

# KINETIC AND FLUID TURBULENT CASCADES IN MAGNETIZED WEAKLY COLLISIONAL ASTROPHYSICAL PLASMAS

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## ABSTRACT

This paper presents a theoretical framework for understanding plasma turbulence in astrophysical plasmas. It is motivated by observations of electromagnetic and density fluctuations in the solar wind, interstellar medium and galaxy clusters, as well as by models of particle heating in accretion disks. All of these plasmas and many others have turbulent motions at weakly collisional and collisionless scales. This paper focuses on turbulence in a strong mean magnetic field (the guide field). The key assumptions behind the theory developed here are that the turbulent fluctuations are anisotropic with respect to the mean field and that their frequency is low compared to the ion cyclotron frequency. The turbulence is assumed to be stirred (forced) at some system-specific outer scale  $L$ . The energy injected at this scale has to be dissipated into heat, which ultimately cannot be accomplished without collisions. A *kinetic cascade* develops that brings the energy to collisional scales both in space and velocity. The nature of the kinetic cascade in various scale ranges depends on the physics of plasma fluctuations that can exist there. There are four special scales that separate physically distinct regimes: the electron gyroscale  $\rho_e$ , the ion gyroscale  $\rho_i$ , the mean free path  $\lambda_{mfp_i}$  and the electron heat diffusion scale  $(m_i/m_e)^{1/2}\lambda_{mfp_i}$  ( $m_e$  and  $m_i$  are electron and ion masses). In each of the scale ranges separated by these scales, a number of physically meaningful and rigorously justifiable simplifications of the fully kinetic plasma description are possible. These are derived systematically via a hierarchy of asymptotic expansions. The result is that, in each scale range, the fully kinetic problem is reduced to a more physically transparent and computationally tractable system of equations, which are derived in a rigorous way. In the “*inertial range*” above the ion gyroscale, the kinetic cascade separates into two parts: a cascade of Alfvénic fluctuations and a passive cascade of density and magnetic-field-strength fluctuations. The former are governed by two fluid-like Reduced Magnetohydrodynamic (RMHD) equations at both the collisional and collisionless scales; the latter obey a linear kinetic equation along the (moving) field lines associated with the Alfvénic component (in the collisional limit, these passive fluctuations become the slow and entropy modes of the conventional MHD). In the “*dissipation range*” between the ion and electron gyroscopes, there are again two cascades: the kinetic-Alfvén-wave (KAW) cascade governed by two fluid-like Electron Reduced Magnetohydrodynamic (ERMHD) equations and a passive cascade of ion entropy fluctuations both in space and velocity. The latter cascade brings the energy of the inertial-range fluctuations that was damped by collisionless wave-particle interaction at the ion gyroscale to collisional scales and leads to ion heating. The KAW energy is similarly damped at the electron gyroscale and is converted into electron heat. The relationship between the theoretical models proposed in this paper and astrophysical applications and observations is discussed in detail.

*Subject headings:* magnetic fields—methods: analytical—MHD—plasmas—turbulence

## 1. INTRODUCTION

As observations of velocity, density and magnetic fields in astrophysical plasmas probe ever smaller scales, turbulence—i.e., broad-band disordered fluctuations usually characterized by power-law energy spectra—is revealed as a fundamental and ubiquitous feature. A property that has recurred in these measurements so frequently that it is sometimes (erroneously) thought of as synonymous with turbulence is the  $k^{-5/3}$  energy spectrum, known as the Kolmogorov spectrum.

One of the earliest appearances of this spectrum in the astrophysical context was its detection for the magnetic fluctuations in the solar wind over a frequency range of about two decades (Matthaeus & Goldstein 1982; Bavassano et al. 1982).

A multitude of subsequent observations have confirmed this behaviour of the magnetic fluctuations to a high degree of accuracy (e.g., Horbury et al. 1996; Leamon et al. 1998; Narita et al. 2006). It has also been found that the spectrum of the fluctuations of the electric field (and, by implication, of the  $\mathbf{E} \times \mathbf{B}$  velocity field) follows very closely the spectrum of magnetic fluctuations (Bale et al. 2005).

The Kolmogorov power law appears also to hold for the electron density spectrum in the interstellar medium (ISM)—in this case it emerges from observations by various methods and famously extends over as many as 10 decades of scales (Armstrong et al. 1995; Lazio et al. 2004), a record that has earned it the name of “the Great Power Law in the Sky.” Magnetic fluctuations observed in the ISM appear to have a Kolmogorov spectrum as well, similar to that of the density fluctuations (Minter & Spangler 1996; Haverkorn et al. 2004). In galaxy clusters, analysis of pressure maps reveals a spectrum that may be consistent with Kolmogorov’s, though still suffers from poor resolution (Schuecker et al. 2004); magnetic fluctuations in cluster cores were also recently discovered to have

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a spectrum consistent with  $k^{-5/3}$  (Vogt & Enßlin 2005).

### 1.1. Kolmogorov and Goldreich–Sridhar Turbulence

It is perhaps useful to remind the reader of the argument that leads to the  $k^{-5/3}$  spectrum (Kolmogorov 1941). Suppose the average energy per unit time per unit volume that the system dissipates is  $\varepsilon$ . This energy has to be transferred from some (large) *outer scale*  $L$  at which it is injected to some (small) *inner scale(s)* at which dissipation occurs (see § 1.3). It is assumed that in the range of scales intermediate between the outer and the inner, the statistical properties of the turbulence are spatially homogeneous and isotropic and the energy transfer is local in scale space. If we are dealing with a neutral incompressible fluid, its properties are fully described by the velocity field. The flux of kinetic energy through scale  $\lambda$  is independent of  $\lambda$ :

$$\frac{u_\lambda^2}{\tau_\lambda} \sim \varepsilon = \text{const}, \quad (1)$$

where the (constant) density of the medium is absorbed into  $\varepsilon$ ,  $u_\lambda$  is the typical velocity fluctuation associated with the scale  $\lambda$ , and  $\tau_\lambda$  is the cascade time. (this is the version of Kolmogorov’s dimensional theory due to Obukhov 1941). It is dimensionally inevitable that if  $\tau_\lambda$  is to be expressed in terms of quantities associated with scale  $\lambda$ , it has to be  $\tau_\lambda \sim \lambda/u_\lambda$  (the nonlinear interaction time, or turnover time), so we get

$$u_\lambda \sim (\varepsilon \lambda)^{1/3}. \quad (2)$$

This corresponds to a  $k^{-5/3}$  spectrum of kinetic energy.

That astronomical data appears to point to a universal nature of what, in its origin, is a dimensional result for a turbulent energy cascade in a neutral fluid, might appear surprising. Indeed, the astrophysical plasmas in question are highly conducting and support magnetic fields whose energy is at least comparable to the kinetic energy of the motions. In the presence of magnetic field there is no dimensionally unique way of determining the cascade time  $\tau_\lambda$  because besides the nonlinear interaction time  $\lambda/u_\lambda$ , there is a second characteristic time associated with the fluctuation of size  $\lambda$ , namely the Alfvén time  $l_{\parallel\lambda}/v_A$ , where  $v_A$  is the Alfvén speed and  $l_{\parallel\lambda}$  is the typical scale of the fluctuation along the magnetic field.

Until the mid-1990s, the accepted view of this magnetohydrodynamic (MHD) turbulence was the theory of Iroshnikov (1963) and Kraichnan (1965), who calculated  $\tau_\lambda$  by assuming an isotropic cascade of weakly interacting Alfvén-wave packets and obtained a  $k^{-3/2}$  spectrum. The failure of the observed spectra to conform to this law was a source of some puzzlement.

The observed  $k^{-5/3}$  spectra for cosmic plasmas only started to make sense when Goldreich & Sridhar (1995, henceforth, GS) removed the isotropy assumption from the theory of Alfvén-wave turbulence and conjectured instead that, in the presence of a dynamically strong background magnetic field (the mean, or guide, field),<sup>7</sup> (a) all electromagnetic perturbations are strongly anisotropic, so that the characteristic wave numbers along the mean field are much smaller than those across it,  $k_{\parallel} \ll k_{\perp}$ , or  $l_{\parallel\lambda} \gg \lambda$ ; (b) the interactions between the Alfvén-wave packets are strong and the turbulence at sufficiently small scales always arranges itself in such a way that the Alfvén time scale and the perpendicular nonlinear interaction time scale are comparable to each other, i.e.,

$$\omega \sim k_{\parallel} v_A \sim k_{\perp} u_{\perp}, \quad (3)$$

<sup>7</sup> See § 8.2.3 for a brief discussion of the no-mean-field case.

where  $\omega$  is the typical frequency of the fluctuations and  $u_{\perp}$  is the velocity fluctuation perpendicular to the mean field. Physically, Eq. (3) means that two planes perpendicular to the mean field are correlated if an Alfvén wave can propagate between them in one perpendicular decorrelation time. Taken scale by scale, this assumption, known as the *critical balance*, gives  $l_{\parallel\lambda}/v_A \sim \lambda/u_{\perp\lambda}$  and the dimensional ambiguity of the MHD turbulence theory is removed. It is reasonable to take the cascade time to be the same as the Alfvén time,  $\tau_\lambda \sim l_{\parallel\lambda}/v_A$ , whence

$$u_\lambda \sim (\varepsilon l_{\parallel\lambda}/v_A)^{1/2} \sim (\varepsilon \lambda)^{1/3}, \quad l_{\parallel\lambda} \sim l_0^{1/3} \lambda^{2/3}, \quad (4)$$

where  $l_0 = v_A^3/\varepsilon$ . The first of the scaling relations (4) is equivalent to a  $k_{\perp}^{-5/3}$  spectrum of kinetic energy, the second quantifies the anisotropy by establishing the relation between the perpendicular and parallel scales. The fluctuations are Alfvénic, so the typical magnetic field fluctuation is  $\delta B_{\perp\lambda} \sim u_{\perp\lambda} \sqrt{4\pi\rho_0}$ , where  $\rho_0$  is the mean density. Other MHD modes—slow waves and entropy fluctuations—turn out to be passively advected by the Alfvénic component of the turbulence (this follows from the anisotropy; see Lithwick & Goldreich 2001, and § 2.3, § 2.6, § 5.5, and § 6.3 for further discussion of the passive component of MHD turbulence).

Along with the  $k^{-5/3}$  scaling of the spectra, the anisotropy of astrophysical plasma turbulence is also an observed phenomenon. It is seen most clearly in the case of the solar wind (Belcher & Davis 1971; Matthaeus et al. 1990; Bieber et al. 1996; Dasso et al. 2005; Bigazzi et al. 2006; Sorriso-Valvo et al. 2006; Horbury et al. 2005, 2007; Osman & Horbury 2007)—in particular, a recent solar-wind data analysis by Horbury et al. (2007) approaches quantitative corroboration of the GS theory by confirming the scaling of the spectrum with the parallel wave number  $\sim k_{\parallel}^{-2}$  [this follows from the first scaling relation in Eq. (4)]. Anisotropy is also observed indirectly in the ISM (Wilkinson et al. 1994; Trotter et al. 1998; Rickett et al. 2002; Dennett-Thorpe & de Bruyn 2003) and, with unambiguous consistency, in numerical simulations of MHD turbulence (Shebalin et al. 1983; Cho & Vishniac 2000; Maron & Goldreich 2001; Cho et al. 2002; Müller et al. 2003).<sup>8</sup>

### 1.2. Kinetic Turbulence

Thus, GS have to some extent reconciled theory with data and made sense of the MHD turbulence in highly-conducting

<sup>8</sup> The numerical evidence is much less clear on the scaling of the spectrum. The fact that the spectrum appears closer to  $k_{\perp}^{-3/2}$  than to  $k_{\perp}^{-5/3}$  (Maron & Goldreich 2001; Müller et al. 2003) has prompted Boldyrev (2006) to propose a scaling argument that allows an anisotropic Alfvénic turbulence with a  $k_{\perp}^{-3/2}$  spectrum. His argument is based on the conjecture that the fluctuating velocity and magnetic fields tend to partially align at small scales, an idea that has had some numerical support (Maron & Goldreich 2001; Beresnyak & Lazarian 2006; Mason et al. 2006). The alignment weakens nonlinear interactions and alters the scalings. However, the basic assumption that the turbulence is strong is retained. This is precisely the assumption that we make in this paper: the critical balance conjecture (3) is used below not as a scaling prescription but in a weaker sense of an ordering assumption, i.e., we simply take the wave propagation terms to be comparable to the nonlinear terms in the equations. It is not hard to show that the results derived in what follows remain valid whether or not the alignment is present. We note that observationally, only in the solar wind does one measure the spectra with sufficient accuracy to state that they are consistent with  $k_{\perp}^{-5/3}$  but *not* with  $k_{\perp}^{-3/2}$  (see § 8.1.1).

cosmic plasmas exhibiting the same statistical scaling as turbulence in a viscous neutral fluid (although the underlying dynamics are very different in these two cases). However, there is an aspect of the observed astrophysical turbulence that undermines the applicability of any type of fluid description: in most cases, the inertial range where the Kolmogorov scaling holds extends to scales far below the mean free path deep into the collisionless regime. For example, in the case of the solar wind, the mean free path is close to 1 AU, so all scales are collisionless—an extreme case, which also happens to be the best studied, thanks to the *in situ* measurements (see § 8.1).

The proper way of treating such plasmas is using kinetic theory, not fluid equations. The basis for the application of the MHD fluid description to them has been the following well known result from the linear theory of plasma waves: while the fast, slow and entropy modes are damped at the mean-free-path scale both by collisional viscosity (Braginskii 1965) and by collisionless wave-particle interactions (Barnes 1966), the Alfvén waves are only damped at the ion gyroscale. It has, therefore, been assumed that the MHD description, inasmuch as it concerns the Alfvén-wave cascade, can be extended to the ion gyroscale, with the understanding that this cascade is decoupled from the damped cascades of the rest of the MHD modes. This approach and its application to the turbulence in the ISM are best explained by Lithwick & Goldreich (2001).

While the fluid description may be sufficient to understand the Alfvénic fluctuations in the inertial range, it is certainly inadequate for everything else. The fundamental challenge that a comprehensive theory of astrophysical plasma turbulence must meet is to give the full account of how the turbulent fluctuation energy injected at the outer scale is cascaded to small scales and deposited into particle heat. We shall see (§ 3.4 and § 3.5) that the familiar concept of an energy cascade can be generalized in the kinetic framework as the *kinetic cascade* of a single quantity that we call the generalized energy. The small scales developed in the process are small scales both in the position and velocity space. The fundamental reason for this is the low collisionality of the plasma: since heating cannot ultimately be accomplished without collisions, large gradients in phase space are necessary for the collisions to be effective.

In order to understand the physics of the kinetic cascade in various scale ranges, we derive in what follows a hierarchy of simplified, yet rigorous kinetic, fluid and hybrid descriptions. While the full kinetic theory of turbulence is very difficult to handle either analytically or numerically, the models we derive are much more tractable, yet rigorous. For all, the regimes of applicability (scale/parameter ranges, underlying assumptions) are clearly articulated. In each of these regimes, the kinetic cascade splits into several channels of energy transfer, some of them familiar (e.g., the Alfvénic cascade, § 5.3 and § 5.4), others conceptually new (e.g., the entropy cascade, §§ 7.4–7.6).

In order to introduce this theoretical framework in a way that is both analytically systematic and physically intelligible, let us first consider the characteristic scales that are relevant to the problem of astrophysical turbulence (§ 1.3). The models we derive are previewed in § 1.4, at the end of which the plan of further developments is given.

### 1.3. Scales in the Problem

*Outer scale.*— It is a generic feature of turbulent systems that energy is injected via some large-scale mechanism: “large scale” in this context means some scale (or

TABLE 1  
REPRESENTATIVE PARAMETERS FOR ASTROPHYSICAL PLASMAS.

Parameter	Solar wind at 1 AU <sup>(a)</sup>	Warm ionized ISM <sup>(b)</sup>	Accretion flow near Sgr A* <sup>(c)</sup>	Galaxy clusters (cores) <sup>(d)</sup>
$n_e = n_i, \text{ cm}^{-3}$	30	0.5	$10^6$	$6 \times 10^{-2}$
$T_e, \text{ K}$	$\sim 10^{5(e)}$	8000	$10^{11}$	$3 \times 10^7$
$T_i, \text{ K}$	$5 \times 10^5$	8000	$\sim 10^{12(f)}$	$?(e)$
$B, \text{ G}$	$10^{-4}$	$10^{-6}$	30	$7 \times 10^{-6}$
$\beta_i$	5	14	4	9
$v_{thi}, \text{ km/s}$	90	10	$10^5$	700
$v_A, \text{ km/s}$	40	3	$7 \times 10^4$	60
$U, \text{ km/s}^{(f)}$	$\sim 10$	$\sim 10$	$\sim 10^4$	$\sim 10^2$
$L, \text{ km}^{(f)}$	$\sim 10^5$	$\sim 10^{15}$	$\sim 10^8$	$\sim 10^{17}$
$(m_i/m_e)^{1/2} \lambda_{mfp i}, \text{ km}$	$10^{10}$	$2 \times 10^8$	$4 \times 10^{10}$	$4 \times 10^{16}$
$\lambda_{mfp i}, \text{ km}^{(g)}$	$3 \times 10^8$	$6 \times 10^6$	$10^9$	$10^{15}$
$\rho_i, \text{ km}$	90	1000	0.4	$10^5$
$\rho_e, \text{ km}$	2	30	0.003	200

<sup>a</sup> Data for slow wind ( $V_{sw} = 350 \text{ km/s}$ ) measured by Cluster spacecraft and taken from Bale et al. (2005), except the value of  $T_e$ , which they do not report, but which is expected to be close to  $T_i$  (Newbury et al. 1998). See Bruno & Carbone (2005) for fast-wind parameters and another set of numbers for the slow wind as measured by Helios 2. See discussion in § 8.1.

<sup>b</sup> Typical values (see, e.g., Norman & Ferrara 1996; Ferrière 2001). See discussion in § 8.2.

<sup>c</sup> Data based on observational constraints for the radio-emitting plasma around the Galactic Center (Sgr A\*) as interpreted by Loeb & Waxman (2007) (see also Quataert 2003). See discussion in § 8.3.

<sup>d</sup> Data for the core region of the Hydra A cluster taken from Enßlin & Vogt (2006); see Schekochihin & Cowley 2006a for a consistent set of numbers for the hot plasmas outside the cores. See discussion in § 8.4.

<sup>e</sup> We assume  $T_i = T_e$  for these estimates.

<sup>f</sup> Rough order-of-magnitude estimate.

<sup>g</sup> Defined  $\lambda_{mfp i} = v_{thi}/\nu_{ii}$ , where  $\nu_{ii}$  is given by Eq. (44).

a range of scales) comparable with the size of the system, depending on its global properties, and much larger than the microphysical scales at which energy can be dissipated and converted into heat. Examples of large-scale stirring of turbulent fluctuations include the solar activity in the corona (launching Alfvén waves to produce turbulence in the solar wind); supernova explosions in the ISM (e.g., Norman & Ferrara 1996; Ferrière 2001); the magnetorotational instability in accretion disks (Balbus & Hawley 1998); merger events, galaxy wakes and active galactic nuclei in galaxy clusters (e.g., Subramanian et al. 2006; Enßlin & Vogt 2006; Chandran 2005a). Since in this paper we are concerned with the local properties of astrophysical plasmas, let us simply assume that energy injection occurs at some characteristic *outer scale*  $L$ . All further considerations will apply to scales that are much smaller than  $L$  and we will assume that the particular character of the energy injection does not matter at these small scales.

In what follows, we assume that a strong uniform mean field is present. Strictly speaking, this model choice is only applicable at scales sufficiently smaller than  $L$ : at the outer scale, magnetic field lines are bent and we would have to discuss the ways in which such a regular field can be created in various astrophysical conditions—a highly nontrivial issue that is beyond the scope of this paper (see also the discussion in § 8.2.3). In a similar vein, we neglect the role of any outer-



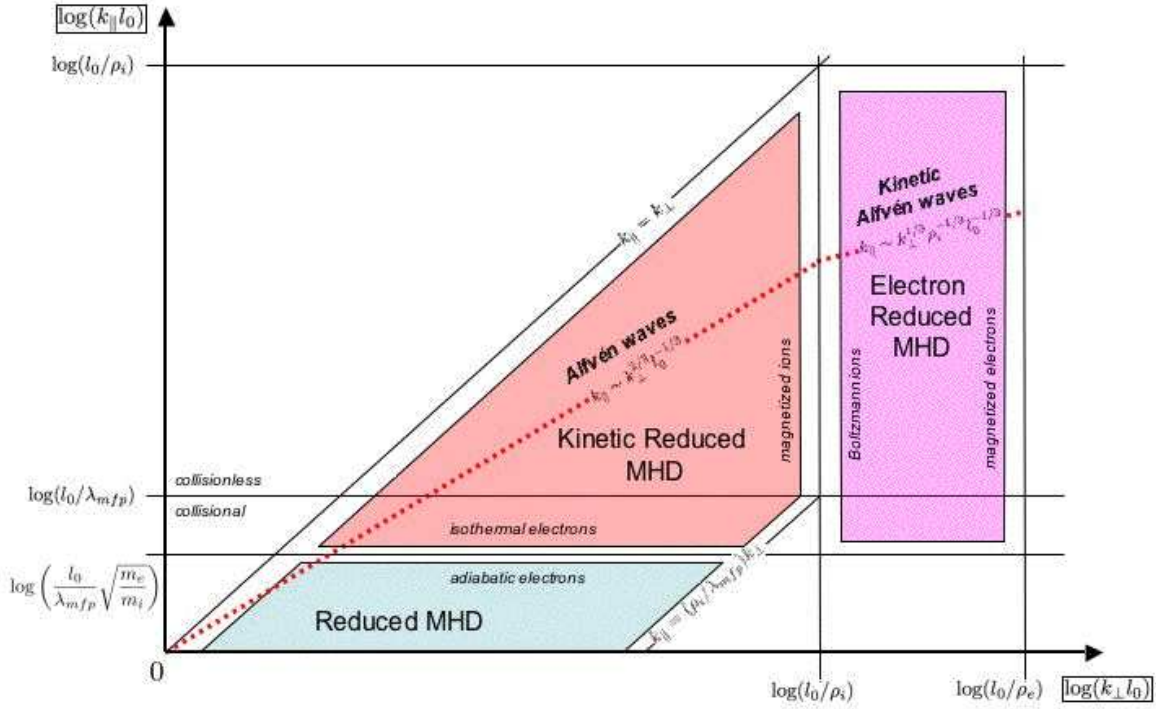


FIG. 1.— Partition of the wave-number space by characteristic scales. The wave numbers are normalized by  $l_0 \sim \nu_A^3/\varepsilon$ , where  $\varepsilon$  is the total power input (see § 1.1). Dotted line shows the path an Alfvén-wave cascade stirred at the outer scale  $L \sim l_0$  takes through the wave-number space. We also show the regions of validity of the three tertiary approximations. They all require  $k_{\parallel} \ll k_{\perp}$  (anisotropic fluctuations) and  $k_{\parallel} \rho_i \sim k_{\parallel} v_{thi}/\Omega_i \ll 1$  (low-frequency limit). Reduced MHD (RMHD, § 2) is valid when  $k_{\perp} \rho_i \ll k_{\parallel} \lambda_{mfp} \ll (m_e/m_i)^{1/2}$  (strongly magnetized collisional limit, adiabatic electrons). The regions of validity of Kinetic Reduced MHD (KRMHD, § 5) and Electron Reduced MHD (ERMHD, § 7) lie within that of the isothermal electron/gyrokinetic ion approximation (Fig. 3) with the additional requirement that  $k_{\perp} \rho_i \ll \min(1, k_{\parallel} \lambda_{mfp})$  (strongly magnetized ions) for KRMHD or  $k_{\perp} \rho_i \gg 1$  (unmagnetized ions) for ERMHD. The collisional limit of KRMHD (§ 6.1 and Appendix D),  $(m_e/m_i)^{1/2} \ll k_{\parallel} \lambda_{mfp} \ll 1$ , is similar to RMHD, except electrons are isothermal. The dotted line is the scaling of  $k_{\parallel}$  vs  $k_{\perp}$  from critical balance in both the Alfvén-wave [§ 1.1, Eq. (4)] and kinetic-Alfvén-wave [§ 7.2, Eq. (209)] regimes.

scale gradients of density or temperature that may exist in the astrophysical plasmas that we are studying.

There are four scales at which dissipation occurs and which, therefore, mark the transitions between distinct physical regimes:

*Electron diffusion scale.* — At  $k_{\parallel} \lambda_{mfp} (m_i/m_e)^{1/2} \gg 1$ , the electron response is isothermal (§ 4.4, Appendix A.4). At  $k_{\parallel} \lambda_{mfp} (m_i/m_e)^{1/2} \ll 1$ , it is adiabatic (§ 4.8.4, Appendix A.3).

*Mean free path.* — At  $k_{\parallel} \lambda_{mfp} \gg 1$ , the plasma is collisionless. In this regime, wave-particle interactions can dissipate some of the turbulent fluctuations via Barnes damping (§ 6.2). At  $k_{\parallel} \lambda_{mfp} \ll 1$ , the plasma is collisional and fluid-like (§ 6.1, Appendices A and D).

*Ion gyroscale.* — At  $k_{\perp} \rho_i \gg 1$ , the ions are unmagnetized and have a Boltzmann response (§ 7.1). At  $k_{\perp} \rho_i \sim 1$ , they are heated by wave-particle interactions (via a kinetic ion-entropy cascade, see §§ 7.4-7.5). At  $k_{\perp} \rho_i \ll 1$ , ions (as well as the electrons) are magnetized and the magnetic field is frozen into the ion flow (the  $\mathbf{E} \times \mathbf{B}$  velocity field). Note that the ion inertial scale  $d_i = \rho_i \beta_i^{-1/2}$  is comparable to the ion gyroscale unless the plasma beta  $\beta_i = 8\pi n_i T_i / B^2$  is very different from unity.

*Electron gyroscale.* — At  $k_{\perp} \rho_e \sim 1$ , the electrons are heated by wave-particle interactions (via a kinetic electron-entropy cascade, see § 7.6). At  $k_{\perp} \rho_e \ll 1$ , electrons are magnetized and the magnetic field is frozen into the electron flow (§ 4,

§ 7, Appendix C).

Typical values of these scales and of several other key parameters are given in Table 1. In Fig. 1, we show how the wave-number space,  $(k_{\perp}, k_{\parallel})$ , is divided by these scales into several domains, where the physics is different. Further partitioning of the wave-number space results from comparing  $k_{\perp} \rho_i$  and  $k_{\parallel} \lambda_{mfp}$  ( $k_{\perp} \rho_i \ll k_{\parallel} \lambda_{mfp}$  is the limit of strong magnetization, see Appendix A.2) and, most importantly, from comparing parallel and perpendicular wave numbers. As we explained above, observational and numerical evidence tells us that Alfvénic turbulence is anisotropic,  $k_{\parallel} \ll k_{\perp}$ . In Fig. 1, we sketch the path the turbulent cascade is expected to take in the wave-number space (we use the scalings of  $k_{\parallel}$  with  $k_{\perp}$  that follow from the GS argument for the Alfvén waves and an analogous argument for the kinetic Alfvén waves, reviewed in § 1.1 and § 7.2, respectively).

#### 1.4. Kinetic and Fluid Models

What is the correct analytical description of the turbulent plasma along the (presumed) path of the cascade? As we promised above, it is going to be possible to simplify the full kinetic theory substantially. These simplifications can be obtained in the form of a hierarchy of approximations.

*Gyrokinetics* (§ 3). — The starting point for these developments and the primary approximation in the hierarchy is *gyrokinetics*, a low-frequency kinetic approximation achieved by averaging over the cyclotron motion of the particles. Gyrokinetics is appropriate for the study of subsonic plasma

turbulence in virtually all astrophysically relevant parameter ranges (Howes et al. 2006). For fluctuations at frequencies lower than the ion cyclotron frequency,  $\omega \ll \Omega_i$ , gyrokinetics can be systematically derived by making use of the two assumptions that underpin the GS theory: (a) anisotropy of the turbulence, so  $\epsilon \sim k_{\parallel}/k_{\perp}$  is used as the small parameter, and (b) strong interactions, i.e., the fluctuation amplitudes are assumed to be such that wave propagation and nonlinear interaction occur on comparable time scales: from Eq. (3),  $u_{\perp}/v_A \sim \epsilon$ . The first of these assumptions implies that fluctuations at Alfvénic frequencies satisfy  $\omega \sim k_{\parallel}v_A \ll \Omega_i$ . This makes gyrokinetics an ideal tool both for analytical theory and for numerical studies of astrophysical plasma turbulence—the numerical approaches are also made attractive by the long experience of gyrokinetic simulations accumulated in the fusion research and by the existence of publicly available gyrokinetic codes (Kotschenreuther et al. 1995; Jenko et al. 2000; Candy & Waltz 2003; Chen & Parker 2003). A concise review of gyrokinetics is provided in § 3. The reader is urged to pay particular attention to § 3.4 and § 3.5, where the concept of the *kinetic cascade* of generalized energy is introduced and the particle heating in gyrokinetics is discussed. The region of validity of gyrokinetics is illustrated in Fig. 2: it covers virtually the entire path of the turbulent cascade, except possibly the largest scales. Note that the two-fluid theory, which is the starting point for the MHD model (see Appendix A), is not an appropriate description at collisionless scales.

*Isothermal Electron Fluid (§ 4).*— While gyrokinetics constitutes a significant simplification, it is still a fully kinetic description. Further progress towards simpler models is achieved by showing that, for parallel scales below the electron diffusion scale,  $k_{\parallel}\lambda_{\text{mfpi}} \gg (m_e/m_i)^{1/2}$ , and perpendicular scales above the electron gyroscale,  $k_{\perp}\rho_e \ll 1$ , the electrons are a magnetized isothermal fluid while ions must be treated kinetically. This is the secondary approximation in our hierarchy, derived in § 4 via an asymptotic expansion in  $(m_e/m_i)^{1/2}$  (see also Appendix C.1). The plasma is described by the ion gyrokinetic equation and two fluid-like equations that contain electron dynamics—these are summarized in § 4.9. The region of validity of this approximation is illustrated in Fig. 3: it does not capture the dissipative effects around the electron diffusion scale or the electron heating, but it remains uniformly valid as the cascade passes from collisional to collisionless scales and also as it crosses the ion gyroscale.

In order to elucidate the nature of the turbulence above and below the ion gyroscale, we derive two tertiary approximations, valid for  $k_{\perp}\rho_i \ll 1$  (§ 5) and for  $k_{\perp}\rho_i \gg 1$  (§ 7; see also Appendix C, which gives a nonrigorous, nongyrokinetic, but perhaps more intuitive, derivation of the results of § 4 and § 7).

*Kinetic Reduced MHD (§ 5 and § 6).*— On scales above the ion gyroscale, known as the “*inertial range*” we demonstrate that the decoupling of the Alfvén-wave cascade and its indifference to both collisional and collisionless damping are explicit and analytically provable properties. We show rigorously the Alfvén-wave cascade is governed by a closed set of two fluid-like equations for the stream and flux functions—the Reduced Magnetohydrodynamics (RMHD)—independently of the collisionality (§ 5.3 and § 5.4; the derivation of RMHD from MHD and its properties are discussed in § 2). The cascade proceeds via interaction of oppositely propagating wave packets and is decoupled from the density and magnetic-field-

strength fluctuations (in the collisional limit, these are the entropy and slow modes; see § 6.1 and Appendix D). The latter are passively mixed by the Alfvén waves, but, unlike in the fluid (collisional) limit, this passive cascade is governed by a (simplified) kinetic equation for the ions (§ 5.5). Together with RMHD, it forms a hybrid fluid-kinetic description of magnetized turbulence in a weakly collisional plasma, which we call *Kinetic Reduced MHD (KRMHD)*. The KRMHD equations are summarized in § 5.7. While the Alfvén waves are undamped in this approximation, the density and magnetic-field-strength fluctuations are subject to damping both in the collisional (Braginskii 1965 viscous damping, § 6.1 and Appendix D) and collisionless (Barnes 1966 damping, § 6.2) limits, provided they develop scales along the magnetic field that are comparable to or smaller than the mean free path. However, the ion kinetic equation is linear along the moving field lines associated with the Alfvén waves, so, in the absence of finite-gyroradius effects, the density and field-strength fluctuations do not develop small parallel scales and their cascade may be undamped above the ion gyroscale—this is discussed in § 6.3.

*Electron Reduced MHD (§ 7).*— At the ion gyroscale, the Alfvénic and the passive cascades are no longer decoupled and their energy is partially damped via collisionless wave-particle interactions. This part of the energy is channelled into ion heat. The rest of it is converted into a cascade of kinetic Alfvén waves (KAW, also sometimes referred to in astrophysical literature as whistlers; see, e.g., Quataert & Gruzinov 1999). This cascade extends through what is known as the “*dissipation range*” to the electron gyroscale, where its energy is also damped by wave-particle interaction and transferred into electron heat. The KAW turbulence is again anisotropic with  $k_{\parallel} \ll k_{\perp}$ . It is governed by a pair of fluid-like equations, also derived from gyrokinetics. We call them *Electron Reduced MHD (ERMHD)* because they are the reduced (anisotropic) form of what is known as the Electron, or Hall, MHD (Kingsep et al. 1990). The ERMHD equations are derived in § 7.1 (see also Appendix C.2) and the KAW cascade is discussed in § 7.2. In § 7.4 and § 7.6, we consider the entropy cascade below the ion gyroscale—a process whereby the collisionless damping occurring at the ion and electron gyroscopes is made irreversible and particles are heated. This part of the cascade is purely kinetic and its salient feature is the particle distribution functions developing small scales in phase space.

The regions of validity of the tertiary approximations—KRMHD and ERMHD—are illustrated in Fig. 1. In this figure, we also show the region of validity of the RMHD system derived from the standard compressible MHD equations by assuming anisotropy of the turbulence and strong interactions. This derivation is the fluid analog of the derivation of gyrokinetics. We present it in § 2, before embarking on the gyrokinetics-based path outlined above, in order to make a connection with the conventional MHD treatment and to demonstrate with particular simplicity how the assumption of anisotropy leads to a reduced fluid system in which the decoupling of the cascades of the Alfvén waves, slow waves and the entropy mode is manifest (Appendix A extends this derivation to Braginskii 1965 two-fluid equations in the limit of strong magnetization; it also works out rigorously the transition from the fluid limit to the KRMHD equations).

The main formal developments of this paper are contained

in §§ 3-7. The outline given above is meant to help the reader navigate these sections. A reader who would rather avoid exposure to technicalities should gain a fairly good practical understanding of what is done there by reading § 2, § 3.1, §§ 3.4-3.5, § 4.9, § 5.7, § 6.3, and § 7.8. In § 8, we discuss at some length how our results apply to various astrophysical plasmas with weak collisionality: the solar wind, the ISM, accretion disks, and galaxy clusters. Finally, in § 9, we provide a brief epilogue and make a few remarks about future directions of inquiry. A number of technical appendices is included—in particular, Appendix B treats in some detail the collision terms in gyrokinetics, a subject that has so far received relatively little analytically thorough and practically useful treatment in the literature.

## 2. REDUCED MHD AND THE DECOUPLING OF TURBULENT CASCADES

Consider the equations of compressible MHD

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{u}, \quad (5)$$

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \left( p + \frac{B^2}{8\pi} \right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi}, \quad (6)$$

$$\frac{ds}{dt} = 0, \quad s = \frac{p}{\rho^\gamma}, \quad \gamma = \frac{5}{3}, \quad (7)$$

$$\frac{d\mathbf{B}}{dt} = \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \nabla \cdot \mathbf{u}, \quad (8)$$

where  $\rho$  is mass density,  $\mathbf{u}$  velocity,  $p$  pressure,  $\mathbf{B}$  magnetic field,  $s$  entropy density, and  $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$  (the conditions under which these equations are valid are discussed in Appendix A). Consider a uniform static equilibrium with a straight mean field in the  $z$  direction, so

$$\rho = \rho_0 + \delta\rho, \quad p = p_0 + \delta p, \quad \mathbf{B} = B_0 \hat{\mathbf{z}} + \delta\mathbf{B}, \quad (9)$$

where  $\rho_0$ ,  $p_0$ , and  $B_0$  are constants. In what follows, the subscripts  $\parallel$  and  $\perp$  will be used to denote the projections of fields, variables and gradients on the mean-field direction  $\hat{\mathbf{z}}$  and onto the plane  $(x, y)$  perpendicular to this direction, respectively.

### 2.1. RMHD Ordering

As we explained in the Introduction, observational and numerical evidence makes it safe to assume that the turbulence in such a system will be anisotropic with  $k_{\parallel} \ll k_{\perp}$ . Let us, therefore, introduce a small parameter  $\epsilon \sim k_{\parallel}/k_{\perp}$  and carry out a systematic expansion of Eqs. (5-8) in  $\epsilon$ . In this expansion, the fluctuations are treated as small, but not arbitrarily so: in order to estimate their size, we shall adopt the critical-balance conjecture (3), which is now treated *not* as a detailed scaling prescription but as an ordering assumption. This allows us to introduce the following ordering:

$$\frac{\delta\rho}{\rho_0} \sim \frac{u_{\perp}}{v_A} \sim \frac{u_{\parallel}}{v_A} \sim \frac{\delta p}{p_0} \sim \frac{\delta B_{\perp}}{B_0} \sim \frac{\delta B_{\parallel}}{B_0} \sim \frac{k_{\parallel}}{k_{\perp}} \sim \epsilon, \quad (10)$$

where  $v_A = B_0/\sqrt{4\pi\rho_0}$  is the Alfvén speed.

In Eq. (10), we made two auxiliary ordering assumptions: that the velocity and magnetic-field fluctuations have the character of Alfvén and slow waves ( $\delta B_{\perp}/B_0 \sim u_{\perp}/v_A$ ,  $\delta B_{\parallel}/B_0 \sim u_{\parallel}/v_A$ ) and that the relative amplitudes of the Alfvén-wave-polarized fluctuations ( $\delta B_{\perp}/B_0$ ,  $u_{\perp}/v_A$ ), slow-wave-polarized fluctuations ( $\delta B_{\parallel}/B_0$ ,  $u_{\parallel}/v_A$ ) and density/pressure/entropy fluctuations ( $\delta\rho/\rho_0$ ,  $\delta p/p_0$ ) are all the same order. Strictly speaking, whether this is the case depends

on the energy sources that drive the turbulence: as we shall see, if no slow waves (or entropy fluctuations) are launched, none will be present. However, in astrophysical contexts, the large-scale energy input may be assumed random and, therefore, comparable power is injected into all types of fluctuations.

We further assume that the characteristic frequency of the fluctuations is  $\omega \sim k_{\parallel} v_A$  [Eq. (3)], meaning that the fast waves, for which  $\omega \simeq k_{\perp} (v_A^2 + c_s^2)^{1/2}$ , where  $c_s^2 = \gamma p_0/\rho_0$  (the sound speed), are ordered out. This restriction must be justified empirically. Observations of the solar wind turbulence confirm that it is primarily Alfvénic (see, e.g., Bale et al. 2005) and that its compressive component is substantially pressure-balanced (Marsch & Tu 1993; Bavassano et al. 2004, see Eq. (17) below). A weak-turbulence calculation of compressible MHD turbulence in low-beta plasmas (Chandran 2005b) suggests that only a small amount of energy is transferred from the fast waves to Alfvén waves with large  $k_{\parallel}$ . A similar conclusion emerges from numerical simulations (Cho & Lazarian 2002, 2003). As the fast waves are also expected to be subject to strong collisionless damping and/or to strong dissipation after they steepen into shocks, eliminating them from the problem and concentrating on low-frequency turbulence appears to be a reasonable course of action.

### 2.2. Alfvén Waves

We start by observing that the Alfvén-wave-polarized fluctuations are two-dimensionally solenoidal: since, from Eq. (5),

$$\nabla \cdot \mathbf{u} = -\frac{d}{dt} \frac{\delta\rho}{\rho_0} = O(\epsilon^2) \quad (11)$$

and  $\nabla \cdot \delta\mathbf{B} = 0$  exactly, separating the  $O(\epsilon)$  part of these divergences gives  $\nabla_{\perp} \cdot \mathbf{u}_{\perp} = 0$  and  $\nabla_{\perp} \cdot \delta\mathbf{B}_{\perp} = 0$ . To lowest order in the  $\epsilon$  expansion, we may, therefore, express  $\mathbf{u}_{\perp}$  and  $\delta\mathbf{B}_{\perp}$  in terms of scalar stream (flux) functions:

$$\mathbf{u}_{\perp} = \hat{\mathbf{z}} \times \nabla_{\perp} \Phi, \quad \frac{\delta\mathbf{B}_{\perp}}{\sqrt{4\pi\rho_0}} = \hat{\mathbf{z}} \times \nabla_{\perp} \Psi. \quad (12)$$

Evolution equations for  $\Phi$  and  $\Psi$  are obtained by substituting the expressions (12) into the perpendicular parts of the induction equation (8) and the momentum equation (6)—of the latter the curl is taken to annihilate the pressure term. Keeping only the terms of the lowest order,  $O(\epsilon^2)$ , we get

$$\frac{\partial \Psi}{\partial t} + \{\Phi, \Psi\} = v_A \frac{\partial \Phi}{\partial z}, \quad (13)$$

$$\frac{\partial}{\partial t} \nabla_{\perp}^2 \Phi + \{\Phi, \nabla_{\perp}^2 \Phi\} = v_A \frac{\partial}{\partial z} \nabla_{\perp}^2 \Psi + \{\Psi, \nabla_{\perp}^2 \Psi\}, \quad (14)$$

where  $\{\Phi, \Psi\} = \hat{\mathbf{z}} \cdot (\nabla_{\perp} \Phi \times \nabla_{\perp} \Psi)$  and we have taken into account that, to lowest order,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_{\perp} \cdot \nabla_{\perp} = \frac{\partial}{\partial t} + \{\Phi, \dots\}, \quad (15)$$

$$\hat{\mathbf{b}} \cdot \nabla = \frac{\partial}{\partial z} + \frac{\delta\mathbf{B}_{\perp}}{B_0} \cdot \nabla_{\perp} = \frac{\partial}{\partial z} + \frac{1}{v_A} \{\Psi, \dots\}. \quad (16)$$

Here  $\hat{\mathbf{b}} = \mathbf{B}/B_0$  is the unit vector along the perturbed field line.

Equations (13-14) are known as the Reduced Magnetohydrodynamics (RMHD). The first derivations of these equations (in the context of fusion plasmas) are due to Kadomtsev & Pogutse (1974) and to Strauss (1976). These were followed by many systematic derivations and generalizations employing various versions and refinements of



the basic expansion, taking into account the non-Alfvénic modes (as we do in § 2.3), and including the effects of spatial gradients of equilibrium fields (see, e.g., Strauss 1977; Montgomery 1982; Hazeltine 1983; Zank & Matthaeus 1992, 1993; Kinney & McWilliams 1997; Bhattacharjee et al. 1998; Kruger et al. 1998, and references therein). A comparative analysis of these expansion schemes and their (often close) relationship to ours falls outside the scope of this paper. One important point we emphasize is that we do not assume the plasma beta [defined in Eq. (26)] to be either large or small.

Equations (13) and (14) form a closed set, meaning that the Alfvén-wave cascade decouples from the slow waves and density fluctuations. It is to the turbulence described by Eqs. (13-14) that the GS theory outlined in § 1.1 applies.<sup>9</sup>

### 2.3. Slow Waves and the Entropy Mode

In order to derive evolution equations for the latter, let us first revisit the perpendicular part of the momentum equation and use Eq. (10) to order terms in it. In the lowest order,  $O(\epsilon)$ , we get the pressure balance

$$\nabla_{\perp} \left( \delta p + \frac{B_0 \delta B_{\parallel}}{4\pi} \right) = 0 \quad \Rightarrow \quad \frac{\delta p}{p_0} = -\gamma \frac{v_A^2}{c_s^2} \frac{\delta B_{\parallel}}{B_0}. \quad (17)$$

Using Eq. (17) and the entropy equation (7), we get

$$\frac{d}{dt} \frac{\delta s}{s_0} = 0, \quad \frac{\delta s}{s_0} = \frac{\delta p}{p_0} - \gamma \frac{\delta \rho}{\rho_0} = -\gamma \left( \frac{\delta \rho}{\rho_0} + \frac{v_A^2}{c_s^2} \frac{\delta B_{\parallel}}{B_0} \right), \quad (18)$$

where  $s_0 = p_0/\rho_0^\gamma$ . Now, substituting Eq. (11) for  $\nabla \cdot \mathbf{u}$  in the parallel component of the induction equation (8), we get

$$\frac{d}{dt} \left( \frac{\delta B_{\parallel}}{B_0} - \frac{\delta \rho}{\rho_0} \right) - \hat{\mathbf{b}} \cdot \nabla u_{\parallel} = 0. \quad (19)$$

Combining Eqs. (18) and (19), we obtain

$$\frac{d}{dt} \frac{\delta \rho}{\rho_0} = -\frac{1}{1 + c_s^2/v_A^2} \hat{\mathbf{b}} \cdot \nabla u_{\parallel}, \quad (20)$$

$$\frac{d}{dt} \frac{\delta B_{\parallel}}{B_0} = \frac{1}{1 + v_A^2/c_s^2} \hat{\mathbf{b}} \cdot \nabla u_{\parallel}. \quad (21)$$

Finally, we take the parallel component of the momentum equation (6) and notice that, due to the pressure balance (17) and to the smallness of the parallel gradients, the pressure term is  $O(\epsilon^3)$ , while the inertial and tension terms are  $O(\epsilon^2)$ . Therefore,

$$\frac{du_{\parallel}}{dt} = v_A^2 \hat{\mathbf{b}} \cdot \nabla \frac{\delta B_{\parallel}}{B_0}. \quad (22)$$

<sup>9</sup> The Alfvén-wave turbulence in the RMHD system has been studied by many authors. Some of the more recent numerical investigations are due to Kinney & McWilliams (1998), Dmitruk et al. (2003), Oughton et al. (2004). Analytical theory has mostly been confined to the weak-turbulence paradigm (Ng & Bhattacharjee 1996, 1997; Bhattacharjee & Ng 2001; Galtier et al. 2002; Lithwick & Goldreich 2003). We note that adopting the critical balance [Eq. (3)] as an ordering assumption for the expansion in  $k_{\parallel}/k_{\perp}$  does not preclude one from subsequently attempting a weak-turbulence approach: the latter should simply be treated as a subsidiary expansion. Indeed, implementing the anisotropy assumption on the level of MHD equations rather than simultaneously with the weak-turbulence closure (Galtier et al. 2000) significantly reduces the amount of algebra. One should, however, bear in mind that the weak-turbulence approximation always breaks down at some sufficiently small scale—namely, when  $k_{\perp} \sim (v_A/U)^2 k_{\parallel}^2 L$ , where  $L$  is the outer scale of the turbulence,  $U$  velocity at the outer scale, and  $k_{\parallel}$  the parallel wave number of the Alfvén waves (see Goldreich & Sridhar 1997 or the review by Schekochihin & Cowley 2006b). Below this scale, interactions cannot be assumed weak.

