5

Transient and Steady-State Response Analyses

5–1 INTRODUCTION

In early chapters it was stated that the first step in analyzing a control system was to derive a mathematical model of the system. Once such a model is obtained, various methods are available for the analysis of system performance.

In practice, the input signal to a control system is not known ahead of time but is random in nature, and the instantaneous input cannot be expressed analytically. Only in some special cases is the input signal known in advance and expressible analytically or by curves, such as in the case of the automatic control of cutting tools.

In analyzing and designing control systems, we must have a basis of comparison of performance of various control systems. This basis may be set up by specifying particular test input signals and by comparing the responses of various systems to these input signals.

Many design criteria are based on the response to such test signals or on the response of systems to changes in initial conditions (without any test signals). The use of test signals can be justified because of a correlation existing between the response characteristics of a system to a typical test input signal and the capability of the system to cope with actual input signals.

Typical Test Signals. The commonly used test input signals are step functions, ramp functions, acceleration functions, impulse functions, sinusoidal functions, and white noise. In this chapter we use test signals such as step, ramp, acceleration and impulse signals. With these test signals, mathematical and experimental analyses of control systems can be carried out easily, since the signals are very simple functions of time.

Which of these typical input signals to use for analyzing system characteristics may be determined by the form of the input that the system will be subjected to most frequently under normal operation. If the inputs to a control system are gradually changing functions of time, then a ramp function of time may be a good test signal. Similarly, if a system is subjected to sudden disturbances, a step function of time may be a good test signal; and for a system subjected to shock inputs, an impulse function may be best. Once a control system is designed on the basis of test signals, the performance of the system in response to actual inputs is generally satisfactory. The use of such test signals enables one to compare the performance of many systems on the same basis.

Transient Response and Steady-State Response. The time response of a control system consists of two parts: the transient response and the steady-state response. By transient response, we mean that which goes from the initial state to the final state. By steady-state response, we mean the manner in which the system output behaves as t approaches infinity. Thus the system response $c(t)$ may be written as

$$c(t) = c_{\mathrm{tr}}(t) + c_{\mathrm{ss}}(t)$$

where the first term on the right-hand side of the equation is the transient response and the second term is the steady-state response.

Absolute Stability, Relative Stability, and Steady-State Error. In designing a control system, we must be able to predict the dynamic behavior of the system from a knowledge of the components. The most important characteristic of the dynamic behavior of a control system is absolute stability—that is, whether the system is stable or unstable. A control system is in equilibrium if, in the absence of any disturbance or input, the output stays in the same state. A linear time-invariant control system is stable if the output eventually comes back to its equilibrium state when the system is subjected to an initial condition. A linear time-invariant control system is critically stable if oscillations of the output continue forever. It is unstable if the output diverges without bound from its equilibrium state when the system is subjected to an initial condition. Actually, the output of a physical system may increase to a certain extent but may be limited by mechanical "stops," or the system may break down or become nonlinear after the output exceeds a certain magnitude so that the linear differential equations no longer apply.

Important system behavior (other than absolute stability) to which we must give careful consideration includes relative stability and steady-state error. Since a physical control system involves energy storage, the output of the system, when subjected to an input, cannot follow the input immediately but exhibits a transient response before a steady state can be reached. The transient response of a practical control system often exhibits damped oscillations before reaching a steady state. If the output of a system at steady state does not exactly agree with the input, the system is said to have steady-state error. This error is indicative of the accuracy of the system. In analyzing a control system, we must examine transient-response behavior and steady-state behavior.

Outline of the Chapter. This chapter is concerned with system responses to aperiodic signals (such as step, ramp, acceleration, and impulse functions of time). The outline of the chapter is as follows: Section 5–1 has presented introductory material for the chapter. Section 5–2 treats the response of first-order systems to aperiodic inputs. Section 5–3 deals with the transient response of the second-order systems. Detailed analyses of the step response, ramp response, and impulse response of the second-order systems are presented. Section 5–4 discusses the transient-response analysis of higher-order systems. Section 5–5 gives an introduction to the MATLAB approach to the solution of transient-response problems. Section 5–6 gives an example of a transient-response problem solved with MATLAB. Section 5–7 presents Routh's stability criterion. Section 5–8 discusses effects of integral and derivative control actions on system performance. Finally, Section 5–9 treats steady-state errors in unity-feedback control systems.

5–2 FIRST-ORDER SYSTEMS

Consider the first-order system shown in Figure 5–1(a). Physically, this system may represent an RC circuit, thermal system, or the like. A simplified block diagram is shown in Figure 5–1(b). The input-output relationship is given by

$$\frac{C(s)}{R(s)} = \frac{1}{Ts + 1} \tag{5–1}$$

In the following, we shall analyze the system responses to such inputs as the unit-step, unit-ramp, and unit-impulse functions. The initial conditions are assumed to be zero.

Note that all systems having the same transfer function will exhibit the same output in response to the same input. For any given physical system, the mathematical response can be given a physical interpretation.

Unit-Step Response of First-Order Systems. Since the Laplace transform of the unit-step function is $1/s$, substituting $R(s) = 1/s$ into Equation (5–1), we obtain

$$C(s) = \frac{1}{Ts + 1}\frac{1}{s}$$

Expanding $C(s)$ into partial fractions gives

$$C(s) = \frac{1}{s} - \frac{T}{Ts + 1} = \frac{1}{s} - \frac{1}{s + (1/T)} \tag{5–2}$$

Taking the inverse Laplace transform of Equation (5–2), we obtain

$$c(t) = 1 - e^{-t/T}, \qquad \text{for } t \geq 0 \tag{5–3}$$

Equation (5–3) states that initially the output $c(t)$ is zero and finally it becomes unity. One important characteristic of such an exponential response curve $c(t)$ is that at $t = T$ the value of $c(t)$ is 0.632, or the response $c(t)$ has reached 63.2% of its total change. This may be easily seen by substituting $t = T$ in $c(t)$. That is,

$$c(T) = 1 - e^{-1} = 0.632$$

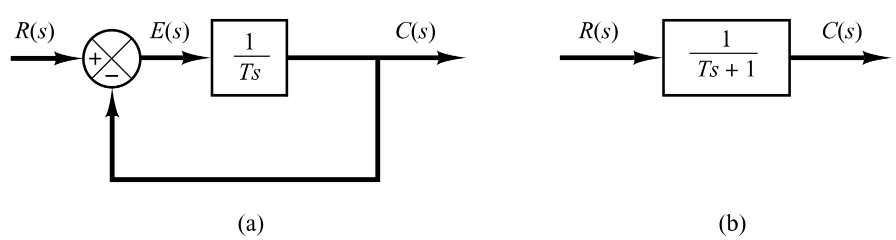


Figure 5–1
(a) Block diagram of a first-order system; (b) simplified block diagram.

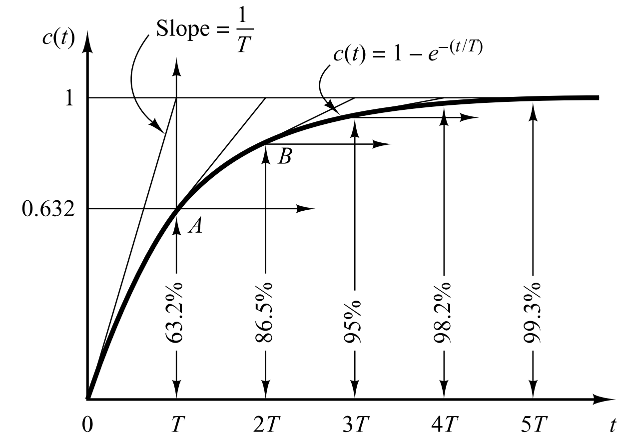


Figure 5–2 Exponential response curve.

Note that the smaller the time constant T, the faster the system response. Another important characteristic of the exponential response curve is that the slope of the tangent line at $t = 0$ is $1/T$, since

$$\left.\frac{dc}{dt}\right|_{t=0} = \left.\frac{1}{T}e^{-t/T}\right|_{t=0} = \frac{1}{T} \tag{5–4}$$

The output would reach the final value at $t = T$ if it maintained its initial speed of response. From Equation (5–4) we see that the slope of the response curve $c(t)$ decreases monotonically from $1/T$ at $t = 0$ to zero at $t = \infty$.

The exponential response curve $c(t)$ given by Equation (5–3) is shown in Figure 5–2. In one time constant, the exponential response curve has gone from 0 to 63.2% of the final value. In two time constants, the response reaches 86.5% of the final value. At $t = 3T, 4T$, and $5T$, the response reaches 95%, 98.2%, and 99.3%, respectively, of the final value. Thus, for $t \geq 4T$, the response remains within 2% of the final value. As seen from Equation (5–3), the steady state is reached mathematically only after an infinite time. In practice, however, a reasonable estimate of the response time is the length of time the response curve needs to reach and stay within the 2% line of the final value, or four time constants.

Unit-Ramp Response of First-Order Systems. Since the Laplace transform of the unit-ramp function is $1/s^2$, we obtain the output of the system of Figure 5–1(a) as

$$C(s) = \frac{1}{Ts + 1}\frac{1}{s^2}$$

Expanding $C(s)$ into partial fractions gives

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts + 1} \tag{5–5}$$

Taking the inverse Laplace transform of Equation (5–5), we obtain

$$c(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0 \tag{5–6}$$

The error signal $e(t)$ is then

$$\begin{aligned} e(t) &= r(t) - c(t) \\ &= T\left(1 - e^{-t/T}\right) \end{aligned}$$

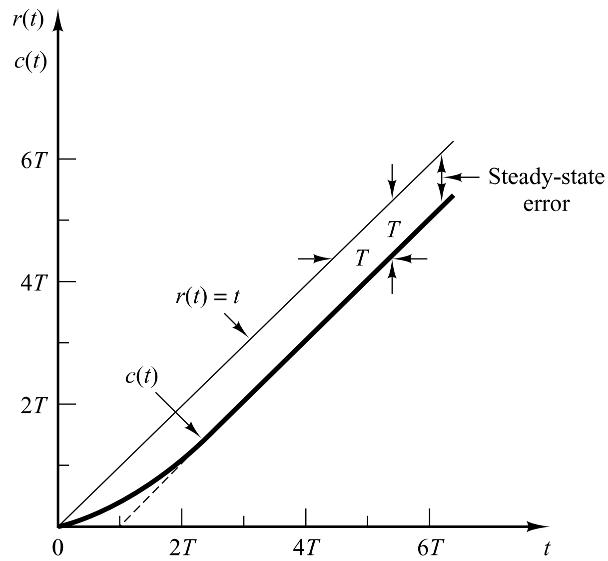


Figure 5–3
Unit-ramp response of the system shown in Figure 5–1(a).

As t approaches infinity, $e^{-t/T}$ approaches zero, and thus the error signal $e(t)$ approaches T or

$$e(\infty) = T$$

The unit-ramp input and the system output are shown in Figure 5–3. The error in following the unit-ramp input is equal to T for sufficiently large t. The smaller the time constant T, the smaller the steady-state error in following the ramp input.

Unit-Impulse Response of First-Order Systems. For the unit-impulse input, $R(s) = 1$ and the output of the system of Figure 5–1(a) can be obtained as

$$C(s) = \frac{1}{Ts + 1} \tag{5–7}$$

The inverse Laplace transform of Equation (5–7) gives

$$c(t) = \frac{1}{T} e^{-t/T}, \qquad \text{for } t \geq 0 \tag{5–8}$$

The response curve given by Equation (5–8) is shown in Figure 5–4.

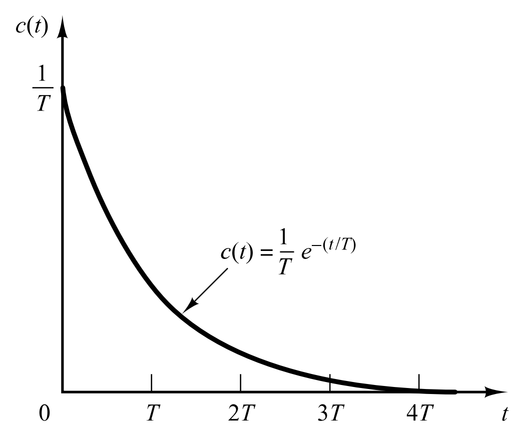


Figure 5–4
Unit-impulse response of the system shown in Figure 5–1(a).

An Important Property of Linear Time-Invariant Systems. In the analysis above, it has been shown that for the unit-ramp input the output $c(t)$ is

$$c(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0 \qquad \text{[See Equation (5–6).]}$$

For the unit-step input, which is the derivative of unit-ramp input, the output $c(t)$ is

$$c(t) = 1 - e^{-t/T}, \qquad \text{for } t \geq 0 \qquad \text{[See Equation (5–3).]}$$

Finally, for the unit-impulse input, which is the derivative of unit-step input, the output $c(t)$ is

$$c(t) = \frac{1}{T}e^{-t/T}, \qquad \text{for } t \geq 0 \qquad \text{[See Equation (5–8).]}$$

Comparing the system responses to these three inputs clearly indicates that the response to the derivative of an input signal can be obtained by differentiating the response of the system to the original signal. It can also be seen that the response to the integral of the original signal can be obtained by integrating the response of the system to the original signal and by determining the integration constant from the zero-output initial condition. This is a property of linear time-invariant systems. Linear time-varying systems and nonlinear systems do not possess this property.

5–3 SECOND-ORDER SYSTEMS

In this section, we shall obtain the response of a typical second-order control system to a step input, ramp input, and impulse input. Here we consider a servo system as an example of a second-order system.

Servo System. The servo system shown in Figure 5–5(a) consists of a proportional controller and load elements (inertia and viscous-friction elements). Suppose that we wish to control the output position c in accordance with the input position r.

The equation for the load elements is

$$J\ddot{c} + B\dot{c} = T$$

where T is the torque produced by the proportional controller whose gain is K. By taking Laplace transforms of both sides of this last equation, assuming the zero initial conditions, we obtain

$$Js^2C(s) + BsC(s) = T(s)$$

So the transfer function between $C(s)$ and $T(s)$ is

$$\frac{C(s)}{T(s)} = \frac{1}{s(Js + B)}$$

By using this transfer function, Figure 5–5(a) can be redrawn as in Figure 5–5(b), which can be modified to that shown in Figure 5–5(c). The closed-loop transfer function is then obtained as

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K} = \frac{K/J}{s^2 + (B/J)s + (K/J)}$$

Such a system where the closed-loop transfer function possesses two poles is called a second-order system. (Some second-order systems may involve one or two zeros.)

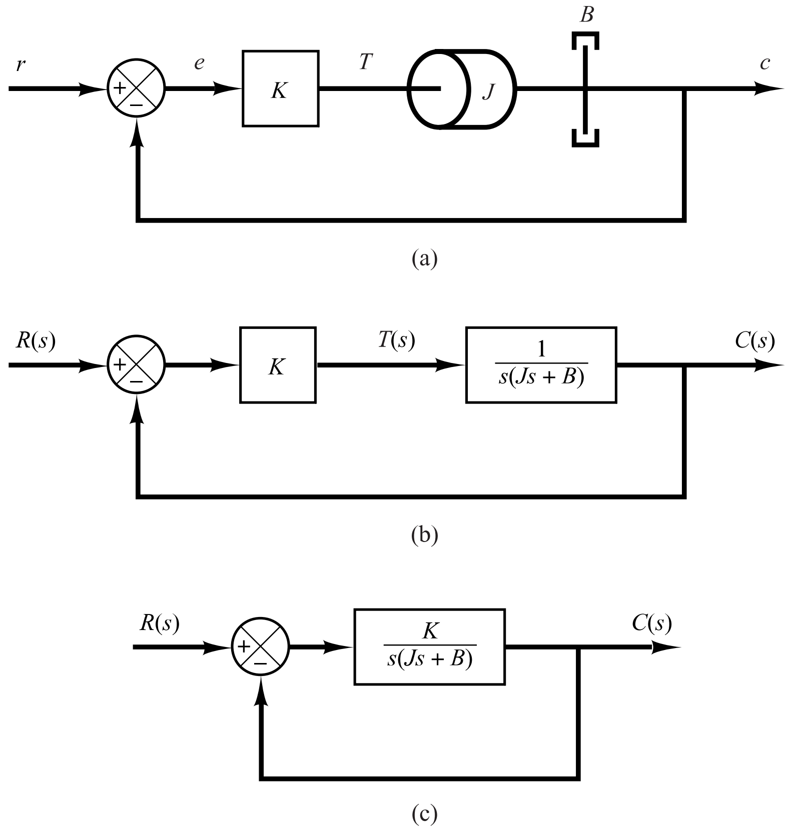


Figure 5–5
(a) Servo system;
(b) block diagram;
(c) simplified block diagram.

Step Response of Second-Order System. The closed-loop transfer function of the system shown in Figure 5–5(c) is

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K} \tag{5–9}$$

which can be rewritten as

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{J}}{\left[s + \dfrac{B}{2J} + \sqrt{\left(\dfrac{B}{2J}\right)^2 - \dfrac{K}{J}}\right]\left[s + \dfrac{B}{2J} - \sqrt{\left(\dfrac{B}{2J}\right)^2 - \dfrac{K}{J}}\right]}$$

The closed-loop poles are complex conjugates if $B^2 - 4JK < 0$ and they are real if $B^2 - 4JK \geq 0$. In the transient-response analysis, it is convenient to write

$$\frac{K}{J} = \omega_n^2, \qquad \frac{B}{J} = 2\zeta\omega_n = 2\sigma$$

where σ is called the *attenuation*; ω_n, the *undamped natural frequency*; and ζ, the *damping ratio* of the system. The damping ratio ζ is the ratio of the actual damping B to the critical damping $B_c = 2\sqrt{JK}$ or

$$\zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

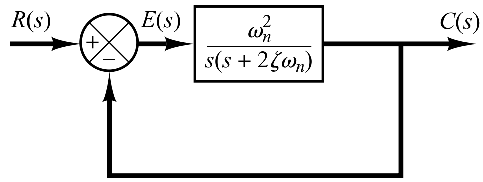


Figure 5–6
Second-order system.

In terms of ζ and ω_n, the system shown in Figure 5–5(c) can be modified to that shown in Figure 5–6, and the closed-loop transfer function $C(s)/R(s)$ given by Equation (5–9) can be written

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5–10}$$

This form is called the *standard form* of the second-order system.

The dynamic behavior of the second-order system can then be described in terms of two parameters ζ and ω_n. If $0 < \zeta < 1$, the closed-loop poles are complex conjugates and lie in the left-half s plane. The system is then called underdamped, and the transient response is oscillatory. If $\zeta = 0$, the transient response does not die out. If $\zeta = 1$, the system is called critically damped. Overdamped systems correspond to $\zeta > 1$.

We shall now solve for the response of the system shown in Figure 5–6 to a unit-step input. We shall consider three different cases: the underdamped $(0 < \zeta < 1)$, critically damped $(\zeta = 1)$, and overdamped $(\zeta > 1)$ cases.

(1) *Underdamped case* $(0 < \zeta < 1)$: In this case, $C(s)/R(s)$ can be written

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$. The frequency ω_d is called the *damped natural frequency*. For a unit-step input, $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s} \tag{5–11}$$

The inverse Laplace transform of Equation (5–11) can be obtained easily if $C(s)$ is written in the following form:

$$\begin{aligned} C(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \end{aligned}$$

Referring to the Laplace transform table in Appendix A, it can be shown that

$$\mathcal{L}^{-1}\left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t}\cos\omega_d t$$

$$\mathcal{L}^{-1}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t}\sin\omega_d t$$

Hence the inverse Laplace transform of Equation (5–11) is obtained as

$$\begin{aligned}\mathcal{L}^{-1}[C(s)] &= c(t) \\ &= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) \\ &= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \qquad \text{for } t \geq 0 \end{aligned} \tag{5–12}$$

From Equation (5–12), it can be seen that the frequency of transient oscillation is the damped natural frequency ω_d and thus varies with the damping ratio ζ. The error signal for this system is the difference between the input and output and is

$$\begin{aligned}e(t) &= r(t) - c(t) \\ &= e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right), \qquad \text{for } t \geq 0\end{aligned}$$

This error signal exhibits a damped sinusoidal oscillation. At steady state, or at $t = \infty$, no error exists between the input and output.

If the damping ratio ζ is equal to zero, the response becomes undamped and oscillations continue indefinitely. The response $c(t)$ for the zero damping case may be obtained by substituting $\zeta = 0$ in Equation (5–12), yielding

$$c(t) = 1 - \cos\omega_n t, \qquad \text{for } t \geq 0 \tag{5–13}$$

Thus, from Equation (5–13), we see that ω_n represents the undamped natural frequency of the system. That is, ω_n is that frequency at which the system output would oscillate if the damping were decreased to zero. If the linear system has any amount of damping, the undamped natural frequency cannot be observed experimentally. The frequency that may be observed is the damped natural frequency ω_d, which is equal to $\omega_n\sqrt{1-\zeta^2}$. This frequency is always lower than the undamped natural frequency. An increase in ζ would reduce the damped natural frequency ω_d. If ζ is increased beyond unity, the response becomes overdamped and will not oscillate.

(2) *Critically damped case* ($\zeta = 1$): If the two poles of $C(s)/R(s)$ are equal, the system is said to be a critically damped one.

For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s+\omega_n)^2 s} \tag{5–14}$$

The inverse Laplace transform of Equation (5–14) may be found as

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t), \qquad \text{for } t \geq 0 \tag{5–15}$$

This result can also be obtained by letting ζ approach unity in Equation (5–12) and by using the following limit:

$$\lim_{\zeta\to 1}\frac{\sin\omega_d t}{\sqrt{1-\zeta^2}} = \lim_{\zeta\to 1}\frac{\sin\omega_n\sqrt{1-\zeta^2}\,t}{\sqrt{1-\zeta^2}} = \omega_n t$$

(3) *Overdamped case* ($\zeta > 1$): In this case, the two poles of $C(s)/R(s)$ are negative real and unequal. For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{\left(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)\left(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)s} \tag{5–16}$$

The inverse Laplace transform of Equation (5–16) is

$$\begin{aligned} c(t) &= 1 + \frac{1}{2\sqrt{\zeta^2 - 1}\left(\zeta + \sqrt{\zeta^2 - 1}\right)} e^{-\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} \\ &\quad - \frac{1}{2\sqrt{\zeta^2 - 1}\left(\zeta - \sqrt{\zeta^2 - 1}\right)} e^{-\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} \\ &= 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right), \qquad \text{for } t \geq 0 \end{aligned} \tag{5–17}$$

where $s_1 = \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$ and $s_2 = \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$. Thus, the response $c(t)$ includes two decaying exponential terms.

When ζ is appreciably greater than unity, one of the two decaying exponentials decreases much faster than the other, so the faster-decaying exponential term (which corresponds to a smaller time constant) may be neglected. That is, if $-s_2$ is located very much closer to the $j\omega$ axis than $-s_1$ (which means $|s_2| \ll |s_1|$), then for an approximate solution we may neglect $-s_1$. This is permissible because the effect of $-s_1$ on the response is much smaller than that of $-s_2$, since the term involving s_1 in Equation (5–17) decays much faster than the term involving s_2. Once the faster-decaying exponential term has disappeared, the response is similar to that of a first-order system, and $C(s)/R(s)$ may be approximated by

$$\frac{C(s)}{R(s)} = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} = \frac{s_2}{s + s_2}$$

This approximate form is a direct consequence of the fact that the initial values and final values of both the original $C(s)/R(s)$ and the approximate one agree with each other.

With the approximate transfer function $C(s)/R(s)$, the unit-step response can be obtained as

$$C(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{\left(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)s}$$

The time response $c(t)$ is then

$$c(t) = 1 - e^{-\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}, \qquad \text{for } t \geq 0$$

This gives an approximate unit-step response when one of the poles of $C(s)/R(s)$ can be neglected.

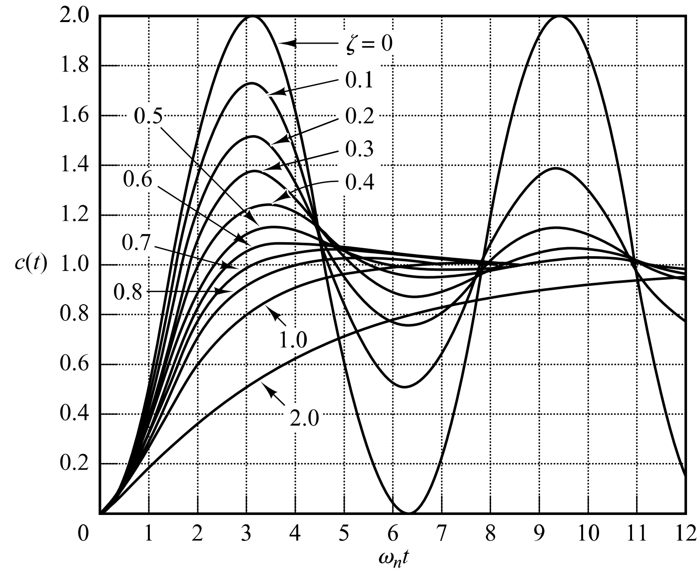


Figure 5–7
Unit-step response curves of the system shown in Figure 5–6.

A family of unit-step response curves $c(t)$ with various values of ζ is shown in Figure 5–7, where the abscissa is the dimensionless variable $\omega_n t$. The curves are functions only of ζ. These curves are obtained from Equations (5–12), (5–15), and (5–17). The system described by these equations was initially at rest.

Note that two second-order systems having the same ζ but different ω_n will exhibit the same overshoot and the same oscillatory pattern. Such systems are said to have the same relative stability.

From Figure 5–7, we see that an underdamped system with ζ between 0.5 and 0.8 gets close to the final value more rapidly than a critically damped or overdamped system. Among the systems responding without oscillation, a critically damped system exhibits the fastest response. An overdamped system is always sluggish in responding to any inputs.

It is important to note that, for second-order systems whose closed-loop transfer functions are different from that given by Equation (5–10), the step-response curves may look quite different from those shown in Figure 5–7.

Definitions of Transient-Response Specifications. Frequently, the performance characteristics of a control system are specified in terms of the transient response to a unit-step input, since it is easy to generate and is sufficiently drastic. (If the response to a step input is known, it is mathematically possible to compute the response to any input.)

The transient response of a system to a unit-step input depends on the initial conditions. For convenience in comparing transient responses of various systems, it is a common practice to use the standard initial condition that the system is at rest initially with the output and all time derivatives thereof zero. Then the response characteristics of many systems can be easily compared.

The transient response of a practical control system often exhibits damped oscillations before reaching steady state. In specifying the transient-response characteristics of a control system to a unit-step input, it is common to specify the following:

1. Delay time, t_d
2. Rise time, t_r

3. Peak time, t_p
4. Maximum overshoot, M_p
5. Settling time, t_s

These specifications are defined in what follows and are shown graphically in Figure 5–8.

1. Delay time, t_d: The delay time is the time required for the response to reach half the final value the very first time.
2. Rise time, t_r: The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value. For underdamped second-order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.
3. Peak time, t_p: The peak time is the time required for the response to reach the first peak of the overshoot.
4. Maximum (percent) overshoot, M_p: The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

   The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.
5. Settling time, t_s: The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system. Which percentage error criterion to use may be determined from the objectives of the system design in question.

The time-domain specifications just given are quite important, since most control systems are time-domain systems; that is, they must exhibit acceptable time responses. (This means that, the control system must be modified until the transient response is satisfactory.)

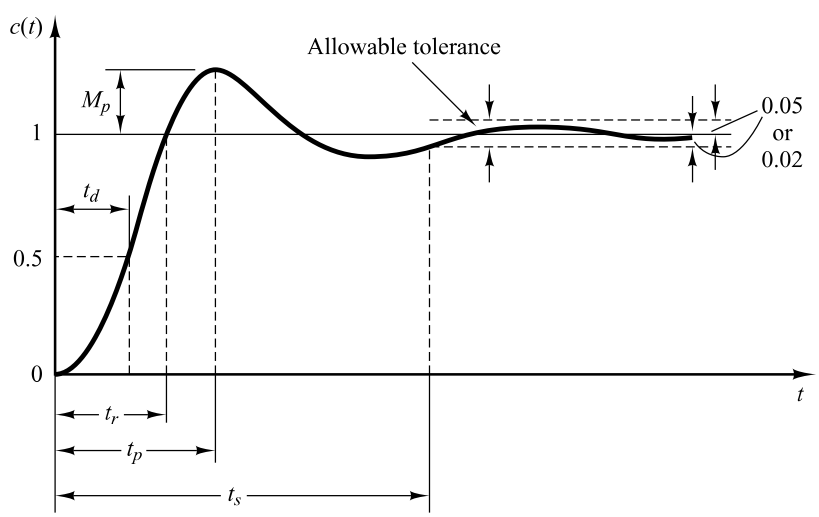


Figure 5–8
Unit-step response curve showing t_d, t_r, t_p, M_p, and t_s.

Note that not all these specifications necessarily apply to any given case. For example, for an overdamped system, the terms peak time and maximum overshoot do not apply. (For systems that yield steady-state errors for step inputs, this error must be kept within a specified percentage level. Detailed discussions of steady-state errors are postponed until Section 5–8.)

A Few Comments on Transient-Response Specifications. Except for certain applications where oscillations cannot be tolerated, it is desirable that the transient response be sufficiently fast and be sufficiently damped. Thus, for a desirable transient response of a second-order system, the damping ratio must be between 0.4 and 0.8. Small values of ζ (that is, $\zeta < 0.4$) yield excessive overshoot in the transient response, and a system with a large value of ζ (that is, $\zeta > 0.8$) responds sluggishly.

We shall see later that the maximum overshoot and the rise time conflict with each other. In other words, both the maximum overshoot and the rise time cannot be made smaller simultaneously. If one of them is made smaller, the other necessarily becomes larger.

Second-Order Systems and Transient-Response Specifications. In the following, we shall obtain the rise time, peak time, maximum overshoot, and settling time of the second-order system given by Equation (5–10). These values will be obtained in terms of ζ and ω_n. The system is assumed to be underdamped.

Rise time t_r: Referring to Equation (5–12), we obtain the rise time t_r by letting $c(t_r) = 1$.

$$c(t_r) = 1 = 1 - e^{-\zeta\omega_n t_r}\left(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right) \tag{5–18}$$

Since $e^{-\zeta\omega_n t_r} \neq 0$, we obtain from Equation (5–18) the following equation:

$$\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r = 0$$

Since $\omega_n\sqrt{1-\zeta^2} = \omega_d$ and $\zeta\omega_n = \sigma$, we have

$$\tan\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$$

Thus, the rise time t_r is

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi - \beta}{\omega_d} \tag{5–19}$$

where angle β is defined in Figure 5–9. Clearly, for a small value of t_r, ω_d must be large.

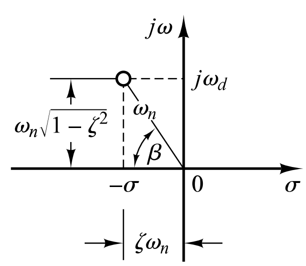


Figure 5–9 Definition of the angle β.

Peak time t_p: Referring to Equation (5–12), we may obtain the peak time by differentiating $c(t)$ with respect to time and letting this derivative equal zero. Since

$$\frac{dc}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$
$$+ e^{-\zeta\omega_n t}\left(\omega_d \sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right)$$

and the cosine terms in this last equation cancel each other, dc/dt, evaluated at $t = t_p$, can be simplified to

$$\left.\frac{dc}{dt}\right|_{t=t_p} = (\sin\omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p} = 0$$

This last equation yields the following equation:

$$\sin\omega_d t_p = 0$$

or

$$\omega_d t_p = 0, \pi, 2\pi, 3\pi, \ldots$$

Since the peak time corresponds to the first peak overshoot, $\omega_d t_p = \pi$. Hence

$$t_p = \frac{\pi}{\omega_d} \tag{5–20}$$

The peak time t_p corresponds to one-half cycle of the frequency of damped oscillation.

Maximum overshoot M_p: The maximum overshoot occurs at the peak time or at $t = t_p = \pi/\omega_d$. Assuming that the final value of the output is unity, M_p is obtained from Equation (5–12) as

$$\begin{aligned} M_p &= c(t_p) - 1 \\ &= -e^{-\zeta\omega_n(\pi/\omega_d)}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right) \\ &= e^{-(\sigma/\omega_d)\pi} = e^{-(\zeta/\sqrt{1-\zeta^2})\pi} \end{aligned} \tag{5–21}$$

The maximum percent overshoot is $e^{-(\sigma/\omega_d)\pi} \times 100\%$.

