Approximation of the distribution of a stationary Markov process with application to option pricing

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Abstract

We build a sequence of empirical measures on the space $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ of $\mathbb{R}^d$-valued càdlàg functions on $\mathbb{R}_+$ in order to approximate the law of a stationary $\mathbb{R}^d$-valued Markov and Feller process (X_t). We obtain some general results of convergence of this sequence. Then, we apply them to Brownian diffusions and solutions to Lévy driven SDE's under some Lyapunov-type stability assumptions.
As a numerical application of this work, we show that this procedure gives an efficient way of option pricing in stochastic volatility models.

Keywords: stationary process ; numerical approximation ; Lévy process ;
Euler scheme ; option pricing, stochastic volatility model.

1 Introduction

1.1 Objectives and Motivations

In this paper, we deal with an $\mathbb{R}^d$-valued Feller Markov process (X_t) with semi-group $(P_t)_{t\geq 0}$ and we assume that (X_t) admits an invariant distribution ν_0. The aim of this work is to propose a way to approximate the whole stationary distribution law $\mathbb{P}_{\nu_0}$ of (X_t). More precisely, we want to construct a sequence of weighted occupation measures $(\nu^{(n)}(\omega,dy))_{n\geq 1}$ on the Skorokhod space $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ such that $\nu^{(n)}(\omega,F)\xrightarrow{n\to+\infty}\int F(y)\mathbb{P}_{\nu_0}(dy)$ *a.s.* for a class of functionals $F:\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ which includes bounded continuous functionals for the Skorokhod topology.
One of our motivations is to develop a new numerical method for option pricing in stationary stochastic volatility models which are slight modifications of the classical stochastic volatility models, where we suppose that the volatility evolves under its stationary regime instead of assuming that the initial value of the volatility is constant.

1.2 Background and construction of the procedure

This work is in the continuity of a series of recent papers due to Lamberton-Pagès ([13, 14]), Lemaire ([15, 16]) and Panloup ([19, 20, 21]) where the problem of the approximation of the

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invariant distribution is investigated for Brownian diffusions and for Lévy driven SDE's.[1]. In these papers, the algorithm is based on an adapted Euler scheme with decreasing step $(\gamma_k)_{k\geq 1}$. To be precise, let (Γ_n) be the sequence of discretization times: $\Gamma_0 = 0$, $\Gamma_n = \sum_{k=1}^n \gamma_k$ for every $n \geq 1$, and assume that $\Gamma_n \to +\infty$ when $n \to +\infty$. Let $(\bar{X}_{\Gamma_n})_{n\geq 0}$ be the Euler scheme obtained by "freezing" the coefficients between the Γ_n's and let $(\eta_n)_{n\geq 1}$ be a sequence of positive weights such that $H_n := \sum_{k=1}^n \eta_k \to +\infty$ when $k \to +\infty$. Then, under some Lyapunov-type stability assumptions adapted to the stochastic processes of interest, one shows that for a large class of steps and weights $(\eta_n, \gamma_n)_{n\geq 1}$,

$$\bar{\nu}_n(\omega, f) := \frac{1}{H_n}\sum_{k=1}^n \eta_k f(\bar{X}_{\Gamma_{k-1}}) \xrightarrow{n\to+\infty} \int f(x)\nu_0(dx) \quad a.s. \tag{1}$$

(at least)[2] for every bounded continuous function f.
Since the problem of the approximation of the invariant distribution has been deeply studied for a wide class of Markov processes (Brownian diffusions and Lévy driven SDE's) and since the proof of (1) can be adapted to other classes of Markov processes under some specific Lyapunov assumptions, we choose in this paper to consider a general Markov process and to assume the existence of an Euler scheme $(\bar{X}_{\Gamma_k})_{k\geq 0}$ such that (1) holds for the class of bounded continuous functions. The aim of this paper is then to investigate the convergence properties of a functional version of the sequence $(\bar{\nu}_n(\omega, dy))_{n\geq 1}$:

Let (X_t) be a Markov and Feller process and let $(\bar{X}_t)_{t\geq 0}$ be a stepwise constant Euler scheme of (X_t) with non-increasing step sequence $(\gamma_n)_{n\geq 1}$ satisfying

$$\lim_{n\to+\infty} \gamma_n = 0, \quad \Gamma_n := \sum_{k=1}^n \gamma_k \xrightarrow{n\to+\infty} +\infty. \tag{2}$$

Setting $\Gamma_0 := 0$ and $\bar{X}_0 = x_0 \in \mathbb{R}^d$, we assume that

$$\bar{X}_t = \bar{X}_{\Gamma_n} \qquad \forall t \in [\Gamma_n, \Gamma_{n+1}[, \tag{3}$$

and that $(\bar{X}_{\Gamma_n})_{n\geq 0}$ can be simulated recursively.
We denote by $(\mathcal{F}_t)_{t\geq 0}$ (resp. $(\bar{\mathcal{F}}_t)_{t\geq 0}$) the usual augmentation of the natural filtration $(\sigma(X_s, 0 \leq s \leq t))_{t\geq 0}$ (resp. $(\sigma(\bar{X}_s, 0 \leq s \leq t))_{t\geq 0}$).
For $k \geq 0$, we denote by $(\bar{X}_t^{(k)})_{t\geq 0}$ the shifted process defined by

$$\bar{X}_t^{(k)} := \bar{X}_{\Gamma_k + t}.$$

In particular, $\bar{X}_t^{(0)} = \bar{X}_t$. We define a sequence of random probabilities $(\nu^{(n)}(\omega, dy))_{n\geq 1}$ on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ by

$$\nu^{(n)}(\omega, dy) = \frac{1}{H_n}\sum_{k=1}^n \eta_k \mathbf{1}_{\{\bar{X}^{(k-1)}(\omega)\in dy\}}$$

[1]Note that computing the invariant distribution is equivalent to computing the marginal laws of the stationary process (X_t) since $\nu_0 P_t = \nu_0$ for every $t \geq 0$.

[2]The class of functions for which (1) holds depends on the stability of the dynamical system. In particular, in the Brownian diffusion case, the convergence may hold for continuous functions with subexponential growth whereas the class of functions strongly depends on the moments of the Lévy process when the stochastic process is a Lévy driven SDE.

where $(\eta_k)_{k\geq 1}$ is a sequence of weights. For $t \geq 0$, $(\nu_t^{(n)}(\omega, dy))_{n\geq 1}$ will denote the sequence of "marginal" empirical measures on $\mathbb{R}^d$ defined by

$$\nu_t^{(n)}(\omega, dy) = \frac{1}{H_n}\sum_{k=1}^{n} \eta_k \mathbf{1}_{\{\bar{X}_t^{(k-1)}(\omega)\in dy\}}.$$

1.3 Simulation of $(\nu^{(n)}(\omega, F))_{n\geq 1}$

For every functional $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$, the following recurrence relation holds for every $n \geq 1$:

$$\nu^{(n+1)}(\omega, F) = \nu^{(n)}(\omega, F) + \frac{\eta_{n+1}}{H_{n+1}}(F(X^{(n)}(\omega)) - \nu^{(n)}(\omega, F)). \tag{4}$$

For $T > 0$, let $\mathbb{D}([0, T], \mathbb{R}^d)$ denote the set of càdlàg functions α defined on $[0, T]$. Based on (4), $(\nu^{(n)}(\omega, F))_{n\geq 1}$ can be simulated for every functional $F : \mathbb{D}([0, T], \mathbb{R}^d) \to \mathbb{R}$ by the following procedure :

Step 0: i. Simulate $(\bar{X}_t^{(0)})_{t\geq 0}$ on $[0, T]$, *i.e.* simulate $(\bar{X}_{\Gamma_k})_{k\geq 0}$ for $k = 0, \ldots, N(0, T)$ where

$$N(n, T) := \inf\{k \geq n, \Gamma_{k+1} - \Gamma_n > T\} = \max\{k \geq 0, \Gamma_k - \Gamma_n \leq T\}) \quad n \geq 0,\ T > 0. \tag{5}$$

Note that $n \mapsto N(n, t)$ is an increasing sequence since (γ_n) is non-increasing and that

$$\Gamma_{N(n,T)} - \Gamma_n \leq T < \Gamma_{N(n,T)+1} - \Gamma_n.$$

ii. Compute $F((\bar{X}_t^{(0)})_{t\geq 0})$ and $\nu^{(1)}(\omega, F)$. Store the values of $(\bar{X}_{\Gamma_k})$ for $k = 1, \ldots, N(0, T)$.

Step n $(\mathbf{n \geq 1})$: i. Since the values $(\bar{X}_{\Gamma_k})_{k\geq 0}$ are stored for $k = n, \ldots, N(n-1, T)$, simulate $(\bar{X}_{\Gamma_k})_{k\geq 0}$ for $k = N(n-1, T)+1, \ldots, N(n, T)$ in order to obtain a path of $(\bar{X}_t^{(n)})$ on $[0, T]$.
ii. Compute $F((\bar{X}_t^{(n)})_{t\geq 0})$ and use (4) to compute $\nu^{(n+1)}(\omega, F)$. Store the values of $(\bar{X}_{\Gamma_k})$ for $k = n+1, \ldots, N(n, T)$.

In general, the computation of $F(\bar{X}^{(n)})$ needs the values of $(\bar{X}_{\Gamma_k})$ for $k = n, \ldots, N(n, T)$. Hence, first, the necessary number of values to store at step n is of order $N(n, T) - n$ and grows to infinity when $n \to +\infty$ since $\gamma_n \to 0$. For instance, when $\gamma_n = Cn^{-\rho}$ with $\rho \in (0, 1)$, one can check that

$$N(n, t) - n \overset{n\to+\infty}{\sim} \frac{T}{C} n^{\rho}.$$

Second, $N(n, T) - n$ also represents the order of the complexity of the algorithm. In particular, the complexity is not linear and depends on the decreasing-rate of the step sequence. However, in some good situations, the computation time can be significantly reduced. Indeed, since $\bar{X}^{(n+1)} = \bar{X}^{(n)}_{\gamma_{n+1}+.}$, it is usually possible to use at step $n + 1$ the preceding computations and to simulate the sequence $(F(\bar{X}^{(n)}))_{n\geq 0}$ in a "quasi-recursive" way. Assume for instance that for a functional $F : \mathbb{D}([0, T], \mathbb{R}^d) \to \mathbb{R}$, there exists a function $g : \mathbb{R}^d \to \mathbb{R}$ such that for every $\alpha \in \mathbb{D}([0, T], \mathbb{R}^d)$

$$F(\alpha) = g(I_T(\alpha)) \quad \text{where} \quad I_T(\alpha) = \int_0^T h(\alpha_s)ds.$$

One checks that for every $n \geq 0$,

$$I_T(\bar{X}^{(n+1)}) = I_T(\bar{X}^{(n)}) - \gamma_{n+1}\bar{X}_{\Gamma_n} + \int_T^{T+\gamma_{n+1}} h(\bar{X}_s^{(n)})ds.$$

Hence, for such a functional, the complexity is lower since the number of operations at each step n is of order $N(n+1,T) - N(n,T)$ (instead of $N(n+1,T) - (n+1)$ in the general case). When $\gamma_n = Cn^{-\rho}$ with $\rho \in (0,1]$,

$$\limsup_{n\to+\infty} \big(N(n+1,T) - N(n,T)\big) \leq C_\rho < +\infty.$$

Before outlining the sequel of the paper, we list some notations: $\mathcal{C}(\mathbb{R}_+,\mathbb{R}^d)$ denotes the space of continuous functions on $\mathbb{R}_+$ with values in $\mathbb{R}^d$ endowed with the topology of uniform convergence on compact sets. As well, $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ denotes the space of càdlàg functions on $\mathbb{R}_+$ with values in $\mathbb{R}^d$ endowed with the Skorokhod topology. We will say that a functional $F : \mathbb{D}(\mathbb{R}_+,\mathbb{R}^d) \to \mathbb{R}$ is Sk-continuous if F is continuous for the Skorokhod topology on $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ and the notation "$\stackrel{(Sk)}{\Longrightarrow}$" will denote the weak convergence on $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$.

In Section 2, we state our main results for a general $\mathbb{R}^d$-valued Feller Markov process. Then, we apply them to Brownian diffusions and Lévy SDE's in Section 3. Section 4 is devoted to the proof of the main general results. Finally, in Section 5, we complete this paper by an application to option pricing in Stationary Stochastic Volatility models.

2 General results

In this section, we state the results of convergence of the sequence $(\nu^{(n)}(\omega,dy))_{n\geq1}$ when (X_t) is a general Feller Markov process. As explained in the introduction, since the *a.s.* convergence of $(\nu_0^{(n)}(\omega,dy))_{n\geq1}$ to the invariant distribution ν_0 has already been deeply studied for a large class of Markov processes (Brownian diffusions and Lévy driven SDE's), our approach will be to derive the convergence of $(\nu^{(n)}(\omega,dy))_{n\geq1}$ toward $\mathbb{P}_{\nu_0}$, from that of $(\nu_0^{(n)}(\omega,dy))_{n\geq1}$ to the invariant distribution ν_0. More precisely, we will assume in Theorem 1 that

$(\mathbf{C_{0,1}})$: (X_t) admits a unique invariant distribution ν_0 and

$$\nu_0^{(n)}(\omega,dy) \stackrel{n\to+\infty}{\Longrightarrow} \nu_0(dy) \quad a.s.$$

whereas in Theorem 2, we will only assume that

$(\mathbf{C_{0,2}})$: $(\nu_0^{(n)}(\omega,dy))_{n\geq1}$ is *a.s.* tight on $\mathbb{R}^d$.