Equations (21-22) describe the slow-wave-polarized fluctuations, while Eq. (18) describes the zero-frequency entropy mode, which is decoupled from the slow waves (for other expansion schemes leading to reduced sets of equations for the non-Alfvénic modes see references in § 2.2. The nonlinearity in Eqs. (21-22) enters via the derivatives defined in Eqs. (15-16) and is due solely to interactions with Alfvén waves. Thus, both the slow-wave and the entropy-fluctuation cascades occur via passive scattering/mixing by Alfvén waves, in the course of which there is no energy exchange between the cascades. The turbulent cascade of the passive fields is discussed in § 2.6 and also in § 6.3, where we examine the implications of the fact that the equations for the slow and entropy modes become linear in the Lagrangian frame associated with the Alfvénic turbulence.

### 2.4. Elsasser Fields for RMHD

The MHD equations (5-8) in the incompressible limit ( $\rho = \text{const}$ ) acquire a symmetric form if written in terms of the Elsasser fields  $\mathbf{z}^{\pm} = \mathbf{u} \pm \delta \mathbf{B}/\sqrt{4\pi\rho}$  (Elsasser 1950). Let us demonstrate how this symmetry manifests itself in the reduced equations derived above.

We introduce *Elsasser potentials*  $\zeta^{\pm} = \Phi \pm \Psi$ , so that  $\mathbf{z}_{\perp}^{\pm} = \hat{\mathbf{z}} \times \nabla_{\perp} \zeta^{\pm}$ . For these potentials, Eqs. (13-14) become

$$\begin{aligned} \frac{\partial}{\partial t} \nabla_{\perp}^2 \zeta^{\pm} \mp v_A \frac{\partial}{\partial z} \nabla_{\perp}^2 \zeta^{\pm} = & -\frac{1}{2} [\{\zeta^{+}, \nabla_{\perp}^2 \zeta^{-}\} + \{\zeta^{-}, \nabla_{\perp}^2 \zeta^{+}\} \\ & \mp \nabla_{\perp}^2 \{\zeta^{+}, \zeta^{-}\}]. \end{aligned} \quad (23)$$

These equations show that the RMHD has a simple set of exact solutions: if  $\zeta^{-} = 0$  or  $\zeta^{+} = 0$ , the nonlinear term vanishes and the other, nonzero, Elsasser potential is simply a fluctuation of arbitrary shape and magnitude propagating along the mean field at the Alfvén speed  $v_A$ :  $\zeta^{\pm} = f^{\pm}(x, y, z \mp v_A t)$ . These solutions are called finite-amplitude Alfvén waves. Only counterpropagating such solutions can interact and thereby give rise to the Alfvén-wave cascade (Kraichnan 1965). Note that these interactions are conservative in the sense that the “+” and “−” waves scatter off each other without exchanging energy.

For the slow-wave variables, we may introduce generalized Elsasser fields:

$$z_{\parallel}^{\pm} = u_{\parallel} \pm \frac{\delta B_{\parallel}}{\sqrt{4\pi\rho_0}} \left( 1 + \frac{v_A^2}{c_s^2} \right)^{1/2}. \quad (24)$$

Straightforwardly, the evolution equation for these fields is

$$\begin{aligned} \frac{\partial z_{\parallel}^{\pm}}{\partial t} \mp \frac{v_A}{\sqrt{1 + v_A^2/c_s^2}} \frac{\partial z_{\parallel}^{\pm}}{\partial z} = & -\frac{1}{2} \left( 1 \mp \frac{1}{\sqrt{1 + v_A^2/c_s^2}} \right) \{\zeta^{+}, z_{\parallel}^{\pm}\} \\ & -\frac{1}{2} \left( 1 \pm \frac{1}{\sqrt{1 + v_A^2/c_s^2}} \right) \{\zeta^{-}, z_{\parallel}^{\pm}\}. \end{aligned} \quad (25)$$

In the limit  $v_A \ll c_s$ , the generalized Elsasser fields (24) become the parallel components of the conventional incompressible Elsasser fields. This is known as the high-beta limit, with the plasma beta defined by

$$\beta = \frac{8\pi p_0}{B_0^2} = \frac{2}{\gamma} \frac{c_s^2}{v_A^2}. \quad (26)$$

We see that only in the limit of  $\beta \gg 1$  do the slow waves interact exclusively with the counterpropagating Alfvén waves. For general  $\beta$ , the phase speed of the slow waves is smaller than that of the Alfvén waves and, therefore, Alfvén waves can “catch up” and interact with the slow waves that travel in the same direction. All of these interactions are of scattering type and involve no exchange of energy.

### 2.5. Five RMHD Cascades

Thus, the anisotropy and critical balance (3) taken as ordering assumptions lead to a neat decomposition of the MHD turbulent cascade into a decoupled Alfvén-wave cascade and cascades of slow waves and entropy fluctuations passively scattered/mixed by the Alfvén waves. More precisely, Eqs. (18), (23) and (25) imply that, for arbitrary  $\beta$ , there are five conserved quantities:

$$W_{\perp}^{\pm} = \frac{1}{2} \int d^3\mathbf{r} \rho_0 |\nabla \zeta^{\pm}|^2 \quad (\text{Alfvén waves}), \quad (27)$$

$$W_{\parallel}^{\pm} = \frac{1}{2} \int d^3\mathbf{r} \rho_0 |z_{\parallel}^{\pm}|^2 \quad (\text{slow waves}), \quad (28)$$

$$W_s = \frac{1}{2} \int d^3\mathbf{r} \frac{\delta s^2}{s_0^2} \quad (\text{entropy fluctuations}). \quad (29)$$

$W_{\perp}^{+}$  and  $W_{\perp}^{-}$  are always cascaded by interaction with each other,  $W_s$  is passively mixed by  $W_{\perp}^{+}$  and  $W_{\perp}^{-}$ ,  $W_{\parallel}^{\pm}$  are passively scattered by  $W_{\perp}^{\mp}$  and, unless  $\beta \gg 1$ , also by  $W_{\perp}^{\pm}$ .

### 2.6. Discussion

The decoupling of the slow- and Alfvén-wave cascades in MHD turbulence was studied in some detail and confirmed in direct numerical simulations by Maron & Goldreich (2001, for  $\beta \gg 1$ ) and by Cho & Lazarian (2002, 2003, for a range of values of  $\beta$ ). The derivation given in § 2.2 and § 2.3 (cf. Lithwick & Goldreich 2001) provides a straightforward theoretical basis for these results, assuming anisotropy of the turbulence (which was also confirmed in these numerical studies).

In astrophysical turbulence,  $\beta$  tends to be order unity or moderately high:  $\beta \sim 1 \dots 10$  by order of magnitude for all of the astrophysical plasmas mentioned in the Introduction (see Table 1; the solar corona, where  $\beta \sim 10^{-6}$ , is one prominent example of the extreme opposite limit). In the high-beta limit, which is equivalent to the incompressible approximation for the slow waves, density fluctuations are due solely to the entropy mode. They decouple from the slow-wave cascade and are passively mixed by the Alfvén-wave turbulence:

$$\frac{d\delta\rho}{dt} = 0 \quad (30)$$

[Eq. (18) or Eq. (20),  $c_s \gg v_A$ ].

The scaling of such passively mixed scalar fields is slaved to the scaling of the Alfvénic fluctuations. As in Kolmogorov–Obukhov theory (see § 1.1), one assumes a local-in-scale-space cascade of scalar variance and a constant flux  $\varepsilon_n$  of this variance. Then, analogously to Eq. (1),  $\delta\rho_{\lambda}^2/\tau_{\lambda} \sim \varepsilon_n$ . The cascade time is  $\tau_{\lambda}^{-1} \sim \mathbf{u}_{\perp} \cdot \nabla_{\perp} \sim v_A/l_{\parallel\lambda} \sim \varepsilon/u_{\perp\lambda}^2$ . This gives

$$\delta\rho_{\lambda} \sim (\varepsilon_n/\varepsilon)^{1/2} u_{\perp\lambda}, \quad (31)$$

so the scalar fluctuations have the same scaling as the turbulence that mixes them (Obukhov 1949; Corrsin 1951), i.e., in the GS turbulence, the scalar variance spectrum should be

$k_{\perp}^{-5/3}$ . The same argument applies to all passive fields, e.g., the magnetic-field-strength fluctuations (see § 6.3 for a discussion of the anisotropy and the parallel cascade of the passive component of MHD turbulence).

It is precisely the electron-density spectrum that provides the main evidence of the  $k^{-5/3}$  scaling in the interstellar turbulence (Armstrong et al. 1995; Lazio et al. 2004). The explanation of this spectrum in terms of passive mixing of the entropy mode, originally conjectured by Higdon (1984), was developed on the basis of the GS theory by Lithwick & Goldreich (2001).

It turns out that the decoupling of the Alfvén-wave cascade that we demonstrated above for the anisotropic MHD turbulence is a uniformly valid property of plasma turbulence at both collisional and collisionless scales and that this cascade is correctly described by the RMHD equations (13–14) all the way down to the ion gyroscale, while the fluctuations of density and magnetic-field strength do not satisfy simple fluid evolution equations anymore and require solving the kinetic equation (see discussion of these cascades and of the relevant solar-wind data in § 6.3). In order to prove this, we adopt a kinetic description and apply to it the same ordering (§ 2.1) as we used to reduce the MHD equations. The kinetic theory that emerges as a result is called gyrokinetics.

### 3. GYROKINETICS

The gyrokinetic formalism was first worked out for linear waves by Rutherford & Frieman (1968) and by Taylor & Hastie (1968) (see also Catto 1978; Antonsen & Lane 1980; Catto et al. 1981) and subsequently extended to the nonlinear regime by Frieman & Chen (1982). Rigorous derivations of the gyrokinetic equation based on the Hamiltonian formalism were developed by Dubin et al. (1983, electrostatic) and Hahm et al. (1988, electromagnetic). This approach is reviewed in Brizard & Hahm (2007). A more pedestrian, but perhaps also more transparent exposition of the gyrokinetics in a straight mean field can be found in Howes et al. (2006), who also provide a detailed explanation of the gyrokinetic ordering in the context of astrophysical plasma turbulence and a treatment of the linear waves and damping rates. Here we review only the main points so as to allow the reader to understand the present paper without referring elsewhere.

In general, a plasma is completely described by the distribution function  $f_s(t, \mathbf{r}, \mathbf{v})$ —the probability density for a particle of species  $s$  ( $= i, e$ ) to be found at the spatial position  $\mathbf{r}$  moving with velocity  $\mathbf{v}$ . This function obeys the kinetic Vlasov–Landau equation

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s} \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \left( \frac{\partial f_s}{\partial t} \right)_c, \quad (32)$$

where  $q_s$  and  $m_s$  are the particle’s charge and mass,  $c$  is the speed of light, and the right-hand side is the collision term (quadratic in  $f$ ). The electric and magnetic fields are

$$\mathbf{E} = -\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A} \quad (33)$$

(the first equality is Faraday’s law uncurled, the second the magnetic-field solenoidality condition). They satisfy the Poisson and the Ampère–Maxwell equations with the charge and



current densities determined by  $f_s(t, \mathbf{r}, \mathbf{v})$ :

$$\nabla \cdot \mathbf{E} = 4\pi \sum_s q_s n_s = 4\pi \sum_s q_s \int d^3\mathbf{v} f_s, \quad (34)$$

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \mathbf{j} = \frac{4\pi}{c} \sum_s q_s \int d^3\mathbf{v} \mathbf{v} f_s. \quad (35)$$

### 3.1. Gyrokinetic Ordering and Dimensionless Parameters

As in § 2 we set up a static equilibrium with a uniform mean field,  $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$ ,  $\mathbf{E}_0 = 0$ , assume that the perturbations will be anisotropic with  $k_{\parallel} \ll k_{\perp}$ , and construct an expansion of the kinetic theory around this equilibrium with respect to the small parameter  $\epsilon \sim k_{\parallel}/k_{\perp}$ . Besides  $\epsilon$ , several other dimensionless parameters are present: the electron-ion mass ratio  $m_e/m_i$ , the charge ratio

$$Z = q_i/|q_e| = q_i/e \quad (36)$$

(for hydrogen, this is 1, which applies to most astrophysical plasmas of interest to us), the temperature ratio<sup>10</sup>

$$\tau = T_i/T_e, \quad (37)$$

and the plasma (ion) beta

$$\beta_i = \frac{v_{\text{thi}}^2}{v_A^2} = \frac{8\pi n_i T_i}{B_0^2} = \beta \left(1 + \frac{Z}{\tau}\right)^{-1}, \quad (38)$$

where  $v_{\text{thi}} = (2T_i/m_i)^{1/2}$  is the ion thermal speed and the total  $\beta$  was defined in Eq. (26) based on the total pressure  $p = n_i T_i + n_e T_e$ .

We now adopt the ordering expressed by Eqs. (3) and (10), i.e., we assume the perturbations to be strongly interacting Alfvén waves plus electron density and magnetic-field-strength fluctuations. As we want our theory to be uniformly valid at all (perpendicular) scales above, at or below the ion gyroscale, we order

$$k_{\perp} \rho_i \sim 1, \quad (39)$$

where  $\rho_i = v_{\text{thi}}/\Omega_i$  is the ion gyroradius,  $\Omega_i = q_i B_0/cm_i$  the ion cyclotron frequency. Similarly, we want our theory to be valid both in the collisional and the collisionless regimes, so we do not assume  $\omega$  to be either smaller or larger than the (ion) collision frequency  $\nu_{ii}$ :

$$\frac{\omega}{\nu_{ii}} \sim \frac{k_{\parallel} \lambda_{\text{mfpi}}}{\sqrt{\beta_i}} \sim 1, \quad (40)$$

where  $\lambda_{\text{mfpi}} = v_{\text{thi}}/\nu_{ii}$  is the ion mean free path (this ordering can actually be inferred from equating the gyrokinetic entropy production terms to the collisional entropy production; see extended discussion in Howes et al. 2006).

Note that

$$\rho_e = \frac{Z}{\sqrt{\tau}} \sqrt{\frac{m_e}{m_i}} \rho_i \quad (41)$$

<sup>10</sup> It can be shown that equilibrium temperatures change on the time scale  $\sim (\epsilon^2 \omega)^{-1}$  (Howes et al. 2006). On the other hand, from standard theory of collisional transport (e.g., Helander & Sigmar 2002), the ion and electron temperatures equalize on the time scale  $\sim \nu_{ie}^{-1} \sim (m_i/m_e)^{1/2} \nu_{ii}^{-1}$  [see Eq. (43)]. Therefore,  $\tau$  can depart from unity by an amount of order  $\epsilon^2 (\omega/\nu_{ii}) (m_i/m_e)^{1/2}$ . In our ordering scheme [Eq. (40)], this is  $O(\epsilon^2)$  and, therefore, we should simply set  $\tau = 1 + O(\epsilon^2)$ . However, we shall carry the parameter  $\tau$  because other ordering schemes are possible that permit arbitrary values of  $\tau$ . These are appropriate to plasmas with very weak collisions. For example, in the solar wind,  $\tau$  appears to be order unity but not exactly 1 (Newbury et al. 1998), while in accretion flows near the black hole, some models predict  $\tau \gg 1$  (see § 8.3).

and (see, e.g., Helander & Sigmar 2002)

$$\nu_{ei} = Z \nu_{ee} = \frac{\tau^{3/2}}{Z^2} \sqrt{\frac{m_i}{m_e}} \nu_{ii}, \quad (42)$$

$$\nu_{ie} = \frac{8}{3\sqrt{\pi}} \frac{\tau^{3/2}}{Z} \sqrt{\frac{m_e}{m_i}} \nu_{ii}, \quad (43)$$

$$\nu_{ii} = \frac{\sqrt{2\pi} Z^4 e^4 n_i \ln \Lambda}{m_i^{1/2} T_i^{3/2}}, \quad (44)$$

where  $\ln \Lambda$  is the Coulomb logarithm and the numerical factor in the definition of  $\nu_{ie}$  has been inserted for future notational convenience (see Appendix A). We always define  $\lambda_{\text{mfpi}} = v_{\text{thi}}/\nu_{ii}$  and  $\lambda_{\text{mfpe}} = v_{\text{the}}/\nu_{ei} = (Z/\tau)^2 \lambda_{\text{mfpi}}$ .

Equation (3) now allows us to order the fluctuations of the electric field: on the one hand, we have from Eq. (3)  $u_{\perp} \sim \epsilon v_A$ ; on the other hand, the plasma mass flow velocity is (to the lowest order) the  $\mathbf{E} \times \mathbf{B}$  drift velocity of the ions,  $u_{\perp} \sim c E_{\perp}/B_0 \sim c k_{\perp} \phi/B_0$ , so

$$\frac{e\phi}{T_e} \sim \frac{\tau}{Z} \frac{1}{k_{\perp} \rho_i \sqrt{\beta_i}} \epsilon. \quad (45)$$

Finally, Eq. (3) implies

$$\frac{\omega}{\Omega_i} \sim \frac{k_{\parallel} v_A}{\Omega_i} \sim \frac{k_{\perp} \rho_i}{\sqrt{\beta_i}} \epsilon. \quad (46)$$

Thus, gyrokinetics is a low-frequency limit that averages over the time scales associated with the particle gyration. Because we have assumed that the fluctuations are anisotropic and have (by order of magnitude) Alfvénic frequencies, we see from Eq. (46) that their frequency remains far below  $\Omega_i$  at all scales, including the ion and even electron gyroscale—the gyrokinetics remains valid at all of these scales and the cyclotron-frequency effects are negligible (cf. Quataert & Gruzinov 1999).

Note that several parameters have been introduced above that are considered to be order unity in the gyrokinetic expansion:  $m_e/m_i$ ,  $\beta_i$ ,  $k_{\perp} \rho_i$  and  $k_{\parallel} \lambda_{\text{mfpi}}$ . Subsidiary expansions in small  $m_e/m_i$  (§ 4) and in small or large values of the other three parameters (§§ 5-7) can be carried out at a later stage as long as their values are not so large or small as to interfere with the primary expansion in  $\epsilon$ .

The region of validity of the gyrokinetic approximation in the wave-number space is illustrated in Fig. 2—it embraces all of the scales that are expected to be transversed by the anisotropic energy cascade.

### 3.2. Gyrokinetic Equation

Given the gyrokinetic ordering introduced above, the expansion of the distribution function up to first order in  $\epsilon$  can be written as

$$f_s(t, \mathbf{r}, \mathbf{v}) = F_{0s}(v) - \frac{q_s \phi(t, \mathbf{r})}{T_{0s}} F_{0s}(v) + h_s(t, \mathbf{R}_s, v_{\perp}, v_{\parallel}). \quad (47)$$

To zeroth order, it is a Maxwellian:

$$F_{0s}(v) = \frac{n_{0s}}{(\pi v_{\text{ths}}^2)^{3/2}} \exp\left(-\frac{v^2}{v_{\text{ths}}^2}\right), \quad v_{\text{ths}} = \sqrt{\frac{2T_{0s}}{m_s}}, \quad (48)$$

with uniform density  $n_{0s}$  and temperature  $T_{0s}$  and no mean flow. As will be explained in more detail in § 3.5,  $F_{0s}$  has a slow time dependence via the equilibrium temperature,  $T_{0s} = T_{0s}(\epsilon^2 t)$ . This reflects the slow heating of the plasma as the turbulent energy is dissipated. However,  $T_{0s}$  can be treated as a

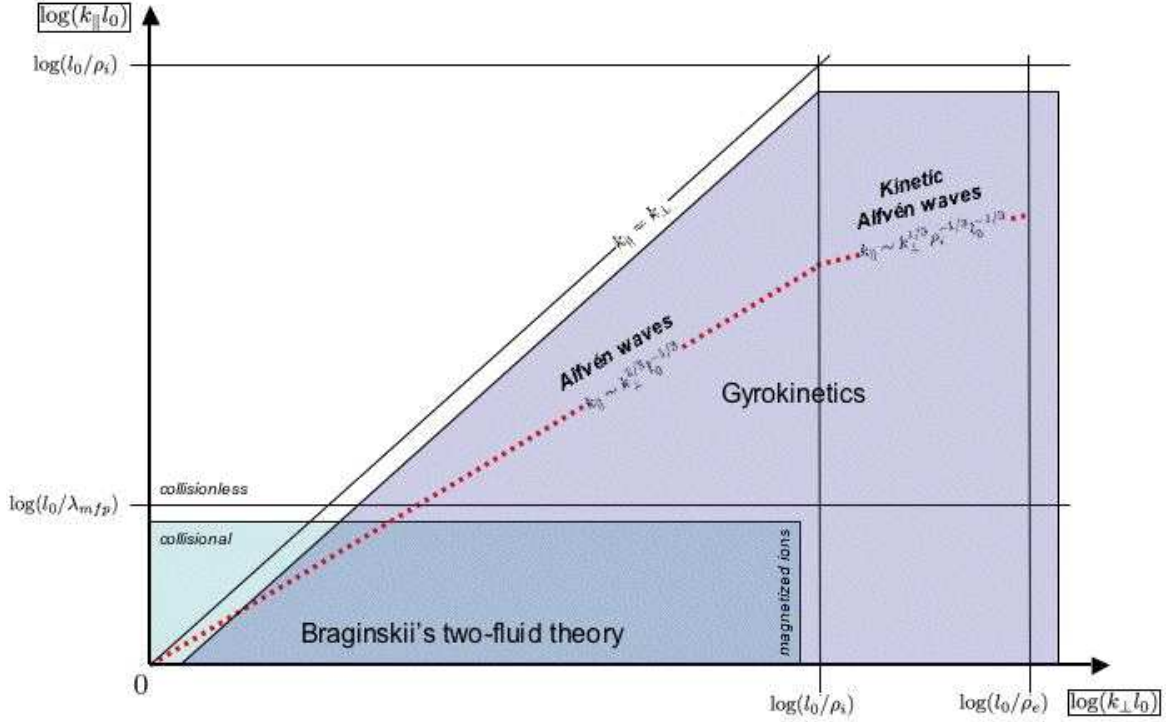


FIG. 2.— The regions of validity in the wave-number space of two primary approximations—the two-fluid (Appendix A.1) and gyrokinetic (§ 3). The gyrokinetic theory holds when  $k_{\parallel} \ll k_{\perp}$  and  $\omega \ll \Omega_i$  [when  $k_{\parallel} \ll k_{\perp} < \rho_i^{-1}$ , the second requirement is automatically satisfied for Alfvén, slow and entropy modes; see Eq. (46)]. The two-fluid equations hold when  $k_{\parallel} \lambda_{mfp} \ll 1$  (collisional limit) and  $k_{\perp} \rho_i \ll 1$  (magnetized plasma). Note that the gyrokinetic theory holds for all but the very largest scales.

constant with respect to the time dependence of the first-order distribution function (the time scale of the turbulent fluctuations). The first-order part of the distribution function is composed of the Boltzmann response [second term in Eq. (47)] and the *gyrocenter distribution function*  $h_s$ . The spatial dependence of the latter is expressed not by the particle position  $\mathbf{r}$  but by the position  $\mathbf{R}_s$  of the particle gyrocenter (or guiding center)—the center of the ring orbit that the particle follows in a strong guide field:

$$\mathbf{R}_s = \mathbf{r} + \frac{\mathbf{v}_{\perp} \times \hat{\mathbf{z}}}{\Omega_s}. \quad (49)$$

Thus, some of the velocity dependence of the distribution function is subsumed in the  $\mathbf{R}_s$  dependence of  $h_s$ . Explicitly,  $h_s$  depends only on two velocity-space variables:  $v_{\perp}$  and  $v_{\parallel}$  (we shall sometimes find an alternative pair,  $v$  and  $\xi = v_{\parallel}/v$ , useful). What has to be constantly kept in mind is that derivatives of  $h_s$  with respect to these variables are taken at constant  $\mathbf{R}_s$ , not at constant  $\mathbf{r}$ .

The function  $h_s$  satisfies the *gyrokinetic equation*:

$$\frac{\partial h_s}{\partial t} + v_{\parallel} \frac{\partial h_s}{\partial z} + \frac{c}{B_0} \{ \langle \chi \rangle_{\mathbf{R}_s}, h_s \} = \frac{q_s F_{0s}}{T_{0s}} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} + \left( \frac{\partial h_s}{\partial t} \right)_c, \quad (50)$$

where

$$\chi(t, \mathbf{r}, \mathbf{v}) = \phi - \frac{v_{\parallel} A_{\parallel}}{c} - \frac{\mathbf{v}_{\perp} \cdot \mathbf{A}_{\perp}}{c}, \quad (51)$$

the Poisson brackets are defined in the usual way:

$$\{ \langle \chi \rangle_{\mathbf{R}_s}, h_s \} = \hat{\mathbf{z}} \cdot \left( \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial \mathbf{R}_s} \times \frac{\partial h_s}{\partial \mathbf{R}_s} \right), \quad (52)$$

and the ring average notation is introduced:

$$\langle \chi(t, \mathbf{r}, \mathbf{v}) \rangle_{\mathbf{R}_s} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \chi \left( t, \mathbf{R}_s - \frac{\mathbf{v}_{\perp} \times \hat{\mathbf{z}}}{\Omega_s}, \mathbf{v} \right), \quad (53)$$

where  $\varphi$  is the angle in the velocity space taken in the plane perpendicular to the guide field  $B_0 \hat{\mathbf{z}}$ . Note that, while  $\chi$  is a function of  $\mathbf{r}$ , its ring average is a function of  $\mathbf{R}_s$ . Note also that the ring averages depend on the species index, as does the gyrocenter variable  $\mathbf{R}_s$ . Equation (50) is derived by transforming the first-order kinetic equation to the gyrocenter variable (49) and ring averaging the result (see Howes et al. 2006, or the references given at the beginning of § 3). The ring-averaged collision integral  $(\partial h_s / \partial t)_c$  is discussed in Appendix B.

### 3.3. Field Equations

To Eq. (50), we must append the equations that determine the electromagnetic field, namely, the potentials  $\phi(t, \mathbf{r})$  and  $\mathbf{A}(t, \mathbf{r})$  that enter the expression for  $\chi$  [Eq. (51)]. In the non-relativistic limit ( $v_{thi} \ll c$ ), these are the plasma quasineutrality constraint [which follows from the Poisson equation (34) to lowest order in  $v_{thi}/c$ ]:

$$0 = \sum_s q_s \delta n_s = \sum_s q_s \left[ -\frac{q_s \phi}{T_{0s}} n_{0s} + \int d^3 \mathbf{v} \langle h_s \rangle_{\mathbf{r}} \right] \quad (54)$$

and the parallel and perpendicular parts of Ampère's law [Eq. (35) to lowest order in  $\epsilon$  and in  $v_{\text{thi}}/c$ ]:

$$\nabla_{\perp}^2 A_{\parallel} = -\frac{4\pi}{c} j_{\parallel} = -\frac{4\pi}{c} \sum_s q_s \int d^3 \mathbf{v} v_{\parallel} \langle h_s \rangle_{\mathbf{r}}, \quad (55)$$

$$\begin{aligned} \nabla_{\perp}^2 \delta B_{\parallel} &= -\frac{4\pi}{c} \hat{\mathbf{z}} \cdot (\nabla_{\perp} \times \mathbf{j}_{\perp}) \\ &= -\frac{4\pi}{c} \hat{\mathbf{z}} \cdot \left[ \nabla_{\perp} \times \sum_s q_s \int d^3 \mathbf{v} \langle \mathbf{v}_{\perp} h_s \rangle_{\mathbf{r}} \right], \end{aligned} \quad (56)$$

where we have used  $\delta B_{\parallel} = \hat{\mathbf{z}} \cdot (\nabla_{\perp} \times \mathbf{A}_{\perp})$  and dropped the displacement current. Since field variables  $\phi$ ,  $A_{\parallel}$  and  $\delta B_{\parallel}$  are functions of the spatial variable  $\mathbf{r}$ , not of the gyrocenter variable  $\mathbf{R}_s$ , we had to determine the contribution from the gyrocenter distribution function  $h_s$  to the charge distribution at fixed  $\mathbf{r}$  by performing a gyroaveraging operation dual to the ring average defined in Eq. (53):

$$\langle h_s(t, \mathbf{R}_s, v_{\perp}, v_{\parallel}) \rangle_{\mathbf{r}} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi h_s \left( t, \mathbf{r} + \frac{\mathbf{v}_{\perp} \times \hat{\mathbf{z}}}{\Omega_s}, v_{\perp}, v_{\parallel} \right). \quad (57)$$

In other words, the velocity-space integrals in Eqs. (54-56) are performed over  $h_s$  at constant  $\mathbf{r}$ , rather than constant  $\mathbf{R}_s$ . If we Fourier-transform  $h_s$  in  $\mathbf{R}_s$ , the gyroaveraging operation takes a simple mathematical form:

$$\begin{aligned} \langle h_s \rangle_{\mathbf{r}} &= \sum_{\mathbf{k}} \langle e^{i\mathbf{k} \cdot \mathbf{R}_s} \rangle_{\mathbf{r}} h_{s\mathbf{k}}(t, v_{\perp}, v_{\parallel}) \\ &= \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \left\langle \exp \left( i\mathbf{k} \cdot \frac{\mathbf{v}_{\perp} \times \hat{\mathbf{z}}}{\Omega_s} \right) \right\rangle_{\mathbf{r}} h_{s\mathbf{k}}(t, v_{\perp}, v_{\parallel}) \\ &= \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} J_0(a_s) h_{s\mathbf{k}}(t, v_{\perp}, v_{\parallel}), \end{aligned} \quad (58)$$

where  $a_s = k_{\perp} v_{\perp} / \Omega_s$  and  $J_0$  is a Bessel function that arose from the angle integral in the velocity space. In Eq. (56), an analogous calculation taking into account the angular dependence of  $\mathbf{v}_{\perp}$  leads to

$$\delta B_{\parallel} = -\frac{4\pi}{B_0} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \sum_s \int d^3 \mathbf{v} m_s v_{\perp}^2 \frac{J_1(a_s)}{a_s} h_{s\mathbf{k}}(t, v_{\perp}, v_{\parallel}). \quad (59)$$

Note that Eq. (56) [and, therefore, Eq. (59)] is the gyrokinetic equivalent of the perpendicular pressure balance that appeared in § 2 [Eq. (17)]:

$$\begin{aligned} \nabla_{\perp}^2 \frac{B_0 \delta B_{\parallel}}{4\pi} &= \frac{B_0}{c} \nabla_{\perp} \cdot \sum_s q_s \int d^3 \mathbf{v} \langle \hat{\mathbf{z}} \times \mathbf{v}_{\perp} h_s \rangle_{\mathbf{r}} \\ &= \nabla_{\perp} \cdot \sum_s \Omega_s m_s \int d^3 \mathbf{v} \frac{\partial \mathbf{v}_{\perp}}{\partial \varphi} h_s \left( t, \mathbf{r} + \frac{\mathbf{v}_{\perp} \times \hat{\mathbf{z}}}{\Omega_s}, v_{\perp}, v_{\parallel} \right) \\ &= -\nabla_{\perp} \nabla_{\perp} : \sum_s \int d^3 \mathbf{v} m_s \langle \mathbf{v}_{\perp} \mathbf{v}_{\perp} h_s \rangle_{\mathbf{r}} = -\nabla_{\perp} \nabla_{\perp} : \delta \mathbf{P}_{\perp}, \end{aligned} \quad (60)$$

where we have integrated by parts with respect to the gyroangle  $\varphi$  and used  $\partial \mathbf{v}_{\perp} / \partial \varphi = \hat{\mathbf{z}} \times \mathbf{v}_{\perp}$ ,  $\partial^2 \mathbf{v}_{\perp} / \partial \varphi^2 = -\mathbf{v}_{\perp}$ .

Once the fields are determined, they have to be substituted into  $\chi$  [Eq. (51)] and the result ring averaged [Eq. (53)]. Again, we emphasize that  $\phi$ ,  $A_{\parallel}$  and  $\delta B_{\parallel}$  are functions of  $\mathbf{r}$ , while  $\langle \chi \rangle_{\mathbf{R}_s}$  is a function of  $\mathbf{R}_s$ . The transformation is accomplished via a calculation analogous to the one that led to

Eqs. (58) and (59):

$$\langle \chi \rangle_{\mathbf{R}_s} = \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{R}_s} \langle \chi \rangle_{\mathbf{R}_s, \mathbf{k}}, \quad (61)$$

$$\langle \chi \rangle_{\mathbf{R}_s, \mathbf{k}} = J_0(a_s) \left( \phi_{\mathbf{k}} - \frac{v_{\parallel} A_{\parallel \mathbf{k}}}{c} \right) + \frac{T_{0s}}{q_s} \frac{2v_{\perp}^2}{v_{\text{th}s}^2} \frac{J_1(a_s)}{a_s} \frac{\delta B_{\parallel \mathbf{k}}}{B_0}. \quad (62)$$

The last equation establishes a correspondence between the Fourier transforms of the fields with respect to  $\mathbf{r}$  and the Fourier transform of  $\langle \chi \rangle_{\mathbf{R}_s}$  with respect to  $\mathbf{R}_s$ .

### 3.4. Generalized Energy

As promised in § 1.2, the central unifying concept of this paper is now introduced.

If we multiply the gyrokinetic equation (50) by  $T_{0s} h_s / F_{0s}$  and integrate over the velocities and gyrocenters, we find that the nonlinear term conserves the variance of  $h_s$  and

$$\begin{aligned} \frac{d}{dt} \int d^3 \mathbf{v} \int d^3 \mathbf{R}_s \frac{T_{0s} h_s^2}{2F_{0s}} &= \int d^3 \mathbf{v} \int d^3 \mathbf{R}_s q_s \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} h_s \\ &\quad + \int d^3 \mathbf{v} \int d^3 \mathbf{R}_s \frac{T_{0s} h_s}{F_{0s}} \left( \frac{\partial h_s}{\partial t} \right)_c. \end{aligned} \quad (63)$$

Let us now sum this equation over all species. The first term on the right-hand side is

$$\begin{aligned} \sum_s q_s \int d^3 \mathbf{v} \int d^3 \mathbf{R}_s \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} h_s &= \int d^3 \mathbf{r} \sum_s q_s \int d^3 \mathbf{v} \left\langle \frac{\partial \chi}{\partial t} h_s \right\rangle_{\mathbf{r}} \\ &= \int d^3 \mathbf{r} \left[ \frac{\partial \phi}{\partial t} \sum_s q_s \int d^3 \mathbf{v} \langle h_s \rangle_{\mathbf{r}} - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \cdot \sum_s q_s \int d^3 \mathbf{v} \langle \mathbf{v} h_s \rangle_{\mathbf{r}} \right] \\ &= \frac{d}{dt} \int d^3 \mathbf{r} \sum_s \frac{q_s^2 \phi^2 n_{0s}}{2T_{0s}} + \int d^3 \mathbf{r} \mathbf{E} \cdot \mathbf{j}, \end{aligned} \quad (64)$$

where we have used Eq. (54) and Ampère's law [Eqs. (55-56)] to express the integrals of  $h_s$ . The second term on the right-hand side is the total work done on plasma per unit time. Using Faraday's law [Eq. (33)], it can be written as

$$\int d^3 \mathbf{r} \mathbf{E} \cdot \mathbf{j} = -\frac{d}{dt} \int d^3 \mathbf{r} \frac{|\delta \mathbf{B}|^2}{8\pi} + P_{\text{ext}}, \quad (65)$$

where  $P_{\text{ext}}$  is the total power injected into the system by the external energy sources (outer-scale stirring; in terms of the Kolmogorov energy flux  $\varepsilon$  used in scaling arguments in § 1.1,  $P_{\text{ext}} = V m_i n_{0i} \varepsilon / 2$ , where  $V$  is the system volume). Combining Eqs. (63-65), we find (Howes et al. 2006)

$$\begin{aligned} \frac{dW}{dt} &\equiv \frac{d}{dt} \int d^3 \mathbf{r} \left[ \sum_s \left( \int d^3 \mathbf{v} \frac{T_{0s} \langle h_s^2 \rangle_{\mathbf{r}}}{2F_{0s}} - \frac{q_s^2 \phi^2 n_{0s}}{2T_{0s}} \right) + \frac{|\delta \mathbf{B}|^2}{8\pi} \right] \\ &= P_{\text{ext}} + \sum_s \int d^3 \mathbf{v} \int d^3 \mathbf{R}_s \frac{T_{0s} h_s}{F_{0s}} \left( \frac{\partial h_s}{\partial t} \right)_c. \end{aligned} \quad (66)$$

We will refer to the quantity  $W$  as the *generalized energy*. It is a positive definite quantity—this becomes explicit if we use Eq. (54) to express it in terms of the total perturbed distribution function  $\delta f_s = -q_s \phi F_{0s} / T_{0s} + h_s$  [see Eq. (47)]:

$$W = \int d^3 \mathbf{r} \left( \sum_s \int d^3 \mathbf{v} \frac{T_{0s} \delta f_s^2}{2F_{0s}} + \frac{|\delta \mathbf{B}|^2}{8\pi} \right). \quad (67)$$



Equation (66) is a conservation law of the generalized energy:  $P_{\text{ext}}$  is the source and the second term on the right-hand side, which is negative definite, represents collisional dissipation. This suggests that we might think of kinetic plasma turbulence in terms of the generalized energy  $W$  injected by the outer-scale stirring and dissipated by collisions. In order for the dissipation to be important, the collisional term in Eq. (66) has to become comparable to  $P_{\text{ext}}$ . This can happen in two ways:

1. at collisional scales ( $k_{\parallel} \lambda_{\text{mfpi}} \sim 1$ ) due to deviations of the perturbed distribution function from a local perturbed Maxwellian (see § 6.1 and Appendix D);
2. at collisionless scales ( $k_{\parallel} \lambda_{\text{mfpi}} \gg 1$ ) due the development of small scales in the velocity space (which is accompanied by the development of small perpendicular scales in the position space; see § 7.4).

Thus, the dissipation is only important at particular (small) scales, which are generally well separated from the outer scale. The generalized energy is transferred from the outer scale to the dissipation scales via a nonlinear cascade. We shall refer to this cascade as *the kinetic cascade*. It is analogous to the energy cascade in fluid or MHD turbulence, but a conceptually new feature is present: the small scales at which dissipation happens can be small scales both in the velocity and position space. How such small scales are accessed will be discussed in more detail in § 7.4.

In the sections that follow, we shall derive particular forms of  $W$  for various limiting cases of the gyrokinetic theory (§ 4.7, § 5.6, § 7.3, Appendix D.2). We shall see that the kinetic cascade of  $W$  is, indeed, a direct generalization of the more familiar fluid cascades (such as the RMHD cascades discussed in § 2) and that  $W$  contains the energy invariants of the fluid models in the appropriate limits. In these limits, the cascade of the generalized energy will split into several decoupled cascades, as it did in the case of RMHD (§ 2.5). Whenever one of the physically important scales (§ 1.3) is crossed and a change of physical regime occurs, these cascades are mixed back together into the overall kinetic cascade of  $W$ , which can then be split in a different way as it emerges on the “opposite side” of the transition region in the scale space. The conversion of the Alfvénic cascade into the KAW cascade and the entropy cascade at  $k_{\perp} \rho_i \sim 1$  is the most interesting example of such a transition, discussed in § 7.

### 3.5. Heating and Entropy

In a stationary state, all of the the turbulent power injected by the external stirring is dissipated and thus transferred into heat. Mathematically, this is expressed as a slow increase in the temperature of the Maxwellian equilibrium. In gyrokinetics, the heating time scale is ordered as  $\sim (\epsilon^2 \omega)^{-1}$ .

Even though the dissipation of turbulent fluctuations may be occurring “collisionlessly” at scales such that  $k_{\parallel} \lambda_{\text{mfpi}} \gg 1$  (e.g., via wave-particle interaction at the ion gyroscale), the resulting heating must ultimately be effected with the help of collisions. This is because heating is an irreversible process and it is a small amount of collisions that make “collisionless” damping irreversible. In other words, slow heating of the Maxwellian equilibrium is equivalent to entropy production and Boltzmann’s  $H$ -theorem rigorously requires collisions to

make this possible. Indeed, the total entropy of species  $s$  is

$$S_s = - \int d^3 \mathbf{r} \int d^3 \mathbf{v} f_s \ln f_s \\ = - \int d^3 \mathbf{r} \int d^3 \mathbf{v} \left( F_{0s} \ln F_{0s} + \frac{\delta f_s^2}{2 F_{0s}} \right) + O(\epsilon^3), \quad (68)$$

where we took  $\int d^3 \mathbf{r} \delta f_s = 0$ . It is then not hard to show that

$$\frac{3}{2} V n_{0s} \frac{1}{T_{0s}} \frac{dT_{0s}}{dt} = \frac{\overline{dS_s}}{dt} = - \int d^3 \mathbf{v} \int d^3 \mathbf{R}_s \frac{T_{0s} h_s}{F_{0s}} \left( \frac{\partial h_s}{\partial t} \right), \quad (69)$$

where the overlines mean averaging over times longer than the characteristic time of the turbulent fluctuations  $\sim \omega^{-1}$  but shorter than the typical heating time  $\sim (\epsilon^2 \omega)^{-1}$  (see Howes et al. 2006 for a detailed derivation of this and related results on heating in gyrokinetics; an earlier discussion of the entropy production in gyrokinetics can be found in Krommes & Hu 1994; Krommes 1999). We have omitted the term describing the interspecies collisional temperature equalization. Note that both sides of Eq. (69) are order  $\epsilon^2 \omega$ .

If we now time average Eq. (66) in a similar fashion, the left-hand side vanishes because it is a time derivative of a quantity fluctuating on the time scale  $\sim \omega^{-1}$  and we confirm that the right-hand side of Eq. (69) is simply equal to the average power  $\overline{P_{\text{ext}}}$  injected by external stirring. The import of Eq. (69) is that it tells us that heating can only be effected by collisions, while Eq. (66) indicates that the injected power gets to the collisional scales in velocity and position space by means of a kinetic cascade of generalized energy. This cascade contains the cascade of entropy: indeed, the first term in the expression for the generalized energy (67) is simply  $-\sum_s T_{0s} \delta S_s$ , where  $\delta S_s$  is the perturbed entropy [see Eq. (68)]. The contribution to it from the gyrocenter distribution is the integral of  $-h_s^2/2F_{0s}$ , whose evolution equation (63) can be viewed as the gyrokinetic version of the  $H$ -theorem. The first term on the right-hand side of this equation represents the wave-particle interaction. Under time average, it is related to  $\overline{P_{\text{ext}}}$  via Eqs. (64-65) and is balanced by the second term, which is the collisional-heating, or entropy-production, term that already appeared in Eq. (69). The entropy cascade to collisional scales will be discussed further in § 7.4 and § 7.5.

This concludes a short primer on gyrokinetics necessary (and sufficient) for adequate understanding of what is to follow. Formally, all further analytical developments in this paper are simply subsidiary expansions of the gyrokinetics in the parameters we listed in § 3.1: in § 4, we expand in  $(m_e/m_i)^{1/2}$ , in § 5 in  $k_{\perp} \rho_i$  (followed by further subsidiary expansions in large and small  $k_{\parallel} \lambda_{\text{mfpi}}$  in § 6), and in § 7 in  $1/k_{\perp} \rho_i$ .

