

FROBENIUS-SCHUR INDICATORS FOR SEMISIMPLE LIE ALGEBRAS

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1. INTRODUCTION

Classically Frobenius-Schur indicators were defined for irreducible representations of finite groups over the field of complex numbers. The interest in doing so came from the second indicator which determines whether an irreducible representation is real, complex or quaternionic. Namely, a classical theorem of Frobenius and Schur asserts that an irreducible representation is real, complex or quaternionic if and only if its second indicator is 1, 0 or -1 , respectively (see e.g. [S]). However, no representation-theoretic interpretation of the higher indicators is known.

Recently, Frobenius-Schur indicators of irreducible representations of complex semisimple finite dimensional (quasi-)Hopf algebras H were defined by Linchenko and Montgomery [LM] and Mason and Ng [MN] (see also [KSZ]), generalizing the definition in the group case. The values of the m th indicator are cyclotomic integers in \mathbb{Q}_m . Moreover, an analog of the Frobenius-Schur theorem on the second indicator was proved, and in general it has been shown that the indicators carry rich information on H , as well as on its representation category (see also [NS2]).

In fact, one can generalize the definition of Frobenius-Schur indicators to simple objects of any semisimple tensor categories which admit a pivotal structure ($=$ tensor isomorphism $\text{id} \rightarrow **$), thus showing in particular that the indicators are categorical invariants (see e.g. [FGSV], [NS1]).

The category of finite dimensional representations of a finite dimensional complex semisimple Lie algebra is a pivotal semisimple tensor category, and hence one can define the Frobenius-Schur indicators of its simple objects. The second indicator was already defined and known to be nonzero if and only if the simple representation is self-dual, and 1 or -1 if and only if the representation is orthogonal or symplectic, respectively. Furthermore, Tits gave an explicit formula for it in representation-theoretic terms (see Section 3).

The purpose of this paper is to study Frobenius-Schur indicators (of all degrees) for semisimple Lie algebras. More specifically to find a closed formula for the indicators in representation-theoretic terms and deduce its asymptotical behavior. In particular we obtain that the indicators take integer values.

The organization of the paper is as follows.

Section 2 is devoted to preliminaries. We recall some basic definitions and facts from Lie theory which we need (e.g. the Weyl integration formula). Next we define the m th Frobenius-Schur indicator of the representation categories of finite dimensional complex semisimple Lie algebras.

In section 3 we recall the properties of the second indicator. For the benefit of the reader we also give a proof of Tits' theorem.

Section 4 is dedicated to the proof of our main results. In 4.1 we prove the formula for the m th Frobenius-Schur indicator ν_m , $m \geq 2$, which is given by the following theorem.

Theorem 1.1. *Let \mathfrak{g} be a finite dimensional complex semisimple Lie algebra. Let $V(\lambda)$ be an irreducible representation of \mathfrak{g} with highest weight λ , \mathcal{W} the Weyl group of \mathfrak{g} , ρ the half sum of positive roots, and $V(\lambda) \left[\frac{\rho - \sigma \cdot \rho}{m} \right]$ the weight space of the weight $\frac{\rho - \sigma \cdot \rho}{m}$ where $m \geq 2$ is an integer. Then the m th Frobenius-Schur indicator $\nu_m(V(\lambda))$ of $V(\lambda)$ is given by*

$$\nu_m(V(\lambda)) = \sum_{\sigma \in \mathcal{W}} sn(\sigma) \dim V(\lambda) \left[\frac{\rho - \sigma \cdot \rho}{m} \right].$$

Our proof of Theorem 1.1 is analytic. Namely, we work with the equivalent representation category of the associated simply connected Lie group and use the Weyl integration formula to obtain our formula.

Next, in 4.2 we prove the following corollary of Theorem 1.1.

Corollary 1.2. *For large enough m , $\nu_m(V(\lambda)) = \dim V(\lambda)[0]$ (which is not zero if and only if λ belongs to the root lattice). In particular for the classical Lie algebras $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(2n, \mathbb{C})$, $\mathfrak{so}(2n+1, \mathbb{C})$ and $\mathfrak{sp}(2n, \mathbb{C})$, $\nu_m(V(\lambda)) = \dim V(\lambda)[0]$ for m greater or equal to $2n-1$, $4n-5$, $4n-3$ and $2n+1$, respectively.*

Finally in 4.3 we use our formula and Kostant's theorem to compute explicitly the Frobenius-Schur indicators for the representation category of $sl(3, \mathbb{C})$. More specifically, we prove:

Theorem 1.3. *Let $V(a, b)$ be an irreducible representation of $sl(3, \mathbb{C})$. Then*

- (1) $\nu_2(V(a, b)) = 1$ if $a = b$, and $\nu_2(V(a, b)) = 0$ if $a \neq b$.
- (2) $\nu_3(V(a, b)) = 1 + \min\{a, b\}$.
- (3) For $m > 3$ we have, $\nu_m(V(a, b)) = 1 + \min\{a, b\}$ if (a, b) is in the root lattice and $\nu_m(V(a, b)) = 0$ otherwise.

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2. PRELIMINARIES

Throughout let \mathfrak{g} be a finite dimensional complex semisimple Lie algebra of rank r , (\cdot, \cdot) its Killing form, \mathfrak{h} a Cartan subalgebra (CSA) of \mathfrak{g} , Φ the root system corresponding to \mathfrak{h} , Δ a fixed base, $\{h_1, \dots, h_r\}$ the corresponding coroot system, and \mathcal{W} the Weyl group.

Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight (i.e. $\lambda(h_i)$ is a nonnegative integer for all i), $V(\lambda)$ the finite dimensional irreducible representation of \mathfrak{g} with highest weight λ and $\Pi(\lambda)$ the set of integral weights occurring in $V(\lambda)$; it is a finite set which is invariant under the action of the Weyl group. For $\mu \in \Pi(\lambda)$, let $m_\lambda(\mu) = \dim V(\lambda)[\mu]$ be the multiplicity of μ in $V(\lambda)$. Recall that the multiplicities are invariant under the Weyl group action. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ (half sum of positive roots); it is a strongly dominant integral weight.

Let us recall Kostant's theorem on the multiplicities of weights (for a proof see [Hu]). Let $\mu \in \mathfrak{h}^*$ and define $p(\mu)$ to be the number of sets of non-negative integers $\{k_\alpha | \alpha \succ 0\}$ for which $\mu = \sum_{\alpha \succ 0} k_\alpha \alpha$ (p is called the Kostant's partition function).

Of course, $p(\mu) = 0$ if μ is not in the root lattice.

Theorem 2.1. (Kostant) *Let λ be a dominant weight and $\mu \in \Pi(\lambda)$. Then the multiplicities of $V(\lambda)$ are given by the formula*

$$m_\lambda(\mu) = \sum_{\sigma \in \mathcal{W}} sn(\sigma) p(\sigma(\lambda + \rho) - \mu - \rho).$$

Let \mathfrak{g}_c be the compact real form of \mathfrak{g} , and G the corresponding simply connected compact matrix Lie group with Lie algebra \mathfrak{g}_c . It is known that $\text{Rep}(\mathfrak{g})$, $\text{Rep}(\mathfrak{g}_c)$ and $\text{Rep}(G)$ are equivalent symmetric tensor categories.

Let \mathfrak{t} be a CSA of \mathfrak{g}_c ; it corresponds to a maximal torus T of G . Then $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{t}$. It is known that $\alpha(h)$ is purely imaginary for all $h \in \mathfrak{t}$ and $\alpha \in \Phi$. If \mathfrak{t}^* denotes the space of real-valued linear functionals on \mathfrak{t} , then the roots are contained in $i\mathfrak{t}^* \subset \mathfrak{h}^*$. It is then convenient to introduce the *real* roots, which are simply $\frac{1}{i}$ times the ordinary roots, the *real* coroots h_α which are the elements of \mathfrak{t} corresponding to the elements $\frac{2\alpha}{(\alpha, \alpha)}$ where α is a real root, and the *real* weights of an irreducible representation of G . An element μ of \mathfrak{t}^* is said to be *integral* if $\mu(h_\alpha) \in \mathbb{Z}$ for each real coroot h_α . The real weights of any finite dimensional representation of \mathfrak{g} are integral. (See [Ha].)

The Weyl denominator is the function $A_\rho : T \rightarrow \mathbb{C}$ given by

$$A_\rho(t) = A_\rho(e^h) = \sum_{\omega \in \mathcal{W}} sn(\omega) e^{i(\omega \cdot \rho)(h)}.$$

Theorem 2.2. (Weyl integration formula) *Let G be a simply connected compact Lie group. Let f be a continuous class function on G , dg the normalized Haar measure on G , and dt the normalized Haar measure on T . Then*

$$\int_G f(g) dg = \frac{1}{|\mathcal{W}|} \int_T f(t) |A_\rho(t)|^2 dt.$$

Let us now define the Frobenius-Schur indicators of an irreducible representation of \mathfrak{g} .

Definition 2.3. Let V be an irreducible representation of \mathfrak{g} and $m \geq 2$ be an integer. The m th Frobenius-Schur indicator of V is the number $\nu_m(V) = \text{tr}(c|_{(V^{\otimes m})_{\mathfrak{g}}})$, where c is the cyclic automorphism of $V^{\otimes m}$ given by $v_1 \otimes \cdots \otimes v_m \mapsto v_m \otimes \cdots \otimes v_{m-1}$.

Remark 2.4. In fact, as we mentioned in the introduction, the indicators can be defined categorically. Applying the categorical definition to $\text{Rep}(\mathfrak{g})$ yields the above definition, while applying it to $\text{Rep}(G)$ yields $\text{tr}(c|_{(V^{\otimes m})_G})$. Since the indicators of V regarded as a \mathfrak{g} -module coincide with the indicators of V regarded as a G -module we have $\nu_m(V) = \text{tr}(c|_{(V^{\otimes m})_G})$.

3. TITS' THEOREM ON THE SECOND INDICATOR

Theorem 3.1. (See [B]) *Let G be a compact Lie group. Let V be an irreducible complex representation of G , and set $\epsilon_V = \int_G \chi(g^2) dg$. Then V is self dual if and only if $\epsilon_V \neq 0$. Furthermore, suppose V is self dual and let B be a (unique up to*

scalar) G -invariant non-degenerate bilinear form on V . Then B is either symmetric or skew-symmetric, and it is such if and only if $\epsilon_V = 1, -1$, respectively.

Remark 3.2. In Proposition 4.4 we will prove that $\epsilon_V = \nu_2(V)$ as defined above. Historically $\nu_2(V)$ was defined by ϵ_V .