We also introduce three other assumptions $(\mathbf{C_1})$, $(\mathbf{C_2})$ and $(\mathbf{C_{3,\varepsilon}})$ relative to the continuity in probability of the flow $x \mapsto (X_t^x)$, to the asymptotic convergence of the shifted Euler scheme to the true process (X_t) and to the steps and weights respectively.

$(\mathbf{C_1})$: For every $x_0 \in \mathbb{R}^d$, $\epsilon > 0$ and $T > 0$,

$$\limsup_{x_0\to x} \mathbb{P}(\sup_{0\leq t\leq T} |X_t^x - X_t^{x_0}| \geq \epsilon) = 0. \tag{6}$$

$(\mathbf{C_2})$: $(\bar{X}_t)$ is a non-homogeneous Markov process and for every $n \geq 0$, it is possible to construct a family of stochastic processes $(\hat{X}_t^{(n,x)})_{x\in\mathbb{R}^d}$ such that
i) $\mathcal{L}(\hat{X}^{(n,x)}) \overset{\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)}{=} \mathcal{L}(\bar{X}^{(n)}|\bar{X}_0^{(n)} = x)$.
ii) For every compact set K of $\mathbb{R}^d$, for every $T \geq 0$,

$$\sup_{x\in K} \sup_{0\leq t\leq T} |\hat{X}_t^{(n,x)} - X_t^x| \xrightarrow{n\to+\infty} 0 \quad \text{in probability.} \tag{7}$$

$(\mathbf{C_{3,\varepsilon}})$: For every $n \geq 1$, $\eta_n \leq C\gamma_n H_n^\varepsilon$.

Remark 1. Assumption $(\mathbf{C_2})$ implies in particular that asymptotically and uniformly on compact sets of $\mathbb{R}^d$, the law of the approximate process $(\bar{X}^{(n)})$ given its initial value is closed to that of the true process.

If there exists a unique invariant distribution ν_0, the second assertion of $(\mathbf{C_2})$ can be relaxed into the less stringent: for all $\epsilon > 0$, there exists a compact set $A_\epsilon \subset \mathbb{R}^d$ such that $\nu_0(A_\epsilon^c) \leq \epsilon$ and such that

$$\sup_{x\in A_\epsilon} \sup_{0\leq t\leq T} |\hat{X}_t^{(n,x)} - X_t^x| \xrightarrow{n\to+\infty} 0 \quad \text{in probability.} \tag{8}$$

This weaker assumption can be needed in the stochastic volatility models like the Heston model (see Section 5 for details).

The preceding assumptions are all what we need for the convergence of $(\nu^{(n)}(\omega, dy))_{n\geq 1}$ toward $\mathbb{P}_{\nu_0}$ along the bounded Sk-continuous functionals, *i.e.* for the *a.s.* weak convergence on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$. However, the integration of non-bounded continuous functionals $F : \mathbb{D}([0,T], \mathbb{R}^d) \to \mathbb{R}$ will need some additional assumptions depending on the stability of the Euler scheme and on the steps and weights sequences. We will suppose that F is dominated (in a sense specified after) by a function $\mathcal{V} : \mathbb{R}^d \to \mathbb{R}_+$ that satisfies the following assumptions with $s \geq 2$ and $\varepsilon < 1$:
$\mathbf{H}(\mathbf{s}, \varepsilon)$: For every $T > 0$,

$$(i) \quad \sup_{n\geq 1} \mathbb{E}\{\sup_{0\leq t\leq T} \mathcal{V}^s(\hat{X}_t^{(n,x)})\} \leq C_T \mathcal{V}^s(x), \qquad (ii) \quad \sup_{n\geq 1} \nu_0^{(n)}(\mathcal{V}) < +\infty,$$

$$(iii) \quad \sum_{k\geq 1} \frac{\eta_k}{H_k^2} \mathbb{E}[\mathcal{V}^2(\bar{X}_{\Gamma_{k-1}})] < +\infty, \quad (iv) \quad \sum_{k\geq 1} \frac{\Delta N(k,T)}{H_k^s} \mathbb{E}[\mathcal{V}^{s(1-\varepsilon)}(\bar{X}_{\Gamma_{k-1}})] < +\infty.$$

where $T \mapsto C_T$ is locally bounded on $\mathbb{R}_+$ and $\Delta N(k,T) = N(k,T) - N(k-1,T)$.

For every $\varepsilon < 1$, we then set

$$\mathcal{K}(\varepsilon) = \Big\{\mathcal{V} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}_+), \mathbf{H}(\mathbf{s}, \varepsilon) \text{ holds with } s \geq 2\Big\}.$$

Remark 2. Except Assumption (i) which is a classical condition on the finite time horizon control of the Euler scheme, the assumptions in $\mathbf{H}(\mathbf{s}, \varepsilon)$ strongly rely on the Lyapunov stability of the Euler scheme. More precisely, we will see when we apply our general results to SDE's that these properties are some consequences of the Lyapunov assumptions needed for the tightness of $(\nu_0^{(n)}(\omega, dy))_{n\geq 1}$.

We can now state our first main result:

Theorem 1. *Assume* ($\mathbf{C_{0,1}}$), ($\mathbf{C_1}$), ($\mathbf{C_2}$) *and* ($\mathbf{C_{3,\varepsilon}}$) *with* $\varepsilon \in (-\infty, 1)$. *Then, a.s., for every bounded Sk-continuous functional* $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$,

$$\nu^{(n)}(\omega, F) \xrightarrow{n\to+\infty} \int F(y)\mathbb{P}_{\nu_0}(dy) \tag{9}$$

where $\mathbb{P}_{\nu_0}$ *denotes the stationary distribution of* (X_t) *(with initial law* ν_0*).*
Furthermore, for every $T > 0$, (9) *holds for every* $F : \mathbb{D}([0,T], \mathbb{R}^d) \to \mathbb{R}$ *satisfying*

$$|F(x_t, 0 \le t \le T)| \le \sup_{0\le t\le T} \mathcal{V}^\rho(x_t) \qquad \forall\ (x_t) \in \mathbb{D}([0,T], \mathbb{R}^d). \tag{10}$$

with $\mathcal{V} \in \mathcal{K}(\varepsilon)$ *and* $\rho \in [0, 1)$.

In the second result, the uniqueness of the invariant distribution is not required and the sequence $(\nu_0^{(n)}(\omega, dy))_{n\ge1}$ is only supposed to be tight:

Theorem 2. *Assume* ($\mathbf{C_{0,2}}$), ($\mathbf{C_1}$), ($\mathbf{C_2}$) *and* ($\mathbf{C_{3,\varepsilon}}$) *with* $\varepsilon \in (-\infty, 1)$. *Assume that* $(\nu_0^{(n)}(\omega, dy))_{n\ge1}$ *is a.s. tight on* $\mathbb{R}^d$. *Then,*
(i) The sequence $(\nu^{(n)}(\omega, dy))_{n\ge1}$ *is a.s. tight on* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ *and a.s., for every convergent subsequence* $(n_k(\omega))_{n\ge1}$, *for every bounded Sk-continuous functional* $F : \mathbb{D}(\mathbb{R}^d, \mathbb{R}_+) \to \mathbb{R}$,

$$(\nu^{(n_k(\omega))}(\omega, F) \xrightarrow{n\to+\infty} \int F(y)\mathbb{P}_{\nu_\infty}(dy) \tag{11}$$

where $\mathbb{P}_{\nu_\infty}$ *is the law of* (X_t) *with initial law* ν_∞ *living in the weak limits of* $(\nu_0^{(n)}(\omega, dy))_{n\ge1}$.
Furthermore, a.s., for every $T > 0$, (11) *holds for every* $F : \mathbb{D}([0,T], \mathbb{R}^d) \to \mathbb{R}$ *satisfying* (10) *with* $\mathcal{V} \in \mathcal{K}(\varepsilon)$ *and* $\rho \in [0, 1)$.
(ii) If moreover,

$$\frac{1}{H_n}\sum_{k=1}^n \max_{l\ge k+1} \frac{|\Delta\eta_\ell|}{\gamma_\ell} < +\infty, \tag{12}$$

then ν_∞ *is necessary an invariant distribution for the Markov process* (X_t).

Remark 3. Condition (12) holds for a large class of steps and weights. For instance, if $\eta_n = C_1 n^{-\rho_1}$ and $\gamma_n = C_2 n^{-\rho_2}$ with $\rho_1 \in [0,1]$ and $\rho_2 \in (0,1]$, then (12) is satisfied if $\rho_1 = 0$ or if $\rho_1 \in (\max(0, 2\rho_2 - 1), 1)$.

3 Application to Brownian diffusions and Lévy driven SDE's

Let $(X_t)_{t\ge0}$ be a càdlàg stochastic process solution to the following SDE:

$$dX_t = b(X_{t^-})dt + \sigma(X_{t^-})dW_t + \kappa(X_{t^-})dZ_t \tag{13}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \mapsto \mathbb{M}_{d,\ell}$ (set of $d \times \ell$ real matrices) and $\kappa : \mathbb{R}^d \mapsto \mathbb{M}_{d,\ell}$ are continuous functions with sublinear growth, $(W_t)_{t\ge0}$ is a ℓ-dimensional Brownian motion and $(Z_t)_{t\ge0}$ is a purely discontinuous $\mathbb{R}^\ell$-valued Lévy process independent of $(W_t)_{t\ge0}$ with Lévy measure π and characteristic function given for every $t \ge 0$ by

$$\mathbb{E}[e^{i<u,Z_t>}] = \exp\big[t\big(\int e^{i<u,y>} - 1 - i < u, y > 1_{\{|y|\le1\}}\pi(dy)\big)\big].$$

Let $(\gamma_n)_{n\ge1}$ be a non-increasing step sequence satisfying (2). Let $(U_n)_{n\ge1}$ be a sequence of i.i.d. random variables such that $U_1 \overset{\mathcal{L}}{=} \mathcal{N}(0, I_\ell)$ and let $\xi := (\xi_n)_{n\ge1}$ be a sequence of independent $\mathbb{R}^\ell$-valued random variables, independent of $(U_n)_{n\geq 1}$. We then denote by $(\bar{X}_t)_{t\geq 0}$ the stepwise constant Euler scheme of (X_t) for which $(\bar{X}_{\Gamma_n})_{n\geq 0}$ is recursively defined by $\bar{X}_0 = x \in \mathbb{R}^d$ and

$$\bar{X}_{\Gamma_{n+1}} = \bar{X}_{\Gamma_n} + \gamma_{n+1} b(\bar{X}_{\Gamma_n}) + \sqrt{\gamma_{n+1}}\sigma(\bar{X}_{\Gamma_n})U_{n+1} + \kappa(\bar{X}_{\Gamma_n})\xi_{n+1}. \tag{14}$$

We recall that the increments of (Z_t) can not be simulated in general. That is why we generally need to construct the sequence (ξ_n) with some approximations of the true increments. We will come back to this construction in subsection 3.2.
As in the general case, we denote by $(\bar{X}_t^{(k)})$ and $(\nu^{(n)}(\omega, dy))$ the sequence of associated shifted Euler schemes and empirical measures respectively.

Let us now introduce some Lyapunov assumptions for the SDE. We denote by $\mathcal{EQ}(\mathbb{R}^d)$ the set of *Essentially Quadratic* $\mathcal{C}^2$-functions $V : \mathbb{R}^d \to \mathbb{R}_+^*$ such that $\lim V(x) = +\infty$ as $|x| \to +\infty$, $|\nabla V| \leq C\sqrt{V}$ and $D^2 V$ is bounded. We also denote by $\tilde{b} := b + \kappa \int_{\{|y|>1\}} y\pi(dy)$, the "real" drift term of the dynamical system resulting from b and the big jumps of the Lévy process.
Finally, let $a \in (0,1]$ denote the intensity of the mean-reversion intensity. The Lyapunov (or mean-reversion) assumption is the following:

$(\mathbf{S_a})$: There exists a function $V \in \mathcal{EQ}(\mathbb{R}^d)$ such that:

$$i. \quad |b|^2 \leq CV^a \quad \mathrm{Tr}(\sigma\sigma^*(x)) + \|\kappa(x)\|^2 \overset{|x|\to+\infty}{=} o(V^a(x)),$$

$$ii. \quad \text{There exist } \beta \in \mathbb{R} \text{ and } \alpha > 0 \text{ such that } \langle \nabla V, \tilde{b}\rangle \leq \beta - \alpha V^a.$$

From now on, we separate the Brownian diffusions and Lévy driven SDE's cases.

3.1 Application to Brownian diffusions

In this part, we assume that $\kappa = 0$. We recall a result of [14].

