Injective Morita Contexts (Revisited)

Dedicated to Prof. Robert Wisbauer

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Abstract

This paper is an exposition of the so-called *injective Morita contexts* (in which the connecting bimodule morphisms are injective) and *Morita α-contexts* (in which the connecting bimodules enjoy some local projectivity in the sense of Zimmermann-Huisgen). Motivated by situations in which only one trace ideal is in action, or the compatibility between the bimodule morphisms is not needed, we introduce the notions of Morita *semi-contexts* and *Morita data*, and investigate them. Injective Morita data will be used (with the help of *static* and *adstatic modules*) to establish equivalences between some *intersecting subcategories* related to subcategories of modules that are localized or colocalized by trace ideals of a Morita datum. We end up with applications of Morita α-contexts to $*$-*modules* and *injective right wide Morita contexts.*

1 Introduction

Morita contexts, in general, and (*semi-*)*strict Morita* contexts (with surjective connecting bilinear morphisms), in particular, were extensively studied and developed exponentially during the last few decades (e.g. [AGH-Z1997]). However, we sincerely feel that there is a gap in the literature on *injective Morita contexts* (i.e. those with injective connecting bilinear morphisms). Apart from the results in [Nau1994-a], [Nau1994-b] (where the second author initially explored this notion) and from an application to Grothendieck groups in the recent paper ([Nau2004]), it seems that injective Morita contexts were not studied *systematically* at all.

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It is noticed in ([Nau1993], [Nau1994-a] and [Nau1994-b]) that in several results related to Morita contexts, only one trace ideal is used. Observing this fact, we introduce the notions of *Morita semi-contexts* and *Morita data* and investigate them. Several results are proved then for *injective* Morita semi contexts and injective Morita data.

Consider a Morita datum $\mathcal{M} = (T, S, P, Q, <,>_T, <,>_S)$, with not necessarily compatible bimodule morphisms $<,>_T$: $P \otimes_S Q \to T$ and $<,>_S$: $Q \otimes_T P \to S$. We say that $\mathcal{M}$ is *injective*, iff $<,>_T$ and $<,>_S$ are injective, and to be a *Morita α-datum*, iff the associated dual pairings $\mathbf{P}_l := (Q, {}_TP)$, $\mathbf{P}_r := (Q, P_S)$, $\mathbf{Q}_l := (P, {}_SQ)$ and $\mathbf{Q}_r := (P, Q_T)$ satisfy the α-condition (which is closely related to the notion of local projectivity in the sense of Zimmermann-Huisgen [Z-H1976]). The α-condition was introduced in [AG-TL2001] and further investigated by the first author in [Abu2005].

While (semi-)strict unital Morita contexts induce equivalences between the whole module categories of the rings under consideration, we show in this paper how injective Morita (semi-)contexts and injective Morita data play an important role in establishing equivalences between suitable *intersecting subcategories* of module categories (e.g. intersections of subcategories that are localized/colocalized by trace ideals of a Morita datum with subcategories of static/adstatic modules, etc.). Our main applications in addition to equivalences related to the Kato-Ohtake-Müller *localization-colocalization theory* (developed in [Kat1978], [KO1979] and [Mül1974]), will be to $*$*-modules* (introduced by Menini and Orsatti [MO1989]) and to *right wide Morita contexts* (introduced by F. Castaño Iglesias and J. Gómez-Torrecillas [C-IG-T1995]).

Most of our results will be stated for *left modules*, while deriving the "dual" versions for right modules is left to the interested reader. Moreover, for Morita contexts, some results are stated/proved for only one of the Morita semi-contexts, as the ones corresponding to the second semi-context can be obtained analogously. For the convenience of the reader, we tried to make the paper self-contained, so that it can serve as a reference on injective *Morita* (*semi-*)*contexts* and their applications. In this respect, and for the sake of completeness, we have included some pervious results of the authors that are (in most cases) either provided with new shorter proofs, or are obtained under weaker conditions.

This paper is organized as follows: After this brief introduction, we give in Section 2 some preliminaries including the basic properties of *dual α-pairings,* which play a central role in rest of the work. The notions of *Morita semi-contexts* and *Morita data* are introduced in Section 3, where we clarified their relations with the *dual pairings* and the so-called *elementary rings*. *Injective Morita* (*semi-*)*contexts* appear in Section 4, where we study their interplay with dual α-pairings and provide some examples and a counter-examples. In Section 5 we include some observations regarding *static* and *adstatic* modules and involve them to obtain equivalences among suitable *intersecting subcategories* of modules related to a Morita (semi-)context. In the last section, more applications are presented, mainly to subcategories of modules that are *localized* or *colocalized* by a trace ideal of an injective Morita (semi-)context, to $*$*-modules* and to *injective right wide Morita contexts.*

2 Preliminaries

Throughout, R denotes a commutative ring with $1_R \neq 0_R$ and A, A', B, B' are unital R-algebras. We have reserved the term "*ring*" for an associative ring with a multiplicative unity, and we will use the term "*rng*" for a general associative ring (not necessarily with unity). All modules of rings are assumed to be unitary, and ring morphisms are assumed to respect multiplicative unities. If $\mathfrak{T}$ and $\mathfrak{S}$ are categories, then we write $\mathfrak{T} \leq \mathfrak{S}$ to mean that $\mathfrak{T}$ is a subcategory of $\mathfrak{S}$, and $\mathfrak{T} \approx \mathfrak{S}$ to indicate that $\mathfrak{T}$ and $\mathfrak{S}$ are equivalent.

Rngs and their modules

2.1. By an A-**rng**, we mean an (A, A)-bimodule T with an (A, A)-bilinear morphism $\mu_T : T \otimes_A T \to T$, such that $\mu_T \circ (\mu_T \otimes_A id_T) = \mu_T \circ (id_T \otimes_A \mu_T)$. We call an A-rng (T, μ_T) an A-**ring**, iff there exists in addition an (A, A)-bilinear morphism $\eta_T : A \to T$, called the **unity map**, such that $\mu_T \circ (\eta_T \otimes_A id_T) = \vartheta_T^l$ and $\mu_T \circ (id_T \otimes_A \eta_T) = \vartheta_T^r$ (where $A \otimes_A T \overset{\vartheta_T^l}{\simeq} A$ and $T \otimes_A A \overset{\vartheta_T^r}{\simeq} T$ are the canonical isomorphisms). So, an A-ring is a unital A-rng; and an A-rng is (roughly speaking) an A-ring not necessarily with unity.

2.2. A morphism of rngs $(\psi : \delta) : (T : A) \to (T' : A')$ consists of a morphism of R-algebras $\delta : A \to A'$ and an (A, A)-bilinear morphism $\psi : T \to T'$, such that $\mu_{T'} \circ \chi_{(T',T')}^{(A,A')} \circ (\psi \otimes_A \psi) = \psi \circ \mu_T$ (where $\chi_{(T',T')}^{(A,A')} : T' \otimes_A T' \to T' \otimes_{A'} T'$ is the canonical map induced by δ). By $\mathbb{RNG}$ we denote the category of associative rngs and rng morphisms and by $\mathbb{URNG} < \mathbb{RNG}$ the (non-full) subcategory of *unital* rings with morphisms being the morphisms in $\mathbb{RNG}$ which respect multiplicative unities.

2.3. Let (T, μ_T) be an A-rng. By a **left** T**-module** we mean a left A-module N with a left A-linear morphism $\phi_T^N : T \otimes_A N \to N$, such that $\phi_T^N \circ (\mu_T \otimes_A id_N) = \phi_T^N \circ (id_T \otimes_A \phi_T^N)$. For left T-modules M, N, we call a left A-linear morphism $f : M \to N$ a T**-linear morphism**, iff $f(tm) = tf(m)$ for all $t \in T$. The category of left T-modules and left T-linear morphisms is denoted by ${}_T\mathbb{M}$. The category $\mathbb{M}_T$ of right T-modules is defined analogously. Let $(T : A)$ and $(T' : A')$ be rngs. We call an (A, A')-bimodule N a (T, T')**-bimodule**, iff (N, ϕ_T^N) is a left T-module and $(N, \phi_{T'}^N)$ is a right T'-module, such that $\phi_{T'}^N \circ (\phi_T^N \otimes_{A'} id_{T'}) = \phi_T^N \circ (id_T \otimes_A \phi_{T'}^N)$. For (T, T')-bimodules M, N, we call an (A, A')-bilinear morphism $f : M \to N$ (T, T')**-bilinear**, provided f is left T-linear and right T'-linear. The category of (T, T')-bimodules is denoted by ${}_T\mathbb{M}_{T'}$. In particular, for any A-rng T, a left (right) T-module M has a canonical structure of a *unitary* right (left) S-module, where $S := \mathrm{End}({}_TM)^{op}$ ($S := \mathrm{End}(M_T)$); and moreover, with this structure M becomes a (T, S)-bimodule (an (S, T)-bimodule).

Notation. Let T be an A-rng. We write ${}_TU$ (U_T) to denote that U is a left (right) T-module. For a left (right) T-module ${}_TU$, we consider the set ${}^*U := \mathrm{Hom}_{T-}(U, T)$ ($U^* := \mathrm{Hom}_{-T}(U, T)$) of all left (right) T-linear morphisms from U to T with the canonical right (left) T-module structure.

Generators and cogenerators

Definition 2.4. Let T be an A-rng. For a left T-module ${}_TU$ consider the following subclasses of ${}_T\mathbb{M}$:

$$\begin{array}{lcl} \mathrm{Gen}({}_TU) & := & \{{}_TV \mid \exists \text{ a set } \Lambda \text{ and an exact sequence } U^{(\Lambda)} \to V \to 0\}; \\ \mathrm{Cogen}({}_TU) & := & \{{}_TW \mid \exists \text{ a set } \Lambda \text{ and an exact sequence } 0 \to W \to U^{\Lambda}\}; \\ \mathrm{Pres}({}_TU) & := & \{{}_TV \mid \exists \text{ sets } \Lambda_1, \Lambda_2 \text{ and an exact sequence } U^{(\Lambda_2)} \to U^{(\Lambda_1)} \to V \to 0\}; \\ \mathrm{Copres}({}_TU) & := & \{{}_TW \mid \exists \text{ sets } \Lambda_1, \Lambda_2 \text{ and an exact sequence } 0 \to W \to U^{\Lambda_1} \to U^{\Lambda_2}\}; \end{array}$$

A left T-module in $\mathrm{Gen}({}_TU)$ (respectively $\mathrm{Cogen}({}_TU)$, $\mathrm{Pres}({}_TU)$, $\mathrm{Copres}({}_TU)$) is said to be **U-generated** (respectively **U-cogenerated**, **U-presented**, **U-copresented**). Moreover, we say that ${}_TU$ is a **generator** (respectively **cogenerator**, **presentor**, **copresentor**), iff $\mathrm{Gen}({}_TU) = {}_T\mathbb{M}$ (respectively $\mathrm{Cogen}({}_TU) = {}_T\mathbb{M}$, $\mathrm{Pres}({}_TU) = {}_T\mathbb{M}$, $\mathrm{Copres}({}_TU) = {}_T\mathbb{M}$).

Throughout, for any A-rng T we denote with ${}_T\mathbf{E}$ an arbitrary, but fixed, injective cogenerator in ${}_T\mathbb{M}$.

Notation. Let T be an A-rng. For any left T-module ${}_TV$, we set ${}^{\#}V := \mathrm{Hom}_T(V, {}_T\mathbf{E})$. If moreover, ${}_TV_S$ is a (T, S)-bimodule for some B-rng S, then we consider ${}^{\#}_{S}V$ with the left S-module structure induced by that of V_S.

Lemma 2.5. (Compare [Col1990, Lemma 3.2.], [CF2004, Lemmas 2.1.2., 2.1.3.]) *Let T be an A-rng, S a B-rng and ${}_TV_S$ an (T, S)-bimodule,*

1. *A left T-module ${}_TK$ is V-generated if and only if the canonical T-linear morphism*

$$\omega^l_{V,K} : V \otimes_S \mathrm{Hom}_T(V, K) \to K \tag{1}$$

is surjective. Moreover, $V \otimes_S W \subseteq \mathrm{Pres}({}_TV) \subseteq \mathrm{Gen}({}_TV)$ for every left S-module ${}_SW$.

2. *A left S-module ${}_SL$ is ${}^{\#}_{S}V$-cogenerated if and only if the canonical S-linear morphism*

$$\eta^l_{V,L} : L \to \mathrm{Hom}_T(V, V \otimes_S L) \tag{2}$$

is injective. Moreover, $\mathrm{Hom}_T(V, M) \subseteq \mathrm{Copres}({}^{\#}_{S}V) \subseteq \mathrm{Cogen}({}^{\#}_{S}V)$ for every left T-module ${}_TM$.

Remark 2.6. Let S be a B-rng, V_S a right S-module and consider for every left S-module ${}_SL$ the annihilator $\mathrm{ann}^{\otimes}_L(V_S) := \{l \in L \mid V \otimes_S l = 0\}$. Following [AF1974, Exercises 19], we say V_S is **L-faithful**, iff $\mathrm{ann}^{\otimes}_L(V_S) = 0$; and to be **completely faithful**, iff V_S is L-faithful for every left S-module ${}_SL$. Let ${}_TV_S$ be a (T, S)-bimodule (e.g. $T := \mathrm{End}(V_S)$). Then $\mathrm{ann}^{\otimes}_L(V_S) = \mathrm{Ker}(\eta^l_{V,L})$, whence (by Lemma 2.5 "2") V_S is L-faithful if and only if ${}_SL$ is ${}^{\#}_{S}V$-cogenerated. It follows then that V_S is completely faithful if and only if ${}^{\#}_{S}V$ is a cogenerator.

Dual α-pairings

In what follows we recall the definition and properties of dual α-pairings introduced in [AG-TL2001, Definition 2.3.] and studied further in [Abu2005].

2.7. Let T be an A-rng. A **dual left T-pairing** $\mathbf{P}_l = (V, {}_TW)$ consists of a left T-module W and a right T-module V with a right T-linear morphism $\kappa_{\mathbf{P}_l} : V \to {}^*W$ (equivalently a left T-linear morphism $\chi_{\mathbf{P}_l} : W \to V^*$). For dual pairings $\mathbf{P}_l = (V, {}_TW)$, $\mathbf{P}_l' = (V', {}_{T'}W')$, a morphism of dual left pairings $(\xi, \theta) : (V', W') \to (V, W)$ consists is a triple

$$(\xi, \theta : \varsigma) : (V,\ {}_TW) \to (V,\ {}_{T'}W'),$$

where $\psi : V \to V'$ and $\theta : W' \to W$ are T-linear and $\varsigma : T \to T'$ is a morphism of rngs, such that considering the induced maps $<,>_T: V \times W \to T$ and $<,>_{T'}: V' \times W' \to T'$ we have

$$< \xi(v), w' >_{T'} = \varsigma(< v, \theta(w') >_T) \text{ for all } v \in V \text{ and } w' \in W'. \qquad (3)$$

The dual left pairings with the morphisms defined above build a category, which we denote by $\mathcal{P}_l$. With $\mathcal{P}_l(T) < \mathcal{P}_l$ we denote the subcategory of dual T-pairings. The category $\mathcal{P}_r$ of dual right pairings and its full subcategory $\mathcal{P}_r(T) < \mathcal{P}_r$ of dual right T-pairings are defined analogously.

