The Complexity of Hamiltonian Cycle Problem in Digraps with Degree Bound Two is Polynomial Time

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Abstract. The incidence matrix of C_{nm} of a simple digraph is mapped into a incidence matrix F of a balanced bipartite undirected graph by divided C into two groups. Based on the mapping, it proves that the complexity is polynomial to determin a Hamiltonian cycle existence or not in a simple digraph with degree bound two and obtain all solution if it exists Hamiltonian cycle. It also proves $P = NP$ with the different results in [1].

1 Introduction

It is well known that the Hamiltonian cycle problem(HCP) is one of the standard NP-complete problem [2]. As for digraph, even limited the digraph on these cases: planar digraphs with indegree 1 or 2 and outdegree 2 or 1 respectively, it is still on $NP - Complete$ which is proved by J.Plesník [1].

Let us named a simple strong connected digraph with at most indegree 1 or 2 and outdegree 2 or 1 as Γ digraph. As this paper proof, the HCP of Γ graph could be solved by following main results.

Theorem 1. *Divided the incidence matrix C_{nm} of Γ digraph to a incidence matrix F, the incidence matrix F represents a undirected balanced bipartite graph $G(X, Y; E)$, which obeys the following properties:*

c1. $|X| = n$,$|Y| = n$,$|E| = m$
c2.

$$\forall x_i \in X \wedge 1 \leq d(x_i) \leq 2$$

$$\forall y_i \in Y \wedge 1 \leq d(y_i) \leq 2$$

c3. G has at most $n/2$ component.

Let us namd the undirected balanced bipartite graph $G(X, Y : E)$ as projector graph of D.

Theorem 2. *Let G be the projector graph of a Γ graph $D(V, A)$, determinning a Hamiltonian cycle in Γ digraph is equivalent to find a perfect match M in G and $rank(C') = n - 1$, where C' is the incidence matrix of $D'(V, L) \subseteq D$ and $L = \{a_i | a_i \in D \wedge e_i \in M\}$.*

Theorem 3. *Given the incidence matrix C_{nm} of a Γ digraph , the complexity of finding a Hamiltonian cycle existing or not is $O(n^4)$*

More in additional, we given the complexity of HCP in a simple (un)directed graph .

Theorem 4. *The complexity of finding a Hamiltonian cycle existing or not in a strong connected (un)directed graph is polynomial time,Which implies $P = NP$.*

The concepts of cycle and rank of graph are given in section 2. Then the follows sections are the proof of above theorems.

2 Definition and properties

Throughout this paper we consider the finite simple (un)directed graph $D = (V, A)$ ($G(V, E)$, respectively), i.e. the graph has no multi-arcs and no self loops. Let n and m denote the number of vertices V and arcs A (edges E, respectively), respectively.

As conventional, let $|S|$ denote the number of a set S. The set of vertices V and set of arcs of A of a digraph $D(V, A)$ are denoted by $V = \{v_i | 1 \leq i \leq n\}$ and $A = \{a_j | (1 \leq j \leq m) \wedge a_j =< v_i, v_k >, (v_i \neq v_k \in V)\}$ respectively, where $< v_i, v_k >$ is a arc from v_i to v_k and a reverse arc is denoted by $\overleftarrow{a_k} =< v_k, v_i >$ if it exists . Let the out degree of vertex v_i denoted by $d^+(v_i)$, which has the in degree by denoted as $d^-(v_i)$ and has the degree $d(v_i)$ which equals $d^+(v_i) + d^-(v_i)$. Let the $N^+(v_i) = \{v_j | < v_i, v_j > \in A\}$, and $N^-(v_i) = \{v_j | < v_j, v_i > \in A\}$.

Let us define a forward relation $\bowtie$ between two arcs as following, $a_i \bowtie a_j = v_k$ iff $a_i =< v_i, v_k > \wedge a_j =< v_k, v_j >$

It is obvious that $|a_i \bowtie a_i| = 0$. A pair of symmetric arcs $< a_i, a_j >$ are two arcs of a simple digraph if and only if $|a_i \bowtie a_j| = 1 \wedge |a_j \bowtie a_i| = 1$.