Proposition 1. *Let $a \in (0,1]$ such that $(\mathbf{S_a})$ holds. Assume that the sequence $(\eta_n/\gamma_n)_{n\geq 1}$ is nonincreasing.*
(a) Let $(\theta_n)_{n\geq 1}$ be a sequence of positive numbers such that $\sum_{n\geq 1}\theta_n\gamma_n < +\infty$ and that there exists $n_0 \in \mathbb{N}$ such that $(\theta_n)_{n\geq n_0}$ is nonincreasing. Assume that there exists $n_0 \in \mathbb{N}$ such that $(\theta_n)_{n\geq n_0}$ is a nonincreasing sequence. Then, for every positive r,

$$\sum_{n\geq 1}\theta_n\gamma_n \mathbb{E}[V^r(\bar{X}_{\Gamma_{n-1}})] < +\infty.$$

(b) For every $r > 0$,

$$\sup_{n\geq 1} \nu_0^{(n)}(\omega, V^r) < +\infty \quad a.s. \tag{15}$$

Hence, the sequence $(\nu_0^{(n)}(\omega, dy))_{n\geq 1}$ is a.s. tight.
(c) Moreover, every weak limit of this sequence is an invariant probability for the SDE (13)*. In particular, if $(X_t)_{t\geq 0}$ admits a unique invariant probability ν_0, for every continuous function f such that $f \leq CV^r$ with $r > 0$, $\lim_{n\to\infty} \nu_0^{(n)}(\omega, f) = \nu_0(f)$ a.s.*

Remark 4. For instance, if $V(x) = 1 + |x|^2$, the preceding convergence holds for every continuous function with polynomial growth. According to Theorem 3.2 of [15], it is possible to extend these results to continuous functions with exponential growth but it then strongly depends on σ. As well, the conditions on steps and weights can be less restrictive and may contain the case $\eta_n = 1$ for instance (see Remark 4 of [14] and [15]).

We then derive the following result from the preceding proposition and from Theorems 1 and 2.

Theorem 3. *Assume that b and σ are locally Lipschitz functions and that $\kappa = 0$. Let $a \in (0,1]$ such that $(\mathbf{S_a})$ holds and assume that (η_n/γ_n) is nonincreasing.*
(a) The sequence $(\nu^{(n)}(\omega, dy))_{n\geq 1}$ is a.s. tight on $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ and every weak limit of $(\nu^{(n)}(\omega, dy))_{n\geq 1}$ is a stationary distribution for the process $(X_t)_{t\geq 0}$ solution to (13). In particular, when uniqueness holds for the invariant distribution ν_0, a.s., for every bounded continuous functional $F : \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$,

$$\nu^{(n)}(\omega, F) \xrightarrow{n\to+\infty} \int F(x)\mathbb{P}_{\nu_0}(dx). \tag{16}$$

(b) Furthermore, if there exists $s \in (2, +\infty)$ and $n_0 \in \mathbb{N}$ such that

$$\left(\frac{\Delta N(k,T)}{\gamma_k H_k^s}\right)_{n\geq n_0} \textit{ is nonincreasing and } \sum_{k\geq 1} \frac{\Delta N(k,T)}{H_k^s} < +\infty, \tag{17}$$

then, for every $T > 0$, (16) holds for every continuous $F : \mathcal{C}([0,T], \mathbb{R}^d) \to \mathbb{R}$ satisfying:

$$\exists r > 0 \quad \textit{such that} \quad |F(x_t, 0 \leq t \leq T)| \leq C \sup_{0\leq t\leq T} V^r(x_t) \quad \forall (x_t) \in \mathbb{D}([0,T], \mathbb{R}^d).$$

Remark 5. If $\eta_n = C_1 n^{-\rho_1}$ and $\gamma_n = C_2 n^{-\rho_2}$ with $0 < \rho_2 \leq \rho_1 \leq 1$, then for $s \in (1, +\infty)$, (17) is fulfilled if and only if $s > 1/(1-\rho_1)$. It follows that there exists $s \in (2, +\infty)$ such that (17) holds as soon as $\rho_1 < 1$.

Proof. We want to apply Theorem 2. First, by Proposition 1, Assumption $(\mathbf{C_{0,2}})$ is fulfilled and every weak limit of $(\nu_0^{(n)}(\omega, dy))$ is an invariant distribution. Second, it is well-known that $(\mathbf{C_1})$ and $(\mathbf{C_2})$ are fulfilled when b and σ are locally Lispchitz sublinear functions. Then, since $(\mathbf{C_{3,\varepsilon}})$ holds with $\varepsilon = 0$, (16) holds for every bounded continuous functional F. Finally, one checks that $\mathbf{H(s,0)}$ holds with $\mathcal{V} := V^r$ $(r > 0)$. It is classical that Assumption (i) is true when b and σ are sublinear. Assumptions (ii) follows from Proposition 1(b). Set $\theta_{n,1} = \eta_n/(\gamma_n H_n^2)$ and $\theta_{n,2} = \Delta N(n,T)/(\gamma_n H_n^s)$. Using that (η_n/γ_n) is nonincreasing and (17) yields that $(\theta_{n,1})$ and $(\theta_{n,2})$ satisfy the conditions of Proposition 1 (see (36) for details). Then, (iii) and (iv) of $\mathbf{H(s,0)}$ are some consequences of Proposition 1(a). This completes the proof. □

3.2 Application to Lévy driven SDE's

When we want to extend the results obtained for Brownian SDE's to Lévy driven SDE's, one of the main difficulties comes from the moments of the jump component (see [19] for details). For simplification, we assume here that (Z_t) has a moment of order $2p \geq 2$, *i.e.* that its Lévy measure π satisfies the following assumption with $p \geq 1$:

$$(\mathbf{H_p^1}) \quad : \quad \int_{|y|>1} \pi(dy)|y|^{2p} < +\infty.$$

We also introduce an assumption about the behaviour of the moments of the Lévy measure at 0:

$$(\mathbf{H^2_q}) \quad : \quad \int_{|y|\le 1} \pi(dy)|y|^{2q} < +\infty \quad q \in [0,1].$$

This assumption ensures that (Z_t) has finite $2q$-variations. Since $\int_{|y|\le 1}|y|^2\pi(dy)$ is finite, this is always satisfied for $q=1$.
Let us now specify the law of (ξ_n) introduced in (14). When the increments of (Z_t) can be exactly simulated, we denote by (E) the Euler scheme and by $(\xi_{n,E})$ the associated sequence:

$$\xi_{n,E} \stackrel{\mathcal{L}}{=} Z_{\gamma_n} \quad \forall n \ge 1.$$

When the increments of (Z_t) can not be simulated, we have to introduce some approximations. The canonical approximation is to truncate the small jumps and is based on the following property: let $(u_n)_{n\ge1}$ be a sequence of positive numbers such that $u_n \to 0$. Let $D_n = \{|y| > u_n\}$ and let $((Z_{t,n})_{t\ge0})_{n\ge1}$ denote the sequence of processes defined by

$$Z_{t,n} := \sum_{0<s\le t} \Delta Z_s 1_{\{\Delta Z_s \in D_n\}} - t\int_{D_n} y\pi(dy) \qquad \forall t \ge 0. \tag{18}$$

For every $n \ge 1$, $(Z_{t,n})_{t\ge0}$ is a compensated compound Poisson process (CCPP) with intensity $\lambda_n = \pi(D_n)$ and jump size distribution $\mu_n(dy) = 1_{D_n}\frac{\pi(dy)}{\pi(D_n)}$. Furthermore, $Z_{.,n} \xrightarrow{n\to+\infty} Z$ in L^2 locally uniformly, *i.e.*

$$\mathbb{E}\big[\sup_{0\le t\le T}|Z_t - Z_{t,n}|^2\big] \xrightarrow{n\to+\infty} 0 \qquad \forall T>0.$$

We then denote by (P) the Euler scheme built with the sequence (ξ_n^P) defined as follows:

$$\xi_{n,P} \stackrel{\mathcal{L}}{=} Z^n_{\gamma_n} \quad \forall n \ge 1.$$

As shown in [21], the error induced by this approximation is very strong when the local behavior of the small jumps component is irregular. However, it is possible to refine this approximation by a *wienerization* of the small jumps. This wienerization consists in replacing the small jumps by a linear transform of a Brownian motion instead of discarding them. This idea introduced by Asmussen-Rosinski (see [2]) is adapted in [21] in a decreasing step framework. This scheme is denoted by (W) and the associated sequence $(\xi_{n,W})$ satisfies

$$\xi_{n,W} \stackrel{\mathcal{L}}{=} \xi_{n,P} + \sqrt{\gamma_n}\Lambda_n \quad \forall n \ge 1,$$

where $(\Lambda_n)_{n\ge1}$ is a sequence of i.i.d. random variables, independent of $(\xi_{n,P})_{n\ge1}$ and $(U_n)_{n\ge1}$, such that $\Lambda_1 \stackrel{\mathcal{L}}{=} \mathcal{N}(0, I_\ell)$ and (Q_n) is a sequence of $\ell\times\ell$ matrices such that

$$(Q_nQ_n^*)_{i,j} = \int_{|y|\le u_k} y_iy_j\pi(dy).$$

We recall the result obtained in [19] in our slightly simplified framework:

Proposition 2. *Let $a \in (0,1]$, $p \ge 1$ and $q \in [0,1]$ such that $(\mathbf{H^1_p})$, $(\mathbf{H^2_q})$ and $(\mathbf{S_a})$ hold. Assume that the sequence $(\eta_n/\gamma_n)_{n\ge1}$ is nonincreasing. Then, the following assertions hold for Schemes (E), (P) and (W).*

(a) Let (θ_n) satisfy conditions of Proposition 1. Then, $\sum_{n\geq 1}\theta_n\gamma_n\mathbb{E}[V^{p+a-1}(\bar{X}_{\Gamma_{n-1}})]<+\infty$.
(b) We have

$$\sup_{n\geq 1}\nu_0^{(n)}(\omega, V^{\frac{p}{2}+a-1})<+\infty \quad a.s. \tag{19}$$

Hence, the sequence $(\nu_0^{(n)}(\omega,dy))_{n\geq 1}$ is a.s. tight as soon as $p/2+a-1>0$.
(b) Moreover, if $\mathrm{Tr}(\sigma\sigma^*)+\|\kappa\|^{2q}\leq CV^{\frac{p}{2}+a-1}$, then every weak limit of this sequence is an invariant probability for the SDE (13). In particular, if $(X_t)_{t\geq 0}$ admits a unique invariant probability ν_0, for every continuous function f such that $f=o(V^{\frac{p}{2}+a-1})$, $\lim_{n\to\infty}\nu_0^{(n)}(\omega,f)=\nu_0(f)$ a.s.

Remark 6. For Schemes (E) and (P), the above proposition is a direct consequence of Theorem 2 and Proposition 2 of [19]. As concerns scheme (W), a straightforward adaptation of the proof yields the result.

Our main functional result for Lévy driven SDE's is then the following:

Theorem 4. *Let $a\in(0,1]$ and $p\geq 1$ such that $p/2+a-1>0$ and let $q\in[0,1]$. Assume $(\mathbf{H_p^1})$, $(\mathbf{H_q^2})$ and $(\mathbf{S_a})$. Assume that b, σ and κ are locally Lipschitz functions. If moreover $(\eta_n/\gamma_n)_{n\geq 1}$ is nonincreasing, then the following result holds for schemes (E), (P) and (W).*
a. The sequence $(\nu^{(n)}(\omega,dy))_{n\geq 1}$ is a.s. tight on $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$. Moreover, if

$$Tr(\sigma\sigma^*)+\|\kappa\|^{2q}\leq CV^{\frac{p}{2}+a-1} \quad or \quad \frac{1}{H_n}\sum_{k=1}^{n}\max_{l\geq k+1}\frac{|\Delta\eta_\ell|}{\gamma_{\ell-1}}<+\infty, \tag{20}$$

every weak limit of $(\nu^{(n)}(\omega,dy))_{n\geq 1}$ is a stationary distribution for the process $(X_t)_{t\geq 0}$ solution to (13) with initial law ν_0.
b. Assume that the invariant distribution is unique. Let $\varepsilon\leq 0$ such that $(\mathbf{C_{3,\varepsilon}})$. Then, a.s., for every $T>0$, for every Sk-continuous functional $F:\mathbb{D}([0,T],\mathbb{R}^d)\to\mathbb{R}$ satisfying,

$$|F(x_t,0\leq t\leq T)|\leq C\sup_{0\leq t\leq T}V^{\frac{\rho(p+a-1)}{s}}(x_t) \quad \forall (x_t)\in\mathbb{D}([0,T],\mathbb{R}^d) \quad with\ \rho<1\ and\ s\geq 2,$$

(16) *holds if*

$$\left(\frac{\Delta N(k,T)}{\gamma_k H_k^{s(1-\varepsilon)}}\right)_{n\geq n_0} \text{ is nonincreasing and } \sum_{k\geq 1}\frac{\Delta N(k,T)}{H_k^{s(1-\varepsilon)}}<+\infty. \tag{21}$$

Remark 7. In (20), both asssumptions imply the invariance of every weak limit of $(\nu_0^{(n)}(\omega,dy))$. These two assumptions are very different. The first one is needed in Proposition 2 for using the Echeverria-Weiss invariance criteria (see [9], p. 238, [13], [15]) whereas the second one appears in Theorem 2, where our functional approach shows that under some mild additional conditions on steps and weights, every weak limit is always invariant.

4 Proofs of Theorems 1 and 2

We first endow $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ with a distance d compatible with the Skorokhod topology and for which $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ is Polish.