If the final value $c(\infty)$ of the output is not unity, then we need to use the following equation:

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

Settling time t_s: For an underdamped second-order system, the transient response is obtained from Equation (5–12) as

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \qquad \text{for } t \geq 0$$

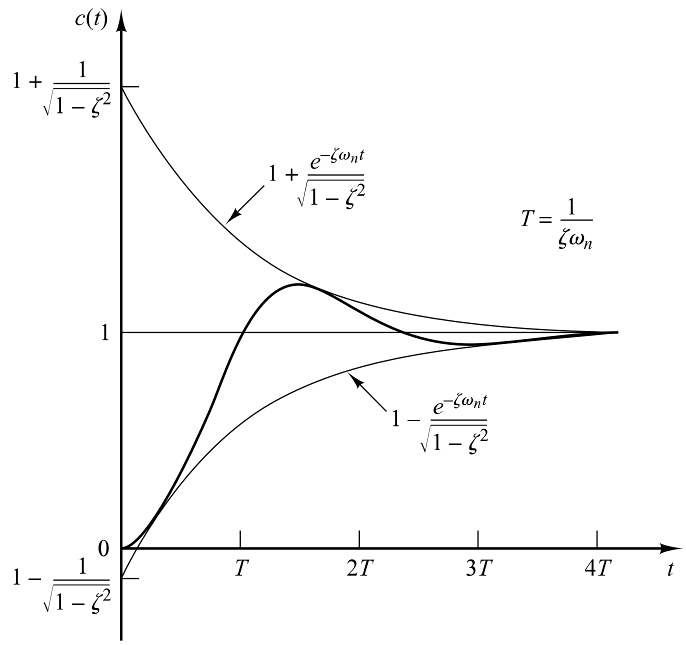


Figure 5–10
Pair of envelope curves for the unit-step response curve of the system shown in Figure 5–6.

The curves $1 \pm \left(e^{-\zeta\omega_n t}/\sqrt{1-\zeta^2}\right)$ are the envelope curves of the transient response to a unit-step input. The response curve $c(t)$ always remains within a pair of the envelope curves, as shown in Figure 5–10. The time constant of these envelope curves is $1/\zeta\omega_n$.

The speed of decay of the transient response depends on the value of the time constant $1/\zeta\omega_n$. For a given ω_n, the settling time t_s is a function of the damping ratio ζ. From Figure 5–7, we see that for the same ω_n and for a range of ζ between 0 and 1 the settling time t_s for a very lightly damped system is larger than that for a properly damped system. For an overdamped system, the settling time t_s becomes large because of the sluggish response.

The settling time corresponding to a $\pm 2\%$ or $\pm 5\%$ tolerance band may be measured in terms of the time constant $T = 1/\zeta\omega_n$ from the curves of Figure 5–7 for different values of ζ. The results are shown in Figure 5–11. For $0 < \zeta < 0.9$, if the 2% criterion is used, t_s is approximately four times the time constant of the system. If the 5% criterion is used, then t_s is approximately three times the time constant. Note that the settling time reaches a minimum value around $\zeta = 0.76$ (for the 2% criterion) or $\zeta = 0.68$ (for the 5% criterion) and then increases almost linearly for large values of ζ. The discontinuities in the curves of Figure 5–11 arise because an infinitesimal change in the value of ζ can cause a finite change in the settling time.

For convenience in comparing the responses of systems, we commonly define the settling time t_s to be

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \qquad \text{(2\% criterion)} \qquad (5\text{–}22)$$

or

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \qquad \text{(5\% criterion)} \qquad (5\text{–}23)$$

Note that the settling time is inversely proportional to the product of the damping ratio and the undamped natural frequency of the system. Since the value of ζ is usually determined from the requirement of permissible maximum overshoot, the settling time

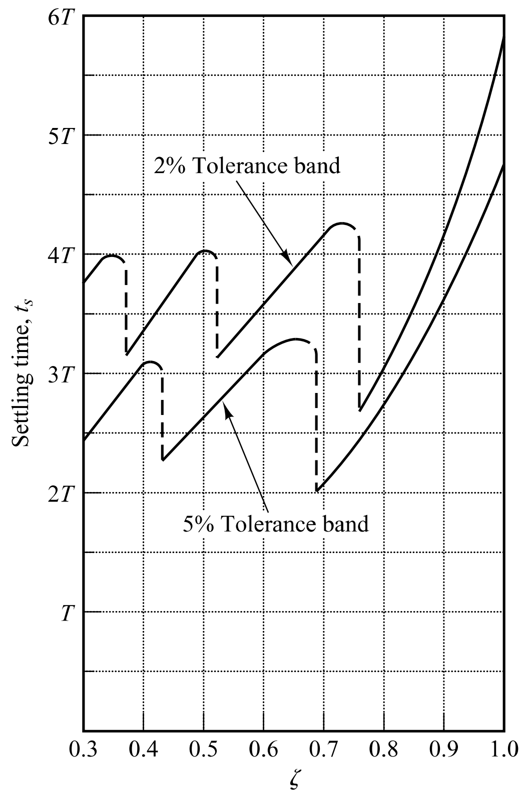


Figure 5–11
Settling time t_s versus ζ curves.

is determined primarily by the undamped natural frequency ω_n. This means that the duration of the transient period may be varied, without changing the maximum overshoot, by adjusting the undamped natural frequency ω_n.

From the preceding analysis, it is evident that for rapid response ω_n must be large. To limit the maximum overshoot M_p and to make the settling time small, the damping ratio ζ should not be too small. The relationship between the maximum percent overshoot M_p and the damping ratio ζ is presented in Figure 5–12. Note that if the damping ratio is between 0.4 and 0.7, then the maximum percent overshoot for step response is between 25% and 4%.

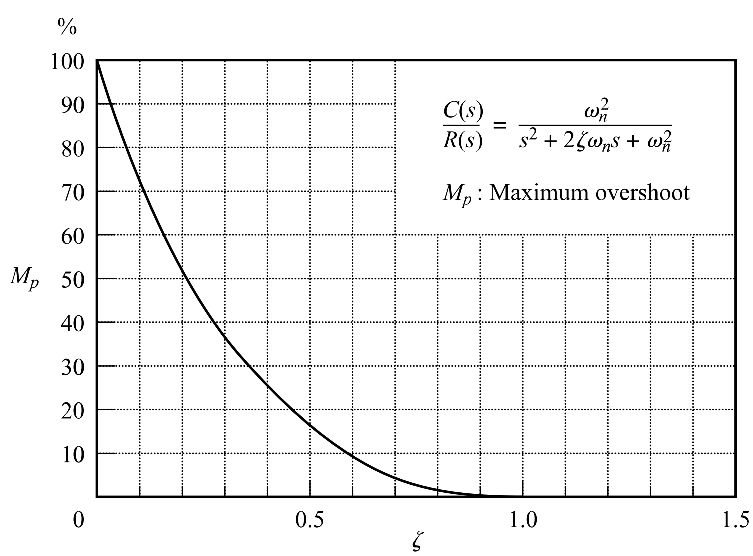


Figure 5–12
M_p versus ζ curve.

It is important to note that the equations for obtaining the rise time, peak time, maximum overshoot, and settling time are valid only for the standard second-order system defined by Equation (5–10). If the second-order system involves a zero or two zeros, the shape of the unit-step response curve will be quite different from those shown in Figure 5–7.

EXAMPLE 5–1 Consider the system shown in Figure 5–6, where $\zeta = 0.6$ and $\omega_n = 5$ rad/sec. Let us obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time t_s when the system is subjected to a unit-step input.

From the given values of ζ and ω_n, we obtain $\omega_d = \omega_n\sqrt{1 - \zeta^2} = 4$ and $\sigma = \zeta\omega_n = 3$.

Rise time t_r: The rise time is

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{4}$$

where β is given by

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{4}{3} = 0.93 \text{ rad}$$

The rise time t_r is thus

$$t_r = \frac{3.14 - 0.93}{4} = 0.55 \text{ sec}$$

Peak time t_p: The peak time is

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \text{ sec}$$

Maximum overshoot M_p: The maximum overshoot is

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\times 3.14} = 0.095$$

The maximum percent overshoot is thus 9.5%.

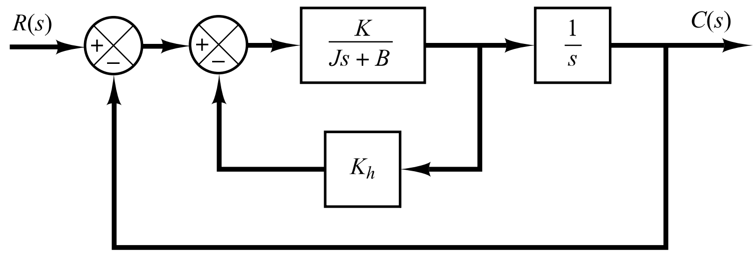
Settling time t_s: For the 2% criterion, the settling time is

$$t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33 \text{ sec}$$

For the 5% criterion,

$$t_s = \frac{3}{\sigma} = \frac{3}{3} = 1 \text{ sec}$$

Servo System with Velocity Feedback. The derivative of the output signal can be used to improve system performance. In obtaining the derivative of the output position signal, it is desirable to use a tachometer instead of physically differentiating the output signal. (Note that the differentiation amplifies noise effects. In fact, if discontinuous noises are present, differentiation amplifies the discontinuous noises more than the useful signal. For example, the output of a potentiometer is a discontinuous voltage signal because, as the potentiometer brush is moving on the windings, voltages are induced in the switchover turns and thus generate transients. The output of the potentiometer therefore should not be followed by a differentiating element.)



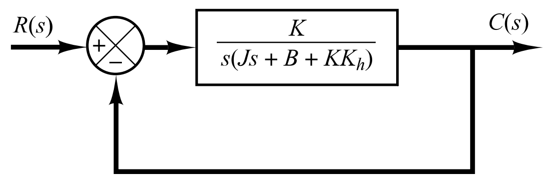


Figure 5–13
(a) Block diagram of a servo system;
(b) simplified block diagram.

The tachometer, a special dc generator, is frequently used to measure velocity without differentiation process. The output of a tachometer is proportional to the angular velocity of the motor.

Consider the servo system shown in Figure 5–13(a). In this device, the velocity signal, together with the positional signal, is fed back to the input to produce the actuating error signal. In any servo system, such a velocity signal can be easily generated by a tachometer. The block diagram shown in Figure 5–13(a) can be simplified, as shown in Figure 5–13(b), giving

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K} \tag{5–24}$$

Comparing Equation (5–24) with Equation (5–9), notice that the velocity feedback has the effect of increasing damping. The damping ratio ζ becomes

$$\zeta = \frac{B + KK_h}{2\sqrt{KJ}} \tag{5–25}$$

The undamped natural frequency $\omega_n = \sqrt{K/J}$ is not affected by velocity feedback. Noting that the maximum overshoot for a unit-step input can be controlled by controlling the value of the damping ratio ζ, we can reduce the maximum overshoot by adjusting the velocity-feedback constant K_h so that ζ is between 0.4 and 0.7.

It is important to remember that velocity feedback has the effect of increasing the damping ratio without affecting the undamped natural frequency of the system.

EXAMPLE 5–2 For the system shown in Figure 5–13(a), determine the values of gain K and velocity-feedback constant K_h so that the maximum overshoot in the unit-step response is 0.2 and the peak time is 1 sec. With these values of K and K_h, obtain the rise time and settling time. Assume that $J = 1$ kg-m^2 and $B = 1$ N-m/rad/sec.

Determination of the values of K and K_h: The maximum overshoot M_p is given by Equation (5–21) as

$$M_p = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$$

This value must be 0.2. Thus,

$$e^{-(\zeta/\sqrt{1-\zeta^2})\pi} = 0.2$$

or

$$\frac{\zeta\pi}{\sqrt{1 - \zeta^2}} = 1.61$$

which yields

$$\zeta = 0.456$$

The peak time t_p is specified as 1 sec; therefore, from Equation (5–20),

$$t_p = \frac{\pi}{\omega_d} = 1$$

or

$$\omega_d = 3.14$$

Since ζ is 0.456, ω_n is

$$\omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}} = 3.53$$

Since the natural frequency ω_n is equal to $\sqrt{K/J}$,

$$K = J\omega_n^2 = \omega_n^2 = 12.5 \text{ N-m}$$

Then K_h is, from Equation (5–25),

$$K_h = \frac{2\sqrt{KJ}\zeta - B}{K} = \frac{2\sqrt{K}\zeta - 1}{K} = 0.178 \text{ sec}$$

Rise time t_r: From Equation (5–19), the rise time t_r is

$$t_r = \frac{\pi - \beta}{\omega_d}$$

where

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1} 1.95 = 1.10$$

Thus, t_r is

$$t_r = 0.65 \text{ sec}$$

Settling time t_s: For the 2% criterion,

$$t_s = \frac{4}{\sigma} = 2.48 \text{ sec}$$

For the 5% criterion,

$$t_s = \frac{3}{\sigma} = 1.86 \text{ sec}$$

Impulse Response of Second-Order Systems. For a unit-impulse input $r(t)$, the corresponding Laplace transform is unity, or $R(s) = 1$. The unit-impulse response $C(s)$ of the second-order system shown in Figure 5-6 is

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The inverse Laplace transform of this equation yields the time solution for the response $c(t)$ as follows:

For $0 \leq \zeta < 1$,

$$c(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1 - \zeta^2}\, t, \qquad \text{for } t \geq 0 \tag{5–26}$$

For $\zeta = 1$,

$$c(t) = \omega_n^2 t e^{-\omega_n t}, \qquad \text{for } t \geq 0 \tag{5–27}$$

For $\zeta > 1$,

$$c(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t} - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t}, \qquad \text{for } t \geq 0 \tag{5–28}$$

Note that without taking the inverse Laplace transform of $C(s)$ we can also obtain the time response $c(t)$ by differentiating the corresponding unit-step response, since the unit-impulse function is the time derivative of the unit-step function. A family of unit-impulse response curves given by Equations (5–26) and (5–27) with various values of ζ is shown in Figure 5–14. The curves $c(t)/\omega_n$ are plotted against the dimensionless variable $\omega_n t$, and thus they are functions only of ζ. For the critically damped and overdamped cases, the unit-impulse response is always positive or zero; that is, $c(t) \geq 0$. This can be seen from Equations (5–27) and (5–28). For the underdamped case, the unit-impulse response $c(t)$ oscillates about zero and takes both positive and negative values.

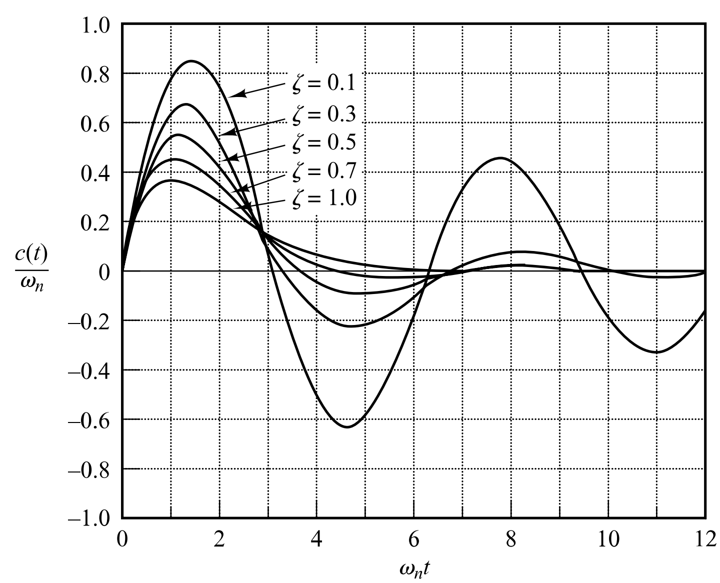


Figure 5–14 Unit-impulse response curves of the system shown in Figure 5–6.

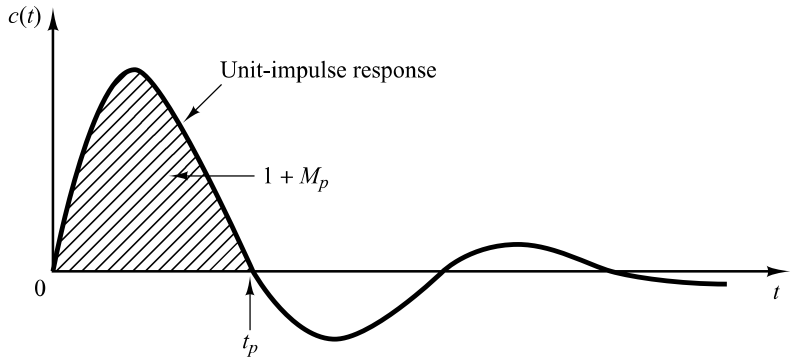


Figure 5–15 Unit-impulse response curve of the system shown in Figure 5–6.

From the foregoing analysis, we may conclude that if the impulse response $c(t)$ does not change sign, the system is either critically damped or overdamped, in which case the corresponding step response does not overshoot but increases or decreases monotonically and approaches a constant value.

The maximum overshoot for the unit-impulse response of the underdamped system occurs at

$$t = \frac{\tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}}, \qquad \text{where } 0 < \zeta < 1 \tag{5–29}$$

[Equation (5–29) can be obtained by equating dc/dt to zero and solving for t.] The maximum overshoot is

$$c(t)_{\max} = \omega_n \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \qquad \text{where } 0 < \zeta < 1 \tag{5–30}$$

[Equation (5–30) can be obtained by substituting Equation (5–29) into Equation (5–26).]

Since the unit-impulse response function is the time derivative of the unit-step response function, the maximum overshoot M_p for the unit-step response can be found from the corresponding unit-impulse response. That is, the area under the unit-impulse response curve from $t = 0$ to the time of the first zero, as shown in Figure 5–15, is $1 + M_p$, where M_p is the maximum overshoot (for the unit-step response) given by Equation (5–21). The peak time t_p (for the unit-step response) given by Equation (5–20) corresponds to the time that the unit-impulse response first crosses the time axis.

5–4 HIGHER-ORDER SYSTEMS

In this section we shall present a transient-response analysis of higher-order systems in general terms. It will be seen that the response of a higher-order system is the sum of the responses of first-order and second-order systems.

Transient Response of Higher-Order Systems. Consider the system shown in Figure 5–16. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{5–31}$$

In general, $G(s)$ and $H(s)$ are given as ratios of polynomials in s, or

$$G(s) = \frac{p(s)}{q(s)} \qquad \text{and} \qquad H(s) = \frac{n(s)}{d(s)}$$

where $p(s)$, $q(s)$, $n(s)$, and $d(s)$ are polynomials in s. The closed-loop transfer function given by Equation (5–31) may then be written

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{p(s)d(s)}{q(s)d(s) + p(s)n(s)} \\ &= \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \qquad (m \leq n)\end{aligned}$$

The transient response of this system to any given input can be obtained by a computer simulation. (See Section 5–5.) If an analytical expression for the transient response is desired, then it is necessary to factor the denominator polynomial. [MATLAB may be used for finding the roots of the denominator polynomial. Use the command roots(den).] Once the numerator and the denominator have been factored, $C(s)/R(s)$ can be written in the form

$$\frac{C(s)}{R(s)} = \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)} \tag{5–32}$$

Let us examine the response behavior of this system to a unit-step input. Consider first the case where the closed-loop poles are all real and distinct. For a unit-step input, Equation (5–32) can be written

$$C(s) = \frac{a}{s} + \sum_{i=1}^{n} \frac{a_i}{s + p_i} \tag{5–33}$$

where a_i is the residue of the pole at $s = -p_i$. (If the system involves multiple poles, then $C(s)$ will have multiple-pole terms.) [The partial-fraction expansion of $C(s)$, as given by Equation (5–33), can be obtained easily with MATLAB. Use the residue command. (See Appendix B.)]

If all closed-loop poles lie in the left-half s plane, the relative magnitudes of the residues determine the relative importance of the components in the expanded form of

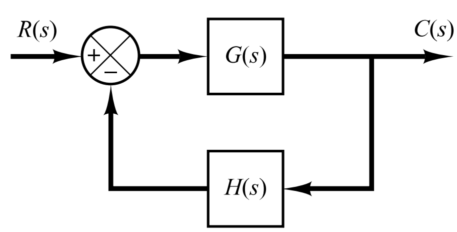


Figure 5–16
Control system.

$C(s)$. If there is a closed-loop zero close to a closed-loop pole, then the residue at this pole is small and the coefficient of the transient-response term corresponding to this pole becomes small. A pair of closely located poles and zeros will effectively cancel each other. If a pole is located very far from the origin, the residue at this pole may be small. The transients corresponding to such a remote pole are small and last a short time. Terms in the expanded form of $C(s)$ having very small residues contribute little to the transient response, and these terms may be neglected. If this is done, the higher-order system may be approximated by a lower-order one. (Such an approximation often enables us to estimate the response characteristics of a higher-order system from those of a simplified one.)

Next, consider the case where the poles of $C(s)$ consist of real poles and pairs of complex-conjugate poles. A pair of complex-conjugate poles yields a second-order term in s. Since the factored form of the higher-order characteristic equation consists of first- and second-order terms, Equation (5–33) can be rewritten

$$C(s) = \frac{a}{s} + \sum_{j=1}^{q} \frac{a_j}{s + p_j} + \sum_{k=1}^{r} \frac{b_k(s + \zeta_k \omega_k) + c_k \omega_k \sqrt{1 - \zeta_k^2}}{s^2 + 2\zeta_k \omega_k s + \omega_k^2} \qquad (q + 2r = n)$$

where we assumed all closed-loop poles are distinct. [If the closed-loop poles involve multiple poles, $C(s)$ must have multiple-pole terms.] From this last equation, we see that the response of a higher-order system is composed of a number of terms involving the simple functions found in the responses of first- and second-order systems. The unit-step response $c(t)$, the inverse Laplace transform of $C(s)$, is then

$$c(t) = a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} b_k e^{-\zeta_k \omega_k t} \cos \omega_k \sqrt{1 - \zeta_k^2}\, t$$
$$+ \sum_{k=1}^{r} c_k e^{-\zeta_k \omega_k t} \sin \omega_k \sqrt{1 - \zeta_k^2}\, t, \qquad \text{for } t \geq 0 \qquad (5\text{–}34)$$

Thus the response curve of a stable higher-order system is the sum of a number of exponential curves and damped sinusoidal curves.

If all closed-loop poles lie in the left-half s plane, then the exponential terms and the damped exponential terms in Equation (5–34) will approach zero as time t increases. The steady-state output is then $c(\infty) = a$.

Let us assume that the system considered is a stable one. Then the closed-loop poles that are located far from the $j\omega$ axis have large negative real parts. The exponential terms that correspond to these poles decay very rapidly to zero. (Note that the horizontal distance from a closed-loop pole to the $j\omega$ axis determines the settling time of transients due to that pole. The smaller the distance is, the longer the settling time.)

Remember that the type of transient response is determined by the closed-loop poles, while the shape of the transient response is primarily determined by the closed-loop zeros. As we have seen earlier, the poles of the input $R(s)$ yield the steady-state response terms in the solution, while the poles of $C(s)/R(s)$ enter into the exponential transient-response terms and/or damped sinusoidal transient-response terms. The zeros of $C(s)/R(s)$ do not affect the exponents in the exponential terms, but they do affect the magnitudes and signs of the residues.

Dominant Closed-Loop Poles. The relative dominance of closed-loop poles is determined by the ratio of the real parts of the closed-loop poles, as well as by the relative magnitudes of the residues evaluated at the closed-loop poles. The magnitudes of the residues depend on both the closed-loop poles and zeros.

If the ratios of the real parts of the closed-loop poles exceed 5 and there are no zeros nearby, then the closed-loop poles nearest the $j\omega$ axis will dominate in the transient-response behavior because these poles correspond to transient-response terms that decay slowly. Those closed-loop poles that have dominant effects on the transient-response behavior are called *dominant closed-loop* poles. Quite often the dominant closed-loop poles occur in the form of a complex-conjugate pair. The dominant closed-loop poles are most important among all closed-loop poles.

Note that the gain of a higher-order system is often adjusted so that there will exist a pair of dominant complex-conjugate closed-loop poles. The presence of such poles in a stable system reduces the effects of such nonlinearities as dead zone, backlash, and coulomb-friction.

Stability Analysis in the Complex Plane. The stability of a linear closed-loop system can be determined from the location of the closed-loop poles in the s plane. If any of these poles lie in the right-half s plane, then with increasing time they give rise to the dominant mode, and the transient response increases monotonically or oscillates with increasing amplitude. This represents an unstable system. For such a system, as soon as the power is turned on, the output may increase with time. If no saturation takes place in the system and no mechanical stop is provided, then the system may eventually be subjected to damage and fail, since the response of a real physical system cannot increase indefinitely. Therefore, closed-loop poles in the right-half s plane are not permissible in the usual linear control system. If all closed-loop poles lie to the left of the $j\omega$ axis, any transient response eventually reaches equilibrium. This represents a stable system.

Whether a linear system is stable or unstable is a property of the system itself and does not depend on the input or driving function of the system. The poles of the input, or driving function, do not affect the property of stability of the system, but they contribute only to steady-state response terms in the solution. Thus, the problem of absolute stability can be solved readily by choosing no closed-loop poles in the right-half s plane, including the $j\omega$ axis. (Mathematically, closed-loop poles on the $j\omega$ axis will yield oscillations, the amplitude of which is neither decaying nor growing with time. In practical cases, where noise is present, however, the amplitude of oscillations may increase at a rate determined by the noise power level. Therefore, a control system should not have closed-loop poles on the $j\omega$ axis.)

Note that the mere fact that all closed-loop poles lie in the left-half s plane does not guarantee satisfactory transient-response characteristics. If dominant complex-conjugate closed-loop poles lie close to the $j\omega$ axis, the transient response may exhibit excessive oscillations or may be very slow. Therefore, to guarantee fast, yet well-damped, transient-response characteristics, it is necessary that the closed-loop poles of the system lie in a particular region in the complex plane, such as the region bounded by the shaded area in Figure 5–17.

Since the relative stability and transient-response performance of a closed-loop control system are directly related to the closed-loop pole-zero configuration in the s plane,

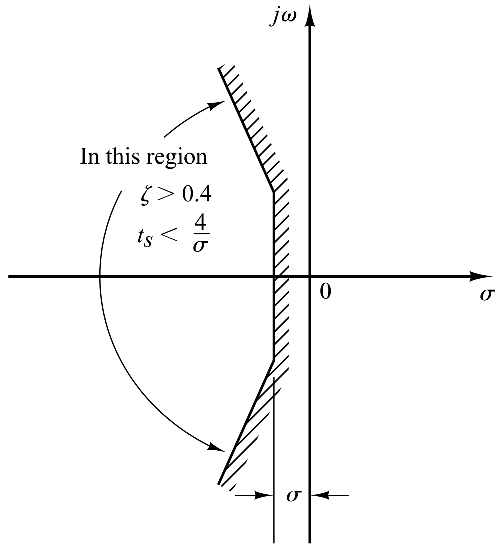


Figure 5–17 Region in the complex plane satisfying the conditions $\zeta > 0.4$ and $t_s < 4/\sigma$.

it is frequently necessary to adjust one or more system parameters in order to obtain suitable configurations. The effects of varying system parameters on the closed-loop poles will be discussed in detail in Chapter 6.

5–5 TRANSIENT-RESPONSE ANALYSIS WITH MATLAB

Introduction. The practical procedure for plotting time response curves of systems higher than second order is through computer simulation. In this section we present the computational approach to the transient-response analysis with MATLAB. In particular, we discuss step response, impulse response, ramp response, and responses to other simple inputs.

MATLAB Representation of Linear Systems. The transfer function of a system is represented by two arrays of numbers. Consider the system

$$\frac{C(s)}{R(s)} = \frac{2s + 25}{s^2 + 4s + 25} \tag{5–35}$$

This system can be represented as two arrays, each containing the coefficients of the polynomials in decreasing powers of s as follows:

```
num = [2  25]
den = [1  4  25]
```

An alternative representation is

```
num = [0  2  25]
den = [1  4  25]
```

In this expression a zero is padded. Note that if zeros are padded, the dimensions of "num" vector and "den" vector become the same. An advantage of padding zeros is that the "num" vector and "den" vector can be directly added. For example,

```
num + den = [0  2  25] + [1  4  25]
          = [1  6  50]
```

If num and den (the numerator and denominator of the closed-loop transfer function) are known, commands such as

```
step(num,den),    step(num,den,t)
```

will generate plots of unit-step responses (t in the step command is the user-specified time.)

For a control system defined in a state-space form, where state matrix **A**, control matrix **B**, output matrix **C**, and direct transmission matrix **D** of state-space equations are known, the command

```
step(A,B,C,D),    step(A,B,C,D,t)
```

will generate plots of unit-step responses. When t is not explicitly included in the step commands, the time vector is automatically determined.

Note that the command step(sys) may be used to obtain the unit-step response of a system. First, define the system by

```
sys = tf(num,den)
```

or

```
sys = ss(A,B,C,D)
```

Then, to obtain, for example, the unit-step response, enter

```
step(sys)
```

into the computer.

When step commands have left-hand arguments such as

```
[y,x,t] = step(num,den,t)
[y,x,t] = step(A,B,C,D,iu)
[y,x,t] = step(A,B,C,D,iu,t)
```

(5–36)

no plot is shown on the screen. Hence it is necessary to use a plot command to see the response curves. The matrices y and x contain the output and state response of the system, respectively, evaluated at the computation time points t. (y has as many columns as outputs and one row for each element in t. x has as many columns as states and one row for each element in t.)

Note in Equation (5–36) that the scalar iu is an index into the inputs of the system and specifies which input is to be used for the response, and t is the user-specified time. If the system involves multiple inputs and multiple outputs, the step command, such as given by Equation (5–36), produces a series of step-response plots, one for each input and output combination of

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

(For details, see Example 5–3.)

EXAMPLE 5–3 Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 6.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Obtain the unit-step response curves.

Although it is not necessary to obtain the transfer-matrix expression for the system to obtain the unit-step response curves with MATLAB, we shall derive such an expression for reference. For the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

the transfer matrix $\mathbf{G}(s)$ is a matrix that relates $\mathbf{Y}(s)$ and $\mathbf{U}(s)$ as follows:

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$$

Taking Laplace transforms of the state-space equations, we obtain

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{5–37}$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \tag{5–38}$$

In deriving the transfer matrix, we assume that $\mathbf{x}(0) = \mathbf{0}$. Then, from Equation (5–37), we get

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \tag{5–39}$$

Substituting Equation (5–39) into Equation (5–38), we obtain

$$\mathbf{Y}(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s)$$

Thus the transfer matrix $\mathbf{G}(s)$ is given by

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

The transfer matrix $\mathbf{G}(s)$ for the given system becomes

$$\begin{aligned} \mathbf{G}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -6.5 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \frac{1}{s^2 + s + 6.5} \begin{bmatrix} s & -1 \\ 6.5 & s+1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \frac{1}{s^2 + s + 6.5} \begin{bmatrix} s-1 & s \\ s+7.5 & 6.5 \end{bmatrix} \end{aligned}$$

Hence

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s-1}{s^2+s+6.5} & \dfrac{s}{s^2+s+6.5} \\ \dfrac{s+7.5}{s^2+s+6.5} & \dfrac{6.5}{s^2+s+6.5} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

Since this system involves two inputs and two outputs, four transfer functions may be defined, depending on which signals are considered as input and output. Note that, when considering the signal u_1 as the input, we assume that signal u_2 is zero, and vice versa. The four transfer functions are

$$\frac{Y_1(s)}{U_1(s)} = \frac{s - 1}{s^2 + s + 6.5}, \qquad \frac{Y_1(s)}{U_2(s)} = \frac{s}{s^2 + s + 6.5}$$

$$\frac{Y_2(s)}{U_1(s)} = \frac{s + 7.5}{s^2 + s + 6.5}, \qquad \frac{Y_2(s)}{U_2(s)} = \frac{6.5}{s^2 + s + 6.5}$$

Assume that u_1 and u_2 are unit-step functions. The four individual step-response curves can then be plotted by use of the command

step(A,B,C,D)

MATLAB Program 5–1 produces four such step-response curves. The curves are shown in Figure 5–18. (Note that the time vector t is automatically determined, since the command does not include t.)

