

On Almost Periodicity Criteria for Morphic Sequences in Some Particular Cases

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Abstract

In some particular cases we give criteria for morphic sequences to be almost periodic (=uniformly recurrent). Namely, we deal with fixed points of non-erasing morphisms and with automatic sequences. In both cases a polynomial-time algorithm solving the problem is found. A result more or less supporting the conjecture of decidability of the general problem is given.

1 Introduction

Different problems of decidability in combinatorics on words are always of great interest and difficulty. Here we deal with two main types of symbolic infinite sequences — morphic and almost periodic — and try to understand connections between them. Namely, we are trying to find an algorithmic criterion which given a morphic sequence decides whether it is almost periodic.

Though the main problem still remains open, we propose polynomial-time algorithms solving the problem in two important particular cases: for pure morphic sequences generated by non-erasing morphisms (Section 3) and for automatic sequences (Section 4). In Section 5 we say a few words about connections with monadic logics. In particular, in a curious result of Corollary 4 we give a reason why the main problem may be decidable.

Some attempts to solve the problem were already done. In [3] A. Cobham gives a criterion for automatic sequence to be almost periodic. But even if his criterion gives some effective procedure solving the problem (which is not clear from his result, and he does not care about it at all), this procedure could not be fast. We construct a polynomial-time algorithm solving the problem. In [5] A. Maes deals with pure morphic sequences and finds a criterion for them to belong to a slightly different class of generalized almost periodic sequences (but he calls them almost periodic — see [9] for different definitions). And again, his algorithm does not seem to be polynomial-time.

All the results of this paper can be found in [10].

2 Preliminaries

Denote the set of natural numbers $\{0, 1, 2, \dots\}$ by \mathbb{N} and the binary alphabet $\{0, 1\}$ by \mathbb{B} . Let A be a finite alphabet. We deal with sequences over this alphabet, i. e., mappings $x: \mathbb{N} \rightarrow A$, and denote the set of these sequences by $A^{\mathbb{N}}$.

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Denote by A^* the set of all finite words over A including the empty word Λ . If $i \leq j$ are natural, denote by $[i, j]$ the segment of \mathbb{N} with ends in i and j , i. e., the set $\{i, i+1, i+2, \dots, j\}$. Also denote by $x[i, j]$ a subword $x(i)x(i+1)\dots x(j)$ of a sequence x . A segment $[i, j]$ is an occurrence of a word $u \in A^*$ in a sequence x if $x[i, j] = u$. We say that $u \neq \Lambda$ is a factor of x if u occurs in x . A word of the form $x[0, i]$ for some i is called prefix of x , and respectively a sequence of the form $x(i)x(i+1)x(i+2)\dots$ for some i is called suffix of x and is denoted by $x[i, \infty)$. Denote by $|u|$ the length of a word u . The occurrence $u = x[i, j]$ in x is k -aligned if $k|i$.

A sequence x is periodic if for some T we have $x(i) = x(i+T)$ for each $i \in \mathbb{N}$. This T is called a period of x . We denote by \mathcal{P} the class of all periodic sequences. Let us consider an extension of this class.

A sequence x is called *almost periodic*¹ if for every factor u of x there exists a number l such that every factor of x of length l contains at least one occurrence of u (and therefore u occurs in x infinitely many times). Obviously, to show almost periodicity of a sequence it is sufficient to check the mentioned condition only for all prefixes but not for all factors (and even for some increasing sequence of prefixes only). Denote by \mathcal{AP} the class of all almost periodic sequences.

