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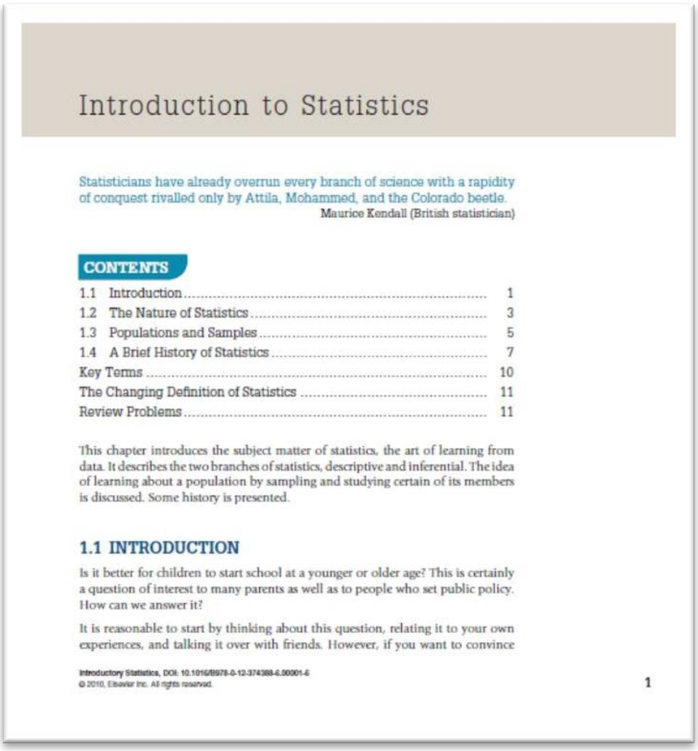
Chapter 1: Introduction to Statistics

This Chapter is Non-Codable (Reason: The chapter contains no example problems and mostly contains definitions.)



Introduction to Statistics

Statisticians have already overrun every branch of science with a rapidity of conquest rivalled only by Attila, Mohammed, and the Colorado beetle.
Maurice Kendall (British statistician)

CONTENTS

This chapter introduces the subject matter of statistics, the art of learning from data. It describes the two branches of statistics, descriptive and inferential. The idea of learning about a population by sampling and studying certain of its members is discussed. Some history is presented.

1.1 INTRODUCTION

Is it better for children to start school at a younger or older age? This is certainly a question of interest to many parents as well as to people who set public policy. How can we answer it?

It is reasonable to start by thinking about this question, relating it to your own experiences, and talking it over with friends. However, if you want to convince

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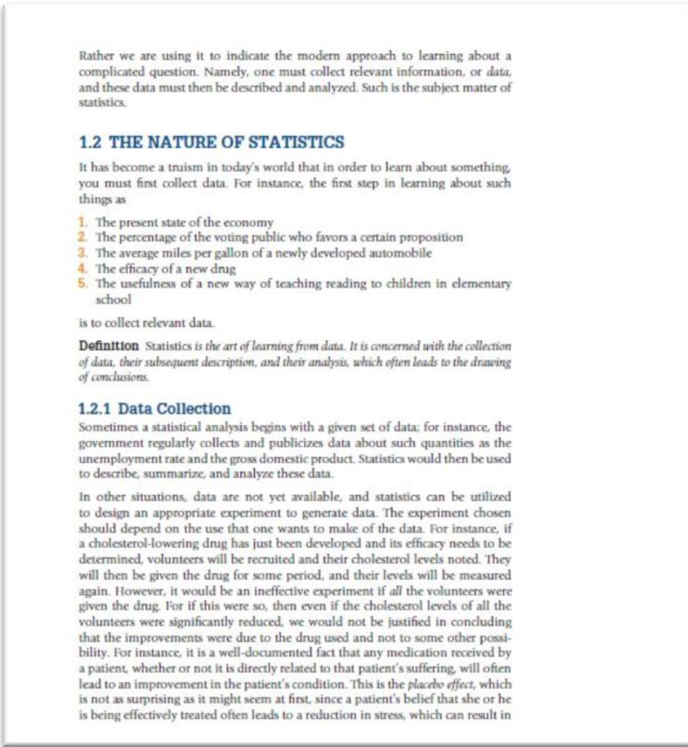

Rather we are using it to indicate the modern approach to learning about a complicated question. Namely, one must collect relevant information, or *data*, and these data must then be described and analyzed. Such is the subject matter of statistics.

1.2 THE NATURE OF STATISTICS

It has become a truism in today's world that in order to learn about something, you must first collect data. For instance, the first step in learning about such things as

1. The present state of the economy
2. The percentage of the voting public who favors a certain proposition
3. The average miles per gallon of a newly developed automobile
4. The efficacy of a new drug
5. The usefulness of a new way of teaching reading to children in elementary school

is to collect relevant data.

Definition Statistics *is the art of learning from data. It is concerned with the collection of data, their subsequent description, and their analysis, which often leads to the drawing of conclusions.*

1.2.1 Data Collection

Sometimes a statistical analysis begins with a given set of data; for instance, the government regularly collects and publicizes data about such quantities as the unemployment rate and the gross domestic product. Statistics would then be used to describe, summarize, and analyze these data.

In other situations, data are not yet available, and statistics can be utilized to design an appropriate experiment to generate data. The experiment chosen should depend on the use that one wants to make of the data. For instance, if a cholesterol-lowering drug has just been developed and its efficacy needs to be determined, volunteers will be recruited and their cholesterol levels noted. They will then be given the drug for some period, and their levels will be measured again. However, it would be an ineffective experiment if *all* the volunteers were given the drug. For if this were so, then even if the cholesterol levels of all the volunteers were significantly reduced, we would not be justified in concluding that the improvements were due to the drug used and not to some other possibility. For instance, it is a well-documented fact that any medication received by a patient, whether or not it is directly related to that patient's suffering, will often lead to an improvement in the patient's condition. This is the *placebo effect*, which is not as surprising as it might seem at first, since a patient's belief that she or he is being effectively treated often leads to a reduction in stress, which can result in

Chapter 2: Describing Data Sets

Example 2.1 – Codable

Example 2.2 – Codable

Example 2.3 – Codable

Example 2.4 – Codable

Example 2.5 – Codable

Example 2.6 – Codable

Example 2.7 – Non-Codable (Reason: The stem and leaf plot is already given and the plot is explained theoretically.)

Example 2.7- The following stem-and-leaf plot represents the weights of 80 attendees at a sporting convention. The stem represents the tens digit, and the leaves are the ones digit.

■ Example 2.7

The following stem-and-leaf plot represents the weights of 80 attendees at a sporting convention. The stem represents the tens digit, and the leaves are the ones digit.

10	2, 3, 3, 4, 7	(5)
11	0, 1, 2, 2, 3, 6, 9	(7)
12	1, 2, 4, 4, 6, 6, 6, 7, 9	(9)
13	1, 2, 2, 5, 5, 6, 6, 8, 9	(9)
14	0, 4, 6, 7, 7, 9, 9	(7)
15	1, 1, 5, 6, 6, 6, 7	(7)
16	0, 1, 1, 1, 2, 4, 5, 6, 8, 8	(10)
17	1, 1, 3, 5, 6, 6, 6	(7)
18	1, 2, 2, 5, 5, 6, 6, 9	(8)
19	0, 0, 1, 2, 4, 5	(6)
20	9, 9	(2)
21	7	(1)
22	1	(1)
23		(0)
24	9	(1)

The numbers in parentheses on the right represent the number of values in each stem class. These summary numbers are often useful. They tell us, for instance, that there are 10 values having stem 16; that is, 10 individuals have weights between 160 and 169. Note that a stem without any leaves (such as stem value 23) indicates that there are no occurrences in that class.

It is clear from this plot that almost all the data values are between 100 and 200, and the spread is fairly uniform throughout this region, with the exception of fewer values in the intervals between 100 and 110 and between 190 and 200. ■

Stem-and-leaf plots are quite useful in showing all the data values in a clear representation that can be the first step in describing, summarizing, and learning from the data. It is most helpful in moderate-size data sets. (If the size of the data set were very large, then, from a practical point of view, the values of all the leaves might be too overwhelming and a stem-and-leaf plot might not be any more informative than a histogram.) Physically this plot looks like a histogram turned on its side, with the additional plus that it presents the original within-group data values. These within-group values can be quite valuable to help you discover patterns in the data, such as that all the data values are multiples of some common value,

Stem-and-leaf plots are quite useful in showing all the data values in a clear representation that can be the first step in describing, summarizing, and learning from the data. It is most helpful in moderate-size data sets. (If the size of the data set were very large, then, from a practical point of view, the values of all the leaves might be too overwhelming and a stem-and-leaf plot might not be any more informative than a histogram.) Physically this plot looks like a histogram turned on its side, with the additional plus that it presents the original within-group data values. These within-group values can be quite valuable to help you discover patterns in the data, such as that all the data values are multiples of some common value, or find out which values occur most frequently within a stem group.