MATLAB Program 5–1

```
A = [–1  –1;6.5  0];
B = [1  1;1  0];
C = [1  0;0  1];
D = [0  0;0  0];
step(A,B,C,D)
```

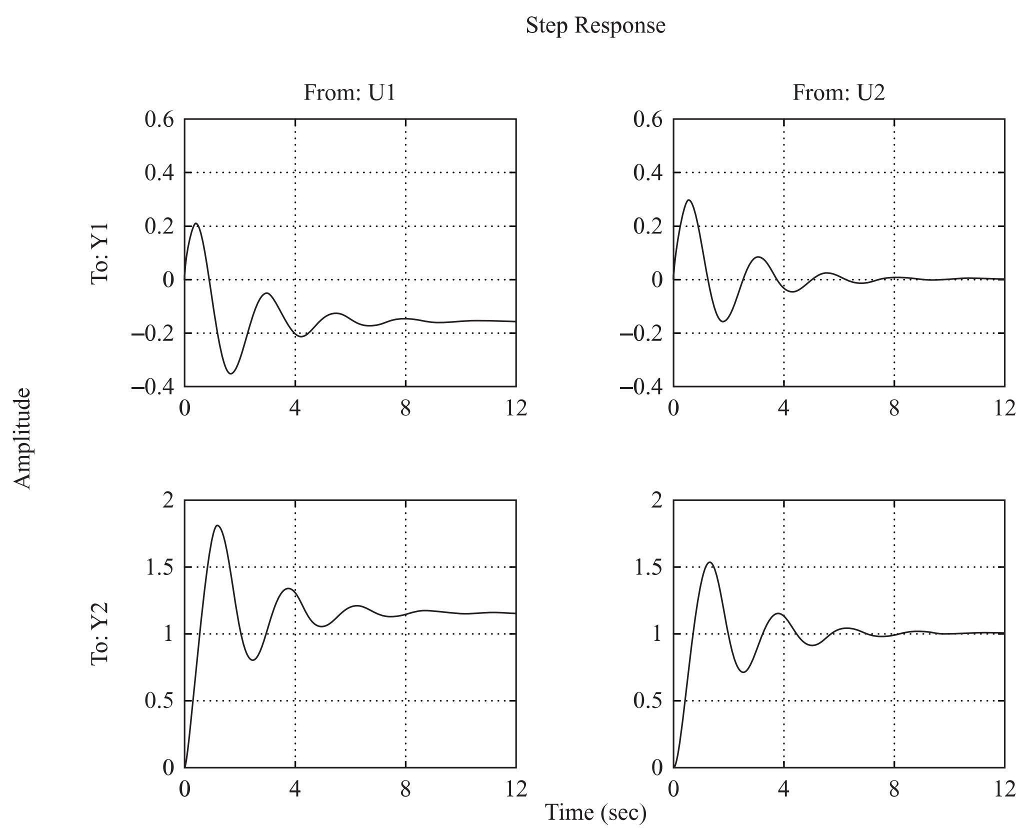


Figure 5–18
Unit-step response curves.

To plot two step-response curves for the input u_1 in one diagram and two step-response curves for the input u_2 in another diagram, we may use the commands

```
step(A,B,C,D,1)
```

and

```
step(A,B,C,D,2)
```

respectively. MATLAB Program 5–2 is a program to plot two step-response curves for the input u_1 in one diagram and two step-response curves for the input u_2 in another diagram. Figure 5–19 shows the two diagrams, each consisting of two step-response curves. (This MATLAB program uses text commands. For such commands, refer to the paragraph following this example.)

MATLAB Program 5–2

```
% ***** In this program we plot step-response curves of a system
% having two inputs (u1 and u2) and two outputs (y1 and y2) *****

% ***** We shall first plot step-response curves when the input is
% u1. Then we shall plot step-response curves when the input is
% u2 *****

% ***** Enter matrices A, B, C, and D *****

A = [-1  -1;6.5  0];
B = [1  1;1  0];
C = [1  0;0  1];
D = [0  0;0  0];

% ***** To plot step-response curves when the input is u1, enter
% the command 'step(A,B,C,D,1)' *****

step(A,B,C,D,1)
grid
title ('Step-Response Plots: Input = u1 (u2 = 0)')
text(3.4, -0.06,'Y1')
text(3.4, 1.4,'Y2')

% ***** Next, we shall plot step-response curves when the input
% is u2. Enter the command 'step(A,B,C,D,2)' *****

step(A,B,C,D,2)
grid
title ('Step-Response Plots: Input = u2 (u1 = 0)')
text(3,0.14,'Y1')
text(2.8,1.1,'Y2')
```

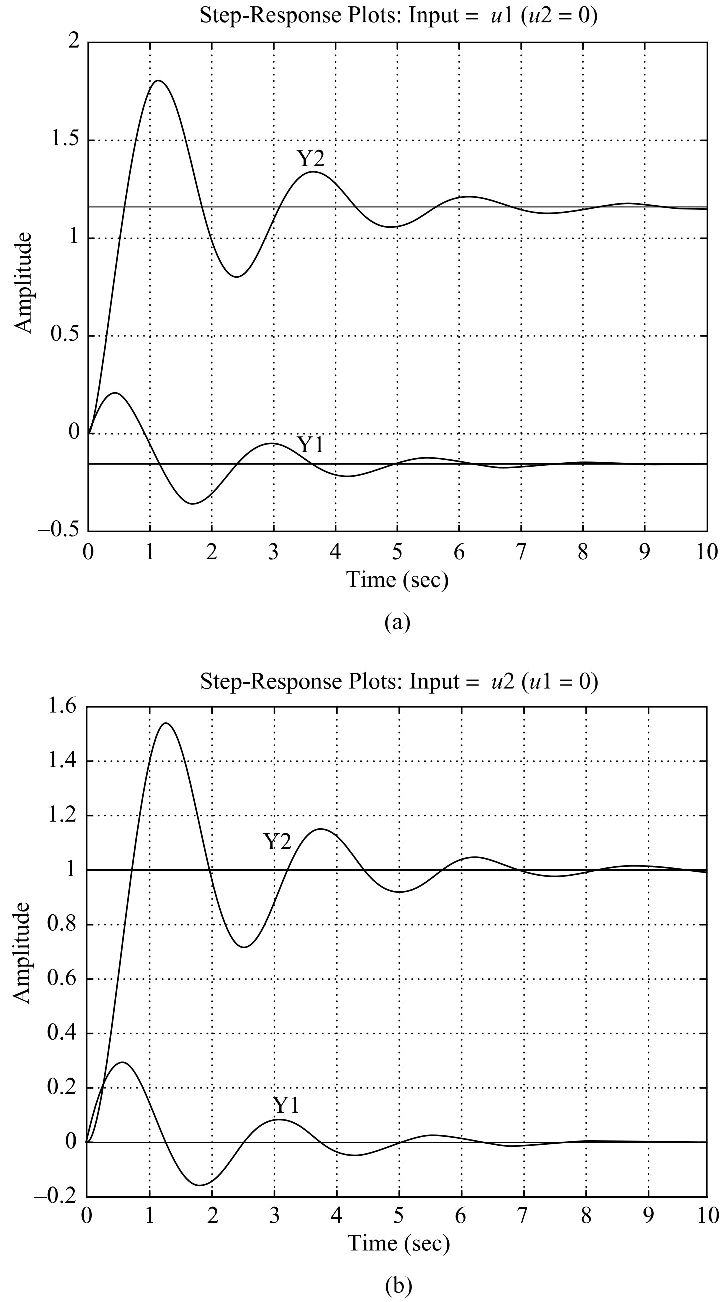


Figure 5–19
Unit-step response curves. (a) u_1 is the input $(u_2 = 0)$; (b) u_2 is the input $(u_1 = 0)$.

Writing Text on the Graphics Screen. To write text on the graphics screen, enter, for example, the following statements:

```
text(3.4, -0.06,'Y1')
```

and

```
text(3.4,1.4,'Y2')
```

The first statement tells the computer to write 'Y1' beginning at the coordinates $x = 3.4$, $y = -0.06$. Similarly, the second statement tells the computer to write 'Y2' beginning at the coordinates $x = 3.4$, $y = 1.4$. [See MATLAB Program 5–2 and Figure 5–19(a).]

Another way to write a text or texts in the plot is to use the gtext command. The syntax is

```
gtext('text')
```

When gtext is executed, the computer waits until the cursor is positioned (using a mouse) at the desired position in the screen. When the left mouse button is pressed, the text enclosed in simple quotes is written on the plot at the cursor's position. Any number of gtext commands can be used in a plot. (See, for example, MATLAB Program 5–15.)

MATLAB Description of Standard Second-Order System. As noted earlier, the second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5–40}$$

is called the standard second-order system. Given ω_n and ζ, the command

```
printsys(num,den)        or        printsys(num,den,s)
```

prints num/den as a ratio of polynomials in s.

Consider, for example, the case where $\omega_n = 5$ rad/sec and $\zeta = 0.4$. MATLAB Program 5–3 generates the standard second-order system, where $\omega_n = 5$ rad/sec and $\zeta = 0.4$. Note that in MATLAB Program 5–3, "num 0" is 1.

MATLAB Program 5–3

```
wn = 5;
damping_ratio = 0.4;
[num0,den] = ord2(wn,damping_ratio);
num = 5^2*num0;
printsys(num,den,'s')
num/den =
                25
          -------------
          S^2 + 4s + 25
```

Obtaining the Unit-Step Response of the Transfer-Function System. Let us consider the unit-step response of the system given by

$$G(s) = \frac{25}{s^2 + 4s + 25}$$

MATLAB Program 5–4 will yield a plot of the unit-step response of this system. A plot of the unit-step response curve is shown in Figure 5–20.

MATLAB Program 5–4

```
% ------------- Unit-step response -------------

% ***** Enter the numerator and denominator of the transfer
% function *****

num = [25];
 den = [1  4  25];

% ***** Enter the following step-response command *****

step(num,den)

% ***** Enter grid and title of the plot *****

grid
title (' Unit-Step Response of G(s) = 25/(s^2+4s+25)')
```

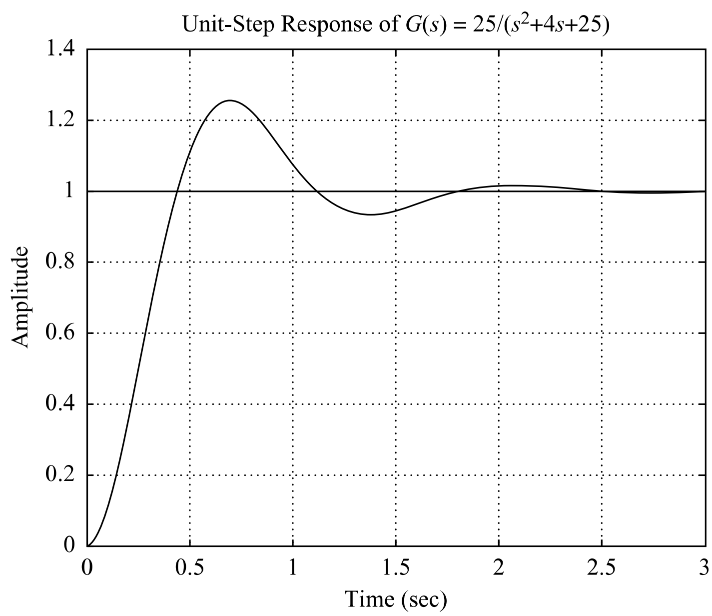


Figure 5–20 Unit-step response curve.

Notice in Figure 5–20 (and many others) that the x-axis and y-axis labels are automatically determined. If it is desired to label the x axis and y axis differently, we need to modify the step command. For example, if it is desired to label the x axis as 't Sec' and the y axis as 'Output,' then use step-response commands with left-hand arguments, such as

```
c = step(num,den,t)
```

or, more generally,

```
[y,x,t] = step(num,den,t)
```

and use plot(t,y) command. See, for example, MATLAB Program 5–5 and Figure 5–21.

MATLAB Program 5–5

```
% ------------- Unit-step response -------------

num = [25];
den = [1  4  25];
t = 0:0.01:3;
[y,x,t] = step(num,den,t);
plot(t,y)
grid
title('Unit-Step Response of G(s)=25/(s^2+4s+25)')
xlabel('t Sec')
ylabel('Output')
```

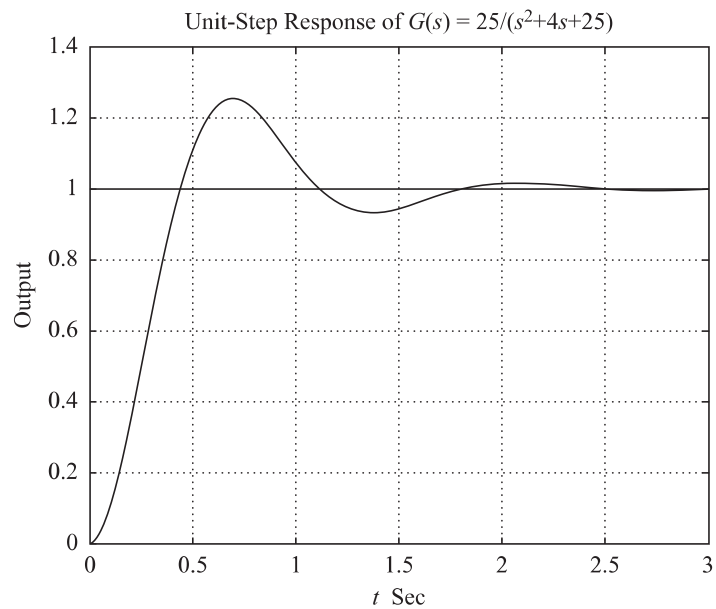


Figure 5–21
Unit-step response curve.

Obtaining Three-Dimensional Plot of Unit-Step Response Curves with MATLAB. MATLAB enables us to plot three-dimensional plots easily. The commands to obtain three-dimensional plots are "mesh" and "surf." The difference between the "mesh" plot and "surf" plot is that in the former only the lines are drawn and in the latter the spaces between the lines are filled in by colors. In this book we use only the "mesh" command.

EXAMPLE 5–4 Consider the closed-loop system defined by

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + 2\zeta s + 1}$$

(The undamped natural frequency ω_n is normalized to 1.) Plot unit-step response curves $c(t)$ when ζ assumes the following values:

$$\zeta = 0,\ 0.2,\ 0.4,\ 0.6.\ 0.8,\ 1.0$$

Also plot a three-dimensional plot.

An illustrative MATLAB Program for plotting a two-dimensional diagram and a three-dimensional diagram of unit-step response curves of this second-order system is given in MATLAB Program 5–6. The resulting plots are shown in Figures 5–22(a) and (b), respectively. Notice that we used the command mesh(t,zeta,y') to plot the three-dimensional plot. We may use a command mesh(y') to get the same result. [Note that command mesh(t,zeta,y) or mesh(y) will produce a three-dimensional plot the same as Figure 5–22(b), except that x axis and y axis are interchanged. See Problem **A–5–15**.]

When we want to solve a problem using MATLAB and if the solution process involves many repetitive computations, various approaches may be conceived to simplify the MATLAB program. A frequently used approach to simplify the computation is to use "for loops." MATLAB Program 5–6 uses such a "for loop." In this book many MATLAB programs using "for loops" are presented for solving a variety of problems. Readers are advised to study all those problems carefully to familiarize themselves with the approach.

MATLAB Program 5–6

```
% ------- Two-dimensional plot and three-dimensional plot of unit-step
% response curves for the standard second-order system with wn = 1
% and zeta = 0, 0.2, 0.4, 0.6, 0.8, and 1. -------

t = 0:0.2:10;
zeta = [0   0.2   0.4   0.6   0.8   1];
    for n = 1:6;
    num = [1];
    den = [1   2*zeta(n)   1];
    [y(1:51,n),x,t] = step(num,den,t);
    end

% To plot a two-dimensional diagram, enter the command plot(t,y).

plot(t,y)
grid
title('Plot of Unit-Step Response Curves with \omega_n = 1 and \zeta = 0, 0.2, 0.4, 0.6, 0.8, 1')
xlabel('t (sec)')
ylabel('Response')
text(4.1,1.86,'\zeta = 0')
text(3.5,1.5,'0.2')
text(3 .5,1.24,'0.4')
text(3.5,1.08,'0.6')
text(3.5,0.95,'0.8')
text(3.5,0.86,'1.0')

% To plot a three-dimensional diagram, enter the command mesh(t,zeta,y').

mesh(t,zeta,y')
title('Three-Dimensional Plot of Unit-Step Response Curves')
xlabel('t Sec')
ylabel('\zeta')
zlabel('Response')
```

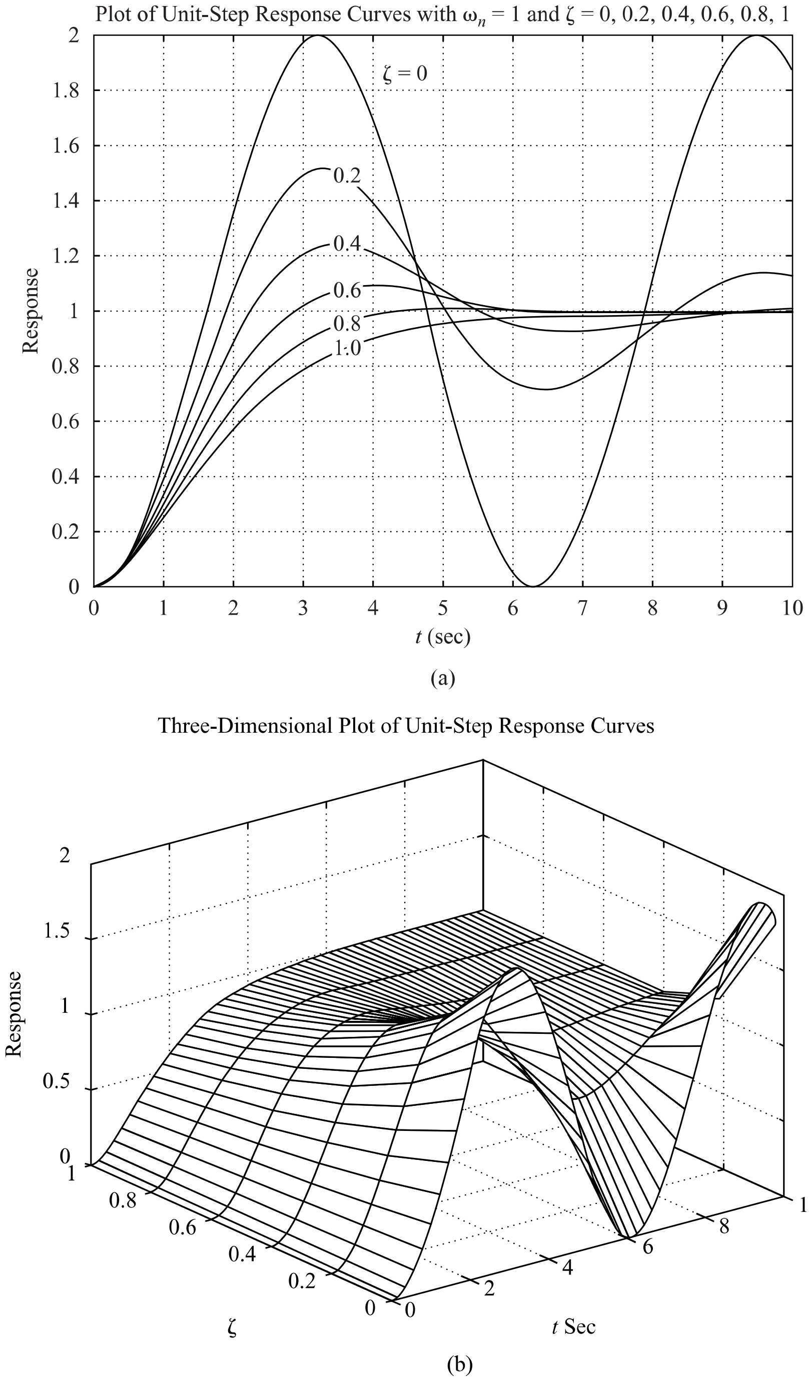


Figure 5–22
(a) Two-dimensional plot of unit-step response curves for $\zeta = 0, 0.2, 0.4, 0.6, 0.8,$ and 1.0; (b) three-dimensional plot of unit-step response curves.

Obtaining Rise Time, Peak Time, Maximum Overshoot, and Settling Time with MATLAB. MATLAB can conveniently be used to obtain the rise time, peak time, maximum overshoot, and settling time. Consider the system defined by

$$\frac{C(s)}{R(s)} = \frac{25}{s^2 + 6s + 25}$$

MATLAB Program 5–7 yields the rise time, peak time, maximum overshoot, and settling time. A unit-step response curve for this system is given in Figure 5–23 to verify the results obtained with MATLAB Program 5–7. (Note that this program can also be applied to higher-order systems. See Problem **A–5–10**.)

MATLAB Program 5–7

```
% ------- This is a MATLAB program to find the rise time, peak time,
% maximum overshoot, and settling time of the second-order system
% and higher-order system -------
% ------- In this example, we assume zeta = 0.6 and wn = 5 -------
num = [25];
den = [1  6  25];
t = 0:0.005:5;
[y,x,t] = step(num,den,t);
r = 1; while y(r) < 1.0001; r = r + 1; end;
rise_time = (r - 1)*0.005
rise_time =
   0.5550
[ymax,tp] = max(y);
peak_time = (tp - 1)*0.005
peak_time =
   0.7850
max_overshoot = ymax-1
max_overshoot =
   0.0948
s = 1001; while y(s) > 0.98 & y(s) < 1.02; s = s - 1; end;
settling_time = (s - 1)*0.005
settling_time =
   1.1850
```

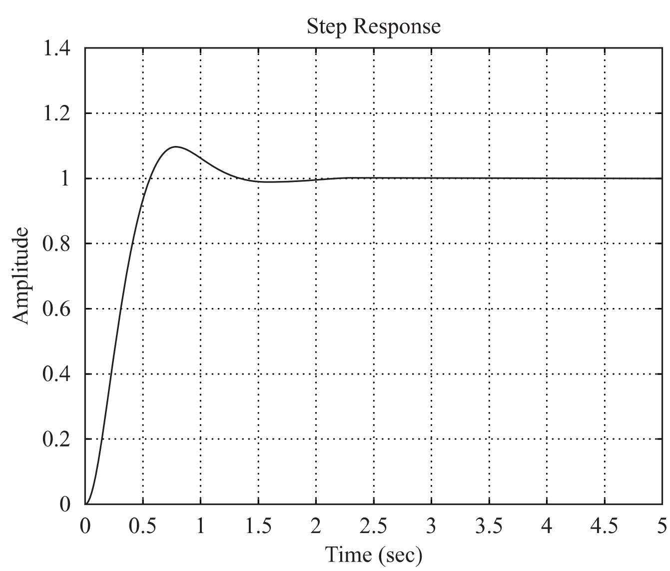


Figure 5–23 Unit-step response curve.

Impulse Response. The unit-impulse response of a control system may be obtained by using any of the impulse commands such as

```
impulse(num,den)

impulse(A,B,C,D)

[y,x,t] = impulse(num,den)

[y,x,t] = impulse(num,den,t)                (5–41)

[y,x,t] = impulse(A,B,C,D)

[y,x,t] = impulse(A,B,C,D,iu)               (5–42)

[y,x,t] = impulse(A,B,C,D,iu,t)             (5–43)
```

The command impulse(num,den) plots the unit-impulse response on the screen. The command impulse(A,B,C,D) produces a series of unit-impulse-response plots, one for each input and output combination of the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

Note that in Equations (5–42) and (5–43) the scalar iu is an index into the inputs of the system and specifies which input to be used for the impulse response.

Note also that if the command used does not include "t" explicitly, the time vector is automatically determined. If the command includes the user-supplied time vector "t", as do the commands given by Equations (5–41) and (5–43)], this vector specifies the times at which the impulse response is to be computed.

If MATLAB is invoked with the left-hand argument [y,x,t], such as in the case of [y,x,t] = impulse(A,B,C,D), the command returns the output and state responses of the system and the time vector t. No plot is drawn on the screen. The matrices y and x contain the output and state responses of the system evaluated at the time points t. (y has as many columns as outputs and one row for each element in t. x has as many columns as state variables and one row for each element in t.) To plot the response curve, we must include a plot command, such as plot(t,y).

EXAMPLE 5–5 Obtain the unit-impulse response of the following system:

$$\frac{C(s)}{R(s)} = G(s) = \frac{1}{s^2 + 0.2s + 1}$$

MATLAB Program 5–8 will produce the unit-impulse response. The resulting plot is shown in Figure 5–24.

MATLAB Program 5–8

```
num = [1];
den = [1  0.2  1];
impulse(num,den);
grid
title('Unit-Impulse Response of G(s) = 1/(s^2 + 0.2s + 1)')
```

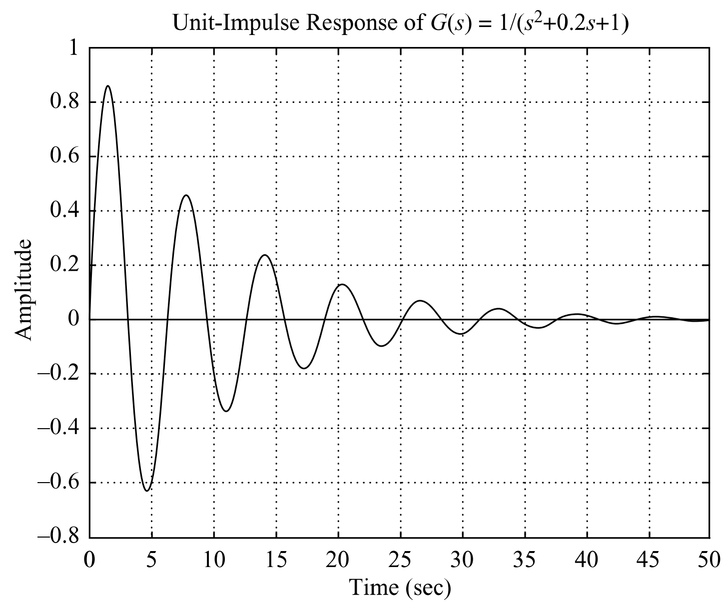


Figure 5–24 Unit-impulse-response curve.

Alternative Approach to Obtain Impulse Response. Note that when the initial conditions are zero, the unit-impulse response of $G(s)$ is the same as the unit-step response of $sG(s)$.

Consider the unit-impulse response of the system considered in Example 5–5. Since $R(s) = 1$ for the unit-impulse input, we have

$$\frac{C(s)}{R(s)} = C(s) = G(s) = \frac{1}{s^2 + 0.2s + 1}$$

$$= \frac{s}{s^2 + 0.2s + 1}\frac{1}{s}$$

We can thus convert the unit-impulse response of $G(s)$ to the unit-step response of $sG(s)$.

If we enter the following num and den into MATLAB,

```
num = [0  1  0]
den = [1  0.2  1]
```

and use the step-response command; as given in MATLAB Program 5–9, we obtain a plot of the unit-impulse response of the system as shown in Figure 5–25.

MATLAB Program 5–9

```
num = [1  0];
den = [1  0.2  1];
step(num,den);
grid
title('Unit-Step Response of sG(s) = s/(s^2 + 0.2s + 1)')
```

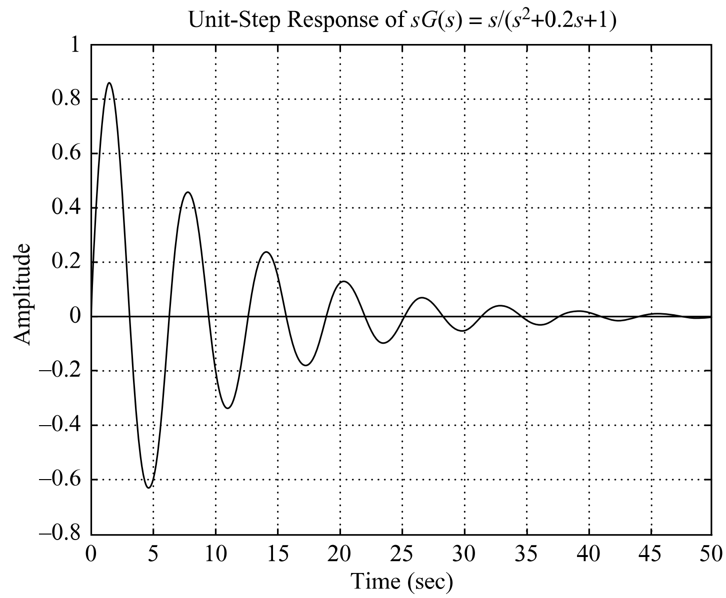


Figure 5–25 Unit-impulse-response curve obtained as the unit-step response of $sG(s) = s/(s^2 + 0.2s + 1)$.

Ramp Response. There is no ramp command in MATLAB. Therefore, we need to use the step command or the lsim command (presented later) to obtain the ramp response. Specifically, to obtain the ramp response of the transfer-function system $G(s)$, divide $G(s)$ by s and use the step-response command. For example, consider the closed-loop system

$$\frac{C(s)}{R(s)} = \frac{2s + 1}{s^2 + s + 1}$$

For a unit-ramp input, $R(s) = 1/s^2$. Hence

$$C(s) = \frac{2s + 1}{s^2 + s + 1}\frac{1}{s^2} = \frac{2s + 1}{(s^2 + s + 1)s}\frac{1}{s}$$

To obtain the unit-ramp response of this system, enter the following numerator and denominator into the MATLAB program:

```
num = [2  1];
den = [1  1  1  0];
```

and use the step-response command. See MATLAB Program 5–10. The plot obtained by using this program is shown in Figure 5–26.

MATLAB Program 5–10

```
% -------------- Unit-ramp response ---------------

% ***** The unit-ramp response is obtained as the unit-step
% response of G(s)/s *****

% ***** Enter the numerator and denominator of G(s)/s *****

num = [2  1];
den = [1  1  1  0];

% ***** Specify the computing time points (such as t = 0:0.1:10)
% and then enter step-response command: c = step(num,den,t) *****

t = 0:0.1:10;
c = step(num,den,t);

% ***** In plotting the ramp-response curve, add the reference
% input to the plot. The reference input is t. Add to the
% argument of the plot command with the following: t,t,'-'. Thus
% the plot command becomes as follows: plot(t,c,'o',t,t,'-') *****

plot(t,c,'o',t,t,'-')

% ***** Add grid, title, xlabel, and ylabel *****

grid
title('Unit-Ramp Response Curve for System G(s) = (2s + 1)/(s^2 + s + 1)')
xlabel('t Sec')
ylabel('Input and Output')
```

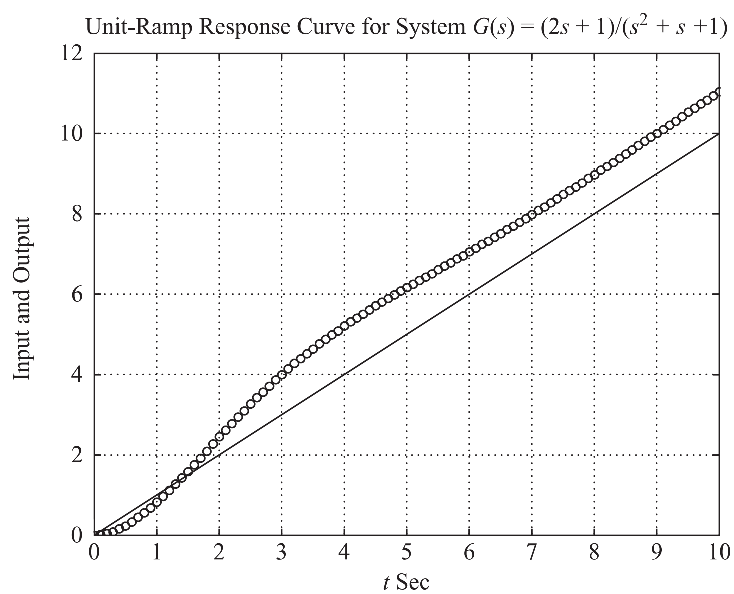


Figure 5–26
Unit-ramp response curve.

