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Total Chapters: 14

Total Examples: 227

Codable Examples: 170

Chapter 1: Introduction

Example 1.1 - Codable

Example 1.2 - Codable

Example 1.3 -Codable

Example 1.4 – Codable

Example 1.5 - Codable

Example 1.6 - Codable

Example 1.7 - Codable

Example 1.8 -Codable

Example 1.9 – Codable

Example 1.10 - Codable

Example 1.11 - Codable

Example 1.12- Codable

Example 1.13 -Codable

Example 1.14 – Codable

Example 1.15 - Codable

Example 1.16 - Codable

Example 1.17 - Codable

Example 1.18 -Codable

Example 1.19 – Codable

Example 1.20 - Codable

Example 1.21 - Codable

Example 1.22 - Codable

Example 1.23 -Codable

Example 1.24 – Codable

Example 1.25 - Codable

Chapter 2: probability

Example 2.1 – Codable

Example 2.2 – Codable

Example 2.3 – Codable

Example 2.4 – Codable

Example 2.5 – Codable

Example 2.6 – Not Codable(problem contains theoretical explaination)

EXAMPLE 6

Construct a sample space for the length of the useful life of a certain electronic component and indicate the subset that represents the event F that the component fails before the end of the sixth year.

25

Probability

Solution

If t is the length of the component's useful life in years, the sample space may be written $S = \{t|t \geqq 0\}$, and the subset $F = \{t|0 \leqq t < 6\}$ is the event that the component fails before the end of the sixth year.

Example 2.7 – Codable

Example 2.8 – Codable

Example 2.9 – Codable

Example 2.10 – Codable

Example 2.11 – Codable

Example 2.12 – Codable

Example 2.13 – Codable

Example 2.14 – Codable

Example 2.15 – Codable

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Example 2.25 – Codable

Example 2.26 – Codable

Example 2.27 – Codable

Example 2.28 – Codable

Example 2.29 – Codable

Chapter 3: Probability Distributions and Probability Densities

Example 3.1 - Codable

Example 3.2 - Codable

Example 3.3 - Codable

Example 3.4 - Codable

Example 3.5 - Codable

Example 3.6 - Codable

Example 3.7 - Codable

Example 3.8 - Not Codable (problem contains theoretical explaination)

EXAMPLE 8

Suppose that we are concerned with the possibility that an accident will occur on a freeway that is 200 kilometers long and that we are interested in the probability that it will occur at a given location, or perhaps on a given stretch of the road. The sample space of this "experiment" consists of a continuum of points, those on the interval from 0 to 200, and we shall assume, for the sake of argument, that the probability that an accident will occur on any interval of length d is $\frac{d}{200}$, with d measured in

kilometers. Note that this assignment of probabilities is consistent with Postulates 1 and 2. (Postulate 1 states that probability of an event is a nonnegative real number; that is, $P(A)$ G 0 for any subset A of S but in Postulate 2 $P(S) = 1$.) The probabilities $\frac{d}{200}$ are all nonnegative and $P(S) = \frac{200}{200} = 1$. So far this assignment of probabilities applies only to intervals on the line segment from 0 to 200, but if we use Postulate 3 (Postulate 3: If $A_1, A_2, A_3, \ldots$, is a finite or infinite sequence of mutually exclusive events of S, then $P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$), we can also obtain probabilities for the union of any finite or countably infinite sequence of nonoverlapping intervals. For instance, the probability that an accident will occur on either of two nonoverlapping intervals of length d_1 and d_2 is

$$\frac{d_1 + d_2}{200}$$

and the probability that it will occur on any one of a countably infinite sequence of nonoverlapping intervals of length $d_1, d_2, d_3, \ldots$ is

$$\frac{d_1 + d_2 + d_3 + \cdots}{200}$$

Example 3.9 - Codable

Example 3.10 - Codable

Example 3.11 - Not Codable (problem contains theoretical explaination)

EXAMPLE 11

Find a probability density function for the random variable whose distribution function is given by

$$F(x) = \begin{cases} 0 & \text{for } x \leqq 0 \\ x & \text{for } 0 < x < 1 \\ 1 & \text{for } x \geqq 1 \end{cases}$$

and plot its graph.

Solution

Since the given density function is differentiable everywhere except at $x = 0$ and $x = 1$, we differentiate for $x < 0, 0 < x < 1$, and $x > 1$, getting 0, 1, and 0. Thus, according to the second part of Theorem 6, we can write

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } 0 < x < 1 \\ 0 & \text{for } x > 1 \end{cases}$$

To fill the gaps at $x = 0$ and $x = 1$, we let $f(0)$ and $f(1)$ both equal zero. Actually, it does not matter how the probability density is defined at these two points, but there are certain advantages for choosing the values in such a way that the probability density is nonzero over an open interval. Thus, we can write the probability density of the original random variable as

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Its graph is shown in Figure 7.

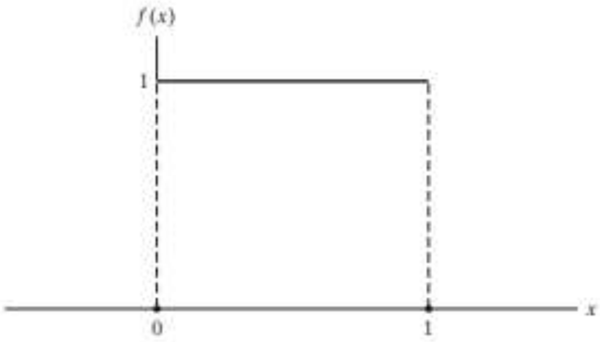

Figure 7. Probability density of Example 11.

Example 3.12 - Codable

Example 3.13 - Codable

Example 3.14 - Codable

Example 3.15 - Codable

Example 3.16 - Not Codable (problem contains theoretical explaination)

EXAMPLE 16

If the joint probability density of X and Y is given by

$$f(x, y) = \begin{cases} x + y & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find the joint distribution function of these two random variables.

Solution

If either $x < 0$ or $y < 0$, it follows immediately that $F(x, y) = 0$. For $0 < x < 1$ and $0 < y < 1$ (Region I of Figure 10), we get

$$F(x, y) = \int_0^y \int_0^x (s + t)\, ds\, dt = \frac{1}{2}xy(x + y)$$

for $x > 1$ and $0 < y < 1$ (Region II of Figure 10), we get

$$F(x, y) = \int_0^y \int_0^1 (s + t)\, ds\, dt = \frac{1}{2}y(y + 1)$$

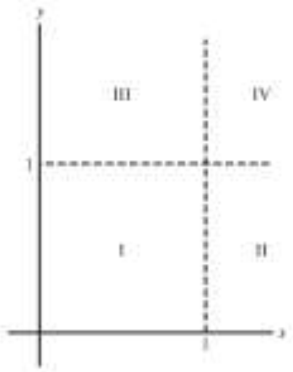

Figure 10. Diagram for Example 16.

for $0 < x < 1$ and $y > 1$ (Region III of Figure 10), we get

$$F(x, y) = \int_0^1 \int_0^x (s + t)\, ds\, dt = \frac{1}{2}x(x + 1)$$

and for $x > 1$ and $y > 1$ (Region IV of Figure 10), we get

$$F(x, y) = \int_0^1 \int_0^1 (s + t)\, ds\, dt = 1$$

Since the joint distribution function is everywhere continuous, the boundaries between any two of these regions can be included in either one, and we can write

$$F(x, y) = \begin{cases} 0 & \text{for } x \leqq 0 \text{ or } y \leqq 0 \\ \frac{1}{2}xy(x + y) & \text{for } 0 < x < 1, 0 < y < 1 \\ \frac{1}{2}y(y + 1) & \text{for } x \geqq 1, 0 < y < 1 \\ \frac{1}{2}x(x + 1) & \text{for } 0 < x < 1, y \geqq 1 \\ 1 & \text{for } x \geqq 1, y \geqq 1 \end{cases}$$

Example 3.17 - Codable

Example 3.18 - Codable

Example 3.19 - Codable

Example 3.20 - Not Codable (problem contains theoretical explaination)

EXAMPLE 20

In Example 12 we derived the joint probability distribution of two random variables X and Y, the number of aspirin caplets and the number of sedative caplets included among two caplets drawn at random from a bottle containing three aspirin, two sedative, and four laxative caplets. Find the probability distribution of X alone and that of Y alone.

Solution

The results of Example 12 are shown in the following table, together with the marginal totals, that is, the totals of the respective rows and columns:

			x		
		0	1	2	
y	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{7}{12}$
	1	$\frac{2}{9}$	$\frac{1}{6}$		$\frac{7}{18}$
	2	$\frac{1}{36}$			$\frac{1}{36}$
		$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{12}$	

The column totals are the probabilities that X will take on the values 0, 1, and 2. In other words, they are the values

$$g(x) = \sum_{y=0}^{2} f(x, y) \qquad \text{for } x = 0, 1, 2$$

of the probability distribution of X. By the same token, the row totals are the values

$$h(y) = \sum_{x=0}^{2} f(x, y) \qquad \text{for } y = 0, 1, 2$$

of the probability distribution of Y.

Example 3.21 - Not Codable (problem contains theoretical explaination)

EXAMPLE 21

Given the joint probability density

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find the marginal densities of X and Y.

Solution

Performing the necessary integrations, we get

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{2}{3}(x+2y)\,dy = \frac{2}{3}(x+1)$$

for $0 < x < 1$ and $g(x) = 0$ elsewhere. Likewise,

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{2}{3}(x+2y)\,dx = \frac{1}{3}(1+4y)$$

for $0 < y < 1$ and $h(y) = 0$ elsewhere.

Example 3.22 - Not Codable (problem contains theoretical explaination)

EXAMPLE 22

Considering again the trivariate probability density of Example 19,

$$f(x_1,x_2,x_3) = \begin{cases} (x_1+x_2)e^{-x_3} & \text{for } 0<x_1<1, 0<x_2<1, x_3>0 \\ 0 & \text{elsewhere} \end{cases}$$

find the joint marginal density of X_1 and X_3 and the marginal density of X_1 alone.

Solution

Performing the necessary integration, we find that the joint marginal density of X_1 and X_3 is given by

$$m(x_1,x_3) = \int_0^1 (x_1+x_2)e^{-x_3}\,dx_2 = \left(x_1+\frac{1}{2}\right)e^{-x_3}$$

for $0<x_1<1$ and $x_3>0$ and $m(x_1,x_3)=0$ elsewhere. Using this result, we find that the marginal density of X_1 alone is given by

$$\begin{aligned} g(x_1) &= \int_0^{\infty}\int_0^1 f(x_1,x_2,x_3)\,dx_2\,dx_3 = \int_0^{\infty} m(x_1,x_3)\,dx_3 \\ &= \int_0^{\infty}\left(x_1+\frac{1}{2}\right)e^{-x_3}\,dx_3 = x_1+\frac{1}{2} \end{aligned}$$

for $0<x_1<1$ and $g(x_1)=0$ elsewhere.

