Pfaffians, hafnians, and products of real linear functionals

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Abstract

We prove pfaffian and hafnian versions of Lieb's inequalities on determinants and permanents of positive semi-definite matrices. We use the hafnian inequality to improve the lower bound of Révész and Sarantopoulos on the norm of a product of linear functionals on a real Euclidean space (this subject is sometimes called the 'real linear polarization constant' problem).

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Old inequalities on determinants and permanents

Recall that the determinant and the permanent of an $n \times n$ matrix $A = (a_{i,j})$ are defined by

$$\det A = \sum_{\pi \in \mathfrak{S}_n} (-1)^\pi \prod_{i=1}^n a_{i,\pi(i)}, \qquad \text{per } A = \sum_{\pi \in \mathfrak{S}_n} \prod_{i=1}^n a_{i,\pi(i)},$$

where $\mathfrak{S}_n$ is the symmetric group on n elements. Throughout this section, we assume that A is a positive semi-definite Hermitian $n \times n$ matrix (we write $A \geq 0$). For such A, Hadamard proved that

$$\det A \leq \prod_{i=1}^n a_{i,i},$$

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with equality if and only if A has a zero row or is a diagonal matrix. Fischer generalized this to

$$\det A \le \det A' \cdot \det A''$$

for

$$A = \begin{pmatrix} A' & B \\ B^* & A'' \end{pmatrix} \ge 0, \tag{1}$$

with equality if and only if $\det A' \cdot \det A'' \cdot B = 0$.

Concerning the permanent of a positive semi-definite matrix, Marcus [Mar1, Mar2] proved that

$$\operatorname{per} A \ge \prod_{i=1}^{n} a_{i,i}, \tag{2}$$

with equality if and only if A has a zero row or is a diagonal matrix. Lieb [L] generalized this to

$$\operatorname{per} A \ge \operatorname{per} A' \cdot \operatorname{per} A'' \tag{3}$$

for A as in (1), with equality if and only if A has a zero row or $B = 0$. Moreover, he proved that in the polynomial $P(\lambda)$ of degree n' (=size of A') defined by

$$P(\lambda) = \operatorname{per} \begin{pmatrix} \lambda A' & B \\ B^* & A'' \end{pmatrix} = \sum_{t=0}^{n'} c_t \lambda^t,$$

all coefficients are real and non-negative. This is indeed a stronger theorem since it implies

$$\operatorname{per} A = P(1) = \sum_{t=0}^{n'} c_t \ge c_{n'} = \operatorname{per} A' \cdot \operatorname{per} A''.$$

Đoković [D, Mi] gave a simple proof of Lieb's inequalities, and showed also that if A' and A'' are positive definite then $c_{n'-t} = 0$ if and only if all subpermanents of B of order t vanish. Lieb [L] also states an analogous (and analogously provable) theorem for determinants: for A as in (1), let

$$D(\lambda) = \det \begin{pmatrix} \lambda A' & B \\ B^* & A'' \end{pmatrix} = \sum_{t=0}^{n'} d_t \lambda^t.$$

If $\det A' \cdot \det A'' = 0$, then $D(\lambda) = 0$. If A' and A'' are positive definite, then $(-1)^t d_{n'-t}$ is positive for $t \le \operatorname{rk} B$ and is zero for $t > \operatorname{rk} B$.

1 New inequalities on pfaffians and hafnians

For an $n \times n$ matrix $A = (a_{i,j})$ and subsets S, T of $N := \{1, \ldots, n\}$, we write $A_{S,T} := (a_{i,j})_{i \in S, j \in T}$. If $|T| = 2t$ is even, we write

$$(-1)^T := (-1)^{t + \sum_{j \in T} j}.$$

Recall that the pfaffian of a $2n \times 2n$ antisymmetric matrix $\tilde{A} = (a_{i,j})$ is defined by

$$\mathrm{pf}\ \tilde{A} = \frac{1}{n! 2^n} \sum_{\pi \in \mathfrak{S}_{2n}} (-1)^\pi a_{\pi(1),\pi(2)} \cdots a_{\pi(2n-1),\pi(2n)}.$$