Sometimes a stem-and-leaf plot appears to have too many leaves per stem line and as a result looks cluttered. One possible solution is to double the number of stems by having two stem lines for each stem value. On the top stem line in the pair we could include all leaves having values 0 through 4, and on the bottom stem line all leaves having values 5 through 9. For instance, suppose one line of a stem-and-leaf plot is as follows:

6 | 0, 0, 1, 2, 2, 3, 4, 4, 4, 4, 5, 5, 6, 6, 7, 7, 7, 7, 8, 9, 9

This could be broken into two lines:

6 | 0, 0, 1, 2, 2, 3, 4, 4, 4, 4
6 | 5, 5, 6, 6, 7, 7, 7, 7, 8, 9, 9

Chapter 3: Using Statistics to Summarize Data Sets

Example 3.1 – Codable

Example 3.2– Codable

Example 3.3 – Codable

Example 3.4 – Codable

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Example 3.23 – Codable

Example 3.24 – Codable

Chapter 4: Probability

Example 4.1 – Non-Codable (Reason: The example problem has examples of experiments and their sample spaces and the final result is arbitrary in each case.)

Example 4.1- Some examples of experiments and their sample spaces are as follows.

■ Example 4.1

Some examples of experiments and their sample spaces are as follows.

(a) If the outcome of the experiment is the gender of a child, then

$$S = \{g, b\}$$

4.2 Sample Space and Events of an E

where outcome g means that the child is a girl and b that it is a boy.

(b) If the experiment consists of flipping two coins and noting whether they land heads or tails, then

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

The outcome is (H, H) if both coins land heads, (H, T) if the first coin lands heads and the second tails, (T, H) if the first is tails and the second is heads, and (T, T) if both coins land tails.

(c) If the outcome of the experiment is the order of finish in a race among 7 horses having positions 1, 2, 3, 4, 5, 6, 7, then

$$S = \{\text{all orderings of } 1, 2, 3, 4, 5, 6, 7\}$$

4 horse comes in first, the number 1 horse comes in second, and so on.

(d) Consider an experiment that consists of rolling two six-sided dice and noting the sides facing up. Calling one of the dice die 1 and the other die 2, we can represent the outcome of this experiment by the pair of upturned values on these dice. If we let (i, j) denote the outcome in which die 1 has value i and die 2 has value j, then the sample space of this experiment is

$$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$$ ■

Any set of outcomes of the experiment is called an *event*. That is, an event is a subset of the sample space. Events will be denoted by the capital letters A, B, C, and so on.

Example 4.2– Non-Codable (Reason: The example problem deals with the definition of events and has a theoretical explanation of Venn diagram)

Example 4.2- In Example 4.1(a), if $A = \{g\}$, then A is the event that the child is a girl. Similarly, if $B = \{b\}$, then B is the event that the child is a boy.

■ Example 4.2

In Example 4.1(a), if $A = \{g\}$, then A is the event that the child is a girl. Similarly, if $B = \{b\}$, then B is the event that the child is a boy.

In Example 4.1(b), if $A = \{(H, H), (H, T)\}$, then A is the event that the first coin lands on heads.

In Example 4.1(c), if

$$A = \{\text{all outcomes in } S \text{ starting with } 2\}$$

then A is the event that horse number 2 wins the race.

In Example 4.1(d), if

$$A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

then A is the event that the sum of the dice is 7. ■

Definition *Any set of outcomes of the experiment is called an* event. *We designate events by the letters A, B, C, and so on. We say that the event A occurs whenever the outcome is contained in A.*

For any two events A and B, we define the new event $A \cup B$, called the *union* of events A and B, to consist of all outcomes that are in A or in B or in both A and B. That is, the event $A \cup B$ will occur if *either* A or B occurs.

For any two events A and B, we define the new event $A \cup B$, called the *union* of events A and B, to consist of all outcomes that are in A or in B or in both A and B. That is, the event $A \cup B$ will occur if *either* A or B occurs.

In Example 4.1(a), if $A = \{g\}$ is the event that the child is a girl and $B = \{b\}$ is the event that it is a boy, then $A \cup B = \{g, b\}$. That is, $A \cup B$ is the whole sample space S.

In Example 4.1(c), let

$$A = \{\text{all outcomes starting with } 4\}$$

be the event that the number 4 horse wins; and let

$$B = \{\text{all outcomes whose second element is } 2\}$$

be the event that the number 2 horse comes in second. Then $A \cup B$ is the event that either the number 4 horse wins or the number 2 horse comes in second or both.

A graphical representation of events that is very useful is the *Venn diagram*. The sample space S is represented as consisting of all the points in a large rectangle, and events are represented as consisting of all the points in circles within the rectangle. Events of interest are indicated by shading appropriate regions of the diagram. The colored region of Fig. 4.1 represents the union of events A and B.

For any two events A and B, we define the *intersection* of A and B to consist of all outcomes that are both in A and in B. That is, the intersection will occur if *both* A and B occur. We denote the intersection of A and B by $A \cap B$. The colored region of Fig. 4.2 represents the intersection of events A and B.

In Example 4.1(b), if $A = \{(H, H), (H, T)\}$ is the event that the first coin lands heads and $B = \{(H, T), (T, T)\}$ is the event that the second coin lands tails, then $A \cap B = \{(H, T)\}$ is the event that the first coin lands heads and the second lands tails.

In Example 4.1(c), if A is the event that the number 2 horse wins and B is the event that the number 3 horse wins, then the event $A \cap B$ does not contain any outcomes and so cannot occur. We call the event without any outcomes the *null* event, and

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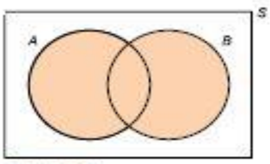

FIGURE 4.1
A Venn diagram: shaded region is $A \cup B$.

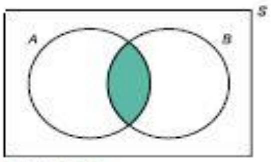

FIGURE 4.2
Shaded region is $A \cap B$.

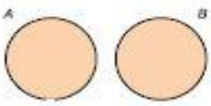

FIGURE 4.3
A and B are disjoint events.

designate it as $\emptyset$. If the intersection of A and B is the null event, then since A and B cannot simultaneously occur, we say that A and B are *disjoint*, or *mutually exclusive*. Two disjoint events are pictured in the Venn diagram of Fig. 4.3.

For any event A we define the event A^c, called the *complement* of A, to consist of all outcomes in the sample space that are not in A. That is, A^c will occur when A does not, and vice versa. For instance, in Example 4.1(a), if $A = \{g\}$ is the event that the child is a girl, then $A^c = \{b\}$ is the event that it is a boy. Also note that the complement of the sample space is the null set, that is, $S^c = \emptyset$. Figure 4.4 indicates A^c, the complement of event A.

We can also define unions and intersections of more than two events. For instance, the union of events A, B, and C, written $A \cup B \cup C$, consists of all the outcomes

Example 4.3 – Codable

Example 4.4 – Codable

Example 4.5 – Codable

Example 4.6 – Codable

Example 4.7 – Codable

Example 4.8 – Codable

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Example 4.19 – Codable

Example 4.20 – Codable

Example 4.21 – Codable

Example 4.22 – Codable

Example 4.23 – Codable

Example 4.24 – Codable

Example 4.25 – Non-Codable (Reason: The example problem is a derivation of a formula.)

Example 4.25- Suppose that *n* + *m* digits, *n* of which are equal to 1 and *m* of which are equal to 0, are to be arranged in a linear order. How many different arrangements are possible? For instance, if *n* = 2 and *m* = 1, then there are 3 possible arrangements:
1, 1, 0 1, 0, 1 0, 1, 1

■ Example 4.25

Suppose that $n + m$ digits, n of which are equal to 1 and m of which are equal to 0, are to be arranged in a linear order. How many different arrangements are possible? For instance, if $n = 2$ and $m = 1$, then there are 3 possible arrangements:

$$1, 1, 0 \quad 1, 0, 1 \quad 0, 1, 1$$

Solution

Each arrangement will have a digit in position 1, another digit in position 2, another in position 3, …, and finally a digit in position $n + m$. Each arrangement can therefore be described by specifying the n positions that contain the digit 1. That is, each different choice of n of the $n + m$ positions will result in a different arrangement. Therefore, there are $\binom{n+m}{n}$ different arrangements.

Of course, we can also describe an arrangement by specifying the m positions that contain the digit 0. This results in the solution $\binom{n+m}{m}$, which is equal to $\binom{n+m}{n}$. ■

Chapter 5: Discrete Random Variables

Example 5.1 – Codable

Example 5.2– Codable

Example 5.3 – Codable

Example 5.4 – Codable

Example 5.5 – Codable

Example 5.6 – Non-Codable (Reason: The example problem is variable based.)