Unit-Ramp Response of a System Defined in State Space. Next, we shall treat the unit-ramp response of the system in state-space form. Consider the system described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

where u is the unit-ramp function. In what follows, we shall consider a simple example to explain the method. Consider the case where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{x}(0) = \mathbf{0}$$
$$\mathbf{C} = [1 \quad 0], \qquad D = [0]$$

When the initial conditions are zeros, the unit-ramp response is the integral of the unit-step response. Hence the unit-ramp response can be given by

$$z = \int_0^t y\,dt \tag{5–44}$$

From Equation (5–44), we obtain

$$\dot{z} = y = x_1 \tag{5–45}$$

Let us define

$$z = x_3$$

Then Equation (5–45) becomes

$$\dot{x}_3 = x_1 \tag{5–46}$$

Combining Equation (5–46) with the original state-space equation, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \tag{5–47}$$

$$z = [0 \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{5–48}$$

where u appearing in Equation (5–47) is the unit-step function. These equations can be written as

$$\dot{\mathbf{x}} = \mathbf{AA}\mathbf{x} + \mathbf{BB}u$$
$$z = \mathbf{CC}\mathbf{x} + DDu$$

where

$$\mathbf{AA} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \left[\begin{array}{c:c} \mathbf{A} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hdashline \mathbf{C} & 0 \end{array}\right]$$

$$\mathbf{BB} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}, \qquad \mathbf{CC} = [0 \quad 0 \quad 1], \qquad DD = [0]$$

Note that x_3 is the third element of $\mathbf{x}$. A plot of the unit-ramp response curve $z(t)$ can be obtained by entering MATLAB Program 5–11 into the computer. A plot of the unit-ramp response curve obtained from this MATLAB program is shown in Figure 5–27.

MATLAB Program 5–11

```
% --------------- Unit-ramp response ---------------

% ***** The unit-ramp response is obtained by adding a new
% state variable x3. The dimension of the state equation
% is enlarged by one *****

% ***** Enter matrices A, B, C, and D of the original state
% equation and output equation *****

A = [0  1;-1  -1];
B = [0;  1];
C = [1  0];
D = [0];

% ***** Enter matrices AA, BB, CC, and DD of the new,
% enlarged state equation and output equation *****

AA = [A  zeros(2,1);C  0];
BB = [B;0];
CC = [0  0  1];
DD = [0];

% ***** Enter step-response command: [z,x,t] = step(AA,BB,CC,DD) *****

[z,x,t] = step(AA,BB,CC,DD);

% ***** In plotting x3 add the unit-ramp input t in the plot
% by entering the following command: plot(t,x3,'o',t,t,'-') *****

x3 = [0  0  1]*x'; plot(t,x3,'o',t,t,'-')
grid
title('Unit-Ramp Response')
xlabel('t Sec')
ylabel('Input and Output')
```

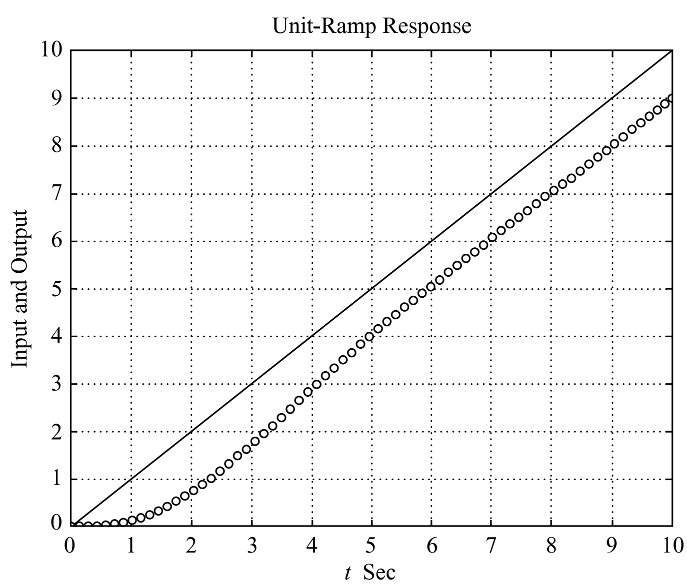


Figure 5–27
Unit-ramp response curve.

Obtaining Response to Arbitrary Input. To obtain the response to an arbitrary input, the command lsim may be used. The commands like

```
lsim(num,den,r,t)
lsim(A,B,C,D,u,t)
y = lsim(num,den,r,t)
y = lsim(A,B,C,D,u,t)
```

will generate the response to input time function r or u. See the following two examples. (Also, see Problems **A–5–14** through **A–5–16**.)

EXAMPLE 5–6 Using the lsim command, obtain the unit-ramp response of the following system:

$$\frac{C(s)}{R(s)} = \frac{2s + 1}{s^2 + s + 1}$$

We may enter MATLAB Program 5–12 into the computer to obtain the unit-ramp response. The resulting plot is shown in Figure 5–28.

MATLAB Program 5–12

```
% ------- Ramp Response -------
num = [2  1];
 den = [1  1  1];
t = 0:0.1:10;
r = t;
y = lsim(num,den,r,t);
plot(t,r,'-',t,y,'o')
grid
title('Unit-Ramp Response Obtained by Use of Command "lsim"')
xlabel('t Sec')
ylabel('Unit-Ramp Input and System Output')
text(6.3,4.6,'Unit-Ramp Input')
text(4.75,9.0,'Output')
```

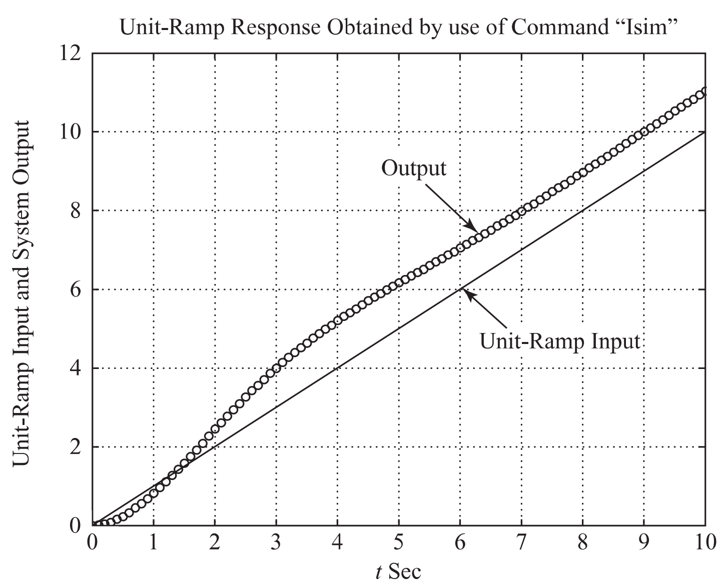


Figure 5–28
Unit-ramp response.

EXAMPLE 5–7 Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0.5 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Using MATLAB, obtain the response curves $y(t)$ when the input u is given by

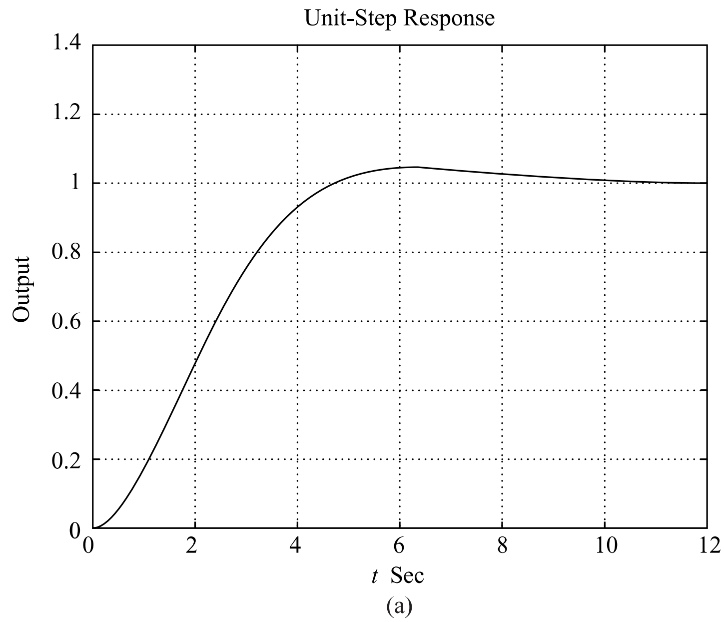
1. $u =$ unit-step input
2. $u = e^{-t}$

Assume that the initial state is $\mathbf{x}(0) = \mathbf{0}$.

A possible MATLAB program to produce the responses of this system to the unit-step input $[u = 1(t)]$ and the exponential input $[u = e^{-t}]$ is shown in MATLAB Program 5–13. The resulting response curves are shown in Figures 5–29(a) and (b), respectively.

MATLAB Program 5–13

```
t = 0:0.1:12;
A = [-1  0.5;-1  0];
B = [0;1];
C = [1  0];
D = [0];

% For the unit-step input u = 1(t), use the command "y = step(A,B,C,D,1,t)".

y = step(A,B,C,D,1,t);
plot(t,y)
grid
title('Unit-Step Response')
xlabel('t Sec')
ylabel('Output')

% For the response to exponential input u = exp(-t), use the command
% "z = lsim(A,B,C,D,u,t)".

u = exp(-t);
z = lsim(A,B,C,D,u,t);
plot(t,u,'-',t,z,'o')
grid
title('Response to Exponential Input u = exp(-t)')
xlabel('t Sec')
ylabel('Exponential Input and System Output')
text(2.3,0.49,'Exponential input')
text(6.4,0.28,'Output')
```



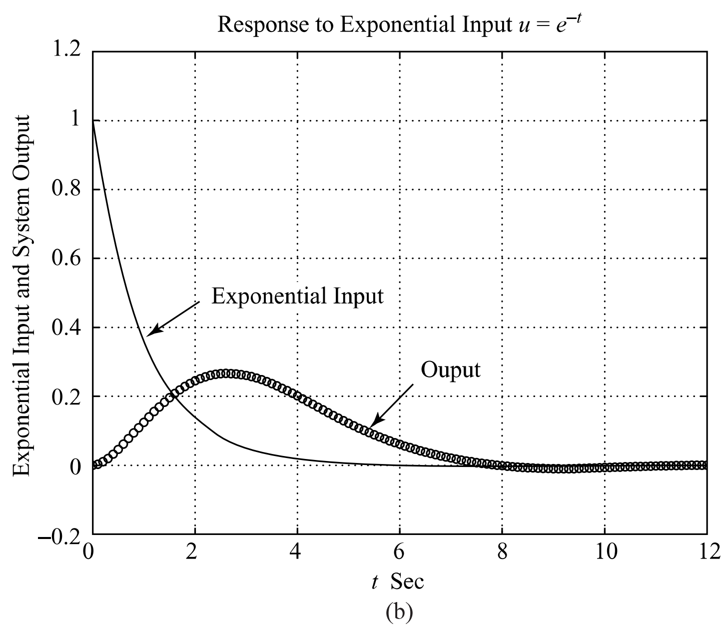


Figure 5–29
(a) Unit-step response;
(b) response to input $u = e^{-t}$.

Response to Initial Condition. In what follows we shall present a few methods for obtaining the response to an initial condition. Commands that we may use are "step" or "initial". We shall first present a method to obtain the response to the initial condition using a simple example. Then we shall discuss the response to the initial condition when the system is given in state-space form. Finally, we shall present a command initial to obtain the response of a system given in a state-space form.

EXAMPLE 5–8 Consider the mechanical system shown in Figure 5–30, where $m = 1$ kg, $b = 3$ N-sec/m, and $k = 2$ N/m. Assume that at $t = 0$ the mass m is pulled downward such that $x(0) = 0.1$ m and $\dot{x}(0) = 0.05$ m/sec. The displacement $x(t)$ is measured from the equilibrium position before the mass is pulled down. Obtain the motion of the mass subjected to the initial condition. (Assume no external forcing function.)

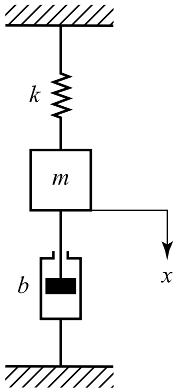


Figure 5–30 Mechanical system.

The system equation is

$$m\ddot{x} + b\dot{x} + kx = 0$$

with the initial conditions $x(0) = 0.1$ m and $\dot{x}(0) = 0.05$ m/sec. (x is measured from the equilibrium position.) The Laplace transform of the system equation gives

$$m[s^2X(s) - sx(0) - \dot{x}(0)] + b[sX(s) - x(0)] + kX(s) = 0$$

or

$$(ms^2 + bs + k)X(s) = mx(0)s + m\dot{x}(0) + bx(0)$$

Solving this last equation for $X(s)$ and substituting the given numerical values, we obtain

$$X(s) = \frac{mx(0)s + m\dot{x}(0) + bx(0)}{ms^2 + bs + k}$$

$$= \frac{0.1s + 0.35}{s^2 + 3s + 2}$$

This equation can be written as

$$X(s) = \frac{0.1s^2 + 0.35s}{s^2 + 3s + 2}\frac{1}{s}$$

Hence the motion of the mass m may be obtained as the unit-step response of the following system:

$$G(s) = \frac{0.1s^2 + 0.35s}{s^2 + 3s + 2}$$

MATLAB Program 5–14 will give a plot of the motion of the mass. The plot is shown in Figure 5–31.

MATLAB Program 5–14

```
% -------------- Response to initial condition ---------------

% ***** System response to initial condition is converted to
% a unit-step response by modifying the numerator polynomial *****

% ***** Enter the numerator and denominator of the transfer
% function G(s) *****

num = [0.1  0.35  0];
den = [1  3  2];

% ***** Enter the following step-response command *****

step(num,den)

% ***** Enter grid and title of the plot *****

grid
title('Response of Spring-Mass-Damper System to Initial Condition')
```

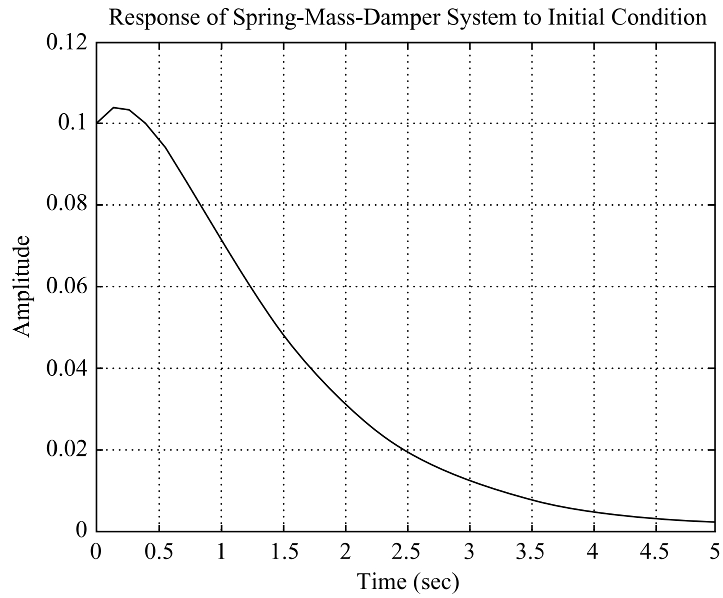


Figure 5–31 Response of the mechanical system considered in Example 5–8.

Response to Initial Condition (State-Space Approach, Case 1). Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{5–49}$$

Let us obtain the response $\mathbf{x}(t)$ when the initial condition $\mathbf{x}(0)$ is specified. Assume that there is no external input function acting on this system. Assume also that $\mathbf{x}$ is an n-vector.

First, take Laplace transforms of both sides of Equation (5–49).

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$$

This equation can be rewritten as

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(\mathrm{s}) + \mathbf{x}(0) \tag{5–50}$$

Taking the inverse Laplace transform of Equation (5–50), we obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{x}(0)\,\delta(t) \tag{5–51}$$

(Notice that by taking the Laplace transform of a differential equation and then by taking the inverse Laplace transform of the Laplace-transformed equation we generate a differential equation that involves the initial condition.)

Now define

$$\dot{\mathbf{z}} = \mathbf{x} \tag{5–52}$$

Then Equation (5–51) can be written as

$$\ddot{\mathbf{z}} = \mathbf{A}\dot{\mathbf{z}} + \mathbf{x}(0)\,\delta(t) \tag{5–53}$$

By integrating Equation (5–53) with respect to t, we obtain

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{x}(0)1(t) = \mathbf{A}\mathbf{z} + \mathbf{B}u \tag{5–54}$$

where

$$\mathbf{B} = \mathbf{x}(0), \qquad u = 1(t)$$

Referring to Equation (5–52), the state $\mathbf{x}(t)$ is given by $\dot{\mathbf{z}}(t)$. Thus,

$$\mathbf{x} = \dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u \tag{5–55}$$

The solution of Equations (5–54) and (5–55) gives the response to the initial condition.

Summarizing, the response of Equation (5–49) to the initial condition $\mathbf{x}(0)$ is obtained by solving the following state-space equations:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

$$\mathbf{x} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

where

$$\mathbf{B} = \mathbf{x}(0), \qquad u = 1(t)$$

MATLAB commands to obtain the response curves, where we do not specify the time vector t (that is, we let the time vector be determined automatically by MATLAB), are given next.

```
% Specify matrices A and B
[x,z,t] = step(A,B,A,B);
x1 = [1  0  0 ... 0]*x';
x2 = [0  1  0 ... 0]*x';
     .
     .
     .
xn = [0  0  0 ... 1]*x';
plot(t,x1,t,x2, ... ,t,xn)
```

If we choose the time vector t (for example, let the computation time duration be from t = 0 to t = tp with the computing time increment of Δt), then we use the following MATLAB commands:

```
t = 0: Δt: tp;
% Specify matrices A and B
[x,z,t] = step(A,B,A,B,1,t);
x1 = [1  0  0 ... 0]*x';
x2 = [0  1  0 ... 0]*x';
     .
     .
     .
xn = [0  0  0 ... 1]*x';
plot(t,x1,t,x2, ... ,t,xn)
```

(See, for example, Example 5–9.)

Response to Initial Condition (State-Space Approach, Case 2). Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{5–56}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{5–57}$$

(Assume that $\mathbf{x}$ is an n-vector and $\mathbf{y}$ is an m-vector.)

Similar to case 1, by defining

$$\dot{\mathbf{z}} = \mathbf{x}$$

we can obtain the following equation:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{x}(0)1(t) = \mathbf{A}\mathbf{z} + \mathbf{B}u \tag{5–58}$$

where

$$\mathbf{B} = \mathbf{x}(0), \qquad u = 1(t)$$

Noting that $\mathbf{x} = \dot{\mathbf{z}}$, Equation (5–57) can be written as

$$\mathbf{y} = \mathbf{C}\dot{\mathbf{z}} \tag{5–59}$$

By substituting Equation (5–58) into Equation (5–59), we obtain

$$\mathbf{y} = \mathbf{C}(\mathbf{A}\mathbf{z} + \mathbf{B}u) = \mathbf{C}\mathbf{A}\mathbf{z} + \mathbf{C}\mathbf{B}u \tag{5–60}$$

The solution of Equations (5–58) and (5–60), rewritten here

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

$$\mathbf{y} = \mathbf{C}\mathbf{A}\mathbf{z} + \mathbf{C}\mathbf{B}u$$

where $\mathbf{B} = \mathbf{x}(0)$ and $u = 1(t)$, gives the response of the system to a given initial condition. MATLAB commands to obtain the response curves (output curves y1 versus t, y2 versus t, ... , ym versus t) are shown next for two cases:

Case A. When the time vector t is not specified (that is, the time vector t is to be determined automatically by MATLAB):

```
% Specify matrices A, B, and C
[y,z,t] = step(A,B,C*A,C*B);
y1 = [1  0  0 ... 0]*y';
y2 = [0  1  0 ... 0]*y';
     .
     .
     .
ym = [0  0  0 ... 1]*y';
plot(t,y1,t,y2, ... ,t,ym)
```

Case B. When the time vector t is specified:

```
t = 0: Δt: tp;
% Specify matrices A, B, and C
[y,z,t] = step(A,B,C*A,C*B,1,t)
y1 = [1  0  0 ... 0]*y';
y2 = [0  1  0 ... 0]*y';
  .
  .
  .
ym = [0  0  0 ... 1]*y';
plot(t,y1,t,y2, ... ,t,ym)
```

EXAMPLE 5–9 Obtain the response of the system subjected to the given initial condition.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

Obtaining the response of the system to the given initial condition resolves to solving the unit-step response of the following system:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$
$$\mathbf{x} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

where

$$\mathbf{B} = \mathbf{x}(0), \qquad u = 1(t)$$

Hence a possible MATLAB program for obtaining the response may be given as shown in MATLAB Program 5–15. The resulting response curves are shown in Figure 5–32.

MATLAB Program 5–15

```
t = 0:0.01:3;
A = [0  1;-10  -5];
B = [2;1];
[x,z,t] = step(A,B,A,B,1,t);
x1 = [1  0]*x';
x2 = [0  1]*x';
plot(t,x1,'x',t,x2,'-')
grid
title('Response to Initial Condition')
xlabel('t Sec')
ylabel('State Variables x1 and x2')
gtext('x1')
gtext('x2')
```

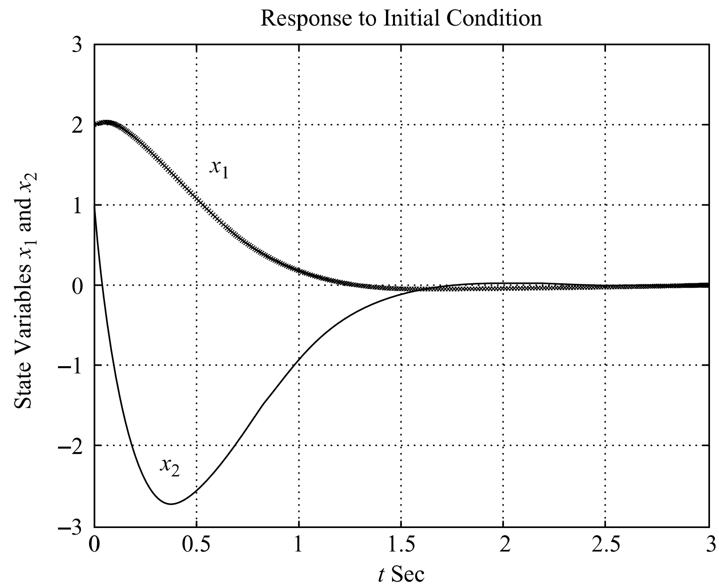


Figure 5–32
Response of system in Example 5–9 to initial condition.

For an illustrative example of how to use Equations (5–58) and (5–60) to find the response to the initial condition, see Problem **A–5–16**.

Obtaining Response to Initial Condition by Use of Command Initial. If the system is given in the state-space form, then the following command

```
initial(A,B,C,D,[initial condition],t)
```

will produce the response to the initial condition.

Suppose that we have the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \mathbf{C} = [0 \quad 0], \qquad D = 0$$

$$\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Then the command "initial" can be used as shown in MATLAB Program 5–16 to obtain the response to the initial condition. The response curves $x_1(t)$ and $x_2(t)$ are shown in Figure 5–33. They are the same as those shown in Figure 5–32.

MATLAB Program 5–16

```
t = 0:0.05:3;
A = [0  1;-10  -5];
B = [0;0];
C = [0  0];
D = [0];
[y,x] = initial(A,B,C,D,[2;1],t);
x1 = [1  0]*x';
x2 = [0  1]*x';
plot(t,x1,'o',t,x1,t,x2,'x',t,x2)
grid
title('Response to Initial Condition')
xlabel('t Sec')
ylabel('State Variables x1 and x2')
gtext('x1')
gtext('x2')
```

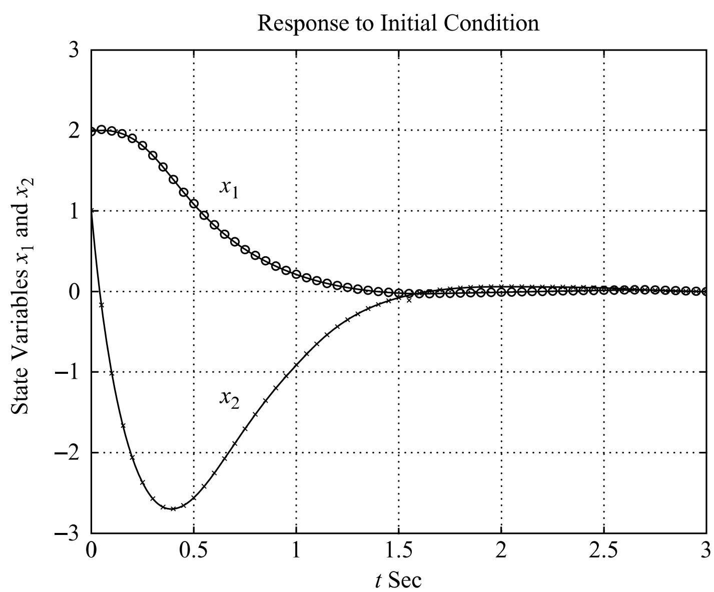


Figure 5–33 Response curves to initial condition.

EXAMPLE 5–10 Consider the following system that is subjected to the initial condition. (No external forcing function is present.)

$$\dddot{y} + 8\ddot{y} + 17\dot{y} + 10y = 0$$

$$y(0) = 2, \qquad \dot{y}(0) = 1, \qquad \ddot{y}(0) = 0.5$$

Obtain the response $y(t)$ to the given initial condition.

By defining the state variables as

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$

we obtain the following state-space representation for the system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0.5 \end{bmatrix}$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A possible MATLAB program to obtain the response $y(t)$ is given in MATLAB Program 5–17. The resulting response curve is shown in Figure 5–34.

MATLAB Program 5–17

```
t = 0:0.05:10;
A = [0  1  0;0  0  1;-10  -17  -8];
B = [0;0;0];
C = [1  0  0];
D = [0];
y = initial(A,B,C,D,[2;1;0.5],t);
plot(t,y)
grid
title('Response to Initial Condition')
xlabel('t (sec)')
ylabel('Output y')
```

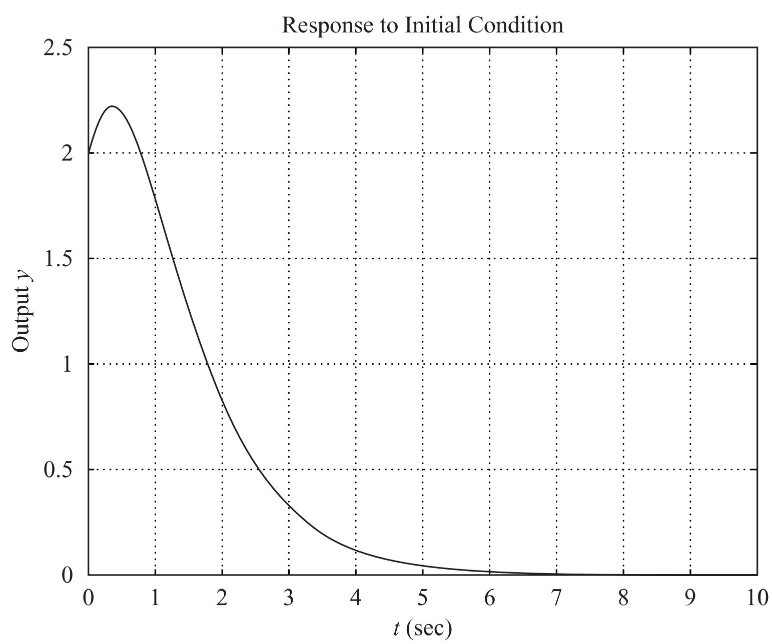


Figure 5–34
Response $y(t)$ to initial condition.

5–6 ROUTH'S STABILITY CRITERION

The most important problem in linear control systems concerns stability. That is, under what conditions will a system become unstable? If it is unstable, how should we stabilize the system? In Section 5–4 it was stated that a control system is stable if and only if all closed-loop poles lie in the left-half s plane. Most linear closed-loop systems have closed-loop transfer functions of the form

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} = \frac{B(s)}{A(s)}$$

where the a's and b's are constants and $m \leq n$. A simple criterion, known as Routh's stability criterion, enables us to determine the number of closed-loop poles that lie in the right-half s plane without having to factor the denominator polynomial. (The polynomial may include parameters that MATLAB cannot handle.)

Routh's Stability Criterion. Routh's stability criterion tells us whether or not there are unstable roots in a polynomial equation without actually solving for them. This stability criterion applies to polynomials with only a finite number of terms. When the criterion is applied to a control system, information about absolute stability can be obtained directly from the coefficients of the characteristic equation.

The procedure in Routh's stability criterion is as follows:

1. Write the polynomial in s in the following form:

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0 \tag{5–61}$$

where the coefficients are real quantities. We assume that $a_n \neq 0$; that is, any zero root has been removed.

2. If any of the coefficients are zero or negative in the presence of at least one positive coefficient, a root or roots exist that are imaginary or that have positive real parts. Therefore, in such a case, the system is not stable. If we are interested in only the absolute stability, there is no need to follow the procedure further. Note that all the coefficients must be positive. This is a necessary condition, as may be seen from the following argument: A polynomial in s having real coefficients can always be factored into linear and quadratic factors, such as $(s + a)$ and $(s^2 + bs + c)$, where a, b, and c are real. The linear factors yield real roots and the quadratic factors yield complex-conjugate roots of the polynomial. The factor $(s^2 + bs + c)$ yields roots having negative real parts only if b and c are both positive. For all roots to have negative real parts, the constants a, b, c, and so on, in all factors must be positive. The product of any number of linear and quadratic factors containing only positive coefficients always yields a polynomial with positive coefficients. It is important to note that the condition that all the coefficients be positive is not sufficient to assure stability. The necessary but not sufficient condition for stability is that the coefficients of Equation (5–61) all be present and all have a positive sign. (If all a's are negative, they can be made positive by multiplying both sides of the equation by -1.)

3. If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern:

$$
\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_1 & & & &
\end{array}
$$

The process of forming rows continues until we run out of elements. (The total number of rows is $n + 1$.) The coefficients b_1, b_2, b_3, and so on, are evaluated as follows:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$\vdots$$

The evaluation of the b's is continued until the remaining ones are all zero. The same pattern of cross-multiplying the coefficients of the two previous rows is followed in evaluating the c's, d's, e's, and so on. That is,

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

$$\vdots$$

and

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

$$d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

$$\vdots$$

This process is continued until the nth row has been completed. The complete array of coefficients is triangular. Note that in developing the array an entire row may be divided or multiplied by a positive number in order to simplify the subsequent numerical calculation without altering the stability conclusion.

Routh's stability criterion states that the number of roots of Equation (5–61) with positive real parts is equal to the number of changes in sign of the coefficients of the first column of the array. It should be noted that the exact values of the terms in the first column need not be known; instead, only the signs are needed. The necessary and sufficient condition that all roots of Equation (5–61) lie in the left-half s plane is that all the coefficients of Equation (5–61) be positive and all terms in the first column of the array have positive signs.

EXAMPLE 5–11 Let us apply Routh's stability criterion to the following third-order polynomial:

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

where all the coefficients are positive numbers. The array of coefficients becomes

$$\begin{array}{lcc} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \dfrac{a_1 a_2 - a_0 a_3}{a_1} & \\ s^0 & a_3 & \end{array}$$

The condition that all roots have negative real parts is given by

$$a_1 a_2 > a_0 a_3$$

EXAMPLE 5–12 Consider the following polynomial:

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Let us follow the procedure just presented and construct the array of coefficients. (The first two rows can be obtained directly from the given polynomial. The remaining terms are obtained from these. If any coefficients are missing, they may be replaced by zeros in the array.)

$$
\begin{array}{lrrr||lrrrl}
s^4 & 1 & 3 & 5 & s^4 & 1 & 3 & 5 & \\
s^3 & 2 & 4 & 0 & s^3 & \cancel{2} & \cancel{4} & \cancel{0} & \text{The second row is divided} \\
 & & & & & 1 & 2 & 0 & \text{by 2.} \\
s^2 & 1 & 5 & & s^2 & 1 & 5 & & \\
s^1 & -6 & & & s^1 & -3 & & & \\
s^0 & 5 & & & s^0 & 5 & & &
\end{array}
$$

In this example, the number of changes in sign of the coefficients in the first column is 2. This means that there are two roots with positive real parts. Note that the result is unchanged when the coefficients of any row are multiplied or divided by a positive number in order to simplify the computation.

Special Cases. If a first-column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number ϵ and the rest of the array is evaluated. For example, consider the following equation:

$$s^3 + 2s^2 + s + 2 = 0 \tag{5–62}$$

The array of coefficients is

$$
\begin{array}{lcc}
s^3 & 1 & 1 \\
s^2 & 2 & 2 \\
s^1 & 0 \approx \epsilon & \\
s^0 & 2 &
\end{array}
$$

If the sign of the coefficient above the zero (ϵ) is the same as that below it, it indicates that there are a pair of imaginary roots. Actually, Equation (5–62) has two roots at $s = \pm j$.