Remark 2.8. The reader should be warned that (in general) for a non-commuative rng T and a dual left T-pairing $\mathbf{P}_l = (V, {}_TW)$, the following map induced by the right T-linear morphism $\kappa_{\mathbf{P}_l} : V \to {}^*W$:

$$<,>_T: V \times W \to T, \quad < v, w >_T := \kappa_{\mathbf{P}_l}(v)(w)$$

is not necessarily T-*balanced*, and so does not induce (in general) a map $V \otimes_T W \to T$. In fact, for all $v \in V$, $w \in W$ and $t \in T$ we have

$$\begin{array}{lllllllll} < vt, w > & = & \kappa_{\mathcal{P}_l}(vt)(w) & = & [\kappa_{\mathcal{P}_l}(v)t](w) & = & [\kappa_{\mathcal{P}_l}(v)(w)]t & = & < v, w >_T t; \\ < v, tw > & = & \kappa_{\mathcal{P}_l}(v)(tw) & = & t[\kappa_{\mathcal{P}_l}(v)(w)] & = & t < v, w >_T . & & \end{array}$$

2.9. Let T be an A-rng, N, W be left T-modules and identify N^W with the set of all mappings from W to N. Considering N with the *discrete topology* and N^W with the product topology, the induced *relative topology* on $\mathrm{Hom}_{T-}(W, N) \hookrightarrow N^W$ is a linear topology (called the **finite topology**), where the *basis of neighborhoods of* 0 is given by the set of annihilator submodules:

$$\mathcal{B}_f(0) := \{F^{\perp(\mathrm{Hom}_{T-}(W,N))} \mid F = \{w_1, ..., w_k\} \subset W \text{ is a finite subset}\}.$$

2.10. Let T be an A-rng, $\mathbf{P}_l = (V, {}_TW)$ a dual left T-pairing and consider for every right T-module U_T the following canonical map

$$\alpha_U^{\mathbf{P}_l} : U \otimes_T W \to \mathrm{Hom}_{-T}(V, U), \ \sum u_i \otimes_T w_i \mapsto [v \mapsto \sum u_i < v, w_i >]. \qquad (4)$$

We say that $\mathbf{P}_l = (V, {}_TW) \in \mathcal{P}_l(T)$ **satisfies the left α-condition** (or is a **dual left α-pairing**), iff $\alpha_U^{\mathbf{P}_l}$ is injective for every right T-module U_T. By $\mathcal{P}_l^{\alpha}(T) < \mathcal{P}_l(T)$ we denote the *full* subcategory of dual left T-pairings satisfying the left α-condition. The full subcategory of **dual right α-pairings** $\mathcal{P}_r^{\alpha}(T) < \mathcal{P}_r(T)$ is defined analogously.

Definition 2.11. Let T be an A-rng, $\mathbf{P}_l = (V, {}_TW)$ be a dual left T-pairing and consider

$$\kappa_{\mathbf{P}_l} : V \to {}^*W \text{ and } \alpha_V^{\mathbf{P}_l} : V \otimes_T W \to \mathrm{End}(V_T).$$

We say $\mathbf{P}_l \in \mathcal{P}_l(T)$ is

dense, iff $\kappa_{\mathbf{P}_l}(V) \subseteq {}^*W$ is dense (w.r.t. the *finite topology* on ${}^*W \hookrightarrow T^W$);
injective (resp. **semi-strict**, **strict**), iff $\alpha_V^{\mathbf{P}_l}$ is injective (resp. surjective, bijective);
non-degenerate, iff $V \overset{\kappa_{\mathbf{P}_l}}{\hookrightarrow} {}^*W$ and $W \overset{\chi_{\mathbf{P}_l}}{\hookrightarrow} V^*$ canonically.

2.12. Let T be an A-rng. We call a T-module W **locally projective** (in the sense of B. Zimmermann-Huisgen [Z-H1976]), iff for every diagram of T-modules

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "\iota"] \arrow[dr, dotted, "g'\circ\iota"'] & W \arrow[d, dotted, "g'"] \arrow[dr, "g"] & & \\
& & L \arrow[r, "\pi"'] & N \arrow[r] & 0
\end{tikzcd}$$

with exact rows and finitely generated T-submodule $F \subseteq W$: for every T-linear morphism $g : W \to N$, there exists a T-linear morphism $g' : W \to L$, such that the entstanding parallelogram is commutative (i.e. $g \circ \iota = \pi \circ g' \circ \iota$).

For proofs of the following basic properties of *locally projective modules* and *dual α-pairings* see [Abu2005] and [Z-H1976]:

Proposition 2.13. *Let T be an A-ring and $\mathbf{P}_l = (V, {}_TW) \in \mathcal{P}_l(T)$.*

1. *The left T-module ${}_TW$ is locally projective if and only if $({}^*W, W)$ is an α-pairing.*
2. *The left T-module ${}_TW$ is locally projective, iff for any finite subset $\{w_1, ..., w_k\} \subseteq W$, there exists $\{(f_i, \widetilde{w}_i)\}_{i=1}^k \subset {}^*W \times W$ such that $w_j = \sum_{i=1}^{k} f_i(w_j)\widetilde{w}_i$ for all $j = 1, ..., k$.*
3. *If ${}_TW$ is a locally projective, then ${}_TW$ is flat and T-cogenerated.*
4. *If $\mathbf{P}_l \in \mathcal{P}_l^{\alpha}(T)$, then ${}_TW$ is locally projective.*
5. *If ${}_TW$ is locally projective and $\kappa_P(V) \subseteq {}^*W$ is dense, then $\mathbf{P}_l \in \mathcal{P}_l^{\alpha}(T)$.*
6. *Let T_T be an injective cogenerator. Then $\mathbf{P}_l \in \mathcal{P}_l^{\alpha}(T)$ if and only if ${}_TW$ is locally projective and $\kappa_{\mathbf{P}_l}(V) \subseteq {}^*W$ is dense.*
7. *If T is a QF ring, then $\mathbf{P}_l \in \mathcal{P}_l^{\alpha}(T)$ if and only if ${}_TW$ is projective and $W \overset{\chi_{\mathbf{P}_l}}{\hookrightarrow} V^*$.*

The following result completes the nice observation [BW2003, 42.13.] about locally projective modules:

Proposition 2.14. *Let T be a ring, ${}_TW$ a left T-module, $S := \mathrm{End}({}_TW)^{op}$ and consider the canonical (S,S)-bilinear morphism*

$$[,]_W : \ {}^*W \otimes_T W \to \mathrm{End}({}_TW),\ f \otimes_T w \mapsto [\widetilde{w} \mapsto f(\widetilde{w})w].$$

1. *${}_TW$ is* finitely generated projective *if and only if $[,]_W$ is surjective.*

2. *${}_TW$ is locally projective if and only if $\mathrm{Im}([,]_W) \subseteq \mathrm{End}({}_TW)$ is dense.*

Proof. 1. This follows by [Fai1981, 12.8.].

2. Assume ${}_TW$ is locally projective and consider for every left T-module N the canonical mapping

$$[,]_N^W :^* W \otimes_T N \to \mathrm{Hom}_T(W,N),\ f \otimes_T n \mapsto [\widetilde{w} \mapsto f(\widetilde{w})n].$$

It follows then by [BW2003, 42.13.], that $\mathrm{Im}([,]_N^W) \subseteq \mathrm{Hom}_T(W,N)$ is dense. In particular, setting $N = W$ we conclude that $\mathrm{Im}([,]_W) \subseteq \mathrm{End}({}_TW)$ is dense. On the other hand, assume $\mathrm{Im}([,]_W) \subseteq \mathrm{End}({}_TW)$ is dense. Then for every finite subset $\{w_1, ..., w_k\} \subseteq W$, there exists $\sum\limits_{i=1}^{n} \widetilde{g}_i \otimes_T \widetilde{w}_i \in {}^*W \otimes_T W$ with

$$w_j = id_W(w_j) = [,]_W(\sum_{i=1}^{n} \widetilde{g}_i \otimes_T \widetilde{w}_i)(w_j) = \sum_{i=1}^{n} \widetilde{g}_i(w_j)\widetilde{w}_i \text{ for } j = 1, ..., k.$$

It follows then by Proposition 2.13 “2” that ${}_TW$ is locally projective.■

3 Morita (Semi)contexts

We noticed, in the proofs of some results on equivalences between subcategories of module categories associated to a given Morita context, that no use is made of the *compatibility* between the connecting bimodule morphisms (or even that only one trace ideal is used and so only one of the two bilinear morphisms is really in action). Some results of this type appeared, for example, in [Nau1993], [Nau1994-a] and [Nau1994-b]. Moreover, in our considerations some Morita contexts will be formed for arbitrary associative rngs (i.e. not necessarily unital rings). These considerations motivate us to make the following general definitions:

3.1. By a **Morita semi-context** we mean a tuple

$$\mathbf{m}_T = ((T:A),(S:B),P,Q,<,>_T,I), \tag{5}$$

where T is an A-rng, S is a B-rng, P a (T,S)-bimodule, Q an (S,T)-bimodule, $<,>_T$: $P \otimes_S Q \to T$ is a (T,T)-bilinear morphism and $I := \mathrm{Im}(<,>_T) \lhd T$ (called the **trace ideal associated to $\mathbf{m}_T$**). We drop the ground rings A, B and the trace ideal $I \lhd T$, if they are not explicitly in action. If $\mathbf{m}_T$ (5) is a Morita semi-context and T, S are rings, then we call $\mathbf{m}_T$ a **unital Morita semi-context**.

3.2. Let $\mathbf{m}_T = ((T:A),(S:B),P,Q,<,>_T)$, $\mathbf{m}_{T'} = ((T':A'),(S':B'),P',Q',<,>_{T'})$ be Morita semi-contexts. By a **morphism of Morita semi-contexts** from $\mathbf{m}_T$ to $\mathbf{m}_{T'}$ we mean a four fold set of morphisms

$$((\beta:\delta),(\gamma:\sigma),\phi,\psi) : ((T:A),(S:B),P,Q) \to ((T':A'),(S':B'),P',Q'),$$

where $(\beta:\delta):(T:A)\to(T':A')$ and $(\gamma:\sigma):(S:B)\to(S':B')$ are rng morphisms, $\phi : P \to P'$ is (T,S)-bilinear and $\psi : Q \to Q'$ is (S,T)-bilinear, such that

$$\beta(< p, q >_T) =< \phi(p), \psi(q) >_{T'} \text{ for all } p \in P, q \in Q\ .$$

Notice that we consider P' as a (T,S)-bimodule and Q' as an (S,T)-bimodule with actions induced by the morphism of rngs $(\beta:\delta)$ and $(\gamma:\sigma)$. By $\mathbb{MSC}$ we denote the *category of Morita semi-contexts* with morphisms defined as above, and by $\mathbb{UMSC} < \mathbb{MSC}$ the (non-full) subcategory of *unital Morita semi-contexts.*

Morita semi-contexts are closely related to *dual pairings* in the sense of [Abu2005]:

3.3. Let $(T,S,P,Q,<,>_T) \in \mathbb{MSC}$ and consider the canonical isomorphisms of Abelian groups

$$\mathrm{Hom}_{(S,T)}(Q,{}^*P) \overset{\xi}{\simeq} \mathrm{Hom}_{(T,T)}(P \otimes_S Q, T) \overset{\zeta}{\simeq} \mathrm{Hom}_{(T,S)}(P, Q^*).$$

This means that we have two dual T-pairings $\mathbf{P}_l := (Q, {}_TP) \in \mathcal{P}_l(T)$ and $\mathbf{Q}_r := (P, Q_T) \in \mathcal{P}_r(T)$, induced by the canonical T-linear morphisms

$$\kappa_{\mathbf{P}_l} := \xi^{-1}(<,>_T) : Q \to {}^*P \text{ and } \kappa_{\mathbf{Q}_r} := \zeta(<,>_T) : P \to Q^*.$$

On the other hand, let $(S,T,Q,P,<,>_S) \in \mathbb{MSC}$ and consider the canonical isomorphisms of Abelian groups

$$\mathrm{Hom}_{(S,T)}(Q,P^*) \overset{\xi'}{\simeq} \mathrm{Hom}_{(S,S)}(Q \otimes_T P, S) \overset{\zeta'}{\simeq} \mathrm{Hom}_{(T,S)}(P, {}^*Q).$$

Then we have two dual S-pairings $\mathbf{P}_r := (Q, P_S) \in \mathcal{P}_r(S)$ and $\mathbf{Q}_l := (P, {}_SQ) \in \mathcal{P}_l(S)$, induced by the canonical morphisms

$$\kappa_{\mathbf{P}_r} := \xi'^{-1}(<,>_S) : Q \to P^* \text{ and } \kappa_{\mathbf{Q}_r} := \zeta'(<,>_S) : P \to {}^*Q.$$

3.4. By a **Morita datum** we mean a tuple

$$\mathcal{M} = ((T:A),(S:B),P,Q,<,>_T,<,>_S,I,J), \tag{6}$$

where the following are Morita semi-contexts.

$$\mathcal{M}_T := ((T:A),(S:B),P,Q,<,>_T,I) \text{ and } \mathcal{M}_S := ((S:B),(T:A),Q,P,<,>_S,J) \quad (7)$$

If, moreover, the bilinear morphisms $<,>_T: P\otimes_S Q \to T$ and $<-,>_S: Q\otimes_T P \to S$ are *compatible,* in the sense that

$$< q,p >_S q' = q < p,q' >_T \text{ and } p < q,p' >_S \ =< p,q >_T p' \ \forall\ p,p' \in P,\ q,q' \in Q, \quad (8)$$

then we call $\mathcal{M}$ a **Morita context**. If T, S in a Morita datum (context) $\mathcal{M}$ are unital, then we call $\mathcal{M}$ a **unital Morita datum (context).**

3.5. Let $\mathcal{M} = ((T:A),(S:B),P,Q,<,>_T,<,>_S)$ and $\mathcal{M}' = ((T':A'),(S':B'),P',Q',<,>_{T'},<,>_{S'})$ be Morita contexts. Extending [Ami1971, Page 275], we mean by a **morphism of Morita contexts** from $\mathcal{M}$ to $\mathcal{M}'$ a four fold set of maps

$$((\beta:\delta),(\gamma:\sigma),\phi,\psi) : ((T:A),(S:B),P,Q) \to ((T':A'),(S':B'),P',Q'),$$

where $(\beta:\delta):(T:A)\to(T':A')$, $(\gamma:\sigma):(S:B)\to(S':B')$ are rng morphisms, $\phi: P\to P'$ is (T,S)-bilinear and $\psi: Q\to Q'$ is (S,T)-bilinear, such that

$$\beta(< p,q >_T) =< \phi(p),\psi(q) >_{T'} \text{ and } \gamma(< q,p >_S) =< \psi(q),\phi(p) >_{S'} \ \ \forall\ p\in P, q\in Q.$$

By $\mathbb{MC}$ we denote the *category of Morita contexts* and morphisms defined as above and by $\mathbb{UMC} < \mathbb{MC}$ the (non-full) subcategory of *unital Morita contexts*.

