Fluctuation-dissipation relation on a Melde string in a turbulent flow, considerations on a "dynamical temperature".

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Abstract. – We report on measurements of the transverse fluctuations of a string in a turbulent air jet flow. Harmonic modes are excited by the fluctuating drag force, at different wave-numbers. This simple mechanical probe makes it possible to measure excitations of the flow at specific scales, averaged over space and time: it is a scale-resolved, global measurement. We also measure the dissipation associated to the string motion, and we consider the ratio of the fluctuations over dissipation (FDR). In an exploratory approach, we investigate the concept of *effective temperature* defined through the FDR. We compare our observations with other definitions of temperature in turbulence. From the theory of Kolmogorov (1941), we derive the exponent $-11/3$ expected for the spectrum of the fluctuations. This simple model and our experimental results are in good agreement, over the range of wave-numbers, and Reynolds number accessible ($74000 \leq Re \leq 170000$).

Introduction. – Turbulent flows exhibit a notoriously complex and unpredictable dynamics: they present a huge number of degrees of freedom, and their dynamics are both far from equilibrium and dissipative [1–3]. Let the flow by itself, kinetic energy will be dissipated into heat by the molecular viscosity, but not directly at the same scale. Instability mechanisms associated with non-linearities generate harmonics, almost without dissipation. An equivalent picture would consist in vortices stretching each other in such a way that a non-zero energy transfer occurs toward smaller scales. This picture of *cascade* process was first proposed by Richardson [4]. The cascade stops approximately in the range of scales where the viscosity becomes efficient to damp velocity gradients. From this qualitative idea, Kolmogorov derived, in the late thirties, a phenomenological theory accounting for the fluctuations of various observables in fully developed turbulence [5]. In the present work, we are neither concerned by the large (energy injection) scales, nor by the small (dissipation) scales, but by the intermediate range. In this intermediate inertial range, we study the transport process through scales, expected to be universal. Instead of scale r, one often refers to the wave-number $k = 2\pi/r$, its conjugate variable by Fourier transformation.

From the point of view of hydrodynamics, the control parameter of the flow is the Reynolds number: $Re = \frac{VL}{\nu}$, where L is the macroscopic scale of the flow (integral scale, or correlation length), V is a characteristic shear velocity at large scale, and ν is the kinematic viscosity of the fluid. It is also the mean ratio of the inertial by the dissipative contribution of the forcing over a fluid particle. Interesting predictions were derived by Kolmogorov (1941), that we use in the following. Especially, the range of scales over which fluctuations occur scales as $Re^{3/4}$. The prediction for the exponent of the power spectral density as $\langle|\tilde{v}|^2\rangle \propto k^{-5/3}$ is among the most famous successes of this theory [1–3].

In this work, we study turbulence from the point of view of statistical physics. Let us recall one important break-through of the twentieth century in this field, the statement of the Fluctuation-Dissipation Theorem (FDT). Consider a couple of conjugate state variables of a small system in thermal contact with a large heat reservoir at equilibrium: one extensive $x(t)$, one intensive $f(t)$. The theorem originates from the idea that spontaneous fluctuations of $x(t)$ should have the same statistical properties as the relaxation of $x(t)$ after the removal of an external perturbation $f(t)$. The main hypothesis needed to derive this theorem are: - linear response between f and x, -thermal equilibrium between the system considered and the thermostat, - thermal equilibrium of the thermostat itself. The response function $H_{x,f}$ is such that: $f(t) = \int_{-\infty}^{t} H_{x,f}(t-t')f(t')dt'$. Equivalently it can be written in the Fourier space as: $\tilde{x}(\omega) = \tilde{H}_{x,f}\ \tilde{f}(\omega)$. Under some hypothesis, the fluctuations of x (its 2-times correlation function) are linked by a very simple relation with the dissipative response of the system to a perturbation of the conjugate intensive variable f (imaginary part of the response function). It is simply proportional, and the coefficient is nothing but the temperature multiplied by the Boltzman constant: $k_{\rm B}T$ [6, 7]. The validity of the hypothesis have to be discussed in each case. If they are satisfied, the correlation function of the spontaneous fluctuations is proportional to the response function, i.e. the factor is unique and constant. Moreover, this factor is the same for all couples of conjugate variables, and this factor is $k_{\rm B}T$, where T is the temperature of the system. The Boltzman constant $k_{\rm B} \simeq 1.38\ 10^{-23} JK^{-1}$ is an universal constant. This relation can be expressed in spectral variables:

$$\langle|\tilde{x}(\omega)|^2\rangle = \frac{4\ k_{\rm B}T}{\omega}\ Im[\tilde{H}_{x,f}(\omega)]. \tag{1}$$

In this expression of the FDT, $\langle|\tilde{x}(\omega)|^2\rangle$ is the power spectral density of the fluctuations of the extensive quantity $x(t)$, as $\tilde{H}_{\rm x,f}(\omega)$ is the response function on x to the conjugate intensive variable f. The imaginary part of the response function $Im[\tilde{H}]$ is the energy dissipation.

In the perspective of constructing a non-equilibrium thermodynamics, the FDT has been reconsidered: can we assert that FDT still holds true for systems *weakly out of equilibrium*, at least approximately ? In this spirit, L. Cugliandolo and J. Kurchan investigated the case of amorphous materials relaxing, after a thermal quench through the glass transition [8, 9].

Let us consider in that case the criteria expressed above. 1- If the excitation is small enough, the response vanishes: linearity of the response is reasonable if the excitation is small enough. But the proportionality might not be rigorously constant, as the system is not stationary. The two other hypothesis have been tested numerically on mean-field spin-glass models [9]. 2- Response and correlation are linked not by a unique constant factor, but a function of frequency and time: high frequency modes relax faster than the others. 3- In many cases, there are not several independent couples of observables available for experimental checking. However, dielectric as well as rheological measurements on Laponite gel were performed, that showed some disagreement [10]. If FDT is not valid in such configuration, it must be that some of the hypothesis are not fulfilled. The time-frequency dependence of the Fluctuation over Dissipation Ratio reflects the slow dynamics of the relaxation process in glasses: not all modes relax at the same rate toward equilibrium. After a thermal quench through the glass transition, the low frequency modes relax slower than the high frequency ones: this phenomenon is called ageing. In this non-stationary context with time and frequency dependence, and in an extended formalism, the Fluctuation-Dissipation Ratio (FDR) can be rewritten:

$$\frac{\omega \ \langle \tilde{x}(\omega)^2 \rangle}{Im[\tilde{H}_{\mathrm{x,f}}(\omega)]} = 4\ k_{\mathrm{B}} T_{\mathrm{eff.}}(\omega, t_w), \tag{2}$$

where the temperature is replaced by an "effective" temperature $T_{\mathrm{eff.}}$, function of frequency ω and waiting time t_w after the quench. The time and frequency dependence of $T_{\mathrm{eff.}}$ expresses that different degrees of freedom are not at equilibrium with each other, resulting in internal heat fluxes.

This theoretical work motivated many experiments on systems for which both fluctuations and response function are measurable. For instance, the FDR has been investigated in polymer or colloidal glasses [10–13], and also in granular gases [14], the later being strongly dissipative. Note that, even though the FDR can be measured, the interpretation of $T_{\mathrm{eff.}}$ as a non-equilibrium temperature in the thermodynamical sense is still questionable.

We propose here to investigate fully-developed turbulence from the point of view of the FDR. We measured the vibrations of a thin Melde string coupled to a turbulent flow by viscous drag. More precisely, each (independent) mode of the string couples to (non-independent) scale of the flow. As our jet flow is stationary, we averaged our measurements on time. Therefore, there is no time dependence of the FDR, but just frequency. Measurements of the fluctuations of the string give Fourier components of the excitation of the flow. We measured independently the fluctuations, and the complex response function to a specified excitation, in a way discussed below. We propose to analyse these measurements with the three criteria discussed above.