For antisymmetric A and symmetric B, both of size $n \times n$, we consider the polynomial

$$(-1)^{\lfloor n/2 \rfloor} \mathrm{pf} \begin{pmatrix} \lambda A & B \\ -B & A \end{pmatrix} = \sum_{t=0}^{\lfloor n/2 \rfloor} p_t \lambda^t.$$

Theorem 1.1 *Let A and B be real $n \times n$ matrices with A antisymmetric and B symmetric. If B is positive semi-definite, then $(-1)^t p_t \geq 0$ for all t. If B is positive definite, then $p_t = 0$ if and only if all $2t \times 2t$ subpfaffians of A vanish.*

Proof. If $B = (b_{i,j})$ is positive semi-definite, then there exist vectors x_1, …, x_n in a real Euclidean space V such that $(x_i, x_j) = b_{i,j}$. Recall that in the exterior tensor algebra $\bigwedge V$ a positive definite inner product (and the corresponding Euclidean norm) is defined by

$$\left(\bigwedge v_i, \bigwedge w_j \right) := \det((v_i, w_j)).$$

We have

$$(-1)^t p_t = \sum_{|S|=2t} \sum_{|T|=2t} (-1)^S (-1)^T \mathrm{pf}\ A_{S,S} \cdot \mathrm{pf}\ A_{T,T} \cdot \det B_{N \setminus S, N \setminus T} =$$

$$= \left| \sum_{|S|=2t} (-1)^S \mathrm{pf}\ A_{S,S} \cdot \bigwedge_{i \notin S} x_i \right|^2 \geq 0.$$

Assume that B is positive definite. Then the vectors x_i are linearly independent. It follows that the tensors $\bigwedge_{i \notin S} x_i$ are also linearly independent as S runs over the subsets of N. Thus $p_t = 0$ if and only if pf $A_{S,S} = 0$ for all $|S| = 2t$. $\square$

Theorem 1.2 *Let A and B be real $n \times n$ matrices with A antisymmetric and B symmetric. Let $\lambda \leq 0$. If B is positive semi-definite, then*

$$(-1)^{\lfloor n/2 \rfloor} \mathrm{pf} \begin{pmatrix} \lambda A & B \\ -B & A \end{pmatrix} \geq \det B.$$

If B is positive definite, then equality occurs if and only if A is a diagonal matrix or $\lambda = 0$.

Proof. The left hand side is

$$p_0 + p_1 \lambda + \cdots + p_{\lfloor n/2 \rfloor} \lambda^{\lfloor n/2 \rfloor}.$$

The right hand side is p_0. □

Recall that the hafnian of a $2n \times 2n$ symmetric matrix $\tilde{A} = (a_{i,j})$ is defined by

$$\mathrm{haf}\ \tilde{A} = \frac{1}{n! 2^n} \sum_{\pi \in \mathfrak{S}_{2n}} a_{\pi(1),\pi(2)} \cdots a_{\pi(2n-1),\pi(2n)}.$$

For symmetric A and B, both of size $n \times n$, we consider the polynomial

$$\mathrm{haf} \begin{pmatrix} \lambda A & B \\ B & A \end{pmatrix} = \sum_{t=0}^{\lfloor n/2 \rfloor} h_t \lambda^t.$$

Theorem 1.3 *Let A and B be symmetric real $n \times n$ matrices. If B is positive semi-definite, then $h_t \geq 0$ for all t. If B is positive definite, then $h_t = 0$ if and only if all $2t \times 2t$ subhafnians of A vanish.*

Proof. If $B = (b_{i,j})$ is positive semi-definite, then there exist vectors $x_1, \ldots, x_n$ in a real Euclidean space V such that $(x_i, x_j) = b_{i,j}$. Recall [MN] that in the symmetric tensor algebra SV a positive definite inner product (and the corresponding Euclidean norm) is defined by

$$\left(\prod v_i, \prod w_j \right) := \mathrm{per}\ ((v_i, w_j)).$$

We have

$$h_t = \sum_{|S|=2t} \sum_{|T|=2t} \mathrm{haf}\ A_{S,S} \cdot \mathrm{haf}\ A_{T,T} \cdot \mathrm{per}\ B_{N \setminus S, N \setminus T} =$$

$$= \left| \sum_{|S|=2t} \mathrm{haf}\ A_{S,S} \cdot \prod_{i \notin S} x_i \right|^2 \geq 0.$$