## 4. ISOTHERMAL ELECTRON FLUID

In this section, we carry out an expansion of the electron gyrokinetic equation in powers of  $(m_e/m_i)^{1/2} \simeq 0.02$  (for hydrogen plasma). In virtually all cases of interest, this expansion can be done while still considering  $\beta_i$ ,  $k_{\perp} \rho_i$ , and  $k_{\parallel} \lambda_{\text{mfpi}}$  to be order unity.<sup>11</sup> Note that the assumption  $k_{\perp} \rho_i \sim 1$  together with Eq. (41) mean that

$$k_{\perp} \rho_e \sim k_{\perp} \rho_i (m_e/m_i)^{1/2} \ll 1, \quad (70)$$

<sup>11</sup> One notable exception is the LAPD device at UCLA, where  $\beta \sim 10^{-4} - 10^{-3}$  (due mostly to the electron pressure because the ions are cold,  $\tau \sim 0.1$ , so  $\beta_i \sim \beta_e/10$ ; see, e.g., Morales et al. 1999). This interferes with the mass-ratio expansion.

i.e., the expansion in  $(m_e/m_i)^{1/2}$  means also that we are considering scales larger than the electron gyroradius. The idea of such an expansion of the electron kinetic equation has been utilized many times in plasma physics literature. The mass-ratio expansion of the gyrokinetic equation in a form very similar to what is presented below is found in Snyder & Hammett (2001).

The primary import of this section will be technical: we shall dispense with the electron gyrokinetic equation and thus prepare the necessary ground for further approximations. The main results are summarized in § 4.9. A reader who is only interested in following qualitatively the major steps in the derivation may skip to this summary.

#### 4.1. Ordering the Terms in the Kinetic Equation

In view of Eq. (70),  $a_e \ll 1$ , so we can expand the Bessel functions arising from averaging over the electron ring motion:

$$J_0(a_e) = 1 - \frac{1}{4}a_e^2 + \dots, \quad \frac{J_1(a_e)}{a_e} = \frac{1}{2} \left( 1 - \frac{1}{8}a_e^2 + \dots \right). \quad (71)$$

Keeping only the lowest-order terms of the above expansions in Eq. (62) for  $\langle \chi \rangle_{\mathbf{R}_e}$ , then substituting this  $\langle \chi \rangle_{\mathbf{R}_e}$  and  $q_e = -e$  in the electron gyrokinetic equation, we get the following kinetic equation for the electrons, accurate up to and including the first order in  $(m_e/m_i)^{1/2}$  (or in  $k_\perp \rho_e$ ):

$$\begin{aligned} & \frac{\partial h_e}{\partial t} + v_{\parallel} \frac{\partial h_e}{\partial z} + \frac{c}{B_0} \left\{ \phi - \frac{v_{\parallel} A_{\parallel}}{c} - \frac{T_{0e}}{e} \frac{v_{\perp}^2}{v_{\text{the}}^2} \frac{\delta B_{\parallel}}{B_0}, h_e \right\} \\ & = - \frac{e F_{0e}}{T_{0e}} \frac{\partial}{\partial t} \left( \phi - \frac{v_{\parallel} A_{\parallel}}{c} - \frac{T_{0e}}{e} \frac{v_{\perp}^2}{v_{\text{the}}^2} \frac{\delta B_{\parallel}}{B_0} \right) + \left( \frac{\partial h_e}{\partial t} \right)_c. \end{aligned} \quad (72)$$

Note that  $\phi$ ,  $A_{\parallel}$ ,  $\delta B_{\parallel}$  in Eq. (72) are taken at  $\mathbf{r} = \mathbf{R}_e$ . We have indicated the lowest order to which each of the terms enters if compared with  $v_{\parallel} \partial h_e / \partial z$ . In order to obtain these estimates, we have assumed that the physical ordering introduced in § 3.1 holds with respect to the subsidiary expansion in  $(m_e/m_i)^{1/2}$  as well as for the primary gyrokinetic expansion in  $\epsilon$ , so we can use Eqs. (3) and (10) to order terms with respect to  $(m_e/m_i)^{1/2}$ . We have also made use of Eqs. (41), (45), and of the following three relations:

$$\frac{k_{\parallel} v_{\parallel}}{\omega} \sim \frac{v_{\text{the}}}{v_A} \sim \sqrt{\frac{\beta_i}{\tau}} \sqrt{\frac{m_i}{m_e}}, \quad (73)$$

$$\frac{(v_{\parallel}/c) A_{\parallel}}{\phi} \sim \frac{v_{\text{the}} \delta B_{\perp}}{c k_{\perp} \phi} \sim \frac{1}{k_{\perp} \rho_e} \frac{T_{0e}}{e \phi} \frac{\delta B_{\perp}}{B_0} \sim \sqrt{\frac{\beta_i}{\tau}} \sqrt{\frac{m_i}{m_e}}, \quad (74)$$

$$\frac{T_{0e}}{e \phi} \frac{v_{\perp}^2}{v_{\text{the}}^2} \frac{\delta B_{\parallel}}{B_0} \sim \frac{Z}{\tau} k_{\perp} \rho_i \sqrt{\beta_i}. \quad (75)$$

The collision term is estimated to be zeroth order because [see Eqs. (40), (42)]

$$\frac{\nu_{ei}}{\omega} \sim \frac{\tau^{3/2} \sqrt{\beta_i}}{Z^2} \sqrt{\frac{m_i}{m_e}} \frac{1}{k_{\parallel} \lambda_{\text{mfpi}}}. \quad (76)$$

The consequences of other possible orderings of the collision terms are discussed in § 4.8. We remind the reader that all dimensionless parameters except  $k_{\parallel}/k_{\perp} \sim \epsilon$  and  $(m_e/m_i)^{1/2}$  are held to be order unity.

We now let  $h_e = h_e^{(0)} + h_e^{(1)} + \dots$  and carry out the expansion to two lowest orders in  $(m_e/m_i)^{1/2}$ .

#### 4.2. Zeroth Order

To zeroth order, the electron kinetic equation is

$$v_{\parallel} \hat{\mathbf{b}} \cdot \nabla h_e^{(0)} = v_{\parallel} \frac{e F_{0e}}{c T_{0e}} \frac{\partial A_{\parallel}}{\partial t} + \left( \frac{\partial h_e^{(0)}}{\partial t} \right)_c, \quad (77)$$

where we have assembled the terms in the left-hand side to take the form of the derivative of the distribution function along the perturbed magnetic field:

$$\hat{\mathbf{b}} \cdot \nabla = \frac{\partial}{\partial z} + \frac{\delta \mathbf{B}_{\perp}}{B_0} \cdot \nabla = \frac{\partial}{\partial z} - \frac{1}{B_0} \{A_{\parallel}, \dots\}. \quad (78)$$

We now multiply Eq. (77) by  $h_e^{(0)}/F_{0e}$  and integrate over  $\mathbf{v}$  and  $\mathbf{r}$  (since we are only retaining lowest-order terms, the distinction between  $\mathbf{r}$  and  $\mathbf{R}_e$  does not matter here). Since  $\nabla \cdot \mathbf{B} = 0$ , the left-hand side vanishes (assuming that all perturbations are either periodic or vanish at the boundaries) and we get

$$\int d^3 \mathbf{r} \int d^3 \mathbf{v} \frac{h_e^{(0)}}{F_{0e}} \left( \frac{\partial h_e^{(0)}}{\partial t} \right)_c = - \frac{e n_{0e}}{c T_{0e}} \int d^3 \mathbf{r} \frac{\partial A_{\parallel}}{\partial t} u_{\parallel e}^{(0)} = 0. \quad (79)$$

The right-hand side of this equation is zero because the electron flow velocity is zero in the zeroth order,  $u_{\parallel e}^{(0)} = (1/n_{0e}) \int d^3 \mathbf{v} v_{\parallel} h_e^{(0)} = 0$ . This is a consequence of the parallel Ampère's law [Eq. (55)], which can be written as follows

$$u_{\parallel e} = \frac{c}{4\pi e n_{0e}} \nabla_{\perp}^2 A_{\parallel} + u_{\parallel i}, \quad (80)$$

where

$$u_{\parallel i} = \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \frac{1}{n_{0i}} \int d^3 \mathbf{v} v_{\parallel} J_0(a_i) h_{i\mathbf{k}}. \quad (81)$$

The three terms in Eq. (80) can be estimated as follows

$$\frac{u_{\parallel e}^{(0)}}{v_A} \sim \frac{\epsilon v_{\text{the}}}{v_A} \sim \sqrt{\frac{\beta_i}{\tau}} \sqrt{\frac{m_i}{m_e}} \epsilon, \quad (82)$$

$$\frac{u_{\parallel i}}{v_A} \sim \epsilon, \quad (83)$$

$$\frac{c \nabla_{\perp}^2 A_{\parallel}}{4\pi e n_{0e} v_A} \sim \frac{k_{\perp} \rho_i}{Z \sqrt{\beta_i}} \epsilon, \quad (84)$$

where we have used the fundamental ordering (10) of the slow waves ( $u_{\parallel i} \sim \epsilon v_A$ ) and Alfvén waves ( $\delta B_{\perp} \sim \epsilon B_0$ ). Thus, the two terms in the right-hand side of Eq. (80) are one order of  $(m_e/m_i)^{1/2}$  smaller than  $u_{\parallel e}^{(0)}$ , which means that to zeroth order, the parallel Ampère's law is  $u_{\parallel e}^{(0)} = 0$ .

The collision operator in Eq. (79) contains electron-electron and electron-ion collisions. To lowest order in  $(m_e/m_i)^{1/2}$ , the electron-ion collision operator is simply the pitch-angle scattering operator [see Eq. (B19) in Appendix B and recall that  $u_{\parallel i}$  is first order]. Therefore, we may then rewrite Eq. (79) as follows

$$\begin{aligned} & \int d^3 \mathbf{r} \int d^3 \mathbf{v} \frac{h_e^{(0)}}{F_{0e}} C_{ee}[h_e^{(0)}] \\ & - \int d^3 \mathbf{r} \int d^3 \mathbf{v} \frac{\nu_{D}^{ei}(\mathbf{v})}{F_{0e}} \frac{1 - \xi^2}{2} \left( \frac{\partial h_e^{(0)}}{\partial \xi} \right)^2 = 0. \end{aligned} \quad (85)$$

Both terms in this expression are negative definite and must, therefore, vanish individually. This implies that  $h_e^{(0)}$  must be a perturbed Maxwellian distribution with zero mean velocity (this follows from the proof of Boltzmann's H theorem; see,

e.g., Lifschitz & Pitaevskii 1981), i.e., the full electron distribution function to zeroth order in the mass-ratio expansion is [see Eq. (47)]:

$$f_e = F_{0e} + \frac{e\phi}{T_{0e}} + h_e^{(0)} = \frac{n_e}{(2\pi T_e/m_e)^{3/2}} \exp\left(-\frac{m_e v^2}{2T_e}\right), \quad (86)$$

where  $n_e = n_{0e} + \delta n_e$ ,  $T_e = T_{0e} + \delta T_e$ . Expanding around the unperturbed Maxwellian  $F_{0e}$ , we get

$$h_e^{(0)} = \left[ \frac{\delta n_e}{n_{0e}} - \frac{e\phi}{T_{0e}} + \left( \frac{v^2}{v_{the}^2} - \frac{3}{2} \right) \frac{\delta T_e}{T_{0e}} \right] F_{0e}, \quad (87)$$

where the fields are taken at  $\mathbf{r} = \mathbf{R}_e$ . Now substitute this solution back into Eq. (77). The collision term vanishes and the remaining equation must be satisfied at arbitrary values of  $v$ . This gives

$$\frac{1}{c} \frac{\partial A_{\parallel}}{\partial t} + \hat{\mathbf{b}} \cdot \nabla \phi = \hat{\mathbf{b}} \cdot \nabla \frac{T_{0e}}{e} \frac{\delta n_e}{n_{0e}}, \quad (88)$$

$$\hat{\mathbf{b}} \cdot \nabla \frac{\delta T_e}{T_{0e}} = 0. \quad (89)$$

#### 4.3. Flux Conservation

Equation (88) establishes a relation between  $A_{\parallel}$ ,  $\phi$  and  $\delta n_e$ . The collision term is neglected because, for  $h_e^{(0)}$  given by Eq. (87), it vanishes to zeroth order. Therefore, Ohmic resistivity is a subdominant effect in the  $(m_e/m_i)^{1/2}$  expansion and the magnetic-field lines cannot be broken. Indeed, we may follow Cowley (1985) and argue that the left-hand side of Eq. (88) is minus the projection of the electric field on the total magnetic field [see Eq. (33)], so we have  $\mathbf{E} \cdot \hat{\mathbf{b}} = -\hat{\mathbf{b}} \cdot \nabla (T_{0e} \delta n_e / en_{0e})$ ; hence the total electric field is

$$\mathbf{E} = (\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) \cdot \left( \mathbf{E} + \nabla \frac{T_{0e}}{e} \frac{\delta n_e}{n_{0e}} \right) - \nabla \frac{T_{0e}}{e} \frac{\delta n_e}{n_{0e}} \quad (90)$$

and Faraday's law becomes

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} = \nabla \times (\mathbf{u}_{\text{eff}} \times \mathbf{B}), \quad (91)$$

$$\mathbf{u}_{\text{eff}} = \frac{c}{B^2} \left( \mathbf{E} + \nabla \frac{T_{0e}}{e} \frac{\delta n_e}{n_{0e}} \right) \times \mathbf{B}, \quad (92)$$

i.e., the magnetic field lines are frozen into the velocity field  $\mathbf{u}_{\text{eff}}$ . In Appendix C.1, we show that this effective velocity is the part of the electron flow velocity  $\mathbf{u}_e$  perpendicular to the total magnetic field  $\mathbf{B}$  [see Eq. (C9)].

#### 4.4. Isothermal Electrons

Equation (89) mandates that the perturbed electron temperature must remain constant along the perturbed field lines. Strictly speaking, this does not preclude  $\delta T_e$  varying across the field lines. However, we shall now assume that the field lines are stochastic and, therefore,  $\delta T_e = \text{const}$  (has no spatial variation). Assuming that no spatially uniform perturbations exist, we may set  $\delta T_e = 0$ . Equation (87) then reduces to

$$h_e^{(0)} = \left( \frac{\delta n_e}{n_{0e}} - \frac{e\phi}{T_{0e}} \right) F_{0e}(v), \quad (93)$$

or, using Eq. (47),

$$\delta f_e = \frac{\delta n_e}{n_{0e}} F_{0e}(v). \quad (94)$$

Hence follows the equation of state for isothermal electrons:

$$\delta p_e = T_{0e} \delta n_e. \quad (95)$$

#### 4.5. First Order

We now subtract Eq. (77) from Eq. (72), integrate the remainder over the velocity space (keeping  $\mathbf{r}$  constant), and retain first-order terms only. Using Eq. (93), we get

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{\delta n_e}{n_{0e}} - \frac{\delta B_{\parallel}}{B_0} \right) + \frac{c}{B_0} \left\{ \phi, \frac{\delta n_e}{n_{0e}} - \frac{\delta B_{\parallel}}{B_0} \right\} \\ + \frac{\partial u_{\parallel e}}{\partial z} - \frac{1}{B_0} \{ A_{\parallel}, u_{\parallel e} \} + \frac{c T_{0e}}{e B_0} \left\{ \frac{\delta n_e}{n_{0e}}, \frac{\delta B_{\parallel}}{B_0} \right\} = 0, \end{aligned} \quad (96)$$

where the parallel electron velocity is first order:

$$u_{\parallel e} = u_{\parallel e}^{(1)} = \frac{1}{n_{0e}} \int d^3 \mathbf{v} v_{\parallel} h_e^{(1)}. \quad (97)$$

The velocity-space integral of the collision term does not enter because it is subdominant by at least one factor of  $(m_e/m_i)^{1/2}$ ; indeed, as shown in Appendix B.1, the velocity integration leads to an extra factor of  $k_{\perp}^2 \rho_e^2$ , so that

$$\begin{aligned} \frac{1}{n_{0e}} \int d^3 \mathbf{v} \left( \frac{\partial h_e}{\partial t} \right)_c \sim \nu_{ei} k_{\perp}^2 \rho_e^2 \frac{\delta n_e}{n_{0e}} \\ \sim \sqrt{\tau} \beta_i \frac{k_{\perp}^2 \rho_i^2}{k_{\parallel} \lambda_{\text{mfpi}}} \sqrt{\frac{m_e}{m_i}} \omega \frac{\delta n_e}{n_{0e}}, \end{aligned} \quad (98)$$

where we have used Eqs. (41) and (76).

#### 4.6. Field Equations

Using Eq. (93) and  $q_i = Ze$ ,  $n_{0e} = Zn_{0i}$ ,  $T_{0e} = T_{0i}/\tau$ , we derive from the quasineutrality equation (54) [see also Eq. (58)]

$$\frac{\delta n_e}{n_{0e}} = \frac{\delta n_i}{n_{0i}} = -\frac{Ze\phi}{T_{0i}} + \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \frac{1}{n_{0i}} \int d^3 \mathbf{v} J_0(a_i) h_{i\mathbf{k}}, \quad (99)$$

and, from the perpendicular part of Ampère's law [Eq. (59), using also Eq. (99)],

$$\begin{aligned} \frac{\delta B_{\parallel}}{B_0} = \frac{\beta_i}{2} \left\{ \left( 1 + \frac{Z}{\tau} \right) \frac{Ze\phi}{T_{0i}} - \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \right. \\ \left. \times \frac{1}{n_{0i}} \int d^3 \mathbf{v} \left[ \frac{Z}{\tau} J_0(a_i) + \frac{2v_{\perp}^2}{v_{thi}^2} \frac{J_1(a_i)}{a_i} \right] h_{i\mathbf{k}} \right\}. \end{aligned} \quad (100)$$

The parallel electron velocity,  $u_{\parallel e}$ , is determined from the parallel part of Ampère's law, Eq. (80).

The ion distribution function  $h_i$  that enters these equations has to be determined by solving the ion gyrokinetic equation: Eq. (50) with  $s = i$ . Note that computing the collision term  $(\partial h_i / \partial t)_c$  in this equation does not require knowledge of  $h_e$  because the ion-electron collisions are subdominant in the  $(m_e/m_i)^{1/2}$  expansion [see Eq. (43)].

#### 4.7. Generalized Energy

The generalized energy (§3.4) for the case of isothermal electrons is calculated by substituting Eq. (94) into Eq. (67):

$$W = \int d^3 \mathbf{r} \left( \int d^3 \mathbf{v} \frac{T_{0i} \delta f_i^2}{2 F_{0i}} + \frac{n_{0e} T_{0e}}{2} \frac{\delta n_e^2}{n_{0e}^2} + \frac{|\delta \mathbf{B}|^2}{8\pi} \right). \quad (101)$$

#### 4.8. Validity of the Mass-Ratio Expansion

Let us examine the range of spatial scales in which the equations derived above are valid. In carrying out the expansion in  $(m_e/m_i)^{1/2}$ , we ordered  $k_{\perp} \rho_i \sim 1$  [Eq. (70)] and  $k_{\parallel} \lambda_{\text{mfpi}} \sim 1$  [Eq. (76)]. Formally, this means that the perpendicular and parallel wavelengths of the perturbations must not be so small or so large as to interfere with the mass ratio expansion. We now discuss the four conditions that this requirement leads to and whether any of them can be violated without destroying the validity of the equations derived above.



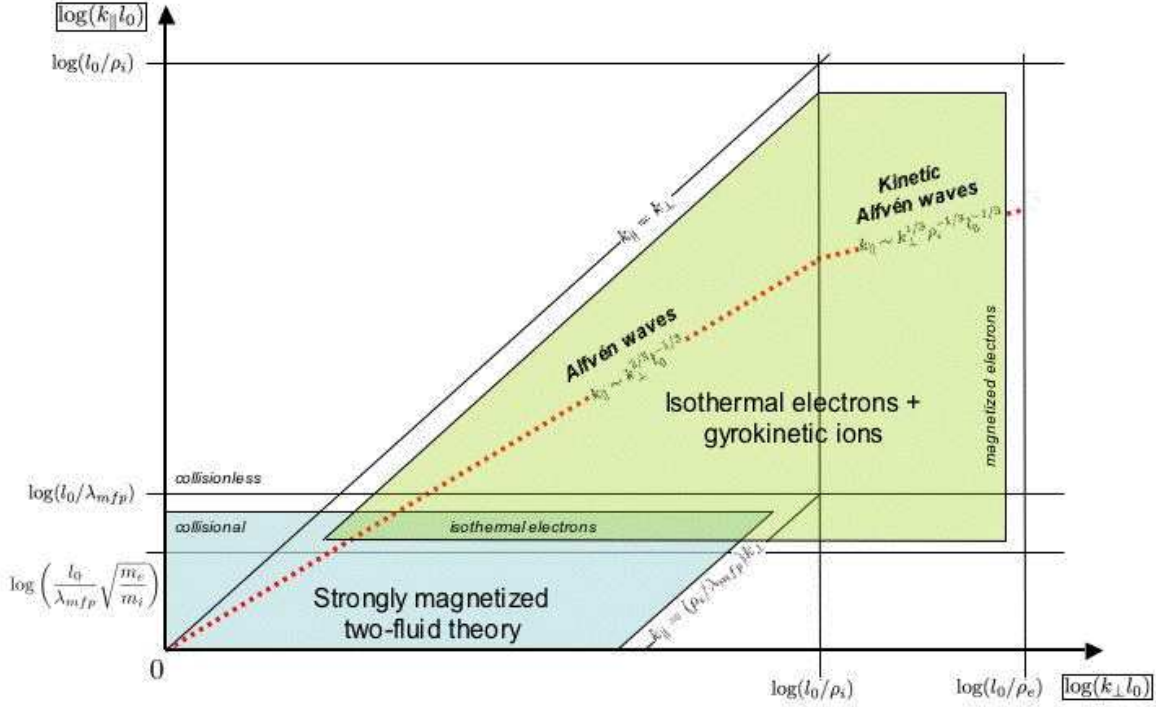


FIG. 3.— The region of validity in the wave-number space of the secondary approximation—*isothermal electrons and gyrokinetic ions* (§4). It is the region of validity of the gyrokinetic approximation (Fig. 2) further circumscribed by two conditions:  $k_{\parallel} \lambda_{mfpi} \gg (m_e/m_i)^{1/2}$  (*isothermal electrons*) and  $k_{\perp} \rho_e \ll 1$  (*magnetized electrons*). The region of validity of the strongly magnetized two-fluid theory (Appendix A.2) is also shown. It is the same as for the full two-fluid theory plus the additional constraint  $k_{\perp} \rho_i \ll k_{\parallel} \lambda_{mfpi}$ . The region of validity of MHD (or one-fluid theory) is the subset of this with  $k_{\parallel} \lambda_{mfpi} \ll (m_e/m_i)^{1/2}$  (*adiabatic electrons*).

#### 4.8.1. $k_{\perp} \rho_i \ll (m_i/m_e)^{1/2}$ .

This is equivalent to demanding that  $k_{\perp} \rho_e \ll 1$ , a condition that was, indeed, essential for the expansion to hold [Eq. (71)]. This is not a serious limitation because electrons can be considered well magnetized at virtually all scales of interest for astrophysical applications. However, we do forfeit the detailed information about some important electron physics at  $k_{\perp} \rho_e \sim 1$ : for example such effects as wave damping at the electron gyroscale and the electron heating (although the total amount of the electron heating can be deduced by subtracting the ion heating from the total energy input). The breaking of the flux conservation (resistivity) is also an effect that requires incorporation of the finite electron gyroscale physics.

#### 4.8.2. $k_{\perp} \rho_i \gg (m_e/m_i)^{1/2}$ .

If this condition is broken, the small- $k_{\perp} \rho_i$  expansion, carried out in §5, must, formally speaking, precede the mass-ratio expansion. However, it turns out that the small- $k_{\perp} \rho_i$  expansion commutes with the mass-ratio expansion (Schekochihin et al. 2007a, see also footnote 13), so we may use the equations derived in §§4.2-4.6 when  $k_{\perp} \rho_i \lesssim (m_e/m_i)^{1/2}$ .

#### 4.8.3. $k_{\parallel} \lambda_{mfpi} \ll (m_i/m_e)^{1/2}$ .

Let us consider what happens if this condition is broken and  $k_{\parallel} \lambda_{mfpi} \gtrsim (m_i/m_e)^{1/2}$ . In this case, the collisions become even weaker and the expansion procedure must be modified. Namely, the collision term picks up one extra order of  $(m_e/m_i)^{1/2}$ , so it is first order in Eq. (72). To zeroth order, the electron kinetic equation no longer contains collisions: in-

stead of Eq. (77), we have

$$v_{\parallel} \hat{\mathbf{b}} \cdot \nabla h_e^{(0)} = v_{\parallel} \frac{e F_{0e}}{c T_{0e}} \frac{\partial A_{\parallel}}{\partial t}. \quad (102)$$

We may seek the solution of this equation in the form  $h_e^{(0)} = H(t, \mathbf{R}_e) F_{0e} + \tilde{h}$ , where  $H(t, \mathbf{R}_e)$  is an unknown function to be determined and  $\tilde{h}$  is the homogeneous solution satisfying

$$\hat{\mathbf{b}} \cdot \nabla \tilde{h} = 0, \quad (103)$$

i.e.,  $\tilde{h}$  must be constant along the perturbed magnetic field. This is a generalization of Eq. (89). Again assuming stochastic field lines, we conclude that  $\tilde{h}$  is independent of space. If we rule out spatially uniform perturbations, we may set  $\tilde{h} = 0$ . The unknown function  $H(t, \mathbf{R}_e)$  is readily expressed in terms of  $\delta n_e$  and  $\phi$ :

$$\frac{\delta n_e}{n_{0e}} = \frac{e\phi}{T_{0e}} + \frac{1}{n_{0e}} \int d^3 \mathbf{v} h_e^{(0)} \Rightarrow H = \frac{\delta n_e}{n_{0e}} - \frac{e\phi}{T_{0e}}, \quad (104)$$

so  $h_e^{(0)}$  is again given by Eq. (93), so the equations derived in §§4.2-4.6 are unaltered. Thus, the mass-ratio expansion remains valid at  $k_{\parallel} \lambda_{mfpi} \gtrsim (m_i/m_e)^{1/2}$ .

#### 4.8.4. $k_{\parallel} \lambda_{mfpi} \gg (m_e/m_i)^{1/2}$ .

If the parallel wavelength of the fluctuations is so long that this is violated,  $k_{\parallel} \lambda_{mfpi} \lesssim (m_e/m_i)^{1/2}$ , the collision term in Eq. (72) is minus first order. This is the lowest-order term in the equation. Setting it to zero obliges  $h_e^{(0)}$  to be a perturbed Maxwellian again given by Eq. (87). Instead of Eq. (77), the zeroth-order kinetic equation is

$$v_{\parallel} \hat{\mathbf{b}} \cdot \nabla h_e^{(0)} = v_{\parallel} \frac{e F_{0e}}{c T_{0e}} \frac{\partial A_{\parallel}}{\partial t} + \left( \frac{\partial h_e^{(1)}}{\partial t} \right)_c. \quad (105)$$

Now the collision term in this order contains  $h_e^{(1)}$ , which can be determined from Eq. (105) by inverting the collision operator. This sets up a perturbation theory that in due course leads to the Reduced MHD version of the general MHD equations—this is what was considered in § 2. Equation (89) no longer needs to hold, so the electrons are not isothermal. In this true one-fluid limit, both electrons and ions are adiabatic with equal temperatures [see Eq. (107) below]. The collisional transport terms in this limit (parallel and perpendicular resistivity, viscosity, heat fluxes, etc.) were calculated [starting not from gyrokinetics but from the general Vlasov–Landau equation (32)] in exhaustive detail by Braginskii (1965). His results and the way RMHD emerges from them are reviewed in Appendix A.

In physical terms, the electrons can no longer be isothermal if the parallel electron diffusion time becomes longer than the characteristic time of the fluctuations (the Alfvén time):

$$\frac{1}{\nu_{\text{the}} \lambda_{\text{mfpi}} k_{\parallel}^2} \gtrsim \frac{1}{k_{\parallel} v_A} \Leftrightarrow k_{\parallel} \lambda_{\text{mfpi}} \lesssim \frac{1}{\sqrt{\beta_i}} \sqrt{\frac{m_e}{m_i}}. \quad (106)$$

Furthermore, under a similar condition, electron and ion temperatures must equalize: this happens if the ion-electron collision time is shorter than the Alfvén time,

$$\frac{1}{\nu_{ie}} \lesssim \frac{1}{k_{\parallel} v_A} \Leftrightarrow k_{\parallel} \lambda_{\text{mfpi}} \lesssim \sqrt{\beta_i} \sqrt{\frac{m_e}{m_i}} \quad (107)$$

(see Lithwick & Goldreich 2001 for a discussion of these conditions in application to the ISM).

#### 4.9. Summary

The original gyrokinetic description introduced in § 3 was a system of two kinetic equations [Eq. (50)] that evolved the electron and ion distribution functions  $h_e$ ,  $h_i$  and three field equations [Eqs. (54–56)] that related  $\phi$ ,  $A_{\parallel}$  and  $\delta B_{\parallel}$  to  $h_e$  and  $h_i$ . In this section, we have taken advantage of the smallness of the electron mass to treat the electrons as an isothermal magnetized fluid, while ions remained fully gyrokinetic.

In mathematical terms, we solved the electron kinetic equation and replaced the gyrokinetics with a simpler closed system of equations that evolve 6 unknown functions:  $\phi$ ,  $A_{\parallel}$ ,  $\delta B_{\parallel}$ ,  $\delta n_e$ ,  $u_{\parallel e}$  and  $h_i$ . These satisfy two fluid-like evolution equations (88) and (96), three integral relations (99), (100), and (80) which involve  $h_i$ , and the kinetic equation (50) for  $h_i$ . The system is simpler because the full electron distribution function has been replaced by two scalar fields  $\delta n_e$  and  $u_{\parallel e}$ . We now summarize this new system of equations: denoting  $a_i = k_{\perp} v_{\perp} / \Omega_i$ , we have

$$\frac{1}{c} \frac{\partial A_{\parallel}}{\partial t} + \hat{\mathbf{b}} \cdot \nabla \phi = \hat{\mathbf{b}} \cdot \nabla \frac{T_{0e}}{e} \frac{\delta n_e}{n_{0e}}, \quad (108)$$

$$\frac{d}{dt} \left( \frac{\delta n_e}{n_{0e}} - \frac{\delta B_{\parallel}}{B_0} \right) + \hat{\mathbf{b}} \cdot \nabla u_{\parallel e} = -\frac{c T_{0e}}{e B_0} \left\{ \frac{\delta n_e}{n_{0e}}, \frac{\delta B_{\parallel}}{B_0} \right\}, \quad (109)$$

$$\frac{\delta n_e}{n_{0e}} = -\frac{Ze\phi}{T_{0i}} + \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \frac{1}{n_{0i}} \int d^3 \mathbf{v} J_0(a_i) h_{i\mathbf{k}}, \quad (110)$$

$$\begin{aligned} \frac{\delta B_{\parallel}}{B_0} = \frac{\beta_i}{2} \left\{ \left( 1 + \frac{Z}{\tau} \right) \frac{Ze\phi}{T_{0i}} - \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \right. \\ \left. \times \frac{1}{n_{0i}} \int d^3 \mathbf{v} \left[ \frac{Z}{\tau} J_0(a_i) + \frac{2v_{\perp}^2}{v_{\text{thi}}^2} \frac{J_1(a_i)}{a_i} \right] h_{i\mathbf{k}} \right\}, \quad (111) \end{aligned}$$

$$u_{\parallel e} = \frac{c}{4\pi e n_{0e}} \nabla_{\perp}^2 A_{\parallel} + \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \frac{1}{n_{0i}} \int d^3 \mathbf{v} v_{\parallel} J_0(a_i) h_{i\mathbf{k}}, \quad (112)$$

and Eq. (50) for  $s = i$  and ion-ion collisions only:

$$\frac{\partial h_i}{\partial t} + v_{\parallel} \frac{\partial h_i}{\partial z} + \frac{c}{B_0} \{ \langle \chi \rangle_{\mathbf{R}_i}, h_i \} = \frac{Ze}{T_{0i}} \frac{\partial \langle \chi \rangle_{\mathbf{R}_i}}{\partial t} F_{0i} + \langle C_{ii}[h_i] \rangle_{\mathbf{R}_i}, \quad (113)$$

where  $\langle C_{ii}[\dots] \rangle_{\mathbf{R}_i}$  is the gyrokinetic ion-ion collision operator (see Appendix B) and the ion-electron collisions have been neglected to lowest order in  $(m_e/m_i)^{1/2}$  [see Eq. (43)]. Note that Eqs. (108–109) have been written in a compact form, where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla = \frac{\partial}{\partial t} + \frac{c}{B_0} \{ \phi, \dots \} \quad (114)$$

is the convective derivative with respect to the  $\mathbf{E} \times \mathbf{B}$  drift velocity,  $\mathbf{u}_E = -c \nabla_{\perp} \phi \times \hat{\mathbf{z}} / B_0$ , and

$$\hat{\mathbf{b}} \cdot \nabla = \frac{\partial}{\partial z} + \frac{\delta \mathbf{B}_{\perp}}{B_0} \cdot \nabla = \frac{\partial}{\partial z} - \frac{1}{B_0} \{ A_{\parallel}, \dots \} \quad (115)$$

is the gradient along the total magnetic field (mean field plus perturbation).

The generalized energy conserved by Eqs. (108–113) is given by Eq. (101).

It is worth observing that the left-hand side of Eq. (108) is simply minus the component of the electric field along the total magnetic field [see Eq. (33)]. This was used in § 4.3 to prove that the magnetic flux described by Eq. (108) is exactly conserved. Equation (108) is the projection of the generalized Ohm's law onto the total magnetic field—the right-hand side of this equation is the so-called thermoelectric term. This is discussed in more detail in Appendix C.1, where we also show that Eq. (109) is the parallel part of Faraday's law and give a qualitative nongyrokinetic derivation of Eqs. (108–109).

We will refer to Eqs. (108–113) as *the equations of isothermal electron fluid*. They are valid in a broad range of scales: the only constraints are that  $k_{\parallel} \ll k_{\perp}$  (gyrokinetic ordering, § 3.1),  $k_{\perp} \rho_e \ll 1$  (electrons are magnetized, § 4.8.1) and  $k_{\parallel} \lambda_{\text{mfpi}} \gg (m_e/m_i)^{1/2}$  (electrons are isothermal, § 4.8.4). The region of validity of Eqs. (108–113) in the wave-number space is illustrated in Fig. 3. A particular advantage of this hybrid fluid-kinetic system is that it is uniformly valid across the transition from magnetized to unmagnetized ions (i.e., from  $k_{\perp} \rho_i \ll 1$  to  $k_{\perp} \rho_i \gg 1$ ).

#### 5. TURBULENCE IN THE INERTIAL RANGE: KINETIC RMHD

Our goal in this section is to derive a reduced set of equations that describe the magnetized plasma in the limit of small  $k_{\perp} \rho_i$ . Before we proceed with an expansion in  $k_{\perp} \rho_i$ , we need to make a formal technical step, the usefulness of which will become clear shortly. A reader with no patience for this or any of the subsequent technical developments may skip to the summary at the end of this section (§ 5.7).

##### 5.1. A Technical Step

Let us formally split the ion gyrocenter distribution function into two parts:

$$\begin{aligned} h_i &= \frac{Ze}{T_{0i}} \left\langle \phi - \frac{\mathbf{v}_{\perp} \cdot \mathbf{A}_{\perp}}{c} \right\rangle_{\mathbf{R}_i} F_{0i} + g \\ &= \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{R}_i} \left[ J_0(a_i) \frac{Ze\phi_{\mathbf{k}}}{T_i} + \frac{2v_{\perp}^2}{v_{\text{thi}}^2} \frac{J_1(a_i)}{a_i} \frac{\delta B_{\parallel \mathbf{k}}}{B_0} \right] F_{0i} + g. \quad (116) \end{aligned}$$

Then  $g$  satisfies the following equation, obtained by substituting Eq. (116) and the expression for  $\partial A_{\parallel} / \partial t$  that follows from

Eq. (108) into the ion gyrokinetic equation (113):

$$\begin{aligned}
& \frac{\partial g}{\partial t} + v_{\parallel} \frac{\partial g}{\partial z} + \frac{c}{B_0} \{ \langle \chi \rangle_{\mathbf{R}_i}, g \} - \langle C_{ii}[g] \rangle_{\mathbf{R}_i} = \\
& \quad \textcircled{0} \\
& \quad - \frac{Ze}{T_{0i}} v_{\parallel} \left\langle \frac{1}{B_0} \{ A_{\parallel}, \phi - \langle \phi \rangle_{\mathbf{R}_i} \} \right\rangle_{\mathbf{R}_i} \\
& \quad \textcircled{1} \\
& + \hat{\mathbf{b}} \cdot \nabla \left( \frac{T_{0e}}{e} \frac{\delta n_e}{n_{0e}} - \left\langle \frac{\mathbf{v}_{\perp} \cdot \mathbf{A}_{\perp}}{c} \right\rangle_{\mathbf{R}_i} \right) \Bigg\rangle_{\mathbf{R}_i} F_{0i} \\
& \quad \textcircled{0} \\
& + \frac{Ze}{T_{0i}} \left\langle C_{ii} \left[ \left\langle \phi - \frac{\mathbf{v}_{\perp} \cdot \mathbf{A}_{\perp}}{c} \right\rangle_{\mathbf{R}_i} F_{0i} \right] \right\rangle_{\mathbf{R}_i}. \quad (117)
\end{aligned}$$

The field equations (110-111) rewritten in terms of  $g$  are

$$\begin{aligned}
& \frac{\delta n_{ke}}{n_{0e}} - \Gamma_1(\alpha_i) \frac{\delta B_{\parallel \mathbf{k}}}{B_0} + [1 - \Gamma_0(\alpha_i)] \frac{Ze\phi_{\mathbf{k}}}{T_{0i}} \\
& \quad \textcircled{0} \quad \quad \textcircled{0} \quad \quad \textcircled{1} \\
& = \frac{1}{n_{0i}} \int d^3 \mathbf{v} J_0(a_i) g_{\mathbf{k}}, \quad (118) \\
& \quad \quad \quad \textcircled{0}
\end{aligned}$$

$$\begin{aligned}
& u_{\parallel \mathbf{k}e} + \frac{c}{4\pi e n_{0e}} k_{\perp}^2 A_{\parallel \mathbf{k}} = \frac{1}{n_{0i}} \int d^3 \mathbf{v} v_{\parallel} J_0(a_i) g_{\mathbf{k}} = u_{\parallel \mathbf{k}i}, \quad (119) \\
& \quad \textcircled{0} \quad \quad \textcircled{1} \quad \quad \quad \textcircled{0}
\end{aligned}$$

$$\begin{aligned}
& \frac{Z}{\tau} \frac{\delta n_{ke}}{n_{0e}} + \left[ \Gamma_2(\alpha_i) + \frac{2}{\beta_i} \right] \frac{\delta B_{\parallel \mathbf{k}}}{B_0} - [1 - \Gamma_1(\alpha_i)] \frac{Ze\phi_{\mathbf{k}}}{T_{0i}} \\
& \quad \textcircled{0} \quad \quad \quad \textcircled{0} \quad \quad \quad \textcircled{1} \\
& = -\frac{1}{n_{0i}} \int d^3 \mathbf{v} \frac{2v_{\perp}^2}{v_{\text{thi}}^2} \frac{J_1(a_i)}{a_i} g_{\mathbf{k}}, \quad (120) \\
& \quad \quad \quad \textcircled{0}
\end{aligned}$$

where  $\alpha_i = k_{\perp}^2 \rho_i^2 / 2$  and we have defined

$$\begin{aligned}
& \Gamma_0(\alpha_i) = \frac{1}{n_{0i}} \int d^3 \mathbf{v} [J_0(a_i)]^2 F_{0i} \\
& = I_0(\alpha_i) e^{-\alpha_i} = 1 - \alpha_i + \dots, \quad (121)
\end{aligned}$$

$$\begin{aligned}
& \Gamma_1(\alpha_i) = \frac{1}{n_{0i}} \int d^3 \mathbf{v} \frac{2v_{\perp}^2}{v_{\text{thi}}^2} J_0(a_i) \frac{J_1(a_i)}{a_i} F_{0i} = -\Gamma_0'(\alpha_i) \\
& = [I_0(\alpha_i) - I_1(\alpha_i)] e^{-\alpha_i} = 1 - \frac{3}{2} \alpha_i + \dots, \quad (122)
\end{aligned}$$

$$\Gamma_2(\alpha_i) = \frac{1}{n_{0i}} \int d^3 \mathbf{v} \left[ \frac{2v_{\perp}^2}{v_{\text{thi}}^2} \frac{J_1(a_i)}{a_i} \right]^2 F_{0i} = 2\Gamma_1(\alpha_i). \quad (123)$$

Underneath each term in Eqs. (117-120), we have indicated the lowest order in  $k_{\perp} \rho_i$  to which this term enters.

### 5.2. Subsidiary Ordering in $k_{\perp} \rho_i$

In order to carry out a subsidiary expansion in small  $k_{\perp} \rho_i$ , we must order all terms in Eqs. (88-96) and (117-120) with respect to  $k_{\perp} \rho_i$ . Let us again assume, like we did when expanding the electron equation (§ 4), that the ordering introduced

for the gyrokinetics in § 3.1 holds also for the subsidiary expansion in  $k_{\perp} \rho_i$ . First note that, in view of Eq. (45), we must regard  $Ze\phi/T_i$  to be minus first order:

$$\frac{Ze\phi}{T_i} \sim \frac{\epsilon}{k_{\perp} \rho_i \sqrt{\beta_i}}. \quad (124)$$

Also, as  $\delta B_{\perp}/B_0 \sim \epsilon$  [Eq. (10)],

$$\frac{(v_{\parallel}/c)A_{\parallel}}{\phi} \sim \frac{v_{\text{thi}} \delta B_{\perp}}{ck_{\perp} \phi} \sim \frac{1}{k_{\perp} \rho_i} \frac{T_i}{Ze\phi} \frac{\delta B_{\perp}}{B_0} \sim \sqrt{\beta_i}, \quad (125)$$

so  $\phi$  and  $(v_{\parallel}/c)A_{\parallel}$  are same order.

Since  $u_{\parallel} = u_{\parallel i}$  (electrons do not contribute to the mass flow), assuming that slow waves and Alfvén waves have comparable energies implies  $u_{\parallel i} \sim u_{\perp}$ . As  $u_{\parallel i}$  is determined by the second equality in Eq. (119), we can order  $g$  [using Eq. (10)]:

$$\frac{g}{F_{0i}} \sim \frac{u_{\parallel}}{v_{\text{thi}}} \sim \frac{u_{\perp}}{v_{\text{thi}}} \sim \frac{\epsilon}{\sqrt{\beta_i}}, \quad (126)$$

so  $g$  is zeroth order in  $k_{\perp} \rho_i$ . Similarly,  $\delta n_e/n_{0e} \sim \delta B_{\parallel}/B_0 \sim \epsilon$  are zeroth order in  $k_{\perp} \rho_i$ —this follows directly from Eq. (10).

Together with Eq. (3), the above considerations allow us to order all terms in our equations. The ordering of the collision term involving  $\phi$  is explained in Appendix B.2.

### 5.3. Alfvén Waves: Kinetic Derivation of RMHD

We shall now show that the RMHD equations (13-14) hold in this approximation. There is a simple correspondence between the stream and flux functions defined in Eq. (12) and the electromagnetic potentials  $\phi$  and  $A_{\parallel}$ :

$$\Phi = \frac{c}{B_0} \phi, \quad \Psi = -\frac{A_{\parallel}}{\sqrt{4\pi m_i n_{0i}}}. \quad (127)$$

The first of these definitions says that the perpendicular flow velocity  $\mathbf{u}_{\perp}$  is the  $\mathbf{E} \times \mathbf{B}$  drift velocity; the second definition is the standard MHD relation between the magnetic flux function and the parallel component of the vector potential.

Deriving Eq. (13) is straightforward: in Eq. (88), we retain only the lowest—minus first—order terms (those that contain  $\phi$  and  $A_{\parallel}$ ). The result is

$$\frac{\partial A_{\parallel}}{\partial t} + c \frac{\partial \phi}{\partial z} - \frac{c}{B_0} \{ A_{\parallel}, \phi \} = 0. \quad (128)$$

Using Eq. (127) and the definition of the Alfvén speed,  $v_A = B_0/\sqrt{4\pi m_i n_{0i}}$ , we get Eq. (13). By the argument of § 4.3, Eq. (128) expresses the fact that that magnetic-field lines are frozen into the  $\mathbf{E} \times \mathbf{B}$  velocity field, which is the mean flow velocity associated with the Alfvén waves (see § 5.4).