Let d_1 denote the Skorokhod distance on $[0,1]$ defined for every α, $\beta \in \mathbb{D}([0,1],\mathbb{R}^d)$ by

$$d_1(\alpha,\beta) = \inf_{\lambda\in\Lambda_1}\Big\{\max\Big(\sup_{t\in[0,1]}|\alpha(t)-\beta(\lambda(t))|, \sup_{0\leq s<t\leq 1}\Big|\log(\frac{\lambda(t)-\lambda(s)}{t-s})\Big|\Big)\Big\}$$

where Λ_1 denotes the set of increasing homeomorphisms of $[0,T]$. Now, let $\alpha\in\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$. For every $T>0$, $\phi_T : \mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)\mapsto\mathbb{D}([0,1],\mathbb{R}^d)$ denotes the function defined by $(\phi_T(\alpha))(s) = \alpha(sT)$ for every $s\in[0,1]$. The distance d is then defined for every $\alpha,\beta\in\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ by,

$$d(\alpha,\beta) = \int_0^{+\infty} e^{-t}\Big(1\wedge d_1(\phi_t(\alpha),\phi_t(\beta))\Big)dt. \tag{22}$$

The space $(\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d),d)$ is Polish and the induced topology is the usual Skorokhod topology on $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ (see [17]). For every $T>0$, we denote by

$$\mathcal{D}_t = \bigcap_{s>t}\sigma(\pi_u, 0\leq u\leq s)$$

where $\pi_u : \mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)\to\mathbb{R}^d$ is defined by $\pi(\alpha)=\alpha(s)$.

We begin the proof by some technical lemmas. In Lemma 1, we show that the $a.s$ weak convergence of the random measures $(\nu^{(n)}(\omega,dy))_{n\geq 1}$ can be characterized by the convergence (9) along the set of bounded Lipschitz functionals F for the distance d. Then, in Lemma 2, we show with some martingale arguments that if the functional F is also $\mathcal{D}_T$-measurable, then, the convergence of $(\nu^{(n)}(\omega,F))_{n\geq 1}$ is equivalent to that of a more regular sequence. This step is fundamental for the sequel of the proof.
Finally, Lemma 4 is needed for the proof of Theorem 2. We show that under some mild conditions on the step and weight sequences, any Markovian weak limit of the sequence $(\nu^{(n)}(\omega,dy))_{n\geq 1}$ is stationary.

4.1 Preliminary lemmas

Lemma 1. *Let (E,d) be a separable metric space and let $\mathcal{P}(E)$ denote the set of probability measures on the Borel σ-field $\mathcal{B}(E)$, endowed with the weak convergence topology. Let $(\mu^{(n)}(\omega,dy))_{n\geq 1}$ be a sequence of random probabilities defined on $\Omega\times\mathcal{B}(E)$.*
(a) Assume that there exists $\mu^{(\infty)}\in\mathcal{P}(E)$ such that for every bounded Lipschitz function $F:E\to\mathbb{R}$,

$$\mu^{(n)}(\omega,F)\xrightarrow{n\to+\infty}\mu^{(\infty)}(F)\quad a.s. \tag{23}$$

Then, a.s., $(\mu^{(n)}(\omega,dy))_{n\geq 1}$ weakly converges to $\mu^{(\infty)}$ on $\mathcal{P}(E)$.
(b) Let $\mathcal{U}$ be a subset of $\mathcal{P}(E)$. Assume that for every sequence $(F_k)_{k\geq 1}$ of Lispschitz and bounded functions, a.s., for every subsequence $(\mu^{(\phi_\omega(n))}(\omega,dy))$, there exists a subsequence $(\mu^{(\phi_\omega\circ\psi_\omega(n))}(\omega,dy))$ and a $\mathcal{U}$-valued random probability $\mu^{(\infty)}(\omega,dy)$ such that for every $k\geq 1$,

$$\mu^{(\psi_\omega\circ\phi_\omega(n))}(\omega,F_k)\xrightarrow{n\to+\infty}\mu^{(\infty)}(\omega,F_k)\quad a.s. \tag{24}$$

Then, $(\mu^{(n)}(\omega,dy))_{n\geq 1}$ is a.s. sequentially relatively compact with weak limits in $\mathcal{U}$. Furthermore, if (E,d) is Polish, then $(\mu^{(n)}(\omega,dy))_{n\geq 1}$ is a.s. tight.

Proof. First, one easily checks that if the assumption of (a) is satisfied, then the assumption of (b) is also fulfilled with $\mathcal{U}=\{\mu^{(\infty)}\}$. It follows that (a) is a particular case of (b) and that we only need to prove (b).

Since (E,d) is a separable metric space, there is a countable basis of open sets $(O_m)_{m\geq 1}$ stable by finite union. For every $m \geq 1$, we denote by $(F_{l,m})_{l\geq 1}$, the sequence of bounded Lispchitz functions on (E,d) defined by

$$F_{l,m}(x) = 1 - (1 - ld(x, O_m^c))_+.$$

Let $\tilde{\Omega}$ be a subset of Ω such that $\mathbb{P}(\tilde{\Omega}) = 1$ on which Assumption (b) holds. Then, let $\omega \in \tilde{\Omega}$ and consider a subsequence $(\mu^{(\phi_\omega(n))}(\omega, dy))_{n\geq 1}$ of $(\mu^{(n)}(\omega, dy))_{n\geq 1}$. Since $(F_{l,m})_{l,m\geq 1}$ is a countable family, it follows from (24) that there exists a subsequence $(\mu^{(\phi_\omega \circ \psi_\omega(n))}(\omega, dy))_{n\geq 1}$ and an $\mathcal{U}$-valued probability measure $\mu^{(\infty)}(\omega, dy)$ such that

$$\mu^{(\phi_\omega\circ\psi_\omega(n))}(\omega, F_{l,m})) \xrightarrow{n\to+\infty} \mu^{(\infty)}(\omega, F_{l,m}) \quad \forall l, m \geq 1. \tag{25}$$

For every $m \geq 1$, $(F_{l,m})_{l\geq 1}$ decreases toward 1_{O_m}. Then, one derives from (25) and from the monotone convergence Theorem that

$$\liminf_{n\to+\infty} \mu^{(\phi_\omega\circ\psi_\omega(n))}(\omega, O_m)) \geq \lim_{l\to+\infty} \mu^{(\infty)}(\omega, F_{l,m}) = \mu^{(\infty)}(\omega, O_m). \tag{26}$$

Finally, let O be an open set of (E,d). The family (O_m) being stable by finite union, there exists a subsequence $(m_k)_{k\geq 1}$ such that $O_{m_k} \subset O_{m_{k+1}}$ and $O = \bigcup_{k\geq 1} O_{m_k}$. Then, by (26) and the monotone convergence Theorem, we deduce that for every open set O,

$$\liminf_{n\to+\infty} \mu^{(n)}(\omega, O) \geq \mu^{(\infty)}(O).$$

It follows from the portmanteau Theorem (see *e.g.* [5]) that $(\mu^{(\phi_\omega\circ\psi_\omega(n))}(\omega, dy))_{n\geq 1}$ weakly converges to $\mu^{\infty}(\omega, dy)$. Hence, for every $\omega \in \tilde{\Omega}$, $(\mu^{(n)}(\omega, dy))_{n\geq 1}$ is sequentially relatively compact with limits in $\mathcal{U}$. In particulary, $(\mu^{(n)}(\omega, dy))_{n\geq 1}$ is tight for every $\omega \in \tilde{\Omega}$ when E is Polish (see *e.g.* [5]). □

For every $n \geq 0$, for every $T > 0$, we introduce $\tau(n,T)$ defined by

$$\tau(n,T) := \min\{k \geq 0, N(k,T) \geq n\} = \min\{k \leq n, \Gamma_k + T \geq \Gamma_n\}. \tag{27}$$

Note that for $k \in \{0, \ldots, \tau(n,T) - 1\}$, $\{\bar{X}_t^{(k)}, 0 \leq t \leq T\}$ is $\bar{\mathcal{F}}_{\Gamma_n}$-measurable and that

$$T - \gamma_{\tau(n,T)-1} \leq \Gamma_n - \Gamma_{\tau(n,T)} \leq T.$$

Lemma 2. *Assume* $(\mathbf{C}_{\mathbf{3},\varepsilon})$ *with* $\varepsilon < 1$. *Let* $T > 0$ *and let* $F : \mathbb{D}([0,T], \mathbb{R}^d) \to \mathbb{R}$, *be a* $\mathcal{D}_T$*-measurable functional. Let* $(\mathcal{G}_k)$ *be a filtration such that* $\bar{\mathcal{F}}_{\Gamma_k} \subset \mathcal{G}_k$ *for every* $k \geq 1$. *Then:*
(a) If F *is bounded,*

$$\frac{1}{H_n}\sum_{k=1}^{n} \eta_k \Big(F(\bar{X}_t^{(k-1)}, 0 \leq t \leq T) - \mathbb{E}[F(\bar{X}_t^{(k-1)}, 0 \leq t \leq T)/\mathcal{G}_{k-1}]\Big) \xrightarrow{n\to+\infty} 0. \tag{28}$$

(b) Let $\mathcal{V} : \mathbb{R}^d \to \mathbb{R}_+$ *satisfy* $\mathbf{H}(\mathbf{s}, \varepsilon)$. *Then* (28) *is true for every measurable* $F : \mathbb{D}([0,T], \mathbb{R}^d) \to \mathbb{R}$ *such that* $|F(x_t, 0 \leq t \leq T)| \leq \sup_{0\leq t\leq T} \mathcal{V}(x_t)$ *for every* $(x_t) \in \mathbb{D}([0,T], \mathbb{R}^d)$. *Furthermore,*

$$\sup_{n\geq 1} \nu^{(n)}(\omega, F) < +\infty \quad a.s. \tag{29}$$

Proof. We prove (a) and (b) simultaneously. Let $Y^{(k)}$ be defined by $Y^{(k)} = F(\bar{X}_t^{(k)}, 0 \leq t \leq T)$. We have

$$\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\Big(Y^{(k-1)} - \mathbb{E}[Y^{(k-1)}/\mathcal{G}_{k-1}]\Big) = \frac{1}{H_n}\sum_{k=1}^{n}\eta_k\Big(Y^{(k-1)} - \mathbb{E}[Y^{(k-1)}/\mathcal{G}_{n}]\Big) \tag{30}$$

$$+\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\Big(\mathbb{E}[Y^{(k-1)}/\mathcal{G}_{n}] - \mathbb{E}[Y^{(k-1)}/\mathcal{G}_{k-1}]\Big). \tag{31}$$

We have to prove that the right-hand side of (30) and (31) tend to 0 $a.s.$ when $n \to +\infty$. We first focus on the right-hand side of (30). Since $\bar{\mathcal{F}}_{\Gamma_n} \subset \mathcal{G}_n$ and $\{\bar{X}_t^{(k)}, 0 \leq t \leq T\}$ is $\bar{\mathcal{F}}_{\Gamma_n}$-measurable for $k \in \{0, \ldots, \tau(n,T)-1\}$, it follows that $Y^{(k)}$ is $\mathcal{G}_n$-measurable and that $Y^{(k)} = \mathbb{E}[Y^{(k)}/\mathcal{G}_n]$ for every $k \leq \tau(n,T)-1$. Then, if F is bounded, we derive from $(\mathbf{C_{3}}_{,\varepsilon})$ that

$$\Big|\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\Big(Y^{(k-1)} - \mathbb{E}[Y^{(k-1)}/\mathcal{G}_{n}]\Big)\Big| \leq \frac{2\|F\|_{\sup}}{H_n}\sum_{k=\tau(n,T)}^{n}\eta_k \leq \frac{C}{H_n}\sum_{k=\tau(n,T)}^{n}\gamma_k H_k^{\varepsilon}$$

$$\leq \frac{C}{H_n^{1-\varepsilon}}\big(\Gamma_n - \Gamma_{\tau(n,T)-1}\big) = \frac{C}{H_n^{1-\varepsilon}}\big(\Gamma_n - \Gamma_{\tau(n,T)} + \gamma_{\tau(n,T)}\big) \leq \frac{C(T+\gamma_1)}{H_n^{1-\varepsilon}} \xrightarrow{n\to+\infty} 0 \quad a.s.$$

where we used that $(H_n)_{n\geq 1}$ and $(\gamma_n)_{n\geq 1}$ are non-decreasing and non-increasing sequences respectively.

