A Rigorous Time-Domain Analysis of Full–Wave Electromagnetic Cloaking (Invisibility) *†

Ricardo Weder‡
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas
Universidad Nacional Autónoma de México
Apartado Postal 20-726, México DF 01000
weder@servidor.unam.mx

Abstract

There is currently a great deal of interest in the theoretical and practical possibility of cloaking objects from the observation by electromagnetic waves. The basic idea of these invisibility devices [4, 5, 7], [12] is to use anisotropic *transformation media* whose permittivity and permeability $\varepsilon^{\lambda\nu}$, $\mu^{\lambda\nu}$, are obtained from the ones, $\varepsilon_0^{\lambda\nu}$, $\mu_0^{\lambda\nu}$, of isotropic media, by singular transformations of coordinates.
In this paper we study electromagnetic cloaking in the time-domain using the formalism of time-dependent scattering theory [17]. This formalism provides us with a rigorous method to analyze the propagation of electromagnetic wave packets with finite energy in *transformation media*. In particular, it allows us to settle in an unambiguous way the mathematical problems posed by the singularities of the inverse of the permittivity and the permeability of the *transformation media* on the boundary of the cloaked objects. Von Neumann's theory of self-adjoint extensions of symmetric operators plays an important role on this issue. We write Maxwell's equations in Schrödinger form with the electromagnetic propagator playing the role of the Hamiltonian. We prove that every self-adjoint extension of the electromagnetic propagator in a *transformation medium* is the direct sum of a fixed self-adjoint extension in the exterior of the cloaked objects, that is unitarily equivalent to the electromagnetic propagator in the homogeneous medium, with some self-adjoint extension of the electromagnetic propagator in the interior of the cloaked objects.

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Each of these self-adjoint extensions corresponds to a possible unitary time evolution for finite energy electromagnetic waves. As is well known, the fact that time evolution is unitary assures us that energy is conserved. It is also well known that choosing a particular self-adjoint extension of the electromagnetic propagator of the cloaked objects amounts to choosing some boundary condition on the inside of the boundary of the cloaked objects. In other words, any possible unitary dynamics implies the existence of some boundary condition on the inside of the boundary of the cloaked objects. The particular boundary condition that has to be taken will depend on the specific properties of the metamaterials used to build the transformation media as well us on the properties of the media inside the cloaked objects.

Our results mean that the electromagnetic waves inside the cloaked objects are not allowed to leave them, and viceversa, electromagnetic waves outside can not go inside. Furthermore, we prove that the scattering operator is the identity. This implies that for any incoming finite-energy electromagnetic wave packet the outgoing wave packet is precisely the same. In other words, it is not possible to detect the cloaked objects in any scattering experiment where a finite energy wave packet is sent towards the cloaked objects, since the outgoing wave packet that is measured after interaction is the same as the incoming one.

Our results give a rigorous proof of cloaking of passive and active devices from observation with electromagnetic waves. A rigorous prof of cloaking has already been given by [3] where fixed frequency waves were studied, i.e., in the frequency domain, and cloaking, at any frequency, with respect to the measurement of the Cauchy data on a surface that encloses the cloaked object was proven. Our results here give an alternative treatment of this problem in the time-domain.

1 Introduction

There is currently a great deal of interest in the theoretical and practical possibility of cloaking objects from the observation by electromagnetic fields. The basic idea of these invisibility devices [4, 5, 7], [12] is to use anisotropic *transformation media* whose permittivity and permeability, $\varepsilon^{\lambda\nu}, \mu^{\lambda\nu}$, are obtained from the ones, $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$, of isotropic media, by singular transformations of coordinates. The singularities lie on the boundary of the objects to be cloaked. Here the *material interpretation* is taken. Namely, the $\varepsilon^{\lambda\nu}, \mu^{\lambda\nu}$ and the $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$, represent the components in flat Cartesian space of the permittivity and the permeability of physical media with *different material properties*. It appears that with existing technology it is possible to construct media as described above using artificially structured metamaterials. In [4, 5] a proof of cloaking was given for the conductivity equation -i.e., in the case of zero frequency- from detection by measurement of the Dirichlet to Neumann map that relates the value of the electric potential on the boundary to its normal derivative. The papers [7] and [12] consider electromagnetic waves in the geometrical optics approximation, i.e. for large frequencies. In [18] a experimental verification of cloaking is presented and [1] gives a numerical simulation. A rigorous prof of cloaking has already been given by [3] where fixed frequency waves were studied, i.e., in the frequency domain, and cloaking, at any frequency, with respect to the measurement of the Cauchy data on a surface that encloses the cloaked object was proven. We give further comments on this paper below. For other results on this problem see [19] and [9]. In [10] cloaking of elastic waves is considered, and the history of invisibility is discussed.

In this paper we study electromagnetic cloaking in the time-domain using the formalism of time-dependent scattering theory [17]. This formalism provides us with a rigorous method to analyze the propagation of electromagnetic wave packets with finite energy in *transformation media*. In particular, it allows us to settle in an unambiguous way the mathematical problems posed by the singularities of the inverse of the permittivity and the permeability of the *transformation media* on the boundary of the cloaked objects. Von Neumann's theory of self-adjoint extensions of symmetric operators plays an important role on this issue. We write Maxwell's equations in Schrödinger form with the electromagnetic propagator playing the role of the Hamiltonian. We prove that every self-adjoint extension of the electromagnetic propagator in a *transformation medium* is the direct sum of a fixed self-adjoint extension in the exterior of the cloaked objects, that is unitarily equivalent to the electromagnetic propagator in the homogeneous medium, with some self-adjoint extension of the electromagnetic propagator in the interior of the cloaked objects. Each of these self-adjoint extensions corresponds to a possible unitary time evolution for finite energy electromagnetic waves. As is well known, the fact that time evolution is unitary assures us that energy is conserved. It is also well known that choosing a particular self-adjoint extension of the electromagnetic propagator of the cloaked objects amounts to choosing some boundary condition on the inside of the boundary of the cloaked objects. In other words, any possible unitary dynamics implies the existence of some boundary condition on the inside of the boundary of the cloaked objects. The particular boundary condition that has to be taken will depend on the specific properties of the metamaterials used to build the transformation media as well us on the properties of the media inside the cloaked objects.

This results implies that the electromagnetic waves inside and outside of the cloaked objects completely decouple from each other. Actually, electromagnetic waves inside the cloaked objects are not allowed to leave them, and viceversa, electromagnetic waves outside can not go inside. This means, in particular, that the presence of active devices inside the cloaked objects has no effect on the cloaking outside.

