) \to (X',d')$ $\mathbb{R}_+$-*nat. transf.*, i.e., $0 \geq d'(fx, f'x)$.

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1. $\mathbb{R}_+$ complete lattice: $\bot = \infty$, $\top = 0$, $\bigvee_i q_i = \inf_i q_i$, $\bigwedge_i q_i = \sup_i q_i$
   $\mathcal{Met}$ has small products and small coproducts
   $\prod_{i:I}(X_i, d_i) = (\prod_{i:I} X_i, d)$ with $d(x,y) = \bigwedge_i d_i(x_i, y_i)$
   $\coprod_{i:I}(X_i, d_i) = (\coprod_{i:I} X_i, d)$ with $d(x_i, y_j) = d_i(x,y)$ if $i = j$ else $\bot$

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2. $\mathbb{R}_+$ is commutative, i.e., $p \otimes q = q \otimes p$
   $\mathcal{M}et$ has $\otimes_{i:n}(X_i, d_i) = (\prod_{i:n} X_i, d)$ with $d(x,y) = \otimes_i d_i(x_i, y_i)$
   if $(X,d) \in \mathcal{M}et$, then $(X, d^o), (X, d^s) \in \mathcal{M}et$, where
   $d^o(x,y) = d(y,x)$ dual of d and
   $d^s(x,y) = d(x,y) \wedge d(y,x)$ symmetrization of d

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2. $\mathbb{R}_+$ is commutative, i.e., $p \otimes q = q \otimes p$
3. $\mathbb{R}_+$ is *closed*, i.e., exists $[p,q] = q-p$ if $q \geq p$ else 0 such that
$x \otimes p \sqsubseteq q \iff x \sqsubseteq [p,q]$ ($[p,q]$ is like an implication $p \implies q$),
$\mathbb{R}$ is $\mathbb{R}_+$-enrichable $(\mathbb{R},d_\mathbb{R}) \in \mathcal{M}et$, where $d_\mathbb{R}(x,y) = [x,y]$

From $\mathbb{R}_+$-metric spaces to Q-metric spaces.

Quantales [Mul86]

An ordered monoid $(Q, \sqsubseteq, \otimes, \mathrm{u})$ is a **quantale** $\iff$ $(Q, \sqsubseteq)$ complete and **distributivity** holds, i.e., $p \otimes \bigvee_i q_i = \bigvee_i p \otimes q_i$ & $(\bigvee_i p_i) \otimes q = \bigvee_i (p_i \otimes q)$
distributivity is equivalent to require Q *bi-closed*

- Q **commutative** $\iff$ $p \otimes q = q \otimes p$ Q **affine** $\iff$ $\mathrm{u} = \top$
- Q **locale** $\iff$ $p \otimes q = p \wedge q$ ($\implies$ Q commutative & affine)
- Q **linear** $\iff$ $(Q, \sqsubseteq)$ linear order

Examples: variations on $\mathbb{R}_+$

- $\mathbb{R}_+$ linear, commutative, affine quantale
- $\mathbb{R}_\wedge = ([0, \infty], \geq, \max, 0)$ locale: ultra-metric spaces are $\mathbb{R}_\wedge$-categories
- $\mathbb{N}_+ = (\{0, 1, \ldots, \infty\}, \geq, +, 0)$: size of data structures
- $\Sigma = (\{0, \infty\}, \geq, +, 0)$ locale: preorders are Σ-categories

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Examples: quantale constructions

- the product $\prod_{i:I} Q_i$ of quantales is a quantale
- the set Q^P of monotonic maps from a poset P to a quantale Q is a sub-quantale of $\prod_{p:P} Q$
- Q/u affine sub-quantale of Q with carrier $\{q{:}\, Q \mid q \sqsubseteq \mathrm{u}\}$
- $(P(X^2), \subseteq, \otimes)$ quantale of binary relations on X
- $(D(V), \subseteq, \otimes)$ quantale of downwards closed subsets of $(V, \sqsubseteq, \otimes)$.

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Examples: quantales for data sizes/cost analyses

- $\mathbb{N}_+ = (\{0, 1, \ldots, \infty\}, \geq, +, 0)$: size of data, number of step
- $\mathbb{N}_+^\omega$: time complexities T of programs, $T(n)$ upper bound on the number of steps to compute result for inputs of size at most $n \in \omega$
- $O(\mathbb{N}_+^\omega)$: ordered monoid of O-classes for time complexity, i.e., replace T with $O(T) = \{T' | \exists m, C. \forall n > m. T'(n) \leq C * T(n)\}$

Use different quantales for different data/cost analyses.

Q-metrics on $P(X)$ - [Goubault-Larrecq2008]

If Q is a quantale and (X, d) is Q-**metric space** (i.e., a Q-category), then one can define the following Q-metrics on the powerset $P(X)$

- $d_{HH}(A, B) = \bigwedge_{x:A} \bigvee_{y:B} d(x, y)$ Hausdorff-Hoare
- $d_{HS}(A, B) = \bigwedge_{y:B} \bigvee_{x:A} d(x, y)$ Hausdorff-Smyth
- $d_H(A, B) = d_{HH}(A, B) \wedge d_{HS}(A, B)$ Hausdorff

P $d_{HH}(\{x\}, \{y\}) = d(x, y) = d_{HS}(\{x\}, \{y\})$

P if $A \subseteq B$, then $\mathrm{u} \sqsubseteq d_{HH}(A, B)$ and $\mathrm{u} \sqsubseteq d_{HS}(B, A)$

P if $\emptyset \subset A$, then $d_{HH}(A, \emptyset) = \bot = d_{HS}(\emptyset, A)$

Problems with ordinary metric (when Q is $\mathbb{R}_+$)

- d_{HH} and d_{HS} are never ordinary metrics (they are not symmetric)
- if (X, d) is an ordinary metric space, then d_H is an ordinary metric only when it is restricted to a subset of $P(X)$, e.g., the set of *compact non-empty* subsets of X.

1. Transforming Q-metric spaces into Q'-metric spaces using *lax-monoidal maps*, i.e., monotonic maps $h\colon Q \to Q'$ such that $\mathrm{u}' \sqsubseteq' h(\mathrm{u})$ and $h(p) \otimes' h(q) \sqsubseteq' h(p \otimes q)$

$$\begin{tikzcd} Q \arrow[r, dashed, shift left=2] \arrow[r, phantom, "\top"] & Q/\mathrm{u} \arrow[l, hook', shift left=2] \arrow[r, dashed, shift left=2] \arrow[r, phantom, "\top"] & \Sigma \arrow[l, hook', shift left=2] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{N}_+ \arrow[r, hook, shift right=2] \arrow[r, phantom, "\top"] & \mathbb{R}_+ \arrow[l, dashed, shift right=2] \end{tikzcd}$$

More Results (see [DFM2023])

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2. Transforming Q-metric spaces (X, d) into topological spaces on X, when Q is a *continuous* quantale (use the **way-below** relation $\ll$)
   - τ_d generated by open balls $B(x, \delta) = \{y | \delta \ll d(x, y)\}$
   - τ_d^o generated by dual open balls $B^o(x, \delta) = \{y | \delta \ll d(y, x)\}$

   where $\delta \ll \mathrm{u}$ (in $\mathbb{R}_+$ the relation $\ll$ is $>$ and in $\mathbb{N}_+$ is $\geq$).