Assume now that F satisfies the assumptions of (b) with $\mathcal{V}$ satisfying $\mathbf{H}(\mathbf{s},\varepsilon)$. By the L^1-Borel-Cantelli lemma, it suffices to show that

$$\sum_{n\geq 1}\mathbb{E}\Big[\Big|\frac{1}{H_n^s}\sum_{k=\tau(n,T)}^{n}\eta_k\big(Y^{(k-1)} - \mathbb{E}[Y^{(k-1)}/\mathcal{G}_n]\big)\Big|^s\Big] < +\infty. \tag{32}$$

Let us prove (32). Set $a_k := \eta_k^{\frac{s-1}{s}}$ and $b_k(\omega) := \eta_k^{\frac{1}{s}}\big(Y^{(k-1)} - \mathbb{E}[Y^{(k-1)}/\mathcal{G}_n]\big)$. The Hölder inequality applied with $\bar{p} = s/(s-1)$ and $\bar{q} = s$ yields

$$\Big|\sum_{k=\tau(n,T)}^{n} a_k b_k(\omega)\Big|^s \leq \Big(\sum_{k=\tau(n,T)}^{n}\eta_k\Big)^{s-1}\Big(\sum_{k=\tau(n,T)}^{n}\eta_k\big|Y^{(k-1)} - \mathbb{E}[Y^{(k-1)}/\mathcal{G}_n]\big|^s\Big).$$

Now, since $F(x_t, 0 \leq t \leq T) \leq \sup_{0\leq t\leq T}\mathcal{V}(x_t)$, it follows from the Markov property and from $\mathbf{H}(\mathbf{s},\varepsilon)(i)$,

$$\mathbb{E}[|F(\bar{X}_t^{(k)}, 0 \leq t \leq T)|^s/\bar{\mathcal{F}}_{\Gamma_k}] \leq C\mathbb{E}[\sup_{0\leq t\leq T}\mathcal{V}^s(\bar{X}_t^{(k)})/\bar{\mathcal{F}}_{\Gamma_k}] \leq C_T\mathcal{V}^s(\bar{X}_{\Gamma_k}).$$

Then, using the two preceding inequalities and $(\mathbf{C_{3}}_{,\varepsilon})$ yields

$$\mathbb{E}\Big[\Big|\sum_{k=\tau(n,T)}^{n}\eta_k\big(Y^{(k-1)} - \mathbb{E}[Y^{(k-1)}/\mathcal{G}_n]\big)\Big|^s\Big] \leq C\Big(\sum_{k=\tau(n,T)}^{n}\eta_k\Big)^{s-1}\Big(\sum_{k=\tau(n,T)}^{n}\eta_k\mathbb{E}[\mathcal{V}^s(\bar{X}_{\Gamma_{k-1}})]\Big)$$

$$\leq C\Big(\sum_{k=\tau(n,T)}^{n}\eta_k\Big)^s\mathbb{E}[\sup_{k=\tau(n,T)}^{n}\mathcal{V}^s(\bar{X}_{\Gamma_{k-1}})] \leq C\Big(\sum_{k=\tau(n,T)}^{n}\gamma_k H_k^{\varepsilon}\Big)^s\mathbb{E}\Big[\sup_{t\in[0,S(n,T)]}\mathcal{V}^s(\bar{X}_t^{\tau(n,T)-1})\Big].$$

where $S(n,T) = \Gamma_{n-1} - \Gamma_{\tau(n,T)-1}$ and C does not depend n. By the definition of $\tau(n,T)$, $\Gamma_n - \Gamma_{\tau(n,T)} \leq T$. Hence, since $(\gamma_n)_{n\geq 1}$ is nonincreasing $\sum_{k=\tau(n,T)}^{n}\gamma_k \leq T + \gamma_{\tau(n,T)} \leq T + \gamma_1$ and $S(n,T) \leq T$. Then, using again $\mathbf{H}(\mathbf{s},\varepsilon)$(i) yields

$$\sum_{n\geq 1}\mathbb{E}\Big[\frac{1}{H_n^s}\Big|\sum_{k=\tau(n,T)}^{n}\eta_k\big(Y^{(k-1)} - \mathbb{E}[Y^{(k-1)}/\mathcal{G}_n]\big)\Big|^s\Big] \leq C\sum_{n\geq 1}\frac{1}{H_n^{s(1-\varepsilon)}}\mathbb{E}[\mathcal{V}^s(\bar{X}_{(\tau(n,T)-1)})].$$

Since $n \mapsto N(n,T)$ is an increasing function, $n \mapsto \tau(n,T)$ is a non-decreasing function and $\mathrm{Card}\{n, \tau(n,T)=k\} = \Delta N(k+1,T) := N(k+1,T) - N(k,T)$. Then, since $n \mapsto H_n$ increases, a change of variable yields

$$\sum_{n\geq 1} \mathbb{E}\Big[\frac{1}{H_n^s}\Big|\sum_{k=\tau(n,T)}^{n} \eta_k\big(Y^{(k-1)} - \mathbb{E}[Y^{(k-1)}/\mathcal{G}_n]\big)\Big|^s\Big] \leq C\sum_{k\geq 1}\frac{\Delta N(k,T)}{H_k^{s(1-\varepsilon)}}\mathbb{E}[\mathcal{V}^s(\bar{X}_{\Gamma_{k-1}})] < +\infty$$

by $\mathbf{H}(\mathbf{s},\varepsilon)$(iv).

Secondly, we prove that (31) tends to 0. For every $n \geq 1$, we set

$$M_n = \sum_{k=1}^{n} \frac{\eta_k}{H_k}\Big(\mathbb{E}[Y^{(k-1)}/\mathcal{G}_n] - \mathbb{E}[Y^{(k-1)}/\mathcal{G}_{k-1}]\Big). \tag{33}$$

The process $(M_n)_{n\geq 1}$ is a $(\mathcal{G}_n)$-martingale and we want to prove that this process is L^2-bounded. Set $\Phi^{(k,n)} = \mathbb{E}[F(\bar{X}_t^{(k)}, 0\leq t\leq T)/\mathcal{G}_n] - \mathbb{E}[F(\bar{X}_t^{(k)}, 0\leq t\leq T)/\mathcal{G}_k]$. The random variable $\Phi^{(k,n)}$ is $\bar{\mathcal{F}}_{\Gamma_{N(k,T)}}$-measurable. Then, for every $i \geq N(k,T)$, $\Phi^{(k,n)}$ is $\mathcal{G}_i$-measurable so that

$$\mathbb{E}\big[\Phi^{(i,n)}\Phi^{(k,n)}\big] = \mathbb{E}\big[\Phi^{(k,n)}\mathbb{E}[\Phi^{(i,n)}/\mathcal{G}_i]\big] = 0.$$

It follows that

$$\mathbb{E}[M_n^2] = \sum_{k\geq 1}\frac{\eta_k^2}{H_k^2}\mathbb{E}\big[(\Phi^{(k-1,n)})^2\big] + 2\sum_{k\geq 1}\frac{\eta_k}{H_k}\sum_{i=k+1}^{N(k-1,T)\wedge n}\frac{\eta_i}{H_i}\mathbb{E}\big[\Phi^{(i-1,n)}\Phi^{(k-1,n)}\big]. \tag{34}$$

Then,

$$\begin{aligned}\sup_{n\geq 1}\mathbb{E}[M_n^2] &\leq C\sum_{k\geq 1}\frac{\eta_k^2}{H_k^2}\mathbb{E}\big[(\Phi^{(k-1,n)})^2\big] + 2\sum_{k\geq 1}\frac{\eta_k}{H_k}\sum_{i=k+1}^{N(k-1,T)}\frac{\eta_i}{H_i}\mathbb{E}\big[\Phi^{(i-1,n)}\Phi^{(k-1,n)}\big]. \qquad (35)\\ &\leq C\sum_{k\geq 1}\frac{\eta_k}{H_k^{2-\varepsilon}}\mathbb{E}\big[(\Phi^{(k-1,n)})^2\big] + 2\sum_{k\geq 1}\frac{\eta_k}{H_k^{2-\varepsilon}}\sum_{i=k+1}^{N(k-1,T)}\gamma_i\mathbb{E}\big[\Phi^{(i-1,n)}\Phi^{(k-1,n)}\big]\end{aligned}$$

where in the second inequality, we used Assumption $(\mathbf{C}_{\mathbf{3},\varepsilon})$ and the decrease of $i \mapsto 1/H_i^{1-\varepsilon}$. Hence, if F is bounded, using that $\sum_{i=k+1}^{N(k-1,T)}\gamma_i \leq T$ yields

$$\sup_{n\geq 1}\mathbb{E}[M_n^2] \leq C\sum_{k\geq 1}\frac{\eta_k}{H_k^{2-\varepsilon}} \leq C\Big(\frac{\eta_1}{H_1^{2-\varepsilon}} + \int_{\eta_1}^{\infty}\frac{du}{u^{2-\varepsilon}}\Big) < +\infty \tag{36}$$

since $\varepsilon < 1$. Assume now that the assumptions of (b) hold and let F be dominated by a function $\mathcal{V}$ satisfying $\mathbf{H}(\mathbf{s},\varepsilon)$. By the Markov property, the Jensen inequality and $\mathbf{H}(\mathbf{s},\varepsilon)$(i),

$$\mathbb{E}\big[(\Phi^{(k,n)})^2\big] \leq C\mathbb{E}[\mathbb{E}[\sup_{0\leq t\leq T}\mathcal{V}^2(\bar{X}_t^{(k)})/\bar{\mathcal{F}}_{\Gamma_k}]] \leq C_T\mathbb{E}[\mathcal{V}^2(\bar{X}_{\Gamma_k})]$$

Then, we derive from the Cauchy-Schwarz inequality that for every $n,k\geq 1$, for every $i \in \{k,\dots,N(k,T)\}$,

$$\big|\mathbb{E}\big[\Phi^{(i,n)}\Phi^{(k,n)}\big]\big| \leq C\sqrt{\mathbb{E}[\mathcal{V}^2(\bar{X}_{\Gamma_i})]}\sqrt{\mathbb{E}[\mathcal{V}^2(\bar{X}_{\Gamma_k})]} \leq C\sup_{t\in[0,T]}\mathbb{E}[\mathcal{V}^2(\bar{X}_t^{(k)})] \leq C\mathbb{E}[\mathcal{V}^2(\bar{X}_{\Gamma_k})]$$

where in the last inequality, we used $\mathbf{H}(\mathbf{s},\varepsilon)$(i) once again. It follows that

$$\sup_{n\geq 1}\mathbb{E}[M_n^2]\leq C\sum_{k\geq 1}\frac{\eta_k}{H_k^{2-\varepsilon}}\mathbb{E}[\mathcal{V}^2(\bar{X}_{\Gamma_{k-1}})]<+\infty$$

by $\mathbf{H}(\mathbf{s},\varepsilon)$(iii). Therefore, (35) is finite and (M_n) is bounded in L^2. Finally, we derive from the Kronecker lemma that

$$\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\Big(\mathbb{E}[F(\bar{X}_t^{(k-1)},0\leq t\leq T)/\mathcal{G}_n]-\mathbb{E}[F(\bar{X}_t^{(k-1)},0\leq t\leq T)/\mathcal{G}_{k-1}]\Big)\xrightarrow{n\to+\infty}0\quad a.s.$$

As a consequence, $\sup_{n\geq 1}\nu^{(n)}(\omega,F)<+\infty$ *a.s.* if and only if

$$\sup_{n\geq 1}\frac{1}{H_n}\sum_{k=1}^{n}\mathbb{E}[F(\bar{X}_t^{(k-1)},0\leq t\leq T)/\mathcal{F}_{k-1}]\Big)<+\infty\quad a.s.$$

This last property is easily derived from $\mathbf{H}(\mathbf{s},\varepsilon)$(i) and (ii). This completes the proof. □

Lemma 3. *(a) Assume* $(\mathbf{C_1})$ *and let* $x_0\in\mathbb{R}^d$*. Then, we have* $\lim_{x\to x_0}\mathbb{E}[d(X^x,X^{x_0})]=0$*. In particular, for every bounded Lispchitz (w.r.t. the distance d) functional* $F:\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)\to\mathbb{R}$*, the function* Φ^F *defined by* $\Phi^F(x)=\mathbb{E}[F(X^x)]$ *is a (bounded) continuous function on* $\mathbb{R}^d$*.*
(b) Assume $(\mathbf{C_2})$*. For every compact set* $K\subset\mathbb{R}^d$*,*

$$\sup_{x\in K}\mathbb{E}[d(\hat{X}^{n,x},X^x)]\xrightarrow{n\to+\infty}0. \tag{37}$$

Set $\Phi_n^F(x)=\mathbb{E}[F(\hat{X}^{n,x})]$*. Then, for every bounded Lispchitz functional* $F:\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)\to\mathbb{R}$*,*

$$\sup_{x\in K}|\Phi^F(x)-\Phi_n^F(x)|\xrightarrow{n\to+\infty}0\quad \textit{for every compact set } K\subset\mathbb{R}^d. \tag{38}$$

Proof. (a) By the definition of d, for every α, $\beta\in\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ and for every $T>0$,

$$d(\alpha,\beta)\leq\big(1\wedge\sup_{0\leq t\leq T}|\alpha(t)-\beta(t)|\big)+e^{-T}. \tag{39}$$

It easily follows from Assumption $(\mathbf{C_1})$ and from the dominated convergence Theorem that

$$\limsup_{x\to x_0}\mathbb{E}[d(X^x,X^{x_0})]\leq e^{-T}\quad\text{for every } T>0.$$

Letting $T\to+\infty$ implies that $\lim_{x\to x_0}\mathbb{E}[d(X^x,X^{x_0})]=0$.
(b) We deduce from (39) and from Assumption $(\mathbf{C_2})$ that for every compact set $K\subset\mathbb{R}^d$, for every $T>0$

$$\limsup_{n\to+\infty}\sup_{x\in K}\mathbb{E}[d(\hat{X}^{n,x},X^x)]\leq e^{-T}.$$

