# In quest of a generalized Callias index theorem

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**Abstract**

We give a prescription for how to compute the Callias index, using as regulator an exponential function. We find agreement with old results in all odd dimensions. We show that the problem of computing the dimension of the moduli space of self-dual strings can be formulated as an index problem in even-dimensional (loop-)space. We think that the regulator used in this Letter can be applied to this index problems.

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## 1 Introduction

We do not know what six-dimensional $(2,0)$ theory really is. It is believed that it can sustain solitonic self-dual strings [1], although no one today knows what a (non-Abelian) self-dual string really is. But if we break the gauge group maximally to $U(1)^r$, then we should be able to define the charges of these mysterious self-dual strings by the asymptotic behaviour of the $U(1)$ gauge fields. One should expect these asymptotic $U(1)$ fields to be (at least isomorphic with) a copy of the familiar abelian two-form gauge potentials (with self-dual field strengths).

It now seems to make sense to ask a question like, what is the dimension of the moduli space of self-dual strings of a given charge?

If the gauge group is $SU(2)$ and is broken to $U(1)$ by the Higgs vacuum expectation value (that should also determine the tension of the string), then the intuitive answer to this question is $4N$ where $N$ is the $U(1)$ charge in a suitable normalization, such that $N = 1$ corresponds to one self-dual string. One may argue that half the supersymmetry is broken by the string. Therefore one string should sustains 4 fermionic zero modes. Since some (half) of the supersymmety is unbroken there should also be 4 corresponding bosonic zero modes. These are naturally identified with the translational zero modes associated with the four transverse directions to the string. Furthermore, the strings being BPS, should be possible to separate at no cost of energy (thus staying in the moduli space approximation). If we take them far from each other, one may suspect that we can just add 4 bosonic zero modes from each string, to get $4N$ bosonic zero modes in total in a configuration of $N$ strings [2].

It would of course be nice to have a proof of this conjecture. Could it be proven if one had some index theorem? We will not provide a full solution to this problem in this Letter. But we will make it plausible that the problem can indeed be solved by computing the index of a certain Dirac operator in loop space.

To address our index problem, we think that one can lend the methods that Callias [3] used to prove his index theorem in odd-dimensional spaces. In our case we have an even number of dimensions (namely the four transverse direction) so it is apparent that we would have to construct a new type of index. This we do in section 3.

In section 2 we recall the Callias method [3] to address index problems in open spaces, though we will modify Callias' regularization, using the more convergent exponential function to obtain the index, as the limit

$$\lim_{s\to\infty} \operatorname{Tr}\left(\gamma e^{-sD^2}\right), \tag{1}$$

(here $D^2 > 0$ and $\gamma =$diag $(1,-1)$) rather than

$$\lim_{M\to 0} \operatorname{Tr}\left(\gamma \frac{M^2}{D^2+M^2}\right), \tag{2}$$

which is the regularization that Callias used. We think that using the more convergent regularization of an exponential function is interesting in itself, as it could possibly extend the Callias index theorem to a wider class of index problems. Therefore we will devote the first part of this Letter on this subject.

But let us at once say that our regulator probably has no advantages when attacking these old problems. It does not provide us with a solution for how to count the number of zero modes in a multimonopole configuration with a non-maximally broken gauge group, where the index can not be reliable computed due to a contribution from the continuum portion of the spectrum. What we hope though, is that our regulatization can be useful when attacking our new index problem associated with the moduli space of self-dual strings.

In section 2 we obtain the index in one and three dimensions. In three dimensions we apply this on the multimonopole moduli space and re-derive the result in [4]. A recent review article on monopoles and supersymmetry is [5]. The one and three-dimensional index problems have also been studied in [6]. We then indicate how our method manages to reproduce the correct results in any odd dimensions. In section 3 we show how one at least in principle should be able to compute the dimension of the moduli space of $N$ self-dual strings by computing a certain index.

# 2 Computing the Callias index in odd-dimensional spaces

For Dirac operators on open $n-1$-dimensional space where $n-1$ is odd, there is an index theorem by Callias [3]. This applies to Dirac equations of the form

$$D\psi = 0 \tag{3}$$

where the Dirac operator $D$ is of the form

$$D = \gamma_i i D_i + \gamma_n \phi. \tag{4}$$

Here $i = 1, ..., n-1$ and $\gamma_\mu \equiv (\gamma_i, \gamma_n)$ denote the Dirac gamma matrices,

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \tag{5}$$

We define the gauge covariant derivative as $iD_{is} = i\partial_{is} + A_{is}$ and all our fields are hermitian. If $n-1$ is odd, the gamma matrices can be represented as

$$\begin{aligned} \gamma_i &= \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \\ \gamma_n &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \end{aligned} \tag{6}$$

One may use the $n$-dimesional notation $A_\mu = (A_i, \phi)$, $D = \gamma_\mu i D_\mu$, but one must then remember that space is really $n-1$ dimensional.

If $n-1$ is even there is no Weyl representation of the gamma matrices (because of the inclusion of the 'gamma-five'), and no index theorem of this form exists.

We define the 'gamma-five' for even $n$ as

$$\gamma \equiv -i^{-\frac{n}{2}} \gamma_{1 \cdots n} \tag{7}$$

which then is hermitian, and we define the projectors

$$P_\pm = \frac{1}{2}\left(1 \mp \gamma\right). \tag{8}$$

In odd dimensions $n-1$, the Dirac operator splits into two Weyl operators

$$\begin{aligned} \mathcal{D} &\equiv P_+ D P_- \\ \mathcal{D}^\dagger &\equiv P_- D P_+ \end{aligned} \tag{9}$$

Because $P_\pm$ and $D$ are all hermitian, it follows that $\mathcal{D}^\dagger$ is the hermitian conjugate of $\mathcal{D}$. Also, because $D$ is already of an off-block diagonal form, it suffices to include just one of the projectors, so we can just as well write this as

$$\begin{aligned} \mathcal{D} = P_+ D = D P_- \\ \mathcal{D}^\dagger = P_- D = D P_+ \end{aligned} \tag{10}$$

The index can now be defined as

$$\dim\ker\mathcal{D} - \dim\ker\mathcal{D}^\dagger \tag{11}$$

Since $\ker\mathcal{D} = \ker\left(\mathcal{D}^\dagger\mathcal{D}\right)$ and $\ker\mathcal{D}^\dagger = \ker\left(\mathcal{D}\mathcal{D}^\dagger\right)$ we can express this as[2]

$$\dim\ker\left(\mathcal{D}^\dagger\mathcal{D}\right) - \dim\ker\left(\mathcal{D}\mathcal{D}^\dagger\right) = \dim\ker\left(\gamma D^2\right). \tag{12}$$

where we have noted that $\gamma = P_- - P_+$.

