Eternal inflation and localization on the landscape

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We model the essential features of eternal inflation on the landscape populated by a dense discretuum of vacua by the potential $V(\phi) = V_0 + \delta V(\phi)$, where $|\delta V(\phi)| \ll V_0$ is random. We find that the diffusion of the distribution function $\rho(\phi, t)$ of the inflaton VEV in different Hubble patches is strongly suppressed, in analogy with the Anderson localization taking place in disordered quantum systems. This leads to the very strong dependence of observable quantities on initial conditions for eternal inflation on the landscape. Thus the generic problem of initial conditions for inflation appears to exist also on the landscape.

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String theory is believed to imply a wide landscape of both metastable vacua with a positive cosmological constant and true vacua with a vanishing or negative cosmological constant [1]. The problem of calculating statistical distributions of these vacua (as well as the estimation of their total number which is considered to be of order $10^{100} \div 10^{1000}$) is incredibly complicated [2] and is even considered to be NP-hard [3]. However, it may be possible to address the dynamics of the low-energy approximation of the string theory on the landscape. In the present paper, we consider how eternal inflation proceeds on the landscape making use of the mere fact that the number of vacua within the landscape is extremely large. The Fokker-Planck equations describing the dynamics of eternal inflation then reveal an interesting parallel to certain problems encountered in condensed matter physics[1].

The dynamics of eternal inflation on the landscape can be modeled as follows [5, 6]. Let $P_i(t)$ be the probability to measure a given (positive) cosmological constant Λ_i in a given Hubble patch, while the time independent matrix Γ_{ij} defining the rates of tunneling between the metastable minima i and j on the landscape. Then the probabilities P_i satisfy the system of "vacuum dynamics" equations [7]

$$\dot{P}_i = \sum_{j\neq i} \Gamma_{ij} P_i + \sum_{j\neq i} \Gamma_{ji} P_j - \Gamma_{is} P_i. \tag{1}$$

The last term corresponds to tunneling between the metastable de Sitter vacuum i and a true AdS vacuum, i.e. tunneling into a collapsing AdS spacetime [8]. The collapse time $t_{\rm col} \sim M_P / V_{\rm AdS}^{1/2}$ is much shorter than the characteristic time $t_{\rm rec} \sim \exp\left(M_P^4/V_{\rm dS}\right)$ for tunneling back into a de Sitter metastable vacuum, so that the AdS true vacua effectively play the role of sinks for the probability currents describing eternal inflation on the landscape [5]. Within the model (1) the tunneling rates Γ_{ij} do not depend on time, so we are interested in what happens at time scales much larger than the typical inverse tunneling rate $\langle \Gamma_{ij}^{-1} \rangle$. In what follows we will assume that the effect of the AdS sinks is relatively small; otherwise the landscape will be divided into almost unconnected "islands" of vacua [6], preventing the population of the whole landscape by eternal inflation. We therefore wish to discuss features of eternal inflation arising at time scales $t \ll t_{\rm AdS}$, where $t_{\rm AdS} \sim \langle \Gamma_{is}^{-1} \rangle$ is the typical time of tunneling into an AdS sink from vacua in a given island.

In the limit of weak tunneling only the vacua closest to each other are important. Thus it is convenient to classify parts (islands) of the landscape according to the typical number of adjacent vacua. Technically, the landscape of vacua of the string theory can be represented as a graph with $10^{100} \div 10^{1000}$ nodes and a number of connections between them of the same order. By an island on the landscape, we mean a subgraph relatively weakly connected to the major tree. The dimensionality of the island can then be defined as the Hausdorff (or Minkowski-Bouligand) dimension N_H of the corresponding subgraph [14]. In a few words, if there are only two adjacent vacua for any given vacuum in a given island, then the Hausdorff dimension of the island is 1 and we denote it as quasi-one-dimensional; a domain of vacua with $N_H = 2$ is quasi-two-dimensional, and so on.