Letting $T\to+\infty$ yields (37). □

Lemma 4. *Assume that* $(\eta_n)_{n\geq 1}$ *and* (γ_n) *satisfy* $(\mathbf{C}_{\mathbf{3},\varepsilon})$ *with* $\varepsilon<1$ *and* (12)*. Then,*
(i) For every $t\geq 0$*, for every bounded continuous function* $f:\mathbb{R}^d\to\mathbb{R}$*,*

$$\nu_t^{(n)}(\omega,f)-\nu_0^{(n)}(\omega,f)\xrightarrow{n\to+\infty}0\quad a.s.$$

(ii) If moreover, a.s., every weak limit $\nu^{(\infty)}(\omega,dy)$ *of* $(\nu^{(n)}(\omega,dy))_{n\geq 1}$ *is the distribution of a Markov process with semi-group* $(Q_t^{\omega})_{t\geq 0}$*, then, a.s.,* $\nu^{(\infty)}(\omega,dy)$ *is a stationary process.*

Proof. (i) Let $f:\mathbb{R}^d\to\mathbb{R}$ be a bounded continuous function. Since $\bar{X}_t^{(k)}=\bar{X}_{\Gamma_{N(k,t)}}$, we have

$$\nu_t^{(n)}(\omega,f)-\nu_0^{(n)}(\omega,f)=\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\Big(f(\bar{X}_{\Gamma_{N(k-1,t)}})-f(\bar{X}_{\Gamma_{k-1}})\Big)$$

From the very definition of $N(n,T)$ and $\tau(n,T)$, one checks that $N(k-1,T)\le n-1$ if and only if $\tau(n,T)\ge k$. Then,

$$\frac{1}{H_n}\sum_{k=1}^{n}\eta_k f(\bar{X}_{\Gamma_{N(k-1,t)}})=\frac{1}{H_n}\sum_{k=1}^{\tau(n,t)}\eta_k(\bar{X}_{\Gamma_{N(k-1,t)}})+\frac{1}{H_n}\sum_{k=1}^{n}\eta_k(\bar{X}_{\Gamma_{N(k-1,t)}}).$$

It follows that

$$\nu_t^{(n)}(\omega,f)-\nu_0^{(n)}(\omega,f)=\frac{1}{H_n}\sum_{k=1}^{\tau(n,t)}(\eta_k-\eta_{N(k-1,t)+1})f(\bar{X}_{\Gamma_{N(k-1,t)}})$$

$$+\frac{1}{H_n}\sum_{\tau(n,t)+1}^{n}\eta_k f(\bar{X}_{\Gamma_{N(k-1,t)}})-\frac{1}{H_n}\sum_{k=1}^{n}\eta_k f(\bar{X}_{\Gamma_{k-1}})1_{\{k-1\notin N(\{0,\ldots,n\},t)\}}.$$

Then, since f is bounded and

$$\sum_{k=1}^{n}\eta_k 1_{\{k-1\notin N(\{0,\ldots,n\},t)\}}=\sum_{k=1}^{n}\eta_k-\sum_{k=1}^{\tau(n,t)}\eta_{N(k-1,t)+1}\le\sum_{k=1}^{\tau(n,t)}|\eta_k-\eta_{N(k-1,t)+1}|+\sum_{k=\tau(n,t)+1}^{n}\eta_k,$$

we deduce that

$$\big|\nu_t^{(n)}(\omega,f)-\nu_0^{(n)}(\omega,f)\big|\le 2\|f\|_\infty\Big(\frac{1}{H_n}\sum_{k=1}^{\tau(n,t)}|\eta_k-\eta_{N(k-1,t)+1}|+\frac{1}{H_n}\sum_{k=\tau(n,t)+1}^{n}\eta_k\Big).$$

Hence, we have to show that the sequences of the right-hand side of the preceding inequality tend to 0. On the one hand, we observe that

$$|\eta_k-\eta_{N(k-1,t)+1}|\le\sum_{\ell=k+1}^{N(k-1,T)+1}|\eta_\ell-\eta_{\ell-1}|\le\max_{\ell\ge k+1}\frac{|\Delta\eta_\ell|}{\gamma_\ell}\sum_{\ell=k}^{N(k-1,T)+1}\gamma_\ell.$$

Using that $\sum_{\ell=k}^{N(k-1,T)+1}\gamma_\ell\le T+\gamma_1$ and Condition (12) yields

$$\frac{1}{H_n}\sum_{k=1}^{\tau(n,t)}|\eta_k-\eta_{N(k-1,t)+1}|\xrightarrow{n\to+\infty}0.$$

On the other hand, by $(\mathbf{C}_{\mathbf{3},\varepsilon})$, we have

$$\frac{1}{H_n}\sum_{k=\tau(n,T)+1}^{n}\eta_k\le\frac{1}{H_n^{1-\varepsilon}}\sum_{k=\tau(n,T)+1}^{n}\gamma_k\le\frac{T}{H_n^{1-\varepsilon}}\xrightarrow{n\to+\infty}0\quad a.s.$$

which completes the proof of (i) .

(ii) Let $\mathbb{Q}_+$ denote the set of non negative rational numbers. Let $(f_\ell)_{\ell\ge1}$ be an everywhere sequence in $\mathcal{C}_K(\mathbb{R}^d)$ endowed with the topology of uniform convergence on compact sets.

Since $\mathbb{Q}_+$ and $(f_\ell)_{\ell\geq 1}$ are countable, we derive from (i) that there exists $\tilde{\Omega}\subset\Omega$ such that $\mathbb{P}(\tilde{\Omega})=1$ and such that for every $\omega\in\tilde{\Omega}$, for every $t\in\mathbb{Q}_+$, for every $\ell\geq 1$,

$$\nu_t^{(n)}(\omega,f_\ell)-\nu_0^{(n)}(\omega,f_\ell)\xrightarrow{n\to+\infty}0.$$

Let $\omega\in\tilde{\Omega}$ and $\nu^{(\infty)}(\omega,dy)$ denote a weak limit of $(\nu^{(n)}(\omega,dy))_{n\geq 1}$. We have

$$\nu_t^{(\infty)}(\omega,f_\ell)=\nu_0^{(\infty)}(\omega,f_\ell)\quad\forall t\in\mathbb{Q}_+\quad\forall\ell\geq 1,$$

and we easily deduce that

$$\nu_t^{(\infty)}(\omega,f)=\nu_0^{(\infty)}(\omega,f)\quad\forall t\in\mathbb{R}_+\quad\forall f\in\mathcal{C}_K(\mathbb{R}^d).$$

Hence, if $\nu^{(\infty)}(\omega,dy)$ is the distribution of a Markov process (Y_t) with semi-group $(Q_t^\omega)_{t\geq 0}$, we have for all $f\in\mathcal{C}_K(\mathbb{R}^d)$,

$$\int Q_t^\omega f(x)\nu_0^{(\infty)}(\omega,dx)=\int f(x)\nu_0^{(\infty)}(\omega,dx)\quad\forall t\geq 0.$$

Then, $\nu_0^{(\infty)}(\omega,dx)$ is an invariant distribution for (Y_t). This completes the proof. $\square$

4.2 Proof of Theorem 1

Thanks to Lemma 1(a) applied with $E=\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$ and d defined by (22),

$$\nu^{(n)}(\omega,dy)\overset{(Sk)}{\Longrightarrow}\mathbb{P}_{\nu_0}(dy)\quad a.s.\Longleftrightarrow\nu^{(n)}(\omega,F)\xrightarrow{n\to+\infty}\int F(x)\mathbb{P}_{\nu_0}(dx)\quad a.s.\tag{40}$$

for every bounded Lipschitz functional $F:\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)\to\mathbb{R}$. Then, consider such a functional. By the assumptions of Theorem 1, we know that $a.s.$, $(\nu_0^{(n)}(\omega,dy))_{n\geq 1}$ converges weakly to ν_0. Set $\Phi^F(x):=\mathbb{E}[F(X^x)]$, $x\in\mathbb{R}^d$. By Lemma 3(a), Φ^F is a bounded continuous function on $\mathbb{R}^d$. Then, it follows that

$$\frac{1}{H_n}\sum_{k=1}^n\eta_k\Phi^F(\bar{X}_0^{(k-1)})\xrightarrow{n\to+\infty}\int\Phi^F(x)\nu_0(dx)=\int F(x)\mathbb{P}_{\nu_0}(dx)\quad a.s.$$

. Hence, the right-hand side of (40) holds for F as soon as

$$\frac{1}{H_n}\sum_{k=1}^n\eta_k\Big(F(\bar{X}^{(k-1)})-\Phi^F(\bar{X}_0^{(k-1)})\Big)\xrightarrow{n\to+\infty}0\qquad a.s.\tag{41}$$

Let us prove (41). For every $T>0$, let $F_T:\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)\to\mathbb{R}$ denote the functional defined by $F_T(\alpha):=F(\alpha^T)$ where α^T satisfies $\alpha^T(t):=\alpha(t\wedge T)$. The functional F_T is $\mathcal{D}_T$-measurable. Then, applying Lemma 2 to F_T, we derives that for every $T>0$,

$$\frac{1}{H_n}\sum_{k=1}^n\eta_kF_T(\bar{X}^{(k-1)})-\frac{1}{H_n}\sum_{k=1}^n\eta_k\mathbb{E}[F_T(\bar{X}^{(k-1)})/\bar{\mathcal{F}}_{\Gamma_{k-1}}]\xrightarrow{n\to+\infty}0\qquad a.s.\tag{42}$$

With the notations of Lemma 3(2), we derive from Assumption $(\mathbf{C_2})$(i) that

$$\mathbb{E}[F_T(\bar{X}^{(k-1)})/\bar{\mathcal{F}}_{\Gamma_{k-1}}]=\Phi_k^{F_T}(\bar{X}_0^{(k-1)}).$$

Let $N \in \mathbb{N}$. On the one hand, by Lemma 3(2),

$$\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\big(\Phi_k^{F_T}(\bar{X}_0^{(k-1)}) - \Phi^{F_T}(\bar{X}_0^{(k-1)})\big)1_{\{|\bar{X}_0^{(k-1)}|\leq N\}} \xrightarrow{n\to+\infty} 0 \qquad a.s. \tag{43}$$

On the other hand, the tightness of $(\nu_0^{(n)}(\omega, dy))_{n\geq 1}$ on $\mathbb{R}^d$ yields

$$\psi(\omega, N) := \sup_{n\geq 1}(\nu_0^{(n)}(\omega, (B(0, N)^c))) \xrightarrow{N\to+\infty} 0 \quad a.s.$$

It follows that $a.s.$,

$$\sup_{n\geq 1}\Big(\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\big|\Phi_k^{F_T}(\bar{X}_0^{(k-1)}) - \Phi^{F_T}(\bar{X}_0^{(k-1)})\big|1_{\{|\bar{X}_0^{(k-1)}|>N\}}\Big) \leq 2\|F\|_\infty\psi(\omega, N) \xrightarrow{N\to+\infty} 0. \tag{44}$$

Keeping in mind that $\bar{X}_0^{(k-1)} = \bar{X}_{k-1}$, a combination of (43) and (44) yields

$$\forall T > 0, \quad \frac{1}{H_n}\sum_{k=1}^{n}\eta_k\big(\Phi_k^{F_T}(\bar{X}_{k-1}) - \Phi^{F_T}(\bar{X}_{k-1})\big) \xrightarrow{n\to+\infty} 0 \quad a.s. \tag{45}$$

Finally, let $(T_\ell)_{\ell\geq 1}$ be a sequence of positive numbers such that $T_\ell \to +\infty$ when $\ell \to +\infty$. Combining (45) and (42), we obtain that $a.s.$, for every $\ell \geq 1$,

$$\begin{aligned}\limsup_{n\to+\infty}\Big|\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\big(F(\bar{X}^{(k-1)}) - \Phi^F(\bar{X}^{(k-1)})\big)\Big| &\leq \limsup_{n\to+\infty}\Big|\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\big(F(\bar{X}^{(k-1)}) - F_{T_\ell}(\bar{X}^{(k-1)})\big)\Big| \\ &+ \limsup_{n\to+\infty}\Big|\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\big(\Phi^{F_{T_\ell}}(\bar{X}^{(k-1)}) - \Phi^F(\bar{X}^{(k-1)})\big)\Big|.\end{aligned}$$

By the definition of d, $|F - F_{T_\ell}| \leq e^{-T_\ell}$. Then, $a.s.$,

$$\limsup_{n\to+\infty}\Big|\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\big(F(\bar{X}^{(k-1)}) - \Phi^F(\bar{X}^{(k-1)})\big)\Big| \leq 2e^{-T_\ell} \quad \forall\ell \geq 1.$$

Letting $\ell \to +\infty$ implies (41).
The generalization to non bounded functionals in Theorem 1 is then derived from (29) and from an uniform integrability argument.