Example 5.6-Consider a random variable *X* that takes on either the value 1 or 0 with respective probabilities *p* and 1 – *p*.

■ Example 5.6

Consider a random variable X that takes on either the value 1 or 0 with respective probabilities p and $1 - p$. That is,

$$P\{X = 1\} = p \quad \text{and} \quad P\{X = 0\} = 1 - p$$

Find $E[X]$.

Solution

The expected value of this random variable is

$$E[X] = 1(p) + 0(1 - p) = p$$ ■

Example 5.7 – Non-Codable (Reason: The example problem is variable based.)

Example 5.7- An insurance company sets its annual premium on its life insurance policies so that it makes an expected profit of 1 percent of the amount it would have to pay out upon death. Find the annual premium on a $200,000 life insurance policy for an individual who will die during the year with probability 0.02.

■ Example 5.7

An insurance company sets its annual premium on its life insurance policies so that it makes an expected profit of 1 percent of the amount it would have to pay out upon death. Find the annual premium on a $200,000 life insurance policy for an individual who will die during the year with probability 0.02.

Solution

In units of $1000, the insurance company will set its premium so that its expected profit is 1 percent of 200, or 2. If we let A denote the annual premium, then the profit of the insurance company will be either

$$A \quad \text{if policyholder lives}$$

or

$$A - 200 \quad \text{if policyholder dies}$$

Therefore, the expected profit is given by

$$\begin{aligned} E[\text{profit}] &= AP\{\text{policyholder lives}\} + (A - 200)P\{\text{policyholder dies}\} \\ &= A(1 - 0.02) + (A - 200)(0.02) \\ &= A - 200(0.02) \\ &= A - 4 \end{aligned}$$

So the company will have an expected profit of $2000 if it charges an annual premium of $6000. ■

As seen in Example 5.7, $E[X]$ is always measured in the same units (dollars in that example) as the random variable X.

Example 5.8 – Codable

Example 5.9 – Codable

Example 5.10 – Codable

Example 5.11 – Codable

Example 5.12 – Non-Codable (Reason The example problem is variable based.)

Example 5.12- Find Var(*X*) when the random variable *X* is such that
X =

_
1 with probability *p*
0 with probability 1 – *p*

■ Example 5.12

Find Var(X) when the random variable X is such that

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

Solution

In Example 5.6 we showed that $E[X] = p$. Therefore, using the computational formula for the variance, we have

$$\text{Var}(X) = E[X^2] - p^2$$

Now,

$$X^2 = \begin{cases} 1^2 & \text{if } X = 1 \\ 0^2 & \text{if } X = 0 \end{cases}$$

Since $1^2 = 1$ and $0^2 = 0$, we see that

$$\begin{aligned} E[X^2] &= 1 \cdot P\{X = 1\} + 0 \cdot P\{X = 0\} \\ &= 1 \cdot p = p \end{aligned}$$

Hence,

$$\text{Var}(X) = p - p^2 = p(1 - p) \quad ■$$

.
Example 5.13 – Codable

Example 5.14– Codable

Example 5.15 – Codable

Example 5.16 – Codable

Example 5.17 – Codable

Example 5.18 – Codable

Example 5.19 – Codable

Example 5.20 – Codable

Example 5.21 – Codable

Example 5.22 – Codable

Example 5.23 – Codable

Chapter 6: Normal Random Variables

Example 6.1 – Non-Codable (Reason: The approximation rule is theoretically explained in this example and hence cannot be coded.)

Example 6.1- Test scores on the Scholastic Aptitude Test (SAT) verbal portion are normally distributed with a mean score of 504. If the standard deviation of a score is 84, then we can conclude that approximately 68 percent of all scores are between 504 – 84 and 504 + 84. That is, approximately 68 percent of the scores are between 420 and 588. Also, approximately 95 percent of them are between 504 – 168 = 336 and 504 + 168 = 672; and approximately 99.7 percent are between 252 and 756.

■ **Example 6.1**

Test scores on the Scholastic Aptitude Test (SAT) verbal portion are normally distributed with a mean score of 504. If the standard deviation of a score is 84, then we can conclude that approximately 68 percent of all scores are between $504 - 84$ and $504 + 84$. That is, approximately 68 percent of the scores are between 420 and 588. Also, approximately 95 percent of them are between $504 - 168 = 336$ and $504 + 168 = 672$; and approximately 99.7 percent are between 252 and 756. ■

The approximation rule is the theoretical basis of the empirical rule of Sec. 3.6. The connection between these rules will become apparent in Chap. 8, when we show how a sample mean and sample standard deviation can be used to estimate the quantities μ and σ.

By using the symmetry of the normal curve about the value μ, we can obtain other facts from the approximation rule. For instance, since the area between μ and $\mu + \sigma$ is the same as that between $\mu - \sigma$ and μ, it follows from this rule that a normal random variable will be between μ and $\mu + \sigma$ with approximate probability $0.68/2 = 0.34$.

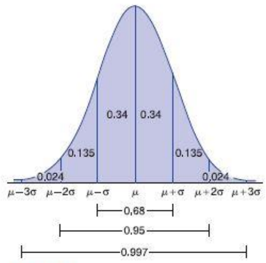


FIGURE 6.6
Approximate areas under a normal curve.

Example 6.2– Codable

Example 6.3 – Codable

Example 6.4 – Codable

Example 6.5 – Codable

Example 6.6 – Codable

Example 6.7 – Codable

Example 6.8 – Codable

Example 6.9 – Codable

Example 6.10 – Codable

Example 6.11 – Codable

Chapter 7: Distributions of Sampling Statistics

Example 7.1– Codable

Example 7.2– Codable

Example 7.3 – Codable

Example 7.4 – Codable

Example 7.5 – Non-Codable (Reason: It is a definition based example problem that has a proof-like solution and the final answer is in the form of a formula/variables.)

Example 7.5

Suppose that 60 out of a total of 900 students of a particular school are left-handed. If left-handedness is the characteristic of interest, then $N = 900$ and $p = 60/900 = 1/15$. ■

A sample of size n is said to be a *random sample* if it is chosen in a manner so that each of the possible population subsets of size n is equally likely to be in the sample. For instance, if the population consists of the three elements a, b, c, then a random sample of size 2 is one chosen so that it is equally likely to be any of the subsets $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$. A random subset can be chosen sequentially by letting its first element be equally likely to be any of the N elements of the population, then letting its second element be equally likely to be any of the remaining $N - 1$ elements of the population, and so on.

Definition *A sample of size* n *selected from a population of* N *elements is said to be a* random sample *if it is selected in such a manner that the sample chosen is equally likely to be any of the subsets of size* n.

The mechanics of using a computer to choose a random sample are explained in App. C. (In addition, Program A-1 on the enclosed disk can be used to accomplish this task.)

Suppose now that a random sample of size n has been chosen from a population of size N. For $i = 1, \dots, n$, let

$$X_i = \begin{cases} 1 & \text{if the } i\text{th member of the sample has the characteristic} \\ 0 & \text{otherwise} \end{cases}$$

Since the term X_i contributes 1 to the sum if the ith member of the sample has the characteristic and contributes 0 otherwise, it follows that the sum is equal to the number of members of the sample that possess the characteristic. (For instance, suppose $n = 3$ and $X_1 = 1, X_2 = 0$, and $X_3 = 1$. Then members 1 and 3 of the sample possess the characteristic, and member 2 does not. Hence, exactly 2 of the sample members possess it, as indicated by $X_1 + X_2 + X_3 = 2$.) Similarly, the sample mean

$$\overline{X} = \frac{X}{n} = \frac{\sum_{i=1}^{n} X_i}{n}$$

will equal the *proportion* of members of the sample who possess the characteristic. Let us now consider the probabilities associated with the statistic $\overline{X}$.

Since the ith member of the sample is equally likely to be any of the N members of the population, of which Np have the characteristic, it follows that

$$P\{X_i = 1\} = \frac{Np}{N} = p$$

Also

$$P\{X_i = 0\} = 1 - P\{X_i = 1\} = 1 - p$$

That is, each X_i is equal to either 1 or 0 with respective probabilities p and $1 - p$.

Note that the random variables $X_1, X_2, \dots, X_n$ are not independent. For instance, since the second selection is equally likely to be any of the N members of the

Indeed, it can be shown that when the population size N is large with respect to the sample size n, then $X_1, X_2, \dots, X_n$ are approximately independent. Now if we think of each X_i as representing the result of a trial that is a success if X_i equals 1 and a failure otherwise, it follows that $\sum_{i=1}^{n} X_i$ can be thought of as representing the total number of successes in n trials. Hence, if the X's are independent, then X represents the number of successes in n independent trials, where each trial is a success with probability p. In other words, X is a binomial random variable with parameters n and p.

