Rigid subsets of symplectic manifolds

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Abstract

We show that there is an hierarchy of intersection rigidity properties of sets in a closed symplectic manifold: some sets cannot be displaced by symplectomorphisms from more sets than the others. We also find new examples of rigidity of intersections involving, in particular, specific fibers of moment maps of Hamiltonian torus actions, monotone Lagrangian submanifolds (following the works of P.Albers and P.Biran-O.Cornea), as well as certain, possibly singular, sets defined in terms of Poisson-commutative subalgebras of smooth functions. In addition, we get some geometric obstructions to semi-simplicity of the quantum homology of symplectic manifolds. The proofs are based on the Floer-theoretical machinery of partial symplectic quasi-states.

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Contents

1 Introduction and main results

1.1 Many facets of displaceability

A well-studied and easy to visualize rigidity property of subsets of a symplectic manifold (M, ω) is the rigidity of intersections: a subset $X \subset M$ cannot be displaced from the closure of a subset $Y \subset M$ by a compactly supported Hamiltonian isotopy:

$$\phi(X) \cap \overline{Y} \neq \emptyset \quad \forall \phi \in Ham(M) \, .$$

We say in such a case that X *cannot be displaced* from Y. If X cannot be displaced from itself we call it *non-displaceable*. These properties become especially interesting and purely symplectic when X can be displaced from itself or from Y by a (compactly supported) smooth isotopy.

One of the main themes of the present paper is that *"some non-displaceable sets are more rigid than others."* To explain this, we need the following ramifications of the notion of a non-displaceable set:

STRONG NON-DISPLACEABILITY: A subset $X \subset M$ is called *strongly non-displaceable* if one cannot displace it by any (not necessarily Hamiltonian) symplectomorphism of (M, ω).

STABLE NON-DISPLACEABILITY: Consider $T^*\mathbb{S}^1 = \mathbb{R} \times \mathbb{S}^1$ with the coordinates (r, θ) and the symplectic form $dr \wedge d\theta$. We say that $X \subset M$ is *stably non-displaceable* if $X \times \{r = 0\}$ is non-displaceable in $M \times T^*\mathbb{S}^1$ equipped with the split symplectic form $\bar{\omega} = \omega \oplus (dr \wedge d\theta)$. Let us mention that detecting stably non-displaceable subsets is useful for studying geometry and dynamics of Hamiltonian flows (see for instance [50] for their role in Hofer's geometry and [51] for their appearance in the context of kick stability in Hamiltonian dynamics).

Formally speaking, the properties of strong and stable non-displaceability are mutually independent and both are strictly stronger than displaceability.

In the present paper we refine the machinery of partial symplectic quasi-states introduced in [23] and get new examples of stably non-displaceable sets, including certain fibers of moment maps of Hamiltonian torus actions as well as monotone Lagrangian submanifolds discussed by Albers [2] and Biran-Cornea [15]. Further, we address the following question: given the class of stably non-displaceable sets, can one distinguish those of them which are also strongly non-displaceable by means of the Floer theory? Or, other way around, what are the Floer-homological features of stably non-displaceable but strongly displaceable sets? Toy examples are given by the equator of the symplectic two-sphere and by the meridian on a symplectic two-torus. Both are stably non-displaceable since their Lagrangian Floer homologies are non-trivial. On the other hand, the equator is strongly non-displaceable, while the meridian is strongly displaceable by a non-Hamiltonian shift. Later on we shall explain the difference between these two examples from the viewpoint of Hamiltonian Floer homology and present various generalizations.

The question on Floer-homological characterization of (strongly) non-displaceable but stably displaceable sets is totally open, see Section 1.7.1 below for an example involving Gromov's packing theorem and discussion.

Leaving Floer-theoretical considerations for the next section, let us outline (in parts, informally) the general scheme of our results: Given a symplectic manifold (M, ω), we shall define (in the language of the Floer theory) two collections of closed subsets of M, *heavy subsets* and *superheavy subsets*. Every superheavy subset is heavy, but, in general, not vice versa. Formally speaking, the hierarchy heavy-superheavy depends in a delicate way on the choice of an idempotent in the quantum homology ring of M. This and other nuances will be ignored in this outline. The key properties of these collections are as follows (see Theorems 1.2 and 1.5 below):

Invariance: Both collections are invariant under the group of all symplectomorphisms of M.

Stable non-displaceability: Every heavy subset is stably non-displaceable.

Intersections: Every superheavy subset intersects every heavy subset. In particular, superheavy subsets are strongly non-displaceable. In contrast to this, heavy subsets can be mutually disjoint and strongly displaceable.

Products: Product of any two (super)heavy subsets is (super)heavy.

What is inside the collections? The collections of heavy and superheavy sets include the following examples:

STABLE STEMS: Let $\mathbb{A} \subset C^\infty(M)$ be a finite-dimensional Poisson-commutative subspace (i.e. any two functions from $\mathbb{A}$ commute with respect to the Poisson brackets). Let $\Phi : M \to \mathbb{A}^*$ be the moment map: $\langle \Phi(x), F \rangle = F(x)$. A non-empty fiber $\Phi^{-1}(p)$, $p \in \mathbb{A}^*$, is called a *stem* of $\mathbb{A}$ (see [23]) if all non-empty fibers $\Phi^{-1}(q)$ with $q \neq p$ are displaceable and a *stable stem* if they are stably displaceable. If a subset of M is a (stable) stem of a finite-dimensional Poisson-commutative subspace of $C^\infty(M)$, it will be called just *a (stable) stem*. Clearly, any stem is a stable stem. **The collection of superheavy subsets includes all stable stems** (see Theorem 1.6 below). One readily shows that a direct product of stable stems is a stable stem and that the image of a stable stem under *any* symplectomorphism is again a stable stem.

The following example of a stable stem is borrowed (with a minor modification) from [23]: Let $X \subset M$ be a closed subset whose complement is a finite disjoint union of stably displaceable sets. Then X is a stable stem. For instance, the codimension-1 skeleton of a sufficiently fine triangulation of any closed symplectic manifold is a stable stem. Another example is given by the equator of $\mathbb{S}^2$: it divides the sphere into two displaceable open discs and hence is a stable stem. By taking products, one can get more sophisticated examples of stable stems. Already the product of equators of the two-spheres gives rise to a Lagrangian Clifford torus in $\mathbb{S}^2 \times \ldots \times \mathbb{S}^2$. To prove its rigidity properties (such as stable non-displaceability) one has to use non-trivial symplectic tools such as Lagrangian Floer homology, see e.g. [44]. Products of the 1-skeletons of fine triangulations of the two-spheres can be considered as *singular Lagrangian submanifolds*, an object which is currently out of reach of the Lagrangian Floer theory.

Another example of stable stems comes from Hamiltonian torus actions. Consider an effective Hamiltonian action $\varphi : \mathbb{T}^k \to Ham(M)$ with the moment map $\Phi = (\Phi_1, \ldots, \Phi_k) : M \to \mathbb{R}^k$. Assume that Φ_i is a normalized Hamiltonian, that is $\int_M \Phi_i = 0$ for all $i = 1, \ldots, k$. A torus action is called *compressible* if the image of the homomorphism $\varphi_\sharp : \pi_1(\mathbb{T}^k) \to \pi_1(Ham(M))$, induced by the action φ, is a finite group. One can show that for compressible actions the fiber $\Phi^{-1}(0)$ is a stable stem (see Theorem 1.7 below).

Special fibers of Hamiltonian torus actions: Consider an effective Hamiltonian torus action φ on a spherically monotone symplectic manifold. Let $I : \pi_1(Ham(M)) \to \mathbb{R}$ be the mixed action-Maslov homomorphism introduced in [49]. Since the target space $\mathbb{R}^k$ of the moment map Φ is naturally identified with $\mathrm{Hom}(\pi_1(\mathbb{T}^k), \mathbb{R})$, the pull back $p_{spec} := -\varphi_\sharp^* I$ of the mixed action-Maslov homomorphism with the reversed sign can be considered as a point of $\mathbb{R}^k$. The preimage $\Phi^{-1}(p_{spec})$ is called *the special fiber* of the action. We shall see below that the special fiber is always non-empty. For monotone symplectic toric manifolds (that is when $2k = \dim M$) the special fiber is a monotone Lagrangian torus. Note that when the action is compressible we have $p_{spec} = 0$ and therefore the special fiber is a stable stem according to the previous example. It is unknown whether the latter property persists for general non-compressible actions. Thus in what follows we treat stable stems and special fibers as separate examples. **The collection of superheavy subsets includes all special fibers** (see Theorem 1.9 below).

For instance, consider $\mathbb{C}P^2$ and the Lagrangian Clifford torus in it (i.e. the torus $\{[z_0 : z_1 : z_2] \in \mathbb{C}P^2 \mid |z_0| = |z_1| = |z_2|\}$). Take the standard Hamiltonian $\mathbb{T}^2$-action on $\mathbb{C}P^2$ preserving the Clifford torus. It has three global fixed points away from the Clifford torus. Make an equivariant symplectic blow-up, M, of $\mathbb{C}P^2$ at k of these fixed points, $0 \leq k \leq 3$, so that the obtained symplectic manifold is spherically monotone. The torus action lifts to a Hamiltonian action on M. One can show that its special fiber is the proper transform of the Clifford torus.

Monotone Lagrangian submanifolds: Let (M^{2n}, ω) be a spherically monotone symplectic manifold, and let $L \subset M$ be a closed monotone Lagrangian submanifold with the minimal Maslov number $N_L \geq 2$. We say that L *satisfies the Albers condition* [2] if the image of the natural morphism $H_*(L; \mathbb{Z}_2) \to H_*(M; \mathbb{Z}_2)$ contains a non-zero element S with

$$\deg S > \dim L + 1 - N_L \ .$$

The collection of heavy sets includes all closed monotone Lagrangian submanifolds satisfying the Albers condition (see Theorem 1.15 below).

Specific examples include the meridian on $\mathbb{T}^2$, $\mathbb{R}P^n \subset \mathbb{C}P^n$ and all Lagrangian spheres in complex projective hypersurfaces of degree d in $\mathbb{C}P^{n+1}$ with $n > 2d - 3$. In the case when the fundamental class $[L]$ of L divides a non-trivial idempotent in the quantum homology algebra of M, L is, in fact, superheavy (see Theorem 1.18 below). For instance, this is the case for $\mathbb{R}P^n \subset \mathbb{C}P^n$. Furthermore, a version of superheaviness holds for any Lagrangian sphere in the complex quadric of even (complex) dimension.

However, there exist examples of heavy, but not superheavy, Lagrangian submanifolds: For instance, the meridian of the 2-torus is strongly displaceable by a (non-Hamiltonian!) shift and hence is not superheavy. Another example of heavy but not superheavy Lagrangian submanifold is the sphere arising as the real part of the Fermat hypersurface

$$M = \{-z_0^d + z_1^d + \ldots + z_{n+1}^d = 0\} \subset \mathbb{C}P^{n+1}$$

with even $d \geq 4$ and $n > 2d-3$. We refer to Section 1.5 for more details on (super)heavy monotone Lagrangian submanifolds.

Motivation: Our motivation for the selection of examples appearing in the list above is as follows. Stable stems provide a playground for studying symplectic rigidity of singular subsets. In particular, no visible analogue of the conventional Lagrangian Floer homology technique is applicable to them.

Detecting (stable) non-displaceability of Lagrangian submanifolds via Lagrangian Floer homology is one of the central themes of symplectic topology. In contrast to this, detecting *strong* non-displaceabilty has at the moment the status of art rather than science. That's why we were intrigued by Albers' observation that monotone Lagrangian submanifolds satisfying his condition are in some situations strongly non-displaceable. In the present work we tried to digest Albers' results [2] and look at them from the viewpoint of theory of partial symplectic quasi-states developed in [23]. In addition, our result on superheaviness of the Lagrangian anti-diagonal in $\mathbb{S}^2 \times \mathbb{S}^2$ allows us to detect an "exotic" monotone Lagrangian torus in this symplectic manifold: this torus does not intersect the anti-diagonal, and hence is not heavy in contrast to the standard Clifford torus, see Example 1.20 below.

In [23] we proved a theorem which roughly speaking states that every (singular) coisotropic foliation has at least one non-displaceable fiber. However, our proof is non-constructive and does not tell us which specific fibers are non-displaceable. The notion of the special fiber arose as an attempt to solve this problem for Hamiltonian circle actions.

Let us mention also that the **product property** enables us to produce even more examples of (super)heavy subsets by taking products of the subsets appearing in the list.

A few comments on the methods involved into our study of heavy and superheavy subsets are in order. These collections are defined in terms of partial symplectic quasi-states which were introduced in [23]. These are certain real-valued functionals on $C^\infty(M)$ with rich algebraic properties which are constructed by means of the Hamiltonian Floer theory and which conveniently encode a part of the information contained in this theory. In general, the definition of a partial symplectic quasi-state involves the choice of an *idempotent element* in the commutative part $QH_\bullet(M)$ of the quantum homology algebra of M. Though the default choice is just the unity of the algebra, there exist some other meaningful choices, in particular in the case when $QH_\bullet(M)$ is semi-simple. This gives rise to another theme discussed in this paper: "visible" topological obstructions to semi-simplicity (see Corollary 1.24 and Theorem 1.25 below). For instance, we shall show that if a monotone symplectic manifold M contains "too many" disjoint monotone Lagrangian spheres whose minimal Maslov numbers exceed $n+1$, the quantum homology $QH_\bullet(M)$ cannot be semi-simple.

Let us pass to the precise set-up. For the reader's convenience, the material presented in this brief outline will be repeated in parts in the next sections in a less compressed form.

1.2 Preliminaries on quantum homology

THE NOVIKOV RING: Let $\mathcal{F}$ denote a base field which in our case will be either $\mathbb{C}$ or $\mathbb{Z}_2$, and let $\Gamma \subset \mathbb{R}$ be a countable subgroup (with respect to the addition). Let s, q be formal variables. Define a field $\mathcal{K}_\Gamma$ whose elements are generalized Laurent series in s of the following form:

$$\mathcal{K}_\Gamma := \left\{ \sum_{\theta \in \Gamma} z_\theta s^\theta,\ z_\theta \in \mathcal{F},\ \sharp\big\{\theta > c \mid z_\theta \neq 0\big\} < \infty,\ \forall c \in \mathbb{R} \right\}.$$

Define a ring $\Lambda_\Gamma := \mathcal{K}_\Gamma[q, q^{-1}]$ as the ring of polynomials in q, q^{-1} with coefficients in $\mathcal{K}_\Gamma$. We turn Λ_Γ into a graded ring by setting the degree of s to be 0 and the degree of q to be 2.

The ring Λ_Γ serves as an abstract model of the Novikov ring associated to a symplectic manifold. Let (M, ω) be a closed connected symplectic manifold. Denote by $H_2^S(M)$ the subgroup of spherical homology classes in the integral homology group $H_2(M; \mathbb{Z})$. Abusing the notation we will write $\omega(A)$, $c_1(A)$ for the results of evaluation of the cohomology classes $[\omega]$ and $c_1(M)$ on $A \in H_2(M; \mathbb{Z})$. Set

$$\bar{\pi}_2(M) := H_2^S(M)/\sim,$$

where by definition

$$A \sim B \text{ iff } \omega(A) = \omega(B) \text{ and } c_1(A) = c_1(B).$$

Denote by $\Gamma(M,\omega) := [\omega](H_2^S(M)) \subset \mathbb{R}$ the subgroup of periods of the symplectic form on M on spherical homology classes. By definition, the Novikov ring of a symplectic manifold (M,ω) is $\Lambda_{\Gamma(M,\omega)}$. In what follows, when (M,ω) is fixed, we abbreviate and write Γ, $\mathcal{K}$ and Λ instead of $\Gamma(M,\omega)$, $\mathcal{K}_{\Gamma(M,\omega)}$ and $\Lambda_{\Gamma(M,\omega)}$ respectively.

