GEOMETRY OF LOCALLY COMPACT GROUPS OF POLYNOMIAL GROWTH AND SHAPE OF LARGE BALLS.

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Abstract. We show that any locally compact group G with polynomial growth is weakly commensurable to some simply connected solvable Lie group S, the Lie shadow of G. We then study the shape of large balls and show, generalizing work of P. Pansu, that after a suitable renormalization, they converge to a limiting compact set which can be interpreted geometrically. As a consequence, we get the volume asymptotics of large balls. We discuss the speed of convergence, treat some examples and give an application to ergodic theory. We also answer a question of Burago [5] and recover some results of Stoll [24].

1. Introduction

1.1. **Groups of polynomial growth.** Let G be a locally compact group with left Haar measure vol_G. We will assume that G is generated by a compact symmetric subset Ω. Classically, G is said to have *polynomial growth* if there exist $C > 0$ and $k > 0$ such that for any integer $n \geq 1$

$$vol_G(\Omega^n) \leq C \cdot n^k$$

Another choice for Ω would only change the constant C, but not the polynomial nature of the bound. One of the consequences of the analysis carried out in this paper is the following theorem:

Theorem 1.1. *Let G be a locally compact group of polynomial growth and Ω a compact symmetric generating subset of G. Then there exists $c(\Omega) > 0$ and an integer $d(G) \geq 0$ depending on G only such that the following holds:*

$$\lim_{n \to +\infty} \frac{vol_G(\Omega^n)}{n^{d(G)}} = c(\Omega)$$

This extends the main result of Pansu [20]. The integer $d(G)$ coincides with the exponent of growth of a naturally associated graded nilpotent Lie group, the asymptotic cone of G, and is given by the Bass-Guivarc'h formula (3) below. The constant $c(\Omega)$ will be interpreted as the volume of the unit ball of a sub-Riemannian Finsler metric on this nilpotent Lie group. Theorem 1.1 is a by-product of our study of the asymptotic behavior of *periodic pseudodistances* on G, that is pseudodistances that are invariant under a co-compact subgroup of G and satisfy a weak kind of the existence of geodesics axiom (see Definition 4.1).

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Our first task is to get a better understanding of the structure of locally compact groups of polynomial growth. It is an old characterization due to Guivarc'h and Jenkins that a connected Lie group has polynomial growth if and only if it is of type (R), that is if for all $x \in Lie(S)$, $ad(x)$ has only purely imaginary eigenvalues. Such groups are solvable-by-compact and any connected nilpotent Lie group is of type (R). However, unlike the case of discrete groups, where by a famous theorem of Gromov polynomial growth implies virtual nilpotence, most groups of polynomial growth, in the category of connected Lie groups, are genuinely not nilpotent (see Example 2.4). We prove the following generalization of Gromov's theorem:

Theorem 1.2. *Let G be a locally compact group of polynomial growth. Then there exists a simply connected solvable Lie group S of type (R), which is weakly commensurable to G. We call such a Lie group a Lie shadow of G.*

Two locally compact groups are said to be weakly commensurable if, up to moding out by a compact kernel, they have a common closed co-compact subgroup. More precisely, we will show that, for some normal compact subgroup K, G/K has a co-compact subgroup H/K which can be embedded as a closed and co-compact subgroup of a simply connected solvable Lie group S of type (R). We must be aware that being weakly commensurable is not an equivalence relation among locally compact groups (unlike among finitely generated groups). Additionally, the Lie shadow S is unfortunately not unique up to isomorphism (e.g. $\mathbb{Z}^3$ is a co-compact lattice in both $\mathbb{R}^3$ and the universal cover of the group of motions of the plane).

Theorem 1.2 gives detailed information on the geometry of G and will enable us to reduce most geometric questions to the connected Lie group case. Observe also that Theorem 1.2 subsumes Gromov's theorem on polynomial growth, since a lattice in a solvable Lie group of polynomial growth must be virtually nilpotent by a simple argument. Of course in the proof we first reduce to the case when G is a closed subgroup of a connected Lie group, and this reduction step (which is the core of Gromov's proof when G is finitely generated) makes use of Gromov's method, in its generalized form for locally compact groups due to Losert. The rest of the proof combines ideas of Y. Guivarc'h, D. Mostow and H.C. Wang.

1.2. **Asymptotic shapes.** The main part of the paper is devoted to the asymptotic behavior of left invariant pseudodistances on G. Theorem 1.2 enables us to assume that G is a co-compact subgroup of a simply connected solvable Lie group S, and rather than looking at pseudodistances on G, we will look at pseudodistances on S that are left invariant under the co-compact subgroup H.

In the case when S is $\mathbb{R}^d$ and H is $\mathbb{Z}^d$ it is a simple exercise to show that any periodic pseudodistance is asymptotic to a norm on $\mathbb{R}^d$, i.e. $\rho(e,x)/\|x\| \to 1$ as $x \to \infty$, where $\|x\| = \lim \frac{1}{n}\rho(e,nx)$ is a well defined norm on $\mathbb{R}^d$. Burago in [4] showed a much finer result, namely that if ρ is coarsely geodesic, then $\rho(e,x) - \|x\|$ is bounded when x ranges over $\mathbb{R}^d$. When S is a nilpotent Lie group and H a lattice in S, then Pansu proved in his thesis [20], that a similar result holds, namely that $\rho(e,x)/|x| \to 1$ for some (unique only after a choice of a one-parameter group of dilations) homogeneous quasi-norm $|x|$ on the nilpotent Lie group. However, we show in Section 8, that it is not true in general that $\rho(e,x) - |x|$ stays bounded, even for finitely generated nilpotent groups, thus answering a question of Burago (see also Gromov [14]). The main purpose of this paper is to extend Pansu's result to solvable Lie groups of polynomial growth.

As was first noticed by Guivarc'h in his thesis [15], when dealing with geometric properties of solvable Lie groups, it is useful to consider the so-called nilshadow of the group, a construction first introduced by Auslander and Green in [1]. According to this construction, it is possible to modify the Lie product on S in a natural way, by so to speak removing the semisimple part of the action on the nilradical, in order to turn S into a nilpotent Lie group, its nilshadow S_N. The two Lie groups have the same underlying manifold, which is diffeomorphic to $\mathbb{R}^n$, only a different Lie product. They also share the same Haar measure. This "semisimple part" is a commutative relatively compact subgroup $T(S)$ of automorphisms of S, image of S under a homomorphism $T : S \to Aut(S)$. The new product is defined by twisting the old one by means of $T(S)$. The two groups S and S_N are easily seen to be quasi-isometric, and this is why any locally compact group of polynomial growth G is quasi-isometric to some nilpotent Lie group. Yet in this paper, we are dealing with a finer relation than quasi-isometry. In particular, we will be able in Corollary 1.7 to identify the asymptotic cone of G up to isometry and not only up to quasi-isometry or bi-Lipschitz equivalence.

The following is our main theorem:

Theorem 1.3. *(Metric comparison) Let S be a simply connected solvable Lie group of polynomial growth and S_N its nilshadow. Suppose that ρ is a periodic pseudodistance on S invariant under a co-compact subgroup H. Let H_K be the closed subgroup of S generated by the $k(h)$'s, $k \in K$ and $h \in H$, where K is the closure of $T(H)$ in $Aut(S)$. It is a co-compact subgroup for both group structures corresponding to S and S_N. Furthermore, there exists a periodic pseudodistance ρ_K on S which is invariant under both left translations by any element of H_K such that*

$$\lim_{x\to\infty} \frac{\rho(e,x)}{\rho_K(e,x)} = 1$$

Moreover, let $\delta = (\delta_t)_{t>0}$ be a K-invariant one-parameter group of dilations on S_N. Then, among all δ-homogeneous quasi norms on S_N there is a unique one, which we denote by $|\cdot|$ and call the asymptotic norm of ρ, such that

$$\lim_{x\to\infty} \frac{\rho(e,x)}{|x|} = 1$$

Finally we have $|x| = d_\infty(e,x)$, where d_∞ is a left-invariant Carnot-Caratheodory Finsler metric on the graded nilpotent Lie group $S_{N,\delta}$ associated to the choice of dilations δ.

The first half of the theorem is proven in Section 5 and makes use of the properties of the nilshadow. The key point is that unipotent automorphisms induce only a small distortion, forcing the metric ρ to be asymptotically invariant under

K. Once we have thus reduced to the nilpotent group case, the second half of the theorem is proven in Section 6 and follows Pansu's strategy.

Homogeneous quasi-norms and one-parameter groups of dilations on nilpotent Lie groups are defined in Section 3 and their basic properties are recalled. The quasi-norm $|x| = d_\infty(e,x)$ from the above theorem is explicitely defined to be the $S_{N,\delta}$-left invariant sub-Riemannian (or Carnot-Caratheodory) Finsler metric induced by the following choice of a norm $\|\cdot\|_0$ on the eigenspace $m_1 = \{v \in Lie(S_N),\, \delta_t(v) = tv\}$:

$$\{v \in m_1, \|v\|_0 \leq 1\} = \overline{CvxHull}\left\{\frac{\pi_1(\log h)}{\rho_K(e,h)}, h \in H\backslash\{e\}\right\}$$

where $\log : S_N \to Lie(S_N)$ is the logarithm map and π_1 is the linear projection $Lie(S_N) \to m_1$ with kernel $[Lie(S_N), Lie(S_N)]$.

If H is nilpotent to begin with and ρ is the word metric associated to a compact generating set Ω of H, the above norm takes the following simple form:

$$\{v \in m_1, \|v\|_0 \leq 1\} = \overline{CvxHull}\,\{\pi_1(\omega), \omega \in \Omega\} \tag{1}$$

For instance, in the special case when H is a finitely generated nilpotent group and Ω a finite symmetric set, the unit ball $\{v \in m_1, \|v\|_0 \leq 1\}$ is a polyhedron in m_1. When H is not nilpotent, the limit norm is K-invariant and its unit ball is contained in the convex hull of $K \cdot \pi_1(\Omega)$. However, its exact shape is more complicated, yet can be computed in some examples (see Paragraph 8.1).

As a corollary, we obtain that large metric balls in groups of polynomial growth have a well defined asymptotic shape given by Carnot-Caratheodory balls on graded nilpotent Lie groups.

Corollary 1.4. *(Asymptotic shape) Let S be a simply connected solvable Lie group with polynomial growth and ρ a periodic pseudodistance on S. Let $B_\rho(t)$ be the ρ-ball of radius t in S, and let $S_{N,\delta}$ be the graded nilshadow associated to a one-parameter group of dilations $(\delta_t)_t$ as in Theorem 1.3. Then there is a compact neighborhood $\mathcal{C}$ of the identity in S, the asymptotic shape of ρ, such that in the Hausdorff metric,*

$$\lim_{t\to+\infty} \delta_{\frac{1}{t}}(B_\rho(t)) = \mathcal{C}$$

Moreover, $\mathcal{C} = \{g \in S, d_\infty(e,g) \leq 1\}$ is the unit ball of the limit Carnot-Caratheodory metric from Theorem 1.3.

Proof. By Theorem 1.3, for every $\varepsilon > 0$ we have $B_{d_\infty}(t-\varepsilon t) \subset B_\rho(t) \subset B_{d_\infty}(t+\varepsilon t)$ if t is large enough. Since $\delta_{\frac{1}{t}}(B_{d_\infty}(t)) = \mathcal{C}$, for all $t > 0$, we are done. $\square$

We also get the following corollary, of which Theorem 1.1 is only a special case with ρ the word metric associated to the generating set Ω.

Corollary 1.5. *(Volume asymptotics) Suppose that G is a locally compact group with polynomial growth and ρ is a periodic pseudodistance on G. Let $B_\rho(t)$ be the ρ-ball of radius t in G, i.e. $B_\rho(t) = \{x \in G, \rho(e,x) \leq t\}$, then there exists a*

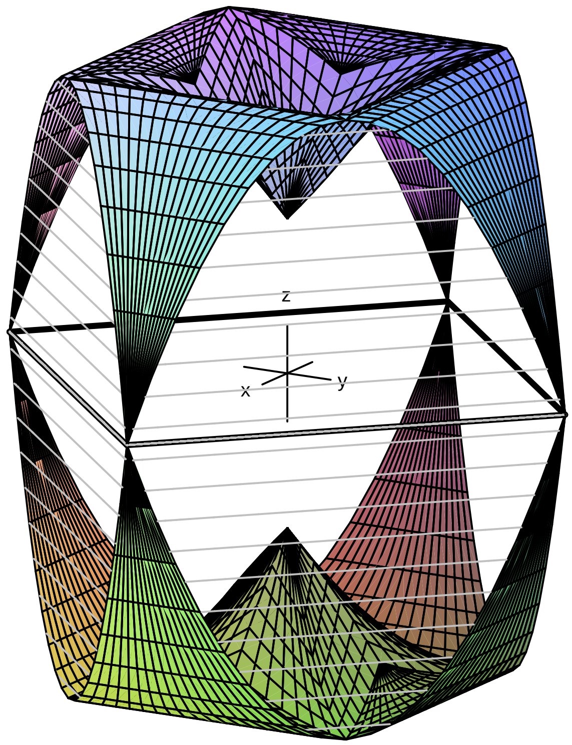

FIGURE 1. The asymptotic shape of large balls in the Cayley graph of the Heisenberg group $H(\mathbb{Z}) = \langle x, y | [x,[x,y]] = [y,[x,y]] = 1\rangle$ viewed in exponential coordinates.

constant $c(\rho) > 0$ such that the following limit exists:

$$\lim_{t\to+\infty} \frac{vol_G(B_\rho(t))}{t^{d(G)}} = c(\rho) \tag{2}$$

Here $d(G)$ is the integer $d(S_N)$, the so-called homogeneous dimension of the nilshadow S_N of a Lie shadow S of G (obtained by Theorem 1.2), and is given by the Bass-Guivarc'h formula:

$$d(S_N) = \sum_{k\geq 0} \dim(C^k(S_N)) \tag{3}$$

where $\{C^k(S_N)\}_k$ is the descending central series of S_N.

The limit $c(\rho)$ is equal to the volume $vol_S(\mathcal{C})$ of the limit shape $\mathcal{C}$ from Corollary 1.4 once we make the right choice of Haar measure on a Lie shadow S of G. Let us explain this choice. Recall that according to Theorem 1.2, G/K admits a co-compact subgroup H/K which embeds co-compactly in S. Starting with a Haar measure vol_G on G, we get a Haar measure on G/K after fixing the Haar measure of K to be of total mass 1, and we may then choose a Haar measure on H/K so that the compact quotient G/H has volume 1. Finally we choose the Haar measure on S so that the other compact quotient $S/(H/K)$ has volume 1. This gives the desired Haar measure vol_S such that $c(\rho) = vol_S(\mathcal{C})$.

In the appendix, we explain Figure 1 and work out explicitely the asymptotic shape of balls for the standard word metric in the discrete Heisenberg groups of dimension 3 and 5. Computing their volume, we can recover the main result of Stoll [24], namely that the growth series of the Heisenberg group of dimension 5 with the standard generators is transcendental.

Another interesting feature is asymptotic invariance:

Corollary 1.6. *(Asymptotic invariance) Let S be a simply connected solvable Lie group with polynomial growth and ρ a periodic pseudodistance on S. Let $(S_N, *)$ be the nilshadow of S. Then there exists a distance d on S, which is invariant under $*$-left translations, such that $\rho(e,x)/d(e,x) = 1$ as $x \to \infty$. In particular $\rho(e, g * x)/\rho(e,x) \to 1$ as $x \to \infty$ for every $g \in S$.*

It is worth noting that we may not replace $*$ by the ordinary product on S. Indeed, let for instance $S = \mathbb{R} \cdot \mathbb{R}^2$ be the universal cover of the group of motions of the Euclidean plane, then S, like its nilshadow $\mathbb{R}^3$, admits a lattice $\Gamma \simeq \mathbb{Z}^3$. The quotient S/Γ is diffeomorphic to the 3-torus $\mathbb{R}^3/\mathbb{Z}^3$ and it is easy to find Riemannian metrics on this torus so that their lift to $\mathbb{R}^3$ is not invariant under rotation around the z-axis. Hence this metric, viewed on the Lie group S will not be asymptotically invariant under left translation by elements of S.

Corollary 1.7. *(Asymptotic cone) Let G be a locally compact group with polynomial growth. Then there is a uniquely defined graded simply connected nilpotent Lie group N such that if ρ a periodic distance on G, then there is a unique (up to isometry) left invariant Carnot-Caratheodory Finsler metric d_∞ on N such that (N, d_∞) is isometric to "the asymptotic cone" associated to (G, ρ). This asymptotic cone is independent of the choice of a ultrafilter used to define it.*

This corollary is a generalization of Pansu's theorem ((10) in [20]).

1.3. **Folner sets and ergodic theory.** A consequence of Corollary 1.5 is that sequences of balls with radius going to infinity are Folner sequences, namely:

Corollary 1.8. *Let G be a locally compact group with polynomial growth and ρ a periodic pseudodistance on G. Let $B_\rho(t)$ be the ρ-ball of radius t in G. Then $\{B_\rho(t)\}_{t>0}$ form a Folner family of subsets of G namely, for any compact set F in G, we have*

$$\lim_{t \to +\infty} \frac{vol_G(FB_\rho(t) \Delta B_\rho(t))}{vol_G(B_\rho(t))} = 0 \tag{4}$$

Proof. Indeed $FB_\rho(t)\Delta B_\rho(t) \subset B_\rho(t+c) \backslash B_\rho(t)$ for some $c >$ depending on F. Hence (4) follows from (2). □

This settles the so-called "localization problem" of Greenleaf for locally compact groups of polynomial growth (see [11]), i.e. determining whether the powers of a compact generating set $\{\Omega^n\}_n$ form a Folner sequence. At the same time it implies that the ergodic theorem for G-actions holds along any sequence of balls with radius going to infinity.

Theorem 1.9. *(Ergodic Theorem) Let be given a locally compact group G with polynomial growth together with a measurable G-space X endowed with a G-invariant ergodic probability measure m. Let ρ be a periodic pseudodistance on G and $B_\rho(t)$ the ρ-ball of radius t in G. Then for any p, $1 \leq p < \infty$, and any function $f \in \mathbb{L}^p(X, m)$ we have*

$$\lim_{t\to+\infty} \frac{1}{vol_G(B_\rho(t))} \int_{B_\rho(t)} f(gx)dg = \int_X f dm$$

for m-almost every $x \in X$ and also in $\mathbb{L}^p(X, m)$.

In fact, Corollary 1.8 above, was the "missing bloc" in the proof of the ergodic theorem on groups of polynomial growth. So far and to my knowledge, Corollary 1.8 and Theorem 1.9 were known only along some subsequence of balls $\{B_\rho(t_n)\}_n$ chosen so that (4) holds (see for instance [6] or [25]). This issue was drawn to my attention by A. Nevo. We refer the reader to the A. Nevo's survey paper [19] Section 5. R. Tessera [26] has a more direct proof of Corollary 1.8, a very slick argument that relies only on the doubling property (see [19]).

