DUALITY AND TAMENESS

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ABSTRACT. We prove a duality theorem for certain graded algebras and show by various examples different kinds of failure of tameness of local cohomology.

INTRODUCTION

The purpose of this paper is to construct examples of strange behavior of local cohomology. In these constructions we follow a strategy that was already used in [CH] and which relates, via a spectral sequence introduced in [HR], the local cohomology for the two distinguished bigraded prime ideals in a standard bigraded algebra.

In the first part we consider algebras with rather general gradings and deduce a similar spectral sequence in this more general situation. A typical example of such an algebra is the Rees algebra of a graded ideal. The proof for the spectral sequence given here is simpler than that of the corresponding spectral sequence in [HR].

In the second part of this paper we construct examples of standard graded rings A, which are algebras over a field K, such that the function

$$(1) \qquad j \mapsto \dim_K(H^i_{A_+}(A)_{-j})$$

is an interesting function for $j \gg 0$. In our examples, this dimension will be finite for all j.

Suppose that A_0 is a Noetherian local ring, $A = \bigoplus_{j \geq 0} A_j$ is a standard graded ring and set $A_+ := \bigoplus_{j > 0} A_j$. Let M be a finitely generated graded A-module and $\mathcal{F} := \tilde{M}$ be the sheafification of M on $Y = \mathrm{Proj}(A)$. We then have graded A-module isomorphisms

$$H^{i+1}_{A_+}(M) \cong \bigoplus_{n \in \mathbf{Z}} H^i(Y, \mathcal{F}(n))$$

for $i \geq 1$, and a similar expression for $i = 0$ and 1.

By Serre vanishing, $H^i_{A_+}(M)_j = 0$ for all i and $j \gg 0$. However, the asymptotic behaviour of $H^i_{A_+}(M)_{-j}$ for $j \gg 0$ is much more mysterious.

In the case when $A_0 = K$ is a field, the function (1) is in fact a polynomial for large enough j. The proof is a consequence of graded local duality ([BS, 13.4.6] or [BH, 3.6.19]) or follows from Serre duality on a projective variety.

For more general A_0, $H_{A_+}(M)_{-j}$ are finitely generated A_0 modules, but need not have finite length.

The following problem was proposed by Brodmann and Hellus [BrHe].

Tameness problem: Are the local cohomology modules $H^i_{A_+}(M)$ tame? That is, is it true that either

$$\{H^i_{A_+}(M)_j \neq 0, \ \forall j \ll 0\} \text{ or } \{H^i_{A_+}(M)_j = 0, \ \forall j \ll 0\}?$$

The problem has a positive solution for A_0 of small dimension (some of the references are Brodmann [Br], Brodmann and Hellus [BrHe], Lim [L], Rotthaus and Sega [RS]).

Theorem 0.1 ([BrHe]). *If* $\dim(A_0) \leq 2$, *then* M *is tame.*

However, it has recently been shown by two of the authors that tameness can fail if $\dim(A_0) = 3$.

Theorem 0.2 ([CH]). *There are examples with* $\dim(A_0) = 3$ *where* M *is not tame.*

The statement of this example is reproduced in Theorem 3.1 of this paper. The function (1) is periodic for large j. Specifically, the function (1) is 2 for large even j and is 0 for large odd j.

In Theorem 3.3 we construct an example of failure of tameness of local cohomology which is not periodic, and is not even a quasi polynomial (in $-j$) for large j. Specifically, we have for $j > 0$,

$$\dim_K(H^2_{A_+}(A)_{-j}) = \begin{cases} 1 & \text{if } j \equiv 0 \ (\text{mod}) \ (p+1), \\ 1 & \text{if } j = p^t \text{ for some odd } t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where the characteristic of K is p. We have $p^t \equiv -1 \ (\text{mod}) \ (p+1)$ for all odd $t \geq 0$.

We also give an example (Theorem 3.5) of failure of tameness where (1) is a quasi polynomial with linear growth in even degree and is 0 in odd degree.

In Theorem 3.6, we give a tame example, but we have

$$\lim_{j \to \infty} \frac{\dim_K(H^2_{A_+}(A)_{-j})}{j^3} = 54\sqrt{2},$$

so (1) is far from being a quasi polynomial in $-j$ for large j.

While the example of [CH] is for $M = \omega_A$, where ω_A is the canonical module of A, the examples of the paper are all for $M = A$. This allows us to easily reinterpret our examples as Rees algebras in Section 4, and thus we have examples of Rees algebras over local rings for which the above failure of tameness holds.

In the final section, Section 5, we give an analysis of the explicit and implicit role of bigraded duality in the construction of the examples, and some comments on how it effects the geometry of the constructions.

1. DUALITY FOR POLYNOMIAL RINGS IN TWO SETS OF VARIABLES

Let K be any commutative ring (with unit). In later applications K will be mostly a field. Furthermore let $S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$, $P = (x_1, \ldots, x_m)$ and $Q = (y_1, \ldots, y_n)$.

The homology of the Čech complex $\mathcal{C}_P(\_)$ (resp. $\mathcal{C}_Q(\_)$) will be denoted by $H_P(\_)$ (resp. $H_Q(\_)$). Notice that for any commutative ring K, this homology is the local cohomology supported in P (resp. Q), as P and Q are generated by a regular sequences.

Assume that S is Γ-graded for some abelian group Γ, and that $\deg(a) = 0$ for $a \in K$. If $x^s y^p \in R$, $\deg(x^s y^p) = l(s) + l'(p)$ with $l(s) := \sum_i s_i \deg(x_i)$ and $l'(p) := \sum_j p_j \deg(y_j)$.

Definition 1.1. Let $I \subset S$ be a Γ-graded ideal. The Γ-grading of S is *I-sharp* if $H^i_I(S)_\gamma$ is a finitely generated K-module, for every i and $\gamma \in \Gamma$.

Lemma 1.2. *The following conditions are equivalent:*

 (i) *the Γ-grading of S is P-sharp.*
 (ii) *the Γ-grading of S is Q-sharp.*
 (iii) *for all $\gamma \in \Gamma$, $|\{(\alpha, \beta)\ :\ \alpha \geq 0, \beta \geq 0, \quad l(\alpha) = \gamma + l'(\beta)\}| < \infty$.*

Note that if K is Noetherian, M is a finitely generated Γ-graded S-module, and the Γ-grading of S is I-sharp, then $H^i_I(M)_\gamma$ is a finite K-module, for every i and $\gamma \in \Gamma$. This follows from the converging Γ-graded spectral sequence $H_{p-q}(H^p_I(\mathbb{F})) \Rightarrow H^q_I(M)$, where $\mathbb{F}$ is a Γ-graded free S-resolution of M with $\mathbb{F}_i$ finite for every i.

We will assume from now on that the Γ-grading of S is P-sharp (equivalently Q-sharp). Set $\sigma = \deg(x_1 \cdots x_m y_1 \cdots y_n)$, and if N is a Γ-graded module, then let $N^\vee = \mathrm{Hom}_S(N, S(-\sigma))$ and $N^* = {}^*\mathrm{Hom}_K(N, K)$ where the Γ-grading of N^* is given by $(N^*)_\gamma = \mathrm{Hom}_K(N_{-\gamma}, K)$. More generally, we always denote the graded K-dual of a graded module N (over what graded K-algebra soever) by N^*. Finally we denote by $\varphi_{\alpha\beta}$ the map $S(-a) \to S(-b)$ induced by multiplication by $x^\alpha y^\beta$ where $a = \deg x^\alpha$ and $b = -\deg y^\beta$. Then

Lemma 1.3. $H^m_P(\varphi_{\alpha\beta})_\gamma \cong H^n_Q(\varphi^\vee_{\alpha\beta})^*$.