Quantum homology: Set $2n = \dim M$. The quantum homology $QH_*(M)$ is defined as follows. First, it is a graded module over Λ given by

$$QH_*(M) := H_*(M;\mathcal{F}) \otimes_{\mathcal{F}} \Lambda,$$

with the grading defined by the gradings on $H_*(M;\mathcal{F})$ and Λ:

$$\deg(a \otimes zs^{\theta}q^{k}) := \deg(a) + 2k\ .$$

Second, and most important, $QH_*(M)$ is equipped with a *quantum product*: if $a \in H_k(M;\mathcal{F})$, $b \in H_l(M;\mathcal{F})$, their quantum product is a class $a * b \in QH_{k+l-2n}(M)$, defined by

$$a * b = \sum_{A \in \bar{\pi}_2(M)} (a*b)_A \otimes s^{-\omega(A)} q^{-c_1(A)},$$

where $(a*b)_A \in H_{k+l-2n+2c_1(A)}(M)$ is defined by the requirement

$$(a*b)_A \circ c = GW_A^{\mathcal{F}}(a,b,c)\ \forall c \in H_*(M;\mathcal{F}).$$

Here $\circ$ stands for the intersection index and $GW_A^{\mathcal{F}}(a,b,c) \in \mathcal{F}$ denotes the Gromov-Witten invariant which, roughly speaking, counts the number of pseudo-holomorphic spheres in M in the class A that meet cycles representing $a,b,c \in H_*(M;\mathcal{F})$ (see [55], [56], [41] for the precise definition).

Extending this definition by Λ-linearity to the whole $QH_*(M)$ one gets a correctly defined graded-commutative associative product operation $*$ on $QH_*(M)$ which is a deformation of the classical $\cap$-product in singular homology [37], [41], [55], [56], [69]. The *quantum homology algebra* $QH_*(M)$ is a ring whose unity is the fundamental class $[M]$ and which is a module of finite rank over Λ. If $a, b \in QH_*(M)$ have graded degrees $deg\,(a)$, $deg\,(b)$ then

$$deg\,(a*b) = deg\,(a) + deg\,(b) - 2n. \tag{1}$$

We will be mostly interested in the commutative part of the quantum homology ring (which in the case $\mathcal{F} = \mathbb{Z}_2$ is, of course, the whole quantum homology ring). For this purpose we introduce the following notation:

We denote by $QH_\bullet(M)$ the whole quantum homology $QH_*(M)$ if $\mathcal{F} = \mathbb{Z}_2$ and the even-degree part of $QH_*(M)$ if $\mathcal{F} = \mathbb{C}$.

In general, given a topological space X, we denote by $H_\bullet(X; \mathcal{F})$ the whole singular homology group $H_*(X; \mathcal{F})$ if $\mathcal{F} = \mathbb{Z}_2$ and the even-degree part of $H_*(X; \mathcal{F})$ if $\mathcal{F} = \mathbb{C}$.

Thus, in our notation the ring $QH_\bullet(M) = H_\bullet(M; \mathcal{F}) \otimes_{\mathcal{F}} \Lambda$ is always a commutative subring with unity of $QH_*(M)$ and a module of finite rank over Λ. We will identify Λ with a subring of $QH_\bullet(M)$ by $\lambda \mapsto [M] \otimes \lambda$.

1.3 An hierarchy of rigid subsets within Floer theory

Fix a non-zero idempotent $a \in QH_{2n}(M)$ (by obvious grading considerations the degree of every idempotent equals $2n$). We shall deal with spectral invariants $c(a, H)$, where $H = H_t : M \to \mathbb{R}$, $t \in \mathbb{R}$, is a smooth time-dependent and 1-periodic in time Hamiltonian function on M, or $c(a, \phi_H)$, where ϕ_H is an element of the universal cover $\widetilde{Ham}(M)$ of $Ham(M)$ represented by an identity-based path given by the time-1 Hamiltonian flow generated by H. If H is *normalized*, meaning that $\int_M H_t \omega^{\dim M/2} = 0$ for all t, then $c(a, H) = c(a, \phi_H)$. These invariants, which nowadays are standard objects of the Floer theory, were introduced in [45] (cf. [59] in the aspherical case; also see [42],[43] for an earlier version of the construction and [22] for a summary of definitions and results in the monotone case).

Disclaimer: Throughout the paper we tacitly assume that (M, ω) (as well as $(M \times \mathbb{T}^2, \bar{\omega})$, when we speak of stable displaceability) belongs to the class $\mathcal{S}$ of closed symplectic manifolds for which the spectral invariants are well defined and enjoy the standard list of properties (see e.g. [41, Theorem 12.4.4]). For instance, $\mathcal{S}$ contains all symplectically aspherical and spherically monotone manifolds. Furthermore, $\mathcal{S}$ contains all symplectic manifolds M^{2n} for which, on one hand, either $c_1 = 0$ or the minimal Chern number (on $H_2^S(M)$) is at least $n - 1$ and, on the other hand, $[\omega](H_2^S(M))$ is a discrete subgroup of $\mathbb{R}$ (cf. [64]). The general belief is that the class $\mathcal{S}$ includes **all** symplectic manifolds.

Define a functional $\zeta : C^\infty(M) \to \mathbb{R}$ by

$$\zeta(H) := \lim_{l \to +\infty} \frac{c(a, lH)}{l} \tag{2}$$

It is shown in [23] that the functional ζ has some very special algebraic properties (see Theorem 3.6) which form the axioms of *a partial symplectic quasi-state* introduced in [23]. The next definition is motivated in part by the work of Albers [2].

Definition 1.1. A closed subset $X \subset M$ is called *heavy* (with respect to ζ or with respect to a used to define ζ) if

$$\zeta(H) \geq \inf_X H \ \ \forall H \in C^\infty(M) \ , \tag{3}$$

and is called *superheavy* (with respect to ζ or a) if

$$\zeta(H) \leq \sup_X H \ \ \forall H \in C^\infty(M) \ . \tag{4}$$

The default choice of an idempotent a is the unity $[M] \in QH_*(M)$. In this case, as we shall see below, the collections of heavy and superheavy sets satisfy the properties listed in Section 1.1 and include the examples therein. In view of potential applications (including geometric obstructions to semi-simplicity of the quantum homology), we shall work, whenever possible, with general idempotents.

The asymmetry between $\sup_X H$ and $\inf_X H$ is related to the fact that the spectral numbers satisfy a triangle inequality $c(a * b, \phi_F \phi_G) \leq c(a, \phi_F) + c(b, \phi_G)$, while there may not be a suitable inequality "in the opposite direction". In the case when such an "opposite" inequality exists (e.g. when $a = b$ is an idempotent and ζ defined by it is a genuine *symplectic quasi-state* – see Section 1.6 below) the symmetry between $\sup_X H$ and $\inf_X H$ gets restored and the classes of heavy and superheavy sets coincide.

Let us emphasize that the notion of (super)heaviness depends on the choice of a coefficient ring for the Floer theory. In this paper the coefficients for the Floer theory will be either $\mathbb{Z}_2$ or $\mathbb{C}$ depending on the situation. Unless otherwise stated, our results on (super)heavy subsets are valid for any choice the coefficients.

The group *Symp*(M) of all symplectomorphisms of M acts naturally on $H_*(M; \mathcal{F})$ and hence on $QH_*(M) = H_*(M; \mathcal{F}) \otimes_{\mathcal{F}} \Lambda$. Clearly, the identity component *Symp*$_0(M)$ of *Symp*(M) acts trivially on $QH_*(M)$ and hence for any idempotent $a \in QH_*(M)$ the corresponding ζ is *Symp*$_0(M)$-invariant. Thus the image of a (super)heavy set under an element of *Symp*$_0(M)$ is again a (super)heavy set with respect to the same idempotent a. If a is invariant under the action of the whole *Symp* (M) (for instance, if $a = [M]$) the classes of heavy and superheavy sets with respect to a are invariant under the action of the whole *Symp* (M) in agreement with the **invariance** property presented in Section 1.1 above.

Let us mention also that the collections of (super)heavy sets enjoy a stability property under inclusions: If X, Y, $X \subset Y$, are closed subsets of M and X is heavy (respectively, superheavy) with respect to an idempotent a then Y is also heavy (respectively, superheavy) with respect to the same a.

We are ready now to formulate the main results of the present section.

Theorem 1.2. *Assume a and ζ are fixed. Then*

(i) Every superheavy set is heavy, but, in general, not vice versa.

(ii) Every heavy subset is stably non-displaceable.

(iii) Every superheavy set intersects every heavy set. In particular, a superheavy set cannot be displaced by a **symplectic** *(not necessarily Hamiltonian) isotopy and if the idempotent a is invariant under the symplectomorphism group of (M, ω) (e.g. if $a = [M]$), every superheavy set is strongly non-displaceable.*

The following theorem discusses the relation between heaviness/superheaviness properties with respect to different idempotents. In particular, it shows that $[M]$ plays a special role among all the other non-zero idempotents in $QH_*(M)$.

Theorem 1.3. *Assume a is a non-zero idempotent in the quantum homology. Then*

(i) Every set that is superheavy with respect to $[M]$ is also superheavy with respect to a.

(ii) Every set that is heavy with respect to a is also heavy with respect to $[M]$.

(iii) Assume that the idempotent a is a sum of non-zero idempotents $e_1, \ldots, e_l$ and assume that a closed subset $X \subset M$ is heavy with respect to a. Then X is heavy with respect to e_i for at least one i.

The next proposition shows that, in general, the heaviness of a set *does depend* on the choice of an idempotent in the quantum homology.

Proposition 1.4. *Consider the torus $\mathbb{T}^{2n}$ equipped with the standard symplectic structure $\omega = dp \wedge dq$. Let $M^{2n} = \mathbb{T}^{2n} \sharp \overline{\mathbb{C}P^n}$ be a symplectic blow-up of $\mathbb{T}^{2n}$ at one point (the blow up is performed in a small ball around the point). Assume that the Lagrangian torus $L \subset \mathbb{T}^{2n}$ given by $q = 0$ does not intersect the ball in $\mathbb{T}^{2n}$, where the blow up was performed.*

Then the proper transform of L (identified with L) is a Lagrangian submanifold of M, which is not heavy with respect to some non-zero idempotent $a \in QH_*(M)$ but heavy with respect to $[M]$. (Here we work with $\mathcal{F} = \mathbb{Z}_2$).

Next, consider direct products of (super)heavy sets. We start with the following convention on tensor products. Let Γ_i, $i = 1, 2$, be two countable subgroups of $\mathbb{R}$. Let E_i be a module over $\mathcal{K}_{\Gamma_i}$. We put

$$E_1 \widehat{\otimes}_{\mathcal{K}} E_2 = \left(E_1 \otimes_{\mathcal{K}_{\Gamma_1}} \mathcal{K}_{\Gamma_1+\Gamma_2} \right) \otimes_{\mathcal{K}_{\Gamma_1+\Gamma_2}} \left(E_2 \otimes_{\mathcal{K}_{\Gamma_2}} \mathcal{K}_{\Gamma_1+\Gamma_2} \right) . \qquad (5)$$

If E_1, E_2 are also rings we automatically assume that the middle tensor product is the tensor product of rings. In simple words, we extend both modules to $\mathcal{K}_{\Gamma_1+\Gamma_2}$-modules and consider the usual tensor product over $\mathcal{K}_{\Gamma_1+\Gamma_2}$.

Given two symplectic manifolds, (M_1, ω_1) and (M_2, ω_2), note that the subgroups of periods of the symplectic forms satisfy

$$\Gamma(M_1 \times M_2, \omega_1 \oplus \omega_2) = \Gamma(M_1, \omega_1) + \Gamma(M_2, \omega_2) \; .$$

Furthermore, due to the Künneth formula for quantum homology (see e.g. [41, Exercise 11.1.15] for the statement in the monotone case; the general case in our algebraic setup can be treated similarly) there exists a natural ring monomorphism linear over $\mathcal{K}_{\Gamma_1+\Gamma_2}$

$$QH_{2n_1}(M_1) \widehat{\otimes}_{\mathcal{K}} QH_{2n_2}(M_2) \hookrightarrow QH_{2n_1+2n_2}(M_1 \times M_2) \; ,$$

We shall fix a pair of idempotents $a_i \in QH_*(M_i)$, $i = 1, 2$. The notions of (super)heaviness in M_1, M_2 and $M_1 \times M_2$ are understood in the sense of idempotents a_1, a_2 and $a_1 \otimes a_2$ respectively.

Theorem 1.5. *Assume that X_i is a heavy (resp. superheavy) subset of M_i with respect to some idempotent a_i, $i = 1, 2$. Then the product $X_1 \times X_2$ is a heavy (resp. superheavy) subset of M with respect to the idempotent $a_1 \otimes a_2 \in QH_\bullet(M_1 \times M_2)$.*

An important class of superheavy sets is given by stable stems introduced and illustrated in Section 1.1.

Theorem 1.6. *Every stable stem is a superheavy subset with respect to any non-zero idempotent $a \in QH_*(M)$. In particular, it is strongly and stably non-displaceable.*

In the next section we present an example of stable stems coming from Hamiltonian torus actions.

1.4 Hamiltonian torus actions

Fibers of the moment maps of Hamiltonian torus actions form an interesting playground for testing the various notions of displaceability and heaviness introduced above. Throughout the paper we deal with *effective* actions only, that is we assume that the map $\varphi : \mathbb{T}^k \to Ham(M)$ defining the action is a monomorphism. Furthermore, we assume that the moment map $\Phi = (\Phi_1, \ldots, \Phi_k) : M \to \mathbb{R}^k$ of the action is normalized: Φ_i is a normalized Hamiltonian for all $i = 1, \ldots, k$. By the Atiyah-Guillemin-Sternberg theorem [6], [30], the image $\Delta = \Phi(M)$ of Φ is a k-dimensional convex polytope, called the *moment polytope.* The subsets $\Phi^{-1}(p)$, $p \in \Delta$, are called *fibers* of the moment map. A torus action is called *compressible* if the image of the homomorphism $\varphi_\sharp : \pi_1(\mathbb{T}^k) \to \pi_1(Ham(M))$, induced by the action φ, is a finite group.

Theorem 1.7. *Assume that (M, ω) is equipped with a compressible Hamiltonian $\mathbb{T}^k$-action with moment map Φ and moment polytope Δ. Let $Y \subset \Delta$ be any closed convex subset which does not contain 0. Then the subset $\Phi^{-1}(Y)$ is stably displaceable. In particular, the fiber $\Phi^{-1}(0)$ is a stable stem.*

Note that for symplectic toric manifolds, that is when $2k = \dim M$, the point 0 is the barycenter of the moment polytope with respect to the Lebesgue measure. This follows from our assumption on the normalization of the moment map.

Theorems 1.6 and 1.7 imply that the fiber $\Phi^{-1}(0)$ of a compressible torus action is stably non-displaceable, and thus we get the complete description of stably displaceable fibers for such actions.

In the case when the action is not compressible, the question of the complete description of stably non-displaceable fibers remains open. We make a partial progress in this direction by presenting at least one such fiber, called *the special fiber*, explicitly in the case when (M,ω) is spherically monotone:

$$[\omega]|_{H_2^S(M)} = \kappa\, c_1(TM)|_{H_2^S(M)}\,, \quad \kappa > 0\,.$$

The special fiber can be described via the mixed action-Maslov homomorphism introduced in [49]: Let (M^{2n},ω) be a spherically monotone symplectic manifold, and let $\{f_t\}, t \in [0,1]$, be any loop of Hamiltonian diffeomorphisms, with $f_0 = f_1 = \mathbf{1}$, generated by a 1-periodic normalized Hamiltonian function $F(x,t)$. The orbits of any Hamiltonian loop are contractible due to the standard Floer theory[1]. Pick any point $x \in M$ and any disc $u : \mathbb{D}^2 \to M$ spanning the orbit $\gamma = \{f_t x\}$. Define the action[2] of the orbit by

$$\mathcal{A}_F(\gamma,u) := \int_0^1 F(\gamma(t),t)dt - \int_{\mathbb{D}^2} u^*\omega\,.$$

Trivialize the symplectic vector bundle $u^*(TM)$ over $\mathbb{D}^2$ and denote by $m_F(\gamma,u)$ the Maslov index of the loop of symplectic matrices corresponding to $\{f_{t*}\}$ with respect to the chosen trivialization. One readily checks that, in view of the spherical monotonicity, the quantity

$$I(F) := -\mathcal{A}_F(\gamma,u) - \frac{\kappa}{2} m_F(\gamma,u)$$

does not depend on the choice of the point x and the disc u, and is invariant under homotopies of the Hamiltonian loop $\{f_t\}$. In fact, I is a well defined homomorphism from $\pi_1(Ham(M))$ to $\mathbb{R}$ (see [49], [68]).