Example 3.23 - Codable

Example 3.24 - Codable

Example 3.25 - Not Codable (problem contains theoretical explaination)

EXAMPLE 25

Given the joint probability density

$$f(x,y) = \begin{cases} 4xy & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find the marginal densities of X and Y and the conditional density of X given $Y = y$.

Solution

Performing the necessary integrations, we get

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 4xy\,dy$$
$$= 2xy^2\Big|_{y=0}^{y=1} = 2x$$

for $0 < x < 1$, and $g(x) = 0$ elsewhere; also

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 4xy\,dx$$
$$= 2x^2y\Big|_{x=0}^{x=1} = 2y$$

for $0 < y < 1$, and $h(y) = 0$ elsewhere. Then, substituting into the formula for a conditional density, we get

$$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{4xy}{2y} = 2x$$

for $0 < x < 1$, and $f(x|y) = 0$ elsewhere.

Example 3.26 - Not Codable(problem contains theoretical explaination)

EXAMPLE 26

Considering n independent flips of a balanced coin, let X_i be the number of heads (0 or 1) obtained in the ith flip for $i = 1, 2, \ldots, n$. Find the joint probability distribution of these n random variables.

Solution

Since each of the random variables X_i, for $i = 1, 2, \ldots, n$, has the probability distribution

$$f_i(x_i) = \frac{1}{2} \qquad \text{for } x_i = 0, 1$$

and the n random variables are independent, their joint probability distribution is given by

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= f_1(x_1) \cdot f_2(x_2) \cdot \ldots \cdot f_n(x_n) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \ldots \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^n \end{aligned}$$

where $x_i = 0$ or 1 for $i = 1, 2, \ldots, n$.

Example 3.27 - Codable

Example 3.28 - Codable

Example 3.29 - Codable

Example 3.30 - Codable

Chapter 4: Mathematical Expectation

Example 4.1 - Codable

Example 4.2 - Codable

Example 4.3 - Codable

Example 4.4 - Codable

Example 4.5 - Codable

Example 4.6 - Codable

Example 4.7 - Not Codable (problem contains theoretical explaination)

EXAMPLE 7

Show that

$$E[(aX+b)^n] = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i E(X^{n-i})$$

Solution

Since $(ax+b)^n = \sum_{i=0}^{n} \binom{n}{i} (ax)^{n-i} b^i$, it follows that

$$E[(aX+b)^n] = E\left[\sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i X^{n-i}\right]$$

$$= \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i E(X^{n-i})$$

Example 4.8 - Codable

Example 4.9 - Codable

Example 4.10 - Codable

Example 4.11 - Codable

Example 4.12 - Codable

Example 4.13 - Not Codable (problem contains theoretical explaination)

EXAMPLE 13

Find the moment-generating function of the random variable whose probability density is given by

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

and use it to find an expression for μ'_r.

Solution

By definition

$$\begin{aligned} M_X(t) = E(e^{tX}) &= \int_0^{\infty} e^{tx} \cdot e^{-x}\, dx \\ &= \int_0^{\infty} e^{-x(1-t)}\, dx \\ &= \frac{1}{1-t} \quad \text{for } t < 1 \end{aligned}$$

As is well known, when $|t| < 1$ the Maclaurin's series for this moment-generating function is

$$\begin{aligned} M_X(t) &= 1 + t + t^2 + t^3 + \cdots + t^r + \cdots \\ &= 1 + 1! \cdot \frac{t}{1!} + 2! \cdot \frac{t^2}{2!} + 3! \cdot \frac{t^3}{3!} + \cdots + r! \cdot \frac{t^r}{r!} + \cdots \end{aligned}$$

and hence $\mu'_r = r!$ for $r = 0, 1, 2, \ldots$.

Example 4.14 - Codable

Example 4.15 - Codable

Example 4.16 - Codable

Example 4.17 - Codable

Example 4.18 - Codable

Example 4.19 - Codable

Example 4.20 - Codable

Example 4.21 – Codable

Chapter 5: Special Probability Distributions

Example 5.1 - Codable

Example 5.2 - Codable

Example 5.3 - Codable

Example 5.4 - Codable

Example 5.5 - Codable

Example 5.6 - Codable

Example 5.7 - Codable

Example 5.8 - Codable

Example 5.9 - Codable

Example 5.10 - Codable

Example 5.11 - Codable

Example 5.12 - Codable

Example 5.13 - Codable

Example 5.14 - Codable

Example 5.15 - Codable

Example 5.16 – Codable

Chapter 6: Special Probability Densities

Example 6.1 - Codable

Example 6.2 - Codable

Example 6.3 - Not Codable (problem contains theoretical explaination)

EXAMPLE 3

With reference to the standard normal distribution table, find the values of z that correspond to entries of

(a) 0.3512;

(b) 0.2533.

Solution

(a) Since 0.3512 falls between 0.3508 and 0.3531, corresponding to $z = 1.04$ and $z = 1.05$, and since 0.3512 is closer to 0.3508 than 0.3531, we choose $z = 1.04$.

(b) Since 0.2533 falls midway between 0.2517 and 0.2549, corresponding to $z = 0.68$ and $z = 0.69$, we choose $z = 0.685$.

Example 6.4 - Codable

Example 6.5 - Codable

Example 6.6 - Codable

Example 6.7 - Codable

Example 6.8 – Codable

Chapter 7: Functions of Random Variables

Example 7.1 - Not Codable (problem contains theoretical explaination)

EXAMPLE 1

If the probability density of X is given by

$$f(x) = \begin{cases} 6x(1-x) & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find the probability density of $Y = X^3$.

Solution

Letting $G(y)$ denote the value of the distribution function of Y at y, we can write

$$\begin{aligned} G(y) &= P(Y \leqq y) \\ &= P(X^3 \leqq y) \\ &= P(X \leqq y^{1/3}) \\ &= \int_0^{y^{1/3}} 6x(1-x)\,dx \\ &= 3y^{2/3} - 2y \end{aligned}$$

and hence

$$g(y) = 2(y^{-1/3} - 1)$$

for $0 < y < 1$; elsewhere, $g(y) = 0$. In Exercise 15 the reader will be asked to verify this result by a different technique.

Example 7.2 - Not Codable

EXAMPLE 2

If $Y = |X|$, show that

$$g(y) = \begin{cases} f(y) + f(-y) & \text{for } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $f(x)$ is the value of the probability density of X at x and $g(y)$ is the value of the probability density of Y at y. Also, use this result to find the probability density of $Y = |X|$ when X has the standard normal distribution.

Solution

For $y > 0$ we have

$$\begin{aligned} G(y) &= P(Y \leqq y) \\ &= P(|X| \leqq y) \\ &= P(-y \leqq X \leqq y) \\ &= F(y) - F(-y) \end{aligned}$$

and, upon differentiation,

$$g(y) = f(y) + f(-y)$$

Also, since $|x|$ cannot be negative, $g(y) = 0$ for $y < 0$. Arbitrarily letting $g(0) = 0$, we can thus write

$$g(y) = \begin{cases} f(y) + f(-y) & \text{for } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

If X has the standard normal distribution and $Y = |X|$, it follows that

$$\begin{aligned} g(y) &= n(y; 0, 1) + n(-y; 0, 1) \\ &= 2n(y; 0, 1) \end{aligned}$$

for $y > 0$ and $g(y) = 0$ elsewhere. An important application of this result may be found in Example 9.

Example 7.3 - Not Codable (problem contains theoretical explaination)

EXAMPLE 3

If the joint density of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} 6e^{-3x_1-2x_2} & \text{for } x_1 > 0, x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find the probability density of $Y = X_1 + X_2$.

Solution

Integrating the joint density over the shaded region of Figure 1, we get

$$F(y) = \int_0^y \int_0^{y-x_2} 6e^{-3x_1-2x_2} dx_1\, dx_2$$

$$= 1 + 2e^{-3y} - 3e^{-2y}$$

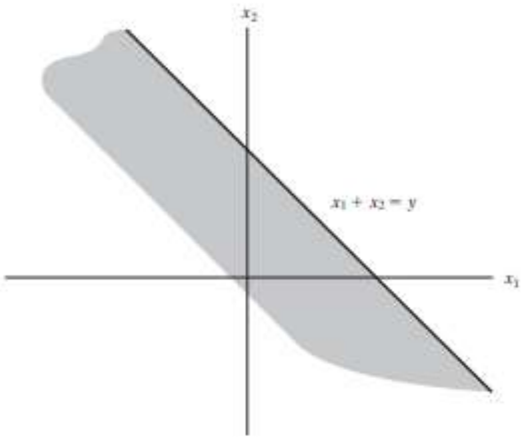

Figure 1. Diagram for Example 3.

and, differentiating with respect to y, we obtain

$$f(y) = 6(e^{-2y} - e^{-3y})$$

for $y > 0$; elsewhere, $f(y) = 0$.

Example 7.4 - Codable

Example 7.5 - Codable

Example 7.6 - Not Codable (problem contains theoretical explaination)

EXAMPLE 6

If X has the exponential distribution given by

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find the probability density of the random variable $Y = \sqrt{X}$.

Solution

The equation $y = \sqrt{x}$, relating the values of X and Y, has the unique inverse $x = y^2$, which yields $w'(y) = \frac{dx}{dy} = 2y$. Therefore,

$$g(y) = e^{-y^2}|2y| = 2ye^{-y^2}$$

for $y > 0$ in accordance with Theorem 1. Since the probability of getting a value of Y less than or equal to 0, like the probability of getting a value of X less than or equal to 0, is zero, it follows that the probability density of Y is given by

$$g(y) = \begin{cases} 2ye^{-y^2} & \text{for } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Example 7.7 - Not Codable (problem contains theoretical explaination)

EXAMPLE 7

If the double arrow of Figure 5 is spun so that the random variable Θ has the uniform density

$$f(\theta) = \begin{cases} \frac{1}{\pi} & \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases}$$

determine the probability density of X, the abscissa of the point on the x-axis to which the arrow will point.

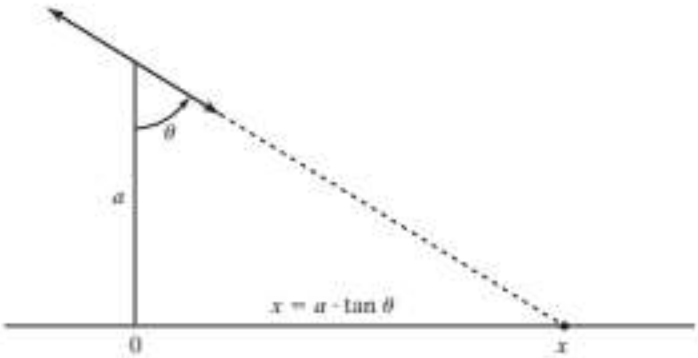

Figure 5. Diagram for Example 7.