Furthermore, we prove that the scattering operator is the identity. In consequence, for any incoming finite-energy electromagnetic wave packet the outgoing wave packet is precisely the same as the incoming one. In other words, it is not possible to detect the cloaked objects in any scattering experiment where a finite energy wave packet is sent towards them, since the outgoing wave packet that is measured after interaction is the same as the incoming one.

Our results give a rigorous proof that the construction of [4, 5, 7], [12] perfectly cloaks passive and active devices from observation by electromagnetic waves. Recall that the fact that the electromagnetic propagator is a direct sum implies that there is always going to be a boundary condition on the inside of the boundary of the cloaked objects. This issue was also discussed in the paper [3], where the fact that for the single coating there has to be boundary conditions on the inside of the boundary of the cloaked objects has already been observed. They call them "hidden boundary conditions". Note that there is no real contradiction between our results and the ones of [3]. Our results imply that there is always a "hidden boundary condition" on the inside of the boundary of the cloaked objects, that is imposed upon us by the fundamental principle of the conservation of the energy of the electromagnetic waves, that implies that time evolution has to be given by a unitary group generated by a self-adjoint extension of the electromagnetic propagator, and this amounts to a boundary condition at the inside of the boundary of the cloaked objects. Note that we do not exclude here the possibility that in some cases the electromagnetic propagator of the cloaked objects could be essentially self-adjoint, and in this situation the dynamics inside the cloaked objects will be uniquely defined. In this case the "hidden boundary condition" will be uniquely determined by the boundary conditions satisfied by the functions in the domain of the unique self-adjoint realization of the electromagnetic propagator in the cloaked objects. Remark that we can also say that there are" hidden boundary conditions" in the outside of the boundary of the cloaked objects, that are determined by the boundary conditions satisfied by the functions in the domain of the unique self-adjoint realization of the electromagnetic propagator outside of the cloaked objects. But, of course, these are unique. We also comment on the relation between the results of [3] and ours after the paragraph below equation (2.55).

Actually, we consider a slightly more general construction than the one of [4, 5, 7], [12] since we allow for a finite number of star-shaped cloaked objects.

In [3] a very general construction for cloaking is introduced. In the case of Maxwell's equations all their constructions are made within the context of the permittivity and the permeability tensor densities being conformal to each other, i.e., multiples of each other by a positive scalar function. In particular, all isotropic media are included in this category. They mention that both for mathematical and practical reasons, it would be very interesting to understand cloaking for general anisotropic materials in the absence of this assumption. In this paper we actually solve this problem, since we prove cloaking for all general anisotropic materials. In particular, our results prove that it is possible to cloak objects inside general crystals.

Note, moreover, that [3] also considers the cases of the Helmholtz equation and Maxwell's equations with an infinite cylinder. We do not discuss these problems here.

Furthermore, remark that the existing theorems in the uniqueness of inverse scattering do not apply under the present conditions.

Finally, let us discuss the relation between the frequency domain approach of [3] and the time-domain method presented here. In the case where the permittivity and the permeability are bounded above and below it is well known that the Cauchy data at a fixed frequency given on a surface that encloses the objects and the scattering matrix at the same frequency are equivalent data. See for example [11] and [20]. This equivalence is, however, not proven in the case where the permittivity and the permeability vanish on the boundary of the objects. In fact, it is perhaps even not true in general, since in this case it is possible that there are finite energy electromagnetic waves that are absorbed by the boundary of the objects as $t \to \pm\infty$. If this is true, the equivalence will not hold since the Cauchy data in a surface that encloses the objects will not contain information on the waves that are asymptotically absorbed by the boundary of the objects. It is a problem of independent interest to see if this actually happens or not. For an example of scattering by a bounded obstacle with a singular boundary and Neumann boundary condition, "the jelly roll", where this happens see [6]. Note that in our time-dependent approach we directly consider the finite-energy electromagnetic wave packets that are used in scattering experiments.

In the analysis of Maxwell's equations with permittivity and permeability that are independent of frequency the dispersion of the medium is not taken into account. This means that cloaking will hold for electromagnetic wave packets with a narrow enough range of frequencies, such that this assumption is valid.

2 Electromagnetic Cloaking

Let us consider Maxwell's equations,

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}, \ \nabla \times \mathbf{H} = \frac{\partial}{\partial t}\mathbf{D}, \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0, \nabla \cdot \mathbf{D} = 0, \tag{2.2}$$

in a domain, $\Omega \subset \mathbb{R}^3$, as follows,

$$\Omega := \mathbb{R}^3 \setminus \cup_{j=1}^N K_j, \ K_j \cap K_l = \emptyset, j \neq l \tag{2.3}$$

where $K_j, j = 1, 2, \cdots, N$, are closed and bounded set, that are the objects to be cloaked. We assume that each K_j is star-shaped with center $\mathbf{c}_j$, i.e.,

$$K_j = \left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \mathbf{c}_j + \rho\, g_j(\hat{\mathbf{z}})\, \hat{\mathbf{z}}, \, 0 \leq \rho \leq 1, \, \hat{\mathbf{z}} \in \mathbb{S}^2\right\}, \tag{2.4}$$

where $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{R}^3$. The $g_j, j = 1, 2, \cdots, N$, are twice continuously differentiable, bounded and positive functions, that are defined on $\mathbb{R}^3$. We suppose that

$$\partial K_j := \left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \mathbf{c}_j + g_j(\hat{\mathbf{z}})\, \hat{\mathbf{z}}, \ \hat{\mathbf{z}} \in \mathbb{S}^2\right\}, \tag{2.5}$$

is a closed C^2 surface that divides $\mathbb{R}^3$ into two components with K_j the bounded one. The cloaked objects are denoted by

$$K := \cup_{j=1}^{N} K_j.$$

We designate the Cartesian coordinates of $\mathbf{x}$ by $x^\lambda, \lambda = 1,2,3$ and by $E_\lambda, H_\lambda, B^\lambda, D^\lambda, \lambda = 1,2,3$, respectively, the components of $\mathbf{E}, \mathbf{H}, \mathbf{B}$, and $\mathbf{D}$. As usual, we denote by $\varepsilon^{\lambda\nu}$ and $\mu^{\lambda\nu}$, respectively, the permittivity and the permeability. We have that,

$$D^\lambda = \varepsilon^{\lambda\nu} E_\nu, \;\; B^\lambda = \mu^{\lambda\nu} H_\nu, \tag{2.6}$$

where we use the standard convention of summing over repeated lower and upper indices.