If, however, the sign of the coefficient above the zero (ϵ) is opposite that below it, it indicates that there is one sign change. For example, for the equation

$$s^3 - 3s + 2 = (s - 1)^2(s + 2) = 0$$

the array of coefficients is

$$
\begin{array}{llcc}
\text{One sign change:} & s^3 & 1 & -3 \\
 & s^2 & 0 \approx \epsilon & 2 \\
 & s^1 & -3 - \dfrac{2}{\epsilon} & \\
\text{One sign change:} & s^0 & 2 &
\end{array}
$$

There are two sign changes of the coefficients in the first column. So there are two roots in the right-half s plane. This agrees with the correct result indicated by the factored form of the polynomial equation.

If all the coefficients in any derived row are zero, it indicates that there are roots of equal magnitude lying radially opposite in the s plane—that is, two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots. In such a case, the evaluation of the rest of the array can be continued by forming an auxiliary polynomial with the coefficients of the last row and by using the coefficients of the derivative of this polynomial in the next row. Such roots with equal magnitudes and lying radially opposite in the s plane can be found by solving the auxiliary polynomial, which is always even. For a $2n$-degree auxiliary polynomial, there are n pairs of equal and opposite roots. For example, consider the following equation:

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

The array of coefficients is

$$\begin{array}{llll} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \quad \leftarrow \text{Auxiliary polynomial } P(s) \\ s^3 & 0 & 0 & \end{array}$$

The terms in the s^3 row are all zero. (Note that such a case occurs only in an odd-numbered row.) The auxiliary polynomial is then formed from the coefficients of the s^4 row. The auxiliary polynomial $P(s)$ is

$$P(s) = 2s^4 + 48s^2 - 50$$

which indicates that there are two pairs of roots of equal magnitude and opposite sign (that is, two real roots with the same magnitude but opposite signs or two complex-conjugate roots on the imaginary axis). These pairs are obtained by solving the auxiliary polynomial equation $P(s) = 0$. The derivative of $P(s)$ with respect to s is

$$\frac{dP(s)}{ds} = 8s^3 + 96s$$

The terms in the s^3 row are replaced by the coefficients of the last equation—that is, 8 and 96. The array of coefficients then becomes

$$\begin{array}{llll} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ s^3 & 8 & 96 & \quad \leftarrow \text{Coefficients of } dP(s)/ds \\ s^2 & 24 & -50 & \\ s^1 & 112.7 & 0 & \\ s^0 & -50 & & \end{array}$$

We see that there is one change in sign in the first column of the new array. Thus, the original equation has one root with a positive real part. By solving for roots of the auxiliary polynomial equation,

$$2s^4 + 48s^2 - 50 = 0$$

we obtain

$$s^2 = 1, \qquad s^2 = -25$$

or

$$s = \pm 1, \qquad s = \pm j5$$

These two pairs of roots of $P(s)$ are a part of the roots of the original equation. As a matter of fact, the original equation can be written in factored form as follows:

$$(s + 1)(s - 1)(s + j5)(s - j5)(s + 2) = 0$$

Clearly, the original equation has one root with a positive real part.

Relative Stability Analysis. Routh's stability criterion provides the answer to the question of absolute stability. This, in many practical cases, is not sufficient. We usually require information about the relative stability of the system. A useful approach for examining relative stability is to shift the s-plane axis and apply Routh's stability criterion. That is, we substitute

$$s = \hat{s} - \sigma \qquad (\sigma = \text{constant})$$

into the characteristic equation of the system, write the polynomial in terms of $\hat{s}$; and apply Routh's stability criterion to the new polynomial in $\hat{s}$. The number of changes of sign in the first column of the array developed for the polynomial in $\hat{s}$ is equal to the number of roots that are located to the right of the vertical line $s = -\sigma$. Thus, this test reveals the number of roots that lie to the right of the vertical line $s = -\sigma$.

Application of Routh's Stability Criterion to Control-System Analysis. Routh's stability criterion is of limited usefulness in linear control-system analysis, mainly because it does not suggest how to improve relative stability or how to stabilize an unstable system. It is possible, however, to determine the effects of changing one or two parameters of a system by examining the values that cause instability. In the following, we shall consider the problem of determining the stability range of a parameter value.

Consider the system shown in Figure 5–35. Let us determine the range of K for stability. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2 + s + 1)(s + 2) + K}$$

The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

The array of coefficients becomes

$$\begin{array}{llll}
s^4 & 1 & 3 & K \\
s^3 & 3 & 2 & 0 \\
s^2 & \frac{7}{3} & K & \\
s^1 & 2 - \frac{9}{7}K & & \\
s^0 & K & &
\end{array}$$

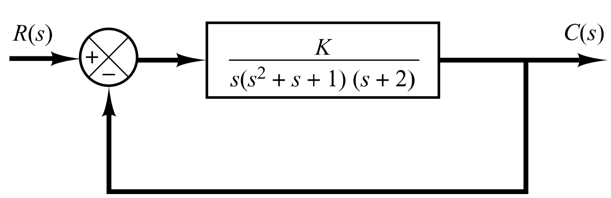


Figure 5–35
Control system.

For stability, K must be positive, and all coefficients in the first column must be positive. Therefore,

$$\frac{14}{9} > K > 0$$

When $K = \frac{14}{9}$, the system becomes oscillatory and, mathematically, the oscillation is sustained at constant amplitude.

Note that the ranges of design parameters that lead to stability may be determined by use of Routh's stability criterion.

5–7 EFFECTS OF INTEGRAL AND DERIVATIVE CONTROL ACTIONS ON SYSTEM PERFORMANCE

In this section, we shall investigate the effects of integral and derivative control actions on the system performance. Here we shall consider only simple systems, so that the effects of integral and derivative control actions on system performance can be clearly seen.

Integral Control Action. In the proportional control of a plant whose transfer function does not possess an integrator $1/s$, there is a steady-state error, or offset, in the response to a step input. Such an offset can be eliminated if the integral control action is included in the controller.

In the integral control of a plant, the control signal—the output signal from the controller—at any instant is the area under the actuating-error-signal curve up to that instant. The control signal $u(t)$ can have a nonzero value when the actuating error signal $e(t)$ is zero, as shown in Figure 5–36(a). This is impossible in the case of the proportional controller, since a nonzero control signal requires a nonzero actuating error signal. (A nonzero actuating error signal at steady state means that there is an offset.) Figure 5–36(b) shows the curve $e(t)$ versus t and the corresponding curve $u(t)$ versus t when the controller is of the proportional type.

Note that integral control action, while removing offset or steady-state error, may lead to oscillatory response of slowly decreasing amplitude or even increasing amplitude, both of which are usually undesirable.

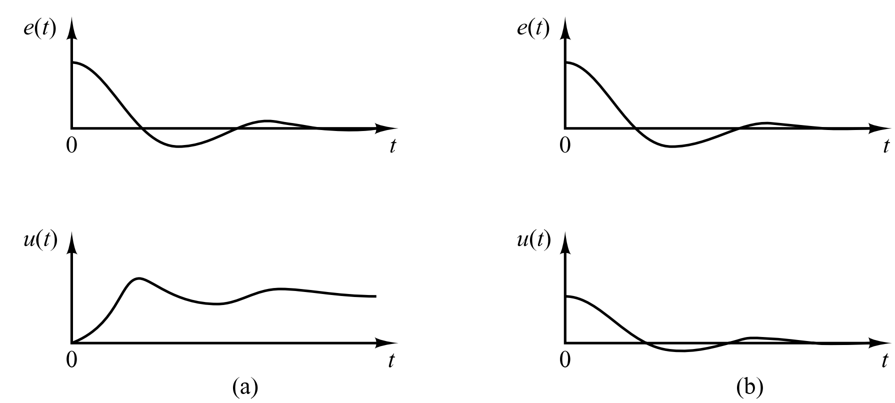


Figure 5–36 (a) Plots of $e(t)$ and $u(t)$ curves showing nonzero control signal when the actuating error signal is zero (integral control); (b) plots of $e(t)$ and $u(t)$ curves showing zero control signal when the actuating error signal is zero (proportional control).

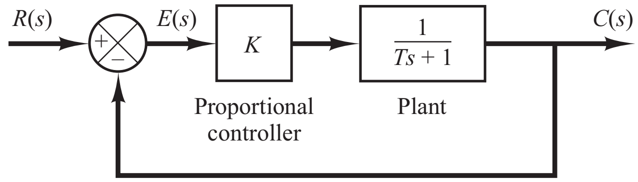


Figure 5–37 Proportional control system.

Proportional Control of Systems. We shall show that the proportional control of a system without an integrator will result in a steady-state error with a step input. We shall then show that such an error can be eliminated if integral control action is included in the controller.

Consider the system shown in Figure 5–37. Let us obtain the steady-state error in the unit-step response of the system. Define

$$G(s) = \frac{K}{Ts + 1}$$

Since

$$\frac{E(s)}{R(s)} = \frac{R(s) - C(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = \frac{1}{1 + G(s)}$$

the error $E(s)$ is given by

$$E(s) = \frac{1}{1 + G(s)} R(s) = \frac{1}{1 + \dfrac{K}{Ts + 1}} R(s)$$

For the unit-step input $R(s) = 1/s$, we have

$$E(s) = \frac{Ts + 1}{Ts + 1 + K} \frac{1}{s}$$

The steady-state error is

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{Ts + 1}{Ts + 1 + K} = \frac{1}{K + 1}$$

Such a system without an integrator in the feedforward path always has a steady-state error in the step response. Such a steady-state error is called an offset. Figure 5–38 shows the unit-step response and the offset.

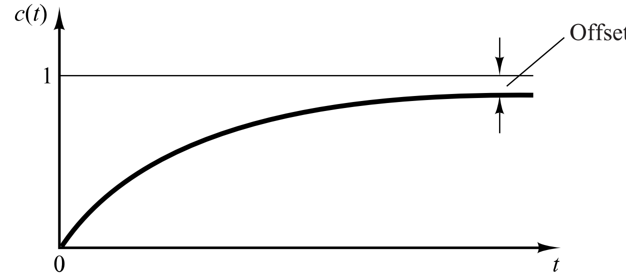


Figure 5–38 Unit-step response and offset.

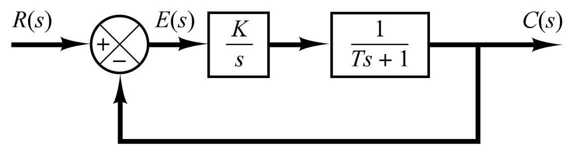


Figure 5–39 Integral control system.

Integral Control of Systems. Consider the system shown in Figure 5–39. The controller is an integral controller. The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{K}{s(Ts + 1) + K}$$

Hence

$$\frac{E(s)}{R(s)} = \frac{R(s) - C(s)}{R(s)} = \frac{s(Ts + 1)}{s(Ts + 1) + K}$$

Since the system is stable, the steady-state error for the unit-step response can be obtained by applying the final-value theorem, as follows:

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} sE(s) \\ &= \lim_{s \to 0} \frac{s^2(Ts + 1)}{Ts^2 + s + K} \frac{1}{s} \\ &= 0 \end{aligned}$$

Integral control of the system thus eliminates the steady-state error in the response to the step input. This is an important improvement over the proportional control alone, which gives offset.

Response to Torque Disturbances (Proportional Control). Let us investigate the effect of a torque disturbance occurring at the load element. Consider the system shown in Figure 5–40. The proportional controller delivers torque T to position the load element, which consists of moment of inertia and viscous friction. Torque disturbance is denoted by D.

Assuming that the reference input is zero or $R(s) = 0$, the transfer function between $C(s)$ and $D(s)$ is given by

$$\frac{C(s)}{D(s)} = \frac{1}{Js^2 + bs + K_p}$$

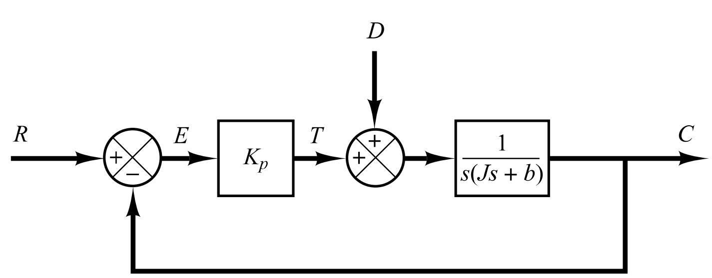


Figure 5–40 Control system with a torque disturbance.

Hence

$$\frac{E(s)}{D(s)} = -\frac{C(s)}{D(s)} = -\frac{1}{Js^2 + bs + K_p}$$

The steady-state error due to a step disturbance torque of magnitude T_d is given by

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} \frac{-s}{Js^2 + bs + K_p} \frac{T_d}{s} \\ &= -\frac{T_d}{K_p} \end{aligned}$$

At steady state, the proportional controller provides the torque $-T_d$, which is equal in magnitude but opposite in sign to the disturbance torque T_d. The steady-state output due to the step disturbance torque is

$$c_{ss} = -e_{ss} = \frac{T_d}{K_p}$$

The steady-state error can be reduced by increasing the value of the gain K_p. Increasing this value, however, will cause the system response to be more oscillatory.

Response to Torque Disturbances (Proportional-Plus-Integral Control). To eliminate offset due to torque disturbance, the proportional controller may be replaced by a proportional-plus-integral controller.

If integral control action is added to the controller, then, as long as there is an error signal, a torque is developed by the controller to reduce this error, provided the control system is a stable one.

Figure 5–41 shows the proportional-plus-integral control of the load element, consisting of moment of inertia and viscous friction.

The closed-loop transfer function between $C(s)$ and $D(s)$ is

$$\frac{C(s)}{D(s)} = \frac{s}{Js^3 + bs^2 + K_p s + \dfrac{K_p}{T_i}}$$

In the absence of the reference input, or $r(t) = 0$, the error signal is obtained from

$$E(s) = -\frac{s}{Js^3 + bs^2 + K_p s + \dfrac{K_p}{T_i}} D(s)$$

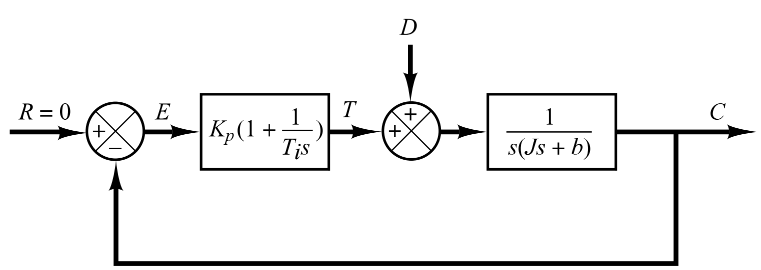
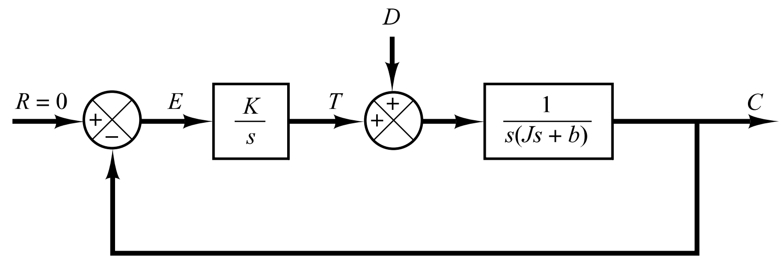


Figure 5–41 Proportional-plus-integral control of a load element consisting of moment of inertia and viscous friction.

Figure 5–42 Integral control of a load element consisting of moment of inertia and viscous friction.



If this control system is stable—that is, if the roots of the characteristic equation

$$Js^3 + bs^2 + K_p s + \frac{K_p}{T_i} = 0$$

have negative real parts—then the steady-state error in the response to a unit-step disturbance torque can be obtained by applying the final-value theorem as follows:

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} sE(s) \\ &= \lim_{s \to 0} \frac{-s^2}{Js^3 + bs^2 + K_p s + \dfrac{K_p}{T_i}} \frac{1}{s} \\ &= 0 \end{aligned}$$

Thus steady-state error to the step disturbance torque can be eliminated if the controller is of the proportional-plus-integral type.

Note that the integral control action added to the proportional controller has converted the originally second-order system to a third-order one. Hence the control system may become unstable for a large value of K_p, since the roots of the characteristic equation may have positive real parts. (The second-order system is always stable if the coefficients in the system differential equation are all positive.)

It is important to point out that if the controller were an integral controller, as in Figure 5–42, then the system always becomes unstable, because the characteristic equation

$$Js^3 + bs^2 + K = 0$$

will have roots with positive real parts. Such an unstable system cannot be used in practice.

Note that in the system of Figure 5–41 the proportional control action tends to stabilize the system, while the integral control action tends to eliminate or reduce steady-state error in response to various inputs.

Derivative Control Action. Derivative control action, when added to a proportional controller, provides a means of obtaining a controller with high sensitivity. An advantage of using derivative control action is that it responds to the rate of change of the actuating error and can produce a significant correction before the magnitude of the actuating error becomes too large. Derivative control thus anticipates the actuating error, initiates an early corrective action, and tends to increase the stability of the system.

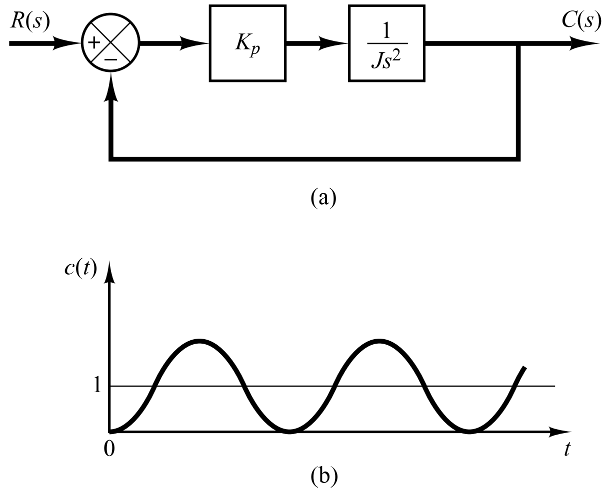


Figure 5–43
(a) Proportional control of a system with inertia load; (b) response to a unit-step input.

Although derivative control does not affect the steady-state error directly, it adds damping to the system and thus permits the use of a larger value of the gain K, which will result in an improvement in the steady-state accuracy.

Because derivative control operates on the rate of change of the actuating error and not the actuating error itself, this mode is never used alone. It is always used in combination with proportional or proportional-plus-integral control action.

Proportional Control of Systems with Inertia Load. Before we discuss further the effect of derivative control action on system performance, we shall consider the proportional control of an inertia load.

Consider the system shown in Figure 5–43(a). The closed-loop transfer function is obtained as

$$\frac{C(s)}{R(s)} = \frac{K_p}{Js^2 + K_p}$$

Since the roots of the characteristic equation

$$Js^2 + K_p = 0$$

are imaginary, the response to a unit-step input continues to oscillate indefinitely, as shown in Figure 5–43(b).

Control systems exhibiting such response characteristics are not desirable. We shall see that the addition of derivative control will stabilize the system.

Proportional-Plus-Derivative Control of a System with Inertia Load. Let us modify the proportional controller to a proportional-plus-derivative controller whose transfer function is $K_p(1 + T_d s)$. The torque developed by the controller is proportional to $K_p(e + T_d \dot{e})$. Derivative control is essentially anticipatory, measures the instantaneous error velocity, and predicts the large overshoot ahead of time and produces an appropriate counteraction before too large an overshoot occurs.

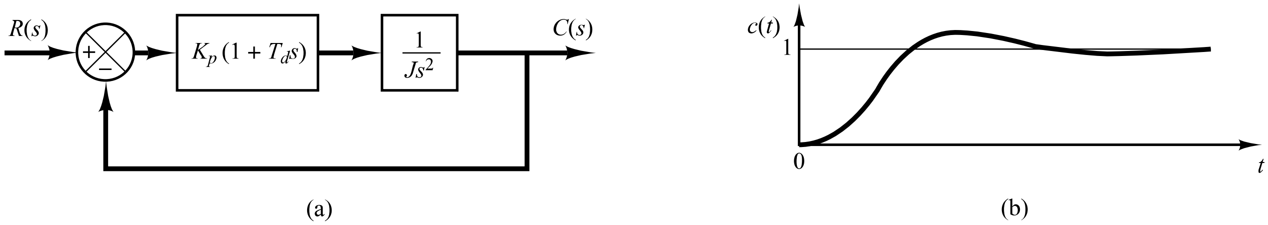


Figure 5–44
(a) Proportional-plus-derivative control of a system with inertia load; (b) response to a unit-step input.

Consider the system shown in Figure 5–44(a). The closed-loop transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{K_p(1 + T_d s)}{Js^2 + K_p T_d s + K_p}$$

The characteristic equation

$$Js^2 + K_p T_d s + K_p = 0$$

now has two roots with negative real parts for positive values of J, K_p, and T_d. Thus derivative control introduces a damping effect. A typical response curve $c(t)$ to a unit-step input is shown in Figure 5–44(b). Clearly, the response curve shows a marked improvement over the original response curve shown in Figure 5–46(b).

Proportional-Plus-Derivative Control of Second-Order Systems. A compromise between acceptable transient-response behavior and acceptable steady-state behavior may be achieved by use of proportional-plus-derivative control action.

Consider the system shown in Figure 5–45. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K_p + K_d s}{Js^2 + (B + K_d)s + K_p}$$

The steady-state error for a unit-ramp input is

$$e_{ss} = \frac{B}{K_p}$$

The characteristic equation is

$$Js^2 + (B + K_d)s + K_p = 0$$

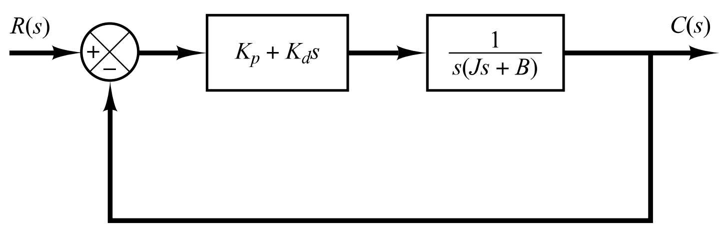


Figure 5–45
Control system.

The effective damping coefficient of this system is thus $B + K_d$ rather than B. Since the damping ratio ζ of this system is

$$\zeta = \frac{B + K_d}{2\sqrt{K_p J}}$$

it is possible to make both the steady-state error e_{ss} for a ramp input and the maximum overshoot for a step input small by making B small, K_p large, and K_d large enough so that ζ is between 0.4 and 0.7.

5–8 STEADY-STATE ERRORS IN UNITY-FEEDBACK CONTROL SYSTEMS

Errors in a control system can be attributed to many factors. Changes in the reference input will cause unavoidable errors during transient periods and may also cause steady-state errors. Imperfections in the system components, such as static friction, backlash, and amplifier drift, as well as aging or deterioration, will cause errors at steady state. In this section, however, we shall not discuss errors due to imperfections in the system components. Rather, we shall investigate a type of steady-state error that is caused by the incapability of a system to follow particular types of inputs.

Any physical control system inherently suffers steady-state error in response to certain types of inputs. A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input. (The only way we may be able to eliminate this error is to modify the system structure.) Whether a given system will exhibit steady-state error for a given type of input depends on the type of open-loop transfer function of the system, to be discussed in what follows.

Classification of Control Systems. Control systems may be classified according to their ability to follow step inputs, ramp inputs, parabolic inputs, and so on. This is a reasonable classification scheme, because actual inputs may frequently be considered combinations of such inputs. The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

Consider the unity-feedback control system with the following open-loop transfer function $G(s)$:

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

It involves the term s^N in the denominator, representing a pole of multiplicity N at the origin. The present classification scheme is based on the number of integrations indicated by the open-loop transfer function. A system is called type 0, type 1, type 2, ... , if $N = 0$, $N = 1$, $N = 2, \ldots$, respectively. Note that this classification is different from that of the order of a system. As the type number is increased, accuracy is improved; however, increasing the type number aggravates the stability problem. A compromise between steady-state accuracy and relative stability is always necessary.

We shall see later that, if $G(s)$ is written so that each term in the numerator and denominator, except the term s^N, approaches unity as s approaches zero, then the open-loop gain K is directly related to the steady-state error.

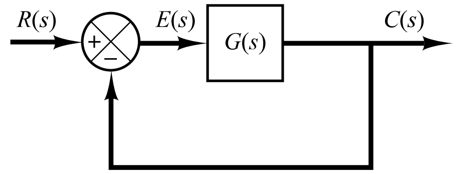


Figure 5–46
Control system.

Steady-State Errors. Consider the system shown in Figure 5–46. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

The transfer function between the error signal $e(t)$ and the input signal $r(t)$ is

$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = \frac{1}{1 + G(s)}$$

where the error $e(t)$ is the difference between the input signal and the output signal.

The final-value theorem provides a convenient way to find the steady-state performance of a stable system. Since $E(s)$ is

$$E(s) = \frac{1}{1 + G(s)} R(s)$$

the steady-state error is

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)}$$

The static error constants defined in the following are figures of merit of control systems. The higher the constants, the smaller the steady-state error. In a given system, the output may be the position, velocity, pressure, temperature, or the like. The physical form of the output, however, is immaterial to the present analysis. Therefore, in what follows, we shall call the output "position," the rate of change of the output "velocity," and so on. This means that in a temperature control system "position" represents the output temperature, "velocity" represents the rate of change of the output temperature, and so on.

Static Position Error Constant K_p. The steady-state error of the system for a unit-step input is

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} \frac{s}{1 + G(s)} \frac{1}{s} \\ &= \frac{1}{1 + G(0)} \end{aligned}$$

The static position error constant K_p is defined by

$$K_p = \lim_{s\to 0} G(s) = G(0)$$

Thus, the steady-state error in terms of the static position error constant K_p is given by

$$e_{ss} = \frac{1}{1 + K_p}$$

For a type 0 system,

$$K_p = \lim_{s\to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

For a type 1 or higher system,

$$K_p = \lim_{s\to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 1$$

Hence, for a type 0 system, the static position error constant K_p is finite, while for a type 1 or higher system, K_p is infinite.

For a unit-step input, the steady-state error e_{ss} may be summarized as follows:

$$e_{ss} = \frac{1}{1 + K}, \qquad \text{for type 0 systems}$$

$$e_{ss} = 0, \qquad \text{for type 1 or higher systems}$$

From the foregoing analysis, it is seen that the response of a feedback control system to a step input involves a steady-state error if there is no integration in the feedforward path. (If small errors for step inputs can be tolerated, then a type 0 system may be permissible, provided that the gain K is sufficiently large. If the gain K is too large, however, it is difficult to obtain reasonable relative stability.) If zero steady-state error for a step input is desired, the type of the system must be one or higher.

Static Velocity Error Constant K_v. The steady-state error of the system with a unit-ramp input is given by

$$e_{ss} = \lim_{s\to 0} \frac{s}{1 + G(s)} \frac{1}{s^2}$$

$$= \lim_{s\to 0} \frac{1}{sG(s)}$$

The static velocity error constant K_v is defined by

$$K_v = \lim_{s\to 0} sG(s)$$

Thus, the steady-state error in terms of the static velocity error constant K_v is given by

$$e_{ss} = \frac{1}{K_v}$$

The term *velocity error* is used here to express the steady-state error for a ramp input. The dimension of the velocity error is the same as the system error. That is, velocity error is not an error in velocity, but it is an error in position due to a ramp input.

For a type 0 system,

$$K_v = \lim_{s\to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

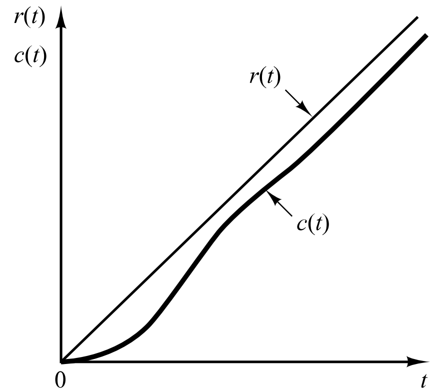


Figure 5–47
Response of a type 1 unity-feedback system to a ramp input.

For a type 1 system,

$$K_v = \lim_{s\to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

For a type 2 or higher system,

$$K_v = \lim_{s\to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 2$$

The steady-state error e_{ss} for the unit-ramp input can be summarized as follows:

$$e_{ss} = \frac{1}{K_v} = \infty, \qquad \text{for type 0 systems}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}, \qquad \text{for type 1 systems}$$

$$e_{ss} = \frac{1}{K_v} = 0, \qquad \text{for type 2 or higher systems}$$

The foregoing analysis indicates that a type 0 system is incapable of following a ramp input in the steady state. The type 1 system with unity feedback can follow the ramp input with a finite error. In steady-state operation, the output velocity is exactly the same as the input velocity, but there is a positional error. This error is proportional to the velocity of the input and is inversely proportional to the gain K. Figure 5–47 shows an example of the response of a type 1 system with unity feedback to a ramp input. The type 2 or higher system can follow a ramp input with zero error at steady state.

Static Acceleration Error Constant K_a. The steady-state error of the system with a unit-parabolic input (acceleration input), which is defined by

$$r(t) = \frac{t^2}{2}, \qquad \text{for } t \geq 0$$

$$= 0, \qquad \text{for } t < 0$$

is given by

$$e_{ss} = \lim_{s\to 0} \frac{s}{1 + G(s)} \frac{1}{s^3}$$

$$= \frac{1}{\lim_{s\to 0} s^2 G(s)}$$

The static acceleration error constant K_a is defined by the equation

$$K_a = \lim_{s\to 0} s^2 G(s)$$

The steady-state error is then

$$e_{ss} = \frac{1}{K_a}$$

Note that the acceleration error, the steady-state error due to a parabolic input, is an error in position.

The values of K_a are obtained as follows:

For a type 0 system,

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

For a type 1 system,

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

For a type 2 system,

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^2(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

For a type 3 or higher system,

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 3$$

Thus, the steady-state error for the unit parabolic input is

$$e_{ss} = \infty, \qquad \text{for type 0 and type 1 systems}$$

$$e_{ss} = \frac{1}{K}, \qquad \text{for type 2 systems}$$

$$e_{ss} = 0, \qquad \text{for type 3 or higher systems}$$

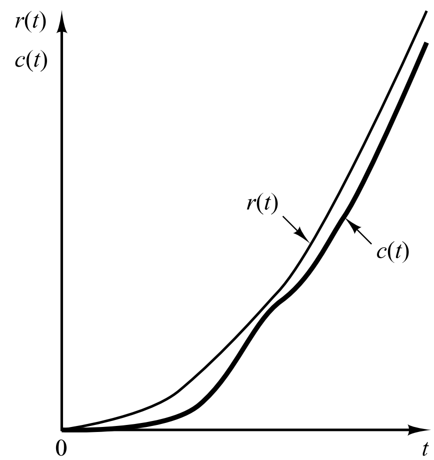


Figure 5–48
Response of a type 2 unity-feedback system to a parabolic input.

Note that both type 0 and type 1 systems are incapable of following a parabolic input in the steady state. The type 2 system with unity feedback can follow a parabolic input with a finite error signal. Figure 5–48 shows an example of the response of a type 2 system with unity feedback to a parabolic input. The type 3 or higher system with unity feedback follows a parabolic input with zero error at steady state.

Summary. Table 5–1 summarizes the steady-state errors for type 0, type 1, and type 2 systems when they are subjected to various inputs. The finite values for steady-state errors appear on the diagonal line. Above the diagonal, the steady-state errors are infinity; below the diagonal, they are zero.

Table 5–1 Steady-State Error in Terms of Gain K

	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acceleration Input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1 + K}$	∞	∞
Type 1 system	0	$\frac{1}{K}$	∞
Type 2 system	0	0	$\frac{1}{K}$

Remember that the terms *position error, velocity error*, and *acceleration error* mean steady-state deviations in the output position. A finite velocity error implies that after transients have died out, the input and output move at the same velocity but have a finite position difference.

The error constants K_p, K_v, and K_a describe the ability of a unity-feedback system to reduce or eliminate steady-state error. Therefore, they are indicative of the steady-state performance. It is generally desirable to increase the error constants, while maintaining the transient response within an acceptable range. It is noted that to improve the steady-state performance we can increase the type of the system by adding an integrator or integrators to the feedforward path. This, however, introduces an additional stability problem. The design of a satisfactory system with more than two integrators in series in the feedforward path is generally not easy.