In the quasi-one-dimensional case (neglecting the AdS sinks) the system (1) reduces to

$$\dot{P}_i = -\Gamma_{i,i+1} P_i + \Gamma_{i+1,i} P_{i+1} - \Gamma_{i,i-1} P_i + \Gamma_{i-1,i} P_{i-1}. \tag{2}$$

While in general $\Gamma_{ij} \neq \Gamma_{ji}$, it is reasonable to take $\langle \Gamma_{ij} \rangle = \langle \Gamma_{ji} \rangle$ on the average. Furthermore, suppose that the initial condition for Eq. (2) is delta-function-like with

$$P_i(0) = 1, \;\; P_{j\neq i}(0) = 0. \tag{3}$$

so that the initial state is well localized. Naively, one may expect that the distribution function $P_i(t)$ would start to spread out according to the usual diffusion law and the system of vacua would exponentially quickly reach a "thermal" equilibrium distribution of probabilities for a given Hubble patch to be in a given dS vacuum. However, there exists a well known result [9] from the theory of diffusion on random lattices, according to which the

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[1] An approach similar to ours in some respect was also presented in [4].

distribution function P_i remains localized near the initial distribution peak for a very long time, with its characteristic width behaving as

$$\langle i^2(t)\rangle \sim \log^4 t\ . \tag{4}$$

This is a surprising result when applied to eternal inflation where the general lore (see for example [10]) is that the initial conditions for eternal inflation will be forgotten almost immediately after its beginning. Instead, in what follows we will argue that the memory about the initial conditions survives during infinitely long time on the quasi-one-dimensional islands of the landscape, as suggested by condensed matter physics.

Instead of the lattice description Eq. (2), we will model the landscape by a continuous inflaton potential

$$V(\phi) = V_0 + \delta V(\phi), \tag{5}$$

where V_0 is constant, and $\delta V(\phi)$ is a random contribution such that $|\delta V(\phi)| \ll V_0$. We assume that the potential has a very large number of local minima separated by relatively low barriers penetrated by rapid tunneling. As in stochastic inflation [13], in different causally connected regions fluctuations have a randomly distributed amplitude and observers living in different Hubble patches see different VEVs of the inflaton. When the stochastic fluctuation is larger than the effect of the classical force, one may find an effective growth of the VEV of ϕ in a given Hubble patch. This behavior can be determined from the Langevin equation [13]

$$\dot{\phi} = -\frac{1}{3H_0}\frac{\partial \delta V}{\partial \phi} + f(t), \tag{6}$$

where the stochastic force $f(\phi, t)$ is Gaussian with correlation properties

$$\langle f(t) f(t')\rangle = \frac{H_0^3}{4\pi^2}\delta(t-t'). \tag{7}$$

From the Langevin equation one can derive the Fokker-Planck equation, which determines the evolution of the probability distribution $\rho(\phi, t)$ describing how the values of ϕ are distributed among different Hubble patches in the multiverse and how they are correlated with each other. One finds [13]

$$\frac{\partial \rho(\phi,t)}{\partial t} = \frac{H_0^3}{8\pi^2}\frac{\partial^2 \rho}{\partial \phi^2} + \frac{1}{3H_0}\frac{\partial}{\partial \phi}\left(\frac{\partial V}{\partial \phi}\rho\right). \tag{8}$$

The general solution to Eq. (8) is given by

$$\rho = e^{-\frac{4\pi^2 \delta V(\phi)}{3H_0^4}} \sum_n c_n \psi_n(\phi) e^{-\frac{E_n H_0^3 (t-t_0)}{4\pi^2}}, \tag{9}$$

where ψ_n and E_n are respectively the eigenfunctions and the eigenvalues of the effective Hamiltonian

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial \phi^2} + W(\phi). \tag{10}$$

Here

$$W(\phi) = \frac{8\pi^4}{9H_0^8}\left(\frac{\partial \delta V}{\partial \phi}\right)^2 - \frac{2\pi^2}{3H_0^4}\frac{\partial^2 \delta V}{\partial \phi^2} \tag{11}$$

is a functional of the scalar field potential $V(\phi)$. It is often denoted as the superpotential due to its "supersymmetric" form: the Hamiltonian Eq. (10) can be rewritten as $\hat{H} = \hat{Q}^\dagger \hat{Q}$, where $\hat{Q} = -\partial/\partial\phi + v'(\phi)$ with $v(\phi) = 4\pi^2 \delta V(\phi)/(3H_0^4)$.

The Hamiltonian (10) satisfies the Schrödinger equation

$$\frac{1}{2}\frac{\partial^2 \psi_n}{\partial \phi^2} + (E_n - W(\phi))\psi_n = 0, \tag{12}$$

the solutions of which have the following well known features [13]:

1. The eigenvalues of the Hamiltonian (10) are positive definite. Assuming normalizable wave functions $\psi_n(\phi)$, the ground state $\psi_0(\phi)$ corresponds to the zero eigenvalue and defines the steady state solution of the Fokker-Planck equation (8).

