# Topology Change of Black Holes

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Abstract.

The topological structure of the event horizon has been considered in terms of the Morse theory. The elementary process of topology changes can be understood as a handle attachment. It has been found that there are certain constraints on the black hole topology changes: (i) There are $n$ kinds of the handle attachment in the $(n+1)$-dimensional black hole space-times. (ii) The handle is further classified into either black or white type and only the black handles appear in real black hole space-times. (iii) The spacial section of the exterior of the black hole region is always connected. As a corollary, it is shown that the formation of black hole with the $S^{n-2}\times S^1$ horizon from that with the $S^{n-1}$ horizon must be non-axisymmetric in asymptotically flat space-times.

## 1. Introduction

The black holes in space-time dimensions $\geq 5$ have rich topological structure. The well-known results by Hawking for topology of black holes in four-dimensional space-time states that the apparent horizon or the spatial section of the stationary event horizon is necessarily diffeomorphic to 2-sphere [1, 2]. This follows from the fact that the total curvature which is the integral of the intrinsic scalar curvature over the horizon becomes positive under the dominant energy condition and from the Gauss-Bonnet theorem. The alternative and improved proofs of the Hawking's theorem have been given by several authors [3, 4, 5, 6]. On the other hand, an apparent horizon or the spatial section of the stationary event horizon in higher dimensional space-time may not be topological sphere [7, 8, 9, 10], because one fails to apply Gauss-Bonnet theorem there. Nevertheless, the positivity of the total curvature of the horizon still holds. This puts a certain topological restrictions on the black hole topology, though they are rather week ones. For example, the apparent horizon in five-dimensional space-time can be finite connected sums of copies of $S^3/\Gamma$ and copies of $S^2\times S^1$. In fact, the exact solutions representing black hole space-time possessing a horizon of nonspherical topology are recently found in five-dimensional general relativity. When such black holes with nontrivial topology are regarded as formed in the course of gravitational collapse, one naturally asks the topology change of black holes. Our purpose here is to understand the time evolution of topology of event horizons in general settings. The relation between the crease set, where the event horizon is nondifferentiable, and the topology of event horizon is previously studied in Refs. [11, 12, 13] with space-time dimensions four. The present work advances the systematic investigation and will find the useful rules to determine the admissible processes of topology change for time slicing of the black hole.

Our approach is to utilize the Morse theory [14, 15] in differential topology. The Morse theory is useful in understanding the topology of smooth manifold. The basic tool is the smooth function on the differentiable manifold. The event horizon however is not differentiable manifold but has a wedgelike structure at the past endpoints of the null geodesic generators of the horizon. Hence we first smoothen the wedge. Then, the smooth time function which is assumed to exist plays a role of the Morse function on the smoothed event horizon. According to the Morse theory, the topology change of the event horizon can be then decomposed into elementary processes called handle attachment. Starting with spherical horizon, one adds several handles each characterized by the index of the critical points of the Morse function, which is an integer ranging from 0 to $n$ (the dimension of the smoothed horizon as differentiable manifold).

The purpose of the present article is to show that there are several constraints on the handle attachments for real black hole space-times.

## 2. The Morse Theory for Event Horizons

Let $M$ be the $(n+1)$-dimensional asymptotically flat space-time. We require the existence of a global time function $t : M \to \mathbb{R}$, which is smooth and of which gradient is everywhere timelike future pointing. The event horizon $H$ is defined to be the boundary of the causal past of the future null infinity $H = \partial J^-(\mathscr{I}^+)$ [2]. We treat the event horizon defined with respect to a single asymptotic end unless otherwise stated. In other words, the future null infinity $\mathscr{I}^+$ is assumed to be connected. The black hole region $\mathscr{B}$ is defined by the inside region of $H$, $\mathscr{B} = M \setminus J^-(\mathscr{I}^+)$ and the exterior region $\mathscr{E}$ of the black hole region is its complement $\mathscr{E} = \mathrm{int}(J^-(\mathscr{I}^+))$. We shall call the intersection of the black hole region and the time slice $\Sigma(t_0) = \{t = t_0\}$, the black hole $\mathscr{B}(t_0) = \mathscr{B} \cap \Sigma(t_0)$ at time $t = t_0$. The exterior region at time $t = t_0$ is accordingly denoted by $\mathscr{E}(t_0) = \mathscr{E} \cap \Sigma(t_0)$.