The paper is organised as follows. The next section describes the experimental setup, turbulent flow properties, and setting of the string. General properties of a vibrating Melde string are also discussed. The measurements are shown in the following section, response, fluctuations, and the Fluctuation Dissipation Ratio of this system. In the next section, we derive from Kolmogorov's theory a simple scaling model for the fluctuations of the drag, and therefore the FDR, which accounts for the exponent observed in the whole range of accessible *Re*. The last section is devoted to a discussion of our results, especially in comparison to several definitions of temperature in turbulence proposed in the literature.

The Melde string and the experimental setup. – The experimental setup is sketched on Fig.1. A turbulent air jet is created from a nozzle of diameter 5 cm. The flow facility we used is thoroughly described in [15]. A stainless steel string of length 60 cm is located 2 m downstream the nozzle, perpendicular to the axis of the flow. At this distance, the length of the string is about the diameter of the turbulent jet. The response of the string to a perturbation is measured using piezoelectric multi-layer ceramics on each end of the string, one as actuator, one as sensor. The amplified white noise signal from a HP3562A source excites the input piezo. The output signal on the other end is amplified. Both are recorded with a 24 bits A/D converter. The acquisition frequency is 50 KHz. We call response the ratio of the voltage amplitudes on each piezo. Voltages *in* and *out* are proportional respectively to the displacement and the constraint (on the piezos). The diameter of the string is 100 μm, less than the viscous scale of the flow which is about $\eta \simeq 170\ \mu$m at the largest *Re* accessible. The equation of motion of a string is a linear wave equation. Its solutions with fixed ends are standing waves $A\ \cos(\omega_n t - k_n x)$, where A is the amplitude, t is time and x is position along the wire. The discrete wave numbers are $k_n = n\frac{\pi}{L}$, where L is the length of the string and n is a positive integer. In a first approximation, the waves are not dispersive: $\omega_n = c\, k_n$,

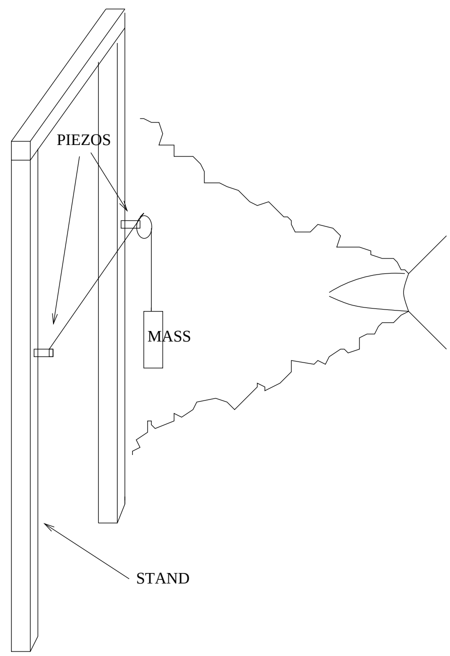

Fig. 1 – Eperimental setup: the thin steel wire is pulled across a turbulent air jet by a 4 Kg weight on a rigid stand. Piezoelectric transducers are in mecanical contact with the wire at each end.