Assume again that $\varphi : \mathbb{T}^k \to Ham(M,\omega)$ is a Hamiltonian torus action. Write $\varphi_\sharp$ for the induced homomorphism of the fundamental groups. Since the target space $\mathbb{R}^k$ of the moment map Φ is naturally identified with $\mathrm{Hom}(\pi_1(\mathbb{T}^k),\mathbb{R})$, the pull back $-\varphi_\sharp^* I$ of the mixed action-Maslov homomorphism with the reversed sign can be considered as a point of $\mathbb{R}^k$. We call it *a special point* and denote by p_{spec}. The preimage $\Phi^{-1}(p_{spec})$ is called *the special fiber* of the moment map. In the case $k=1$, when Φ is a real-valued function on M, we will call p_{spec} *the special value* of Φ.

[1]The Floer theory guarantees the existence of at least one contractible periodic orbit – this is not obvious *a priori* if $\{f_t\}$ is not an autonomous flow. Since all the orbits of $\{f_t\}$ are homotopic, all of them are contractible.

[2]Note that our action functional and the one in [49] are of opposite signs.

If $k = n$ and M is a symplectic toric manifold, then p_{spec} can be defined in purely combinatorial terms involving only the polytope Δ. Namely, pick a vertex $\mathbf{x}$ of Δ. Since Δ in this case is a *Delzant polytope* [20], there is a unique (up to a permutation) choice of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ which

- originate at $\mathbf{x}$;
- span the n rays containing the edges of Δ adjacent to $\mathbf{x}$;
- form a basis of $\mathbb{Z}^n$ over $\mathbb{Z}$.

Proposition 1.8.

$$p_{spec} = \mathbf{x} + \kappa \sum_{i=1}^{n} \mathbf{v}_i. \tag{6}$$

Proof. The vertices of the moment polytope are in one-to-one correspondence with the fixed points of the action. Let $x \in M$ be the fixed point corresponding to the vertex $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$. Then the vectors $\mathbf{v}_j = (v_j^1, \ldots, v_j^n)$, $j = 1, \ldots, n$, are simply the weights of the isotropy $\mathbb{T}^n$-action on T_xM. Since the definition of the mixed action-Maslov invariant of a Hamiltonian circle action does not depend on the choice of a 1-periodic orbit and a disc spanning it, let us compute all I_i, $l = 1, \ldots, n$, using the constant periodic orbit concentrated at the fixed point x and the constant disc u spanning it. Clearly,

$$\mathcal{A}_{\Phi_i}(x, u) = \Phi_i(x) = \mathbf{x}_i \ \text{ and } \ m_{\Phi_i}(x, u) = 2 \sum_{j=1}^{n} v_j^i \ \ \forall i = 1, \ldots, n,$$

which readily yields formula (6). □

E.Shelukhin pointed out to us that by summing up equations (6) over all the vertices $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)} \in \mathbb{R}^n$ of the moment polytope, one readily gets that $p_{spec} = \frac{1}{m} \sum_i \mathbf{x}^{(i)}$.

Theorem 1.9. *Assume M^{2n} is a spherically monotone symplectic manifold equipped with a Hamiltonian $\mathbb{T}^k$-action. Then the special fiber of the moment map is superheavy with respect to any (non-zero) idempotent $a \in QH_{2n}(M)$. In particular, it is stably and strongly non-displaceable.*

Let us mention that, in particular, the special fiber is non-empty and so $p_{spec} \in \Delta$. Moreover p_{spec} is an interior point of Δ – otherwise $\Phi^{-1}(p_{spec})$ is isotropic of dimension $< n$ and hence displaceable (see e.g. [9]).

Remark 1.10. If $\dim M = 2 \dim \mathbb{T}^k$ (that is we deal with a symplectic toric manifold), the special fiber, say L, is a Lagrangian torus. In fact, this torus is monotone: for every $D \in \pi_2(M, L)$ we have

$$\int_D \omega = \kappa \cdot m^L(D) \ ,$$

where m^L stands for the Maslov class of L. This is an immediate consequence of the definitions.

Remark 1.11. Note that when M is spherically monotone and the action is compressible Theorems 1.7 and 1.9 match each other: in this case $p_{spec} = 0$ and therefore the special fiber is a stable stem by Theorem 1.7. It is unknown whether this property persists for the special fibers of non-compressible actions.

Example 1.12. Let M be the monotone symplectic blow up of $\mathbb{C}P^2$ at k points ($0 \le k \le 3$) which is equivariant with respect to the standard $\mathbb{T}^2$-action and which is performed away from the Clifford torus in $\mathbb{C}P^2$. Since the blow-up is equivariant, M comes equipped with a Hamiltonian $\mathbb{T}^2$-action extending the $\mathbb{T}^2$-action on $\mathbb{C}P^2$. The Clifford torus is a fiber of the moment map of the $\mathbb{T}^2$-action on $\mathbb{C}P^2$. Let $L \subset M$ be the Lagrangian torus which is the proper transform of the Clifford torus under the blow-up – it is a fiber of the moment map of the $\mathbb{T}^2$-action on M. Using Proposition 1.8 it is easy to see that L is the special fiber of M. According to Theorem 1.9, it is stably and strongly non-displaceable. In fact, it is a stem: the displaceability of all the other fibers was checked for $k = 0$ in [10], for $k = 1$ in [23] and for $k = 2, 3$ in [40].

We refer to Section 1.7.2 for further discussion of related problems and very recent advances.

Digression: Calabi vs. action-Maslov. The method used to prove Theorem 1.9 also allows to prove the following result involving the mixed action-Maslov homomorphism. Denote by $\mathrm{vol}\,(M)$ the symplectic volume of M. Consider the function $\mu : \widetilde{Ham}\,(M) \to \mathbb{R}$ defined by

$$\mu(\phi_H) := -\mathrm{vol}\,(M) \lim_{l \to +\infty} c(a, \phi_H^l)/l.$$

In the case when a is the unity in a field that is a direct summand in the decomposition of the $\mathcal{K}$-algebra $QH_{2n}(M,\omega)$, as an algebra, into a direct sum of subalgebras, μ is a homogeneous quasi-morphism on $\widetilde{Ham}(M)$ called *Calabi quasi-morphism* [22],[24],[46]; in the general case it has weaker properties [23]. With this language the functional ζ (on normalized functions) is induced (up to a constant factor) by the pull-back of μ to the Lie algebra of $\widetilde{Ham}(M)$.

Following P.Seidel we described in [22] the restriction of μ (in fact, for *any* spherically monotone M) on $\pi_1(Ham(M)) \subset \widetilde{Ham}(M)$ in terms of the Seidel homomorphism $\pi_1(Ham(M)) \to QH_*^{inv}(M)$, where $QH_*^{inv}(M)$ denotes the group of invertible elements in the ring $QH_*(M)$. Here we give an alternative description of $\mu|_{\pi_1(Ham(M))}$ in terms of the mixed action-Maslov homomorphism I which, in turn, also provides certain information about the Seidel homomorphism.

Theorem 1.13. *Assume M is spherically monotone and let μ be defined as above for some non-zero idempotent $a \in QH_*(M)$. Then*

$$\mu|_{\pi_1(Ham(M))} = \mathrm{vol}\,(M) \cdot I.$$

Note that, in particular, $\mu|_{\pi_1(Ham(M))}$ does not depend on a used to define μ. The theorem also implies that μ descends to a quasi-morphism on $Ham(M)$ if and only if $I : \pi_1(Ham(M)) \to \mathbb{R}$ vanishes identically (since μ descends to a quasi-morphism on $Ham(M)$ if and only if $\mu|_{\pi_1(Ham(M))} \equiv 0$ – see e.g. [22], Prop. 3.4). The proof of the theorem is given in Section 9.1.

Let us mention also that, interestingly enough, the homomorphism I coincides with the restriction to $\pi_1(Ham(M))$ of yet another quasi-morphism on $\widetilde{Ham}(M)$ constructed by P.Py (see [52, 53]).

Digression: Action-Maslov homomorphism and Futaki invariant. This remark grew from an observation pointed out to us by Chris Woodward – we are grateful to him for that. Assume that our symplectic manifold M is complex Kähler (i.e. the symplectic structure on M is induced by the Kähler one) and Fano (by this we mean here that $[\omega] = c_1$). Assume also that a Hamiltonian $\mathbb{S}^1$-action $\{f_t\}$ preserves the Kähler metric and the complex structure. For instance, if M^{2n} is a symplectic toric manifold it can be equipped canonically with a complex structure and a Kähler metric invariant under the $\mathbb{T}^n$-action on M, hence under the action of any $\mathbb{S}^1$-subgroup $\{f_t\}$ of $\mathbb{T}^n$.

Let V be the Hamiltonian vector field generating the Hamiltonian flow $\{f_t\}$. Since $\{f_t\}$ preserves the complex structure, one can associate to V its *Futaki invariant* $\mathbb{F}(V) \in \mathbb{C}$ [29]. It has been checked by E.Shelukhin [63] that, up to a universal constant factor, this Futaki invariant is equal to the value of the mixed action-Maslov homomorphism on the loop $\{f_t\}$:

$$\mathbb{F}(V) = const \cdot I(\{f_t\}).$$

Note that if such an M admits a Kähler-Einstein metric then the Futaki invariant has to vanish [29] – thus if $I(\{f_t\}) \neq 0$ the manifold does not admit a Kähler-Einstein metric. Moreover, if M^{2n} is toric the opposite is also true: if the Futaki invariant vanishes for any V generating a subgroup of the torus $\mathbb{T}^n$ acting on M then M admits a Kähler-Einstein metric – this follows from a theorem by Wang and Zhu [67], combined with a previous result of Mabuchi [38]. In terms of the moment polytope, the vanishing of the Futaki invariant, and accordingly the existence of a Kähler-Einstein metric, on a Kähler Fano toric manifold means precisely that the special point of the polytope coincides with the barycenter.

1.5 Super(heavy) monotone Lagrangian submanifolds

Let (M^{2n}, ω) be a closed spherically monotone symplectic manifold with $[\omega] = \kappa \cdot c_1(TM)$ on $\pi_2(M)$, $\kappa > 0$. Let $L \subset M$ be a closed monotone Lagrangian submanifold with the minimal Maslov number $N_L \geq 2$. As usually, we put $N_L = +\infty$ if $\pi_2(M, L) = 0$. As before, we work with the basic field $\mathcal{F}$ which is either $\mathbb{Z}_2$ or $\mathbb{C}$. In the case $\mathcal{F} = \mathbb{C}$, we assume that L is relatively spin, that is L is orientable and the 2nd Stiefel-Whitney class of L is the restriction of some integral cohomology class of M.

Disclaimer: In the case $\mathcal{F} = \mathbb{C}$ the results of this section are conditional: We take for granted that Proposition 8.1 below, which was proved by Biran and Cornea [15] for homologies with $\mathbb{Z}_2$-coefficients, extends to homologies with $\mathbb{C}$-coefficients. In each of the specific examples below we will explicitly state which $\mathcal{F}$ we are using and whenever we use $\mathcal{F} = \mathbb{C}$ we assume that L is relatively spin.

Denote by j the natural morphism $j : H_\bullet(L; \mathcal{F}) \to H_\bullet(M; \mathcal{F})$. We say that L *satisfies the Albers condition* [2] if there exists an element $S \in H_\bullet(L; \mathcal{F})$ so that $j(S) \neq 0$ and

$$\deg S > \dim L + 1 - N_L\ .$$

We shall refer to such S as to *an Albers element* of L.

Example 1.14. Assume $[L] \in H_\bullet(L;\mathcal{F})$ and $j([L]) \in H_\bullet(M;\mathcal{F})$ is non-zero. This means precisely that $[L]$ is an Albers element of L.

A closed monotone Lagrangian submanifold L which satisfies this condition (and whose minimal Maslov number is greater than 1) will be called *homologically non-trivial* in M.

Theorem 1.15. *Let L be a closed monotone Lagrangian submanifold satisfying the Albers condition. Then L is heavy with respect to $[M]$. In particular, any homologically non-trivial Lagrangian submanifold is heavy with respect to $[M]$.*

Example 1.16. Assume that $\pi_2(M, L) = 0$. Then the homology class of a point is an Albers element of L, and hence L is heavy. Note that in this case heaviness cannot be improved to superheaviness: the meridian on the two-torus is heavy but not superheavy. Here we took $\mathcal{F} = \mathbb{Z}_2$.

Example 1.17 (Lagrangian spheres in Fermat hypersurfaces)**.** More examples of heavy (but not necessarily superheavy) monotone Lagrangian submanifolds can be constructed as follows[3].

Let $M \subset \mathbb{C}P^{n+1}$ be a smooth complex hypersurface of degree d. The pull-back of the standard symplectic structure from $\mathbb{C}P^{n+1}$ turns M into a symplectic manifold (of real dimension $2n$). If $d \geq 2$, then, as it is explained, for instance, in [12], M contains a Lagrangian sphere: M can be included into a family of algebraic hypersurfaces of $\mathbb{C}P^{n+1}$ with quadratic degenerations at isolated points and the vanishing cycle of such a degeneration can be realized by a Lagrangian sphere following [5], [21], [60], [61], [62].

Let $M \subset \mathbb{C}P^{n+1}$ be a projective hypersurface of degree d, $2 \leq d < n+2$. The minimal Chern number of M equals $N := n + 2 - d > 0$. Let $L^n \subset M^{2n}$ be a simply connected Lagrangian submanifold (for instance, a Lagrangian sphere).

First, consider the case when n is even, L is relatively spin and the Euler characteristics of L does not vanish (this is the case for a sphere). Then the

[3] We thank P.Biran for his indispensable help with these examples.

homology class $j([L]) \in H_n(M;\mathbb{Z})$ is non-zero: its self-intersection number in M up to the sign equals the Euler characteristic. Thus $[L]$ is an Albers element. (Here we use $\mathcal{F} = \mathbb{C}$). In view of Theorem 1.15, L is heavy with respect to $[M]$.

Second, suppose that n is of arbitrary parity but $n > 2d-3$, and no restriction on the Euler characteristics of L is assumed anymore. This yields $N_L = 2N > n+1$ and thus L satisfies the Albers condition with the class of a point P as an Albers element. Thus L is heavy with respect to $[M]$ – here we use $\mathcal{F} = \mathbb{Z}_2$.

Finally, fix $n \geq 3$ and an even number d such that $4 \leq d < n+2$. Consider a Fermat hypersurface of degree d

$$M = \{-z_0^d + z_1^d + \ldots + z_{n+1}^d = 0\} \subset \mathbb{C}P^{n+1} .$$

Its real part $L := M \cap \mathbb{R}P^{n+1}$ lies in the affine chart $z_0 \neq 0$ and is given by the equation

$$x_1^d + \ldots + x_{n+1}^d = 1,$$

where $x_j := \mathrm{Re}(z_j/z_0)$. Since d is even, L is an n-dimensional sphere. As it was explained above, L is heavy with respect to $[M]$ if either n is even (and $\mathcal{F} = \mathbb{C}$) or $n > 2d-3$ (and $\mathcal{F} = \mathbb{Z}_2$). However, in either case L *is not superheavy* with respect to $[M]$. Indeed, let $\Sigma_d \approx \mathbb{Z}_d$ be the group of complex roots of unity. Given a vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\Sigma_d)^{n+1}$, denote by f_α the symplectomorphism of M given by

$$f_\alpha(z_0 : z_1 : \ldots : z_{n+1}) = (z_0 : \alpha_1 z_1 : \ldots : \alpha_{n+1} z_{n+1}) . \tag{7}$$

If all $\alpha_j \in \mathbb{C} \setminus \mathbb{R}$, then $\alpha_j x \notin \mathbb{R}$ whenever $x \in \mathbb{R} \setminus \{0\}$, and thus $f_\alpha(L) \cap L = \emptyset$. Therefore L is strongly displaceable and the claim follows from the part (iii) of Theorem 1.2.