Callias, Weinberg and others used the regulator

$$I(M^2) = \mathrm{Tr}\left(\gamma \frac{M^2}{D^2 + M^2}\right) \tag{13}$$

to obtain the index as the limit $M^2 \to 0$. In this Letter we will be slightly more general. We define

$$J_i(x,y) \equiv \mathrm{tr}\left\langle x\left|\gamma\gamma_i f(D)\right|y\right\rangle, \tag{14}$$

for any function $f$ (and of course $D$ is not dimensionless, so $D$ has to be accompanied by $M$ in a suitable way). Then we notice that

$$\begin{aligned} W(x,y) &\equiv \left(i\gamma_i\partial_{x^i} + \gamma_\mu A_\mu(x) + M\right)\left\langle x\left|f\left(D\right)\right|y\right\rangle \\ &= \left\langle x\left|f\left(D\right)\right|y\right\rangle\left(-i\gamma_i\partial_{y^i} + \gamma_\mu A_\mu(y) + M\right) \end{aligned} \tag{15}$$

where (manifestly)

$$W(x,y) = \left\langle x\left|(D+M) f\left(D\right)\right|y\right\rangle. \tag{16}$$

From this, we obtain the following identity

$$\begin{aligned} i\left(\partial_{x^i} + \partial_{y^i}\right) J_i(x,y) &= 2\mathrm{tr}\left\langle x\left|\gamma D f(D)\right|y\right\rangle \\ &\quad + \mathrm{tr}\left(A_\mu(y) - A_\mu(x)\right)\left\langle x\left|\gamma\gamma_\mu f(D)\right|y\right\rangle \end{aligned} \tag{17}$$

In odd dimensions, the second term in the right hand side vanishes as $x$ approaches $y$. This can be seen as being equivalent to the statement that there is no chiral anomaly in odd dimensions (by using point-splitting and inserting a Wilson line). So we get

$$i\partial_i J_i(x,x) = 2\mathrm{tr}\left\langle x\left|\gamma D f(D)\right|x\right\rangle \tag{18}$$

[2] To see this that $\ker\mathcal{D} = \ker\mathcal{D}^\dagger\mathcal{D}$ we apply the definition of hermitian conjugate with respect to the inner product $(\psi,\chi) = \int dx \psi^\dagger\chi$ and the property of the norm, to $0 = (\psi, \mathcal{D}^\dagger\mathcal{D}\psi) = (\mathcal{D}\psi, \mathcal{D}\psi)$.

If we wish to compute the index as in Eq (13), then we can take

$$f(D) = \frac{1}{D}\frac{M^2}{D^2+M^2} \tag{19}$$

(however there is no unique choice of $J_i$). We then get

$$\begin{aligned} J_i(x,y) &= \mathrm{tr}\left\langle x \left| \gamma\gamma_i \frac{1}{D}\frac{M^2}{D^2+M^2} \right| y \right\rangle \\ &= \mathrm{tr}\left\langle x \left| \gamma\gamma_i \frac{1}{D}\left(-D^2+D^2+M^2\right)\frac{1}{D^2+M^2} \right| y \right\rangle \\ &= -\mathrm{tr}\left\langle x \left| \gamma\gamma_i D \frac{1}{D^2+M^2} \right| y \right\rangle . \end{aligned} \tag{20}$$

The virtue of expressing Eq (13) as a total divergence, is that we then can compute the index as a boundary integral over an $(n-2)$-sphere at infinity as

$$I(M^2) = \frac{i}{2}\int_{S_\infty^{n-2}} d\Omega_{n-2} r^{n-2}\hat{x}_i J_i(x,x). \tag{21}$$

where $r$ is the radius of the sphere and $d\Omega_{n-2}$ denotes the volume element of the unit sphere.

If instead we wish to compute the index as the limit of

$$I(s) = \mathrm{Tr}\left(\gamma e^{-sD^2}\right). \tag{22}$$

as $s \to \infty$, then we get

$$J_i(x,y) = \mathrm{tr}\left\langle x \left| \gamma\gamma_i \frac{1}{D} e^{-sD^2} \right| y \right\rangle . \tag{23}$$

We will now illustrate how one can use this $J_i$ to compute the index in odd dimensions.

## One dimension

We choose our gamma matrices as

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{24}$$

and we have

$$\gamma = i\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{25}$$

The Dirac operator reads

$$D = i\gamma_1\partial + \gamma_2\phi \tag{26}$$

We need the square of the Dirac operator,

$$D^2 = -\partial^2 + \phi^2 + \gamma\partial\phi. \tag{27}$$

We make the choice

$$J_1(x,y) = -\mathrm{tr}\left\langle x \left| \gamma\gamma_1 D \frac{1}{D^2+M^2} \right| y \right\rangle \tag{28}$$

We assume that $\phi(x)$ converges towards some constant values at $x = -\infty$ and $x = +\infty$. That means that we may ignore $\partial\phi(x)$ for sufficiently large $|x|$, where we then get

$$J_1(x,x) = \mathrm{tr}\left(\gamma\gamma_1\gamma_2\right) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\phi}{k^2+\phi^2+M^2} = -i\frac{\phi}{\sqrt{\phi^2+M^2}} \tag{29}$$

The index is now given by

$$\lim_{M\to 0} \frac{i}{2}\left(J_1(+\infty) - J_1(-\infty)\right) = \pm 1 \tag{30}$$

if $\phi$ flips the sign an odd number of times when going from $-\infty$ to $+\infty$, and 0 otherwise.