One of most basic properties of the event horizon is that it is generated by null geodesics without future endpoints. In general, the event horizon is not smoothly imbedded into the space-time manifold $M$, but it has a wedgelike structure at the past endpoints of null geodesic generators, where distinct null geodesic generators intersect. We call the set of past endpoints of null geodesic generators of $H$, where two or more null geodesic generators emanate, the crease set $S$ [11, 12]. When the crease set $S$ is absent between the time slices $t = t_1$ and $t = t_2$, the null geodesic generators of $H$ naturally define diffeomorphism $\partial\mathscr{B}(t_1) \approx \partial\mathscr{B}(t_2)$. Hence, the topology change of the black hole can take place only when the time slice has intersection with the crease set $S$. Of course, the event horizon itself is a gauge independent object. Nevertheless, we often understand the dynamics of space-time by scanning it along time slices. Then, the topology change of the black hole depends on the time function.

It is expected that the Morse theory [14] gives useful techniques to analyze such a topology change process. Since the Morse theory calls functions on smooth manifold in question, we first regularize $H$ around the crease set $S$. The event horizon is not necessarily smooth even on $H \setminus S$, when the future null infinity $\mathscr{I}^+$ has a pathological structure [16]. Here it is assumed that $H$ is smooth on $H \setminus S$. Then, small deformation of $H$ near the crease set $S$ will make $H$ smooth hypersurface $\widetilde{H}$ in $M$ while keeping accordingly deformed black hole $\widetilde{\mathscr{B}}(t_0)$, defined such that $\partial\widetilde{\mathscr{B}}(t_0) = \widetilde{H} \cup \Sigma(t_0)$ holds, homeomorphic to the original one $\approx \mathscr{B}(t_0)$ for all $t_0 \in \mathbb{R}$. This deformation is assumed to be arranged such that the time function $t|_{\widetilde{H}}$, which is the restriction of $t$ on $\widetilde{H}$, gives a Morse function on $\widetilde{H}$, which has only nondegenerate critical points, where the gradient of $t|_{\widetilde{H}}$ defined on $\widetilde{H}$ becomes zero and also the Hessian matrix $(\partial_i\partial_j t|_{\widetilde{H}})$ of $t|_{\widetilde{H}}$ is nondegenerate. This assumption is not always justified, while the set of the Morse funtion is open dense. The Fig. 1 gives such an example, where any smoothing procedure does not make the induced time function $t|_{\widetilde{H}}$ a Morse function on $\widetilde{H}$ because the intersection of the crease set $S$ of the event horizon and $t = t_0$ hypersurface has an accumulating point. Nevertheless, we shall just exclude such complicated situations, because our conclusion will not be affected by this simplification.

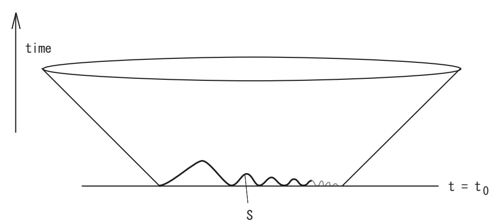


FIGURE 1. An example where any smoothing procedure does not make $t|_{\widetilde{H}}$ a Morse function on $\widetilde{H}$. The intersection of the crease set $S$ of the event horizon and $t = t_0$ hypersurface has an accumulating point.

According to the Morse Lemma, there is a local coordinate system $\{x^1, \cdots, x^n\}$ on $\widetilde{H}$ around the critical point $p \in \widetilde{H}$ such that the restriction $t|_{\widetilde{H}}$ of the time function $t$ on $\widetilde{H}$ takes the form

$$t|_{\widetilde{H}}(x^1, \cdots, x^n) = t(p) - (x^1)^2 - \cdots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \cdots + (x^n)^2. \tag{1}$$

The integer $\lambda$ ranging from 0 to $n$ is called the index of the critical point $p$. The topology of the black hole $\widetilde{\mathscr{B}}(t)$ changes when $\Sigma(t)$ pass through critical points, or equivalently, when the time function $t$ takes the critical values. This implies that the critical points appears only near the crease set $S$.

The gradient-like vector field for $t|_{\widetilde{H}}$ is defined to be the tangent vector field $X$ on $\widetilde{H}$ such that $Xt|_{\widetilde{H}} > 0$ holds, namely it is future directed, on $\widetilde{H}$ except for critical points and has the form

$$X = -2x^1\frac{\partial}{\partial x^1} - \cdots - 2x^\lambda\frac{\partial}{\partial x^\lambda} + 2x^{\lambda+1}\frac{\partial}{\partial x^{\lambda+1}} + \cdots + 2x^n\frac{\partial}{\partial x^n} \tag{2}$$

near the critical point of index $\lambda$ in terms of the standard coordinate system appeared in the Morse Lemma. We choose a gradient-like vector field $X$ such that it coincides with the future directed tangent vector field of null geodesic generators of $H$ except for the small neibourhood of the crease set $S$ (Fig. 2).