The next result gives a user-friendly sufficient condition of superheaviness.

Theorem 1.18. *Assume L is homologically non-trivial in M and assume $a \in QH_{2n}(M)$ is a non-zero idempotent divisible by $j([L])$ in $QH_\bullet(M)$, that is $a \in j([L]) * QH_\bullet(M)$. Then L is superheavy with respect to a.*

The homological non-triviality of L in the hypothesis of the theorem means just that $[L]$ is an Albers element of L (see Example 1.14). In fact, the theorem can be generalized to the cases when L has other Albers elements – see Remark 8.3 (ii).

Example 1.19 (Lagrangian spheres in quadrics)**.** Here we work with $\mathcal{F} = \mathbb{C}$. Let M be the real part of the Fermat quadric $M = \{-z_0^2 + \sum_{j=1}^{n+1} z_j^2 = 0\}$. Assume that n is even and L is a simply connected Lagrangian submanifold with non-vanishing Euler characteristic (e.g. a Lagrangian sphere). Under this assumption, $[L] \in H_\bullet(L)$ and $j([L]) \neq 0$, since L has non-vanishing self-intersection. Denote by $p \in H_*(M; \mathcal{F})$ the class of a point. The quantum homology ring of M was described by Beauville in [8]. In particular, $p * p = w^{-2}[M]$, where $w = s^{\kappa n} q^n$. Thus

$$a_{\pm} := \frac{[M] \pm pw}{2}$$

are idempotents. One can show that $j([L])$ divides a_- and hence L is a_--superheavy. Since a_- is invariant under the action of *Symp*(M), the manifold L is strongly non-displaceable.

For simplicity, we present the calculation in the case $n = 2$ – the general case is absolutely analogous. The 2-dimensional quadric is symplectomorphic to $(\mathbb{S}^2 \times \mathbb{S}^2, \omega \oplus \omega)$. Denote by A and B the classes of $[\mathbb{S}^2] \times$ [point] and [point] $\times [\mathbb{S}^2]$ respectively. Since the symplectic form vanishes on $j([L])$ we get that $j([L]) = l(B - A)$ with $l \neq 0$. It is known that $A * B = p$ and $B * B = w^{-1}[M]$. Thus $j([L]) * \frac{1}{2l} wB = a_-$, that is $j([L])$ divides a_-.

In particular, the Lagrangian anti-diagonal

$$\Delta := \{(x, y) \in \mathbb{S}^2 \times \mathbb{S}^2 \ : \ x = -y\} \ ,$$

which is diffeomorphic to the 2-sphere, is superheavy with respect to a_-. It is unknown whether Δ is super-heavy with respect to a_+. Further information on superheavy Lagrangian submanifolds in the quadrics can be extracted from [15].

Example 1.20 (A non-heavy monotone Lagrangian torus in $\mathbb{S}^2 \times \mathbb{S}^2$)**.** Consider the quadric $M = \mathbb{S}^2 \times \mathbb{S}^2$ from the previous example. We will think of $\mathbb{S}^2$ as of the unit sphere in $\mathbb{R}^3$ whose symplectic form is the area form divided by 4π. We will work again with $\mathcal{F} = \mathbb{C}$. Interestingly enough, such an M contains a monotone Lagrangian torus that is not heavy with respect to a_-.

Namely, consider a submanifold K given by equations[4]

$$K = \{(x, y) \in \mathbb{S}^2 \times \mathbb{S}^2 \ : \ x_1 y_1 + x_2 y_2 + x_3 y_3 = -\frac{1}{2}, x_3 + y_3 = 0\} \ .$$

[4]We thank Frol Zapolsky for his help with calculations in this example.

One readily checks that K is a monotone Lagrangian torus with $N_K = 2$ which represents a zero element in $H_2(M; \mathcal{F})$ (both with $\mathcal{F} = \mathbb{C}$ and $\mathcal{F} = \mathbb{Z}_2$). Thus $H_\bullet(K; \mathcal{F})$ does not contain any Albers element. Furthermore, K is disjoint from the Lagrangian anti-diagonal Δ and hence is not heavy with respect to a_- since, as it was shown above, Δ is superheavy with respect to a_-. In particular, K is *an exotic monotone torus*: it is not symplectomorphic to the Clifford torus which is a stem and hence a_--superheavy. A further study of exotic tori in products of spheres is currently being carried out by Y.Chekanov and F.Schlenk.

It is an interesting problem to understand whether K is superheavy with respect to a_+, or at least non-displaceable. Identify $M \setminus$ {the diagonal} with the unit co-ball bundle of the 2-sphere. After such an identification Δ corresponds to the zero section, while K corresponds to a monotone Lagrangian torus, say K'. Interestingly enough, the Lagrangian Floer homology of K' in $T^*\mathbb{S}^2$ (with $\mathcal{F} = \mathbb{Z}_2$) does not vanish as was shown by Albers and Frauenfelder in [3], and thus K is not displaceable in $M \setminus$ {the diagonal}. Thus the question on (non)-displaceability of K is related to understanding of the effect of the compactification of the unit co-ball bundle to $\mathbb{S}^2 \times \mathbb{S}^2$.

The proofs of theorems above are based on spectral estimates due to Albers [2] and Biran-Cornea [15]. Furthermore, the results above admit various generalizations in the framework of Biran-Cornea theory of quantum invariants for monotone Lagrangian submanifolds, see [15] and the discussion in Section 8 below.

1.6 An effect of semi-simplicity

Recall that a commutative (finite-dimensional) algebra Q over a field $\mathcal{A}$ is called *semi-simple* if it splits into a direct sum of fields as follows: $Q = Q_1 \oplus \ldots \oplus Q_d$, where

- each $Q_i \subset Q$ is a finite-dimensional linear subspace over $\mathcal{A}$;
- each Q_i is a field with respect to the induced ring structure;
- the multiplication in Q respects the splitting:

$$(a_1, \ldots, a_d) \cdot (b_1, \ldots, b_d) = (a_1 b_1, \ldots, a_d b_d).$$

A classical theorem of Wedderburn (see e.g. [66], §96) implies that the semi-simplicity is equivalent to the absence of nilpotents in the algebra.

Remark 1.21. Assume that the $\mathcal{K}$-algebra $QH_{2n}(M,\omega)$ splits, as an algebra, into a direct sum of two algebras, at least one of which is a field, and let e be the unity in that field. In particular, this is the case when $QH_{2n}(M,\omega) = Q_1 \oplus \ldots \oplus Q_d$ is semi-simple and e is the unity in one of the fields Q_i. A slight generalization of the argument in [23, 46] (see [24], the remark on pp. 56-57) shows that the partial quasi-state $\zeta(e,\cdot)$ associated to e is $\mathbb{R}$-homogeneous (and not just $\mathbb{R}_+$-homogeneous as in the general case).

This immediately yields that *every set which is heavy with respect to e is automatically superheavy with respect to e.*

In fact, in this situation ζ is a genuine *symplectic quasi-state* in the sense of [23] and, in particular, a *topological quasi-state* in the sense of Aarnes [1] (see [23] for details). In [1] Aarnes proved an analogue of the Riesz representation theorem for topological quasi-states which generalizes the correspondence between genuine states (that is positive linear functionals on $C(M)$) and measures. The object τ_ζ corresponding to a quasi-state ζ is called a *quasi-measure* (or a *topological measure*). With this language in place, the sets that are (super)heavy with respect to ζ are nothing else but the closed sets of the full quasi-measure τ_ζ. Any two such sets have to intersect for the following basic reason: any quasi-measure is finitely additive on disjoint closed subsets and therefore if two closed subsets of M of the full quasi-measure do not intersect, the quasi-measure of their union must be greater than the total quasi-measure of M, which is impossible.

Example 1.22. In this example we again assume that $\mathcal{F} = \mathbb{Z}_2$. Let $M = \mathbb{C}P^n$ be equipped with the Fubini-Study symplectic structure ω, normalized so that $[\omega] = c_1$, and let $A \in H_{2n-2}(M)$ be the homology class of the hyperplane. One readily verifies the following $\mathcal{K}$-algebra isomorphism

$$QH_{2n}(M) \cong \mathcal{K}[X]/\langle X^{n+1} - u^{-1}\rangle,$$

where

$$\mathcal{K} = \mathbb{Z}_2[[u] = \{z_k u^k + z_{k-1}u^{k-1} + \ldots, z_i \in \mathbb{Z}_2 \ \forall i\}$$

is the field of Laurent-type series in $u := s^{n+1}$ with coefficients in $\mathbb{Z}_2$ and $X = qA$. Since no root of degree 2 or more of u^{-1} is contained in $\mathcal{K}$, the polynomial P is irreducible over $\mathcal{K}$ for any n (see e.g. [34], Theorem 9.1) and therefore $QH_{2n}(M)$ is a field. Hence the collections of heavy and superheavy sets with respect to the fundamental class coincide.

We claim that $L := \mathbb{R}P^n \subset \mathbb{C}P^n$ is superheavy. The case $n = 1$ corresponds to the equator of the sphere, which is known to be a stable stem. For $n \geq 2$, note that $N_L = n+1$ and $S = [\mathbb{R}P^2]$ is an Albers element of L. Therefore, L is $[M]$-heavy by Theorem 1.15, and hence superheavy.

The next result follows directly from Theorem 1.3 (iii) and Remark 1.21:

Theorem 1.23. *Assume that $QH_{2n}(M)$ is semi-simple and splits into a direct sum of d fields whose unities will be denoted by $e_1, \ldots, e_d$. Assume that a closed subset $X \subset M$ is heavy with respect to a non-zero idempotent a – as one can easily see, such an idempotent has to be of the form $a = e_{j_1} + \ldots + e_{j_l}$ for some $1 \leq j_1 < \ldots < j_l \leq d$. Then X is superheavy with respect to some e_{j_i}, $1 \leq i \leq l$.*

The theorem yields the following geometric characterization of non-semi-simplicity of $QH_{2n}(M)$. Namely, define the *symplectic Torelli group* as the group of all symplectomorphisms of M which induce the identity map on $H_\bullet(M;\mathcal{F})$. For instance, this group contains $Symp_0(M)$. Note that any element of the symplectic Torelli group acts trivially on the quantum homology of M and hence maps sets (super)heavy with respect to an idempotent a to sets (super)heavy with respect to a.

Now Theorem 1.23 readily implies the following

Corollary 1.24. *Assume that (M,ω) contains a closed subset X which is heavy with respect to a non-zero idempotent and displaceable by a symplectomorphism from the symplectic Torelli group. Then $QH_{2n}(M)$ is not semi-simple.*

The simplest examples are provided by sets of the form $X \times \{\text{a meridian}\}$ in $M \times \mathbb{T}^2$ with a heavy X.

Another result in the same vein is as follows[5]. Given a set Y of positive integers, put $\beta_Y(M) = \sum_{i \in Y} \beta_i(M)$, where $\beta_i(M)$ stands for the i-th Betti number of M over $\mathcal{F}$.

[5]In the case $\mathcal{F} = \mathbb{C}$, Theorem 1.25 is conditional, see the disclaimer in the previous section.

Theorem 1.25. *Assume that either of the following (not mutually excluding) conditions holds:*

(a) M contains $m > \beta_Y(M) + 1$ pair-wise disjoint closed monotone Lagrangian submanifolds whose minimal Maslov numbers are greater than $n+1$ and belong to a set Y of positive integers.

(b) M contains pair-wise disjoint homologically non-trivial Lagrangian submanifolds[6] whose fundamental classes, viewed as (non-zero) elements of $H_\bullet(M;\mathcal{F})$, are linearly dependent over $\mathcal{F}$.

(In the case $\mathcal{F} = \mathbb{C}$ assume that all the Lagrangian submanifolds above are also relatively spin.)

Then $QH_{2n}(M)$ is not semi-simple.

The proof is given in Section 8.

Example 1.26. For instance, if all the Lagrangian submanifolds from part (a) of the theorem are simply connected, their minimal Maslov numbers are equal to $2N$, so that the set Y consists of one element: $Y = \{2N\}$. Thus if $2N > n + 1$ and $QH_{2n}(M)$ is semi-simple, M cannot contain more than $\beta_{2N}(M) + 1$ pair-wise disjoint simply-connected Lagrangians (provided all of them are relatively spin if we work with $\mathcal{F} = \mathbb{C}$).

Example 1.27. Set $\mathcal{F} = \mathbb{C}$. Fix $n \geq 11$ and an even number d such that $6 \leq d < (n + 3)/2$. Consider a Fermat hypersurface of degree d

$$M = \{-z_0^d + z_1^d + \ldots + z_{n+1}^d = 0\} \subset \mathbb{C}P^{n+1}.$$

As we already saw in Example 1.17, the manifold $L := M \cap \mathbb{R}P^{n+1}$ is an n-dimensional Lagrangian sphere. Consider the images $f_\alpha(L)$, where symplectomorphisms f_α are defined by (7). Note that, as long as $\alpha_j/\beta_j \neq \pm 1$ for all j, the Lagrangian spheres $f_\alpha(L)$ and $f_\beta(L)$ are disjoint. Using this observation, it is easy to find $d/2$ disjoint Lagrangian spheres in M.

The minimal Chern number N of M equals $n+2-d$, and so $2N$ lies in the interval $[n+2, 2n-4]$. In this case $\beta_{2N}(M) = 1$ (see e.g. [31]). Since $d/2 > 2$, we conclude from the previous example that $QH_{2n}(M)$ is not semi-simple. This conclusion agrees with the computation of $QH_*(M)$ by Beauville [8].

[6] See Example 1.14 for the definition. As in that example we again assume that all our Lagrangian submanifolds are closed, monotone and have minimal Maslov number greater than 1.

It would be interesting to find examples of symplectic manifolds where the quantum homology is not known *a priori* and where the above theorems are applicable. Let us mention that different obstructions to the semi-simplicity of $QH_\bullet(M)$ coming from Lagrangian submanifolds were recently found by Biran and Cornea [14].

1.7 Discussion and open questions

1.7.1 Strong displaceability beyond Floer theory?

Clearly, displaceability implies stable displaceability. The converse is not true, as the next example shows:

Example 1.28. Consider the complex projective space $\mathbb{C}P^n$ equipped with the Fubini-Study symplectic form (in our normalization the area of a line equals 1). Identify $\mathbb{C}P^n$ with the symplectic cut of the Euclidean ball $B(1) \subset \mathbb{C}^n$ (that is the boundary of $B(1)$ is collapsed to $\mathbb{C}P^{n-1}$ along the fibers of the Hopf fibration, see [36]), where $B(r) := \{\pi|z|^2 \leq r\}$. Then $B(r) \subset \mathbb{C}P^n$ is:

(i) displaceable for $r < 1/2$;

(ii) strongly non-displaceable but stably displaceable for $r \in [1/2, n/n+1)$;

(iii) strongly and stably non-displaceable for $r \geq n/n+1$.

It is instructive to analyze the techniques involved in the proofs: The strong non-displaceability result in (ii) is an immediate consequence of Gromov's packing-by-two-balls theorem, which is proved via the J-holomorphic variant of the theorem which states that there exists a J-holomorphic line in $\mathbb{C}P^n$ passing through any two points. In the case (iii) the ball $B(r)$ contains the Clifford torus, which is stably non-displaceable. This follows either from the fact that the Clifford torus is a stem (see [10]), or from non-vanishing of its Lagrangian Floer homology [16].

The displaceability of $B(r)$ in (i) follows from the explicit construction of the two balls packing (see [33]). The stable displaceability in (ii) is a direct consequence of Theorem 1.7 above: Indeed, consider the standard $\mathbb{T}^n$-action on $\mathbb{C}P^n$. The normalized moment polytope $\Delta \subset \mathbb{R}^n$ has the form $\Delta = \Delta_{stand} + w$ where Δ_{stand} is the standard simplex $\{\rho_i \geq 0, \sum \rho_i \leq 1\}$ in

$\mathbb{R}^n$, where $(\rho_1, \ldots, \rho_n)$ denote coordinates in $\mathbb{R}^n$, and $w = -\frac{1}{n+1}(1, \ldots, 1)$. Note that the ball $B(r)$ equals to $\Phi^{-1}(\Delta_r)$ where $\Delta_r := r \cdot \Delta_{stand} + w$. Note that Δ_r does not contain the origin exactly when $r \leq \frac{n}{n+1}$ which yields the stable displaceability in (ii) above.