Example 3.3. Let us use Theorem 1.1 to calculate $\nu_m(V)$ in the representation category of $sl(2, \mathbb{C})$. Let $sl(2, \mathbb{C}) = sp\{h, x, y\}$, where $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The root system is $\Phi = \{\alpha, -\alpha\}$, where $\alpha(h) = 2$. The Weyl group is $\mathcal{W} = \{1, \sigma_\alpha\}$, $\rho = \frac{1}{2}\alpha$ and $\sigma_\alpha(\rho) = -\frac{1}{2}\alpha$. Let $V(n) = \bigoplus_{j=0}^n V[n-2j]$ be the irreducible representation of highest weight $\lambda(h) = n$ with its weight space decomposition. By Theorem 1.1,

$$\nu_m(V(n)) = \dim V(n)[0] - \dim V(n) \left\lfloor \frac{\alpha}{m} \right\rfloor.$$

Let $m = 2$. By the formula above, if n is odd, then $\dim V(n)[0] = 0$ and $\dim V(n)[\frac{\alpha}{2}] = 1$. Hence $\nu_2(V(n)) = -1$. Similarly, if n is even, $\nu_2(V(n)) = 1$. Consequently $\nu_2(V(n)) = (-1)^n = (-1)^{\lambda(h)}$.

For $m \geq 3$, $\frac{\alpha(h)}{m} = \frac{2}{m}$ is not an integer and hence $\nu_m(V(n)) = \dim V(n)[0]$. Therefore we have

$$\nu_m(V(n)) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Let $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha)$ be the root space decomposition of \mathfrak{g} and $\Delta = \{\alpha_1, \dots, \alpha_r\}$ a fixed base. Fix a standard set of generators for \mathfrak{g} : $x_i \in \mathfrak{g}_{\alpha_i}$, $y_i \in \mathfrak{g}_{-\alpha_i}$ so that $[x_i, y_i] = h_i$. Let $\check{\rho} := 1/2 \sum_{\alpha \in \Phi^+} h_\alpha$ be the half sum of positive coroots.

Proposition 3.4. Let $E := x_1 + \dots + x_r$ and $H := 2\check{\rho}$. Then there exist constants a_1, \dots, a_r such that the subalgebra P generated by $H, E, F := a_1 y_1 + \dots + a_r y_r$ is isomorphic to $sl(2, \mathbb{C})$.

The Lie subalgebra $P \subseteq \mathfrak{g}$ is called a principal $sl(2, \mathbb{C})$ -subalgebra of \mathfrak{g} (see [K] or [D]).

Lemma 3.5. Let $V = V(\lambda)$ be an irreducible representation of \mathfrak{g} . Let P be a principal $sl(2, \mathbb{C})$ -subalgebra of \mathfrak{g} . Consider V as a P -module. Then its highest weight is $\lambda(H)$, and it contains the irreducible $sl(2, \mathbb{C})$ -representation $V(\lambda(H))$ with multiplicity one.

Proof. Let v^+ be a highest weight vector of V considered as a \mathfrak{g} -module. Then obviously we have $Hv^+ = \lambda(H)v^+$ and $Ev^+ = 0$. Hence v^+ is a highest weight vector with weight $\lambda(H)$ for V considered as a P -module. Therefore we can write $V = V(\lambda(H)) \oplus V(n_j)$. Now it remains to show that $\lambda(H) > n_j$ for any j . Let $V = V[\lambda] \oplus \bigoplus V[\mu]$ be the weight space decomposition of V as a \mathfrak{g} -module. It is also a weight space decomposition of V considered as a P -module, so $V[\mu]$ is a weight space of P with weight $\mu(H)$. Recall that $\mu = \lambda - \sum_{j=1}^r k_j \alpha_j$ where $k_j \in \mathbb{Z}^+$. Note that $\lambda(H) > \mu(H)$ if and only if $\lambda(H) > \lambda(H) - \sum_{j=1}^r k_j \alpha_j(H)$ if and only if $\sum_{j=1}^r k_j \alpha_j(2\check{\rho}) > 0$ if and only if $\sum_{j=1}^r k_j (\alpha_j, 2\rho) > 0$. But 2ρ is strongly dominant, i.e., $(\alpha_j, 2\rho) > 0$ for all $1 \leq j \leq r$. The proof is complete. \square

Let $\omega_0 \in \mathcal{W}$ be the unique element sending Δ to $-\Delta$.

Theorem 3.6. (Tits) *Let $V = V(\lambda)$ be a finite dimensional irreducible representation of \mathfrak{g} . If $\lambda + \omega_0\lambda \neq 0$ then $\nu_2(V) = 0$. Otherwise, $\nu_2(V) = (-1)^{\lambda(2\check{\rho})}$.*

Proof. It is known that the dual of $V(\lambda)$ is $V(-\omega_0\lambda)$, so if $V(\lambda)$ is not self dual (i.e., $\lambda + \omega_0\lambda \neq 0$) then $\nu_2(V) = 0$.

Suppose that V is self dual as a \mathfrak{g} -module. Then V admits a non-degenerate \mathfrak{g} -invariant bilinear form, and we have to decide if it is symmetric or skew symmetric. To do so, consider the principal $sl(2, \mathbb{C})$ -subalgebra P as in Lemma 3.5. The restriction of V to P has a unique copy of the largest representation of P occurring in V , with highest weight $\lambda(2\check{\rho})$. We already proved that this representation has indicator $(-1)^{\lambda(2\check{\rho})}$. Now we can use Theorem 3.1 to prove that V has a symmetric (skew-symmetric) \mathfrak{g} -invariant form if and only if it has a symmetric (skew-symmetric) P -invariant form. The first direction is obvious. Conversely, suppose that V has a symmetric P -invariant form and suppose on the contrary that V admits a skew-symmetric \mathfrak{g} -invariant form. Then if we restrict the bilinear \mathfrak{g} -form to P we get that V has a skew-symmetric P -invariant form which is a contradiction. Similar considerations are applied when V has a skew-symmetric P -invariant form. We conclude that $\nu_2(V) = (-1)^{\lambda(2\check{\rho})}$. \square

4. THE MAIN RESULTS

4.1. Proof of Theorem 1.1. Let G be the associated simply connected compact Lie group. From now on we will consider $V(\lambda)$ as a G -module. For convenience set $V = V(\lambda)$, $N = V(\lambda)^{\otimes m}$, and let $\pi : G \longrightarrow GL(V)$ be the irreducible representation.