Let A, B be finite alphabets. A mapping $\phi: A^* \rightarrow B^*$ is called a *morphism* if $\phi(uv) = \phi(u)\phi(v)$ for all $u, v \in A^*$. A morphism is obviously determined by its values on single-letter words. A morphism is *non-erasing* if $|\phi(a)| \geq 1$ for each $a \in A$. A morphism is k -uniform if $|\phi(a)| = k$ for each $a \in A$. A 1-uniform morphism is called a coding. For $x \in A^{\mathbb{N}}$ denote

$$\phi(x) = \phi(x(0))\phi(x(1))\phi(x(2))\dots$$

Further we consider only morphisms of the form $A^* \rightarrow A^*$ (but codings are of the form $A \rightarrow B$, which in fact does not matter, they can be also of the form $A \rightarrow A$ without loss of generality). Let $\phi(s) = su$ for some $s \in A, u \in A^*$. Then for all natural $m < n$ the word $\phi^n(s)$ begins with the word $\phi^m(s)$, so $\phi^\infty(s) = \lim_{n \rightarrow \infty} \phi^n(s) = su\phi(u)\phi^2(u)\phi^3(u)\dots$ is well-defined. If $\forall n \phi^n(u) \neq \Lambda$, then $\phi^\infty(s)$ is infinite. In this case we say that ϕ is *prolongable* on s . Sequences of the form $h(\phi^\infty(s))$ for a coding $h: A \rightarrow B$ are called *morphic*, of the form $\phi^\infty(a)$ are called *pure morphic*.

Notice that there exist almost periodic sequences that are not morphic (in fact, the set of almost periodic sequences has cardinality continuum, while the set of morphic sequences is obviously countable), as well as there exist morphic sequences that are not almost periodic (you will find examples later). Our goal is to determine whether a morphic sequence is almost periodic or not given its constructive definition.

First of all, observe the following

Lemma 1. *A sequence $\phi^\infty(s)$ is almost periodic iff s occurs in this sequence infinitely many times with bounded distances.*

Proof. In one direction the statement is obviously true by definition.

Suppose now that s occurs in $\phi^\infty(s)$ infinitely many times with bounded distances. Then for every m the word $\phi^m(s)$ also occurs in $\phi^\infty(s)$ infinitely many times with bounded distances. But every word u occurring in $\phi^\infty(s)$ occurs in some prefix $\phi^m(s)$ and thus occurs infinitely many times with bounded distances. \square

For a morphism $\phi: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ we can define a corresponding matrix $M(\phi)$, such that $M(\phi)_{ij}$ is a number of occurrences of symbol i into $\phi(j)$. One can easily check that for each l we have $M(\phi)^l = M(\phi^l)$.

Morphism ϕ is called *primitive* if for some l all the numbers in $M(\phi^l)$ are positive.

¹It was called *strongly* or *strictly almost periodic* in [7, 8].

Let us construct an oriented graph G corresponding to a morphism. Let its set of vertices be A . In G edges go from $b \in A$ to all the symbols occurring in $\phi(b)$.

For $\phi^\infty(s)$ it can easily be found using the graph corresponding to ϕ which symbols from A really occur in this sequence. Indeed, these symbols form the set of all vertices that can be reached from s . So without loss of generality from now on we assume that all the symbols from A occur in $\phi^\infty(s)$.

A morphism is primitive if and only if its corresponding graph is strongly connected, i. e., there exists an oriented path between every two vertices. This reformulation of the primitiveness notion seems to be more appropriate for computational needs.

By Lemma 1 (and the observation that codings preserve almost periodicity) morphic sequences obtained by primitive morphisms are always almost periodic. Moreover, in the case of increasing morphisms (such that $|\phi(b)| \geq 2$ for each b) this sufficient condition is also necessary (and this is a polynomial-time algorithmic criterion). However when we generalize this case even on non-erasing morphisms, it is not enough to consider only the corresponding graph or even the matrix of morphism (which has more information), as it can be seen from the following example.

Let ϕ_1 be as follows: $0 \rightarrow 01$, $1 \rightarrow 120$, $2 \rightarrow 2$, and ϕ_2 be as follows: $0 \rightarrow 01$, $1 \rightarrow 210$, $2 \rightarrow 2$. Then these two morphisms have identical matrices of morphism, but $\phi_1^\infty(0)$ is almost periodic, while $\phi_2^\infty(0)$ is not. Indeed, in $\phi_2^\infty(0)$ there are arbitrary long segments like $222 \dots 22$, so $\phi_2^\infty(0) \notin \mathcal{AP}$. There is no such problem in $\phi_1^\infty(0)$. Since 0 occurs in both $\phi_1(0)$ and $\phi_1(1)$, and 22 does not occur in $\phi_1^\infty(0)$, it follows that 0 occurs in $\phi_1^\infty(0)$ with bounded distances. Thus $\phi_1^m(0)$ for every $m \geq 0$ occurs in $\phi_1^\infty(0)$ with bounded distances, so $\phi_1^\infty(0) \in \mathcal{AP}$. See Theorem 1 for a general criterion of almost periodicity in the case of fixed points of non-erasing morphisms.

