Towards Minimal Resources of Measurement-based Quantum Computation

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Abstract. We improve the upper bound on the minimal resources required for measurement-based quantum computation [4, 3, 6]. Minimizing the resources required for this model is a key issue for experimental realization of a quantum computer based on projective measurements. This new upper bound allows also to reply in the negative to the open question presented in [5] about the existence of a trade-off between observable and ancillary qubits in measurement-based quantum computation.

1. Introduction

The discovery of new models of quantum computation (QC), such as the one-way quantum computer [7] and the projective measurement-based model [4], have opened up new experimental avenues toward the realisation of a quantum computer in laboratories. At the same time they have challenged the traditional view about the nature of quantum computation.

Since the introduction of the quantum Turing machine by Deutsch [1], unitary transformations plays a central rôle in QC. However, it is known that the action of unitary gates can be simulated using the process of quantum teleportation based on projective measurements-only [4]. Characterizing the minimal resources that are sufficient for this model to be universal, is a key issue.

Resources of measurement-based quantum computations are composed of two ingredients: (i) a universal family of observables, which describes the measurements that can be performed (ii) the number of ancillary qubits used to simulate any unitary transformation.

Successive improvements of the upper bounds on these minimal resources have been made by Leung and others [2, 3]. These bounds have been significantly reduced when the state transfer (which is a restricted form of teleportation) has been introduced: one two-qubit observable $(Z \otimes X)$ and three one-qubit observables $(X, Z$ and $(X + Y)/\sqrt{2})$, associated with only one ancillary qubit, are sufficient for simulating any unitary-based QC [6]. Are these resources minimal ? In [5], a sub-family of observables $(Z \otimes X, Z$, and $(X - Y)/\sqrt{2})$ is proved to be universal, however two ancillary qubits are used to make this sub-family universal.

These two results lead to an open question : is there a trade-off between observables and ancillary qubits in measurement-based QC ? In this paper, we reply in the negative to this

open question by proving that the sub-family $\{Z \otimes X, Z, (X - Y)/\sqrt{2}\}$ is universal using only one ancillary qubit, improving the upper bound on the minimal resources required for measurement-based QC.

2. Measurement-based QC

The simulation of a given unitary transformation U by means of projective measurements can be decomposed into:

- (*Step of simulation*) First, U is probabilistically simulated up to a Pauli operator, leading to σU, where σ is either identity or a Pauli operator σ_x, σ_y, or σ_z.

- (*Correction*) Then, a corrective strategy consisting in combinig conditionally steps of simulation is used to obtain a non-probabilistic simulation of U.

Teleportation can be realized by two successive Bell measurements (figure 1), where a Bell measurement is a projective measurement in the basis of the Bell states $\{|B_{ij}\rangle\}_{i,j\in\{0,1\}}$, where $|B_{ij}\rangle = \frac{1}{\sqrt{2}}(\sigma_z^i \otimes \sigma_x^j)(|00\rangle + |11\rangle)$. A step of simulation of U is obtained by replacing the second measurement by a measurement in the basis $\{(U^\dagger \otimes Id)|B_{ij}\rangle\}_{i,j\in\{0,1\}}$ (figure 2).
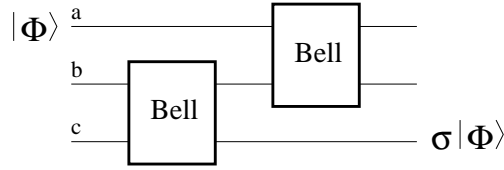

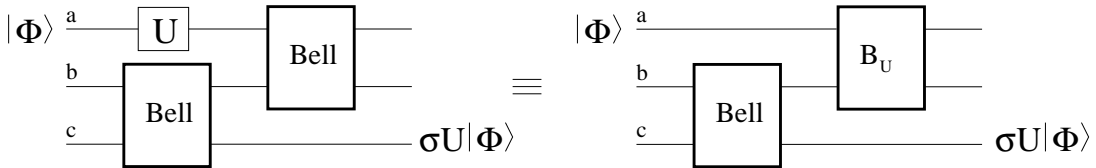**Figure 1.** Bell measurement-based teleportation

Figure 2. Simulation of U up to a Pauli operator

If a step of simulation is represented as a probabilistic black box (figure 3, left), there exists a corrective strategy (figure 3, right) which leads to a full simulation of U. This strategy consists in conditionally composing steps of simulation of U, but also of each Pauli operator. A similar step of simulation and strategy are given for the two-qubit unitary transformation ΛX (*Controlled-X*) in [4]. Notice that this simulation uses four ancillary qubits.