1.4. **Organization of the paper.** Sections 2-4 are devoted to preliminaries. A full proof of the Bass-Guivarc'h formula is given in Section 3. Section 5-7 contain the core of the proof of the main theorems. In Section 5, we assume that G is a simply connected solvable Lie group and reduce the problem to the nilpotent case. In Section 6, we assume that G is a simply connected nilpotent Lie group and prove Theorem 1.3 in this case following the strategy used by Pansu in [20]. In Section 7, we prove Theorem 1.2 and reduce the proof of the results of the introduction to the Lie case. In the last section we make further comments about the speed of convergence and we answer a question of Burago. The Appendix is devoted to discrete Heisenberg groups of dimension 3 and 5, we compute their limit balls, explain Figure 1, and recover the main result of Stoll [24].

The reader who is mainly interested in the nilpotent group case can read directly Section 6 while keeping an eye on Sections 3 and 4 for background notations and elementary facts.

Finally, let us mention that the results and methods of this paper were largely inspired by the works of Y. Guivarc'h [15] and P. Pansu [20].

Contents

2. The nilshadow

In this section, we assume that G is a simply connected solvable Lie group. The nilshadow of G is a simply connected nilpotent Lie group G_N associated to G in a natural way. This notion was first introduced by Auslander and Green in [1] in their study of flows on solvmanifolds. They defined it as the unipotent radical of a *semi-simple splitting* of G. However, we are going to follow a different approach for its construction by working first at the Lie algebra level. We refer the reader to the book [8] where this approach is taken up.

Let $\mathfrak{g}$ be a solvable real Lie algebra and $\mathfrak{n}$ the nilradical of $\mathfrak{g}$. We have $[\mathfrak{g},\mathfrak{n}] \subset \mathfrak{n}$. If $x \in \mathfrak{g}$, we write $ad(x) = ad_s(x) + ad_n(x)$ the Jordan decomposition of $ad(x)$ in $GL(\mathfrak{g})$. Since $ad(x) \in Der(\mathfrak{g})$, the space of derivations of $\mathfrak{g}$, and $Der(\mathfrak{g})$ is the Lie algebra of the *algebraic* group $Aut(\mathfrak{g})$, the Jordan components $ad_s(x)$ and $ad_n(x)$ also belong to $Der(\mathfrak{g})$. Moreover, for each $x \in \mathfrak{g}$, $ad_s(x)$ sends $\mathfrak{g}$ into $\mathfrak{n}$ (because so does $ad(x)$ and $ad_s(x)$ is a polynomial in $ad(x)$). Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Since $\mathfrak{g}/\mathfrak{n}$ is abelian, it follows that $\mathfrak{h} + \mathfrak{n} = \mathfrak{g}$. Moreover $ad_s(x)_{|\mathfrak{h}} = 0$ if $x \in \mathfrak{h}$, because $\mathfrak{h}$ is nilpotent. Hence, there is a real vector subspace $\mathfrak{v}$ of $\mathfrak{h}$ such that the following two conditions hold:

(i) $\mathfrak{v} \oplus \mathfrak{n} = \mathfrak{g}$.

(ii) $ad_s(x)(y) = 0$ for all $x, y \in \mathfrak{v}$.

From (i) and (ii), it follows easily that $ad_s(x)$ commutes with $ad(y)$, $ad_s(y)$ and $ad_n(y)$, for all x, y in $\mathfrak{v}$. We have:

Lemma 2.1. *The map $\mathfrak{v} \to Der(\mathfrak{g})$ defined by $x \to ad_s(x)$ is a linear map.*

Proof. Let $x, y \in \mathfrak{v}$. By the above $ad_s(y)$ and $ad_s(x)$ commute with each other (hence their sum is semi-simple) and commute with $ad_n(x) + ad_n(y)$. From the uniqueness of the Jordan decomposition it remains to check that $ad_n(x) + ad_n(y)$ is nilpotent if x, y in $\mathfrak{v}$. To see this, apply the following obvious remark twice to $a = ad_n(x)$ and $V = ad(\mathfrak{n})$ first and then to $a = ad_n(y)$ and $V = span\{ad_n(x), ad((ad(y))^n x), n \geq 1\}$: *Let V be a nilpotent subspace of $GL(\mathfrak{g})$ and $a \in GL(\mathfrak{g})$ nilpotent, i.e. $V^n = 0$ and $a^m = 0$ for some $n, m \in \mathbb{N}$ and assume $[a, V] \subset V$. Then $(a + V)^{nm} = 0$.* □

We define a new Lie bracket on $\mathfrak{g}$ by setting:

$$[x, y]_N = [x, y] - ad_s(x_v)(y) + ad_s(y_v)(x) \tag{5}$$

where x_v is the linear projection of x on $\mathfrak{v}$ according to the direct sum $\mathfrak{v} \oplus \mathfrak{n} = \mathfrak{g}$. The Jacobi identity is checked by a straightforward computation where the following fact is needed: $ad_s(ad_s(x)(y)) = 0$ for all $x, y \in \mathfrak{g}$. This holds because, as we just saw, $ad_s(x)(\mathfrak{g}) \subset \mathfrak{n}$ for all $x \in \mathfrak{g}$, and $ad_s(a) = 0$ if $a \in \mathfrak{n}$.

Definition 2.2. *Let $\mathfrak{g}_N$ be the vector space $\mathfrak{g}$ endowed with the new Lie algebra structure $[\cdot,\cdot]_N$ given by (5). The* ***nilshadow*** *G_N of G is defined to be the simply connected Lie group with Lie algebra $\mathfrak{g}_N$.*

It is easy to check that $\mathfrak{g}_N$ *is a nilpotent Lie algebra.* To see this, note first that $[\mathfrak{g}_N,\mathfrak{g}_N]_N \subset \mathfrak{n}$, and if $x \in \mathfrak{g}_N$ and $y \in \mathfrak{n}$ then $[x,y]_N = (ad_n(x_v) + ad(x_n))(y)$. However, $ad_n(x_v) + ad(x_n)$ is a nilpotent endomorphism of $\mathfrak{n}$ as follows from the same remark used in the proof of Lemma 2.1. Hence $\mathfrak{g}_N$ is a nilpotent.

Besides, it is also possible to define G_N by modifying the Lie group structure on G as follows. First note that the map $\mathfrak{g} \to Der(\mathfrak{g})$ given by $x \mapsto ad_s(x_v)$ is a morphism of Lie algebras with an abelian image. This is easily checked from the fact that all $ad_s(x)$, $x \in \mathfrak{v}$ commute with one another and $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{n}$. Hence we can lift this map to the simply connected Lie group G to obtain a Lie group homomorphism $\mathcal{T} : G \to Aut(\mathfrak{g})$ such that $\mathcal{T}(e^a)(b) = e^{ad_s(a_v)}b$ for all $a,b \in \mathfrak{g}$. Since G is simply connected $Aut(G) \simeq Aut(\mathfrak{g})$ and $\mathcal{T}$ gives rise to a Lie group homomorphism $T : G \to Aut(G)$ such that $T(e^a)(e^b) = \exp(e^{ad_s(a_v)}b)$ for all $a,b \in \mathfrak{g}$.

Then one can define a new Lie group multiplication $\overline{*}$ on G setting $g\overline{*}h = g \cdot T(g^{-1})h$ for all $g,h \in G$. It is easy to check that this defines a new Lie group structure on G and that the Lie bracket on the corresponding Lie algebra is exaclty $[\cdot,\cdot]_N$ above. Hence it is isomorphic to G_N and we write $(G_N,\overline{*})$ to denote the nilshadow of G.

In the construction, we made a choice of a complement $\mathfrak{v}$ for $\mathfrak{n}$ inside a Cartan subalgebra of $\mathfrak{g}$. Since two Cartan subalgebras are conjugate by a inner automorphism from $\mathfrak{n}$, it can be shown that two different choices lead to isomorphic Lie algebras. Hence by abuse of language, we speak of *the* nilshadow of $\mathfrak{g}$. See [8] III for details.

We now list some easy to check properties of the nilshadow for further use.

(A) $T(gh) = T(hg)$ et $T(T(g)h) = T(h)$ for any $g,h \in G$.

(B) $ad_s(x_v)$ is a derivation of $\mathfrak{g}_N$ for all $x \in \mathfrak{g}$, $\mathcal{T}(g) \in Aut(\mathfrak{g}_N)$ for every g, and T is also a Lie group homomorphism $T : (G_N,\overline{*}) \to Aut(G_N,\overline{*})$.

(C) the inverse $g^{\overline{*}-1}$ equals $T(g)g^{-1}$.

(D) we have $T(x)(e^b) = e^b$ for all $x \in G$ and all $b \in \mathfrak{v}$. Hence

$$gxg^{-1} = g\overline{*}T(g)x\overline{*}g^{\overline{*}-1}$$

for every $g \in \exp(\mathfrak{v})$ and for all $x \in G$.

(E) a vector subspace of $\mathfrak{n}$ is an ideal of $\mathfrak{g}$ if and only if it is an ideal of $\mathfrak{g}_N$ and is invariant under all $ad_s(x)$, $x \in \mathfrak{v}$.

(F) if $\mathfrak{d} \subseteq \mathfrak{n}$ is an ideal of $\mathfrak{g}$, then $[\mathfrak{g},\mathfrak{d}]_N$ is also an ideal of $\mathfrak{g}$. Indeed,

$$[\mathfrak{g},\mathfrak{d}]_N = [\mathfrak{v},\mathfrak{d}]_N + [\mathfrak{n},\mathfrak{d}]_N = \langle ad_n(y)\mathfrak{d}, y \in \mathfrak{v}\rangle + [\mathfrak{n},\mathfrak{d}]$$

and both terms in the above sum are invariant under $ad_s(x)$ for $x \in \mathfrak{v}$ because $ad_s(x)$ and $ad_n(y)$, commute.

(G) the central descending series of $\mathfrak{g}_N$ is made of ideals of $\mathfrak{g}$ and these are invariant under $ad_s(x)$, $x \in \mathfrak{v}$.

(H) the brackets $[\cdot,\cdot]$ and $[\cdot,\cdot]_N$ coincide on $\mathfrak{n}$ and so do the Lie products on the nilradical of G.

(I) if $a \in \mathfrak{n}$ or if $a \in \mathfrak{v}$, then $e^a = \exp_N(a)$ where $\exp_N$ is the exponential map in G_N.

Remark 2.3. *The exponential map is not always a diffeomorphism, as the example of the universal cover $\widetilde{E}$ of the group E of motions of the plane shows (indeed any 1-parameter subgroup of E is either a translation subgroup or a rotation subgroup, but the rotation subgroup is compact hence a torus, so its lift will contain the (discrete) center of E, hence will miss every lift of a non trivial translation). In fact, it is easy to see that if $\mathfrak{g}$ is the Lie algebra of a solvable (non-nilpotent) Lie group of polynomial growth, then $\mathfrak{g}$ maps surjectively on the Lie algebra of E. Hence, for a simply connected solvable and non-nilpotent Lie group of polynomial growth, the exponential map is never onto. Nevertheless its image is easily seen to be dense.*

However, exponential coordinates of the second kind behave nicely:

(J) the following map is a diffeomorphism (see [21]).

$$\begin{array}{rcl} \mathfrak{n} \oplus \mathfrak{v} & \to & G \\ (n, v) & \mapsto & e^n \cdot e^v \end{array}$$

and $e^n \cdot e^v = \exp_N(n) \overline{*} \exp_N(v)$ for all $n \in \mathfrak{n}$ and $v \in \mathfrak{v}$.

(K) for any choice of vector subspaces (m_i)'s such that $\mathfrak{n} = m_r \oplus ... \oplus m_1$ where $m_r \oplus ... \oplus m_i$ is the i-th term in the central descending series of $\mathfrak{g}_N$, we can define exponential coordinates of the second kind by setting

$$\begin{array}{rcl} m_r \oplus ... \oplus m_1 \oplus \mathfrak{v} & \to & G \\ (\xi_r, ..., \xi_1, v) & \mapsto & \exp_N(\xi_r) \overline{*} ... \overline{*} \exp_N(\xi_1) \overline{*} \exp_N(v) \end{array}$$

which is a diffeomorphism (by (J) and the existence of such coordinates for nilpotent Lie groups, see [7] Ch. 1.) Moreover, it follows from (H) and (I) that $\exp_N(\xi_r) \overline{*} ... \overline{*} \exp_N(\xi_1) \overline{*} \exp_N(v) = e^{\xi_r} \cdot ... \cdot e^{\xi_1} \cdot e^v$ for all choices of $v \in \mathfrak{v}$ and $\xi_i \in m_i$.

(L) there exists a Riemannian metric on G which is left invariant under both Lie structure. Indeed it suffices to pick a scalar product on $\mathfrak{g}$ which is invariant under the relatively compact subgroup $\mathcal{T}(G) \subset Aut(\mathfrak{g})$.

The following example shows several of the features of a typical solvable Lie group of polynomial growth.

Example 2.4. *Let $G = \mathbb{R} \ltimes_\phi \mathbb{R}^n$ where $\phi_t \in GL_n(\mathbb{R})$ is some one parameter subgroup given by $\phi_t = \exp(tA) = k_t u_t$ where A is some matrix in $M_n(\mathbb{R})$ and $A = A_s + A_u$ is its Jordan decomposition, giving rise to $k_t = \exp(tA_s)$ and $u_t = \exp(tA_u)$. The group G is diffeomorphic to $\mathbb{R}^{n+1}$, hence simply connected. If all eigenvalues of A_s are purely imaginary, then G has polynomial growth. However G is not nilpotent unless $A_s = 0$. So let us assume that neither A_s nor A_u is zero. Then the nilshadow G_N is the semi-direct product $\mathbb{R} \ltimes_u \mathbb{R}^n$ where u_t is the unipotent part of ϕ_t.*

It is easy to compute the homogeneous dimension of G (or G_N) in terms of the dimension of the Jordan blocs of A_u. If n_k is the number of Jordan blocks of A_u of size k, then

$$d(G) = 1 + \sum_{k \geq 1} \frac{k(k+1)}{2} n_k$$

3. Quasi-norms and analysis on nilpotent Lie groups

In this section, we review the necessary background material on nilpotent Lie groups. In paragraph 3.4, we give some crucial properties of homogeneous quasi norms and reproduce some lemmas originally due to Y. Guivarc'h which will be used in the sequel. Meanwhile, we prove the Bass-Guivarc'h formula for the degree of polynomial growth of nilpotent Lie groups, following Guivarc'h's original argument.

3.1. **Carnot-Caratheodory metrics.** Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let m_1 be a vector subspace of $\mathfrak{g}$. We denote by $\|\cdot\|$ a norm on m_1.

We now recall the definition of a *Carnot-Carathéodory metric* on G. Let $x, y \in G$. We consider all possible piecewise smooth paths $\xi : [0,1] \to G$ going from $\xi(0) = x$ to $\xi(1) = y$. Let $\xi'(u)$ be the tangent vector which is pulled back to the identity by a left translation, i.e.

$$\frac{d\xi}{du} = \xi(u) \cdot \xi'(u) \tag{6}$$

where $\xi'(u) \in \mathfrak{g}$ and the notation $\xi(u) \cdot \xi'(u)$ means the image of $\xi'(u)$ under the differential at the identity of the left translation by the group element $\xi(u)$. We say that the path ξ is *horizontal* if the vector $\xi'(u)$ belongs to m_1 for all $u \in [0,1]$. We denote by $\mathcal{H}$ the set of piecewise smooth horizontal paths. The Carnot-Carathéodory metric associated to the norm $\|\cdot\|$ is defined by:

$$d(x,y) = \inf\{\int_0^1 \|\xi'(u)\| \, du, \xi \in \mathcal{H},\ \xi(0) = x, \xi(1) = y\}$$

where the infimum is taken over all piecewise smooth paths $\xi : [0,1] \to N$ with $\xi(0) = x, \xi(1) = y$ that are horizontal in the sense that $\xi'(u) \in m_1$ for all u. If $m_1 = \mathfrak{g}$, and $\|\cdot\|$ is a Euclidean (resp. arbitrary) norm on $\mathfrak{g}$ then d is simply the classical left invariant Riemannian (resp. Finsler) metric associated to $\|\cdot\|$.

Chow's theorem (e.g. see [13]) tells us that $d(x,y)$ is finite for all x and y in G if and only if the vector subspace m_1, together with all brackets of elements of m_1, generates the full Lie algebra $\mathfrak{g}$. If this condition is satisfied then d is a distance on G which induces the original topology of G.

In this paper, we will only be concerned with Carnot-Caratheodory metrics on a simply connected nilpotent Lie group N. In the sequel, whenever we speak of a Carnot-Carathéodory metric on N, we mean one that is associated to a norm $\|\cdot\|$ on a subspace m_1 such that $\mathfrak{n} = m_1 \oplus [\mathfrak{n}, \mathfrak{n}]$ where $\mathfrak{n} = Lie(N)$. It is easy to check that any such m_1 generates the Lie algebra $\mathfrak{n}$.

Remark 3.1. *Let us observe here that for such a metric d on N, we have the following description of the unit ball for $\|\cdot\|$*

$$\{v \in m_1, \|v\| \leq 1\} = \left\{ \frac{\pi_1(x)}{d(e,x)}, x \in N \backslash \{e\} \right\}$$

where π_1 is the linear projection from $\mathfrak{n}$ (identified with N) to m_1 with kernel $[\mathfrak{n}, \mathfrak{n}]$. Indeed, π_1 gives rise to a homomorphism from N to the vector space m_1. And if $\xi(u)$ is a horizontal path from e to x, then applying π_1 to (6) we get $\frac{d}{du}\pi_1(\xi(u)) = \xi'(u)$, hence $\pi_1(x) = \int_0^1 \xi'(u)du$. Finally we obtain $\|\pi_1(x)\| \leq d(e,x)$ with equality if $x \in m_1$.

3.2. **Dilations on a nilpotent Lie group and the associated graded group.** We now focus on the case of simply connected nilpotent Lie groups. Let N be such a group with Lie algebra $\mathfrak{n}$ and nilpotency class r. For background about analysis on such groups, we refer the reader to the book [7]. The exponential map is a diffeomorphism between $\mathfrak{n}$ and N. Most of the time, if $x \in \mathfrak{n}$, we will abuse notations and denote the group element $\exp(x)$ simply by x. We denote by $\{C^p(\mathfrak{n})\}_p$ the central descending series for $\mathfrak{n}$, i.e. $C^{p+1}(\mathfrak{n}) = [\mathfrak{n}, C^p(\mathfrak{n})]$ with $C^0(\mathfrak{n}) = \mathfrak{n}$ and $C^r(\mathfrak{n}) = \{0\}$.

Let $(m_p)_{p\geq 1}$ be a collection of vector subspaces of $\mathfrak{n}$ such that for each $p \geq 1$,

$$C^{p-1}(\mathfrak{n}) = C^p(\mathfrak{n}) \oplus m_p. \tag{7}$$

Then $\mathfrak{n} = \oplus_{p\geq 1} m_p$ and in this decomposition, any element x in $\mathfrak{n}$ (or N by abuse of notation) will be written in the form

$$x = \sum_{p\geq 1} \pi_p(x)$$

where $\pi_p(x)$ is the linear projection onto m_p.