Example 3.6. If R is commutative, then any Morita semi-context $(R,R,P,Q,<,>_R)$ yields a Morita context $(R,R,P,Q,<,>_R,[,]_R)$, where $[,]_R := Q\otimes_R P \simeq P\otimes_R Q \overset{<,>_R}{\longrightarrow} R$.$\blacksquare$

3.7. We call a Morita semi-context $\mathbf{m}_T = (T,S,P,Q,<,>_T)$ **semi-derived** (**derived**), iff $S := \mathrm{End}({}_TP)^{op}$ (and $Q = {}^*P$). We call a Morita datum, or a Morita context, $\mathcal{M} = (T,S,P,Q,<,>_T,<,>_S)$ **semi-derived (derived)**, iff $S = \mathrm{End}({}_TP)^{op}$, or $T = \mathrm{End}(P_S)$ ($S = \mathrm{End}({}_TP)^{op}$ and $Q = {}^*P$, or $T = \mathrm{End}(P_S)$ and $Q = P^*$).

Remark 3.8. Following [Cae1998, 1.2.] (however, dropping the condition that the bilinear map $<,>_T: P\otimes_S Q \to T$ is surjective), Morita semi-contexts $(T,S,P,Q,<>_T)$ in our sense were called *dual pairs* in [Ver2006]. However, we think the terminology we are using is more informative and avoids confusion with other notions of dual pairings in the literature (e.g. the ones studied by the first author in [Abu2005]). The reason for this specific terminology (i.e. Morita semi-contexts is that every Morita context contains two Morita semi-contexts as clear from the definition; and that any Morita semi-context can be *extended* to a (not necessarily unital) Morita context in a natural way as explained below.

Elementary rngs

In what follows we demonstrate how to build new Morita (semi-)contexts from a given Morita semi-context. These constructions are inspired by the notion of *elementary rngs* in [Cae1998, 1.2.] (and [Ver2006, Remark 3.8.]):

Lemma 3.9. *Let* $\mathbf{m}_T := ((T:A),(S:B),P,Q,<,>_T) \in \mathbb{MSC}$.

1. *The* (T,T)*-bimodule* $\mathbb{T} := P \otimes_S Q$ *has a structure of a* T*-rng with multiplication*

$$(p \otimes_S q) \cdot_{\mathbb{T}} (p' \otimes_S q') :=< p, q >_T p' \otimes_S \ q' \ \forall \ p, p' \in P, \ q, q' \in Q,$$

such that $<,>_T: \mathbb{T} \to T$ *is a morphism of* A*-rngs,* P *is a* $(\mathbb{T}, S)$*-bimodule and* Q *is an* $(S,\mathbb{T})$*-bimodule, where*

$$(p \otimes_S q) \rightharpoonup \widetilde{p} :=< p, q >_T \widetilde{p} \text{ and } \widetilde{q} \leftharpoonup (p \otimes_S q) := \widetilde{q} < p, q >_T .$$

Moreover, we have morphisms of T*-rngs*

$$\begin{array}{lllllll}\psi & : & \mathbb{T} & \to & \mathrm{End}(P_S), & p \otimes_S q & \mapsto & [\widetilde{p} \mapsto < p, q >_T \widetilde{p}]; \\ \phi & : & \mathbb{T} & \to & \mathrm{End}({}_SQ)^{op}, & p \otimes_S q & \mapsto & [\widetilde{q} \mapsto \widetilde{q} < p, q >_T],\end{array}$$

$((\mathbb{T}:A),(S:B),P,Q,id_{\mathbb{T}}) \in \mathbb{MSC}$ *and we have a morphism of Morita semi-contexts*

$$(<,>_T, id_S, , id_P, id_Q) : (\mathbb{T}, S, P, Q, id_{\mathbb{T}}) \to (T, S, P, Q, <,>_T).$$

2. *The* (S,S)*-bimodule* $\mathbf{S} := Q \otimes_T P$ *has a structure of an* S*-rng with multiplication*

$$(q \otimes_T p) \cdot_{\mathbf{S}} (q' \otimes_T p') := q < p, q' >_T \otimes_T p' = q \otimes_T < p, q' >_T p' \ \forall \ p, p' \in P, \ q, q' \in Q,$$

P *is a* $(T,\mathbf{S})$*-bimodule and* Q *is an* $(\mathbf{S},T)$*-bimodule, where*

$$\widetilde{p} \leftharpoonup (q \otimes_T p) :=< \widetilde{p}, q >_T p \text{ and } (q \otimes_T p) \rightharpoonup \widetilde{q} := q < p, \widetilde{q} >_T .$$

Moreover, we have morphisms of S*-rngs*

$$\begin{array}{lllllll}\Psi & : & S & \to & \mathrm{End}({}_TP)^{op}, & q \otimes_T p & \mapsto & [\widetilde{p} \mapsto < \widetilde{p}, q >_T p], \\ \Phi & : & S & \to & \mathrm{End}(Q_T), & q \otimes_T p & \mapsto & [\widetilde{q} \mapsto q < p, \widetilde{q} >_T],\end{array}$$

and $\mathcal{M} := ((T:A),(\mathbf{S}:B),P,Q,<,>_T, id_{\mathbf{S}})$ *is a Morita context.*

Remarks 3.10. 1. Given $((S:B),(T:A),Q,P,<,>_S) \in \mathbb{MSC}$, the (S,S)-bimodule $\mathbb{S} := Q \otimes_T P$ becomes an S-rng with multiplication

$$(q \otimes_T p) \cdot_{\mathbb{S}} (q' \otimes_T p') :=< q, p >_S q' \otimes_T \ p' \ \forall \ p, p' \in P, \ q, q' \in Q;$$

and the (T,T)-bimodule $\mathbf{T} := P \otimes_S Q$ becomes a T-rng with multiplication

$$(p \otimes_S q) \cdot_{\mathbf{T}} (p' \otimes_S q') := p < q, p' >_S \otimes_S q' = p \otimes_S \ < q, p' >_S q' \ \forall \ p, p' \in P, \ q, q' \in Q.$$

Analogous results to those in Lemma 3.9 can be obtained for the S-rng $\mathbb{S}$ and the T-rng $\mathbf{T}$.

2. Given a Morita semi-context $(T, S, P, Q, <, >_T)$ several equivalent conditions for the T-rng $\mathbf{T} := P \otimes_S Q$ to be unital and the modules ${}_{\mathbf{T}}P$, $Q_{\mathbf{T}}$ to be *firm* can be found in [Ver2006, Theorem 3.3.]. Analogous results can be formulated for the S-rng $Q \otimes_T P$ and the S-modules $P_{\mathbf{S}}$, ${}_{\mathbf{S}}Q$ corresponding to any $(S, T, Q, P, <, >_S) \in \mathbb{MSC}$.

Proposition 3.11. *1. Let $\mathbf{m}_T = (T, S, P, Q, <, >_T) \in \mathbb{UMSC}$ and assume the A-rng $\mathbb{T} := P \otimes_S Q$ to be unital. If $<, >_T : \mathbb{T} \to T$ respects unities (and $\mathbf{m}_T$ is injective), then $<, >_T$ is surjective ($\mathbb{T} \overset{<,>_T}{\simeq} T$ as A-rings).*

2. Let $\mathbf{m}_S = (S, T, Q, P, <, >_S) \in \mathbb{UMSC}$ and assume the B-rng $\mathbb{S} := Q \otimes_S P$ to be unital. If the morphism $<, >_S : \mathbb{S} \to S$ respects unities (and $\mathbf{m}_S$ is injective), then $<, >_S$ is surjective ($\mathbb{S} \overset{<,>_S}{\simeq} S$ as B-rings).

3. Let $\mathcal{M} = (T, S, P, Q, <, >_T, <, >_S) \in \mathbb{UMC}$ and assume the rngs $\mathbb{T} := P \otimes_S Q$, T, $\mathbb{S} := Q \otimes_S P$ to be unital. If $<, >_T : P \otimes_S Q \to T$ and $<, >_S : \mathbb{S} \to S$ respect unities, then $\mathbb{T} \overset{<,>_T}{\simeq} T$ as A-ring, $\mathbb{S} \overset{<,>_S}{\simeq} S$ as B-rings and we have equivalences of categories ${}_{\mathbb{T}}\mathbb{M} \approx {}_{\mathbb{S}}\mathbb{M}$ (and $\mathbb{M}_{\mathbb{T}} \approx \mathbb{M}_{\mathbb{S}}$).

Proof. Assume $\mathbb{T}$ is unital with $1_{\mathbb{T}} = \sum_{i=1}^{n} p_i \otimes_S q_i$. If $<, >_T$ respects unities, then we have $\sum_{i=1}^{n} < p_i, q_i >_T = 1_T$ and so for any $t \in T$ we get $t = t1_T = \sum_{i=1}^{n} t < p_i, q_i >_T = \sum_{i=1}^{n} < tp_i, q_i >_T \in \mathrm{Im}(<, >_T)$. One can prove "2" analogously. As for "3", it is well known that a unital Morita context with surjective connecting bimodule morphisms is strict (e.g. [Fai1981, 12.7.]), hence $\mathbb{T} \overset{<,>_T}{\simeq} T$, $\mathbb{S} \overset{<,>_S}{\simeq} S$. The equivalences of categories ${}_{\mathbb{T}}\mathbb{M} \simeq {}_T\mathbb{M} \approx {}_S\mathbb{M} \simeq {}_{\mathbb{S}}\mathbb{M}$ (and $\mathbb{M}_{\mathbb{T}} \simeq \mathbb{M}_T \approx \mathbb{M}_S \simeq \mathbb{M}_{\mathbb{S}}$) follow then by classical Morita Theory (e.g. [Fai1981, Chapter 12]).■

Under suitable conditions, the following result characterizes the Morita data, which are Morita contexts:

Proposition 3.12. *Let $\mathcal{M} = (T, S, P, Q, <, >_T, <, >_S)$ be a Morita datum.*

1. If $\mathcal{M} \in \mathbb{MC}$, then $\mathbf{S} \overset{id}{\simeq} \mathbb{S}$ and $\mathbf{T} \overset{id}{\simeq} \mathbb{T}$ as rngs.

2. Assume ${}_TP$ is Q-faithful and Q_T is P-faithful. Then $\mathcal{M} \in \mathbb{MC}$ if and only if $\mathbf{S} \overset{id}{\simeq} \mathbb{S}$ and $\mathbf{T} \overset{id}{\simeq} \mathbb{T}$ as rngs.

Proof. 1. Obvious.

2. Assume $\mathbf{S} \overset{id}{\simeq} \mathbb{S}$ and $\mathbf{T} \overset{id}{\simeq} \mathbb{T}$ as rngs. If $p \in P$ and $q, q' \in Q$ are arbitrary, then we have for any $\widetilde{p} \in P$:

$< q, p >_S q' \otimes_T \ \widetilde{p} = (q \otimes_T p) \cdot_{\mathbb{S}} (q' \otimes_T \widetilde{p}) = (q \otimes_T p) \cdot_{\mathbf{S}} (q' \otimes_T \widetilde{p}) = q < p, q' >_T \otimes_T \widetilde{p},$

hence $< q, p >_S q' - q < p, q' >_T \in \mathrm{ann}_Q(P) = 0$ (since ${}_TP$ is Q-faithful), i.e. $< q, p >_S q' = q < p, q' >_T$ for all $p \in P$ and $q, q' \in Q$. Assuming Q_T is P-faithful, one can prove analogously that $< p, q >_T p' = p < q, p' >_S$ for all $p, p' \in P$ and $q \in Q$. Consequently, $\mathcal{M}$ is a Morita context.■

4 Injective Morita (Semi-)Contexts

Definition 4.1. We call a Morita context $\mathbf{m}_T = (T, S, P, Q, <, >_T, I)$:

injective (resp. **semi-strict**, **strict**), iff $<, >_T : P \otimes_S Q \to T$ is injective (resp. surjective, bijective);

non-degenerate, iff $Q \hookrightarrow {}^*P$ and $P \hookrightarrow Q^*$ canonically;

Morita α-semi-context, iff $\mathbf{P}_l := (Q, {}_TP) \in \mathcal{P}_l^\alpha(T)$ and $\mathbf{Q}_r := (P, Q_T) \in \mathcal{P}_r^\alpha(T)$.

Notation. By $\mathbb{MSC}^\alpha < \mathbb{MSC}$ ($\mathbb{UMSC}^\alpha < \mathbb{UMSC}$) we denote the full subcategory of (unital) Morita semi-contexts satisfying the α-condition. Moreover, we denote by $\mathbb{IMSC} < \mathbb{MSC}$ ($\mathbb{IUMSC} < \mathbb{UMSC}$) the full subcategory of injective (unital) Morita semi-contexts.

Definition 4.2. We say a Morita datum (context) $\mathcal{M} = (T, S, P, Q, <, >_T, <, >_S, I, J)$:

is **injective** (resp. **semi-strict**, **strict**), iff $<, >_T : P \otimes_S Q \to T$ and $<, >_S : Q \otimes_T P \to S$ are injective (resp. surjective, bijective);

is **non-degenerate**, iff $Q \hookrightarrow {}^*P$, $P \hookrightarrow Q^*$, $Q \hookrightarrow P^*$ and $P \hookrightarrow {}^*Q$ canonically;

satisfies the left α-condition, iff $\mathbf{P}_l := (Q, {}_TP) \in \mathcal{P}_l^\alpha(T)$ and $\mathbf{Q}_l := (P, {}_SQ) \in \mathcal{P}_l^\alpha(S)$;

satisfies the right α-condition, iff $\mathbf{Q}_r := (P, Q_T) \in \mathcal{P}_r^\alpha(T)$ and $\mathbf{P}_r := (Q, P_S) \in \mathcal{P}_r^\alpha(S)$;

satisfies the α-condition, or $\mathcal{M}$ is a **Morita α-datum (Morita α-context**), iff $\mathcal{M}$ satisfies both the left and the right α-conditions.