Solution

As is apparent from the diagram, the relationship between x and θ is given by $x = a \cdot \tan\theta$, so that

$$\frac{d\theta}{dx} = \frac{a}{a^2 + x^2}$$

and it follows that

$$\begin{aligned} g(x) &= \frac{1}{\pi} \cdot \left| \frac{a}{a^2 + x^2} \right| \\ &= \frac{1}{\pi} \cdot \frac{a}{a^2 + x^2} \quad \text{for } -\infty < x < \infty \end{aligned}$$

according to Theorem 1

Example 7.8 - Not Codable (problem contains theoretical explaination)

EXAMPLE 8

If $F(x)$ is the value of the distribution function of the continuous random variable X at x, find the probability density of $Y = F(X)$.

Solution

As can be seen from Figure 6, the value of Y corresponding to any particular value of X is given by the area under the curve, that is, the area under the graph of the density of X to the left of x. Differentiating $y = F(x)$ with respect to x, we get

$$\frac{dy}{dx} = F'(x) = f(x)$$

and hence

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{f(x)}$$

provided $f(x) \neq 0$. It follows from Theorem 1 that

$$g(y) = f(x) \cdot \left| \frac{1}{f(x)} \right| = 1$$

for $0 < y < 1$, and we can say that y has the uniform density with $\alpha = 0$ and $\beta = 1$.

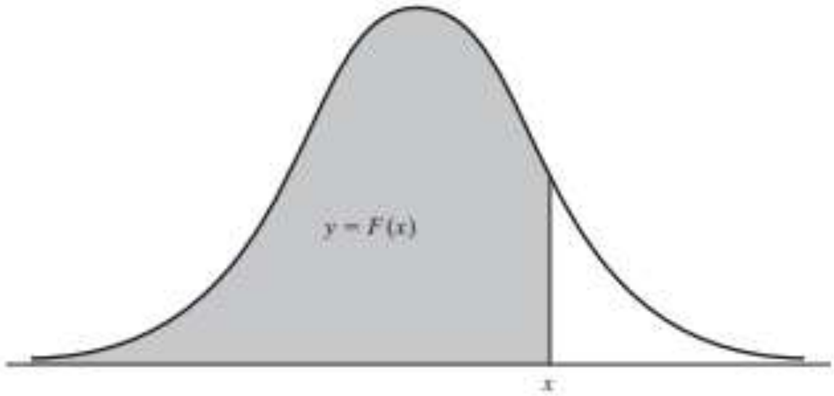

Figure 6. Diagram for Example 8.

Example 7.9 - Not Codable(problem contains theoretical explaination)

EXAMPLE 9

If X has the standard normal distribution, find the probability density of $Z = X^2$.

Solution

Since the function given by $z = x^2$ is decreasing for negative values of x and increasing for positive values of x, the conditions of Theorem 1 are not met. However, the transformation from X to Z can be made in two steps: First, we find the probability density of $Y = |X|$, and then we find the probability density of $Z = Y^2 (= X^2)$.

As far as the first step is concerned, we already studied the transformation $Y = |X|$ in Example 2; in fact, we showed that if X has the standard normal distribution, then $Y = |X|$ has the probability density

$$g(y) = 2n(y; 0, 1) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

for $y > 0$, and $g(y) = 0$ elsewhere. For the second step, the function given by $z = y^2$ is increasing for $y > 0$, that is, for all values of Y for which $g(y) \neq 0$. Thus, we can use Theorem 1, and since

$$\frac{dy}{dz} = \frac{1}{2} z^{-\frac{1}{2}}$$

we get

$$h(z) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}z} \left| \frac{1}{2} z^{-\frac{1}{2}} \right|$$

$$= \frac{1}{\sqrt{2\pi}} z^{-\frac{1}{2}} e^{-\frac{1}{2}z}$$

for $z > 0$, and $h(z) = 0$ elsewhere. Observe that since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, the distribution we have arrived at for Z is a chi-square distribution with $\nu = 1$.

Example 7.10 - Not Codable (problem contains theoretical explaination)

EXAMPLE 10

If X_1 and X_2 are independent random variables having Poisson distributions with the parameters λ_1 and λ_2, find the probability distribution of the random variable $Y = X_1 + X_2$.

Solution

Since X_1 and X_2 are independent, their joint distribution is given by

$$f(x_1, x_2) = \frac{e^{-\lambda_1}(\lambda_1)^{x_1}}{x_1!} \cdot \frac{e^{-\lambda_2}(\lambda_2)^{x_2}}{x_2!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1)^{x_1}(\lambda_2)^{x_2}}{x_1!x_2!}$$

for $x_1 = 0, 1, 2, \ldots$ and $x_2 = 0, 1, 2, \ldots$. Since $y = x_1 + x_2$ and hence $x_1 = y - x_2$, we can substitute $y - x_2$ for x_1, getting

$$g(y, x_2) = \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_2)^{x_2}(\lambda_1)^{y-x_2}}{x_2!(y-x_2)!}$$

for $y = 0, 1, 2, \ldots$ and $x_2 = 0, 1, \ldots, y$, for the joint distribution of Y and X_2. Then, summing on x_2 from 0 to y, we get

$$h(y) = \sum_{x_2=0}^{y} \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_2)^{x_2}(\lambda_1)^{y-x_2}}{x_2!(y-x_2)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{y!} \cdot \sum_{x_2=0}^{y} \frac{y!}{x_2!(y-x_2)!}(\lambda_2)^{x_2}(\lambda_1)^{y-x_2}$$

after factoring out $e^{-(\lambda_1+\lambda_2)}$ and multiplying and dividing by $y!$. Identifying the summation at which we arrived as the binomial expansion of $(\lambda_1 + \lambda_2)^y$, we finally get

$$h(y) = \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1 + \lambda_2)^y}{y!} \quad \text{for } y = 0, 1, 2, \ldots$$

and we have thus shown that the sum of two independent random variables having Poisson distributions with the parameters λ_1 and λ_2 has a Poisson distribution with the parameter $\lambda = \lambda_1 + \lambda_2$.

Example 7.11 - Codable

Example 7.12 - Codable

Example 7.13 - Not Codable (problem contains theoretical explaination)

EXAMPLE 13

If the joint density of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} 1 & \text{for } 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find

(a) the joint density of $Y = X_1 + X_2$ and $Z = X_2$;

(b) the marginal density of Y.

Note that in Exercise 6 the reader was asked to work the same problem by the distribution function technique.

Solution

(a) Solving $y = x_1 + x_2$ and $z = x_2$ for x_1 and x_2, we get $x_1 = y - z$ and $x_2 = z$, so that

$$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

Because this transformation is one-to-one, mapping the region $0 < x_1 < 1$ and $0 < x_2 < 1$ in the x_1x_2-plane into the region $z < y < z + 1$ and $0 < z < 1$ in the yz-plane (see Figure 7), we can use Theorem 2 and we get

$$g(y, z) = 1 \cdot |1| = 1$$

for $z < y < z + 1$ and $0 < z < 1$; elsewhere, $g(y, z) = 0$.

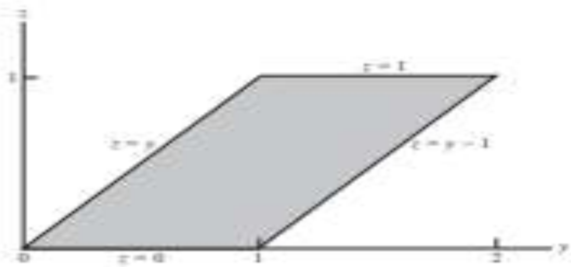

Figure 7. Transformed sample space for Example 13.

(b) Integrating out z separately for $y \leqq 0$, $0 < y < 1$, $1 < y < 2$, and $y \geqq 2$, we get

$$h(y) = \begin{cases} 0 & \text{for } y \leqq 0 \\ \int_0^y 1 \cdot dz = y & \text{for } 0 < y < 1 \\ \int_{y-1}^1 1 \cdot dz = 2 - y & \text{for } 1 < y < 2 \\ 0 & \text{for } y \geqq 2 \end{cases}$$

and to make the density function continuous, we let $h(1) = 1$. We have thus shown that the sum of the given random variables has the **triangular probability density** whose graph is shown in Figure 8.

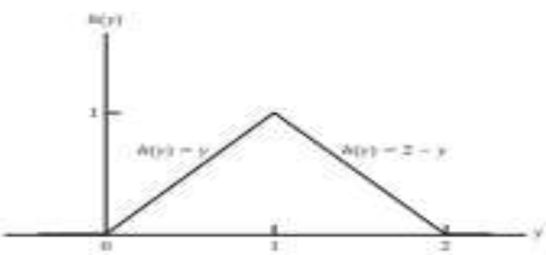

Figure 8. Triangular probability density.

Example 7.14 - Not Codable (problem contains theoretical explaination)

EXAMPLE 14

If the joint probability density of X_1, X_2, and X_3 is given by

$$f(x_1, x_2, x_3) = \begin{cases} e^{-(x_1+x_2+x_3)} & \text{for } x_1 > 0, x_2 > 0, x_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find

(a) the joint density of $Y_1 = X_1 + X_2 + X_3$, $Y_2 = X_2$, and $Y_3 = X_3$;

(b) the marginal density of Y_1.

Solution

(a) Solving the system of equations $y_1 = x_1 + x_2 + x_3$, $y_2 = x_2$, and $y_3 = x_3$ for x_1, x_2, and x_3, we get $x_1 = y_1 - y_2 - y_3$, $x_2 = y_2$, and $x_3 = y_3$. It follows that

$$J = \begin{vmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

and, since the transformation is one-to-one, that

$$g(y_1, y_2, y_3) = e^{-y_1} \cdot |1|$$
$$= e^{-y_1}$$

for $y_2 > 0$, $y_3 > 0$, and $y_1 > y_2 + y_3$; elsewhere, $g(y_1, y_2, y_3) = 0$.

(b) Integrating out y_2 and y_3, we get

$$h(y_1) = \int_0^{y_1} \int_0^{y_1 - y_3} e^{-y_1} \, dy_2 \, dy_3$$
$$= \frac{1}{2} y_1^2 \cdot e^{-y_1}$$

for $y_1 > 0$; $h(y_1) = 0$ elsewhere. Observe that we have shown that the sum of three independent random variables having the gamma distribution with $\alpha = 1$ and $\beta = 1$ is a random variable having the gamma distribution with $\alpha = 3$ and $\beta = 1$.