To such a decomposition is associated a semi-group of dilations $(\delta_t)_{t>0}$. These are linear endomorphisms of $\mathfrak{n}$ defined by

$$\delta_t(x) = t^p x$$

for any $x \in m_p$ and for every p. Conversely, the semi-group $(\delta_t)_{t\geq 0}$ determines the $(m_p)_{p\geq 1}$ since they appear as eigenspaces of each δ_t, $t \neq 1$. The dilations δ_t do not preserve *a priori* the Lie bracket on $\mathfrak{n}$. This is the case if and only if

$$[m_p, m_q] \subseteq m_{p+q} \tag{8}$$

for every p and q (where $[m_p, m_q]$ is the subspace generated by all commutators of elements of m_p with elements of m_q). If (8) holds, we say that the $(m_p)_{p\geq 1}$ form a *gradation* of the Lie algebra $\mathfrak{n}$, and that $\mathfrak{n}$ is a *graded* (or homogeneous) Lie algebra. It is an exercise to check that (8) is equivalent to require $[m_1, m_p] = m_{p+1}$ for all p.

If (8) does not hold, we can however consider a new Lie algebra structure on the real vector space $\mathfrak{n}$ by defining the new Lie bracket as $[x,y]_\infty = \pi_{p+q}([x,y])$ if $x \in m_p$ and $y \in m_q$. This new Lie algebra $\mathfrak{n}_\infty$ is graded and has the same underlying vector space as $\mathfrak{n}$. We denote by N_∞ the associated Lie group. Moreover the $(\delta_t)_{t>0}$ form a semi-group of automorphisms of $\mathfrak{n}_\infty$. In fact the original Lie bracket $[x,y]$ on $\mathfrak{n}$ can be deformed continuously to $[x,y]_\infty$ through a continuous family of Lie algebra structures by setting

$$[x,y]_t = \delta_{\frac{1}{t}}([\delta_t x, \delta_t y]) \tag{9}$$

and letting $t \to +\infty$. Note that conversely, if the δ_t's are automorphisms of $\mathfrak{n}$, then $[x,y] = \pi_{p+q}([x,y])$ for all $x \in m_p$ and $y \in m_q$, and $\mathfrak{n} = \mathfrak{n}_\infty$.

The graded Lie algebra associated to $\mathfrak{n}$ is by definition

$$gr(\mathfrak{n}) = \bigoplus_{p \geq 0} C^p(\mathfrak{n})/C^{p+1}(\mathfrak{n})$$

endowed with the Lie bracket induced from that of $\mathfrak{n}$. The quotient map $m_p \to C^p(\mathfrak{n})/C^{p+1}(\mathfrak{n})$ gives rise to a linear isomorphism between $\mathfrak{n}$ and $gr(\mathfrak{n})$, which is a Lie algebra isomorphism between the new Lie algebra structure $\mathfrak{n}_\infty$ and $gr(\mathfrak{n})$. Hence graded Lie algebra structures induced by a choice of supplementary subspaces $(m_p)_{p\geq 1}$ as in (7) are all isomorphic to $gr(\mathfrak{n})$.

Assume now that N_∞ is a graded simply connected nilpotent Lie group. This means that there is a direct sum decomposition

$$\mathfrak{n}_\infty = m_1 \oplus ... \oplus m_r \tag{10}$$

of the Lie algebra $\mathfrak{n}_\infty$, called a gradation, such that $m_i = [m_1, m_{i-1}]$ for $i \geq 2$. In particular $\sum_{k\geq i} m_k = C^i(\mathfrak{n}_\infty)$. We denote by π_i the projection on m_i associated to (10). On such groups the Carnot-Carathéodory metrics d_∞ associated to m_1 are of special interest and we will restrict our attention to those throughout the rest of these notes. If we let $\delta_t : N_\infty \to N_\infty$ be the dilation of ratio t, i.e. the endomorphism of $\mathfrak{n}_\infty$ given on m_i by $x \mapsto t^i x$, then it is easy to check that δ_t is an automorphism of N_∞ and that

$$d_\infty(\delta_t x, \delta_t y) = t d_\infty(x,y) \tag{11}$$

for any $x, y \in N_\infty$.

If N is a general, not necessarily graded, simply connected nilpotent Lie group, then the semi-group of dilations $(\delta_t)_t$ associated to a choice of supplementary vector subspaces m_i's as in (7) will not consist of automorphisms of N and the relation (11) will not hold.

Note also that if we are given two different choices of supplementary subspaces m_i's and m_i''s as in (7), then the Carnot-Caratheodory metrics on the corresponding graded Lie groups are isometric if and only if $(m_1, \|\cdot\|)$ and $(m_1', \|\cdot\|')$ are isometric (i.e. there exists a linear isomorphism from m_1 to m_1' that sends $\|\cdot\|$ to $\|\cdot\|'$.

3.3. **The Campbell-Hausdorff formula.** The exponential map $\exp : \mathfrak{n} \to N$ is a diffeomorphism and, in the sequel, we will abuse notations and identify N and $\mathfrak{n}$ without further notice. In particular, for two elements x and y of $\mathfrak{n}$ (or N equivalently) xy will denote their product in N, while $x+y$ denotes the sum in $\mathfrak{n}$. Let $(\delta_t)_t$ be a semi-group of dilations associated to a choice of supplementary subspaces m_i's as in (7). We denote the corresponding graded Lie algebra by $\mathfrak{n}_\infty$

as above and the Lie group by N_∞. The product on N_∞ is denoted by $x * y$. On N_∞ the dilations $(\delta_t)_t$ are automorphisms.

The Campbell-Hausdorff formula (see [7]) allows to give a more precise form of the product in N. Let $(e_i)_{1\leq i\leq d}$ be a basis of $\mathfrak{n}$ adapted to the decomposition into m_i's, that is $m_i = span\{e_j, e_j \in m_i\}$. Let $x = x_1e_1 + ... + x_de_d$ the corresponding decomposition of an element $x \in \mathfrak{n}$. Then define the degree $d_i = \deg(e_i)$ to be the largest j such that $e_i \in C^{j-1}(\mathfrak{n})$. If $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ is a multi-index, then let $d_\alpha = \deg(e_1)\alpha_1 + ... + \deg(e_d)\alpha_d$. Then, by the Campbell-Hausdorff formula, we have

$$(xy)_i = x_i + y_i + \sum C_{\alpha,\beta}x^\alpha y^\beta \tag{12}$$

where $C_{\alpha,\beta}$ are real constants and the sum is over all multi-indices α and β such that $d_\alpha + d_\beta \leq \deg(e_i)$, $d_\alpha \geq 1$ and $d_\beta \geq 1$.

From (9), it is easy to give the form of the associated graded Lie group law:

$$(x * y)_i = x_i + y_i + \sum C_{\alpha,\beta}x^\alpha y^\beta \tag{13}$$

where the sum is restricted to those α's and β's such that $d_\alpha + d_\beta = \deg(e_i)$, $d_\alpha \geq 1$ and $d_\beta \geq 1$.

3.4. **Homogeneous quasi-norms and Guivarc'h's theorem on polynomial growth.** Let $\mathfrak{n}$ be a finite dimensional real nilpotent Lie algebra and consider a decomposition

$$\mathfrak{n} = m_1 \oplus ... \oplus m_r$$

by supplementary vector subspaces as in (7). Let $(\delta_t)_{t>0}$ be the one parameter semi-group of dilations associated to this decomposition, that is $\delta_t(x) = t^i x$ if $x \in m_i$. We now introduce the following definition.

Definition 3.2. *A continuous function $|\cdot| : \mathfrak{n} \to \mathbb{R}_+$ is called a homogeneous quasi-norm associated to the dilations $(\delta_t)_t$, if it satisfies the following properties:*

(i) $|x| = 0 \Leftrightarrow x = 0$.

(ii) $|\delta_t(x)| = t|x|$ for all $t > 0$.

Example 3.3. *(1) Quasi-norms of supremum type, i.e. $|x| = \max_p \|\pi_p(x)\|_p^{1/p}$ where $\|\cdot\|_p$ are ordinary norms on the vector space m_p and π_p is the projection on m_p as above.*

(2) $|x| = d_\infty(e, x)$, where d_∞ is a Carnot-Carathéodory metric on a graded nilpotent Lie group (as the relation (11) shows).

Clearly, a quasi-norm is determined by its sphere of radius 1 and two quasi-norms (which are homogeneous with respect to the same semi-group of dilations) are always equivalent in the sense that

$$\frac{1}{c}\,|\cdot|_1 \leq |\cdot|_2 \leq c\,|\cdot|_1 \tag{14}$$

for some constant $c > 0$ (indeed, by continuity, $|\cdot|_2$ admits a maximum on the "sphere" $\{|x|_1 = 1\}$). If the two quasi-norms are homogeneous with respect to two distinct semi-groups of dilations, then the inequalities (14) continue to hold outside a neighborhood of 0, but may fail near 0.

Homogeneous quasi-norms satisfy the following properties:

Proposition 3.4. *Let $|\cdot|$ be a homogeneous quasi-norm on $\mathfrak{n}$, then there are constants $C, C_1, C_2 > 0$ such that*
(*a*) $|x_i| \leq C \cdot |x|^{\deg(e_i)}$ *if* $x = x_1 e_1 + ... + x_n e_n$ *in an adapted basis* $(e_i)_i$.
(*b*) $|x^{-1}| \leq C \cdot |x|$.
(*c*) $|x + y| \leq C \cdot (|x| + |y|)$
(*d*) $|xy| \leq C_1(|x| + |y|) + C_2$.

Properties (a), (b) and (c) are straightforward from the fact that $|x| = \max_p \|\pi_p(x)\|_p^{1/p}$ is a homogeneous quasi-norm and from (14). Property (d) justifies the term "quasi-norm" and follows from Lemma 3.5 below. It can be a problem that the constant C_1 in (d) may not be equal to 1. In fact, this is why we use the word quasi-norm instead of just norm, because we do not require the triangle inequality axiom to hold. However the following lemma of Guivarc'h is often a good enough remedy to this situation. Let $\|\cdot\|_p$ be an arbitrary norm on the vector space m_p.

Lemma 3.5. *(Guivarc'h, [15] lemme II.1) Let $\varepsilon > 0$. Up to rescaling each $\|\cdot\|_p$ into a proportional norm $\lambda_p \|\cdot\|_p$ ($\lambda_p > 0$) if necessary, the quasi-norm $|x| = \max_p \|\pi_p(x)\|_p^{1/p}$ satisfies*

$$|xy| \leq |x| + |y| + \varepsilon \tag{15}$$

for all $x, y \in N$. If N is graded with respect to $(\delta_t)_t$ we can take $\varepsilon = 0$.

This lemma is crucial also for computing the coarse asymptotics of volume growth. For the reader's convenience, we reproduce here Guivarc'h's argument, which is based on the Campbell-Hausdorff formula (12).

Proof. We fix $\lambda_1 = 1$ and we are going to give a condition on the λ_i's so that (15) holds. The λ_i's will be taken to be smaller and smaller as i increases. We set $|x| = \max_p \|\pi_p(x)\|_p^{1/p}$ and let $|x|_\lambda = \max_p \|\lambda_p \pi_p(x)\|_p^{1/p}$ for any r-tuple of λ_i's. We want that for any index $p \leq r$,

$$\lambda_p \|\pi_p(xy)\|_p \leq (|x|_\lambda + |y|_\lambda + \varepsilon)^p \tag{16}$$

By (12) we have $\pi_p(xy) = \pi_p(x) + \pi_p(y) + P_p(x, y)$ where P_p is a polynomial map into m_p depending only on the $\pi_i(x)$ and $\pi_i(y)$ with $i \leq p - 1$ such that

$$\|P_p(x, y)\|_p \leq C_p \cdot \sum_{l,m \geq 1, l+m \leq p} M_{p-1}(x)^l M_{p-1}(y)^m$$

where $M_k(x) := \max_{i \leq k} \|\pi_i(x)\|_i^{1/i}$ and $C_p > 0$ is a constant depending on P_p and on the norms $\|\cdot\|_i$'s. Since $\varepsilon > 0$, when expanding the right hand side of (16) all terms of the form $|x|_\lambda^l |y|_\lambda^m$ with $l + m \leq p$ appear with some positive coefficient, say $\varepsilon_{l,m}$. The terms $|x|_\lambda^p$ and $|y|_\lambda^p$ appear with coefficient 1 and cause no trouble since we always have $\lambda_p \left\|\pi_p(x)\right\|_p \leq |x|_\lambda^p$ and $\lambda_p \left\|\pi_p(y)\right\|_p \leq |y|_\lambda^p$. Therefore, for (16) to hold, it is sufficient that

$$\lambda_p C_p M_{p-1}(x)^l M_{p-1}(y)^m \leq \varepsilon_{l,m} |x|_\lambda^l |y|_\lambda^m$$

for all remaining l and m. However, clearly $M_k(x) \leq \Lambda_k \cdot |x|_\lambda$ where $\Lambda_k := \max_{i \leq k}\{1/\lambda_i^{1/i}\} \geq 1$. Hence a sufficient condition for (16) to hold is

$$\lambda_p \leq \frac{\overline{\varepsilon}}{C_p \Lambda_{p-1}^p}$$

where $\overline{\varepsilon} = \min \varepsilon_{l,m}$. Since Λ_{p-1} depends only on the first $p-1$ values of the λ_i's, it is obvious that such a set of conditions can be fulfilled by a suitable r-tuple λ. $\square$

Remark 3.6. *The constant C_2 in Property (d) above can be taken to be 0 when N is graded with respect to the m_i's (i.e. the δ_t's are automorphisms), as is easily seen after changing x and y into their image under δ_t. And conversely, if $C_2 = 0$ for some δ_t-homogeneous quasi-norm on N, then N is graded. Indeed, from (12) and (13), we see that if N is not graded, then one can find $x, y \in N$ such that, when t is small enough, $|\delta_t(xy) - \delta_t(x)\delta_t(y)| \geq ct^{(r-1)/r}$ for some $c > 0$. However, combining Properties (c) and Property (d) with $C_2 = 0$ above we must have $|\delta_t(xy) - \delta_t(x)\delta_t(y)| = O(t)$ near $t = 0$. A contradiction.*

Guivarc'h's lemma enables us to show:

Theorem 3.7. *(Guivarc'h ibid.) Let Ω be a compact neighborhood of the identity in N and $\rho_\Omega(x,y) = \inf\{n \geq 1, x^{-1}y \in \Omega^n\}$. Then for any homogeneous quasi-norm $|\cdot|$ on N, there is a constant $C > 0$ such that*

$$\frac{1}{C}|x| \leq \rho_\Omega(e,x) \leq C|x| + C \tag{17}$$

Proof. Since any two homogeneous quasi-norms are equivalent, it is enough to do the proof for one of them, so we consider the quasi-norm obtained in Lemma 3.5 with the extra property (15). The lower bound in (17) is a direct consequence of (15) and one can take there C to be $\max\{|x|, x \in \Omega\} + \varepsilon$. For the upper bound, it suffices to show that there is $C \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $|x| \leq n$ then $x \in \Omega^{Cn}$. To achieve this, we proceed by induction of the nilpotency length of N. The result is clear when N is abelian. Otherwise, by induction we obtain $C_0 \in \mathbb{N}$ such that $x = \omega_1 \cdot ... \cdot \omega_{C_0 n} \cdot z$ where $\omega_i \in \Omega$ and $z \in C^{r-1}(N)$ whenever $|x| \leq n$. Hence $|z| \leq |x| + C_0 n \cdot \max|\omega_i^{-1}| + \varepsilon C_0 \cdot n \leq C_1 n$ for some other constant $C_1 \in \mathbb{N}$. So we have reduced the problem to $x = z \in m_r = C^{r-1}(N)$ which is central in N. We have $z = z_1^{n^r}$ where $|z_1| = |z|/n \leq C_1$. Since Ω is a neighborhood of the identity in N, the set $\mathcal{U}$ of all products of at most $\dim(m_r)$ simple commutators of length r of elements in Ω is a neighborhood of the identity in $C^{r-1}(N)$ (e.g. see [13], p113). It follows that there is a constant $C_2 \in \mathbb{N}$ such that z_1 is in $\mathcal{U}^{C_2}$, hence the product of at most $C_2 \dim(m_r)$ simple commutators. Then we are done because z itself will be equal to the same product of commutators where each letter $x_i \in \Omega$ is replaced by x_i^n. This last fact follows from the following lemma:

Lemma 3.8. *Let G be a nilpotent group of nilpotency class r and $n_1, ..., n_r$ be positive integers. Then for any $x_1, ..., x_r \in G$*

$$[x_1^{n_1}, [x_2^{n_2}, [..., x_r^{n_r}]...] = [x_1, [x_2, [..., x_r]...]^{n_1 \cdot ... \cdot n_r}$$

To prove the lemma it suffices to use induction and the following obvious fact: if $[x, y]$ commutes to x and y then $[x^n, y] = [x, y]^n$. □

Finally, we obtain:

Corollary 3.9. *Let Ω be a compact neighborhood of the identity in N. Then there are positive constants C_1 and C_2 such that for all $n \in \mathbb{N}$,*

$$C_1 n^d \leq vol_N(\Omega^n) \leq C_2 n^d$$

where d is given by the Bass-Guivarc'h formula:

$$d = \sum_{i \geq 1} i \cdot \dim m_i \tag{18}$$

Proof. By Theorem 3.7, it is enough to estimate the volume of the quasi-norm balls. By homogeneity of the quasi-norm, we have $vol_N\{x, |x| \leq t\} = t^d vol_N\{x, |x| \leq 1\}$. □

Remark 3.10. *The use of Malcev's embedding theorem allows, as Guivarc'h observed, to deduce immediately that the analogous result holds for virtually nilpotent finitely generated groups (a fact that was proven also independently by H. Bass* [2] *by a direct combinatorial argument).*

4. Periodic metrics

In this section, unless otherwise stated, G will denote an arbitrary locally compact group.

4.1. **Definitions.** By a *pseudodistance* (or metric) on a topological space X, we mean a function $\rho : X \times X \to \mathbb{R}_+$ satisfying $\rho(x, y) = \rho(y, x)$ and $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for any triplet of points of X. However $\rho(x, y)$ may be equal to 0 even if $x \neq y$.

We will also require the pseudodistance to be *locally bounded*, meaning that the image under ρ of any compact subset of $G \times G$ is a bounded subset of $\mathbb{R}_+$. To avoid irrelevant cases (for instance $\rho \equiv 0$) we will also assume that ρ is *proper*, i.e. the map $y \mapsto \rho(e, y)$ is a proper map. When ρ is locally bounded then it is proper if and only if $y \mapsto \rho(x, y)$ is proper for any $x \in G$.

A pseudodistance ρ on G is said to be *asymptotically geodesic* if for every $\varepsilon > 0$ there exists $s > 0$ such that for any $x, y \in G$ one can find a sequence of points $x_1 = x$, $x_2, ..., x_n = y$ in G such that

$$\sum_{i=1}^{n-1} \rho(x_i, x_{i+1}) \leq (1 + \varepsilon)\rho(x, y) \tag{19}$$

and $\rho(x_i, x_{i+1}) \leq s$ for all $i = 1, ..., n - 1$.

We will consider exclusively pseudodistances on a group G that are *invariant* under left translations by all elements of a fixed closed and co-compact subgroup H of G, meaning that for all $x, y \in G$ and all $h \in H$, $\rho(hx, hy) = \rho(x, y)$.