EXAMPLE PROBLEMS AND SOLUTIONS

A–5–1. In the system of Figure 5–49, $x(t)$ is the input displacement and $\theta(t)$ is the output angular displacement. Assume that the masses involved are negligibly small and that all motions are restricted to be small; therefore, the system can be considered linear. The initial conditions for x and θ are zeros, or $x(0-) = 0$ and $\theta(0-) = 0$. Show that this system is a differentiating element. Then obtain the response $\theta(t)$ when $x(t)$ is a unit-step input.

Solution. The equation for the system is

$$b(\dot{x} - L\dot{\theta}) = kL\theta$$

or

$$L\dot{\theta} + \frac{k}{b}L\theta = \dot{x}$$

The Laplace transform of this last equation, using zero initial conditions, gives

$$\left(Ls + \frac{k}{b}L\right)\Theta(s) = sX(s)$$

And so

$$\frac{\Theta(s)}{X(s)} = \frac{1}{L}\frac{s}{s + (k/b)}$$

Thus the system is a differentiating system.

For the unit-step input $X(s) = 1/s$, the output $\Theta(s)$ becomes

$$\Theta(s) = \frac{1}{L}\frac{1}{s + (k/b)}$$

The inverse Laplace transform of $\Theta(s)$ gives

$$\theta(t) = \frac{1}{L}e^{-(k/b)t}$$

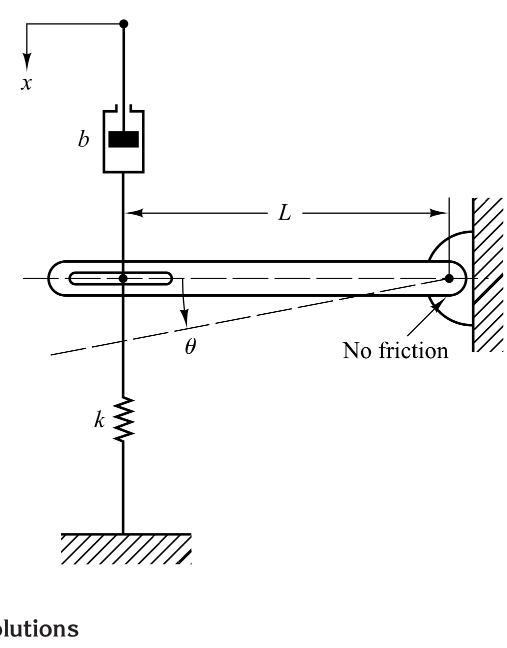


Figure 5–49
Mechanical system.

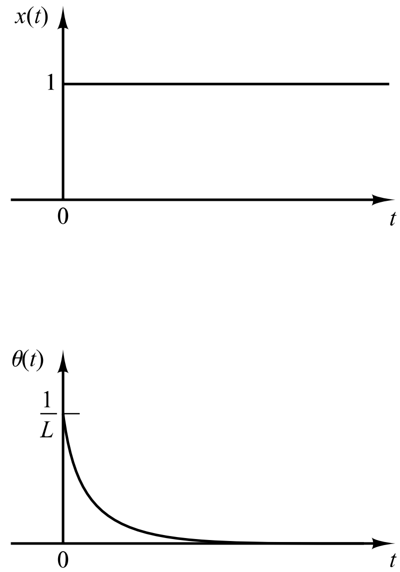


Figure 5–50 Unit-step input and the response of the mechanical system shown in Figure 5–49.

Note that if the value of k/b is large, the response $\theta(t)$ approaches a pulse signal, as shown in Figure 5–50.

A–5–2. Gear trains are often used in servo systems to reduce speed, to magnify torque, or to obtain the most efficient power transfer by matching the driving member to the given load.

Consider the gear-train system shown in Figure 5–51. In this system, a load is driven by a motor through the gear train. Assuming that the stiffness of the shafts of the gear train is infinite (there is neither backlash nor elastic deformation) and that the number of teeth on each gear is proportional to the radius of the gear, obtain the equivalent moment of inertia and equivalent viscous-friction coefficient referred to the motor shaft and referred to the load shaft.

In Figure 5–51 the numbers of teeth on gears 1, 2, 3, and 4 are N_1, N_2, N_3, and N_4, respectively. The angular displacements of shafts, 1, 2, and 3 are θ_1, θ_2, and θ_3, respectively. Thus, $\theta_2/\theta_1 = N_1/N_2$ and $\theta_3/\theta_2 = N_3/N_4$. The moment of inertia and viscous-friction coefficient of each gear-train component are denoted by $J_1, b_1; J_2, b_2;$ and $J_3, b_3;$ respectively. (J_3 and b_3 include the moment of inertia and friction of the load.)

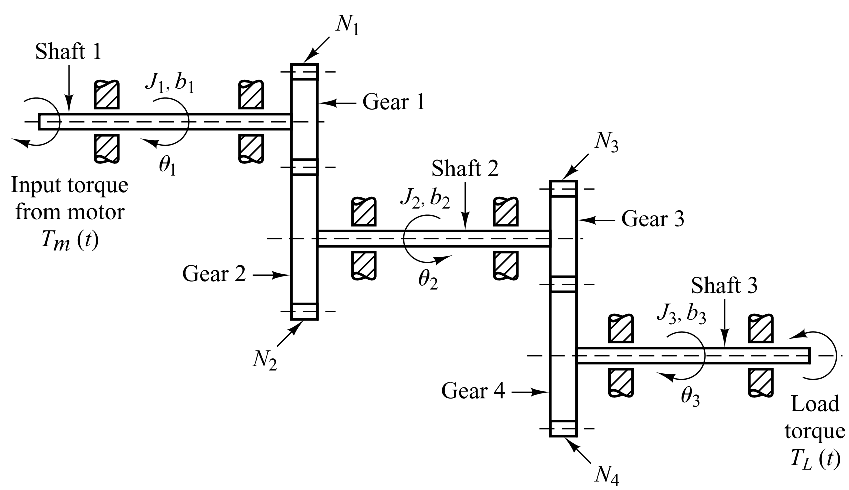


Figure 5–51 Gear-train system.

Solution. For this gear-train system, we can obtain the following equations: For shaft 1,

$$J_1\ddot{\theta}_1 + b_1\dot{\theta}_1 + T_1 = T_m \tag{5–63}$$

where T_m is the torque developed by the motor and T_1 is the load torque on gear 1 due to the rest of the gear train. For shaft 2,

$$J_2\ddot{\theta}_2 + b_2\dot{\theta}_2 + T_3 = T_2 \tag{5–64}$$

where T_2 is the torque transmitted to gear 2 and T_3 is the load torque on gear 3 due to the rest of the gear train. Since the work done by gear 1 is equal to that of gear 2,

$$T_1\theta_1 = T_2\theta_2 \qquad \text{or} \qquad T_2 = T_1\frac{N_2}{N_1}$$

If $N_1/N_2 < 1$, the gear ratio reduces the speed as well as magnifies the torque. For shaft 3,

$$J_3\ddot{\theta}_3 + b_3\dot{\theta}_3 + T_L = T_4 \tag{5–65}$$

where T_L is the load torque and T_4 is the torque transmitted to gear 4. T_3 and T_4 are related by

$$T_4 = T_3\frac{N_4}{N_3}$$

and θ_3 and θ_1 are related by

$$\theta_3 = \theta_2\frac{N_3}{N_4} = \theta_1\frac{N_1}{N_2}\frac{N_3}{N_4}$$

Eliminating T_1, T_2, T_3, and T_4 from Equations (5–63), (5–64), and (5–65) yields

$$J_1\ddot{\theta}_1 + b_1\dot{\theta}_1 + \frac{N_1}{N_2}\left(J_2\ddot{\theta}_2 + b_2\dot{\theta}_2\right) + \frac{N_1N_3}{N_2N_4}\left(J_3\ddot{\theta}_3 + b_3\dot{\theta}_3 + T_L\right) = T_m$$

Eliminating θ_2 and θ_3 from this last equation and writing the resulting equation in terms of θ_1 and its time derivatives, we obtain

$$\begin{aligned}&\left[J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2 + \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 J_3\right]\ddot{\theta}_1 \\ &\quad + \left[b_1 + \left(\frac{N_1}{N_2}\right)^2 b_2 + \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 b_3\right]\dot{\theta}_1 + \left(\frac{N_1}{N_2}\right)\left(\frac{N_3}{N_4}\right)T_L = T_m\end{aligned} \tag{5–66}$$

Thus, the equivalent moment of inertia and viscous-friction coefficient of the gear train referred to shaft 1 are given, respectively, by

$$J_{1\text{eq}} = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2 + \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 J_3$$

$$b_{1\text{eq}} = b_1 + \left(\frac{N_1}{N_2}\right)^2 b_2 + \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 b_3$$

Similarly, the equivalent moment of inertia and viscous-friction coefficient of the gear train referred to the load shaft (shaft 3) are given, respectively, by

$$J_{3\text{eq}} = J_3 + \left(\frac{N_4}{N_3}\right)^2 J_2 + \left(\frac{N_2}{N_1}\right)^2\left(\frac{N_4}{N_3}\right)^2 J_1$$

$$b_{3\text{eq}} = b_3 + \left(\frac{N_4}{N_3}\right)^2 b_2 + \left(\frac{N_2}{N_1}\right)^2\left(\frac{N_4}{N_3}\right)^2 b_1$$

The relationship between J_{1eq} and J_{3eq} is thus

$$J_{1eq} = \left(\frac{N_1}{N_2}\right)^2 \left(\frac{N_3}{N_4}\right)^2 J_{3eq}$$

and that between b_{1eq} and b_{3eq} is

$$b_{1eq} = \left(\frac{N_1}{N_2}\right)^2 \left(\frac{N_3}{N_4}\right)^2 b_{3eq}$$

The effect of J_2 and J_3 on an equivalent moment of inertia is determined by the gear ratios N_1/N_2 and N_3/N_4. For speed-reducing gear trains, the ratios, N_1/N_2 and N_3/N_4 are usually less than unity. If $N_1/N_2 \ll 1$ and $N_3/N_4 \ll 1$, then the effect of J_2 and J_3 on the equivalent moment of inertia J_{1eq} is negligible. Similar comments apply to the equivalent viscous-friction coefficient b_{1eq} of the gear train. In terms of the equivalent moment of inertia J_{1eq} and equivalent viscous-friction coefficient b_{1eq}, Equation (5–66) can be simplified to give

$$J_{1eq}\ddot{\theta}_1 + b_{1eq}\dot{\theta}_1 + nT_L = T_m$$

where

$$n = \frac{N_1 N_3}{N_2 N_4}$$

A–5–3. When the system shown in Figure 5–52(a) is subjected to a unit-step input, the system output responds as shown in Figure 5–52(b). Determine the values of K and T from the response curve.

Solution. The maximum overshoot of 25.4% corresponds to $\zeta = 0.4$. From the response curve we have

$$t_p = 3$$

Consequently,

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_n\sqrt{1-0.4^2}} = 3$$

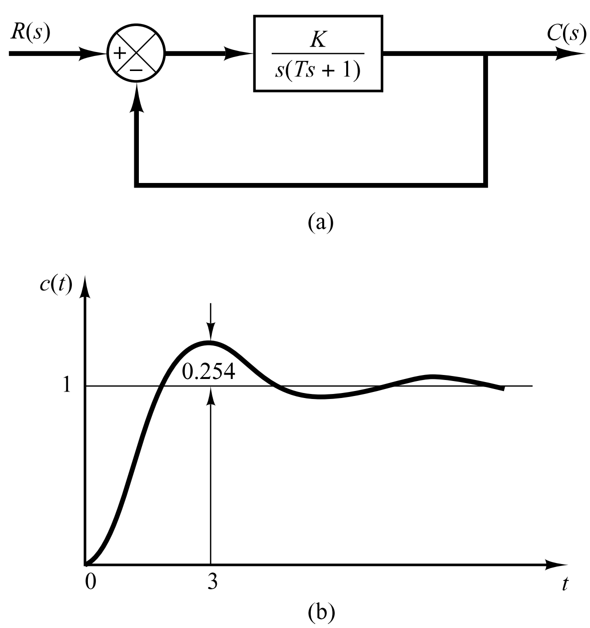


Figure 5–52
(a) Closed-loop system; (b) unit-step response curve.

It follows that

$$\omega_n = 1.14$$

From the block diagram we have

$$\frac{C(s)}{R(s)} = \frac{K}{Ts^2 + s + K}$$

from which

$$\omega_n = \sqrt{\frac{K}{T}}, \qquad 2\zeta\omega_n = \frac{1}{T}$$

Therefore, the values of T and K are determined as

$$T = \frac{1}{2\zeta\omega_n} = \frac{1}{2 \times 0.4 \times 1.14} = 1.09$$

$$K = \omega_n^2 T = 1.14^2 \times 1.09 = 1.42$$

A–5–4. Determine the values of K and k of the closed-loop system shown in Figure 5–53 so that the maximum overshoot in unit-step response is 25% and the peak time is 2 sec. Assume that $J = 1$ kg-m^2.

Solution. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Kks + K}$$

By substituting $J = 1$ kg-m^2 into this last equation, we have

$$\frac{C(s)}{R(s)} = \frac{K}{s^2 + Kks + K}$$

Note that in this problem

$$\omega_n = \sqrt{K}, \qquad 2\zeta\omega_n = Kk$$

The maximum overshoot M_p is

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

which is specified as 25%. Hence

$$e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.25$$

from which

$$\frac{\zeta\pi}{\sqrt{1 - \zeta^2}} = 1.386$$

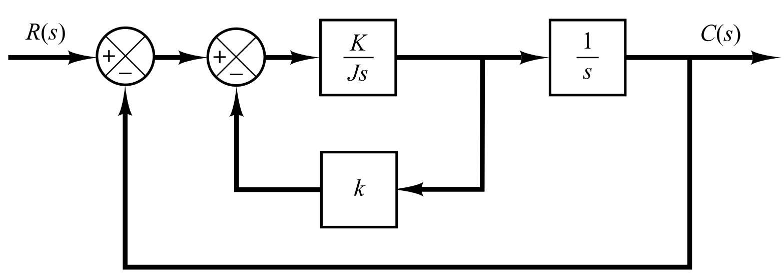


Figure 5–53
Closed-loop system.

or

$$\zeta = 0.404$$

The peak time t_p is specified as 2 sec. And so

$$t_p = \frac{\pi}{\omega_d} = 2$$

or

$$\omega_d = 1.57$$

Then the undamped natural frequency ω_n is

$$\omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}} = \frac{1.57}{\sqrt{1 - 0.404^2}} = 1.72$$

Therefore, we obtain

$$K = \omega_n^2 = 1.72^2 = 2.95 \text{ N-m}$$

$$k = \frac{2\zeta\omega_n}{K} = \frac{2 \times 0.404 \times 1.72}{2.95} = 0.471 \text{ sec}$$

A–5–5. Figure 5–54(a) shows a mechanical vibratory system. When 2 lb of force (step input) is applied to the system, the mass oscillates, as shown in Figure 5–54(b). Determine m, b, and k of the system from this response curve. The displacement x is measured from the equilibrium position.

Solution. The transfer function of this system is

$$\frac{X(s)}{P(s)} = \frac{1}{ms^2 + bs + k}$$

Since

$$P(s) = \frac{2}{s}$$

we obtain

$$X(s) = \frac{2}{s(ms^2 + bs + k)}$$

It follows that the steady-state value of x is

$$x(\infty) = \lim_{s \to 0} sX(s) = \frac{2}{k} = 0.1 \text{ ft}$$

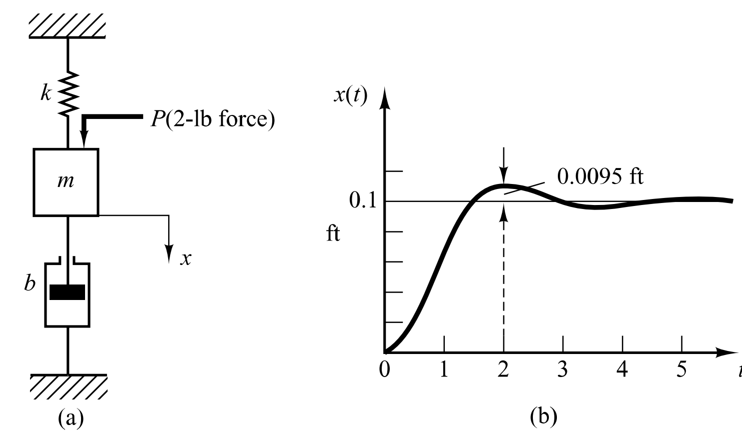


Figure 5–54
(a) Mechanical vibratory system; (b) step-response curve.

Hence

$$k = 20 \text{ lb}_f/\text{ft}$$

Note that $M_p = 9.5\%$ corresponds to $\zeta = 0.6$. The peak time t_p is given by

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1 - \zeta^2}} = \frac{\pi}{0.8\omega_n}$$

The experimental curve shows that $t_p = 2$ sec. Therefore,

$$\omega_n = \frac{3.14}{2 \times 0.8} = 1.96 \text{ rad/sec}$$

Since $\omega_n^2 = k/m = 20/m$, we obtain

$$m = \frac{20}{\omega_n^2} = \frac{20}{1.96^2} = 5.2 \text{ slugs} = 167 \text{ lb}$$

(Note that 1 slug $= 1$ lb$_f$-sec^2/ft.) Then b is determined from

$$2\zeta\omega_n = \frac{b}{m}$$

or

$$b = 2\zeta\omega_n m = 2 \times 0.6 \times 1.96 \times 5.2 = 12.2 \text{ lb}_f/\text{ft/sec}$$

A–5–6. Consider the unit-step response of the second-order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The amplitude of the exponentially damped sinusoid changes as a geometric series. At time $t = t_p = \pi/\omega_d$, the amplitude is equal to $e^{-(\sigma/\omega_d)\pi}$. After one oscillation, or at $t = t_p + 2\pi/\omega_d = 3\pi/\omega_d$, the amplitude is equal to $e^{-(\sigma/\omega_d)3\pi}$; after another cycle of oscillation, the amplitude is $e^{-(\sigma/\omega_d)5\pi}$. The logarithm of the ratio of successive amplitudes is called the *logarithmic decrement*. Determine the logarithmic decrement for this second-order system. Describe a method for experimental determination of the damping ratio from the rate of decay of the oscillation.

Solution. Let us define the amplitude of the output oscillation at $t = t_i$ to be x_i, where $t_i = t_p + (i - 1)T(T =$ period of oscillation). The amplitude ratio per one period of damped oscillation is

$$\frac{x_1}{x_2} = \frac{e^{-(\sigma/\omega_d)\pi}}{e^{-(\sigma/\omega_d)3\pi}} = e^{2(\sigma/\omega_d)\pi} = e^{2\zeta\pi/\sqrt{1-\zeta^2}}$$

Thus, the logarithmic decrement δ is

$$\delta = \ln\frac{x_1}{x_2} = \frac{2\zeta\pi}{\sqrt{1 - \zeta^2}}$$

It is a function only of the damping ratio ζ. Thus, the damping ratio ζ can be determined by use of the logarithmic. decrement.

In the experimental determination of the damping ratio ζ from the rate of decay of the oscillation, we measure the amplitude x_1 at $t = t_p$ and amplitude x_n at $t = t_p + (n - 1)T$. Note that it is necessary to choose n large enough so that the ratio x_1/x_n is not near unity. Then

$$\frac{x_1}{x_n} = e^{(n-1)2\zeta\pi/\sqrt{1-\zeta^2}}$$

or

$$\ln \frac{x_1}{x_n} = (n - 1) \frac{2\zeta\pi}{\sqrt{1 - \zeta^2}}$$

Hence

$$\zeta = \frac{\frac{1}{n-1}\left(\ln \frac{x_1}{x_n}\right)}{\sqrt{4\pi^2 + \left[\frac{1}{n-1}\left(\ln \frac{x_1}{x_n}\right)\right]^2}}$$

A–5–7. In the system shown in Figure 5–55, the numerical values of m, b, and k are given as $m = 1$ kg, $b = 2$ N-sec/m, and $k = 100$ N/m. The mass is displaced 0.05 m and released without initial velocity. Find the frequency observed in the vibration. In addition, find the amplitude four cycles later. The displacement x is measured from the equilibrium position.

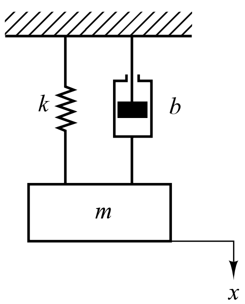


Figure 5–55 Spring-mass-damper system.

Solution. The equation of motion for the system is

$$m\ddot{x} + b\dot{x} + kx = 0$$

Substituting the numerical values for m, b, and k into this equation gives

$$\ddot{x} + 2\dot{x} + 100x = 0$$

where the initial conditions are x(0) = 0.05 and $\dot{x}(0) = 0$. From this last equation the undamped natural frequency ω_n and the damping ratio ζ are found to be

$$\omega_n = 10, \qquad \zeta = 0.1$$

The frequency actually observed in the vibration is the damped natural frequency ω_d.

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 10\sqrt{1 - 0.01} = 9.95 \text{ rad/sec}$$

In the present analysis, $\dot{x}(0)$ is given as zero. Thus, solution $x(t)$ can be written as

$$x(t) = x(0)e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}}\sin\omega_d t\right)$$

It follows that at $t = nT$, where $T = 2\pi/\omega_d$,

$$x(nT) = x(0)e^{-\zeta\omega_n nT}$$

Consequently, the amplitude four cycles later becomes

$$x(4T) = x(0)e^{-\zeta\omega_n 4T} = x(0)e^{-(0.1)(10)(4)(0.6315)}$$
$$= 0.05e^{-2.526} = 0.05 \times 0.07998 = 0.004 \text{ m}$$

A–5–8. Obtain both analytically and computationally the unit-step response of tbe following higher-order system:

$$\frac{C(s)}{R(s)} = \frac{3s^3 + 25s^2 + 72s + 80}{s^4 + 8s^3 + 40s^2 + 96s + 80}$$

[Obtain the partial-fraction expansion of $C(s)$ with MATLAB when $R(s)$ is a unit-step function.]

Solution. MATLAB Program 5–18 yields the unit-step response curve shown in Figure 5–56. It also yields the partial-fraction expansion of $C(s)$ as follows:

$$
\begin{aligned}
C(s) &= \frac{3s^3 + 25s^2 + 72s + 80}{s^4 + 8s^3 + 40s^2 + 96s + 80}\frac{1}{s} \\
&= \frac{-0.2813 - j0.1719}{s + 2 - j4} + \frac{-0.2813 + j0.1719}{s + 2 + j4} \\
&\quad + \frac{-0.4375}{s + 2} + \frac{-0.375}{(s + 2)^2} + \frac{1}{s} \\
&= \frac{-0.5626(s + 2)}{(s + 2)^2 + 4^2} + \frac{(0.3438) \times 4}{(s + 2)^2 + 4^2} \\
&\quad - \frac{0.4375}{s + 2} - \frac{0.375}{(s + 2)^2} + \frac{1}{s}
\end{aligned}
$$

MATLAB Program 5–18

```
% ------- Unit-Step Response of C(s)/R(s) and Partial-Fraction Expansion of C(s) -------

num = [3  25  72  80];
den = [1  8  40  96  80];
step(num,den);
v = [0  3  0  1.2]; axis(v), grid

% To obtain the partial-fraction expansion of C(s), enter commands
%      num1 = [3  25  72  80];
%       den1 = [1  8  40  96  80  0];
%      [r,p,k] = residue(num1,den1)

num1 = [25  72  80];
den1 = [1  8  40  96  80  0];
[r,p,k] = residue(num1,den1)

r =

   -0.2813- 0.1719i
   -0.2813+ 0.1719i
   -0.4375
   -0.3750
    1.0000

p =

   -2.0000+ 4.0000i
   -2.0000- 4.0000i
   -2.0000
   -2.0000
    0

k =

     []
```

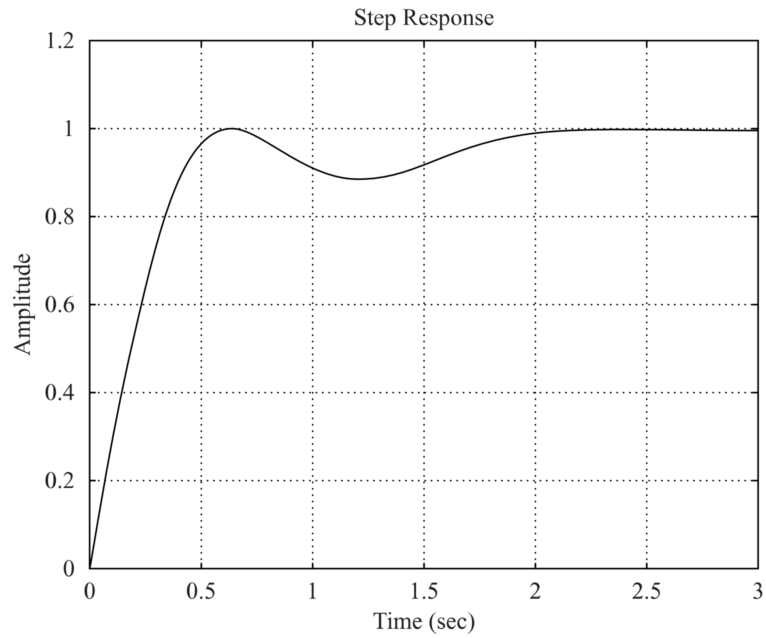


Figure 5–56
Unit-step response curve.

Hence, the time response $c(t)$ can be given by

$$c(t) = -0.5626e^{-2t}\cos 4t + 0.3438e^{-2t}\sin 4t$$
$$- 0.4375e^{-2t} - 0.375te^{-2t} + 1$$

The fact that the response curve is an exponential curve superimposed by damped sinusoidal curves can be seen from Figure 5–56.

A–5–9. When the closed-loop system involves a numerator dynamics, the unit-step response curve may exhibit a large overshoot. Obtain the unit-step response of the following system with MATLAB:

$$\frac{C(s)}{R(s)} = \frac{10s + 4}{s^2 + 4s + 4}$$

Obtain also the unit-ramp response with MATLAB.

Solution. MATLAB Program 5–19 produces the unit-step response as well as the unit-ramp response of the system. The unit-step response curve and unit-ramp response curve, together with the unit-ramp input, are shown in Figures 5–57(a) and (b), respectively.

Notice that the unit-step response curve exhibits over 215% of overshoot. The unit-ramp response curve leads the input curve. These phenomena occurred because of the presence of a large derivative term in the numerator.

MATLAB Program 5–19

```
num = [10  4];
den = [1  4  4];
t = 0:0.02:10;
y = step(num,den,t);
plot(t,y)
grid
title('Unit-Step Response')
xlabel('t (sec)')
ylabel('Output')

num1 = [10  4];
den1 = [1  4  4  0];
y1 = step(num1,den1,t);
plot(t,t,'--',t,y1)
v = [0  10  0  10]; axis(v);
grid
title('Unit-Ramp Response')
xlabel('t (sec)')
ylabel('Unit-Ramp Input and Output')
text(6.1,5.0,'Unit-Ramp Input')
text(3.5,7.1,'Output')
```

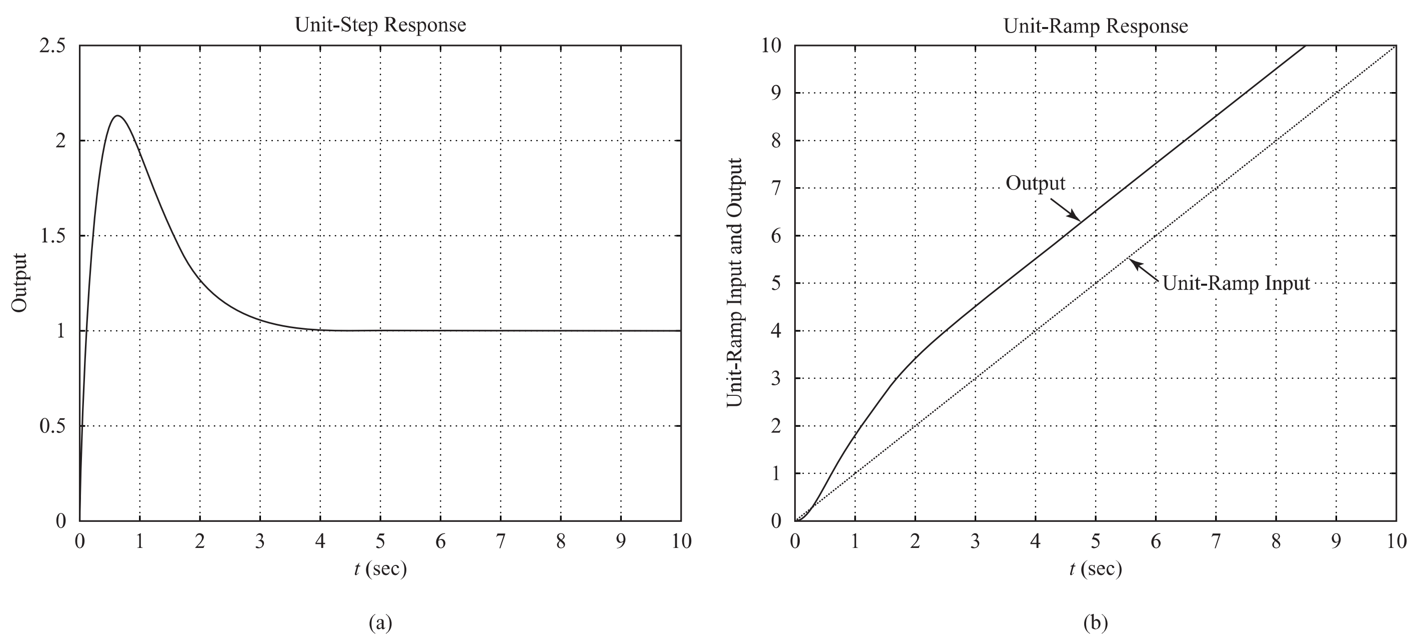


Figure 5–57
(a) Unit-step response curve; (b) unit-ramp response curve plotted with unit-ramp input.

A–5–10. Consider a higher-order system defined by

$$\frac{C(s)}{R(s)} = \frac{6.3223s^2 + 18s + 12.811}{s^4 + 6s^3 + 11.3223s^2 + 18s + 12.811}$$

Using MATLAB, plot the unit-step response curve of this system. Using MATLAB, obtain the rise time, peak time, maximum overshoot, and settling time.

Solution. MATLAB Program 5–20 plots the unit-step response curve as well as giving the rise time, peak time, maximum overshoot, and settling time. The unit-step response curve is shown in Figure 5–58.

MATLAB Program 5–20

```
% ------- This program is to plot the unit-step response curve, as well as to
% find the rise time, peak time, maximum overshoot, and settling time.
% In this program the rise time is calculated as the time required for the
% response to rise from 10% to 90% of its final value. -------

num = [6.3223  18  12.811];
den = [1  6  11.3223  18  12.811];
t = 0:0.02:20;
[y,x,t] = step(num,den,t);
plot(t,y)
grid
title('Unit-Step Response')
xlabel('t (sec)')
ylabel('Output y(t)')

r1 = 1; while y(r1) < 0.1, r1 = r1+1; end;
r2 = 1; while y(r2) < 0.9, r2 = r2+1; end;
rise_time = (r2-r1)*0. 02

rise_time =

   0.5800

[ymax,tp] = max(y);
peak_time = (tp-1)*0.02

peak_time =

   1.6600

max_overshoot = ymax-1

max_overshoot =

   0.6182

s = 1001; while y(s) > 0.98 & y(s) < 1.02; s = s-1; end;
settling_time = (s-1)*0.02

settling_time =

   10.0200
```

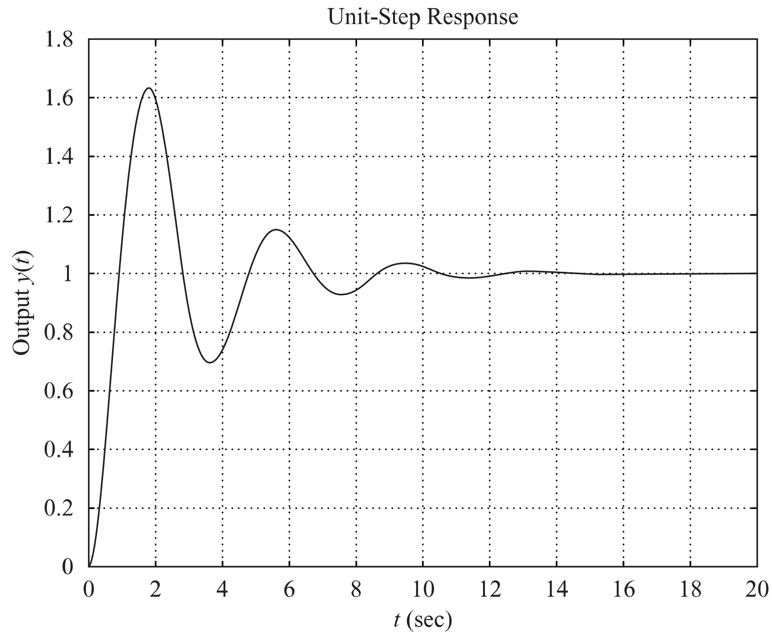


Figure 5–58
Unit-step response curve.