To introduce a bit the notion of almost periodicity, let us formulate an interesting result on this topic. It seems to be first proved in [3], but also follows from the results of [9]. For $x \in A^\mathbb{N}$, $y \in B^\mathbb{N}$ define $x \times y \in (A \times B)^\mathbb{N}$ such that $(x \times y)(i) = \langle x(i), y(i) \rangle$.

Proposition 1. *If x is almost periodic and y is periodic, then $x \times y$ is almost periodic.*

3 Pure Morphic Sequences Generated by Non-erasing Morphisms

Here we consider the case of morphic sequence of the form $\phi^\infty(s)$ for non-erasing ϕ . We present an algorithm that determines whether a morphic sequence $\phi^\infty(s)$ is almost periodic given an alphabet A , a morphism ϕ and a symbol $s \in A$.

Suppose we have A , ϕ and $s \in A$, such that $|A| = n$, $\max_{b \in A} |\phi(b)| = k$, s begins $\phi(s)$. Remember that we suppose that all the symbols from A appear in $\phi^\infty(s)$.

Divide A into two parts. Let I be the set of all symbols $b \in A$ such that $|\phi^m(b)| \rightarrow \infty$ as $m \rightarrow \infty$. Denote $F = A \setminus I$, it is the set of all symbols b such that $|\phi^m(b)|$ is bounded. Also define $E \subseteq F$ to be the set of all symbols b such that $|\phi(b)| = 1$.

We can find a decomposition $A = I \sqcup F$ in $\text{poly}(n, k)$ -time as follows.

Find E . Then find all the cycles in G with all the vertices lying in E . Join all the vertices of all these cycles in a set D . This set is stabilizing: F is the set of all vertices in G such that all infinite paths starting from them stabilize in D . Polynomiality can be checked easily.

Construct “a graph of left tails” L with marked edges. Its set of vertices is I . From each vertex b exactly one edge goes off. To construct this edge, find a representation $\phi(b) = uv$, where $c \in I$, u is the maximal prefix of $\phi(b)$ containing only symbols from F . It follows from the definitions of I and F that u does not coincide with $\phi(b)$, that is why this representation is correct. Then construct in L an edge from b to c and write u on it.

Analogously we construct “a graph of right tails” R . (In this case we consider representations $\phi(b) = vu$ where $u \in F^*$, $c \in I$.)

Now we formulate a general criterion.

Theorem 1. *A sequence $\phi^\infty(s)$ is almost periodic iff*

- 1) *G restricted to I is strongly connected;*
- 2) *in graphs L and R on each edge of each cycle an empty word Λ is written.*

It seems that full and detailed proof of this theorem can only confuse a reader, rather than a proof sketch.

Proof sketch. By Lemma 1 for almost periodicity it is necessary and sufficient to check whether symbol s occurs infinitely many times with bounded distances.

For every symbol $b \in I$ the symbol s should occur in some $\phi^l(b)$, that is what the 1st part of the criterion says.

Furthermore, in the sequence $\phi^\infty(s)$ all the segments of consecutive symbols from F should be bounded. Indeed, every such segment consists only of symbols from F , but $s \notin F$. That is what the 2nd part of the criterion means, let us explain why.