Example 7.15 - Not Codable (problem contains theoretical explaination)

EXAMPLE 15

Find the probability distribution of the sum of n independent random variables X_1, $X_2, \ldots, X_n$ having Poisson distributions with the respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Solution

By the theorem "The moment-generating function of the Poisson distribution is given by $M_X(t) = e^{\lambda(e^t-1)}$" we have

$$M_{X_i}(t) = e^{\lambda_i(e^t-1)}$$

hence, for $Y = X_1 + X_2 + \cdots + X_n$, we obtain

$$M_Y(t) = \prod_{i=1}^{n} e^{\lambda_i(e^t-1)} = e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)(e^t-1)}$$

which can readily be identified as the moment-generating function of the Poisson distribution with the parameter $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Thus, the distribution of the sum of n independent random variables having Poisson distributions with the parameters λ_i is a Poisson distribution with the parameter $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Note that in Example 10 we proved this for $n = 2$.

Example 7.16 - Not Codable (problem contains theoretical explaination)

EXAMPLE 16

If $X_1, X_2, \ldots, X_n$ are independent random variables having exponential distributions with the same parameter θ, find the probability density of the random variable $Y = X_1 + X_2 + \cdots + X_n$.

Solution

Since the exponential distribution is a gamma distribution with $\alpha = 1$ and $\beta = \theta$, we have

$$M_{X_i}(t) = (1 - \theta t)^{-1}$$

and hence

$$M_Y(t) = \prod_{i=1}^{n}(1 - \theta t)^{-1} = (1 - \theta t)^{-n}$$

Identifying the moment-generating function of Y as that of a gamma distribution with $\alpha = n$ and $\beta = \theta$, we conclude that the distribution of the sum of n independent random variables having exponential distributions with the same parameter θ is a gamma distribution with the parameters $\alpha = n$ and $\beta = \theta$. Note that this agrees with the result of Example 14, where we showed that the sum of three independent random variables having exponential distributions with the parameter $\theta = 1$ has a gamma distribution with $\alpha = 3$ and $\beta = 1$.

Example 7.17 - Not Codable (problem contains theoretical explaination)

EXAMPLE 17

Suppose the resistance in a simple circuit varies randomly in response to environmental conditions. To determine the effect of this variation on the current flowing through the circuit, an experiment was performed in which the resistance (R) was varied with equal probabilities on the interval $0 < R \le A$ and the ensuing voltage (E) was measured. Find the distribution of the random variable I, the current flowing through the circuit.

Solution

Using the well-known relation $E = IR$, we have $I = u(R) = \frac{E}{R}$. The probability distribution of R is given by $f(R) = \frac{1}{A}$ for $0 < R \le A$. Thus, $w(I) = \frac{E}{I}$, and the probability density of I is given by

$$g(I) = f(R) \cdot |w'(I)| = \frac{1}{A}\left|-\frac{E}{R^2}\right| = \frac{E}{AR^2} \qquad R > 0$$

Example 7.18 - Not Codable (problem contains theoretical explaination)

EXAMPLE 18

What underlying distribution of the data is assumed when the square-root transformation is used to obtain approximately normally distributed data? (Assume the data are nonnegative, that is, the probability of a negative observation is zero.)

Solution

A simple alternate way to use the distribution-function technique is to write down the differential element of the density function, $f(x)\,dx$, of the transformed observations, y, and to substitute x^2 for y. (When we do this, we must remember that the differential element, dy, must be changed to $dx = 2x\,dx$.) We obtain

$$f(x)\,dx = \frac{1}{\sqrt{2\pi}\sigma} \cdot 2x \cdot e^{-\frac{1}{2}(x^2-\mu)^2/\sigma^2}\,dx$$

The required density function is given by

$$f(x) = \sqrt{\frac{2}{\pi\sigma^2}}xe^{-\frac{1}{2}(x^2-\mu)^2/\sigma^2}$$

This distribution is not immediately recognizable, but it can be graphed quickly using appropriate computer software.

Example 7.19 – Codable

Chapter 8: Sampling Distributions

Example 8.1 - Codable

Example 8.2 - Codable

Example 8.3 - Codable

Example 8.4 - Not Codable (problem contains theoretical explaination)

EXAMPLE 4

Show that for random samples of size n from an exponential population with the parameter θ, the sampling distributions of Y_1 and Y_n are given by

$$g_1(y_1) = \begin{cases} \frac{n}{\theta} \cdot e^{-ny_1/\theta} & \text{for } y_1 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

and

$$g_n(y_n) = \begin{cases} \frac{n}{\theta} \cdot e^{-y_n/\theta}[1 - e^{-y_n/\theta}]^{n-1} & \text{for } y_n > 0 \\ 0 & \text{elsewhere} \end{cases}$$

and that, for random samples of size $n = 2m+1$ from this kind of population, the sampling distribution of the median is given by

$$h(\tilde{x}) = \begin{cases} \frac{(2m+1)!}{m!m!\theta} \cdot e^{-\tilde{x}(m+1)/\theta}[1 - e^{-\tilde{x}/\theta}]^m & \text{for } \tilde{x} > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Solution

The integrations required to obtain these results are straightforward, and they will be left to the reader in Exercise 45.

Chapter 9: Decision Theory

Example 9.1 - Codable

Example 9.2 - Codable

Example 9.3 - Codable

Example 9.4 - Codable

Example 9.5 - Codable

Example 9.6 - Not Codable (problem contains theoretical explaination)

EXAMPLE 6

With reference to the game of Example 5, suppose that Player A uses a gambling device (dice, cards, numbered slips of paper, a table of random numbers) that leads to the choice of Strategy I with probability x and to the choice of Strategy II with probability $1-x$. Find the value of x that will minimize Player A's maximum expected loss.

Solution

If Player B chooses Strategy 1, Player A can expect to lose

$$E = 8x - 5(1-x)$$

dollars, and if Player B chooses Strategy 2, Player A can expect to lose

$$E = 2x + 6(1-x)$$

dollars. Graphically, this situation is described in Figure 1, where we have plotted the lines whose equations are $E = 8x - 5(1-x)$ and $E = 2x + 6(1-x)$ for values of x from 0 to 1.

Applying the minimax criterion to the expected losses of Player A, we find from Figure 1 that the greater of the two values of E for any given value of x is smallest where the two lines intersect, and to find the corresponding value of x, we have only to solve the equation

$$8x - 5(1-x) = 2x + 6(1-x)$$

which yields $x = \frac{11}{17}$. Thus, if Player A uses 11 slips of paper numbered I and 6 slips of paper numbered II, shuffles them thoroughly, and then acts according to which kind he randomly draws, he will be holding his maximum expected loss down to $8 \cdot \frac{11}{17} - 5 \cdot \frac{6}{17} = 3\frac{7}{17}$, or \$3.41 to the nearest cent.

Example 9.7 - Not Codable (problem contains theoretical explaination)

EXAMPLE 7

A random variable has the uniform density

$$f(x) = \begin{cases} \frac{1}{\theta} & \text{for } 0 < x < \theta \\ 0 & \text{elsewhere} \end{cases}$$

and we want to estimate the parameter θ (the "move" of Nature) on the basis of a single observation. If the decision function is to be of the form $d(x) = kx$, where $k \geqq 1$, and the losses are proportional to the absolute value of the errors, that is,

$$L(kx, \theta) = c|kx - \theta|$$

where c is a positive constant, find the value of k that will minimize the risk.

Solution

For the risk function we get

$$\begin{aligned} R(d, \theta) &= \int_0^{\theta/k} c(\theta - kx) \cdot \frac{1}{\theta}\, dx + \int_{\theta/k}^{\theta} c(kx - \theta) \cdot \frac{1}{\theta}\, dx \\ &= c\theta \left(\frac{k}{2} - 1 + \frac{1}{k} \right) \end{aligned}$$

and there is nothing we can do about the factor θ; but it can easily be verified that $k = \sqrt{2}$ will minimize $\frac{k}{2} - 1 + \frac{1}{k}$. Thus, if we actually took the observation and got $x = 5$, our estimate of θ would be $5\sqrt{2}$, or approximately 7.07.

Example 9.8 - Not Codable (problem contains theoretical explaination)

EXAMPLE 8

Use the minimax criterion to estimate the parameter θ of a binomial distribution on the basis of the random variable X, the observed number of successes in n trials, when the decision function is of the form

$$d(x) = \frac{x+a}{n+b}$$

where a and b are constants, and the loss function is given by

$$L\left(\frac{x+a}{n+b}, \theta\right) = c\left(\frac{x+a}{n+b} - \theta\right)^2$$

where c is a positive constant.

Solution

The problem is to find the values of a and b that will minimize the corresponding risk function after it has been maximized with respect to θ. After all, we have control over the choice of a and b, while Nature (our presumed opponent) has control over the choice of θ.

Since $E(X) = n\theta$ and $E(X^2) = n\theta(1-\theta+n\theta)$ it follows that

$$\begin{aligned} R(d,\theta) &= E\left[c\left(\frac{X+a}{n+b} - \theta\right)^2\right] \\ &= \frac{c}{(n+b)^2}[\theta^2(b^2-n) + \theta(n-2ab) + a^2] \end{aligned}$$

and, using calculus, we could find the value of θ that maximizes this expression and then minimize $R(d,\theta)$ for this value of θ with respect to a and b. This is not particularly difficult, but it is left to the reader in Exercise 6 as it involves some tedious algebraic detail.

Example 9.9 - Not Codable (problem contains theoretical explaination)

EXAMPLE 9

With reference to Example 7, suppose that the parameter of the uniform density is looked upon as a random variable with the probability density

$$h(\theta) = \begin{cases} \theta \cdot e^{-\theta} & \text{for } \theta > 0 \\ 0 & \text{elsewhere} \end{cases}$$

If there is no restriction on the form of the decision function and the loss function is quadratic, that is, its values are given by

$$L[d(x), \theta] = c\{d(x) - \theta\}^2$$

find the decision function that minimizes the Bayes risk.

Solution

Since Θ is now a random variable, we look upon the original probability density as the conditional density

$$f(x|\theta) = \begin{cases} \dfrac{1}{\theta} & \text{for } 0 < x < \theta \\ 0 & \text{elsewhere} \end{cases}$$

and, letting $f(x,\theta) = f(x|\theta)\cdot h(\theta)$ we get

$$f(x,\theta) = \begin{cases} e^{-\theta} & \text{for } 0 < x < \theta \\ 0 & \text{elsewhere} \end{cases}$$

As the reader will be asked to verify in Exercise 8, this yields

$$g(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

for the marginal density of X and

$$\varphi(\theta|x) = \begin{cases} e^{x-\theta} & \theta > x \\ 0 & \text{elsewhere} \end{cases}$$

for the conditional density of Θ given $X = x$.