If instead we choose

$$J(x,y) \quad = \quad \mathrm{tr}\left\langle x \left| \gamma\gamma_1 D \frac{1}{D^2} e^{-sD^2} \right| y \right\rangle \tag{31}$$

then we get

$$J(x,x) = \mathrm{tr}\left(\gamma\gamma_1\gamma_2\right) \int \frac{dk}{2\pi} \frac{\phi}{k^2+\phi^2} e^{-s(k^2+\phi^2)} \tag{32}$$

If we compute the integral over $k$ in the most natural way, then we get a result that vanishes in the limit $s \to \infty$. Could there be another way of defining this integral, such that we do not get zero as the result? We notice that the integral

$$A(s) \equiv \int dk \frac{e^{-s(k^2+1)}}{k^2+1} \tag{33}$$

for $s > 0$ is convergent only if we integrate $k$ along a line in the complex plane which is such that it asymptotically is such that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ where $k = |k|e^{i\theta}$. Integrating along any such line in the complex plane, we get the same value of this integral. If on the other hand we integrate over a line that asymptotically lies outside this cone, then we get a divergent integral for $s > 0$. But we get a convergent integral for $s < 0$. We then define the value of the integral for $s > 0$ as the analytic continuation of the same integral for $s < 0$. It remains to compute this convergent integral. Replacing $k$ by $ik$ and $s$ by $-s$, we get the integral

$$A(-s) = i\int_{-\infty}^{\infty} dk \frac{e^{-s(k^2-1)}}{k^2-1} \tag{34}$$

We can compute its derivative

$$A'(-s) = i\int_{-\infty}^{\infty} dk e^{-s(k^2-1)} = i\sqrt{\frac{\pi}{s}}e^{s} \tag{35}$$

The right-hand side can obviously be analytically continued to $-s$, and that is how we will define $A(s)$ where the integral representation does not converge. We can then integrate up $A'(s)$,

$$A(\infty) = A(0) + \int_o^\infty ds \sqrt{\frac{\pi}{s}} e^{-s} = A(0) + \sqrt{\pi}\Gamma\left(\frac{1}{2}\right) = A(0) + \pi \tag{36}$$

and we then need to compute

$$A(0) \quad = \quad i \int_{-\infty}^{\infty} dk \frac{1}{k^2 - 1} \tag{37}$$

We define this as the principal value. This is ad hoc – we have no argument why one should define it like this. But if we accept this, then we get $A(0) = 0$. We conclude that we could just as well define the integral that we had, as

$$\lim_{s\to\infty} \int dk \frac{e^{-s(k^2+1)}}{k^2+1} = \pi. \tag{38}$$

But this requires us to perform the integration of $k$ in the cone where it diverges for $s > 0$, and then define this integral by analytic continuation. This seem to be rather ad hoc. We have three rather week arguments why one should Wick rotate. First, if we keep $x - y$ as a small number, then we get the factor $e^{ik(x-y)}$ and this can act as a convergence factor only if we Wick rotate. (We illustrate this in the Appendix where we compute the corresponding integral in any complex number of dimensions.) Second, it seems to be the only way that we could produce a non-trivial answer. Third, with this prescription we will manage to reproduce the right answer in any odd number of dimensions, where we can check our result against the safer regularization used by Callias.

If we compute the integral by this prescription, then we get

$$J(x,x) = \mathrm{tr}\,(\gamma\gamma_1\gamma_2) \lim_{s\to\infty} \int \frac{dk}{2\pi} \frac{\phi}{k^2+\phi^2} e^{-s(k^2+\phi^2)} = -i\frac{\phi}{\sqrt{\phi^2}} \tag{39}$$

and we see that we indeed get the right answer.

## Three dimensions and magnetic monopoles

The physics problem that we will consider in three dimensions, is to compute number of zero modes of the Bogomolnyi equation

$$F_{ij} = \epsilon_{ijk} D_k \phi \tag{40}$$

We choose the convention that our fields are hermitian. It is convenient to group the fields into 'gauge potential'

$$A_\mu = (A_i, \phi) \tag{41}$$

We define $D_\mu = (D_i, \phi)$ such that $iD_\mu = i\partial_\mu + A_\mu$ and we let $G_{\mu\nu} = i[D_\mu, D_\nu]$ be the associated 'field strength'. Then the Bogomolnyi equation reads

$$G_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma}. \tag{42}$$

Linearizing this, we get

$$D_\mu \delta A_\nu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} D_\rho \delta A_\sigma \tag{43}$$

Contracting with $\gamma_{\mu\nu}$, we get

$$(1+\gamma)\gamma_{\mu\nu} D_\mu \delta A_\nu = 0 \tag{44}$$

and if we impose the background gauge condition

$$D_\mu \delta A_\mu = 0 \tag{45}$$

which is to say that zero modes are orthogonal to gauge variations with respect to the moduli space metric, then we can write this linearized equation as a Dirac equation

$$D\psi \equiv \gamma_\mu D_\mu \psi = 0 \tag{46}$$

where

$$\psi := (1+\gamma)\gamma_\mu \delta A_\mu . \tag{47}$$

We compute

$$D^2 = -D_i^2 + \phi^2 + \frac{1}{2} i \gamma_{\mu\nu} G_{\mu\nu} \tag{48}$$

Inserting the Bogomolnyi configuration we can write this, thus using the fact that $G_{\mu\nu}$ is selfdual,

$$D^2 = -D_i^2 + \phi^2 + \frac{1}{4}(1+\gamma) i \gamma_{\mu\nu} G_{\mu\nu} . \tag{49}$$

and get a vanishing theorem. Namely, $\dim\ker \mathcal{D}\mathcal{D}^\dagger = 0$ as $\mathcal{D}\mathcal{D}^\dagger > 0$ is strictly postive. Hence we can compute the dimension of the moduli space $\dim\ker\mathcal{D} \equiv \dim\ker\mathcal{D}^\dagger\mathcal{D}$ just by computing the index of $\mathcal{D}$. To compute the index, we now wish to compute

$$J_i(x,x) = \mathrm{tr}\left\langle x \left| \gamma\gamma_i\gamma_k D_k \frac{1}{D^2} e^{-sD^2} \right| x \right\rangle \tag{50}$$