The effect of the critical point $p$ of index $\lambda$ is equivalent to the attachment of $\lambda$-handle [14, 15]. The handlebody is just a topological $n$-disk $D^n \approx I^n$ $(I = [0, 1])$, but it is regarded as the product space $D^n \approx D^\lambda \times D^{n-\lambda}$ (Fig. 3).

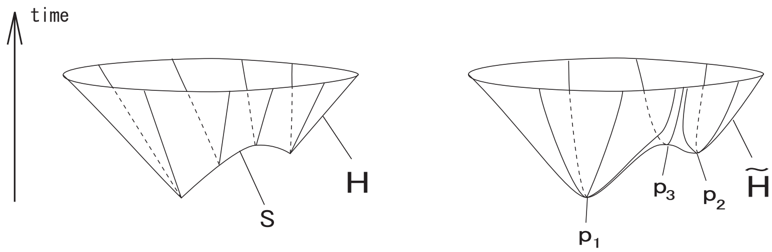


FIGURE 2. The smoothing procedure of the event horizon $H$. The gradient-like vector field on $\widetilde{H}$ can be made from slight deformation of null geodesic generators of $H$. Here the effect of the crease set $S$ has been replaced by that of the critical points $p_1$, $p_2$ and $p_3$.

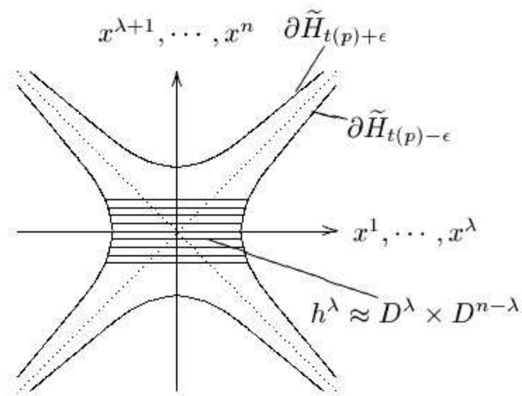


FIGURE 3. The local structure around the critical point $p$ of index $\lambda$. It can be seen that $\widetilde{H}_{t(p)+\epsilon}$ is homeomorphic to $\widetilde{H}_{t(p)+\epsilon}$ with $\lambda$-handle attached.

The $\lambda$-handle attachment to an $n$-dimensional manifold $N$ with boundary consists of the set $h^\lambda = (D^\lambda \times D^{n-\lambda}, f)$, where the attaching map $f$ is imbedding of $\partial D^\lambda \times D^{n-\lambda} \subset \partial D^n$ into $\partial N$ (Fig. 4). The new manifold obtained by the $\lambda$-handle attachment to $N$ is given by

$$N \cup h^\lambda = N \cup (D^\lambda \times D^{n-\lambda})/(x \sim f(x)), \ (x \in \partial D^\lambda \times D^{n-\lambda}). \tag{3}$$

Let us denote by $\widetilde{H}_{t_0}$ the $t \leq t_0$ part of $\widetilde{H}$. Then, $\widetilde{H}_{t(p)+\epsilon}$ $(\epsilon > 0)$ just above the critical point $p$ of index $\lambda$ is homeomorphic (in fact diffeomorphic taking account of smoothing procedure) to that just below $p$, $\widetilde{H}_{t(p)-\epsilon}$ attached with $\lambda$-handle

$$\widetilde{H}_{t(p)+\epsilon} \approx \widetilde{H}_{t(p)-\epsilon} \cup h^\lambda, \tag{4}$$

if there is no other critical points between $t(p) - \epsilon \leq t \leq t(p) + \epsilon$. The handlebody itself is denoted by $h^\lambda$ as well.

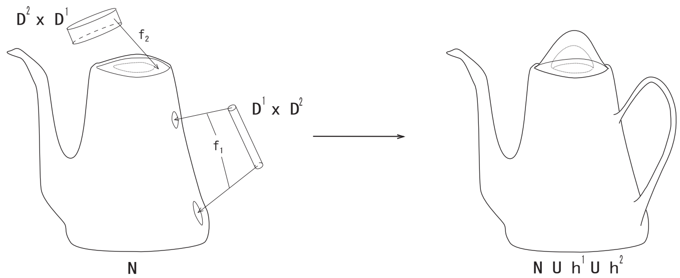


FIGURE 4. The attachment of 1-handle and 2-handle to 3-manifold $N$ makes a new a 3-manifold $N \cup h^1 \cup h^2$.

Consider several examples. The 0-handle attachment does not need attaching map $f$. It just corresponds to the emergence of $(n-1)$-sphere $S^{n-1} \approx \partial D^n$ as a black hole horizon $\partial\mathscr{B}(t)$. A typical example is the creation of a black hole (Fig. 5).

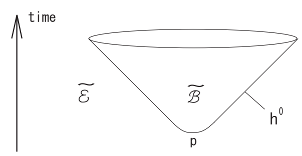


FIGURE 5. The emergence of a black hole by 0-handle attachment.