4.3 Proof of Theorem 2

(i) We want to prove that the conditions of Lemma 1(ii) are fulfilled: since $(\nu_0^{(n)}(\omega, dy))_{n\geq 1}$ is supposed to be $a.s.$ tight, one can check that for every bounded Lipschitz functional $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$, (41) is still valid. Then, let $(F_\ell)_{\ell\geq 1}$ be a sequence of bounded Lipschitz functionals. There exists $\tilde{\Omega} \subset \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that for every $\omega \in \tilde{\Omega}$, $(\nu_0^{(n)}(\omega, dy))_{n\geq 1}$ is tight and

$$\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\Big(F_\ell(\bar{X}^{(k-1)}(\omega)) - \Phi^{F_\ell}(\bar{X}_0^{(k-1)}(\omega))\Big) \xrightarrow{n\to+\infty} 0 \quad \forall\ell \geq 1. \tag{46}$$

Let $\omega \in \tilde{\Omega}$ and let $\phi_\omega : \mathbb{N} \mapsto \mathbb{N}$ be an increasing function. As $(\nu_0^{(\phi_\omega(n))}(\omega, dy))_{n\geq 1}$ is tight, there exists a convergent subsequence $(\nu_0^{(\phi_\omega\circ\psi_\omega(n))}(\omega, dy))_{n\geq 1}$. We denote its weak limit by ν_∞. Since Φ^{F_ℓ} is continuous for every $\ell \geq 1$ (see Lemma 3(i)),

$$\nu_0^{(\phi_\omega\circ\psi_\omega(n))}(\omega, \Phi^{F_\ell}) \xrightarrow{n\to+\infty} \nu_\infty(\Phi^{F_\ell}) = \int F_\ell d\mathbb{P}_{\nu_\infty} \quad \forall\ell \geq 1.$$

Then, we derive from (46) that for every $\ell \geq 1$

$$\nu^{(\phi_\omega \circ \psi_\omega(n))}(\omega, F_\ell) \xrightarrow{n\to+\infty} \int F_\ell d\mathbb{P}_{\nu_\infty}.$$

It follows that the conditions of Lemma 1(ii) are fulfilled with $\mathcal{U} = \{\mathbb{P}_\mu, \mu \in \mathcal{I}\}$ where

$$\mathcal{I} = \Big\{\mu \in \mathcal{P}(\mathbb{R}^d),\ \exists \omega \in \tilde{\Omega} \text{ and an increasing function } \phi : \mathbb{N} \mapsto \mathbb{N},\ \mu = \lim_{n\to+\infty} \nu^{(\phi(n))}(\omega, dy)\Big\}.$$

Hence, by Lemma 1(ii), we deduce that $(\nu^{(n)}(\omega, dy))_{n\geq 1}$ is *a.s.* tight with $\mathcal{U}$-valued limits. Finally, Theorem 2(ii) is a consequence of Condition (12) and Lemma 4(ii).

5 Path-dependent option pricing in Stationary Stochastic Volatility Models

In this section, we propose a simple and efficient method to price option in Stationary Stochastic Volatility (SSV) models. In most Stochastic Volatility (SV) models, the volatility is a mean-reverting process. These processes are generally ergodic with a unique invariant distribution (see below the case of the Heston model but also the SABR model [10], the Barndorff-Nielsen and Shephard model [3],...). However, they are usually considered in SV models under a non stationary regime, starting from a deterministic value (which usually turns out to be the mean of their invariant distribution). In fact, the instantaneous volatility is not easy to observe on the market since it is not a traded asset (see however [8]) and it seems there is no real reason to consider it as an observable random variable at $t = 0$.

From a purely calibration viewpoint, considering an SV model in its SSV regime will not modify the set of parameters used to generate the implied volatility surface, although it will modify its shape, mainly for short maturities. This effect can in fact be an asset of the SSV approach since it may correct some observed drawbacks of some models (see *e.g.* the Heston model below).

From a numerical point of view, considering SSV models is no longer an obstacle, especially when considering multi-asset models, (in the unidimensional case, the stationary distribution can be more or less explicited like in the Heston model, see below), since our algorithm is precisely devised to compute by simulation some expectations of functionals of processes under their stationary regime even if this stationary regime cannot be directly simulated.

As an illustration (and a benchmark) of the method, we will describe in detail the algorithm for the pricing of Asian options in a Heston model. Then, we will show in our numerical results to what extend it differs, in terms of smile and skew, from the usual SV Heston model for short maturities. The adaptation of this procedure to other SV models under their SSV regime like SABR or Barndorff-Nielsen and Shephard model is straightforward (see [20] for other illustrations). Let us also mention that this method can be applied to other fields of finance like interest rates and to commodities and energy derivatives where mean-reverting process play an important role.

5.1 Option Pricing in the Heston SSV model

In the following, we consider a Heston stochastic volatility model. The dynamic of the asset price process $(S_t)_{t\geq 0}$ is given by: $S_0 = s_0$ and

$$dS_t = S_t(rdt + \sqrt{(1-\rho^2)v_t}dW_t^1 + \rho\sqrt{v_t}dW_t^2)$$
$$dv_t = k(\theta - v_t)dt + \varsigma\sqrt{v_t}dW_t^2$$

where r denotes the interest rate, (W^1, W^2) is a standard two-dimensional Brownian motion, $\rho \in [-1,1]$ and k, θ and ς are some non-negative numbers. This model was introduced by Heston in 1993 (see [11]). The equation for (v_t) has a unique (strong) pathwise continuous solution living in $\mathbb{R}_+$. If moreover, $2k\theta > \varsigma^2$ then, (v_t) is a positive process (see [12]). In this case, (v_t) has a unique invariant probability ν_0. Moreover, $\nu_0 = \gamma(a,b)$ with $a = (2k)/\varsigma^2$ and $b = (2k\theta)/\varsigma^2$. In the following, we will assume that (v_t) is in its stationary regime, *i.e.* that

$$\mathcal{L}(v_0) = \nu_0.$$

5.1.1 Option price and stationary processes

Using our procedure to price options in this model needs naturally to express the option price as the expectation of a functional of a stationary stochastic process.

Naïve method: (may work) Since $(v_t)_{t\geq 0}$ is stationary, the first idea is to express the option price as the expectation of a functional of $(v_t)_{t\geq 0}$: by Itô calculus, we have

$$S_t = s_0 \exp\Big((rt - \frac{1}{2}\int_0^t v_s ds)t + \rho\int_0^t \sqrt{v_s}dW_s^2\Big) + \sqrt{1-\rho^2}\int_0^t \sqrt{v_s}dW_s^1\Big). \tag{47}$$

Since

$$\int_0^t \sqrt{v_s}dW_s^2 = \Lambda(t,(v_t)) := \frac{v_t - v_0 - k\theta t + k\int_0^t v_s ds}{\varsigma},$$

it follows by setting $M_t = \int_0^t \sqrt{v_s}dW_s^1$ that

$$S_t = \Psi(t,(v_s),(M_s)) \tag{48}$$

where Ψ is given for every $t \geq 0$, u and $w \in \mathcal{C}(\mathbb{R}_+,\mathbb{R})$ by

$$\Psi(t,u,w) = s_0 \exp\Big((rt - \frac{1}{2}\int_0^t u(s)ds) + \rho\Lambda(t,u) + \sqrt{1-\rho^2}w(t)\Big).$$

Then, let $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$ be a non-negative measurable functional. Conditioning by $\mathcal{F}_T^{W^2}$ yields

$$\mathbb{E}[F(S_t, 0 \leq t \leq T)] = \mathbb{E}[\tilde{F}((v_t)_{t\leq T})]$$

where for every $u \in \mathcal{C}([0,T],\mathbb{R})$,

$$\tilde{F}(u) = \mathbb{E}[F(\Psi(t,u,\int_0^t u(s)dW_s^1), 0 \leq t \leq T)].$$

Except some particular options as the European call or put (thanks to the Black-Scholes formula), the functional $\tilde{F}$ is not explicit. Hence, if we implement the algorithm using this way, the computation of $\tilde{F}$ will need some Monte-Carlo methods at each step. This method is then very consuming in general. That is why we are going to introduce another representation of the option as a functional of a stationary process.

General method: (always works) We express the option premium as the expectation of a functional of a two-dimensional stationary stochastic process. This method is based on the following idea. Even if (v_t, M_t) is not stationary, (S_t) can be expressed as a functional of a stationary process (v_t, y_t). Indeed, consider the following SDE given by

$$\begin{cases} dy_t = -y_t dt + \sqrt{v_t} d\tilde{W}_t^1 \\ dv_t = k(\theta_t - v_t)dt + \varsigma\sqrt{v_t} dW_t^2. \end{cases} \tag{49}$$

Firstly, one checks that the SDE has a unique strong solution and that Assumption $(\mathbf{S_1})$ is fulfilled with $V(x_1, x_2) = 1 + x_1^2 + x_2^2$. This ensures the existence of an invariant distribution $\tilde{\nu}_0$ for the SDE (see *e.g.* [18]). Then, since (v_t) is positive and has a unique invariant distribution, the uniqueness of the invariant distribution follows. Then, assume that $\mathcal{L}(y_0, v_0) = \tilde{\nu}_0$. Since $(v_t, M_t) = (v_t, y_t - y_0 + \int_0^t y_s ds)$, we have for every positive measurable functional $F : \mathcal{C}(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}$,

$$\begin{aligned} \mathbb{E}[F(S_t, 0 \le t \le T)] &= \mathbb{E}[F(\psi(t, v_t, M_t), 0 \le t \le T)] \\ &= \mathbb{E}_{\tilde{\nu_0}}[F(\psi(t, v_t, y_t - y_0 + \int_0^t y_s ds), 0 \le t \le T)] \end{aligned} \tag{50}$$

where $\mathbb{P}_{\tilde{\nu_0}}$ is the stationary distribution of the process (v_t, y_t). Then, every option price can be expressed as the expectation of an explicit functional of a stationary process. We will develop this second general approach in the numerical tests below.

Remark 8. The idea of the second method holds for every stochastic volatility model for which (S_t) can be written as follows :

$$S_t = \Phi(t, v_t, \sum_{i=1}^{p} \int_0^t h_i(|v_s|) dY_s^i) \tag{51}$$

where for every $i \in \{1, \dots, p\}$, $h_i : \mathbb{R}_+ \to \mathbb{R}$ is a positive function such that $h_i(x) = o(|x|)$ as $|x| \to +\infty$ and (Y_t^i) is a square-integrable centered Lévy process, and (v_t) is a mean-reverting stochastic process solution to a Lévy driven SDE.
In some complex models, showing the uniqueness of the invariant distribution may be difficult. In fact, it is important to note at this stage that the uniqueness of the invariant distribution for the couple (v_t, y_t) is not required. Indeed, by construction, the local martingale (M_t) does not depend on the choice of y_0. It follows that if $\mathcal{L}(y_0, v_0) = \tilde{\mu}$ with $\tilde{\mu}$ constructed such that $\mathcal{L}(v_0) = \nu_0$, Equality (50) still holds true. This implies that it is only necessary that the uniqueness of the invariant distribution holds for the stochastic volatility process.

5.1.2 Numerical tests on Asian options

We recall that (v_t) is a Cox-Ingersoll-Ross process. For this type of processes, it is well-known that the genuine Euler scheme can not be implemented since it does not preserve the non negativity of the (v_t). That is why some specific discretization schemes have been studied by several authors: Alfonsi [1], Deelstra and Delbaen [6] and Berkaoui, Bossy and Diop (see [4, 7]). In this paper, we consider in a decreasing step framework the natural scheme studied by Berkaoui, Bossy and Diop that we denote by $(\bar{v}_t)$. We set $\bar{v}_0 = x > 0$ and

$$\bar{v}_{\Gamma_{n+1}} = \left|\bar{v}_{\Gamma_n} + k\gamma_{n+1}(\theta - \bar{v}_{\Gamma_n}) + s\sqrt{\bar{v}_{\Gamma_n}}(W^2_{\Gamma_{n+1}} - W^2_{\Gamma_n})\right|.$$

We also introduce the stepwise constant Euler scheme $(\bar{y}_t)$ of $(y_t)_{t\geq 0}$ defined

$$\bar{y}_{\Gamma_{n+1}} = \bar{y}_{\Gamma_n} - \gamma_{n+1}\bar{y}_{\Gamma_n} + \sqrt{\bar{v}_{\Gamma_n}}(\tilde{W}^1_{\Gamma_{n+1}} - \tilde{W}^1_{\Gamma_n}), \quad \bar{y}_0 = y \in \mathbb{R}^d.$$

Denote by $(\bar{v}^{(k)}_t)$ and $(\bar{y}^{(k)}_t)$ the shifted processes defined by $\bar{v}^{(k)}_t := \bar{v}_{\Gamma_k+t}$ and $\bar{y}^{(k)}_t = \bar{y}_{\Gamma_k+t}$ and let $(\nu^{(n)}(\omega, dy))_{n\geq 1}$ be the sequence of empirical measures defined by

$$\nu^{(n)}(\omega, dy) = \frac{1}{H_n}\sum_{k=1}^{n}\eta_k 1_{\{(\bar{v}^{(k-1)}, \bar{y}^{(k-1)})\in dy\}}.$$

The specifity of both the model and the Euler scheme implies that Theorems 1 and 2 can not be directly applied here. However, a specific study using that (7) holds for every compact of $\mathbb{R}^*_+ \times \mathbb{R}$ when $2k\theta/\varsigma^2 > 1 + 2\sqrt{6}/\varsigma$ (see Theorem 2.2 of [4] and Remark 7) shows that

$$\nu^{(n)}(\omega, dy) \overset{n\to+\infty}{\Longrightarrow} \mathbb{P}_{\tilde{\nu}_0}(dy) \quad a.s.$$

when $2k\theta/\varsigma^2 > 1 + 2\sqrt{6}/\varsigma$. Details are left to the reader.

