

Geometric Complexity Theory V: On deciding nonvanishing of a generalized Littlewood-Richardson coefficient

Dedicated to Sri Ramakrishna

Ketan D. Mulmuley
The University of Chicago and I.I.T., Mumbai*

Hariharan Narayanan
The university of Chicago

June 21, 2013

Abstract

In this note it is observed that nonvanishing of a generalized Littlewood-Richardson coefficient of any type can be decided in polynomial time assuming the conjecture in [2, 6] that the coefficients of the associated stretching quasi-polynomial are nonnegative.

1 Introduction

Given dominant weights λ, μ, ν of a semi-simple Lie algebra \mathcal{G} , the generalized Littlewood-Richardson coefficient $C_{\lambda, \mu}^{\nu}$ is defined to be the multiplicity of V_{ν} in $V_{\lambda} \otimes V_{\mu}$, where V_{λ} denotes an irreducible representation of \mathcal{G} with highest weight λ . It has been observed in [2, 8, 9] independently that, when \mathcal{G} is simple of type A , nonvanishing of $C_{\lambda, \mu}^{\nu}$ can be decided in polynomial time; i.e., in time that is polynomial in the bitlengths of λ, μ and ν . Furthermore, the algorithm in [9] is strongly polynomial—i.e., the number of arithmetic steps in the algorithm depends only on the total number of parts

*Visiting faculty member

of λ, μ and ν , but not their bitlengths. One crucial ingredient in this algorithm is the saturation theorem of Knutson and Tao [7], which does not hold for simple Lie algebras of type B, C or D [13].

The main observation in this note is:

Theorem 1.1 *Nonvanishing of a generalized Littlewood-Richardson coefficient $C_{\lambda, \mu}^{\nu}$ for any semi-simple Lie algebra \mathcal{G} can be decided in strongly polynomial time, assuming the following positivity conjecture.*

Let $\tilde{C}(n) = \tilde{C}_{\lambda, \mu}^{\mu}(n) = C_{n\lambda, n\mu}^{n\nu}$ denote the stretching function associated with $C_{\lambda, \mu}^{\nu}$. Assume that the type of \mathcal{G} is B, C or D . Then $\tilde{C}(n)$ is a quasi-polynomial of period at most two [2]. That is, there exist polynomials $\tilde{C}_1(n)$ and $\tilde{C}_2(n)$ such that

$$C_{n\lambda, n\mu}^{n\nu} = \begin{cases} \tilde{C}_1(n), & \text{if } n \text{ is odd;} \\ \tilde{C}_2(n), & \text{if } n \text{ is even.} \end{cases}$$

Conjecture 1.2 (Positivity conjecture) *(De Loera, McAllister [2])*

The quasi-polynomial $\tilde{C}(n) = \tilde{C}_{\lambda, \mu}^{\nu}(n)$ is positive—i.e., the coefficients of $\tilde{C}_i(n)$, $i = 1, 2$, are nonnegative.

This is an extension of an analogous earlier conjecture of King, Tollu and Toumazet [6] for type A . Considerable experimental evidence for these conjectures has been given in these papers. Analogous conjecture can also be made for arbitrary (semi)simple \mathcal{G} , though this is not needed for the proof of Theorem 1.1.

Here it is assumed that each highest weight is specified by giving its coordinates in the basis of fundamental weights. The bitlength $\langle \lambda \rangle$ is defined to the total bitlength of all coordinates. Strongly polynomial means the number of arithmetic steps is polynomial in the rank of \mathcal{G} , and the bitlength of every intermediate operand is polynomial in $\langle \lambda \rangle, \langle \mu \rangle$ and $\langle \nu \rangle$.

2 Proof

Let $P = P_{\lambda, \mu}^{\nu}$ denote the BZ-polytope [1] whose Ehrhart quasi-polynomial coincides with $\tilde{C}_{\lambda, \mu}^{\nu}(n)$. Let $\text{Aff}(P)$ denote its affine span. Let $\mathbb{Z}_{\langle 2 \rangle}$ denote the subring of \mathbb{Q} obtained by localizing \mathbb{Z} at 2—i.e., the subring of fractions with odd denominators.

Lemma 2.1 *Assume that \mathcal{G} is simple of type B, C or D . If the positivity conjecture is true, the following are equivalent:*

- (1) $C_{\lambda\mu}^\nu \geq 1$.
- (2) *There exists an odd integer n such that $C_{n\lambda n\mu}^{n\nu} \geq 1$.*
- (3) *P contains a point in $\mathbb{Z}_{<2>}^d$.*
- (4) *$\text{Aff}(P)$ contains a point in $\mathbb{Z}_{<2>}^d$.*

Proof: Clearly, the first three statements are equivalent, and (3) implies (4). It remains to show that (4) implies (3). When $\text{Aff}(P)$ is one-dimensional, this holds because of a simple density argument. The general case can be easily reduced to the one-dimensional case. Q.E.D.

Now we turn to the proof of Theorem 1.1. First, let us assume that \mathcal{G} is simple of type B, C or D .