We assume that asymptotically $\phi$ approaches a constant value at infinity. This corresponds to a gauge choice where we have a Dirac string singularity. Some further examination reveals that we get a non-negligible contribution to $J_i$, for a sufficiently large two-sphere, only from the term

$$J_i(x,x) = \mathrm{tr}\left(\gamma\gamma_i\gamma_4\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2+\phi^2+\frac{1}{2}i\gamma_{\mu\nu}G_{\mu\nu}} e^{-s\left(k^2+\phi^2+\frac{1}{2}i\gamma_{\mu\nu}G_{\mu\nu}\right)}\right) \tag{51}$$

We thus need to perform an integral of the form

$$A(s) = \int dk \frac{k^2}{k^2+1} e^{-s(k^2+1)} \tag{52}$$

If we choose the same prescription as we did in one dimension, then we get the result

$$A(+\infty) = -\pi. \tag{53}$$

For details of such a computation we refer to appendix $A$.

If we apply this result to the integral that we had, we get

$$J_i(x,x) = -\frac{1}{2\pi}\mathrm{tr}\left(\gamma\gamma_i\gamma_4\phi\sqrt{\phi^2 + \frac{1}{2}i\gamma_{\mu\nu}G_{\mu\nu}}\right) \tag{54}$$

We expand the square root,

$$\sqrt{\phi^2 + \frac{1}{2}i\gamma_{\mu\nu}G_{\mu\nu}} = \phi + \frac{1}{4\sqrt{\phi^2}}i\gamma_{\mu\nu}G_{\mu\nu} + ... \tag{55}$$

In the far distance, in a charge $Q$ monopole configuration, we find that

$$\gamma_{\mu\nu}G_{\mu\nu} = 2\gamma_i\gamma_4(1-\gamma)\frac{\hat{x}_k}{r^2}Q \tag{56}$$

and so when we trace over the gamma matrices, we get

$$J_i(x,x) = -\frac{i}{2\pi r^2}\epsilon_{ijk}\mathrm{tr}\left(\frac{\phi Q}{\sqrt{\phi^2}}\right). \tag{57}$$

If we now for instance assume $SU(2)$ gauge group, broken to $U(1)$, then if we integrate $\frac{i}{2}J_i$ over $S^2$, we get the index $2Q$. The number of bosonic zero modes is twice the index, i.e. $4Q$ [4, 5].

## $(2m+1)$ dimensions

In $2m+1$ dimensions we get the integral

$$A(\mu) \equiv \lim_{s\to\infty}\int dk\frac{k^{2m}}{k^2+\mu^2}e^{-s(k^2+\mu^2)} \tag{58}$$

if we use our regulator. Here

$$\mu^2 \equiv v^2 + G \tag{59}$$

(and $G$ is an abbreviation for $\frac{1}{2}i\gamma_{\mu\nu}G_{\mu\nu}$.) This should be compared to the integral

$$B(\mu) \equiv \lim_{M\to 0}(-1)^m\int dk\frac{k^{2m}}{(k^2+v^2+M^2)^{m+1}}G^m \tag{60}$$

that we get using the Callias regulator. [3] In order to compare these integrals, we rewrite them as

$$A(\mu) \quad = \quad \mu^{2m-1}a$$

[3] This integral comes from expanding

$$\frac{1}{k^2+v^2+G+M^2} = \frac{1}{k^2+v^2+M^2} + ... \tag{61}$$

in powers of $G$ as a geometric series [4].

$$B(\mu) = v^{-1} b G^m \tag{62}$$

where

$$\begin{aligned} a &= \lim_{\tilde{s}\to\infty} \int d\xi \frac{\xi^{2m}}{\xi^2+1} e^{-\tilde{s}\left(\xi^2+1\right)} \\ b &= \lim_{\tilde{M}\to 0} (-1)^m \int d\xi \frac{\xi^{2m}}{\left(\xi^2+1+\tilde{M}^2\right)^{m+1}} \end{aligned} \tag{63}$$

We compute $a$ according the prescription introduced above in one and three dimensions, that is by Wick rotating $\xi$ and continue analytically in $s$. (Details are in appendix $A$.) We can compute $b$ using residue calculus (introducing a regulator so that we can close the contour on a semi-circle at infinity). The result is

$$\begin{aligned} a &= (-1)^m \pi \\ b &= -(-1)^m \frac{1}{2} \pi \frac{\Gamma\left(m-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \end{aligned} \tag{64}$$

We next expand

$$\begin{aligned} vA(\mu) &= v\left(v^2+G\right)^{m-\frac{1}{2}} a \\ &= v^{2m} a + ... + \frac{\Gamma\left(m-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)} a G^m + ... \\ vB(\mu) &= b G^m \end{aligned} \tag{65}$$

and we find that the coefficient of $G^m$ becomes equal to

$$(-1)^m \frac{\Gamma\left(m-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)} \pi \tag{66}$$

if one uses our regularization, and equal to

$$-(-1)^m \frac{1}{2} \frac{\Gamma\left(m-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \pi \tag{67}$$

if one uses the Callias regularization. We see that the two expressions coincide for all $m$.

We have now showed that if we use our prescription of Wick rotating $k$ to compute the integrals over the exponential, then we get the right answer for all cases that can be safely computed using a regulator that is less convergent. We are inclined to think that our prescription for how to compute the integral, will also work for index problems where the Callias regulator diverges. But we have no proof. It is perhaps not so obvious that more general index problems can be formulated. In the next section we will give one example of a more general type of index problem.