Assume that B is positive definite. Then the vectors x_i are linearly independent. It follows that the tensors $\prod_{i \notin S} x_i$ are also linearly independent as S runs over the subsets of N. Thus $h_t = 0$ if and only if $\operatorname{haf} A_{S,S} = 0$ for all $|S| = 2t$. $\square$

Theorem 1.4 *Let A and B be symmetric real $n \times n$ matrices. Let $\lambda \geq 0$. If B is positive semi-definite, then*

$$\operatorname{haf} \begin{pmatrix} \lambda A & B \\ B & A \end{pmatrix} \geq \operatorname{per} B.$$

If B is positive definite, then equality occurs if and only if A is a diagonal matrix or $\lambda = 0$.

Proof. The left hand side is

$$h_0 + h_1 \lambda + \cdots + h_{\lfloor n/2 \rfloor} \lambda^{\lfloor n/2 \rfloor}.$$

The right hand side is h_0. $\square$

Setting $A = B$ and $\lambda = 1$, and combining with Marcus's inequality (2), we arrive at case $p = 1$ of

Conjecture 1.5 *If $A = (a_{i,j})$ is a positive semi-definite symmetric real $n \times n$ matrix, then the hafnian of the $2pn \times 2pn$ matrix consisting of $2p \times 2p$ blocks A is at least $(2p-1)!!^n \prod a_{i,i}^p$, with equality if and only if A has a zero row or is a diagonal matrix.*

2 Products of linear functionals

Let $\xi_1, \ldots, \xi_d$ denote independent random variables with standard Gaussian distribution, i.e., with joint density function $(2\pi)^{-d/2} \exp(-|\xi|^2/2)$, where $|\xi|^2 = \sum \xi_k^2$. We write $Ef(\xi)$ for the expectation of a function $f = f(\xi) = f(\xi_1, \ldots, \xi_d)$. Recall that

$$E\xi_k^{2p} = (2p-1)!! = (2p-1)(2p-3) \cdots 3 \cdot 1$$

for $k = 1, \ldots, d$ (easy inductive proof via integration by parts), and thus

$$E \prod_{k=1}^{d} \xi_k^{2p_k} = \prod_{k=1}^{d} (2p_k - 1)!!.$$

On $\mathbb{R}^d$, we write $(\cdot, \cdot)$ for the standard Euclidean inner product. We recall the well-known [S, Z]

Wick formula *Let $x_1, \ldots, x_n$ be vectors in $\mathbb{R}^d$ with Gram matrix $A = ((x_i, x_j))$. Then*

$$E \prod_{i=1}^{n} (x_i, \xi) = \operatorname{haf} A. \tag{4}$$

(For odd n, we define $\operatorname{haf} A = 0$.)

Proof. Both sides are multilinear in the x_i, so we may assume that each x_i is an element of the standard orthonormal basis $e_1, \ldots, e_d$. If there is an e_k that occurs an odd number of times among the x_i, then both sides are zero. If each e_k occurs $2p_k$ times, then the left hand side is $E \prod_{k=1}^{d} \xi_k^{2p_k}$, and the right hand side is $\prod_{k=1}^{d}(2p_k - 1)!!$, which are equal. $\square$

The following theorems are easy corollaries of Theorem 1.4 together with the Wick formula (4) and Marcus's theorem (2).

Theorem 2.1 *If $X_1, \ldots, X_n$ are jointly normal random variables with zero expectation, then*

$$E\left(X_1^2 \cdots X_n^2\right) \geq EX_1^2 \cdots EX_n^2.$$

Equality holds if and only if they are independent or at least one of them is almost surely zero.