If we let X denote the number of members of the population who have the characteristic, then it follows from the preceding that if the population size N is large in relation to the sample size n, then the distribution of X is approximately a binomial distribution with parameters n and p.

For the remainder of this text we will suppose that the underlying population is large in relation to the sample size, and we will take the distribution of X to be binomial.

By using the formulas given in Sec. 5.5.1 for the mean and standard deviation of a binomial random variable, we see that

$$E[X] = np \quad \text{and} \quad SD(X) = \sqrt{np(1-p)}$$

Since $\overline{X}$, the proportion of the sample that has the characteristic, is equal to X/n, we see that

$$E[\overline{X}] = \frac{E[X]}{n} = p$$

and

$$SD(\overline{X}) = \frac{SD(X)}{n} = \sqrt{\frac{p(1-p)}{n}}$$

Example 7.6 – Codable

Example 7.7 – Codable

Chapter 8: Estimation

Example 8.1 – Codable

Example 8.2– Codable

Example 8.3 – Codable

Example 8.4 – Codable

Example 8.5 – Codable

Example 8.6 – Codable

Example 8.7 – Codable

Example 8.8 – Codable

Example 8.9 – Codable

Example 8.10 – Codable

Example 8.11 – Codable

Example 8.12 – – Non-Codable (Reason: The example problem is variable based and hence not codeable.)

Example 8.12- Find $t_{8,0.05}$.

■ Example 8.12

Find $t_{8,0.05}$.

Solution

The value of $t_{8,0.05}$ can be obtained from Table D.2. The following is taken from that table.

Values of $t_{n,\alpha}$

n	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.025$
6	1.440	1.943	2.447
7	1.415	1.895	2.365
→ 8	1.397	1.860	2.306
9	1.383	1.833	2.262

Reading down the $\alpha = 0.05$ column for the row $n = 8$ shows that $t_{8,0.05} = 1.860$. ■

By the symmetry of the t distribution about zero, it follows (see Fig. 8.10) that

$$P\{|T_n| \le t_{n,\alpha/2}\} = P\{-t_{n,\alpha/2} \le T_n \le t_{n,\alpha/2}\} = 1 - \alpha$$

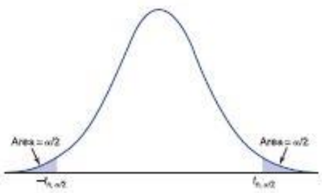


FIGURE 8.10
$P\{|T_n| \le t_{n,\alpha/2}\} = P\{-t_{n,\alpha/2} \le T_n \le t_{n,\alpha/2}\} = 1 - \alpha$.

Hence, upon using the result that $\sqrt{n}(\overline{X} - \mu)/S$ has a t distribution with $n - 1$ degrees of freedom, we see that

$$P\left\{\sqrt{n}\frac{|\overline{X} - \mu|}{S} \le t_{n-1,\alpha/2}\right\} = 1 - \alpha$$

In exactly the same manner as we did when σ was known, we can show that the preceding equation is equivalent to

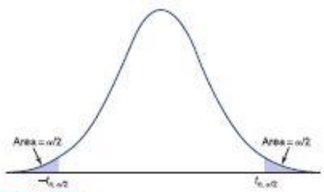


FIGURE 8.10
$P\{|T_n| \le t_{n,\alpha/2}\} = P\{-t_{n,\alpha/2} \le T_n \le t_{n,\alpha/2}\} = 1 - \alpha$.

Hence, upon using the result that $\sqrt{n}(\overline{X} - \mu)/S$ has a t distribution with $n - 1$ degrees of freedom, we see that

$$P\left\{\sqrt{n}\frac{|\overline{X} - \mu|}{S} \le t_{n-1,\alpha/2}\right\} = 1 - \alpha$$

In exactly the same manner as we did when σ was known, we can show that the preceding equation is equivalent to

$$P\left\{\overline{X} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}} \le \mu \le \overline{X} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}\right\} = 1 - \alpha$$

Therefore, we showed the following.

A $100(1 - \alpha)$ percent confidence interval estimator for the population mean μ is given by the interval

$$\overline{X} \pm t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}$$

Program 8-3 will compute the desired confidence interval estimate for a given data set.

Example 8.13 – Codable

Example 8.14– Codable

Example 8.15 – Codable

Example 8.16 – Codable

Example 8.17 – Codable

Example 8.18 – Codable

Chapter 9: Testing Statistical Hypotheses

Example 9.1 – Codable

Example 9.2– Codable

Example 9.3 – Codable

Example 9.4 – Codable

Example 9.5 – Codable

Example 9.6 – Codable

Example 9.7 – Codable

Example 9.8 – Codable

Example 9.9 – Codable

Example 9.10 – Codable

Example 9.11 – Codable

Chapter 10: Hypothesis Tests Concerning Two Populations

Example 10.1 – Codable

Example 10.2– Codable

Example 10.3 – Codable

Example 10.4 – Codable

Example 10.5 – Codable

Example 10.6 – Codable

Example 10.7 – Non-Codable (Reason: The example problem is variable based and the final answer is also in the form of variables.)

Example 10.7- Suppose we are interested in learning about the effect of a newly developed gasoline detergent additive on automobile mileage. To gather information, seven cars have been assembled, and their gasoline mileages (in units of miles per gallon) have been determined. For each car this determination is made both when gasoline without the additive is used and when gasoline with the additive is used. The data can be represented as follows:

■ Example 10.7

Suppose we are interested in learning about the effect of a newly developed gasoline detergent additive on automobile mileage. To gather information, seven cars have been assembled, and their gasoline mileages (in units of miles per gallon) have been determined. For each car this determination is made both when gasoline without the additive is used and when gasoline with the additive is used. The data can be represented as follows:

Car	Mileage without additive	Mileage with additive
1	24.2	23.5
2	30.4	29.6
3	32.7	32.3
4	19.8	17.6
5	25.0	25.3
6	24.9	25.4
7	22.2	20.6

For instance, car 1 got 24.2 miles per gallon by using gasoline without the additive and only 23.5 miles per gallon by using gasoline with the additive, whereas car 4 obtained 19.8 miles per gallon by using gasoline without the additive and 17.6 miles per gallon by using gasoline with the additive.

Now, it is easy to see that two factors will determine a car's mileage per gallon. One factor is whether the gasoline includes the additive, and the second factor is the car itself. For this reason we should not treat the two samples as being independent; rather, we should consider paired data. ■

Suppose we want to test

$$H_0: \mu_x = \mu_y \quad \text{against} \quad H_1: \mu_x \neq \mu_y$$

where the two samples consist of the paired data $X_i, Y_i, = 1, \ldots, n$. We can test this null hypothesis that the population means are equal by looking at the differences between the data values in a pairing. That is, let

$$D_i = X_i - Y_i \quad i = 1, \ldots, n$$

Now,

$$E[D_i] = E[X_i] - E[Y_i]$$

or, with $\mu_d = E[D_i]$,

$$\mu_d = \mu_x - \mu_y$$

The hypothesis that $\mu_x = \mu_y$ is therefore equivalent to the hypothesis that $\mu_d = 0$. Thus we can test the hypothesis that the population means are equal by testing

$$H_0: \mu_d = 0 \quad \text{against} \quad H_1: \mu_d \neq 0$$

Assuming that the random variables $D_1, \ldots, D_n$ constitute a sample from a normal population, we can test this null hypothesis by using the t test described in Sec. 9.4. That is, if we let $\overline{D}$ and S_d denote, respectively, the sample mean and sample standard deviation of the data $D_1, \ldots, D_n$, then the test statistic TS is given by

$$\text{TS} = \sqrt{n}\frac{\overline{D}}{S_d}$$

The significance-level-α test will be to

$$\begin{array}{ll} \text{Reject } H_0 & \text{if } |\text{TS}| \geq t_{n-1,\alpha/2} \\ \text{Not reject } H_0 & \text{otherwise} \end{array}$$

where the value of $t_{n-1,\alpha/2}$ can be obtained from Table D.2.

Equivalently, the test can be performed by computing the value of the test statistic TS, say it is equal to v, and then computing the resulting p value, given by

$$p \text{ value} = P\{|T_{n-1}| \geq |v|\} = 2P\{T_{n-1} \geq |v|\}$$

where T_{n-1} is a t random variable with $n-1$ degrees of freedom. If a personal computer is available, then Program 9-1 can be used to determine the value of the test statistic and the resulting p value. The successive data values entered in this program should be $D_1, D_2, \ldots, D_n$ and the value of μ_0 (the null hypothesis value for the mean of D) entered should be 0.

Example 10.8 – Codable

Example 10.9 – Codable

Example 10.10 – Codable

Example 10.11 – Codable

Example 10.12 – Non-Codable (Reason: The example problem is variable based and the final answer is also in the form of variables.)