Combining all previous axioms, we set the following definition.

Definition 4.1. *Let G be a locally compact group. A pseudodistance ρ on G will be said to be a* **periodic metric** *(or H-periodic metric) if it satisfies the following properties:*

(i) ρ is invariant under left translations by some closed co-compact subgroup $H \leq G$.

(ii) ρ is locally bounded and proper.

(iii) ρ is asymptotically geodesic.

Remark 4.2. *The assumption that ρ is symmetric, i.e. $\rho(x, y) = \rho(y, x)$ is here only for the sake of simplicity, and most of what is proven in this paper can be done without this hypothesis.*

4.2. **Basic properties.** Let ρ be a periodic metric on G and H some co-compact subgroup of G. The following properties are straighforward.

(1) ρ is at a bounded distance from its restriction to H. This means that if F is a bounded fundamental domain for H in G and for an arbitrary $x \in G$, if h_x denotes the element of H such that $x \in h_x F$, then $|\rho(x, y) - \rho(h_x, h_y)| \leq C$ for some constant $C > 0$.

(2) $\forall t > 0$ there exists a compact subset K_t of G such that, $\forall x, y \in G$, $\rho(x, y) \leq t \Rightarrow x^{-1}y \in K_t$. And conversely, if K is a compact subset of G, $\exists t(K) > 0$ s.t. $x^{-1}y \in K \Rightarrow \rho(x, y) \leq t(K)$.

(3) If $\rho(x, y) \geq s$, the x_i's in (19) can be chosen in such a way that $s \leq \rho(x_i, x_{i+1}) \leq 2s$ (one can take a suitable subset of the original x_i's).

(4) The restriction of ρ to $H \times H$ is a periodic pseudodistance on H. This means that the x_i's in (19) can be chosen in H.

(5) Conversely, given a periodic pseudodistance ρ_H on H, it is possible to extend it to a periodic pseudodistance on G by setting $\rho(x, y) = \rho_H(h_x, h_y)$ where $x = h_x F$ for some bounded fundamental domain F for H in G.

4.3. **Examples.** Let us give a few examples of periodic pseudodistances.

(1) Let Γ be a finitely generated torsion free nilpotent group which is embedded as a co-compact discrete subgroup of a simply connected nilpotent Lie group N. Given a finite symmetric generating set S of Γ, we can consider the corresponding word metric d_S on Γ which gives rise to a periodic metric on N given by $\rho(x, y) = d_S(\gamma_x, \gamma_y)$ where $x \in \gamma_x F$ and $y \in \gamma_y F$ if F is some fixed fundamental domain for Γ in N.

(2) Another example, given in [20], is as follows. Let N/Γ be a nilmanifold with universal cover N and fundamental group Γ. Let g be a Riemannian metric on N/Γ. It can be lifted to the universal cover and thus gives rise to a Riemannian metric $\widetilde{g}$ on N. This metric is Γ-invariant, proper and locally bounded. Since Γ is co-compact in N, it is easy to check that it is also asymptotically geodesic hence periodic.

(3) Any word metric on G. That is, if Ω is a compact symmetric generating subset of G, let $\Delta_\Omega(x) = \inf\{n \geq 1, x \in \Omega^n\}$. Then define $\rho(x,y) = \Delta_\Omega(x^{-1}y)$. Clearly ρ is a pseudodistance (although not a distance) and it is G-invariant on the left, it is also proper, locally bounded and asymptotically geodesic, hence periodic.

(4) If G is a connected Lie group, any left invariant Riemannian metric on G. Here again $H = G$ and we obtain a periodic distance. Similarly, any left invariant Carnot-Carathéodory metric on G will do.

4.4. **Coarse equivalence between invariant pseudodistances.** The following proposition is basic:

Proposition 4.3. *Let ρ_1 and ρ_2 be two periodic pseudodistances on G. Then there is a constant $C > 0$ such that for all $x, y \in G$*

$$\frac{1}{C}\rho_2(x,y) - C \leq \rho_1(x,y) \leq C\rho_2(x,y) + C \tag{20}$$

Proof. Clearly it suffices to prove the upper bound. Let $s > 0$ be the number corresponding to the choice $\varepsilon = 1$ in (19) for ρ_2. From 4.2 (2), there exists a compact subset K_s in G such that $\rho_2(x,y) \leq 2s \Rightarrow x^{-1}y \in K_{2s}$, and there is a constant $t = t(K_{2s}) > 0$ such that $x^{-1}y \in K_{2s} \Rightarrow \rho_1(x,y) \leq t$. Let $C = \max\{2t/s, t\}$, and let $x, y \in G$. If $\rho_2(x,y) \leq s$ then $\rho_1(x,y) \leq t$ so the right hand side of (20) holds. If $\rho_2(x,y) \geq s$ then, from (19) and 4.2 (3), we get a sequence of x_i's in G from x to y such that $s \leq \rho_2(x_i, x_{i+1}) \leq 2s$ and $\sum_1^N \rho_2(x_i, x_{i+1}) \leq 2\rho_2(x,y)$. It follows that $\rho_1(x_i, x_{i+1}) \leq t$ for all i. Hence $\rho_1(x,y) \leq \sum \rho_1(x_i, x_{i+1}) \leq Nt \leq \frac{2}{s}t\rho_2(x,y)$ and the right hand side of (20) holds. $\square$

In the particular case when $G = N$ is a simply connected nilpotent Lie group, the distance to the origin $x \mapsto \rho(e,x)$ is also coarsely equivalent to any homogeneous quasi-norm on N. We have,

Proposition 4.4. *Suppose N is a simply connected nilpotent Lie group. Let ρ_1 be a periodic pseudodistance on N and $|\cdot|$ be a homogeneous quasi-norm, then there exists $C > 0$ such that for all $x \in N$*

$$\frac{1}{C}|x^{-1}y| - C \leq \rho_1(x,y) \leq C|x^{-1}y| + C \tag{21}$$

Moreover, if ρ_2 is a periodic pseudodistance on the graded nilpotent group N_∞ associated to N, then again, there is a constant $C > 0$ such that

$$\frac{1}{C}\rho_2(e,x) - C \leq \rho_1(e,x) \leq C\rho_2(e,x) + C \tag{22}$$

The proposition follows at once from Guivarc'h's theorem (see Corollary 3.7 above), the equivalence of homogeneous quasi-norms, and the fact that Carnot-Caratheodory distances on N_∞ are homogeneous quasi norms. However, since the group structures on N and N_∞ differ, (22) cannot in general be replaced by the stronger relation (20) as simple examples show.

The next proposition is of fundamental importance for the study of metrics on Lie groups of polynomial growth:

Proposition 4.5. *Let G be a simply connected solvable Lie group of polynomial growth and G_N its nilshadow. Let ρ and ρ_N be arbitrary periodic pseudodistances on G and G_N respectively. Then there is a constant $C > 0$ such that for all $x, y \in G$*

$$\frac{1}{C}\rho_N(x,y) - C \leq \rho(x,y) \leq C\rho_N(x,y) + C \tag{23}$$

Proof. According to Proposition 4.3, it is enough to show (23) for *some* choice of periodic metrics on G and G_N. But in Section 2 (L) we constructed a Riemannian metric on G which is left invariant for both G and G_N. We are done. □

4.5. **Right invariance under a compact subgroup.** Here we verify that, given a compact subgroup of G, any periodic metric is at bounded distance from another periodic metric which is invariant on the right by this compact subgroup. Let K be a compact subgroup of G and ρ a periodic pseudodistance on G. We average ρ with the help of the normalized Haar measure on K to get:

$$\rho^K(x,y) = \int_{K\times K} \rho(xk_1, yk_2)dk_1dk_2 \tag{24}$$

Then the following holds:

Lemma 4.6. *There is a constant $C_0 > 0$ depending only on ρ and K such that for all $k_1, k_2 \in K$ and all $x, y \in G$*

$$|\rho(xk_1, yk_2) - \rho(x,y)| \leq C_0 \tag{25}$$

Proof. From 4.2 (2), $\exists t = t(K) > 0$ s.t. $\forall x \in G$, $\rho(x, xk) \leq t$. Applying the triangle inequality, we are done. □

Hence we obtain:

Proposition 4.7. *The pseudodistance ρ^K is periodic and lies at a bounded distance from ρ. In particular, as x tends to infinity in G the following limit holds*

$$\lim_{x\to\infty} \frac{\rho^K(e,x)}{\rho(e,x)} = 1 \tag{26}$$

Proof. From Lemma 4.6 and 4.2 (3), it is easy to check that ρ^K must be asymptotically geodesic, and periodic. Integrating (25) we get that ρ^K is at a bounded distance from ρ and (26) is obvious. □

If K is normal in G, we thus obtain a periodic metric ρ^K on G/K such that $\rho^K(p(x), p(y))$ is at a bounded distance from $\rho(x,y)$, where p is the quotient map $G \to G/K$.

5. Reduction to the nilpotent case

In this section, G denotes a *simply connected* solvable Lie group of polynomial growth. We are going to reduce the proof of the theorems of the Introduction to the case of a nilpotent G. This is performed by showing that any *periodic* pseudodistance ρ on G is asymptotic to some associated *periodic* pseudodistance

ρ_N on the nilshadow G_N. The key step is Proposition 5.2 below, which shows the asymptotic invariance of ρ under the "semisimple part" of G. The crucial fact there is that the displacement of a distant point under a fixed unipotent automorphism is negligible compared to the distance from the identity (see Lemmas 5.4, 5.6), so that the action of the semisimple part of large elements can be simply approximated by their action by left translation.

5.1. **Asymptotic invariance under a compact group of automorphisms of G.** Let $T : G \to Aut(G)$ be the homomorphism introduced in Section 2 and let $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{n}$ be the associated decomposition of the Lie algebra of G. If N is the nilpotent radical of G, then $\mathfrak{n} = Lie(N)$ and $\mathfrak{v}$ is a supplementary vector subspace such that $ad_s(x)(\mathfrak{v}) = 0$ for all $x \in \mathfrak{g}$. Recall that T satisfies $T(e^a)(e^b) = \exp(e^{ad_s(a_v)}b)$ for all $a, b \in \mathfrak{g}$, where a_v is the projection of a to $\mathfrak{v}$.

Let $\mathcal{T}(g)$ be the differential of $T(g)$ at the identity, so that $\mathcal{T} : G \to Aut(\mathfrak{g})$ and $\mathcal{T}(e^a) = e^{ad_s(a_v)}$. Recall (see Section 2 above) that the group structure on the nilshadow $(G_N, \overline{*})$ is defined by $g \overline{*} h = g \cdot T(g^{-1})h$, and the corresponding Lie bracket satisfies $[a, b]_N = [a, b] - (ad_s(a_v)b - ad_s(b_v)a)$.

Moreover, since G has polynomial growth, the eigenvalues of $ad_s(x)$ are purely imaginary, hence $\{T(g), g \in G\}$ is relatively compact in $Aut(G)$. The main result of this section is the following:

Proposition 5.1. *Let H be a closed co-compact subgroup of G and ρ an H-periodic pseudodistance on G. There exist a closed subset H_K containing H which is a co-compact subgroup for both G and G_N, and an H_K-periodic pseudodistance ρ_K on the nilshadow G_N such that*

$$\lim_{x \to \infty} \frac{\rho_K(e, x)}{\rho(e, x)} = 1 \tag{27}$$

The new pseudodistance ρ_K is obtained by averaging ρ in the following way. Suppose that ρ is invariant under the co-compact subgroup H of G. Let $K = \overline{\{T(h), h \in H\}}$. It is a compact commutative subgroup of $Aut(G)$ and we denote by dk its normalized Haar measure. Let H_K be the closed subgroup of G generated by all $k(h)$ with $k \in K$ and $h \in H$. Clearly H_K is a closed and co-compact subgroup of G. The homogeneous space $H \backslash H_K$ is compact and bears a right invariant probability measure μ. We can now define a new pseudodistance on G by setting

$$\rho_K(x, y) = \int_{H \backslash H_K} \int_K \rho(gk(x), gk(y)) dk d\mu(g) \tag{28}$$

An essential part of the proof of Proposition 5.1 is enclosed in the following statement:

Proposition 5.2. *Let ρ be a periodic pseudodistance on G which is invariant under a co-compact subgroup H. Then ρ is asymptotically invariant under the action of $K = \overline{\{T(h), h \in H\}} \subseteq Aut(G)$. Namely, for all $k \in K$,*

$$\lim_{x \to \infty} \frac{\rho(e, k(x))}{\rho(e, x)} = 1 \tag{29}$$

The proof of Proposition 5.2 splits into two steps. First we show that it is enough to prove (29) for a dense subset of k's. This is a consequence of the following continuity statement:

Lemma 5.3. *Let $\varepsilon > 0$, then there is a neighborhood U of the identity in K such that, for all $k \in U$,*

$$1 - \varepsilon < \underline{\lim}_{x \to \infty} \frac{\rho(e, k(x))}{\rho(e, x)} \leq \overline{\lim}_{x \to \infty} \frac{\rho(e, k(x))}{\rho(e, x)} < 1 + \varepsilon$$

Then we show that the action of $T(g)$ can be approximated by the conjugation by g, essentially because the unipotent part of this conjugation does not move x very much when x is far. This is the content of the following lemma:

Lemma 5.4. *Let ρ be a periodic pseudodistance on G which is invariant under a co-compact subgroup H. Then for any $\varepsilon > 0$, and any compact subset F in H there is $s_0 > 0$ such that*

$$|\rho(e, T(h)x) - \rho(e, hx)| \leq \varepsilon \rho(e, x)$$

for any $h \in F$ and as soon as $\rho(e, x) > s_0$.

Proof of Proposition 5.2 modulo Lemmas (5.3) and (5.4): As ρ is assumed to be H-invariant, for every $h \in H$, we have $\rho(e, h^{-1}x)/\rho(e, x) \to 1$. The proof of the proposition then follows immediately from the combination of the last two lemmas. $\square$

5.2. **Proof of Lemmas (5.3) and (5.4).** We identify $K = \overline{\{T(g), g \in G\}}$ with its image in $Aut(\mathfrak{g})$ under the canonical isomorphism between $Aut(G)$ and $Aut(\mathfrak{g})$. Recall that, according to property (7) in Section 2, the central descending series of $\mathfrak{g}_N$ is invariant under $ad_s(x)$ for all $x \in \mathfrak{v}$ and consists of ideals of $\mathfrak{g}$. It follows that it is also invariant under the action of G via $\mathcal{T}$. Lifting to the group level, we see that the central descending series of G_N is invariant under the action of K. Since K is compact, its representation on $\mathfrak{g}$ via $\mathcal{T}$ is completely reducible. Therefore, it is possible to choose supplementary subspaces m_i's in $\mathfrak{g}_N$ as in (7) in such a way that they are invariant under K. Since $\mathfrak{n}$ too is invariant under K, we may also choose a K-invariant supplementary subspace $\mathfrak{l}$ that $\mathfrak{n} = [\mathfrak{g}_N, \mathfrak{g}_N] \oplus \mathfrak{l}$.

Recall that, according to property (10) in Section 2, if $\xi = \xi_r + ... + \xi_1 + \xi_0$ is the expression of $\xi \in \mathfrak{g}_N$ associated to the K-invariant decomposition

$$\mathfrak{g}_N = m_r \oplus ... \oplus m_2 \oplus \mathfrak{l} \oplus \mathfrak{v} \tag{30}$$

then $e^{\xi_r} \cdot ... \cdot e^{\xi_0} = \exp_N(\xi_r) \overline{*} ... \overline{*} \exp_N(\xi_0)$. So we can read off the action of $k \in K \subset Aut(G)$ on those exponential coordinates of the second kind:

$$\begin{aligned} k\left(e^{\xi_r} \cdot ... \cdot e^{\xi_0}\right) &= k(e^{\xi_r}) \cdot ... \cdot k(e^{\xi_0}) = e^{k(\xi_r)} \cdot ... \cdot e^{k(\xi_0)} \\ &= \exp_N(k(\xi_r)) \overline{*} ... \overline{*} \exp_N(k(\xi_0)) \end{aligned}$$

and $k(\xi) = k(\xi_r) + ... + k(\xi_0)$ is the decomposition of $k(\xi)$ in the direct sum (30). Let $(\delta_t)_t$ be the 1-parameter group of dilations on $\mathfrak{g}_N$ associated to the m_i's as defined in Section 3.2. Let $\|\cdot\|_i$ be some K-invariant norm on m_i and define

$|x|' = \max_i \|\xi_i\|_i^{1/d_i}$ if $x = \exp_N(\xi_r)\overline{*}...\overline{*}\exp_N(\xi_0)$ and $d_i = i$ if $i > 0$ and $d_0 = 1$. Observe also that for all $i = 0, ..., r$ we have $\delta_t(\exp_N(\xi_i)) = \exp_N(t^{d_i}\xi_i)$. Although $|\cdot|'$ is not homogeneous with respect to $(\delta_t)_t$, it is easy to check from (12) and (13) that $|\cdot|'$ is equivalent (in the sense of (14)) to any homogeneous quasi norm on G_N outside a neighborhood of zero.

Lemma 5.5. *Let $|\cdot|$ be a homogeneous quasi norm on G_N. For any $\varepsilon > 0$, there exists $s_1 > 0$ and a neighborhood U of the identity in K such that if $|x| \geq s_1$ then*

$$|x^{\overline{*}-1}\overline{*}k(x)| \leq \varepsilon|x|$$

for every $k \in U$.

Proof. We write

$$|x^{\overline{*}-1}\overline{*}k(x)| = |x| \cdot \left|\delta_{\frac{1}{|x|}}\left(x^{\overline{*}-1}\overline{*}k(x)\right)\right|$$

Observe that, as follows from (12) and (13), if $x, y \in G_N$ and $|x|, |y|$ are $O(t)$, then $|\delta_{\frac{1}{t}}(x\overline{*}y) - \delta_{\frac{1}{t}}(x) * \delta_{\frac{1}{t}}(y)| = O(t^{-1/r})$, where $*$ is the product in the graded nilshadow $G_{N,\infty}$ associated to $(\delta_t)_t$. When writing $x = \exp_N(\xi_r)\overline{*}...\overline{*}\exp_N(\xi_0)$, and setting $t = |x|$, we thus obtain that the following quantity

$$\left|\delta_{\frac{1}{t}}(x^{\overline{*}-1}\overline{*}k(x)) - \prod_{0\leq i\leq r}^{*} \exp_N(-t^{-d_i}\xi_i) * \prod_{0\leq i\leq r}^{*} \exp_N(t^{-d_{r-i}}k(\xi_{r-i}))\right|$$

is a $O(t^{-1/r})$. Recall that $|\cdot|'$ is equivalent to $|\cdot|$, hence as x gets larger, each $t^{-d_i}\xi_i$ remains in a compact subset of m_i. Therefore, there is a neighborhood U of the identity in K such that each $t^{-d_i}k(\xi_i)$ is very close to $t^{-d_i}\xi_i$ independently of the choice of a (large) $x \in G_N$. Hence, as soon as $t = |x|$ is large enough, for all k in U,

$$\left|\exp_N(-t^{-d_0}\xi_0) * ... * \exp_N(-t^{-d_r}\xi_r) * \exp_N(t^{-d_r}k(\xi_r)) * ... * \exp_N(t^{-d_0}k(\xi_0))\right| \leq \varepsilon$$

Combining the last two estimates, and using Proposition 3.4 (c), we obtain the desired result. ☐

Lemma 5.6. *Let N be a simply connected nilpotent Lie group and let $|\cdot|$ be a homogeneous quasi norm on N associated to some 1-parameter group of dilations $(\delta_t)_t$. For any $\varepsilon > 0$ and any compact subset F of N, there is a constant $s_2 > 0$ such that*

$$|x^{-1}gx| \leq \varepsilon|x|$$

for all $g \in F$ and as soon as $|x| > s_2$.