Let us now state our numerical results obtained for the pricing of asian options with this discretization. We denote by $C_{as}(\nu_0, K, T)$ and $P_{as}(\nu_0, K, T)$ the asian call and put prices in the SSV Heston model. We have

$$C_{as}(\nu_0, K, T) = e^{-rT}\mathbb{E}_{\nu_0}[(\frac{1}{T}\int_0^T S_s ds - K)_+]$$

$$\text{and} \quad P_{as}(\nu_0, K, T) = e^{-rT}\mathbb{E}_{\nu_0}[(K - \frac{1}{T}\int_0^T S_s ds)_+]$$

With the notations of (50), approximating $C_{as}(\nu_0, K, T)$ and $P_{as}(\nu_0, K, T)$ by our procedure needs to simulate the sequences $(C^n_{as})_{n\geq 1}$ and $(P^n_{as})_{n\geq 1}$ defined by

$$C^n_{as} = \frac{1}{H_n}\sum_{k=1}^{n}\eta_k\Big(\frac{1}{T}\int_0^T \Psi(t, \bar{v}^{(k-1)}, \bar{M}^{(k-1)})ds - K\Big)_+,$$

$$P^n_{as} = \frac{1}{H_n}\sum_{k=1}^{n}\eta_k\Big(K - \frac{1}{T}\int_0^T \Psi(t, \bar{v}^{(k-1)}, \bar{M}^{(k-1)})ds\Big)_+.$$

These sequences can be computed by the method developed in subsection 1.3. Note that the specific properties of the exponential function and the linearity of the integral imply that $(\int_0^T \Psi(t, \bar{v}^{(n-1)}, \bar{M}^{(n-1)})ds)$ can be computed quasi-recursively.
Let us state our numerical results for the Asian call with parameters:

$$s_0 = 50, \quad r = 0.05, \quad T = 1, \quad \rho = 0.05, \quad \theta = 0.01, \quad \varsigma = 0.1, \quad k = 2. \tag{52}$$

We also assume that $K \in \{44, \ldots, 56\}$ and choose the following steps and weights: $\gamma_n = \eta_n = n^{-\frac{1}{3}}$. In Table 1, we first state the reference value for the Asian call price obtained for $N = 10^8$ iterations. In the two following lines, we state our results for $N = 5.10^4$ and $N = 5.10^5$ iterations. Then, in the last ones, we present the numerical results obtained using the Call-Put parity:

$$C_{as}(\nu_0, K, T) - P_{as}(\nu_0, S_0, K, T) = \frac{s_0}{rT}(e^{rT} - 1) - Ke^{-rT} \tag{53}$$

as a way of variance reduction. The computation times for $N = 5.10^4$ and $N = 5.10^5$ (using MATLAB with a processor Xeon 2.4 GHz) are about $5s$ and $51s$. In particular, the complexity is quasi-linear and the additional computations needed when we use the Call-Put parity are negligible.

K	44	45	46	47	48	49	50
Asian call (ref)	**6.92**	**5.97**	**5.04**	**4.12**	**3.25**	**2.46**	**1.78**
$N = 5.10^4$	6.89	6.07	5.07	4.13	3.18	2.49	1.77
$N = 5.10^5$	6.90	6.02	5.00	4.11	3.24	2.46	1.79
$N = 5.10^4$ (CP parity)	6.92	5.96	5.04	4.13	3.26	2.46	1.78
$N = 5.10^5$ (CP parity)	6.92	5.97	5.04	4.12	3.25	2.47	1.78

K	51	52	53	54	55	56
Asian call (ref)	**1.23**	**0.82**	**0.53**	**0.33**	**0.21**	**0.12**
$N = 5.10^4$	1.21	0.81	0.51	0.34	0.22	0.11
$N = 5.10^5$	1.23	0.82	0.53	0.33	0.21	0.13
$N = 5.10^4$ (CP parity)	1.23	0.82	0.53	0.31	0.21	0.12
$N = 5.10^5$(CP parity)	1.23	0.82	0.53	0.33	0.21	0.13

Table 1: Approximation of the Asian call price

5.2 Implied volatility surfaces of SSV models and SV models

Given a particular pricing model (with initial value s_0 and interest rate r) and its associated european call prices denoted by $C_{\text{eur}}(K,T)$, we recall that the implied volatility surface is the graph of the function $(K,T) \mapsto \sigma_{\text{imp}}(K,T)$ where $\sigma_{\text{imp}}(K,T)$ is defined for every maturity $T > 0$ and strike K as the unique solution of

$$C_{BS}(s_0, K, T, r, \sigma_{\text{imp}}(K,T)) = C_{\text{eur}}(K,T).$$

where $C_{BS}(s_0, K, T, r, \sigma)$ is the price of the european call in the Black-Scholes model with parameters s_0, r and σ. When $C_{\text{eur}}(K,T)$ is known, the value of $\sigma_{\text{imp}}(K,T)$ can be numerically computed using the Newton method or by dichotomy if the first method is not convergent.
In this last part, we compare the implied volatility surfaces induced by the SSV and SV Heston models where suppose that the initial value of (v_t) in the SV Heston model is the mean of the invariant distribution, *i.e.* we suppose that $v_0 = \theta$[3]. We also assume that the parameters are those of (52) except the correlation coefficient ρ.
In Figures 1 and 2, the volatility curves obtained when $T = 1$ are depicted whereas in Figures 3 and 4, we set the strike K at $K = 50$ and let the time vary. These representations show that when the maturity is long, the differences between the SSV and SV Heston models vanish. This is a consequence of the convergence of the stochastic volatility to its stationary regime when $T \to +\infty$.
The main differences between these models then appear for short maturities. That is why we complete this part by a representation of the volatility curve when $T = 0.1$ for $\rho = 0$ and $\rho = 0.5$ in Figures 5 and 6 respectively. We observe that for short maturities, the volatility smile is more curved and the skew is steeper. These phenomenons seem interesting for calibration since one well-known drawback of the standard Heston model is to have some too flat volatility curves for short maturities.

[3]This choice is the most usual in practice.

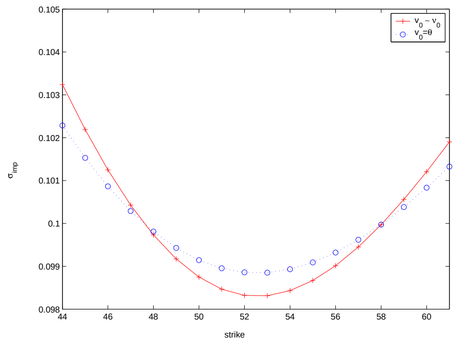

Figure 1: $\rho = 0$, $K \mapsto \sigma_{\text{imp}}(K, 1)$

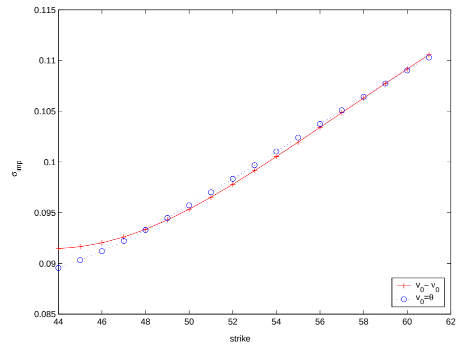

Figure 2: $\rho = 0.5$, $K \mapsto \sigma_{\text{imp}}(K, 1)$

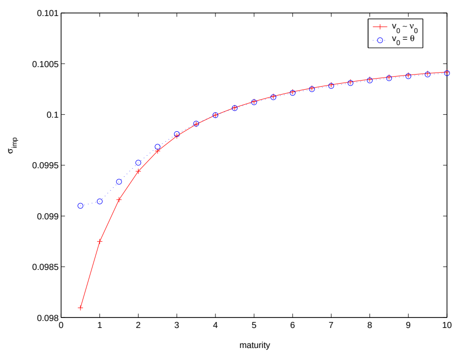

Figure 3: $\rho = 0$, $T \mapsto \sigma_{\text{imp}}(50, T)$

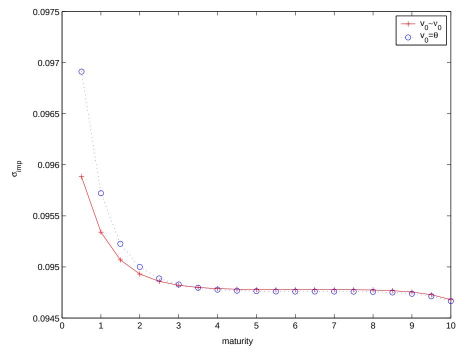

Figure 4: $\rho = 0.5$, $T \mapsto \sigma_{\text{imp}}(50, T)$

References

[1] Alfonsi A. (2005), On the discretization schemes for the CIR (and Bessel squared) processes, rapport CERMICS [2005-279].

[2] Asmussen S., Rosinski J. (2001), Approximations of small jumps of Lévy processes with a view towards simulation, *J. Appl. Probab.*, **38**, pp. 482-493.

[3] Barndorff-Nielsen O.E., Shephard N. (2001), Modelling by Lévy processess for financial economics, *Lévy Processes, Theory and Applications*, pp. 283-318.

[4] Berkaoui A., Bossy M., Diop A. (2005), Euler scheme for SDE's with non-Lipschitz diffusion coefficient: strong convergence, *Rapport de recherche de l'INRIA*.

[5] Billingsley P. (1968), *Convergence of Probability Measures*, Wiley.

[6] Deelstra G., Delbaen F. (1998), Convergence of Discretized Stochastic (Interest Rate) Processes with Stochastic Drift Term, *Appl. Stochastic Models Data Anal.*, **14**, pp. 77-84.

[7] Diop, A. (2003), *Sur la discrétisation et le comportement à petit bruit d'EDS unidimensionnelles dont les coefficients sont à dérivées singulières*, PhD Thesis, Université de Nice Sophia Antipolis.

[8] Durrleman V. (2005), From Spot to implied volatilities, *working paper*, Standford University.

[9] Ethier S., Kurtz T. (1986), *Markov processes, characterization and convergence*, Wyley series in probability and mathematical statistics, Wiley, New York.

[10] Hagan D., Kumar D., Lesniewsky A., Woodward D. (2002), Managing smile risk,*Wilmott Magazine*, **9**, pp. 84-108.

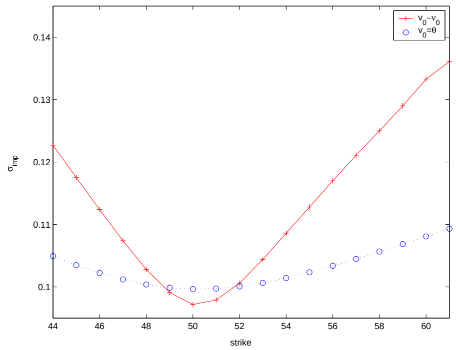

Figure 5: $\rho = 0$, $T \mapsto \sigma_{\text{imp}}(50, T)$

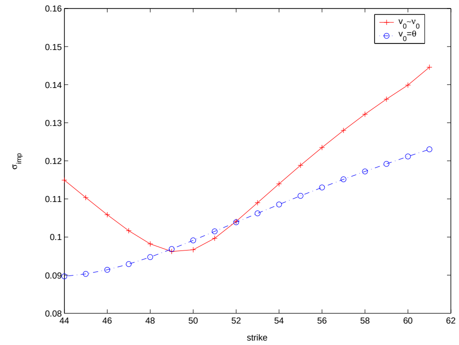

Figure 6: $\rho = 0.5$, $T \mapsto \sigma_{\text{imp}}(50, T)$

[11] Heston, S. (1993), A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Review of Financial Studies*, **6**, pp. 327-343.

[12] Lamberton D., Lapeyre B. (1996), *Introduction to Stochastic Calculus Applied to Finance*, Chapman and Hall/CRC, New York.

[13] Lamberton D., Pagès G. (2002), Recursive computation of the invariant distribution of a diffusion, *Bernoulli*, **8**, pp. 367-405.

[14] Lamberton D., Pagès G. (2003), Recursive computation of the invariant distribution of a diffusion: the case of a weakly mean reverting drift, *Stoch. Dynamics*, **4**, pp. 435-451.

[15] Lemaire V. (2006), An adaptative scheme for the approximation of dissipative systems, *Stoch. Proc. Appl.*, to appear.

[16] Lemaire V. (2005), *Estimation numérique de la mesure invariante d'un processus de diffusion*, PhD Thesis, Université de Marne-La Vallée.

[17] Pagès G. (1985), *Théorèmes limites pour les semi-martingales*, PhD Thesis, Université Paris VI.

[18] Pagès G. (2001), Sur quelques algorithmes récursifs pour les probabilités numériques, ESAIM: P.S. **5** (2001), pp.141-170.

[19] Panloup F. (2005), Recursive computation of the invariant measure of a SDE driven by a Lévy process, accepted for publication at *Annals Appl. Probab.*

[20] Panloup F. (2006), Computation of the invariant measure of a Lévy driven SDE: Rate of convergence, Preprint.

[21] Panloup F. (2006), *Approximation du régime stationnaire d'une EDS avec sauts*, PhD Thesis, Université Paris VI.