We consider now a transformation from $\Omega_0 := \mathbb{R}^3 \setminus \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_N\}$ onto Ω that is a generalization of the transformation first used to obtain cloaking for the conductivity equation, i.e. at zero frequency, by [4, 5] and then by [12] for cloaking electromagnetic waves (for a related result in two dimensions using conformal mappings see [7]).

We define,

$$G_{j,\delta} := \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \mathbf{c}_j + \rho\, g_j(\hat{\mathbf{z}})\, \hat{\mathbf{z}},\, 1 \leq \rho \leq \delta,\, \hat{\mathbf{z}} \in \mathbb{S}^2 \right\}. \tag{2.7}$$

Clearly, $G_{j,1} = \partial K_j$.

For any $\mathbf{y} \in \mathbb{R}^3$ we denote, $\hat{\mathbf{y}} := \mathbf{y}/|\mathbf{y}|$. Let $y^\lambda, \lambda = 1,2,3$, designate the cartesian coordinates of $\mathbf{y} \in \Omega_0$. Then, for $0 < |\mathbf{y} - \mathbf{c}_j| \leq \delta - 1$, with $\delta > 1$, we define,

$$\mathbf{x} = \mathbf{x}(\mathbf{y}) = f(\mathbf{y}) := \mathbf{c}_j + (|\mathbf{y} - \mathbf{c}_j| + 1)\, g_j(\widehat{\mathbf{y} - \mathbf{c}_j})\; \widehat{\mathbf{y} - \mathbf{c}_j}. \tag{2.8}$$

Note that this transformation blows up the point $\mathbf{c}_j$ onto ∂K_j and that it sends the punctuated ball $\tilde{B}_{\mathbf{c}_j}(\delta - 1) := \{\mathbf{y} \in \mathbb{R}^3 : 0 < |\mathbf{y} - \mathbf{c}_j| \leq \delta - 1\}$ onto $G_{j,\delta}$. We take δ so close to one that,

$$\tilde{B}_{\mathbf{c}_j}(\delta - 1) \cap \tilde{B}_{\mathbf{c}_l}(\delta - 1) = \emptyset, \;\; G_{j,\delta} \cap G_{l,\delta} = \emptyset\,, j \neq l, 1 \leq j, l \leq N. \tag{2.9}$$

For $\mathbf{y} \in \mathbb{R}^3 \setminus \cup_{j=1}^{N} \tilde{B}_{\mathbf{c}_j}(\delta - 1)$ we define the transformation to be the identity, $\mathbf{x} = \mathbf{x}(\mathbf{y}) = f(\mathbf{y}) := \mathbf{y}$. Our transformation is a bijection from Ω_0 onto Ω. By $\mathbf{y} = \mathbf{y}(\mathbf{x}) := f^{-1}(\mathbf{x})$ we designate the inverse transformation. We denote the elements of the Jacobian matrix by $A^{\lambda}_{\lambda'}$,

$$A^{\lambda}_{\lambda'} := \frac{\partial x^{\lambda}}{\partial y^{\lambda'}}. \tag{2.10}$$

Note that the $A^{\lambda}_{\lambda'} \in C^1\left(\Omega_0 \setminus \cup_{j=1}^N \partial \tilde{B}_{\mathbf{c}_j}(\delta-1)\right)$ and that they have jump discontinuities at $\cup_{j=1}^N \partial \tilde{B}_{\mathbf{c}_j}(\delta-1)$. This, however, will pose no problem for us. We designate by $A^{\lambda'}_{\lambda}$ the elements of the Jacobian of the inverse bijection, $\mathbf{y} = \mathbf{y}(\mathbf{x}) = f^{-1}(\mathbf{x})$,

$$A^{\lambda'}_{\lambda} := \frac{\partial y^{\lambda'}}{\partial x^{\lambda}} \in C^1\left(\Omega \setminus \cup_{j=1}^N \partial G_{j,\delta}\right), \tag{2.11}$$

with jump discontinuities at $\cup_{j=1}^N \partial G_{j,\delta}$. [4, 5] and [12] considered the case where $N=1, \mathbf{c}_1 = 0$ and $g_1 \equiv 1$.

We take here the so called *material interpretation* and we consider our transformation as a bijection between two different spaces, Ω_0 and Ω. However, our transformation can be considered, as well, as a change of coordinates in Ω_0. Of course, these two point of view are mathematically equivalent. This means, in particular, that under our transformation the Maxwell equations in Ω_0 and in Ω will have the same invariance that they have under change of coordinates in three-space. See, for example, [15]. Let us denote by Δ the determinant of the Jacobian matrix (2.10). Then,

$$\Delta := \left(\frac{1+|\mathbf{y}-\mathbf{c}_j|}{|\mathbf{y}-\mathbf{c}_j|}\right)^2 \left(g(\widehat{\mathbf{y}-\mathbf{c}_j})\right)^3, \text{ for } 0 < |\mathbf{y}-\mathbf{c}_j| \leq \delta - 1. \tag{2.12}$$

This result is easily obtained rotating into a coordinate system such that, $\mathbf{y}-\mathbf{c}_j = (|\mathbf{y}-\mathbf{c}_j|, 0, 0)$. For $\mathbf{y} \in \Omega_0 \setminus \cup_{j=1}^N \tilde{B}_{\mathbf{c}_j}(\delta-1), \Delta \equiv 1$.

Let us denote by $\mathbf{E}_0, \mathbf{H}_0, \mathbf{B}_0, \mathbf{D}_0, \varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$, respectively, the electric and magnetic fields, the magnetic induction, the electric displacement, and the permittivity and permeability of Ω_0. The, $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$, are positive, hermitian matrices that are constant in Ω_0.