Proof. Recall, as in the proof of the last lemma, that for any $c_1 > 0$ there is a $c_2 > 0$ such that if $t > 1$ and $x, y \in N$ are such that $|x|, |y| \leq c_1 t$, then $|\delta_{\frac{1}{t}}(xy) - \delta_{\frac{1}{t}}(x) * \delta_{\frac{1}{t}}(y)| \leq c_2 t^{-1/r}$ where $*$ is the product in the graded Lie group N_∞ associated to $(\delta_t)_t$. In particular, if we set $t = |x|$, then

$$\left|\delta_{\frac{1}{t}}(x^{-1}gx) - \delta_{\frac{1}{t}}(x)^{-1} * \delta_{\frac{1}{t}}(g) * \delta_{\frac{1}{t}}(x)\right| \leq c_2 t^{-1/r}$$

On the other hand, as g remains in the compact set F, $\delta_{\frac{1}{t}}(g)$ tends uniformly to the identity when $t = |x|$ goes to infinity, and $\delta_{\frac{1}{t}}(x)$ remains in a compact set. By continuity, we see that $\delta_{\frac{1}{t}}(x)^{-1} * \delta_{\frac{1}{t}}(g) * \delta_{\frac{1}{t}}(x)$ becomes arbitrarily small as t increases. We use again Proposition 3.4 (c) to conclude. $\square$

We are now ready to finish the proof of Lemmas 5.3 and 5.4, hence the proof of Proposition 5.2.

Proof of Lemma 5.3. Let $|\cdot|$ be a homogeneous quasi norm on G_N. Combining Propositions 4.4 and 4.5, there is a constant $C > 0$ such that for all $x, y \in G$, $\rho(x,y) \leq C|x^{\overline{*}-1}\overline{*}y| + C$. Now, applying Lemma 5.5, for every $\varepsilon > 0$ one can find a neighborhood U of the identity in K such that

$$\rho(x, k(x)) \leq C\varepsilon\rho(e,x) + C$$

as soon as $k \in U$ and $\rho(e,x)$ is large enough. Writing $\rho(e,k(x)) \leq \rho(e,x) + \rho(x,k(x))$ and letting x tend to infinity, we obtain the desired upper bound. For the lower bound we replace k by k^{-1}. $\square$

Proof of Lemma 5.4. Let $|\cdot|$ be a homogeneous quasi norm on G_N. Recall that $hx = h\overline{*}T(h)x$ for all $x, h \in G$. From Propositions 4.4 and 4.5, there is a constant $C > 0$ such that, if $h, y \in G$,

$$|\rho(e,y) - \rho(e,h\overline{*}y)| \leq \rho(y, h\overline{*}y) \leq C \cdot |y^{\overline{*}-1}\overline{*}h\overline{*}y| + C$$

where we have set $y = T(h)x$. By Remark 5.8 below we have $\frac{1}{C}\rho(e,x) - C \leq |y| \leq C \cdot \rho(e,x) + C$ (up to taking a larger C if necessary). Now Lemma 5.6, applied to G_N, yields, for any $\varepsilon > 0$ and any compact subset F of G a constant $s_2 > 0$ such that

$$|\rho(e,y) - \rho(e,h\overline{*}y)| \leq C \cdot \varepsilon \cdot |y| + C$$

as soon as $|y| > s_2$ and $h \in F$, hence as soon as $\rho(e,x) \geq C(s_2 + C)$. We are done. $\square$

5.3. **Proof of Proposition 5.1.** First we prove the following continuity statement:

Lemma 5.7. *Let ρ be a periodic pseudodistance on G and $\varepsilon > 0$. Then there exists a neighborhood of the identity U in G and $s_3 > 0$ such that*

$$1 - \varepsilon \leq \frac{\rho(e,gx)}{\rho(e,x)} \leq 1 + \varepsilon$$

as soon $g \in U$ and $\rho(e,x) > s_3$.

Proof. As in the proof of Lemma 5.4, we write

$$|\rho(e,x) - \rho(e,gx)| \leq \rho(x,gx) \leq C \cdot |x^{\overline{*}-1}\overline{*}gx| + C$$

However $gx = g\overline{*}T(g)x = (g\overline{*}x)\overline{*}(x^{\overline{*}-1}\overline{*}T(g)x)$. Hence

$$|\rho(e,x) - \rho(e,gx)| \leq C \cdot |x^{\overline{*}-1}\overline{*}g\overline{*}x| + C \cdot |x^{\overline{*}-1}\overline{*}T(g)x| + C$$

for some $C > 0$ (maybe larger, see Proposition 3.4). To complete the proof, we apply Lemmas 5.5 and 5.6 to the right hand side above. $\square$

We proceed with the proof of Proposition 5.1. Let L be the set of all $g \in G$ such that $\rho(e, gx)/\rho(e, x)$ tends to 1 as x tends to infinity in G. Clearly L is a subgroup of G. Lemma 5.7 shows that L is closed. The H-invariance of ρ insures that L contains H. Moreover, Proposition 5.2 implies that L is invariant under K. Consequently L contains H_K, the closed subgroup generated by all $k(h)$, $k \in K$, $h \in H$. This, together with Proposition 5.2, grants pointwise convergence of the integrand in (27). Convergence of the integral follows by applying Lebesgue's dominated convergence theorem. Remark 5.8 below shows that it is legitimate to do so.

The fact that ρ_K is invariant under left multiplication by H and invariant under precomposition by automorphisms from K insures that ρ_K is invariant under $\overline{*}$-left multiplication by any element $h \in H$, where $\overline{*}$ is the multiplication in the nilshadow G_N. Moreover we check that $T(g) \in K$ if $g \in H_K$, hence H_K is a *subgroup* of G_N. It is clearly co-compact in G_N too (if F is compact and $HF = G$ then $H \overline{*} F_K = G$ where F_K is the union of all $k(F)$, $k \in K$).

Clearly ρ_K is proper and locally bounded, so in order to finish the proof, we need only to check that ρ_K is asymptotically geodesic. By H-invariance of ρ_K and since H is co-compact in G, it is enough to exhibit a pseudogeodesic between e and a point $x \in H$. Let $x = z_1 \cdot \ldots \cdot z_n$ with $z_i \in H$ and $\sum \rho(e, z_i) \leq (1+\varepsilon) \cdot \rho(e, x)$. Fix a compact fundamental domain F for H in H_K so that integration in (27) over $H \backslash H_K$ is replaced by integration over F. Then for some constant $C_F > 0$ we have $|\rho(g, gz) - \rho(e, gz)| \leq C_F$ for $g \in F$ and $z \in H$. Moreover, it follows from Proposition 5.2, Lemma 5.7 and the fact that $H_K \subset L$, that

$$\rho(e, gk(z)) \leq (1+\varepsilon) \cdot \rho(e, z) \tag{31}$$

for all $g \in F$, $k \in K$ and as soon as $z \in G$ is large enough. Fix s large enough so that $C_F \leq \varepsilon s$ and so that (31) holds when $\rho(e, z) \geq s$. As already observed in the discussion following Definition 4.1 (property 4.2 (3)) we may take the z_i's so that $\frac{s}{2} \leq \rho(e, z_i) \leq s$. Then $nC_F \leq ns\varepsilon \leq 3\varepsilon\rho(e, x)$. Finally we get for $\varepsilon < 1$ and x large enough

$$\begin{aligned}
\sum \rho_K(e, z_i) &\leq C_F n + (1+\varepsilon)^2 \rho(e, x) \\
&\leq C_F n + (1+\varepsilon)^3 \rho_K(e, x) \\
&\leq (1+10\varepsilon) \cdot \rho_K(e, x)
\end{aligned}$$

where we have used the convergence $\rho_K/\rho \to 1$ that we just proved. $\square$

Remark 5.8. *If F is a compact subset of G, there is a constant $C > 0$ such that*

$$\frac{1}{C} \leq \frac{\rho(e, gk(x))}{\rho(e, x)} \leq C$$

for all $k \in K$, g in F and x large enough. This is a direct consequence of Proposition 4.4 and the fact that there exists a left invariant Riemannian metric on G which is also invariant under K.

6. The nilpotent case

In this section, we prove Theorem 1.3 and Corollary 1.5 for a simply connected nilpotent Lie group. Throughout the section, this Lie group will be denoted by N, and its Lie algebra by $\mathfrak{n}$.

Let m_1 be any vector subspace of $\mathfrak{n}$ such that $\mathfrak{n} = m_1 \oplus [\mathfrak{n}, \mathfrak{n}]$. Let π_1 the associated linear projection of $\mathfrak{n}$ onto m_1. Let H be a closed co-compact subgroup of N. To every H-periodic metric ρ on N we associate a norm $\|\cdot\|_0$ on m_1 which is the norm whose unit ball is defined to be the closed convex hull of all elements $\pi_1(h)/\rho(e,h)$ for all $h \in H \backslash \{e\}$. In other words,

$$E := \{x \in m_1, \|x\|_0 \leq 1\} = \overline{CvxHull} \left\{ \frac{\pi_1(h)}{\rho(e,h)}, h \in H \backslash \{e\} \right\} \tag{32}$$

The set E is clearly a convex subset of m_1 which is symmetric around 0 (since ρ is symmetric). To check that E is indeed the unit ball of a norm on m_1 it remains to see that E is bounded and that 0 lies in its interior. The first fact follows immediately from (21) and Example 3.3. If 0 does not lie in the interior of E, then E must be contained in a proper subspace of m_1, contradicting the fact that H is co-compact in N. It is easy to see that the following holds:

Proposition 6.1. *For $s > 0$ let E_s be the closed convex hull of all $\pi_1(x)/\rho(e,x)$ with $x \in N$ and $\rho(e,x) > s$. Then $E = \bigcap_{s>0} E_s$.*

Proof. Since ρ is H-periodic, we have $\rho(e,h^n) \leq n\rho(e,h)$ for all $n \in \mathbb{N}$ and $h \in H$. This shows $E \subset \bigcap_{s>0} E_s$. The opposite inclusion follows easily from the fact that ρ is at a bounded distance from its restriction to H, i.e. from 4.2 (1). ☐

We now choose a set of supplementary subspaces (m_i) starting with m_1 as in Paragraph 3.2. This defines a new Lie product $*$ on N so that $N_\infty = (N, *)$ is graded. We can then consider the $*$-left invariant Carnot-Carathéodory metric associated to the norm $\|\cdot\|_0$ as defined in Paragraph 3.1 on the graded nilpotent Lie group N_∞. The main theorem of this section reads:

Theorem 6.2. *Let ρ be a periodic metric on N and d_∞ the Carnot-Carathéodory metric defined above, then as $x^{-1}y$ tends to infinity in N*

$$\lim \frac{\rho(x,y)}{d_\infty(e, x^{-1}y)} = 1 \tag{33}$$

Note that d_∞ is left-invariant for the N_∞ Lie product, but not the original Lie product on N. Hence we cannot replace $d_\infty(e, x^{-1}y)$ above by $d_\infty(x,y)$. In fact easy examples show that in general there is no constant C such that $d_\infty(e, x^{-1}y) \leq Cd_\infty(x,y) + C$.

Before going further, let us draw some simple consequences.

(1) In Theorem 6.2 we may replace $d_\infty(e, x^{-1}y)$ by $d(x,y)$, where d is the left invariant Carnot-Caratheodory metric on N (rather than N_∞) defined by the norm $\|\cdot\|_0$ (as opposed to d_∞ which is $*$-left invariant). Hence ρ, d and d_∞ are asymptotic. This follows from the combination of Theorem 6.2 and Remark 3.1.

(2) Observe that the choice of m_1 was arbitrary. Hence two Carnot-Carathéodory metrics corresponding to two different choices of a supplementary subspace m_1 with the same induced norm on $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$, are asymptotically equivalent (i.e. their ratio tends to 1), and in fact isometric (see Remark 3.1). Conversely, if two Carnot-Carathéodory metrics are associated to the same supplementary subspace m_1 and are asymptotically equivalent, they must be equal. This shows that the set of all possible norms on the quotient vector space $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ is in bijection with the set of all classes of asymptotic equivalence of Carnot-Carathéodory metrics on N_∞, and also in bijection with the set of Carnot-Caratheodory metrics on N_∞ up to isometry.

(3) As another consequence we see that if a locally bounded proper and asymptotically geodesic left-invariant pseudodistance on N is also homogeneous with respect to the 1-parameter group $(\delta_t)_t$ (i.e. $\rho(e,\delta_t x) = t\rho(e,x)$) then it has to be of the form $\rho(x,y) = d_\infty(e,x^{-1}y)$ where d_∞ is a Carnot-Carathéodory metric on N_∞.

6.1. **Volume asymptotics.** Theorem 6.2 also yields a formula for the asymptotic volume of ρ-balls of large radius. Let us fix a Haar measure on N (for example Lebesgue measure on $\mathfrak{n}$ gives rise to a Haar measure on N under exp). Since d_∞ is homogeneous, it straightforward to compute the volume of a d_∞-ball:

$$vol(\{x \in N, d_\infty(e,x) \leq t\}) = t^{d(N)} vol(\{x \in N, d_\infty(e,x) \leq 1\})$$

where $d(N) = \sum_{i\geq 1} \dim(C^i(\mathfrak{n}))$ is the *homogeneous dimension* of N. For a pseudodistance ρ as in the statement of Theorem 6.2, we can define the *asymptotic volume of* ρ to be the volume of the unit ball for the associated Carnot-Carathéodory metric d_∞.

$$AsVol(\rho) = vol(\{x \in N, d_\infty(e,x) \leq 1\})$$

Then we obtain as an immediate corollary of Theorem 6.2:

Corollary 6.3. *Let ρ be periodic metric on N. Then*

$$\lim_{t\to+\infty} \frac{1}{t^{d(N)}} vol(\{x \in N, \rho(e,x) \leq t\}) = AsVol(\rho) > 0$$

Finally, if Γ is an arbitrary finitely generated nilpotent group, we need to take case of the torsion elements. They form a normal finite subgroup T and applying Theorem 6.2 to Γ/T, we obtain:

Corollary 6.4. *Let S be a finite symmetric generating set of Γ and S^n the ball of radius n is the word pseudodistance ρ_S associated to S, then*

$$\lim_{n\to+\infty} \frac{1}{n^{d(N)}} \#S^n = \#T \cdot \frac{AsVol(\rho_{\overline{S}})}{vol(N/\overline{\Gamma})} > 0$$

where N is the Malcev closure of $\overline{\Gamma} = \Gamma/T$, the torsion free quotient of Γ, and $d_{\overline{S}}$ is the word pseudodistance associated to $\overline{S}$, the projection of S in $\overline{\Gamma}$.

Moreover, it is possible to be a bit more precise about $AsVol(\rho_{\overline{S}})$. In fact, the norm $\|\cdot\|_0$ on m_1 used to define the limit Carnot-Carathéodory distance d_∞ associated to $\rho_{\overline{S}}$ is a simple polyhedral norm defined by

$$\{\|x\|_0 \leq 1\} = CvxHull\,(\pi_1(\overline{s}), s \in S)$$

More generally the following holds. Let H be any closed, co-compact subgroup of N. Choose a Haar measure on H so that $vol_N(N/H) = 1$. Theorem 6.2 yields:

Corollary 6.5. *Let Ω be a compact symmetric (i.e. $\Omega = \Omega^{-1}$) neighborhood of the identity, which generates H. Let $\|\cdot\|_0$ be the norm on m_1 whose unit ball is $\overline{CvxHull}\{\pi_1(\Omega)\}$ and let d_∞ be the corresponding Carnot-Carathéodory metric on N_∞. Then we have the following limit in the Hausdorff topology*

$$\lim_{n\to+\infty} \delta_{\frac{1}{n}}(\Omega^n) = \{g \in N, d_\infty(e,g) \leq 1\}$$

and

$$\lim_{n\to+\infty} \frac{vol_H(\Omega^n)}{n^{d(N)}} = vol_N\left(\{g \in N, d_\infty(e,g) \leq 1\}\right)$$

Remark 6.6. *We can remove the hypothesis that Ω (or S in 6.4) is symmetric, and the analogous result holds, but then $\|\cdot\|_0$ is a convex gauge instead of a norm, and d_∞ is no longer symmetric. We made this assumption to keep things simple, but one can check that the proof below remains valid for "non-symmetric metrics" as well.*

6.2. **Outline of the proof.** We first devise some standard lemmas about piecewise approximations of horizontal paths (Lemmas 6.7, 6.8, 6.11). Then it is shown (Lemma 6.12) that the original product on N and the product in the associated graded Lie group are asymptotic to each other, namely, if $(\delta_t)_t$ is a 1-parameter group of dilations of N, then after renormalization by $\delta_{\frac{1}{t}}$, the product of $O(t)$ elements lying in some bounded subset of N, is very close to the renormalized product of the same elements in the graded Lie group N_∞. This is why all complications due to the fact that N may not be *a priori* graded and the δ_t's may not be automorphisms disappear when looking at the large scale geometry of the group. Finally, we observe (Lemma 6.15), as follows from the very definition of the unit ball E for the limit norm $\|\cdot\|_0$, that any vector in the boundary of E, can be approximated, after renormalizing by $\delta_{\frac{1}{s}}$ by some element $x \in N$ lying in a fixed annulus $s(1-\varepsilon) \leq \rho(e,x) \leq s(1+\varepsilon)$. This enables to assert that any ρ-quasi geodesic gives rise, after renormalization to a d_∞-geodesic (lower bound). And vice-versa, that any d_∞-geodesic can be approximated uniformly by some renormalized ρ-quasi geodesic (upper bound).