A–5–11. Consider the closed-loop system defined by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Using a "for loop," write a MATLAB program to obtain unit-step response of this system for the following four cases:

Case 1: $\zeta = 0.3, \quad \omega_n = 1$

Case 2: $\zeta = 0.5, \quad \omega_n = 2$

Case 3: $\zeta = 0.7, \quad \omega_n = 4$

Case 4: $\zeta = 0.8, \quad \omega_n = 6$

Solution. Define $\omega_n^2 = a$ and $2\zeta\omega_n = b$. Then, a and b each have four elements as follows:

$$a = [1 \quad 4 \quad 16 \quad 36]$$

$$b = [0.6 \quad 2 \quad 5.6 \quad 9.6]$$

Using vectors a and b, MATLAB Program 5–21 will produce the unit-step response curves as shown in Figure 5–59.

MATLAB Program 5–21

```
a = [1  4  16  36];
b = [0.6  2  5.6  9.6];
t = 0:0.1:8;
y = zeros(81,4);
      for i = 1:4;
      num = [a(i)];
      den = [1  b(i)  a(i)];
      y(:,i) = step(num,den,t);
      end
plot(t,y(:,1),'o',t,y(:,2),'x',t,y(:,3),'-',t,y(:,4),'-.')
grid
title('Unit-Step Response Curves for Four Cases')
xlabel('t Sec')
ylabel('Outputs')
gtext('1')
gtext('2')
gtext('3')
gtext('4')
```

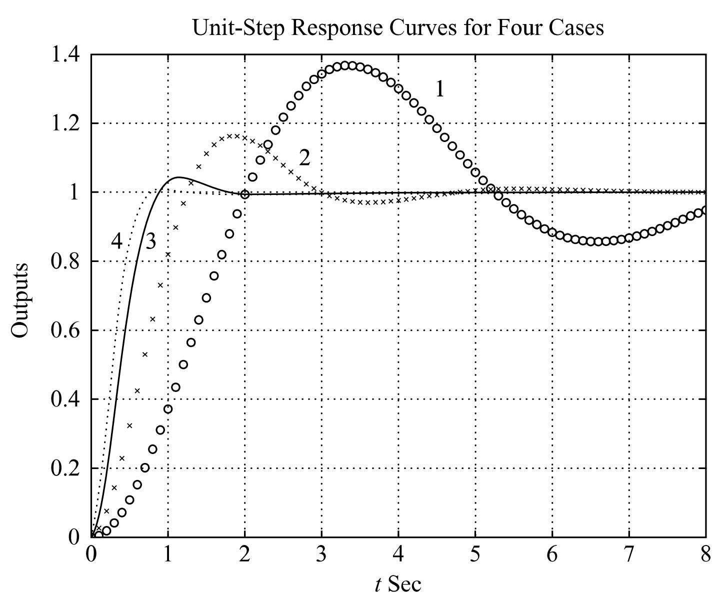


Figure 5–59
Unit-step response curves for four cases.

A–5–12. Using MATLAB, obtain the unit-ramp response of the closed-loop control system whose closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{s + 10}{s^3 + 6s^2 + 9s + 10}$$

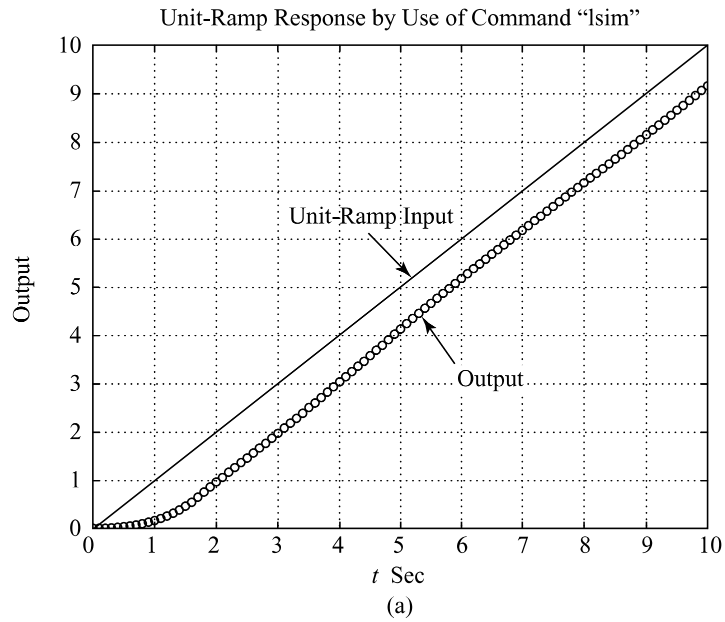
Also, obtain the response of this system when the input is given by

$$r = e^{-0.5t}$$

Solution. MATLAB Program 5–22 produces the unit-ramp response and the response to the exponential input $r = e^{-0.5t}$. The resulting response curves are shown in Figures 5–60(a) and (b), respectively.

MATLAB Program 5–22

```
% --------- Unit-Ramp Response ---------

num = [1  10];
den = [1  6  9  10];
t = 0:0.1:10;
r = t;
y = lsim(num,den,r,t);
plot(t,r,'-',t,y,'o')
grid
title('Unit-Ramp Response by Use of Command "lsim"')
xlabel('t Sec')
ylabel('Output')
text(3.2,6.5,'Unit-Ramp Input')
text(6.0,3.1,'Output')

% --------- Response to Input r1 = exp(-0.5t). ---------

num = [0  0  1  10];
den = [1  6  9  10];
t = 0:0.1:12;
r1 = exp(-0.5*t);
y1 = lsim(num,den,r1,t);
plot(t,r1,'-',t,y1,'o')
grid
title('Response to Input r1 = exp(-0.5t)')
xlabel('t Sec')
ylabel('Input and Output')
text(1.4,0.75,'Input r1 = exp(-0.5t)')
text(6.2,0.34,'Output')
```



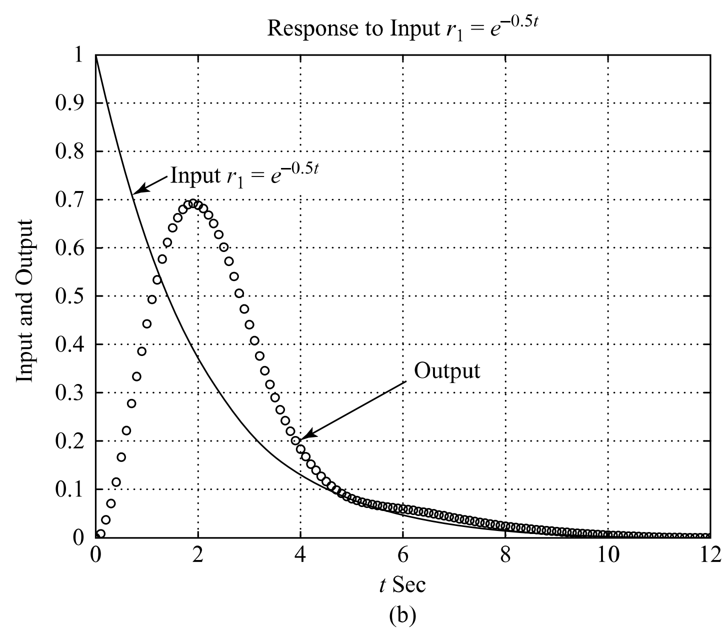


Figure 5–60
(a) Unit-ramp response curve; (b) response to exponential input $r_1 = e^{-0.5t}$.

A–5–13. Obtain the response of the closed-loop system defined by

$$\frac{C(s)}{R(s)} = \frac{5}{s^2 + s + 5}$$

when the input $r(t)$ is given by

$$r(t) = 2 + t$$

[The input $r(t)$ is a step input of magnitude 2 plus unit-ramp input.]

Solution. A possible MATLAB program is shown in MATLAB Program 5–23. The resulting response curve, together with a plot of the input function, is shown in Figure 5–61.

MATLAB Program 5–23

```
num = [5];
den = [1  1  5];
t = 0:0.05:10;
r = 2+t;
c = lsim(num,den,r,t);
plot(t,r,'-',t,c,'o')
grid
title('Response to Input r(t) = 2 + t')
xlabel('t Sec')
ylabel('Output c(t) and Input r(t) = 2 + t')
```

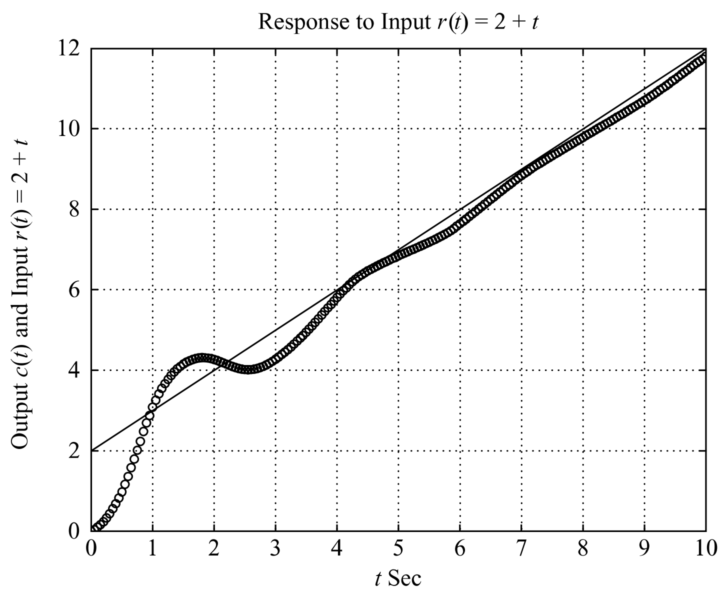


Figure 5–61 Response to input $r(t) = 2 + t$.

A–5–14. Obtain the response of the system shown in Figure 5–62 when the input $r(t)$ is given by

$$r(t) = \frac{1}{2}t^2$$

[The input $r(t)$ is the unit-acceleration input.]

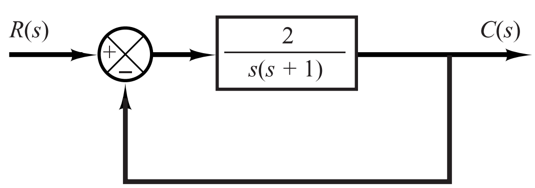


Figure 5–62 Control system.

Solution. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{2}{s^2 + s + 2}$$

MATLAB Program 5–24 produces the unit-acceleration response. The resulting response, together with the unit-acceleration input, is shown in Figure 5–63.

MATLAB Program 5–24

```
num = [2];
den = [1  1  2];
t = 0:0.2:10;
r = 0.5*t.^2;
y = lsim(num,den,r,t);
plot(t,r,'-',t,y,'o',t,y,'-')
grid
title('Unit-Acceleration Response')
xlabel('t Sec')
ylabel('Input and Output')
text(2.1,27.5,'Unit-Acceleration Input')
text(7.2,7.5,'Output')
```

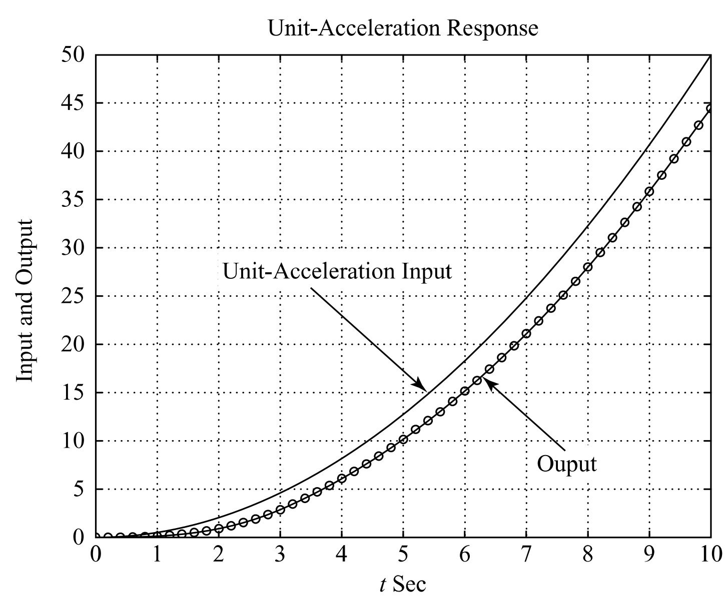


Figure 5–63 Response to unit-acceleration input.

A–5–15. Consider the system defined by

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + 2\zeta s + 1}$$

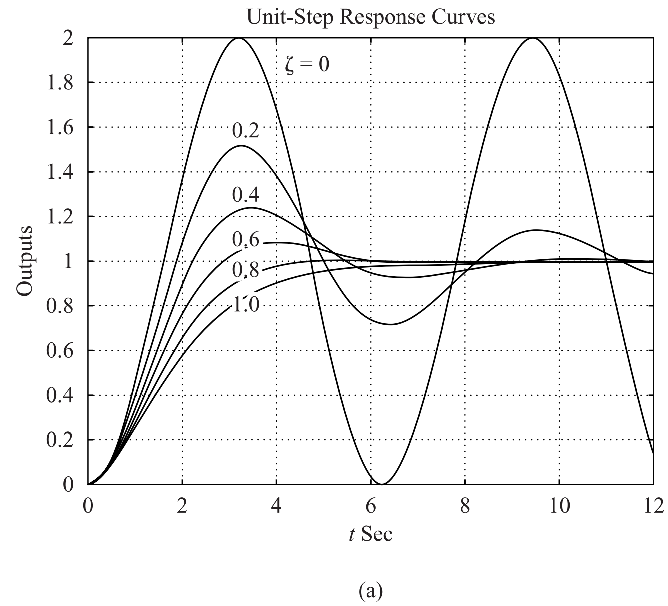
where $\zeta = 0, \quad 0.2, \quad 0.4, \quad 0.6, \quad 0.8$, and 1.0. Write a MATLAB program using a "for loop" to obtain the two-dimensional and three-dimensional plots of the system output. The input is the unit-step function.

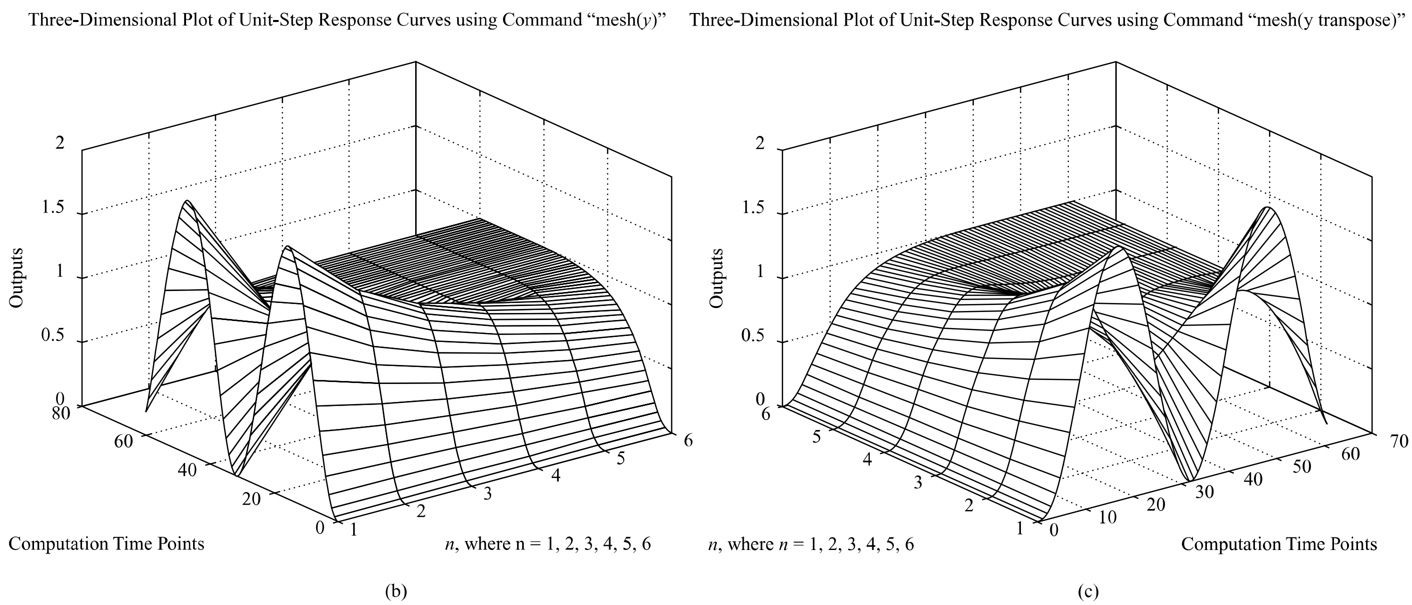
Solution. MATLAB Program 5–25 is a possible program to obtain two-dimensional and three-dimensional plots. Figure 5–64(a) is the two-dimensional plot of the unit-step response curves for various values of ζ. Figure 5–64(b) is the three-dimensional plot obtained by use of the command "mesh(y)" and Figure 5–64(c) is obtained by use of the command "mesh(y′)". (These two three-dimensional plots are basically the same. The only difference is that x axis and y axis are interchanged.)

MATLAB Program 5–25

```
t = 0:0.2:12;
    for n = 1:6;
    num = [1];
    den = [1  2*(n-1)*0.2  1];
    [y(1:61,n),x,t] = step(num,den,t);
    end
plot(t,y)
grid
title('Unit-Step Response Curves')
xlabel('t Sec')
ylabel('Outputs')
gtext('\zeta = 0'),
gtext('0.2')
gtext('0.4')
gtext('0.6')
gtext('0.8')
gtext('1.0')

% To draw a three-dimensional plot, enter the following command: mesh(y) or mesh(y').
% We shall show two three-dimensional plots, one using "mesh(y)" and the other using
% "mesh(y')". These two plots are the same, except that the x axis and y axis are
% interchanged.

mesh(y)
title('Three-Dimensional Plot of Unit-Step Response Curves using Command "mesh(y)"')
xlabel('n, where n = 1,2,3,4,5,6')
ylabel('Computation Time Points')
zlabel('Outputs')

mesh(y')
title('Three-Dimensional Plot of Unit-Step Response Curves using Command "mesh(y transpose)"')
xlabel('Computation Time Points')
ylabel('n, where n = 1,2,3,4,5,6')
zlabel('Outputs')
```

Figure 5–64
(a) Two-dimensional plot of unit-step response curves;
(b) three-dimensional plot of unit-step response curves using command "mesh(y)";
(c) three-dimensional plot of unit-step response curves using command "mesh(y′)".





A–5–16. Consider the system subjected to the initial condition as given below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0.5 \end{bmatrix}$$

$$y = [1 \;\; 0 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(There is no input or forcing function in this system.) Obtain the response y(t) versus t to the given initial condition by use of Equations (5–58) and (5–60).

Solution. A possible MATLAB program based on Equations (5–58) and (5–60) is given by MATLAB program 5–26. The response curve obtained here is shown in Figure 5–65. (Notice that this problem was solved by use of the command "initial" in Example 5–16. The response curve obtained here is exactly the same as that shown in Figure 5–34.)

MATLAB Program 5–26

```
t = 0:0.05:10;
A = [0 1 0;0 0 1;-10 -17 -8];
B = [2;1;0.5];
C=[1 0 0];
[y,x,t] = step(A,B,C*A,C*B,1,t);
plot(t,y)
grid;
title('Response to Initial Condition')
xlabel('t (sec)')
ylabel('Output y')
```

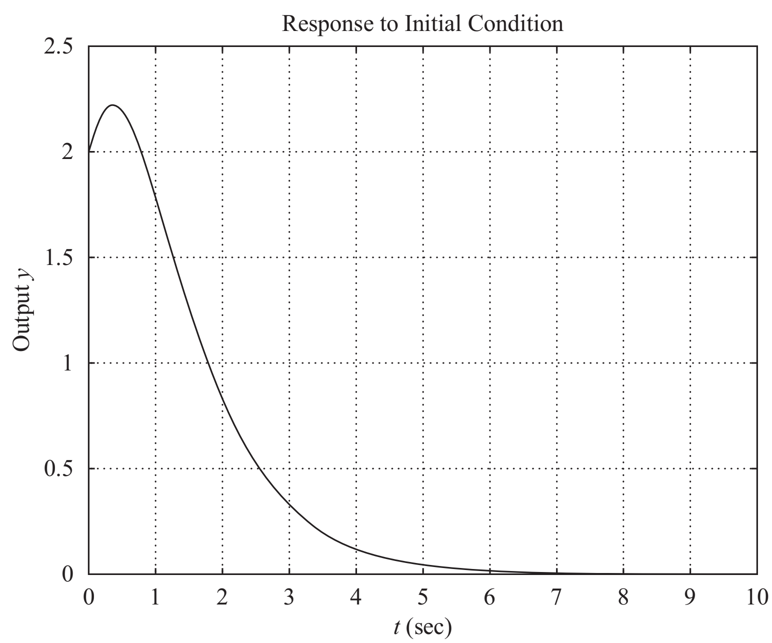


Figure 5–65 Response y(t) to the given initial condition.

A–5–17. Consider the following characteristic equation:

$$s^4 + Ks^3 + s^2 + s + 1 = 0$$

Determine the range of K for stability.

Solution. The Routh array of coefficients is

$$\begin{array}{llll} s^4 & 1 & 1 & 1 \\ s^3 & K & 1 & 0 \\ s^2 & \dfrac{K-1}{K} & 1 & \\ s^1 & 1 - \dfrac{K^2}{K-1} & & \\ s^0 & 1 & & \end{array}$$

For stability, we require that

$$K > 0$$

$$\frac{K-1}{K} > 0$$

$$1 - \frac{K^2}{K-1} > 0$$

From the first and second conditions, K must be greater than 1. For $K > 1$, notice that the term $1 - \left[K^2/(K-1)\right]$ is always negative, since

$$\frac{K - 1 - K^2}{K-1} = \frac{-1 + K(1-K)}{K-1} < 0$$

Thus, the three conditions cannot be fulfilled simultaneously. Therefore, there is no value of K that allows stability of the system.

A–5–18. Consider the characteristic equation given by

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0 \qquad (5\text{–}67)$$

The Hurwitz stability criterion, given next, gives conditions for all the roots to have negative real parts in terms of the coefficients of the polynomial. As stated in the discussions of Routh's stability criterion in Section 5–6, for all the roots to have negative real parts, all the coefficients a's must be positive. This is a necessary condition but not a sufficient condition. If this condition is not satisfied, it indicates that some of the roots have positive real parts or are imaginary or zero. A sufficient condition for all the roots to have negative real parts is given in the following Hurwitz stability criterion: If all the coefficients of the polynomial are positive, arrange these coefficients in the following determinant:

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & \cdots & \cdot & \cdot & \cdot \\ 0 & a_1 & a_3 & \cdots & a_n & 0 & 0 \\ 0 & a_0 & a_2 & \cdots & a_{n-1} & 0 & 0 \\ \cdot & \cdot & \cdot & & a_{n-2} & a_n & 0 \\ \cdot & \cdot & \cdot & & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & a_{n-4} & a_{n-2} & a_n \end{vmatrix}$$

where we substituted zero for a_s if $s > n$. For all the roots to have negative real parts, it is necessary and sufficient that successive principal minors of Δ_n be positive. The successive principal minors are the following determinants:

$$\Delta_i = \begin{vmatrix} a_1 & a_3 & \cdots & a_{2i-1} \\ a_0 & a_2 & \cdots & a_{2i-2} \\ 0 & a_1 & \cdots & a_{2i-3} \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & a_i \end{vmatrix} \qquad (i = 1, 2, \ldots, n-1)$$

where $a_s = 0$ if $s > n$. (It is noted that some of the conditions for the lower-order determinants are included in the conditions for the higher-order determinants.) If all these determinants are positive, and $a_0 > 0$ as already assumed, the equilibrium state of the system whose characteristic equation is given by Equation (5–67) is asymptotically stable. Note that exact values of determinants are not needed; instead, only signs of these determinants are needed for the stability criterion.

Now consider the following characteristic equation:

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$$

Obtain the conditions for stability using the Hurwitz stability criterion.

Solution. The conditions for stability are that all the a's be positive and that

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = a_1 a_2 - a_0 a_3 > 0$$

$$\begin{aligned} \Delta_3 &= \begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} \\ &= a_1(a_2 a_3 - a_1 a_4) - a_0 a_3^2 \\ &= a_3(a_1 a_2 - a_0 a_3) - a_1^2 a_4 > 0 \end{aligned}$$

It is clear that, if all the a's are positive and if the condition $\Delta_3 > 0$ is satisfied, the condition $\Delta_2 > 0$ is also satisfied. Therefore, for all the roots of the given characteristic equation to have negative real parts, it is necessary and sufficient that all the coefficients a's are positive and $\Delta_3 > 0$.

A–5–19. Show that the first column of the Routh array of

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

is given by

$$1, \quad \Delta_1, \quad \frac{\Delta_2}{\Delta_1}, \quad \frac{\Delta_3}{\Delta_2}, \ldots, \quad \frac{\Delta_n}{\Delta_{n-1}}$$

where

$$\Delta_r = \begin{vmatrix} a_1 & 1 & 0 & 0 & \cdot & 0 \\ a_3 & a_2 & a_1 & 1 & \cdot & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ a_{2r-1} & \cdot & \cdot & \cdot & \cdot & a_r \end{vmatrix}, \qquad (n \geq r \geq 1)$$

$$a_k = 0 \qquad \text{if } k > n$$

Solution. The Routh array of coefficients has the form

$$\begin{matrix} 1 & a_2 & a_4 & a_6 & \cdots & a_n \\ a_1 & a_3 & a_5 & \cdots & & \\ b_1 & b_2 & b_3 & \cdots & & \\ c_1 & c_2 & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \end{matrix}$$

The first term in the first column of the Routh array is 1. The next term in the first column is a_1, which is equal to Δ_1. The next term is b_1, which is equal to

$$\frac{a_1a_2 - a_3}{a_1} = \frac{\Delta_2}{\Delta_1}$$

The next term in the first column is c_1, which is equal to

$$\frac{b_1a_3 - a_1b_2}{b_1} = \frac{\left[\dfrac{a_1a_2 - a_3}{a_1}\right]a_3 - a_1\left[\dfrac{a_1a_4 - a_5}{a_1}\right]}{\left[\dfrac{a_1a_2 - a_3}{a_1}\right]}$$

$$= \frac{a_1a_2a_3 - a_3^2 - a_1^2a_4 + a_1a_5}{a_1a_2 - a_3}$$

$$= \frac{\Delta_3}{\Delta_2}$$

In a similar manner the remaining terms in the first column of the Routh array can be found.

The Routh array has the property that the last nonzero terms of any columns are the same; that is, if the array is given by

$$\begin{array}{llll}
a_0 & a_2 & a_4 & a_6 \\
a_1 & a_3 & a_5 & a_7 \\
b_1 & b_2 & b_3 & \\
c_1 & c_2 & c_3 & \\
d_1 & d_2 & & \\
e_1 & e_2 & & \\
f_1 & & & \\
g_1 & & &
\end{array}$$

then

$$a_7 = c_3 = e_2 = g_1$$

and if the array is given by

$$\begin{array}{llll}
a_0 & a_2 & a_4 & a_6 \\
a_1 & a_3 & a_5 & 0 \\
b_1 & b_2 & b_3 & \\
c_1 & c_2 & 0 & \\
d_1 & d_2 & & \\
e_1 & 0 & & \\
f_1 & & &
\end{array}$$

then

$$a_6 = b_3 = d_2 = f_1$$

In any case, the last term of the first column is equal to a_n, or

$$a_n = \frac{\Delta_{n-1}a_n}{\Delta_{n-1}} = \frac{\Delta_n}{\Delta_{n-1}}$$

For example, if $n = 4$, then

$$\Delta_4 = \begin{vmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix} = \begin{vmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{vmatrix} = \Delta_3 a_4$$

Thus it has been shown that the first column of the Routh array is given by

$$1, \quad \Delta_1, \quad \frac{\Delta_2}{\Delta_1}, \quad \frac{\Delta_3}{\Delta_2}, \quad \cdots, \quad \frac{\Delta_n}{\Delta_{n-1}}$$

A–5–20. Show that the Routh's stability criterion and Hurwitz stability criterion are equivalent.

Solution. If we write Hurwitz determinants in the triangular form

$$\Delta_i = \begin{vmatrix} a_{11} & & & & * \\ & a_{22} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & a_{ii} \end{vmatrix}, \qquad (i = 1, 2, \ldots, n)$$

where the elements below the diagonal line are all zeros and the elements above the diagonal line any numbers, then the Hurwitz conditions for asymptotic stability become

$$\Delta_i = a_{11}a_{22}\cdots a_{ii} > 0, \qquad (i = 1, 2, \ldots, n)$$

which are equivalent to the conditions

$$a_{11} > 0, \qquad a_{22} > 0, \qquad \ldots, \qquad a_{nn} > 0$$

We shall show that these conditions are equivalent to

$$a_1 > 0, \qquad b_1 > 0, \qquad c_1 > 0, \qquad \ldots$$

where $a_1, b_1, c_1, \ldots,$ are the elements of the first column in the Routh array.

Consider, for example, the following Hurwitz determinant, which corresponds to $i = 4$:

$$\Delta_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

The determinant is unchanged if we subtract from the ith row k times the jth row. By subtracting from the second row a_0/a_1 times the first row, we obtain

$$\Delta_4 = \begin{vmatrix} a_{11} & a_3 & a_5 & a_7 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

where

$$a_{11} = a_1$$

$$a_{22} = a_2 - \frac{a_0}{a_1} a_3$$

$$a_{23} = a_4 - \frac{a_0}{a_1} a_5$$

$$a_{24} = a_6 - \frac{a_0}{a_1} a_7$$

Similarly, subtracting from the fourth row a_0/a_1 times the third row yields

$$\Delta_4 = \begin{vmatrix} a_{11} & a_3 & a_5 & a_7 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_1 & a_3 & a_5 \\ 0 & 0 & \hat{a}_{43} & \hat{a}_{44} \end{vmatrix}$$

where

$$\hat{a}_{43} = a_2 - \frac{a_0}{a_1} a_3$$

$$\hat{a}_{44} = a_4 - \frac{a_0}{a_1} a_5$$

Next, subtracting from the third row a_1/a_{22} times the second row yields

$$\Delta_4 = \begin{vmatrix} a_{11} & a_3 & a_5 & a_7 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & \hat{a}_{43} & \hat{a}_{44} \end{vmatrix}$$

where

$$a_{33} = a_3 - \frac{a_1}{a_{22}} a_{23}$$

$$a_{34} = a_5 - \frac{a_1}{a_{22}} a_{24}$$

Finally, subtracting from the last row $\hat{a}_{43}/a_{33}$ times the third row yields

$$\Delta_4 = \begin{vmatrix} a_{11} & a_3 & a_5 & a_7 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{vmatrix}$$

where

$$a_{44} = \hat{a}_{44} - \frac{\hat{a}_{43}}{a_{33}} a_{34}$$

From this analysis, we see that

$$\Delta_4 = a_{11}a_{22}a_{33}a_{44}$$

$$\Delta_3 = a_{11}a_{22}a_{33}$$

$$\Delta_2 = a_{11}a_{22}$$

$$\Delta_1 = a_{11}$$

The Hurwitz conditions for asymptotic stability

$$\Delta_1 > 0, \qquad \Delta_2 > 0, \qquad \Delta_3 > 0, \qquad \Delta_4 > 0, \qquad \ldots$$

reduce to the conditions

$$a_{11} > 0, \qquad a_{22} > 0, \qquad a_{33} > 0, \qquad a_{44} > 0, \qquad \ldots$$

The Routh array for the polynomial

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$$

where $a_0 > 0$ and $n = 4$, is given by

$$\begin{array}{lll} a_0 & a_2 & a_4 \\ a_1 & a_3 & \\ b_1 & b_2 & \\ c_1 & & \\ d_1 & & \end{array}$$

From this Routh array, we see that

$$a_{11} = a_1$$

$$a_{22} = a_2 - \frac{a_0}{a_1} a_3 = b_1$$

$$a_{33} = a_3 - \frac{a_1}{a_{22}} a_{23} = \frac{a_3 b_1 - a_1 b_2}{b_1} = c_1$$

$$a_{44} = \hat{a}_{44} - \frac{\hat{a}_{43}}{a_{33}} a_{34} = a_4 = d_1$$

(The last equation is obtained using the fact that $a_{34} = 0$, $\hat{a}_{44} = a_4$, and $a_4 = b_2 = d_1$.) Hence the Hurwitz conditions for asymptotic stability become

$$a_1 > 0, \qquad b_1 > 0, \qquad c_1 > 0, \qquad d_1 > 0$$

Thus we have demonstrated that Hurwitz conditions for asymptotic stability can be reduced to Routh's conditions for asymptotic stability. The same argument can be extended to Hurwitz determinants of any order, and the equivalence of Routh's stability criterion and Hurwitz stability criterion can be established.