Example 10.12- In 1970, the researchers Herbst,Ulfelder, and Poskanzer (H-U-P) suspected that vaginal cancer in young women, a rather rare disease, might be caused by one's

mother having taken the drug diethylstilbestrol (usually referred to as DES)
while pregnant……..

■ Example 10.12

In 1970, the researchers Herbst, Ulfelder, and Poskanzer (H-U-P) suspected that vaginal cancer in young women, a rather rare disease, might be caused by one's mother having taken the drug diethylstilbestrol (usually referred to as DES) while pregnant. To study this possibility, the researchers could have performed an observational study by searching for a (treatment) group of women whose mothers took DES when pregnant and a (control) group of women whose mothers did not. They could then observe these groups for a period of time and use the resulting data to test the hypothesis that the probabilities of contracting vaginal cancer are the same for both groups. However, because vaginal cancer is so rare (in both groups), such a study would require a large number of individuals in both groups and would probably have to continue for many years to obtain significant results. Consequently, H-U-P decided on a different type of observational study. They uncovered 8 women between the ages of 15 and 22 who had vaginal cancer. Each of these women (called cases) was then matched with 4 others, called *referents* or *controls*. Each of the referents of a case was free of the cancer and was born within 5 days in the same hospital and in the same type of room (either private or public) as the case. Arguing that if DES had no effect on vaginal cancer then the probability, call it p_c, that the mother of a case took DES would be the same as the probability, call it p_r, that the mother of a referent took DES, the researchers H-U-P decided to test

$$H_0: p_c = p_r \quad \text{against} \quad H_1: p_c \neq p_r$$

Discovering that 7 of the 8 cases had mothers who took DES while pregnant whereas none of the 32 referents had mothers who took the drug, the researchers concluded that there was a strong association between

DES and vaginal cancer (see Herbst, A., Ulfelder, H., and Poskanzer, D., "Adenocarcinoma of the Vagina: Association of Maternal Stilbestrol Therapy with Tumor Appearance in Young Women," *New England Journal of Medicine*, **284**, 878–881, 1971). (The p value for these data is approximately 0.) ■

DES and vaginal cancer (see Herbst, A., Ulfelder, H., and Poskanzer, D., "Adenocarcinoma of the Vagina: Association of Maternal Stilbestrol Therapy with Tumor Appearance in Young Women," *New England Journal of Medicine*, **284**, 878–881, 1971). (The p value for these data is approximately 0.) ■

If we are interested in verifying the one-sided hypothesis that p_1 is larger than p_2, then we should take that to be the alternative hypothesis and so test

$$H_0: p_1 \leq p_2 \quad \text{against} \quad H_1: p_1 > p_2$$

The same test statistic TS as used before is still employed, but now we reject H_0 only when TS is large (since this occurs when $\hat{p}_1 - \hat{p}_2$ is large). Thus, the one-sided significance-level-α test is to

$$\begin{array}{ll} \text{Reject } H_0 & \text{if } TS \geq z_\alpha \\ \text{Not reject } H_0 & \text{otherwise} \end{array}$$

Alternatively, if the value of the test statistic TS is v, then the resulting p value is

$$p \text{ value} = P\{Z \geq v\}$$

where Z is a standard normal.

Remark *The test of*

$$H_0: p_1 \leq p_2 \quad \textit{against} \quad H_1: p_1 > p_2$$

is the same as

$$H_0: p_1 = p_2 \quad \textit{against} \quad H_1: p_1 > p_2$$

This is so because in both cases we want to reject H_0 when $\hat{p}_1 - \hat{p}_2$ is so large that such a large value would have been highly unlikely if p_1 were not greater than p_2.

Example 10.13 – Codable

Chapter 11: Analysis of Variance

Example 11.1 – Codable

Example 11.2– Non-Codable (Reason: The example is the derivation of the value of Test Statistic and the result is in the form of variables/formula.)

Example 11.2- Let us do the computations of Example 11.1 by using Program 11-1. After the

data have been entered, we get the following output.

■ Example 11.2

Let us do the computations of Example 11.1 by using Program 11-1. After the data have been entered, we get the following output.

The denominator estimate is 165.967

The numerator estimate is 431.667

The value of the f-statistic is 2.6009

The p-value is 0.11525 ■

Table 11.2 summarizes the results of this section.

Remark *When $m = 2$, the preceding is a test of the null hypothesis that two independent samples, having a common population variance, have the same mean. The reader might*

Table 11.2 One-Factor ANOVA Table

Variables $\overline{X}_i$ and $S_i^2, i = 1, \ldots, m$, are the sample means and sample variances, respectively, of independent samples of size n from normal populations having means μ_i and a common variance σ^2.

Source of estimator	Estimator of σ^2	Value of test statistic
Between samples	$n\overline{S}^2 = \frac{n\sum_{i=1}^{m}(\overline{X}_i - \overline{\overline{X}})^2}{m-1}$	$\text{TS} = \frac{n\overline{S}^2}{\sum_{i=1}^{m}\frac{S_i^2}{m}}$
Within samples	$\sum_{i=1}^{m}\frac{S_i^2}{m}$	

Significance-level-α test of H_0: all μ_1 values are equal:

Reject H_0 if $\text{TS} \geq F_{m-1,m(n-1),\alpha}$

Table 11.2 One-Factor ANOVA Table

Variables $\overline{X}_i$ and $S_i^2, i = 1, \ldots, m$, are the sample means and sample variances, respectively, of independent samples of size n from normal populations having means μ_i and a common variance σ^2.

Source of estimator	Estimator of σ^2	Value of test statistic
Between samples	$n\overline{S}^2 = \frac{n\sum_{i=1}^{m}(\overline{X}_i - \overline{\overline{X}})^2}{m-1}$	$\text{TS} = \frac{n\overline{S}^2}{\sum_{i=1}^{m}\frac{S_i^2}{m}}$
Within samples	$\sum_{i=1}^{m}\frac{S_i^2}{m}$	

Significance-level-α test of H_0: all μ_1 values are equal:

	Reject H_0	if $\text{TS} \geq F_{m-1,m(n-1),\alpha}$
	Do not reject H_0	otherwise
If $\text{TS} = v$, then		p value $= P\{F_{m-1,m(n-1)} \geq v\}$

where $F_{m-1,m(n-1)}$ is an F random variable with $m-1$ numerator and $m(n-1)$ denominator degrees of freedom.

wonder how this compares with the one presented in Chap. 10. It turns out that the tests are exactly the same. That is, assuming the same data are used, they always give rise to exactly the same p value.

Example 11.3 – Non-Codable (Reason: The example problem explains the concept of Parameter Estimation using variables and the final result is also in the form of variables.)

Example 11.3- Four different standardized reading achievement tests were administered to

each of five students. Their scores were as follows:

■ Example 11.3

Four different standardized reading achievement tests were administered to each of five students. Their scores were as follows:

	Student				
Examination	**1**	**2**	**3**	**4**	**5**
1	75	73	60	70	86
2	78	71	64	72	90
3	80	69	62	70	85
4	73	67	63	80	92

Each value in this set of 20 data points is affected by two factors: the examination and the student whose score on that examination is being recorded. The examination factor has four possible values, or *levels*, and the student factor has five possible levels. ■

In general, let us suppose that there are m possible levels of the first factor and n possible levels of the second. Let X_{ij} denote the value of the data obtained when the first factor is at level i and the second factor is at level j. We often portray the data set in the following array of rows and columns:

$$\begin{array}{ccccccc} X_{11} & X_{12} & \cdots & X_{1j} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2j} & \cdots & X_{2n} \\ \cdots & & & & & \cdots \\ X_{i1} & X_{i2} & \cdots & X_{ij} & \cdots & X_{in} \\ \cdots & & & & & \cdots \\ X_{m1} & X_{m2} & \cdots & X_{mj} & \cdots & X_{mn} \end{array}$$

Because of this we refer to the first factor as the *row* factor and the second factor as the *column* factor. Also, the data value X_{ij} is the value in row i and column j.