Notation. By $\mathbb{MC}_l^\alpha < \mathbb{MC}$ ($\mathbb{UMC}_l^\alpha < \mathbb{UMC}$) we denote the full subcategory of Morita contexts satisfying the left α-condition, and by $\mathbb{MC}_r^\alpha < \mathbb{MC}$ ($\mathbb{UMC}_r^\alpha < \mathbb{UMC}$) the full subcategory of (unital) Morita contexts satisfying the right α-condition. Moreover, we set $\mathbb{MC}^\alpha := \mathbb{MC}_l^\alpha \cap \mathbb{MC}_r^\alpha$ and $\mathbb{UMC}^\alpha := \mathbb{UMC}_l^\alpha \cap \mathbb{UMC}_r^\alpha$.

The following results show that Morita α-contexts are injective:

Lemma 4.3. *Let $\mathcal{M} = (T, S, P, Q, <, >_T, <, >_S, I, J) \in \mathbb{MC}$. Consider the Morita semi-context $\mathcal{M}_S := (S, T, Q, P, <, >_S)$, the dual pairings $\mathbf{P}_l := (Q, {}_TP) \in \mathcal{P}_l(T)$, $\mathbf{Q}_r := (P, Q_T) \in \mathcal{P}_r(T)$ and the canonical morphisms of rings*

$$\rho_P : S \to \mathrm{End}({}_TP)^{op} \text{ and } \lambda_Q : S \to \mathrm{End}(Q_T).$$

1. *If $\mathbf{Q}_r$ is injective (semi-strict), then $\mathcal{M}_S$ is injective ($\rho_P : S \to \mathrm{End}({}_TP)^{op}$ is a surjective morphism of B-rngs).*

2. *Assume P_S is faithful. If $\mathbf{Q}_r$ is semi-strict (and S is unital, or $\mathbf{Q}_r$ is strict), then $\mathcal{M}_S$ is semi-strict (strict) and $S \simeq \mathrm{End}({}_TP)^{op}$.*

3. *If $\mathbf{P}_l$ is injective (semi-strict), then $\mathcal{M}_S$ is injective ($\lambda_Q : S \to \mathrm{End}(Q_T)$ is a surjective morphism of B-rngs).*

4. *Assume ${}_SQ$ is faithful. If $\mathbf{P}_l$ is semi-strict (and S is unital, or $\mathbf{P}_l$ is strict), then $\mathcal{M}_S$ is semi-strict (strict) and $S \simeq \mathrm{End}(Q_T)$.*

Proof. We prove only "1" and "2", as "3" and "4" can be proved analogously. Consider the following butterfly diagram with canonical morphisms

$$
\begin{tikzcd}
Q \otimes_T Q^* \arrow[dddd, "{[,]_Q^r}"'] & & & & {}^*P \otimes_T P \arrow[dddd, "{[,]_P^l}"] \\
 & & Q \otimes_T P \arrow[ull, "id_Q \otimes_T \kappa_{\mathbf{Q}_r}"'] \arrow[urr, "\kappa_{\mathbf{P}_l} \otimes_T id_P"] \arrow[dl, "\alpha_Q^P"'] \arrow[dr, "\alpha_P^Q"] \arrow[dd, "{<,>_S}"'] \arrow[dddll, dotted, "\alpha_Q^{\mathbf{P}_l}"] \arrow[dddrr, dotted, "\alpha_P^{\mathbf{Q}_r}"'] & & \\
 & \mathrm{Hom}_{-T}({}^*P, Q) \arrow[ddl, "{(\kappa_{\mathbf{P}_r}, Q)}"'] & & \mathrm{Hom}_{-T}(Q^*, P) \arrow[ddr, "{(\kappa_{\mathbf{Q}_r}, P)}"] & \\
 & & S \arrow[dll, "\lambda_Q"] \arrow[drr, "\rho_P"'] & & \\
\mathrm{End}(Q_T) & & & & \mathrm{End}({}_TP)^{op}
\end{tikzcd}
\tag{9}
$$

Let $\sum q_i \otimes_T p_i \in Q \otimes_T P$ be arbitrary. For every $\widetilde{p} \in P$ we have

$$
\begin{array}{rcl}
[(\kappa_{\mathbf{Q}_r}, P) \circ \alpha_P^Q)(\sum q_i \otimes_T p_i)](\widetilde{p}) & = & \sum < \widetilde{p}, q_i >_T p_i \\
 & = & \sum \widetilde{p} < q_i, p_i >_S \\
 & = & \rho_P(\sum < q_i, p_i >_S)(\widetilde{p}) \\
 & = & (\rho_P \circ <, >_S)(\sum q_i \otimes_T p_i)(\widetilde{p}),
\end{array}
$$

i.e. $\alpha_P^{\mathbf{Q}_r} := (\kappa_{\mathbf{Q}_r}, P) \circ \alpha_P^Q = \rho_P \circ <, >_S$; and

$$
\begin{array}{rcl}
[,]_P^l \circ (\kappa_{\mathbf{P}_l} \otimes_T id_P))(\sum q_i \otimes_T p_i)](\widetilde{p}) & = & \sum \kappa_{\mathbf{P}_l}(q_i)(\widetilde{p}) p_i \\
 & = & \sum < \widetilde{p}, q_i >_T p_i \\
 & = & \sum \widetilde{p} < q_i, p_i >_S \\
 & = & \rho_P(\sum < q_i, p_i >_S)(\widetilde{p}) \\
 & = & [(\rho_P \circ <, >_S)(\sum q_i \otimes_T p_i)](\widetilde{p}),
\end{array}
$$

i.e. $[,]_P^l \circ (\kappa_{\mathbf{P}_l} \otimes_T id_P) = \rho_P \circ <, >_S$. On the other hand, for every $\widetilde{q} \in Q$ we have

$$
\begin{array}{rcl}
((\kappa_{\mathbf{P}_l}, Q) \circ \alpha_Q^{\mathbf{P}_l})(\sum q_i \otimes_T p_i)(\widetilde{q}) & = & \sum q_i < p_i, \widetilde{q} >_T \\
 & = & (\sum < q_i, p_i >_S)\widetilde{q} \\
 & = & \lambda_Q(\sum < q_i, p_i >_S)(\widetilde{q}) \\
 & = & (\lambda_Q \circ <, >_S)(\sum q_i \otimes_T p_i),
\end{array}
$$

i.e. $\alpha_Q^{\mathbf{P}_l} := (\kappa_{\mathbf{P}_l}, Q) \circ \alpha_Q^{\mathbf{P}_l} = \lambda_Q \circ <,>_S$ and

$$\begin{array}{rcl}([,]_Q^r \circ (id_Q \otimes_T \kappa_{\mathbf{Q}_r}))(\sum q_i \otimes_T p_i)](\widetilde{q}) & = & \sum q_i \kappa_{\mathbf{Q}_r}(p_i)(\widetilde{q}) \\ & = & \sum q_i < p_i, \widetilde{q} >_T \\ & = & \sum < q_i, p_i >_S \widetilde{q} \\ & = & \lambda_Q(\sum < q_i, p_i >_S)(\widetilde{q}) \\ & = & [(\lambda_Q \circ <,>_S)(\sum q_i \otimes_T p_i)](\widetilde{q}),\end{array}$$

i.e. $[,]_Q^r \circ (id_Q \otimes_T \kappa_{\mathbf{Q}_r}) = \lambda_Q \circ <,>_S$. Hence Diagram (9) is commutative. By assumption $\alpha_P^{\mathbf{Q}_r} = \rho_P \circ <,>_S$ is injective, whence $<,>_S$ is injective. Assume now that P_S is faithful, so that the canonical left S-linear map $\rho_P : S \to \mathrm{End}(_TP)^{op}$ is injective. The surjectivity of $\alpha_P^{\mathbf{Q}_r} = \rho_P \circ <,>_S$ implies $<,>_S$ is surjective (since ρ_P is injective). Assume now that S is unital and $1_S = \sum_j < \widetilde{q}_j, \widetilde{p}_j >_S$ for some $\{(\widetilde{q}_j, \widetilde{p}_j)\}_J \subseteq Q \times P$. If $\sum_i q_i \otimes_T p_i \in \mathrm{Ker}(<,>_S)$, then

$$\begin{array}{rclcl}\sum_i q_i \otimes_T p_i & = & (\sum_i q_i \otimes_T p_i) \cdot 1_S & = & \sum_i (q_i \otimes_T p_i) \cdot (\sum_j < \widetilde{q}_j, \widetilde{p}_j >_S) \\ & = & \sum_{i,j} q_i \otimes_T p_i < \widetilde{q}_j, \widetilde{p}_j >_S & = & \sum_{i,j} q_i \otimes_T < p_i, \widetilde{q}_j >_T \widetilde{p}_j \\ & = & \sum_{i,j} q_i < p_i, \widetilde{q}_j >_T \otimes_T \widetilde{p}_j & = & \sum_{i,j} < q_i, p_i >_S \widetilde{q}_j \otimes_T \widetilde{p}_j \\ & = & \sum_j (\sum_i < q_i, p_i >_S) \widetilde{q}_j \otimes_T \widetilde{p}_j & = & 0,\end{array}$$

i.e. $<,>_S$ is injective, hence an isomorphism.■

Corollary 4.4. $\mathbb{MC}_l^\alpha \cup \mathbb{MC}_r^\alpha \subseteq \mathbb{IMC}$.

Example 4.5. Let $\mathbf{m}_T = (T, S, P, Q, <,>_T)$ be a non-degenerate Morita semi-context. If T is a QF ring and the T-modules $_TP$, Q_T are projective, then by Proposition 2.13 "7" $\mathbf{P}_l := (Q, {}_TP) \in \mathcal{P}_l^\alpha(T)$ and $\mathbf{Q}_r := (P, Q_T) \in \mathcal{P}_r^\alpha(T)$ (i.e. $\mathbf{m}_T$ is a Morita α-semi-context, whence injective). On the other hand, Let $\mathcal{M} = (T, S, P, Q, <,>_T, <,>_S)$ be a non-degenerate Morita datum. If T, S are QF rings and the modules $_TP$, Q_T, P_S, $_SQ$ are projective, then $\mathcal{M}$ is an Morita α-datum (whence injective).

Every semi-strict *unital* Morita context is injective (whence strict, e.g. [Fai1981, 12.7.]). The following example shows that the converse is not necessarily true:

Example 4.6. Let $\mathcal{M} = (F, F', P, Q, <,>_F, <,>_{F'})$ be a non-degenerate Morita context, where F and F' are *non-isomorphic* commutative QF rings (e.g. fields). Then $\mathcal{M}$ is a Morita α-context by Example 4.5. However, $\mathcal{M}$ is not strict (otherwise, it would follow by [Fai1981, 12.10.] that the centers of F and F' are isomorphic, i.e. $F \simeq F'$, a contradiction).

Definition 4.7. Let T be a ring and $I \lhd T$ an ideal. For every left T-module $_TV$ consider the canonical T-linear map

$$\zeta_{I,V} : V \to \mathrm{Hom}_T(I, V),\ v \mapsto [t \mapsto tv].$$

We say ${}_TI$ is **strongly V-faithful**, iff $\mathrm{ann}_V(I) := \mathrm{Ker}(\zeta_{I,V}) := 0$. Moreover, we say I is **strongly faithful**, if ${}_TI$ is V-faithful for every left T-module ${}_TV$. Strong faithfulness w.r.t. right T-modules can be analogously defined.

Remark 4.8. Let T be a ring, $I \lhd T$ an ideal and ${}_TU$ a left ideal. It's clear that $\mathrm{ann}_U^{\otimes}(I_T) \subseteq \mathrm{ann}_U(I) := \mathrm{Ker}(\zeta_{I,U})$. Hence, if ${}_TI$ is *strongly U-faithful*, then I_T is *U-faithful* (which justifies our terminology). In particular, if ${}_TI$ is *strongly faithful*, then I_T is *completely faithful*.

Morita α-contexts are injective by Corollary 4.4. The following result gives a partial converse:

Lemma 4.9. *Let $\mathcal{M} = (T, S, P, Q, <,>_T, <,>_S, I, J) \in \mathbb{MC}$ and assume the Morita semi-context $\mathcal{M}_S := (S, T, Q, P, <,>_S, J)$ is injective.*

1. *If ${}_SJ$ is strongly faithful, then $\mathbf{Q}_r := (P, Q_T) \in \mathcal{P}_r^{\alpha}(T)$ (in particular, Q_T is locally projective).*
2. *If J_S is strongly faithful, then $\mathbf{P}_l := (Q, {}_TP) \in \mathcal{P}_l^{\alpha}(T)$ (in particular, ${}_TP$ is locally projective).*

Proof. We prove only "1", since "2" can be proved similarly. Assume $\mathcal{M}_S$ is injective and consider for every left T-module U the following diagram

$$\begin{tikzcd} Q \otimes_T U \arrow[rr, "\alpha_U^{\mathbf{Q}_r}"] \arrow[dr, "\zeta_{J, Q\otimes_T U}"'] & & \mathrm{Hom}_{T-}(P, U) \arrow[dl, "\psi_{Q,U}"] \\ & \mathrm{Hom}_{S-}(J, Q \otimes_T U) & \end{tikzcd} \tag{10}$$

where for all $f \in \mathrm{Hom}_{T-}(P, U)$ and $\sum < q_j, p_j >_S \in J$ we define

$$\psi_{Q,U}(f)(\sum < q_j, p_j >_S) := \sum q_j \otimes_T f(p_j).$$

Then we have for every $\sum \widetilde{q}_i \otimes_T \widetilde{u}_i \in Q \otimes_T U$ and $s \in J$:

$$\begin{aligned} (\psi_{Q,U} \circ \alpha_U^{\mathbf{Q}_r})(\textstyle\sum_i \widetilde{q}_i \otimes_T \widetilde{u}_i)(s) &= \sum_j q_j \otimes_T [\alpha_U^{\mathbf{Q}_r}(\sum_i \widetilde{q}_i \otimes_T \widetilde{u}_i)](p_j) \\ &= \sum_j q_j \otimes_T \sum_i < p_j, \widetilde{q}_i >_T \widetilde{u}_i] \\ &= \sum_{i,j} q_j \otimes_T < p_j, \widetilde{q}_i >_T \widetilde{u}_i \\ &= \sum_{i,j} q_j < p_j, \widetilde{q}_i >_T \otimes_T \widetilde{u}_i \\ &= \sum_{i,j} < q_j, p_j >_S \widetilde{q}_i \otimes_T \widetilde{u}_i \\ &= \zeta_{Q\otimes_T U, J}(\textstyle\sum_i \widetilde{q}_i \otimes_T \widetilde{u}_i)(s), \end{aligned}$$

i.e. diagram (10) is commutative. If ${}_SJ$ is strongly faithful, then $\mathrm{Ker}(\zeta_{J,Q\otimes_T U}) = \mathrm{ann}_{Q\otimes_T U}(J) = 0$, hence $\zeta_{J,Q\otimes_T U}$ is injective and it follows then that $\alpha_U^{\mathbf{Q}_r}$ is injective.■

Proposition 4.10. *Let $\mathcal{M} = (T, S, P, Q, <,>_T, <,>_S, I, J) \in \mathbb{IMC}$. If ${}_TI$, I_T, ${}_SJ$ and J_S are strongly faithful, then $\mathcal{M} \in \mathbb{MC}^{\alpha}$.*

5 Equivalences of Categories

In this section we give some applications of *injective Morita* (*semi-*)*contexts* and *injective Morita data* to equivalences between suitable subcategories of modules arising in the Kato-Müller-Ohtake localization-colocalization theory (as developed in (e.g. [Kat1978], [KO1979], [Mül1974]). All rings, hence all Morita (semi-)contexts and data, considered in this section are unital.