As we are about to see, in order to derive Eq. (132), we have to separate the first-order part of the  $k_{\perp} \rho_i$  expansion. The easiest way to achieve this, is to integrate Eq. (117) over the velocity space (keeping  $\mathbf{r}$  constant) and expand the resulting equation in small  $k_{\perp} \rho_i$ . Using Eqs. (118) and (119) to express



the velocity-space integrals of  $g$ , we get

$$\begin{aligned} \frac{\partial}{\partial t} [1 - \Gamma_0(\alpha_i)] \frac{Ze\phi_{\mathbf{k}}}{T_{0i}} + \frac{\partial}{\partial t} \left[ \frac{\delta n_{\mathbf{k}e}}{n_{0e}} - \Gamma_1(\alpha_i) \frac{\delta B_{\parallel \mathbf{k}}}{B_0} \right] & \quad (129) \\ \textcircled{1} & \quad \textcircled{0} \\ + \frac{\partial}{\partial z} \left( u_{\parallel \mathbf{k}e} + \frac{c}{4\pi en_{0e}} k_{\perp}^2 A_{\parallel \mathbf{k}} \right) & \quad \textcircled{0} \quad \textcircled{1} \\ + \frac{c}{B_0} \frac{1}{n_{0i}} \int d^3 \mathbf{v} J_0(a_i) \{ \langle \chi \rangle_{\mathbf{R}_i}, g \}_{\mathbf{k}} & \quad \textcircled{0} \\ = \frac{1}{n_{0i}} \int d^3 \mathbf{v} J_0(a_i) \left\langle C_{ii} \left[ \frac{Ze}{T_{0i}} \left\langle \phi - \frac{\mathbf{v}_{\perp} \cdot \mathbf{A}_{\perp}}{c} \right\rangle_{\mathbf{R}_i} F_{0i} + g \right] \right\rangle_{\mathbf{R}_i, \mathbf{k}} & \quad \textcircled{3} \quad \textcircled{2} \quad \textcircled{2} \end{aligned}$$

Underneath each term, the lowest order in  $k_{\perp} \rho_i$  to which it enters is shown. We see that terms containing  $\phi$  are all first order, so it is up to this order that we shall retain terms. The collision term integrated over the velocity space picks up two extra orders of  $k_{\perp} \rho_i$  (see Appendix B.1), so it is second order and can, therefore, be dropped. As a consequence of quasineutrality, the zeroth-order part of the above equation exactly coincides with Eq. (96), i.e.,  $\delta n_i/n_{0i} = \delta n_e/n_{0e}$  satisfy the same equation. Indeed, neglecting second-order terms (but not first-order ones!), the nonlinear term in Eq. (129) (the last term on the left-hand side) is

$$\begin{aligned} \frac{c}{B_0} \left\{ \phi, \frac{1}{n_{0i}} \int d^3 \mathbf{v} g \right\} - \frac{1}{B_0} \left\{ A_{\parallel}, \frac{1}{n_{0i}} \int d^3 \mathbf{v} v_{\parallel} g \right\} \\ + \frac{c T_{0i}}{Ze B_0} \left\{ \frac{\delta B_{\parallel}}{B_0}, \frac{1}{n_{0i}} \int d^3 \mathbf{v} \frac{v_{\perp}^2}{v_{\text{thi}}^2} g \right\}, \quad (130) \end{aligned}$$

and, using Eqs. (118-120) to express velocity-space integrals of  $g$  in the above expression, we find that the zeroth-order part of the nonlinearity is the same as the nonlinearity in Eq. (96), while the first-order part is

$$-\frac{c}{B_0} \left\{ \phi, \frac{1}{2} \rho_i^2 \nabla_{\perp}^2 \frac{Ze\phi}{T_{0i}} \right\} + \frac{1}{B_0} \left\{ A_{\parallel}, \frac{c}{4\pi en_{0e}} \nabla_{\perp}^2 A_{\parallel} \right\}, \quad (131)$$

where we have used the expansion (121) of  $\Gamma_0(\alpha_i)$  and converted it back into  $x$  space.

Thus, if we subtract Eq. (96) from Eq. (129), the remainder is first order and reads

$$\begin{aligned} \frac{\partial}{\partial t} \frac{1}{2} \rho_i^2 \nabla_{\perp}^2 \frac{Ze\phi}{T_{0i}} + \frac{c}{B_0} \left\{ \phi, \frac{1}{2} \rho_i^2 \nabla_{\perp}^2 \frac{Ze\phi}{T_{0i}} \right\} \\ + \frac{\partial}{\partial z} \frac{c}{4\pi en_{0e}} \nabla_{\perp}^2 A_{\parallel} - \frac{1}{B_0} \left\{ A_{\parallel}, \frac{c}{4\pi en_{0e}} \nabla_{\perp}^2 A_{\parallel} \right\} = 0. \quad (132) \end{aligned}$$

Multiplying Eq. (132) by  $2T_{0i}/Ze\rho_i^2$  and using Eq. (127), we get the second RMHD equation (14).

We have established that the Alfvén-wave component of the turbulence is decoupled and fully described by the RMHD equations (13) and (14). This result is the same as that in § 2.2 but now we have proven that collisions do not affect the Alfvén waves and that a fluid-like description only requires  $k_{\perp} \rho_i \ll 1$  to be valid.

#### 5.4. Why Alfvén Waves Ignore Collisions

Let us write explicitly the distribution function of the ion gyrocenters [Eq. (116)] to two lowest orders in  $k_{\perp} \rho_i$ :

$$h_i = \frac{Ze}{T_{0i}} \langle \phi \rangle_{\mathbf{R}_i} F_{0i} + \frac{v_{\perp}^2}{v_{\text{thi}}^2} \frac{\delta B_{\parallel}}{B_0} F_{0i} + g + \dots, \quad (133)$$

where, up to corrections of order  $k_{\perp}^2 \rho_i^2$ , the ring-averaged scalar potential is  $\langle \phi \rangle_{\mathbf{R}_i} = \phi(\mathbf{R}_i)$ , the scalar potential taken at the position of the ion gyrocenter. Note that in Eq. (133), the first term is minus first order in  $k_{\perp} \rho_i$  [see Eq. (124)], the second and third terms are zeroth order [Eq. (126)], and all terms of first and higher orders are omitted. In order to compute the full ion distribution function given by Eq. (47), we have to convert  $h_i$  to the  $\mathbf{r}$  space. Keeping terms up to zeroth order, we get

$$\begin{aligned} \frac{Ze}{T_{0i}} \langle \phi \rangle_{\mathbf{R}_i} & \simeq \frac{Ze}{T_{0i}} \phi(\mathbf{R}_i) = \frac{Ze}{T_{0i}} \left[ \phi(\mathbf{r}) + \frac{\mathbf{v}_{\perp} \times \hat{\mathbf{z}}}{\Omega_i} \cdot \nabla \phi(\mathbf{r}) + \dots \right] \\ & = \frac{Ze}{T_{0i}} \phi(\mathbf{r}) + \frac{2\mathbf{v}_{\perp} \cdot \mathbf{u}_E}{v_{\text{thi}}^2} + \dots, \quad (134) \end{aligned}$$

where  $\mathbf{u}_E = -c\nabla\phi(\mathbf{r}) \times \hat{\mathbf{z}}/B_0$ , the  $\mathbf{E} \times \mathbf{B}$  drift velocity. Substituting Eq. (134) into Eq. (133) and then Eq. (133) into Eq. (47), we find

$$f_i = F_{0i} + \frac{2\mathbf{v}_{\perp} \cdot \mathbf{u}_E}{v_{\text{thi}}^2} F_{0i} + \frac{v_{\perp}^2}{v_{\text{thi}}^2} \frac{\delta B_{\parallel}}{B_0} F_{0i} + g + \dots \quad (135)$$

The first two terms can be combined into a Maxwellian with mean perpendicular flow velocity  $\mathbf{u}_{\perp} = \mathbf{u}_E$ . These are the terms responsible for the Alfvén waves. The remaining terms, which we shall denote  $\delta \tilde{f}_i$ , are the perturbation of the Maxwellian in the moving frame of the Alfvén waves—they describe the passive component of the turbulence (see § 5.5). Thus, the ion distribution function is

$$f_i = \frac{n_{0i}}{(\pi v_{\text{thi}}^2)^{3/2}} \exp \left[ -\frac{(\mathbf{v}_{\perp} - \mathbf{u}_E)^2 + v_{\parallel}^2}{v_{\text{thi}}^2} \right] + \delta \tilde{f}_i. \quad (136)$$

This sheds some light on the indifference of Alfvén waves to collisions: Alfvénic perturbations do not change the Maxwellian character of the ion distribution. Unlike in a neutral fluid or gas, where viscosity arises when particles transport the local mean momentum a distance  $\sim \lambda_{\text{mfpi}}$ , the particles in a magnetized plasma instantaneously take on the local  $\mathbf{E} \times \mathbf{B}$  velocity (they take a cyclotron period to adjust, so, roughly speaking,  $\rho_i$  plays the role of the mean free path). Thus, there is no memory of the mean perpendicular motion and, therefore, no perpendicular momentum transport.

Some readers may find it illuminating to notice that Eq. (132) can be interpreted as stating simply  $\nabla \cdot \mathbf{j} = 0$ : the first two terms represent the divergence of the polarization current, which is perpendicular to the magnetic field;<sup>12</sup> the last two terms are  $\hat{\mathbf{b}} \cdot \nabla j_{\parallel}$ . No contribution to the current arises from the collisional term in Eq. (129) as ion-ion collisions cause no particle transport.

#### 5.5. Density and Magnetic-Field-Strength Fluctuations

The equations that describe the density ( $\delta n_e$ ) and magnetic-field-strength ( $\delta B_{\parallel}$ ) fluctuations follow immediately from

<sup>12</sup> The polarization-drift velocity is formally higher-order than  $\mathbf{u}_E$  in the gyrokinetic expansion. However, since  $\mathbf{u}_E$  does not produce any current, the lowest-order contribution to the perpendicular current comes from the polarization drift. The higher-order contributions to the gyrocenter distribution function did not need to be calculated explicitly because the information about the polarization charge is effectively carried by the quasineutrality condition (54). We do not dwell on this point because, in our approach, the notion of polarization charge is only ever brought in for interpretative purposes, but is not needed to carry out calculations. For further qualitative discussion of the role of the polarization charge and polarization drift in gyrokinetics, we refer the reader to Krommes 2006 and references therein.

Eqs. (117-120) if only zeroth-order terms are kept. In these equations, terms that involve  $\phi$  and  $A_{\parallel}$  also contain factors  $\sim k_{\perp}^2 \rho_i^2$  and are, therefore, first-order [with the exception of the nonlinearity on the left-hand side of Eq. (117)]. The fact that  $\langle C_{ii}[\langle \phi \rangle_{\mathbf{R}_i} F_{0i}] \rangle_{\mathbf{R}_i}$  in Eq. (117) is first order is proved in Appendix B.2. Dropping these terms along with all other contributions of order higher than zeroth and making use of Eq. (62) to write out  $\langle \chi \rangle_{\mathbf{R}_i}$ , we find that Eq. (117) takes the form

$$\begin{aligned} \frac{dg}{dt} + v_{\parallel} \hat{\mathbf{b}} \cdot \nabla \left[ g + \left( \frac{Z}{\tau} \frac{\delta n_e}{n_{0e}} + \frac{v_{\perp}^2}{v_{\text{thi}}^2} \frac{\delta B_{\parallel}}{B_0} \right) F_{0i} \right] \\ = \left\langle C_{ii} \left[ g + \frac{v_{\perp}^2}{v_{\text{thi}}^2} \frac{\delta B_{\parallel}}{B_0} F_{0i} \right] \right\rangle_{\mathbf{R}_i}, \end{aligned} \quad (137)$$

where we have used definitions (114-115) of the convective time derivative  $d/dt$  and the total gradient along the magnetic field  $\hat{\mathbf{b}} \cdot \nabla$  to write our equation in a compact form. Note that, in view of the correspondence between  $\Phi$ ,  $\Psi$  and  $\phi$ ,  $A_{\parallel}$  [Eq. (127)], these nonlinear derivatives are the same as those defined in Eqs. (15-16). The collision term in the right-hand side of the above equation is the zeroth-order limit of the gyrokinetic ion-ion collision operator: a useful model form of it is given in Appendix B.3 [Eq. (B17)].

To zeroth order, Eqs. (118-120) are

$$\frac{\delta n_e}{n_{0e}} - \frac{\delta B_{\parallel}}{B_0} = \frac{1}{n_{0i}} \int d^3 \mathbf{v} g, \quad (138)$$

$$u_{\parallel} = \frac{1}{n_{0i}} \int d^3 \mathbf{v} v_{\parallel} g, \quad (139)$$

$$\frac{Z}{\tau} \frac{\delta n_e}{n_{0e}} + 2 \left( 1 + \frac{1}{\beta_i} \right) \frac{\delta B_{\parallel}}{B_0} = -\frac{1}{n_{0i}} \int d^3 \mathbf{v} \frac{v_{\perp}^2}{v_{\text{thi}}^2} g. \quad (140)$$

Note that  $u_{\parallel}$  is not an independent quantity—it can be computed from the ion distribution but is not needed for the determination of the latter.

Equations (137-140) evolve the ion distribution function  $g$ , the “slow-wave quantities”  $u_{\parallel}$ ,  $\delta B_{\parallel}$ , and the density fluctuations  $\delta n_e$ . The nonlinearities in Eq. (137), contained in  $d/dt$  and  $\hat{\mathbf{b}} \cdot \nabla$ , involve the Alfvén-wave quantities  $\Phi$  and  $\Psi$  (or, equivalently,  $\phi$  and  $A_{\parallel}$ ) determined separately and independently by the RMHD equations (13-14). The situation is qualitatively similar to that in MHD (§ 2.3), except now a kinetic description is necessary—Eqs. (137-140) replace Eqs. (20-22)—and the nonlinear scattering/mixing of the slow waves and the entropy mode by the Alfvén waves takes the form of passive advection of the distribution function  $g$ . The density and magnetic-field-strength fluctuations are velocity-space moments of  $g$ .

Another way to understand the passive nature of the kinetic component of the turbulence discussed above is to think of it as the perturbation of a local Maxwellian equilibrium associated with the Alfvén waves. Indeed, in § 5.4, we split the full ion distribution function [Eq. (136)] into such a local Maxwellian and its perturbation

$$\delta \tilde{f}_i = g + \frac{v_{\perp}^2}{v_{\text{thi}}^2} \frac{\delta B_{\parallel}}{B_0} F_{0i}. \quad (141)$$

It is this perturbation that contains all the information about the passive component; the second term in the above expression enforces to lowest order the conservation of the first adiabatic invariant  $\mu_i = m_i v_{\perp}^2 / 2B$ . In terms of the function (141), Eqs. (137-140) take a somewhat more compact form

(cf. Schekochihin et al. 2007a):

$$\frac{d}{dt} \left( \delta \tilde{f}_i - \frac{v_{\perp}^2}{v_{\text{thi}}^2} \frac{\delta B_{\parallel}}{B_0} \right) + v_{\parallel} \hat{\mathbf{b}} \cdot \nabla \left( \delta \tilde{f}_i + \frac{Z}{\tau} \frac{\delta n_e}{n_{0e}} F_{0i} \right) = \langle C_{ii}[\delta \tilde{f}_i] \rangle_{\mathbf{R}_i}, \quad (142)$$

$$\frac{\delta n_e}{n_{0e}} = \frac{1}{n_{0i}} \int d^3 \mathbf{v} \delta \tilde{f}_i, \quad (143)$$

$$\frac{\delta B_{\parallel}}{B_0} = -\frac{\beta_i}{2} \frac{1}{n_{0i}} \int d^3 \mathbf{v} \left( \frac{Z}{\tau} + \frac{v_{\perp}^2}{v_{\text{thi}}^2} \right) \delta \tilde{f}_i. \quad (144)$$

### 5.6. Generalized Energy: Three KRMHD Cascades

The generalized energy (§ 3.4) in the limit  $k_{\perp} \rho_i \ll 1$  is calculated by substituting into Eq. (101) the perturbed ion distribution function  $\delta f_i = 2\mathbf{v}_{\perp} \cdot \mathbf{u}_E F_{0i} / v_{\text{thi}}^2 + \delta \tilde{f}_i$  [see Eqs. (135) and (141)]. After performing velocity integration, we get

$$\begin{aligned} W = \int d^3 \mathbf{r} \left[ \frac{m_i n_{0i} u_E^2}{2} + \frac{\delta B_{\perp}^2}{8\pi} \right. \\ \left. + \frac{n_{0i} T_{0i}}{2} \left( \frac{Z}{\tau} \frac{\delta n_e^2}{n_{0e}^2} + \frac{2}{\beta_i} \frac{\delta B_{\parallel}^2}{B_0^2} + \frac{1}{n_{0i}} \int d^3 \mathbf{v} \frac{T_{0i} \delta \tilde{f}_i^2}{2F_{0i}} \right) \right] \\ = W_{\text{AW}} + W_{\text{compr}}. \end{aligned} \quad (145)$$

We see that the kinetic energy of the Alfvénic fluctuations has emerged from the ion entropy part of the generalized energy. The first two terms in Eq. (145) are the total (kinetic plus magnetic) energy of the Alfvén waves, denoted  $W_{\text{AW}}$ . As we learned from § 5.3, it cascades independently of the rest of the generalized energy,  $W_{\text{compr}}$ , which contains the passive (compressive) component of the turbulence (§ 5.5) and is the invariant conserved by Eqs. (142-144) (see Appendix D.2 for its form in the fluid limit).

In terms of the potentials used in our discussion of RMHD in § 2, we have

$$\begin{aligned} W_{\text{AW}} &= \int d^3 \mathbf{r} \frac{m_i n_{0i}}{2} (|\nabla \Phi|^2 + |\nabla \Psi|^2) \\ &= \int d^3 \mathbf{r} \frac{m_i n_{0i}}{2} (|\nabla \zeta^+|^2 + |\nabla \zeta^-|^2) = W_{\perp}^+ + W_{\perp}^-, \end{aligned} \quad (146)$$

where  $W_{\perp}^+$  and  $W_{\perp}^-$  are the energies of the “+” and “−” waves [Eq. (27)], which, as we know from § 2.4, cascade by scattering off each other but without exchanging energy.

Thus, the kinetic cascade in the limit  $k_{\perp} \rho_i \ll 1$  is split, independently of the collisionality, into three cascades: of  $W_{\perp}^+$ ,  $W_{\perp}^-$  and  $W_{\text{compr}}$ .

### 5.7. Summary

In § 4, gyrokinetics was reduced to a hybrid fluid-kinetic system by means of an expansion in the electron mass, which was valid for  $k_{\perp} \rho_e \ll 1$ . In this section, we have further restricted the scale range by taking  $k_{\perp} \rho_i \ll 1$  and as a result have been able to achieve a further reduction in the complexity of the kinetic theory describing the turbulent cascades. The reduced theory derived here evolves 5 unknown functions:  $\Phi$ ,  $\Psi$ ,  $\delta B_{\parallel}$ ,  $\delta n_e$  and  $g$ . The stream and flux functions,  $\Phi$  and  $\Psi$  are related to the fluid quantities (perpendicular velocity and magnetic field perturbations) via Eq. (12) and to the electromagnetic potentials  $\phi$ ,  $A_{\parallel}$  via Eq. (127). They satisfy a closed system of equations, Eqs. (13-14), which describe the decoupled cascade of Alfvén waves. These are the same equations that arise from the MHD approximations, but we have now

proven that their validity does not depend on the assumption of high collisionality (the fluid limit) and extends to scales well below the mean free path, but above the ion gyroscale. The physical reasons for this are explained in § 5.4. The density and magnetic-field-strength fluctuations (the “compressive” fluctuations, or the slow waves and the entropy mode in the MHD limit) now require a kinetic description in terms of the ion distribution function  $g$  [or  $\delta\tilde{f}_i$ , Eq. (141)], evolved by the kinetic equation (137) [or Eq. (142)]. The kinetic equation contains  $\delta n_e$  and  $\delta B_{\parallel}$ , which are, in turn calculated in terms of the velocity-space integrals of  $g$  via Eqs. (138) and (140) [or Eqs. (143) and (144)]. The nonlinear evolution (turbulent cascade) of  $g$ ,  $\delta B_{\parallel}$  and  $\delta n_e$  is due solely to passive advection of  $g$  by the Alfvén-wave turbulence.

Let us summarize the new set of equations:

$$\frac{\partial \Psi}{\partial t} = v_A \hat{\mathbf{b}} \cdot \nabla \Phi, \quad (147)$$

$$\frac{d}{dt} \nabla_{\perp}^2 \Phi = v_A \hat{\mathbf{b}} \cdot \nabla \nabla_{\perp}^2 \Psi, \quad (148)$$

$$\begin{aligned} \frac{dg}{dt} + v_{\parallel} \hat{\mathbf{b}} \cdot \nabla \left[ g + \left( \frac{Z}{\tau} \frac{\delta n_e}{n_{0e}} + \frac{v_{\perp}^2}{v_{thi}^2} \frac{\delta B_{\parallel}}{B_0} \right) F_{0i} \right] \\ = \left\langle C_{ii} \left[ g + \frac{v_{\perp}^2}{v_{thi}^2} \frac{\delta B_{\parallel}}{B_0} F_{0i} \right] \right\rangle_{\mathbf{R}_i}, \end{aligned} \quad (149)$$

$$\begin{aligned} \frac{\delta n_e}{n_{0e}} = - \left[ \frac{Z}{\tau} + 2 \left( 1 + \frac{1}{\beta_i} \right) \right]^{-1} \frac{1}{n_{0i}} \int d^3 \mathbf{v} \left[ \frac{v_{\perp}^2}{v_{thi}^2} \right. \\ \left. - 2 \left( 1 + \frac{1}{\beta_i} \right) \right] g, \end{aligned} \quad (150)$$

$$\frac{\delta B_{\parallel}}{B_0} = - \left[ \frac{Z}{\tau} + 2 \left( 1 + \frac{1}{\beta_i} \right) \right]^{-1} \frac{1}{n_{0i}} \int d^3 \mathbf{v} \left( \frac{v_{\perp}^2}{v_{thi}^2} + \frac{Z}{\tau} \right) g, \quad (151)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \{ \Phi, \dots \}, \quad \hat{\mathbf{b}} \cdot \nabla = \frac{\partial}{\partial z} + \frac{1}{v_A} \{ \Psi, \dots \}. \quad (152)$$

An explicit form of the collision term in the right-hand side of Eq. (149) is provided in Appendix B.3 [Eq. (B17)].

Equations (147-151) conserve the generalized energy given by Eq. (145). The kinetic cascade is split, with the Alfvénic cascade proceeding independently of the compressive fluctuations.

As already noted above, the decoupling of the Alfvénic cascade is manifested by Eqs. (147-148) forming a closed subset. We have already noted in § 4.9 that Eq. (147) is the component of Ohm’s law along the total magnetic field,  $\mathbf{B} \cdot \mathbf{E} = 0$ . Equation (148) can be interpreted as the evolution equation for the vorticity of the perpendicular plasma flow velocity, which is the  $\mathbf{E} \times \mathbf{B}$  drift velocity.

We shall refer to the system of equations (147-151) as *Kinetic Reduced Magnetohydrodynamics (KRMHD)*.<sup>13</sup> It is a

<sup>13</sup> The term is introduced by analogy with a popular fluid-kinetic system known as Kinetic MHD, or KMHD (see Kulsrud 1964, 1983). KMHD is derived for magnetized plasmas ( $\rho_i \ll \lambda_{mfp i}$ ) under the assumption that  $k \rho_s \ll 1$  and  $\omega \ll \Omega_s$  but without assuming either strong anisotropy ( $k_{\parallel} \ll k_{\perp}$ ) or small fluctuations ( $|\delta \mathbf{B}| \ll B_0$ ). The KRMHD equations (147-151) can be recovered from KMHD by applying to it the RMHD ordering (10) and an expansion in  $(m_e/m_i)^{1/2}$  (Schekochihin et al. 2007a). This means that the  $k_{\perp} \rho_i$  expansion (§ 5), which for KMHD is the primary expansion, commutes with the gyrokinetic expansion (§ 3) and the  $(m_e/m_i)^{1/2}$  expansion (§ 4), both of which preceded it in this paper.

hybrid fluid-kinetic description of low-frequency turbulence in strongly magnetized weakly collisional plasma that is uniformly valid at all scales satisfying  $k_{\perp} \rho_i \ll \min(1, k_{\parallel} \lambda_{mfp i})$  (ions are strongly magnetized)<sup>14</sup> and  $k_{\parallel} \lambda_{mfp i} \gg (m_e/m_i)^{1/2}$  (electrons are isothermal), as illustrated in Fig. 1. Therefore, it smoothly connects the collisional and collisionless regimes and is the appropriate theory for the study of the turbulent cascades in the inertial range. It does not, however, describe what happens to these cascades at or below the ion gyroscale—we shall move on to these scales in § 7, but first we would like to discuss the turbulent cascades of density and magnetic-field-strength fluctuations and their damping by collisional and collisionless mechanisms.

## 6. COMPRESSIVE FLUCTUATIONS WITH AND WITHOUT COLLISIONS

Here we first derive the (nonlinear) equations that govern the evolution of the density and magnetic-field-strength fluctuations in the collisional ( $k_{\parallel} \lambda_{mfp i} \ll 1$ , § 6.1) and collisionless ( $k_{\parallel} \lambda_{mfp i} \gg 1$ , § 6.2) limits and discuss the linear damping that these fluctuations undergo in the two limits. As in previous sections, an impatient reader may skip to § 6.3 where the results of the previous two subsections are summarized and the implications for the structure of the turbulent cascades of the density and field-strength fluctuations are discussed.

### 6.1. Collisional Regime

In the collisional regime,  $k_{\parallel} \lambda_{mfp i} \ll 1$ , the fluid limit is recovered by expanding Eqs. (147-151) in small  $k_{\parallel} \lambda_{mfp i}$ . The calculation that is necessary to achieve this is done in Appendix D (see also Appendix A.4). The result is a closed set of three fluid equations that evolve  $\delta B_{\parallel}$ ,  $\delta n_e$  and  $u_{\parallel}$ :

$$\frac{d}{dt} \frac{\delta B_{\parallel}}{B_0} = \hat{\mathbf{b}} \cdot \nabla u_{\parallel} + \frac{d}{dt} \frac{\delta n_e}{n_{0e}}, \quad (153)$$

$$\frac{du_{\parallel}}{dt} = v_A^2 \hat{\mathbf{b}} \cdot \nabla \frac{\delta B_{\parallel}}{B_0} + \nu_{\parallel i} \hat{\mathbf{b}} \cdot \nabla (\hat{\mathbf{b}} \cdot \nabla u_{\parallel}), \quad (154)$$

$$\frac{d}{dt} \frac{\delta T_i}{T_{0i}} = \frac{2}{3} \frac{d}{dt} \frac{\delta n_e}{n_{0e}} + \kappa_{\parallel i} \hat{\mathbf{b}} \cdot \nabla \left( \hat{\mathbf{b}} \cdot \nabla \frac{\delta T_i}{T_{0i}} \right), \quad (155)$$

where

$$\left( 1 + \frac{Z}{\tau} \right) \frac{\delta n_e}{n_{0e}} = - \frac{\delta T_i}{T_{0i}} - \frac{2}{\beta_i} \left( \frac{\delta B_{\parallel}}{B_0} + \frac{1}{3 v_A^2} \nu_{\parallel i} \hat{\mathbf{b}} \cdot \nabla u_{\parallel} \right), \quad (156)$$

and  $\nu_{\parallel i}$  and  $\kappa_{\parallel i}$  are the coefficients of parallel ion viscosity and thermal diffusivity, respectively. The viscous and thermal diffusion are anisotropic because plasma is magnetized,  $\lambda_{mfp i} \gg \rho_i$  (Braginskii 1965). The method of calculation of  $\nu_{\parallel i}$  and  $\kappa_{\parallel i}$  is explained in Appendix D.3. Here we shall ignore numerical prefactors of order unity and give order-of-magnitude values for these coefficients:

$$\nu_{\parallel i} \sim \kappa_{\parallel i} \sim \frac{v_{thi}^2}{v_{ii}} \sim v_{thi} \lambda_{mfp i}. \quad (157)$$

If we set  $\nu_{\parallel i} = \kappa_{\parallel i} = 0$ , Eqs. (153-156) are the same as the RMHD equations of § 2 with the sound speed defined as

$$c_s = v_A \sqrt{\frac{\beta_i}{2} \left( \frac{Z}{\tau} + \frac{5}{3} \right)} = \sqrt{\frac{Z T_{0e}}{m_i} + \frac{5 T_{0i}}{3 m_i}}. \quad (158)$$

<sup>14</sup> The condition  $k_{\perp} \rho_i \ll k_{\parallel} \lambda_{mfp i}$  must be satisfied because in our estimates of the collision terms (Appendix B.2) we took  $k_{\perp} \rho_i \ll 1$  while assuming that  $k_{\parallel} \lambda_{mfp i} \sim 1$ .



This is the natural definition of  $c_s$  for the case of adiabatic ions, whose specific heat ratio is  $\gamma_i = 5/3$ , and isothermal electrons, whose specific heat ratio is  $\gamma_e = 1$  [because  $\delta p_e = T_{0e} \delta n_e$ ; see Eq. (95)]. Note that Eq. (156) is equivalent to the pressure balance [Eq. (17) of § 2] with  $p = n_i T_i + n_e T_e$  and  $\delta p_e = T_{0e} \delta n_e$ .

As in § 2, the fluctuations described by Eqs. (153-156) separate into the zero-frequency entropy mode and the left- and right-propagating slow waves with

$$\omega = \pm \frac{k_{\parallel} v_A}{\sqrt{1 + v_A^2/c_s^2}} \quad (159)$$

[see Eq. (25)]. All three are cascaded independently of each other via nonlinear interaction with the Alfvén waves [in Appendix D.2, we show that the generalized energy  $W_{\text{compr}}$  for this system, given in § 5.6, splits into the three familiar invariants  $W_{\parallel}^+$ ,  $W_{\parallel}^-$ , and  $W_s$ , defined by Eqs. (28-29)].

The diffusion terms add dissipation to the equations. Because diffusion occurs along the field lines of the total magnetic field (mean field plus perturbation), the diffusive terms are nonlinear and the dissipation process also involves interaction with the Alfvén waves. We can estimate the characteristic parallel scale at which the diffusion terms become important by balancing the nonlinear cascade time and the typical diffusion time:

$$k_{\parallel} v_A \sim v_{\text{thi}} \lambda_{\text{mfpi}} k_{\parallel}^2 \Leftrightarrow k_{\parallel} \lambda_{\text{mfpi}} \sim 1/\sqrt{\beta_i}, \quad (160)$$

where we have used Eq. (157).

Technically speaking, the cutoff given by Eq. (160) always lies in the range of  $k_{\parallel}$  that is outside the region of validity of the small- $k_{\parallel} \lambda_{\text{mfpi}}$  expansion adopted in the derivation of Eqs. (153-155). In fact, in the low-beta limit, the collisional cutoff falls manifestly in the collisionless scale range, i.e., the collisional (fluid) approximation breaks down before the slow-wave and entropy cascades are damped and one must use the collisionless (kinetic) limit to calculate the damping (see § 6.2). The situation is different in the high-beta limit: in this case, the expansion in small  $k_{\parallel} \lambda_{\text{mfpi}}$  can be reformulated as an expansion in small  $1/\sqrt{\beta_i}$  and the cutoff falls within the range of validity of the fluid approximation. Equations (153-155) in this limit are

$$\frac{d}{dt} \frac{\delta B_{\parallel}}{B_0} = \hat{\mathbf{b}} \cdot \nabla u_{\parallel}, \quad (161)$$

$$\frac{du_{\parallel}}{dt} = v_A^2 \hat{\mathbf{b}} \cdot \nabla \frac{\delta B_{\parallel}}{B_0} + \nu_{\parallel i} \hat{\mathbf{b}} \cdot \nabla (\hat{\mathbf{b}} \cdot \nabla u_{\parallel}), \quad (162)$$

$$\frac{d}{dt} \frac{\delta n_e}{n_{0e}} = \frac{1 + Z/\tau}{5/3 + Z/\tau} \kappa_{\parallel i} \hat{\mathbf{b}} \cdot \nabla \left( \hat{\mathbf{b}} \cdot \nabla \frac{\delta n_e}{n_{0e}} \right). \quad (163)$$

As in § 2 [Eq. (30)], the density fluctuations [Eq. (163)] have decoupled from the slow waves [Eqs. (161-162)]. The former are damped by thermal diffusion, the latter by viscosity. The corresponding linear dispersion relations are

$$\omega = -i \frac{1 + Z/\tau}{5/3 + Z/\tau} \kappa_{\parallel i} k_{\parallel}^2, \quad (164)$$

$$\omega = \pm k_{\parallel} v_A \sqrt{1 - \left( \frac{\nu_{\parallel i} k_{\parallel}}{2v_A} \right)^2} - i \frac{\nu_{\parallel i} k_{\parallel}^2}{2}. \quad (165)$$

Equation (164) describes strong diffusive damping of the density fluctuations. The slow-wave dispersion relation (165) has two distinct regimes:

1. When  $k_{\parallel} < 2v_A/\nu_{\parallel i}$ , it describes damped slow waves. In particular, in the limit  $k_{\parallel} \lambda_{\text{mfpi}} \ll 1/\sqrt{\beta_i}$ , we have

$$\omega \simeq \pm k_{\parallel} v_A - i \frac{\nu_{\parallel i} k_{\parallel}^2}{2}. \quad (166)$$

2. For  $k_{\parallel} > 2v_A/\nu_{\parallel i}$ , both solutions become purely imaginary, so the slow waves are converted into aperiodic decaying fluctuations. The stronger-damped (diffusive) branch has  $\omega \simeq -i\nu_{\parallel i} k_{\parallel}^2$ , the weaker-damped one has

$$\omega \simeq -i \frac{v_A^2}{\nu_{\parallel i}} \sim -\frac{i}{\beta_i} \frac{v_{\text{thi}}}{\lambda_{\text{mfpi}}} \sim -\frac{i}{\sqrt{\beta_i}} \frac{v_A}{\lambda_{\text{mfpi}}}. \quad (167)$$

This is valid until  $k_{\parallel} \lambda_{\text{mfpi}} \sim 1$ , where it is replaced by the collisionless damping discussed in § 6.2 [see Eq. (184)].

## 6.2. Collisionless Regime

In the collisionless regime,  $k_{\parallel} \lambda_{\text{mfpi}} \gg 1$ , the collision integral in the right-hand side of the kinetic equation (149) can be neglected. The  $v_{\perp}$  dependence can then be integrated out of Eq. (149). Indeed, let us introduce the following two auxiliary functions:

$$G_n(v_{\parallel}) = - \left[ \frac{Z}{\tau} + 2 \left( 1 + \frac{1}{\beta_i} \right) \right]^{-1} \times \frac{2\pi}{n_{0i}} \int_0^{\infty} dv_{\perp} v_{\perp} \left[ \frac{v_{\perp}^2}{v_{\text{thi}}^2} - 2 \left( 1 + \frac{1}{\beta_i} \right) \right] g, \quad (168)$$

$$G_B(v_{\parallel}) = - \left[ \frac{Z}{\tau} + 2 \left( 1 + \frac{1}{\beta_i} \right) \right]^{-1} \times \frac{2\pi}{n_{0i}} \int_0^{\infty} dv_{\perp} v_{\perp} \left( \frac{v_{\perp}^2}{v_{\text{thi}}^2} + \frac{Z}{\tau} \right) g. \quad (169)$$

In terms of these functions,

$$\frac{\delta n_e}{n_{0e}} = \int_{-\infty}^{+\infty} dv_{\parallel} G_n, \quad (170)$$

$$\frac{\delta B_{\parallel}}{B_0} = \int_{-\infty}^{+\infty} dv_{\parallel} G_B \quad (171)$$

and Eq. (149) reduces to the following two coupled one-dimensional kinetic equations

$$\frac{dG_n}{dt} + v_{\parallel} \hat{\mathbf{b}} \cdot \nabla G_n = - \left[ \frac{Z}{\tau} + 2 \left( 1 + \frac{1}{\beta_i} \right) \right]^{-1} v_{\parallel} F_M(v_{\parallel}) \times \hat{\mathbf{b}} \cdot \nabla \left[ \frac{Z}{\tau} \left( 1 + \frac{2}{\beta_i} \right) \frac{\delta n_e}{n_{0e}} + \frac{2}{\beta_i} \frac{\delta B_{\parallel}}{B_0} \right], \quad (172)$$

$$\frac{dG_B}{dt} + v_{\parallel} \hat{\mathbf{b}} \cdot \nabla G_B = \left[ \frac{Z}{\tau} + 2 \left( 1 + \frac{1}{\beta_i} \right) \right]^{-1} v_{\parallel} F_M(v_{\parallel}) \times \hat{\mathbf{b}} \cdot \nabla \left[ \frac{Z}{\tau} \left( 1 + \frac{Z}{\tau} \right) \frac{\delta n_e}{n_{0e}} + \left( 2 + \frac{Z}{\tau} \right) \frac{\delta B_{\parallel}}{B_0} \right], \quad (173)$$

where  $F_M(v_{\parallel}) = (1/\sqrt{\pi} v_{\text{thi}}) \exp(-v_{\parallel}^2/v_{\text{thi}}^2)$  is a one-dimensional Maxwellian. This system can be diagonalized, so it splits into two decoupled equations

$$\frac{dG^-}{dt} + v_{\parallel} \hat{\mathbf{b}} \cdot \nabla G^- = \frac{v_{\parallel} F_M(v_{\parallel})}{\Lambda^-} \hat{\mathbf{b}} \cdot \nabla \int_{-\infty}^{+\infty} dv'_{\parallel} G^-(v'_{\parallel}), \quad (174)$$

$$\frac{dG^+}{dt} + v_{\parallel} \hat{\mathbf{b}} \cdot \nabla G^+ = \frac{v_{\parallel} F_M(v_{\parallel})}{\Lambda^+} \hat{\mathbf{b}} \cdot \nabla \int_{-\infty}^{+\infty} dv'_{\parallel} G^+(v'_{\parallel}), \quad (175)$$

where

$$\Lambda^\pm = -\frac{\tau}{Z} + \frac{1}{\beta_i} \pm \sqrt{\left(1 + \frac{\tau}{Z}\right)^2 + \frac{1}{\beta_i^2}} \quad (176)$$

and we have introduced a new pair of functions

$$G^- = G_n + \frac{\tau}{Z} \frac{2}{\beta_i} \left[ 1 + \frac{\tau}{Z} + \frac{1}{\beta_i} + \sqrt{\left(1 + \frac{\tau}{Z}\right)^2 + \frac{1}{\beta_i^2}} \right]^{-1} G_B, \quad (177)$$

$$G^+ = G_B + \left(1 + \frac{Z}{\tau}\right) \left[ 1 + \frac{\tau}{Z} + \frac{1}{\beta_i} + \sqrt{\left(1 + \frac{\tau}{Z}\right)^2 + \frac{1}{\beta_i^2}} \right]^{-1} G_n. \quad (178)$$

Equations (174-175) describe two decoupled kinetic cascades.

These cascades are subject to collisionless damping. Indeed, let us linearize Eqs. (174-175), Fourier transform in time and space, divide through by  $-i(\omega - k_\parallel v_\parallel)$ , and integrate over  $v_\parallel$ . This gives the following dispersion relation (the “-” branch is for  $G^-$ , the “+” branch for  $G^+$ )

$$\zeta_i Z(\zeta_i) = \Lambda^\pm - 1, \quad (179)$$

where  $\zeta_i = \omega/|k_\parallel|v_{thi} = \omega/|k_\parallel|v_A\sqrt{\beta_i}$  and we have used the plasma dispersion function (Fried & Conte 1961)

$$Z(\zeta_i) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \frac{e^{-x^2}}{x - \zeta_i} \quad (180)$$

(the integration is along the Landau contour). This function is not to be confused with the parameter  $Z$  [Eq. (36)],

Formally, Eq. (179) has an infinite number of solutions. When  $\beta_i \sim 1$ , they are all strongly damped with damping rates  $\text{Im}(\omega) \sim |k_\parallel|v_{thi} \sim |k_\parallel|v_A$ , so the damping time is comparable to the characteristic time scale on which the Alfvén waves cause these fluctuations to cascade to smaller scales.

It is interesting to consider the high- and low-beta limits.

#### 6.2.1. High-Beta Limit

When  $\beta_i \gg 1$ , we have in Eq. (179)

$$\Lambda^- - 1 \simeq -2 \left(1 + \frac{\tau}{Z}\right), \quad G^- \simeq G_n, \quad (181)$$

$$\Lambda^+ - 1 \simeq \frac{1}{\beta_i}, \quad G^+ \simeq G_B + \frac{1}{2} \frac{Z}{\tau} G_n. \quad (182)$$

The “-” branch corresponds to the density fluctuations. The solution of Eq. (179) has  $\text{Im}(\zeta_i) \sim 1$ , so these fluctuations are strongly damped:

$$\omega \sim -i|k_\parallel|v_A\sqrt{\beta_i}. \quad (183)$$

The damping rate is much greater than the Alfvénic rate  $k_\parallel v_A$  of the nonlinear cascade. In contrast, for the “+” branch, the damping rate is small: it can be obtained by expanding  $Z(\zeta_i) = i\sqrt{\pi} + O(\zeta_i)$ , which gives<sup>15</sup>

$$\omega = -i \frac{|k_\parallel|v_{thi}}{\sqrt{\pi}\beta_i} = -i \frac{|k_\parallel|v_A}{\sqrt{\pi}\beta_i}. \quad (184)$$

<sup>15</sup> This is the gyrokinetic limit ( $k_\parallel/k_\perp \ll 1$ ) of the more general damping effect known in astrophysics as the Barnes (1966) damping and in plasma physics as transit-time damping. We remind the reader that our approach was to carry out the gyrokinetic expansion (in small  $k_\parallel/k_\perp$ ) first, and then take the high-beta limit as a subsidiary expansion. A more standard approach in the linear theory of plasma waves is to take the limit of high  $\beta_i$  while treating  $k_\parallel/k_\perp$  as an arbitrary quantity. A detailed calculation of the damping rates done in this way can be found in Foote & Kulsrud (1979).

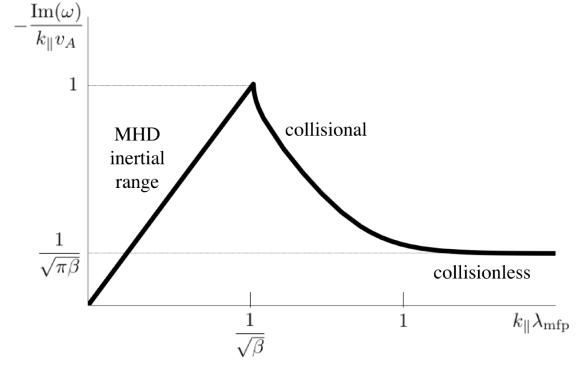


FIG. 4.— A schematic log-log plot (artist’s impression) of the ratio of the damping rate of magnetic-field-strength fluctuations to the Alfvén frequency  $k_\parallel v_A$  in the high-beta limit [see Eqs. (165) and (184)].

Since  $G_n$  is strongly damped, Eq. (182) implies  $G^+ \simeq G_B$ , i.e., the fluctuations that are damped at the rate (184) are predominantly of the magnetic-field strength. The damping rate is a constant (independent of  $k_\parallel$ ) small fraction  $\sim 1/\sqrt{\beta_i}$  of the Alfvénic cascade rate.

In Fig. 4, we give a schematic plot of the damping rate of the magnetic-field-strength fluctuations (slow waves) connecting the fluid and kinetic limits for  $\beta_i \gg 1$ .

#### 6.2.2. Low-Beta Limit

When  $\beta_i \ll 1$ , we have

$$\Lambda^- - 1 \simeq -\left(1 + \frac{\tau}{Z}\right), \quad G^- \simeq G_n + \frac{\tau}{Z} G_B, \quad (185)$$

$$\Lambda^+ - 1 \simeq \frac{2}{\beta_i}, \quad G^+ \simeq G_B. \quad (186)$$

For the “-” branch, we again have  $\text{Im}(\zeta_i) \sim 1$ , so

$$\omega \sim -i|k_\parallel|v_A\sqrt{\beta_i}, \quad (187)$$

which now is much smaller than the Alfvénic cascade rate  $k_\parallel v_A$ . For the “+” branch (predominantly the field-strength fluctuations), we seek a solution with  $\zeta = -i\tilde{\zeta}_i$  and  $\tilde{\zeta}_i \gg 1$ . Then Eq. (179) becomes  $\zeta_i Z(\zeta_i) \simeq 2\sqrt{\pi}\tilde{\zeta}_i \exp(\tilde{\zeta}_i) = 2/\beta_i$ . Up to logarithmically small corrections, this gives  $\tilde{\zeta}_i \simeq \sqrt{|\ln \beta_i|}$ , whence

$$\omega \sim -i|k_\parallel|v_A\sqrt{\beta_i|\ln \beta_i|}. \quad (188)$$

While this damping rate is slightly greater than that of the “-” branch, it is still much smaller than the Alfvénic cascade rate.