2. The contributions from eigenfunctions of excited states $\psi_{n>0}(\phi)$ to the solution Eq. (9) become exponentially quickly damped with time. However, if one is interested in what happens at time scales $\Delta t \lesssim 1/E_n$, the first n eigenfunctions should be taken into account. In particular, if the spectrum of the Hamiltonian (10) *is very dense*, as in the case of the string theory landscape, knowing the ground state alone is not enough for understanding dynamics of eternal inflation.

We now recall that the potential $V(\phi)$ (and therefore the superpotential $W(\phi)$) is now a random function of the inflaton field and has extremely large number of minima. This allows us to make several conclusions regarding the form of the eigenfunctions $\psi_n(\phi)$ using the formal analogy between Eq. (12) and the time-independent Schrödinger equation describing the motion of carriers in disordered quantum systems such as semiconductors, where the carriers (electrons) move in the potential of impurity atoms (dopants). The physical quantities in disordered systems can be calculated by averaging over the random potential of impurities[2]. A famous consequence of the random potential generated by impurities in crystalline materials is the strong suppression of the conductivity, known as Anderson localization [11, 12]. This effect is very essential in dimensions lower than 3

[2] The typical number of these impurities varies between 10^{12} to 10^{17} per cm^3 while the number of vacua on the string theory landscape is $10^{100} \div 10^{1000}$. Therefore, it is perfectly legitimate to describe the dynamics of eternal inflation on the landscape at long time scales by averaging over the disorder.

and completely defines the kinetics of carriers in one-dimensional systems. Impurities create a random potential for Bloch waves with correlation properties

$$\langle u(r)u(r')\rangle \;=\; \frac{1}{\nu\tau}\delta(r-r'),\ \langle u(r)\rangle = 0\ . \qquad (13)$$

(Strictly speaking, the Gaussian correlation properties are realized only in the limit $a_0 \to 0$, $n \to \infty$ where a_0 is the scattering length on a single impurity and n is the concentration of impurities [12]). As a consequence, in the one-dimensional case all eigenstates of the electron hamiltonian become localized with

$$\psi_n(r) \sim \exp\left(-\frac{|r-r_0|}{L}\right) \qquad (14)$$

at $t \to \infty$ (here r_0 is the initial position of the wave package). The localization length L remains of the order of the mean free path $l_\tau = \langle v\rangle\tau$ for all eigenstates. As a result, the probability density $\rho(R,t)$ to find electron at the point R at time t asymptotically approaches the limit $\rho(R) \sim \exp(-R/L)$ for $R \gg L$, or $\rho(R) \sim \mathrm{Const}$ for $R \ll L$ at $t \to \infty$. An important feature of the one-dimensional Anderson localization is that it takes place for *an arbitrarily weak disorder* and *arbitrary correlation properties* of the random potential $u(r)$.

Also in a two-dimensional case all electron eigenstates in a random potential remain localized. However, the localization length grows exponentially with energy, the rate of growth being related to the strength of the disorder. In three-dimensional case, the localization properties of eigenstates are defined by the Ioffe-Regel-Mott criterion: if the corresponding eigenvalue of the Hamiltonian of electrons E_n satisfies the condition $E_n < E_g$ where E_g is so called mobility edge, then the eigenstate is localized. Otherwise it has the form of the Bloch wave function. The mobility edge E_g is a function of the strength of the disorder. In higher dimensional cases the situation is unknown.

Let us now return to the discussion of eternal inflation described by the Fokker-Planck equation (8). Since the localization is the property of the eigenfunctions of the *time-independent* hamiltonian (10), localization is also a natural consequence of the effective randomness of the potential of the string theory landscape. In fact, the diffusion of the probability distribution (4) is suppressed due to the localization of the eigenfunctions $\psi_n(\phi)$ contributing to the overall solution (9). Physically, this leads to a very strong dependence on the initial conditions for eternal inflation on the landscape, counteracting the general wisdom that eternal inflation rapidly washes out any information of the initial conditions. Indeed, examining the form of the solution (9) one would easily be led to conclude that the distribution function $\rho(\phi,t)$ settles exponentially quickly down to its equilibrium form defined by

$$\rho_{\mathrm{eq}} \sim \exp\left(-\frac{8\pi^2\delta V(\phi)}{3H_0^4}\right). \qquad (15)$$

Instead, in the quasi-one-dimensional case all the wave functions $\psi_n(\phi)$ are localized, i.e., behave as

$$\psi_n(\phi) \sim \exp\left(-\frac{|\phi-\phi_0|}{L}\right) \qquad (16)$$

where ϕ_0 is the initial value of the inflaton field with the initial condition $\rho_0(\phi) \sim \delta(\phi-\phi_0)$, and L is the localization length. The latter is of the same order of magnitude as the "mean free path" related to the strength of the disorder in the superpotential $W(\phi)$.