A *cycle* L is a set of arcs $(a_1, a_2, \ldots, a_q)$ in a digraph D, which obeys two conditions:

c1. $\forall a_i \in L, \exists a_j, a_k \in L \setminus \{a_i\},\ a_i \bowtie a_j \neq a_j \bowtie a_k \in V$

c2. $|\bigcup_{a_i \neq a_j \in L} a_i \bowtie a_j| = |L|$

If a cycle L obeys the following conditions, it is a *simple cycle*.

c3. $\forall L' \subset L$, L' does not satisfy both conditions $c1$ and $c2$.

A *Hamiltonian cycle* L is also a simple cycle of length $n = |V| \geq 2$ in digraph. As for simplify, this paper given a sufficient condition of Hamiltonian cycle in digraph.

Lemma 1. *If a digraph $D(V, A)$ include a sub graph $D'(V, L)$ with following two properties, the D is a Hamiltonian graph.*

c1. $\forall v_i \in D' \rightarrow d^+(v_i) = 1 \wedge d^-(v_i) = 1$,
c2. $|L| = |V| \geq 2$ *and* D' *is a strong connected digraph.*

A graph that has at least one Hamiltonian cycle is called a *Hamiltonian graph.* A graph G=$(V; E)$ is bipartite if the vertex set V can be partitioned into two sets X and Y (the bipartition) such that $\exists e_i \in E, x_j \in X, \forall x_k \in X \setminus \{x_j\}$, $(e_i \bowtie x_j \neq \emptyset \rightarrow e_i \bowtie x_k = \emptyset)$ $(e_i, Y$, respectively). if $|X| = |Y|$, We call that G is a balanced bipartite graph. A matching $M \subseteq E$ is a collection of edges such that every vertex of V is incident to at most one edge of M, a matching of balanced bipartite graph is perfect if $|M| = |X|$. Hopcroft and Karp shows that constructs a perfect matching of bipartite in $O((m+n)\sqrt(n))$ [3]. The matching of bipartite has a relation with neighbour of X.

Theorem 5. *[4] A bipartite graph $G = (X, Y; E)$ has a matching from X into Y if and only if $|N(S)| \geq S$, for any $S \subseteq X$.*

Two matrices representation for graphs are defined as follows.

Definition 1. *[5] The incidence matrix C of undirected graph G is a two dimensional $n \times m$ table, each row represents one vertex, each column represents one edge, the c_{ij} in C are given by*

$$c_{ij} = \begin{cases} 1, \text{ if } v_i \in e_j; \\ 0, \text{ otherwise.} \end{cases} \tag{1}$$

It is obvious that every column of an incidence matrix has exactly two 1 entries.

Definition 2. *[5] The incidence matrix C of directed graph D is a two dimensional $n \times m$ table, each row represents one vertex, each column represents one arc the c_{ij} in C are given by*

$$c_{ij} = \begin{cases} 1, & \text{if} < v_i, v_i > \bowtie a_j = v_i; \\ -1, & \text{if } a_j \bowtie < v_i, v_i >= v_i; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

It is obvious to obtain a corollary of the incidence matrix as following.

Corollary 1. *Each column of an incidence matrix of digraph has exactly one* 1 *and one* -1 *entries.*

Theorem 6. *[5] The C is the incidence matrix of a directed graph with k components the rank of C is given by*

$$rank(C) = n - k \tag{3}$$

In order to convince to describe the graph D properties, in this paper, we denotes the $rank(D) = rank(C)$.

3 Divided incidence matrix and Projector incidence matrix

Firstly, let us divided the matrix of C into two groups.

$$C^+ = \{c_{ij} | c_{ij} \geq 0 \text{ otherwise } 0 \ \} \tag{4}$$

$$C^- = \{c_{ij} | c_{ij} \leq 0 \text{ otherwise } 0 \ \} \tag{5}$$

It is obvious that the matrix of C^+ represents the forward arc of a digraph and C^- matrix represents the backward arc respectively. A corollary is deduced as following.