As a consequence, since any unitary transformation can be decomposed into ΛX and one-qubit unitary transformations, any unitary transformation can be simulated by means of projective measurements only. More precisely, for any circuit C of size n – with basis ΛX and all one-qubit unitary transformations – and for any $\epsilon > 0$, $O(n \log(n/\epsilon))$ projective measurements are enough to simulate C with probability greater than $1 - \epsilon$.
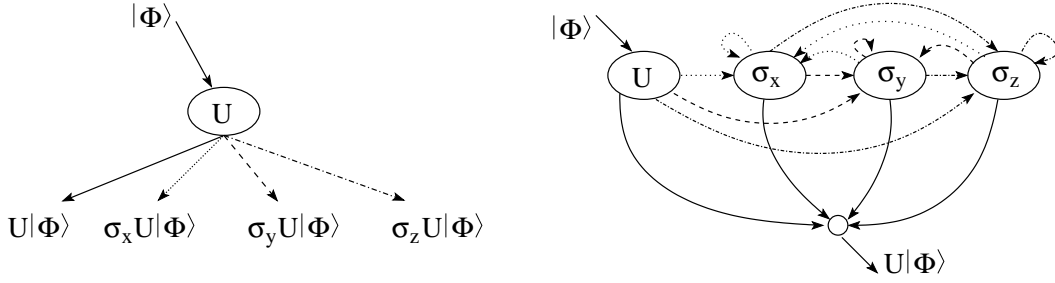
Figure 3. Left: step of simulation abstracted into a probabilistic black box representation – Rigth: conditional composition of steps of simulation.

Approximative universality, based on a finite family of projective measurements, can also be considered. Leung [3] has shown that a family composed of five observables $\mathcal{F}_0 = \{Z, X \otimes X, Z \otimes Z, X \otimes Z, \frac{1}{\sqrt{2}}(X - Y) \otimes X\}$ is approximatively universal, using four ancillary qubits. It means that for any unitary transformation U, any $\epsilon > 0$ and any $\delta > 0$, there exists a conditional composition of projective measurements from $\mathcal{F}_0$ leading to the simulation of a unitary transformation $\tilde{U}$ with probability greater than $1 - \epsilon$ and such that $||U - \tilde{U}|| < \delta$.
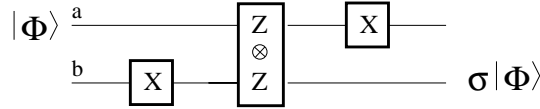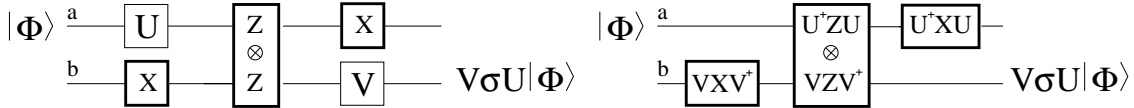