A black hole always emerges as 0-handle attachment. The other possiblity is the creation of a bubble in black hole region, such that this bubble is subset of $J^-(\mathscr{I}^+)$ (Fig. 6).

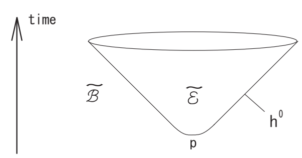


FIGURE 6. The emergence of a bubble in the black hole region by 0-handle attachment, which does not occur in the real black hole space-times.

One might think that this corresponds to the wormhole creation between inside and outside region of the event horizon. Although in the framework of the standard Morse theory on $\widetilde{H}$, these two examples are indistinguishable, we will see that the latter process is impossible.

Next, we shall look at the 1-handle attachment. The typical example is the collision of two black holes. Then, the 1-handle serves as the bridge connecting black holes, or it corresponds to taking the connected sum of each component of black holes (Fig. 7).

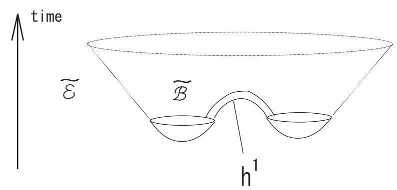


FIGURE 7. The collision of a pair of black holes into a single one is realized by 1-handle attachment.

The time reversal of the black hole collision is the bifurcation of a black hole into two. This is realized by $(n-1)$-handle attachment, if possible (Fig. 8). It is however well known that such a process is forbidden [2]. In general, the time reversal of the $\lambda$-handle attachment corresponds to $(n-\lambda)$-handle attachment, which we shall call the Poincaré duality since it is essentially the Poincaré duality between the $\lambda$-th and the $(n-\lambda)$-th Betti numbers of $\widetilde{H}$, if $\widetilde{H}$ were the closed manifold.

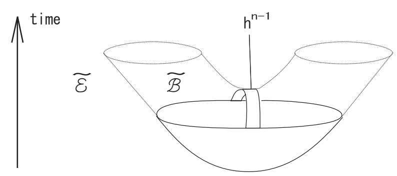


FIGURE 8. The bifurcation of a black hole into two is represeted by $(n-1)$-handle attachment. This however never occurs in the real black hole space-times.

In advance of discussing general cases, consider the structure of handlebody. Recall that the $\lambda$-handle consists of the product space $D^\lambda \times D^{n-\lambda}$. The subset $D^\lambda \times \{0\} \subset D^\lambda \times D^{n-\lambda}$ is called core of the handlebody, and $\{0\} \times D^{n-\lambda} \subset D^\lambda \times D^{n-\lambda}$ is called co-core.

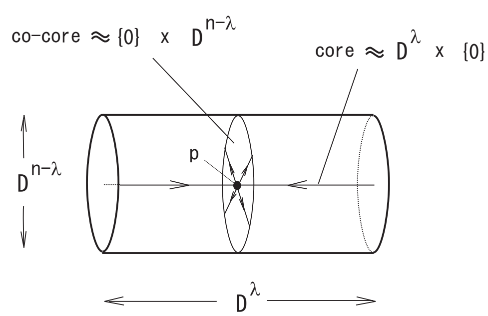


FIGURE 9. The structure of $\lambda$-handle. The core $D^\lambda \times \{0\}$ corresponds to the stable submanifold against the flow generated by gradient-like vector field and the co-core $\{0\} \times D^{n-\lambda}$ corresponds to the unstable submanifold.

The core and the co-core intersect transversely at a point. This point may be regarded as coinciding with a critical point $p$. Let us call the subset $W_p(p)$ of $\widetilde{H}$ consisting of points which converge to $p$ along the flow generated by the gradient-like vector field $X$, the past directed disk with respect to the critical point $p$. The past directed disk $W_p(p)$ is homeomorphic to $\mathbb{R}^\lambda$ if the index of $p$ is given by $\lambda$. In the same way, let us call by the future directed disk with respect to $p$ the subset $W_f(p) \subset \widetilde{H}$ consisting of point which converge to $p$ along the flow generated by $(-X)$. For the future directed disk, $W_f(p) \approx \mathbb{R}^{n-\lambda}$ holds. $W_p(p)$ and $W_f(p)$ are also called stable and unstable manifold, respectively, of the fixed point $p$ in the literature. The portion of the past and future directed disks in the handlebody may just be regarded as corresponding to that of core and co-core, respectively.