where c is the phase velocity. T is the tension of the string and μ its mass per unit length, $c = \sqrt{T/\mu} \simeq 300$ m/s. With a 4 kg weight on one end, the string's fundamental frequency is $f_0 = 344$ Hz.
Dissipation is mainly due to friction on air, and causes little dispersion. More precise treatment would require terms of dissipation in the wire itself and in the piezoelectric transducers that fix the ends. We neglect this, as the amplitude remains small (a few tens of micrometers) if compared to the length of the ceramic pile (3 mm), or even the wire diameter ($100\,\mu m$). The possible coupling with compression wave is not relevant, as the range of frequency is distinct. (Compression wave speed in steel is a few thousands of m/s, larger than what we consider here: $c \simeq 300$ m/s.) When this wire is immersed into the turbulent flow, the resonant modes are excited by the drag forcing. This device probes drag forcing fluctuations, at those discrete frequencies. The quantities measured are averaged along the wire. They are therefore global in space but local in scale, or more precisely in Fourier-space. The vortices at scale r are expected to excite modes of wave-number $k = 2\pi/r$. In that sense, the string is acting like a mechanical spectrometer, almost exactly like a Fabry-Perot interferometer.

Measurements. – Modulus of the response function is plotted on Fig.2. It shows that the resonance peaks are indeed very narrow: the quality factor is approximately $Q \simeq 4000$, ensuring a very precise selection of wave-numbers. The imaginary part of the response function is giving the dissipation. The width of the peaks in the modulus is also linked to the dissipation, as well as the damping time after a perturbation. We used in the following the measurement of the imaginary part of the response, but checked that these different methods coincide. Only the resonant frequencies are considered in this study.

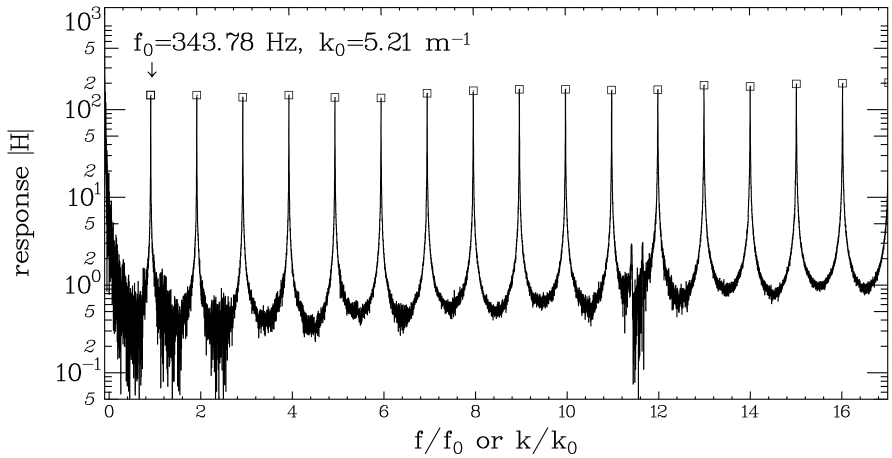

Fig. 2 – Modulus of the response function versus the harmonic number, at $Re = 154000$. The abscissa is given in non-dimensional coordinates, normalised by the fundamental frequency.

Spectrum of the fluctuation excited by the turbulent drag is shown on Fig.3. Fluctuations resonance peaks are clearly identified. Spurious vibrations are visible, mainly caused by the vibrations of the stand. Because the peaks are very thin, long acquisitions are necessary, as well as large windows for the FFT calculations (150000 points), in order to achieve a sufficient resolution (0.33 Hz). The protocol we used to find the resonance frequencies, the value of the amplitude of fluctuations, and imaginary part of the response, is the following. Resonance frequency is obtained by spline smoothing each peak around the maximum amplitude of the response. Then, imaginary part is measured after being also smoothed. The amplitude of the fluctuations peaks are collected on the spectrum, after local smoothing around the maxima.