Proof. The free K-module $H^m_P(S)_\gamma$ is generated by the elements $x^{-s-1} y^p$ with $s, p \geq 0$ and $-l(s) - l(1) + l'(p) = \gamma$, and $H^n_Q(S)_{\gamma'}$ is generated by the elements $x^t y^{-q-1}$ with $t, q \geq 0$ and $l(t) - l'(q) - l'(1) = \gamma'$.

Let $d_\gamma \colon H^m_P(S)_\gamma \to (H^n_Q(S^\vee)^*)_\gamma = H^n_Q(S)_{-\gamma-\sigma}$ be the K-linear map defined by

$$d_\gamma(x^{-s-1} y^p)(x^t y^{-q-1}) = \begin{cases} 1, & \text{if } s = t \text{ and } p = q, \\ 0, & \text{else.} \end{cases}$$

Then d_γ is an isomorphism (because the Γ-grading of R is Q-sharp) and there is a commutative diagram

$$
\begin{array}{ccc}
H^m_P(S)_{\gamma-a} & \xrightarrow{\ H^m_P(\varphi_{\alpha\beta})_\gamma\ } & H^m_P(S)_{\gamma-b} \\
{\scriptstyle d_{\gamma-a}} \downarrow & & {\scriptstyle d_{\gamma-b}} \downarrow \\
(H^n_Q(S)_{-\gamma+a-\sigma})^* & \xrightarrow{\ H^n_Q((\varphi^\vee_{\alpha\beta})_{-\gamma})^*\ } & (H^n_Q(S)_{-\gamma+b-\sigma})^*.
\end{array}
$$

The assertion follows. $\qquad\square$

As an immediate consequence we obtain

Corollary 1.4. (a) *Let $f \in S$ be an homogeneous element of degree $a-b$, and $\varphi\colon S(-a) \to S(-b)$ the graded degree zero map induced by multiplication with f. Then*

$$H_P^m(\varphi) \simeq H_Q^n(\varphi^\vee)^*.$$

(b) *Let $\mathbb{F}$ be a Γ-graded complex of finitely generated free S-modules. Then*
  (i) $H_P^i(\mathbb{F}) = 0$ *for $i \neq m$ and $H_Q^j(\mathbb{F}) = 0$ for $j \neq n$,*
  (ii) $H_P^m(\mathbb{F}) \simeq H_Q^n((\mathbb{F})^\vee)^*.$

As the main result of this section we have

Theorem 1.5. *Assume that K is Noetherian, the Γ-grading of S is P-sharp (equivalently Q-sharp) and M is a finitely generated Γ-graded S-module. Set $\omega_{S/K} := S(-\sigma)$. Let $\mathbb{F}$ be a minimal Γ-graded S-resolution of M. Then,*
  (a) *For all i, there is a functorial isomorphism*

$$H_P^i(M) \simeq H_{m-i}(H_P^m(\mathbb{F})).$$

  (b) *There is a convergent Γ-graded spectral sequence,*

$$H_Q^i(\mathrm{Ext}_S^j(M, \omega_{S/K})) \Rightarrow H^{i+j-n}(H_P^m(\mathbb{F})^*).$$

  In particular, if K is a field, there is a convergent Γ-graded spectral sequence,

$$H_Q^i(\mathrm{Ext}_S^j(M, \omega_S)) \Rightarrow H_P^{\dim S - (i+j)}(M)^*.$$

Proof. Claim (a) is an immediate consequence of Corollary 1.4 via the Γ-graded spectral sequence $H_{p-i}(H_P^p(\mathbb{F})) \Rightarrow H_P^i(M)$. For (b), the two spectral sequences arising from the double complex $\mathcal{C}_Q \mathbb{F}^\vee$ have as second terms respectively $'E_2^{ij} = H_Q^i(\mathrm{Ext}_S^j(M, \omega_{S/K}))$, $''E_2^{ij} = 0$ for $i \neq n$ and $''E_2^{nj} = H^j(H_Q^n(\mathbb{F}^\vee)) \simeq H^j(H_P^m(\mathbb{F})^*)$. If further K is a field, $H^j(H_P^m(\mathbb{F})^*) \simeq (H_j(H_P^m(\mathbb{F})))^* \simeq H_P^{m-j}(\mathbb{F})^*$. $\square$

Corollary 1.6. *Under the hypotheses of the theorem, if K is a field, then for any $\gamma \in \Gamma$, there are convergent spectral sequences of finite dimensional K-vector spaces*

$$H_Q^i(\mathrm{Ext}_S^j(M, \omega_R))_\gamma \Rightarrow H_P^{\dim S - (i+j)}(M)_{-\gamma},$$

$$H_P^i(\mathrm{Ext}_S^j(M, \omega_R))_\gamma \Rightarrow H_Q^{\dim S - (i+j)}(M)_{-\gamma}.$$

We now consider the special case that $\Gamma = \mathbb{Z}^2$, $S := K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ with $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (d_j, 1)$ with $d_j \geq 0$. Set $T := K[x_1, \ldots, x_m]$ and let M be a Γ-graded S-module. We view M as a $\mathbb{Z}$-graded module by defining $M_k = \bigoplus_j M_{(j,k)}$. Observe that each M_k itself is a graded T-module with $(M_k)_j = M_{(j,k)}$ for all j. We also note that $H_P^i(M)_k \cong H_{P_0}^i(M_k)$, as can been seen from the definition of local cohomology using the Čech complex. Here $P_0 = (x_1, \ldots, x_m)$ is the graded maximal ideal of T.

Corollary 1.7. *With the notation introduced, let $s := \dim S = m + n$ and $d := \dim M$. Then*
  (a) $H_P^0(\mathrm{Ext}_S^{s-d}(M, \omega_S)) \cong H_Q^d(M)^*$ *for any k,*
  (b) *there is an exact sequence*

$$0 \to H_P^1(\mathrm{Ext}_S^{s-d}(M, \omega_S)) \to H_Q^{d-1}(M)^* \to H_P^0(\mathrm{Ext}_S^{s-d+1}(M, \omega_S)).$$

(c) *Let $i \geq 2$. If $\text{Ext}_S^j(M, \omega_S)$ is annihilated by a power of P for all $s - d < j < s - d + i$, then there is an exact sequence*

$$\text{Ext}_S^{s-d+i-1}(M, \omega_S) \to H_P^i(\text{Ext}_S^{s-d}(M, \omega_S)) \to H_Q^{d-i}(M)^* \to H_P^0(\text{Ext}_S^{s-d+i}(M, \omega_S)).$$

In particular, if $\text{Ext}_S^j(M, \omega_S)$ has finite length for all $s - d < j \leq s - d + i_0$, for some integer i_0, then

$$H_{P_0}^i(\text{Ext}_S^{s-d}(M, \omega_S)_k) \cong (H_Q^{d-i}(M)_{-k})^* \quad \text{for all } i \leq i_0 \text{ and } k \gg 0.$$

Consequently, if M is a generalized Cohen-Macaulay module (i.e. $\text{Ext}_S^{s-i}(M, \omega_S)$ has finite length for all $i \neq d$), and if we set $N = \text{Ext}_S^{s-d}(M, \omega_S)$, then

$$H_{P_0}^i(N_k) \cong (H_Q^{d-i}(M)_{-k})^* \quad \text{for all } i \text{ and all } k \gg 0.$$

Proof. (a), (b) and (c) are direct consequences of Corollary 1.6. For the application, notice that if $\gamma = (\ell, k) \in \Gamma$ with $k \gg 0$ one has $\text{Ext}_S^j(M, \omega_S)_\gamma = 0$ for all $s - d < j \leq s - d + i_0$. Therefore, for such γ, the desired conclusion follows. $\square$

A typical example to which this situation applies is the Rees algebra of a graded ideal I in the standard graded polynomial ring $T = K[x_1, \ldots, x_m]$. Say, I is generated be the homogeneous polynomials $f_1, \ldots, f_n$ with $\deg f_j = d_j$ for $j = 1, \ldots, n$. Then the Rees algebra $\mathcal{R}(I) \subset T[t]$ is generated the elements $f_j t$. If we set $\deg f_j t = (d_j, 1)$ for all j and $\deg x_i = (1, 0)$ for all i, then $\mathcal{R}(I)$ becomes a Γ-graded S-module via the K-algebra homomorphism $S \to \mathcal{R}(I)$ with $x_i \mapsto x_i$ and $y_j \mapsto f_j t$.