Corollary 2. *For a strong connected digraph $D = (V, A)$, the rank of divided incidence matrix satisfies $rank(C^+) = rank(C^-) = n$.*

Secondly Let us combined the the C^+ and C^- as following matrix.

$$F = \begin{pmatrix} C^+ \\ -C^- \end{pmatrix} \tag{6}$$

In more additional, let F represents as a incidence of matrix of undirected graph $G(X, Y; E)$, and the F is named as *projector incidence matrix* of C and G is named as *projector graph* where X represents the vertices of C^+, Y represents the vertices of $-C^-$ respectively. In another words we build a mapping $F : D \to G$ and denotes it as $G = F(D)$. So the $F(D)$ has $2n$ vertices and m edges if D has n vertices and m arcs. We also build up a reverse mapping: $F^{-1} : G \to D$ When G is a projector graph. To simplify, we also denotes the vertex $x_i = F^{-1}(v_i)$ in graph D v_i in G.

4 Proof of Theorem 1

Firstly ,let us prove the theorem 1.

Proof. c1. Since Γ digraph is strong connected, then each vertices of Γ digraph has at least one forward arcs, each row of C^+ has at least one 1 entries, and the U represents the C^+ , so

$$|U| = n$$

the same principle of C^-, each row of C^- has at least one -1 entries, and the V represents the C^- , so

$$|V| = n$$

Since the columns of F equal to the columns of C,

$$|E| = m$$

c2. Since the degree of each v_i of Γ digraph is $1 \leq d^+(v_i) \leq 2$,

$$\forall u_i \in U \wedge 1 \leq d(u_i) \leq 2$$

Since the degree of each v_i of Γ digraph is $1 \leq d^-(v_i) \leq 2$,

$$\forall v_i \in V \wedge 1 \leq d(v_i) \leq 2$$

c3. Since the F is $2n \times m$ matrix, where $m \leq 3n/2$, $rank(F) \leq 3n/2$. Suppose the G has k disjoint sub connected graph and $k > n/2$, then the $rank(F) = 3n/2 - k < n$, since $rank(F) \geq rank(C^+)$, then $rank(C^+) < n$ it is contradiction with the Γ digraph is strong connected graph.

Secondly ,let us given the properties after mapping Hamiltonian cycle L of D into the sub graph M of projector graph G.

Lemma 2. *If a Hamiltonian cycle L of D mapping into a forest M of projector graph G, the forest M consist of $|L|$ number of trees which has only two node and one edge, and M has a unique perfect matching.*

Proof. Let the Γ digraph $D(V, A)$ has a sub digraph $D'(V, L)$ which exists one Hamiltonian cycle and $|L| = n$, the incidence matrix C of L could be permutation as follows.

$$C = \begin{pmatrix} 1 & 0 & 0 & \dots 0 & -1 \\ -1 & 1 & 0 & \dots 0 & 0 \\ 0 & -1 & 1 & \dots 0 & 0 \\ 0 & 0 & -1 & \dots 0 & 0 \\ 0 & 0 & 0 & \dots 0 & 0 \\ 0 & 0 & 0 & \dots -1 & 1 \end{pmatrix}. \tag{7}$$

Let

$$F = \begin{pmatrix} C^+ \\ -C^- \end{pmatrix}$$

It is obvious that each row of F has only one 1 entry and each column of F has two 1 entries.

According to theorem 1, F represents a balanced bipartite graph $G(X, Y; E)$ that each vertex has one edge connected, and each edge e_i connect on vertex $x_i \in X$, another in Y, in another words, $\exists e_i \in E\, x_j \in X$,$\forall x_k \in X \setminus \{x_j\}$, $e_i \bowtie x_j \neq \emptyset \rightarrow e_i \bowtie x_k = \emptyset(e_i, Y$,respectively). According the matching definition, M is a matching, since $|E| = |L|$, E is a perfect matching. and pair of vertices between X and Y only has one edge, so M is a forest, and each tree has only two node with one edge.

5 Proof of Theorem 2

Proof. $\Rightarrow$ Let the Γ digraph $D(V, A)$ has a sub digraph $D'(V, L)$ which exists one Hamiltonian cycle and $|L| = n$, let matrix C' represents the incidence matrix of D', so $rank(C') = n-1$; According to lemma 2, the projector graph $F(D')$ has a perfect matching, thus $F(D)$ also has a perfect matching.