Now, the Bayes risk $E[R(d,\Theta)]$ that we shall want to minimize is given by the double integral

$$\int_0^\infty \left\{ \int_0^\theta c[d(x)-\theta]^2 f(x|\theta)\,dx \right\} h(\theta)\,d\theta$$

which can also be written as

$$\int_0^\infty \left\{ \int_0^\infty c[d(x)-\theta]^2 \varphi(\theta|x)\,d\theta \right\} g(x)\,dx$$

making use of the fact that $f(x|\theta)\cdot h(\theta) = \varphi(\theta|x)\cdot g(x)$ and changing the order of integration. To minimize this double integral, we must choose $d(x)$ for each x so that the integral

$$\int_x^\infty c[d(x)-\theta]^2 \varphi(\theta|x)\,d\theta = \int_x^\infty c[d(x)-\theta]^2 e^{x-\theta}\,d\theta$$

is as small as possible. Differentiating with respect to $d(x)$ and putting the derivative equal to 0, we get

$$2ce^x \cdot \int_x^\infty [d(x)-\theta]e^{-\theta}\,d\theta = 0$$

This yields

$$d(x)\cdot \int_x^\infty e^{-\theta}\,d\theta - \int_x^\infty \theta e^{-\theta}\,d\theta = 0$$

and, finally,

$$d(x) = \frac{\displaystyle\int_x^\infty \theta e^{-\theta}\,d\theta}{\displaystyle\int_x^\infty e^{-\theta}\,d\theta} = \frac{(x+1)e^{-x}}{e^{-x}} = x+1$$

Thus, if the observation we get is $x = 5$, this decision function gives the Bayes estimate $5+1 = 6$ for the parameter of the original uniform density.

Example 9.10 - Not Codable (problem contains theoretical explaination)

EXAMPLE 10

Suppose a manufacturer incurs warranty costs of C_w for every defective unit shipped and it costs C_d to detail an entire lot. The sampling inspection procedure is to inspect n items chosen at random from a lot containing N units, and to make the decision to accept or reject on the basis of the number of defective units found in the sample. Two strategies are to be compared, as follows:

Number of Sample Defectives, x	*Strategy 1*	*Strategy 2*
0	Accept	Accept
1	Accept	Reject
2	Accept	Reject
3 or more	Reject	Reject

In other words, the acceptance number is 2 under the first strategy, and 0 under the second.

(a) Find the risk function for these two strategies.

(b) Under what conditions is either strategy preferable?

Solution

The decision function d_1 accepts the lot if x, the number of defective units found in the sampling inspection, does not exceed 2, and rejects the lot otherwise. The decision function d_2 accepts the lot if $x = 0$ and rejects it otherwise. Thus, the loss functions are

$$
\begin{aligned}
L(d_1,\theta) &= C_w \cdot x \cdot P(x = 0,1,2|\theta) + C_d \cdot P(x > 2|\theta) \\
&= C_w \cdot x \cdot B(2; n,\theta) + C_d \cdot [1 - B(2; n,\theta)] \\
L(d_2,\theta) &= C_w \cdot x \cdot P(x = 0|\theta) + C_d \cdot P(x > 0|\theta) \\
&= C_w \cdot x \cdot B(0; n,\theta) + C_d \cdot [1 - B(0; n,\theta)]
\end{aligned}
$$

where $B(x; n,\theta)$ represents the cumulative binomial distribution having the parameters n and θ. The corresponding risk functions are found by taking the expected values of the loss functions with respect to x, obtaining

$$
\begin{aligned}
R(d_1,\theta) &= C_w \cdot n\theta \cdot B(2; n,\theta) + C_d \cdot [1 - B(2; n,\theta)] \\
R(d_2,\theta) &= C_w \cdot n\theta \cdot B(0; n,\theta) + C_d \cdot [1 - B(0; n,\theta)]
\end{aligned}
$$

Either the minimax or the Bayes criterion could be used to choose between the two decision functions. However, if we use the minimax criterion, we need to maximize the risk functions with respect to θ and then minimize the results. This is a somewhat daunting task for this example, and we shall not attempt it here. On the other hand, use of the Bayes criterion requires that we assume a prior distribution for θ, thus introducing a new assumption that may not be warranted. It is not too difficult, however, to examine the difference between the two risk functions as a function of θ and to determine for which values of θ one is associated with less risk than the other. Experience with the proportions of defectives in prior lots can guide us in determining for which "reasonable" values of θ we should compare the two risks.

To illustrate, suppose the sample size is chosen to be $n = 10$, the warranty cost per defective unit shipped is $C_w = \$100$, and the cost of detailing a rejected lot is $C_d = \$2{,}000$. The risk functions become

$$R(d_1, \theta) = 1{,}000 \cdot \theta \cdot B(2; 10, \theta) + 2{,}000 \cdot [1 - B(2; 10, \theta)]$$
$$R(d_2, \theta) = 1{,}000 \cdot \theta \cdot B(0; 10, \theta) + 2{,}000 \cdot [1 - B(0; 10, \theta)]$$

Collecting coefficients of $B(2; 10, \theta)$ in the first equation and $B(2; 10, \theta)$ in the second, then subtracting, we obtain

$$\delta(\theta) = R(d_1, \theta) - R(d_2, \theta) = (1{,}000\theta - 2{,}000)[B(2; 10, \theta) - B(0; 10, \theta)]$$

Since $\theta \leq 1$, the quantity $(1{,}000\theta - 2{,}000) \leq 0$. Also, it is straight forward to show that $B(2; 10, \theta) \geq B(0; 10, \theta)$. Thus, $\delta(\theta)$ is never positive and, since the risk for Strategy 1 is less than or equal to that for Strategy 2 for all values of θ, we choose Strategy 1, for which the acceptance number is 2.

Chapter 10: Point Estimation

Example 10.1 - Not Codable (problem contains theoretical explaination)

EXAMPLE 1

Definition 2 requires that $E(\Theta) = \theta$ for all possible values of θ. To illustrate why this statement is necessary, show that unless $\theta = \frac{1}{2}$, the minimax estimator of the binomial parameter θ is biased.

Solution

Since $E(X) = n\theta$, it follows that

$$E\left(\frac{X + \frac{1}{2}\sqrt{n}}{n + \sqrt{n}}\right) = \frac{E\left(X + \frac{1}{2}\sqrt{n}\right)}{n + \sqrt{n}} = \frac{n\theta + \frac{1}{2}\sqrt{n}}{n + \sqrt{n}}$$

and it can easily be seen that this quantity does not equal θ unless $\theta = \frac{1}{2}$.

Example 10.2 - Not Codable

EXAMPLE 2

If X has the binomial distribution with the parameters n and θ, show that the sample proportion, $\frac{X}{n}$, is an unbiased estimator of θ.

Solution

Since $E(X) = n\theta$, it follows that

$$E\left(\frac{X}{n}\right) = \frac{1}{n} \cdot E(X) = \frac{1}{n} \cdot n\theta = \theta$$

and hence that $\frac{X}{n}$ is an unbiased estimator of θ.

Example 10.3 - Not Codable (problem contains theoretical explaination)

EXAMPLE 3

If $X_1, X_2, \ldots, X_n$ constitute a random sample from the population given by

$$f(x) = \begin{cases} e^{-(x-\delta)} & \text{for } x > \delta \\ 0 & \text{elsewhere} \end{cases}$$

show that $\overline{X}$ is a biased estimator of δ.

Solution

Since the mean of the population is

$$\mu = \int_{\delta}^{\infty} x \cdot e^{-(x-\delta)} dx = 1 + \delta$$

it follows from the theorem "If $\overline{X}$ is the mean of a random sample of size n taken without replacement from a finite population of size N with the mean μ and the variance σ^2, then $E(\overline{X}) = \mu$ and $\text{var}(\overline{X}) = \frac{\sigma^2}{n} \cdot \frac{N-n}{N-1}$" that $E(\overline{X}) = 1 + \delta \neq \delta$ and hence that $\overline{X}$ is a biased estimator of δ.

Example 10.4 - Not Codable (problem contains theoretical explaination)

EXAMPLE 4

If $X_1, X_2, \ldots, X_n$ constitute a random sample from a uniform population with $\alpha = 0$, show that the largest sample value (that is, the nth order statistic, Y_n) is a biased estimator of the parameter β. Also, modify this estimator of β to make it unbiased.

Solution

Substituting into the formula for $g_n(y_n) = \begin{cases} \frac{n}{\theta} \cdot e^{-y_n/\theta}[1-e^{-y_n/\theta}]^{n-1} & \text{for } y_n > 0 \\ 0 & \text{elsewhere} \end{cases}$,

we find that the sampling distribution of Y_n is given by

$$g_n(y_n) = n \cdot \frac{1}{\beta} \cdot \left(\int_0^{y_n} \frac{1}{\beta} dx \right)^{n-1}$$
$$= \frac{n}{\beta^n} \cdot y_n^{n-1}$$

for $0 < y_n < \beta$ and $g_n(y_n) = 0$ elsewhere, and hence that

$$E(Y_n) = \frac{n}{\beta^n} \cdot \int_0^{\beta} y_n^n \, dy_n$$
$$= \frac{n}{n+1} \cdot \beta$$

Thus, $E(Y_n) \neq \beta$ and the nth order statistic is a biased estimator of the parameter β. However, since

$$E\left(\frac{n+1}{n} \cdot Y_n\right) = \frac{n+1}{n} \cdot \frac{n}{n+1} \cdot \beta$$
$$= \beta$$

it follows that $\frac{n+1}{n}$ times the largest sample value is an unbiased estimator of the parameter β.

Example 10.5 - Not Codable (problem contains theoretical explaination)

EXAMPLE 5

Show that $\overline{X}$ is a minimum variance unbiased estimator of the mean μ of a normal population.

Solution

Since

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad \text{for } -\infty < x < \infty$$

it follows that

$$\ln f(x) = -\ln \sigma\sqrt{2\pi} - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

so that

$$\frac{\partial \ln f(x)}{\partial \mu} = \frac{1}{\sigma}\left(\frac{x-\mu}{\sigma}\right)$$

and hence

$$E\left[\left(\frac{\partial \ln f(X)}{\partial \mu}\right)^2\right] = \frac{1}{\sigma^2} \cdot E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = \frac{1}{\sigma^2} \cdot 1 = \frac{1}{\sigma^2}$$

Thus,

$$\frac{1}{n \cdot E\left[\left(\frac{\partial \ln f(X)}{\partial \mu}\right)^2\right]} = \frac{1}{n \cdot \frac{1}{\sigma^2}} = \frac{\sigma^2}{n}$$

and since $\overline{X}$ is unbiased and $\text{var}(\overline{X}) = \frac{\sigma^2}{n}$, it follows that $\overline{X}$ is a minimum variance unbiased estimator of μ.