As in Sec. 11.2, we suppose that all the data values $X_{ij}, i = 1, \ldots, m, j = 1, \ldots, n$, are independent normal random variables with common variance σ^2. However, whereas in Sec. 11.2 we supposed that only a single factor affected the mean value of a data point—namely, the sample to which it belonged—in this section we will suppose that the mean value of the data point depends on both its row and its column. However, before specifying this model, we first recall the model of Sec. 11.2. If we let X_{ij} represent the value of the jth member of sample i, then this model supposes that

$$E[X_{ij}] = \mu_i$$

If we now let μ denote the average value of the μ_i, that is,

$$\mu = \frac{\sum_{i=1}^{m}\mu_i}{m}$$

then we can write the preceding as

$$E[X_{ij}] = \mu + \alpha_i$$

where $\alpha_i = \mu_i - \mu$. With this definition of α_i equal to the deviation of μ_i from the average of the means μ, it is easy to see that

$$\sum_{i=1}^{m}\alpha_i = 0$$

In the case of two factors, we write our model in terms of row and column deviations. Specifically, we suppose that the expected value of variable X_{ij} can be expressed as follows:

$$E[X_{ij}] = \mu + \alpha_i + \beta_j$$

The value μ is referred to as the *grand mean*, α_i is the *deviation from the grand mean due to row* i, and β_j is the *deviation from the grand mean due to column* j. In addition, these quantities satisfy the following equalities:

$$\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j = 0$$

Let us start by determining estimators for parameters μ, α_i, and β_j, $i = 1, \ldots, m$, $j = 1, \ldots, n$. To do so, we will find it convenient to introduce the following "dot" notation. Let

$$X_{i.} = \frac{\sum_{j=1}^{n} X_{ij}}{n} = \text{average of all values in row } i$$

$$X_{.j} = \frac{\sum_{i=1}^{m} X_{ij}}{m} = \text{average of all values in column } j$$

$$X_{..} = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}}{nm} = \text{average of all } nm \text{ data values}$$

It is not difficult to show that

$$E[X_{i.}] = \mu + \alpha_i$$
$$E[X_{.j}] = \mu + \beta_j$$
$$E[X_{..}] = \mu$$

Since the preceding is equivalent to

$$E[X_{..}] = \mu$$
$$E[X_{i.} - X_{..}] = \alpha_i$$
$$E[X_{.j} - X_{..}] = \beta_j$$

we see that unbiased estimators of μ, α_i and β_j—call them $\hat{\mu}, \hat{\alpha}_i$, and $\hat{\beta}_j$—are given by

$$\hat{\mu} = X_{..}$$
$$\hat{\alpha}_i = X_{i.} - X_{..}$$
$$\hat{\beta}_j = X_{.j} - X_{..}$$

Example 11.4 – Codeable

Example 11.5 – Codable

Chapter 12: Linear Regression

Example 12.1 – Codable

Example 12.2– Codable

Example 12.3 – Codable

Example 12.4 – Codable

Example 12.5 – Codable

Example 12.6 – Codable

Example 12.7 – Codable

Example 12.8 – Codable

Example 12.9 – Codable

Example 12.10 – Codable

Example 12.11 – – Non-Codable (Reason: The problem is definition based and uses the given values to illustrate the theoretical concept of multiple linear regression.)

Example 12.11-In laboratory experiments two factors that often affect the percentage yield of the experiment are the temperature and the pressure at which the experiment is conducted. The following data detail the results of four independent experiments. For each experiment, we have the temperature (in degrees Fahrenheit) at which the experiment is run, the pressure (in pounds per square inch), and the percentage yield.

■ Example 12.11

In laboratory experiments two factors that often affect the percentage yield of the experiment are the temperature and the pressure at which the experiment is conducted. The following data detail the results of four independent experiments. For each experiment, we have the temperature (in degrees Fahrenheit) at which the experiment is run, the pressure (in pounds per square inch), and the percentage yield.

Experiment	Temperature	Pressure	Percentage yield
1	140	210	68
2	150	220	82
3	160	210	74
4	130	230	80

■

Suppose that we are interested in predicting the response value Y on the basis of the values of the k input variables $x_1, x_2, \ldots, x_k$.

Definition *The multiple linear regression model supposes that the response Y is related to the input values* x_i, $i = 1, \ldots, k$, *through the relationship*

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + e$$

Definition *The multiple linear regression model supposes that the response Y is related to the input values* x_i, $i = 1, \ldots, k$, *through the relationship*

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + e$$

In this expression, $\beta_0, \beta_1, \ldots, \beta_k$ are *regression parameters* and e is an *error* random variable that has mean 0. The regression parameters will not be initially known and must be estimated from a set of data.

Suppose that we have at our disposal a set of n responses corresponding to n different sets of the k input values. Let y_i denote the ith response, and let the k input values corresponding to this response be $x_{i1}, x_{i2}, \ldots, x_{ik}$, $i = 1, \ldots, n$. Thus, for instance, y_1 was the response when the k input values were $x_{11}, x_{12}, \ldots, x_{1k}$. The data set is presented in Fig. 12.10.

Example 12.12 –Non-Codable (Reason: The problem uses the given values to illustrate the theoretical concept to estimate regression parameters.)

Example 12.12-In Example 12.11 there are two input variables, the temperature and the pressure, and so k = 2. There are four experimental results, and so n = 4. The value

■ Example 12.12

In Example 12.11 there are two input variables, the temperature and the pressure, and so $k = 2$. There are four experimental results, and so $n = 4$. The value

Set	Input 1	Input 2	...	Input k	Response
1	x_{11}	x_{12}	...	x_{1k}	y_1
2	x_{21}	x_{22}	...	x_{2k}	y_2
3	x_{31}	x_{32}	...	x_{3k}	y_3
⋮					
n	x_{n1}	x_{n2}	...	x_{nk}	y_n

FIGURE 12.10
Data on n experiments.

x_{i1} refers to the temperature and x_{i2} to the pressure of experiment i. The value y_i is the percentage yield (response) of experiment i. Thus, for instance,

$$x_{31} = 160 \quad x_{32} = 210 \quad y_3 = 74$$

■

To estimate the regression parameters again, as in the case of simple linear regres-

To estimate the regression parameters again, as in the case of simple linear regres-sion, we use the method of least squares. That is, we start by noting that if $B_0, B_1, \ldots, B_k$ are estimators of the regression parameters $\beta_0, \beta_1, \ldots, \beta_k$, then the estimate of the response when the input values are $x_{i1}, x_{i2}, \ldots, x_{ik}$ is given by

$$\text{Estimated response} = B_0 + B_1 x_{i1} + B_2 x_{i2} + \cdots + B_k x_{ik}$$

Since the actual response was y_i, we see that the difference between the actual response and what would have been predicted if we had used the estimators $B_0, B_1, \ldots, B_k$ is

$$\epsilon_i = y_i - (B_0 + B_1 x_{i1} + B_2 x_{i2} + \cdots + B_k x_{ik})$$

Thus, ϵ_i can be regarded as the *error* that would have resulted if we had used the estimators B_i, $i = 0, \ldots, k$. The estimators that make the sum of the squares of the errors as small as possible are called the *least-squares estimators*.

The least-squares estimators of the regression parameters are the choices of B_i that make

$$\sum_{i=1}^{n} \epsilon_i^2$$

as small as possible.

The actual computations needed to obtain the least-squares estimators are algebraically messy and will not be presented here. Instead we refer to Program 12-2 to do the computations for us. The outputs of this program are the estimates of the regression parameters. In addition, the program provides predicted response values for specified sets of input values. That is, if the user enters the values $x_1, x_2, \ldots, x_k$, then the computer will print out the value of $B(0) + B(1)x_1 + \cdots + B(k)x_k$, where $B(0), B(1), \ldots, B(k)$ are the least-squares estimators of the regression parameters.

Example 12.13 – Codable

Chapter 13: Chi-Squared Goodness-of-Fit Tests

Example 13.1 – Non-Codable (Reason: The example problem verifies the null hypothesis theoretically and the final answer is in the form of a variable.)

■ Example 13.1

It is known that 41 percent of the U.S. population has type A blood, 9 percent has type B, 4 percent has type AB, and 46 percent has type O. Suppose that we suspect that the blood type distribution of people suffering from stomach cancer is different from that of the overall population.

To verify that the blood type distribution is different for those suffering from stomach cancer, we could test the null hypothesis

$$H_0: P_1 = 0.41,\ P_2 = 0.09,\ P_3 = 0.04,\ P_4 = 0.46$$

where P_1 is the proportion of all those with stomach cancer who have type A blood, P_2 is the proportion of those who have type B blood, P_3 is the proportion who have type AB blood, and P_4 is the proportion who have type O blood. A rejection of H_0 would then enable us to conclude that the blood type distribution is indeed different for those suffering from stomach cancer.

In the preceding scenario, each member of the population of individuals who are suffering from stomach cancer is given one of four possible values according to his or her blood type. We are interested in testing the hypothesis that $P_1 = 0.41, P_2 = 0.09, P_3 = 0.04, P_4 = 0.46$ represent the proportions of this population having each of the different values. ■

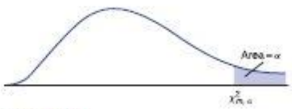


FIGURE 13.3
Chi-squared percentile $P(\chi^2_m \geq \chi^2_{m,\alpha}) = \alpha$.