Figure 4. State transfer

Figure 5. Step of simulation based on state transfer

In order to decrease the number of two-qubit measurements – four in $\mathcal{F}_0$ – and the number of ancillary, an new scheme called *state transfer* has been introduced [6]. The state transfer (figure 4) replaces the teleportation scheme for realizing a step of simulation. Composed of one two-qubit measurements, two one-qubit measurements, and using only one ancillary qubit, the state transfer can be used to simulate any one-qubit unitary transformation up to a Pauli operator (figure 5). For instance, simulations of H and HT – see section 3 for definitions of H and T – are represented in figure 6. Moreover a scheme composed of two two-qubit measurements, two one-qubit measurements, and using only one ancillary qubit can be used to simulated ΛX up to a Pauli operator (figure 7). Since $\{H, T, \Lambda X\}$ is a universal family of unitary transformations, the family $\mathcal{F}_1 = \{Z \otimes X, X, Z, \frac{1}{\sqrt{2}}(X - Y)\}$ of observables is approximatively universal, using one ancillary qubit [6]. This result improves the result by Leung, since only one two-qubit measurement and one ancillary qubit are used, instead of four two-qubit measurements and four ancillary qubits. Moreover, one can prove that at least one

two-qubit measurement and one ancillary qubit are required for approximative universality. Thus, it turns out that the upper bound on the minimal resources for measurement-based QC differs form the lower bound, on the number of one-qubit measurements only.
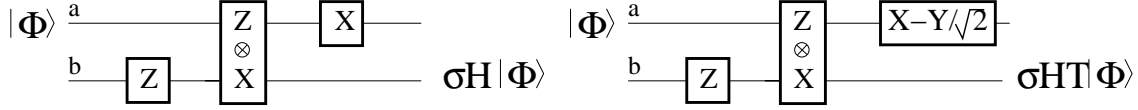

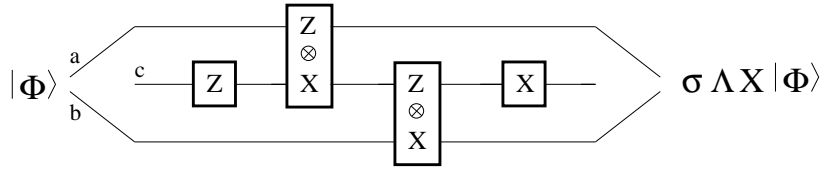**Figure 6.** Simulation of H and HT up to a Pauli operator.

Figure 7. Simulation of ΛX up to a Pauli operator.

In [5], it has been shown that the number of these one-qubit measurements can be decreased, since the family $F_2 = \{Z \otimes X, Z, \frac{1}{\sqrt{2}}(X - Y)\}$, composed of one two-qubit and only two one-qubit measurements, is also approximatively universal, using *two* ancillary qubit. The proof is based on the simulation of X-measurements by means of Z and $Z \otimes X$ measurements (figure 8). This result leads to a possible *trade-off* between the number of one-qubit measurements and the number of ancillary qubits required for approximative universality.
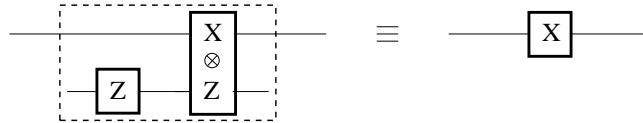

Figure 8. X-measurement simulation

In this paper, we meanly prove that the family $\mathcal{F}_2$ is approximatively universal, using only one ancillary qubit. Thus, the upper bound on the minimal resources required for approximative universality is improved, and moreover we answer the open question of the trade-off between observables and ancillary qubits. Notice that we prove that the trade-off conjectured in [5] does not exist, but another trade-off between observables and ancillary qubits may exist since the bounds on the minimal resources for measurement-based quantum computation are not tight.

3. Universal family of unitary transformations

There exist several universal families of unitary transformations, for instance $\{H, T, \Lambda X\}$ is one of them:

$$H = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix}, \Lambda X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Lambda Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We prove that the family $\{HT, \sigma_y, \Lambda Z\}$ is also approximatively universal:

Theorem 1 $\mathcal{U} = \{HT, \sigma_y, \Lambda Z\}$ *is approximatively universal.*

The proof is based on the following properties. Let $R_{\mathbf{n}}(\alpha)$ be the rotation of the Bloch sphere about the axis $\mathbf{n}$ through an angle α.

Proposition 1 *If* $\mathbf{n} = (a, b, c)$ *is a real unit vector, then for any* α, $R_{\mathbf{n}}(\alpha) = \cos(\alpha/2)I - i\sin(\alpha/2)(a\sigma_x + b\sigma_y + c\sigma_z)$.