# 3 Four dimensions and self-dual strings

To introduce the notation, we first consider the free Abelian tensor multiplet theory in $1+5$ dimensions. The on-shell field content is a two-form gauge potential $B_{\mu\nu}$, five scalar fields $\phi^A$ and corresponding Weyl fermions $\psi$. The field strength $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}$ is selfdual. The supersymmetry variation of the Weyl fermions is

$$\delta\psi = \left(\frac{1}{12}\Gamma^{\mu\nu\rho}H_{\mu\nu\rho} + \Gamma^\mu\Gamma_A\partial_\mu\phi^A\right)\epsilon \tag{68}$$

where we use eleven-dimensional gamma matrices splitted into $SO(1,5)\times SO(5)$, so that in particular

$$\{\Gamma^\mu, \Gamma_A\} \quad = \quad 0. \tag{69}$$

In a static and $x^5$ independent field configuration, in which only $\phi^5 =: \phi$ is non-zero, we find the SUSY variation

$$\delta\psi = \left(\Gamma^{0i5}H_{0i5} + \Gamma^i\Gamma_{A=5}\partial_i\phi\right)\epsilon \tag{70}$$

If we assume that the classical bosonic field configuration is such that

$$\partial_i\phi = H_{0i5} \tag{71}$$

then the SUSY variation reduces to

$$\delta\psi = \partial_i\phi\Gamma^i\left(\Gamma^{05} + \Gamma_{A=5}\right)\epsilon \tag{72}$$

and we find the condition for unbroken SUSY as

$$\left(1 + \Gamma^{05}\Gamma_{A=5}\right)\epsilon = 0 \tag{73}$$

If we use the Weyl condition

$$\Gamma\epsilon = -\epsilon \tag{74}$$

of the $(2,0)$ supersymmetry parameter $\epsilon$, then we can also write this as

$$\left(1 + \Gamma^{1234}\Gamma_{A=5}\right)\epsilon = 0. \tag{75}$$

We may represent the gamma matrices as

$$\begin{aligned} \Gamma_\mu &= (\Gamma_0, \Gamma_i, \Gamma_5) = \left(1\otimes i\sigma^2\otimes 1, \gamma_i\otimes\sigma^1\otimes 1, \gamma\otimes\sigma^1\otimes 1\right) \\ \Gamma_A &= 1\otimes i\sigma^2\otimes\sigma_A \end{aligned} \tag{76}$$

where $\sigma^{1,2,3}$ are the Pauli sigma matrices, $\gamma = \gamma_{1234}$. Then the condition for unbroken SUSY is

$$(1 + \gamma\otimes\sigma)\epsilon = 0 \tag{77}$$

where $\sigma = \sigma_{1234} = \sigma_{A=5}$.

We have found that if

$$H_{ijk} = \epsilon_{ijkl}\partial_l\phi \tag{78}$$

then half SUSY is unbroken. This equation is the Bogomolnyi equation for selfdual strings [1]. We are interested in finding the number of parameters needed to describe solutions of this equation. We can linearize it and get the equation

$$\gamma_i\partial_i\chi = 0 \tag{79}$$

for the bosonic zero modes, that we have gathered into a matrix

$$\chi \equiv \gamma_{ij}\delta B_{ij} + \gamma\delta\phi. \tag{80}$$

For this to work we must also assume the background gauge condition

$$\partial_i B_{ij} = 0. \tag{81}$$

Now this linearized equation Eq (79) does not make any reference to the gauge field. So there is no way that we could count the number of parameters of a multi-string configuration just using this equation. This should of course not be a surprise. The strings that we have in the Abelian theory are not solutions of the field equations. They have to be inserted by hand, that is we need to insert delta function sources by hand, in the same spirit as for Dirac monopoles.

To be able to count the number of zero modes, we must consider some interacting theory which (at the classical level) has solitonic string solutions.

To pass to non-Abelian theory we begin by rewriting the Abelian theory in loop space. Loop space consists of parametrized loops $C$: $s \mapsto C^\mu(s)$. We introduce the Abelian 'loop fields' [7]

$$\begin{array}{rcl} A_{\mu s} & = & B_{\mu\nu}(C(s))\dot{C}^\nu(s) \\ \phi^{\mu s} & = & \phi(C(s))\dot{C}^\mu(s) \\ \psi^{\mu s} & = & \psi(C(s))\dot{C}^\mu(s) \end{array} \tag{82}$$

With these definitions, a short computation reveals that $A_{\mu s}$ transforms as a vector and $\phi^{\mu s}$ a contra-variant vector under diffeomorphisms in loop space induced by diffeomorphisms in space-time. One may then extend these transformation properties to any diffeomorphism in loop space. Space-time diffeomorphism and reparametrizations of the loops then get unified and are both diffemorphisms in loop space. The only thing to remember is what is kept fixed under the variation. If it is the parameter of the loop, or the loop itself.

The field strength becomes

$$F_{\mu s,\nu t} = H_{\mu\nu\rho}(C(s))\dot{C}^\rho(s)\delta(s-t) \tag{83}$$

In terms of these fields, the Bogomolnyi equation will read[4]

$$F_{is,jt} = \epsilon_{ijkl}\partial_{k(s}\phi_{lt)}. \tag{84}$$

We pass to the non-Abelian theory by letting these loop fields become non-Abelian, in the sense that $A_{\mu s} = A^a_{\mu s}\lambda^a(s)$ where $\lambda^a(s)$ are generators of a loop algebra associated to the gauge group [7]. We introduce a covariant derivative

$$D_{\mu s} \quad = \quad \partial_{\mu s} + A_{\mu s}. \tag{85}$$

Local gauge transformations act as

$$\begin{array}{rcl} \delta_\Lambda A_{\mu s} & = & D_{\mu s}\Lambda \\ \delta_\Lambda \phi^{\mu s} & = & [\phi^{\mu s}, \Lambda]. \end{array} \tag{86}$$