Consider some $v = buc$ occurring somewhere in $\phi^\infty(a)$, where $b, c \in I$, $u \in F^*$. Every element of sequence of words $v, \phi(v), \phi^2(v), \phi^3(v), \dots$ occurs in $\phi^\infty(s)$. Somewhere in the middle of $\phi^l(v) = \phi^l(b)\phi^l(u)\phi^l(c)$ a word $\phi^l(u)$ occurs. As l increases, some words from F^* might stick to $\phi^l(u)$ from left or right for these words can come from $\phi^l(b)$ or $\phi^l(c)$. These words exactly correspond to those written on edges of L or R . The 2nd part of the criterion exactly says that this situation can happen only finitely many times, until we get to some cycle in L or R . \square

Let us consider examples with ϕ_1 and ϕ_2 from the end of Section 2. In both cases $I = \{0, 1\}$, $F = \{2\}$. On every edge of R in both cases Λ is written. Almost the same is true for L : the only difference is about the edge going from 1 to 1. In the case of ϕ_1 an empty word is written on this edge, while in the case of ϕ_2 a word 2 is written. That is why $\phi_1^\infty(0)$ is almost periodic, while $\phi_2^\infty(0)$ is not.

Corollary 1. *If for all $b \in A$ we have $|\phi(b)| \geq 2$, then $\phi^\infty(s)$ is almost periodic iff ϕ is primitive.*

Proof. Follows from Theorem 1. In that case $A = I$, and on all the edges of L and R the empty word is written. \square

Corollary 2. *There exists a $\text{poly}(n, k)$ -algorithm that says whether $\phi^\infty(s)$ is almost periodic.*

Proof. Conditions from Theorem 1 can be checked in polynomial time. \square

It also seems useful to formulate an explicit version of the criterion for the binary case. We do it without any additional assumptions, opposite to the previous.

Corollary 3. *For non-erasing $\phi: \mathbb{B} \rightarrow \mathbb{B}$ that is prolongable on 0 a sequence $\phi^\infty(0)$ is almost periodic iff one of the following conditions holds:*

- 1) *$\phi(0)$ contains only 0s;*
- 2) *$\phi(1)$ contains 0;*
- 3) *$\phi(1) = \Lambda$;*
- 4) *$\phi(1) = 1$ and $\phi(0) = 0u0$ for some word u .*

4 Uniform Morphisms

Now we deal with morphic sequences obtained by uniform morphisms. Again we present a polynomial-time algorithm for solving the problem in this situation.

Suppose we have an alphabet A , a morphism $\phi: A^* \rightarrow A^*$, a coding $h: A \rightarrow B$, and $s \in A$, such that $|A| = n$, $|B| \leq n$, $\forall b \in A \ |\phi(b)| = k$, s begins $\phi(s)$. We are interested in whether $h(\phi^\infty(s))$ is almost periodic. Sequences of the form $h(\phi^\infty(s))$ with ϕ being k -uniform are also called k -automatic (see [1]).

4.1 Equivalence Relations and Uniform Morphisms

For each $l \in \mathbb{N}$ define an equivalence relation on A : $b \sim_l c$ iff $h(\phi^l(b)) = h(\phi^l(c))$. We can easily continue this relation on A^* : $u \sim_l v$ iff $h(\phi^l(u)) = h(\phi^l(v))$. In fact, this means $|u| = |v|$ and $u(i) \sim_l v(i)$ for all i , $1 \leq i \leq |u|$.

Let B_m be the Bell number, i. e., the number of all possible equivalence relations on a finite set with exactly m elements, see [13]. As it follows from this article, we can estimate B_m in the following way.

Lemma 2. $2^m \leq B_m \leq 2^{Cm \log m}$ for some constant C .

Thus the number of all possible relations \sim_l is not greater than $B_n = 2^{O(n \log n)}$. Moreover, the following lemma gives a simple description for the behavior of these relations as l tends to infinity.

Lemma 3. If \sim_r equals \sim_s , then \sim_{r+p} equals \sim_{s+p} for all p .