Proposition 2 *For a given vector* $\mathbf{n}$ *of the Bloch sphere, if* θ *is an irrational multiple of* π, *then for any* α *and any* $\epsilon > 0$, *there exists* k *such that*

$$||R_{\mathbf{n}}(\alpha) - R_{\mathbf{n}}(\theta)^k)|| < \epsilon/3$$

Proposition 3 *If* $\mathbf{n}$ *and* $\mathbf{m}$ *are non parallel vectors of the Bloch sphere, then for any one-qubit unitary transformation* U, *there exists* $\alpha, \beta, \gamma, \delta$ *such that:*

$$U = e^{i\alpha} R_{\mathbf{n}}(\beta) R_{\mathbf{m}}(\gamma) R_{\mathbf{n}}(\delta)$$

Proposition 4 (Włodarski [8]) *If* α *is not an integer multiple of* $\pi/4$ *and* $\cos\beta = \cos^2\alpha$, *then either* α *or* β *is an irrational multiple of* π.

Proof of theorem 1:

First we prove that any 1-qubit unitary transformation can be approximated by HT and $\sigma_y HT$. Consider the unitary transformations $U_1 = T, U_2 = HTH, U_3 = \sigma_y HTH\sigma_y$. Notice that T is, up to an unimportant global phase, a rotation by $\pi/4$ radians around z axis on the Block sphere:

$$\begin{aligned} U_1 &= T & &= e^{-i\pi/8}(\cos(\pi/8)I - i\sin(\pi/8)\sigma_z) \\ U_2 &= HTH & &= e^{-i\pi/8}(\cos(\pi/8)I - i\sin(\pi/8)\sigma_x) \\ U_3 &= \sigma_y HTH\sigma_y & &= e^{-i\pi/8}(\cos(\pi/8)I + i\sin(\pi/8)\sigma_x) \end{aligned}$$

Composing U_1 and U_2 gives, up to a global phase:

$$\begin{aligned} U_2 U_1 &= (\cos(\pi/8)I - i\sin(\pi/8)\sigma_x)(\cos(\pi/8)I - i\sin(\pi/8)\sigma_z) \\ &= \cos^2(\pi/8)I - i[\cos(\pi/8)(\sigma_x + \sigma_z) - \sin(\pi/8)\sigma_y]\sin(\pi/8) \end{aligned}$$

According to proposition 1, $U_2 U_1$ is a rotation of the Bloch sphere about an axis along $\mathbf{n} = (\cos(\pi/8), -\sin(\pi/8), \cos(\pi/8))$ and through an angle θ defined as a solution of $\cos(\theta/2) = \cos^2(\pi/8)$. Since $\pi/8$ is not an integer multiple of $\pi/4$ but a rational multiple of π, according to proposition 4, a such θ is an irrational multiple of π. This irrationality implies that for any angle α, the rotation around $\mathbf{n}$ about angle α can be approximated to arbitrary accuracy by repeating rotations around $\mathbf{n}$ about angle θ (see proposition 3). For any α and any $\epsilon > 0$, there exists k such that

$$||R_{\mathbf{n}}(\alpha) - R_{\mathbf{n}}(\theta)^k)|| < \epsilon/3$$

Moreover, composing U_1 and U_3 gives, up to a global phase:

$$
\begin{aligned}
U_3 U_1 &= (\cos(\pi/8)I + i\sin(\pi/8)\sigma_x)(\cos(\pi/8)I - i\sin(\pi/8)\sigma_z) \\
&= \cos^2(\pi/8)I - i[\cos(\pi/8)(-\sigma_x + \sigma_z) + \sin(\pi/8)\sigma_y]\sin(\pi/8)
\end{aligned}
$$