#### 6.3. Parallel and Perpendicular Cascades

Let us return to the kinetic equation (149) and transform it to the Lagrangian frame associated with the velocity field  $\mathbf{u}_\perp = \hat{\mathbf{z}} \times \nabla_\perp \Phi$  of the Alfvén waves:  $(t, \mathbf{r}) \rightarrow (t, \mathbf{r}_0)$ , where

$$\mathbf{r}(t, \mathbf{r}_0) = \mathbf{r}_0 + \int_0^t dt' \mathbf{u}_\perp(t', \mathbf{r}(t', \mathbf{r}_0)). \quad (189)$$

In this frame, the convective derivative  $d/dt$  defined in Eq. (152) turns into  $\partial/\partial t$ , while the parallel spatial gradient  $\hat{\mathbf{b}} \cdot \nabla$  can be calculated by employing the Cauchy solution for the perturbed magnetic field  $\delta \mathbf{B}_\perp = \hat{\mathbf{z}} \times \nabla_\perp \Psi$ :

$$\hat{\mathbf{b}}(t, \mathbf{r}) = \hat{\mathbf{z}} + \frac{\delta \mathbf{B}_\perp(t, \mathbf{r})}{B_0} = \hat{\mathbf{b}}(0, \mathbf{r}_0) \cdot \nabla_0 \mathbf{r}, \quad (190)$$

where  $\mathbf{r}$  is given by Eq. (189) and  $\nabla_0 = \partial/\partial\mathbf{r}_0$ . Then

$$\hat{\mathbf{b}} \cdot \nabla = \hat{\mathbf{b}}(0, \mathbf{r}_0) \cdot (\nabla_0 \mathbf{r}) \cdot \nabla = \hat{\mathbf{b}}(0, \mathbf{r}_0) \cdot \nabla_0 = \frac{\partial}{\partial s_0}, \quad (191)$$

where  $s_0$  is the arc length along the perturbed magnetic field taken at  $t = 0$  [if  $\delta\mathbf{B}_\perp(0, \mathbf{r}_0) = 0$ ,  $s_0 = z_0$ ]. Thus, in the Lagrangian frame associated with the Alfvénic component of the turbulence, Eq. (149) is linear. This means that, if the effect of finite ion gyroradius is neglected, the KRMHD system does not give rise to a cascade of density and magnetic-field-strength fluctuations to smaller scales along the moving (perturbed) field lines, i.e.,  $\hat{\mathbf{b}} \cdot \nabla \delta n_e$  and  $\hat{\mathbf{b}} \cdot \nabla \delta B_\parallel$  do not increase. In contrast, there is a perpendicular cascade (cascade in  $k_\perp$ ): the perpendicular wandering of field lines due to the Alfvénic turbulence causes passive mixing of  $\delta n_e$  and  $\delta B_\parallel$  in the direction transverse to the magnetic field.<sup>16</sup>

We emphasize that this lack of nonlinear refinement of the scale of  $\delta n_e$  and  $\delta B_\parallel$  along the moving field lines is a particular property of the passive component of the turbulence, not shared by the Alfvén waves. Indeed, unlike Eq. (149), the RMHD equations (147-148), do not reduce to a linear form under the Lagrangian transformation (189), so the Alfvén waves should develop small scales both across and along the perturbed magnetic field.

Whether the density and magnetic-field-strength fluctuations develop small scales along the magnetic field has direct physical and observational consequences. Damping of these fluctuations, both in the collisional and collisionless regimes, discussed in § 6.1 and § 6.2, respectively, depends precisely on their scale along the perturbed field: indeed, the linear results derived there are exact in the Lagrangian frame (189). To summarize these results, the damping rate of  $\delta n_e$  and  $\delta B_\parallel$  at  $\beta_i \sim 1$  is

$$\gamma \sim \nu_{\text{thi}} \lambda_{\text{mfpi}} k_{\parallel 0}^2, \quad k_{\parallel 0} \lambda_{\text{mfpi}} \ll 1, \quad (192)$$

$$\gamma \sim \nu_{\text{thi}} k_{\parallel 0}, \quad k_{\parallel 0} \lambda_{\text{mfpi}} \gg 1, \quad (193)$$

where  $k_{\parallel 0} \sim \hat{\mathbf{b}} \cdot \nabla$  is the wave number along the perturbed field (i.e., if there is no parallel cascade, the wave number of the large-scale stirring).

Whether this damping cuts off the cascades of  $\delta n_e$  and  $\delta B_\parallel$  depends on the relative magnitudes of the damping rate  $\gamma$  for a given  $k_\perp$  and the characteristic rate at which the Alfvén waves cause  $\delta n_e$  and  $\delta B_\parallel$  to cascade to higher  $k_\perp$ . This rate is  $\omega_A \sim k_{\parallel A} v_A$ , where  $k_{\parallel A}$  is the parallel wave number of the Alfvén waves that have the same  $k_\perp$ . Since the Alfvén waves do have a parallel cascade, assuming scale-by-scale critical balance (3) leads to [Eq. (4)]

$$k_{\parallel A} \sim k_\perp^{2/3} l_0^{-1/3}. \quad (194)$$

If, in contrast to the Alfvén waves,  $\delta n_e$  and  $\delta B_\parallel$  have no parallel cascade,  $k_{\parallel 0}$  does not grow with  $k_\perp$ , so, for large enough  $k_\perp$ ,  $k_{\parallel 0} \ll k_{\parallel A}$  and  $\gamma \ll \omega_A$ . This means that, despite the damping, the density and field-strength fluctuations should have perpendicular cascades extending to the ion gyroscale.

The validity of the argument at the beginning of this section that ruled out the parallel cascade of  $\delta n_e$  and  $\delta B_\parallel$  is not

<sup>16</sup> See § 2.6 for a quick recapitulation of the standard scaling argument on the passive cascade that leads to a  $k_\perp^{-5/3}$  in the perpendicular direction. Note that effectively, there is also a cascade in  $k_\parallel$  if the latter is measured along the unperturbed field, i.e., a cascade in  $k_z$ . This is due to the perpendicular deformation of the perturbed magnetic field by the Alfvén-wave turbulence: since  $\nabla_\perp$  grows while  $\hat{\mathbf{b}} \cdot \nabla$  remains the same, we have from Eq. (115)  $\partial/\partial z \simeq -(\delta\mathbf{B}_\perp/B_0) \cdot \nabla_\perp$ .

quite as obvious as it might appear. Lithwick & Goldreich (2001) argued that the dissipation of  $\delta n_e$  and  $\delta B_\parallel$  at the ion gyroscale would cause these fluctuations to become uncorrelated at the same parallel scales as the Alfvénic fluctuations by which they are mixed, i.e.,  $k_{\parallel 0} \sim k_{\parallel A}$ . The damping rate then becomes comparable to the cascade rate, cutting off the cascades of density and field-strength fluctuations at  $k_{\parallel} \lambda_{\text{mfpi}} \sim 1$ . The corresponding perpendicular cutoff wave number is [see Eq. (194)]

$$k_\perp \sim l_0^{1/2} \lambda_{\text{mfpi}}^{-3/2}. \quad (195)$$

Asymptotically speaking, in a weakly collisional plasma, this cutoff is far above the ion gyroscale,  $k_\perp \rho_i \ll 1$ . However, the relatively small value of  $\lambda_{\text{mfpi}}$  in the warm ISM (which was the main focus of Lithwick & Goldreich 2001) meant that the numerical value of the perpendicular cutoff scale given by Eq. (195) was, in fact, quite close both to the ion gyroscale (see Table 1) and to the observational estimates for the inner scale of the electron-density fluctuations in the ISM (Spangler & Gwinn 1990; Armstrong et al. 1995). Thus, it was not possible to tell whether Eq. (195), rather than  $k_\perp \sim \rho_i^{-1}$ , represented the correct prediction.

The situation is rather different in the nearly collisionless case of the solar wind, where the cutoff given by Eq. (195) would mean that very little density or field-strength fluctuations should be detected above the ion gyroscale. Observations do not support such a conclusion: the density fluctuations appear to follow a  $k^{-5/3}$  law at all scales larger than a few times  $\rho_i$  (Celnikier et al. 1983, 1987; Coles & Harmon 1989; Marsch & Tu 1990; Coles et al. 1991), consistently with the expected behavior of an undamped passive scalar field (see § 2.6). It is interesting, in view of the above discussion, that in near-the-Sun measurements, these fluctuations are reported to have the form of highly anisotropic density filaments aligned with the magnetic field (Armstrong et al. 1990; Grall et al. 1997; Woo & Habbal 1997). An extended range of  $k^{-5/3}$  scaling above the ion gyroscale is also observed for the fluctuations of the magnetic-field strength (Marsch & Tu 1990; Bershadskii & Sreenivasan 2004; Hnat et al. 2005).

These observational facts suggest that the cutoff formula (195) does not apply. This does not, however, conclusively vitiate the Lithwick & Goldreich (2001) Theory. Heuristically, their argument is plausible, although it is, perhaps, useful to note that in order for the effect of the perpendicular dissipation terms, not present in the KRMHD equations (149-151), to be felt, the density and field-strength fluctuations should reach the ion gyroscale in the first place. Quantitatively, the failure of the density fluctuations in the solar wind to be damped could still be consistent with the Lithwick & Goldreich (2001) theory because of the relative weakness of the collisionless damping, especially at low beta (§ 6.2.2)—the explanation they themselves favor. The way to check observationally whether this explanation suffices would be to make a comparative study of the density and field-strength fluctuations for solar-wind data with different values of  $\beta_i$ . If the strength of the damping is the decisive factor, one should always see cascades of both  $\delta n_e$  and  $\delta B_\parallel$  at low  $\beta_i$ , no cascades at  $\beta_i \sim 1$ , and a cascade of  $\delta B_\parallel$  but not  $\delta n_e$  at high  $\beta_i$  (in this limit, the damping of the density fluctuations is strong, of field-strength weak; see § 6.2.1). If, on the other hand, the parallel cascade of the compressive fluctuations is intrinsically inefficient, very little  $\beta_i$  dependence is expected and a perpendicular cascade should be seen in all cases.



Obviously, an even more direct observational (or numerical) test would be the detection or non-detection of near-perfect alignment of the density and field-strength structures with the moving field lines (*not* with the mean magnetic field—see footnote 16), but it is not clear how to measure this reliably.

#### 7. TURBULENCE IN THE DISSIPATION RANGE: ELECTRON RMHD AND THE ENTROPY CASCADE

The validity of the theory discussed in § 5 and § 6 breaks down when  $k_{\perp}\rho_i \sim 1$ . As the ion gyroscale is approached, the Alfvén waves are no longer decoupled from the rest of the plasma dynamics. All modes now contain perturbations of the density and magnetic-field strength and can, therefore, be damped. Because of the low-frequency nature of the Alfvén-wave cascade,  $\omega \ll \Omega_i$  even at  $k_{\perp}\rho_i \sim 1$  [Eq. (46)], so ion cyclotron damping is not important, while the Landau damping is. The linear theory of this collisionless damping in the gyrokinetic approximation is worked out in detail in Howes et al. (2006).

The nonlinear theory of what happens at  $k_{\perp}\rho_i \sim 1$  is very poorly understood. It is, however, possible to make progress by examining what kind of fluctuations emerge on “the other side” of the transition, at  $k_{\perp}\rho_i \gg 1$ . It turns out that another turbulent cascade—this time of kinetic Alfvén waves (KAW, sometimes called whistlers in astrophysical literature)—is possible in this so-called “dissipation range.” It can transfer the turbulent energy down to the electron gyroscale, where it is dissipated (probably by the electron Landau damping; see Howes et al. 2006). Some observational evidence of such a cascade is, indeed, available in the solar wind (Bale et al. 2005; Grison et al. 2005, see further discussion in § 8.1.5). Below, we derive the equations that describe KAW-like fluctuations in the scale range  $k_{\perp}\rho_i \gg 1$ ,  $k_{\perp}\rho_e \ll 1$  and discuss this cascade.

Because of the presence of the collisionless damping at the ion gyroscale, only a certain fraction of the turbulent power arriving there from the inertial range is converted into the KAW cascade, while the rest is damped. The damping leads to the heating of the ions, an aspect of the  $k_{\perp}\rho_i \sim 1$  physics that is of considerable astrophysical interest: e.g., the efficiency of ion heating has been studied in the context of advection-dominated accretion flows (Quataert & Gruzinov 1999, see discussion in § 8.3) and of the solar corona (e.g., Cranmer & van Ballegoijen 2003). The process of depositing the collisionlessly damped fluctuation energy into the ion heat is not trivial because, as we explained in § 3.5, collisions do need to play a role in order for heating to occur. That is brought about by the development of a purely kinetic entropy cascade in phase space, discussed in § 7.4 and § 7.5.

A short summary of this section is given in § 7.8.

##### 7.1. Equations of Electron Reduced MHD

The derivation is straightforward: when  $a_i \sim k_{\perp}\rho_i \gg 1$ , all Bessel functions in Eqs. (110-111) are small, so the integrals of the ion distribution function vanish and Eqs. (110-111) become

$$\frac{\delta n_e}{n_{0e}} = -\frac{Ze\phi}{T_{0i}} = -\frac{2}{\sqrt{\beta_i}} \frac{\Phi}{\rho_i v_A}, \quad (196)$$

$$\frac{\delta B_{\parallel}}{B_0} = \frac{\beta_i}{2} \left(1 + \frac{Z}{\tau}\right) \frac{Ze\phi}{T_{0i}} = \sqrt{\beta_i} \left(1 + \frac{Z}{\tau}\right) \frac{\Phi}{\rho_i v_A}, \quad (197)$$

$$u_{\parallel e} = \frac{c}{4\pi en_{0e}} \nabla_{\perp}^2 A_{\parallel} = -\frac{\rho_i \nabla_{\perp}^2 \Psi}{\sqrt{\beta_i}}, \quad u_{\parallel i} = 0, \quad (198)$$

where we used the definitions (127) of the stream and flux functions  $\Phi$  and  $\Psi$ . These equations are a reflection of the fact that, for  $k_{\perp}\rho_i \gg 1$ , the ion response is effectively purely Boltzmann, with the gyrokinetic part  $h_i$  contributing nothing to the fields or flows [see Eq. (47) with  $h_i$  omitted;  $h_i$  does play an important role, however, as explained in §§ 7.3-7.5 below].

Substituting Eqs. (196-198) into Eqs. (108-109), we obtain the following closed system of equations

$$\frac{\partial \Psi}{\partial t} = v_A (1 + Z/\tau) \hat{\mathbf{b}} \cdot \nabla \Phi, \quad (199)$$

$$\frac{\partial \Phi}{\partial t} = -\frac{v_A}{2 + \beta_i (1 + Z/\tau)} \hat{\mathbf{b}} \cdot \nabla (\rho_i^2 \nabla_{\perp}^2 \Psi). \quad (200)$$

Note that, in view of Eq. (197), Eqs. (199) and (200) are evolution equations for the perpendicular and parallel components of the perturbed magnetic field, respectively.

We shall refer to Eqs. (199-200) as *Electron Reduced MHD* (ERMHD). They are related to the Electron, or Hall, Magnetohydrodynamics (EMHD)—a fluid-like approximation that evolves the magnetic field only and arises if one assumes that the magnetic field is frozen into the electron flow velocity  $\mathbf{u}_e$ , while the ions are immobile,  $\mathbf{u}_i = 0$  (Kingsep et al. 1990):

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{c}{4\pi en_{0e}} \nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}]. \quad (201)$$

It is not hard to show that if the gyrokinetic ordering (§ 3.1) is applied to Eq. (201), we have

$$\frac{\delta \mathbf{B}}{B_0} = \frac{1}{v_A} \hat{\mathbf{z}} \times \nabla_{\perp} \Psi + \hat{\mathbf{z}} \frac{\delta B_{\parallel}}{B_0}, \quad (202)$$

where the second term can be written in terms of  $\Phi$  according to Eq. (197) and the equations for  $\Psi$  and  $\Phi$  then coincide with the  $\beta_i \ll 1$  limit of Eqs. (199) and (200). When  $\beta_i$  is arbitrary,  $\mathbf{u}_i = 0$  does not quite follow from assuming  $k_{\perp}\rho_i \gg 1$ , because density fluctuations are important [Eq. (196)] and, consequently, the divergence of the ion velocity field  $\nabla \cdot \mathbf{u}_i$  gives rise to a nonnegligible contribution in Eq. (200) (see Appendix C.2 for details).

##### 7.2. Kinetic-Alfvén-Wave Cascade

The linear fluctuation modes supported by ERMHD are kinetic Alfvén waves with frequencies

$$\omega = \pm \sqrt{\frac{1 + Z/\tau}{2 + \beta_i (1 + Z/\tau)}} k_{\perp} \rho_i k_{\parallel} v_A. \quad (203)$$

The corresponding eigenfunctions are

$$\Theta_{\mathbf{k}}^{\pm} = \sqrt{(1 + Z/\tau) [2 + \beta_i (1 + Z/\tau)]} \frac{\Phi_{\mathbf{k}}}{\rho_i} \mp k_{\perp} \Psi_{\mathbf{k}}. \quad (204)$$

As in the case of Alfvén waves, there are two counter-propagating modes, but it is no longer true that copropagating waves do not interact, nor do individual finite-amplitude KAW-like fluctuations constitute exact solutions of the non-linear problem (cf. § 2.4). Since we have derived Eqs. (199-200) from gyrokinetics, the anisotropy of the fluctuations is hard-wired. However, it is not guaranteed that the actual physical cascade below the ion gyroscale is indeed anisotropic (although analysis of solar-wind measurements does seem to indicate that it is; see Leamon et al. 1998).

There is no definitive *a priori* argument that the EMHD turbulence should be analogous to the Alfvénic one, i.e., that it should consist of KAW-like fluctuations with  $k_{\parallel} \ll k_{\perp}$

and that the interactions between them should be strong. In fact, the EMHD turbulence was thought to be weak by several authors, who predicted a  $k^{-2}$  spectrum of magnetic energy assuming isotropy (Goldreich & Reisenegger 1992) or  $k_{\perp}^{-5/2}$  for the anisotropic case (Galtier & Bhattacharjee 2003; Galtier 2006; see also Voitenko 1998). However, numerical simulations based on Eq. (201) (Biskamp et al. 1996, 1999; Ng et al. 2003; Cho & Lazarian 2004) have revealed that the spectrum of magnetic fluctuations scales as  $k_{\perp}^{-7/3}$ , the outcome consistent with assuming that the turbulence is anisotropic (Quataert & Gruzinov 1999) and critically balanced (Cho & Lazarian 2004), i.e., that the propagation time and nonlinear interaction time are comparable at every scale.

Let us review the argument that leads to this scaling.

First assume that the fluctuations are KAW-like and that  $\Theta^+$  and  $\Theta^-$  [Eq. (204)] have similar scaling. This implies

$$\Psi_{\lambda} \sim \sqrt{1+\beta_i} \frac{\lambda}{\rho_i} \Phi_{\lambda} \quad (205)$$

(for the purposes of scaling arguments and order-of-magnitude estimates, we set  $Z/\tau = 1$ , but keep the  $\beta_i$  dependence so low- and high-beta limits could be recovered if necessary).

Assuming scale-space locality of interactions implies, analogously to Eq. (1),

$$\frac{(\Psi_{\lambda}/\lambda)^2}{\tau_{\text{KAW}\lambda}} \sim \frac{(1+\beta_i)(\Phi_{\lambda}/\rho_i)^2}{\tau_{\text{KAW}\lambda}} \sim \varepsilon_{\text{KAW}} = \text{const}, \quad (206)$$

where  $\tau_{\text{KAW}\lambda}$  is the cascade time and  $\varepsilon_{\text{KAW}}$  is the KAW energy flux proportional to the fraction of the total flux  $\varepsilon$  (or the total turbulent power  $\overline{P}_{\text{ext}}$ ; see § 3.4) that was converted into the KAW cascade at the ion gyroscale.

Using Eqs. (199-200) and Eq. (205), it is not hard to see that the characteristic nonlinear decorrelation time is  $\lambda^2/\Phi_{\lambda}$ . If the turbulence is strong, then this time is comparable to the inverse KAW frequency [Eq. (203)] scale by scale and we may assume the cascade time is comparable to either:

$$\tau_{\text{KAW}\lambda} \sim \frac{\lambda^2}{\Phi_{\lambda}} \sim \frac{1}{\sqrt{1+\beta_i}} \frac{\rho_i}{\lambda} \frac{v_A}{l_{\parallel\lambda}}. \quad (207)$$

In other words, this says that  $\partial/\partial z \sim (\delta \mathbf{B}_{\perp}/B_0) \cdot \nabla_{\perp}$  and so  $\delta B_{\perp\lambda}/B_0 \sim \lambda/l_{\parallel\lambda}$  (note that the last relation confirms that our scaling arguments do not violate the gyrokinetic ordering; see § 2.1 and § 3.1). Equation (207) is the critical balance assumption for KAW. As in the case of the Alfvén waves (§ 1.1), we might argue physically that the critical balance is set up because the parallel correlation length  $l_{\parallel\lambda}$  of the KAW turbulence is such that a wave can propagate the distance  $l_{\parallel\lambda}$  in one nonlinear decorrelation time corresponding to the perpendicular correlation length  $\lambda$ .

Combining Eqs. (206) and (207), we get the desired scaling relations for the KAW turbulence:

$$\Phi_{\lambda} \sim \left( \frac{\varepsilon_{\text{KAW}}}{\varepsilon} \right)^{1/3} \frac{v_A}{(1+\beta_i)^{1/3}} l_0^{-1/3} \rho_i^{2/3} \lambda^{2/3}, \quad (208)$$

$$l_{\parallel\lambda} \sim \left( \frac{\varepsilon}{\varepsilon_{\text{KAW}}} \right)^{1/3} l_0^{1/3} \rho_i^{1/3} \lambda^{1/3} \frac{1}{(1+\beta_i)^{1/6}}, \quad (209)$$

where  $l_0 = v_A^3/\varepsilon$ , as before. The first of these scaling relations is equivalent to a  $k_{\perp}^{-7/3}$  spectrum of magnetic energy, the second quantifies the anisotropy (which is stronger than for the

GS turbulence). Both scalings were confirmed in the numerical simulations of Cho & Lazarian (2004)—it is their detection of the scaling (209) that makes a particularly strong case that KAW turbulence is not weak and that the critical balance hypothesis applies.

For KAW-like fluctuations, the density [Eq. (196)] and magnetic field [Eqs. (197) and (204)] have the same spectrum as the scalar potential, i.e.,  $k_{\perp}^{-7/3}$ , while the electric field  $E \sim k_{\perp} \phi$  has a  $k_{\perp}^{-1/3}$  spectrum. The solar-wind fluctuation spectra below the ion gyroscale reported by Bale et al. (2005) are consistent with KAW turbulence: the  $k^{-7/3}$  scaling for the magnetic fluctuations and  $k^{-1/3}$  for the electric ones. However, not all solar-wind observations are quite as straightforwardly supportive of the notion of the KAW cascade and much steeper magnetic-fluctuation spectra have also been reported (e.g., Leamon et al. 1998; Smith et al. 2006). Possible reasons for this will emerge in § 7.5.

### 7.3. Generalized Energy: KAW and Entropy Cascades

The generalized energy (§ 3.4) in the limit  $k_{\perp} \rho_i \gg 1$  is calculated by substituting Eqs. (196) and (197) into Eq. (101):

$$\begin{aligned} W &= \int d^3 \mathbf{r} \left\{ \int d^3 \mathbf{v} \frac{T_{0i} \langle h_i^2 \rangle_{\mathbf{r}}}{2F_{0i}} + \frac{\delta B_{\perp}^2}{8\pi} \right. \\ &\quad \left. + \frac{n_{0i} T_{0i}}{2} \left( 1 + \frac{Z}{\tau} \right) \left[ 1 + \frac{\beta_i}{2} \left( 1 + \frac{Z}{\tau} \right) \right] \left( \frac{Ze\phi}{T_{0i}} \right)^2 \right\} \\ &= W_h + W_{\text{KAW}}. \end{aligned} \quad (210)$$

Here the first term,  $W_h$ , is the total variance of  $h_i$ , which is proportional to minus the entropy of the ion gyrocenter distribution (see § 3.5) and whose cascade to collisional scales will be discussed in § 7.4 and § 7.5. The remaining two terms are the independently cascaded KAW energy:

$$\begin{aligned} W_{\text{KAW}} &= \int d^3 \mathbf{r} \frac{m_i n_{0i}}{2} \left\{ |\nabla \Psi|^2 \right. \\ &\quad \left. + \left( 1 + \frac{Z}{\tau} \right) \left[ 1 + \frac{\beta_i}{2} \left( 1 + \frac{Z}{\tau} \right) \right] \frac{\Phi^2}{\rho_i^2} \right\} \\ &= \int d^3 \mathbf{r} \frac{m_i n_{0i}}{2} (|\Theta^+|^2 + |\Theta^-|^2). \end{aligned} \quad (211)$$

While we can write  $W_{\text{KAW}}$  as the sum of the energies of the “+” and “−” linear KAW eigenmodes [Eq. (204)], the two do not cascade independently and can exchange energy.

Comparing the way the generalized energy is split above and below the ion gyroscale (see § 5.6 for the  $k_{\perp} \rho_i \ll 1$  limit), we might interpret what happens at the  $k_{\perp} \rho_i \sim 1$  transition as a redistribution of the power that arrived from large scales between a cascade of KAW and a cascade of the (minus) gyrocenter entropy in the phase space. The latter cascade is the way in which the energy diverted from the electromagnetic fluctuations by the collisionless damping (wave-particle interaction) can be transferred to the collisional scales and deposited into heat. The concept of entropy cascade as the key agent in the heating of the plasma was introduced in § 3.5, where we promised a more detailed discussion later on. We now proceed to this discussion.

### 7.4. Kinetic Ion-Entropy Cascade in the Presence of KAW Turbulence

The ion-gyrocenter distribution function  $h_i$  satisfies the ion gyrokinetic equation (113), where ion-electron collisions are

neglected under the mass-ratio expansion. At  $k_\perp \rho_i \gg 1$ , the dominant contribution to  $\langle \chi \rangle_{\mathbf{R}_i}$  comes from the the electromagnetic fluctuations associated with KAW turbulence. Since the KAW cascade is decoupled from the entropy cascade,  $h_i$  is a passive tracer of the ring-averaged KAW turbulence in phase space. Expanding the Bessel functions in the expression for  $\langle \chi \rangle_{\mathbf{R}_i, \mathbf{k}}$  [ $a_i \gg 1$  in Eq. (62) with  $s = i$ ] and making use of Eqs. (197-198) and of the KAW scaling  $\Psi \sim \Phi/k_\perp \rho_i$  [Eq. (204)], it is not hard to show that

$$\frac{Ze}{T_{0i}} \langle \chi \rangle_{\mathbf{R}_i, \mathbf{k}} \simeq \frac{Ze}{T_{0i}} \langle \phi \rangle_{\mathbf{R}_i, \mathbf{k}} = \frac{2}{\sqrt{\beta_i}} \frac{J_0(a_i) \Phi_{\mathbf{k}}}{\rho_i v_A}, \quad (212)$$

where

$$J_0(a_i) \simeq \sqrt{\frac{2}{\pi a_i}} \cos\left(a_i - \frac{\pi}{4}\right), \quad a_i = k_\perp \rho_i \frac{v_\perp}{v_{\text{thi}}}, \quad (213)$$

so  $h_i$  satisfies [Eq. (113)]

$$\frac{\partial h_i}{\partial t} + v_\parallel \frac{\partial h_i}{\partial z} + \{\langle \Phi \rangle_{\mathbf{R}_i}, h_i\} = \frac{2}{\sqrt{\beta_i} \rho_i v_A} \frac{\partial \langle \Phi \rangle_{\mathbf{R}_i}}{\partial t} F_{0i} + \langle C_{ii}[h_i] \rangle_{\mathbf{R}_i} \quad (214)$$

with the conservation law [Eq. (63),  $s = i$ ]

$$\begin{aligned} \frac{1}{T_{0i}} \frac{dW_h}{dt} &\equiv \frac{d}{dt} \int d^3 \mathbf{v} \int d^3 \mathbf{R}_i \frac{h_i^2}{2F_{0i}} \\ &= \frac{2}{\sqrt{\beta_i} \rho_i v_A} \int d^3 \mathbf{v} \int d^3 \mathbf{R}_i \frac{\partial \langle \Phi \rangle_{\mathbf{R}_i}}{\partial t} h_i \\ &\quad + \int d^3 \mathbf{v} \int d^3 \mathbf{R}_i \frac{h_i \langle C_{ii}[h_i] \rangle_{\mathbf{R}_i}}{F_{0i}}. \end{aligned} \quad (215)$$

#### 7.4.1. Small Scales in Phase Space

The wave-particle interaction term (the first term on the right hand sides of these two equations) will shortly be seen to be subdominant at  $k_\perp \rho_i \gg 1$ . It represents the source of the invariant  $W_h$  due to the collisionless damping at the ion gyroscale of some fraction of the energy arriving from the inertial range. In a stationary turbulent state, we should have  $dW_h/dt = 0$  and this source should be balanced on average by the (negative definite) collisional dissipation term [= heating; see Eq. (69)]. This balance can only be achieved if  $h_i$  develops small scales in the velocity space and carries the generalized energy, or, in this case, entropy, to scales in the phase space at which collisions are important. A quick way to see this is by recalling that the collision operator has two velocity derivatives and can only balance the terms on the left-hand side of Eq. (214) if

$$\nu_{ii} v_{\text{thi}}^2 \left( \frac{\partial}{\partial \mathbf{v}} \right)^2 \sim \omega \quad \Rightarrow \quad \frac{\delta v}{v_{\text{thi}}} \sim \left( \frac{\nu_{ii}}{\omega} \right)^{1/2}, \quad (216)$$

where  $\omega$  is the characteristic frequency of the fluctuations of  $h_i$ . If  $\nu_{ii} \ll \omega$ ,  $\delta v/v_{\text{thi}} \ll 1$ . This is certainly true for  $k_\perp \rho_i \sim 1$ : taking  $\omega \sim k_\parallel v_A$  and using  $k_\parallel \lambda_{\text{mfpi}} \gg 1$  (which is the appropriate limit at and below the ion gyroscale for most of the plasmas of interest; cf. footnote 14), we have  $\nu_{ii}/\omega \sim \sqrt{\beta_i}/k_\parallel \lambda_{\text{mfpi}} \ll 1$ .

The condition (216) means that the collision rate can be arbitrarily small—this will always be compensated by the sufficiently fine velocity-space structure of the distribution function to produce a finite amount of entropy production (heating) independent of  $\nu_{ii}$  in the limit  $\nu_{ii} \rightarrow +0$ . The situation bears some resemblance to the emergence of small spatial scales in neutral-fluid turbulence with arbitrarily small

but nonzero viscosity (Kolmogorov 1941). The analogy is not perfect, however, because the ion gyrokinetic equation (214) does not contain a nonlinear interaction term that would explicitly cause a cascade in the velocity space. Instead, the (ring-averaged) KAW turbulence mixes  $h_i$  in the gyrocenter space via the nonlinear term in Eq. (214), so  $h_i$  will have small-scale structure in  $\mathbf{R}_i$  on characteristic scales much smaller than  $\rho_i$ . Let us assume that the dominant nonlinear effect is a local interaction of the small-scale fluctuations of  $h_i$  with the similarly small-scale component of  $\langle \Phi \rangle_{\mathbf{R}_i}$ . Since ring averaging is involved and  $k_\perp \rho_i$  is large, the values of  $\langle \Phi \rangle_{\mathbf{R}_i}$  corresponding to two velocities  $\mathbf{v}$  and  $\mathbf{v}'$  will come from spatially decorrelated electromagnetic fluctuations if  $k_\perp v_\perp/\Omega_i$  and  $k_\perp v'_\perp/\Omega_i$  [the argument of the Bessel function in Eq. (212)] differ by order unity, i.e., for

$$\frac{\delta v_\perp}{v_{\text{thi}}} = \frac{|v_\perp - v'_\perp|}{v_{\text{thi}}} \sim \frac{1}{k_\perp \rho_i}. \quad (217)$$

This relation gives a correspondence between the decorrelation scales of  $h_i$  in the position and velocity space. Combining Eqs. (217) and (216), we see that there is a collisional cutoff scale determined by  $k_\perp \rho_i \sim (\omega/\nu_{ii})^{1/2} \gg 1$ .<sup>17</sup> The cutoff scale is much smaller than the ion gyroscale. In the range between these scales, collisional dissipation is small. The ion entropy fluctuations are transferred across this scale range by means of a cascade, for a which we now construct a scaling theory.

#### 7.4.2. Scalings and the Phase-Space Cutoff

Since entropy is a conserved quantity, we will follow the well trodden Kolmogorov path, assume locality of interactions in scale space and constant entropy flux, and conclude, analogously to Eq. (1),

$$\frac{\nu_{\text{thi}}^8}{n_{0i}^2} h_{i\lambda}^2 \sim \varepsilon_h = \text{const}, \quad (218)$$

where  $\varepsilon_h$  is the entropy flux proportional to the fraction of the total turbulent power  $\varepsilon$  (or  $P_{\text{ext}}$ ; see § 3.4) that was diverted into the entropy cascade at the ion gyroscale, and is the cascade time that we now need to find.

By the critical-balance assumption, the decorrelation time of the electromagnetic fluctuations in KAW turbulence is comparable at each scale to the KAW period at that scale and to the nonlinear interaction time (Eq. (207)):

$$\tau_{\text{KAW}\lambda} \sim \frac{\lambda^2}{\Phi_\lambda} \sim \left( \frac{\varepsilon}{\varepsilon_{\text{KAW}}} \right)^{1/3} \frac{(1 + \beta_i)^{1/3} v_0^{1/3} \rho_i^{-2/3} \lambda^{4/3}}{v_A}. \quad (219)$$

The characteristic time associated with the nonlinear term in Eq. (214) is longer than  $\tau_{\text{KAW}\lambda}$  by a factor of  $(\rho_i/\lambda)^{1/2}$  due to the ring averaging, which reduces the strength of the nonlinear interaction. This weakness of the nonlinearity makes it possible to develop a systematic analytical theory of the entropy cascade (Schekochihin et al. 2007b). It is also possible to estimate the cascade time via a more qualitative argument analogous to that first devised by Kraichnan (1965) for the weak turbulence of Alfvén waves: during each KAW correlation time  $\tau_{\text{KAW}\lambda}$ , the nonlinearity changes the amplitude of  $h_i$  by only a small amount:

$$\Delta h_{i\lambda} \sim (\lambda/\rho_i)^{1/2} h_{i\lambda} \ll h_{i\lambda}; \quad (220)$$

<sup>17</sup> Another source of small-scale spatial smoothing comes from the perpendicular gyrocenter-diffusion terms  $\sim \nu_{ii}(v/v_{\text{thi}})^2 k_\perp^2 \rho_i^2 h_{i\mathbf{k}}$  that arise in the ring-averaged collision operators, e.g., the second term in the model operator (B12). These terms again enforce a cutoff wave number such that  $k_\perp \rho_i \sim (\omega/\nu_{ii})^{1/2} \gg 1$ .



these changes accumulate with time as a random walk, so after time  $t$ , the cumulative change in amplitude is  $\Delta h_{i\lambda}(t/\tau_{\text{KAW}\lambda})^{1/2}$ ; finally, the cascade time  $t =$  is the time after which the cumulative change in amplitude is comparable to the amplitude itself, which gives, using Eq. (219),

$$\sim \frac{\rho_i}{\lambda} \tau_{\text{KAW}\lambda} \sim \left( \frac{\varepsilon}{\varepsilon_{\text{KAW}}} \right)^{1/3} (1 + \beta_i)^{1/3} \frac{l_0^{1/3} \rho_i^{1/3} \lambda^{1/3}}{v_A}. \quad (221)$$

Substituting this into Eq. (218), we get

$$h_{i\lambda} \sim \frac{n_{0i}}{v_{\text{thi}}^3} \left( \frac{\varepsilon_h}{\varepsilon} \right)^{1/2} \left( \frac{\varepsilon}{\varepsilon_{\text{KAW}}} \right)^{1/6} \frac{(1 + \beta_i)^{1/6}}{\sqrt{\beta_i}} l_0^{-1/3} \rho_i^{1/6} \lambda^{1/6}, \quad (222)$$

which corresponds to a  $k_{\perp}^{-4/3}$  spectrum of entropy.

To work out the cutoff scales both in the gyrocenter and velocity space, we use Eqs. (216) and (217): in Eq. (216),  $\omega \sim 1/\tau$ , where  $\tau$  is the characteristic decorrelation time of  $h_i$  given by Eq. (221); using Eq. (217), we find the cutoffs:

$$\frac{\delta v_{\perp}}{v_{\text{thi}}} \sim \frac{1}{k_{\perp} \rho_i} \sim (\nu_{ii} \tau_{h\rho_i})^{3/5}, \quad (223)$$

where  $\tau_{h\rho_i}$  is the cascade time (221) taken at  $\lambda = \rho_i$ .

In the argument presented above, we assumed that the scaling of  $h_i$  was determined by the nonlinear mixing of  $h_i$  by the ring-averaged KAW fluctuations rather than by the wave-particle-interaction term on the right-hand side of Eq. (214). We can now confirm the validity of this assumption. The change in amplitude of  $h_i$  in one KAW correlation time  $\tau_{\text{KAW}\lambda}$  due to the wave-particle-interaction term is

$$\begin{aligned} \Delta h_{i\lambda} &\sim \frac{n_{0i}}{v_{\text{thi}}^3} \left( \frac{\lambda}{\rho_i} \right)^{1/2} \frac{\Phi_{\lambda}}{\sqrt{\beta_i} \rho_i v_A} \\ &\sim \frac{n_{0i}}{v_{\text{thi}}^3} \left( \frac{\varepsilon_{\text{KAW}}}{\varepsilon} \right)^{1/3} \frac{1}{\sqrt{\beta_i} (1 + \beta_i)^{1/3}} l_0^{-1/3} \rho_i^{-5/6} \lambda^{7/6}, \end{aligned} \quad (224)$$

where we have used Eq. (208). Comparing this with Eq. (220) and using Eq. (222), we see that  $\Delta h_{i\lambda}$  in Eq. (224) is a factor of  $(\lambda/\rho_i)^{1/2}$  smaller than  $\Delta h_{i\lambda}$  due to the nonlinear mixing.

#### 7.4.3. Parallel Phase Mixing

Another assumption, which was made implicitly, was that the parallel phase mixing due to the second term on the left-hand side of Eq. (214) could be ignored. This requires justification, especially because it is with this “ballistic” term that one traditionally associates the emergence of small-scale structure in the velocity space (e.g., Krommes 1999). The effect of the parallel phase mixing is to produce small scales in velocity space  $\delta v_{\parallel} \sim 1/k_{\parallel} t$ . Let us assume that the KAW turbulence imparts its parallel decorrelation scale to  $h_i$  and use the scaling relation (209) to estimate  $k_{\parallel} \sim l_{\parallel}^{-1}$ . Then, after one cascade time [Eq. (221)],  $h_i$  is decorrelated on the parallel velocity scales

$$\frac{\delta v_{\parallel}}{v_{\text{thi}}} \sim \frac{l_{\parallel\lambda}}{v_{\text{thi}}} \sim \frac{1}{\sqrt{\beta_i(1 + \beta_i)}} \sim 1. \quad (225)$$

We conclude that the nonlinear perpendicular phase mixing [Eq. (223)] is more efficient than the linear parallel one. Note that up to a  $\beta_i$ -dependent factor Eq. (225) is equivalent to a critical-balance-like assumption for  $h_i$  in the sense that the propagation time is comparable to the cascade time, or  $k_{\parallel} v_{\parallel} \sim -1$  [see Eq. (214)].

### 7.5. Kinetic Ion-Entropy Cascade in the Absence of KAW Turbulence

It is not currently known how one might determine analytically what fraction of the turbulent power arriving from the inertial range to the ion gyroscale is channelled into the KAW cascade and what fraction is dissipated via the kinetic ion-entropy cascade introduced in § 7.4 (perhaps it can only be determined by direct numerical simulations). It is certainly a fact that in many solar-wind measurements, the relatively shallow magnetic-energy spectra associated with the KAW cascade (§ 7.2) fail to appear and much steeper spectra are detected (close to  $k^{-4}$ ; see Leamon et al. 1998; Smith et al. 2006). In view of this evidence, it is interesting to ask what would be the nature of electromagnetic fluctuations below the ion gyroscale if the KAW cascade failed to be launched, i.e., if all (or most) of the turbulent power were directed into the entropy cascade (i.e., if  $W \simeq W_h$  in § 7.3).

#### 7.5.1. Equations

It is again possible to derive a closed set of equations for all fluctuating quantities.

Let us assume (and verify *a posteriori*; § 7.5.4) that the characteristic frequency of such fluctuations is much lower than the KAW frequency [Eq. (203)] so that the first term in Eq. (108) is small and the equation reduces to the balance of the other two terms. This gives

$$\frac{\delta n_e}{n_{0e}} = \frac{e\phi}{T_{0e}}, \quad (226)$$

meaning that the electrons are purely Boltzmann [ $h_e = 0$  to lowest order; see Eq. (93)]. Then, from Eq. (110),

$$\frac{Ze\phi}{T_{0i}} \equiv \frac{2}{\sqrt{\beta_i}} \frac{\Phi}{\rho_i v_A} = \left(1 + \frac{\tau}{Z}\right)^{-1} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \cdot \frac{1}{n_{0i}} \int d^3\mathbf{v} J_0(a_i) h_{i\mathbf{k}} \quad (227)$$

Using Eq. (227), we find from Eq. (111) that the field-strength fluctuations are

$$\frac{\delta B_{\parallel}}{B_0} = -\frac{\beta_i}{2} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{1}{n_{0i}} \int d^3\mathbf{v} \frac{2v_{\perp}^2}{v_{\text{thi}}^2} \frac{J_1(a_i)}{a_i} h_{i\mathbf{k}}, \quad (228)$$

which is smaller than  $Ze\phi/T_{0i}$  by a factor of  $\beta_i/k_{\perp}\rho_i$ .

Therefore, we can neglect  $\delta B_{\parallel}/B_0$  compared to  $\delta n_e/n_{0e}$  in Eq. (109). Using Eq. (226), we get what is physically the electron continuity equation:

$$\frac{\partial}{\partial t} \frac{e\phi}{T_{0e}} + \hat{\mathbf{b}} \cdot \nabla \left( \frac{c}{4\pi e n_{0e}} \nabla_{\perp}^2 A_{\parallel} + u_{\parallel i} \right) = 0, \quad (229)$$

$$u_{\parallel i} = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{1}{n_{0i}} \int d^3\mathbf{v} v_{\parallel} J_0(a_i) h_{i\mathbf{k}}. \quad (230)$$

Note that in terms of the stream and flux functions, Eq. (229) takes the form

$$\frac{\partial}{\partial z} \rho_i^2 \nabla_{\perp}^2 \Psi = \frac{2\tau}{Z} \frac{1}{v_A} \frac{\partial \Phi}{\partial t} + \sqrt{\beta_i} \rho_i \frac{\partial u_{\parallel i}}{\partial z}, \quad (231)$$

where we have approximated  $\hat{\mathbf{b}} \cdot \nabla \simeq \partial/\partial z$ , which will, indeed, be shown to be correct in § 7.5.4.

Together with ion gyrokinetic equation, which determines  $h_i$ , Eqs. (226-229) form a closed set. They describe low-frequency fluctuations of the density and electromagnetic field due solely to the presence of fluctuations of  $h_i$  below the ion gyroscale.

It follows from Eq. (228) that  $\delta B_{\parallel}/B_0$  contributes subdominantly to  $\langle \chi \rangle_{\mathbf{R}_i}$  [Eq. (62) with  $s = i$  and  $a_i \gg 1$ ]. It will be verified *a posteriori* (§ 7.5.4) that the same is true for  $A_{\parallel}$ . Therefore, Eqs. (212) and (214) continue to hold true, as in the case with KAW. This means that Eqs. (214) and (227) form a closed subset. Thus the kinetic ion-entropy cascade is self-regulating in the sense that  $h_i$  is no longer passive (as it was in the presence of KAW turbulence; § 7.4) but is mixed by the ring-averaged “electrostatic” fluctuations of the scalar potential, which themselves are produced by  $h_i$  according to Eq. (227).

The magnetic fluctuations are passive and determined by the electrostatic and entropy fluctuations via Eqs. (228) and (229).

### 7.5.2. Scalings for the Electrostatic Fluctuations

From Eq. (227), we can establish a correspondence between  $\Phi_{\lambda}$  and  $h_{i\lambda}$  (the electrostatic fluctuations and the fluctuations of the ion-gyrocenter distribution function):

$$\Phi_{\lambda} \sim \rho_i v_A \sqrt{\beta_i} \left( \frac{\lambda}{\rho_i} \right)^{1/2} \frac{h_{i\lambda} v_{\text{thi}}^3}{n_{0i}} \left( \frac{\delta v_{\perp}}{v_{\text{thi}}} \right)^{1/2} \sim \frac{v_{\text{thi}}^4}{n_{0i}} h_{i\lambda} \lambda, \quad (232)$$

where the factor of  $(\lambda/\rho_i)^{1/2}$  comes from the Bessel function [Eq. (213)] and the factor of  $(\delta v_{\perp}/v_{\text{thi}})^{1/2}$  results from the  $v_{\perp}$  integration of the oscillatory factor in the Bessel function times  $h_i$ , which decorrelates on small scales in the velocity space and, therefore, its integral accumulates in a random-walk-like fashion. The velocity-space scales are related to the spatial scales via Eq. (217), which was arrived at by an argument not specific to KAW-like fluctuations and, therefore, continues to hold.

Using Eq. (232), we find that the wave-particle interaction term in the right-hand side of Eq. (214) is subdominant: comparing it with  $\partial h_i / \partial t$  shows that it is smaller by a factor of  $(\lambda/\rho_i)^{3/2} \ll 1$ . Therefore, it is the nonlinear term in Eq. (214) that controls the scalings of  $h_{i\lambda}$  and  $\Phi_{\lambda}$ .