Proof. Indeed, suppose \sim_r equals \sim_s . Then $b \sim_{r+1} c$ iff $\phi(b) \sim_r \phi(c)$ iff $\phi(b) \sim_s \phi(c)$ iff $b \sim_{s+1} c$. So if \sim_r equals \sim_s , then \sim_{r+1} equals \sim_{s+1} , which implies the lemma statement. \square

This lemma means that the sequence $(\sim_l)_{l \in \mathbb{N}}$ turns out to be ultimately periodic with a period and a preperiod both not greater than B_n . Thus we obtain the following

Lemma 4. For some $p, q \leq B_n$ we have for all i and all $t > p$ that \sim_t equals \sim_{t+iq} .

4.2 Criterion

Now we are trying to get a criterion which we could check in polynomial time. Notice that the situation is much more difficult than in the pure case because of a coding allowed. In particular, the analogue of Lemma 1 for non-pure case does not hold.

We will move step by step to the appropriate version of the criterion reformulating it several times.

This proposition is quite obvious and follows directly from the definition of almost periodicity since all $h(\phi^m(a))$ are the prefixes of $h(\phi^\infty(a))$.

Proposition 2. A sequence $h(\phi^\infty(s))$ is almost periodic iff for all m the word $h(\phi^m(s))$ occurs in $h(\phi^\infty(s))$ infinitely often with bounded distances.

And now a bit more complicated version.

Proposition 3. A sequence $h(\phi^\infty(s))$ is almost periodic iff for all m the symbols that are \sim_m -equivalent to s occur in $\phi^\infty(s)$ infinitely often with bounded distances.

Proof. \Leftarrow . If the distance between two consecutive occurrences in $\phi^\infty(s)$ of symbols that are \sim_m -equivalent to s is not greater than t , then the distance between two consecutive occurrences of $h(\phi^m(s))$ in $h(\phi^\infty(s))$ is not greater than tk^m .

\Rightarrow . Suppose $h(\phi^\infty(s))$ is almost periodic. Let $y_m = 012 \dots (k^m - 2)(k^m - 1)01 \dots (k^m - 1)0 \dots$ be a periodic sequence with a period k^m . Then by Proposition 1 a sequence $h(\phi^\infty(s)) \times y_m$ is almost periodic, which means that the distances between consecutive k^m -aligned occurrences of $h(\phi^m(s))$ in $h(\phi^\infty(s))$ are bounded. It only remains to notice that if $h(\phi^\infty(s))[ik^m, (i+1)k^m - 1] = h(\phi^m(s))$, then $\phi^\infty(s)(i) \sim_m s$. \square

Let Y_m be the following statement: symbols that are \sim_m -equivalent to s occur in $\phi^\infty(s)$ infinitely often with bounded distances.

Suppose for some T that Y_T is true. This implies that $h(\phi^T(s))$ occurs in $h(\phi^\infty(s))$ with bounded distances. Therefore for all $m \leq T$ a word $h(\phi^m(s))$ occurs in $h(\phi^\infty(s))$ with bounded distances since $h(\phi^m(s))$ is a prefix of $h(\phi^T(s))$. Thus we do not need to check the statements Y_m for all m , but only for all $m \geq T$ for some T .

Furthermore, it follows from Lemma 4, that we are sufficient to check the only one such statement as in the following

Proposition 4. *For all $r \geq B_n$: a sequence $h(\phi^\infty(s))$ is almost periodic iff the symbols that are \sim_r -equivalent to s occur in $\phi^\infty(s)$ infinitely often with bounded distances.*

And now the final version of our criterion.

Proposition 5. *For all $r \geq B_n$: a sequence $h(\phi^\infty(s))$ is almost periodic iff for some m the symbols that are \sim_r -equivalent to s occur in $\phi^m(b)$ for all $b \in A$.*

Indeed, if the symbols of some set occur with bounded distances, then they occur on each k^m -aligned segment for some sufficiently large m .

4.3 Polynomiality

Now we explain how to check a condition from Proposition 5 in polynomial time. We need to show two things: first, how to choose some $r \geq B_n$ and to find in polynomial time the set of all symbols that are \sim_r -equivalent to s (and this is a complicated thing keeping in mind that B_n is exponential), and second, how to check whether for some m the symbols from this set for all $b \in A$ occur in $\phi^m(b)$.