Example 10.6 - Not Codable (problem contains theoretical explaination)

EXAMPLE 6

In Example 4 we showed that if $X_1, X_2, \ldots, X_n$ constitute a random sample from a uniform population with $\alpha = 0$, then $\frac{n+1}{n} \cdot Y_n$ is an unbiased estimator of β.

(a) Show that $2\overline{X}$ is also an unbiased estimator of β.

(b) Compare the efficiency of these two estimators of β.

Solution

(a) Since the mean of the population is $\mu = \frac{\beta}{2}$ according to the theorem "The mean and the variance of the uniform distribution are given by $\mu = \frac{\alpha+\beta}{2}$ and $\sigma^2 = \frac{1}{12}(\beta-\alpha)^2$" it follows from the theorem "If $X_1, X_2, \ldots, X_n$ constitute a random sample from an infinite population with the mean μ and the variance σ^2, then $E(\overline{X}) = \mu$ and $\operatorname{var}(\overline{X}) = \frac{\sigma^2}{n}$" that $E(\overline{X}) = \frac{\beta}{2}$ and hence that $E(2\overline{X}) = \beta$. Thus, $2\overline{X}$ is an unbiased estimator of β.

(b) First we must find the variances of the two estimators. Using the sampling distribution of Y_n and the expression for $E(Y_n)$ given in Example 4, we get

$$E(Y_n^2) = \frac{n}{\beta^n} \cdot \int_0^\beta y_n^{n+1}\, dy_n = \frac{n}{n+2} \cdot \beta^2$$

and

$$\operatorname{var}(Y_n) = \frac{n}{n+2} \cdot \beta^2 - \left(\frac{n}{n+1} \cdot \beta\right)^2$$

If we leave the details to the reader in Exercise 27, it can be shown that

$$\operatorname{var}\left(\frac{n+1}{n} \cdot Y_n\right) = \frac{\beta^2}{n(n+2)}$$

Since the variance of the population is $\sigma^2 = \frac{\beta^2}{12}$ according to the first stated theorem in the example, it follows from the above (second) theorem that $\operatorname{var}(\overline{X}) = \frac{\beta^2}{12n}$ and hence that

$$\operatorname{var}(2\overline{X}) = 4 \cdot \operatorname{var}(\overline{X}) = \frac{\beta^2}{3n}$$

Therefore, the efficiency of $2\overline{X}$ relative to $\frac{n+1}{n} \cdot Y_n$ is given by

$$\frac{\operatorname{var}\left(\frac{n+1}{n} \cdot Y_n\right)}{\operatorname{var}(2\overline{X})} = \frac{\frac{\beta^2}{n(n+2)}}{\frac{\beta^2}{3n}} = \frac{3}{n+2}$$

and it can be seen that for $n > 1$ the estimator based on the nth order statistic is much more efficient than the other one. For $n = 10$, for example, the relative efficiency is only 25 percent, and for $n = 25$ it is only 11 percent.

Example 10.7 - Not Codable (problem contains theoretical explaination)

EXAMPLE 7

When the mean of a normal population is estimated on the basis of a random sample of size $2n+1$, what is the efficiency of the median relative to the mean?

Solution

From the theorem on the previous page we know that $\overline{X}$ is unbiased and that

$$\text{var}(\overline{X}) = \frac{\sigma^2}{2n+1}$$

As far as $\widetilde{X}$ is concerned, it is unbiased by virtue of the symmetry of the normal distribution about its mean, and for large samples

$$\text{var}(\widetilde{X}) = \frac{\pi\sigma^2}{4n}$$

Thus, for large samples, the efficiency of the median relative to the mean is approximately

$$\frac{\text{var}(\overline{X})}{\text{var}(\widetilde{X})} = \frac{\dfrac{\sigma^2}{2n+1}}{\dfrac{\pi\sigma^2}{4n}} = \frac{4n}{\pi(2n+1)}$$

and the **asymptotic efficiency** of the median with respect to the mean is

$$\lim_{n\to\infty} \frac{4n}{\pi(2n+1)} = \frac{2}{\pi}$$

or about 64 percent.

Example 10.8 - Not Codable (problem contains theoretical explaination)

EXAMPLE 8

Show that for a random sample from a normal population, the sample variance S^2 is a consistent estimator of σ^2.

Solution

Since S^2 is an unbiased estimator of σ^2 in accordance with Theorem 3, it remains to be shown that $\text{var}(S^2) \to 0$ as $n \to \infty$. Referring to the theorem "the random variable $\frac{(n-1)S^2}{\sigma^2}$ has a chi-square distribution with $n-1$ degrees of freedom", we find that for a random sample from a normal population

$$\text{var}(S^2) = \frac{2\sigma^4}{n-1}$$

t follows that $\text{var}(S^2) \to 0$ as $n \to \infty$, and we have thus shown that S^2 is a consistent stimator of the variance of a normal population.

Example 10.9 - Not Codable (problem contains theoretical explaination)

EXAMPLE 9

With reference to Example 3, show that the smallest sample value (that is, the first order statistic Y_1) is a consistent estimator of the parameter δ.

Solution

Substituting into the formula for $g_1(y_1)$, we find that the sampling distribution of Y_1 is given by

$$\begin{aligned} g_1(y_1) &= n \cdot e^{-(y_1-\delta)} \cdot \left[\int_{y_1}^{\infty} e^{-(x-\delta)}\, dx \right]^{n-1} \\ &= n \cdot e^{-n(y_1-\delta)} \end{aligned}$$

for $y_1 > \delta$ and $g_1(y_1) = 0$ elsewhere. Based on this result, it can easily be shown that $E(Y_1) = \delta + \frac{1}{n}$ and hence that Y_1 is an asymptotically unbiased estimator of δ. Furthermore,

$$\begin{aligned} P(|Y_1 - \delta| < c) &= P(\delta < Y_1 < \delta + c) \\ &= \int_{\delta}^{\delta+c} n \cdot e^{-n(y_1-\delta)}\, dy_1 \\ &= 1 - e^{-nc} \end{aligned}$$

Since $\lim_{n \to \infty} (1 - e^{-nc}) = 1$, it follows from Definition 5 that Y_1 is a consistent estimator of δ.

Example 10.10 - Not Codable (problem contains theoretical explaination)

EXAMPLE 10

If $X_1, X_2, \ldots, X_n$ constitute a random sample of size n from a Bernoulli population, show that

$$\hat{\Theta} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

is a sufficient estimator of the parameter θ.

Solution

By the definition "**BERNOULLI DISTRIBUTION.** *A random variable* X *has a* ***Bernoulli distribution*** *and it is referred to as a Bernoulli random variable if and only if its probability distribution is given by* $f(x;\theta) = \theta^x(1-\theta)^{1-x}$ for $x = 0, 1$",

$$f(x_i;\theta) = \theta^{x_i}(1-\theta)^{1-x_i} \qquad \text{for } x_i = 0, 1$$

so that

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} \\ &= \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n-\sum_{i=1}^{n} x_i} \\ &= \theta^x(1-\theta)^{n-x} \\ &= \theta^{n\hat{\theta}}(1-\theta)^{n-n\hat{\theta}} \end{aligned}$$

for $x_i = 0$ or 1 and $i = 1, 2, \ldots, n$. Also, since

$$X = X_1 + X_2 + \cdots + X_n$$

is a binomial random variable with the parameters θ and n, its distribution is given by

$$b(x; n, \theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$$

and the transformation-of-variable technique yields

$$g(\hat{\theta}) = \binom{n}{n\hat{\theta}}\theta^{n\hat{\theta}}(1-\theta)^{n-n\hat{\theta}} \qquad \text{for } \hat{\theta} = 0, \frac{1}{n}, \ldots, 1$$

Now, substituting into the formula for $f(x_1, x_2, \ldots, x_n|\hat{\theta})$ on the previous page, we get

$$\begin{aligned}
\frac{f(x_1, x_2, \ldots, x_n, \hat{\theta})}{g(\hat{\theta})} &= \frac{f(x_1, x_2, \ldots, x_n)}{g(\hat{\theta})} \\
&= \frac{\theta^{n\hat{\theta}}(1-\theta)^{n-n\hat{\theta}}}{\binom{n}{n\hat{\theta}}\theta^{n\hat{\theta}}(1-\theta)^{n-n\hat{\theta}}} \\
&= \frac{1}{\binom{n}{n\hat{\theta}}} \\
&= \frac{1}{\binom{n}{x}} \\
&= \frac{1}{\binom{n}{x_1 + x_2 + \cdots + x_n}}
\end{aligned}$$

for $x_i = 0$ or 1 and $i = 1, 2, \ldots, n$. Evidently, this does not depend on θ and we have shown, therefore, that $\hat{\Theta} = \frac{X}{n}$ is a sufficient estimator of θ.

Example 10.11 - Not Codable (problem contains theoretical explaination)

EXAMPLE 11

Show that $Y = \frac{1}{6}(X_1 + 2X_2 + 3X_3)$ is not a sufficient estimator of the Bernoulli parameter θ.

Solution

Since we must show that

$$f(x_1, x_2, x_3|y) = \frac{f(x_1, x_2, x_3, y)}{g(y)}$$

is not independent of θ for some values of X_1, X_2, and X_3, let us consider the case where $x_1 = 1, x_2 = 1$, and $x_3 = 0$. Thus, $y = \frac{1}{6}(1 + 2 \cdot 1 + 3 \cdot 0) = \frac{1}{2}$ and

$$f\left(1,1,0|Y = \frac{1}{2}\right) = \frac{P\left(X_1 = 1, X_2 = 1, X_3 = 0, Y = \frac{1}{2}\right)}{P\left(Y = \frac{1}{2}\right)}$$

$$= \frac{f(1,1,0)}{f(1,1,0) + f(0,0,1)}$$

where

$$f(x_1, x_2, x_3) = \theta^{x_1+x_2+x_3}(1-\theta)^{3-(x_1+x_2+x_3)}$$

for $x_1 = 0$ or 1 and $i = 1,2,3$. Since $f(1,1,0) = \theta^2(1-\theta)$ and $f(0,0,1) = \theta(1-\theta)^2$, it follows that

$$f\left(1,1,0|Y = \frac{1}{2}\right) = \frac{\theta^2(1-\theta)}{\theta^2(1-\theta) + \theta(1-\theta)^2} = \theta$$

and it can be seen that this conditional probability depends on θ. We have thus shown that $Y = \frac{1}{6}(X_1 + 2X_2 + 3X_3)$ is not a sufficient estimator of the parameter θ of a Bernoulli population.

Example 10.12 - Not Codable (problem contains theoretical explaination)

EXAMPLE 12

Show that $\overline{X}$ is a sufficient estimator of the mean μ of a normal population with the known variance σ^2.