The effect of smoothing the event horizon $H$ to $\widetilde{H}$ is to deform the null vector field generating $H$ into gradient-like vector field $X$. Primary difference between the null geodesic generators and the flow generated by $X$ is that the former does not have future endpoints, but the latter can have them. The critical points of smoothed manifold $\widetilde{H}$ consisting of the divergences and the saddle points are regarded as isolation of the crease set $S$, which as a whole is a source of divergence of the null vector field generating $H$. In this way, an integral curve of the gradient-like vector field $X$ on $\widetilde{H}$ may have a future endpoint due to the effect of isolation of $S$.

Then, there would be admissible and inadmissible processes for the smoothed manifold $\tilde{H}$. The admissible process is given by $\widetilde{H}$ which is obtained from in priciple realizable event horizon, while inadmissible one is made from a fake event horizon, in the sense of the null hypersurface containing null geodesic generators with a future endpoint.

## 3. The Structure of the Critical Points

The spatial topology of the black hole changes only when the time function takes the critical value. The time evolution of the black hole topology is made clear by looking at its local structure around critical points. To distinguish whether given topology change process is admissible or inadmissible, it is not enough to solely consider the intrinsic structure of the event horizon. Rather, it is required to take account of its imbedding structure relative to the space-time.

In a time slice, any point separate from $\widetilde{H}$ belongs to either black hole or exterior of the black hole region. It is useful to observe the local behavior of of black hole region or exterior region near the critical point $p$. Let us call the exterior of the black hole region $\mathscr{E}$ simply the exterior region, for brevity. The deformed exterior region is denoted by $\widetilde{\mathscr{E}}$ and the deformed exterior region at the time $t$ by $\widetilde{\mathscr{E}}(t) = \widetilde{\mathscr{E}} \cap \Sigma(t)$, accordingly. The 0-handle is placed at $t \geq t(p)$, since the past directed disk $W_p(p)$ and the core consist of a single point $p$. It describes the emergence of the black hole region at the critical point $p$ and its expansion with time. The emergence of a bubble in the background of the black hole region, where the bubble consists of a part of $J^-(\mathscr{I}^+)$, would be also described by 0-handle attachment. This however never occurs, because the emergence of the bubble inside the black hole region means the existence of the causal curve escaping from the black hole region, which is impossible. Hence, the 0-handle attachment always describes the creation of the black hole homeomorphic to the $n$-disk.

The $n$-handle attachment corresponds to the time reversal of the 0-handle attachement from the Poincaré duality. This process however never occurs in the real black hole space-time. The $n$-handle is defined for $t \le t(p)$, which means that it terminates at the critical point $p$. The crease set is isolated into critical points in the course of the smoothing procedure. Hence, one can arrange the smoothing procedure such that the critical point appears in the crease set. The gradient-like vector field, which can be regarded as being tangent to the generator of the deformed event horizon $\widetilde{H}$, may have several inward (converging) directions at the critical point due to this smoothing procedure, while the original null generator of the event horizon does not have the inword direction at the crease set. In the case of the $n$-handle, all the directions become outword (diverging) at the critical point. This implies that the null generators of the event horizon $H$ must have future endpoint at the critical point, which is of course impossible. In this way, it turns out that the $n$-handle attachment never occurs in the real black hole space-time.

The remaining cases are the $\lambda$-handle attachment for $1 \le \lambda \le n-1$. In these cases, the $\lambda$-handle lies around the critical point $p$ both in the future $(t > t(p))$ and in the past $(t < t(p))$. Then, we consider that the handle stands during the sufficiently small time interval $t \in [t(p)-\delta, t(p)+\delta]$ $(\delta > 0)$, to see the change of the topology of the black hole region at the critical point $p$.

Introduce a coordinate system $\{t, x^i\}$ $(i = 1, \cdots, n)$ in the neibourhood $U$ of $p$, where $t$ is a time function already given and $\{x^i\}$ is the extention over $U$ of the cannonical coordinate appeared in the Morse Lemma such that each curve $(x^1, \cdots, x^n) = \text{const.}$ is timelike in $U$. We regard that $U$ is the solid cylinder (the world disk) given by $t \in [t(p)-\delta, t(p)+\delta]$, $\sum (x^i)^2 \le \delta$. In this coordinate system, the $\lambda$-handle $h^\lambda$ is given by the saddle surface

$$t = t(p) - (x^1)^2 - \cdots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \cdots + (x^n)^2 \tag{5}$$

in $U$, which is an acausal set if the constant $\delta$ is taken sufficiently small, since $h^\lambda$ is tangent to the space-like hypersurface $t = t(p)$ at $p$. Therefore, $h^\lambda$ separates $U$ into two open subsets, the future (past) region of $U^\pm$ of $U$, respectively, where $U^\pm$ is the subset lying chronological future (past) of $h^\lambda$, $U^\pm = I^\pm(h^\lambda) \cap U$. Explicitly, the future and past regions $U^\pm$ are the regions

$$t \gtrless t(p) - (x^1)^2 - \cdots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \cdots + (x^n)^2 \tag{6}$$

in $U$, respectively (Fig. 10).