Given a loop $C$, we automatically get a tangent vector $\dot{C}^\mu(s)$ that makes no reference to space-time. We can therefore impose the loop space constraints

$$\dot{C}^\mu(s)A_{\mu s} = 0 \tag{87}$$

[4]We denote by $\partial_{is}$ the usual functional derivative with respect to $C^\mu(s)$.

for each $s$, and also

$$\phi^{\mu s} = \dot{C}^\mu(s)\phi(s;C) \tag{88}$$

for some subtle field $\phi(s;C)$ on loop space. As a consequence, we find that

$$A_{\mu s}\phi^{\mu s} = 0. \tag{89}$$

These constraints are covariant under diffeomorphisms of space-time and reparametrizations of loops. They are invariant also under local gauge transformations, provided that the gauge parameter is subject to the condition

$$\dot{C}^\mu(s)\partial_{\mu s}\Lambda = 0 \tag{90}$$

which is the condition of reparametrization invariance. With the assumption made that $\lambda^a(s)$ are generators of a loop algebra, we find that the constraint can also be written as

$$[A_{\mu s}, \phi^{\mu t}] = 0 \tag{91}$$

A local gauge variation of this constraint is

$$\begin{aligned}
[D_{\mu s}\Lambda, \phi^{\mu t}] + [A_{\mu s}, [\phi^{\mu t}, \Lambda]] &= [\partial_{\mu s}\Lambda, \phi^{\mu t}] + [[A_{\mu s}, \Lambda], \phi^{\mu t}] + [A_{\mu s}, [\phi^{\mu t}, \Lambda]] \\
&= [\partial_{\mu s}\Lambda, \phi^{\mu t}] + [\Lambda, [\phi^{\mu t}, A_{\mu s}]]
\end{aligned} \tag{92}$$

The last term vanishes by the constraint. The first term gives us the constraint Eq (91) that we must impose on the gauge parameter

$$\Lambda = \int ds \Lambda^a(s,C)\lambda^a(s). \tag{93}$$

We have now introduced non-local non-Abelian fields with infinitely many components. It is also likely that consisteny of the theory requires an infinite set of constraints on these fields. Maybe then, it could be that we may in the end descend to a finite degrees of freedom. But this is just a speculation. The problem appears to be difficult and ill-defined – How should one define a degree of freedom in a strongly coupled non-local theory?

The non-Abelian generalization of the Bogomolnyi equation should be given by [7]

$$F_{is,jt} = \pm\epsilon_{ijkl}D_{ks}\phi_{lt}. \tag{94}$$

This equation is gauge invariant and invariant under the residual $SO(4)$ Lorentz group that is preserved by the strings. We can not think of any reasonable modification of this equation that would preserve these symmetries, so on this grounds alone one could suspect this equation to be correct. Of course this is not the only requirement that the BPS condition imposes. We also get conditions on the $0s$ and the $5s$ components. But these BPS equations will be of no interest to us right now.

We will show below that the linearized Bogomolnyi equation can be written as

$$\gamma_i \left(D_{i(s} + \sigma\phi_{i(s}\right)\chi_{t)} = 0 \tag{95}$$

We will also see below that we (presumably) can actually drop the symmetrization in $s$ and $t$ in this equation. The fields transform in the adjoint representation of the loop algebra, by which we mean that $\phi_{is}\chi_t = [\phi_{is}, \chi_t]$. We define the Dirac operator

$$D_s = \gamma_i \left(D_{is} + \sigma\phi_{is}\right) \tag{96}$$

and the projectors

$$P_\pm \equiv \frac{1}{2}\left(1 \mp \gamma\sigma\right), \tag{97}$$

We can now formulate an index problem, in an even-dimensional (loop-)space. The even-dimensional space in this case is given by the 4-dimensional transverse space to the strings, and the index is given by

$$\dim\ker\mathcal{D}_s - \dim\ker\mathcal{D}_s^\dagger \tag{98}$$

where

$$\begin{aligned} \mathcal{D}_s &= P_+ D_s = D_s P_- \\ \mathcal{D}_s^\dagger &= P_- D_s = D_s P_+. \end{aligned} \tag{99}$$

Since $D_s$ and $P_\pm$ are hermitian, it is manifest that $\mathcal{D}_s^\dagger$ defined this way will be the hermitian conjugate of $\mathcal{D}_s$, thus justifying the notation.

Computing the index alone is not sufficient in order to obtain the dimension of the moduli space of self-dual strings. We also need a vanishing theorem that says that $\dim\ker\mathcal{D}_s^\dagger = 0$.

Linearizing the Bogomolnyi equation, we get

$$2D_{[is}\delta A_{jt]} = \pm\epsilon_{ijkl}\left(D_{ks}\delta\phi_{lt} + \phi_{ks}\delta A_{lt}\right) \tag{100}$$

Contracting by $\gamma^{ij}$, we get

$$\gamma^{ij}\tilde{D}_{is}\chi_{jt} = 0 \tag{101}$$

where we have defined

$$\begin{aligned} \tilde{D}_{is} &\equiv D_{is} \mp \gamma\phi_{is} \\ \chi_{is} &\equiv \delta A_{is} \mp \gamma\delta\phi_{is} \end{aligned} \tag{102}$$

To see that the linearized BPS equation can be written like this, one must use the constraint

$$\gamma^{ij}\phi_{is}\delta\phi_{jt} = 0. \tag{103}$$

We can avoid having explicit $\pm$ signs by introducing the other chiraly matrix at our disposal, namely $\sigma$ that lives in a different vector space than $\gamma$. We can then hide the $\pm$ signs in the tensor product

$$\gamma \otimes \sigma = \pm 1 \tag{104}$$

which amounts to

$$\tilde{D}_{is} \equiv D_{is} + \sigma\phi_{is}$$

$$\chi_{is} \quad \equiv \quad \delta A_{is} + \sigma\delta\phi_{is} \tag{105}$$

without any $\pm$.[5] If we define

$$\chi_s \equiv \gamma^i \chi_{is} \tag{107}$$

then we can write the zero mode equation as

$$\gamma^i \tilde{D}_{is}\chi_t + \tilde{D}^i_s \chi_{it} = 0. \tag{108}$$

Let us analyze the second term in this equation. It is given by

$$\begin{aligned} & D^i_s \delta A_{it} + \phi^i_s \delta\phi_{it} \\ - \quad & \gamma\left(\phi^i_s \delta A_{it} + D^i_s \delta\phi_{it}\right) \end{aligned} \tag{109}$$