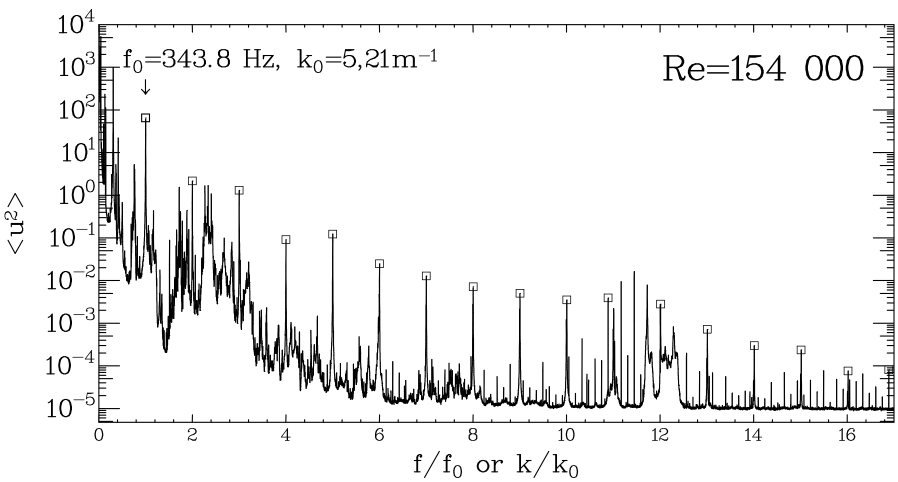

Fig. 3 – Spectrum of the resonance modes of the string excited by turbulent drag fluctuations, at $Re = 154000$.

One can see the FDR in Fig.4, called $k_\mathrm{B}T_{\mathrm{eff.}}$, for several values of Re. Uncertainties on this ratio have multiple origins. Errors indicated by the size of the symbols are those coming from the determination of the resonances. Spurious vibrations of the stand are difficult to handle: we perform measurements of response and fluctuations in the same conditions, to reduce its influence on the ratio. We believe the scattering of the points on Fig.4 comes mainly from the weakening of signal/noise ratio for large frequencies, simply because there is less energy in the flow at large k. The only possible escape on this point is to improve the coupling between the string and the sensors.

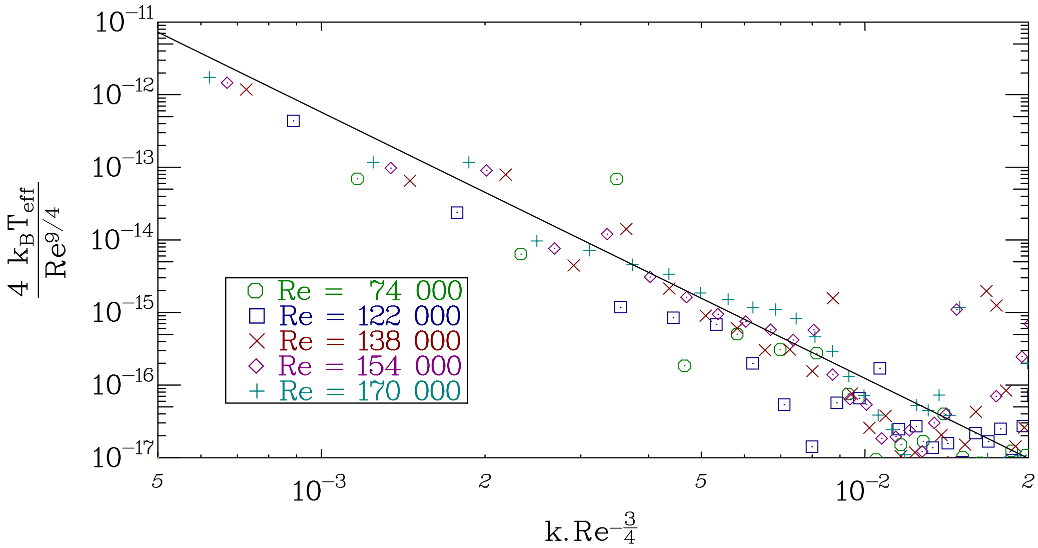

Fig. 4 – Spectrum of the FDR, labelled as *thermal agitation* per degree of freedom. Axis are rescaled with proper Reynolds number dependence, between 74000 and 170000. The size of the symbols represents the uncertainty in the determination of the maxima of the peaks. The solid line is a $k^{-11/3}$ power-law given as an eye guide.