$\Leftarrow$ Let $G(X,Y;E)$ be a projector graph of the Γ graph $D(V,A)$,M is a perfect matching in G. Let $D'(V,L)$ be a sub graph of $D(V,A)$ and $L = \{a_i | a_i \in D \wedge e_i \in M\}$. Since $rank(L) = n-1$, $D'(V,L)$ is a strong connected digraph. it deduces that $\forall v_i \in D'$,$d^+(v_i) \geq 1 \wedge d^-(v_i) \geq 1$. Suppose $\exists v_i \in D'$, $d^+(v_i) > 1$ ($d^-(v_i) > 1$ respectively), Since $|M| = n$, it deduces that $\sum_{i=1}^{n} d(v_i) > 2n+1$, which imply that $|L| > n$. this is contradiction with $L = \{a_i | a_i \in D \wedge e_i \in M\}$ and $|M| = n$. So $\forall v_i \in D'$, $d^+(v_i) = d^-(v_i) = 1$, According the lemma 1, D' has a Hamiltonian cycle.

6 Proof of Theorem 3

Firstly ,let us considering if there exists a cycle in a projector graph $F(D)$, then what is equivalent to the original digraph D.

Now let proof the theorem 3.

Proof. Let G be a project balanced bipartition of D. According theorem 1, the Γ graph is equivalent to find a perfect match M in a project G.

Assume there exists k component in forest G and $G_1, \ldots, G_i, \ldots, G_k \wedge k \leq n/2$ is disconnected subgraph $G(V_i, E_i)$ of G. Let us give two equation to obtain a perfect matching M from $G_i \wedge i \in [1..k]$ which $rank(F^{-1}(M)) = n-1$ by greedy approach.

$$M' = M \otimes E_i \tag{8}$$

where $G_i(V_i, E_i)$ is a simple cycle with degree 2 and $|E_i| \geq 4$.

$$M = \begin{cases} M', & \text{if } rank(F^{-1}(M')) > rank(F^{-1}(M)) \\ M(0), & M(0) \text{ is a initial perfect matching in } G. \end{cases} \tag{9}$$

When $rank(F^{-1}(M)) = n-1$(including $M(0)$), According the theorem 1, the $A = F^{-1}(M)$ is a Hamiltonian cycle solution. If all of $rank(F^{-1}(M(t))) < n-1$(including $M(0)$), then there has no Hamiltonian cycle in D.

Since each component in G_i has three cases:

case 1. $G_i(V_i, E_i)$ is a path, since each degree $d(V_i) \geq 2$, there has no more small subgraph is cycle. Each perfect matching satisfies $M \cap E_i = M_0 \cap E_i$.

case 2 $G_i(V_i, E_i)$ is a $2q$ length cycle,
Let $M \cap E_i = \{e_l, e_l + 1, \ldots, e_l + q\}$, and

$$M' \cap E_i = (M \otimes E_i) \cap E_i = E_i \setminus \{e_l, e_l + 1, \ldots, e_l + q\}$$

Then

$$M = M \cap M' \cup \{e_l, e_l + 1, \ldots, e_l + q\}$$

$$M' = M \cap M' \cup E_i \setminus \{e_l, e_l + 1, \ldots, e_l + q\}$$

Suppose $e_l, e_l+1, \ldots, e_l+q$ belongs to a perfect matching M_t with $rank(F^{-1}(M_t)) = n-1$, then $F^{-1}(M \cap M')$ and $F^{-1}(\{e_l, e_l+1, \ldots, e_l+q\})$ are linear independence. So then $rank(F^{-1}(M) = rank(F^{-1}(M \cap M')) + q \geq rank(M')$. Suppose $\{e_l, e_l+1, \ldots, e_l+q\}$ does not belong to any perfect matching M_t with $rank(F^{-1}(M_t)) = n-1$, but $E_i \setminus \{e_l, e_l+1, \ldots, e_l+q\}$ belongs to one of this matching. $e_l, e_l+1, \ldots, e_l+q$ and M are linear dependence. So then $rank(F^{-1}(M) < rank(F^{-1}(M \cap M')) + q = rank(M')$.

case 3 $G_i(V_i, E_i)$ consists of a path with a short cycle subgraph, it is impossible since $d(v_i) \leq 2$.