The electric field is a covariant vector that transforms as,

$$E_{\lambda}(\mathbf{x}) = A^{\lambda'}_{\lambda}(\mathbf{y}) E_{0,\lambda'}(\mathbf{y}). \tag{2.13}$$

The magnetic field $\mathbf{H}$ is a covariant pseudo-vector, but as we only consider space transformations with positive determinant, it also transforms as in (2.13). The magnetic induction $\mathbf{B}$ and the electric displacement $\mathbf{D}$ are contravariant vector densities of weight one that transform as

$$B^{\lambda}(\mathbf{x}) = (\Delta(\mathbf{y}))^{-1} A^{\lambda}_{\lambda'}(\mathbf{y}) B_0^{\lambda'}(\mathbf{y}), \tag{2.14}$$

with the same transformation for $\mathbf{D}$. The permittivity and permeability are contravariant tensor densities of weight one that transform as,

$$\varepsilon^{\lambda\nu}(\mathbf{x}) = (\Delta(\mathbf{y}))^{-1} A^{\lambda}_{\lambda'}(\mathbf{y})\, A^{\nu}_{\nu'}(\mathbf{y})\, \varepsilon_0^{\lambda'\nu'}(\mathbf{y}), \tag{2.15}$$

with the same transformation for $\mu^{\lambda\nu}$. The Maxwell equations (2.1, 2.2) are the same in both spaces Ω and Ω_0. Let us denote by $\varepsilon_{\lambda\nu}, \mu_{\lambda\nu}, \varepsilon_{0\lambda\nu}, \mu_{0\lambda\nu}$, respectively, the inverses of the corresponding permittivity and permeability. They are covariant tensor densities of weight minus one that transform as,

$$\varepsilon_{\lambda\nu}(\mathbf{x}) = \Delta(\mathbf{y}) A^{\lambda'}_{\lambda}(\mathbf{y})\, A^{\nu'}_{\nu}(\mathbf{y})\, \varepsilon_{0\lambda'\nu'}(\mathbf{y}),\ \mu_{\lambda\nu}(\mathbf{x}) = \Delta(\mathbf{y}) A^{\lambda'}_{\lambda}(\mathbf{y})\, A^{\nu'}_{\nu}(\mathbf{y})\, \mu_{0\lambda'\nu'}(\mathbf{y}). \tag{2.16}$$

Note that

$$\det \varepsilon^{\lambda\nu} = \Delta^{-1} \det \varepsilon_0^{\lambda\nu},\ \det \mu^{\lambda\nu} = \Delta^{-1} \det \mu_0^{\lambda\nu}, \tag{2.17}$$

$$\det \varepsilon_{0\lambda\nu} = \Delta \det \varepsilon_{0\lambda\nu},\ \det \mu_{0\lambda\nu} = \Delta \det \mu_{0\lambda\nu}. \tag{2.18}$$

We now introduce the Hilbert spaces of electric and magnetic fields with finite energy. The $\mathbf{E}_0, \mathbf{H}_0, \mathbf{B}_0, \mathbf{D}_0$, were defined in Ω_0, but since $\mathbb{R}^3 \setminus \Omega_0 = \{\mathbf{c}_j\}_{j=1}^N$ is of measure zero, we can consider them as defined in $\mathbb{R}^3$, what we do below.

We denote by $\mathcal{H}_{0E}$ the Hilbert space of all measurable, square integrable, $\mathbf{C}^3-$ valued functions defined on $\mathbb{R}^3$ with the scalar product,

$$\left(\mathbf{E}_0^{(1)}, \mathbf{E}_0^{(2)}\right)_{0E} := \int_{\mathbb{R}^3} E_{0\lambda}^{(1)}\, \varepsilon_0^{\lambda\nu}\, \overline{E_{0\nu}^{(2)}}\, d\mathbf{y}^3. \tag{2.19}$$

We similarly define the Hilbert space, $\mathcal{H}_{0H}$, of all measurable, square integrable, $\mathbf{C}^3-$ valued functions defined on $\mathbb{R}^3$ with the scalar product,

$$\left(\mathbf{H}_0^{(1)}, \mathbf{H}_0^{(2)}\right)_{0H} := \int_{\mathbb{R}^3} H_{0\lambda}^{(1)}\, \mu_0^{\lambda\nu}\, \overline{H_{0\nu}^{(2)}}\, d\mathbf{y}^3. \tag{2.20}$$

The Hilbert space of finite energy fields in $\mathbb{R}^3$ is the direct sum

$$\mathcal{H}_0 := \mathcal{H}_{0E} \oplus \mathcal{H}_{0H}. \tag{2.21}$$

Moreover, we designate by $\mathcal{H}_{\Omega E}$ the Hilbert space of all measurable, $\mathbf{C}^3-$ valued functions defined on Ω that are square integrable with the weight $\varepsilon^{\lambda\nu}$, with the scalar product,

$$\left(\mathbf{E}^{(1)}, \mathbf{E}^{(2)}\right)_{\Omega E} := \int_{\Omega} E_{\lambda}^{(1)}\, \varepsilon^{\lambda\nu}\, \overline{E_{\nu}^{(2)}}\, d\mathbf{x}^3. \tag{2.22}$$

Finally, we denote by $\mathcal{H}_{\Omega H}$ the Hilbert space of all measurable, $\mathbf{C}^3-$ valued functions defined on Ω that are square integrable with the weight $\mu^{\lambda\nu}$, with the scalar product,

$$\left(\mathbf{H}^{(1)}, \mathbf{H}^{(2)}\right)_{\Omega H} := \int_{\Omega} H_{\lambda}^{(1)}\, \mu^{\lambda\nu}\, \overline{H_{\nu}^{(2)}}\, d\mathbf{x}^3. \tag{2.23}$$

The Hilbert space of finite energy fields in Ω is the direct sum

$$\mathcal{H}_\Omega := \mathcal{H}_{\Omega E} \oplus \mathcal{H}_{\Omega H}. \tag{2.24}$$

We now write the Maxwell's equations (2.1) in Schrödinger form. We first consider the case of $\mathbb{R}^3$. We denote by ε_0 and μ_0, respectively, the matrices with entries $\varepsilon_{0\lambda\nu}$ and $\mu_{0\lambda\nu}$. Recall that $(\nabla \times \mathbf{E})^\lambda = s^{\lambda\nu\rho}\left(\frac{\partial}{\partial x_\nu} E_\rho - \frac{\partial}{\partial x_\rho} E_\nu\right)$ where $s^{\lambda\nu\rho}$ is the permutation contravariant pseudo-density of weight -1 (see section 6 of chapter II of [15], where a different notation is used). By a_0 we denote the following formal differential operator,

$$a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} = i \begin{pmatrix} \varepsilon_0 \nabla \times \mathbf{H}_0 \\ -\mu_0 \nabla \times \mathbf{E}_0 \end{pmatrix}. \tag{2.25}$$

Here, as usual, we denote, $\varepsilon_0 \nabla \times \mathbf{H}_0 := \varepsilon_{0\lambda\nu}(\nabla \times \mathbf{H}_0)^\nu$, and $\mu_0 \nabla \times \mathbf{E}_0 = \mu_{0\lambda\nu}(\nabla \times \mathbf{E}_0)^\nu$. Then, equations (2.1) are equivalent to,

$$i \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} = a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}. \tag{2.26}$$