6.3. **Preliminary lemmas.**

Lemma 6.7. *Let G be a Lie group and let $\|\cdot\|_e$ be a Euclidean norm on the Lie algebra of G and $d_e(\cdot,\cdot)$ the associated left invariant Riemannian metric on G. Then there is a constant $C_0 = C_0(d_e) > 0$ such that whenever $d_e(e,u) \leq 1$ and $x, y \in G$*

$$|d_e(xu, yu) - d_e(x,y)| \leq C_0 d_e(x,y) d_e(e,u)$$

Proof. The proof reduces to the case when u and $x^{-1}y$ are in a small neighborhood of e. Then the inequality boils down to the following $\|[X,Y]\|_e \leq c\|X\|_e \|Y\|_e$ for some $c > 0$ and every X, Y in $Lie(G)$. □

Lemma 6.8. *Let G be a Lie group, let $\|\cdot\|$ be some norm on the Lie algebra of G and let $d_e(\cdot,\cdot)$ be a left invariant Riemannian metric on G. Then for every $L > 0$ there is a constant $C = C(d_e, \|\cdot\|, L) > 0$ with the following property. Assume $\xi_1, \xi_2 : [0,1] \to G$ are two piecewise smooth paths in the Lie group G with $\xi_1(0) = \xi_2(0) = e$. Let $\xi_i' \in Lie(G)$ be the tangent vector pulled back at the identity by a left translation of G. Assume that $\sup_{t\in[0,1]} \|\xi_1'(t)\| \leq L$, and that $\int_0^1 \|\xi_1'(t) - \xi_2'(t)\| dt \leq \varepsilon$. Then*

$$d_e(\xi_1(1), \xi_2(1)) \leq C\varepsilon$$

Proof. The function $f(t) = d_e(\xi_1(t), \xi_2(t))$ is piecewise smooth. For small dt we may write, using Lemma 6.7

$$\begin{aligned} f(t+dt) - f(t) &\leq d(\xi_1(t)\xi_1'(t)dt, \xi_1(t)\xi_2'(t)dt) + d(\xi_1(t)\xi_2'(t)dt, \xi_2(t)\xi_2'(t)dt) - f(t) + o(dt) \\ &\leq \left\|\xi_1'(t) - \xi_2'(t)\right\|_e dt + C_0 f(t) \left\|\xi_2'(t)dt\right\|_e + o(dt) \\ &\leq \varepsilon(t)dt + Lf(t)dt + o(dt) \end{aligned}$$

where $\varepsilon(t) = \|\xi_1'(t) - \xi_2'(t)\|_e$.

$$f'(t) \leq \varepsilon(t) + C_0 L f(t)$$

Since $f(0) = 0$, Gronwall's lemma implies that $f(1) \leq e^{C_0 L} \int_0^1 \varepsilon(s) e^{-C_0 L s} ds \leq C\varepsilon$. □

From now on, we will take G to be the graded group N_∞, and $d_e(\cdot,\cdot)$ will denote a left invariant Riemannian metric on N_∞ while $d_\infty(\cdot,\cdot)$ is a left invariant Carnot-Caratheodory Finsler metric on N_∞ associated to some norm $\|\cdot\|$ on m_1.

Remark 6.9. *There is $c_0 > 0$ such that $c_0^{-1} d_e(e,x) \leq d_\infty(e,x) \leq c_0 d_e(e,x)^{\frac{1}{r}}$ in a neighborhood of e. Hence in the situation of the lemma we get $d_\infty(\xi_1(1), \xi_2(1)) \leq C_1 \varepsilon^{\frac{1}{r}}$ for some other constant $C_1 = C_1(L, d_\infty, d_e)$.*

Lemma 6.10. *Let $N \in \mathbb{N}$ and $d_N(x,y)$ be the function in N_∞ defined in the following way:*

$$d_N(x,y) = \inf\{\int_0^1 \left\|\xi'(u)\right\| du, \xi \in \mathcal{H}_{PL(N)},\ \xi(0) = x, \xi(1) = y\}$$

where $\mathcal{H}_{PL(N)}$ is the set of horizontal paths ξ which are piecewise linear with at most N possible values for ξ'. Then we have $d_N \to d_\infty$ uniformly on compact subsets of N_∞.

Proof. Note that it follows from Chow's theorem that there exists $K_0 \in \mathbb{N}$ such that $A := \sup_{d_\infty(e,x)=1} d_{K_0}(e,x) < \infty$. Moreover, since PL-paths are dense in L^1, it follows for example from Lemma 6.8 that for each fixed x, $d_n(e,x) \to d_\infty(e,x)$. We need to show that $d_N(e,x) \to d_\infty(e,x)$ uniformly in x satisfying $d_\infty(e,x) = 1$. By contradiction, suppose there is a sequence $(x_n)_n$ such that $d_\infty(e,x_n) = 1$ and $d_n(e,x_n) \geq 1 + \varepsilon_0$ for some $\varepsilon_0 > 0$. We may assume that $(x_n)_n$ converges to say x.

Let $y_n = x^{-1} * x_n$ and $t_n = d_\infty(e, y_n)$. Then $d_{K_0}(e, y_n) = t_n d_{K_0}(e, \delta_{\frac{1}{t_n}}(y_n)) \leq A t_n$. Thus $d_n(e, x_n) \leq d_n(e, x) + d_n(e, y_n) \leq d_n(e, x) + A t_n$ as soon as $n \geq K_0$. As n tends to ∞, we get a contradiction. □

This lemma prompts the following notation. For $\varepsilon > 0$, we let $N_\varepsilon \in \mathbb{N}$ be the first integer such that $1 \leq d_{N_\varepsilon}(e, x) \leq 1 + \varepsilon$ for all x with $d_\infty(e, x) = 1$. Then we have:

Lemma 6.11. *For every $x \in N_\infty$ with $d_\infty(e, x) = 1$, and all $\varepsilon > 0$ there exists a path $\xi : [0, 1] \to N_\infty$ in $\mathcal{H}_{PL(N_\varepsilon)}$ with unit speed (i.e. $\|\xi'\| = 1$) such that $\xi(0) = e$ and $d_\infty(x, \xi(1)) \leq C_2 \varepsilon$ and ξ' has at most one discontinuity on any subinterval of $[0, 1]$ of length $\varepsilon^r / N_\varepsilon$.*

Proof. We know that there is a path in $\mathcal{H}_{PL(N_\varepsilon)}$ connecting e to x with length $\ell \leq 1 + \varepsilon$. Reparametrizing the path so that it has unit speed, we get a path $\xi_0 : [0, \ell] \to N_\infty$ in $\mathcal{H}_{PL(N_\varepsilon)}$ with $d_\infty(x, \xi_0(1)) = d_\infty(\xi_0(\ell), \xi_0(1)) \leq \varepsilon$. The derivative ξ_0' is constant on at most N_ε different intervals say $[u_i, u_{i+1})$. Let us remove all such intervals of length $\leq \varepsilon^r / N_\varepsilon$ by merging them to an adjacent interval and let us change the value of ξ_0' on these intervals to the value on the adjacent interval (it doesn't matter if we choose the interval on the left or on the right). We obtain a new path $\xi : [0, 1] \to N_\infty$ in $\mathcal{H}_{PL(N_\varepsilon)}$ with unit speed and such that ξ' has at most one discontinuity on any subinterval of $[0, 1]$ of length $\varepsilon^r / N_\varepsilon$. Moreover $\int_0^1 \|\xi'(t) - \xi_0'(t)\| \, dt \leq \varepsilon^r$. By Lemma 6.8 and Remark 6.3, we have $d_\infty(\xi(1), \xi_0(1)) \leq C_1 \varepsilon$, hence

$$d_\infty(\xi(1), x) \leq d_\infty(x, \xi_0(1)) + d_\infty(\xi_0(1), \xi(1)) \leq (C_1 + 1)\varepsilon$$

□

Lemma 6.12. *Let $x * y$ denote the product inside the graded Lie group N_∞ and $x \cdot y$ the ordinary product in N. Let $n \in \mathbb{N}$ and $t \geq n$. Then for any compact subset K of N, and any $x_1, ..., x_n$ elements of K, we have*

$$d_e(\delta_{\frac{1}{t}}(x_1 \cdot ... \cdot x_n), \delta_{\frac{1}{t}}(x_1 * ... * x_n)) \leq c_1 \frac{1}{t}$$

and

$$d_e(\delta_{\frac{1}{t}}(x_1 * ... * x_n), \delta_{\frac{1}{t}}(\pi_1(x_1) * ... * \pi_1(x_n))) \leq c_2 \frac{1}{t}$$

where c_1, c_2 depend on K and d_e only.

Proof. Let $\|\cdot\|$ be a norm on the Lie algebra of N. For $k = 1, ..., n$ let $z_k = x_1 \cdot ... \cdot x_{k-1}$ and $y_k = x_{k+1} * ... * x_n$. Since all x_i's belong to K, it follows from (22) that as soon as $t \geq n$, all $\delta_{\frac{1}{t}}(z_k)$ and $\delta_{\frac{1}{t}}(y_k)$ for $k = 1, ..., n$ remain in a bounded set depending only on K. Comparing (13) and (12), we see that whenever $y = O(1)$ and $\delta_{\frac{1}{t}}(x) = O(1)$, we have

$$\left\| \delta_{\frac{1}{t}}(xy) - \delta_{\frac{1}{t}}(x * y) \right\| = O(\frac{1}{t^2}) \tag{34}$$

On the other hand, from (13) it is easy to verify that right $*$-multiplication by a bounded element is Lipschitz for $\|\cdot\|$ and the Lipschitz constant is locally bounded.

It follows that there is a constant $C_1 > 0$ (depending only on K and $\|\cdot\|$) such that for all $k \leq n$

$$\left\|\delta_{\frac{1}{t}}((z_k \cdot x_k) * y_k) - \delta_{\frac{1}{t}}(z_k * x_k * y_k)\right\| \leq C_1 \left\|\delta_{\frac{1}{t}}(z_k \cdot x_k) - \delta_{\frac{1}{t}}(z_k * x_k)\right\|$$

Applying n times the relation (34) with $x = x_1 \cdot ... \cdot x_{k-1}$ and $y = x_k$, we finally obtain

$$\left\|\delta_{\frac{1}{t}}(x_1 \cdot ... \cdot x_n) - \delta_{\frac{1}{t}}(x_1 * ... * x_n)\right\| = O(\frac{n}{t^2}) = O(\frac{1}{t})$$

where $O()$ depends only on K. On the other hand, using (12), it is another simple verification to check that if x, y lie in a bounded set, then $\frac{1}{c_2} d_e(x,y) \leq \|x - y\| \leq c_2 d_e(x,y)$ for some constant $c_2 > 0$. The first inequality follows.

For the second inequality, we apply Lemma 6.8 to the paths ξ_1 and ξ_2 starting at e and with derivative equal on $[\frac{k}{n}, \frac{k+1}{n})$ to $n\delta_{\frac{1}{t}}(x_k)$ for ξ_1 and to $n\frac{\pi_1(x_k)}{t}$ for ξ_2. We get

$$d_e(\delta_{\frac{1}{t}}(x_1 * ... * x_n), \delta_{\frac{1}{t}}(\pi_1(x_1) * ... * \pi_1(x_n)) = O(\frac{1}{t}).$$

$\square$

Remark 6.13. *From Remark 6.3 we see that if we replace d_e by d_∞ in the above lemma, we get the same result with $\frac{1}{t}$ replaced by $t^{-\frac{1}{r}}$.*

Remark 6.14. *The dependence of the constants on K can easily be made explicit. One sees that if the $d_\infty(e, x_i)$'s are $O(s)$, then c_1 and c_2 are $O(s^{O(1)})$.*

Lemma 6.15. *Recall that $\|\cdot\|_0$ is the norm on m_1 defined in (32). For any $\varepsilon > 0$, there exists $s_0 > 0$ such that for every $s > s_0$ and every $v \in m_1$ such that $\|v\|_0 = 1$, there exists $h \in H$ such that*

$$(1 - \varepsilon)s \leq \rho(e,h) \leq (1 + \varepsilon)s$$

and

$$\left\|\frac{\pi_1(h)}{\rho(e,h)} - v\right\|_0 \leq \varepsilon$$

Proof. Let $\varepsilon > 0$ be fixed. Considering a finite ε-net in E, we see that there exists a finite symmetric subset $\{g_1, ..., g_p\}$ of $H \backslash \{e\}$ such that, if we consider the closed convex hull of $\mathfrak{F} = \{f_i = \pi_1(g_i)/\rho(e, g_i) | i = 1, ..., p\}$ and $\|\cdot\|_\varepsilon$ the associated norm on m_1, then $\|\cdot\|_0 \leq \|\cdot\|_\varepsilon \leq (1 + 2\varepsilon) \|\cdot\|_0$. Up to shrinking $\mathfrak{F}$ if necessary, we may assume that $\|f_i\|_\varepsilon = 1$ for all i's. We may also assume that the f_i's generate m_1 as a vector space. The sphere $\{x, \|x\|_\varepsilon = 1\}$ is a symmetric polyhedron in m_1 and to each of its facets corresponds $d = \dim(m_1)$ vertices lying in $\mathfrak{F}$ and forming a vector basis of m_1. Let $f_1, ..., f_d$, say, be such vertices for a given facet. If $x \in m_1$ is of the form $x = \sum_{i=1}^d \lambda_i f_i$ with $\lambda_i \geq 0$ for $1 \leq i \leq d$ then we see that $\|x\|_\varepsilon = \sum_{i=1}^d \lambda_i$, because the convex hull of $f_1, ..., f_d$ is precisely that facet, hence lies on the sphere $\{x, \|x\|_\varepsilon = 1\}$.

Now let $v \in m_1$, $\|v\|_0 = 1$, and let $s > 0$. The half line tv, $t > 0$, hits the sphere $\{x, \|x\|_\varepsilon = 1\}$ in one point. This point belongs to some facet and there are d linearly independent elements of $\mathfrak{F}$, say $f_1, ..., f_d$, the vertices of that facet, such that this point belongs to the convex hull of $f_1, ..., f_d$. The point sv then lies in the convex cone generated by $\pi_1(g_1), ..., \pi_1(g_d)$. Moreover, there is a constant $C_\varepsilon > 0$ ($C_\varepsilon \leq \frac{d}{2} \max_{1\leq i\leq p} \rho(e, g_i)$) such that

$$\left\| sv - \sum_{i=1}^{d} n_i \pi_1(g_i) \right\|_\varepsilon \leq C_\varepsilon$$

for some non-negative integers $n_1, ..., n_d$ depending on $s > 0$. Hence

$$\begin{aligned} \frac{1}{s} \sum_{i=1}^{d} n_i \rho(e, g_i) &= \frac{1}{s} \left\| \sum_{i=1}^{d} n_i \pi_1(g_i) \right\|_\varepsilon \leq \frac{1}{s} (\|sv\|_\varepsilon + C_\varepsilon) \\ &\leq 1 + 2\varepsilon + \frac{C_\varepsilon}{s} \leq 1 + 3\varepsilon \end{aligned}$$

where the last inequality holds as soon as $s > C_\varepsilon/\varepsilon$.

Now let $h = g_1^{n_1} \cdot ... \cdot g_d^{n_d} \in H$. We have $\pi_1(h) = \sum_{i=1}^{d} n_i \pi_1(g_i)$

$$\rho(e, h) \geq \|\pi_1(h)\|_0 \geq s - C_\varepsilon \geq s(1 - \varepsilon)$$

Moreover

$$\rho(e, h) \leq \sum_{i=1}^{d} n_i \rho(e, g_i) \leq s(1 + 3\varepsilon)$$

Changing ε into say $\frac{\varepsilon}{5}$ and for say $\varepsilon < \frac{1}{2}$, we get the desired result with $s_0(\varepsilon) = \frac{d}{\varepsilon} \max_{1\leq i\leq p} \rho(e, g_i)$. $\square$

6.4. **Proof of Theorem 6.2.** We need to show that as $x \to \infty$ in N

$$1 \leq \underline{\lim} \frac{\rho(e, x)}{d_\infty(e, x)} \leq \overline{\lim} \frac{\rho(e, x)}{d_\infty(e, x)} \leq 1$$

First note that it is enough to prove the bounds for $x \in H$. This follows from (4.2) (1).

Let us begin with the lower bound. We fix $\varepsilon > 0$ and $s = s(\varepsilon)$ as in the definition of an asymptotically geodesic metric (see (19)). We know by 4.2 (3) and (4) that as soon as $\rho(e, x) \geq s$ we may find $x_1, ..., x_n$ in H with $s \leq \rho(e, x_i) \leq 2s$ such that $x = \prod x_i$ and $\sum \rho(e, x_i) \leq (1 + \varepsilon)\rho(e, x)$. Let $t = d_\infty(e, x)$, then $n \leq \frac{1+\varepsilon}{s} \rho(e, x)$, hence $n \leq \frac{C}{s(\varepsilon)} t$ where C is a constant depending only on ρ (see (21)). We may then apply Lemma 6.12 (and the remark following it) to get, as $t \geq n$ as soon as $s(\varepsilon) \geq C$,

$$d_\infty(\delta_{\frac{1}{t}}(x), \delta_{\frac{1}{t}}(\pi_1(x_1) * ... * \pi_1(x_n))) \leq c_1' t^{-\frac{1}{r}}$$

But for each i we have $\|\pi_1(x_i)\|_0 \leq \rho(e, x_i)$ by definition of the norm, hence

$$t = d_\infty(e, x) \leq \sum \|\pi_1(x_i)\|_0 + d_\infty(x, \pi_1(x_1) * ... * \pi_1(x_n)) \leq (1 + \varepsilon)\rho(e, x) + c_1' t^{1-\frac{1}{r}}$$

Since ε was arbitrary, letting $t \to \infty$ we obtain

$$\underline{\lim} \frac{\rho(e, x)}{d_\infty(e, x)} \geq 1$$

We now turn to the upper bound. Let $t = d_\infty(e,x)$ and $\varepsilon > 0$. According to Lemma 6.11, there is a horizontal piecewise linear path $\{\xi(u)\}_{u\in[0,1]}$ with unit speed such that $d_\infty(\delta_{\frac{1}{t}}(x),\xi(1)) \leq C_2\varepsilon$ and no interval of length $\geq \frac{\varepsilon^r}{N_\varepsilon}$contains more than one change of direction. Let $s_0(\varepsilon)$ be given by Lemma 6.15 and assume $t > s_0(\varepsilon^r)N_\varepsilon/\varepsilon^r$. We split $[0,1]$ into n subintervals of length $u_1,...,u_n$ such that ξ' is constant equal to y_i on the i-th subinterval and $s_0(\varepsilon^r) \leq tu_i \leq 2s_0(\varepsilon^r)$. We have $\xi(1) = u_1y_1 * ... * u_ny_n$. Lemma 6.15 yields points $x_i \in H$ such that

$$\left\| y_i - \frac{\pi_1(x_i)}{tu_i} \right\| \leq \varepsilon^r$$

and $\rho(e,x_i) \in [(1-\varepsilon^r)tu_i,(1+\varepsilon^r)tu_i]$ (note that $tu_i > s_0(\varepsilon^r)$). Let $\overline{\xi}$ be the piecewise linear path $[0,1] \to N_\infty$ with the same discontinuities as ξ and where the value y_i is replaced by $\frac{\pi_1(x_i)}{tu_i}$. Then according to Lemma 6.8, $d_\infty(\xi(1),\overline{\xi}(1)) \leq C\varepsilon$. Since $\rho(e,x_i) \leq 4s_0(\varepsilon^r)$ for each i, we may apply Lemma 6.12 (and the remark following it) and see that if $y = x_1 \cdot ... \cdot x_n$,

$$d_\infty(\overline{\xi}(1),\delta_{\frac{1}{t}}(y)) \leq c_1'(\varepsilon)t^{-\frac{1}{r}}$$

Hence $d_\infty(\delta_{\frac{1}{t}}(x),\delta_{\frac{1}{t}}(y)) \leq (C_2+C)\varepsilon + c_1'(\varepsilon)t^{-\frac{1}{r}}$ and $\rho(e,y) \leq \sum \rho(e,x_i) \leq (1+\varepsilon^r)t$ while $\rho(x,y) \leq C'td_\infty(e,\delta_{\frac{1}{t}}(x^{-1}y)) + C' \leq t(Cd_\infty(\delta_{\frac{1}{t}}(x),\delta_{\frac{1}{t}}(y)) + o_\varepsilon(1))$. Hence

$$\rho(e,x) \leq t + o_\varepsilon(t)$$

□

Remark 6.16. *In the last argument we used the fact that* $\left\|\delta_{\frac{1}{t}}(xu) - \delta_{\frac{1}{t}}(x*u)\right\| = O(\frac{1}{t^{\frac{1}{r}}})$ *if* $\delta_{\frac{1}{t}}(x)$ *and* $\delta_{\frac{1}{t}}(u)$ *are bounded, in order to get for* $y = xu$,

$$\begin{aligned} d_\infty(e,\delta_{\frac{1}{t}}(u)) &\leq d_\infty(\delta_{\frac{1}{t}}(x),\delta_{\frac{1}{t}}(xu)) + d_\infty(\delta_{\frac{1}{t}}(xu),\delta_{\frac{1}{t}}(x*u)) \\ &\leq d_\infty(\delta_{\frac{1}{t}}(x),\delta_{\frac{1}{t}}(y)) + o(1). \end{aligned}$$

7. Locally compact G

In this section, we prove Theorem 1.2 and complete the proof of Corollary 1.5.