Since the $\lambda$-handle is the subset of the black hole boundary $\widetilde{H}$, one of $U^\pm$ is contained in black hole region $\widetilde{\mathscr{B}}$ and the other in exterior region $\widetilde{\mathscr{E}}$. However, the future region $U^+$ of $U$ is always included in the black hole region $U^+ \subset \widetilde{\mathscr{B}}$ and hence $U^- \subset \widetilde{\mathscr{E}}$, since otherwise there is causal curves in $U$ escaping from the black hole region through $h^\lambda$.

Therefore, the black hole region $\widetilde{\mathscr{B}}(t(p)-\epsilon) \cap U$ in $U$ at the time $t = t(p) - \epsilon$ just before the critical time is given by

$$(x^1)^2 + \cdots + (x^\lambda)^2 > (x^{\lambda+1})^2 + \cdots + (x^n)^2 + \epsilon, \tag{7}$$

which is homotopic to the $(\lambda-1)$-sphere $S^{\lambda-1}$. (For $\lambda = 1$, $S^0$ just consists of two points.) While $\widetilde{\mathscr{B}}(t(p)+\epsilon) \cap U$ just after the critical time is given by

$$(x^1)^2 + \cdots + (x^\lambda)^2 + \epsilon > (x^{\lambda+1})^2 + \cdots + (x^n)^2, \tag{8}$$

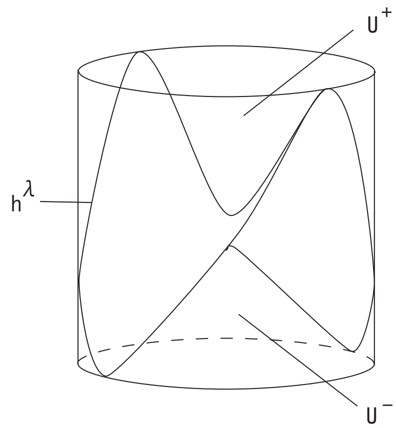


FIGURE 10. The neibourhood $U$ of $p$ is separated by $h^\lambda$ into the future region $U^+$ and the past region $U^-$.

which is homotopic to the $n$-disk. This region is homotopic to $(\lambda-1)$-sphere $S^{\lambda-1}$. In this way, the black hole region restricted to the small neibourhood of the critical point $p$ is initially homotopic to sphere. Then, the inside region of the sphere is filled up at the critical time $t = t(p)$ and become homotopically trivial finally. On the other hand, the exterior region $\widetilde{\mathscr{E}}(t) \cap U$ in $U$ is at first homotopic to $n$-disk for $t = t(p) - \epsilon$. Next, its $(n-\lambda)$-dimensional direction is penetrated by black hole region at $t = t(p)$. Then, it becomes homotopic to $(n-\lambda-1)$-sphere $S^{n-\lambda-1}$ for $t = t(p)+\epsilon$. If the fake event horizon is also taken into account, the future region $U^+$ might be the subset of $\widetilde{\mathscr{E}}$ and therefore the past region $U^-$ might be the subset of $\widetilde{\mathscr{B}}$. Then, the black hole region in the $\lambda$-handle might be homotopic to $n$-disk initially and become homotopic to $(n-\lambda-1)$-sphere finally, and the exterior region vice versa. Let us call such topology change of black hole region $\widetilde{\mathscr{B}}(t) \cap U$ from the region homotopic to the sphere to the disk the black handle attachment, and that from the disk to the region homotopic to the sphere the white handle attachment. The observation above shows that only the black handle attachment occurs if the sufficiently small neibourhood of the critical point is considered. For example, the collision of black holes corresponds to the black 1-handle attachment, while the bifurcation of the black hole does not, which rather corresponds to white $(n-1)$-handle attachment in the sense that the homotopy type of the exterior region $\widetilde{\mathscr{E}}(t) \cap U$ changes from that of $S^{n-2}$ to $D^n$. This local argument also gives the reasoning that the black hole collision is admissible, while the black hole bifurcation, which is its time reversal, is inadmissible. We also note that the effect of the time reversal, which we call the Poincaré duality, is to convert the black $\lambda$-handle attachment into the white $(n-\lambda)$-handle attachment.

It will be appropriate to call the 0-handle attachment corresponding to the creation of a black hole, the black 0-handle attachment . Then, the proposition above is also applied to the 0-handle attachment.

## 4. Connectedness of the Exterior Region

There still are the processes inadimissible due to global conditions. Let us, for a moment, consider the event horizon in maximally extended Schwarzschild space-time. Though we are interested in the event horizon defined with respect to a specific asymptotic end, we examine the event horizon defined with respect to a pair of asymptotic ends in Schwarzschild space-time, for explanation.