According to this definition we have $\mathcal{R}(I)_k = I^k$ for all k.

Since $\dim \mathcal{R}(I) = m + 1$, the module $\omega_{\mathcal{R}(I)} = \text{Ext}_S^{n-1}(\mathcal{R}(I), \omega_S)$ is the canonical module of $\mathcal{R}(I)$ (in the sense of [HK, 5. Vortrag]). Recall that if a ring R is a finite S-module of dimension $m + 1$, the natural finite map $R \to \text{Hom}(\omega_R, \omega_R) \cong \text{Ext}_S^{n-1}(\omega_R, \omega_S)$ is an isomorphism if and only if R is S_2. Thus in combination with Corollary 1.7 we obtain

Corollary 1.8. *Let $R := \mathcal{R}(I)$. Suppose that $R_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p} \neq (\mathfrak{m}, R_+)$ where $\mathfrak{m} = (x_1, \ldots, x_m)$ and $R_+ = \bigoplus_{k>0} I^k t^k$. Then*

$$H_{\mathfrak{m}}^i(I^k) \cong (H_{R_+}^{m+1-i}(\omega_R)_{-k})^* \quad \text{for all } i \text{ and all } k \gg 0.$$

Proof. Since ω_R localizes, the conditions imply that $(\omega_R)_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p} \neq (\mathfrak{m}, R_+)$. Hence the natural into map $R \to R' := \text{Ext}_S^{n-1}(\omega_R, \omega_S)$ has a cokernel of finite length. In particular, $R'_k = R_k = I^k$ for $k \gg 0$. Thus Corollary 1.7 applied to $M = \omega_R$ gives the desired conclusion. $\square$

Remark 1.9. Let $R := \mathcal{R}(I)$. If the cokernel of $R \to \text{Hom}(\omega_R, \omega_R)$ is annihilated by a power of R_+ (in other words, the blow-up is S_2, as a projective scheme over $\text{Spec}(T)$), then $R'_k = I^k$ for $k \gg 0$ and therefore one has an exact sequence

$$0 \to H_{\mathfrak{m}}^0(T/I^k) \to (H_{R_+}^m(\omega_R)_{-k})^* \to H_{\mathfrak{m}}^0(\text{Ext}_S^n(\omega_R, \omega_S)_k) \to H_{\mathfrak{m}}^1(T/I^k) \to (H_{R_+}^{m-1}(\omega_R)_{-k})^*$$

for such a k.

2. A method of constructing examples

Suppose that $R = \bigoplus_{i,j \geq 0} R_{ij}$ is a standard bigraded algebra over a ring $K = R_{00}$. Define $R^i = \bigoplus_{j \geq 0} R_{ij}$ and $R_j = \bigoplus_{i \geq 0} R_{ij}$. Define ideals $P = \bigoplus_{i > 0} R^i$ and $Q = \bigoplus_{j > 0} R_j$ in R. Suppose that $M = \bigoplus_{ij \in \mathbf{Z}} M_{ij}$ is a finitely generated, bigraded R-module. Define $M^i = \bigoplus_{j \in \mathbf{Z}} M_{ij}$ and $M_j = \bigoplus_{i \in \mathbf{Z}} M_{ij}$. M^i is a graded R^0-module and M_j is a graded R_0-module. Let $Q_0 = R_{01} R^0$, so that $Q = Q_0 R$. Let $P_0 = R_{10} R_0$ so that $P = R_{10} R$. We have K module isomorphisms

$$H_Q^l(M)_{m,n} \cong H_{Q_0}^l(M^m)_n$$

for $m, n \in \mathbf{Z}$. Let $\widetilde{M^m}$ be the sheafification of the graded R^0-module M^m on $\mathrm{Proj}(R^0)$. We have K module isomorphisms

$$H_{Q_0}^l(M^m)_n \cong H^{l-1}(\mathrm{Proj}(R^0), \widetilde{M^m}(n))$$

for $l \geq 2$ and exact sequences

$$0 \to H_{Q_0}^0(M^m)_n \to (R^m)_n = R_{m,n} \to H^0(\mathrm{Proj}(R^0), \widetilde{M^m}(n)) \to H_{Q_0}^1(M^m)_n \to 0.$$

We have similar formulas for the calculation of $H_P^l(M)$.

Now assume that X is a projective scheme over K and $\mathcal{F}_1$ and $\mathcal{F}_2$ are very ample line bundles on X. Let

$$R_{m,n} = \Gamma(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}).$$

We require that $R = \bigoplus_{m,n \geq 0} R_{m,n}$ be a standard bigraded K-algebra. We have

$$X \cong \mathrm{Proj}(R_0) \cong \mathrm{Proj}(R^0).$$

The sheafification of the graded R^0-module R^m on X is $\widetilde{R^m} = \mathcal{F}_1^{\otimes m}$, and the sheafification of the graded R_0-module R_n on X is $\widetilde{R_n} \cong \mathcal{F}_2^{\otimes n}$ (Exercise II.5.9 [Ha]).

For $l \geq 2$ we have bigraded isomorphisms

$$H_Q^l(R) \cong \bigoplus_{m \geq 0} H_{Q_0}^l(R^m)_n \cong \bigoplus_{m \geq 0, n \in \mathbf{Z}} H^{l-1}(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}).$$

Viewing R as a graded R_0 algebra, we thus have graded isomorphisms

$$(2) \qquad H_Q^l(R)_n \cong \bigoplus_{m \geq 0} H^{l-1}(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}),$$

for $l \geq 2$ and $n \in \mathbf{Z}$. Let $d = \dim(R) = \dim(X) + 2$.

We now further assume that K is an algebraically closed field and X is a nonsingular K variety. Let

$$V = \mathbf{P}(\mathcal{F}_1 \oplus \mathcal{F}_2),$$

a projective space bundle over X with projection $\pi : V \to X$. Since $\mathcal{F}_1 \oplus \mathcal{F}_2$ is an ample bundle on X, $\mathcal{O}_V(1)$ is ample on V. Since

$$R \cong \bigoplus_{t \geq 0} \Gamma(V, \mathcal{O}_V(t))$$

with

$$\Gamma(V, \mathcal{O}_V(t)) \cong \Gamma(X, S^t(\mathcal{F}_1 \oplus \mathcal{F}_2)) \cong \bigoplus_{i+j=t} R_{ij}$$

and R is generated in degree 1 with respect to this grading, $\mathcal{O}_V(1)$ is very ample on V and R is the homogeneous coordinate ring of the nonsingular projective variety V, so that R is generalized Cohen Macaulay (all local cohomology modules $H^i_{R_+}(R)$ of R with respect to the maximal bigraded ideal R_+ of R have finite length for $i < d$). We further have that V is projectively normal by this embedding (Exercise II.5.14 [Ha]) so that R is normal.