7.1. **Proof of Theorem 1.2.** Let G be a locally compact group of polynomial growth. It follows from Losert's refinement of Gromov's theorem ([17] Theorem 2) that there exists a normal compact subgroup K of G such that G/K is a Lie group. So we may now assume that G is a Lie group (not necessarily connected) of polynomial growth. The connected component G_0 of G is a connected Lie group of polynomial growth, hence is of the form $G_0 = MQ$ where Q is the solvable radical of G_0 and M is a compact Levi subgroup. The nilradical N (i.e. maximal connected normal nilpotent closed subgroup of Q) is characteristic in Q, and so is the compact part C of the center of N. Since Q is a characteristic subgroup of G_0, and G_0 a characteristic subgroup of G, it follows that C is also characteristic in G. Moding out by C, we may then assume that C is trivial, so that N is simply connected.

Let us consider the action of G by conjugation on $G_0/Q \simeq M$. So we get a homomorphism $\phi : G \to Aut(M)$. Now, since M is a compact semi-simple Lie group, $Aut(M)$ is a compact group. The kernel $\ker\phi$ is a co-compact subgroup of G such that $(G_0 \cap \ker\phi)/Q$ identifies with the center of M, hence is finite. Hence the identity connected component of $\ker\phi$ equals Q. Considering $\ker\phi$ instead of G, we may then assume that G is a Lie group of polynomial growth with $G_0 = Q$ and N simply connected. We are going to show that some co-compact subgroup of G can be embedded co-compactly in a simply connected solvable group. Since G is compactly generated, G/G_0 is finitely generated and has polynomial growth. Hence by Gromov's theorem, up to taking a subgroup of finite index, we may assume that G/G_0 is nilpotent. Since every finitely generated nilpotent group has a torsion free subgroup of finite index, we may even assume that G/G_0 is torsion free.

Since any subgroup of a finitely generated nilpotent group is finitely generated, and any closed subgroup of a connected solvable Lie group is compactly generated, it follows that any closed subgroup of G is compactly generated. It is easy to see that any solvable Lie group with this property admits a unique maximal normal closed nilpotent subgroup G_N, that we call the nilpotent radical of G. Moreover, it is clear that the identity connected component of G_N is exactly N. Arguing as in [21] 4.10, we see that *in a connected solvable Lie group, the maximal closed nilpotent normal subgroup is connected*. Hence $N = G_N \cap G_0$. It follows that G_N/N is a subgroup of G/G_0, hence is torsion free. Since N is simply connected, we get that G_N itself is torsion free. Again, arguing exactly as in [21] 4.10 and 4.11, we see that G has a finite index subgroup G' whose commutator group $[G', G']$ is nilpotent, hence lies in $G_{N.}$ Thus G'/G_N is abelian (and compactly generated). We can therefore take a co-compact subgroup of G'' of G' containing G_N so that $G''/G_N \simeq \mathbb{Z}^n$. To conclude, we can now appeal to Wang's theorem [27]: *If $1 \to A \to B \to \mathbb{Z}^n \to 1$ is an exact sequence where A is a nilpotent Lie group with A_0 simply connected and A/A_0 finitely generated and torsion free, then B can be embedded in a co-compact manner in a solvable Lie group C with finitely many connected components, and such that C_0 is simply connected.* □

7.2. **Proof of Corollary 1.5.** Let G be an arbitrary locally compact group of polynomial growth and ρ a periodic pseudodistance on G.

Claim 1: *Corollary 1.5 holds for a co-compact subgroup H of G, if and only if it holds for G.* Indeed, let F be a bounded Borel fundamental domain for H inside G. And let $\overline{\rho}$ be the periodic metric on G induced by the restriction of ρ to H, that is $\overline{\rho}(x, y) := \rho(h_x, h_y)$ where h_x is the unique element of H such that $x \in h_xF$. By 4.2 (1) and (4), ρ and $\overline{\rho}$ are at a bounded distance from each other. In particular, $B_{\overline{\rho}}(r-C) \subset B_\rho(r) \subset B_{\overline{\rho}}(r+C)$. Hence if the limit (2) holds for $\overline{\rho}$, it also holds for ρ with the same limit. However, $B_{\overline{\rho}}(r) = \{x \in G, \rho(e, h_x) \leq r\} = B_{\rho_H}(r)F$ where ρ_H is the restriction of ρ to H. Hence $vol_G(B_{\overline{\rho}}(r)) = vol_H(B_{\rho_H}(r)) \cdot vol_G(F)$. By 4.2 (4), ρ_H is a periodic metric on H. So the result holds for (H, ρ_H) if and only if it holds for (G, ρ). Conversely, if ρ_0 is a periodic metric on H, then $\overline{\rho_0}(x, y) :=$

$\rho_0(h_x, h_y)$ is a periodic metric on G, hence again $vol_G(B_{\overline{\rho}_0}(r)) = vol_H(B_{\rho_0}(r)) \cdot vol_G(F)$ and the result will hold for (H, ρ_0) if and only if it holds for $(G, \overline{\rho_0})$.

Claim 2: *If Corollary 1.5 holds for G/K, where K is some compact normal subgroup, then it holds for G as well.* Indeed, if ρ is a periodic pseudodistance on G, then the K-average ρ^K, as defined in (24), is at a bounded distance from G according to Lemma 4.6. Now ρ^K induces a periodic pseudodistance $\overline{\rho^K}$ on G/K and $B_{\rho^K}(r) = B_{\overline{\rho^K}}(r)K$. Hence, $vol_G(B_{\rho^K}(r)) = vol_{G/K}(B_{\overline{\rho^K}}(r)) \cdot vol_K(K)$. And if the limit (2) holds for $\overline{\rho^K}$, it also holds for ρ^K, hence for ρ too.

Thus the discussion above combined with Theorem 1.2 reduces Corollary 1.5 to the case when G is simply connected and solvable, which was treated in Section 5 and 6. □

7.3. **Proof of Corollary 1.6.** According to Remark 3.1 and Theorem 1.3, the $*$-left invariant Carnot-Caratheodory metric d on S_N induced by the norm $\|\cdot\|$ associated to ρ by Theorem 1.3 is asymptotic to d_∞, the corresponding Carnot-Caratheodory metric on the graded nilshadow $S_{N,\delta}$, hence also asymptotic to ρ by Theorem 1.3. Since d is $*$-left invariant, ρ will be asymptotically invariant under $*$-left translations. □

7.4. **Proof of Corollary 1.7.** Let $\mathcal{U}$ be a ultrafilter and $(\varepsilon_n)_n$ some sequence of positive numbers going to 0. Let (Y, d) be the corresponding asymptotic cone of (G, ρ), i.e. (Y, d) is the $\mathcal{U}$-ultraproduct of the marked metric spaces $\{(G, \varepsilon_n\rho, id)\}_{n\geq 0}$. The space Y is the set of equivalence classes $Y = X/\tilde{}$, where X is the space of sequences $(x_n)_n$ in G such that $\varepsilon_n\rho(e, x_n)$ remains bounded, and two sequences $(x_n)_n$ and $(y_n)_n$ are said to be equivalent if $\lim_{\mathcal{U}} \varepsilon_n\rho(x_n, y_n) = 0$. For two points $x = (x_n)_n$ and $y = (y_n)_n$ in Y we have $d(x, y) = \lim_{\mathcal{U}} \varepsilon_n\rho(x_n, y_n)$. This makes (Y, d) into a metric space.

Let S be a Lie shadow of G and let d_∞ be the limit Carnot-Caratheodory metric on S obtained in Theorem 1.3 (which is left invariant for the graded nilshadow Lie structure on S). We want to show that (Y, d) is isometric to (S, d_∞). Recall that by Theorem 1.2, G has normal compact subgroup K and a co-compact subgroup H containing K such that H/K can be realized as a closed co-compact subgroup of S. Hence to every $s \in S$, we can associate a point $[s] \in H$ such that $[s]K \in sF$ where F is a fundamental domain for H/K in S. By Lemma 4.6, ρ induces a periodic metric ρ^K on H/K, which in turn induces a periodic metric on S by setting $\rho_S(x, y) = \rho^K(h_x, h_y)$, where $x \in h_xF, y \in h_yF$ and $h_x, h_y \in H/K$. It follows that $|\rho([x], [y]) - \rho_S(x, y)|$ is bounded independently of $x, y \in S$.

We now build an isometry between (S, d_∞) and (Y, d). Let $\phi : S \to Y$ send s to $(\delta_{\frac{1}{\varepsilon_n}}(s))_n$. Applying Theorem 1.3 we check successively that ϕ is well defined, surjective and preserves distances, i.e. $d_\infty(s_1, s_2) = d(\phi(s_1), \phi(s_2))$, hence is the desired isometry.

The statement about the uniqueness of the graded group follows from Pansu's theorem that two quasi-isometric graded nilpotent groups are isomorphic. □

8. Speed of convergence and coarsely geodesic distances

Under no further assumption on the periodic metric ρ, the speed of convergence in the volume asymptotics can be made arbitrarily small. This is easily seen if we consider examples of the following type: define $\rho(x,y) = |x-y| + |x-y|^\alpha$ on $\mathbb{R}$ where $\alpha \in (0,1)$. It is periodic and $vol(B_\rho(t)) = t - t^\alpha + o(t^\alpha)$.

However, many natural examples of periodic metrics, such as word metrics or Riemannian metrics, are in fact coarsely geodesic. A pseudodistance on G is said to be *coarsely geodesic*, if there is a constant $C > 0$ such that any two points can be connected by a C-coarse geodesic, that is, for any $x, y \in G$ there is a map $g : [0,t] \to G$ with $t = \rho(x,y)$, $g(0) = x$ and $g(t) = y$, such that

$$|\rho(g(u), g(v)) - |u - v|| \leq C$$

for all $u, v \in [0,t]$.

This is a stronger requirement than to say that ρ is asymptotically geodesic (see 19). This notion is invariant under coarse isometry. In the case when G is abelian, it was proved by D. Burago [4] that any coarsely geodesic periodic metric on G is at a bounded distance from its asymptotic norm. In particular $vol_G(B_\rho(t)) = c \cdot t^d + O(t^{d-1})$ in this case. In the remarkable paper [23], M. Stoll proved that such an error term in $O(t^{d-1})$ holds for any finitely generated 2-step nilpotent group. Whether $O(t^{d-1})$ is the right error term for any finitely generated nilpotent group remains an open question.

The example below shows on the contrary that in an arbitrary Lie group of polynomial growth no universal error term can be expected.

Theorem 8.1. *Let $\varepsilon_n > 0$ be an arbitrary sequence of positive numbers tending to* 0*. Then there exists a group G of polynomial growth of degree* 3 *and a compact generating set Ω in G and $c > 0$ such that*

$$\frac{vol_G(\Omega^n)}{c \cdot n^3} \leq 1 - \varepsilon_n \tag{35}$$

holds for infinitely many n, although $\frac{1}{c \cdot n^3} vol_G(\Omega^n) \to 1$ as $n \to +\infty$.

The example we give below is a semi-direct product of $\mathbb{Z}$ by $\mathbb{R}^2$ and the metric is a word metric. However, many similar examples can be constructed as soon as the map $T : G \to K$ defined in Paragraph 5.1 in not onto. For example, one can consider left invariant Riemannian metrics on $G = \mathbb{R} \cdot (\mathbb{R}^2 \times \mathbb{R}^2)$ where $\mathbb{R}$ acts by via a dense one-parameter subgroup of the 2-torus $S^1 \times S^1$. This group G is known as the *Mautner group* and is an example of a *wild* group in representation theory.

8.1. An example with arbitrarily small speed.

In this paragraph we describe the example of Theorem 8.1. Let $G_\alpha = \mathbb{Z} \cdot \mathbb{R}^2$ where the action of $\mathbb{Z}$ is given by the rotation R_α of angle $\pi\alpha$, $\alpha \in [0,1)$. The group G_α is quasi-isometric to $\mathbb{R}^3$ and hence of polynomial growth of order 3 and it is co-compact in the analogously defined Lie group $\widetilde{G_\alpha} = \mathbb{R} \cdot \mathbb{R}^2$. Its nilshadow is isomorphic to $\mathbb{R}^3$. The point is

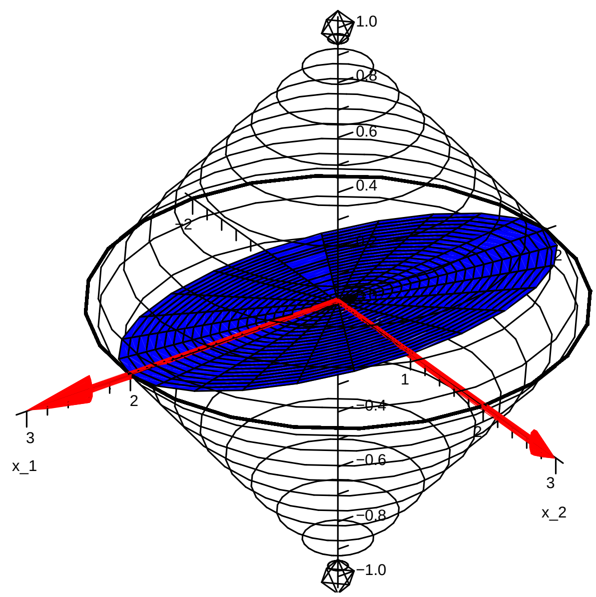

FIGURE 2. The union of the two cones, with basis the disc of radius 2, represents the limit shape of the balls Ω^n in the group $\mathbb{Z} \cdot \mathbb{R}^2$, where $\mathbb{Z}$ acts by an irrational rotation, with generating set $\Omega = \{(\pm 1, 0, 0)\} \cup \{(0, x_1, x_2), \frac{1}{4}x_1^2 + x_2^2 \leq 1\}$.

that if α is a suitably chosen Liouville number, then the balls in G_α will not be well approximated by the limit norm balls.

Elements of G_α are written (k, x) where $k \in \mathbb{Z}$ and $x \in \mathbb{R}^2$. Let $\|x\|^2 = \frac{1}{4}x_1^2 + x_2^2$ be a Euclidean norm on $\mathbb{R}^2$, and let Ω be the symmetric compact generating set given by $\{(\pm 1, 0)\} \cup \{(0, x), \|x\| \leq 1\}$. It induces a word metric ρ_Ω on G. It follows from Theorem 1.3 and the definition of the asymptotic norm that $\rho_\Omega(e, (k, x))$ is asymptotic to the norm on $\mathbb{R}^3$ given by $\rho_0(e, (k, x)) := |k| + \|x\|_0$ where $\|x\|_0$ is the rotation invariant norm on $\mathbb{R}^2$ defined by $\|x\|_0^2 = \frac{1}{4}(x_1^2 + x_2^2)$. The unit ball of $\|\cdot\|_0$ is the convex hull of the union of all images of the unit ball of $\|\cdot\|$ under all rotations $R_{k\alpha}$, $k \in \mathbb{Z}$.

We are going to choose α as a suitable Liouville number so that (35) holds. Let $\delta_n = (4\varepsilon_n)^{1/3}$ and choose α so that the following holds for infinitely many n's:

$$d(k\alpha, \mathbb{Z} + \frac{1}{2}) \geq 2\delta_n \tag{36}$$

for all $k \in \mathbb{Z}$, $|k| \leq n$. This is easily seen to be possible if we choose α of the form $\sum 1/3^{n_i}$ for some suitable lacunary increasing sequence of $(n_i)_i$.

Note that, since $\|x\|_0 \geq \|x\|$, we have $\rho_\Omega \geq \rho_0$. Let S_n be the piece of $\mathbb{R}^2$ defined by $S_n = \{|\theta| \leq \delta_n\}$ where θ is the angle between the point x and the vertical axis

$\mathbb{R}e_2$. We *claim* that if $x \in S_n$, $\rho_0(e,(k,x)) \leq n$ and n satisfies (36), then

$$\rho_\Omega(e,(k,x)) \geq |k| + (1+\frac{\delta_n^2}{4})\,\|x\|_0$$

It follows easily from the claim that $vol_G(\Omega^n) \leq (1-\varepsilon_n)\cdot vol_G(B_{\rho_0}(n))$. Moreover $vol_G(B_{\rho_0}(n)) = c\cdot n^3 + O(n^2)$, where $c = \frac{4\pi}{3}$ if vol_G is given by the Lebesgue measure.

Proof of claim. Here is the idea to prove the claim. To find a short path between the identity and a point on the vertical axis, we have to rotate by a $R_{k\alpha}$ such that $k\alpha$ is close to $\frac{1}{2}$, hence go up from $(0,0)$ to $(k,0)$ first, thus making the vertical direction shorter. However if (36) holds, the vertical direction cannot be made as short as it could after rotation by any of the $R_{k\alpha}$ with $|k| \leq n$.

Note that if $\rho_0(e,(k,x)) \leq n$ then $|k| \leq n$ and $\rho_\Omega(e,(k,x)) \geq |k| + \inf \sum \|R_{k_i\alpha}x_i\|$ where the infimum is taken over all paths $x_1,...,x_N$ such that $x = \sum x_i$ and all rotations $R_{k_i\alpha}$ with $|k_i| \leq n$. Note that if δ_n is small enough and (36) holds then for every $x \in S_n$ we have $\|R_{k\alpha}x\| \geq (1+\delta_n^2)\,\|x\|_0$. On the other hand $\|x\|_0 = \sum \|x_i\|_0 \cos(\theta_i)$ where θ_i is the angle between x_i and the x. Hence

$$\begin{aligned}
\sum \|R_{k_i\alpha}x_i\| &\geq \sum_{|\theta_i|\leq\delta_n} \|R_{k_i\alpha}x_i\| + \sum_{|\theta_i|>\delta_n} \|R_{k_i\alpha}x_i\| \\
&\geq (1+\delta_n^2)\sum_{|\theta_i|\leq\delta_n} \|x_i\|_0\cos(\theta_i) + \frac{1}{\cos(\delta_n)}\sum_{|\theta_i|>\delta_n} \|x_i\|_0\cos(\theta_i) \\
&\geq (1+\frac{\delta_n^2}{4})\cdot\|x\|_0
\end{aligned}$$

□

Remark 8.2. *The limit shape of Figure 2 was easily determined due to the simple form of the generating set Ω. It would be interesting to determine the limit shape for more general generating sets of G_α.*

8.2. **Bounded distance versus asymptotic metrics.** In this paragraph we answer a question of D. Burago (see [5]). We give an example (A) of a finitely generated nilpotent group endowed with two left invariant coarsely geodesic metrics ρ_1 and ρ_2 that are asymptotic to each other, i.e. $\rho_1(e,x)/\rho_2(e,x) \to 1$ as $x \to \infty$ but such that $|\rho_1(e,x) - \rho_2(e,x)|$ is not uniformly bounded. We also exhibit (B) a word metric that is not at a bounded distance from any homogeneous quasi-norm.