The wave-number has been rescaled with the internal viscous scale $\eta \propto Re^{-3/4}$. The ordinates have been rescaled by an estimated number of degrees of freedom: $(L/\eta)^3 \propto Re^{9/4}$. In other words, the *"thermal energy"* $k_\mathrm{B}T_{\mathrm{eff.}}$ that the FDR is representing in the framework of Cugliandolo *et al.*'s theory, is given per degree of freedom. Assuming the number of degrees of freedom is the total number of particles of size η in the total volume is usual, but crude. A more realistic description should involve correlations between them, reducing this number. However, all the curves collapse to a single power-law with this scaling. The exponent is discussed in the following section.

Please note that the equipartition of energy at equilibrium would require this spectrum to be constant. There is no equilibrium between the Fourier modes, because of the energy flux through scales. Moreover, they are not independent, and probably not Gaussian. There is no reason to expect equipartition.

Scaling law. – Because the susceptibility of the string is very high at resonance, the half-wave-length modes $n\lambda/2$ match with velocity structures of scale r (n is an integer). Therefore, the wave number of the standing wave in the string $k = n\ 2\pi/\lambda$ is the same as $k = 2\pi/r$. The necessary condition for this matching is resonance. It also ensures that velocities of the string

and fluid equalise, which is crucial for the following argument.
Displacement is proportional to the drag forcing, itself proportional to velocity, as drag is viscous: the string diameter-based Reynolds number is small (about 10). The Melde string is not dispersive: $\omega = 2\pi f = ck$, c being the wave velocity. Therefore, the displacement is $x = v/\omega = v/(ck)$, and its power spectrum is: $\langle \tilde{x}(\omega)^2 \rangle = \langle \tilde{v}(\omega)^2 \rangle (ck)^{-2} \propto k^{-11/3}$. Because the viscous dissipation is proportional to frequency, the FDR of Eq.2 is simply proportional to $c\,k\,\langle \tilde{x}(\omega)^2 \rangle \propto k^{-11/3}$. Following Eq. 2, an effective "thermal agitation" defined by the FDR would be: $k_{\mathrm{B}} T_{\mathrm{eff.}} \propto k^{-11/3}$, in the inertial range of fully developed turbulence. This exponent is compatible with the spectrum we measured, as can be seen in Fig. 4.

Discussion. – Theoretical characterisation of turbulence in terms of temperature were proposed in the past by several authors. The temperatures as defined by T. M. Brown [16] and B. Castaing [18] do not depend on k through the inertial range. The qualitative idea is that the cascade transport process is efficient enough to equalise a quantity they call temperature. In an other model invoking an extremum principle, B. Castaing proposed a definition of temperature, which might depend on scale [19]. In any case, none of these theories invoke the FDR. On different basis, R. Robert and J. Sommeria proposed a definition of temperature [20], only valid for 2D turbulence. It is not expected to apply in a 3D flow.
Now, let's consider our experimental results from the perspective of the three points of reflexion we proposed in the first section, in relation with the FDT. 1- Linear response: as we mentioned, the coupling between the string and the flow is purely viscous. Therefore, drag force is proportional to velocity: $f(t) = \gamma\, v(t)$, γ being a friction coefficient. It is also the time-derivative of the position $f(t) = \gamma\, \omega\, x(t)$. Response is linear in x, but the coefficient already depends on frequency. 2- Are fluctuations and dissipation proportional ? As we have seen, the measurements of the FDR are consistent with a $k^{-11/3}$ scaling, it is definitely not constant with respect to k. As our system is out of equilibrium but stationary, there is no time evolution like the relaxation of glasses. 3- Setting a string in a turbulent flow allows to perform measurements on a couple of conjugate force-displacement variables. We have no other variables to compare with, for now.
Now we may ask whether what we measured is actually a temperature, in the dynamical sense. If one assumes that each mode of the string is a harmonic oscillator, and that a harmonic oscillator at equilibrium with a bath gives the temperature of this bath through the FDR, then equilibrium between modes of the string and modes of the flow means the temperature is equal: measurements give the temperature of the flow at this corresponding scale. This interpretation still rely on the assumption that FDR on the oscillator gives the temperature of the oscilaror: this is our working hypothesis. By equilibrium between modes of the string and the flow, we mean a 'no-flux' condition on energy. This is ensured by the high susceptibility of the string at resonance. In other words, the probe and the reservoir are at equilibrium at each k.
We have performed measurements on a turbulent flow, coupling to it a set of harmonic oscillators: a Melde string. At equilibrium with the flow, in the sense that each mode of the string couples with the fluid at scale $r = \pi c/\omega$. It gives informations much like a spectrometer, even though the flow itself is strongly out of equilibrium. This is true, of course, as long as the response of the string is fast enough compared to the frequencies of the velocity fluctuations. The displacement spectra are recorded at different values of Re, as well as the complex response of the string over an excitation (contributions of all the standing waves).
The matching of the string's modes and hydrodynamic structures, what we call equilibrium between the string and the flow, is still a questionable working hypothesis. However, drawing inspiration from Cugliandolo *et al.*'s theory of non-equilibrium temperature based on the