Table 13.1 Some Values of $\chi^2_{m,\alpha}$

m	$\alpha = 0.99$	$\alpha = 0.95$	$\alpha = 0.05$	$\alpha = 0.01$
1	0.000157	0.00393	3.841	6.635
2	0.0201	0.103	5.991	9.210
3	0.115	0.352	7.815	11.345
4	0.297	0.711	9.488	13.277
5	0.554	1.145	11.070	15.086
6	0.872	1.635	12.592	16.812
7	1.239	2.167	14.067	18.475

Values of $\chi^2_{m,\alpha}$ for various values of m and α are given in App. Table D.3. A portion of this table is represented in Table 13.1.

To test the null hypothesis that $P_i = p_i, i = 1, \ldots, k$, first we need to draw a random sample of elements from the population. Suppose this sample is of size n. Let N_i denote the number of elements of the sample that have value i, for $i = 1, \ldots, k$. Now, if the null hypothesis is true, then each element of the sample will have value i with probability p_i. Also, since the population is assumed to be very large, it follows that the successive values of the members of the sample will be independent. Thus, if the null hypothesis is true, then N_i will have the same distribution as the number of successes in n independent trials, when each trial is a success with probability p_i. That is, if H_0 is true, then N_i will be a binomial random variable with parameters n and p_i. Since the expected value of a binomial is the product of its parameters, we see that when H_0 is true,

$$E[N_i] = np_i \qquad i = 1, \ldots, k$$

For each i, let e_i denote this expected number of outcomes that equal i when H_0 is true. That is,

$$e_i = np_i$$

Thus, when H_0 is true, we expect that N_i would be relatively close to e_i. That is, when the null hypothesis is true, the quantity $(N_i - e_i)^2$ should not be too large, say, in relation to e_i. Since this is true for each value of i, a reasonable way of testing H_0 would be to compute the value of the test statistic

$$TS = \sum_{i=1}^{k} \frac{(N_i - e_i)^2}{e_i}$$

and then reject H_0 when TS is sufficiently large.

To determine how large TS need be to justify rejection of the null hypothesis, we use a result that was proved by Karl Pearson in 1900. This result states that for large values of n, TS will have an approximately chi-squared distribution with $k-1$ degrees of freedom. Let $\chi^2_{k-1,\alpha}$ denote the $100(1-\alpha)$th percentile of this distribution; that is, a chi-squared random variable having $k-1$ degrees of freedom will exceed this value with probability α (Fig. 13.3). Then the approximate significance-level-α test of the null hypothesis H_0 against the alternative H_1 is as follows:

$$\begin{array}{ll} \text{Reject } H_0 & \text{if } TS \geq \chi^2_{k-1,\alpha} \\ \text{Do not reject } H_0 & \text{otherwise} \end{array}$$

The preceding is called the *chi-squared goodness-of-fit test*. For reasonably large values of n, it results in a hypothesis test of H_0 whose significance level is approximately equal to α. An accepted rule of thumb is that this approximation will be quite good provided n is large enough so that $e_i \geq 1$ for each i and at least 80 percent of the values e_i exceed 5.

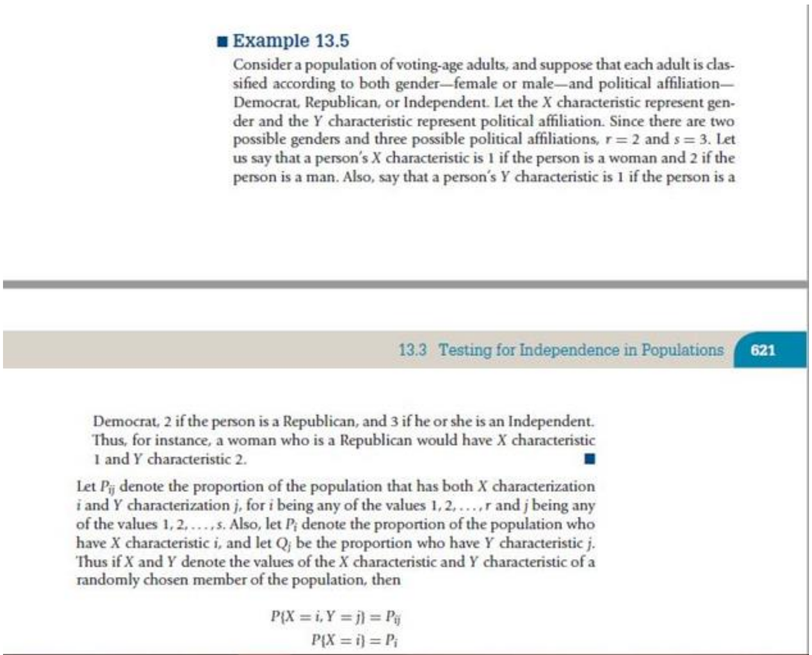
Example 13.2– Codable

Example 13.3 – Codable

Example 13.4 – Codable

Example 13.5 – Non-Codable (Reason: The problem depicts testing for independents in population and the final answer is in the form of variables.)

Example 13.5- Consider a population of voting-age adults, and suppose that each adult is classified according to both gender—female or male—and political affiliation—
Democrat, Republican, or Independent.



Example 13.5

Consider a population of voting-age adults, and suppose that each adult is classified according to both gender—female or male—and political affiliation—Democrat, Republican, or Independent. Let the X characteristic represent gender and the Y characteristic represent political affiliation. Since there are two possible genders and three possible political affiliations, $r = 2$ and $s = 3$. Let us say that a person's X characteristic is 1 if the person is a woman and 2 if the person is a man. Also, say that a person's Y characteristic is 1 if the person is a

13.3 Testing for Independence in Populations 621

Democrat, 2 if the person is a Republican, and 3 if he or she is an Independent. Thus, for instance, a woman who is a Republican would have X characteristic 1 and Y characteristic 2. ■

Let P_{ij} denote the proportion of the population that has both X characterization i and Y characterization j, for i being any of the values $1, 2, \ldots, r$ and j being any of the values $1, 2, \ldots, s$. Also, let P_i denote the proportion of the population who have X characteristic i, and let Q_j be the proportion who have Y characteristic j. Thus if X and Y denote the values of the X characteristic and Y characteristic of a randomly chosen member of the population, then

$$P\{X = i, Y = j\} = P_{ij}$$
$$P\{X = i\} = P_i$$

Example 13.6 – Non-Codable (Reason: Non-Codable (Reason: The problem depicts testing for independents in population and the final answer is in the form of variables.)

Example 13.6- For the situation described in Example 13.5, P_{11} represents the proportion of the population consisting of women who classify themselves as Democrats, P_{12} is the proportion of the population consisting of women who classify themselves as Republicans, and P_{13} is the proportion of the population consisting of women who classify themselves as Independents

■ Example 13.6

For the situation described in Example 13.5, P_{11} represents the proportion of the population consisting of women who classify themselves as Democrats, P_{12} is the proportion of the population consisting of women who classify themselves as Republicans, and P_{13} is the proportion of the population consisting of women who classify themselves as Independents. The proportions P_{21}, P_{22}, and P_{23} are defined similarly, with *men* replacing *women* in the definitions. The quantities P_1 and P_2 are the proportions of the population that are, respectively, women and men; Q_1, Q_2, and Q_3 are the proportions of the population that are, respectively, Democrats, Republicans, and Independents. ■

We will be interested in developing a test of the hypothesis that the X characteristic and Y characteristic of a randomly chosen member of the population are independent. Recalling that X and Y are independent if

$$P\{X = i, Y = j\} = P\{X = i\}P\{Y = j\}$$

it follows that we want to test the null hypothesis

$$\mathrm{H}_0\colon P_{ij} = P_i Q_j \quad \text{for all } i = 1, \ldots, r,\ j = 1, \ldots, s$$

against the alternative

$$\mathrm{H}_1\colon P_{ij} \neq P_i Q_j \quad \text{for some values of } i \text{ and } j$$

To test this hypothesis of independence, we start by choosing a random sample of size n of members of the population. Let N_{ij} denote the number of elements of the sample that have both X characteristic i and Y characteristic j.

Example 13.7 – Non-Codable (Reason: Non-Codable (Reason: Non-Codable (Reason: The problem depicts testing for independents in population and the final answer is in the form of variables.)

Example 13.7- Consider Example 13.5, and suppose that a random sample of 300 people were chosen from the population, with the following data resulting:

■ Example 13.7

Consider Example 13.5, and suppose that a random sample of 300 people were chosen from the population, with the following data resulting:

i	Democrat	Republican	Independent	Total
	j			
Women	68	56	32	156
Men	52	72	20	144
Total	120	128	52	300

Thus, for instance, the random sample of size 300 contained 68 women who classified themselves as Democrats, 56 women who classified themselves as Republicans, and 32 women who classified themselves as Independents; that is, $N_{11} = 68$, $N_{12} = 56$, and $N_{13} = 32$. Similarly, $N_{21} = 52$, $N_{22} = 72$, and $N_{23} = 20$.