Let us denote by $\mathbf{C}_0^1(\mathbb{R}^3)$ the set of all $\mathbf{C}^6-$valued continuously differentiable functions on $\mathbb{R}^3$ that have compact support. Then, a_0 with domain $\mathbf{C}_0^1(\mathbb{R}^3)$ is a symmetric operator in $\mathcal{H}_0$, i.e., $a_0 \subset a_0^*$. Moreover, it is essentially self-adjoint in $\mathcal{H}_0$, i.e., it has only one self-adjoint extension, that we denote by A_0. Its domain is given by,

$$D(A_0) = \left\{ \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} : a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in \mathcal{H}_0 \right\}, \tag{2.27}$$

and,

$$A_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} = a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}, \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in D(A_0), \tag{2.28}$$

where the derivatives are taken in distribution sense. These results follow easily from the fact that -via the Fourier transform- a_0 is unitarily equivalent to multiplication by a matrix valued function that is symmetric with respect to the scalar product of $\mathcal{H}_0$. Moreover, it follows from explicit computation that the only eigenvalue of A_0 is zero, that it has infinite multiplicity, and that,

$$\mathcal{H}_{0\perp} := (\text{kernel}\, A_0)^{\perp} = \left\{ \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in \mathcal{H}_0 : \frac{\partial}{\partial x_\lambda} \varepsilon_0^{\lambda\nu} E_{0\nu} = 0, \frac{\partial}{\partial x_\lambda} \mu_0^{\lambda\nu} H_{0\nu} = 0 \right\}. \tag{2.29}$$

Furthermore, A_0 has no singular-continuous spectrum and its absolutely-continuous spectrum is $\mathbb{R}$. See, for example, [21, 22].

Taking any,

$$\begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in \mathcal{H}_{0\perp} \cap D(A_0) \tag{2.30}$$

we obtain a finite energy solution to the Maxwell equations (2.1,2.2) as follows

$$\begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}(t) = e^{-itA_0} \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}. \tag{2.31}$$

This is the unique finite energy solution with initial value at $t = 0$ given by (2.30). Note that as $e^{-itA_0}\mathcal{H}_{0\perp} \subset \mathcal{H}_{0\perp}$ equations (2.2) are satisfied for all times if they are satisfied at $t = 0$.

Let us now consider the case of Ω. We denote by ε and μ, respectively, the matrices with entries $\varepsilon_{\lambda\nu}$ and $\mu_{\lambda\nu}$.

We now define the following formal differential operator,

$$a_\Omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = i \begin{pmatrix} \varepsilon \nabla \times \mathbf{H} \\ -\mu \nabla \times \mathbf{E} \end{pmatrix}. \tag{2.32}$$

Equations (2.1) are equivalent to,

$$i \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = a_\Omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}.$$

Let us denote by $\mathbf{C}^1_0(\Omega)$ the set of all $\mathbf{C}^6-$valued continuously differentiable functions on Ω that have compact support. Then, a_Ω with domain $\mathbf{C}^1_0(\Omega)$ is a symmetric operator in $\mathcal{H}_\Omega$. To construct a unitary dynamics that preserves energy we have to analyse the self-adjoint extensions of a_Ω.

We denote by U_E the following unitary operator from $\mathcal{H}_{0E}$ onto $\mathcal{H}_{\Omega E}$,

$$(U_E \mathbf{E}_0)_\lambda (\mathbf{x}) := A_\lambda^{\lambda'} E_{0\lambda'}(\mathbf{y}), \tag{2.33}$$

and by U_H the unitary operator from $\mathcal{H}_{0H}$ onto $\mathcal{H}_{\Omega H}$,

$$(U_H \mathbf{H}_0)_\lambda (\mathbf{x}) := A_\lambda^{\lambda'} H_{0\lambda'}(\mathbf{y}). \tag{2.34}$$

Then,

$$U := U_E \oplus U_H \tag{2.35}$$

is a unitary operator from $\mathcal{H}_0$ onto $\mathcal{H}_\Omega$.

Moreover, U sends $\mathbf{C}^1_0(\Omega_0)$ onto $\mathbf{C}^1_0(\Omega)$, and, furthermore, by the invariance of Maxwell's equations,

$$a_\Omega = U\, a_{00}\, U^*, \tag{2.36}$$

where we denote by a_{00} the restriction of a_0 to $\mathbf{C}^1_0(\Omega_0)$. The operator a_{00} is essentially self-adjoint and its only self-adjoint extension is A_0. This follows from the essential self-adjointness of a_0 and from the fact that any function in $\mathbf{C}^1_0(\mathbb{R}^3)$ can be approximated in the graph norm of a_0 by functions in $\mathbf{C}^1_0(\Omega_0)$. To prove this take any continuously differentiable real-valued function, ϕ, defined on $\mathbb{R}$ such that, $\phi(y) = 0, |y| \leq 1$ and $\phi(y) = 1, |y| \geq 2$. Then, for any

$$\begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in \mathbf{C}_0^1(\mathbb{R}^3),$$

we have that,

$$\prod_{j=1}^{N} \phi(n|\mathbf{y} - \mathbf{c}_j|) \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in \mathbf{C}_0^1(\Omega_0)$$

and moreover,

$$\text{s-}\lim_{n\to\infty} \prod_{j=1}^{N} \phi(n|\mathbf{y} - \mathbf{c}_j|) \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} = \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix},$$

$$\text{s-}\lim_{n\to\infty} a_0 \prod_{j=1}^{N} \phi(n|\mathbf{y} - \mathbf{c}_j|) \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} = a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix},$$

where by s- lim we designate the strong limit in $\mathcal{H}_0$.

Then, as a_{00} is essentially self-adjoint, it follows from (2.36) that a_Ω is essentially self-adjoint, and that its unique self-adjoint extension, that we denote by A_Ω, satisfies

$$A_\Omega = U\, A_0\, U^*. \tag{2.37}$$

Hence, we have proven the following theorem.