Let $\mathscr{I}_1^+$ and $\mathscr{I}_2^+$ be the pair of future null infinities of the maximally extended Schwarzschild space-time. The event horizon here is defined by $H = \partial J^-(\mathscr{I}_1^+ \cup \mathscr{I}_2^+)$, which is nondifferentiable at the bifurcate horizon $F = \partial J^-(\mathscr{I}_1^+) \cap \partial J^-(\mathscr{I}_2^+)$. Let $t$ be a global time function and $\chi$ be a global radial coordinate function such that each two surface $t$, $\chi =$ const. is invariant under the $SO(3)$ isometry. These coordinates are chosen such that the bifurcation surface $F$ is located at $t = \chi = 0$ and the event horizon $H$ is determined by $t = |\chi|$ around $F$. The smoothed event horizon $\widetilde{H}$ is also taken invariant under $SO(3)$ isometry. Due to the symmetry of the configuration, the time function $t$ has critical points of degenerate type. In fact, any point on bifurcate horizon $F$ is critical. Here, we are not interested in such a nongeneric situation. Instead, we consider slightly different time slicing determined by the new time function

$$t' = t + \epsilon \sin^2 \frac{\vartheta}{2}, \tag{9}$$

where $\epsilon > 0$ is a sufficiently small positive constant and $0 \leq \vartheta \leq \pi$ is the usual polar coordinate of the 2-sphere. Then, there appears only a pair of isolated critical points at the North pole ($\vartheta = 0$) and the South pole ($\vartheta = \pi$) on the bifurcate horizon $F$, and the time function $t'$ becomes the Morse function on $\widetilde{H}$. At the time $t' = 0$, the black hole appears at the North pole $\vartheta = 0$. This is the 0-handle attachment. The black hole formed there grows into geometrically thick spherical shell with a hole at the South pole, which is nevertheless a topological 3-disk. At the time $t' = \epsilon$, the puncture at the South pole is filled up and the black hole region becomes topologically $S^2 \times [0, 1]$. The deformed event horizon $\widetilde{H}$ splits into disjoint union of a pair of 2-spheres. This is the 2-handle attachment.

This kind of 2-handle attachment occurs because the event horizon is defined with respect to the two asymptotic ends, which is in general inadmissible if the future null infinity is connected, as we assume in the rest of the article. To see the statement above, it should be noted that there is no process, in which the several connected components of the exterior region $\widetilde{\mathscr{E}}(t) = \widetilde{\mathscr{E}} \cap \Sigma(t)$ at time $t$ merge together at a later time since such a handle attachment is not admissible. It is also seen that any connected component of $\widetilde{\mathscr{E}}(t)$ does not disappear because possible $n$-handle attachments are inadmissible. These imply that the number of the connected components of the exterior region $\widetilde{\mathscr{E}}(t)$ cannot decrease with time function $t$. On the other hand, there is only one connected component of the exterior region $\widetilde{\mathscr{E}}(t)$ for sufficiently large $t$, because of the connectedness of $\mathscr{I}^+$. This observation shows that the connected component of the exterior region $\widetilde{\mathscr{E}}(t)$ does not change in number in any process.

The only possible process where the number of the connected components of the exterior region $\widetilde{\mathscr{E}}(t)$ changes is by the $(n-1)$-handle attachment, as constructed above in the Schwarzschild space-time. This is because the subset $D^\lambda \times \partial D^{n-\lambda}$ of the boundary of the $\lambda$-handle

$$\partial h^{\lambda} \approx (\partial D^{\lambda} \times D^{n-\lambda}) \cup (D^{\lambda} \times \partial D^{n-\lambda}), \tag{10}$$

namely the part of $\partial h^{\lambda}$ which is the complement of the preimage of the attaching map

$$f : \partial h^{\lambda} \supset \partial D^{\lambda} \times D^{n-\lambda} \to \widetilde{H}_t, \tag{11}$$

is disconnected only when $\lambda = n-1$. In this case, the homotopy type of the exterior region $\widetilde{\mathscr{E}}(t)$ varies from that of $n$-disk to that of $S^0$, namely the two points. Note that it however does not imply that the exterior region $\widetilde{\mathscr{E}}(t)$ is always separated into two disconnected parts through the $(n-1)$-handle attachment. For example, the transition from the black ring horizon $\approx S^{n-2} \times S^1$ to the spherical black hole horizon $\approx S^{n-1}$ is realized by the black $(n-1)$-handle attachment, which is such a process as to pinch the longitude $\{pt\} \times S^1 \subset S^{n-2} \times S^1$ into a point, while the number of the connected components of the exterior region $\widetilde{\mathscr{E}}(t)$ remains one. Hence, only the $(n-1)$-handle attachment which separates the exterior region $\widetilde{\mathscr{E}}(t)$ is inadmissible.