$U_3 U_1$ is a rotation of the Bloch sphere about an axis along $\mathbf{m} = (-\cos(\pi/8), \sin(\pi/8), \cos(\pi/8))$ and through the angle θ. Thus, for any α and any $\epsilon > 0$, there exists k such that

$$||R_{\mathbf{m}}(\alpha) - R_{\mathbf{m}}(\theta)^k)|| < \epsilon/3$$

Since $\mathbf{n}$ and $\mathbf{m}$ are non-parallel, any one-qubit unitary transformation U, according to proposition 2, can be decomposed into rotations around $\mathbf{n}$ and $\mathbf{m}$: There exist $\alpha, \beta, \gamma, \delta$ such that

$$U = e^{i\alpha} R_{\mathbf{n}}(\beta) R_{\mathbf{m}}(\gamma) R_{\mathbf{n}}(\delta)$$

Finally, for any U and $\epsilon > 0$, there exist k_1, k_2, k_3 such that

$$||U - R_{\mathbf{n}}(\theta)^{k_1} R_{\mathbf{m}}(\theta)^{k_2} R_{\mathbf{n}}(\theta)^{k_3}|| < \epsilon$$

Thus, any one-qubit unitary transformation can be approximated by means of $U_2 U_1$, and $U_3 U_1$. Since $U_2 U_1 = (HT)(HT)$ and $U_3 U_1 = \sigma_y HTH\sigma_y T = -(\sigma_y HT)(\sigma_y HT)$, the family $\{HT, \sigma_y\}$ approximates any one-qubit unitary transformation.

With the additional ΛZ gate, the family $\mathcal{U}$ is approximatively universal. $\qquad\square$

4. Universal family of projective measurements

In [5], the family of observables $\mathcal{F}_2 = \{Z \otimes X, Z, \frac{X-Y}{\sqrt{2}}\}$ is proved to be approximatively universal using two ancillary qubits. We prove that this family requires only one ancillary qubit to be universal:

Theorem 2 $\mathcal{F}_2 = \{Z \otimes X, Z, \frac{X-Y}{\sqrt{2}}\}$ *is approximatively universal, using one ancillary qubit only.*

The proof consists in simulating the unitary transformations of the universal family $\mathcal{U}$. First, one can notice that HT can be simulated up to a Pauli operator, using measurements of $\mathcal{F}_2$, as it is depicted in figure 6. So, the universality is reduced to the ability to simulate ΛZ and the Pauli operators – Pauli operators are needed to simulated $\sigma_y \in \mathcal{F}$, but also to perform the corrections required by the corrective strategy (figure 3).

Lemma 5 *For a given 2-qubit register a, b and one ancillary qubit c, the sequence of measurements according to Z_c, $Z_a \otimes X_c$, $Z_c \otimes X_b$, and Z_b (see figure 9) simulates $\Lambda Z(Id \otimes H)$ on qubits a, b, up to a Pauli operator. The resulting state is located on qubits a and c.*
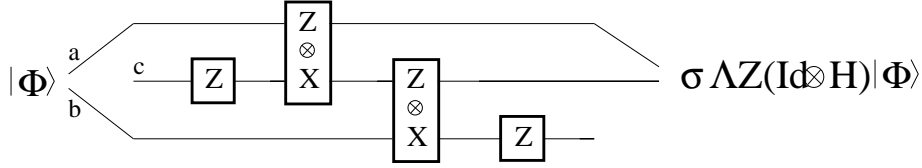

Figure 9. Simulation of $\Lambda Z(Id \otimes H)$

Proof: One can show that, if the state of the register a, b is $|\Phi\rangle$ before the sequence of measurements, the state of the register a, b after the measurements is $\sigma \Lambda Z(Id \otimes H)|\Phi\rangle$, where $\sigma = \sigma_z^{s_1} \otimes \sigma_x^{s_3} \sigma_z^{s_2+s_4}$ and s_i's are the classical outcomes of the sequence of measurements. $\square$

In order to simulate Pauli operators, a new scheme, different from the state transfer, is introduced.