Let us start from the second. Suppose we have found the set H of all the symbols that are \sim_r -equivalent to s . For $m \in \mathbb{N}$ let us denote by $P_m^{(b)}$ the set of all the symbols that occur in $\phi^m(b)$. Our aim is to check whether exists m such that for all b we have $P_m^{(b)} \cap H \neq \emptyset$. First of all, notice that if $\forall b P_m^{(b)} \cap H \neq \emptyset$, then $\forall b P_l^{(b)} \cap H \neq \emptyset$ for all $l \geq m$. Second, notice that the sequence of tuples of sets $((P_m^{(b)})_{b \in \Sigma})_{m=0}^\infty$ is ultimately periodic. Indeed, the sequence $(P_m^{(b)})_{m=0}^\infty$ is obviously ultimately periodic with both period and preperiod not greater than 2^n (recall that n is the size of the alphabet Σ). Thus the period of $((P_m^{(b)})_{b \in \Sigma})_{m=0}^\infty$ is not greater than the least common divisor of that for $(P_m^{(b)})_{m=0}^\infty$, $b \in A$, and the preperiod is not greater than the maximal that of $(P_m^{(b)})_{m=0}^\infty$. So the period is not greater than $(2^n)^n = 2^{n^2}$ and the preperiod is not greater than 2^n . Third, notice that there is a polynomial-time-procedure that given a graph corresponding to some morphism ψ (see Section 2 to recall what is the graph corresponding to a morphism) outputs a graph corresponding to morphism ψ^2 . Thus after repeating this procedure $n^2 + 1$ times we obtain a graph by which we can easily find $(P_{2^{n^2} + 2^n}^{(b)})_{b \in \Sigma}$, since $2^{n^2+1} > 2^{n^2} + 2^n$.

Similar arguments, even described with more details, are used in deciding our next problem. Here we present a polynomial-time algorithm that finds the set of all symbols that are \sim_r -equivalent to s for some $r \geq B_n$.

We recursively construct a series of graphs T_i . Let its common set of vertices be the set of all unordered pairs (b, c) such that $b, c \in A$ and $b \neq c$. Thus the number of vertices is $\frac{n(n-1)}{2}$. The set of all vertices connected with (b, c) in the graph T_i we denote by $V_i(b, c)$.

Define a graph T_0 . Let $V_0(b, c)$ be the set $\{(\phi(b)(j), \phi(c)(j)) \mid j = 1, \dots, k, \phi(b)(j) \neq \phi(c)(j)\}$. In other words, $b \sim_{l+1} c$ if and only if $x \sim_l y$ for all $(x, y) \in V_0(b, c)$.

Thus $b \sim_2 c$ if and only if for all $(x, y) \in V_0(b, c)$ for all $(z, t) \in V_0(x, y)$ we have $z \sim_0 t$. For the graph T_1 let $V_1(b, c)$ be the set of all (x, y) such that there is a path of length 2 from (b, c) to (x, y) in T_0 . The graph T_1 has the following property: $b \sim_2 c$ if and only if $x \sim_0 y$ for all $(x, y) \in V_1(b, c)$. And even more generally: $b \sim_{l+2} c$ if and only if $x \sim_l y$ for all $(x, y) \in V_1(b, c)$.

Now we can repeat operation made with T_0 to obtain T_1 . Namely, in T_2 let $V_2(b, c)$ be the set of all (x, y) such that there is a path of length 2 from (b, c) to (x, y) in T_1 . Then we obtain: $b \sim_{l+4} c$ if and only if $x \sim_l y$ for all $(x, y) \in V_2(b, c)$.

It follows from Lemma 2 that $\log_2 B_n \leq Cn \log n$. Thus after we repeat our procedure $r = \lceil Cn \log n \rceil$ times, we will obtain the graph T_l such that $b \sim_{2^r} c$ if and only if $x \sim_0 y$ for all $(x, y) \in V_2(b, c)$. Recall that $x \sim_0 y$ means $h(x) = h(y)$, so now we can easily compute the set of symbols that are \sim_{2^r} -equivalent to s .