We should not count variations that are gauge variations as bosonic zero modes. We can insure this by demanding the zero modes to be orthogonal to gauge variations, with respect to the metric on the moduli space,

$$(\delta_\Lambda A_{is}, \delta A_{it}) + (\delta_\Lambda \phi_{is}, \delta\phi_{jt}) = 0 \tag{110}$$

This leads to the background gauge condition

$$D^i_s \delta A_{it} + \phi^i_s \delta\phi_{it} = 0. \tag{111}$$

This condition implies that the gauge variation of the zero modes vanishes,

$$\delta_\Lambda \delta A_{is} = 0 = \delta_\Lambda \delta\phi_{is} \tag{112}$$

To see this, we make a gauge variation $\delta_\Lambda \delta A_{is} = D_{is}\Lambda$, $\delta_\Lambda \phi_{is} = \phi_{is}\Lambda$, and ask which gauge parameters $\Lambda$ will respect the background gauge condition. Inserting this gauge variation into the background gauge condition, we get

$$\left(D^i_s D_{it} + \phi^i_s \phi_{it}\right)\Lambda = 0. \tag{113}$$

For this to work nicely, it seems that we must constrain the non-locality of our loop field such that $\partial^i_{(s}\partial_{it)} < 0$. Then the only solution to this equation is $\Lambda = 0$. In other words all gauge variations of the zero modes have to vanish.

Furthermore we want the variation to preserve the orthogonality between $A_{is}$ and $\phi_{is}$,

$$(A_{is}, \delta\phi_{it}) + (\delta A_{is}, \phi_{it}) = 0 \tag{114}$$

If we make a gauge variation of this, then we get the condition

$$(\delta_\Lambda A_{is}, \delta\phi_{it}) + (\delta A_{is}, \delta_\Lambda \phi_{it}) = 0 \tag{115}$$

which amounts to

$$\phi^i_s \delta A_{it} + D^i_s \delta\phi_{it} = 0. \tag{116}$$

[5] To really understand what is going on, one should apply $(1 \pm \gamma\sigma)$ on everything, on $\psi_s$ and on $D_s$. Then one notices that

$$\mp\gamma\,(1 \mp \gamma\sigma) = \sigma\,(1 \mp \gamma\sigma)\,. \tag{106}$$

That is, we can trade $\mp\gamma$ for $\sigma$, once we apply $(1 \pm \gamma\sigma)$ on everything. This is what we really should do, but to keep the notation simple, we do not spell this out.

We conclude that the zero mode equation can be written as

$$D_s \chi_t = 0 \tag{117}$$

where

$$D_s = \gamma_i \left( D_{is} + \sigma \phi_{is} \right) \tag{118}$$

We are interested in counting the number of such modes in a background of $k$ BPS strings. We compute

$$D^2 \quad = \quad (D_{is})^2 + (\phi_{is})^2 + \frac{1}{2}\gamma^{ij}\left(F_{is,js} + \gamma\sigma\epsilon_{ijkl} D_{ks}\phi_{ls}\right) \tag{119}$$

(Here $D^2 \equiv D_s D_s \equiv \int \frac{ds}{2\pi} D_s D_s$, and analogously for the other fields or operators.) In a BPS configuration, we get is

$$D^2 \quad = \quad (D_{is})^2 + (\phi_{is})^2 + \frac{1}{2}\gamma^{ij}\left(1 + \gamma\sigma\right) F_{is,js} \tag{120}$$

Furthermore, in the subspace where $1 + \gamma\sigma = 0$, we find that

$$D^2 \quad = \quad (D_{is})^2 + (\phi_{is})^2 \tag{121}$$

is a strictly negative operator, hence has no zero modes. This means that we have a vanishing theorem, $\dim \ker \mathcal{D}^\dagger = 0$.

## A small comment

The zero mode equation was really

$$D_{(s}\chi_{t)} = 0 \tag{122}$$

where we should symmetrize in $s$ and $t$. That means that we should rather consider

$$\begin{aligned} D_s D_{(s}\chi_{t)} &= \frac{1}{2}\left(D_s D_s \chi_t + D_s D_t \chi_s\right) \\ &= \frac{1}{2}\left(D_s D_s \chi_t + D_t D_s \chi_s + [D_s, D_t]\chi_s\right). \end{aligned} \tag{123}$$

If now $D_{[s}D_{t]} = 0$ and $D_s \chi_s = 0$, then we get

$$D_s D_s \chi_t = 0 \tag{124}$$

The latter condition, $D_s \chi_s = 0$ is of course a consequence of $D_{(s}\chi_{t)} = 0$ with $s = t$. The former condition reads

$$\begin{aligned} 0 &= D_{[s}D_{t]} \\ &= D_{i[s}D_{it]} + \phi_{i[s}\phi_{it]} + \sigma D_{i[s}\phi_{it]} \end{aligned} \tag{125}$$

In the Abelian case we find that $\partial_{i[s}\partial_{|i|t]} = 0$ and $\phi_{i[s}\phi_{|i|t]} = 0$. A delicate computation also shows that $\partial_{i[s}\phi_{|i|t]}\chi_s = 0$. The philosophy we have is that any Abelian constraints are lifted to non-Abelian ones by covariantizing.