Note that the group G_α with ρ_0 and ρ_Ω from the last paragraph also provides an example of asymptotic metrics which are not at a bounded distance (but this group was not discrete).

(A) Let $N = \mathbb{R}\times H_3(\mathbb{R})$ where H_3 is classical Heisenberg group and $\Gamma = \mathbb{Z}\times H_3(\mathbb{Z})$ a lattice in N. In the Lie algebra $\mathfrak{n} = \mathbb{R}V \oplus \mathfrak{h}_3$ we pick two different supplementary subspaces of $[\mathfrak{n},\mathfrak{n}] = \mathbb{R}Z$, i.e. $m_1 = span\{V,X,Y\}$ and $m_1' = span\{V+Z,X,Y\}$, where $\mathfrak{h}_3$ is the Lie algebra of $H_3(\mathbb{R})$ spanned by X,Y and $Z = [X,Y]$. We consider the L^1-norm on m_1 (resp. m_1') corresponding to the basis

(V, X, Y) (resp. $(V + Z, X, Y)$). Both norms induce the same norm on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$. They give rise to left invariant Carnot-Caratheodory Finsler metrics on N, say d_∞ (resp. d'_∞). We use the coordinates $(v, x, y, z) = \exp(vV + xX + yY + zZ)$.

According to Remark (2) after Theorem 6.2, d_∞ and d'_∞ are asymptotic. Let us show that they are not at a bounded distance. First observe that, since V is central, $d_\infty(e, (v, x, y, z)) = |v| + d_{H_3}(e, (x, y, z))$ where d_{H_3} is the Carnot-Caratheodory Finsler metric on $H_3(\mathbb{R})$ defined by the standard L^1-norm on the $span\{X, Y\}$. Similarly $d'_\infty(e, (v, x, y, z)) = |v| + d_{H_3}(e, (x, y, z - v)))$. If d_∞ and d'_∞ were at a bounded distance, we would have a $C > 0$ such that for all $t > 0$

$$|d_\infty(e, (t, 0, 0, t)) - t| \leq C$$

Hence $|d_{H_3}(e, (0, 0, t))| \leq C$, which is a contradiction. □

Remark 8.3. *Observe that in the above example it is crucial to use the L^1-norm. The same example with a Euclidean norm instead would fail. However one can still build an example with a Euclidean norm (consider the free step-3 group with 2 generators and use a so-called "abnormal" geodesic).*

(B) Now let $\Omega = \{(1, 0, 0, 1)^{\pm 1}, (1, 0, 0, -1)^{\pm 1}, (0, 1, 0, 0)^{\pm 1}, (0, 0, 1, 0)^{\pm 1}\}$ be a generating set for Γ and ρ_Ω the word metric associated to it. Let $|\cdot|$ be a homogeneous quasi-norm on N which is at a bounded distance from ρ_Ω, i.e. $|\rho_\Omega(e, g) - |g||$ is bounded. Then $|\cdot|$ is asymptotic to ρ_Ω, hence is equal to the Carnot-Caratheodory Finsler metric d asymptotic to ρ_Ω and homogeneous with respect to the same one parameter group of dilations $\{\delta_t\}_{t>0}$. Let $m_1 = \{v \in \mathfrak{n}, \delta_t(v) = tv\}$. Then d is induced by some norm $\|\cdot\|_0$ on m_1, whose unit ball is given, according to Theorem 1.3 by the convex hull of the projections to m_1 of the generators in Ω. There is a unique vector in m_1 of the form $V + z_0 Z$. Its $\|\cdot\|_0$-norm is 1 and $d(e, (1, 0, 0, z_0)) = 1$. However $d(e, (v, x, y, z)) = |v| + d_{H_3}(e, (x, y, z - vz_0))$. Since $\rho_\Omega(e, (n, 0, 0, n)) = n$, we get

$$d(e, (n, 0, 0, n)) - \rho_\Omega(e, (n, 0, 0, n)) = d_{H_3}(e, (0, 0, n(1 - z_0)))$$

If this is bounded, this forces $z_0 = 1$. But we can repeat the same argument with $(n, 0, 0, -n)$ which would force $z_0 = -1$. A contradiction. □

8.3. **Speed of convergence for nilpotent groups.** The phenomenon in Theorem 8.1 relied crucially on the presence of a semisimple part in G_α ; this doesn't occur in nilpotent groups. To get an error term for $\frac{\rho}{d_\infty}$, let us look more closely at the proof of Theorem 6.2. Observe that the proof was entirely effective, except for Lemma 6.10, where a compactness argument was used in the proof. Thus to derive an error term, we need only to compute N_ε explicitely in Lemma 6.10 and also to compute $s_0(\varepsilon)$ in Lemma 6.15.

As one can see by looking carefully at the proof (and assuming of course that ρ is periodic and coarsely geodesic), the error term $\frac{\rho}{d_\infty} - 1$ is bounded from below by $O(d_\infty^{-\alpha})$ ($\alpha > 0$ depending only on the Lie group). The corresponding upper bound (of the order of $O(d_\infty^{-\beta})$ for some $\beta > 0$) would follow from a corresponding estimate for $s_0(\varepsilon)$ ($s_0(\varepsilon) = O(\frac{1}{\varepsilon})$ comes for free for finitely generated nilpotent groups) and also an estimate for N_ε, i.e. $N_\varepsilon = O(\varepsilon^{-c})$ for some $c > 0$. Estimating

N_ε is more delicate and requires a better understanding of the geometry the CC-balls, and CC-geodesics. We will address this issue in a subsequent paper.

Conversely, observe that an upper bound of the form $\rho \leq d_\infty(1 + O(d_\infty^{-\beta}))$, where ρ is the word metric on N_∞ associated to a generating set equal to the unit ball in m_1 for the norm $\|\cdot\|_0$ giving rise to d_∞, implies that $N_\varepsilon = O(\varepsilon^{-\beta})$ and also $s_0(\varepsilon) = O(\varepsilon^{-1})$.

9. Appendix: the Heisenberg groups

Here we show how to compute the asymptotic shape of balls in the Heisenberg groups $H_3(\mathbb{Z})$ and $H_5(\mathbb{Z})$ and their volume, thus giving another approach to the main result of Stoll [24]. The leading term for the growth of $H_3(\mathbb{Z})$ is rational for all generating sets (Prop. 9.1 below), whereas in $H_5(\mathbb{Z})$ with its standard generating set, it is transcendental. This explains how our Figure 1 was made (compare with the odd [16] Fig. 1).

9.1. **3-dim Heisenberg group.** Let us first consider the Heisenberg group

$$H_3(\mathbb{Z}) = \langle a, b | [a, [a, b]] = [b, [a, b]] = 1 \rangle .$$

We see it as the lattice generated by $a = \exp(X)$ and $b = \exp(Y)$ in the real Heisenberg group $H_3(\mathbb{R})$ with Lie algebra $\mathfrak{h}_3$ generated by X, Y and spanned by $X, Y, Z = [X, Y]$. Let ρ_Ω be the standard word metric on $H_3(\mathbb{Z})$ associated to the generating set $\Omega = \{a^{\pm 1}, b^{\pm 1}\}$. According to Theorem 1.3, the limit shape of the n-ball Ω^n in $H_3(\mathbb{Z})$ coincides with the unit ball $\mathcal{C}_3 = \{g \in H_3(\mathbb{R}), d_\infty(e, g) \leq 1\}$ for the Carnot-Caratheodory metric d_∞ induced on $H_3(\mathbb{R})$ by the ℓ^1-norm $\|xX + yY\|_0 = |x| + |y|$ on $m_1 = span\{X, Y\} \subset \mathfrak{h}_3$.

Computing this unit ball is a rather simple task. Exchanging the roles of X and Y, we see that $\mathcal{C}_3$ is invariant under the reflection $z \mapsto -z$. Then clearly $\mathcal{C}_3$ is of the form $\{xX + yY + zZ$, with $|x| + |y| \leq 1$ and $|z| \leq z(x, y)\}$. Changing X to $-X$ and Y to $-Y$, we get the symmetries $z(x, y) = z(-x, y) = z(x, -y) = z(y, x)$. Hence when determining $z(x, y)$, we may assume $0 \leq y \leq x \leq 1$, $x + y \leq 1$.

The following well known observation is crucial for computing $z(x, y)$. If $\xi(t)$ is a horizontal path in $H_3(\mathbb{R})$ starting from id, then $\xi(t) = \exp(x(t)X + y(t)Y + z(t)Z)$, where $\xi'(t) = x(t)X + y(t)Y$ and $z(t)$ is the "balayage" area of the between the path $\{x(s)X + y(s)Y\}_{0 \leq s \leq t}$ and the chord joining 0 to $x(t)X + y(t)Y$.

Therefore, $z(x, y)$ is given by the solution to the "Dido isoperimetric problem" (see [18]): find a path in the X, Y-plane between 0 and $xX + yY$ of $\|\cdot\|_0$-length 1 that maximizes the "balayage area". Since $\|\cdot\|_0$ is the ℓ^1-norm in the X, Y-plane, as is well-known, such extremal curves are given by arcs of square with sides parallel to the X, Y-axes. There is therefore a dichotomy: the arc of square has either 3 or 4 sides (it may have 1 or 2 sides, but these are included are limiting cases of the previous ones).

If there are 3 sides, they have length ℓ, x and $y + \ell$ with $y + \ell \leq x$. Hence $1 = \ell + x + y + \ell$ and $z(x, y) = \ell x + \frac{1}{2}xy$. Therefore this occurs when $y \leq 3x - 1$ and we then have $z(x, y) = \frac{x(1-x)}{2}$.

If there are 4 sides, they have length $\ell, x+u, y+\ell$ and u, with $\ell + y = x + u$. Hence $1 = 2\ell + 2u + x + y$ and $z(x,y) = (\ell+y)(x+u) - \frac{xy}{2}$. This occurs when $y \geq 3x - 1$ and we then have $z(x,y) = \frac{(1+x+y)^2}{16} - \frac{xy}{2}$.

Hence if $0 \leq y \leq x \leq 1$ and $x + y \leq 1$

$$z(x,y) = 1_{y \leq 3x-1} \frac{x(1-x)}{2} + 1_{y > 3x-1} \frac{(1+x+y)^2}{16} - \frac{xy}{2} \tag{37}$$

The unit ball $\mathcal{C}_3$ drawn in Figure 1 is the solid body $\mathcal{C}_3 = \{xX + yY + zZ$, with $|x| + |y| \leq 1$ and $|z| \leq z(x,y)\}$.

A simple calculation shows that $vol(\mathcal{C}_3) = \frac{31}{72}$ in the Lebesgue measure $dxdydz$. Since $H_3(\mathbb{Z})$ is easily seen to have co-volume 1 for this Haar measure on $H_3(\mathbb{R})$ (actually $\{xX + yY + zZ, x \in [0,1), y \in [0,1), z \in [0,1)\}$ is a fundamental domain), it follows that

$$\lim_{n \to \infty} \frac{\#(\Omega^n)}{n^4} = vol(\mathcal{C}_3) = \frac{31}{72}$$

We thus recover a well-known result (see [3], [22] where even the full growth series is computed and shown to be rational).

One can also determine exactly which points of the sphere $\partial \mathcal{C}_3$ are joined to id by a unique geodesic horizontal path. The reader will easily check that uniqueness fails exactly at the points $(x, y, \pm z(x,y))$ with $|x| < \frac{1}{3}$ and $y = 0$, or $|y| < \frac{1}{3}$ and $x = 0$, or else at the points (x,y,z) with $|x| + |y| = 1$ and $|z| < z(x,y)$.

The above method also yields the following result.

Proposition 9.1. *Let Ω be any symmetric generating set for $H_3(\mathbb{Z})$. Then the leading coefficient in $\#(\Omega^n)$ is rational, i.e.*

$$\lim_{n \to \infty} \frac{\#(\Omega^n)}{n^4} = r$$

is a rational number.

Proof. We only sketch the proof here. We can apply the method above and compute r as the volume of the unit CC-ball $\mathcal{C}(\Omega)$ of the limit CC-metric d_∞ defined in Theorem 1.3. Since we know what is the norm $\|\cdot\|$ in the (x,y)-plane $m_1 = span\,\langle X, Y\rangle$ that generates d_∞ (it is the polygonal norm given by the convex hull of the points of Ω), we can compute $\mathcal{C}(\Omega)$ explicitely. We need to know the solution to Dido's isoperimetric problem for $\|\cdot\|$ in m_1, and as is well known it is given by polygonal lines from the dual polygon rotated by $90°$. Since the polygon defining $\|\cdot\|$ is made of rational lines (points in Ω have integer coordinates), any vector with rational coordinates has rational $\|\cdot\|$-length, and the dual polygon is also rational. The equations defining $z(x,y)$ will therefore have only rational coefficients, and $z(x,y)$ will be piecewisely given by a rational quadratic form in x and y, where the pieces are rational triangles in the (x,y)-plane. The total volume of $\mathcal{C}(\Omega)$ will therefore be rational.

□

9.2. **5-dim Heisenberg group.** The Heisenberg group $H_5(\mathbb{Z})$ is the group generated by a_1, b_1, a_2, b_2,c with relations $c = [a_1, b_1] = [a_2, b_2]$, a_1 and b_1 commute with a_2 and b_2 and c is central. Let $\Omega = \{a_i^{\pm 1}, b_i^{\pm 1}, i = 1, 2\}$. Let us describe the limit shape of Ω^n. Again, we see $H_5(\mathbb{Z})$ as a lattice of co-volume 1 in the group $H_5(\mathbb{R})$ with Lie algebra $\mathfrak{h}_5$ spanned by $X_1, Y_1 X_2, Y_2$ and $Z = [X_i, Y_i]$. By Theorem 1.3, the limit shape is the unit ball $\mathcal{C}_5$ for the Carnot-Caratheodory metric on $H_5(\mathbb{R})$ induced by the ℓ^1-norm $\|x_1X_1 + y_1Y_1 + x_2X_2 + y_2Y_2\|_0 = |x_1| + |y_1| + |x_2| + |y_2|$.

Since X_1, Y_1 commute with X_2, Y_2, in any piecewise linear horizontal path in $H_5(\mathbb{R})$, we can swap the pieces tangent to X_1 or Y_1 with those tangent to X_2 or Y_2 without changing the end point of the path. Therefore if $\xi(t) = \exp(x_1(t)X_1 + y_1(t)Y_1 + x_2(t)X_2 + y_2(t)Y_2 + z(t)Z)$ is a horizontal path, then $z(t) = z_1(t) + z_2(t)$, where $z_i(t)$, $i = 1, 2$, is the "balayage area" of the plane curve $\{x_i(s)X_i + y_i(s)Y_i\}_{0\leq s\leq t}$.

Since, just like for $H_3(\mathbb{Z})$, we know the curve maximizing this area, we can compute the unit ball $\mathcal{C}_5$ explicitely. In exponential coordinates it will take the form $\mathcal{C}_5 = \{\exp(x_1X_1 + y_1Y_1 + x_2X_2 + y_2Y_2 + zZ), |x_1| + |y_1| + |x_2| + |y_2| \leq 1$ and $|z| \leq z(x_1, y_1, x_2, y_2)\}$. Then $z(x_1, y_1, x_2, y_2) = \sup_{0\leq t\leq 1}\{z_t(x_1, y_1) + z_{1-t}(x_2, y_2)\}$, where $z_t(x, y)$ is the maximum "balayage area" of a path of length t between 0 and $xX + yY$. It is easy to see that $z_t(x, y) = t^2 z(x/t, y/t)$ where z is given by (37). Hence z_t is a piecewise quadratic function of t. Again $z(x_1, y_1, x_2, y_2)$ is invariant under changing the signs of the x_i,y_i's, and swapping x and y, or else swapping 1 and 2. We may thus assume that the x_i,y_i's lie in $D = \{0 \leq y_i \leq x_i \leq 1$ and $x_1+y_1+x_2+y_2 \leq 1$, and $x_2-y_2 \geq x_1-y_1\}$. We may therefore determine explicitely the supremum $z(x_1, y_1, x_2, y_2)$, which after some straightforward calculations takes on D the following form:

$$z(x_1, y_1, x_2, y_2) = 1_A \max\{d_1, d_2\} + 1_B \max\{d_1, c_1\} + 1_C \max\{c_1, c_2\}$$

where $d_1 = \frac{x_1y_1}{2} + \frac{x_2}{2}(1-x_1-y_1-x_2)$, $c_1 = \frac{1}{16}(1+x_1+y_1-x_2-y_2)^2 + \frac{x_2y_2-x_1y_1}{2}$, and d_2 and c_2 are obtained from d_1 and c_1 by swapping the indices 1 and 2. The sets A, B and C form the following partition of $D : A = D \cap \{m \leq x_1 - y_1\}$, $B = D \cap \{x_1 - y_1 < m < x_2 - y_2\}$ and $C = D \cap \{x_2 - y_2 \leq m\}$, where $m = (1 - x_1 - x_2 - y_1 - y_2)/2$.

Since $\mathcal{C}_5$ has such an explicit form, it is possible to compute its volume. The fact that $z(x_1, y_1, x_2, y_2)$ is piecewisely given by the maximum of two quadratic forms makes the computation of the integral somewhat cumbersome but tractable. Our equations coincide (fortunately!) with those of Stoll (appendix of [24]), where he computed the main term of the asymptotics of $\#(\Omega^n)$ by a different method. Stoll did calculate that integral and obtained

$$\lim_{n\to\infty} \frac{\#(\Omega^n)}{n^6} = vol(\mathcal{C}_5) = \frac{2009}{21870} + \frac{\log(2)}{32805}$$

which is transcendental. It is also easy to see by this method that if we change the generating set to $\Omega_0 = \{a_1^{\pm 1}b_1^{\pm 1}a_2^{\pm 1}b_2^{\pm 1}\}$, then we get a rational volume. Hence the rationality of the growth series of $H_5(\mathbb{Z})$ depends on the choice of generating set, which is Stoll's theorem.

One advantage of our method is that it can also apply to fancier generating sets. The case of Heisenberg groups of higher dimension with the standard generating set is analogous: the function $z(\{x_i\},\{y_i\})$ is again piecewisely defined as the maximum of finitely many explicit quadratic forms on a linear partition of the ℓ^1-unit ball $\sum |x_i| + |y_i| \leq 1$.

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