We now assume again the scale-space locality and constancy of the entropy flux, so Eq. (218) holds. The cascade (decorrelation) time is equal to the characteristic time associated with the nonlinear term in Eq. (214):  $\sim (\rho_i/\lambda)^{1/2} \lambda^2 / \Phi_{\lambda}$ . Substituting this into Eq. (218) and using Eq. (232), we arrive at the desired scaling relations for the entropy cascade:

$$h_{i\lambda} \sim \frac{n_{0i}}{v_{\text{thi}}^3} \left( \frac{\varepsilon_h}{\varepsilon} \right)^{1/3} \frac{1}{\sqrt{\beta_i}} l_0^{-1/3} \rho_i^{1/6} \lambda^{1/6}, \quad (233)$$

$$\Phi_{\lambda} \sim \left( \frac{\varepsilon_h}{\varepsilon} \right)^{1/3} v_A l_0^{-1/3} \rho_i^{1/6} \lambda^{7/6}, \quad (234)$$

$$\sim \left( \frac{\varepsilon}{\varepsilon_h} \right)^{1/3} \frac{l_0^{1/3} \rho_i^{1/3} \lambda^{1/3}}{v_A}. \quad (235)$$

Note that since the existence of this cascade depends on it not being overwhelmed by the KAW fluctuations, we should have  $\varepsilon_{\text{KAW}} \ll \varepsilon$  and  $\varepsilon_h = \varepsilon - \varepsilon_{\text{KAW}} \approx \varepsilon$ .

The scaling for the ion-gyrocenter distribution function, Eq. (233), implies a  $k_{\perp}^{-4/3}$  spectrum—the same as for the KAW turbulence [Eq. (222)]. The scaling for the cascade time, Eq. (235), is also similar to that for the KAW turbulence [Eq. (221)]. Therefore the velocity- and gyrocenter-space cut-offs are still given by Eq. (223), where  $\tau_{h\rho_i}$  is now given by Eq. (235) taken at  $\lambda = \rho_i$ .

A new feature is the scaling of the scalar potential, given by Eq. (234), which corresponds to a  $k_{\perp}^{-10/3}$  spectrum (unlike the

KAW spectrum, § 7.2). This is a measurable prediction for the electrostatic fluctuations: the implied electric-field spectrum is  $k_{\perp}^{-4/3}$ . From Eq. (226), we also conclude that the density fluctuations should have the same spectrum as the scalar potential,  $k_{\perp}^{-10/3}$ —another measurable prediction.

### 7.5.3. Parallel Cascade and Parallel Phase Mixing

We have again ignored the “ballistic” term (the second on the left-hand side) in Eq. (214). We will estimate the efficiency of the parallel spatial cascade of the ion entropy and of the associated parallel phase mixing by making a conjecture analogous to the critical balance: assuming that any two perpendicular planes only remain correlated provided particles can stream between them in one nonlinear decorrelation time (cf. § 1.1 and § 7.4.3), we conclude that the parallel particle-streaming frequency  $k_{\parallel} v_{\parallel}$  should be comparable at each scale to the inverse nonlinear time<sup>-1</sup>, so

$$k_{\parallel} v_{\text{thi}} \sim 1. \quad (236)$$

As we explained in § 7.4.3, the parallel scales in the velocity space generated via the ballistic term are related to the parallel wave numbers by  $\delta v_{\parallel} \sim 1/k_{\parallel} t$ . From Eq. (236), we find that after one cascade time, the typical parallel velocity scale is  $\delta v_{\parallel}/v_{\text{thi}} \sim 1$ , so the parallel phase mixing is again much less efficient than the perpendicular one.

### 7.5.4. Scalings for the Magnetic Fluctuations

The scaling law for the fluctuations of the magnetic-field strength follows immediately from Eqs. (228) and (234):

$$\frac{\delta B_{\parallel\lambda}}{B_0} \sim \sqrt{\beta_i} \frac{\lambda}{\rho_i} \frac{\Phi_{\lambda}}{\rho_i v_A} \sim \sqrt{\beta_i} l_0^{-1/3} \rho_i^{-11/6} \lambda^{13/6}, \quad (237)$$

whence the spectrum of these fluctuations is  $k_{\perp}^{-16/3}$ .

The scaling of  $A_{\parallel}$  (the perpendicular magnetic fluctuations) depends on the relation between  $k_{\parallel}$  and  $k_{\perp}$ . Indeed, the ratio between the first and the third terms on the left-hand side of Eq. (229) [or, equivalently, between the first and second terms on the right-hand side of Eq. (231)] is  $\sim (k_{\parallel} v_{\text{thi}})^{-1}$ . For a critically balanced cascade, this makes the two terms comparable [Eq. (236)]. Using the first term to work out the scaling for the perpendicular magnetic fluctuations, we get, using Eq. (234),

$$\frac{\delta B_{\perp\lambda}}{B_0} \sim \frac{1}{\lambda} \frac{\Psi_{\lambda}}{v_A} \sim \sqrt{\beta_i} \frac{\lambda}{\rho_i} \frac{\Phi_{\lambda}}{\rho_i v_A} \sim \sqrt{\beta_i} l_0^{-1/3} \rho_i^{-11/6} \lambda^{13/6}, \quad (238)$$

which is the same scaling as for  $\delta B_{\parallel}/B_0$  [Eq. (237)].

Using Eq. (238) together with Eqs. (234) and (235), it is now straightforward to confirm the three assumptions made in § 7.5.1 that we promised to verify *a posteriori*:

1. In Eq. (108),  $\partial A_{\parallel}/\partial t \ll c \hat{\mathbf{b}} \cdot \nabla \phi$ , so Eq. (226) holds (the electrons remain Boltzmann). This means that no KAW can be excited by the cascade.
2.  $\delta B_{\perp}/B_0 \ll k_{\parallel}/k_{\perp}$ , so  $\hat{\mathbf{b}} \cdot \nabla \simeq \partial/\partial z$  in Eq. (229). This means that field lines are not significantly perturbed.
3. In the expression for  $\langle \chi \rangle_{\mathbf{R}_i}$  [Eq. (62)],  $v_{\parallel} A_{\parallel}/c \ll \phi$ , so Eq. (214) holds. This means that the electrostatic fluctuations dominate the cascade.

## 7.5.5. Cascades Superposed?

The spectra of magnetic fluctuations obtained in § 7.5.4 are very steep—steeper, in fact, than those normally observed in the dissipation range of the solar wind. One might speculate that the observed spectra may be due to a superposition of the two cascades realizable below the ion gyroscale: a high-frequency cascade of KAW (§ 7.2) and a low-frequency cascade of electrostatic fluctuations due to the ion entropy fluctuations (§ 7.5). Such a superposition could happen if the power going into the KAW cascade is relatively small,  $\varepsilon_{\text{KAW}} \ll \varepsilon$ . One then expects an electrostatic cascade to be set up just below the ion gyroscale with the KAW cascade superceding it deeper into the dissipation range. Comparing Eqs. (208) and (234), we can estimate the position of the spectral break:

$$k_{\perp} \rho_i \sim (\varepsilon / \varepsilon_{\text{KAW}})^{2/3}. \quad (239)$$

Since  $\rho_i / \rho_e \sim (\tau m_i / m_e)^{1/2} / Z$  is not a very large number, the dissipation range is not very wide. It then conceivable that the observed spectra are not true power laws but simply nonasymptotic superpositions of the electrostatic and KAW spectra with the observed range of “effective” spectral exponents due to varying values of the spectral break (239) between the two cascades.<sup>18</sup>

The value of  $\varepsilon_{\text{KAW}} / \varepsilon$  specific to any particular set of parameters ( $\beta_i$ ,  $\tau$ , etc.) is set by what happens at  $k_{\perp} \rho_i \sim 1$ —clearly a crucial place for determining the properties of the turbulence in the dissipation range, just as it is for solving the problem of ion heating mentioned at the beginning of § 7 (see § 8.1.5, § 8.3 for further discussion).

## 7.6. Below the Electron Gyroscale: The Last Cascade

Finally, let us consider what happens when  $k_{\perp} \rho_e \gg 1$ . At these scales, we have to return to the full gyrokinetic system of equations. The quasineutrality [Eq. (54)], parallel [Eq. (55)] and perpendicular [Eq. (59)] Ampère’s law become

$$\frac{e\phi}{T_{0e}} = - \left(1 + \frac{Z}{\tau}\right)^{-1} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \frac{1}{n_{0e}} \int d^3 \mathbf{v} J_0(a_e) h_{e\mathbf{k}}, \quad (240)$$

$$\frac{c}{4\pi e n_{0e}} \nabla_{\perp}^2 A_{\parallel} = \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \frac{1}{n_{0e}} \int d^3 \mathbf{v} v_{\parallel} J_0(a_e) h_{e\mathbf{k}}, \quad (241)$$

$$\frac{\delta B_{\parallel}}{B_0} = - \frac{\beta_i Z}{2 \tau} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \frac{1}{n_{0e}} \int d^3 \mathbf{v} \frac{2v_{\perp}^2}{v_{\text{the}}^2} \frac{J_1(a_e)}{a_e} h_{e\mathbf{k}}, \quad (242)$$

where we have discarded the velocity integrals of  $h_i$  both because the gyroaveraging makes them subdominant in powers of  $(m_e / m_i)^{1/2}$  and because the fluctuations of  $h_i$  are damped by collisions [assuming the collisional cutoff given by Eq. (223) lies above the electron gyroscale]. To Eqs. (240–242), we must append the gyrokinetic equation for  $h_e$  [Eq. (50) with  $s = e$ ], thus closing the system.

The type of turbulence described by these equations is very similar to that discussed in § 7.5. It is easy to show from Eqs. (240–242) that

$$\frac{\delta B_{\perp}}{B_0} \sim \frac{\delta B_{\parallel}}{B_0} \sim \frac{\beta_i}{k_{\perp} \rho_e} \frac{e\phi}{T_{0e}}. \quad (243)$$

Hence the magnetic fluctuations are subdominant in the expression for  $\langle \chi \rangle_{\mathbf{R}_e}$  [Eq. (62) with  $s = e$  and  $a_e \gg 1$ ], so

<sup>18</sup> Several alternative theories that aim to explain the dissipation-range spectra exist (e.g., Voitenko 1998; Stawicki et al. 2001; Krishan & Mahajan 2004; Galtier & Buchlin 2007; Howes et al. 2007).

$\langle \chi \rangle_{\mathbf{R}_e} \simeq \langle \phi \rangle_{\mathbf{R}_e}$ . The electron gyrokinetic equation then is

$$\frac{\partial h_e}{\partial t} + v_{\parallel} \frac{\partial h_e}{\partial z} + \frac{c}{B_0} \{ \langle \phi \rangle_{\mathbf{R}_e}, h_e \} = \left( \frac{\partial h_e}{\partial t} \right)_c, \quad (244)$$

where the wave-particle interaction term in the right-hand side has been dropped because it can be shown to be small via the same argument as in § 7.5.2.

Together with Eq. (240), Eq. (244) describes the kinetic cascade of electron entropy from the electron gyroscale down to the scale at which electron collisions can dissipate it into heat. This cascade the result of collisionless damping of KAW at  $k_{\perp} \rho_e \sim 1$ , whereby the power in the KAW cascade is converted into the electron-entropy fluctuations: indeed, in the limit  $k_{\perp} \rho_e \gg 1$ , the generalized energy is simply

$$W = \int d^3 \mathbf{v} \int d^3 \mathbf{R}_e \frac{T_{0e} h_e^2}{2 F_{0e}}. \quad (245)$$

The same scaling arguments as in § 7.5.2 apply and scaling relations analogous to Eqs. (233–235), and (237) duly follow:

$$h_e \lambda \sim \frac{n_{0e}}{v_{\text{the}}^3} \sqrt{\frac{m_e}{m_i}} \left( \frac{\varepsilon_{\text{KAW}}}{\varepsilon} \right)^{1/3} \frac{1}{\sqrt{\beta_i}} l_0^{-1/3} \rho_e^{1/6} \lambda^{1/6}, \quad (246)$$

$$\Phi_{\lambda} \sim \left( \frac{\varepsilon_{\text{KAW}}}{\varepsilon} \right)^{1/3} v_A l_0^{-1/3} \rho_e^{1/6} \lambda^{7/6}, \quad (247)$$

$$\sim \left( \frac{\varepsilon}{\varepsilon_{\text{KAW}}} \right)^{1/3} \frac{l_0^{1/3} \rho_e^{1/3} \lambda^{1/3}}{v_A}, \quad (248)$$

$$\frac{\delta B_{\lambda}}{B_0} \sim \sqrt{\beta_i \frac{m_e}{m_i}} \left( \frac{\varepsilon_{\text{KAW}}}{\varepsilon} \right)^{1/3} l_0^{-1/3} \rho_e^{-11/6} \lambda^{13/6}. \quad (249)$$

The formula for the collisional cutoffs in the wavenumber and velocity space is analogous to Eq. (223):

$$\frac{\delta v_{\perp}}{v_{\text{thi}}} \sim \frac{1}{k_{\perp} \rho_i} \sim (\nu_{ei} \tau_{\rho_e})^{3/5}, \quad (250)$$

where  $\tau_{\rho_e}$  is the cascade time (248) taken at  $\lambda = \rho_e$ .

## 7.7. Validity of Gyrokinetics in the Dissipation Range

As the kinetic cascade takes the (generalized) energy to ever smaller scales, the frequency  $\omega$  of the fluctuations increases. In applying the gyrokinetic theory, one must be mindful of the need for this frequency to stay smaller than  $\Omega_i$ . Using the scaling formulae for the characteristic times of the fluctuations derived above [Eqs. (219), (235) and (248)], we can determine the conditions for  $\omega \ll \Omega_i$ . Thus, for the gyrokinetic theory to be valid everywhere in the inertial range, we must have

$$k_{\perp} \rho_i \ll \beta_i^{3/4} (l_0 / \rho_i)^{1/2} \quad (251)$$

at all scales down to  $k_{\perp} \rho_i \sim 1$ , i.e.,  $\rho_i / l_0 \ll \beta_i^{3/2}$ , not a very stringent condition.

Below the ion gyroscale, the KAW cascade (§ 7.2) remains in the gyrokinetic regime as long as

$$k_{\perp} \rho_i \ll (\varepsilon / \varepsilon_{\text{KAW}})^{1/4} \beta_i^{3/8} (1 + \beta_i)^{1/4} (l_0 / \rho_i)^{1/4}. \quad (252)$$

The condition for this still to be true at the electron gyroscale is

$$\frac{\rho_i}{l_0} \ll \frac{\varepsilon}{\varepsilon_{\text{KAW}}} \beta_i^{3/2} (1 + \beta_i) \left( \frac{m_e}{m_i} \right)^2. \quad (253)$$



The ion entropy fluctuations passively mixed by the KAW turbulence (§ 7.4) satisfy Eq. (252) at all scales down to the ion collisional cutoff [Eq. (223)] if

$$\frac{\lambda_{\text{mfpi}}}{l_0} \ll \left( \frac{\varepsilon}{\varepsilon_{\text{KAW}}} \right)^{3/4} \beta_i^{9/8} (1 + \beta_i)^{3/4} \left( \frac{\rho_i}{l_0} \right)^{1/4}. \quad (254)$$

Note that the condition for the ion collisional cutoff to lie above the electron gyroscale is

$$\frac{\lambda_{\text{mfpi}}}{l_0} \ll \left( \frac{\varepsilon}{\varepsilon_{\text{KAW}}} \right)^{1/3} \sqrt{\beta_i} (1 + \beta_i)^{1/3} \left( \frac{m_i}{m_e} \right)^{5/6} \left( \frac{\rho_i}{l_0} \right)^{2/3}. \quad (255)$$

In the absence of KAW turbulence, the pure ion-entropy cascade (§ 7.5) remains gyrokinetic for

$$k_{\perp} \rho_i \ll \beta_i^{3/2} (l_0 / \rho_i). \quad (256)$$

This is valid at all scales down to the ion collisional cutoff provided  $\lambda_{\text{mfpi}}/l_0 \ll \beta_i^3 (l_0 / \rho_i)$ , an extremely weak condition, which is always satisfied. This is because the ion-entropy fluctuations in this case have much lower frequencies than in the KAW regime. The ion collisional cutoff lies above the electron gyroscale if, similarly to Eq. (255),

$$\frac{\lambda_{\text{mfpi}}}{l_0} \ll \sqrt{\beta_i} \left( \frac{m_i}{m_e} \right)^{5/6} \left( \frac{\rho_i}{l_0} \right)^{2/3}. \quad (257)$$

If the condition (255) is satisfied, all fluctuations of the ion distribution function are damped out above the electron gyroscale. This means that below this scale, we only need the electron gyrokinetic equation to be valid, i.e.,  $\omega \ll \Omega_e$ . The electron-entropy cascade (§ 7.6) satisfies this condition for

$$k_{\perp} \rho_e \ll (\varepsilon / \varepsilon_{\text{KAW}}) \beta_i^{3/2} (m_i / m_e)^{3/2} (l_0 / \rho_e). \quad (258)$$

This is valid at all scales down to the electron collisional cutoff [Eq. (250)] provided  $\lambda_{\text{mfpe}}/l_0 \ll (\varepsilon / \varepsilon_{\text{KAW}})^2 \beta_i^3 (m_i / m_e)^3 (l_0 / \rho_e)$ , which is always satisfied.

Within the formal expansion we have adopted ( $k_{\perp} \rho_i \sim 1$  and  $k_{\parallel} \lambda_{\text{mfpi}} \sim \sqrt{\beta_i}$ ), it is not hard to see that  $\lambda_{\text{mfpi}}/l_0 \sim \epsilon^2$  and  $\rho_i/l_0 \sim \epsilon^3$ . Since all other parameters ( $m_e/m_i$ ,  $\beta_i$ , etc.) are order unity with respect to  $\epsilon$ , all of the above conditions for the validity of the gyrokinetics are asymptotically correct by construction. However, in application to real astrophysical plasmas, one should always check whether this construction holds. For example, substituting the relevant parameters for the solar wind shows that the gyrokinetic approximation is, in fact, likely to start breaking down somewhere between the ion and electron gyroscopes (Howes et al. 2007, see this paper also for a more detailed discussion of this point, a set of numerical tests of the validity of gyrokinetics in the dissipation range, and a linear theory of the conversion of KAW into ion-cyclotron-damped Bernstein waves).

### 7.8. Summary

In this section, we have analyzed the turbulence in the dissipation range, which turned out to have many more essentially kinetic features than the inertial range.

At the ion gyroscale,  $k_{\perp} \rho_i \sim 1$ , the kinetic cascade rearranged itself into two distinct components: part of the (generalized) energy arriving from the inertial range was collisionlessly damped, giving rise to a purely kinetic cascade of ion-entropy fluctuations, the rest was converted into a cascade of Kinetic Alfvén Waves (KAW).

The KAW cascade is described by two fluid-like equations for two scalar functions, the magnetic flux function  $\Psi$  and the scalar potential, expressed, for continuity with the results of § 5, in terms of the function  $\Phi = (c/B_0)\phi$ . The equations are (see § 7.1)

$$\frac{\partial \Psi}{\partial t} = v_A (1 + Z/\tau) \hat{\mathbf{b}} \cdot \nabla \Phi, \quad (259)$$

$$\frac{\partial \Phi}{\partial t} = -\frac{v_A}{2 + \beta_i (1 + Z/\tau)} \hat{\mathbf{b}} \cdot \nabla (\rho_i^2 \nabla_{\perp}^2 \Psi), \quad (260)$$

where  $\hat{\mathbf{b}} \cdot \nabla = \partial/\partial z + (1/v_A)\{\Psi, \dots\}$ . The density and magnetic-field-strength fluctuations are directly related to the scalar potential:

$$\frac{\delta n_e}{n_{0e}} = -\frac{2}{\sqrt{\beta_i}} \frac{\Phi}{\rho_i v_A}, \quad \frac{\delta B_{\parallel}}{B_0} = \sqrt{\beta_i} \left( 1 + \frac{Z}{\tau} \right) \frac{\Phi}{\rho_i v_A}. \quad (261)$$

We call Eqs. (259-261) the *Electron Reduced Magnetohydrodynamics (ERMHD)*.

The ion-entropy cascade is described by the ion gyrokinetic equation:

$$\frac{\partial h_i}{\partial t} + v_{\parallel} \frac{\partial h_i}{\partial z} + \{\langle \Phi \rangle_{\mathbf{R}}, h_i\} = \langle C_{ii}[h_i] \rangle_{\mathbf{R}}. \quad (262)$$

The ion distribution function is mixed by the ring-averaged scalar potential and undergoes a cascade both in the velocity and gyrocenter space—this phase-space cascade is essential for the conversion of the turbulent energy into the ion heat, which can ultimately only be done by collisions (see § 7.4).

If the KAW cascade is strong (its power  $\varepsilon_{\text{KAW}}$  is an order-unity fraction of the total injected turbulent power  $\varepsilon$ ), it determines  $\Phi$  in Eq. (262), so the ion-entropy cascade is passive with respect to the KAW turbulence. Equations (259-260) and (262) form a closed system that determines the three functions  $\Phi$ ,  $\Psi$ ,  $h_i$ , of which the latter is slaved to the first two. One can also compute  $\delta n_e$  and  $\delta B_{\parallel}$ , which are proportional to  $\Phi$  [Eq. (261)]. The generalized energy conserved by these equations is given by Eq. (210).

If the KAW cascade is weak ( $\varepsilon_{\text{KAW}} \ll \varepsilon$ ), the ion-entropy cascade dominates the turbulence in the dissipation range and drives low-frequency mostly electrostatic fluctuations, with a subdominant magnetic component. These are given by the following relations (see § 7.5)

$$\Phi = \frac{\rho_i v_A \sqrt{\beta_i}}{2(1 + \tau/Z)} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \frac{1}{n_{0i}} \int d^3 \mathbf{v} J_0(a_i) h_{i\mathbf{k}}, \quad (263)$$

$$\frac{\delta n_e}{n_{0e}} = \frac{2}{\sqrt{\beta_i}} \frac{Z}{\tau} \frac{\Phi}{\rho_i v_A}, \quad (264)$$

$$\Psi = \rho_i \sqrt{\beta_i} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \times \frac{1}{n_{0i}} \int d^3 \mathbf{v} \left( \frac{1}{1 + Z/\tau} \frac{i}{k_{\parallel}} \frac{\partial}{\partial t} - v_{\parallel} \right) \frac{J_0(a_i)}{k_{\perp}^2 \rho_i^2} h_{i\mathbf{k}}, \quad (265)$$

$$\frac{\delta B_{\parallel}}{B_0} = -\frac{\beta_i}{2} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \frac{1}{n_{0i}} \int d^3 \mathbf{v} \frac{2v_{\perp}^2}{v_{\text{th}}^2} \frac{J_1(a_i)}{a_i} h_{i\mathbf{k}}, \quad (266)$$

where  $a_i = k_{\perp} v_{\perp} / \Omega_i$ , Equations (262) and (263) form a closed system for  $\Phi$  and  $h_i$ . The rest of the fields, namely  $\delta n_e$ ,  $\Psi$  and  $\delta B_{\parallel}$ , are slaved to  $h_i$  via Eqs. (264-266).

The fluid and kinetic models summarized above are valid between the ion and electron gyroscopes. Below the electron gyroscale, the collisionless damping of the KAW cascade converts it into a cascade of electron entropy, similar in nature to the ion-entropy cascade (§ 7.6).

The KAW cascade and the low-frequency turbulence associated with the ion-entropy cascade have distinct scaling behaviors. For the KAW cascade, the spectra of the electric, density and magnetic fluctuations are (§ 7.2)

$$E_E(k_\perp) \propto k_\perp^{-1/3}, \quad E_n(k_\perp) \propto k_\perp^{-7/3}, \quad E_B(k_\perp) \propto k_\perp^{-7/3}. \quad (267)$$

For the ion- and electron-entropy cascades (§ 7.4 and § 7.6),

$$E_E(k_\perp) \propto k_\perp^{-4/3}, \quad E_n(k_\perp) \propto k_\perp^{-10/3}, \quad E_B(k_\perp) \propto k_\perp^{-16/3}. \quad (268)$$

We argued in § 7.5.5 that the observed spectra in the dissipation range of the solar wind could be the result of a superposition of these two cascades.

## 8. DISCUSSION OF ASTROPHYSICAL APPLICATIONS

We have so far only occasionally referred to some relevant observational evidence for the solar wind and the ISM. We now discuss in more detail how the theoretical framework laid out above applies to these and other astrophysical plasmas.

### 8.1. Solar Wind

The solar wind is a unique laboratory for studying plasma turbulence in astrophysical conditions because it is the only case in which direct *in situ* measurements of all interesting quantities are possible. Observations of the fluctuating magnetic and velocity fields have been possible since the 1960s (Coleman 1968) and a vast literature now exists on their spectra, anisotropy, Alfvénic character and many other aspects (a short recent review is Horbury et al. 2005; two long ones are Bruno & Carbone 2005; Marsch 2006). It is not our aim here to provide a comprehensive survey of what is known about plasma turbulence in the solar wind. We shall limit our discussion to a few points that we consider important. A more extended and quantitative discussion of the application of the gyrokinetic theory to the turbulence in the slow solar wind was given by Howes et al. (2007).

#### 8.1.1. Alfvénic Nature of Turbulence

The presence of Alfvén waves in the solar wind has been a confirmed fact since Belcher & Davis (1971). These are detected at low frequencies (large scales)—their spectral signature is a  $f^{-1}$  spectrum of magnetic-field fluctuations, where, by the Taylor (1938) hypothesis, the frequency-to-wave-number relationship is  $f \sim \mathbf{k} \cdot \mathbf{V}_{\text{sw}}$ , where  $\mathbf{V}_{\text{sw}}$  is the solar-wind mean velocity (it is highly supersonic and super-Alfvénic and far exceeds the fluctuating velocities:  $V_{\text{sw}} \sim 800$  km/s in the fast wind,  $V_{\text{sw}} \sim 300$  km/s in the slow wind). The  $f^{-1}$  spectrum corresponds to a uniform distribution of scales/frequencies of waves launched by the coronal activity near the Sun. Nonlinear interaction of these waves gives rise to an Alfvénic turbulent cascade of the type that was discussed above. The effective outer scale of this cascade can be detected as a break point where the  $f^{-1}$  scaling steepens to the Kolmogorov slope  $f^{-5/3}$  (Bavassano et al. 1982; Horbury et al. 1996). The particular scale at which this happens increases with the distance from the Sun (Bavassano et al. 1982), reflecting the more developed state of the turbulence at later stages of evolution. At 1 AU, the outer scale is roughly in the range of  $10^5 - 10^6$  km; the  $f^{-5/3}$  range extends down to frequencies that correspond to a few times the ion gyroradius ( $10^2 - 10^3$  km; see Table 1).

The strongest confirmation of the Alfvénic nature of the turbulence in the  $f^{-5/3}$  range is achieved by comparing the

spectra of electric and magnetic fluctuations—these follow each other with remarkable precision (Bale et al. 2005), as they should do for Alfvénic fluctuations. Indeed, as we showed in § 5.3, for  $k_\perp \rho_i \gg 1$ , these fluctuations are rigorously described by the RMHD equations and we should have  $c\phi/B_0 \equiv \Phi = \pm\Psi$  (see § 2.2 and § 2.4), whence

$$|\mathbf{E}_\mathbf{k}|^2 = k_\perp^2 |\phi_\mathbf{k}|^2 = \frac{B_0^2}{c^2} k_\perp^2 |\Psi_\mathbf{k}|^2 = \frac{v_A^2}{c^2} |\delta\mathbf{B}_\perp|_\mathbf{k}|^2 \quad (269)$$

(in other words, to lowest order in  $\epsilon$  and  $k_\perp \rho_i$ , the plasma velocity is the  $\mathbf{E} \times \mathbf{B}$  drift velocity,  $\mathbf{u}_\perp = c\mathbf{E} \times \hat{\mathbf{z}}/B_0$ , which, in an Alfvén wave, should be  $\mathbf{u}_\perp = \pm\delta\mathbf{B}_\perp/\sqrt{4\pi m_i n_0}$ ).

It is an important point that very high accuracy is claimed for the determination of the  $f^{-5/3}$  scaling in the solar wind observations: the measured spectral exponent is between 1.6 and 1.7; agreement with Kolmogorov value 1.67 is often reported to be within  $\pm 0.01$  (e.g., Horbury et al. 1996; Leamon et al. 1998; Bale et al. 2005; Horbury et al. 2007; see, however, a somewhat wider scatter of spectral indices given by Smith et al. 2006 and a puzzling recent result by Podesta et al. 2006, who claim different spectral indices for velocity and magnetic fluctuations— $3/2$  and  $5/3$ , respectively!). Thus, the data does not appear to be consistent with a  $k_\perp^{-3/2}$  spectrum observed in the MHD simulations with a strong mean field (Maron & Goldreich 2001; Müller et al. 2003) and defended on theoretical grounds in a recent modification of the GS theory by Boldyrev (2006). This discrepancy between observations and simulations, as well as the validity of Boldyrev’s polarization-alignment hypothesis, remain an unresolved theoretical issue. It is probably best addressed by higher-resolution numerical modeling of the RMHD equations (§ 2.2) and by a detailed analysis of the structure of the Alfvénic fluctuations measured in the solar wind.

#### 8.1.2. Anisotropy

The solar wind presents the best hope for a quantitative check of the GS relation  $k_\parallel \sim k_\perp^{2/3}$  [see Eq. (4)] in a real astrophysical turbulent plasma. Studies of anisotropy of turbulent fluctuations have progressed from merely detecting their elongation along the magnetic field (Belcher & Davis 1971)—to fitting data to an *ad hoc* model mixing a 2D perpendicular and a 1D parallel (“slab”) turbulent component in some proportion (Matthaeus et al. 1990; Bieber et al. 1996; Dasso et al. 2005)—to formal systematic unbiased analyses showing the persistent presence of anisotropy at all scales (Bigazzi et al. 2006; Sorriso-Valvo et al. 2006)—and finally to computing spectral exponents at fixed angles between  $\mathbf{k}$  and  $\mathbf{B}_0$  (Horbury et al. 2007). The latter authors appear to have achieved the first direct confirmation of the GS theory by demonstrating that the magnetic-energy spectrum scales as  $k_\perp^{-5/3}$  in wave numbers perpendicular to the mean field and as  $k_\parallel^{-2}$  in wave numbers parallel to it [consistent with the first relation in Eq. (4)].

Latest measurements done by the four-spacecraft Cluster mission offer a promise of direct detailed studies of the three-dimensional spatial structure of the turbulent fluctuations in the solar wind (Osman & Horbury 2007; Sahraoui et al. 2006; Narita et al. 2006).

#### 8.1.3. Density and Magnetic-Field-Strength Fluctuations

The density and magnetic-field-strength fluctuations constitute the passive component of the low (Alfvénic) frequency

turbulence, energetically decoupled from and mixed by the Alfvénic cascade (§ 5.5; these are slow and entropy modes in the collisional MHD limit—see § 2.3). These fluctuations ought to be pressure-balanced, as expressed by Eq. (17) or, more generally in gyrokinetics, by Eq. (60). There is, indeed, strong evidence that magnetic and thermal pressures in the solar wind are anticorrelated, although there are some indications of the presence compressive, fast-wave-like fluctuations as well (see Marsch & Tu 1993; Bavassano et al. 2004, and references therein).

Measurements of density and field-strength fluctuations done by a variety of different methods both at 1 AU and near the Sun (Celnikier et al. 1983, 1987; Coles & Harmon 1989; Marsch & Tu 1990; Coles et al. 1991; Bershadskii & Sreenivasan 2004; Hnat et al. 2005; Kellogg & Horbury 2005) consistently show fluctuation levels of order 10% and spectra that have a  $k^{-5/3}$  scaling above scales of order  $10^2 - 10^3$  km, which approximately corresponds to the ion gyroscale. The Kolmogorov value of the spectral exponent is, as in the case of Alfvénic fluctuations, measured quite accurately ( $1.67 \pm 0.03$  in Celnikier et al. 1987). Considering that in the collisionless regime these fluctuations are supposed to be subject to strong kinetic damping (§ 6.2), the presence of well-developed Kolmogorov-like and apparently undamped turbulent spectra should be more surprising than has perhaps been acknowledged. A discussion of this issue was given in § 6.3. Without the inclusion of the dissipation effects associated with the finite ion gyroscale, the passive cascade of the density and field strength is purely perpendicular to the (exact) local magnetic field and does not lead to any scale refinement along the field. This implies highly anisotropic field-aligned structures, for which the kinetic damping is inefficient. While this would seem to explain the presence of fully-fledged Kolmogorov spectra, it is not entirely obvious that the parallel cascade is really absent once dissipation is taken into account (Lithwick & Goldreich 2001), so the issue is not yet settled. This said, we note that there is plenty of evidence of a high degree of anisotropy and field alignment of the density microstructure in the inner solar wind and outer corona (e.g., Armstrong et al. 1990; Grall et al. 1997; Woo & Habbal 1997). Further suggestions for an observational study that could settle this issue are outlined at the end of § 6.3.

#### 8.1.4. Dissipation at the Ion Gyroscale

At scales of the order of the ion gyroscale, several dissipation mechanisms become important. Observations are not consistent with cyclotron damping being the main mechanism for the dissipation of the inertial-range Alfvénic turbulence at the ion gyroscale (Leamon et al. 1998)—not a surprising result in view of the theory presented above, in which the Alfvén frequency is much smaller than the cyclotron frequency even for  $k_{\perp} \rho_i \sim 1$  [see Eq. (46)]. While the cyclotron resonance is not important, the Barnes (1966) one (§ 6.2) should be. Indeed, when  $k_{\perp} \rho_i \sim 1$ , the Alfvénic fluctuations are no longer decoupled from the density and magnetic-field-strength fluctuations. It should be possible to convert at least some fraction of the energy of the Alfvénic cascade into the latter, and then dissipate it via collisionless damping (the gyrokinetic limit of this with finite  $k_{\perp} \rho_i$  is calculated in Howes et al. 2006). We might argue that it is the inflow of energy from the Alfvénic cascade that accounts for a pronounced local flattening of the spectrum of density fluctuations in the solar wind observed just above the ion gyroscale (Celnikier et al. 1983,

1987; Coles & Harmon 1989; Marsch & Tu 1990; Coles et al. 1991; Kellogg & Horbury 2005).<sup>19</sup>

Since the neglect of the effect of collisions on the Alfvén waves in § 5.3 was predicated on the smallness of  $k_{\perp} \rho_i$  (see Appendices B.1 and B.2), collisional dissipation may also play a role at the ion gyroscale unless the collision frequency is much smaller than the fluctuation frequency and the role of collisions is merely to enable the irreversibility of the “collisionless” damping (see § 3.5 and §§ 7.4-7.5)—the latter limit is probably more relevant to the highly collisionless case of the solar wind.

#### 8.1.5. Fluctuations in the Dissipation Range

If not all of the energy is dissipated at the ion gyroscale (into ion heat), a kinetic-Alfvén-wave cascade (§ 7.1 and § 7.2) is possible below this scale—in what is known as “the dissipation range” in the solar-wind literature. The cascade is damped at the electron gyroscale via the electron Landau damping and converted into electron heat (the expression for the damping rate in the gyrokinetic limit is given in Howes et al. 2006).

KAW have, indeed, been detected in the solar wind (Grison et al. 2005). The spectra of electric and magnetic fluctuations below the ion gyroscale reported by Bale et al. (2005) are consistent with  $k^{-1/3}$  and  $k^{-7/3}$ —a result that gives some observational backing to the theoretical predictions based on assuming an anisotropic critically balanced KAW cascade (§ 7.2; see Schekochihin et al. 2007a for theoretical scaling fits superimposed on a plot taken from Bale et al. 2005; note, however, that Bale et al. 2005 themselves interpreted their data in a somewhat different way and that their resolution was in any case not sufficient to be sure of the scalings). The anisotropy of the fluctuations below the ion gyroscale was corroborated by Leamon et al. 1998, who, however, reported steeper magnetic-fluctuation spectra (with spectral exponent  $\simeq -4$ ), although they too thought they were seeing KAW.

Analysis of many different measurements of the magnetic-fluctuation spectra in the dissipation range of the solar wind reveals a wide spread in the spectral indices (roughly between  $-1$  and  $-4$ ; see Smith et al. 2006). There is evidence of a weak positive correlation between steeper dissipation-range spectra and higher ion temperatures (Leamon et al. 1998) or higher dissipation rates calculated from the inertial range (Smith et al. 2006). This suggests that a larger amount of ion heating corresponds to a fully or partially suppressed KAW cascade, which is in line with the view of the ion heating and the KAW cascade as the two competing channels of the overall kinetic cascade (§ 7.3). With a weakened KAW cascade, all or part of the dissipation range is dominated by the ion entropy cascade—a purely kinetic phenomenon manifested by predominantly electrostatic fluctuations and very steep magnetic-energy spectra (§ 7.5). This might account both for the steepness of the observed spectra and for the spread in their indices (§ 7.5.5).

It is fair to admit, however, that there is currently no satisfactory quantitative or even qualitative theory that would allow us to predict when the KAW cascade is present and when it is not. Resolution both of this issue and of the related ques-

<sup>19</sup> Celnikier et al. (1987) proposed that the flattening might be a  $k^{-1}$  spectrum analogous to Batchelor’s spectrum of passive scalar variance in the viscous-convective range. We think this explanation is incorrect because density is not passive at or below the ion gyroscale.



tion of which fraction of the turbulent power injected at the outer scale ends up in the ion vs. the electron heat requires careful numerical modeling—the minimal model appropriate for such simulations is the system of equations for isothermal electrons and gyrokinetic ions derived in § 4.

The density spectra measured by Celnikier et al. (1983, 1987) steepen below the ion gyroscale following the flattened segment around  $k_{\perp}\rho_i \sim 1$  discussed in § 8.1.4. For a KAW cascade, the density spectrum should be  $k^{-7/3}$  (§ 7.2); without KAW,  $k^{-10/3}$  (§ 7.5.2). The observed slope appears to be somewhat shallower even than  $k^{-2}$ , but, given imperfect resolution, neither seriously in contradiction with the KAW cascade, nor sufficient to corroborate it. Unfortunately, we could not find published simultaneous measurements of density- and magnetic-fluctuation spectra—such measurements would provide a better description of the dissipation range and a possibly a way to distinguish between various theories.

Finally, a caveat is in order that a properly asymptotic behavior in the dissipation range is probably impossible in nature because the scale separation is only about  $(m_i/m_e)^{1/2} \simeq 43$  between the ion and electron gyroscscales [Eq. (41)]. We refer the reader to the paper by Howes et al. (2007) for a discussion of how this lack of asymptoticity, namely the fact that the kinetic damping is not very small anywhere in the dissipation range, could be responsible for the observed spectra.

#### 8.1.6. Is the Equilibrium Distribution Isotropic?

Throughout this paper, we have emphasized that we consider weakly collisional plasmas. The collision frequency was ordered similar to the fluctuation frequency and, therefore,  $k_{\parallel}\lambda_{\text{mfpi}}/\sqrt{\beta_i} \sim 1$  [Eq. (40)]. This meant that we could proceed to treat both collisional and collisionless scales via subsidiary expansions in  $k_{\parallel}\lambda_{\text{mfpi}}$  or in  $1/\sqrt{\beta_i}$  (see § 6). It turns out that this degree of collisionality is sufficient to prove that an isotropic Maxwellian equilibrium distribution  $F_{0s}(v)$  does indeed emerge in the lowest order of the gyrokinetic expansion (Howes et al. 2006).

This argument works very well for plasmas such as the ISM, where collisions are weak ( $\lambda_{\text{mfpi}} \gg \rho_i$ ) but nonnegligible ( $\lambda_{\text{mfpi}} \ll L$ ). In the solar wind, the mean free path is of the order of 1 AU—the distance between the Sun and the Earth (see Table 1). Strictly speaking, in so highly collisionless a plasma, the equilibrium distribution does not have to be either Maxwellian or isotropic. The conservation of the first adiabatic invariant,  $\mu = v_{\perp}^2/2B$ , suggests that temperature anisotropy with respect to the magnetic-field direction ( $T_{0\perp} \neq T_{0\parallel}$ ) may exist. Such anisotropy gives rise to several high-frequency plasma instabilities: most prominently the firehose ( $T_{0\perp} < T_{0\parallel}$ ), mirror and ion cyclotron ( $T_{0\perp} > T_{0\parallel}$ ) modes (e.g., Gary et al. 1976). Their growth rates peak around the ion gyroscale, giving rise to small-scale fluctuations at this scale and possibly an energy cascade below it. Thus, a broad spectrum of mirror structures at and below the ion gyroscale has been clearly identified by some of the spacecraft measurements (e.g., in the terrestrial magnetosheath; see Lucek et al. 2001; Sahraoui et al. 2006).<sup>20</sup>

No definitive analytical theory of such fluctuations has been

proposed, although there does exist a vast space-physics literature on the subject, which will not be reviewed here. It appears to be a plausible assumption that the fluctuations resulting from temperature anisotropy will scatter particles and limit the anisotropy. This idea has some support in the solar-wind observations: the core particle distribution is only moderately anisotropic and consistent with a marginal state with respect to the instability conditions (Gary et al. 2001; Marsch et al. 2004; Kasper et al. 2002; Hellinger et al. 2006, see also Kellogg et al. 2006 for measurements of the electric-field fluctuations in the ion-cyclotron frequency range and estimates of the resulting velocity-space diffusion—claimed to be sufficient to isotropize the ion distribution). This means that assuming a Maxwellian equilibrium is probably an acceptable simplification in treating the turbulence above the ion gyroscale. The status of this simplification for fluctuations at or below the ion gyroscale is more problematic. Further theoretical work is necessary to correctly describe this scale range in the parts of the solar wind or the terrestrial magnetosphere where temperature anisotropy is observed and also to have a clear understanding of the circumstances under which this anisotropy occurs. The particle scattering by high-frequency fluctuations may also affect the turbulent cascades of the low-frequency fluctuations discussed in this paper by changing the effective value of the mean free path—another matter requiring further investigation.

## 8.2. Interstellar Medium

While the solar wind is unmatched by any other astrophysical plasma in the level of detail with which turbulence in it can be measured, the interstellar medium also provides an observer with a fair number of ways to diagnose plasma turbulence, which, in the case of ISM, is probably stirred by the supernova explosions (Norman & Ferrara 1996). The accuracy and resolution of this analysis are due to improve rapidly thanks to many new observatories, e.g., LOFAR,<sup>21</sup> Planck (Enßlin et al. 2006), and, in more distant future, the SKA (Lazio et al. 2004).

The ISM is a spatially inhomogeneous environment consisting of several phases that have different temperatures, densities and degrees of ionization (Ferrière 2001) (and, therefore, different degrees of importance of the neutral particles and the associated ambipolar damping effects—these will not be discussed here; see Lithwick & Goldreich 2001). We will use the Warm ISM phase (see Table 1) as our fiducial interstellar plasma and discuss briefly what is known about the two main observationally accessible quantities—the electron density and magnetic fields—and what this information means.

### 8.2.1. Electron Density Fluctuations

The electron-density fluctuations inferred from the interstellar scintillation measurements appear to have a spectrum with an exponent  $\simeq -1.7$ , consistent with the Kolmogorov scaling (Armstrong et al. 1995; Lazio et al. 2004; see, however, dissenting evidence by Smirnova et al. 2006, who claim a spectral exponent closer to  $-1.5$ ). This holds over about 5 decades of scales  $\lambda \in (10^5, 10^{10})$  km. Other observational evidence at larger and smaller scales supports the case for this presumed inertial range to be extended over as many as 12 decades:  $\lambda \in (10^2, 10^{15})$  km, a fine example of scale separation that prompted the impressed astrophysics community to dub the density scaling “The Great Power Law in the Sky.”

<sup>20</sup> Remarkably, these structures are highly anisotropic with  $k_{\parallel} \ll k_{\perp}$ . They cannot, however, be described by the gyrokinetic theory in its present form because  $\delta B_{\parallel}/B_0$  is very large ( $\sim 40\%$ , occasionally reaching unity) and because the particle trapping effect, which is likely to be essential in the nonlinear physics of the mirror instability (Kivelson & Southwood 1996; Schekochihin et al. 2007d), is ordered out in gyrokinetics.

<sup>21</sup> <http://www.lofar.org>

The upper cutoff here is consistent with the estimates of the supernova scale of order 100 pc—presumably the outer scale of the turbulence (Norman & Ferrara 1996) and also roughly the scale height of the galactic disk, hence the absolute upper limit of validity of any homogeneous model of the ISM turbulence. The lower cutoff is an estimate for the inner scale below which the logarithmic slope of the density spectrum steepens to values around  $-2$  (Spangler & Gwinn 1990).