3. STRANGE BEHAVIOR OF LOCAL COHOMOLOGY

In [CH], we constructed the following example of failure of tameness of local cohomology. In the example, R_0 has dimension 3, which is the lowest possible for failure of tameness [Br].

Theorem 3.1. *Suppose that K is an algebraically closed field. Then there exists a normal standard graded K-algebra R_0 with $\dim(R_0) = 3$, and a normal standard graded R_0-algebra R with $\dim(R) = 4$ such that for $j \gg 0$,*

$$\dim_K(H^2_Q(\omega_R)_{-j}) = \begin{cases} 2 & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd,} \end{cases}$$

where ω_R is the canonical module of R, $Q = \bigoplus_{n>0} R_n$.

We first show that the above theorem is also true for the local cohomology of R.

Theorem 3.2. *Suppose that K is an algebraically closed field. Then there exists a normal standard graded K-algebra R_0 with $\dim(R_0) = 3$, and a normal standard graded R_0-algebra R with $\dim(R) = 4$ such that for $j > 0$,*

$$\dim_K(H^2_Q(R)_{-j}) = \begin{cases} 2 & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd,} \end{cases}$$

where $Q = \bigoplus_{n>0} R_n$.

Proof. We compute this directly for the R of Theorem 3.1 from (2) and the calculations of [CH]. Translating from the notation of this paper to the notation of [CH], we have $X = S$ is an Abelian surface, $\mathcal{F}_1 = \mathcal{O}_S(r_2laH)$ and $\mathcal{F}_2 = \mathcal{O}_S(r_2(D + alH))$.

By (2) of this paper, for $n \in \mathbf{N}$, we have

$$\begin{aligned} \dim_K(H^2_Q(R)_n) &= \sum_{m \geq 0} h^1(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}) \\ &= \sum_{m \geq 0} h^1(S, \mathcal{O}_S((m+n)r_2alH + nr_2D)). \end{aligned}$$

Formula (1) of [CH] tells us that for $m, n \in \mathbf{Z}$,

$$\text{(3)} \qquad h^1(S, \mathcal{O}_S(mH + nD)) = \begin{cases} 2 & \text{if } m = 0 \text{ and } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus for $n < 0$, we have

$$\dim_K(H^2_Q(R)_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

giving the conclusions of the theorem. $\qquad \square$

The following example shows non periodic failure of tameness.

Theorem 3.3. *Suppose that p is a prime number such that $p \equiv 2$ (mod) 3 and $p \geq 11$. Then there exists a normal standard graded K-algebra R_0 over a field K of characteristic p with $\dim(R_0) = 4$, and a normal standard graded R_0-algebra R with $\dim(R) = 5$ such that for $j > 0$,*

$$\dim_K(H_Q^2(R)_{-j}) = \begin{cases} 1 & \text{if } j \equiv 0 \text{ (mod) } (p+1), \\ 1 & \text{if } j = p^t \text{ for some odd } t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $Q = \bigoplus_{n>0} R_n$. We have $p^t \equiv -1 \pmod{(p+1)}$ for all odd $t \geq 0$.

To establish this, we need the following simple lemma.

Lemma 3.4. *Suppose that C is a non singular curve of genus g over an algebraically closed field K, and $\mathcal{M}$, $\mathcal{N}$ are line bundles on C. If $\deg(\mathcal{M}) \geq 2(2g+1)$ and $\deg(\mathcal{N}) \geq 2(2g+1)$, then the natural map*

$$\Gamma(C, \mathcal{M}) \otimes \Gamma(C, \mathcal{N}) \to \Gamma(C, \mathcal{M} \otimes \mathcal{N})$$

is a surjection.

Proof. If $\mathcal{L}$ is a line bundle on C, then $H^1(C, \mathcal{L}) = 0$ if $\deg(\mathcal{L}) > 2g - 2$ and $\mathcal{L}$ is very ample if $\deg(\mathcal{L}) \geq 2g + 1$ (Chapter IV, Section 3 [Ha]).

Suppose that $\mathcal{L}$ is very ample and $\mathcal{G}$ is another line bundle on C. If $\deg(\mathcal{G}) > 2g - 2 - \deg(\mathcal{L})$, then $\mathcal{G}$ is 2-regular for $\mathcal{L}$ (Lecture 14, [M1]). Thus if $\deg(\mathcal{G}) > 2g - 2 + \deg(\mathcal{L})$, then

$$\Gamma(C, \mathcal{G}) \otimes \Gamma(C, \mathcal{L}) \to \Gamma(C, \mathcal{G} \otimes \mathcal{L})$$

is a surjection by Castelnuovo's Proposition, Lecture 14, page 99 [M1].

We now apply the above to prove the lemma. Write $\mathcal{M} \cong \mathcal{A}^{\otimes q} \otimes \mathcal{B}$ where $\mathcal{A}$ is a line bundle such that $\deg(\mathcal{A}) = 2g + 1$, and $2g + 1 \leq \deg(\mathcal{B}) < 2(2g+1)$. $\deg(\mathcal{N}) > 2g - 2 + \deg(\mathcal{A})$. Thus there exists a surjection

$$\Gamma(C, \mathcal{N}) \otimes \Gamma(C, \mathcal{A}) \to \Gamma(C, \mathcal{A} \otimes \mathcal{N}).$$

We iterate to get surjections

$$\Gamma(C, \mathcal{A}^{\otimes i} \otimes \mathcal{N}) \otimes \Gamma(C, \mathcal{A}) \to \Gamma(C, \mathcal{A}^{\otimes(i+1)} \otimes \mathcal{N})$$

for $i \leq q$, and a surjection

$$\Gamma(C, \mathcal{A}^{\otimes q} \otimes \mathcal{N}) \otimes \Gamma(C, \mathcal{B}) \to \Gamma(C, \mathcal{M} \otimes \mathcal{N}).$$

$\square$

We now prove Theorem 3.3. For the construction, we start with an example from Section 6 of [CS]. There exists an algebraically closed field K of characteristic p, a curve C of genus 2 over K, a point $q \in C$ and a line bundle $\mathcal{M}$ on C of degree 0, such that for $n \geq 0$,

$$H^1(C, \mathcal{O}_C(q) \otimes \mathcal{M}^{\otimes n}) = \begin{cases} 1 & \text{if } n = p^t \text{ for some } t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Further, $H^1(C, \mathcal{O}_C(2q) \otimes \mathcal{M}^{\otimes n}) = 0$ for all $n > 0$.