Static and Adstatic Modules

5.1. ([C-IG-TW2003]) Let $\mathcal{A}$ and $\mathcal{B}$ be two complete cocomplete Abelian categories, $\mathbf{R}: \mathcal{A} \to \mathcal{B}$ an additive covariant functor with left adjoint $\mathbf{L}: \mathcal{B} \to \mathcal{A}$ and let

$$\omega : \mathbf{LR} \to 1_{\mathcal{A}} \text{ and } \eta : 1_{\mathcal{B}} \to \mathbf{RL}$$

be the induced natural transformations (called the *counit* and the *unit* of the adjunction, respectively). Related to the adjoint pair $(\mathbf{L}, \mathbf{R})$ are two *full* subcategories of $\mathcal{A}$ and $\mathcal{B}$:

$$\mathrm{Stat}(\mathbf{R}) := \{X \in \mathcal{A} \mid \mathbf{LR}(X) \overset{\omega_X}{\simeq} X\} \text{ and } \mathrm{Adstat}(\mathbf{R}) := \{Y \in \mathcal{B} \mid Y \overset{\eta_Y}{\simeq} \mathbf{RL}(Y)\},$$

whose members are called **R-static objects** and **R-adstatic objects,** respectively. It's evident (from definition) that we have equivalence of categories $\mathrm{Stat}(\mathbf{R}) \approx \mathrm{Adstat}(\mathbf{R})$.

A typical situation, in which static and adstatic objects arise naturally is the following:

5.2. Let T, S be rings, ${}_TU_S$ a (T,S)-bimodule and consider the covariant functors

$$\mathbf{H}_U^l := \mathrm{Hom}_T(U, -) : {}_T\mathbb{M} \to {}_S\mathbb{M} \text{ and } \mathbf{T}_U^l := U \otimes_S - : {}_S\mathbb{M} \to {}_T\mathbb{M}.$$

It is well-known that $(\mathbf{T}_U^l, \mathbf{H}_U^l)$ is an adjoint pair of covariant functors via the *natural isomorphisms*

$$\mathrm{Hom}_T(U \otimes_S M, N) \simeq \mathrm{Hom}_S(M, \mathrm{Hom}_T(U, N)) \text{ for all } M \in {}_S\mathbb{M} \text{ and } N \in {}_T\mathbb{M}$$

and the natural transformations

$$\omega_U^l : U \otimes_S \mathrm{Hom}_T(U, -) \to 1_{{}_T\mathbb{M}} \text{ and } \eta_U^l : 1_{{}_S\mathbb{M}} \to \mathrm{Hom}_T(U, U \otimes_S -)$$

yield for every ${}_TK$ and ${}_SL$ the canonical morphisms

$$\omega_{U,K}^l : U \otimes_S \mathrm{Hom}_T(U, K) \to K \text{ and } \eta_{U,L}^l : L \to \mathrm{Hom}_T(U, U \otimes_S L). \tag{11}$$

We call the $\mathbf{H}_U^l$-*static* modules U**-static w.r.t.** S and set

$$\mathrm{Stat}^l({}_TU_S) := \mathrm{Stat}(\mathbf{H}_U^l) = \{{}_TK \mid U \otimes_S \mathrm{Hom}_{T-}(U, K) \overset{\omega_{U,K}^l}{\simeq} K\};$$

and the $\mathbf{H}_U^l$-*adstatic* modules U**-adstatic w.r.t.** S and set

$$\mathrm{Adstat}^l({}_TU_S) := \mathrm{Adstat}(\mathbf{H}_U^l) = \{{}_SL \mid L \overset{\eta_{U,L}^l}{\simeq} \mathrm{Hom}_{T-}(U, U \otimes_S L)\}.$$

By [Nau1990a] and [Nau1990b], there are equivalences of categories

$$\mathrm{Stat}^l({}_TU_S) \approx \mathrm{Adstat}^l({}_TU_S). \tag{12}$$

On the other hand, one can define the full subcategories $\mathrm{Stat}^r({}_TU_S) \approx \mathrm{Adstat}^r({}_TU_S)$:

$$\begin{array}{lcl} \mathrm{Stat}^r({}_TU_S) & := & \{K_S \mid \mathrm{Hom}_{-S}(U, K) \otimes_T U \simeq K\}; \\ \mathrm{Adstat}^r({}_TU_S) & := & \{L_T \mid L \simeq \mathrm{Hom}_{-S}(U, L \otimes_T U)\}. \end{array}$$

In particular, setting

$$\begin{array}{lcllcl} \mathrm{Stat}({}_TU) & := & \mathrm{Stat}^l({}_TU_{\mathrm{End}({}_TU)^{op}}); & \mathrm{Adstat}({}_TU) & := & \mathrm{Adstat}^l({}_TU_{\mathrm{End}({}_TU)^{op}}); \\ \mathrm{Stat}(U_S) & := & \mathrm{Stat}^r({}_{\mathrm{End}({}_SU)}U_S); & \mathrm{Adstat}(U_S) & := & \mathrm{Adstat}^r({}_{\mathrm{End}({}_SU)}U_S), \end{array}$$

there are equivalences of categories:

$$\mathrm{Stat}({}_TU) \simeq \mathrm{Adstat}({}_TU) \text{ and } \mathrm{Stat}(U_S) \simeq \mathrm{Adstat}(U_S). \tag{13}$$

Remark 5.3. The theory of static and adstatic modules was developed in a series of papers by the second author (see the references). They were also considered by several other authors (e.g. [Alp1990], [CF2004]). For other terminologies used by different authors, the interested reader may refer to a comprehensive treatment of the subject by R. Wisbauer in [Wis2000].

Intersecting subcategories

Several intersecting subcategories related to Morita contexts were introduced in the literature (e.g. [Nau1993], [Nau1994-b]). In what follows we introduce more and we show that many of these coincide, if one starts with an injective Morita semi-context. Moreover, other results on equivalences between some intersecting subcategories related to an injective Morita context will be reframed for arbitrary (not necessarily compatible) injective Morita data.

Definition 5.4. 1. For a right T-module Y, a T-submodule $X' \subseteq X$ is called K**-pure** for some left T-module ${}_TK$, iff the following sequence of Abelian groups is exact

$$0 \to X' \otimes_T K \to X \otimes_T K \to X/X' \otimes_T K \to 0;$$

2. For a left T-module Y, a T-submodule $Y' \subseteq Y$ is called L**-copure** for some left T-module ${}_TL$, iff the following sequence of Abelian groups is exact

$$0 \to \mathrm{Hom}_T(Y/Y', L) \to \mathrm{Hom}_T(Y, L) \to \mathrm{Hom}_T(Y', L) \to 0.$$

Definition 5.5. (Compare [KO1979, Theorems 1.3., 2.3.]) Let T be a ring, $I \lhd T$ an ideal, U a left T-module and consider the canonical T-linear morphisms

$$\zeta_{I,U} : U \to \mathrm{Hom}_T(I,U) \text{ and } \xi_{I,U} : I \otimes_T U \to U.$$

1. We say U is **I-divisible**, iff $\xi_{I,U}$ is surjective, equivalently, iff $IU = U$.
2. We say ${}_TU$ is **I-localized**, iff $U \overset{\zeta_{I,U}}{\simeq} \mathrm{Hom}_T(I,U)$ canonically, equivalently iff ${}_TI$ is strongly U-faithful and ${}_TI \subseteq T$ is U-copure).
3. We say a left T-module K is **I-colocalized**, iff $I \otimes_T U \overset{\xi_{I,U}}{\simeq} K$ canonically (equivalently, iff ${}_TU$ is I-divisible and $I_T \subseteq T$ is U-pure).

Notation. For a ring T and a ideal $I \lhd T$, set

$$\begin{array}{lllllll}
{}_I\mathfrak{D} & := & \{{}_TU \mid IU = U\}; & & {}_I\mathfrak{F} & := & \{{}_TU \mid U \overset{\zeta_{I,U}}{\hookrightarrow} \mathrm{Hom}_{T-}(I,U)\}; \\
{}_I\mathcal{L} & := & \{{}_TU \mid U \simeq \mathrm{Hom}_T(I,U\}; & & {}_I\mathcal{C} & := & \{{}_TU \mid I \otimes_T U \simeq U\}; \\
\mathfrak{D}_I & := & \{U_T \mid UI = U\}; & & \mathfrak{F}_I & := & \{U_T \mid \mathrm{ann}_U(I_T) = 0\}; \\
\mathcal{L}_I & := & \{U_T \mid U \simeq \mathrm{Hom}_T(I,U\}; & & \mathcal{C}_I & := & \{U_T \mid U \overset{\zeta_{I,U}}{\hookrightarrow} \mathrm{Hom}_{-T}(I,U)\}.
\end{array}$$

The following result is duo to T. Kato, K. Ohtake and B. Müller (e.g. [Mül1974], [Kat1978], [KO1979]):

Proposition 5.6. *Let $\mathcal{M} = (T,S,P,Q,<,>_T,<,>_S,I,J) \in \mathbb{UMC}$. Then there are equivalences of categories*

$${}_I\mathcal{C} \approx {}_J\mathcal{C},\ \mathcal{C}_I \approx \mathcal{C}_J,\ {}_I\mathcal{L} \approx {}_J\mathcal{L} \text{ and } \mathcal{L}_I \approx \mathcal{L}_J.$$

5.7. Let $\mathbf{m}_T = (T,S,P,Q,<,>_T,I) \in \mathbb{UMSC}$ and consider the dual pairings $\mathbf{P}_l := (Q, {}_TP) \in \mathcal{P}_l(T)$ and $\mathbf{Q}_r := (P, Q_T) \in \mathcal{P}_r(T)$. For every left (right) T-module U consider the canonical S-linear morphism induced by $<,>_T$:

$$\alpha_U^{\mathbf{P}_r} : Q \otimes_T U \to \mathrm{Hom}_{T-}(P,U)\ (\alpha_U^{\mathbf{P}_l} : U \otimes_T P \to \mathrm{Hom}_{-T}(Q,U)).$$

We define

$$\begin{array}{lll}
\mathcal{D}_l(\mathbf{m}_T) & := & \{{}_TU \mid Q \otimes_T U \overset{\alpha_U^{\mathbf{Q}_r}}{\simeq} \mathrm{Hom}_{T-}(P,U)\}; \\
\mathcal{D}_r(\mathbf{m}_T) & := & \{U_T \mid U \otimes_T P \overset{\alpha_U^{\mathbf{P}_l}}{\simeq} \mathrm{Hom}_{-T}(Q,U)\}.
\end{array}$$

and set

$$
\begin{array}{lcllcl}
\mathcal{U}_l(\mathbf{m}_T) & := & \mathrm{Stat}^l({}_TP_S)\cap\mathrm{Adstat}^l({}_SQ_T); & \mathcal{U}_r(\mathbf{m}_T) & := & \mathrm{Stat}^r({}_SQ_T)\cap\mathrm{Adstat}^r({}_TP_S);\\
&&&&&\\
\mathrm{V}_l(\mathbf{m}_T) & := & \mathrm{Stat}^l({}_TP_S)\cap\mathcal{D}_l(\mathbf{m}_T); & \mathrm{V}_r(\mathbf{m}_T) & := & \mathrm{Stat}^r({}_SQ_T)\cap\mathcal{D}_r(\mathbf{m}_T);\\
\mathbb{V}_l(\mathbf{m}_T) & := & {}_I\mathcal{C}\cap\mathcal{D}_l(\mathbf{m}_T); & \mathbb{V}_r(\mathbf{m}_T) & := & \mathcal{C}_I\cap\mathcal{D}_r(\mathbf{m}_T);\\
\widehat{\mathcal{V}}_l(\mathbf{m}_T) & := & \mathcal{V}_l(\mathbf{m}_T)\cap\,{}_I\mathcal{L}; & \widehat{\mathcal{V}}_r(\mathbf{m}_T) & := & \mathcal{V}_r(\mathbf{m}_T)\cap\mathcal{L}_I;\\
&&&&:=&\\
\mathcal{W}_l(\mathbf{m}_T) & := & \mathrm{Adstat}^l({}_SQ_T)\cap\mathcal{D}_l(\mathbf{m}_T); & \mathcal{W}_r(\mathbf{m}_T) & := & \mathrm{Adstat}^r({}_TP_S)\cap\mathcal{D}_r(\mathbf{m}_T);\\
\mathbb{W}_l(\mathbf{m}_T) & := & {}_I\mathcal{L}\cap\mathcal{D}_l(\mathbf{m}_T); & \mathbb{W}_r(\mathbf{m}_T) & := & \mathcal{L}_I\cap\mathcal{D}_r(\mathbf{m}_T);\\
\widehat{\mathcal{W}}_l(\mathbf{m}_T) & := & \mathcal{W}_l(\mathbf{m}_T)\cap\,{}_I\mathcal{C}; & \widehat{\mathcal{W}}_r(\mathbf{m}_T) & := & \mathcal{W}_r(\mathbf{m}_T)\cap\mathcal{C}_I;\\
&&&&&\\
\mathcal{X}_l(\mathbf{m}_T) & := & \mathcal{V}_l(\mathbf{m}_T)\cap\mathcal{W}_l(\mathbf{m}_T); & \mathcal{X}_r(\mathbf{m}_T) & := & \mathcal{V}_r(\mathbf{m}_T)\cap\mathcal{W}_r(\mathbf{m}_T);\\
\mathbb{X}_l(\mathbf{m}_T) & := & \mathbb{V}_l(\mathbf{m}_T)\cap\mathbb{W}_l(\mathbf{m}_T); & \mathbb{X}_r(\mathbf{m}_T) & := & \mathbb{V}_r(\mathbf{m}_T)\cap\mathbb{W}_r(\mathbf{m}_T).\\
&&&&&\\
\mathcal{X}_l^*(\mathbf{m}_T) & := & \{{}_S(Q\otimes_T U)\mid V\in\mathcal{X}_l(\mathbf{m}_T)\}; & \mathcal{X}_r^*(\mathbf{m}_T) & := & \{(U\otimes_T P)_S\mid V\in\mathcal{X}_l(\mathbf{m}_T)\};\\
\mathbb{X}_l^*(\mathbf{m}_T) & := & \{{}_S(Q\otimes_T U)\mid V\in\mathbb{X}_l(\mathbf{m}_T)\}; & \mathbb{X}_r^*(\mathbf{m}_T) & := & \{(U\otimes_T P)_S\mid V\in\mathbb{X}_r(\mathbf{m}_T)\}.
\end{array}
\tag{14}
$$

Given $\mathbf{m}_S=(S,T,Q,P,<,>_S,J)\in\mathbb{UMSC}$ one can define analogously, the corresponding intersecting subcategories of ${}_S\mathbb{M}$ and $\mathbb{M}_S$.