THEOREM 2.1. *The operator a_Ω is essentially self-adjoint, and its unique self-adjoint extension, A_Ω, satisfies (2.37).*

The unitary equivalence given by (2.37) implies that A_Ω has the same spectral properties that A_0. Namely, it has no singular-continuous spectrum, the absolutely-continuous spectrum is $\mathbb{R}$ and the only eigenvalue is zero and it has infinite multiplicity. Moreover,

$$\mathcal{H}_{\Omega\perp} := (\text{kernel}\, A_\Omega)^\perp = \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in \mathcal{H}_\Omega : \frac{\partial}{\partial x_\lambda} \varepsilon^{\lambda\nu} E_\nu = 0, \frac{\partial}{\partial x_\lambda} \mu^{\lambda\nu} H_\nu = 0 \right\}. \tag{2.38}$$

Furthermore, taking any

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in \mathcal{H}_{\Omega\perp} \cap D(A_\Omega) \tag{2.39}$$

we obtain a finite energy solution to the Maxwell equations (2.1,2.2) as follows

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(t) = e^{-itA_\Omega} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}. \tag{2.40}$$

This is the unique finite energy solution with initial value at $t = 0$ given by (2.39). Note that as $e^{-itA_\Omega}\mathcal{H}_{\Omega\perp} \subset \mathcal{H}_{\Omega\perp}$ equations (2.2) are satisfied for all times if they are satisfied at $t = 0$. We can consider more general solutions by considering the scale of spaces associated with A_Ω, but we do not go into this direction here.

The facts that a_Ω is essentially self-adjoint and that its unique self-adjoint extension A_Ω is unitarily equivalent to the propagator A_0 of the homogeneous medium are strong statements. They mean that the only possible unitary dynamics in Ω that preserves energy is given by (2.40) and that this dynamics is unitarily equivalent to the free dynamics in $\mathbb{R}^3$ given by (2.31). In fact, $\partial\Omega$ acts like a horizon for electromagnetic waves propagating in Ω in the sense that the dynamics is uniquely defined without any need to consider the cloaked objects $K = \cup_{j=1}^N K_j$. As we will prove below this implies electromagnetic cloaking for all frequencies in the strong sense that the scattering operator is the identity.

Let us now consider the propagation of electromagnetic waves in the cloaked objects. For this purpose we assume that in each K_j the permittivity and the permeability are given by $\varepsilon_j^{\lambda\nu}, \mu_j^{\lambda\nu}$, with inverses $\varepsilon_{j\lambda\nu}, \mu_{j\lambda\nu}$ and where ε_j, μ_j are the matrices with entries $\varepsilon_{j\lambda\nu}, \mu_{j\lambda\nu}$. Furthermore, we assume that $0 < \varepsilon^{\lambda\nu}, \mu^{\lambda\nu} \leq C, \mathbf{x} \in K_j$ and that for any compact set Q contained in the interior of K_j there is a positive constant C_Q such that $\det\varepsilon^{\lambda\nu} > C_Q, \det\mu^{\lambda\nu} > C_Q, \mathbf{x} \in Q$. In other words, we only allow for possible singularities of ε_j, μ_j on the boundary of K_j.

We designate by $\mathcal{H}_{jE}$ the Hilbert space of all measurable, $\mathbf{C}^3-$ valued functions defined on K_j that are square integrable with the weight $\varepsilon_j^{\lambda\nu}$, with the scalar product,

$$\left(\mathbf{E}_j^{(1)}, \mathbf{E}_j^{(2)}\right)_{jE} := \int_{K_j} E_{j\lambda}^{(1)}\, \varepsilon_j^{\lambda\nu}\, \overline{E_{j\nu}^{(2)}}\, d\mathbf{x}^3. \tag{2.41}$$

Similarly, we denote by $\mathcal{H}_{jH}$ the Hilbert space of all measurable, $\mathbf{C}^3-$ valued functions defined on K_j that are square integrable with the weight $\mu_j^{\lambda\nu}$, with the scalar product,

$$\left(\mathbf{H}_j^{(1)}, \mathbf{H}_j^{(2)}\right)_{jH} := \int_{K_j} H_{j\lambda}^{(1)}\, \mu_j^{\lambda\nu}\, \overline{H_{j\nu}^{(2)}}\, d\mathbf{x}^3. \tag{2.42}$$

The Hilbert space of finite energy fields in K_j is the direct sum

$$\mathcal{H}_j := \mathcal{H}_{jE} \oplus \mathcal{H}_{jH}, \tag{2.43}$$

and the Hilbert space in the cloaked objects K is the direct sum,

$$\mathcal{H}_K := \oplus_{j=1}^N \mathcal{H}_j.$$

The complete Hilbert space of finite energy fields including the cloaked objects is,

$$\mathcal{H} := \mathcal{H}_\Omega \oplus \mathcal{H}_K. \tag{2.44}$$

We now write (2.1) as a Schrödinger equation in each K_j as before. We define the following formal differential operator,

$$a_j \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} = i \begin{pmatrix} \varepsilon_j \nabla \times \mathbf{H}_j \\ -\mu_j \nabla \times \mathbf{E}_j \end{pmatrix}. \tag{2.45}$$

Equation (2.1) in K_j is equivalent to,

$$i \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} = a_j \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix}. \tag{2.46}$$

Let us denote by $\mathbf{C}_0^1(\hat{K}_j)$ the set of all $\mathbf{C}^6-$valued continuously differentiable functions on K_j that have compact support in the interior of K_j, that we denote by $\hat{K}_j := K_j \setminus \partial K_j$. Then, a_j with domain $C_0^1(\hat{K}_j)$ is a symmetric operator in $\mathcal{H}_j$. We denote,

$$a := a_\Omega \oplus_{j=1}^N a_j, \tag{2.47}$$

with domain,

$$D(a) := \left\{ \begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^N \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \in \mathbf{C}_0^1(\Omega) \oplus_{j=0}^N \mathbf{C}_0^1(\hat{K}_j) \right\}. \tag{2.48}$$

The operator a is symmetric in $\mathcal{H}$. The possible unitary dynamics that preserve energy for the whole system including the cloaked objects K are given by the self-adjoint extensions of a. Let us denote $\overline{a}$ the closure of a, with similar notation for $a_\Omega, a_j, j = 1, \cdots, N$. Then,

$$\overline{a} = A_\Omega \oplus_{j=1}^N \overline{a_j},$$

where we used the fact that as a_Ω is essentially self-adjoint, $\overline{a_\Omega} = A_\Omega$. The adjoint of a is given by,

$$D(a^*) = \left\{ \begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^{N} \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \in \mathcal{H} : \begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \in D(A_\Omega), a_j \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \in \mathcal{H}_j \right\}, \tag{2.49}$$

and

$$a^* \left(\begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^{N} \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \right) = A_\Omega \begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^{N} a_j \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix}, \tag{2.50}$$

for

$$\begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^{N} \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \in D(a^*). \tag{2.51}$$