## 5. Concluding Remarks

The preceding arguments are summerized in the following rules. Assume that (i) *the $(n+1)$-dimensional space-time $M$ is asymptotically flat and the future null infinity $\mathscr{I}^+$ is connected, or the event horizon $H = \partial J^-(\mathscr{I}^+)$ is defined with respect to a single asymptotic end,* (ii) *the space-time $M$ admits a smooth global time function $t$,* (iii) *the event horizon $H$ can be deformed so that the accordingly deformed black hole $\widetilde{\mathscr{B}}(t)$ at each time $t$ is smooth and homeomorphic to original one $\mathscr{B}(t)$ and the time function $t$ becomes the Morse function on $\widetilde{H}$.* Then, the topology change of the event horizon is regarded as $\lambda$-handle attachment $(0 \leq \lambda \leq n)$ subject to the rules:

(1) *The $n$-handle attachment is inadmissible.*
(2) *Only the black $\lambda$-handle attachment $(0 \leq \lambda \leq n-1)$, where the black hole region in the neibourhood of the critical point varies from the region homotopic to the sphere $S^{\lambda-1}$ (regarded as the empty set for $\lambda = 0$) to the $n$-disk $D^n$, is admissible.*
(3) *The $(n-1)$-handle attachment which separates the spacial section of the exterior region of the black hole is inadmissible.*

The first rule just states that any connected component of the black hole does not disappear. It also indicates that the bubble of the exterior region with in the black hole region, if it is formed, does not vanish.

The second rule is concerned with the imbedding structure of the event horizon relative to the space-time manifold. The small neighbourhood of the critical point is separated into two regions by the event horizon. One changes homotopically from sphere to disk and the other from disk to sphere. We call it the black handle attachment when the former corresponds to the black hole region and the white handle attachment otherwise. Then, it states that the white handle attachment never occurs. The reverse process where the black hole region homotopically changes from disk to sphere is ruled out. The white 0-handle attachment, which describes the emergence of exterior region, is also forbidden. It gives another reasoning of the well known result that a black hole cannot bifurcate, because it corresponds to white

$(n-1)$-handle attachment. It also controles more general situations. For example, let us consider the topology change of the event horizon from $S^{n-1}$ to $S^{n-2} \times S^1$ in $(n+1)$-dimensional space-time ($n \geq 3$). When it is realized with a single critical point, it coresponds to the 1-handle attachment. Here, one might expect two possibilities if the second rule is not taken care. One possibility is that the 1-handle is attached in the exterior region of the black hole. The other possibility is that it is attached from inside. Only the latter includes axisymmetric process with the $SO(n-1)$ isometry in asymptotically flat space-times, such that the spherical black hole is pinched out along the symmetric axis. However, the latter corresponds to the white 1-handle attachment for to be inadmissible and only the former, which is the black 1-handle attachment, is possible. In particular, the transition from spherical event horizon ($\approx S^{n-1}$) to black ring horizon ($\approx S^{n-2} \times S^1$) turns out to be always non-axisymmetric in the sense that such a space-time cannot admit $SO(n-1)$ isometry (Fig. 11).

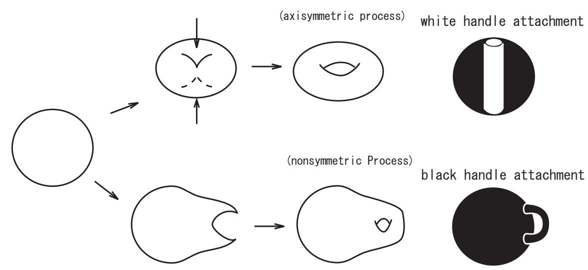


FIGURE 11. The black ring formation from spherical black hole must be non-axisymmetric in the real black hole space-times.

The third rule is not directly determined by local structure of the critical point. It states that the exterior region $\mathscr{E}(t) = \mathscr{E} \cap \Sigma(t)$ at each time is always connected under the assumption that $\mathscr{I}^+$ is connected. Then, the possibility that there forms a bubble of exterior region inside the black hole horizon is ruled out. It should, however, be noted that such a process is possible if $\mathscr{I}^+$ consists of several connected components. This can be also related to the topological censorship theorem [18]. The topological censorship theorem states that any causal curve from $\mathscr{I}^-$ to $\mathscr{I}^+$ is homotopic with each other under the null energy condition. This also forbids the formation of the bubble of exterior region inside the black hole, because otherwise there would be two nonhomotopic causal curve from $\mathscr{I}^-$ to $\mathscr{I}^+$, one passes inside the horizon and the other stays outside. Our argument however does not depend on energy conditions.

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