A–5–21. Consider the characteristic equation

$$s^4 + 2s^3 + (4 + K)s^2 + 9s + 25 = 0$$

Using the Hurwitz stability criterion, determine the range of K for stability.

Solution. Comparing the given characteristic equation

$$s^4 + 2s^3 + (4 + K)s^2 + 9s + 25 = 0$$

with the following standard fourth-order characteristic equation:

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$$

we find

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 4 + K, \quad a_3 = 9, \quad a_4 = 25$$

The Hurwitz stability criterion states that Δ_4 is given by

$$\Delta_4 = \begin{vmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

For all the roots to have negative real parts, it is necessary and sufficient that succesive principal minors of Δ_4 be positive. The successive principal minors are

$$\Delta_1 = |a_1| = 2$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} 2 & 9 \\ 1 & 4 + K \end{vmatrix} = 2K - 1$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 2 & 9 & 0 \\ 1 & 4 + K & 25 \\ 0 & 2 & 9 \end{vmatrix} = 18K - 109$$

For all principal minors to be positive, we require that $\Delta_i (i = 1, 2, 3)$ be positive. Thus, we require

$$2K - 1 > 0$$

$$18K - 109 > 0$$

from which we obtain the region of K for stability to be

$$K > \frac{109}{18}$$

A–5–22. Explain why the proportional control of a plant that does not possess an integrating property (which means that the plant transfer function does not include the factor $1/s$) suffers offset in response to step inputs.

Solution. Consider, for example, the system shown in Figure 5–66. At steady state, if c were equal to a nonzero constant r, then $e = 0$ and $u = Ke = 0$, resulting in $c = 0$, which contradicts the assumption that $c = r =$ nonzero constant.

A nonzero offset must exist for proper operation of such a control system. In other words, at steady state, if e were equal to $r/(1 + K)$, then $u = Kr/(1 + K)$ and $c = Kr/(1 + K)$, which results in the assumed error signal $e = r/(1 + K)$. Thus the offset of $r/(1 + K)$ must exist in such a system.

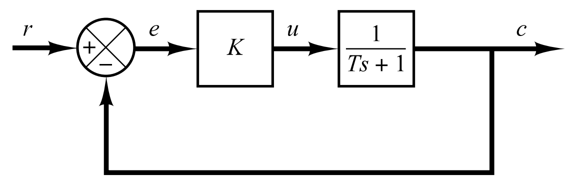


Figure 5–66
Control system.

A–5–23. The block diagram of Figure 5–67 shows a speed control system in which the output member of the system is subject to a torque disturbance. In the diagram, $\Omega_r(s)$, $\Omega(s)$, $T(s)$, and $D(s)$ are the Laplace transforms of the reference speed, output speed, driving torque, and disturbance torque, respectively. In the absence of a disturbance torque, the output speed is equal to the reference speed.

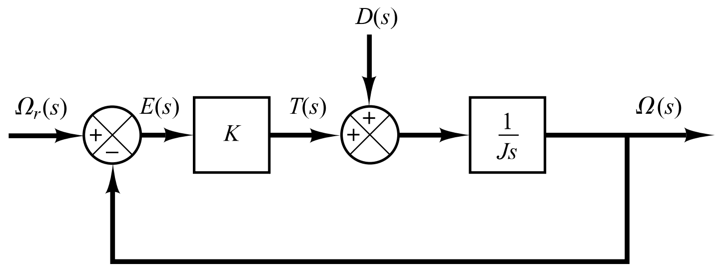


Figure 5–67
Block diagram of a speed control system.

Investigate the response of this system to a unit-step disturbance torque. Assume that the reference input is zero, or $\Omega_r(s) = 0$.

Solution. Figure 5–68 is a modified block diagram convenient for the present analysis. The closed-loop transfer function is

$$\frac{\Omega_D(s)}{D(s)} = \frac{1}{Js + K}$$

where $\Omega_D(s)$ is the Laplace transform of the output speed due to the disturbance torque. For a unit-step disturbance torque, the steady-state output velocity is

$$\begin{aligned}\omega_D(\infty) &= \lim_{s \to 0} s\Omega_D(s) \\ &= \lim_{s \to 0} \frac{s}{Js + K} \frac{1}{s} \\ &= \frac{1}{K}\end{aligned}$$

From this analysis, we conclude that, if a step disturbance torque is applied to the output member of the system, an error speed will result so that the ensuing motor torque will exactly cancel the disturbance torque. To develop this motor torque, it is necessary that there be an error in speed so that nonzero torque will result. (Discussions continue to Problem **A–5–24**.)

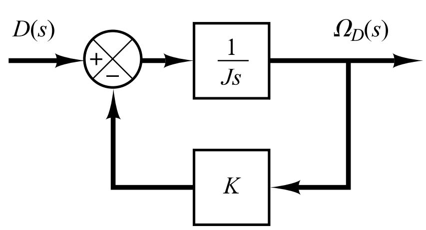


Figure 5–68
Block diagram of the speed control system of Figure 5–67 when $\Omega_r(s) = 0$.

A–5–24. In the system considered in Problem **A–5–23**, it is desired to eliminate as much as possible the speed errors due to torque disturbances.

Is it possible to cancel the effect of a disturbance torque at steady state so that a constant disturbance torque applied to the output member will cause no speed change at steady state?

Solution. Suppose that we choose a suitable controller whose transfer function is $G_c(s)$, as shown in Figure 5–69. Then in the absence of the reference input the closed-loop transfer function between the output velocity $\Omega_D(s)$ and the disturbance torque $D(s)$ is

$$\frac{\Omega_D(s)}{D(s)} = \frac{\frac{1}{Js}}{1 + \frac{1}{Js}G_c(s)}$$

$$= \frac{1}{Js + G_c(s)}$$

The steady-state output speed due to a unit-step disturbance torque is

$$\omega_D(\infty) = \lim_{s \to 0} s\Omega_D(s)$$

$$= \lim_{s \to 0} \frac{s}{Js + G_c(s)} \frac{1}{s}$$

$$= \frac{1}{G_c(0)}$$

To satisfy the requirement that

$$\omega_D(\infty) = 0$$

we must choose $G_c(0) = \infty$. This can be realized if we choose

$$G_c(s) = \frac{K}{s}$$

Integral control action will continue to correct until the error is zero. This controller, however, presents a stability problem, because the characteristic equation will have two imaginary roots.

One method of stabilizing such a system is to add a proportional mode to the controller or choose

$$G_c(s) = K_p + \frac{K}{s}$$

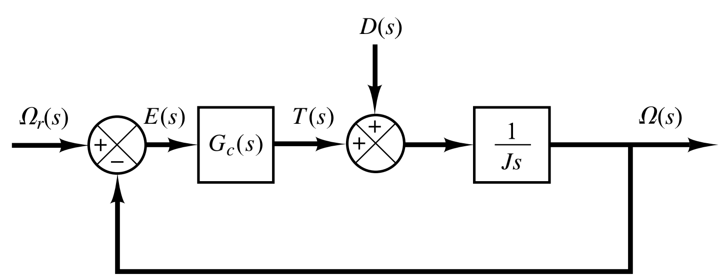


Figure 5–69
Block diagram of a speed control system.

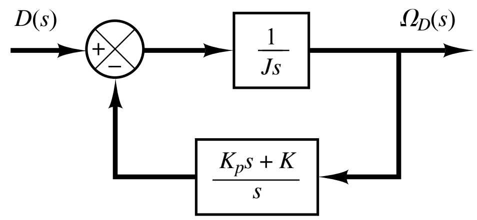


Figure 5–70
Block diagram of the speed control system of Figure 5–69 when $G_c(s) = K_p + (K/s)$ and $\Omega_r(s) = 0$.

With this controller, the block diagram of Figure 5–69 in the absence of the reference input can be modified to that of Figure 5–70. The closed-loop transfer function $\Omega_D(s)/D(s)$ becomes

$$\frac{\Omega_D(s)}{D(s)} = \frac{s}{Js^2 + K_p s + K}$$

For a unit-step disturbance torque, the steady-state output speed is

$$\omega_D(\infty) = \lim_{s\to 0} s\Omega_D(s) = \lim_{s\to 0} \frac{s^2}{Js^2 + K_p s + K}\frac{1}{s} = 0$$

Thus, we see that the proportional-plus-integral controller eliminates speed error at steady state.

The use of integral control action has increased the order of the system by 1. (This tends to produce an oscillatory response.)

In the present system, a step disturbance torque will cause a transient error in the output speed, but the error will become zero at steady state. The integrator provides a nonzero output with zero error. (The nonzero output of the integrator produces a motor torque that exactly cancels the disturbance torque.)

Note that even if the system may have an integrator in the plant (such as an integrator in the transfer function of the plant), this does not eliminate the steady-state error due to a step disturbance torque. To eliminate this, we must have an integrator before the point where the disturbance torque enters.

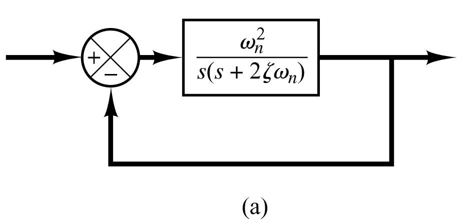
A–5–25. Consider the system shown in Figure 5–71(a). The steady-state error to a unit-ramp input is $e_{ss} = 2\zeta/\omega_n$. Show that the steady-state error for following a ramp input may be eliminated if the input is introduced to the system through a proportional-plus-derivative filter, as shown in Figure 5–71(b), and the value of k is properly set. Note that the error $e(t)$ is given by $r(t) - c(t)$.

Solution. The closed-loop transfer function of the system shown in Figure 5–71(b) is

$$\frac{C(s)}{R(s)} = \frac{(1 + ks)\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Then

$$R(s) - C(s) = \left(\frac{s^2 + 2\zeta\omega_n s - \omega_n^2 ks}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)R(s)$$



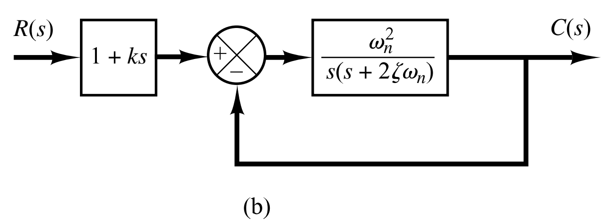


Figure 5–71
(a) Control system; (b) control system with input filter.

If the input is a unit ramp, then the steady-state error is

$$e(\infty) = r(\infty) - c(\infty)$$
$$= \lim_{s \to 0} s \left(\frac{s^2 + 2\zeta\omega_n s - \omega_n^2 ks}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) \frac{1}{s^2}$$
$$= \frac{2\zeta\omega_n - \omega_n^2 k}{\omega_n^2}$$

Therefore, if k is chosen as

$$k = \frac{2\zeta}{\omega_n}$$

then the steady-state error for following a ramp input can be made equal to zero. Note that, if there are any variations in the values of ζ and/or ω_n due to environmental changes or aging, then a nonzero steady-state error for a ramp response may result.

A–5–26. Consider the stable unity-feedback control system with feedforward transfer function $G(s)$. Suppose that the closed-loop transfer function can be written

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{(T_1 s + 1)(T_2 s + 1)\cdots(T_n s + 1)} \qquad (m \le n)$$

Show that

$$\int_0^{\infty} e(t)\,dt = (T_1 + T_2 + \cdots + T_n) - (T_a + T_b + \cdots + T_m)$$

where $e(t) = r(t) - c(t)$ is the error in the unit-step response. Show also that

$$\frac{1}{K_v} = \frac{1}{\lim_{s \to 0} sG(s)} = (T_1 + T_2 + \cdots + T_n) - (T_a + T_b + \cdots + T_m)$$

Solution. Let us define

$$(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1) = P(s)$$

and

$$(T_1 s + 1)(T_2 s + 1)\cdots(T_n s + 1) = Q(s)$$

Then

$$\frac{C(s)}{R(s)} = \frac{P(s)}{Q(s)}$$

and

$$E(s) = \frac{Q(s) - P(s)}{Q(s)} R(s)$$

For a unit-step input, $R(s) = 1/s$ and

$$E(s) = \frac{Q(s) - P(s)}{sQ(s)}$$

Since the system is stable, $\int_0^\infty e(t)\,dt$ converges to a constant value. Noting that

$$\int_0^\infty e(t)\,dt = \lim_{s\to 0} s\frac{E(s)}{s} = \lim_{s\to 0} E(s)$$

we have

$$\begin{aligned}\int_0^\infty e(t)\,dt &= \lim_{s\to 0} \frac{Q(s) - P(s)}{sQ(s)} \\ &= \lim_{s\to 0} \frac{Q'(s) - P'(s)}{Q(s) + sQ'(s)} \\ &= \lim_{s\to 0} \left[Q'(s) - P'(s)\right]\end{aligned}$$

Since

$$\lim_{s\to 0} P'(s) = T_a + T_b + \cdots + T_m$$

$$\lim_{s\to 0} Q'(s) = T_1 + T_2 + \cdots + T_n$$

we have

$$\int_0^\infty e(t)\,dt = \left(T_1 + T_2 + \cdots + T_n\right) - \left(T_a + T_b + \cdots + T_m\right)$$

For a unit-step input $r(t)$, since

$$\int_0^\infty e(t)\,dt = \lim_{s\to 0} E(s) = \lim_{s\to 0} \frac{1}{1 + G(s)} R(s) = \lim_{s\to 0} \frac{1}{1 + G(s)} \frac{1}{s} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{K_v}$$

we have

$$\frac{1}{K_v} = \frac{1}{\lim_{s\to 0} sG(s)} = \left(T_1 + T_2 + \cdots + T_n\right) - \left(T_a + T_b + \cdots + T_m\right)$$

Note that zeros in the left half-plane (that is, positive $T_a, T_b, \ldots, T_m$) will improve K_v. Poles close to the origin cause low velocity-error constants unless there are zeros nearby.

PROBLEMS

B–5–1. A thermometer requires 1 min to indicate 98% of the response to a step input. Assuming the thermometer to be a first-order system, find the time constant.

If the thermometer is placed in a bath, the temperature of which is changing linearly at a rate of 10°/min, how much error does the thermometer show?

B–5–2. Consider the unit-step response of a unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{1}{s(s + 1)}$$

Obtain the rise time, peak time, maximum overshoot, and settling time.

B–5–3. Consider the closed-loop system given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Determine the values of ζ and ω_n so that the system responds to a step input with approximately 5% overshoot and with a settling time of 2 sec. (Use the 2% criterion.)

B–5–4. Consider the system shown in Figure 5–72. The system is initially at rest. Suppose that the cart is set into motion by an impulsive force whose strength is unity. Can it be stopped by another such impulsive force?

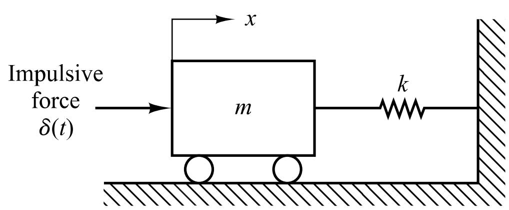


Figure 5–72
Mechanical system.

B–5–5. Obtain the unit-impulse response and the unit-step response of a unity-feedback system whose open-loop transfer function is

$$G(s) = \frac{2s + 1}{s^2}$$

B–5–6. An oscillatory system is known to have a transfer function of the following form:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Assume that a record of a damped oscillation is available as shown in Figure 5–73. Determine the damping ratio ζ of the system from the graph.

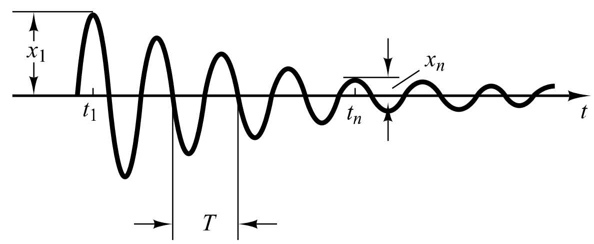


Figure 5–73
Decaying oscillation.

B–5–7. Consider the system shown in Figure 5–74(a). The damping ratio of this system is 0.158 and the undamped natural frequency is 3.16 rad/sec. To improve the relative stability, we employ tachometer feedback. Figure 5–74(b) shows such a tachometer-feedback system.

Determine the value of K_h so that the damping ratio of the system is 0.5. Draw unit-step response curves of both the original and tachometer-feedback systems. Also draw the error-versus-time curves for the unit-ramp response of both systems.

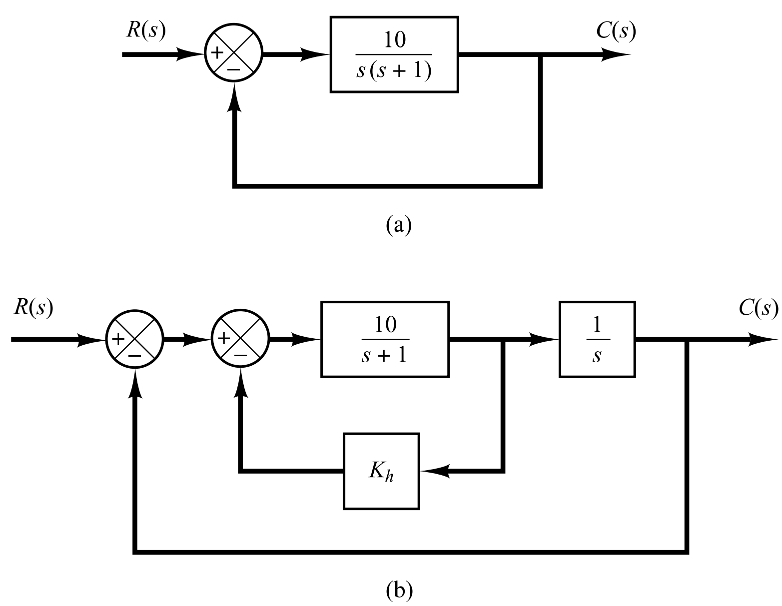


Figure 5–74
(a) Control system; (b) control system with tachometer feedback.

B–5–8. Referring to the system shown in Figure 5–75, determine the values of K and k such that the system has a damping ratio ζ of 0.7 and an undamped natural frequency ω_n of 4 rad/sec.

B–5–9. Consider the system shown in Figure 5–76. Determine the value of k such that the damping ratio ζ is 0.5. Then obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time t_s in the unit-step response.

B–5–10. Using MATLAB, obtain the unit-step response, unit-ramp response, and unit-impulse response of the following system:

$$\frac{C(s)}{R(s)} = \frac{10}{s^2 + 2s + 10}$$

where $R(s)$ and $C(s)$ are Laplace transforms of the input $r(t)$ and output $c(t)$, respectively.

B–5–11. Using MATLAB, obtain the unit-step response, unit-ramp response, and unit-impulse response of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where u is the input and y is the output.

B–5–12. Obtain both analytically and computationally the rise time, peak time, maximum overshoot, and settling time in the unit-step response of a closed-loop system given by

$$\frac{C(s)}{R(s)} = \frac{36}{s^2 + 2s + 36}$$

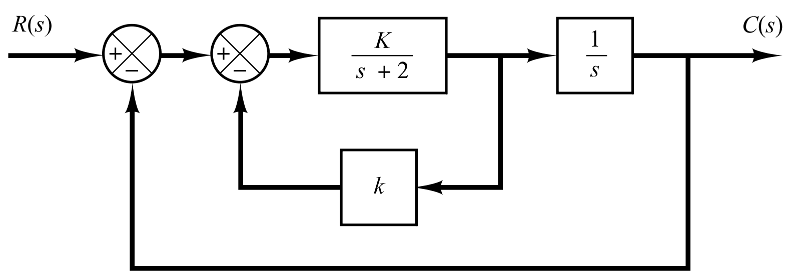


Figure 5–75
Closed-loop system.

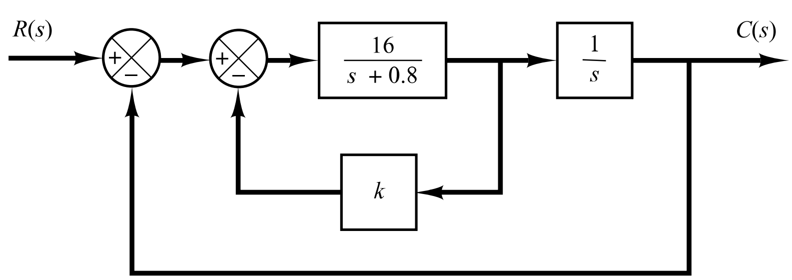


Figure 5–76
Block diagram of a system.

B–5–13. Figure 5–77 shows three systems. System I is a positional servo system. System II is a positional servo system with PD control action. System III is a positional servo system with velocity feedback. Compare the unit-step, unit-impulse, and unit-ramp responses of the three systems. Which system is best with respect to the speed of response and maximum overshoot in the step response?

B–5–14. Consider the position control system shown in Figure 5–78. Write a MATLAB program to obtain a unit-step response and a unit-ramp response of the system. Plot curves $x_1(t)$ versus t, $x_2(t)$ versus t, $x_3(t)$ versus t, and $e(t)$ versus t $[\text{where } e(t) = r(t) - x_1(t)]$ for both the unit-step response and the unit-ramp response.

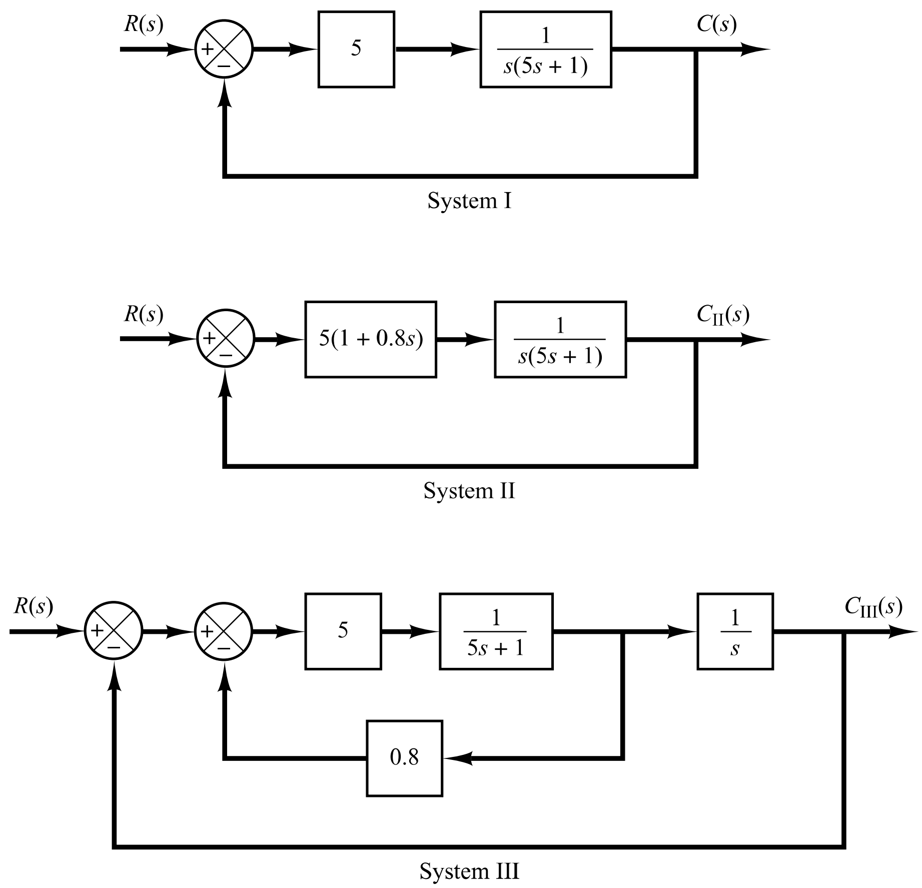


Figure 5–77
Positional servo system (System I), positional servo system with PD control action (System II), and positional servo system with velocity feedback (System III).

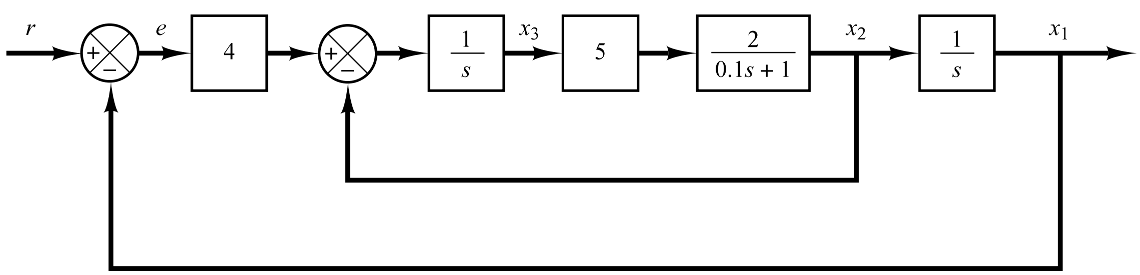


Figure 5–78
Position control system.

B–5–15. Using MATLAB, obtain the unit-step response curve for the unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{10}{s(s+2)(s+4)}$$

Using MATLAB, obtain also the rise time, peak time, maximum overshoot, and settling time in the unit-step response curve.

B–5–16. Consider the closed-loop system defined by

$$\frac{C(s)}{R(s)} = \frac{2\zeta s + 1}{s^2 + 2\zeta s + 1}$$

where $\zeta = 0.2, 0.4, 0.6, 0.8,$ and 1.0. Using MATLAB, plot a two-dimensional diagram of unit-impulse response curves. Also plot a three-dimensional plot of the response curves.

B–5–17. Consider the second-order system defined by

$$\frac{C(s)}{R(s)} = \frac{s + 1}{s^2 + 2\zeta s + 1}$$

where $\zeta = 0.2, 0.4, 0.6, 0.8, 1.0$. Plot a three-dimensional diagram of the unit-step response curves.

B–5–18. Obtain the unit-ramp response of the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where u is the unit-ramp input. Use the lsim command to obtain the response.

B–5–19. Consider the differential equation system given by

$$\ddot{y} + 3\dot{y} + 2y = 0, \qquad y(0) = 0.1, \qquad \dot{y}(0) = 0.05$$

Using MATLAB, obtain the response $y(t)$, subject to the given initial condition.

B–5–20. Determine the range of K for stability of a unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

B–5–21. Consider the following characteristic equation:

$$s^4 + 2s^3 + (4 + K)s^2 + 9s + 25 = 0$$

Using the Routh stability criterion, determine the range of K for stability.

B–5–22. Consider the closed-loop system shown in Figure 5–79. Determine the range of K for stability. Assume that $K > 0$.

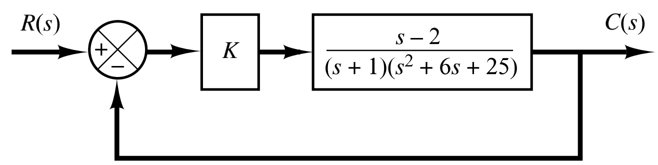


Figure 5–79 Closed-loop system.

B–5–23. Consider the satellite attitude control system shown in Figure 5–80(a). The output of this system exhibits continued oscillations and is not desirable. This system can be stabilized by use of tachometer feedback, as shown in Figure 5–80(b). If $K/J = 4$, what value of K_h will yield the damping ratio to be 0.6?

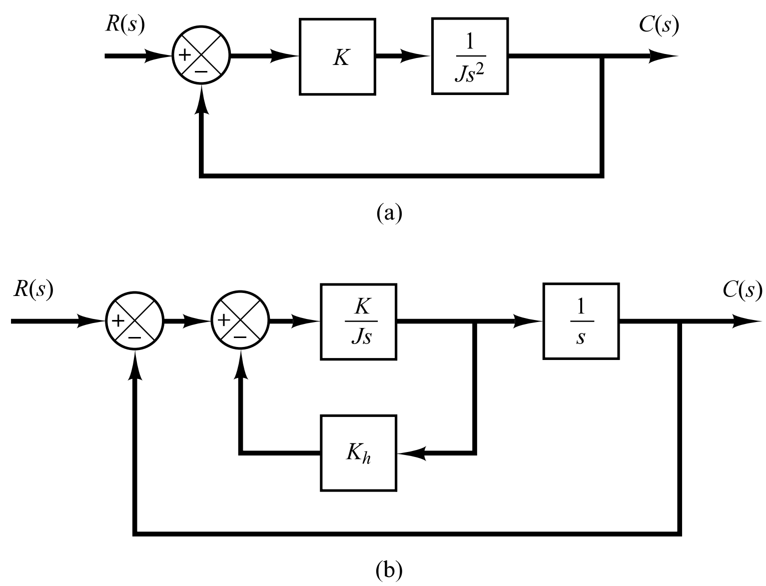


Figure 5–80
(a) Unstable satellite attitude control system;
(b) stabilized system.

B–5–24. Consider the servo system with tachometer feedback shown in Figure 5–81. Determine the ranges of stability for K and K_h. (Note that K_h must be positive.)

B–5–25. Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -b_3 & 0 & 1 \\ 0 & -b_2 & -b_1 \end{bmatrix}$$

($\mathbf{A}$ is called Schwarz matrix.) Show that the first column of the Routh's array of the characteristic equation $|s\mathbf{I} - \mathbf{A}| = 0$ consists of $1, b_1, b_2$, and $b_1 b_3$.

B–5–26. Consider a unity-feedback control system with the closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{Ks + b}{s^2 + as + b}$$

Determine the open-loop transfer function $G(s)$.

Show that the steady-state error in the unit-ramp response is given by

$$e_{ss} = \frac{1}{K_v} = \frac{a - K}{b}$$

B–5–27. Consider a unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{K}{s(Js + B)}$$

Discuss the effects that varying the values of K and B has on the steady-state error in unit-ramp response. Sketch typical unit-ramp response curves for a small value, medium value, and large value of K, assuming that B is constant.

B–5–28. If the feedforward path of a control system contains at least one integrating element, then the output continues to change as long as an error is present. The output stops when the error is precisely zero. If an external disturbance enters the system, it is desirable to have an integrating element between the error-measuring element and the point where the disturbance enters, so that the effect of the external disturbance may be made zero at steady state.

Show that, if the disturbance is a ramp function, then the steady-state error due to this ramp disturbance may be eliminated only if two integrators precede the point where the disturbance enters.

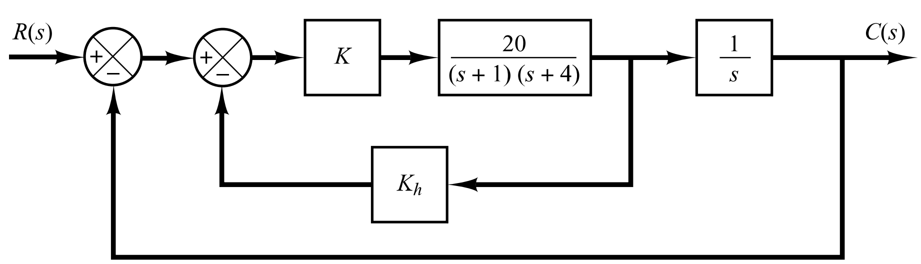


Figure 5–81
Servo system with tachometer feedback.