As an immediate consequence of Proposition 5.6 we get

Corollary 5.8. *Let $\mathcal{M}=(T,S,P,Q,<,>_T,<,>_S,I,J)\in\mathbb{IUMC}$ and consider the associated Morita semi-contexts $\mathcal{M}_T$ and $\mathcal{M}_S$ (7).*

1. *If ${}_I\mathcal{C}\leq\mathcal{D}_l(\mathcal{M}_T)$ and ${}_J\mathcal{C}\leq\mathcal{D}_l(\mathcal{M}_S)$, then $\mathcal{V}_l(\mathcal{M}_T)\approx\mathcal{V}_l(\mathcal{M}_S)$. Similarly, if $\mathcal{C}_I\leq\mathcal{D}_r(\mathcal{M}_T)$ and $\mathcal{C}_J\leq\mathcal{D}_r(\mathcal{M}_S)$, then $\mathcal{V}_r(\mathcal{M}_T)\approx\mathcal{V}_r(\mathcal{M}_S)$.*
2. *If ${}_I\mathcal{L}\leq\mathcal{D}_l(\mathcal{M}_T)$ and ${}_J\mathcal{L}\leq\mathcal{D}_l(\mathcal{M}_S)$, then $\mathcal{W}_l(\mathcal{M}_T)\approx\mathcal{W}_l(\mathcal{M}_S)$. Similarly, if $\mathcal{L}_I\leq\mathcal{D}_r(\mathcal{M}_T)$ and $\mathcal{L}_J\leq\mathcal{D}_r(\mathcal{M}_S)$, then $\mathcal{W}_r(\mathcal{M}_T)\approx\mathcal{W}_r(\mathcal{M}_S)$.*

Starting with a Morita context, the following result was obtained in [Nau1993, Theorem 3.2.]. We restate the result for an arbitrary (not necessarily compatible) Morita datum and *sketch* its proof:

Lemma 5.9. *Let $\mathcal{M}=(T,S,P,Q,<,>_T,<,>_S,I,J)$ be a unital Morita datum and consider the associated Morita semi-contexts $\mathcal{M}_T$ and $\mathcal{M}_S$ (7). Then there are equivalences of categories*

$$\mathcal{X}_l(\mathcal{M}_T)\underset{\mathrm{Hom}_{S-}(Q,-)}{\overset{\mathrm{Hom}_{T-}(P,-)}{\approx}}\mathcal{X}_l(\mathcal{M}_S)\ \textit{and}\ \mathcal{X}_r(\mathcal{M}_T)\underset{\mathrm{Hom}_{-S}(P,-)}{\overset{\mathrm{Hom}_{-T}(Q,-)}{\approx}}\mathcal{X}_r(\mathcal{M}_S).$$

Proof. Let ${}_TV\in\mathcal{X}_l(\mathcal{M}_T)$. By the equivalence $\mathrm{Stat}^l({}_TP_S)\overset{\mathrm{Hom}_T(P,-)}{\approx}\mathrm{Adstat}^l({}_TP_S)$ in Lemma 13 we have $\mathrm{Hom}_{T-}(P,V)\in\mathrm{Adstat}^l({}_TP_S)$. Moreover, $V\in\mathcal{D}_l(M)$, hence $\mathrm{Hom}_{T-}(P,V)\simeq$

$Q\otimes_T V$ canonically and it follows then from the equivalence $\mathrm{Adstat}^l({}_SQ_T) \overset{Q\otimes_T -}{\approx} \mathrm{Stat}^l({}_SQ_T)$ that $\mathrm{Hom}_{T-}(P,V) \in \mathrm{Stat}^l({}_SQ_T)$. Moreover, we have the following *natural* isomorphisms

$$P \otimes_S \mathrm{Hom}_{T-}(P,V) \simeq V \simeq \mathrm{Hom}_{S-}(Q, Q\otimes_T V) \simeq \mathrm{Hom}_{S-}(Q, \mathrm{Hom}_{T-}(P,V)), \qquad (15)$$

i.e. $\mathrm{Hom}_{T-}(P,V) \in \mathcal{D}_l(\mathcal{M}_S)$. Consequently, $\mathrm{Hom}_{T-}(P,V) \in \mathcal{X}_l(\mathcal{M}_S)$. Moreover, (15) yields a natural isomorphism $V \simeq \mathrm{Hom}_{S-}(Q, \mathrm{Hom}_{T-}(P,V))$. Analogously, one can show for every $W \in \mathcal{X}_l(\mathcal{M}_S)$ that $\mathrm{Hom}_{S-}(Q,W) \in \mathcal{X}_l(\mathcal{M}_T)$ and that $W \simeq \mathrm{Hom}_{T-}(P, \mathrm{Hom}_{S-}(Q,W))$ naturally. Consequently, $\mathcal{X}_l(\mathcal{M}_T) \approx \mathcal{X}_l(\mathcal{M}_S)$. The equivalences $\mathcal{X}_r(\mathcal{M}_T) \approx \mathcal{X}_r(\mathcal{M}_S)$ can be proved analogously.■

Proposition 5.10. *Let* $\mathcal{M} = (T,S,P,Q,<,>_T,<,>_S,I,J)$ *be a unital injective Morita datum and consider the associated Morita semi-contexts* $\mathcal{M}_T$ *and* $\mathcal{M}_S$ (7).

1. *There are equivalences of categories*

$$\begin{array}{lllllll} \mathrm{Stat}^l({}_TI_T) & \approx & \mathrm{Adstat}^l({}_TI_T); & \quad & \mathrm{Stat}^l({}_SJ_S) & \approx & \mathrm{Adstat}^l({}_SJ_S); \\ \mathrm{Stat}^r({}_TI_T) & \approx & \mathrm{Adstat}^r({}_TI_T); & \quad & \mathrm{Stat}^r({}_SJ_S) & \approx & \mathrm{Adstat}^r({}_SJ_S). \end{array}$$

2. *If* $\mathrm{Stat}^l({}_TI_T) \leq \mathcal{X}_l^*(\mathcal{M}_S)$ *and* $\mathrm{Stat}^l({}_SJ_S) \leq \mathcal{X}_l^*(\mathcal{M}_T)$, *then there are equivalences of categories*

$$\mathrm{Stat}^l({}_TI_T) \approx \mathrm{Stat}^l({}_SJ_S) \text{ and } \mathrm{Adstat}^l({}_TI_T) \approx \mathrm{Adstat}^l({}_SJ_S).$$

3. *If* $\mathrm{Stat}^r({}_TI_T) \leq \mathcal{X}_r^*(\mathcal{M}_S)$ *and* $\mathrm{Stat}^r({}_SJ_S) \leq \mathcal{X}_r^*(\mathcal{M}_T)$, *then there are equivalences of categories*

$$\mathrm{Stat}^r({}_TI_T) \approx \mathrm{Stat}^r({}_SJ_S) \text{ and } \mathrm{Adstat}^r({}_TI_T) \approx \mathrm{Adstat}^r({}_SJ_S).$$

Proof. To prove "1", notice that since $\mathcal{M}$ is an injective Morita context, $P \otimes_S Q \overset{<,>_T}{\simeq} I$ and $Q \otimes_T P \overset{<,>_S}{\simeq} J$ as bimodules and so the four equivalences of categories result from 5.2. To prove "2", one can use an argument similar to that in [Nau1994-b, Theorem 3.9.] to show that the inclusion $\mathrm{Stat}^l({}_TI_T) = \mathrm{Stat}^l({}_T(P\otimes_S Q)_T) \leq \mathcal{X}_l^*(\mathcal{M}_S)$ implies $\mathrm{Stat}^l({}_TI_T) = \mathrm{Stat}^l({}_T(P\otimes_S Q)_T) = \mathcal{X}_l(\mathcal{M}_T)$ and that the inclusion $\mathrm{Stat}^l({}_SJ_S) = \mathrm{Stat}^l({}_S(Q\otimes_T P)_S) \leq \mathcal{X}_l^*(\mathcal{M}_T)$ implies $\mathrm{Stat}^l({}_SJ_S) = \mathrm{Stat}^l({}_S(Q\otimes_T P)_S) = \mathcal{X}_l(\mathcal{M}_S)$. The result follows then by Lemma 5.9. The proof of "3" is analogous to that of "2".■

For injective Morita semi-contexts, several subcategories in (14) are shown in the following result to be equal:

Theorem 5.11. *Let* $\mathbf{m}_T = (T,S,P,Q,<,>_T,I) \in \mathbb{IUMS}$. *Then*

1. $\mathcal{V}_l(\mathbf{m}_T) = \mathbb{V}_l(\mathbf{m}_T)$, $\mathcal{W}_l(\mathbf{m}_T) = \mathbb{W}_l(\mathbf{m}_T)$, *whence*

$$\widehat{\mathcal{V}}_l(\mathbf{m}_T) = \widehat{\mathcal{W}_l}(\mathbf{m}_T) = \mathcal{X}_l(\mathbf{m}_T) = \mathbb{X}_l(\mathbf{m}_T) = {}_I\mathcal{C} \cap \mathcal{D}_l(\mathbf{m}_T) \cap {}_I\mathcal{L} \text{ and } \mathcal{X}_l^*(\mathbf{m}_T) = \mathbb{X}_l^*(\mathbf{m}_T).$$

2. $\mathcal{V}_r(\mathbf{m}_T) = \mathbb{V}_r(\mathbf{m}_T)$, $\mathcal{W}_r(\mathbf{m}_T) = \mathbb{W}_r(\mathbf{m}_T)$, *whence*

$\widehat{\mathcal{V}}_r(\mathbf{m}_T) = \widehat{\mathcal{W}}_r(\mathbf{m}_T) = \mathcal{X}_r(\mathbf{m}_T) = \mathbb{X}_r(\mathbf{m}_T) = \mathcal{C}_I \cap \mathcal{D}_r(\mathbf{m}_T) \cap \mathcal{L}_I$ *and* $\mathcal{X}_r^*(\mathbf{m}_T) = \mathbb{X}_r^*(\mathbf{m}_T)$.

Proof. We prove only "1" as "2" can be proved analogously. Assume the Morita semi-context $\mathbf{m}_T = (T, S, P, Q, <,>_T, I)$ is injective. By our assumption we have for every $V \in \mathcal{D}_l(\mathbf{m}_T)$ the commutative diagram

$$\begin{tikzcd}
P \otimes_S (Q \otimes_T V) \arrow[rr, "can", "\simeq"'] \arrow[d, "id_P \otimes_S (\alpha_V^{\mathbf{Q}_r})"', "\simeq"] & & (P \otimes_S Q) \otimes_T V \arrow[d, "\simeq"', "<{,}>_T \otimes_T id_V"] \\
P \otimes_S \mathrm{Hom}_{T-}(P, V) \arrow[r, "\omega^l_{P,V}"'] & V & I \otimes_T V \arrow[l, "\xi_{I,V}"]
\end{tikzcd} \tag{16}$$

Then it becomes obvious that $\omega^l_{P,V} : P \otimes_S \mathrm{Hom}_T(P, V)$ is an isomorphism if and only if $\xi_{I,V} : I \otimes_T V \to V$ is an isomorphism. Consequently

$$\mathcal{V}(\mathbf{m}_T) = \mathcal{D}_l(\mathbf{m}_T) \cap \mathrm{Stat}^l({}_T P_S) = \mathcal{D}_l(\mathbf{m}_T) \cap {}_I\mathcal{C} = \mathbb{V}(\mathbf{m}_T).$$

On the other hand, we have for every $V \in \mathcal{D}_l(\mathbf{m}_T)$ the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{S-}(Q, \mathrm{Hom}_{T-}(P, V)) \arrow[rr, "can", "\simeq"'] & & \mathrm{Hom}_{T-}(P \otimes_S Q, V) \\
\mathrm{Hom}_{S-}(Q, Q \otimes_T V) \arrow[u, "(Q{,}\alpha_V^{\mathbf{Q}_r})", "\simeq"'] & V \arrow[l, "\eta^l_{P,L}"'] \arrow[r, "\zeta_{I,V}"'] & \mathrm{Hom}_{T-}(I, V) \arrow[u, "\simeq", "(<{,}>_T{,}V)"']
\end{tikzcd} \tag{17}$$

It follows then that $\eta^l_{P,L} : V \to \mathrm{Hom}_S(Q, Q \otimes_T P)$ is and isomorphism if and only if $\zeta_{I,V} : V \to \mathrm{Hom}_T(I, V)$ is an isomorphism. Consequently,

$$\mathcal{W}(\mathbf{m}_T) = \mathcal{D}_l(\mathbf{m}_T) \cap \mathrm{Adstat}^l({}_T P_S) = \mathcal{D}_l(\mathbf{m}_T) \cap_I \mathcal{L} = \mathbb{W}(\mathbf{m}_T).$$

Moreover, we have

$$\begin{array}{lllllll}
\widehat{\mathcal{V}}_l(\mathbf{m}_T) & := & \mathcal{V}_l(\mathbf{m}_T) \cap {}_I\mathcal{L} & = & \mathbb{V}_l(\mathbf{m}_T) \cap {}_I\mathcal{L} & = & {}_I\mathcal{C} \cap \mathcal{D}_l(\mathbf{m}_T) \cap {}_I\mathcal{L} \\
& = & {}_I\mathcal{C} \cap \mathbb{W}_l(\mathbf{m}_T) & = & {}_I\mathcal{C} \cap \mathcal{W}_l(\mathbf{m}_T) & = & \widehat{\mathcal{W}}_l(\mathbf{m}_T).
\end{array}$$

On the other hand, we have

$$\mathcal{X}_l(\mathbf{m}_T) = \mathcal{V}_l(\mathbf{m}_T) \cap \mathcal{W}_l(\mathbf{m}_T) = \mathbb{V}_l(\mathbf{m}_T) \cap \mathbb{W}_l(\mathbf{m}_T) = \mathbb{X}_l(\mathbf{m}_T)$$

and so the equalities $\widehat{\mathcal{V}}_l(\mathbf{m}_T) = \widehat{\mathcal{W}}_l(\mathbf{m}_T) = \mathcal{X}_l(\mathbf{m}_T) = \mathbb{X}_l(\mathbf{m}_T)$ and $\mathcal{X}_l^*(\mathbf{m}_T) = \mathbb{X}_l^*(\mathbf{m}_T)$ are established.■