The following lemma is easily derived from linear algebra.

Lemma 4.1. *Let $T \in \text{End}(V)$ be a projection, $W = \text{Im}T$ and $S \in \text{End}(V)$ an operator preserving W . Then $\text{tr}|_W(S) = \text{tr}|_V(S \circ T)$.*

Proof. Fix a basis $A = \{w_1, \dots, w_k\}$ for W , and let $\tilde{A} = \{w_1, \dots, w_k, w_{k+1}, \dots, w_n\}$ be a completion to a basis for V . Let $C = [S|_W]_A$ be the matrix representing $S|_W$ with respect to the basis A . Since $T|_W = \text{id}_W$ and $S(W) \subseteq W$ we find out that $[T]_{\tilde{A}} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$, $[S]_{\tilde{A}} = \begin{pmatrix} C & * \\ 0 & * \end{pmatrix}$, and hence $[S]_{\tilde{A}}[T]_{\tilde{A}} = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$. The lemma follows easily now. \square

Proposition 4.2. *We have,*

$$\nu_m(V) = \text{tr}|_{N^G}(c) = \int_G \text{tr}|_V(c \circ \pi^{\otimes m}(g)) dg.$$

Proof. We follow the lines of the proof of the first formula for Frobenius-Schur indicators in the Hopf case, given in Section 2.3 of [KSZ].

Set $\tau = \pi^{\otimes m}$. Consider the operator $\int_G \tau(g) dg : N \longrightarrow N$. Let us first show that the image of this operator is N^G . Indeed, by the invariance of the Haar measure, $\tau(h) \int_G \tau(g) v dg = \int_G \tau(hg) v dg = \int_G \tau(g) v dg$ for all $h \in G$ and $v \in N$. Hence $\text{Im}(\int_G \tau(g) dg) \subseteq N^G$.

Conversely, suppose that $u \in N^G$, then $\int_G \tau(g) u dg = \int_G u dg = u \int_G dg = u$. Hence $N^G \subseteq \text{Im}(\int_G \tau(g) dg)$ and we are done.

In fact, the above shows also that the operator $\int_G \tau(g)dg$ is a projection onto N^G .

Finally, $c \in \text{Aut}(N^G)$, so by Lemma 4.1,

$$\text{tr}|_{N^G}(c) = \text{tr}|_N \left(c \circ \int_G \tau(g)dg \right) = \int_G \text{tr}|_N(c \circ \tau(g))dg,$$

as claimed. \square

The following lemma is a particular case of a lemma in Section 2.3 of [KSZ] and its proof replicates the proof of that lemma.

Lemma 4.3. *Let $f_1, \dots, f_m \in \text{End}(V)$. Then,*

$$\text{tr}|_{V^{\otimes m}}(c \circ (f_1 \otimes \dots \otimes f_m)) = \text{tr}|_V(f_1 \circ \dots \circ f_m).$$

Proof. Let v_1, \dots, v_n be a basis of V with dual basis v_1^*, \dots, v_n^* . For $l = 1, \dots, m$, f_l is presented by the matrix $(a_{ij}^l)_{i,j=1}^n$, where $a_{ij}^l = (v_i^*, f_l(v_j))$. Therefore, $\text{tr}(a_{ij}^l) = \sum_{i=1}^n (v_i^*, f_l(v_i))$. We now have

$$\begin{aligned} \text{tr}|_{V^{\otimes m}}(c \circ (f_1 \otimes \dots \otimes f_m)) &= \\ \sum_{i_1, \dots, i_m=1}^n (v_{i_1}^* \otimes v_{i_2}^* \otimes \dots \otimes v_{i_m}^*, c(f_1(v_{i_1}) \otimes f_2(v_{i_2}) \otimes \dots \otimes f_m(v_{i_m}))) &= \\ \sum_{i_1, \dots, i_m=1}^n (v_{i_1}^*, f_2(v_{i_2})) \cdots (v_{i_{m-1}}^*, f_m(v_{i_m}))(v_{i_m}^*, f_1(v_{i_1})) &= \\ \sum_{i_1, \dots, i_m=1}^n a_{i_1, i_2}^2 a_{i_2, i_3}^3 \cdots a_{i_{m-1}, i_m}^m a_{i_m, i_1}^1 = \text{tr}|_V(f_2 \circ f_3 \circ \dots \circ f_m \circ f_1) &= \\ \text{tr}|_V(f_1 \circ f_2 \circ \dots \circ f_m), \end{aligned}$$

as desired. \square

Consequently we have the following proposition which is analogous to the finite group case.

Proposition 4.4. *Let χ be the irreducible character of V . Then*

$$\nu_m(V) = \int_G \chi(g^m)dg.$$

Proof. We follow the lines of the proof of the first formula for Frobenius-Schur indicators in the Hopf case, given in Section 2.3 of [KSZ].

It follows immediately from Proposition 4.2 and Lemma 4.3 that

$$\begin{aligned} \nu_m(V) &= \int_G \text{tr}|_N(c \circ \pi^{\otimes m}(g))dg = \int_G \text{tr}|_N(c \circ (\pi(g) \otimes \dots \otimes \pi(g)))dg = \\ &= \int_G \text{tr}|_V(\pi(g) \circ \dots \circ \pi(g))dg = \int_G \chi(g^m)dg. \end{aligned}$$

\square

Recall the integral real elements which are those elements μ of \mathfrak{t}^* for which $\frac{2(\mu, \alpha)}{(\alpha, \alpha)}$ is an integer for any simple real root α . For each real integral element μ , there is a function $\tilde{\mu}$ on T given by

$$\tilde{\mu}(e^h) = e^{i\mu(h)}$$

for all h in \mathfrak{t} . Functions of this form are called torus characters and they have the following property.