A mysterious feature of Example 1.28 is as follows. On the one hand, we believe in the following general empiric principle: whenever one can establish the non-displaceability of a subset by means of the Floer homology theory, one gets for free the stable non-displaceability. On the other hand, we believe, following a philosophical explanation provided by Biran, that Gromov's packing-by-two-balls theorem may be extracted from some "operations" in Floer homology. Example 1.28 shows that at least one of these beliefs is wrong. It would be interesting to clarify this issue.

1.7.2 Heavy fibers of Poisson-commutative subspaces

It was shown in [23] that for any finite-dimensional Poisson-commutative subspace $\mathbb{A} \subset C^\infty(M)$ at least one of the fibers of its moment map Φ has to be non-displaceable.

Question. Is it true that at least one fiber of Φ has to be heavy (with respect to some non-zero idempotent $a \in QH_*(M)$)?

It is easy to construct an example of $\mathbb{A}$ whose moment map Φ has no superheavy fibers: take $\mathbb{T}^2$ with the coordinates p, q mod 1 on it and take $\mathbb{A}$ to be the set of all smooth functions depending only on p – the corresponding Φ defines the fibration of $\mathbb{T}^2$ by meridians none of which is superheavy.

Here is another question which concerns fibers of symplectic toric manifolds, i.e. fibers of a moment map Φ of an effective Hamiltonian $\mathbb{T}^n$-action on (M^{2n}, ω). Assume M is (spherically) monotone. Theorem 1.9 shows that in such a case the special fiber of M is superheavy, hence stably and strongly non-displaceable. In all the examples where it has been checked this turns out to be the only non-displaceable fiber of M.

Question. Is the special fiber for a monotone symplectic toric M always a stem? In particular, is it the only non-displaceable fiber of the moment map?

In the monotone case the special fiber is clearly the only heavy fiber of the moment map, because it is superheavy and any other heavy fiber would have had to intersect it. On the other hand, if we consider a Hamiltonian $\mathbb{T}^k$-action on M^{2n} with $k < n$ there can be more than one non-displaceable fiber of the moment map – for instance, because of purely topological obstructions: the simplest Hamiltonian $\mathbb{T}^1$-action on $\mathbb{C}P^2$ provides such an example. In the case of monotone symplectic toric manifolds of dimension bigger than 4 the question above is absolutely open.

After the first draft of this paper appeared, a remarkable progress in this direction has been achieved in the works by Cho [17] and Fukaya, Oh, Ohta and Ono [28]: In particular, it turns out that a non-monotone symplectic toric manifold can have more than one non-displaceable fiber – this happens already for certain equivariant blowups of $\mathbb{C}P^2$.

Organization of the paper:

In Section 2 we prove Theorem 1.7 which in particular states that the special fiber of a compressible torus action is a stable stem.

In Section 3 we sum up various preliminaries from Floer theory including basic properties of spectral invariants and partial symplectic quasi-states. In addition we spell out a useful property of the Conley-Zehnder index: it is a quasi-morphism on the universal cover of the symplectic group (see Proposition 3.5). For completeness we extract a proof of this property from [54]; alternatively, one can use the results of [19].

In Section 4 we prove parts (i) and (iii) of Theorem 1.2 and Theorem 1.3 on basic properties of (super)heavy sets.

In Section 5 we prove Theorem 1.5 on products of (super)heavy sets. Our approach is based on a quite general product formula for spectral invariants (Theorem 5.1), which is proved by a fairly lengthy algebraic argument.

In Section 6 we prove Theorem 1.2 (ii) on stable non-displaceability of heavy subsets. The argument involves a "baby version" of the above-mentioned product formula.

In Section 7 we prove superheaviness of stable stems.

In Section 8 we bring together the proofs of various results related to (super)heaviness of monotone Lagrangian submanifolds satisfying the Albers condition, including Theorems 1.15, 1.18, 1.25 and Proposition 1.4.

In Section 9 we prove Theorem 1.9 on superheaviness of special fibers of Hamiltonian torus actions on monotone symplectic manifolds. The proof is quite involved. In fact, two tricks enabled us to shorten our original argument: First, we use the Fourier transform on the space of rapidly decaying functions on the Lie coalgebra of the torus in order to reduce the problem to the case of Hamiltonian circle actions. Second, we systematically use the quasi-morphism property of the Conley-Zehnder index for asymptotic calculations with Hamiltonian spectral invariants. Finally, in Section 9.1 we prove Theorem 1.13.

Figure 1 sums up the hierarchy of the non-displaceability properties discussed above.

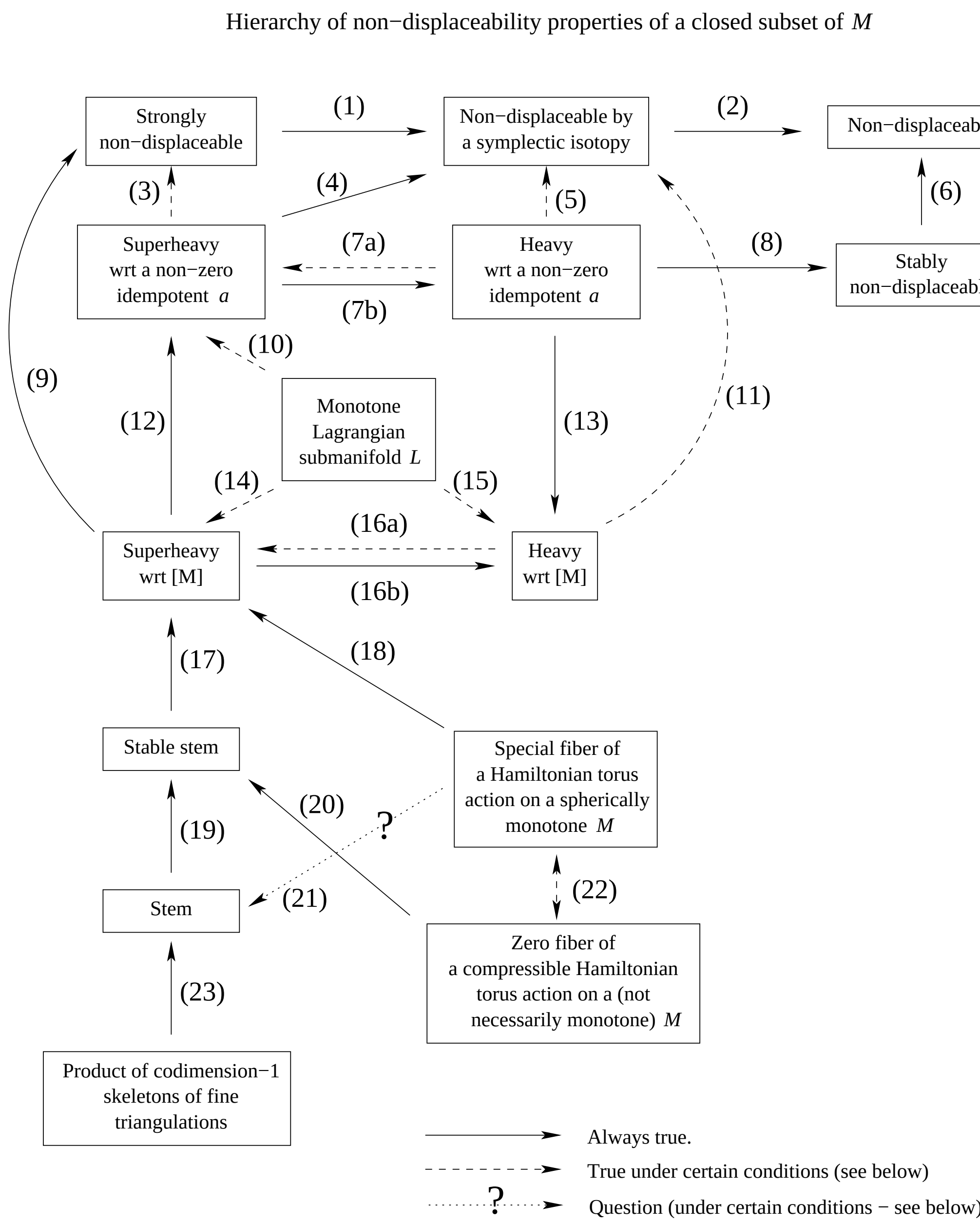

Figure 1: Hierarchy of non-displaceability properties

(1),(2),(6),(19) - Trivial.
(3) True if a is invariant under the action of the whole group $Symp(M)$ – Theorem 1.2, part (iii).
(4), (9) Theorem 1.2, part (iii).
(5) True if the algebra $QH_{2n}(M)$ is semi-simple – see Corollary 1.24.
(7a) True if the algebra $QH_{2n}(M)$ splits, as an algebra, into a direct sum of two algebras, at least one of which is a field, and a is the unity element in that field – see Remark 1.21.
(7b), (16b) Theorem 1.2, part (i).
(8) Theorem 1.2, part (ii).
(10) Theorem 1.18 (see the assumptions on L there).
(11) True if the algebra $QH_{2n}(M)$ is semi-simple – see Corollary 1.24.
(12) Theorem 1.3, part (i).
(13) Theorem 1.3, part (ii).
(14) Theorem 1.18 (see the assumptions on L there) with $a = [M]$ – i.e. $j(L)$ is invertible in $QH_\bullet(M)$.
(15) L satisfies the Albers condition – see Theorem 1.15.
(16a) True if $QH_{2n}(M)$ is a field – see Remark 1.21.
(17) Theorem 1.6.
(18) Theorem 1.9.
(20) Theorem 1.7.
(21) Is the special fiber for a monotone *symplectic toric* M always a stem? See Section 1.7.2.
(22) True if M is spherically monotone and the torus action is compressible – see Remark 1.11.
(23) See [23].

2 Detecting stable displaceability

For detecting stable displaceability of a subset of a symplectic manifold we shall use the following result (cf. [48, Chapter 6]).

Theorem 2.1. *Let X be a closed subset of a closed symplectic manifold (M, ω). Assume that there exists a contractible loop of Hamiltonian diffeomorphisms of (M, ω) generated by a normalized time-periodic Hamiltonian $H_t(x)$ so that $H_t(x) \neq 0$ for all $t \in [0, 1]$ and $x \in X$. Then X is stably displaceable.*

Proof. Denote by h_t the Hamiltonian loop generated by H. Let $h_t^{(s)}$ be its homotopy to the constant loop: $h_t^{(1)} = h_t$ and $h_t^{(0)} = \mathbf{1}$. Write $H^{(s)}(x,t)$ for the corresponding normalized Hamiltonians. Consider the family of diffeomorphisms Ψ_s of $M \times T^*\mathbb{S}^1$ given by

$$\Psi_s(x,r,\theta) = (h_\theta^{(s)} x, r - H^{(s)}(h_\theta^{(s)} x, \theta), \theta) \; .$$

One readily checks that $\Psi_s, s \in [0,1]$, is a Hamiltonian isotopy (not compactly supported). We claim that Ψ_1 displaces $Y := X \times \{r = 0\}$. Indeed, if $\Psi_1(x,0,\theta) \in Y$ we have $h_\theta x \in X$ and $H_\theta(h_\theta x) = 0$ which contradicts the assumption of the theorem. This completes the proof. □

Proof of Theorem 1.7: Choose a linear functional $F : \mathbb{R}^k \to \mathbb{R}$ with rational coefficients which is strictly positive on Y. Then for some sufficiently large positive integer N the Hamiltonian $H := N\Phi^*F$ generates a contractible Hamiltonian circle action on M and H is strictly positive on $X := \Phi^{-1}(Y)$. Thus X is stably displaceable in view of the previous theorem. □

3 Preliminaries on Hamiltonian Floer theory

3.1 Valuation on $QH_*(M)$

Define a function $\nu : \mathcal{K} \to \Gamma$ by

$$\nu(\sum z_\theta s^\theta) = \max\{ \; \theta \mid z_\theta \neq 0\} \; .$$

The convention is that $\nu(0) = -\infty$. In algebraic terms, $\exp \nu$ is a non-Archimedean absolute value on $\mathcal{K}$.

The function ν admits a natural extension to Λ and then to $QH_*(M)$ – abusing the notation we will denote all of them by ν. Namely, any element of $\lambda \in \Lambda$ can be uniquely represented as $\lambda = \sum_\theta u_\theta s^\theta$, where each u_θ belongs to $\mathcal{F}[q, q^{-1}]$, and any non-zero $a \in QH_*(M)$ can be uniquely represented as $a = \sum_i \lambda_i b_i$, $0 \neq \lambda_i \in \Lambda$, $0 \neq b_i \in H_*(M; \mathcal{F})$. Define

$$\nu(\lambda) := \max \big\{ \theta \mid u_\theta \neq 0 \big\},$$

$$\nu(a) := \max_i \nu(\lambda_i).$$

3.2 Hamiltonian Floer theory

We briefly recall the notation and conventions for the setup of the Hamiltonian Floer theory that will be used in the proofs.

Let $\mathcal{L}$ be the space of all smooth contractible loops $\gamma : \mathbb{S}^1 = \mathbb{R}/\mathbb{Z} \to M$. We will view such a γ as a 1-periodic map $\gamma : \mathbb{R} \to M$. Let $\mathbb{D}^2$ be the standard unit disk in $\mathbb{R}^2$. Consider a covering $\widetilde{\mathcal{L}}$ of $\mathcal{L}$ whose elements are equivalence classes of pairs (γ, u), where $\gamma \in \mathcal{L}$, $u : \mathbb{D}^2 \to M$, $u|_{\partial\mathbb{D}^2} = \gamma$ (i.e. $u(e^{2\pi\sqrt{-1}t}) = \gamma(t)$), is a (piecewise smooth) disk spanning γ in M and the equivalence relation is defined as follows: $(\gamma_1, u_1) \sim (\gamma_2, u_2)$ if and only if $\gamma_1 = \gamma_2$ and the 2-sphere $u_1\#(-u_2)$ vanishes in $H_2^S(M)$. The equivalence class of a pair (γ, u) will be denoted by $[\gamma, u]$. The group of deck transformations of the covering $\widetilde{\mathcal{L}} \to \mathcal{L}$ can be naturally identified with $H_2^S(M)$. An element $A \in H_2^S(M)$ acts by the transformation

$$A([\gamma, u]) = [\gamma, u\#(-A)]. \tag{8}$$

Let $F : M \times [0, 1] \to \mathbb{R}$ be a Hamiltonian function (which is time-periodic as we always assume). Set $F_t := F(\cdot, t)$. We will denote by f_t the Hamiltonian flow generated by F, meaning the flow of the time-dependent Hamiltonian vector field X_t defined by the formula

$$\omega(\cdot, X_t) = dF_t(\cdot) \ \ \forall t.$$

(Note our sign convention!)

Let $\mathcal{P}_F \subset \mathcal{L}$ be the set of all contractible 1-periodic orbits of the Hamiltonian flow generated by F, i.e. the set of all $\gamma \in \mathcal{L}$ such that $\gamma(t) = f_t(\gamma(0))$. Denote by $\widetilde{\mathcal{P}}_F$ the full lift of $\mathcal{P}_F$ to $\widetilde{\mathcal{L}}$.

Denote by $\mathrm{Fix}\,(F)$ the set of those fixed points of f that are endpoints of contractible periodic orbits of the flow:

$$\mathrm{Fix}\,(F) := \{x \in M \mid \exists \gamma \in \mathcal{P}_F, \quad x = \gamma(0)\}.$$

We say that F is *regular* if for any $x \in \mathrm{Fix}\,(F)$ the map $d_x f : T_xM \to T_xM$ does not have eigenvalue 1.

Recall that the *action functional* is defined on $\widetilde{\mathcal{L}}$ by the formula:

$$\mathcal{A}_F([\gamma, u]) = \int_0^1 F(\gamma(t), t)dt - \int_{\mathbb{D}^2} u^*\omega.$$

Note that

$$\mathcal{A}_F(Ay) = \mathcal{A}_F(y) + \omega(A) \tag{9}$$

for all $y \in \widetilde{\mathcal{L}}$ and $A \in H_2^S(M)$.