Let us denote by $\mathcal{K}_{\Omega\pm} := \text{kernel}(i \mp a_\Omega^*), \mathcal{K}_{j\pm} := \text{kernel}(i \mp a_j^*)$ the deficiency subspaces of a_Ω and $a_j, j = 1, \cdots, N$. Since a_Ω is essentially self-adjoint $\mathcal{K}_{\Omega\pm} = \{0\}$. Let $\mathcal{K}_\pm := \oplus_{j=1}^{N} \mathcal{K}_{j\pm}$ be the deficiency subspaces of $a_K := \oplus_{j=1}^{N} a_j$. Suppose that $\mathcal{K}_\pm$ have the same dimension. Then, it follows from Corollary 1 in page 141 of [16] that there is a one-to-one correspondence between self-adjoint extensions of a_K and unitary maps from $\mathcal{K}_+$ into $\mathcal{K}_-$. If V is such a unitary, then the corresponding self-adjoint extension A_{KV} is given by,

$$D(A_{KV}) = \{\varphi + \varphi_+ + V\varphi_+ : \varphi \in D(\overline{a_K}), \varphi_+ \in \mathcal{K}_+\},$$

and

$$A_K \varphi = \overline{a_K} \varphi + i\varphi_+ - iV\varphi_+.$$

Hence, since $\mathcal{K}_{\Omega\pm} = \{0\}$ and $\overline{a} = A_\Omega \oplus \overline{a_K}$ there is a one-to-one correspondence between self-adjoint extensions of a and unitary maps, V, from $\mathcal{K}_+$ into $\mathcal{K}_-$. The self-adjoint extension A_V corresponding to V is given by,

$$A_V = A_\Omega \oplus A_{KV}.$$

Thus, we have proven the following theorem.

THEOREM 2.2. *Every self-adjoint extension, A, of a is the direct sum of A_Ω and of some self-adjoint extension, A_K of a_K, i.e.,*

$$A = A_\Omega \oplus A_K. \tag{2.52}$$

This theorem tells us that the cloaked objects K and the exterior Ω are completely decoupled and that we are free to choose any boundary condition inside the cloaked objects K that makes a_K self-adjoint without disturbing the cloaking effect in Ω. Boundary conditions that make A_K self-adjoint are well known. See for example, [13], [14], [8] and [2].

It follows from explicit computation that zero is an eigenvalue of every A_K with infinite multiplicity and that,

$$\mathcal{H}_{K\perp} := (\text{kernel}\, A_K)^{\perp} = \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in \mathcal{H}_K : \frac{\partial}{\partial x_\lambda} \varepsilon_K^{\lambda\nu} E_\nu = 0, \frac{\partial}{\partial x_\lambda} \mu_K^{\lambda\nu} H_\nu = 0 \right\}, \tag{2.53}$$

where by $\varepsilon_K^{\lambda\nu}(\mathbf{x}) := \varepsilon_j^{\lambda\nu}(\mathbf{x})$ for $\mathbf{x} \in K_j$, and $\mu_K^{\lambda\nu}(\mathbf{x}) := \mu_j^{\lambda\nu}(\mathbf{x})$ for $\mathbf{x} \in K_j, j = 1, 2, \cdots, N$. It follows that zero is an eigenvalue of A with infinite multiplicity and that,

$$\mathcal{H}_{\perp} := (\text{kernel}\, A)^{\perp} = \mathcal{H}_{\Omega\perp} \oplus \mathcal{H}_{K\perp}. \tag{2.54}$$

For any $\varphi = \varphi_\Omega \oplus \varphi_K \in \mathcal{H}_\perp \cap D(A)$,

$$e^{-itA}\varphi = e^{-itA_\Omega}\, \varphi_\Omega \oplus e^{-itA_K}\, \varphi_K \tag{2.55}$$

is the unique solution of Maxwell's equations (2.1,2.2) with finite energy that is equal to φ at $t = 0$. This shows once again that the dynamics in Ω and in K are completely decoupled. If at $t = 0$ the electromagnetic fields are zero in Ω, they remain equal to zero for all times, and viceversa. Actually, electromagnetic waves inside the cloaked objects are not allowed to leave them, and viceversa, electromagnetic waves outside can not go inside. This implies, in particular, that the presence of active devices inside the cloaked objects has no effect on the cloaking outside.

The fact that for the single coating there has to be boundary conditions on the inside of ∂K has already been observed by [3]. They call them "hidden boundary conditions". In the case of Maxwell's equations they propose two solutions to this issue. One of them is a lining, i.e., a physical material on the boundary of the cloaked objects that enforces a particular boundary condition, for example, they propose a lining by a perfect electric conductor, what is fine for cloaking. Note that this raises now the question of what is the boundary condition between the lining and the cloaking metamaterial. In fact, we face the same problem as before, since we can always consider that the lining is part of the cloaked objects, and then, the question of what is the appropriate boundary condition remains. As we already mentioned, it is our opinion that this is, *a priori*, not known and that it will depend on the specific properties of the cloaking metamaterial and of the cloaked objects or of the lining, if there is any. The second proposal of [3] is a *double coating* that corresponds to surrounding both the inner and the outersurface of the cloaked objects with appropriately matched metamaterials. As our permittivities and permeabilities inside K are allowed to vanish as they approach ∂K the *double coating* fits in our formalism. Our analysis shows that energy conservation implies that the dynamics inside the cloaked objects with the *double coating* is given by some self-adjoint extension of the electromagnetic propagator in K, and that the corresponding boundary conditions can be considered as "hidden", including the particular case where the electromagnetic propagator in K would be essentially self-adjoint, in which case the "hidden boundary conditions" would be unique.

Let χ_Ω be the characteristic function of Ω, i.e., $\chi_\Omega(\mathbf{x}) = 1, \mathbf{x} \in \Omega, \chi_\Omega(\mathbf{x}) = 0, \mathbf{x} \in \mathbb{R}^3 \setminus \Omega$. We define,

$$\left(J \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \right) (\mathbf{x}) := \chi_\Omega(\mathbf{x}) \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} (\mathbf{x}). \tag{2.56}$$

By (2.8, 2.12, 2.15),

$$\left|\varepsilon^{\lambda\nu}(\mathbf{x})\right| \leq C, \quad \left|\mu^{\lambda\nu}(\mathbf{x})\right| \leq C, \quad \mathbf{x} \in \Omega.$$

Then, J is a bounded operator from $\mathcal{H}_0$ into $\mathcal{H}_\Omega$.