Lemma 4.5.

$$\int_T \tilde{\mu}(t) dt = \int_T e^{i\mu(h)} de^h = \begin{cases} 1 & \mu = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose that $\mu \neq 0$, then there exists $t_0 \in \mathfrak{t}$ such that $\tilde{\mu}(t_0) \neq 1$. Therefore

$$\int_T \tilde{\mu}(t) dt = \int_T \tilde{\mu}(t_0 t) dt = \tilde{\mu}(t_0) \int_T \tilde{\mu}(t) dt,$$

hence $\int_T \tilde{\mu}(t) dt = 0$. \square

Let χ be the character of V . Before we begin the proof of Theorem 1.1, recall that if $t = e^h \in T$ then for all $t \in T$,

$$(1) \quad \chi(t) = \chi(e^h) = \sum_{\mu \in \Pi(V)} \dim(V[\mu]) e^{i\mu(h)}.$$

We can now prove our main result.

Proof of Theorem 1.1: By Proposition 4.4 and the Weyl integration formula we have,

$$(2) \quad \nu_m(V) = \int_G \chi(g^m) dg = \frac{1}{|\mathcal{W}|} \int_T \chi(t^m) |A_\rho(t)|^2 dt.$$

On the other hand,

$$(3) \quad \chi(t^m) = \chi(e^{mh}) = \sum_{\mu \in \Pi(V)} \dim(V[\mu]) e^{im\mu(h)}.$$

Hence by (2) and (3) we have,

$$(4) \quad \nu_m(V) = \frac{1}{|\mathcal{W}|} \sum_{\mu \in \Pi(V)} \dim V[\mu] \int_T e^{im\mu(h)} |A_\rho(e^h)|^2 de^h.$$

Now let us calculate the last integral. We have

$$\begin{aligned} \int_T e^{im\mu(h)} |A_\rho(e^h)|^2 de^h &= \int_T e^{im\mu(h)} A_\rho(e^h) \overline{A_\rho(e^h)} de^h = \\ &= \int_T e^{im\mu(h)} \left(\sum_{\omega \in \mathcal{W}} sn(\omega) e^{i(\omega \cdot \rho)(h)} \right) \left(\sum_{\tau \in \mathcal{W}} sn(\tau) e^{-i(\tau \cdot \rho)(h)} \right) de^h = \\ &= \sum_{\omega, \tau \in \mathcal{W}} sn(\omega\tau) \int_T e^{i(m\mu + \omega \cdot \rho - \tau \cdot \rho)(h)} de^h. \end{aligned}$$

But from Lemma 4.5 we have

$$(5) \quad \int_T e^{i(m\mu + \omega \cdot \rho - \tau \cdot \rho)(h)} de^h = \begin{cases} 1 & \text{if } m\mu + \omega \cdot \rho - \tau \cdot \rho = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence (4) becomes,

$$\begin{aligned}\nu_m(V) &= \frac{1}{|\mathcal{W}|} \sum_{\omega, \tau \in \mathcal{W}} \sum_{\mu = \frac{\tau \cdot \rho - \omega \cdot \rho}{m}} sn(\omega\tau) \dim V[\mu] \\ &= \frac{1}{|\mathcal{W}|} \sum_{\omega, \tau \in \mathcal{W}} sn(\omega\tau) \dim V \left[\frac{\tau \cdot \rho - \omega \cdot \rho}{m} \right].\end{aligned}$$

Since $\dim V[\zeta] = \dim V[\tau \cdot \zeta]$ for all $\zeta \in \Pi(V)$ and $\tau \in \mathcal{W}$, we can write,

$$\nu_m(V) = \frac{1}{|\mathcal{W}|} \sum_{\omega, \tau \in \mathcal{W}} sn(\omega\tau) \dim V \left[\frac{\rho - \tau^{-1}\omega \cdot \rho}{m} \right].$$

Now if we fix $\omega \in \mathcal{W}$, substitute $\sigma = \tau^{-1}\omega$ and use the fact that $sn(\omega\tau) = sn(\tau^{-1}\omega)$, we get

$$\begin{aligned}\sum_{\tau \in \mathcal{W}} sn(\omega\tau) \dim V \left[\frac{\rho - \tau^{-1}\omega \cdot \rho}{m} \right] &= \sum_{\tau \in \mathcal{W}} sn(\tau^{-1}\omega) \dim V \left[\frac{\rho - \tau^{-1}\omega \cdot \rho}{m} \right] = \\ &= \sum_{\sigma \in \mathcal{W}} sn(\sigma) \dim V \left[\frac{\rho - \sigma \cdot \rho}{m} \right].\end{aligned}$$

Consequently,

$$\nu_m(V) = \frac{1}{|\mathcal{W}|} \sum_{\omega, \sigma \in \mathcal{W}} sn(\sigma) \dim V \left[\frac{\rho - \sigma \cdot \rho}{m} \right] = \sum_{\sigma \in \mathcal{W}} sn(\sigma) \dim V \left[\frac{\rho - \sigma \cdot \rho}{m} \right],$$

as desired. \square

It may be interesting to state the following immediate consequence of Theorem 1.1 and Theorem 3.6.

Corollary 4.6. *Let $V(\lambda)$ be an irreducible self dual representation of \mathfrak{g} , then*

$$\sum_{\sigma \in \mathcal{W}} sn(\sigma) \dim V(\lambda) \left[\frac{\rho - \sigma \cdot \rho}{2} \right] = (-1)^{\lambda(2\bar{\rho})}.$$

If $V(\lambda)$ is not self dual, the sum equals 0.