In addition to establishing several other equivalences of intersecting subcategories, the following results reframe the equivalence of categories $\widehat{\mathcal{V}} \approx \widehat{\mathcal{W}}$ in [Nau1990b, Theorem 4.9.] for an arbitrary (not necessarily compatible) injective Morita datum:

Theorem 5.12. *Let $\mathcal{M} = (T, S, P, Q, <,>_T, <,>_S, I, J)$ be an injective Morita datum and consider the associated Morita semi-contexts $\mathcal{M}_T$ and $\mathcal{M}_S$ (7).*

1. *The following subcategories are mutually equivalent:*

$$\widehat{\mathcal{V}}_l(\mathcal{M}_T) = \widehat{\mathcal{W}}_l(\mathcal{M}_T) = \mathbb{X}_l(\mathcal{M}_T) = \mathcal{X}_l(\mathcal{M}_T) \approx \mathcal{X}_l(\mathcal{M}_S) = \mathbb{X}_l(\mathcal{M}_S) = \widehat{\mathcal{W}}_l(\mathcal{M}_S) = \widehat{\mathcal{V}}_l(\mathcal{M}_S). \tag{18}$$

2. *If $\mathcal{V}_l(\mathcal{M}_T) \leq {}_I\mathcal{L}$ and $\mathcal{W}_l(\mathcal{M}_S) \leq {}_J\mathcal{C}$, then $\mathcal{V}_l(\mathcal{M}_T) \approx \mathcal{W}_l(\mathcal{M}_S)$. If $\mathcal{W}_l(\mathcal{M}_T) \leq {}_I\mathcal{C}$ and $\mathcal{V}_l(\mathcal{M}_S) \leq {}_J\mathcal{L}$, then $\mathcal{W}_l(\mathcal{M}_T) \approx \mathcal{V}_l(\mathcal{M}_S)$.*

3. *The following subcategories are mutually equivalent:*

$$\widehat{\mathcal{V}}_r(\mathcal{M}_T) = \widehat{\mathcal{W}}_r(\mathcal{M}_T) = \mathbb{X}_r(\mathcal{M}_T) = \mathcal{X}_r(\mathcal{M}_T) \approx \mathcal{X}_r(\mathcal{M}_S) = \mathbb{X}_r(\mathcal{M}_S) = \widehat{\mathcal{W}}_r(\mathcal{M}_S) = \widehat{\mathcal{V}}_r(\mathcal{M}_S). \tag{19}$$

4. *If $\mathcal{V}_r(\mathcal{M}_T) \leq \mathcal{L}_I$ and $\mathcal{W}_r(\mathcal{M}_T) \leq \mathcal{C}_J$, then $\mathcal{V}_r(\mathcal{M}_T) \approx \mathcal{W}_r(\mathcal{M}_S)$. If $\mathcal{W}_r(\mathcal{M}_T) \leq \mathcal{C}_J$ and $\mathcal{V}_r(\mathcal{M}_S) \leq \mathcal{L}_I$, then $\mathcal{V}_r(\mathcal{M}_S) \approx \mathcal{W}_r(\mathcal{M}_T)$.*

Proof. By Lemma 5.9, $\mathcal{X}_l(\mathcal{M}_T) \approx \mathcal{X}_l(\mathcal{M}_S)$ and so "1" follows by Theorem 5.11. If $\mathcal{V}_l(\mathcal{M}_T) \leq {}_I\mathcal{L}$ and $\widehat{\mathcal{W}}_l(\mathcal{M}_S) \leq {}_J\mathcal{C}$, then we have

$$\mathcal{V}_l(\mathcal{M}_T) = \mathcal{V}_l(\mathcal{M}_T) \cap {}_I\mathcal{L} = \widehat{\mathcal{V}}_l(\mathcal{M}_T) \approx \widehat{\mathcal{W}}_l(\mathcal{M}_S) = \mathcal{W}_l(\mathcal{M}_S) \cap {}_J\mathcal{C} = \mathcal{W}_l(\mathcal{M}_S).$$

On the other hand, if $\mathcal{W}_l(\mathcal{M}_T) \leq {}_I\mathcal{L}$ and $\mathcal{V}_l(\mathcal{M}_S) \leq {}_J\mathcal{C}$, then

$$\mathcal{W}_l(\mathcal{M}_T) = \mathcal{W}_l(\mathcal{M}_T) \cap {}_I\mathcal{C} = \widehat{\mathcal{W}}_l(\mathcal{M}_T) \approx \widehat{\mathcal{V}}_l(\mathcal{M}_S) = \mathcal{V}_l(\mathcal{M}_S) \cap {}_J\mathcal{L} = \mathcal{V}_l(\mathcal{M}_S).$$

So we have established "2". The results in "3" and "4" can be obtained analogously.■

6 More applications

In this final section we give more applications of Morita α-(semi-)contexts and injective Morita (semi-)context. All rings in this section, whence all Morita (semi-)contexts, are unital.

Localization and colocalization

In what follows we clarify the relations between static (adstatic) modules and subcategories colocalized (localized) by a trace ideal of a Morita context satisfying the α-condition.

Recall that for any (T,S)-bimodule ${}_TP_S$ we have by Lemma 2.5:

$$\mathrm{Stat}^l({}_TP_S) \subseteq \mathrm{Gen}({}_TP) \text{ and } \mathrm{Adstat}^l({}_TP_S) \subseteq \mathrm{Cogen}({}^{\#}_{S}P). \tag{20}$$

Theorem 6.1. *Let $\mathcal{M} = (T,S,P,Q,<,>_T,<,>_S,I,J) \in \mathbb{UMC}$. Then we have*

$$_I\mathcal{C} \subseteq \ _I\mathfrak{D} \subseteq \mathrm{Gen}({}_TP). \tag{21}$$

Assume $\mathbf{P}_r := (Q,P_S) \in \mathcal{P}^{\alpha}_r(S)$. Then

1. $\mathrm{Gen}({}_TP) = \mathrm{Stat}^l({}_TP_S) \subseteq \mathfrak{F}({}_TI)$.

2. *If* $\mathrm{Gen}({}_TP) \subseteq {}_I\mathcal{C}$, *then* ${}_I\mathcal{C} = {}_I\mathfrak{D} = \mathrm{Gen}({}_TP) = \mathrm{Stat}^l({}_TP_S)$.

3. *If* $\mathbf{Q}_r := (P,Q_T) \in \mathcal{P}^{\alpha}_r(T)$, *then ${}_TI \subseteq {}_TT$ is pure and ${}_I\mathcal{C} = {}_I\mathfrak{D}$.*

Proof. For every left T-module ${}_TK$, consider the commutative diagram with canonical morphisms and let α_1 and α_2 be so defined, that the induced triangles commute:

$$\begin{tikzcd}
P \otimes_S Q \otimes_T K \arrow[r, "id_P \otimes_S \alpha^{\mathbf{Q}_r}_K"] \arrow[dd, "<{,}>_T \otimes_T id_K"'] & P \otimes_S \mathrm{Hom}_T(P,K) \arrow[r, "\alpha^{\mathbf{P}_r}_{\mathrm{Hom}_T(P,K)}"] \arrow[dd, "\omega^l_{P,K}"] \arrow[ddr, dotted, "\alpha_2"] & \mathrm{Hom}_S(Q, \mathrm{Hom}_T(P,K)) \\
 & & \mathrm{Hom}_T(P \otimes_S Q, K) \arrow[u, "\simeq"'] \\
I \otimes_T K \arrow[r, "\xi_{I,K}"'] \arrow[uur, dotted, "\alpha_1"] & K \arrow[r, "\zeta_{I,K}"'] & \mathrm{Hom}_T(I,K) \arrow[u, "(<{,}>_T{,}K)"']
\end{tikzcd} \tag{22}$$

It follows directly from the definitions that ${}_I\mathcal{C} \subseteq {}_I\mathfrak{D}$ and $\mathrm{Stat}^l({}_TP_S) \subseteq \mathrm{Gen}({}_TP)$. If ${}_TK$ is I-divisible, then $\xi_{I,K} = \omega^l_{P,K} \circ \alpha_1$ is surjective whence $\omega^l_{P,K}$ is surjective and we conclude that ${}_TK$ is P-generated by Lemma 2.5 "1". Consequently, ${}_I\mathfrak{D} \subseteq \mathrm{Gen}({}_TP)$.

Assume now that $\mathbf{P}_r \in \mathcal{P}^{\alpha}_r(S)$. Considering the canonical map $\rho_Q : T \to \mathrm{End}({}_SQ)^{op}$, the map $\rho_Q \circ <,>_T = \alpha^{\mathbf{P}_r}_Q$ is injective and so the bilinear map $<,>_T$ is injective (i.e. $P \otimes_S Q \overset{<,>_T}{\simeq} I$). Moreover, $\alpha^{\mathbf{P}_r}_{\mathrm{Hom}_T(P,K)}$ is injective and the commutativity of the upper right triangle in Diagram (22) implies that α_2 is injective (whence $\omega^l_{P,K}$ is injective by the commutativity of the lower right triangle).

1. If $K \in \mathrm{Stat}^l({}_TP_S)$, then the commutativity of the lower right triangle (22) and the injectivity of α_2 show that $\zeta_{I,K}$ is injective; hence, $\mathrm{Stat}^l({}_TP_S) \subseteq {}_I\mathfrak{F}$. On the other hand, if ${}_TK$ is P-generated, then $\omega^l_{P,K}$ is surjective by Lemma 2.5 (1), thence bijective, i.e. $K \in \mathrm{Stat}^l({}_TP_S)$. Consequently, $\mathrm{Gen}({}_TP) = \mathrm{Stat}^l({}_TP_S)$.

2. This follows directly from the inclusions in (21) and “1”.

3. Assume $\mathbf{Q}_r := (P, Q_T) \in \mathcal{P}_r^{\alpha}(T)$. Since $\mathbf{P}_r \in \mathcal{P}_r^{\alpha}(S)$, it follows by analogy to Proposition 2.13 “3” that P_S is flat, hence $id_P \otimes_S \alpha_K^{\mathbf{Q}_r}$ is injective. The commutativity of the upper left triangle in Diagram (22) implies then that α_1 is injective, thence $\xi_{I,K}$ is injective by commutativity of the lower left triangle (i.e. ${}_T I \subseteq {}_T T$ is K-pure). If ${}_T K$ is divisible, then $K \otimes_T I \overset{\xi_{I,K}}{\simeq} K$ (i.e. $K \in {}_I\mathcal{C}$).■

Theorem 6.2. *Let* $\mathcal{M} = (T, S, P, Q, <,>_T, <,>_S, I, J) \in \mathbb{UMC}$. *Then we have*

$${}_J\boldsymbol{\mathcal{L}} \subseteq {}_J\mathfrak{F} \subseteq \mathrm{Cogen}({}_S^{\#}P) \text{ and } \mathrm{Adtat}^l({}_T P_S) \subseteq \mathrm{Cogen}({}_S^{\#}P).$$

Assume $\mathbf{Q}_r := (P, Q_T) \in \mathcal{P}_r^{\alpha}(T)$. *Then*

1. $J_S \subseteq S_S$ *is pure and* ${}_J\mathcal{C} \subseteq \mathrm{Cogen}({}_S^{\#}P)$.

2. *If* $\mathbf{P}_r := (Q, P_S) \in \mathcal{P}_r^{\alpha}(S)$, *then* ${}_J\boldsymbol{\mathcal{L}} \subseteq \mathrm{Adstat}^l({}_T P_S) \subseteq \mathrm{Cogen}({}_S^{\#}P) \subseteq {}_J\mathfrak{F}$.

3. *If* $\mathbf{P}_r \in \mathcal{P}_r^{\alpha}(S)$ *and* $\mathrm{Cogen}({}_S^{\#}P) \subseteq {}_J\boldsymbol{\mathcal{L}}$, *then* ${}_J\boldsymbol{\mathcal{L}} = \mathrm{Cogen}({}_S^{\#}P) = \mathrm{Adstat}^l({}_T P_S)$.

Proof. For every right S-module L consider the commutative diagram with canonical morphisms and let α_3 and α_4 be so defined, that the entstanding triangles become commutative

$$\begin{tikzcd}
J \otimes_S L \arrow[r, "\xi_{J,L}"] \arrow[ddr, dotted, "\alpha_3"] & L \arrow[r, "\zeta_{J,L}"] \arrow[dd, "\eta^l_{P,L}"] & \mathrm{Hom}_S(J, L) \arrow[d, "(<,>_S),L)"] \\
 & & \mathrm{Hom}_S(Q \otimes_T P, L) \arrow[d, "\simeq"] \\
Q \otimes_T P \otimes_S L \arrow[uu, "(<,>_S)\otimes_S id_L"] \arrow[r, "\alpha^{\mathbf{Q}_r}_{P \otimes_S L}"'] & \mathrm{Hom}_T(P, P \otimes_S L) \arrow[r, "(P, \alpha_L^{\mathbf{P}_r})"'] \arrow[uur, dotted, "\alpha_4"] & \mathrm{Hom}_T(P, \mathrm{Hom}_S(Q, L))
\end{tikzcd} \tag{23}$$

By definition ${}_J\boldsymbol{\mathcal{L}} \subseteq {}_J\mathfrak{F}$ and $\mathrm{Adstat}^l({}_T P_S) \subseteq \mathrm{Cogen}({}_S^{\#}P)$. If ${}_S L \in {}_J\mathfrak{F}$, then $\zeta_{J,L}$ is injective and it follows by commutativity of the right rectangle in Diagram (23) that $\eta^l_{P,L}$ is injective, hence ${}_S L$ is ${}_S^{\#}P$-cogenerated by Lemma 2.5 “2”. Consequently, ${}_J\mathfrak{F} \subseteq \mathrm{Cogen}({}_S^{\#}P)$.

Assume now that $\mathbf{Q}_r \in \mathcal{P}_r^{\alpha}(T)$. Then it follows from Lemma 4.3 that $<,>_S$ is injective (hence $Q \otimes_T P \overset{<,>_S}{\simeq} J$) and α_3 is injective.

1. Since α_3 is injective, $\xi_{J,L}$ is also injective for every ${}_S L$, i.e. $J_S \subseteq S_S$ is pure. If ${}_S L \in {}_J\mathcal{C}$, then it follows from the commutativity of the left rectangle in Diagram (23) that $\eta^l_{P,L}$ is injective, hence $L \in \mathrm{Cogen}({}_S^{\#}P)$ by Lemma 2.5 (2).