The wave operators are defined as follows,

$$W_\pm = \text{s-}\lim_{t \to \pm\infty} e^{itA_\Omega} J e^{-itA_0} P_{0\perp}, \tag{2.57}$$

where $P_{0\perp}$ denotes the projector onto $\mathcal{H}_{0\perp}$.

We denote by I the identity operator on $\mathcal{H}_0$. Then,

LEMMA 2.3.

$$W_\pm = U P_{0\perp}. \tag{2.58}$$

Proof: Denote,

$$W(t) := e^{itA_\Omega}\, J\, e^{-itA_0} P_{0\perp}.$$

By (2.37), for any $\varphi \in \mathcal{H}_0$

$$W(t)\varphi = \psi(t) + U P_{0\perp}\varphi, \tag{2.59}$$

with

$$\psi(t) := U\, e^{itA_0}\ (U^* J - I)\ e^{-itA_0} P_{0\perp}\varphi.$$

Let B_R denote the ball of center zero and radius R in $\mathbb{R}^3$. Since for $|y| \geq R$, with R large enough, our transformation, $\mathbf{x} = f(\mathbf{y})$, is the identity, $\mathbf{x} = \mathbf{y}$, and in consequence, $A^{\lambda}_{\lambda'}(\mathbf{y}) = \delta^{\lambda}_{\lambda'}$ for $|\mathbf{y}| \geq R$, we have that,

$$(U^* J - I) = (U^* J - I)\, \chi_{B_R}.$$

It follows that,

$$\text{s-}\lim_{t\to\pm\infty} \psi(t) = U\, \text{s-}\lim_{t\to\pm\infty} e^{itA_0} \vartheta(t)$$

with,

$$\vartheta(t) := (U^* J - I)\, \chi_{B_R} e^{-itA_0} P_{0\perp}\varphi.$$

We have that,

$$\|\vartheta(t)\|_{\mathcal{H}_0} \leq \left\| J\chi_{B_R} e^{-itA_0} P_{0\perp}\varphi \right\|_{\mathcal{H}} + \left\| \chi_{B_R} e^{-itA_0} P_{0\perp}\varphi \right\|_{\mathcal{H}_0} \leq C \left\| \chi_{B_R} e^{-itA_0} P_{0\perp}\varphi \right\|_{\mathcal{H}_0}. \tag{2.60}$$

Then, as $(A_0 + i)^{-1} P_{0\perp}$ is bounded from $\mathcal{H}_0$ into $W_{1,2}(\mathbb{R}^3)$ [21] [22], it follows from the Rellich local compactness theorem that

$$\chi_{B_R}\, (A_0 + i)^{-1} P_{0\perp}$$

is a compact operator in $\mathcal{H}_0$. Suppose that $\varphi \in D(A_0) \cap \mathcal{H}_{0\perp}$. Then,

$$\text{s-}\lim_{t\to\pm\infty}\chi_{B_R} e^{-itA_0} P_{0\perp}\varphi = \text{s-}\lim_{t\to\pm\infty}\chi_{B_R}(A_0+i)^{-1}P_{0\perp}e^{-itA_0}(A_0+i)\varphi = 0,$$

and whence, by (2.60),

$$\text{s-}\lim_{t\to\pm\infty}\vartheta(t) = 0,$$

and it follows that in this case,

$$\text{s-}\lim_{t\to\pm\infty}\psi(t) = 0. \tag{2.61}$$

By continuity, this is also true for $\varphi \in \mathcal{H}_{0\perp}$.

Then, (2.58) follows from (2.59) and (2.61).

□

The scattering operator is defined as

$$S := W_+^* W_-.$$

COROLLARY 2.4.

$$S = P_{0\perp}. \tag{2.62}$$

Proof: This is immediate from (2.58) because $U^* U = I$.

□

Let us denote by $S_\perp$ the restriction of S to $\mathcal{H}_{0\perp}$. $S_\perp$ is the physically relevant scattering operator that acts in the Hilbert space $\mathcal{H}_{0\perp}$ of finite energy fields that satisfy equations (2.2). We designate by $I_\perp$ the identity operator on $\mathcal{H}_{0\perp}$. We have that,

COROLLARY 2.5.

$$S_\perp = I_\perp. \tag{2.63}$$

Proof: This follows from Corollary 2.4.

□

The fact that $S_\perp$ is the identity operator on $\mathcal{H}_{0\perp}$ means that there is perfect cloaking for all frequencies. Suppose that for very negative times we are given an incoming wave packet

$e^{-itA_0}\varphi_-$, with $\varphi_- \in \mathcal{H}_{0\perp}$. Then, for large positive times the outgoing wave packet is given by $e^{-itA_0}\varphi_+$ with $\varphi_+ = S_\perp \varphi_-$. But, as $S = I$, we have that $\varphi_+ = \varphi_-$ and then,

$$e^{-itA_0}\varphi_- = e^{-itA_0}\varphi_+.$$

Since the incoming and the outgoing wave packets are the same there is no way to detect the cloaked objects K from scattering experiments performed in Ω.

In this paper we considered *transformation media* obtained from a singular transformation that blows up a finite number of points, by simplicity, and since this is the situation in the applications. Suppose that we have a transformation that is singular in a set of points that we call M and denote now $\Omega_0 := \mathbb{R}^3 \setminus M$. What we really used in the proofs is that $W_{1,2}(\Omega_0) = W_{1,2}(\mathbb{R}^3)$. We also assumed that $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$ are constant. What was actually needed is that a_0 is essentially self-adjoint. All our results hold under this more general conditions provided that in (2.57, 2.58) and (2.62) we replace $P_{0\perp}$ by the projector onto the absolutely-continuous subspace of A_0 and that we assume that $D(A_0) \cap \mathcal{H}_{0ac} \subset W_{12}(\mathbb{R}^3)$, where we have denoted the absolutely-continuous subspace of A_0 by $\mathcal{H}_{0ac}$. Moreover, $S_\perp$ has to be defined as the restriction of S to $\mathcal{H}_{0ac}$ and in (2.63) $I_\perp$ has to be the identity operator on $\mathcal{H}_{0ac}$. Note that under these general assumptions A_0 could have non-zero eigenvalues and singular-continuous spectrum. For example, these conditions will be satisfied if the permittivity and the permeability tensor densities $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$ are bounded below and above.

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