If we accept these constraints, then we have now seen that the zero mode equation Eq (122) implies

$$\int ds D_s^\dagger D_s \chi_t = 0 \tag{126}$$

because $D_s$ is anti-self-adjoint with respect to the inner product

$$(\psi_s, \chi_t) = \int \mathcal{D}C \mathrm{tr} \left(\psi_s^\dagger(C)\chi_t(C)\right) \tag{127}$$

on loop space. We can also go in the opposite direction. Assuming that Eq (126) holds, we get

$$0 = \left(\chi_t, D_s^\dagger D_s \chi_t\right) = (D_s\chi_t, D_s\chi_t) \tag{128}$$

and we conclude that (122) implies

$$D_s \chi_t = 0 \tag{129}$$

with no symmetrization in $s, t$.

## How to compute the index

We should now be able to compute an index associated to self-dual strings, as the limit

$$I(s) = \mathrm{Tr}\left(\gamma\sigma e^{sD^2}\right) \tag{130}$$

when $s \to \infty$. We define the quantity

$$J_{is}(C, C') = -\mathrm{tr}\left\langle C \left| \gamma\sigma\gamma_k \left(D_{ks} + \sigma\phi_{is}\right) \frac{1}{D^2} e^{sD^2} \right| C' \right\rangle \tag{131}$$

(it should be clear that the two $s$'s involved in this formula are totally unrelated) and find that

$$I(s) = \int \frac{ds}{2\pi} \int \mathcal{D}C \partial_{is} J^{is}(C, C) \tag{132}$$

We can separate the functional integral over parametrized loops $C$ into several pieces. We can keep a point on the loops $C(s) = x$ fixed, and separate it as

$$\int \mathcal{D}C = \int d^4x \int \mathcal{D}_x C \tag{133}$$

Then we can write $I(s)$ as an integral over a large three-sphere at spatial infinity,

$$\int \frac{ds}{2\pi} \int d^4x \int \mathcal{D}_x C \frac{\partial J_{is}(C)}{\partial C^i(s)} = \int \frac{ds}{2\pi} \int_{S^3} d\Omega_3 \hat{x}^i \int \mathcal{D}_x C J_{is}(C, C) \tag{134}$$

where thus $x = C(s)$.

If we assume that the gauge group is maximally broken to a product of $U(1)$'s by the Higgs vacuum expectation values, then we should have $U(1)$ loop fields at spatial infinity.

If we assume that the gauge group is $SU(2)$ and that it is broken to $U(1)$, then we need only the asymptotic form of the $U(1)$ fields at spatial infinity,

$$\begin{aligned} F_{is,jt} &= H_{ijl}(x)\dot{C}^l(s)\delta(s-t) \\ \phi_{ks} &= v\dot{C}_k(s) \end{aligned} \tag{135}$$

Without doing any computations, we can guess what the outcome of the index calculation should be. A term like

$$\epsilon_{ijkl}\int \mathcal{D}_x C \mathrm{tr}\left(F_{is,jt}(C)F_{ks,jt}(C)\right) \tag{136}$$

can (and hence probably does) arise, but here this term vanishes identically if one uses the Bogomolnyi equation and notices the constraint

$$F_{is,jt}D_{is}\phi_{jt} = 0 \tag{137}$$

For the $U(1)$ fields, this is just the fact that

$$F_{is,jt}\partial_{is}\phi_{jt} \sim H_{ijk}(C(s))\partial_i\phi(C(s))\dot{C}^k(s)\dot{C}^j(s)\delta(s-t)^2 \equiv 0. \tag{138}$$

Then there can be a term

$$\epsilon_{ijkl}\int \mathcal{D}_x C \mathrm{tr}\left(\frac{F_{is,jt}\phi_{ks}}{v}\right) \tag{139}$$

that should arise in a very similar way as the corresponding term arose for monopoles. If we insert the asymptotic $U(1)$ fields, this term becomes proportional to

$$\epsilon_{ijkl}H_{ijk}(x) \tag{140}$$

That means that the index should be given by some numerical constant, times the magnetic charge

$$\int_{S^3_\infty} H. \tag{141}$$

# A Integrals over the exponential

The integral we will analyze here is

$$a(s) = \int_{-\infty}^{+\infty} dk \frac{k^{2\zeta}}{k^2+1} e^{-s\left(k^2+1\right)} e^{i\epsilon k} \tag{142}$$

for any complex number $\zeta$. (The $\epsilon > 0$, say, will be taken towards zero. It arose from $\epsilon = x - y$ and we keep it here just as a convergence factor.) We first compute

$$a(0) = \int_{-\infty}^{+\infty} dk \frac{k^{2\zeta}}{k^2+1} e^{i\epsilon k} \tag{143}$$

In order to make this integral converge for any $\zeta$, we should Wick rotate $k$ to $ik$, and henceforth we will always mean by $i$ the branch $e^{i\pi/2}$, and by $-1$ we mean $e^{i\pi}$. Then we get

$$a(0) = -i^{2\zeta+1} \int_{-\infty}^{+\infty} dk \frac{k^{2\zeta}}{k^2-1} e^{-\epsilon k} \tag{144}$$

and this integral we evaluate as a principal value. That means to evaluate the residues along the real axis and multiply them not by $2\pi i$, but by half of it, that is, by $\pi i$. We get

$$a(0) = (-1)^{\zeta} \pi \frac{1-(-1)^{2\zeta}}{2}. \tag{145}$$

Next we turn to our integral $a(s)$. It is easier to first compute the derivative. We should still work with the Wick rotated integral. Making the substitution $\xi = k^2$ we can put it on the form of two gamma functions. The result is that

$$a'(-s) \quad = \quad e^{i\pi\left(\zeta+\frac{1}{2}\right)} \frac{1}{2} \left(1+(-1)^{2\zeta}\right) \Gamma\left(\zeta+\frac{1}{2}\right) s^{-\zeta-\frac{1}{2}} e^{s} \tag{146}$$

which we can trivially continue analytically to $+s$, and then integrate up. The result is

$$a(+\infty) = \pi \frac{1+(-1)^{2\zeta}}{2} \frac{1}{\cos(\pi\zeta)} + \pi(-1)^{\zeta} \frac{1-(-1)^{2\zeta}}{2}. \tag{147}$$

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