Solution

Making use of the fact that

$$f(x_1, x_2, \ldots, x_n; \mu) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2}\cdot\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

and that

$$\begin{aligned}\sum_{i=1}^{n}(x_i-\mu)^2 &= \sum_{i=1}^{n}[(x_i-\overline{x})-(\mu-\overline{x})]^2 \\ &= \sum_{i=1}^{n}(x_i-\overline{x})^2+\sum_{i=1}^{n}(\overline{x}-\mu)^2 \\ &= \sum_{i=1}^{n}(x_i-\overline{x})^2+n(\overline{x}-\mu)^2\end{aligned}$$

we get

$$\begin{aligned}f(x_1, x_2, \ldots, x_n; \mu) &= \left\{\frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{\overline{x}-\mu}{\sigma/\sqrt{n}}\right)^2}\right\} \\ &\times \left\{\frac{1}{\sqrt{n}}\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{n-1} \cdot e^{-\frac{1}{2}\cdot\sum_{i=1}^{n}\left(\frac{x_i-\overline{x}}{\sigma}\right)^2}\right\}\end{aligned}$$

where the first factor on the right-hand side depends only on the estimate $\overline{x}$ and the population mean μ, and the second factor does not involve μ. According to Theorem 4, it follows that $\overline{X}$ is a sufficient estimator of the mean μ of a normal population with the known variance σ^2.

Example 10.13 - Not Codable (problem contains theoretical explaination)

EXAMPLE 13

Given a random sample of size n from a uniform population with $\beta = 1$, use the method of moments to obtain a formula for estimating the parameter α.

Solution

The equation that we shall have to solve is $m_1' = \mu_1'$, where $m_1' = \bar{x}$ and $\mu_1' = \frac{\alpha+\beta}{2} = \frac{\alpha+1}{2}$. Thus,

$$\bar{x} = \frac{\alpha+1}{2}$$

and we can write the estimate of α as

$$\hat{\alpha} = 2\bar{x} - 1$$

Example 10.14 - Not Codable (problem contains theoretical explaination)

EXAMPLE 14

Given a random sample of size n from a gamma population, use the method of moments to obtain formulas for estimating the parameters α and β.

Solution

The system of equations that we shall have to solve is

$$m_1' = \mu_1' \quad \text{and} \quad m_2' = \mu_2'$$

where $\mu_1' = \alpha\beta$ and $\mu_2' = \alpha(\alpha+1)\beta^2$. Thus,

$$m_1' = \alpha\beta \quad \text{and} \quad m_2' = \alpha(\alpha+1)\beta^2$$

and, solving for α and β, we get the following formulas for estimating the two parameters of the gamma distribution:

$$\hat{\alpha} = \frac{(m_1')^2}{m_2' - (m_1')^2} \quad \text{and} \quad \hat{\beta} = \frac{m_2' - (m_1')^2}{m_1'}$$

Since $m_1' = \dfrac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$ and $m_2' = \dfrac{\sum_{i=1}^{n} x_i^2}{n}$, we can write

$$\hat{\alpha} = \frac{n\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \quad \text{and} \quad \hat{\beta} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n\bar{x}}$$

in terms of the original observations.

Example 10.15 - Not Codable (problem contains theoretical explaination)

EXAMPLE 15

Given x "successes" in n trials, find the maximum likelihood estimate of the parameter θ of the corresponding binomial distribution.

Solution

To find the value of θ that maximizes

$$L(\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

it will be convenient to make use of the fact that the value of θ that maximizes $L(\theta)$ will also maximize

$$\ln L(\theta) = \ln \binom{n}{x} + x \cdot \ln \theta + (n-x) \cdot \ln(1-\theta)$$

Thus, we get

$$\frac{d[\ln L(\theta)]}{d\theta} = \frac{x}{\theta} - \frac{n-x}{1-\theta}$$

and, equating this derivative to 0 and solving for θ, we find that the likelihood function has a maximum at $\theta = \frac{x}{n}$. This is the maximum likelihood estimate of the binomial parameter θ, and we refer to $\hat{\Theta} = \frac{X}{n}$ as the corresponding **maximum likelihood estimator**.

Example 10.16 - Not Codable (problem contains theoretical explaination)

EXAMPLE 16

If $x_1, x_2, \ldots, x_n$ are the values of a random sample from an exponential population, find the maximum likelihood estimator of its parameter θ.

Solution

Since the likelihood function is given by

$$\begin{aligned} L(\theta) &= f(x_1, x_2, \ldots, x_n; \theta) \\ &= \prod_{i=1}^{n} f(x_i; \theta) \\ &= \left(\frac{1}{\theta}\right)^n \cdot e^{-\frac{1}{\theta}\left(\sum_{i=1}^{n} x_i\right)} \end{aligned}$$

differentiation of $\ln L(\theta)$ with respect to θ yields

$$\frac{d[\ln L(\theta)]}{d\theta} = -\frac{n}{\theta} + \frac{1}{\theta^2} \cdot \sum_{i=1}^{n} x_i$$

Equating this derivative to zero and solving for θ, we get the maximum likelihood estimate

$$\hat{\theta} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i = \bar{x}$$

Hence, the maximum likelihood estimator is $\hat{\Theta} = \overline{X}$.

Example 10.17 - Not Codable (problem contains theoretical explaination)

EXAMPLE 17

If $x_1, x_2, \ldots, x_n$ are the values of a random sample of size n from a uniform population with $\alpha = 0$ (as in Example 4), find the maximum likelihood estimator of β.

Solution

The likelihood function is given by

$$L(\beta) = \prod_{i=1}^{n} f(x_i; \beta) = \left(\frac{1}{\beta}\right)^n$$

for β greater than or equal to the largest of the x's and 0 otherwise. Since the value of this likelihood function increases as β decreases, we must make β as small as possible, and it follows that the maximum likelihood estimator of β is Y_n, the nth order statistic.

Example 10.18 - Not Codable (problem contains theoretical explaination)

EXAMPLE 18

If $X_1, X_2, \ldots, X_n$ constitute a random sample of size n from a normal population with the mean μ and the variance σ^2, find joint maximum likelihood estimates of these two parameters.

Solution

Since the likelihood function is given by

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} n(x_i; \mu, \sigma)$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma^2}\cdot\sum_{i=1}^{n}(x_i-\mu)^2}$$

partial differentiation of $\ln L(\mu, \sigma^2)$ with respect to μ and σ^2 yields

$$\frac{\partial[\ln L(\mu, \sigma^2)]}{\partial\mu} = \frac{1}{\sigma^2}\cdot\sum_{i=1}^{n}(x_i - \mu)$$

and

$$\frac{\partial[\ln L(\mu, \sigma^2)]}{\partial\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\cdot\sum_{i=1}^{n}(x_i - \mu)^2$$

Equating the first of these two partial derivatives to zero and solving for μ, we get

$$\hat{\mu} = \frac{1}{n}\cdot\sum_{i=1}^{n} x_i = \overline{x}$$

and equating the second of these partial derivatives to zero and solving for σ^2 after substituting $\mu = \overline{x}$, we get

$$\hat{\sigma}^2 = \frac{1}{n}\cdot\sum_{i=1}^{n}(x_i - \overline{x})^2$$

Example 10.19 - Codable

Example 10.20 - Codable

Example 10.21 – Codable

Chapter 11: Interval Estimation

Example 11.1 - Codable

Example 11.2 - Codable

Example 11.3 - Codable

Example 11.4 - Codable

Example 11.5 - Codable

Example 11.6 - Codable

Example 11.7 - Codable

Example 11.8 - Codable

Example 11.9 - Codable

Example 11.10 - Codable

Example 11.11 - Codable

Example 11.12 – Codable

Chapter 12: Hypothesis Testing

Example 12.1 - Codable

Example 12.2 - Not Codable(problem contains theoretical explaination)

EXAMPLE 2

Suppose that we want to test the null hypothesis that the mean of a normal population with $\sigma^2 = 1$ is μ_0 against the alternative hypothesis that it is μ_1, where $\mu_1 > \mu_0$. Find the value of K such that $\bar{x} > K$ provides a critical region of size $\alpha = 0.05$ for a random sample of size n.

Solution

Referring to Figure 1 and the Standard Normal Distribution table of "Statistical Tables", we find that $z = 1.645$ corresponds to an entry of 0.4500 and hence that

$$1.645 = \frac{K - \mu_0}{1/\sqrt{n}}$$

It follows that

$$K = \mu_0 + \frac{1.645}{\sqrt{n}}$$

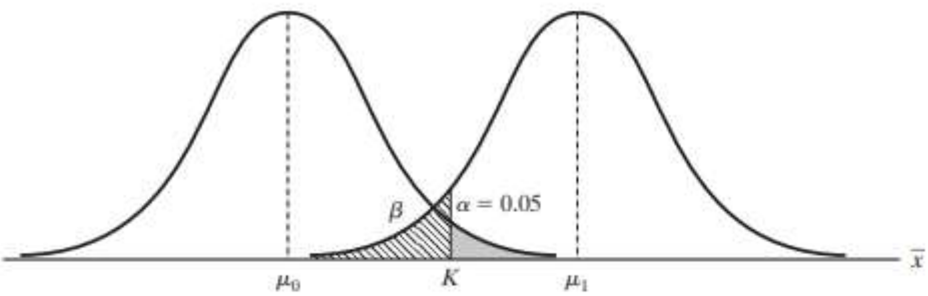

Figure 1. Diagram for Examples 2 and 3.

Example 12.3 - Not Codable(problem contains theoretical explaination)

EXAMPLE 3

With reference to Example 2, determine the minimum sample size needed to test the null hypothesis $\mu_0 = 10$ against the alternative hypothesis $\mu_1 = 11$ with $\beta \leqq 0.06$.

Solution

Since β is given by the area of the ruled region of Figure 1, we get

$$\beta = P\left(\overline{X} < 10 + \frac{1.645}{\sqrt{n}}; \mu = 11\right)$$

$$= P\left[Z < \frac{\left(10 + \frac{1.645}{\sqrt{n}}\right) - 11}{1/\sqrt{n}}\right]$$

$$= P(Z < -\sqrt{n} + 1.645)$$

and since $z = 1.555$ corresponds to an entry of $0.5000 - 0.06 = 0.4400$ in the Standard Normal Distribution table of "Statistical Tables", we set $-\sqrt{n} + 1.645$ equal to -1.555. It follows that $\sqrt{n} = 1.645 + 1.555 = 3.200$ and $n = 10.24$, or 11 rounded up to the nearest integer.

Example 12.4 - Not Codable(problem contains theoretical explaination)

EXAMPLE 4

A random sample of size n from a normal population with $\sigma^2 = 1$ is to be used to test the null hypothesis $\mu = \mu_0$ against the alternative hypothesis $\mu = \mu_1$, where $\mu_1 > \mu_0$. Use the Neyman–Pearson lemma to find the most powerful critical region of size α.