4.2. Proof of Corollary 1.2. Since ρ is strongly dominant, $\sigma \cdot \rho = \rho$ only when $\sigma = 1$. Write

$$\nu_m(V) = \dim V[0] + \sum_{\sigma \neq 1} sn(\sigma) \dim V \left[\frac{\rho - \sigma \cdot \rho}{m} \right].$$

We wish to show that for large enough m , $\frac{\rho - \sigma \cdot \rho}{m}$ is not a weight of V when $\sigma \neq 1$. Indeed, suppose that $\sigma \neq 1$. Recall that $\rho - \sigma \cdot \rho$ is an integral element, hence if we fix some coroot h_α , we have the following set of integers: $U_\alpha = \{(\rho - \sigma \cdot \rho)(h_\alpha) | \sigma \in \mathcal{W}, \sigma \neq 1\}$. Therefore if we take $m_\alpha = 1 + u_\alpha$, where u_α is the maximal element of U_α , then $\frac{\rho - \sigma \cdot \rho}{m_\alpha} \notin \Pi(V)$. Hence $\dim V \left[\frac{\rho - \sigma \cdot \rho}{m_\alpha} \right] = 0$ for all $\sigma \neq 1$, and therefore $\nu_m(V) = \dim V[0]$, for all $m \geq m_\alpha$. \square

Note that by the procedure of the above proof, $m =: \min\{m_\alpha | \alpha \in \Delta\}$ is a better bound. Let us now give an explicit such lower bound.

Lemma 4.7. *If $\omega \in \mathcal{W}$ then*

$$\omega \cdot \rho = \rho - \sum_{\substack{\alpha \in \Phi^+ \\ \omega^{-1}(\alpha) \in \Phi^-}} \alpha.$$

In particular, $s_\alpha(\rho) = \rho - \alpha$ for $\alpha \in \Delta$.

Proof. Evidently, $\omega \cdot \rho$ is half sum of the set $\{\omega(\alpha) | \alpha \in \Phi^+\}$. Like Φ^+ , this is a set of exactly half of the roots, containing each root or its negative but not both. More precisely, this set is obtained from Φ^+ by replacing each $\alpha \in \Phi^+$ such that $\omega^{-1} \cdot \alpha \in \Phi^-$ by its negative. Now,

$$\omega \cdot \rho = \rho - \sum_{\substack{\alpha \in \Phi^+ \\ \omega^{-1}(\alpha) \in \Phi^-}} \alpha$$

is evident, and $s_\alpha(\rho) = \rho - \alpha$ is a special case since one shows that if $\alpha \in \Delta$ and $\beta \in \Phi^+$, then either $\beta = \alpha$ or $s_\alpha(\beta) \in \Phi^+$. \square

Proposition 4.8. *Let V be an irreducible representation of G . Then $\nu_m(V) = \dim V[0]$ for all $m \geq M := \min_{\alpha \in \Delta} \left\{ \sum_{\beta \in \Phi^+} |\beta(h_\alpha)| + 1 \right\}$.*

Proof. Let $h = h_\alpha$ be a simple coroot. For all $1 \neq \omega \in \mathcal{W}$ we have,

$$|(\rho - \omega \cdot \rho)(h)| = \left| \sum_{\substack{\beta \in \Phi^+, \\ \omega^{-1}(\beta) \in \Phi^-}} \beta(h) \right| \leq \sum_{\substack{\beta \in \Phi^+, \\ \omega^{-1}(\beta) \in \Phi^-}} |\beta(h)| \leq \sum_{\beta \in \Phi^+} |\beta(h)|.$$

Therefore if we choose $m = \sum_{\beta \in \Phi^+} |\beta(h)| + 1$ then $\frac{\rho - \omega \cdot \rho}{m}(h) \notin \mathbb{Z}$, namely, $\frac{\rho - \omega \cdot \rho}{m}$ is not a weight. Consequently, $\sum_{\sigma \neq 1} sn(\sigma) \dim V \left[\frac{\rho - \sigma \cdot \rho}{m} \right] = 0$, and we are done. \square

Let us calculate the bound M defined in Proposition 4.8 for $\mathfrak{sl}(n, \mathbb{C})$. Let the Cartan subalgebra be the set of diagonal matrices in $\mathfrak{sl}(n, \mathbb{C})$. Let the set of positive roots be $\Phi^+ = \{\beta_{i,j} | 1 \leq i < j \leq n\}$, where $\beta_{i,j}(\text{diag}(a_1, \dots, a_n)) = a_i - a_j$. The subset $\Delta = \{\beta_{i,i+1} | 1 \leq i \leq n-1\}$ is a base. With respect to this base the simple coroots are $\{h_i | 1 \leq i \leq n-1\}$, where h_i is the matrix with 1 in the (i, i) position, -1 in the $(i+1, i+1)$ position and 0 elsewhere. Then, by an elementary calculation, we get that for any simple coroot h ,

$$\sum_{1 \leq i < j \leq n} |\beta_{i,j}(h)| + 1 = 2n - 1.$$

Consequently we obtain that $M = 2n - 1$.