5 Monadic Theories

Combinatorics on words is closely connected with the theory of second order monadic logics. Here we just want to show some examples of these connections. More details can be found, e. g., in [11, 12].

We consider monadic logics on \mathbb{N} with the relation “ $<$ ”, that is, first-order logics where also unary finite-value function variables and quantifiers over them are allowed. We also suppose that we know some fixed finite-value function $x: \mathbb{N} \rightarrow \Sigma$ and can use it in our formulas. Such a theory is denoted by $\text{MT}\langle \mathbb{N}, <, x \rangle$ and is called *monadic theory of x* .

The main question here can be the question of decidability, that is, does there exist an algorithm that given a sentence in a theory says whether this sentence is true or false.

The criterion of decidability for monadic theories of almost periodic sequences can be formulated in terms of some their very natural characteristic, namely, almost periodicity regulator. An almost periodicity regulator of an almost periodic sequence x is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that every factor u of x of length n occurs in each factor of x of length $f(n)$. So an almost periodicity regulator somehow regulates how periodic a sequence is. Notice that an almost periodicity regulator of a sequence is not unique: every function greater than regulator is also a regulator.

Theorem 2 (Semenov 1983 [12]). *If x is almost periodic, then $\text{MT}\langle \mathbb{N}, <, x \rangle$ is decidable iff x and some its almost periodicity regulator are computable.*

The following result was obtained recently, but uses the technics already used in [11, 12].

Theorem 3 (Carton, Thomas 2002 [2]). *If x is morphic, then $\text{MT}\langle \mathbb{N}, <, x \rangle$ is decidable.*

A curious result can be implied from two these theorems.

Corollary 4. *If x is both morphic and almost periodic, then some its regulator is computable.*

Proof. Indeed, if x is morphic, then by Theorem 3 the theory $MT(\mathbb{N}, <, x)$ is decidable. Since x is almost periodic, from Theorem 2 it follows that some almost periodicity regulator of x is computable. \square

Notice that Corollary 4 does not imply the existence of an algorithm that given a morphic sequence computes some almost periodicity regulator of this sequence whenever it is almost periodic (but probably this algorithm can be constructed after deep analyzing the proofs of Theorems 2 and 3 and showing uniformity in a sense). And it also does not imply the decidability of almost periodicity for morphic sequences. This decidability also does not imply Corollary 4.

By the way, Corollary 4 allows us to hope that these algorithms exist. Though the formulation of this statement uses only combinatorics on words, the proof also involves the theory of monadic logics. Of course, it would be interesting to find a simple combinatorial proof of the result.

And the last remark here is that Corollary 4 (and its probable uniform version) seems to be the best progress that we can obtain by this monadic approach. One could try to express in the monadic theory of morphic sequence (which is decidable by Theorem 3) the property of almost periodicity, but it turns out to be impossible.

6 In General Case

We have described two polynomial-time algorithms, but without any precise bound for their working time. Of course, it can be done after deep analyzing of all the previous, but is probably not so interesting.

It is not still known whether the problem of determining almost periodicity of arbitrary morphic sequence is decidable. Corollary 4 somehow supports the conjecture of decidability (but even does not follow from this conjecture!).

Theorem 7.5.1 from [1] allows us to represent an arbitrary morphic sequence $h(\phi^\infty(s))$ as $g(\psi^\infty(b))$ where ψ is non-erasing. So it is sufficient to solve our main problem for $h(\phi^\infty(s))$ with non-erasing ϕ .

It seems that the general problem is tightly connected with a particular case of $h(\phi^\infty(a))$ where $|\phi(b)| \geq 2$ for each $b \in A$. There is no strict reduction to this case but solving problem in this case can help to deal with general situation.

The problem of finding an effective periodicity criterion in the case of arbitrary morphic sequences is also of great interest, as well as criteria for variations with periodicity and almost periodicity: ultimate periodicity, generalized almost periodicity, ultimate almost periodicity (see [9] for definitions). If one notion is a particular case of another, it does not mean that corresponding criterion for the first case is more difficult (or less difficult) than for the second.

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