For a regular Hamiltonian F define a vector space $C(F)$ over $\mathcal{F}$ as the set of all formal sums

$$\sum_{i=1}^{k} \lambda_i y_i,\ \lambda_i \in \Lambda, y_i \in \widetilde{\mathcal{P}}_F,$$

modulo the relations

$$Ay = s^{-\omega(A)} q^{-c_1(A)} y,$$

for all $y \in \widetilde{\mathcal{P}}_F, A \in H_2^S(M)$. The grading on Λ together with the Conley-Zehnder index on elements of $\widetilde{\mathcal{P}}_F$ (see Section 3.3) defines a $\mathbb{Z}$-grading on $C(F)$. We will denote the i-th graded component by $C_i(F)$.

Given a loop $\{J_t\},\ t \in \mathbb{S}^1$, of ω-compatible almost complex structures, define a Riemannian metric on $\mathcal{L}$ by

$$(\xi_1, \xi_2) = \int_0^1 \omega(\xi_1(t), J_t \xi_2(t)) dt,$$

where $\xi_1, \xi_2 \in T_\gamma \mathcal{L}$. Lift this metric to $\widetilde{\mathcal{L}}$ and consider the negative gradient flow of the action functional $\mathcal{A}_F$. For a generic choice of the Hamiltonian F and the loop $\{J_t\}$ (such a pair (F, J) is called *regular*) the count of isolated gradient trajectories connecting critical points of $\mathcal{A}_F$ gives rise in the standard way [26], [32], [58] to a Morse-type differential

$$d : C(F) \to C(F),\ d^2 = 0. \tag{10}$$

The differential d is Λ-linear and has the graded degree -1. It strictly decreases the action. The homology, defined by d, is called the *Floer homology* and will be denoted by $HF_*(F, J)$. It is a Λ-module. Different choices of a regular pair (F, J) lead to natural isomorphisms between the Floer homology groups.

The following proposition summarizes a few basic algebraic properties of Floer complexes and Floer homology that will be important for us further. The proof is straightforward and we omit it.

Proposition 3.1.

1) Each $C_i(F)$ and each $HF_i(F,J)$, $i \in \mathbb{Z}$, is a finite-dimensional vector space over $\mathcal{K}$.

2) Multiplication by q defines isomorphisms $C_i(F) \to C_{i+2}(F)$ and $HF_i(F,J) \to HF_{i+2}(F,J)$ of $\mathcal{K}$-vector spaces.

3) For each $i \in \mathbb{Z}$ there exists a basis of $C_i(F)$ over $\mathcal{K}$ consisting of the elements of the form $q^l[\gamma, u]$, with $[\gamma, u] \in \widetilde{\mathcal{P}}_F$.

4) A finite collection of elements of the form $q^l[\gamma, u]$, $[\gamma, u] \in \widetilde{\mathcal{P}}_F$, lying in $C_0(F) \cup C_1(F)$ is a basis of the vector space $C_0(F) \oplus C_1(F)$ over the field $\mathcal{K}$ if and only if it is a basis of the module $C(F)$ over the ring Λ.

3.3 Conley-Zehnder and Maslov indices

In this section we briefly outline the definition and recall the relevant properties of the Conley-Zehnder index referring to [54, 58, 57] for details. In particular, we show that *the Conley-Zehnder index is a quasi-morphism on the universal cover $\widetilde{Sp\,(2k)}$ of the symplectic group $Sp(2k)$* (see Proposition 3.5 below), a fact which will be useful for asymptotic calculations with Floer homology in the next sections. There are several routes leading to this fact, which is quite natural since all homogeneous quasi-morphisms on $\widetilde{Sp\,(2k)}$ are proportional, and hence the same quasi-morphism admits quite dissimilar definitions [7]. We extract the quasi-morphism property from the paper of Robbin and Salamon [54] by bringing together several statements contained therein[7].

The Conley-Zehnder index assigns to each $[\gamma, u] \in \widetilde{\mathcal{P}}_F$ a number. Originally the Conley-Zehnder index was defined only for regular Hamiltonians [18] – in this case it is integer-valued and gives rise to a grading of the homology groups in Floer theory. Later the definition was extended in different ways by different authors to arbitrary Hamiltonians. We will use such an extension introduced in [54] (also see [57, 58]). In this case the Conley-Zehnder index may take also half-integer values.

Let k be a natural number. Consider the symplectic vector space $\mathbb{R}^{2k}$ with a symplectic form ω_{2k} on it. Denote by $p = (p_1, \dots, p_k), q = (q_1, \dots, q_k)$ the corresponding Darboux coordinates on the vector space $\mathbb{R}^{2k}$.

[7]We thank V.L. Ginzburg for stimulating discussions on the material of this section.

Robbin-Salamon index of Lagrangian paths: Let $V \subset \mathbb{R}^{2k}$ be a Lagrangian subspace. Consider the Grassmannian $Lagr\,(k)$ of all Lagrangian subspaces in $\mathbb{R}^{2k}$ and consider the hypersurface $\Sigma_V \subset Lagr\,(k)$ formed by all the Lagrangian subspaces that are *not* transversal to V. To such a V and to any smooth path $\{L_t\}$, $0 \leq t \leq 1$, in $Lagr\,(k)$ Robbin and Salamon [54] associate an index, which may take integer or half-integer values and which we will denote by $RS(\{L_t\}, V)$. The definition of the index can be outlined as follows.

A number $t \in [0, 1]$ is called a *crossing* if $L_t \in \Sigma_V$. To each crossing t one associates a certain quadratic form Q_t on the space $L(t) \cap V$ – see [54] for the precise definition. The crossing t is called *regular* if the quadratic form Q_t is non-degenerate. The *index* of such a regular crossing t is defined as the signature of Q_t if $0 < t < 1$ and as half of the signature of Q_t if $t = 0, 1$. One can show that regular crossings are isolated. For a path $\{L_t\}$ with only regular crossings the index $RS(\{L_t\}, V)$ is defined as the sum of the indices of its crossings. An arbitrary path can be perturbed, keeping the endpoints fixed, into a path with only regular crossings and the index of the perturbed path does not depend on the perturbation – in fact, it depends only on the fixed endpoints homotopy class of the path. Moreover, it is additive with respect to the concatenation of paths and satisfies the naturality property: $RS(\{AL_t\}, AV) = RS(\{L_t\}, V)$ for any symplectic matrix A.

Indices of paths in $Sp\,(2k)$: Consider the group $Sp\,(2k)$ of symplectic $2k \times 2k$-matrices. Denote by $\widetilde{Sp\,(2k)}$ its universal cover. One can use the index RS in order to define two indices on the space of smooth paths in $Sp\,(2k)$.

The first index, denoted by Ind_{2k}, is defined as follows. Fix a Lagrangian subspace $V \subset \mathbb{R}^{2k}$. For each smooth path $\{A_t\}$, $0 \leq t \leq 1$, in $Sp\,(2k)$ define $Ind_{2k}\,(\{A_t\}, V)$ as

$$Ind_{2k}\,(\{A_t\}, V) := RS(\{A_t V\}, V).$$

The naturality of the RS index implies that

$$RS(\{BA_tB^{-1}(BV)\}, BV) = RS(\{BA_tV)\}, BV) =$$

$$= RS(\{A_tV)\}, V) \text{ for any } B \in Sp\,(2k)$$

and thus we get the following naturality condition for Ind_{2k}:

$$Ind_{2k}\,(\{BA_tB^{-1}\}, BV) = Ind_{2k}\,(\{A_t\}, V) \text{ for any } B \in Sp\,(2k). \tag{11}$$

The second index, which we will call the *Conley-Zehnder index of a matrix path* and which will be denoted by CZ_{matr}, is defined as follows. For each $A \in Sp\,(2k)$ denote by $Gr\,A$ the graph of A which is a Lagrangian subspace of the symplectic vector space $\mathbb{R}^{4k} = \mathbb{R}^{2k} \times \mathbb{R}^{2k}$ equipped with the symplectic structure $\omega_{4k} = -\omega_{2k} \oplus \omega_{2k}$. Denote by Δ the diagonal in $\mathbb{R}^{4k} = \mathbb{R}^{2k} \times \mathbb{R}^{2k}$ – it is a Lagrangian subspace with respect to ω_{4k}. Now for any smooth path $\{A_t\}$, $0 \leq t \leq 1$, in $Sp\,(2k)$ define CZ_{matr} as

$$CZ_{matr}(\{A_t\}) := RS(\{Gr\,A_t\}, \Delta).$$

Equivalently, one can define $CZ_{matr}(\{A_t\})$ similarly to the index RS by looking at the intersections of $\{A(t)\}$ with the hypersurface $\Sigma \subset Sp\,(2k)$ formed by all the symplectic $2k \times 2k$-matrices with eigenvalue 1 and translating the notions of a regular crossing and the corresponding quadratic form to this setup.

Both indices $Ind_{2k}\,(\{A_t\}, V)$ and $CZ_{matr}(\{A_t\})$ depend only on the fixed endpoints homotopy class of the path $\{A_t\}$ and are additive with respect to the concatenation of paths in $Sp\,(2k)$. The relation between the two indices is as follows. Denote by I_{2k} the $2k \times 2k$ identity matrix. Given a smooth path $\{A_t\}$, $0 \leq t \leq 1$, in $Sp\,(2k)$, set $\widehat{A}_t := I_{2k} \oplus A_t \in Sp\,(4k)$. Then

$$CZ_{matr}(\{A_t\}) = Ind_{4k}(\{\widehat{A}_t\}, \Delta). \tag{12}$$

Remark 3.2. Note that near each $W \in \Sigma_V$ there exists a local coordinate chart (on *Lagr* (k)) in which Σ_V can be defined by an algebraic equation of degree bounded from above by a constant C depending only on k and W. Moreover, since for any two $V, V' \in$ *Lagr* (k) there exists a diffeomorphism of *Lagr* (k) mapping Σ_V into $\Sigma_{V'}$ we can assume that $C = C(k)$ is independent of W and depends only on k. Therefore for any V, for any point $W \in \Sigma_V$ and for any sufficiently small open neighborhood U_W of W in *Lagr* (k) the number of connected components of $U_W \setminus (U_W \cap \Sigma_V)$ is bounded by a constant depending only on k.

Using these observations and the fact that regular crossings are isolated it is easy to show that there exists a constant $C(k)$, depending only on k, such that for any Lagrangian subspace $V \subset \mathbb{R}^{2k}$ and any path $\{A_t\} \subset Sp\,(2k)$, $0 \leq t \leq 1$, there exists a $\delta > 0$ such that for any smooth path $\{A'_t\} \subset Sp\,(2k)$, $0 \leq t \leq 1$, which is δ-close to $\{A_t\}$ in the C^0-metric, one has

$$|Ind_{2k}(\{A_t\}, V) - Ind_{2k}(\{A'_t\}, V| < C(k),$$

$$|CZ_{matr}(\{A_t\}) - CZ_{matr}(\{A'_t\}| < C(k).$$

LERAY THEOREM ON THE INDEX Ind_{2k}: The following result follows from Theorem 5.1 in [54] which Robbin and Salamon credit to Leray [35], p.52. Denote by L the Lagrangian $(q_1, \dots, q_k)$-coordinate plane in $\mathbb{R}^{2k}$. Any symplectic matrix $S \in Sp\,(2k)$ can be decomposed into $k \times k$ blocks as

$$S = \begin{pmatrix} E & F \\ G & H \end{pmatrix},$$

where the blocks satisfy, in particular, the condition that

$$EF^T - FE^T = 0. \tag{13}$$

If $SL \cap L = 0$ then the $k \times k$-matrix F is invertible and multiplying (13) by F^{-1} on the left and $(F^T)^{-1} = (F^{-1})^T$ on the right, we get that $F^{-1}E - E^T(F^{-1})^T = 0$. Therefore the matrix $Q_S := F^{-1}E$ is symmetric.

Theorem 3.3 ([54], Theorem 5.1; [35], p.52)**.** *Assume* $\{A_t\}$, $\{B_t\}$, $0 \leq t \leq 1$, *are two smooth paths in* $Sp\,(2k)$, *such that* $A_0 = B_0 = I_{2k}$ *and* $A_1 L \cap L = 0$, $B_1 L \cap L = 0$, $A_1 B_1 L \cap L = 0$. *Then*

$$Ind_{2k}(\{A_t B_t\}, L) = Ind_{2k}(\{A_t\}, L) + Ind_{2k}(\{B_t\}, L) + \frac{1}{2}\mathrm{sign}\,(Q_{A_1} + Q_{B_1}),$$

where $\mathrm{sign}\,(Q_{A_1} + Q_{B_1})$ *is the signature of the quadratic form defined by the symmetric* $k \times k$*-matrix* $Q_{A_1} + Q_{B_1}$.

Corollary 3.4. *Let* V *be any Lagrangian subspace of* $\mathbb{R}^{2k}$. *Then there exists a positive constant* C, *depending only on* k, *such that for any smooth paths* $\{X_t\}$, $\{Y_t\}$, $0 \leq t \leq 1$, *in* $Sp\,(2k)$, *such that* $X_0 = Y_0 = I_{2k}$ *(there are no assumptions on* X_1, Y_1*!),*

$$|Ind_{2k}(\{X_t Y_t\}, V) - Ind_{2k}(\{X_t\}, V) - Ind_{2k}(\{Y_t\}, V)| < C.$$

Proof. We will write $C_1, C_2, \dots$ for (possibly different) positive constants depending only on k.

Pick a map $\Psi \in Sp\,(2k)$ such that $\Psi V = L$. Denote $A_t = \Psi X_t \Psi^{-1}$, $B_t = \Psi Y_t \Psi^{-1}$. Note that the paths $\{A_t\}$, $\{B_t\}$ are based at the identity.

Using the naturality property (11) of Ind_{2k} we get

$$|Ind_{2k}(\{X_t Y_t\}, V) - Ind_{2k}(\{X_t\}, V) - Ind_{2k}(\{Y_t\}, V)| =$$

$$= |Ind_{2k}(\{\Psi X_t Y_t \Psi^{-1}\}, \Psi V) - Ind_{2k}(\{\Psi X_t \Psi^{-1}\}, \Psi V) -$$
$$- Ind_{2k}(\{\Psi Y_t \Psi^{-1}\}, \Psi V)| =$$
$$= |Ind_{2k}(\{(\Psi X_t \Psi^{-1})(\Psi Y_t \Psi^{-1})\}, L) - Ind_{2k}(\{\Psi X_t \Psi^{-1}\}, L) -$$
$$- Ind_{2k}(\{\Psi Y_t \Psi^{-1}\}, L)| =$$
$$= |Ind_{2k}(\{A_t B_t\}, L) - Ind_{2k}(\{A_t\}, L) - Ind_{2k}(\{B_t\}, L)|.$$

Thus

$$|Ind_{2k}(\{X_t Y_t\}, V) - Ind_{2k}(\{X_t\}, V) - Ind_{2k}(\{Y_t\}, V)| =$$
$$= |Ind_{2k}(\{A_t B_t\}, L) - Ind_{2k}(\{A_t\}, L) - Ind_{2k}(\{B_t\}, L)|. \tag{14}$$

Further on, Remark 3.2 implies that we can find sufficiently C^0-close identity-based perturbations $\{A'_t\}$, $\{B'_t\}$ of $\{A_t\}$, $\{B_t\}$ such that

$$A'_1 L \cap L = 0,\ B'_1 L \cap L = 0,\ A'_1 B'_1 L \cap L = 0. \tag{15}$$

and

$$|Ind_{2k}(\{A_t B_t\}, L) - Ind_{2k}(\{A_t\}, L) - Ind_{2k}(\{B_t\}, L)| -$$
$$- |Ind_{2k}(\{A'_t B'_t\}, L) - Ind_{2k}(\{A'_t\}, L) - Ind_{2k}(\{B'_t\}, L)| < C_1, \tag{16}$$

for some C_1. On the other hand, since the three identity-based paths $\{A'_t\}$, $\{B'_t\}$, $\{A'_t B'_t\}$, satisfy the conditions (15), we can apply to them Theorem 3.3. Hence there exists C_2 such that

$$|Ind_{2k}(\{A'_t B'_t\}, L) - Ind_{2k}(\{A'_t\}, L) - Ind_{2k}(\{B'_t\}, L)| < C_2.$$

Combining it with (14) and (16) we get that there exists C_3 such that

$$|Ind_{2k}(\{X_t Y_t\}, V) - Ind_{2k}(\{X_t\}, V) - Ind_{2k}(\{Y_t\}, V)| < C_3,$$

which finishes the proof. $\square$

Conley-Zehnder index as a quasi-morphism: Recall that $2n = \dim M$. Restricting CZ_{matr} to the identity-based paths in $Sp\,(2n)$ one gets a function on $\widetilde{Sp\,(2n)}$ that will be still denoted by CZ_{matr}.