Higdon (1984) was the first to realize that the electron-density fluctuations in the ISM could be explained as due to a cascade of passive tracer [the MHD entropy mode, see Eq. (18)] mixed by the ambient turbulence. This idea was brought to maturity by Lithwick & Goldreich (2001), who studied the passive cascades of the slow and entropy modes in the framework of the GS theory (see also Maron & Goldreich 2001). If the turbulence is assumed anisotropic, as in the GS theory, the passive nature of the density fluctuations with respect to the decoupled Alfvén-wave cascade becomes a rigorous result both in MHD (§ 2.3) and, as we showed above, in the more general gyrokinetic description appropriate for weakly collisional plasmas (§ 5.5). Anisotropy of the electron-density fluctuations in the ISM is, indeed, observationally supported (Wilkinson et al. 1994; Trotter et al. 1998; Rickett et al. 2002; Dennett-Thorpe & de Bruyn 2003, see also Lazio et al. 2004 for a concise discussion of these results), although detailed scale-by-scale measurements are not currently possible.

If the underlying Alfvén-wave turbulence in the ISM has a  $k_{\perp}^{-5/3}$  spectrum, as predicted by GS, so should the electron density. As we discussed in § 6.3, the physical nature of the inner scale for the density fluctuations depends on whether they have a cascade in  $k_{\parallel}$  and are efficiently damped when  $k_{\parallel} \lambda_{\text{mfp}i} \sim 1$  or fail to develop small parallel scales and can, therefore, reach  $k_{\perp} \rho_i \sim 1$ . The observationally estimated inner scale is consistent with being the ion gyroscale,  $\rho_i \sim 10^3$  km (see Table 1; note that the ion inertial scale  $d_i = \rho_i \beta_i^{-1/2}$  is similar to  $\rho_i$  at the moderate values of  $\beta_i$  characteristic of the ISM). However, since the mean free path in the ISM is not huge (Table 1), it is not possible to distinguish this from the perpendicular cutoff  $\lambda_{\text{mfp}i}^{3/2} L^{-1/2} \sim 500$  km implied by the parallel cutoff at  $k_{\parallel} \lambda_{\text{mfp}i} \sim 1$  [see Eq. (195)], as advocated by Lithwick & Goldreich (2001). Note that the relatively short mean free path means that much of the scale range spanned by the Great Power Law in the Sky is, in fact, well described by the MHD approximation either with adiabatic (§ 2) or isothermal (§ 6.1) electrons.

Below the ion gyroscale, the  $-2$  spectral exponent reported by Spangler & Gwinn (1990) is measured sufficiently imprecisely to be consistent with the  $-7/3$  expected for the density fluctuations in the KAW cascade (§ 7.2). However, given the high degree of uncertainty about what happens in this “dissipation range” even in the much better resolved case of the solar wind (§ 8.1.5), it would probably be wise to reserve judgement until better data is available.

### 8.2.2. Magnetic Fluctuations

The second main observable type of turbulent fluctuations in the ISM are the magnetic fluctuations, accessible indirectly via the measurements of the Faraday rotation of the polarization angle of the pulsar light travelling through the ISM. The structure function of the rotation measure (RM) should have the Kolmogorov slope of  $2/3$  if the magnetic fluctuations are due to Alfvénic turbulence described by the GS the-

ory. There is a considerable uncertainty in interpreting the available data, primarily due to insufficient spatial resolution (rarely better than a few parsec). Structure function slopes consistent with  $2/3$  have been reported (Minter & Spangler 1996), but, depending on where one looks, shallower structure functions that seem to steepen at scales of a few parsec are also observed (Haverkorn et al. 2004). A recent study by Haverkorn et al. (2005) detected an interesting trend: the RM structure functions computed for several regions that lie in the galactic spiral arms are nearly perfectly flat down to the resolution limit, while in the interarm regions, they have detectable slopes (although these are mostly shallower than  $2/3$ ). Observations of magnetic fields in external galaxies also reveal a marked difference in the magnetic-field structure between arms and interarms: the spatially regular (mean) fields are stronger in the interarms, while in the arms, the stochastic fields dominate (Beck 2006). These facts prompt us to return a very important physics issue, which was briefly raised in § 1.3 but otherwise ignored in this paper.

### 8.2.3. MHD Turbulence with and without a Mean Field

The theory of MHD turbulence as turbulence of Alfvénic fluctuations depends on the presence of a mean (guide) field  $\mathbf{B}_0$  that is strong compared to the magnetic fluctuations,  $\delta B/B_0 \sim u/v_A \ll 1$  (§ 2.1). Kraichnan (1965) argued that in a generic situation in which all magnetic fields are produced by the turbulence itself via the dynamo mechanism, one might assume that the strongest field will be at the outer scale and then view this outer-scale field as the approximately uniform guide field for the inertial-range Alfvén waves. It is, however, by no means obvious that this should be true. When a strong mean field is imposed by some external mechanism, the turbulent motions cannot bend it significantly, so only small Alfvénic (or slow-wave-like) perturbations are possible and  $\delta B_{\text{rms}} \ll B_0$ . In contrast, without a strong imposed field, the energy density of the magnetic fluctuations is at most comparable to the kinetic-energy density of the plasma motions, which are then sufficiently energetic to randomly tangle the field, so  $\delta B_{\text{rms}} \gg B_0$ .

In the weak-mean-field case, the dynamically strong stochastic magnetic field is a result of saturation of the *small-scale dynamo*—amplification of magnetic field due to random stretching by the turbulent motions (see review by Schekochihin & Cowley 2006b). The definitive theory of the saturated state remains to be discovered. Both physical arguments and numerical evidence (Schekochihin et al. 2004; Yousef et al. 2007) suggest that magnetic field is organized in folded flux sheets/ribbons. The length of the folds is comparable to the stirring scale, while the scale of the field-direction reversals transverse to the fold is determined by the dissipation physics: in MHD with isotropic viscosity and resistivity, it is the resistive scale.<sup>22</sup> The structure functions of such magnetic fields are flat (Yousef et al. 2007), with the magnetic energy dominantly at the reversal scale (see Haugen et al. 2004 for an alternative view). Although Alfvén waves propagating along the folds may exist (Schekochihin et al. 2004; Schekochihin & Cowley 2006b), the presence of the small-scale direction reversals means that there is no scale-by-scale equipartition between the velocity and magnetic fields: while

<sup>22</sup> In weakly collisional astrophysical plasmas, such a description is not applicable: the field reversal scale is most probably determined by more complicated and as yet poorly understood kinetic plasma effects; below this scale, an Alfvénic turbulence of the kind discussed in this paper may exist (Schekochihin & Cowley 2006a).

the magnetic energy is small-scale dominated due to the direction reversals, the kinetic energy (the velocity dispersion) should be contained primarily at the supernova-forcing scale, with some scaling law (probably Kolmogorov) in the inertial range. The density fluctuations, being passive, should follow the velocity scaling.

#### 8.2.4. *Spiral Arms vs. Interarm Regions*

Thus, there are two asymptotic regimes of MHD turbulence: (I) anisotropic Alfvénic turbulence with  $\delta B_{\text{rms}} \gg B_0$  and (II) isotropic MHD turbulence with small-scale field reversals and  $\delta B_{\text{rms}} \ll B_0$ . The qualitative difference between the magnetic-field structure in the Galactic spiral arms and interarms discovered by Haverkorn et al. (2005) might then be attributed to the turbulence in these regions being of two different kinds: it is closer to the anisotropic variety (I) in the interarms and to the isotropic one (II) in the arms. There are several conspiring reasons for this simplest and, therefore, tempting explanation to also be a plausible one:

(i) the arms are denser than the interarms by a factor of a few (roughly 4; see Roberts & Hausman 1984) and observed to support stronger turbulence (roughly a factor of 2; see Rohlfs & Kreitschmann 1987), which is not surprising as the concentration of supernovae is higher;

(ii) the presence of the spiral mean field in galaxies is usually attributed to some form of mean-field dynamo (Krause & Rädler 1980) and it is possible to argue that this mechanism produces stronger mean fields in the interarms than in the arms (Shukurov 1998; Shukurov & Sokoloff 1998; Rohde & Elstner 1998; Rohde et al. 1998);

(iii) the mean field should be pushed out of the more turbulent region (arms) into the less turbulent one (interarms) by the turbulent diamagnetism (Zeldovich 1957; Krause & Rädler 1980).

Thus,  $u_{\text{rms}}$  is larger and  $B_0$  smaller in the arms, so  $\delta B_{\text{rms}}/B_0 \sim u_{\text{rms}}/v_A$  should be smaller in the arms than in the interarms. Numerical simulations of an inhomogeneously turbulent MHD fluid (Iskakov et al. 2007) lend qualitative support to these ideas.

#### 8.3. *Accretion Disks*

Accretion of plasma onto a central black hole or neutron star is responsible for many of the most energetic phenomena observed in astrophysics (see, e.g., Narayan & Quataert 2005 for a review). It is now believed that a linear instability of differentially rotating plasmas—the magnetorotational instability (MRI)—amplifies magnetic fields and gives rise to MHD turbulence in astrophysical disks (Balbus & Hawley 1998). Magnetic stresses due to this turbulence transport angular momentum, allowing plasma to accrete. The MRI converts the gravitational potential energy of the inflowing plasma into turbulence at the outer scale that is comparable to the scale height of the disk. This energy is then cascaded to small scales and dissipated into heat—powering the radiation that we see from accretion flows. Fluid MHD simulations show that the MRI-generated turbulence in disks is subsonic and has  $\beta \sim 10-100$ . Thus, on scales much smaller than the scale height of the disk, the models developed in this paper should provide a good description of the turbulent fluctuations and a step forward compared to the purely fluid approach. Turbulence is not yet directly observable in disks, so models of turbulence are mostly used to produce testable predictions of observable properties of disks such as their X-ray and radio emission. One of the best observed cases is the (presumed)

accretion flow onto the black hole coincident with the radio source Sgr A\* in the center of our Galaxy (see review by Quataert 2003).

Depending on the rate of heating and cooling in the inflowing plasma (which in turn depend on accretion rate and other properties of the system under consideration), there are different models that describe the physical properties of accretion flows onto a central object. In one class of models, a geometrically thin optically thick accretion disk (Shakura & Sunyaev 1973), the inflowing plasma is cold and dense and well described as an MHD fluid. When applied to Sgr A\*, these models produce a prediction for its total luminosity that is several orders of magnitude larger than observed. Another class of models, which appears to be more consistent with the observed properties of Sgr A\*, is called radiatively inefficient accretion flows (RIAFs; see Rees et al. 1982; Narayan & Yi 1995 and review by Quataert 2003 of the applications and observational constraints in Sgr A\*). In these models, the inflowing plasma near the black hole is believed to adopt a two-temperature configuration, with the ions ( $T_i \sim 10^{11}-10^{12}$  K) hotter than the electrons ( $T_e \sim 10^9-10^{11}$  K).<sup>23</sup> The electron and ion thermodynamics decouple because the densities are so low that the temperature equalization time  $\sim \nu_e^{-1}$  is longer than the time for the plasma to flow into the black hole. Thus, like the solar wind, RIAFs are macroscopically collisionless plasmas (see Table 1 for plasma parameters in the Galactic center; note that these parameters are so extreme that the gyrokinetic description, while probably better than the fluid one, cannot be expected to be rigorously valid; at the very least, it needs to be reformulated in a relativistic form). At the high temperatures appropriate to RIAFs, electrons radiate energy much more efficiently than the ions (by virtue of their much smaller mass) and are, therefore, expected to contribute dominantly to the observed emission, while the thermal energy of the ions is swallowed by the black hole. Since the plasma is collisionless, the electron heating by turbulence largely determines the thermodynamics of the electrons and thus the observable properties of RIAFs. The question of which fraction of the turbulent energy goes into ion and which into electron heating is, therefore, crucial for understanding accretion flows—and the answer to this question depends on the detailed properties of the small-scale kinetic turbulence (e.g., Quataert & Gruzinov 1999; Sharma et al. 2007).

Since all of the turbulent power coming down the cascade must be dissipated into either ion or electron heat, it is really the amount of ion heating at the ion gyroscale that decides how much energy is available to heat the electrons at their (smaller) gyroscale. Again, as in the case of the solar wind (§ 8.1.4 and § 8.1.5), the transition around the ion gyroscale from the Alfvénic turbulence at  $k_{\perp} \rho_i \ll 1$  to the KAW turbulence at  $k_{\perp} \rho_i \gg 1$  emerges as a key unsolved problem.

#### 8.4. *Galaxy Clusters*

Galaxy clusters are the largest plasma objects in the Universe. Like the other examples discussed above, the intracluster plasma is in the weakly collisional regime (see Table 1). Fluctuations of electron density, temperature and of magnetic fields are measured in clusters by X-ray and radio observatories, but the resolution is only just enough to claim that a fairly broad scale range of fluctuations exists (Schuecker et al. 2004; Vogt & Enßlin 2005). No power-law scalings have yet

<sup>23</sup> It is partly with this application in mind that we carried the general temperature ratio in our calculations; see footnote 10.



been established beyond reasonable doubt.

What fundamentally hampers quantitative modeling of turbulence and related effects in clusters is that we do not have a definite theory of the basic properties of the intracluster medium: its (effective) viscosity, magnetic diffusivity or thermal conductivity. In a weakly collisional and strongly magnetized plasma, all of these depend on the structure of the magnetic field (Braginskii 1965), which is shaped by the turbulence. If (or at scales where) a reasonable *a priori* assumption can be made about the field structure, further analytical progress is possible: thus, the theoretical models presented in this paper assume that the magnetic field is a sum of a slowly varying in space “mean field” and small low-frequency perturbations ( $\delta B \ll B_0$ ).

In fact, since clusters do not have mean fields of any magnitude that could be considered dynamically significant, but do have stochastic fields, the MHD turbulence in clusters falls into the weak-mean-field category (see § 8.2.3). The magnetic field should be highly filamentary, organized in long folded direction-reversing structures. It is not currently known what determines the reversal scale.<sup>24</sup> Observations, while tentatively confirming the existence of very long filaments (Clarke & Enßlin 2006), suggest that the reversal scale is much larger than the ion gyroscale: thus, the magnetic-energy spectrum for the Hydra A cluster core reported by Vogt & Enßlin (2005) peaks at around 1 kpc, compared to  $\rho_i \sim 10^5$  km. Below this scale, an Alfvén-wave cascade should exist (as is, indeed, suggested by Vogt & Enßlin’s spectrum being roughly consistent with  $k^{-5/3}$  at scales below the peak). As these scales are collisionless ( $\lambda_{mfp_i} \sim 100$  pc in the cores and  $\sim 10$  kpc in the bulk of the clusters), it is to this turbulence that the theory developed in this paper should be applicable.

Another complication exists, similar to that discussed in § 8.1.6: pressure anisotropies could give rise to fast plasma instabilities whose growth rate peaks just above the ion gyroscale. As was pointed out by Schekochihin et al. (2005), these are, in fact, an inevitable consequence of any large-scale fluid motions that change the strength of the magnetic field. Although a number of interesting and plausible arguments can be made about the way the instabilities might determine the magnetic-field structure (Schekochihin & Cowley 2006a), it is not currently understood with any degree of certainty how the small-scale fluctuations resulting from these instabilities coexist with the Alfvénic cascade.

The uncertainties that result from this imperfect understanding of the nature of the intracluster medium are exemplified by the problem of its thermal conductivity. The magnetic-field reversal scale in clusters is certainly not larger than the electron diffusion scale,  $(m_i/m_e)^{1/2} \lambda_{mfp_i}$ , which varies from a few kpc in the cores to a few hundred kpc in the bulk. Therefore, the approximation of isothermal electron fluid (§ 4) should apply at all scales below the reversal scale, where  $\delta B \ll B_0$  presumably holds. Even this, however, is not definitive. One could imagine the electrons being effectively adiabatic if (or in the regions where) the plasma instabilities give rise to large fluctuations of the magnetic field ( $\delta B/B_0 \sim 1$ ) at the ion gyroscale (Schekochihin et al. 2007c,d), reducing the mean free path to  $\lambda_{mfp_i} \sim \rho_i$ .

<sup>24</sup> See Schekochihin & Cowley (2006a) for a detailed presentation of our views on the interplay between turbulence, magnetic field and plasma effects in cluster; for further discussions and disagreements, see Enßlin & Vogt (2006); Subramanian et al. (2006); Brunetti & Lazarian (2007).

The current state of the observational evidence does not allow one to exclude either of these possibilities. Both isothermal (Fabian et al. 2006; Sanders & Fabian 2006) and nonisothermal (Markevitch & Vikhlinin 2007) coherent structures that appear to be shocks are observed. Disordered fluctuations of temperature can also be detected, which allows one to infer an upper limit for the scale at which the isothermal approximation can start being valid: thus, Markevitch et al. (2003) find temperature variations at all scales down to  $\sim 100$  kpc, which is the statistical limit that defines the spatial resolution of their temperature map. In none of these or similar measurements is the magnetic field data available that would make possible a pointwise comparison of the magnetic and thermal structure.

Because of this lack of information about the state of the magnetized plasma in clusters, theories of the intracluster medium are not sufficiently constrained by observations, so no one theory is in a position to prevail. This uncertain state of affairs might be improved by analyzing the observationally much better resolved case of the solar wind.

## 9. CONCLUSION

In this paper, we have considered magnetized plasma turbulence in the astrophysically prevalent regime of weak collisionality. We have shown how the energy injected at the outer scale cascades in phase space eventually to increase the entropy of the system and heat the particles. In the process, we have explained how one combines plasma physics tools—in particular, the gyrokinetic theory—with ideas on turbulent cascades to arrive at a hierarchy of tractable models of turbulence in various physically distinct scale intervals. These models represent the branching pathways of the energy cascade and make clear the “fluid” and “kinetic” aspects of plasma turbulence.

A detailed outline of these developments was given in the Introduction. Intermediate technical summaries were provided in § 4.9, § 5.7, and § 7.8. Our view of how the transformation of the large-scale turbulent energy into heat occurs was encapsulated in the concept of kinetic cascade. It was previewed in § 1.2 and developed quantitatively in §§ 3.4–3.5, § 4.7, § 5.6, and §§ 7.3–7.6. In § 8, we discussed at length the ways in which all this is useful in understanding turbulence in real astrophysical plasmas.

Following a series of analytical contributions that set up the theoretical framework for astrophysical gyrokinetics, (Howes et al. 2006, 2007; Schekochihin et al. 2007a,b; Abel et al. 2007, and this paper) an extensive programme of fluid (T. A. Yousef et al. 2007, in preparation), mixed fluid-kinetic (T. Tatsuno et al. 2007, in preparation), and gyrokinetic<sup>25</sup> (G. G. Howes et al. 2007, in preparation) numerical simulations of magnetized plasma turbulence is now underway. Careful comparisons of the fully gyrokinetic simulations with simulations based on the computable models derived in this paper (RMHD—§ 2, isothermal electron fluid—§ 4, KRMHD—§ 5, ERMHD—§ 7) as well as with the numerical studies based on various Landau fluid (Snyder et al. 1997; Goswami et al. 2005; Ramos 2005; Sharma et al. 2006, 2007; Passot & Sulem 2007) and gyrofluid (Snyder & Hammett 2001) closures appear to be the way forward in developing a comprehensive numerical model of the turbulent cascade from the outer scale to the electron gyroscale. The objective is a quantitative characterization of the scaling-range properties (spectra, anisotropy, types of modes and interactions), the ion and electron heating, and the transport properties of the

magnetized plasma turbulence.

Of the many astrophysical plasmas to which these results apply, the solar wind, due to the high quality of turbulence measurements possible in it, appears to be best suited for direct and detailed quantitative comparisons of the theory and simulation results with observational evidence.

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<sup>25</sup> Using the GS2 code; see <http://gs2.sourceforge.net>.

## APPENDIX

### A. BRAGINSKII'S TWO-FLUID EQUATIONS AND REDUCED MHD

Here we explain how the standard one-fluid MHD equations used in § 2 and the collisional limit of the KRMHD system (§ 6.1, derived in Appendix D) both emerge as limiting cases of the two-fluid theory. For the case of anisotropic fluctuations,  $k_{\parallel}/k_{\perp} \ll 1$ , all of this can, of course, be derived from gyrokinetics, but it is useful to provide a connection to the more well known fluid description of collisional plasmas.

#### A.1. Two-Fluid Equations

The rigorous derivation of the fluid equations for a collisional plasma was done in the classic paper of Braginskii (1965). His equations, valid for  $\omega/\nu_{ii} \ll 1$ ,  $k_{\parallel}\lambda_{mfp_i} \ll 1$ ,  $k_{\perp}\rho_i \ll 1$  (see Fig. 2), evolve the densities  $n_s$ , mean velocities  $\mathbf{u}_s$  and temperatures  $T_s$  of each plasma species ( $s = i, e$ ):

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_s \cdot \nabla\right) n_s = -n_s \nabla \cdot \mathbf{u}_s, \quad (A1)$$

$$m_s n_s \left(\frac{\partial}{\partial t} + \mathbf{u}_s \cdot \nabla\right) \mathbf{u}_s = -\nabla p_s - \nabla \cdot \hat{\Pi}_s + q_s n_s \left(\mathbf{E} + \frac{\mathbf{u}_s \times \mathbf{B}}{c}\right) + \mathbf{F}_s, \quad (A2)$$

$$\frac{3}{2} n_s \left(\frac{\partial}{\partial t} + \mathbf{u}_s \cdot \nabla\right) T_s = -p_s \nabla \cdot \mathbf{u}_s - \nabla \cdot \hat{\Gamma}_s - \hat{\Pi}_s : \nabla \mathbf{u}_s + Q_s, \quad (A3)$$

where  $p_s = n_s T_s$  and the expressions for the viscous stress tensor  $\hat{\Pi}_s$ , the friction force  $\mathbf{F}_s$ , the heat flux  $\hat{\Gamma}_s$  and the interspecies heat exchange  $Q_s$  are given in Braginskii (1965). Equations (A1-A3) are complemented with the quasineutrality condition,  $n_e = Zn_i$ , and the Faraday and Ampère laws, which are (in the nonrelativistic limit)

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}, \quad \mathbf{j} = en_e(\mathbf{u}_i - \mathbf{u}_e) = \frac{c}{4\pi} \nabla \times \mathbf{B}. \quad (A4)$$

Because of quasineutrality, we only need one of the continuity equations, say the ion one. We can also use the electron momentum equation [Eq. (A2),  $s = e$ ] to express  $\mathbf{E}$ , which we then substitute into the ion momentum equation and the Faraday law. The resulting system is

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{u}, \quad (A5)$$

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \left(p + \frac{B^2}{8\pi}\right) - \nabla \cdot \hat{\Pi} + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi} - \frac{Zm_e}{m_i} \rho \left(\frac{\partial}{\partial t} + \mathbf{u}_e \cdot \nabla\right) \mathbf{u}_e, \quad (A6)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[ \mathbf{u} \times \mathbf{B} - \frac{\mathbf{j} \times \mathbf{B}}{en_e} + \frac{c \nabla p_e}{en_e} + \frac{c \nabla \cdot \hat{\Pi}_e}{en_e} - \frac{c \mathbf{F}_e}{en_e} + \frac{cm_e}{e} \left(\frac{\partial}{\partial t} + \mathbf{u}_e \cdot \nabla\right) \mathbf{u}_e \right], \quad (A7)$$

where  $\rho = m_i n_i$ ,  $\mathbf{u} = \mathbf{u}_i$ ,  $p = p_i + p_e$ ,  $\hat{\Pi} = \hat{\Pi}_i + \hat{\Pi}_e$ ,  $\mathbf{u}_e = \mathbf{u} - \mathbf{j}/en_e$ ,  $n_e = Zn_i$ ,  $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ . The ion and electron temperatures continue to satisfy Eq. (A3).

#### A.2. Strongly Magnetized Limit

In this form, the two-fluid theory starts resembling the standard one-fluid MHD, which was our starting point in § 2: Eqs. (A5-A7) already look similar to the continuity, momentum and induction equations. The additional terms that appear in these equations and the temperature equations (A3) are brought under control by considering how they depend on a number of dimensionless parameters that emerge:  $\omega/\nu_{ii}$ ,  $k_{\parallel}\lambda_{mfp_i}$ ,  $k_{\perp}\rho_i$ ,  $(m_e/m_i)^{1/2}$ . While all these are small in Braginskii's calculation, no assumption is made as to how they compare to each other. We now specify that

$$\frac{\omega}{\nu_{ii}} \sim \frac{k_{\parallel}\lambda_{mfp_i}}{\sqrt{\beta_i}}, \quad k_{\perp}\rho_i \ll k_{\parallel}\lambda_{mfp_i} \sim \sqrt{\frac{m_e}{m_i}} \ll 1 \quad (A8)$$

(see Fig. 3). Note that the first of these relations is equivalent to assuming that the fluctuation frequencies are Alfvénic—the same assumption as in gyrokinetics [Eq. (40)]. The second relation in Eq. (A8) will be referred to by us as the *strongly magnetized limit*. Under the assumptions (A8), the two-fluid equations reduce to the following closed set:<sup>26</sup>

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{u}, \quad (\text{A10})$$

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \left[ p + \frac{B^2}{8\pi} + \frac{1}{3} \rho \nu_{\parallel i} \left( \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{u} - \frac{1}{3} \nabla \cdot \mathbf{u} \right) \right] + \nabla \cdot \left[ \hat{\mathbf{b}}\hat{\mathbf{b}} \rho \nu_{\parallel i} \left( \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{u} - \frac{1}{3} \nabla \cdot \mathbf{u} \right) \right] + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi}, \quad (\text{A11})$$

$$\frac{d\mathbf{B}}{dt} = \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \nabla \cdot \mathbf{u}, \quad (\text{A12})$$

$$\frac{dT_i}{dt} = -\frac{2}{3} T_i \nabla \cdot \mathbf{u} + \frac{1}{\rho} \nabla \cdot (\hat{\mathbf{b}} \rho \kappa_{\parallel i} \hat{\mathbf{b}} \cdot \nabla T_i) - \nu_{ie} (T_i - T_e) + \frac{2}{3} m_i \nu_{\parallel i} \left( \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{u} - \frac{1}{3} \nabla \cdot \mathbf{u} \right)^2, \quad (\text{A13})$$

$$\frac{dT_e}{dt} = -\frac{2}{3} T_e \nabla \cdot \mathbf{u} + \frac{1}{\rho} \nabla \cdot (\hat{\mathbf{b}} \rho \kappa_{\parallel e} \hat{\mathbf{b}} \cdot \nabla T_e) - \frac{1}{Z} \nu_{ie} (T_e - T_i), \quad (\text{A14})$$

where  $\nu_{\parallel i} = 0.90 v_{\text{th}i} \lambda_{\text{mf}pi}$  is the parallel ion viscosity,  $\kappa_{\parallel i} = 2.45 v_{\text{th}i} \lambda_{\text{mf}pi}$  parallel ion thermal diffusivity,  $\kappa_{\parallel e} = 1.40 v_{\text{th}e} \lambda_{\text{mf}pe} \sim (Z^2 / \tau^{5/2}) (m_i / m_e)^{1/2} \kappa_{\parallel i}$  parallel electron thermal diffusivity [here  $\lambda_{\text{mf}pi} = v_{\text{th}i} / \nu_{ii}$  with  $\nu_{ii}$  defined in Eq. (44)], and  $\nu_{ie}$  ion-electron collision rate [defined in Eq. (43)]. Note that the last term in Eq. (A13) represents the viscous heating of the ions.

### A.3. One-Fluid Equations (MHD)

If we now restrict ourselves to the low-frequency regime where ion-electron collisions dominate over all other terms in the ion-temperature equation (A13),

$$\frac{\omega}{\nu_{ie}} \sim \frac{k_{\parallel} \lambda_{\text{mf}pi}}{\sqrt{\beta_i}} \sqrt{\frac{m_i}{m_e}} \ll 1 \quad (\text{A15})$$

[see Eqs. (A8) and (43)], we have, to lowest order in this new subsidiary expansion,  $T_i = T_e = T$ . We can now write  $p = (n_i + n_e)T = (1 + Z)\rho T / m_i$  and, adding Eqs. (A13) and (A14), find the equation for pressure:

$$\frac{dp}{dt} + \frac{5}{3} p \nabla \cdot \mathbf{u} = \nabla \cdot (\hat{\mathbf{b}} n_e \kappa_{\parallel e} \hat{\mathbf{b}} \cdot \nabla T) + \frac{2}{3} m_i \nu_{\parallel i} \left( \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{u} - \frac{1}{3} \nabla \cdot \mathbf{u} \right)^2, \quad (\text{A16})$$

where we have neglected the ion thermal diffusivity compared to the electron one, but kept the ion heating term to maintain energy conservation. Equation (A16) together with Eqs. (A10-A12) constitutes the conventional one-fluid MHD system. With the dissipative terms [which are small because of Eq. (A15)] neglected, this was the starting point for our fluid derivation of RMHD in § 2.

Note that the electrons in this regime are adiabatic because the electron thermal diffusion is small

$$\frac{\kappa_{\parallel e} k_{\parallel}^2}{\omega} \sim k_{\parallel} \lambda_{\text{mf}pi} \sqrt{\beta_i} \sqrt{\frac{m_i}{m_e}} \ll 1, \quad (\text{A17})$$

provided Eq. (A15) holds and  $\beta_i$  is ordered to be order unity. If we take  $\beta_i \gg 1$  instead, we can still satisfy Eq. (A15), so  $T_i = T_e$  follows from the ion temperature equation (A13) and the one-fluid equations emerge as an expansion in high  $\beta_i$ . However, these equations now describe two physical regimes: the adiabatic long-wavelength regime that satisfies Eq. (A17) and the shorter-wavelength regime in which  $(m_e / m_i)^{1/2} / \sqrt{\beta_i} \ll k_{\parallel} \lambda_{\text{mf}pi} \ll (m_e / m_i)^{1/2} \sqrt{\beta_i}$ , so the fluid is isothermal,  $T = T_0 = \text{const}$ ,  $p = [(1 + Z)T_0 / m_i] \rho = c_s^2 \rho$  [Eq. (7) holds with  $\gamma = 1$ ].

### A.4. Two-Fluid Equations with Isothermal Electrons

Let us now consider the regime in which the coupling between the ion and electron temperatures is small and the electron diffusion is large [the limit opposite to Eqs. (A15) and (A17)]:

$$\frac{\omega}{\nu_{ie}} \sim \frac{k_{\parallel} \lambda_{\text{mf}pi}}{\sqrt{\beta_i}} \sqrt{\frac{m_i}{m_e}} \gg 1, \quad \frac{\kappa_{\parallel e} k_{\parallel}^2}{\omega} \sim k_{\parallel} \lambda_{\text{mf}pi} \sqrt{\beta_i} \sqrt{\frac{m_i}{m_e}} \gg 1, \quad (\text{A18})$$

<sup>26</sup> The structure of the momentum equation (A11) is best understood by realizing that  $\rho \nu_{\parallel i} (\hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{u} - \nabla \cdot \mathbf{u} / 3) = p_{\perp} - p_{\parallel}$ , the difference between the perpendicular and parallel (ion) pressures. Since the total pressure is  $p = (2/3)p_{\perp} + (1/3)p_{\parallel}$ , Eq. (A11) can be written

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \left( p_{\perp} + \frac{B^2}{8\pi} \right) + \nabla \cdot [\hat{\mathbf{b}}\hat{\mathbf{b}} (p_{\perp} - p_{\parallel})] + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi}. \quad (\text{A9})$$

This is the general form of the momentum equation that is also valid for collisionless plasmas, when  $k_{\perp} \rho_i \ll 1$  but  $k_{\parallel} \lambda_{\text{mf}pi}$  is order unity or even large. Equation (A9) together with the continuity equation (A11), the induction equation (A12) and a kinetic equation for the particle distribution function (from the solution of which  $p_{\perp}$  and  $p_{\parallel}$  are determined) form the system known as Kinetic MHD (KMHD, see Kulsrud 1964, 1983). The collisional limit,  $k_{\parallel} \lambda_{\text{mf}pi} \ll 1$ , of KMHD is again Eqs. (A10-A14).



Then the electrons are isothermal,  $T_e = T_{0e} = \text{const}$  (with the usual assumption of stochastic field lines, so  $\hat{\mathbf{b}} \cdot \nabla T_e = 0$  implies  $\nabla T_e = 0$ , as in § 4.4), while the ion temperature satisfies

$$\frac{dT_i}{dt} = -\frac{2}{3} T_i \nabla \cdot \mathbf{u} + \frac{1}{\rho} \nabla \cdot (\hat{\mathbf{b}} \rho \kappa_{\parallel i} \hat{\mathbf{b}} \cdot \nabla T_i) + \frac{2}{3} m_i \nu_{\parallel i} \left( \hat{\mathbf{b}} \hat{\mathbf{b}} : \nabla \mathbf{u} - \frac{1}{3} \nabla \cdot \mathbf{u} \right)^2. \quad (\text{A19})$$

Equation (A19) together with Eqs. (A10-A12) and  $p = \rho(T_i + ZT_{0e})/m_i$  are a closed system that describes an MHD-like fluid of adiabatic ions and isothermal electrons. Applying the ordering of § 2.1 to these equations and carrying out an expansion in  $k_{\parallel}/k_{\perp} \ll 1$  entirely analogously to the way it was done in § 2, we arrive at the RMHD equations (13-14) for the Alfvén waves and the following system for the passive fields (slow and entropy modes):

$$\frac{d}{dt} \left( \frac{\delta \rho}{\rho_0} - \frac{\delta B_{\parallel}}{B_0} \right) + \hat{\mathbf{b}} \cdot \nabla u_{\parallel} = 0, \quad (\text{A20})$$

$$\frac{du_{\parallel}}{dt} - \nu_A^2 \hat{\mathbf{b}} \cdot \nabla \frac{\delta B_{\parallel}}{B_0} = \nu_{\parallel i} \hat{\mathbf{b}} \cdot \nabla \left( \hat{\mathbf{b}} \cdot \nabla u_{\parallel} + \frac{1}{3} \frac{d}{dt} \frac{\delta \rho}{\rho_0} \right), \quad (\text{A21})$$

$$\frac{d}{dt} \frac{\delta T_i}{T_{0i}} - \frac{2}{3} \frac{d}{dt} \frac{\delta \rho}{\rho_0} = \kappa_{\parallel i} \hat{\mathbf{b}} \cdot \nabla \left( \hat{\mathbf{b}} \cdot \nabla \frac{\delta T_i}{T_{0i}} \right), \quad (\text{A22})$$

and the pressure balance

$$\left( 1 + \frac{Z}{\tau} \right) \frac{\delta \rho}{\rho_0} = -\frac{\delta T_i}{T_{0i}} - \frac{2}{\beta_i} \left[ \frac{\delta B_{\parallel}}{B_0} + \frac{1}{3\nu_A^2} \nu_{\parallel i} \left( \hat{\mathbf{b}} \cdot \nabla u_{\parallel} + \frac{1}{3} \frac{d}{dt} \frac{\delta \rho}{\rho_0} \right) \right]. \quad (\text{A23})$$

Recall that these equations, being the consequence of Braginskii's two-fluid equations (§ A.1), are an expansion in  $k_{\parallel} \lambda_{\text{mfpi}} \ll 1$  correct up to first order in this small parameter. Since the dissipative terms are small, we can replace  $(d/dt)\delta\rho/\rho_0$  in the viscous terms of Eqs. (A21) and (A23) by its value computed from Eqs. (A20), (A22) and (A23) in neglect of dissipation:  $(d/dt)\delta\rho/\rho_0 = -\hat{\mathbf{b}} \cdot \nabla u_{\parallel} / (1 + c_s^2/\nu_A^2)$  [cf. Eq. (20)], where the speed of sound  $c_s$  is defined by Eq. (158). Substituting this into Eqs. (A21) and (A23), we recover the collisional limit of KRMHD derived in Appendix D, see Eqs. (D21-D23) and (D25).

## B. COLLISIONS IN GYROKINETICS

The general collision operator that appears in Eq. (32) is (Landau 1936)

$$\left( \frac{\partial f_s}{\partial t} \right)_c = 2\pi \ln \Lambda \sum_{s'} \frac{q_s^2 q_{s'}^2}{m_s} \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3 \mathbf{v}' \frac{1}{w} \left( \hat{\mathbf{I}} - \frac{\mathbf{w} \mathbf{w}}{w^2} \right) \cdot \left[ \frac{1}{m_s} f_{s'}(\mathbf{v}') \frac{\partial f_s(\mathbf{v})}{\partial \mathbf{v}} - \frac{1}{m_{s'}} f_s(\mathbf{v}) \frac{\partial f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'} \right], \quad (\text{B1})$$

where  $\mathbf{w} = \mathbf{v} - \mathbf{v}'$  and  $\ln \Lambda$  is the Coulomb logarithm. We now take into account the expansion of the distribution function (47), use the fact that the collision operator vanishes when it acts on a Maxwellian, and retain only first-order terms in the gyrokinetic expansion. This gives us the general form of the collision term in Eq. (50): it is the ring-averaged linearized form of the Landau collision operator (B1),  $(\partial h_s / \partial t)_c = \langle C_s[h] \rangle_{\mathbf{R}_s}$ , where

$$C_s[h] = 2\pi \ln \Lambda \sum_{s'} \frac{q_s^2 q_{s'}^2}{m_s} \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3 \mathbf{v}' \frac{1}{w} \left( \hat{\mathbf{I}} - \frac{\mathbf{w} \mathbf{w}}{w^2} \right) \cdot \left[ F_{0s'}(\mathbf{v}') \left( \frac{\mathbf{v}'}{T_{0s'}} + \frac{1}{m_s} \frac{\partial}{\partial \mathbf{v}} \right) h_s(\mathbf{v}) - F_{0s}(\mathbf{v}) \left( \frac{\mathbf{v}}{T_{0s}} + \frac{1}{m_{s'}} \frac{\partial}{\partial \mathbf{v}'} \right) h_{s'}(\mathbf{v}') \right]. \quad (\text{B2})$$

Note that the velocity derivatives are taken at constant  $\mathbf{r}$ , i.e., the gyrocenter distribution functions that appear in the integrand should be understood as  $h_s(\mathbf{v}) \equiv h_s(t, \mathbf{r} + \mathbf{v}_{\perp} \times \hat{\mathbf{z}}/\Omega_s, v_{\perp}, v_{\parallel})$ . The explicit form of the gyrokinetic collision operator can be derived in  $k$  space as follows:

$$\left( \frac{\partial h_s}{\partial t} \right)_c = \left\langle C_s \left[ \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{R}} h_{\mathbf{k}} \right] \right\rangle_{\mathbf{R}_s} = \sum_{\mathbf{k}} \langle e^{i\mathbf{k} \cdot \mathbf{r}} C_s [e^{i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}] \rangle_{\mathbf{R}_s} = \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{R}_s} \langle e^{-i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} C_s [e^{i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}] \rangle, \quad (\text{B3})$$

where  $\boldsymbol{\rho}_s(\mathbf{v}) = \mathbf{v}_{\perp} \times \hat{\mathbf{z}}/\Omega_s$ . Angle brackets with no subscript refer to averages over the gyroangle  $\varphi$  of quantities that do not depend on spatial coordinates. Note that inside the operator  $C_s[\dots]$ ,  $h$  occurs both with index  $s$  and velocity  $\mathbf{v}$  and with index  $s'$  and velocity  $\mathbf{v}'$  (over which summation/integration is done). In the latter case,  $\boldsymbol{\rho} = \boldsymbol{\rho}_{s'}(\mathbf{v}') = \mathbf{v}'_{\perp} \times \hat{\mathbf{z}}/\Omega_{s'}$  in the exponential factor inside the operator.

Most of the properties of the collision operator that are used in the main body of this paper to order the collision terms can be established in general, already on the basis of Eq. (B3) (§§ B.1-B.2). If the explicit form of the collision operator is required, we could, in principle, perform the ring average on the linearized operator  $C$  [Eq. (B2)] and derive an explicit form of  $(\partial h_s / \partial t)_c$ . In practice, in gyrokinetics, as in the rest of plasma physics, the full collision operator is only used when it is absolutely unavoidable. In most problems of interest, further simplifications are possible: the same-species collisions are often modeled by simpler operators that share the full collision operator's conservation properties (§ B.3), while the interspecies collision operators are expanded in the electron-ion mass ratio (§ B.4).

### B.1. Velocity-Space Integral of the Gyrokinetic Collision Operator

Many of our calculations involve integrating the gyrokinetic equation (50) over the velocity space while keeping  $\mathbf{r}$  constant. Here we estimate the size of the integral of the collision term when  $k_\perp \rho_s \ll 1$ . Using Eq. (B3),

$$\begin{aligned} \int d^3 \mathbf{v} \left\langle \left( \frac{\partial h_s}{\partial t} \right)_c \right\rangle_{\mathbf{r}} &= \sum_{\mathbf{k}} \int d^3 \mathbf{v} e^{i\mathbf{k} \cdot \mathbf{r} + i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} \langle e^{-i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} C_s [e^{i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}] \rangle \\ &= \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} 2\pi \int_0^\infty dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_\parallel \langle e^{i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} \rangle \langle e^{-i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} C_s [e^{i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}] \rangle \\ &= \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \int d^3 \mathbf{v} \langle e^{i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} \rangle e^{-i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} C_s [e^{i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}] = \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \int d^3 \mathbf{v} J_0(a_s) e^{-i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} C_s [e^{i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}] \\ &= \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \int d^3 \mathbf{v} \left[ 1 - i\mathbf{k} \cdot \frac{\mathbf{v}_\perp \times \hat{\mathbf{z}}}{\Omega_s} - \frac{1}{2} \left( \mathbf{k} \cdot \frac{\mathbf{v}_\perp \times \hat{\mathbf{z}}}{\Omega_s} \right)^2 - \frac{1}{4} \left( \frac{k_\perp v_\perp}{\Omega_s} \right)^2 + \dots \right] C_s [e^{i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}]. \end{aligned} \quad (\text{B4})$$

Since the (linearized) collision operator  $C_s$  conserves particle number, the first term in the expansion vanishes. The operator  $C_s = C_{ss} + C_{ss'}$  is a sum of the same-species collision operator [the  $s' = s$  part of the sum in Eq. (B2)] and the interspecies collision operator (the  $s' \neq s$  part). The former conserves total momentum of the particles of species  $s$ , so it gives no contribution to the second term in the expansion in Eq. (B4). Therefore,

$$\int d^3 \mathbf{v} \langle \langle C_{ss} [h_s] \rangle_{\mathbf{R}_s} \rangle_{\mathbf{r}} \sim \nu_{ss} k_\perp^2 \rho_s^2 \delta n_s. \quad (\text{B5})$$

The interspecies collisions do contribute to the second term in Eq. (B4) due to momentum exchange with the species  $s'$ . This contribution is readily inferred from the standard formula for the linearized friction force (see, e.g., Helander & Sigmar 2002):

$$m_s \int d^3 \mathbf{v} \mathbf{v} C_{ss'} [e^{i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}] = - \int d^3 \mathbf{v} \mathbf{v} \left[ m_s \nu_s^{ss'}(v) e^{i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} h_{s\mathbf{k}} + m_{s'} \nu_{s'}^{s's}(v) e^{i\mathbf{k} \cdot \boldsymbol{\rho}_{s'}(\mathbf{v})} h_{s'\mathbf{k}} \right], \quad (\text{B6})$$

$$\nu_s^{ss'}(v) = \frac{\sqrt{2} \pi n_{0s} q_s^2 q_{s'}^2 \ln \Lambda}{m_s^{1/2} T_s^{3/2}} \left( \frac{v_{\text{ths}}}{v} \right)^3 \left( 1 + \frac{m_s}{m_{s'}} \right) \left[ \text{erf} \left( \frac{v}{v_{\text{ths}'}} \right) - \frac{v}{v_{\text{ths}'}} \text{erf}' \left( \frac{v}{v_{\text{ths}'}} \right) \right], \quad (\text{B7})$$

where  $\text{erf}(x) = (2/\sqrt{\pi}) \int_0^x dy \exp(-y^2)$  is the error function. From this, via a calculation of ring averages analogous to Eq. (B16), we get

$$\begin{aligned} \int d^3 \mathbf{v} \left( -i\mathbf{k} \cdot \frac{\mathbf{v}_\perp \times \hat{\mathbf{z}}}{\Omega_s} \right) C_{ss'} [e^{i\mathbf{k} \cdot \boldsymbol{\rho}} h_{\mathbf{k}}] &= \int d^3 \mathbf{v} \left[ \nu_s^{ss'}(v) \langle i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v}) e^{i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} \rangle h_{s\mathbf{k}} + \frac{m_{s'}}{m_s} \frac{\Omega_{s'}}{\Omega_s} \nu_{s'}^{s's}(v) \langle i\mathbf{k} \cdot \boldsymbol{\rho}_{s'}(\mathbf{v}) e^{i\mathbf{k} \cdot \boldsymbol{\rho}_{s'}(\mathbf{v})} \rangle h_{s'\mathbf{k}} \right] \\ &= - \int d^3 \mathbf{v} \left[ \nu_s^{ss'}(v) a_s J_1(a_s) h_{s\mathbf{k}} + \frac{q_{s'}}{q_s} \nu_{s'}^{s's}(v) a_{s'} J_1(a_{s'}) h_{s'\mathbf{k}} \right] \\ &\sim \nu_{ss'} k_\perp^2 \rho_s^2 \delta n_s + \nu_{s's} k_\perp^2 \rho_{s'}^2 \delta n_{s'}. \end{aligned} \quad (\text{B8})$$

For the ion-electron collisions ( $s = i, s' = e$ ), using Eqs. (41) and (43), we find that both terms are  $\sim (m_e/m_i)^{1/2} \nu_{ii} k_\perp^2 \rho_i^2 \delta n_i$ . Thus, besides an extra factor of  $k_\perp^2 \rho_i^2$ , the ion-electron collisions are also subdominant by one order in the mass-ratio expansion compared to the ion-ion collisions. The same estimate holds for the interspecies contributions to the third and fourth terms in Eq. (B4). In a similar fashion, the integral of the electron-ion collision operator ( $s = e, s' = i$ ), is  $\sim \nu_{ei} k_\perp^2 \rho_e^2 \delta n_e$ , which is the same order as the integral of the electron-electron collisions.