This table, which specifies the number of members of the sample that fall in each of the rs cells, is called a *contingency table*. ■

If the hypothesis is true that the X and Y characteristics of a randomly chosen member of the population are independent, then each element of the sample will have X characteristic i and Y characteristic j with probability P_iQ_j. Hence, if these probabilities were known then, from the results of Sec. 13.2, we could test H_0 by using the test statistic

$$TS = \sum_i \sum_j \frac{(N_{ij} - e_{ij})^2}{e_{ij}}$$

where

$$e_{ij} = nP_iQ_j$$

The quantity e_{ij} represents the expected number, when H_0 is true, of elements in the sample that have both X characteristic i and Y characteristic j. In computing TS we must calculate the sum of the terms for all rs possible values of the pair i, j. When H_0 is true, TS will have an approximately chi-squared distribution with $rs - 1$ degrees of freedom.

The trouble with using this approach directly is that the $r + s$ quantities P_i and Q_j, $i = 1, \ldots, r$, $j = 1, \ldots, s$, are not specified by the null hypothesis. Thus, we need first to estimate them. To do so, let N_i and M_j denote the number of elements of the sample that have, respectively, X characteristic i and Y characteristic j. Because N_i/n and M_j/n are the proportions of the sample having, respectively, X characteristic i and Y characteristic j, it is natural to use them as estimators of P_i and Q_j.

That is, we estimate P_i and Q_j by

$$\hat{P}_i = \frac{N_i}{n} \quad \hat{Q}_j = \frac{M_j}{n}$$

This leads to the following estimate of e_{ij}:

$$\hat{e}_{ij} = n\hat{P}_i\hat{Q}_j = \frac{N_iM_j}{n}$$

In words, $\hat{e}_{ij}$ is equal to the product of the number of members of the sample that have X characteristic i (that is, the sum of row i of the contingency table) and the number of members of the sample that have Y characteristic j (that is, the sum of column j of the contingency table) divided by the sample size n.

Thus, it seems that a reasonable test statistic to use in testing the independence of the X characteristic and the Y characteristic is the following:

$$TS = \sum_i \sum_j \frac{(N_{ij} - \hat{e}_{ij})^2}{\hat{e}_{ij}}$$

where $\hat{e}_{ij}$, $i = 1, \ldots, r$, $j = 1, \ldots, s$, are as just given.

To specify the set of values of TS that will result in rejection of the null hypothesis, we need to know the distribution of TS when the null hypothesis is true. It can be shown that when H_0 is true, the distribution of the test statistic TS is approximately a chi-squared distribution with $(r-1)(s-1)$ degrees of freedom. From this, it follows that the significance-level-α test of H_0 is as follows:

Reject H_0	if $TS \geq \chi^2_{(r-1)(s-1),\alpha}$
Do not reject H_0	otherwise

A technical remark: It is not difficult to see why the test statistic TS should have $(r-1)(s-1)$ degrees of freedom. Recall from Sec. 13.2 that if all the values P_i and Q_j are specified in advance, then the test statistic has $rs - 1$ degrees of freedom. (This is so since k, the number of different types of elements in the population, is equal to rs.) Now, at first glance it may seem that we have to use the data to estimate $r + s$ parameters. However, since the P_i's and the Q_j's both sum to 1—that is, $\Sigma_i P_i = \Sigma_j Q_j = 1$—we really only need to estimate $r - 1$ of the P_i's and $s - 1$ of the Q_j's. (For instance, if r is equal to 2, then an estimate of P_1 will automatically provide an estimate of P_2 since $P_2 = 1 - P_1$.) Hence, we actually need to estimate $r - 1 + s - 1 = r + s - 2$ parameters. Since a degree of freedom is lost for each parameter estimated, it follows that the resulting test statistic has $rs - 1 - (r + s - 2) = rs - r - s + 1 = (r-1)(s-1)$ degrees of freedom.

Example 13.8 – Codable

Example 13.9 – Codable

Example 13.10 – Codable

Example 13.11 – Codable

Chapter 14: Nonparametric Hypotheses Tests

Example 14.1 – Codable

Example 14.2– Codable

Example 14.3 – Codable

Example 14.4 – Codable

Example 14.5 – Codable

Example 14.6 – Codable

Example 14.7 – Codable

Example 14.8 – Codable

Example 14.9 – Codable

Example 14.10 – Codable

Example 14.11 – Non-Codable (Reason: The problem is same as Example 14.9)

Example 14.11- Let us reconsider Example 14.9, this time using Program 14-2 to compute the *p* value…….

■ Example 14.11

Let us reconsider Example 14.9, this time using Program 14-2 to compute the *p* value. This program runs best if you designate the sample having the smaller sum of ranks as the first sample. The size of the first sample is 8. The size of the second sample is 9. The sum of the ranks of the first sample is 50. Program 14-2 computes the *p* value as 3.595229E-02.

Thus the exact *p* value is 0.0359, which is reasonably close to the approximate value of 0.0385 obtained by using the normal approximation in Example 14.9. ■

Example 14.12 –Codable

Example 14.13 – Codable

Example 14.14– Codable

Example 14.15 – Codable

Example 14.16 –Codable

Example 14.17 – Codable

Example 14.18 – Codable

Example 14.19 – Codable

Chapter 15: Quality Control

Example 15.1 – Codable

Example 15.2– Codable

Example 15.3 – Codable

Example 15.4 – Codable

Example 15.5 – Codable

Example 15.6 – Codable

Example 15.7 – Non-Codable (Reason: The problem uses the given values to illustrate a theoretical concept.)

Example 15.7- Suppose that the mean and standard deviation of a subgroup average are, respectively, μ = 30 and σ/
√ *n* = 8, and consider the cumulative-sum control
chart with *d* = 0.5, *B* = 5. If the first eight subgroup averages are
29, 33, 35, 42, 36, 44, 43, 45.

■ Example 15.7

Suppose that the mean and standard deviation of a subgroup average are, respectively, $\mu = 30$ and $\sigma/\sqrt{n} = 8$, and consider the cumulative-sum control chart with $d = 0.5$, $B = 5$. If the first eight subgroup averages are

$$29, 33, 35, 42, 36, 44, 43, 45$$

then the successive values of $Y_j = \overline{X}_j - 30 - 4 = \overline{X}_j - 34$ are

$$Y_1 = -5, \quad Y_2 = -1, \quad Y_3 = 1, \quad Y_4 = 8, \quad Y_5 = 2,$$
$$Y_6 = 10, \quad Y_7 = 9, \quad Y_8 = 11$$

Therefore,

$$S_1 = \max\{-5, 0\} = 0$$
$$S_2 = \max\{-1, 0\} = 0$$
$$S_3 = \max\{1, 0\} = 1$$

$$S_4 = \max\{9, 0\} = 9$$
$$S_5 = \max\{11, 0\} = 11$$
$$S_6 = \max\{21, 0\} = 21$$
$$S_7 = \max\{30, 0\} = 30$$
$$S_8 = \max\{41, 0\} = 41$$

Since the control limit is

$$\frac{B\sigma}{\sqrt{n}} = 5(8) = 40$$

Since the control limit is

$$\frac{B\sigma}{\sqrt{n}} = 5(8) = 40$$

the cumulative-sum chart would declare that the mean has increased after observing the eighth subgroup average. ■

To detect either a positive or a negative change in the mean, we employ two one-sided cumulative-sum charts simultaneously. We begin by noting that a decrease in $E[X_i]$ is equivalent to an increase in $E[-X_i]$. Hence, we can detect a decrease in the mean value of an item by running a one-sided cumulative-sum chart on the negatives of the subgroup averages. That is, for specified values d and B, not only do we plot the quantities S_j as before, but, in addition, we let

$$W_j = -\overline{X}_j - (-\mu) - \frac{d\sigma}{n} = \mu - \overline{X}_j - \frac{d\sigma}{\sqrt{n}}$$

and then also plot the values T_j, where

$$T_0 = 0$$
$$T_{j+1} = \max\{T_j + W_{j+1}, 0\}, \quad j \geq 0$$

The first time that either S_j or T_j exceeds $B\sigma/\sqrt{n}$, the process is said to be out of control.

Summing up: The following steps result in a cumulative-sum control chart for detecting a change in the mean value of a produced item: Choose positive constants d and B; use the successive subgroup averages to determine the values of S_j and T_j; declare the process out of control the first time that either exceeds $B\sigma/\sqrt{n}$. Three common choices of the pair of values d and B are: $d = 0.25$, $B = 8.00$; $d = 0.50$, $B = 4.77$; and $d = 1$, $B = 2.49$. Any of these choices results in a control rule that has approximately the same false alarm rate as does the $\overline{X}$ control chart that declares the process out of control the first time a subgroup average differs from μ by more than $3\sigma/\sqrt{n}$. As a general rule of thumb, the smaller the change in mean one wants to guard against, the smaller should be the chosen value of d.