Proposition 3.5 (cf. [19])**.** *The function $CZ_{matr} : \widetilde{Sp\,(2n)} \to \mathbb{R}$ is a quasi-morphism. It means that there exists a constant $C > 0$ such that*

$$|CZ_{matr}(ab) - CZ_{matr}(a) - CZ_{matr}(b)| \leq C \quad \forall a, b \in \widetilde{Sp\,(2n)}.$$

Proof. Represent a and b by identity-based paths $\{A_t\}$, $\{B_t\}$, $0 \leq t \leq 1$, in $Sp\,(2n)$. Then use (12) and apply Corollary 3.4 for $k = 2n$, $V = \Delta$ to $\{\widehat{A}_t\}$, $\{\widehat{B}_t\}$ in $Sp\,(4n)$. ☐

Maslov index of symplectic loops: The Conley-Zehnder index for identity-based loops in $Sp\,(2n)$ is called the *Maslov index* of a loop. Its original definition, going back to [4], is the following: it is the intersection number of an identity-based loop with the stratified hypersurface Σ whose principal stratum is equipped with a certain co-orientation. Note that we do not divide the intersection number by 2 and thus in our case the Maslov index takes only even values; for instance, the Maslov index of a counterclockwise 2π-twist of the standard symplectic $\mathbb{R}^2$ is 2. We denote the Maslov index of a loop $\{B(t)\}$ by $Maslov(\{B(t)\})$.

Conley-Zehnder and Maslov indices of periodic orbits: The Conley-Zehnder index for periodic orbits is defined by means of the Conley-Zehnder index for matrix paths as follows. Given $[\gamma, u] \in \widetilde{\mathcal{P}}_F$, build an identity-based path $\{A(t)\}$ in $Sp\,(2n)$ as follows: take a symplectic trivialization of the bundle $u^*(TM)$ over $\mathbb{D}^2$ and use the trivialization to identify the linearized flow $d_{\gamma(0)} f_t$, $0 \leq t \leq 1$, along γ with a symplectic matrix $\{A(t)\}$. Then the Conley-Zehnder index $CZ_F([\gamma, u])$ is defined as

$$CZ_F([\gamma, u]) := n - CZ_{matr}\,(\{A(t)\}). \tag{17}$$

With such a normalization of CZ_F for any sufficiently C^2-small autonomous Morse Hamiltonian F, the Conley-Zehnder index of an element of $\widetilde{\mathcal{P}}_F$, represented by a pair $[x, u]$ consisting of a critical point x of F (viewed as a constant path in M) and the trivial disk u, is equal to the Morse index of x. Note that with such a normalization $CZ_F(Sy) = CZ_F(y) + 2\int_S c_1(M)$ for every $y \in \widetilde{\mathcal{P}}_F$ and $S \in H_2^S(M)$.

Similarly, if the time-1 flow generated by F defines a loop in $Ham(M)$ then to each $[\gamma, u] \in \widetilde{\mathcal{P}}_F$ one can associate its Maslov index. Namely, trivialize the bundle $u^*(TM)$ over $\mathbb{D}^2$ and identify the linearized flow $\{d_x f_t\}$ along γ with an identity-based loop of symplectic $2n \times 2n$-matrices. Define the Maslov index $m_F([\gamma, u])$ as the Maslov index for the loop of symplectic matrices. Under the action of $H_2^S(M)$ on $\widetilde{\mathcal{P}}_F$ the Maslov index changes as follows:

$$m_F(S \cdot [\gamma, u]) = m_F([\gamma, u]) - 2 \int_S c_1(M), \quad S \in H_2^S(M).$$

Let us make the following remark. Assume $\gamma \in \mathcal{P}_F$ and assume that a symplectic trivialization of the bundle $\gamma^*(TM)$ over $\mathbb{S}^1$ identifies $\{d_{\gamma(0)} f_t\}$ with an identity-based path $\{A(t)\}$ of symplectic matrices. Assume there is another symplectic trivialization of the same bundle, coinciding with the first one at $\gamma(0)$, and denote by $\{B(t)\}$ the identity-based loop of transition matrices from the first symplectic trivialization to the second one. Use the second trivialization to identify $\{d_{\gamma(0)} f_t\}$ with an identity-based path $\{A'(t)\}$. Then

$$CZ_{matr}\left(\{A'(t)\}\right) = CZ_{matr}\left(\{A(t)\}\right) + \mathit{Maslov}\left(\{B(t)\}\right), \tag{18}$$

and if $\{A(t)\}$ is a loop then so is $\{A'(t)\}$ and

$$\mathit{Maslov}\left(\{A'(t)\}\right) = \mathit{Maslov}\left(\{A(t)\}\right) + \mathit{Maslov}\left(\{B(t)\}\right). \tag{19}$$

3.4 Spectral numbers

Given the algebraic setup as above, the construction of the Piunikhin-Salamon-Schwarz (PSS) isomorphism [47] yields a Λ-linear isomorphism (*PSS-isomorphism*) $\phi_M : QH_*(M) \to HF_*(F, J)$ which preserves the grading and which is actually a ring isomorphism (the pair-of-pants product defines a ring structure on $HF_*(F, J)$).

Using the PSS-isomorphism one defines the *spectral numbers* $c(a, F)$, where $0 \neq a \in QH_*(M)$, in the usual way [45]. Namely, the action functional $\mathcal{A}_F$ defines a filtration on $C(F)$ which induces a filtration $HF_*^\alpha(F, J)$, $\alpha \in \mathbb{R}$, on $HF_*(F, J)$, with $HF_*^\alpha(F, J) \subset HF_*^\beta(F, J)$ as long as $\alpha < \beta$. Then

$$c(a, F) := \inf \{\alpha \mid \phi_M(a) \in HF_*^\alpha(F, J)\}.$$

Such spectral number is finite and well-defined (does not depend on J). Here is a brief account of the relevant properties of spectral numbers – for details see [45] (see also [65, 42, 59, 43] for earlier versions of this theory).

(Spectrality) $c(a, H) \in \mathit{spec}\,(H)$, where *the spectrum* $\mathit{spec}\,(H)$ *of* H is defined as the set of critical values of the action functional $\mathcal{A}_H$, i.e. $\mathit{spec}\,(H) := \mathcal{A}_H(\widetilde{\mathcal{P}}_H) \subset \mathbb{R}$.

(Quantum homology shift property) $c(\lambda a, H) = c(a, H) + \nu(\lambda)$ for all $\lambda \in \Lambda$, where ν is the valuation defined in Section 3.1.

(Hamiltonian shift property) $c(a, H + \lambda(t)) = c(a, H) + \int_0^1 \lambda(t)\, dt$ for any Hamiltonian H and function $\lambda : \mathbb{S}^1 \to \mathbb{R}$.

(Monotonicity) If $H_1 \le H_2$, then $c(a, H_1) \le c(a, H_2)$.

(Lipschitz property) The map $H \mapsto c(a, H)$ is Lipschitz on the space of (time-dependent) Hamiltonians $H : M \times \mathbb{S}^1 \to \mathbb{R}$ with respect to the C^0-norm.

(Symplectic invariance) $c(a, \phi^* H) = c(a, H)$ for every $\phi \in Symp_0(M)$, $H \in C^\infty(M)$; more generally, $Symp\,(M)$ acts on $H_*(M; \mathcal{F})$, and hence on $QH_*(M)$, and $c(a, \phi^* H) = c(\phi_* a, H)$ for any $\phi \in Symp\,(M)$.

(Normalization) $c(a, 0) = \nu(a)$ for every $a \in QH_*(M)$.

(Homotopy invariance) $c(a, H_1) = c(a, H_2)$ for any *normalized* H_1, H_2 generating the same $\phi \in \widetilde{Ham}\,(M)$. Thus one can define $c(a, \phi)$ for any $\phi \in \widetilde{Ham}\,(M)$ as $c(a, H)$ for any normalized H generating ϕ.

(Triangle inequality) $c(a * b, \phi\psi) \le c(a, \phi) + c(b, \psi)$.

The commutative ring $QH_\bullet(M)$ admits a $\mathcal{K}$-bilinear and $\mathcal{K}$-valued form Ω on $QH_\bullet(M)$ which associates to a pair of quantum homology classes $a, b \in QH_\bullet(M)$ the coefficient (belonging to $\mathcal{K}$) at the class $[point] = [point] \cdot q^0$ of a point in their quantum product $a * b \in QH_\bullet(M)$ (*the Frobenius structure*). Let $\tau : \mathcal{K} \to \mathcal{F}$ be the map sending each series $\sum_{\theta \in \Gamma} z_\theta s^\theta$, $z_\theta \in \mathcal{F}$, to its free term z_0. Define a non-degenerate $\mathcal{F}$-valued $\mathcal{F}$-linear pairing on $QH_\bullet(M)$ by

$$\Pi(a, b) := \tau\Omega(a, b) = \tau\Omega(a * b, [M]) \ . \tag{20}$$

Note that Π is symmetric and

$$\Pi(a * b, c) = \Pi(a, b * c)\ \forall a, b, c \in QH_\bullet(M). \tag{21}$$

With this notion at hand, we can present another important property of spectral numbers:

(Poincaré duality) $c(b,\phi) = -\inf_{a\in\Upsilon(b)} c(a,\phi^{-1})$ for all $b \in QH_\bullet(M)\setminus\{0\}$ and ϕ. Here $\Upsilon(b)$ denotes the set of all $a \in QH_\bullet(M)$ with $\Pi(a,b) \neq 0$.

The Poincaré duality can be extracted from [47] (cf. [22]) – for a proof see [46].

The next property is an immediate consequence of the definitions (see [22] for a discussion in the monotone case):

(Characteristic exponent property) Given $0 \neq \lambda \in \mathcal{F}$, $a,b \in QH_*(M)$, $a,b,a+b \neq 0$, and a (time-dependent) Hamiltonian H, one has $c(\lambda\cdot a, H) = c(a,H)$ and $c(a+b,H) \leq \max(c(a,H), c(b,H))$.

3.5 Partial symplectic quasi-states

Given a non-zero idempotent $a \in QH_{2n}(M)$ and a time-independent Hamiltonian $H : M \to \mathbb{R}$, define

$$\zeta(a,H) := \lim_{l\to+\infty} \frac{c(a,lH)}{l}\ . \tag{22}$$

When a is fixed, we shall often abbreviate $\zeta(H)$ instead of $\zeta(a,H)$. The limit in the formula (22) always exists and thus the functional $\zeta : C^\infty(M) \to \mathbb{R}$ is well-defined. The functional ζ on $C^\infty(M)$ is Lipschitz with respect to the C^0-norm $\|H\| = \max_M |H|$ and therefore extends to a functional $\zeta : C(M) \to \mathbb{R}$, where $C(M)$ is the space of all continuous functions on M. These facts were proved in [23] in the case $a = [M]$ but the proofs actually go through for any non-zero idempotent $a \in QH_{2n}(M)$.

Here we will list the properties of ζ for such an M. Again, these properties were proved in [23] in the case $a = [M]$ but the proof goes through for any non-zero idempotent $a \in QH_{2n}(M)$. The additivity with respect to constants property was not explicitly listed in [23] but follows immediately from the definition of ζ and the Hamiltonian shift property of spectral numbers. The triangle inequality follows readily from the definition of ζ and from the triangle inequality for the spectral numbers.

Theorem 3.6. *The functional $\zeta : C(M) \to \mathbb{R}$ satisfies the following properties:*

Semi-homogeneity: $\zeta(\alpha F) = \alpha\zeta(F)$ *for any F and any $\alpha \in \mathbb{R}_{\geq 0}$.*

Triangle inequality: If $F_1, F_2 \in C^\infty(M)$, $\{F_1, F_2\} = 0$ then $\zeta(F_1 + F_2) \leq \zeta(F_1) + \zeta(F_2)$.

Partial additivity and vanishing: If $F_1, F_2 \in C^\infty(M)$, $\{F_1, F_2\} = 0$ and the support of F_2 is displaceable, then $\zeta(F_1 + F_2) = \zeta(F_1)$; in particular, if the support of $F \in C(M)$ is displaceable, $\zeta(F) = 0$.

Additivity with respect to constants and normalization: $\zeta(F + \alpha) = \zeta(F) + \alpha$ for any F and any $\alpha \in \mathbb{R}$. In particular, $\zeta(1) = 1$.

Monotonicity: $\zeta(F) \leq \zeta(G)$ for $F \leq G$.

Symplectic invariance: $\zeta(F) = \zeta(F \circ f)$ for every symplectic diffeomorphism $f \in Symp_0(M)$.

Characteristic exponent property: $\zeta(a_1 + a_2, F) \leq \max(\zeta(a_1, F), \zeta(a_2, F))$ for each pair of non-zero idempotents a_1, a_2 with $a_1 * a_2 = 0$, $a_1 + a_2 \neq 0$ (in this case $a_1 + a_2$ is also a non-zero idempotent), and for all $F \in C(M)$.

We will call the functional $\zeta : C(M) \to \mathbb{R}$ satisfying all the properties listed in Theorem 3.6 *a partial symplectic quasi-state*.

4 Basic properties of (super)heavy sets

In this section we prove parts (i) and (iii) of Theorem 1.2, as well as Theorem 1.3. We shall use that a partial symplectic quasi-state ζ extends by continuity in the uniform norm to a monotone functional on the space of **continuous** functions $C(M)$, see Section 3.5 above. In particular, one can use continuous functions instead of the smooth ones in the definition of (super)heaviness in formulae (3) and (4).

Assume a partial quasi-state ζ defined by a non-zero idempotent is fixed and we consider heaviness and superheaviness with respect to ζ. We start with the following elementary

Proposition 4.1. *A closed subset $X \subset M$ is heavy if and only if for every $H \in C^\infty(M)$ with $H|_X = 0$, $H \leq 0$ one has $\zeta(H) = 0$. A closed subset $X \subset M$ is superheavy if and only if for every $H \in C^\infty(M)$ with $H|_X = 0$, $H \geq 0$ one has $\zeta(H) = 0$.*

Proof. The "only if" parts follow readily from the monotonicity property of ζ. Let us prove the "if" part in the "heavy case" – the "superheavy" case is similar. Take a function H on M and put

$$F = \min(H - \inf_X H, 0) .$$

Note that $F|_X = 0$ and $F \le 0$. Thus $\zeta(F) = 0$ by the assumption of the proposition. Thus

$$0 = \zeta(F) \le \zeta(H - \inf_X H) = \zeta(H) - \inf_X H ,$$

which yields heaviness of X. $\square$

The following proposition proves part (i) of Theorem 1.2.

Proposition 4.2. *Every superheavy set is heavy.*

Proof. Let $X \subset M$ be a superheavy subset. Assume that $H|_X = 0$, $H \le 0$. By the triangle inequality for ζ we have $\zeta(H) + \zeta(-H) \ge 0$. Note that $-H|_X = 0$, $-H \ge 0$. Superheaviness yields $\zeta(-H) = 0$, so $\zeta(H) \ge 0$. But by monotonicity $\zeta(H) \le 0$. Thus $\zeta(H) = 0$ and the claim follows from Proposition 4.1. $\square$

Superheavy sets have the following user-friendly property.

Proposition 4.3. *Let $X \subset M$ be a superheavy set. Then for every $\alpha \in \mathbb{R}$ and $H \in C^\infty(M)$ with $H|_X \equiv \alpha$ one has $\zeta(H) = \alpha$.*

Proof. Since $\zeta(H + \alpha) = \zeta(H) + \alpha$ it suffices to prove the proposition for $\alpha = 0$. Take any function H with $H|_X = 0$. Since X is superheavy and, by Proposition 4.2, also heavy, we have

$$0 = \zeta(-|H|) \le \zeta(H) \le \zeta(|H|) = 0 ,$$

which yields $\zeta(H) = 0$. $\square$

As an immediate consequence we get part (iii) of Theorem 1.2.