Let $a = p + 1$. Let E be an elliptic curve over K, and let $T = E \times E$, with projections $\pi_i : T \to E$. Let $b \in E$ be a point and let $\mathcal{A} = \pi_1^*(\mathcal{O}_E(b)) \otimes \pi_2^*(\mathcal{O}_E(b))$. Let $X = T \times C$, with projections $\varphi_1 : X \to T$, $\varphi_2 : X \to C$. Let $\mathcal{L} = \mathcal{O}_C(q)$. Let

$$\mathcal{F}_1 = \varphi_1^*(\mathcal{A})^{\otimes a} \otimes \varphi_2^*(\mathcal{L})^{\otimes a},$$

$$\mathcal{F}_2 = \varphi_1^*(\mathcal{A})^{\otimes(1+a)} \otimes \varphi_2^*(\mathcal{L}^{\otimes(1+a)} \otimes \mathcal{M}^{-1}).$$

For $m, n \geq 0$, we have

(4)
$$
\begin{aligned}
\Gamma(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}) &= \Gamma(T, \mathcal{A}^{\otimes(ma+n(1+a))}) \otimes \Gamma(C, \mathcal{L}^{\otimes(ma+n(1+a))} \otimes \mathcal{M}^{-\otimes n}) \\
&= \Gamma(T, \mathcal{A}^{\otimes a})^{\otimes m} \otimes \Gamma(T, \mathcal{A}^{\otimes(1+a)})^{\otimes n} \otimes \Gamma(C, \mathcal{L}^a)^{\otimes m} \otimes \Gamma(C, \mathcal{L}^{\otimes(1+a)} \otimes \mathcal{M}^{-1})^{\otimes n} \\
&= \Gamma(X, \mathcal{F}_1)^{\otimes m} \otimes \Gamma(X, \mathcal{F}_2)^{\otimes n}
\end{aligned}
$$

by the Künneth formula (IV of Lecture 11 [M1]) and Lemma 3.4.

Let $R_{m,n} = \Gamma(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n})$. $R = \bigoplus_{m,n \geq 0} R_{m,n}$ is a standard bigraded K-algebra by (4). Thus (2) holds.

By the Riemann Roch Theorem, we compute,

(5)
$$h^0(C, \mathcal{L}^{\otimes r} \otimes \mathcal{M}^{-\otimes s}) = h^1(C, \mathcal{L}^{\otimes r} \otimes \mathcal{M}^{-\otimes s}) + r - 1,$$

and for $s < 0$,

(6)
$$
h^1(C, \mathcal{L}^{\otimes r} \otimes \mathcal{M}^{-\otimes s}) = \begin{cases}
1 - r & r < 0, \\
1 & r = 0, s < 0, \\
1 & r = 1, s = -p^t, \text{ for some } t \in \mathbf{N}, \\
0 & r = 1, s \neq -p^t \text{ for some } t \in \mathbf{N}, \\
0 & r = 2, s < 0, \\
0 & r \geq 3.
\end{cases}
$$

We further have

(7)
$$
h^1(T, \mathcal{A}^{\otimes r}) = \begin{cases} 0 & r \neq 0, \\ 2 & r = 0, \end{cases}
$$

and

(8)
$$
h^0(T, \mathcal{A}^{\otimes r}) = \begin{cases} 0 & r < 0, \\ 1 & r = 0, \\ r^2 & r > 0. \end{cases}
$$

By (2), for $n \in \mathbf{Z}$, we have

$$\dim_K(H_Q^2(R)_n) = \sum_{m \geq 0} h^1(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}).$$

By the Künneth formula,

$$
\begin{aligned}
H^1(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}) &\cong H^0(T, \mathcal{A}^{\otimes(ma+n(1+a))}) \otimes H^1(C, \mathcal{L}^{\otimes(ma+n(1+a))} \otimes \mathcal{M}^{-\otimes n}) \\
&\quad \oplus H^1(T, \mathcal{A}^{\otimes(ma+n(1+a))}) \otimes H^0(C, \mathcal{L}^{\otimes(ma+n(1+a))} \otimes \mathcal{M}^{-\otimes n}).
\end{aligned}
$$

Thus by (5) - (8), we have for $j > 0$,

$$\dim_K(H_Q^2(R)_{-j}) = \begin{cases} 1 & j \equiv 0 \ (\mathrm{mod}) \ a, \\ 1 & j = p^t \text{ for some odd } t \in \mathbf{N}, \\ 0 & \text{otherwise.} \end{cases}$$

and we have the conclusions of Theorem 3.3.

Theorem 3.5 gives an example of failure of tameness of local cohomology with larger growth.

Theorem 3.5. *Suppose that K is an algebraically closed field. Then there exists a normal standard graded K-algebra R_0 over K with $\dim(R_0) = 4$, and a normal standard graded R_0-algebra R with $\dim(R) = 5$ such that for $j > 0$,*

$$\dim_K(H_Q^3(R)_{-j}) = \begin{cases} 6j & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd,} \end{cases}$$

where $Q = \bigoplus_{n>0} R_n$.

Proof. Let E be an elliptic curve over K, and let $q \in E$ be a point. Let $\mathcal{L} = \mathcal{O}_E(3q)$. By Proposition IV.4.6 [Ha], $\mathcal{L}$ is very ample on E, and

(9) $$\bigoplus_{n \geq 0} \Gamma(E, \mathcal{L}^{\otimes n})$$

is generated in degree 1 as a K-algebra. For $n \in \mathbf{N}$,

(10) $$h^0(C, \mathcal{L}^{\otimes n}) = \begin{cases} 0 & n < 0, \\ 1 & n = 0, \\ 3n & n > 0. \end{cases}$$

and

(11) $$h^1(C, \mathcal{L}^{\otimes n}) = \begin{cases} -3n & n < 0, \\ 1 & n = 0, \\ 0 & n > 0. \end{cases}$$

Let $X = E^3$, with the three canonical projections $\pi_i : X \to E$. Define

$$\mathcal{F}_1 = \pi_1^*(\mathcal{L}^{\otimes 2}) \otimes \pi_2^*(\mathcal{L}^{\otimes 2}) \otimes \pi_3^*(\mathcal{L}^{\otimes 2})$$

and

$$\mathcal{F}_2 = \pi_1^*(\mathcal{L}) \otimes \pi_2^*(\mathcal{L}) \otimes \pi_3^*(\mathcal{L}^{\otimes 2}).$$

Let

$$R_{m,n} = \Gamma(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}),$$
$$R = \bigoplus_{m,n \geq 0} R_{m,n}.$$

By (9) and the Künneth formula, R is standard bigraded. By (2), the fact that $\omega_X \cong \mathcal{O}_X$ and Serre duality,

$$\dim_K(H_Q^3(R)_{-j}) = \sum_{m \geq 0} h^2(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{-\otimes j}) = \sum_{m \leq 0} h^1(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes j})$$

for $j \in \mathbf{Z}$.

Now by (10), (11) and the Künneth formula, we have that for $n > 0$,

$$h^1(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}) = \begin{cases} 0 & \text{if } 2m + n \neq 0, \\ 2h^0(X, \mathcal{L}^{\otimes n}) & \text{if } 2m + n = 0. \end{cases}$$

Thus the conclusions of Theorem 3.5 hold. $\qquad\square$

The following theorem gives an example of tame, but still rather strange local cohomology. Let $[x]$ be the greatest integer in a real number x.

Theorem 3.6. *Suppose that K is an algebraically closed field. Then there exists a normal standard graded K-algebra R_0 with $\dim(R_0) = 3$, and a normal standard graded R_0-algebra R with $\dim(R) = 4$ such that for $j > 0$,*

$$\dim_K(H_Q^2(R)_{-j}) = 162 \left(j^2 \left(\left[\frac{j}{\sqrt{2}} \right] + \frac{1}{2} \right) - \frac{1}{3} \left[\frac{j}{\sqrt{2}} \right] \left(\left[\frac{j}{\sqrt{2}} \right] + 1 \right) \left(2 \left[\frac{j}{\sqrt{2}} \right] + 1 \right) \right)$$

and

$$\lim_{j \to \infty} \frac{\dim_K(H_Q^2(R)_{-j})}{j^3} = 54\sqrt{2}$$

where $Q = \bigoplus_{n>0} R_n$.