Specification of an explicit linear program of the form $Ax \leq b$ defining the BZ-polytope $P = P_{\lambda,\mu}^\nu$ can be easily computed in strongly polynomial time using its description in [1]. It is also clear from [1] that the entries of A here have constant bit lengths. In the terminology of [12], this linear program is combinatorial. Hence, we can determine if P is nonempty in strongly polynomial time by the combinatorial linear programming algorithm in [12]. If P is nonempty, this algorithm can also be extended to find an integral matrix C and an integral vector D so that $\text{Aff}(P)$ is defined by the linear system $Cx = D$. (Usual linear programming algorithms [4, 5] here, in place of Tardos' algorithm [12], will yield a polynomial time algorithm, instead of a strongly polynomial time algorithm.)

By Lemma 2.1 (4), it remains to check if $\text{Aff}(P)$ contains a point in $\mathbb{Z}_{<2>}^d$. This can be done as follows. By padding, if necessary, we can assume that C is square. Using [3], we find the Smith normal form S of C and unimodular matrices U and V such that $C = USV$; here S is a diagonal integer matrix, whose i -th diagonal entry divides the $(i + 1)$ -st diagonal entry. Since the entries of C have constant bit lengths, the algorithm in [3] works in strongly polynomial time. The question now reduces to checking if $USVx = D$ has a solution $x \in \mathbb{Z}_{<2>}^d$. This is so iff $Sy = U^{-1}D$ has a solution $y \in \mathbb{Z}_{<2>}^d$. Since S is diagonal, this can be verified in (strongly) polynomial time by checking each coordinate.

This proves Theorem 1.1 for types B, C, D .

Now let \mathcal{G} be any semisimple algebra. A generalized Littlewood-Richardson coefficient for \mathcal{G} is the product of corresponding generalized Littlewood-Richardson coefficients for each of its simple factors. Hence, without loss of generality, we can assume that \mathcal{G} is simple. If it is of type A , then Theorem 1.1 holds unconditionally by [2, 8, 9]. If it is an exceptional simple Lie algebra, then a Littlewood-Richardson coefficient can be computed in $O(1)$ arithmetic steps. Because, when the rank of \mathcal{G} is constant, the chambers of quasi-polynomiality [11] of the generalized Littlewood-Richardson coefficient, considered as a vector partition function, are $O(1)$ in number, and are fixed once \mathcal{G} is fixed. Q.E.D.

Remark 2.2 *For Theorem 1.1 to hold, we do not need the full statement of the Positivity Conjecture, but only the following analogue of saturation for Lie groups of types B , C , D :*

$$C_{\lambda\mu}^{\nu} \geq 1 \implies \forall \text{ odd } n, C_{n\lambda, n\mu}^{n\nu} \geq 1.$$

Acknowledgement: We are grateful to T. McAllister for bringing the positivity conjecture in [2] to our attention.

References

- [1] A. D. Berenstein and A. V. Zelevinsky. Tensor product multiplicities and convex polytopes in partition space, *J. Geom. Phys.* **5** (1988), no. 3, 453-472.
- [2] J. De Loera and T. McAllister. On the computation of Clebsch-Gordan coefficients and the dilation effect, *Experiment. Math.* **15** (2006), no. 1, 7-20.
- [3] R. Kannan, A. Bachem. Polynomial algorithms for computing the Smith and Hermite normal forms of an integer matrix, *SIAM J. Comput.*, 8(4), 1979.
- [4] N. Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica*, 4(4):373–395, 1984.
- [5] L. G. Khachian. A polynomial algorithm for linear programming. *Doklady Akedamii Nauk SSSR*, 244:1093–1096, 1979. In Russian.

- [6] R. King, C. Tollu, F. Toumazet Stretched Littlewood-Richardson coefficients and Kostka coefficients. In, Winternitz, P., Harnard, J., Lam, C.S. and Patera, J. (eds.) *Symmetry in Physics: In Memory of Robert T. Sharp*. Providence, USA, AMS OUP, 99-112., CRM Proceedings and Lecture Notes 34, 2004.
- [7] A. Knutson, T. Tao, The Honeycomb model of $GL_n(\mathbb{C})$ tensor products I: proof of the saturation conjecture, *J. Amer. Math. Soc.*, 12, 1999, pp. 1055-1090.
- [8] A. Knutson, T. Tao, Honeycombs and sums of Hermitian matrices, *Notices Amer. Math. Soc.* 48, (2001) No. 2, 175-186.
- [9] K. Mulmuley, M. Sohoni, Geometric complexity theory III, on deciding positivity of Littlewood-Richardson coefficients, cs. ArXiv preprint cs. CC/0501076 v1 26 Jan 2005.
- [10] E. Rassart, A polynomiality property for Littlewood-Richardson coefficients, arXiv:math.CO/0308101, 16 Aug. 2003.
- [11] B. Sturmfels, On vector partition functions, *Journal of Combinatorial Theory Series A*, Volume 72 , Issue 2, November 1995.
- [12] E. Tardos, A strongly polynomial algorithm to solve combinatorial linear programs, *Operations Research* 34 (1986), 250-256.
- [13] A. Zelevinsky, Littlewood-Richardson semigroups, arXiv:math.CO/9704228 v1 30 Apr 1997.