2. Assume that $\mathbf{P}_r \in \mathcal{P}_r^{\alpha}(S)$, so that α_4 is injective. If ${}_S L \in {}_J\boldsymbol{\mathcal{L}}$, then $\zeta_{J,L}$ is an isomorphism, thence $\eta^l_{P,L}$ is surjective (notice that α_4 is injective). Consequently, ${}_J\boldsymbol{\mathcal{L}} \subseteq \mathrm{Adstat}^l({}_T P_S)$.

3. This follows directly from the assumptions and “2”.■

$*$-Modules

To the end of this section, we fix a ring T, a left T-module ${}_TP$ and set $S := \mathrm{End}({}_TP)^{op}$.

By definition, $\mathrm{Stat}^l({}_TP_S) \leq {}_T\mathbb{M}$ and $\mathrm{Adtat}^l({}_TP_S) \leq {}_S\mathbb{M}$ are the *smallest* subcategories between which the adjunction $(P \otimes_S -, \mathrm{Hom}_T(P,-))$ induces an equivalence. On the other hand, Lemma 2.5 shows that $\mathrm{Gen}({}_TP) \leq {}_T\mathbb{M}$ and $\mathrm{Cogen}({}_S^{\#}P) \leq {}_S\mathbb{M}$ are the *largest* possible subcategories, between which the adjunction $(P \otimes_S -, \mathrm{Hom}_T(P,-))$ may induces an equivalence (see [Col1990, Section 3] for more details).

Definition 6.3. ([MO1989]) We call ${}_TP$ a **$*$-module,** iff $\mathrm{Gen}({}_TP) \approx \mathrm{Cogen}({}_S^{\#}P)$.

Remark 6.4. It was shown by J. Trlifaj [Trl1994] that all $*$-modules are finitely generated.

Definition 6.5. A left T-module ${}_TU$ is said to be

semi-$\sum$-quasi-projective (abbr. *s*-**$\sum$-quasi-projective**), iff for any left T-module ${}_TV \in \mathrm{Pres}({}_TU)$ and any U*-presentation*

$$U^{(\Lambda)} \to U^{(\Lambda')} \to V \to 0$$

of ${}_TV$(if any), the following induced sequence is exact:

$$\mathrm{Hom}_T(U, U^{(\Lambda)}) \to \mathrm{Hom}_T(U, U^{(\Lambda')}) \to \mathrm{Hom}_T(U, V) \to 0;$$

weakly-$\sum$-quasi-projective (abbr. *w*-**$\sum$-quasi-projective**), iff for any left T-module ${}_TV$ and any short exact sequence

$$0 \to K \to U^{(\Lambda')} \to V \to 0$$

with $K \in \mathrm{Gen}({}_TU)$ (if any), the following induced sequence is exact:

$$0 \to \mathrm{Hom}_T(U, K) \to \mathrm{Hom}_T(U, U^{(\Lambda')}) \to \mathrm{Hom}_T(U, V) \to 0;$$

self-tilting, iff ${}_TU$ is w-$\sum$-quasi-projective and $\mathrm{Gen}({}_TU) = \mathrm{Pres}({}_TU)$;

$\sum$-self-static, iff any direct sum $U^{(\Lambda)}$ is U-static.

(**self**)**-small,** iff $\mathrm{Hom}_T(U,-)$ commutes with direct sums (of ${}_TU$);

Proposition 6.6. *Assume $\mathcal{M} = (T, S, P, Q, <,>_T, <,>_S)$ is a unital Morita context.*

1. *If $\mathbf{P}_r := (Q, P_S) \in \mathcal{P}_r^{\alpha}(S)$, then:*

   (a) $\mathrm{Gen}({}_TP) = \mathrm{Stat}^l({}_TP_S)$;

   (b) there is an equivalence of categories $\mathrm{Gen}({}_TP) \approx \mathrm{Cop}({}_S^{\#}P)$;

   (c) ${}_TP$ is $\sum$-self-static and $\mathrm{Stat}^l({}_TP_S)$ *is closed under factor modules.*

   (d) $\mathrm{Gen}({}_TP) = \mathrm{Pres}({}_TP)$;

2. If $\mathcal{M} \in \mathbb{UMC}_r^\alpha$ and $\mathrm{Cogen}({}_S^{\#}P) \subseteq {}_J\mathcal{L}$, then:

(a) $\mathrm{Gen}({}_TP) = \mathrm{Stat}^l({}_TP_S)$ *and* $\mathrm{Cogen}({}_S^{\#}P) = \mathrm{Adstat}^l({}_TP_S)$;

(b) there is an equivalence of categories $\mathrm{Cogen}({}_S^{\#}P) \approx \mathrm{Gen}({}_TP)$;

(c) ${}_TP$ *is a* $*$*-module;*

(d) ${}_TP$ *is self-tilting and self-small.*

Proof. 1. If $\mathbf{P}_r \in \mathcal{P}_r^\alpha(S)$, then it follows by Theorem 6.1 that $\mathrm{Gen}({}_TP) = \mathrm{Stat}^l({}_TP_S)$, which is equivalent to each of "b" and "c" by [Wis2000, 4.4.] and to "d" by [Wis2000, 4.3.].

2. It follows by the assumptions, Theorems 6.1, 6.2 and 13 that $\mathrm{Gen}({}_TP) = \mathrm{Stat}^l({}_TP_S) \approx \mathrm{Adstat}^l({}_TP_S) = \mathrm{Cogen}({}_S^{\#}P)$, hence $\mathrm{Gen}({}_TP) \approx \mathrm{Cogen}({}_S^{\#}P)$ (which is the definition of $*$-modules). Hence "a" $\Leftrightarrow$"b" $\Leftrightarrow$"c". The equivalence "a" $\Leftrightarrow$ "d" is evident by [Wis2000, Corollary 4.7.] and we are done.■

Wide Morita Contexts

Wide Morita contexts were introduced by F. Castaño Iglesias and J. Gómez-Torrecillas [C-IG-T1995] and [C-IG-T1996] as an extension of classical *Morita contexts* to Abelian categories.

Definition 6.7. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories. A **right** (**left**) **wide Morita context** between $\mathcal{A}$ and $\mathcal{B}$ is a datum $\mathcal{W}_r = (G, \mathcal{A}, \mathcal{B}, F, \eta, \rho)$, where $G : \mathcal{A} \rightleftarrows \mathcal{B} : F$ are right (left) exact covariant functors and $\eta : F \circ G \longrightarrow 1_{\mathcal{A}}$, $\rho : G \circ F \longrightarrow 1_{\mathcal{B}}$ ($\eta : 1_{\mathcal{A}} \longrightarrow F \circ G$, $\rho : 1_{\mathcal{B}} \longrightarrow G \circ F$) are natural transformations, such that for every pair of objects $(A, B) \in \mathcal{A} \times \mathcal{B}$ the compatibility conditions $G(\eta_A) = \rho_{G(A)}$ and $F(\rho_B) = \eta_{F(B)}$ hold.

Definition 6.8. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories and $\mathcal{W} = (G, \mathcal{A}, \mathcal{B}, F, \eta, \rho)$ be a right (left) wide Morita context. We call $\mathcal{W}$ **injective** (respectively **semi-strict**, **strict**), iff η and ρ are monomorphisms (respectively epimorphisms, isomorphisms)

Remarks 6.9. Let $\mathcal{W} = (G, \mathcal{A}, \mathcal{B}, F, \eta, \rho)$ be a right (left) wide Morita context.

1. It follows by [CDN2005, Propositions 1.1., 1.4.] that if either η or ρ is an epimorphism (monomorphism), then $\mathcal{W}$ is strict, whence $\mathcal{A} \approx \mathcal{B}$.

2. The resemblance of *injective* left wide Morita contexts is with the Morita-Takeuchi contexts for comodules of coalgebras, i.e. the so called *pre-equivalence data* for categories of comodules introduced in [Tak1977] (see [C-IG-T1998] for more details).

Injective Right wide Morita contexts

In a recent work [CDN2005, 5.1.], Chifan, et. al. clarified (for module categories) the relation between *classical Morita contexts* and *right wide Morita contexts.* For the convenience of the reader and for later reference, we include in what follows a brief description of this relation.

6.10. Let T, S be rings, $\mathcal{A} := {}_T\mathbb{M}$ and $\mathcal{B} := {}_S\mathbb{M}$. Associated to each Morita context $\mathcal{M} = (T, S, P, Q, <,>_T, <,>_S)$ is a wide Morita context as follows: Define $G : \mathcal{A} \rightleftarrows \mathcal{B} : F$ by $G(-) = Q \otimes_T -$ and $F(-) = P \otimes_S -$. Then there are natural transformations $\eta : F \circ G \longrightarrow 1_{{}_T\mathbb{M}}$ and $\rho : G \circ F \longrightarrow 1_{{}_S\mathbb{M}}$ such that for each ${}_TV$ and W_S :

$$\begin{array}{lcllllll} \eta_V & : & P \otimes_S (Q \otimes_T V) & \to & V, & \sum p_i \otimes_S (q_i \otimes_T v_i) & \mapsto & \sum < p_i, q_i >_T v_i, \\ \rho_W & : & Q \otimes_T (P \otimes_S W) & \to & W, & \sum q_i \otimes_T (p_i \otimes_S w_i) & \mapsto & \sum < q_i, p_i >_S w_i. \end{array} \tag{24}$$

Then the datum $\mathcal{W}_r(\mathcal{M}) := (G, {}_T\mathbb{M}, {}_S\mathbb{M}, F, \eta, \rho)$ is a right wide Morita context.

Conversely, let T', S' be two rings and $\mathcal{W}'_r = (G', {}_{T'}\mathbb{M}, {}_{S'}\mathbb{M}, F', \eta', \rho')$ *be a right wide* Morita context *between* ${}_{T'}\mathbb{M}$ and ${}_{S'}\mathbb{M}$ such that *the right exact functors* $G' : {}_{T'}\mathbb{M} \rightleftarrows {}_{S'}\mathbb{M} : F'$ *commute with direct sums. By* Watts' Theorems (e.g. [Gol1979]), there exists a (T, S)-bimodule P' (e.g. $F'(S')$) such that $F' \simeq P' \otimes_{S'} -$, an (S, T)-bimodule Q' such that $G' \simeq Q' \otimes_{T'} -$ and there must exits two bilinear forms

$$<,>_{T'}: P' \otimes_{S'} Q' \to T' \text{ and } <,>_{S'}: Q' \otimes_{T'} P' \to S',$$

such that the natural transformations $\eta' : F' \circ G' \to 1_{{}_{T'}\mathbb{M}}$, $\rho : G' \circ F' \to 1_{{}_{S'}\mathbb{M}}$ are given by

$$\eta'_{V'}(p' \otimes_{S'} q' \otimes_{T'} v') =< p', q' >_{T'} v' \text{ and } \rho'_{W'}(q' \otimes_T p' \otimes_S w') =< q', p' >_{S'} w'$$

for all $V' \in {}_{T'}\mathbb{M}$, $W' \in {}_{S'}\mathbb{M}$, $p' \in P'$, $q' \in Q'$, $v' \in V'$ and $w' \in W'$. It can be shown that in this way one obtains a Morita context $\mathcal{M}' = \mathcal{M}'(\mathcal{W}'_r) := (T', S', P', Q', <,>_{T'}, <,>_{S'})$. *Moreover, it turns out that given a wide Morita context* $\mathcal{W}_r$, we have $\mathcal{W}_r \simeq \mathcal{W}_r(\mathcal{M}(\mathcal{W}_r))$.

The following result clarifies the relation between *injective Morita contexts* and *injective right wide Morita contexts.*

Theorem 6.11. *Let* $\mathcal{M} = (T, S, P, Q, <,>_T, <,>_S)$ *be a Morita context,* $\mathcal{A} := {}_T\mathbb{M}$, $\mathcal{B} := {}_S\mathbb{M}$ *and consider the induced right wide Morita context* $\mathcal{W}_r(\mathcal{M}) := (G, \mathcal{A}, \mathcal{B}, F, \eta, \rho)$.

1. *If* $\mathcal{W}_r(\mathcal{M})$ *is an injective right wide Morita context, then* $\mathcal{M}$ *is an injective Morita context.*

2. *If* $\mathcal{M} \in \mathbb{UMC}^{\alpha}_r$, *then* $\mathcal{W}_r(\mathcal{M})$ *is an injective right wide Morita context.*

Proof. 1. Let $\mathcal{W}_r(\mathcal{M})$ be an injective right wide Morita context. Then in particular, $<,>_T = \eta_T$ and $<,>_S = \rho_S$ are injective, i.e. $\mathcal{M}$ is an injective Morita context.

2. Assume that $\mathcal{M}$ satisfies the right α-condition. Suppose there exists some ${}_TV$ and $\sum p_i \otimes_S (q_i \otimes_T v_i) \in \mathrm{Ker}(\eta_V)$. Then for any $q \in Q$ we have

$$\begin{array}{rclcl} 0 & = & q \otimes_T \eta_V(\sum(p_i \otimes_S q_i) \otimes_T v_i) & = & \sum q \otimes_T < p_i, q_i >_T v_i \\ & = & \sum q < p_i, q_i >_T \otimes_T v_i & = & \sum < q, p_i >_S q_i \otimes_T v_i \\ & = & \sum < q, p_i >_S (q_i \otimes_T v_i) & = & \alpha^{\mathbf{P}_r}_{Q \otimes_T V}(\sum p_i \otimes_S (q_i \otimes_T v_i)(q). \end{array}$$

Since $\mathbf{P}_r := (Q, P_S) \in \mathcal{P}^\alpha_r(S)$, the morphism $\alpha^{\mathbf{P}_r}_{Q \otimes_T V}$ is injective and so $\sum p_i \otimes_S (q_i \otimes_T v_i) = 0$, i.e. η_V is injective. Analogously, suppose $\sum q_i \otimes_T (p_i \otimes_S w_i) \in \mathrm{Ker}(\rho_W)$. Then for any $p \in P$ we have

$$\begin{array}{rclcl} 0 & = & p \otimes_S \rho_W(\sum q_i \otimes_T (p_i \otimes_S w_i) & = & \sum p \otimes_S < q_i, p_i >_S w_i \\ & = & \sum p < q_i, p_i >_S \otimes_S w_i & = & \sum < p, q_i >_T p_i \otimes_S w_i \\ & = & \sum < p, q_i >_T (p_i \otimes_S w_i) & & \alpha^{\mathbf{Q}_r}_{P \otimes_S W}(\sum q_i \otimes_T (p_i \otimes_S w_i))(p). \end{array}$$

Since $\mathbf{Q}_r := (P, Q_T) \in \mathcal{P}^\alpha_r(T)$, the morphism $\alpha^{\mathbf{Q}_r}_{P \otimes_S W}$ is injective and so $\sum q_i \otimes_T (p_i \otimes_S w_i) = 0$, i.e. ρ_W is injective. Consequently, the induced right wide Morita context $\mathcal{W}_r(\mathcal{M})$ is injective.■

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