Proof. We use the method of Example 1.6 [Cu]. Let E be an elliptic curve over an algebraically closed field K, and let $p \in E$ be a point. Let $X = E \times E$ with projections $\pi_i : X \to E$. Let $C_1 = \pi_1^*(p)$, $C_2 = \pi_2^*(p)$ and

$$\Delta = \{(q,q) \mid q \in E\}$$

be the diagonal of X. We compute (as in [Cu]) that

$$(12) \qquad\qquad (C_1^2) = (C_2^2) = (\Delta^2) = 0$$

and

$$(13) \qquad\qquad (\Delta \cdot C_1) = (\Delta \cdot C_2) = (C_1 \cdot C_2) = 1.$$

If $\mathcal{N}$ is an ample line bundle on X, then

$$(14) \qquad\qquad H^i(X, \mathcal{N}) = 0 \text{ for } i > 0$$

by the vanishing theorem of Section 16 [M2].

Suppose that $\mathcal{L}$ is a very ample line bundle on X, and $\mathcal{M}$ is a numerically effective (nef) line bundle. Then $\mathcal{M}$ is 3 regular for $\mathcal{L}$, so that

$$\Gamma(X, \mathcal{M} \otimes \mathcal{L}^{\otimes n}) \otimes \Gamma(X, \mathcal{L}) \to \Gamma(X, \mathcal{M} \otimes \mathcal{L}^{\otimes(n+1)})$$

is a surjection if $n \geq 3$. $C_1 + 2C_2$ is an ample divisor by the Moishezon Nakai criterion (Theorem V.1.10 [Ha]), so that $3(C_1 + 2C_2)$ is very ample by Lefschetz's theorem (Theorem, Section 17 [M2]). Let

$$\mathcal{F}_1 = \mathcal{O}_X(9(C_1 + 2C_2)).$$

Then $\mathcal{O}_X$ is 3 regular for $\mathcal{O}_X(3(C_1 + 2C_2))$, so we have surjections

$$\Gamma(X, \mathcal{F}_1^{\otimes n}) \otimes \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_1^{\otimes(n+1)})$$

for all $n \geq 1$.

$\Delta + C_2$ is ample by the Moishezon Nakai criterion. Let $D = 3(\Delta + C_2)$. D is very ample by Lefschetz's theorem, and thus $\mathcal{O}_X(D) \otimes \mathcal{F}_1$ is very ample. Let

$$\mathcal{F}_2 = \mathcal{O}_X(3D) \otimes \mathcal{F}_1^{\otimes 3}.$$

$\mathcal{O}_X$ is 3 regular for $\mathcal{O}_X(D) \otimes \mathcal{F}_1$, so we have surjections

$$\Gamma(X, \mathcal{F}_2^{\otimes n}) \otimes \Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_2^{\otimes(n+1)})$$

for all $n \geq 1$.

for $n > 0$ and $m \geq 0$, we have

$$\mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n} \cong \mathcal{O}_X(3nD) \otimes \mathcal{F}_1^{\otimes(m+3n)}.$$

Since D is nef, it is 3 regular for $\mathcal{F}_1$, and we have a surjection for all $m \geq 0$, $n > 0$,

$$\Gamma(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}) \otimes \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_1^{\otimes(m+1)} \otimes \mathcal{F}_2^{\otimes n}).$$

Let

$$R_{m,n} = \Gamma(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}).$$

We have shown that $\oplus_{m,n \geq 0} R_{m,n}$ is a standard bigraded K-algebra. Thus (2) holds.

For $m, n \in \mathbf{Z}$, let $\mathcal{G} = \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}$. As in Example 1.6 [Cu], and by (14) and Serre Duality ($\omega_X \cong \mathcal{O}_X$ since X is an Abelian variety), we deduce that

1. $(\mathcal{G}^2) > 0$ and $(\mathcal{G} \cdot \mathcal{F}_1) > 0$ imply $\mathcal{G}$ is ample and $h^1(X, \mathcal{G}) = h^2(X, \mathcal{G}) = 0$.
2. $(\mathcal{G}^2) < 0$ implies $h^0(X, \mathcal{G}) = h^2(X, \mathcal{G}) = 0$.
3. $(\mathcal{G}^2) > 0$ and $(\mathcal{G} \cdot \mathcal{F}_1) < 0$ imply $\mathcal{G}^{-1}$ is ample and $h^0(X, \mathcal{G}) = h^1(X, \mathcal{G}) = 0$.

Let $\tau_2 = -4 - \frac{\sqrt{2}}{2}$ and $\tau_1 = -4 + \frac{\sqrt{2}}{2}$.

Using (12) and (13), we compute

$$(\mathcal{F}_1^2) = 2 \cdot 162, (\mathcal{F}_2)^2 = 31 \cdot 162, (\mathcal{F}_1 \cdot \mathcal{F}_2) = 8 \cdot 162.$$

We have

$$
\begin{aligned}
(\mathcal{G}^2) &= 324(m^2 + 8mn + \tfrac{31}{2}n^2) \\
&= 324(m - \tau_1 n)(m - \tau_2 n).
\end{aligned}
$$

and

$$(\mathcal{G} \cdot \mathcal{F}_1) = 324(m + 4n).$$

Since $\tau_2 < -4 < \tau_1 < 0$, for $n < 0$ and $m \in \mathbf{Z}$, we have

1. $m > \tau_2 n$ if and only if $\mathcal{G}^2 > 0$ and $\mathcal{G} \cdot \mathcal{F}_1 > 0$
2. $\tau_1 n < m < \tau_2 n$ if and only if $(\mathcal{G}^2) < 0$
3. $m < \tau_1 n$ if and only if $(\mathcal{G}^2) > 0$ and $(\mathcal{G} \cdot \mathcal{F}_1) < 0$.

By the Riemann Roch Theorem for an Abelian surface (Section 16 [M2]),

$$\chi(\mathcal{G}) = \frac{1}{2}(\mathcal{G}^2).$$

Thus for $m \in \mathbf{Z}$ and $n < 0$,

$$h^1(X, \mathcal{G}) = \begin{cases} -\frac{1}{2}(\mathcal{G}^2) = -162(m^2 + 8mn + \frac{31}{2}n^2) & \text{if } \tau_1 n < m < \tau_2 n, \\ 0 & \text{otherwise.} \end{cases}$$

For $n \in \mathbf{Z}$, let $\sigma(n) = \dim_K(H_Q^2(R_n))$. By (2),

$$\sigma(n) = \sum_{m \geq 0} h^1(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n}).$$

For $n < 0$, we have

$$\sigma(n) = -162\big(\sum_{\tau_1 n < m < \tau_2 n} (m^2 + 8mn + \frac{31}{2}n^2)\big).$$

Setting $r = m + 4n$, we have

$$
\begin{aligned}
\sigma(n) &= -162\big(\sum_{\frac{\sqrt{2}}{2}n < r < -\frac{\sqrt{2}}{2}n} (r^2 - \tfrac{1}{2}n^2)\big) \\
&= -324 \sum_{r=1}^{\left[-\frac{n}{\sqrt{2}}\right]} (r^2 - \tfrac{1}{2}n^2) + 81n^2 \\
&= -324 \left(\tfrac{1}{6}\left[-\tfrac{n}{\sqrt{2}}\right]\left(\left[-\tfrac{n}{\sqrt{2}}\right]+1\right)\left(2\left[-\tfrac{n}{\sqrt{2}}\right]+1\right) - \tfrac{1}{2}n^2\left[-\tfrac{n}{\sqrt{2}}\right]\right) + 81n^2 \\
&= 162\left(n^2\left(\left[-\tfrac{n}{\sqrt{2}}\right]+\tfrac{1}{2}\right) - \tfrac{1}{3}\left[-\tfrac{n}{\sqrt{2}}\right]\left(\left[-\tfrac{n}{\sqrt{2}}\right]+1\right)\left(2\left[-\tfrac{n}{\sqrt{2}}\right]+1\right)\right).
\end{aligned}
$$