The conclusion of this section is that, both for ion and for electron collisions, the velocity-space integral (at constant  $\mathbf{r}$ ) of the gyrokinetic collision operator is higher order than the collision operator itself by two orders of  $k_\perp \rho_s$ . This is the property that we relied on in neglecting collision terms in Eqs. (96) and (129).

### B.2. Ordering of Collision Terms in Eqs. (117) and (129)

In § 5, we claimed that the contribution to the ion-ion collision term due to the  $(Ze\langle\phi\rangle_{\mathbf{R}_i}/T_{0i})F_{0i}$  part of the ion distribution function [Eq. (116)] was one order of  $k_\perp \rho_i$  smaller than the contributions from the rest of  $h_i$ . This was used to order collision terms in Eqs. (117) and (129). Indeed, from Eq. (B3),

$$\begin{aligned} \left\langle C_{ii} \left[ \frac{Ze\langle\phi\rangle_{\mathbf{R}_i}}{T_{0i}} F_{0i} \right] \right\rangle_{\mathbf{R}_i} &= \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{R}_i} \langle e^{-i\mathbf{k} \cdot \boldsymbol{\rho}_i} C_{ii} [e^{i\mathbf{k} \cdot \boldsymbol{\rho}_i} J_0(a_i) F_{0i}] \rangle \frac{Ze\phi_{\mathbf{k}}}{T_{0i}} \\ &= \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{R}_i} \left\langle e^{-i\mathbf{k} \cdot \boldsymbol{\rho}_i} C_{ii} \left[ \left( 1 + i\mathbf{k} \cdot \boldsymbol{\rho}_i - \frac{1}{2} (\mathbf{k} \cdot \boldsymbol{\rho}_i)^2 - \frac{a_i^2}{4} + \dots \right) F_{0i} \right] \right\rangle \frac{Ze\phi_{\mathbf{k}}}{T_{0i}} \sim \nu_{ii} k_\perp^2 \rho_i^2 \frac{Ze\phi}{T_{0i}} F_{0i}. \end{aligned} \quad (\text{B9})$$

This estimate holds because, as it is easy to ascertain using Eq. (B2), the operator  $C_{ii}$  annihilates the first two terms in the expansion and only acts nontrivially on an expression that is second order in  $k_\perp \rho_i$ . With the aid of Eq. (45), the desired ordering of the term (B9) in Eq. (117) follows. When Eq. (B9) is integrated over velocity space, the result picks up two extra orders in  $k_\perp \rho_i$  [a general effect of velocity-space integration; see Eq. (B4)], so the resulting term in Eq. (129) is third order, as stated in § 5.3.

## B.3. Model Pitch-Angle-Scattering Operator for Same-Species Collisions

A popular model operator for same-species collisions that conserves particle number, momentum, and energy is constructed by taking the test-particle pitch-angle-scattering operator and correcting it with an additional term that ensures momentum conservation (Rosenbluth et al. 1972; see also Helander & Sigmar 2002):

$$C_M[h_s] = \nu_D^{ss}(v) \left\{ \frac{1}{2} \left[ \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial h_s}{\partial \xi} + \frac{1}{1 - \xi^2} \frac{\partial^2 h_s}{\partial \varphi^2} \right] + \frac{2\mathbf{v} \cdot \mathbf{U}[h_s]}{v_{\text{ths}}^2} F_{0s} \right\}, \quad \mathbf{U}[h_s] = \frac{3}{2} \frac{\int d^3 \mathbf{v} \nu_D^{ss}(v) h_s}{\int d^3 \mathbf{v} (v/v_{\text{ths}})^2 \nu_D^{ss}(v) F_{0s}(v)}, \quad (\text{B10})$$

$$\nu_D^{ss}(v) = \nu_{ss} \left( \frac{v_{\text{ths}}}{v} \right)^3 \left[ \left( 1 - \frac{1}{2} \frac{v_{\text{ths}}^2}{v^2} \right) \text{erf} \left( \frac{v}{v_{\text{ths}}} \right) + \frac{1}{2} \frac{v_{\text{ths}}}{v} \text{erf}' \left( \frac{v}{v_{\text{ths}}} \right) \right], \quad \nu_{ss} = \frac{\sqrt{2} \pi n_{0s} q_s^4 \ln \Lambda}{m_s^{1/2} T_{0s}^{3/2}}, \quad (\text{B11})$$

where the velocity derivatives are at constant  $\mathbf{r}$ . The gyrokinetic version of this operator is (cf. Catto & Tsang 1977; Dimits & Cohen 1994)<sup>27</sup>

$$\langle C_M[h_s] \rangle_{\mathbf{R}_s} = \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{R}_s} \nu_D^{ss}(v) \left\{ \frac{1}{2} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial h_{s\mathbf{k}}}{\partial \xi} - \frac{v^2(1 + \xi^2)}{4v_{\text{ths}}^2} k_{\perp}^2 \rho_s^2 h_{s\mathbf{k}} + 2 \frac{v_{\perp} J_1(a_s) U_{\perp}[h_{s\mathbf{k}}] + v_{\parallel} J_0(a_s) U_{\parallel}[h_{s\mathbf{k}}]}{v_{\text{ths}}^2} F_{0s} \right\}, \quad (\text{B12})$$

$$U_{\perp}[h_{s\mathbf{k}}] = \frac{3}{2} \frac{\int d^3 \mathbf{v} v_{\perp} J_1(a_s) \nu_D^{ss}(v) h_{s\mathbf{k}}(v_{\perp}, v_{\parallel})}{\int d^3 \mathbf{v} (v/v_{\text{ths}})^2 \nu_D^{ss}(v) F_{0s}(v)}, \quad U_{\parallel}[h_{s\mathbf{k}}] = \frac{3}{2} \frac{\int d^3 \mathbf{v} v_{\parallel} J_0(a_s) \nu_D^{ss}(v) h_{s\mathbf{k}}(v_{\perp}, v_{\parallel})}{\int d^3 \mathbf{v} (v/v_{\text{ths}})^2 \nu_D^{ss}(v) F_{0s}(v)},$$

where  $a_s = k_{\perp} v_{\perp} / \Omega_s$ . The velocity derivatives are now at constant  $\mathbf{R}_s$ . The spatial diffusion term appearing in the ring-averaged collision operator is physically due to the fact that a change in a particle's velocity resulting from a collision can lead to a change in the spatial position of its gyrocenter.

In order to derive Eq. (B12), we use Eq. (B3). Since,  $\boldsymbol{\rho}_s(\mathbf{v}) = (\hat{\mathbf{x}} v \sqrt{1 - \xi^2} \sin \varphi - \hat{\mathbf{y}} v \sqrt{1 - \xi^2} \cos \varphi) / \Omega_s$ , it is not hard to see that

$$\frac{\partial}{\partial \xi} e^{i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} h_{s\mathbf{k}} = e^{i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} \left[ \frac{\partial}{\partial \xi} - \frac{\xi}{1 - \xi^2} \frac{i\mathbf{k}_{\perp} \cdot (\mathbf{v}_{\perp} \times \hat{\mathbf{z}})}{\Omega_s} \right] h_{s\mathbf{k}}, \quad \frac{\partial}{\partial \varphi} e^{i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} h_{s\mathbf{k}} = e^{i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} \left( \frac{\partial}{\partial \varphi} + \frac{i\mathbf{k}_{\perp} \cdot \mathbf{v}_{\perp}}{\Omega_s} \right) h_{s\mathbf{k}}. \quad (\text{B13})$$

Therefore,

$$\left\langle e^{-i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial}{\partial \xi} e^{i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} h_{s\mathbf{k}} \right\rangle = \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial h_{s\mathbf{k}}}{\partial \xi} - \frac{v^2 \xi^2}{2\Omega_s^2} k_{\perp}^2 h_{s\mathbf{k}}, \quad \left\langle e^{-i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} \frac{\partial^2}{\partial \varphi^2} e^{i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} h_{s\mathbf{k}} \right\rangle = -\frac{v^2 (1 - \xi^2)}{2\Omega_s^2} k_{\perp}^2 h_{s\mathbf{k}}. \quad (\text{B14})$$

Combining these formulae, we obtain the first two terms in Eq. (B12). Now let us work out the  $\mathbf{U}$  term:

$$\left\langle e^{-i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} \mathbf{v} \cdot \int d^3 \mathbf{v}' \mathbf{v}' \nu_D^{ss}(v') e^{i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v}')} h_{s\mathbf{k}}(v'_{\perp}, v'_{\parallel}) \right\rangle = \left\langle \mathbf{v} e^{-i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} \right\rangle \cdot 2\pi \int_0^{\infty} dv'_{\perp} v'_{\perp} \int_{-\infty}^{+\infty} dv'_{\parallel} \nu_D^{ss}(v') \left\langle \mathbf{v}' e^{i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v}')} \right\rangle h_{s\mathbf{k}}(v'_{\perp}, v'_{\parallel}). \quad (\text{B15})$$

Since  $\langle \mathbf{v} e^{\pm i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} \rangle = \hat{\mathbf{z}} v_{\parallel} \langle e^{\pm i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} \rangle + \langle \mathbf{v}_{\perp} e^{\pm i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} \rangle$ , where  $\langle e^{\pm i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} \rangle = J_0(a_s)$  and

$$\langle \mathbf{v}_{\perp} e^{\pm i\mathbf{k} \cdot \boldsymbol{\rho}_s(\mathbf{v})} \rangle = \hat{\mathbf{z}} \times \left\langle \left( \mathbf{v}_{\perp} \times \hat{\mathbf{z}} \right) \exp \left( \pm i\mathbf{k}_{\perp} \cdot \frac{\mathbf{v}_{\perp} \times \hat{\mathbf{z}}}{\Omega_s} \right) \right\rangle = \mp i \Omega_s \hat{\mathbf{z}} \times \frac{\partial}{\partial \mathbf{k}_{\perp}} \left\langle \exp \left( \pm i\mathbf{k}_{\perp} \cdot \frac{\mathbf{v}_{\perp} \times \hat{\mathbf{z}}}{\Omega_s} \right) \right\rangle = \pm i \frac{\hat{\mathbf{z}} \times \mathbf{k}_{\perp}}{k_{\perp}} v_{\perp} J_1(a_s), \quad (\text{B16})$$

we obtain the third term in Eq. (B12).

It is useful to give the lowest-order form of the operator (B12) in the limit  $k_{\perp} \rho_s \ll 1$ :

$$\langle C_M[h_s] \rangle_{\mathbf{R}_s} = \nu_D^{ss}(v) \left[ \frac{1}{2} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial h_s}{\partial \xi} + \frac{3v_{\parallel}}{2} \frac{\int d^3 \mathbf{v}' v'_{\parallel} \nu_D^{ss}(v') h_s(v'_{\perp}, v'_{\parallel})}{\int d^3 \mathbf{v}' v'^2 \nu_D^{ss}(v') F_{0s}(v')} F_{0s} \right] + O(k_{\perp}^2 \rho_s^2). \quad (\text{B17})$$

This is the operator that can be used in the right-hand side of Eq. (137) (as, e.g., is done in the calculation of collisional transport terms in Appendix D.3).

In practical numerical computations of gyrokinetic turbulence, the pitch-angle scattering operator is not sufficient because the distribution function develops small scales not only in  $\xi$  but also in  $v$  (M. A. Barnes and W. Dorland 2006, unpublished). This is, indeed, expected because the phase-space entropy cascade produces small scales in  $v_{\perp}$ , rather than just in  $\xi$  (see § 7.4.1). In order to provide a cut off in  $v$ , an energy-diffusion operator must be added to the pitch-angle-scattering operator derived above. A tractable model gyrokinetic energy-diffusion operator was proposed by Abel et al. (2007) on the basis of the approximate collision operator of Hirshman & Sigmar (1976).

## B.4. Electron-Ion Collision Operator

This operator can be expanded in  $m_e/m_i$  and to the lowest order is (see, e.g., Helander & Sigmar 2002)

$$C_{ei}[h] = \nu_D^{ei}(v) \left\{ \frac{1}{2} \left[ \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial h_e}{\partial \xi} + \frac{1}{1 - \xi^2} \frac{\partial^2 h_e}{\partial \varphi^2} \right] + \frac{2\mathbf{v} \cdot \mathbf{u}_i}{v_{\text{the}}^2} F_{0e} \right\}, \quad \nu_D^{ei}(v) = \nu_{ei} \left( \frac{v_{\text{the}}}{v} \right)^3. \quad (\text{B18})$$

<sup>27</sup> This is the collision operator that is implemented in the current version of the publicly available GS2 code (W. Dorland and T. Tatsuno 2006, unpublished).



The corrections to this form are  $O(m_e/m_i)$ . This is second order in the expansion of § 4 and, therefore, we need not keep these corrections. The operator (B18) is mathematically similar to the model operator for the same-species collisions [Eq. (B12)]. The gyrokinetic version of this operator is derived in the way analogous to the calculation in Appendix B.3. The result is

$$\begin{aligned} \langle C_{ei}[h] \rangle_{\mathbf{R}_e} = \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{R}_e} \nu_D^{ei}(\nu) & \left[ \frac{1}{2} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial h_{e\mathbf{k}}}{\partial \xi} - \frac{\nu^2(1 + \xi^2)}{4\nu_{the}^2} k_{\perp}^2 \rho_e^2 h_{e\mathbf{k}} \right. \\ & \left. - \frac{Zm_e}{m_i} \frac{\nu_{\perp}^2}{\nu_{the}^2} \frac{J_1(a_e)}{a_e} F_{0e} k_{\perp}^2 \rho_i^2 \frac{1}{n_{0i}} \int d^3 \mathbf{v}' \frac{2\nu_{\perp}^2}{\nu_{thi}^2} \frac{J_1(a'_i)}{a'_i} h_{i\mathbf{k}} + \frac{2\nu_{\parallel} J_0(a_e) u_{\parallel \mathbf{k}i}}{\nu_{the}^2} F_{0e} \right]. \end{aligned} \quad (\text{B19})$$

The second and third terms are manifestly second order in  $(m_e/m_i)^{1/2}$ , so have to be neglected along with other  $O(m_e/m_i)$  contributions to the electron-ion collisions. The remaining two terms are first order in the mass-ratio expansion: the first term vanishes for  $h_e = h_e^{(0)}$  [Eq. (93)], so its contribution is first order; in the fourth term, we can use Eq. (80) to express  $u_{\parallel i}$  in terms of quantities that are also first order. Keeping only the first-order terms, the gyrokinetic electron-ion collision operator is

$$\langle C_{ei}[h] \rangle_{\mathbf{R}_e} = \nu_D^{ei}(\nu) \left[ \frac{1}{2} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial h_e^{(1)}}{\partial \xi} + \frac{2\nu_{\parallel}}{\nu_{the}^2} \left( u_{\parallel e} - \frac{c}{4\pi en_{0e}} \nabla_{\perp}^2 A_{\parallel} \right) F_{0e} \right]. \quad (\text{B20})$$

### C. A NON-GYROKINETIC DERIVATION OF THE ELECTRON EQUATIONS

Here we first show how the evolution equations (108-109) of § 4 can be derived from electron-fluid dynamics, without the use of gyrokinetics (§ C.1). We then discuss the physical assumptions that allow these equations to be reduced to the EMHD equations (199-200) of § 7 (§ C.2).

#### C.1. Derivation of Eqs. (108-109)

We start with the following three equations:

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}, \quad (\text{C1})$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \quad (\text{C2})$$

$$\mathbf{E} + \frac{\mathbf{u}_e \times \mathbf{B}}{c} = -\frac{\nabla p_e}{en_e}. \quad (\text{C3})$$

Equation (C1) is Faraday's law, Eq. (C2) is the electron continuity equation, and Eq. (C3) is the generalized Ohm's law, which is the electron momentum equation with all electron inertia terms neglected (i.e., effectively, the lowest order in the expansion in the electron mass  $m_e$ ). The electron-pressure term in the right-hand of Eq. (C3) is sometimes called the thermoelectric term. We now assume the same static uniform equilibrium,  $\mathbf{E}_0 = 0$ ,  $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$ , that we have used throughout this paper and apply to Eqs. (C1-C3) the fundamental ordering discussed in § 3.1.

First consider the projection of Ohm's law (C3) onto the *total* magnetic field  $\mathbf{B}$ , use the definition of  $\mathbf{E}$  [Eq. (33)], and keep the leading-order terms in the  $\epsilon$  expansion:

$$\mathbf{E} \cdot \hat{\mathbf{b}} = -\frac{1}{en_e} \hat{\mathbf{b}} \cdot \nabla p_e \quad \Rightarrow \quad \frac{1}{c} \frac{\partial A_{\parallel}}{\partial t} + \hat{\mathbf{b}} \cdot \nabla \phi = \hat{\mathbf{b}} \cdot \nabla \frac{\delta p_e}{en_{0e}}. \quad (\text{C4})$$

This turns into Eq. (108) if we also assume isothermal electrons,  $\delta p_e = T_{0e} \delta n_e$  [see Eq. (95)].

With the aid of Eq. (C3), Faraday's law (C1) turns into

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u}_e \times \mathbf{B}) = -\mathbf{u}_e \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{u}_e - \mathbf{B} \nabla \cdot \mathbf{u}_e. \quad (\text{C5})$$

Keeping the leading-order terms, we find, for the components of Eq. (C5) perpendicular and parallel to the guide field,

$$\left( \frac{\partial}{\partial t} + \mathbf{u}_{\perp e} \cdot \nabla_{\perp} \right) \frac{\delta \mathbf{B}_{\perp}}{B_0} = \hat{\mathbf{b}} \cdot \nabla \mathbf{u}_{\perp e}, \quad (\text{C6})$$

$$\left( \frac{\partial}{\partial t} + \mathbf{u}_{\perp e} \cdot \nabla_{\perp} \right) \left( \frac{\delta B_{\parallel}}{B_0} - \frac{\delta n_e}{n_{0e}} \right) = \hat{\mathbf{b}} \cdot \nabla u_{\parallel e}. \quad (\text{C7})$$

In the last equation, where we have used the electron continuity equation (C2) to write

$$\nabla \cdot \mathbf{u}_e = - \left( \frac{\partial}{\partial t} + \mathbf{u}_{\perp e} \cdot \nabla_{\perp} \right) \frac{\delta n_e}{n_{0e}}. \quad (\text{C8})$$

From Ohm's law (C3), we have, to lowest order,

$$\mathbf{u}_{\perp e} = -\hat{\mathbf{z}} \times \frac{c}{B_0} \left( \mathbf{E}_{\perp} + \nabla_{\perp} \frac{\delta p_e}{en_{0e}} \right) = \hat{\mathbf{z}} \times \nabla_{\perp} \frac{c}{B_0} \left( \phi - \frac{\delta p_e}{en_{0e}} \right). \quad (\text{C9})$$

Using this expression in Eq. (C7) gives

$$\frac{d}{dt} \left( \frac{\delta B_{\parallel}}{B_0} - \frac{\delta n_e}{n_{0e}} \right) - \hat{\mathbf{b}} \cdot \nabla u_{\parallel e} = \frac{c}{B_0} \left\{ \frac{\delta p_e}{en_{0e}}, \frac{\delta B_{\parallel}}{B_0} \right\} - \frac{c}{B_0} \left\{ \frac{\delta p_e}{en_{0e}}, \frac{\delta n_e}{n_{0e}} \right\}, \quad (\text{C10})$$

where  $d/dt$  is defined in the usual way [Eq. (114)]. Assuming isothermal electrons,  $\delta p_e = T_{0e} \delta n_e$ , annihilates the second term on the right-hand side and turns the above equation into Eq. (109). As for Eq. (C6), the use of Eq. (C9) and substitution of  $\delta \mathbf{B}_{\perp} = -\hat{\mathbf{z}} \times \nabla_{\perp} A_{\parallel}$  turns it into the previously derived Eq. (C4), whence follows Eq. (108).

Thus, we have shown that Eqs. (108-109) can be derived as a direct consequence of Faraday's law, electron-fluid dynamics (electron continuity equation and the electron force balance, a. k. a. the generalized Ohm's law), and the assumption of isothermal electrons—all taken to the leading order in the gyrokinetic ordering given in § 3.1 (i.e., assuming strongly interacting anisotropic fluctuations with  $k_{\parallel} \ll k_{\perp}$ ).

We have just proved that Eqs. (108) and (109) are simply the perpendicular and parallel part, respectively, of Eq. (C5). The latter equation means that the magnetic-field lines are frozen into the electron flow velocity  $\mathbf{u}_e$ , i.e., the flux is conserved, the result formally proven in § 4.3 [see Eq. (91)].

### C.2. Electron MHD and the Derivation of Eqs. (199-200)

The standard approach to deriving the Electron MHD approximation is to start with Eq. (C5), which expresses the assumption that the magnetic field is frozen into the electron flow. The electron velocity can be written in terms of the ion velocity and the current density:

$$\mathbf{u}_e = \mathbf{u}_i - \frac{\mathbf{j}}{en_e} = \mathbf{u}_i - \frac{c}{4\pi en_e} \nabla \times \mathbf{B}, \quad (\text{C11})$$

where Ampère's law has been used.<sup>28</sup> It is then assumed that, at scales much smaller than the ion gyroscale ( $k_{\perp} \rho_i \gg 1$ ), the ions can be considered immobile,  $\mathbf{u}_i = 0$ . Substituting the remainder of Eq. (C11) into Eq. (C5) and assuming  $n_e = n_{0e} = \text{const}$  gives Eq. (201), the equation of Electron MHD. Let us see what happens if such an approach is applied in combination with the  $\epsilon$  expansion. To the leading order in  $\epsilon$ , Eq. (C11) gives

$$\mathbf{u}_{\perp e} = \mathbf{u}_{\perp i} + \frac{c}{4\pi en_{0e}} \hat{\mathbf{z}} \times \nabla_{\perp} \delta B_{\parallel}, \quad (\text{C12})$$

$$u_{\parallel e} = u_{\parallel i} + \frac{c}{4\pi en_{0e}} \nabla_{\perp}^2 A_{\parallel}. \quad (\text{C13})$$

The perpendicular and parallel parts of Eq. (C5) to the leading order in  $\epsilon$  are Eqs. (C6) and (C7), respectively. Using Eqs. (C12-C13) and taking  $u_{\parallel i} = 0$ ,  $\mathbf{u}_{\perp i} = 0$ , we get

$$\frac{\partial A_{\parallel}}{\partial t} = -\frac{cB_0^2}{4\pi en_{0e}} \hat{\mathbf{b}} \cdot \nabla \frac{\delta B_{\parallel}}{B_0}, \quad (\text{C14})$$

$$\frac{\partial}{\partial t} \left( \frac{\delta B_{\parallel}}{B_0} - \frac{\delta n_e}{n_{0e}} \right) = \frac{c}{4\pi en_{0e}} \hat{\mathbf{b}} \cdot \nabla \nabla_{\perp}^2 A_{\parallel} + \frac{cB_0}{4\pi en_{0e}} \left\{ \frac{\delta B_{\parallel}}{B_0}, \frac{\delta n_e}{n_{0e}} \right\}. \quad (\text{C15})$$

If we further assume incompressibility,  $\delta n_e = 0$ , use  $n_{0e} = Zn_{0i}$ , and define formally [cf. Eqs. (127) and (197)]

$$\Psi = -\frac{A_{\parallel}}{\sqrt{4\pi m_i n_{0i}}}, \quad \Phi = \frac{1}{\sqrt{\beta_i}} \left( 1 + \frac{Z}{\tau} \right)^{-1} \rho_i v_A \frac{\delta B_{\parallel}}{B_0}, \quad (\text{C16})$$

Eqs. (C14-C15) become

$$\frac{\partial \Psi}{\partial t} = v_A \left( 1 + Z/\tau \right) \hat{\mathbf{b}} \cdot \nabla \Phi, \quad (\text{C17})$$

$$\frac{\partial \Phi}{\partial t} = -\frac{v_A}{\beta_i \left( 1 + Z/\tau \right)} \hat{\mathbf{b}} \cdot \nabla \left( \rho_i^2 \nabla_{\perp}^2 \Psi \right). \quad (\text{C18})$$

These are the same as Eqs. (199-200) of § 7 in the limit of  $\beta_i \gg 1$ . As we have assumed the density perturbation to be zero, it is not surprising that we have recovered what is effectively the incompressible limit of the correct equations. In order Eqs. (199-200) to be reproduced fully in this derivation, we must keep the terms containing  $\delta n_e$  in Eq. (C15) and find a way to relate  $\delta n_e$  to  $\delta B_{\parallel}$ . This can be done by adopting the pressure balance, which is a fundamental feature of the gyrokinetic approximation [cf. Eq. (60)] and assuming both ions and electrons to be isothermal [cf. Eq. (95) and Eq. (47) for ions with  $h_i$  omitted]:

$$\frac{B_0 \delta B_{\parallel}}{4\pi} = -\delta p_i - \delta p_e = -T_{0i} \delta n_i - T_{0e} \delta n_e \quad \Rightarrow \quad \frac{\delta B_{\parallel}}{B_0} = -\frac{\beta_i}{2} \left( 1 + \frac{Z}{\tau} \right) \frac{\delta n_e}{n_{0e}}, \quad (\text{C19})$$

which is consistent with Eqs. (196) and (197). Using Eq. (C19) and the definitions (C16) in Eq. (C15), we recover the desired Eq. (200) [Eq. (C17) is already the same as Eq. (199)]. It should be clear from this derivation that the ERMHD equations (199-200) are the appropriate generalization of the standard EMHD equation (201) to the case of arbitrary  $\beta_i$  under the assumption of the gyrokinetic ordering (§ 3.1). As we have seen, when  $\beta_i$  is not large, the ion motion must be taken into account to the extent that it affects the plasma density: namely, when substituting the expression (C11) for  $\mathbf{u}_e$  into Eq. (C5), we may assume  $\mathbf{u}_i = 0$  in all terms except  $\nabla \cdot \mathbf{u}_e$ , which is calculated from the electron continuity equation [see Eq. (C8)].

<sup>28</sup> Note that if Eq. (C11) is substituted into the generalized Ohm's law (C3), the second term gives rise to what is known as the Hall term [cf. Eq. (A7)].

## D. FLUID LIMIT OF THE KINETIC RMHD

Taking the fluid (collisional) limit of the KRMHD system (summarized in § 5.7) means carrying out another subsidiary expansion—this time in  $k_{\parallel} \lambda_{\text{mfpi}} \ll 1$ . The expansion only affects the equations for the density and magnetic-field-strength fluctuations (§ 5.5) because the Alfvén waves are indifferent to collisional effects.

The calculation presented below follows the Enskog procedure—a standard perturbation algorithm used in the kinetic theory of gases and in plasma physics to derive fluid equations with collisional transport coefficients (Chapman & Cowling 1970). For magnetized plasma, this calculation was carried out in full generality by Braginskii (1965), whose starting point was the full plasma kinetic theory [Eqs. (32-35)]. While what we do below is, strictly speaking, merely a particular case of his calculation, it has the advantage of relative simplicity and also serves to show how the fluid limit is recovered from the gyrokinetic formalism—a demonstration that we believe to be of value.

It is convenient to rewrite the KRMHD system [Eqs. (149-151)] in terms of the function  $\delta \tilde{f}_i = g + (v_{\perp}^2 / v_{\text{thi}}^2)(\delta B_{\parallel} / B_0) F_{0i}$ , which is the perturbation of the local Maxwellian in the frame of the Alfvén waves (see § 5.4). We have

It will be convenient to use KRMHD system written in terms of the function  $\delta \tilde{f}_i = g + (v_{\perp}^2 / v_{\text{thi}}^2)(\delta B_{\parallel} / B_0) F_{0i}$ , which is the perturbation of the local Maxwellian in the frame of the Alfvén waves [Eqs. (142-144)]. We want to expand Eq. (142) in powers of  $k_{\parallel} \lambda_{\text{mfpi}}$ , so we let  $\delta \tilde{f}_i = \delta \tilde{f}_i^{(0)} + \delta \tilde{f}_i^{(1)} + \dots$ ,  $\delta B_{\parallel} = \delta B_{\parallel}^{(0)} + \delta B_{\parallel}^{(1)} + \dots$ , etc.

## D.1. Zeroth Order: Ideal Fluid Equations

Since [see Eq. (40)]

$$\frac{\omega}{\nu_{ii}} \sim \frac{k_{\parallel} v_A}{\nu_{ii}} \sim \frac{k_{\parallel} \lambda_{\text{mfpi}}}{\sqrt{\beta_i}}, \quad \frac{k_{\parallel} v_{\parallel}}{\nu_{ii}} \sim \frac{k_{\parallel} v_{\text{thi}}}{\nu_{ii}} \sim k_{\parallel} \lambda_{\text{mfpi}}, \quad (\text{D1})$$

to zeroth order Eq. (142) becomes  $\left\langle C_{ii} [\delta \tilde{f}_i^{(0)}] \right\rangle_{\mathbf{R}_i} = 0$ . The zero mode of the collision operator is a Maxwellian. Therefore, we may write the full ion distribution function up to zeroth order in  $k_{\parallel} \lambda_{\text{mfpi}}$  as follows [see Eq. (136)]

$$f_i = \frac{n_i}{(2\pi T_i / m_i)^{3/2}} \exp \left\{ -\frac{m_i [(\mathbf{v}_{\perp} - \mathbf{u}_E)^2 + (v_{\parallel} - u_{\parallel})^2]}{2T_i} \right\}, \quad (\text{D2})$$

where  $n_i = n_{0i} + \delta n_i$  and  $T_i = T_{0i} + \delta T_i$  include both the unperturbed quantities and their perturbations. The  $\mathbf{E} \times \mathbf{B}$  drift velocity  $\mathbf{u}_E$  comes from the Alfvén waves (see § 5.4) and does not concern us here. Since the perturbations  $\delta n_i$ ,  $u_{\parallel}$  and  $\delta T_i$  are small in the original gyrokinetic expansion, Eq. (D2) is equivalent to

$$\delta \tilde{f}_i^{(0)} = \left[ \frac{\delta n_e^{(0)}}{n_{0e}} + \left( \frac{v^2}{v_{\text{thi}}^2} - \frac{3}{2} \right) \frac{\delta T_i^{(0)}}{T_{0i}} + \frac{2v_{\parallel}}{v_{\text{thi}}^2} u_{\parallel}^{(0)} \right] F_{0i}, \quad (\text{D3})$$

where we have used quasineutrality to replace  $\delta n_i / n_{0i} = \delta n_e / n_{0e}$ . This automatically satisfies Eq. (143), while Eq. (144) gives us an expression for the ion-temperature perturbation:

$$\frac{\delta T_i^{(0)}}{T_{0i}} = - \left( 1 + \frac{Z}{\tau} \right) \frac{\delta n_e^{(0)}}{n_{0e}} - \frac{2}{\beta_i} \frac{\delta B_{\parallel}^{(0)}}{B_0}. \quad (\text{D4})$$

Note that this is consistent with the interpretation of the perpendicular Ampère's law [Eq. (56), which is the progenitor of Eq. (144)] as the pressure balance [see Eq. (60)]: indeed, recalling that the electron pressure perturbation is  $\delta p_e = T_{0e} \delta n_e$  [Eq. (95)], we have

$$\delta \frac{B^2}{8\pi} = \frac{B_0^2}{4\pi} \frac{\delta B_{\parallel}}{B_0} = -\delta p_e - \delta p_i = -\delta n_e T_{0e} - \delta n_i T_{0i} - n_{0i} \delta T_i, \quad (\text{D5})$$

whence follows Eq. (D4) by way of quasineutrality ( $Zn_i = n_e$ ) and the definitions of  $Z$ ,  $\tau$ ,  $\beta_i$  [Eqs. (36-38)].

Since the collision operator conserves particle number, momentum and energy, we can obtain evolution equations for  $\delta n_e^{(0)} / n_{0e}$ ,  $u_{\parallel}^{(0)}$  and  $\delta B_{\parallel}^{(0)} / B_0$  by multiplying Eq. (142) by 1,  $v_{\parallel}$ ,  $v^2 / v_{\text{thi}}^2$ , respectively, and integrating over the velocity space. The three moments that emerge this way are

$$\frac{1}{n_{0i}} \int d^3 \mathbf{v} \delta \tilde{f}_i^{(0)} = \frac{\delta n_e^{(0)}}{n_{0e}}, \quad (\text{D6})$$

$$\frac{1}{n_{0i}} \int d^3 \mathbf{v} v_{\parallel} \delta \tilde{f}_i^{(0)} = u_{\parallel}^{(0)}, \quad (\text{D7})$$

$$\frac{1}{n_{0i}} \int d^3 \mathbf{v} \frac{v^2}{v_{\text{thi}}^2} \delta \tilde{f}_i^{(0)} = \frac{3}{2} \left( \frac{\delta n_e^{(0)}}{n_{0e}} + \frac{\delta T_i^{(0)}}{T_{0i}} \right). \quad (\text{D8})$$



The three evolution equations for these moments are

$$\frac{d}{dt} \left( \frac{\delta n_e^{(0)}}{n_{0e}} - \frac{\delta B_{\parallel}^{(0)}}{B_0} \right) + \hat{\mathbf{b}} \cdot \nabla u_{\parallel}^{(0)} = 0, \quad (\text{D9})$$

$$\frac{du_{\parallel}^{(0)}}{dt} - v_A^2 \hat{\mathbf{b}} \cdot \nabla \frac{\delta B_{\parallel}^{(0)}}{B_0} = 0, \quad (\text{D10})$$

$$\frac{d}{dt} \left[ \frac{3}{2} \left( \frac{\delta n_e^{(0)}}{n_{0e}} + \frac{\delta T_i^{(0)}}{T_{0i}} \right) - \frac{5}{2} \frac{\delta B_{\parallel}^{(0)}}{B_0} \right] + \frac{5}{2} \hat{\mathbf{b}} \cdot \nabla u_{\parallel}^{(0)} = 0. \quad (\text{D11})$$

These allow us to recover the fluid equations we derived in § 2.3: Eq. (D10) is the parallel component of the MHD momentum equation (22); combining Eqs. (D9), (D11) and (D4), we obtain the continuity equation and the parallel component of the induction equation—these are the same as Eqs. (20) and (21):

$$\frac{d}{dt} \frac{\delta n_e^{(0)}}{n_{0e}} = - \frac{1}{1 + c_s^2/v_A^2} \hat{\mathbf{b}} \cdot \nabla u_{\parallel}^{(0)}, \quad (\text{D12})$$

$$\frac{d}{dt} \frac{\delta B_{\parallel}^{(0)}}{B_0} = \frac{1}{1 + v_A^2/c_s^2} \hat{\mathbf{b}} \cdot \nabla u_{\parallel}^{(0)}, \quad (\text{D13})$$

where the sound speed  $c_s$  is defined by Eq. (158). From Eqs. (D9) and (D11), we also find the analog of the entropy equation (18):

$$\frac{d}{dt} \frac{\delta T_i^{(0)}}{T_{0i}} = \frac{2}{3} \frac{d}{dt} \frac{\delta n_e^{(0)}}{n_{0e}} \Leftrightarrow \frac{d}{dt} \frac{\delta s^{(0)}}{s_0} = 0, \quad \frac{\delta s^{(0)}}{s_0} = \frac{\delta T_i^{(0)}}{T_{0i}} - \frac{2}{3} \frac{\delta n_e^{(0)}}{n_{0e}} = - \left( \frac{5}{3} + \frac{Z}{\tau} \right) \left( \frac{\delta n_e^{(0)}}{n_{0e}} + \frac{v_A^2}{c_s^2} \frac{\delta B_{\parallel}^{(0)}}{B_0} \right). \quad (\text{D14})$$

This implies that the temperature changes due to compressional heating only.

### D.2. Generalized Energy: Five RMHD Cascades Recovered

We now calculate the generalized energy by substituting  $\delta \tilde{f}_i$  from Eq. (D3) into Eq. (145) and using Eqs. (D4) and (D14):

$$W = \int d^3 \mathbf{r} \left[ \frac{m_i n_{0i} u_E^2}{2} + \frac{\delta B_{\perp}^2}{8\pi} + \frac{m_i n_{0i} u_{\parallel}^2}{2} + \frac{\delta B_{\parallel}^2}{8\pi} \left( 1 + \frac{v_A^2}{c_s^2} \right) + \frac{3}{4} n_{0i} T_{0i} \frac{1 + Z/\tau}{5/3 + Z/\tau} \frac{\delta s^2}{s_0^2} \right] = W_{\perp}^+ + W_{\perp}^- + W_{\parallel}^+ + W_{\parallel}^- + \frac{3}{2} n_{0i} T_{0i} \frac{1 + Z/\tau}{5/3 + Z/\tau} W_s. \quad (\text{D15})$$

The first two terms are the Alfvén-wave energy [Eq. (146)]. The following two terms are the slow-wave energy, which splits into the independently cascaded energies of “+” and “−” waves (see § 2.4):

$$W_{\text{SW}} = W_{\parallel}^+ + W_{\parallel}^- = \int d^3 \mathbf{r} \frac{m_i n_{0i}}{2} \left( |z_{\parallel}^+|^2 + |z_{\parallel}^-|^2 \right). \quad (\text{D16})$$

The last term is the total variance of the entropy mode. Thus, we have recovered the five cascades of the RMHD system (§ 2.5).

### D.3. First Order: Collisional Transport

Now let us compute the collisional transport terms for the equations derived above. In order to do this, we have to determine the first-order perturbed distribution function  $\delta \tilde{f}_i^{(1)}$ , which satisfies [see Eq. (142)]

$$\left\langle C_{ii} \left[ \delta \tilde{f}_i^{(1)} \right] \right\rangle_{\mathbf{r}_i} = \frac{d}{dt} \left( \delta \tilde{f}_i^{(0)} - \frac{v_{\perp}^2}{v_{\text{th}i}^2} \frac{\delta B_{\parallel}^{(0)}}{B_0} \right) + v_{\parallel} \hat{\mathbf{b}} \cdot \nabla \left( \delta \tilde{f}_i^{(0)} + \frac{Z}{\tau} \frac{\delta n_e^{(0)}}{n_{0e}} F_{0i} \right). \quad (\text{D17})$$

We now use Eq. (D3) to substitute for  $\delta \tilde{f}_i^{(0)}$  and Eqs. (D12-D14) and (D10) to compute the time derivatives. Equation (D17) becomes

$$\left\langle C_{ii} \left[ \delta \tilde{f}_i^{(1)} \right] \right\rangle_{\mathbf{r}_i} = \left[ - (1 - 3\xi^2) \frac{v^2}{v_{\text{th}i}^2} \frac{2/3 + c_s^2/v_A^2}{1 + c_s^2/v_A^2} \hat{\mathbf{b}} \cdot \nabla u_{\parallel}^{(0)} + \xi v \left( \frac{v^2}{v_{\text{th}i}^2} - \frac{5}{2} \right) \hat{\mathbf{b}} \cdot \nabla \frac{\delta T_i^{(0)}}{T_{0i}} \right] F_{0i}(v), \quad (\text{D18})$$

where  $\xi = v_{\parallel}/v$ . Note that the right-hand side gives zero when multiplied by 1,  $v_{\parallel}$  or  $v^2$  and integrated over the velocity space, as it must do because the collision operator in the left-hand side conserves particle number, momentum and energy.

Solving Eq. (D18) requires inverting the collision operator. While this can be done for the general Landau collision operator (see Braginskii 1965), for our purposes, it is sufficient to use the model operator given in Appendix B.3, Eq. (B17). This simplifies calculations at the expense of an order-one inaccuracy in the numerical values of the transport coefficients. As the exact value of these coefficients will never be crucial for us, this is an acceptable loss of precision. Inverting the collision operator in Eq. (D18) then gives

$$\delta \tilde{f}_i^{(1)} = \frac{1}{v_D^i(v)} \left[ \frac{1 - 3\xi^2}{3} \frac{v^2}{v_{\text{th}i}^2} \frac{2/3 + c_s^2/v_A^2}{1 + c_s^2/v_A^2} \hat{\mathbf{b}} \cdot \nabla u_{\parallel}^{(0)} - \xi v \left( \frac{v^2}{v_{\text{th}i}^2} - \frac{5}{2} \right) \hat{\mathbf{b}} \cdot \nabla \frac{\delta T_i^{(0)}}{T_{0i}} \right] F_{0i}(v), \quad (\text{D19})$$

where  $\nu_D^{ii}(v)$  is a collision frequency defined in Eq. (B11) and we have chosen the constants of integration in such a way that the three conservation laws are respected:  $\int d^3\mathbf{v} \delta f_i^{(1)} = 0$ ,  $\int d^3\mathbf{v} v_{\parallel} \delta f_i^{(1)} = 0$ ,  $\int d^3\mathbf{v} v^2 \delta f_i^{(1)} = 0$ . These relations mean that  $\delta n_e^{(1)} = 0$ ,  $u_{\parallel}^{(1)} = 0$ ,  $\delta T_i^{(1)} = 0$  and that, in view of Eq. (144), we have

$$\frac{\delta B_{\parallel}^{(1)}}{B_0} = -\frac{1}{3v_A^2} \frac{2/3 + c_s^2/v_A^2}{1 + c_s^2/v_A^2} \nu_{\parallel i} \hat{\mathbf{b}} \cdot \nabla u_{\parallel}, \quad (\text{D20})$$

where  $\nu_{\parallel i}$  is defined below [Eq. (D24)]. Equations (D19-D20) are now used to calculate the first-order corrections to the moment equations (D9-D11). They become

$$\frac{d}{dt} \left( \frac{\delta n_e}{n_{0e}} - \frac{\delta B_{\parallel}}{B_0} \right) + \hat{\mathbf{b}} \cdot \nabla u_{\parallel} = 0, \quad (\text{D21})$$

$$\frac{du_{\parallel}}{dt} - v_A^2 \hat{\mathbf{b}} \cdot \nabla \frac{\delta B_{\parallel}}{B_0} = \frac{2/3 + c_s^2/v_A^2}{1 + c_s^2/v_A^2} \nu_{\parallel i} \hat{\mathbf{b}} \cdot \nabla (\hat{\mathbf{b}} \cdot \nabla u_{\parallel}), \quad (\text{D22})$$

$$\frac{d}{dt} \frac{\delta T_i}{T_{0i}} - \frac{2}{3} \frac{d}{dt} \frac{\delta n_e}{n_{0e}} = \kappa_{\parallel i} \hat{\mathbf{b}} \cdot \nabla \left( \hat{\mathbf{b}} \cdot \nabla \frac{\delta T_i}{T_{0i}} \right), \quad (\text{D23})$$

where we have introduced the coefficients of parallel viscosity and parallel thermal diffusivity:

$$\nu_{\parallel i} = \frac{2}{15} \frac{1}{n_{0i}} \int d^3\mathbf{v} \frac{v^4}{\nu_D^{ii}(v) v_{\text{thi}}^2} F_{0i}(v), \quad \kappa_{\parallel i} = \frac{2}{9} \frac{1}{n_{0i}} \int d^3\mathbf{v} \frac{v^4}{\nu_D^{ii}(v) v_{\text{thi}}^2} \left( \frac{v^2}{v_{\text{thi}}^2} - \frac{5}{2} \right) F_{0i}(v). \quad (\text{D24})$$

All perturbed quantities are now accurate up to first order in  $k_{\parallel} \lambda_{\text{mfp}i}$ . Note that in Eq. (D22), we used Eq. (D20) to express  $\delta B_{\parallel}^{(0)} = \delta B_{\parallel} - \delta B_{\parallel}^{(1)}$ . We do the same in Eq. (D4) and obtain

$$\left( 1 + \frac{Z}{\tau} \right) \frac{\delta n_e}{n_{0e}} = -\frac{\delta T_i}{T_{0i}} - \frac{2}{\beta_i} \left( \frac{\delta B_{\parallel}}{B_0} + \frac{1}{3v_A^2} \frac{2/3 + c_s^2/v_A^2}{1 + c_s^2/v_A^2} \nu_{\parallel i} \hat{\mathbf{b}} \cdot \nabla u_{\parallel} \right). \quad (\text{D25})$$

This equation completes the system (D21-D23), which allows us to determine  $\delta n_e$ ,  $u_{\parallel}$ ,  $\delta T_i$  and  $\delta B_{\parallel}$ . In § 6.1, we use the equations derived above, but absorb the prefactor  $(2/3 + c_s^2/v_A^2)/(1 + c_s^2/v_A^2)$  into the definition of  $\nu_{\parallel i}$ . The same system of equations can also be derived from Braginskii's two-fluid theory (Appendix A.4), from which we can borrow the quantitatively correct values of the viscosity and ion thermal diffusivity:  $\nu_{\parallel i} = 0.90 v_{\text{thi}}^2 / \nu_{ii}$ ,  $\kappa_{\parallel i} = 2.45 v_{\text{thi}}^2 / \nu_{ii}$ , where  $\nu_{ii}$  is defined in Eq. (44).

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