Solution

The two likelihoods are

$$L_0 = \left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2}\Sigma(x_i-\mu_0)^2} \quad \text{and} \quad L_1 = \left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2}\Sigma(x_i-\mu_1)^2}$$

where the summations extend from $i = 1$ to $i = n$, and after some simplification their ratio becomes

$$\frac{L_0}{L_1} = e^{\frac{n}{2}(\mu_1^2-\mu_0^2)+(\mu_0-\mu_1)\cdot\Sigma x_i}$$

Thus, we must find a constant k and a region C of the sample space such that

$$e^{\frac{n}{2}(\mu_1^2-\mu_0^2)+(\mu_0-\mu_1)\cdot\Sigma x_i} \leqq k \quad \text{inside } C$$
$$e^{\frac{n}{2}(\mu_1^2-\mu_0^2)+(\mu_0-\mu_1)\cdot\Sigma x_i} \geqq k \quad \text{outside } C$$

and after taking logarithms, subtracting $\frac{n}{2}(\mu_1^2 - \mu_0^2)$, and dividing by the negative quantity $n(\mu_0 - \mu_1)$, these two inequalities become

$$\bar{x} \geqq K \quad \text{inside } C$$
$$\bar{x} \leqq K \quad \text{outside } C$$

where K is an expression in k, n, μ_0, and μ_1.

In actual practice, constants like K are determined by making use of the size of the critical region and appropriate statistical theory. In our case (see Example 2) we obtain $K = \mu_0 + z_\alpha \cdot \frac{1}{\sqrt{n}}$. Thus, the most powerful critical region of size α for testing the null hypothesis $\mu = \mu_0$ against the alternative $\mu = \mu_1$ (with $\mu_1 > \mu_0$) for the given normal population is

$$\bar{x} \geqq \mu_0 + z_\alpha \cdot \frac{1}{\sqrt{n}}$$

and it should be noted that it does not depend on μ_1. This is an important property, to which we shall refer again in Section 5.

Example 12.5 - Not Codable(problem contains theoretical explaination)

EXAMPLE 5

With reference to Example 1, suppose that we had wanted to test the null hypothesis $\theta \geqq 0.90$ against the alternative hypothesis $\theta < 0.90$. Investigate the power function corresponding to the same test criterion as in Exercises 3 and 4, where we accept the null hypothesis if $x > 14$ and reject it if $x \leqq 14$. As before, x is the observed number of successes (recoveries) in $n = 20$ trials.

Solution

Choosing values of θ for which the respective probabilities, $\alpha(\theta)$ or $\beta(\theta)$, are available from the Binomial Probabilities table of "Statistical Tables", we find the

probabilities $\alpha(\theta)$ of getting at most 14 successes for $\theta = 0.90$ and 0.95 and the probabilities $\beta(\theta)$ of getting more than 14 successes for $\theta = 0.85, 0.80, \ldots, 0.50$. These are shown in the following table, together with the corresponding values of the power function, $\pi(\theta)$:

θ	*Probability of type I error* $\alpha(\theta)$	*Probability of type II error* $\beta(\theta)$	*Probability of rejecting* H_0 $\pi(\theta)$
0.95	0.0003		0.0003
0.90	0.0114		0.0114
0.85		0.9326	0.0674
0.80		0.8042	0.1958
0.75		0.6171	0.3829
0.70		0.4163	0.5837
0.65		0.2455	0.7545
0.60		0.1255	0.8745
0.55		0.0553	0.9447
0.50		0.0207	0.9793

Example 12.6 - Not Codable(problem contains theoretical explaination)

EXAMPLE 6

Find the critical region of the likelihood ratio test for testing the null hypothesis

$$H_0\colon \quad \mu = \mu_0$$

against the composite alternative

$$H_1\colon \quad \mu \neq \mu_0$$

on the basis of a random sample of size n from a normal population with the known variance σ^2.

Solution

Since ω contains only μ_0, it follows that $\hat{\mu} = \mu_0$, and since Ω is the set of all real numbers, it follows that $\hat{\hat{\mu}} = \overline{x}$. Thus,

$$\max\ L_0 = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma^2}\cdot\Sigma(x_i-\mu_0)^2}$$

and

$$\max\ L = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma^2}\cdot\Sigma(x_i-\overline{x})^2}$$

where the summations extend from $i = 1$ to $i = n$, and the value of the likelihood ratio statistic becomes

$$\begin{aligned}\lambda &= \frac{e^{-\frac{1}{2\sigma^2}\cdot\Sigma(x_i-\mu_0)^2}}{e^{-\frac{1}{2\sigma^2}\cdot\Sigma(x_i-\overline{x})^2}} \\ &= e^{-\frac{n}{2\sigma^2}(\overline{x}-\mu_0)^2}\end{aligned}$$

after suitable simplifications, which the reader will be asked to verify in Exercise 19. Hence, the critical region of the likelihood ratio test is

$$e^{-\frac{n}{2\sigma^2}(\overline{x}-\mu_0)^2} \leqq k$$

and, after taking logarithms and dividing by $-\dfrac{n}{2\sigma_2}$, it becomes

$$(\overline{x}-\mu_0)^2 \geqq -\frac{2\sigma^2}{n}\cdot\ln k$$

$$|\bar{x} - \mu_0| \geqq K$$

where K will have to be determined so that the size of the critical region is α. Note that $\ln k$ is negative in view of the fact that $0 < k < 1$.

Since $\overline{X}$ has a normal distribution with the mean μ_0 and the variance $\frac{\sigma^2}{n}$, we find that the critical region of this likelihood ratio test is

$$|\bar{x} - \mu_0| \geqq z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

or, equivalently,

$$|z| \geqq z_{\alpha/2}$$

where

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

In other words, the null hypothesis must be rejected when Z takes on a value greater than or equal to $z_{\alpha/2}$ or a value less than or equal to $-z_{\alpha/2}$.

Example 12.7 - Codable

Chapter 13: Tests of Hypothesis Involving Means, Variances, and Proportions

Example 13.1 - Codable

Example 13.2 - Codable

Example 13.3 - Codable

Example 13.4 - Codable

Example 13.5 - Codable

Example 13.6 - Codable

Example 13.7 - Codable

Example 13.8 - Codable

Example 13.9 - Codable

Example 13.10 - Codable

Example 13.11 - Codable

Example 13.12 - Codable

Example 13.13 – Codable

Chapter 14: Regression and Correlation:

Example 14.1 - Not Codable(problem contains theoretical explaination)

EXAMPLE 1

Given the two random variables X and Y that have the joint density

$$f(x,y) = \begin{cases} x \cdot e^{-x(1+y)} & \text{for } x > 0 \text{ and } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find the regression equation of Y on X and sketch the regression curve.

Solution
Integrating out y, we find that the marginal density of X is given by

$$g(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

and hence the conditional density of Y given $X = x$ is given by

$$w(y|x) = \frac{f(x,y)}{g(x)} = \frac{x \cdot e^{-x(1+y)}}{e^{-x}} = x \cdot e^{-xy}$$

for $y > 0$ and $w(y|x) = 0$ elsewhere, which we recognize as an exponential density with $\theta = \frac{1}{x}$. Hence, by evaluating

$$\mu_{Y|x} = \int_0^{\infty} y \cdot x \cdot e^{-xy}\, dy$$

or by referring to the corollary of a theorem given here "The mean and the variance of the exponential distribution are given by $\mu = \theta$ and $\sigma^2 = \theta^2$," we find that the regression equation of Y on X is given by

$$\mu_{Y|x} = \frac{1}{x}$$

Example 14.2 - Not Codable(problem contains theoretical explaination)

EXAMPLE 2

If X and Y have the multinomial distribution

$$f(x, y) = \binom{n}{x, y, n-x-y} \cdot \theta_1^x \theta_2^y (1-\theta_1-\theta_2)^{n-x-y}$$

for $x = 0, 1, 2, \ldots, n$, and $y = 0, 1, 2, \ldots, n$, with $x + y \leqq n$, find the regression equation of Y on X.

Solution

The marginal distribution of X is given by

$$\begin{aligned} g(x) &= \sum_{y=0}^{n-x} \binom{n}{x, y, n-x-y} \cdot \theta_1^x \theta_2^y (1-\theta_1-\theta_2)^{n-x-y} \\ &= \binom{n}{x} \theta_1^x (1-\theta_1)^{n-x} \end{aligned}$$

for $x = 0, 1, 2, \ldots, n$, which we recognize as a binomial distribution with the parameters n and θ_1. Hence,

$$w(y|x) = \frac{f(x, y)}{g(x)} = \frac{\binom{n-x}{y} \theta_2^y (1-\theta_1-\theta_2)^{n-x-y}}{(1-\theta_1)^{n-x}}$$

for $y = 0, 1, 2, \ldots, n-x$, and, rewriting this formula as

$$w(y|x) = \binom{n-x}{y} \left(\frac{\theta_2}{1-\theta_1}\right)^y \left(\frac{1-\theta_1-\theta_2}{1-\theta_1}\right)^{n-x-y}$$

we find by inspection that the conditional distribution of Y given $X = x$ is a binomial distribution with the parameters $n - x$ and $\frac{\theta_2}{1-\theta_1}$, so that the regression equation of Y on X is

$$\mu_{Y|x} = \frac{(n-x)\theta_2}{1-\theta_1}$$

Example 14.3 - Not Codable(problem contains theoretical explaination)

EXAMPLE 3

If the joint density of X_1, X_2, and X_3 is given by

$$f(x_1, x_2, x_3) = \begin{cases} (x_1 + x_2)e^{-x_3} & \text{for } 0 < x_1 < 1, 0 < x_2 < 1, x_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find the regression equation of X_2 on X_1 and X_3.

Solution

The joint marginal density of X_1 and X_3 is given by

$$m(x_1, x_3) = \begin{cases} \left(x_1 + \frac{1}{2}\right)e^{-x_3} & \text{for } 0 < x_1 < 1, x_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Therefore,

$$\mu_{X_2|x_1,x_3} = \int_{-\infty}^{\infty} x_2 \cdot \frac{f(x_1, x_2, x_3)}{m(x_1, x_3)}\, dx_2 = \int_0^1 \frac{x_2(x_1 + x_2)}{\left(x_1 + \frac{1}{2}\right)}\, dx_2$$

$$= \frac{x_1 + \frac{2}{3}}{2x_1 + 1}$$

Example 14.4 - Codable

Example 14.5 - Codable

Example 14.6 - Codable

Example 14.7 - Codable

Example 14.8 - Codable

Example 14.9 - Codable

Example 14.10 - Codable

Example 14.11 - Codable

Example 14.12 - Codable

Example 14.13 - Codable