We thus have the conclusions of the theorem. $\square$

4. Strange examples of Rees Algebras

Let notation and assumptions be as in Section 2. Since $\mathcal{F}_1$ is ample, there exists $l > 0$ such that $\Gamma(X, \mathcal{F}_1^{\otimes l} \otimes \mathcal{F}_2^{-1}) \neq 0$. Thus we have an embedding $\mathcal{F}_2 \otimes \mathcal{F}_1^{-l} \subset \mathcal{O}_X$. Let $\mathcal{A} = \mathcal{F}_2 \otimes \mathcal{F}_1^{-l}$, which we have embedded as an ideal sheaf of X. For $j \geq 0$ and $i \geq jl$, let

$$T_{ij} = \Gamma(X, \mathcal{F}_1^{\otimes i} \otimes \mathcal{A}^{\otimes j}) = R_{i-jl,j}.$$

For $j \geq 0$, let $T_j = \bigoplus_{i \geq jl} T_{ij}$ and $T = \bigoplus_{j \geq 0} T_j$. Let $B = \bigoplus_{j>0} T_j$. $R \cong T$ as graded rings over $R_0 \cong T_0$, although they have different bigraded structures. Thus for all i, j we have

$$(15) \qquad\qquad H_B^i(T)_j \cong H_Q^i(R)_j.$$

T_1 is a homogeneous ideal of T_0, and T is the Rees algebra of T_1. Thus all of the examples of Section 3 can be interpreted as Rees algebras over normal rings T_0 with isolated singularities.

We thus obtain the following theorems from Theorems 3.2 - 3.6. Theorems 4.1, 4.2 and 4.3 give examples of Rees algebras with non tame local cohomology.

Theorem 4.1. *Suppose that K is an algebraically closed field. Then there exists a normal, standard graded K algebra T_0 with $\dim(T_0) = 3$ and a graded ideal $A \subset T_0$ such that the Rees algebra $T = T_0[At]$ of A is normal, and for $j > 0$,*

$$\dim_K(H_B^2(T)_{-j}) = \begin{cases} 2 & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd.} \end{cases}$$

where B is the graded ideal AtT of T.

Theorem 4.2. *Suppose that p is a prime number such that $p \equiv 2(\mathrm{mod})3$ and $p \geq 11$. Then there exists a normal standard graded K-algebra T_0 over a field K of characteristic p with $\dim(T_0) = 4$, and a graded ideal $A \subset T_0$ such that the Rees algebra $T = T_0[At]$ of A is normal, and for $j > 0$,*

$$\dim_K(H_Q^2(T)_{-j}) = \begin{cases} 1 & \text{if } j \equiv 0(\text{mod})(p+1), \\ 1 & \text{if } j = p^t \text{ for some odd } t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where B is the graded ideal AtT of T. We have $p^t \equiv -1(\text{mod})(p+1)$ for all odd $t \geq 0$.

Theorem 4.3. *Suppose that K is an algebraically closed field. Then there exists a normal, standard graded K-algebra T_0 with $\dim(T_0) = 4$ and a graded ideal $A \subset T_0$ such that the Rees algebra $T = R_0[At]$ of A is normal, and for $j > 0$,*

$$\dim_K(H_B^3(T)_{-j}) = \begin{cases} 6j & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd,} \end{cases}$$

where B is the graded ideal AtT of T.

Theorem 4.4. *Suppose that K is an algebraically closed field. Then there exists a normal standard graded K algebra T_0 with $\dim(T_0) = 3$, and a graded ideal $A \subset T_0$ such that the Rees algebra $T = T_0[At]$ of A is normal, and for $j > 0$,*

$$\dim_K(H_B^2(T)_{-j}) = 162 \left(j^2 \left(\left[\frac{j}{\sqrt{2}} \right] + \frac{1}{2} \right) - \frac{1}{3} \left[\frac{j}{\sqrt{2}} \right] \left(\left[\frac{j}{\sqrt{2}} \right] + 1 \right) \left(2 \left[\frac{j}{\sqrt{2}} \right] + 1 \right) \right)$$

and

$$\lim_{j \to \infty} \frac{\dim_K(H_B^2(T)_{-j})}{j^3} = 54\sqrt{2}$$

where B is the graded ideal AtT of T.

By localizing at the graded maximal ideal of T_0, we obtain examples of Rees algebras of local rings with strange local cohomology.

In all of these examples, T_0 is generalized Cohen Macaulay, but is not Cohen Macaulay. This follows since in all of these examples,

$$H_{P_0}^2(R_0)_0 \cong H^1(X, \mathcal{O}_X) \neq 0.$$

5. Local duality in the examples

The example of [CH], giving failure of tameness of local cohomology, is stated in Theorem 3.1 of this paper. The proof of [CH] uses the bigraded local duality theorem of [HR], which now follows from the much more general bigraded local duality theorem, Theorem 1.5 and Corollary 1.7 of this paper, to conclude that in our situation, where R is generalized Cohen Macaulay,

$$(16) \qquad\qquad (H_Q^{d-i}(\omega_R)_{-j})^* \cong H_P^i(R)_j$$

for $j \gg 0$.

In [CH], the formula

$$(17) \qquad \begin{aligned} H_P^i(R)_j &\cong H_{P_0}^i(R_j) \\ &\cong \bigoplus_{m \in \mathbf{Z}} H^{i-1}(X, \widetilde{R_j}(m)) \\ &\cong \bigoplus_{m \in \mathbf{Z}} H^{i-1}(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes j}) \end{aligned}$$

for $i \geq 2$ and $j \geq 0$ is then used with formula (1) of [CH] ((3) of this paper) to prove Theorem 3.1.

In Section 2 we derive (2) from which we directly compute the local cohomology in the examples of this paper. We make essential use of Serre duality on X in computing the examples.

In this section, we show how (16) can be obtained directly from the geometry of X and V, and how this formula can be directly interpreted as Serre duality on X.

Let notation be as in Section 2, so that K is an algebraically closed field, $\mathcal{F}_1$ and $\mathcal{F}_2$ are very ample line bundles on the nonsingular variety X.

Let ω_R be the dualizing module of R, and let ω_X be the canonical bundle of X (which is a dualizing sheaf on X). For a K module W, let $W' = \mathrm{Hom}_K(W, K)$.

Lemma 5.1. *We have that*

$$(\omega_R)_{ij} = \begin{cases} \Gamma(X, \mathcal{F}_1^{\otimes i} \otimes \mathcal{F}_2^{\otimes j} \otimes \omega_X) & \text{if } i \geq 1 \text{ and } j \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Set $(\omega_R)^i = \bigoplus_{j \in \mathbf{Z}} (\omega_R)_{i,j}$, a graded R^0 module. The sheafification of $(\omega_R)^i$ on X is

$$(18) \qquad \widetilde{(\omega_R)^i} = \begin{cases} \mathcal{F}_1^{\otimes i} \otimes \omega_X & \text{if } i \geq 1 \\ 0 & \text{if } i \leq 0. \end{cases}$$

Set $(\omega_R)_j = \bigoplus_{i \in \mathbf{Z}} (\omega_R)_{i,j}$, a graded R_0 module. The sheafification of $(\omega_R)_j$ on X is

$$(19) \qquad \widetilde{(\omega_R)_j} = \begin{cases} \mathcal{F}_2^{\otimes j} \otimes \omega_X & \text{if } j \geq 1 \\ 0 & \text{if } j \leq 0. \end{cases}$$

Proof. Give R the grading where the elements of degree e in R are $[R]_e = \sum_{i+j=e} R_{ij}$.