Proof. The variables can be written as $X_i = (x_i, \xi)$ with ξ of standard normal distribution and the x_i constant vectors with a positive semi-definite Gram matrix $A = (a_{i,j}) = ((x_i, x_j))$. Then

$$\begin{aligned} E \prod_{i=1}^{n} X_i^2 = E \prod_{i=1}^{n} (x_i, \xi)^2 = \\ = \operatorname{haf} \begin{pmatrix} A & A \\ A & A \end{pmatrix} \geq \operatorname{per} A \geq \prod_{i=1}^{n} a_{i,i} = \\ = \prod_{i=1}^{n} E(x_i, \xi)^2 = \prod_{i=1}^{n} EX_i^2, \end{aligned}$$

with equality if and only if A is a diagonal matrix or has a zero row, i.e., the x_i are pairwise orthogonal or at least one of them is zero. $\square$

The generalization of Theorem 2.1 to an arbitrary even exponent $2p$ is equivalent to Conjecture 1.5.

Theorem 2.2 *For any $x_1, \dots, x_n \in \mathbb{R}^d$, $|x_i| = 1$, the average of $\prod (x_i, \xi)^2$ on the unit sphere $\{|\xi| = 1\}$ is at least*

$$\frac{\Gamma(d/2)}{2^n \Gamma(d/2 + n)} = \frac{(d-2)!!}{(d+2n-2)!!} = \frac{1}{d(d+2)(d+4)\dots(d+2n-2)},$$

with equality if and only if the vectors x_i are pairwise orthogonal.

Proof. The average on the unit sphere is the constant in the theorem times the expectation w.r.t. the standard Gaussian measure (see e.g. [B1]). By Theorem 2.1, the latter expectation is minimal if and only if the x_i are pairwise orthogonal, in which case it is 1. $\square$

Theorem 2.3 *For real linear functionals f_i on a real Euclidean space,*

$$||f_1 \cdots f_n|| \geq \frac{||f_1|| \cdots ||f_n||}{\sqrt{n(n+2)(n+4)\cdots(3n-2)}}.$$

Here $|| \cdot ||$ means supremum of the absolute value on the unit sphere.

Proof. We may assume that the space is $\mathbb{R}^d$ with $d \leq n$, and the functionals are given by $f_i(\xi) = (x_i, \xi)$ with $||f_i|| = |x_i| = 1$. Then $||f_1 \cdots f_n||^2$ is at least the average of $\prod f_i^2(\xi) = \prod (x_i, \xi)^2$ on the unit sphere, which by Theorem 2.2 and $d \leq n$ is at least $1/(n(n+2)(n+4)\cdots(3n-2))$. $\square$

It is an unsolved problem, raised by Benítez, Sarantopoulos and Tonge [BST] (1998), whether Theorem 2.3 is true with n^n under the square root sign in the denominator on the right hand side. This is called the 'real linear polarization constant' problem. In the complex case, the affirmative answer was proved by Arias-de-Reyna [A] in 1998, based on the complex analog of the Wick formula [A, B2] and on Lieb's inequality (3). In the real case, the affirmative answer for $n \leq 5$ was proved by Pappas and Révész [PR] in 2004. For general n, the best result known before the present paper was that of Révész and Sarantopoulos [RS] (2004), with $(2n)^n/4$ under the square root sign. See [Mat1, Mat2, MM, R] for accounts on this and related questions.

Note that

$$
\begin{aligned}
n(n+2)(n+4)\cdots(3n-2) &= \\
= \exp\left(\log n + \log(n+2) + \log(n+4) + \cdots + \log(3n-2)\right) &< \\
< \exp\left(\frac{1}{2}\int_n^{3n} \log u \cdot \mathrm{d}u\right) &= \\
= \exp\left([u(\log u - 1)]_n^{3n}/2\right) = \exp((3n\log 3n - 3n - n\log n + n)/2) &= \\
= \exp\frac{n(2\log n + 3\log 3 - 2)}{2} = \left(\frac{3\sqrt{3}}{e}n\right)^n &,
\end{aligned}
$$

and $3\sqrt{3}/e < 3 \cdot 1.8/2.7 = 2$, so Theorem 2.3 is an improvement. Note also that the statement with n^n would follow from Conjecture 1.5.

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