Let us calculate the bound M defined in Proposition 4.8 for $\mathfrak{so}(2n+1, \mathbb{C})$. Let the Cartan subalgebra be the set of diagonal matrices in $\mathfrak{so}(2n+1, \mathbb{C})$. Let the set of positive roots be $\Phi^+ = \{\beta_i \pm \beta_j | 1 \leq i < j \leq n\} \cup \{\beta_i | 1 \leq i \leq n\}$, where $\beta_i(h_j) = \delta_{ij}$. The subset $\Delta = \{\beta_i - \beta_{i+1}, \beta_n | 1 \leq i \leq n-1\}$ is a base. With respect to this base the simple coroots are $\{h_i - h_{i+1}, 2h_n | 1 \leq i \leq n-1\}$, where h_i is the matrix with 1 in the (i, i) position, -1 in the $(n+i, n+i)$ position and 0

elsewhere. Then, by an elementary calculation, we get that for any simple coroot $h := h_k - h_{k+1}$, $1 \leq k \leq n-1$, the sum $\sum_{\beta \in \Phi^+} |\beta(h)| + 1$ equals

$$\sum_{1 \leq i < j \leq n} |(\beta_i + \beta_j)(h)| + \sum_{1 \leq i < j \leq n} |(\beta_i - \beta_j)(h)| + \sum_{i=1}^n |\beta_i(h)| + 1 = 4n - 3,$$

while for the simple coroot $h := 2h_n$ it equals $4n - 1$. Consequently we obtain that $M = 4n - 3$.

Applying similar arguments to the other classical simple Lie algebras yields the following result.

Proposition 4.9. *The bound M for $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(2n, \mathbb{C})$, $\mathfrak{so}(2n+1, \mathbb{C})$ and $\mathfrak{sp}(2n, \mathbb{C})$ is equal to $2n - 1$, $4n - 5$, $4n - 3$ and $2n + 1$, respectively.*

4.3. The proof of Theorem 1.3. Let \mathfrak{h} be the CSA of $sl(3, \mathbb{C})$ generated by the two elements $h_1 = \text{diag}(1, -1, 0)$ and $h_2 = \text{diag}(0, 1, -1)$. We will identify any functional α on \mathfrak{h} with the pair $(\alpha(h_1), \alpha(h_2))$. Under this identification the six roots of $sl(3, \mathbb{C})$ are $\alpha_1 = (2, -1)$, $\alpha_2 = (-1, 2)$, $\alpha_1 + \alpha_2 = (1, 1)$, $-\alpha_1 = (-2, 1)$, $-\alpha_2 = (1, -2)$ and $-\alpha_1 - \alpha_2 = (-1, -1)$. The roots $\alpha_1 = (2, -1)$, $\alpha_2 = (-1, 2)$ form a base and the corresponding simple coroots are h_1, h_2 , respectively.

Recall that if $V = V(\lambda)$ is an irreducible representation of $sl(3, \mathbb{C})$ of highest weight λ , then λ is of the form (a, b) with a and b non-negative integers.

Recall that $\mathcal{W} \cong S_3$ and it acts on \mathfrak{h} by $\sigma \cdot \text{diag}(d_1, d_2, d_3) = \text{diag}(d_{\sigma(1)}, d_{\sigma(2)}, d_{\sigma(3)})$. Therefore, $(12) \cdot \alpha_1 = -\alpha_1$, $(12) \cdot \alpha_2 = \alpha_1 + \alpha_2$; $(13) \cdot \alpha_1 = -\alpha_2$, $(13) \cdot \alpha_2 = -\alpha_1$; $(23) \cdot \alpha_1 = \alpha_1 + \alpha_2$, $(23) \cdot \alpha_2 = -\alpha_2$; $(123) \cdot \alpha_1 = -\alpha_1 - \alpha_2$, $(123) \cdot \alpha_2 = \alpha_1$; and $(132) \cdot \alpha_1 = \alpha_2$, $(132) \cdot \alpha_2 = -\alpha_1 - \alpha_2$.

The half sum of positive roots is $\rho = \frac{1}{2}(2\alpha_1 + 2\alpha_2) = \alpha_1 + \alpha_2$. We have, $\rho - (12)\rho = (2, -1)$, $\rho - (13)\rho = (2, 2)$, $\rho - (23)\rho = (-1, 2)$, $\rho - (123)\rho = (0, 3)$, and $\rho - (132)\rho = (3, 0)$.

Let $m = 2$. Considering our formula, we cancel all the summands which include roots that one of their two components is not divisible by 2. Consequently we get

$$\nu_2(V) = \dim V[(0, 0)] - \dim V[(1, 1)].$$

Recall that an irreducible representation $V(a, b)$ is self dual if and only if $a = b$. Since $\lambda = (s, s) = s\alpha_1 + s\alpha_2$, $(\lambda, 2\rho) = (\lambda, 2h_1 + 2h_2) = (\lambda, 2h_1) + (\lambda, 2h_2) = 4s$, it follows from Tits' theorem that

$$\nu_2(V(a, b)) = \begin{cases} 0 & a \neq b, \\ 1 & \text{otherwise.} \end{cases}$$

Similar considerations for $m \geq 3$ yield,

$$\nu_3(V) = \dim V[(0, 0)] + \dim V[(1, 0)] + \dim V[(0, 1)]$$

and

$$\nu_{m \geq 4}(V) = \dim V[(0, 0)].$$

In particular, if λ does not belong to the root lattice, $\nu_{m \geq 4}(V) = 0$.

We now calculate $\dim V[(0, 0)]$, $\dim V[(1, 0)]$ and $\dim V[(0, 1)]$. Recall that for $\eta \in \mathfrak{h}^*$, $p(\eta) \geq 1$ if and only if η belongs to the root lattice and $\eta \succ 0$. If $\eta = k\alpha_1 + l\alpha_2$ with nonnegative integers k and l , then $p(\eta) = 1 + \min\{k, l\}$. Write $\lambda = k\alpha_1 + l\alpha_2$ where k and l are real numbers and identify it with the pair $(\lambda(h_1), \lambda(h_2)) = (a, b) = (2k - l, 2l - k)$.