We have realized R (with this grading) as the coordinate ring of the projective embedding of $V = \mathbf{P}(\mathcal{F}_1 \oplus \mathcal{F}_2)$ by the very ample divisor $\mathcal{O}_V(1)$, with projection $\pi : V \to X$.

Let ω_V be the canonical line bundle on V. We first calculate ω_V. Let f be a fiber of the map $\pi : V \to X$. By adjunction, we have that $(f \cdot \omega_V) = -2$. Since

$$\mathrm{Pic}(V) \cong \mathbf{Z}\mathcal{O}_V(1) \oplus \pi^*(\mathrm{Pic}(X)),$$

we see that there exists a line bundle $\mathcal{G}$ on X such that

$$\omega_V \cong \mathcal{O}_V(-2) \otimes \pi^*(\mathcal{G}).$$

The natural split exact sequence

$$(20) \qquad 0 \to \mathcal{F}_2 \to \mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{F}_1 \to 0$$

determines a section X_0 of X, such that π_* of the exact sequence

$$0 \to \mathcal{O}_V(1) \otimes \mathcal{O}_V(-X_0) \to \mathcal{O}_V(1) \to \mathcal{O}_V(1) \otimes \mathcal{O}_{X_0} \to 0$$

is (20) (Proposition II.7.12 [Ha]). Thus

$$\mathcal{O}_V(1) \otimes \mathcal{O}_V(-X_0) \cong \pi^*(\mathcal{F}_2)$$

and

$$\mathcal{O}_V(1) \otimes \mathcal{O}_{X_0} \cong \mathcal{F}_1.$$

By adjunction, we have that the canonical line bundle of X_0 is

$$\omega_{X_0} \cong \omega_V \otimes \mathcal{O}_V(X_0) \otimes \mathcal{O}_{X_0}.$$

Putting the above together, we see that

$$\mathcal{G} \cong \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \omega_X.$$

Thus

$$\omega_V \cong \mathcal{O}_V(-2) \otimes \pi^*(\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \omega_X).$$

We realize R as a bigraded quotient of a bigraded polynomial ring

$$S = K[x_1, \ldots, x_m, y_1, \ldots, y_n],$$

with $\deg(x_i) = (1, 0)$ for all i and $\deg(y_j) = (0, 1)$ for all j. Viewing S as a graded K-algebra with the grading determined by $d(x_i) = d(y_j) = 1$ for all i, j, we have a projective embedding $V \subset \mathbf{P} = \mathrm{Proj}(S)$. Since V is nonsingular, we see from Section III.7 of [Ha] that

$$\omega_V \cong \mathcal{E}xt^r_{\mathbf{P}}(\mathcal{O}_V, \mathcal{O}_{\mathbf{P}}(-e))$$

where $e = m + n$ is the dimension of S, and $r = e - \dim(R)$. ω_R is defined as

$$\omega_R = {}^*\mathrm{Ext}^r_S(R, S(-e)) \cong \bigoplus_{m \in \mathbf{Z}} \mathrm{Ext}^r_{\mathbf{P}}(\mathcal{O}_V, \mathcal{O}_{\mathbf{P}}(m - e)).$$

For $m \gg 0$,

$$\Gamma(\mathbf{P}, \mathcal{E}xt^r_{\mathbf{P}}(\mathcal{O}_V, \mathcal{O}_{\mathbf{P}}(m - e))) \cong \mathrm{Ext}^r_{\mathbf{P}}(\mathcal{O}_V, \mathcal{O}_{\mathbf{P}}(m - e))$$

(by Proposition III.6.9 [Ha]). Thus ω_R and

$$\Gamma_*(\omega_V) = \bigoplus_{m \in \mathbf{Z}} \Gamma(V, \omega_V(m))$$

are isomorphic in high degree. Since both modules have depth ≥ 2 at the maximal bigraded ideal of R, we see that

$$\omega_R \cong \Gamma_*(\omega_V).$$

Thus

$$\begin{aligned} \omega_R &= \bigoplus_{m \in \mathbf{Z}} \Gamma(V, \omega_V(m)) \\ &= \bigoplus_{m \in \mathbf{Z}} \Gamma(V, \mathcal{O}_V(m - 2) \otimes \pi^*(\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \omega_X)). \end{aligned}$$

Since a fiber f of π satisfies $(f \cdot \mathcal{O}_V(m - 2) \otimes \pi^*(\mathcal{F}_1 \otimes \mathcal{F}_2)) < 0$ if $m < 2$, we see that (with this grading) $[\omega_R]_m = 0$ if $m < 2$ and For $m \geq 2$, we have

$$\begin{aligned} [\omega_R]_m &= \Gamma(X, S^{m-2}(\mathcal{F}_1 \oplus \mathcal{F}_2) \otimes \mathcal{F}_1 \otimes \mathcal{F}_2) \\ &= \bigoplus_{i+j=m-2} \Gamma(X, \mathcal{F}_1^{\otimes(i+1)} \otimes \mathcal{F}_2^{\otimes(j+1)} \otimes \omega_X). \end{aligned}$$

The conclusions of the lemma now follow. $\qquad \square$

Suppose that $2 \leq i \leq d - 2$. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are ample, and $d - (i + 1) > 0$, there exists a natural number n_0 such that

(21) $$H^{d-(i+1)}(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^n \otimes \omega_X) = 0$$

for $n \geq n_0$ and all $m \geq 0$.

By (18), we have graded isomorphisms

(22) $$H^i_Q(\omega_R)_n \cong \bigoplus_{m \geq 1} H^{i-1}(X, \mathcal{F}_1^{\otimes m} \otimes \mathcal{F}_2^{\otimes n} \otimes \omega_X)$$

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for $n \in \mathbf{Z}$.

By Serre duality,

$$(23) \qquad H_Q^i(\omega_R)_n \cong \bigoplus_{m \geq 1} (H^{d-i-1}(X, \mathcal{F}_1^{-\otimes m} \otimes \mathcal{F}_2^{-\otimes n}))'.$$

By (21), there exists n_0 such that

$$(24) \qquad H_Q^i(\omega_R)_{-n} \cong \bigoplus_{m \in \mathbf{Z}} (H^{d-i-1}(X, \mathcal{F}_1^{-\otimes m} \otimes \mathcal{F}_2^{\otimes n}))'$$

for $n \geq n_0$.

Now apply the functor $L^* = \mathrm{Hom}_K(L, K)$ on graded R_0-modules, with the grading

$$(L^*)_i = \mathrm{Hom}_K(L_{-i}, K)$$

to (24), and compare with (17), to obtain

$$(25) \qquad H_P^{d-i}(R)_n \cong (H_Q^i(\omega_R)_{-n})^*$$

for $n \geq n_0$, from which (16) immediately follows.

We can now verify that Theorem 3.1 is in fact true for all $j > 0$, using (22) and (3).

We finally comment that an alternate proof of Theorem 3.2 for $j \gg 0$ is obtained from Theorem 3.1, Formulas (2) and (22), the fact that X is an Abelian variety so that $\omega_X \cong \mathcal{O}_X$, and the observation that

$$h^1(X, \mathcal{F}_2^{-\otimes n}) = h^1(X, \mathcal{F}_2^{\otimes n}) = 0$$

for $n > 0$.

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