FDR, we measured the Fluctuation over Dissipation Ratio of our string in a turbulent flow, for different values of Re. The FDR, multiplied by an appropriate power of the Reynolds number exhibits a unique power law, when Reynolds number is between 74000 and 170000. The exponent is consistent with a value $-11/3$ given by a very simple model derived from Kolmogorov 1941 theory.

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REFERENCES

[1] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics: Fluid mechanics, Vol. 6*, (Mir, Moscow, 1971),
[2] A.S. Monin and A.M. Yaglom, *Statistical fluid mechanics*, (MIT Press, Cambridge, 1975),
[3] U. Frisch, *Turbulence: the legacy of A. N. Kolmogorov*, (Cambridge Univ. Press., 1995),
[4] L. Richardson, *Weather prediction by numerical process*, (Cambridge Univ. Press, 1922),
[5] A. N. Kolmogorov, C. R. Acad. Sci. U.S.S.R., **30**, (1941), 301,
[6] R. Kubo, *Statistical Physics II: Nonequilibrium Statistical Mechanics Vol. II*, (1985),
[7] S. R. De Groot and P. Mazur, *Non-Equilibrium Thermodynamics*, (1962, reissued 1984)
[8] L. Cugliandolo and J. Kurchan, Phys. Rev. Lett., **71**, (1993), 173-176,
[9] L. Cugliandolo, J. Kurchan, and L. Peliti, Phys. Rev. E, **55**, (1997), 3898,
[10] L. Bellon and S. Ciliberto, Physica D, **168-69**, (2002), 325,
[11] N.S. Grigera and N.E. Israeloff, Phys. Rev. Lett., **83**, (1999), 5038,
[12] L. Buisson, L. Belon and S. Ciliberto, J. Phys.: Condens. Matter, **15**, (2003), S1163,
[13] B. Abou and F. Gallet, Phys. Rev. Lett., **93**, (2004), 160603,
[14] G. D'Anna, P. Mayor, A. Barrat, V. Loreto, and F. Nori, Nature, **424**, (2003), 909-912,
[15] P. Marcq and A. Naert, Phys. of Fluids, **13**, (2001), 9,
[16] T. M. Brown, J. Phys. I, **15**, (1982), 2285,
[17] B. Castaing, Y. Gagne and E. Hopfinger, Physica D, **46**, (1990), 177-200,
[18] B. Castaing, J. Phys. II, **6**, (1996), 105,
[19] B. Castaing, J. Phys. II, **50**, (1989), 147-156, (in French)
[20] J. Sommeria and R. Robert, J. Fluid Mech, **229**, (1991), 291