Note that $(0, 1) = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$ and $(1, 0) = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$. Therefore by Kostant's formula (see Theorem 2.1),

$$\dim V[(0, 0)] = \sum_{\omega \in \mathcal{W}} sn(\omega) p((k+1)\omega \cdot \alpha_1 + (l+1)\omega \cdot \alpha_2 - \alpha_1 - \alpha_2),$$

$$\dim V[(0, 1)] = \sum_{\omega \in \mathcal{W}} sn(\omega) p\left((k+1)\omega \cdot \alpha_1 + (l+1)\omega \cdot \alpha_2 - \frac{4}{3}\alpha_1 - \frac{5}{3}\alpha_2\right),$$

and

$$\dim V[(1, 0)] = \sum_{\omega \in \mathcal{W}} sn(\omega) p\left((k+1)\omega \cdot \alpha_1 + (l+1)\omega \cdot \alpha_2 - \frac{5}{3}\alpha_1 - \frac{4}{3}\alpha_2\right).$$

It is straightforward to verify that in each of the three cases the surviving terms correspond to $\omega = 1, (12), (23)$. For example, in the first case calculating $(k+1)\omega \cdot \alpha_1 + (l+1)\omega \cdot \alpha_2 - \alpha_1 - \alpha_2$ for $\omega = (12), (13), (23), (123), (132)$, yields $(l-k-1)\alpha_1 + l\alpha_2$, $-(l+2)\alpha_1 - (k+2)\alpha_2$ (hence $p = 0$), $k\alpha_1 + (k-l-1)\alpha_2$, $(l-k-1)\alpha_1 - (k+2)\alpha_2$ (hence $p = 0$), and $-(l+2)\alpha_1 + (k-l-1)\alpha_2$ (hence $p = 0$), respectively.

Therefore we have that $\dim V[(0, 0)]$ equals

$$\begin{aligned} & p\left(\left(\frac{b+2a}{3}\right)\alpha_1 + \left(\frac{2b+a}{3}\right)\alpha_2\right) - p\left(\left(\frac{b-a-3}{3}\right)\alpha_1 + \left(\frac{2b+a}{3}\right)\alpha_2\right) \\ & - p\left(\left(\frac{b+2a}{3}\right)\alpha_1 + \left(\frac{-b+a-3}{3}\right)\alpha_2\right), \end{aligned}$$

$\dim V[(0, 1)]$ equals

$$\begin{aligned} & p\left(\left(\frac{b+2a-1}{3}\right)\alpha_1 + \left(\frac{2b+a-2}{3}\right)\alpha_2\right) - p\left(\left(\frac{b-a-4}{3}\right)\alpha_1 + \left(\frac{2b+a-2}{3}\right)\alpha_2\right) \\ & - p\left(\left(\frac{b+2a-1}{3}\right)\alpha_1 + \left(\frac{-b+a-5}{3}\right)\alpha_2\right), \end{aligned}$$

and $\dim V[(1, 0)]$ equals

$$\begin{aligned} & p\left(\left(\frac{b+2a-2}{3}\right)\alpha_1 + \left(\frac{2b+a-1}{3}\right)\alpha_2\right) - p\left(\left(\frac{b-a-5}{3}\right)\alpha_1 + \left(\frac{2b+a-1}{3}\right)\alpha_2\right) \\ & - p\left(\left(\frac{b+2a-2}{3}\right)\alpha_1 + \left(\frac{-b+a-4}{3}\right)\alpha_2\right). \end{aligned}$$

Now, modulo 3, exactly one of the following holds: 1) $b+2a = 0$ and $2b+a = 0$ (in this case λ belongs to the root lattice), 2) $b+2a = 1$ and $2b+a = 2$ and 3) $b+2a = 2$ and $2b+a = 1$. Hence by the above and elementary calculations, we obtain that in the first case $\dim V[(0, 1)] = \dim V[(1, 0)] = 0$, in the second case $\dim V[(0, 0)] = \dim V[(1, 0)] = 0$ and in the third case $\dim V[(0, 0)] = \dim V[(0, 1)] = 0$. Therefore, in the first case $\nu_3(V(a, b))$ equals

$$\begin{aligned} & \max\left\{1 + \min\left\{\frac{b+2a}{3}, \frac{2b+a}{3}\right\}, 0\right\} - \max\left\{1 + \min\left\{\frac{b-a-3}{3}, \frac{2b+a}{3}\right\}, 0\right\} \\ & - \max\left\{1 + \min\left\{\frac{b+2a}{3}, \frac{-b+a-3}{3}\right\}, 0\right\}, \end{aligned}$$

in the second case it equals

$$\max \left\{ 1 + \min \left\{ \frac{b+2a-1}{3}, \frac{2b+a-2}{3} \right\}, 0 \right\} - \max \left\{ 1 + \min \left\{ \frac{b-a-4}{3}, \frac{2b+a-2}{3} \right\}, 0 \right\} \\ - \max \left\{ 1 + \min \left\{ \frac{b+2a-1}{3}, \frac{-b+a-5}{3} \right\}, 0 \right\},$$

and in the third case it equals

$$\max \left\{ 1 + \min \left\{ \frac{b+2a-2}{3}, \frac{2b+a-1}{3} \right\}, 0 \right\} - \max \left\{ 1 + \min \left\{ \frac{b-a-5}{3}, \frac{2b+a-1}{3} \right\}, 0 \right\} \\ - \max \left\{ 1 + \min \left\{ \frac{b+2a-2}{3}, \frac{-b+a-4}{3} \right\}, 0 \right\}.$$

Finally, it is easy to check that in each case the sum equals $1 + \min\{a, b\}$, as claimed. This completes the proof of the theorem. \square

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