Now let considering the complexity of the equation, since the complexity of rank of matrix is $O(n^3)$, finding a simple cycle in a subgraph with degree 2 is $O(n^2)$, and obtaining a perfect matching of a bipartite graph is $O(m+n)\sqrt(n) < O(n^2)$ [3]. Based on therorem 1, there are only at most $n/2$ component in G, the complexity of calculate the equation 8 and equation 9 are $O(n^4)$.

Considering the equation 9, let

$$M(t+1) = M' \text{if } rank(F^{-1}(M')) \geq rank(F^{-1}(M(t)))$$

Saving the $M(t)$ into a stack, when a Hamiltonian cycle in D is obtain, then track back $M(t)$, we can obtain another Hamiltonian cycle if it existence.

Corollary 3. *. Given a Hamiltonian $\mathit{\Gamma}$ digraph , the complexity of obtaining all of a Hamiltonian cycle is polynomial time.*

7 Proof of Theorem 4

Let us prove the theorem 4 the complexity of HCP in graph. Since each edge e_i in undirected graph could be substitute with a pair of symmetric arcs $a_k, a_j \wedge |a_k \bowtie a_j| = 2$, it only need considering the strong connected digraph $D(V, A)$.

Proof. $\Rightarrow$

According the theorem 3, the complexity of HCP in $\mathit{\Gamma}$ digraph is P problem, and as the result of .[1], the complexity of HCP in $\mathit{\Gamma}$ digraph is NP problem. According the results in [2], the complexity of HCP problem of *grid* is NP problem instead of P problem. So $P \subseteq NP$.

$\Leftarrow$ Suppose that in HCP problem in a digraph $D(V, A)$ with $d(v_i) \geq 4$ which $\exists v_i \in$V$, k = d(v_i) \geq 4 \wedge d^-(v_i) \geq 1$ is shown as figure 1

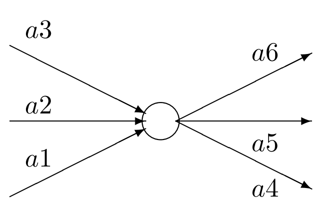
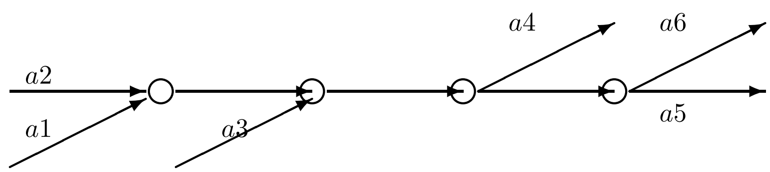


Figure 1. A vertex with degree than 2

Let this vertex spilt to a k vertices which each vertex has degree 3, as shown in figure 2.



Figrue 2 vertex with after mapping to Γ digraph

It is obvious that each vertex in the split Γ graph S has increase $k-2$ vertices and $k-2$ arcs. Suppose the worst cases is each vertex in D has degree $2n-2$, the total vertices in S reduces to $2n^2-2n$ vertices, so the complexity of HCP in S is no more then $O(n^6)$ and if S has a Hamiltonian cycle L, then the D will has a Hamiltonian cycle $L \cap A$ as well. It is means the HCP in digraph D is not *difficult* then S. In additional, suppose that D has a Hamiltonian cycle including a_3, a_4, it can not reduces that S has a Hamiltonian cycle. It is means the HCP in digraph D is *easy* then S. So $D \preceq_p S$, that implies $NP \subseteq P$.

In fact, the [1] proves that $3SAT \preceq_p HCP\ of\ \Gamma\ digraph$, which also implies that $NP = P$.

Based on corollary 3, the apporach reducation of figure 1 to figure 2 can obtain all Hamiltonian cycle is in polynomial time if given Hamiltonian graph have $d(v) \leq 4$, otherwise it only can find second Hamiltonian cycle.

Corollary 4. *. Given a Hamiltonian directed graph with degree bound two, finding all Hamiltonian cycle is polynomial time; Given a Hamiltonian un(directed) graph with degree ≥ 3, the complexity of obtaining second of a Hamiltonian cycle is polynomial time.*

8 Conclusion

According to the theorem 4, not only the complexity of determining a Hamiltonian cycle existence or not is in polynomial time, but also complexity of obtained second Hamiltonian cycle in Hamiltonian graph is in polynomial time by the corollary 4, which proves that $P = NP$ again based on the result in [6].

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