Thus in a quasi-one-dimensional island of the landscape we find complete localization. Let us now discuss how eternal inflation proceeds on islands where the typical number of adjacent vacua is larger than two. In the quasi-two-dimensional case the network of vacua within a given island is described by two indexes i, j, or by a composite index $\vec{i} = (i,j)$. The distribution function ρ for finding a given value of the cosmological constant in a given Hubble patch is a two-dimensional matrix. Again, all the eigenstates of the corresponding tunneling hamiltonian $\hat{H}$ are localized. However, since the localization length exponentially quickly grows with energy, the distribution function effectively spreads out in the regime close to the usual diffusion

$$\langle \vec{i}^2(t)\rangle \sim t\left(1 + c_1\frac{1}{\log^\alpha t} + \cdots\right) \qquad (17)$$

where $\alpha > 0$ are constants depending on the correlation properties of the disorder on the landscape [15]. Since the low energy eigenstates define the long time asymptotic behavior of the distribution function ρ and since such states are localized with a relatively small localization length, there exists a time t_g such that for $t_{\mathrm{AdS}} \gg t \gg t_g$ the distribution function behaves as

$$\rho \sim \exp\left(-\frac{|\vec{i}-\vec{i}_0|}{L_*}\right)\ , \qquad (18)$$

where L_* has the same order of magnitude as the localization length for low energy eigenstates. We will denote this effect as *"recalling initial conditions"* on the landscape. If the disorder in the given part of the landscape is not sufficiently strong, i.e., if $t_g > t_{\mathrm{AdS}}$, then the spreading law for the probability distribution as given by Eq. (17) is valid until the island of the landscape crumbles into AdS sinks.

In the quasi-higher-dimensional cases the distribution function spreads out according to the usual linear diffusion law at intermediate times, although there exists a mobility edge E_g such that the eigenstates of the tunneling Hamiltonian with energies $E < E_g$ are localized. These low energy eigenstates again define the late time asymptotics of the distribution function ρ and at

$$t \gg E_g^{-1} \qquad (19)$$

the distribution function of the cosmological constant values over different Hubble patches can be well approximated by the asymptotics (18). Eventually, the initial conditions for eternal inflation may be recalled even in quasi-higher-dimensional parts of the landscape. Again, if $t_{\rm AdS} \ll E_g^{-1}$, the spreading of the probability distribution ρ proceeds according to the normal diffusion law until the given island submerges into AdS sinks. The value of the mobility edge E_g strongly depends on the dimensionality of the island and the strength of the disorder, and the higher is the dimensionality, the lower is the mobility edge. This generic situation will be discussed more fully in [14].

Finally, let us discuss the effect of the AdS sinks on the dynamics of tunneling between the vacua. On the string theory landscape, dS metastable vacua are typically realized by uplifting stable AdS vacua (as, for example, in the well known KKLT model [16]). The probability to tunnel from the uplifted dS state i back into the AdS vacuum is related to the value of gravitino mass $m_{3/2}$ in the dS state [8] and given by

$$t_{\rm AdS} \sim \Gamma_{is}^{-1} \sim \exp\left(\frac{{\rm Const.}M_P^2}{m_{3/2,i}^2}\right) . \qquad (20)$$

The gravitino mass after uplifting [17] has the order of magnitude

$$m_{3/2,i} \sim |V_{{\rm AdS},i}|^{1/2}/M_P. \qquad (21)$$

Since at long time scales $V_{{\rm AdS},i}$ can also be regarded as a random quantity, our analysis of the general solution of "vacuum dynamics" equations (1) does not have to be modified in any essential way [14]. The most important effect of sinks is that the tunneling hamiltonian $\hat{H}$ is not hermitian in their presence and therefore the evolution of the probability distribution ρ is not unitary. The characteristic time scale of the unitarity breakdown is $t_{\rm AdS} \sim \langle \Gamma_{is}^{-1} \rangle$ and our conclusions remain valid at time scales $t \ll t_{\rm AdS}$.

In summary, we have argued that eternal inflation on the landscape leads to a strong localization of the inflaton VEV in different Hubble patches. Physically, this means that the initial condition problem of the slow roll inflation, cured by eternal inflation in the case of a single potential, reappears in the dense discretuum of the landscape.

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