Lemma 6 *For a given qubit b and one ancillary qubit a, the sequence of measurements Z_a, $X_a \otimes Z_b$, and Z_a (figure 10) simulates, on qubit b, the application of σ_z with probability $1/2$ and Id with probability $1/2$.*
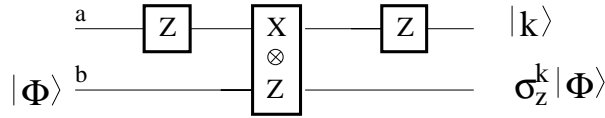

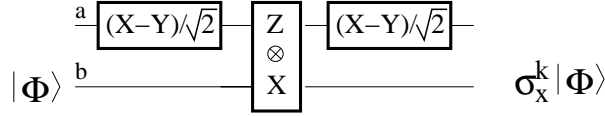**Figure 10.** Simulation of σ_z

Proof: Let $|\Phi\rangle = \alpha|0\rangle + \beta|1\rangle$ be the state of qubit b. After the first measurement, the state of the register a, b is $|\psi_1\rangle = (\sigma_x^{s_1} \otimes Id)|0\rangle \otimes |\Phi\rangle$ where $s_1 \in \{0, 1\}$ is the classical outcome of the measurement.

Since $\langle \psi_1|X \otimes Z|\psi_1\rangle = 0$, the state of the register a, b is:

$$\begin{aligned}|\psi_2\rangle &= \tfrac{\sqrt{2}}{2}(\sigma_x^{s_1} \otimes Id)(Id + (-1)^{s_2} X \otimes Z)|0\rangle \otimes |\Phi\rangle \\ &= \tfrac{\sqrt{2}}{2}(\sigma_x^{s_1}\sigma_z^{s_2} \otimes Id)(|0\rangle \otimes |\Phi\rangle + |1\rangle \otimes (\sigma_z|\Phi\rangle))\end{aligned}$$

Let $s_3 \in \{0, 1\}$ be the outcome of the last measurement, on qubit a. If $s_1 = s_3$ then state of the qubit b is $|\Phi\rangle$, and $\sigma_z|\Phi\rangle$ otherwise. One can prove that these two possibilities occur with equal probabilities. $\square$

Lemma 7 *For a given qubit b and one ancillary qubit a, the sequence of measurements $\left(\frac{X-Y}{\sqrt{2}}\right)_a$, $Z_a \otimes X_b$, and $\left(\frac{X-Y}{\sqrt{2}}\right)_a$ (figure 11) simulates, on qubit b, the application of σ_x with probability $1/2$ and Id with probability $1/2$.*

Figure 11. Simulation of σ_x

The proof of lemma 7 is similar to the proof of lemma 6.

Proof of theorem 2:

First notice that the family of unitary transformations $\mathcal{U}' = \{HT, \sigma_y, \Lambda Z(I \otimes H)\}$ is approximatively universal since $\mathcal{U} = \{HT, \sigma_y, \Lambda Z\}$ is universal.

HT and $\Lambda Z(I \otimes H)$ can be simulated up to a Pauli operator (lemmas 5). The universality of the family of observables $\mathcal{F}_2 = \{Z \otimes X, Z, \frac{X-Y}{\sqrt{2}}\}$ is reduced to the ability to simulate any Pauli operators. Lemma 7 (resp. lemma 6), shows that σ_x (σ_z) can be simulated with probability $1/2$, moreover if the simulation fails, the resulting state is same as the original one. Thus, this simulation can be repeated until a full simulation of σ_x (σ_z). Finally, $\sigma_y = i\sigma_z\sigma_x$ can be simulated, up to a global phase, by combining simulations of σ_x and σ_z. Thus, $\mathcal{F}_2 = \{Z \otimes X, Z, \frac{X-Y}{\sqrt{2}}\}$ is approximatively universal using only one ancillary qubit.   □

5. Conclusion

We have proved a new upper bound on the minimal resources required for measurement-based QC: one two-qubit, and two one-qubit observables are universal, using one ancillary qubit only. This new upper bound has experimental applications, but allows also to prove that the trade-off between observables and ancillary qubits, conjectured in [5], does not exist. This new upper bound is not tight since the lower bound on the minimal resources for this model is one two-qubit observable and one ancillary qubit.

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