Proposition 4.4. *Every superheavy set intersects with every heavy set.*

Proof. Let X be a superheavy set and Y be a heavy set. Assume on the contrary that $X \cap Y = \emptyset$. Take a function $H \le 0$ with $H|_Y \equiv 0$ and $H|_X \equiv -1$. Then $\zeta(H) = -1$ by Proposition 4.3. On the other hand, $\zeta(H) = 0$ since Y is heavy, and we get a contradiction. $\square$

Note that two heavy sets do not necessarily intersect each other: a meridian of $\mathbb{T}^2$ is heavy (see Corollary 6.4 below), while two meridians can be disjoint.

Proof of Theorem 1.3 (i) and (ii): The triangle inequality yields

$$c(a,H) = c(a*[M], 0+H) \leq c(a,0) + c([M],H) = \nu(a) + c([M],H).$$

Passing to the partial quasi-states $\zeta(a,H)$ and $\zeta([M],H)$ we get:

$$\zeta(a,H) = \lim_{k\to+\infty} c(a,kH)/k \leq$$

$$\leq \lim_{k\to+\infty} (\nu(a) + c([M],kH))/k = \lim_{k\to+\infty} c([M],kH)/k = \zeta([M],H).$$

The result now follows from the definition of heavy and superheavy sets (see Definition 1.1). □

Proof of Theorem 1.3 (iii): By the characteristic exponent property of spectral invariants,

$$\zeta(a,F) \leq \max_{i=1,\ldots,l} \zeta(e_i,F) \ \ \forall F \in C^\infty(M)\ . \tag{23}$$

Choose a sequence of functions $G_j \in C^\infty(M)$, $j \to +\infty$, with the following properties: $G_k \leq G_j$ for $k > j$, $G_j = 0$ on X, $G_j \leq 0$ and for every function $F \leq 0$ which vanishes *on an open neighborhood* of X there exists j so that $G_j \leq F$ (existence of such a sequence can be checked easily). In view of inequality (23), we have that for every j there exists i so that $\zeta(a,G_j) \leq \zeta(e_i,G_j)$. Passing, if necessary, to a subsequence $G_{j_k}, j_k \to +\infty$, we can assume without loss of generality that i is *the same* for all j. In view of heaviness of X with respect to a, we have that $\zeta(a,G_j) = 0$. Therefore $\zeta(e_i,G_j) \geq 0$.

Choose any function $F \leq 0$ on M which vanishes *on an open neighborhood* of X. Then there exists j large enough so that $F \geq G_j$. By monotonicity combined with the previous estimate we have

$$0 \geq \zeta(e_i,F) \geq \zeta(e_i,G_j) \geq 0\ ,$$

which yields $\zeta(e_i,F) = 0$.

Now let F be any continuous function on M that vanishes *on* X. Take a sequence of continuous functions F_j, converging to F in the C^0-norm, so that each F_j vanishes on an open neighborhood of X. Then $\zeta(e_i,F_j) = \lim_{j\to+\infty} \zeta(e_i,F_j) = 0$, because $\zeta(e_i,\cdot)$ is Lipschitz with respect to the C^0-norm. The heaviness of X with respect to e_i now follows from Proposition 4.1. This finishes the proof of the theorem. □

5 Products of (super)heavy sets

In this section we prove Theorem 1.5 on products of (super)heavy subsets.

5.1 Product formula for spectral invariants

The proof of Theorem 1.5 is based on the following general result.

Theorem 5.1. *For every pair of time-dependent Hamiltonians G_1, G_2 on M_1 and M_2, and all non-zero $a_1 \in QH_{i_1}(M_1)$, $a_2 \in QH_{i_2}(M_2)$ we have*

$$c(a_1 \otimes a_2, G_1(z_1,t) + G_2(z_2,t)) = c(a_1, G_1) + c(a_1, G_2) \ .$$

Here $G_1(z_1,t) + G_2(z_2,t)$ is a time-dependent Hamiltonian on $M_1 \times M_2$.

Let us deduce Theorem 1.5 from Theorem 5.1.

Proof of Theorem 1.5: We show that the product of superheavy sets is superheavy (the proof for heavy sets goes without any changes). We denote by ζ_1, ζ_2 and ζ the partial quasi-states on M_1, M_2 and $M := M_1 \times M_2$ associated to the idempotents a_1, a_2 and $a_1 \otimes a_2$ respectively. Let $X_i \subset M_i$, $i = 1, 2$, be a superheavy set. By Proposition 4.1 it suffices to show that if a non-negative function $G \in C^\infty(M)$ vanishes on some neighborhood, say U, of $X := X_1 \times X_2$ then $\zeta(G) = 0$. (Since ζ is Lipschitz with respect to the C^0-norm this would imply that $\zeta(G) = 0$ for any non-negative $G \in C(M)$ that vanishes *on* X). Put $K := \max_M G$. Choose neighborhoods U_i of X_i so that $U_1 \times U_2 \subset U$. Choose non-negative functions G_i on M_i which vanish on X_i and such that $G_i(z) > K$ for all $z \in M_i \setminus U_i$. Observe that $G \leq G_1 + G_2$. But, in view of Theorem 5.1 and superheaviness of X_i, we have

$$\zeta(G_1 + G_2) = \zeta_1(G_1) + \zeta_2(G_2) = 0 \ .$$

By monotonicity

$$0 \leq \zeta(G) \leq \zeta(G_1 + G_2) = 0 \ ,$$

and thus $\zeta(G) = 0$. □

It remains to prove Theorem 5.1. Note that the left-hand side of the equality stated in the theorem does not exceed the right-hand side: this is an immediate consequence of the triangle inequality for spectral invariants. However, we were unable to use this observation for proving the theorem. Our approach is based on a rather lengthy algebraic analysis which enables us to calculate separately the left and the right-hand sides "on the chain level". A simple inspection of the results of this calculation yields the desired equality.

5.2 Decorated $\mathbb{Z}_2$-graded complexes

A $\mathbb{Z}_2$*-complex* is a $\mathbb{Z}_2$-graded finite-dimensional vector space V over a field $\mathcal{K}$ equipped with a $\mathcal{K}$-linear differential $\partial : V \to V$ satisfying $\partial^2 = 0$ and shifting the grading. *A decorated complex* over $\mathcal{K} = \mathcal{K}_\Gamma$ includes the following data:

- a countable subgroup $\Gamma \subset \mathbb{R}$;
- a $\mathbb{Z}_2$-graded complex (V, d) over $\mathcal{K}_\Gamma$;
- a preferred basis $x_1, \ldots, x_n$ of V;
- a function $F : \{x_1, \ldots, x_n\} \to \mathbb{R}$ (called *the filter*) which extends to V by
$$F(\sum \lambda_j x_j) = \max\{\nu(\lambda_j) + F(x_j) \;\Big|\; \lambda_j \neq 0\},$$
and satisfies $F(dv) < F(v)$ for all $v \in V \setminus \{0\}$. The convention is that $F(0) = -\infty$. Here ν is the valuation defined in Section 3.1 above.

We shall use the notation
$$\mathbf{V} := (V, \{x_i\}_{i=1,\ldots,n}, F, d, \Gamma)$$
for a decorated complex.

The $\widehat{\otimes}_\mathcal{K}$*-tensor product* $\mathbf{V} = \mathbf{V_1} \widehat{\otimes}_\mathcal{K} \mathbf{V_2}$ of decorated complexes
$$\mathbf{V_i} = (V_i, \{x_j^{(i)}\}_{j=1,\ldots,n_i}, F_i, d_i, \Gamma_i)\ ,\ i = 1, 2$$
is defined as follows. Consider the space $V = V_1 \widehat{\otimes}_\mathcal{K} V_2$ (see formula (5) above) with the natural $\mathbb{Z}_2$-grading. Define the differential d on V by
$$d(x \otimes y) = d_1 x \otimes y + (-1)^{\deg x} x \otimes d_2 y\ .$$
The preferred basis in V is given by $\{x_{pq} := x_p^{(1)} \otimes x_q^{(2)}\}$ and the filter F is defined by
$$F(x_{pq}) = F_1(x_p^{(1)}) + F_2(x_q^{(2)}).$$
Finally, we put $\mathbf{V} := (V, \{x_{pq}\}, F, d, \Gamma_1 + \Gamma_2)$.

The ($\mathbb{Z}_2$-graded) homology of decorated complexes are denoted by $H_*(\mathbf{V})$ – they are $\mathcal{K}$-vector spaces. By the Künneth formula, $H(\mathbf{V_1} \widehat{\otimes}_\mathcal{K} \mathbf{V_2}) = H(\mathbf{V_1}) \widehat{\otimes}_\mathcal{K} H(\mathbf{V_2})$.

Next we define *spectral invariants* associated to a decorated complex $\mathbf{V} := (V, \{x_{pq}\}, F, d)$. Namely, for $a \in H(\mathbf{V})$ put

$$c(a) := \inf\{F(v) \mid a = [v], v \in \operatorname{Ker} d\} \ .$$

We shall see below that $c(a) > -\infty$ for each $a \neq 0$.

The purpose of this algebraic digression is to state the following result:

Theorem 5.2. *For any two decorated complexes* $\mathbf{V_1}, \mathbf{V_2}$

$$c(a_1 \otimes a_2) = c(a_1) + c(a_2) \ \ \forall a_1 \in H(\mathbf{V_1}), a_2 \in H(\mathbf{V_2})$$

5.3 Reduced Floer and Quantum homology

The 2-periodicity of the Floer complex and Floer homology defined by the multiplication by q (see Proposition 3.1 above) allows to encode their algebraic structure in a decorated $\mathbb{Z}_2$-complex. Consider a regular pair (G, J) consisting of a Hamiltonian function and a compatible almost-complex structure on M (both, in general, are time-dependent). Let $(C_*(G), d_{G,J})$ be the corresponding Floer complex. Let us associate to it a $\mathbb{Z}_2$-complex: a $\mathbb{Z}_2$-graded vector space V_G over $\mathcal{K}_\Gamma$, defined as

$$V_G := C_0(G) \oplus C_1(G),$$

with the obvious $\mathbb{Z}_2$-grading, and a differential $\partial_{G,J} : V_G \to V_G$, defined as the direct sum of $d_{G,J} : C_1(G) \to C_0(G)$ and $qd_{G,J} : C_0(G) \to C_1(G)$. One readily checks that this is indeed a $\mathbb{Z}_2$-complex because $d_{G,J} : C(G) \to C(G)$ is Λ_Γ-linear. We will call $(V_G, \partial_{G,J})$ the *$\mathbb{Z}_2$-complex associated to* (G, J).

Note that the cycles and the boundaries of (V_G, ∂_G) having $\mathbb{Z}_2$-degree $i \in \{0, 1\}$ in V_G coincide, respectively, with the cycles and the boundaries having $\mathbb{Z}$-degree i of $(C(G), d_{G,J})$. Therefore the Floer homology $HF_i(G, J)$ is isomorphic, as a vector space over $\mathcal{K}_\Gamma$, to the i-th degree component of the homology of the complex $(V_G, \partial_{G,J})$.

The $\mathbb{Z}_2$-complex $(V_G, \partial_{G,J})$ carries a structure of the decorated complex $\mathbf{V}_{G,J}$ as follows. Let $\gamma_i(t), i = 1, \ldots, m$, be the collection of all contractible 1-periodic orbits of the Hamiltonian flow generated by G. Choose disc u_i in M spanning γ_i. For each i there exists unique integer, say r_i, so that the Conley-Zehnder index of the element $x_i := q^{r_i} \cdot [\gamma_i, u_i]$ lies in the set

$\{0,1\}$. Clearly, the collection $\{x_i\}$ forms a basis of V_G over $\mathcal{K}_\Gamma$. We shall consider it as a preferred basis. Note that the preferred basis is unique up to multiplication of x_i's by elements of the form $s^{\alpha_i}, \alpha_i \in \Gamma$. Finally, the action functional associated to G defines a filtration on V_G.

The homology of $(V_G, \partial_{G,J})$ can be canonically identified via the PSS-isomorphism with the object which we call *reduced* quantum homology:

$$QH_{red}(M) := QH_0(M) \oplus QH_1(M) .$$

We call this isomorphism *the reduced* PSS-isomorphism and denote it by $\psi_{G,J}$.

Note that we have a natural projection $p : QH_*(M) \to QH_{red}(M)$ which sends any degree homogeneous element a to aq^r with $\deg a + 2r \in \{0,1\}$. With this notation, the usual Floer-homological spectral invariant $c(a, G)$ coincides with the spectral invariant $c(p(a))$ of the decorated complex $\mathbf{V}_{G,J}$.

5.4 Proof of Theorem 5.1

By the Lipschitz property of spectral numbers it is enough to consider the case when G_1 and G_2 belong to regular pairs (G_i, J_i), $i = 1, 2$. Set

$$G(z_1, z_2, t) := G_1(z_1, t) + G(z_2, t)$$

and $J := J_1 \times J_2$. Then (G, J) is also a regular pair. Put $\Gamma_i = \Gamma(M_i, \omega_i)$. It is straightforward to see that the decorated complex $\mathbf{V}_{G,J}$ is the $\widehat{\otimes}_{\mathcal{K}}$-tensor product of the decorated complexes $\mathbf{V}_{G_i,J_i}$ for $i = 1, 2$.

Put $(M, \omega) = (M_1 \times M_2, \omega_1 \oplus \omega_2)$. An obvious modification of the Künneth formula for quantum homology (see e.g. [41, Exercise 11.1.15] for the statement in the monotone case) yields a natural monomorphism

$$\imath : QH_{i_1}(M_1, \omega_1) \widehat{\otimes}_{\mathcal{K}} QH_{i_2}(M_1, \omega_1) \to QH_{i_1+i_2}(M, \omega) .$$

Since in our setting quantum homologies are 2-periodic, the collection of these isomorphisms for all pairs (i_1, i_2) from the set $\{0,1\}$ induces an isomorphism

$$j : QH_{red}(M_1) \widehat{\otimes}_{\mathcal{K}} QH_{red}(M_2) \to QH_{red}(M) .$$

It has the following properties: First, given two elements $a_1 \in QH_{i_1}(M_1, \omega_1)$ and $a_2 \in QH_{i_2}(M_2, \omega_2)$ we have that

$$p(a_1) \otimes p(a_2) = p(a_1 \otimes a_2) .$$

Second, the following diagram commutes:

$$\begin{tikzcd}
H(V_{G_1},\partial_{G_1,J_1})\widehat{\otimes}_{\mathcal{K}}H(V_{G_2},\partial_{G_2,J_2}) \arrow[r, "k"] \arrow[d, "\psi_{G_1,J_1}\otimes\psi_{G_2,J_2}"] & H(V_G,\partial_{G,J}) \arrow[d, "\psi_{G,J}"] \\
QH_{red}(M_1)\widehat{\otimes}_{\mathcal{K}}QH_{red}(M_2) \arrow[r, "j"] & QH_{red}(M)
\end{tikzcd}$$

Here k is the isomorphism coming from the Künneth formula for $\mathbb{Z}_2$-complexes, and $\psi_{G_i,J_i},\psi_{G,J}$ stand for the reduced PSS-isomorphisms. It follows that the definition of $c(a_i,G_i)$, $c(a_1\otimes a_2,G)$ matches the definition of $c(p(a_i))$ and $c(p(a_1)\otimes p(a_2))$. By Theorem 5.2 we get that

$$c(a_1\otimes a_2,G)=c(p(a_1)\otimes p(a_2))=c(p(a_1))+c(p(a_2))=c(a_1,G_1)+c(a_2,G_2)\ .$$

This proves Theorem 5.1 modulo Theorem 5.2. $\square$

5.5 Proof of algebraic Theorem 5.2

A decorated complex is called *generic* if $F(x_i)-F(x_j)\notin\Gamma$ for all $i\neq j$ (recall that under our assumptions Γ, the group of periods of the symplectic form ω over $\pi_2(M)$, is a countable subgroup of $\mathbb{R}$). We start from some auxiliary facts from linear algebra. Let $\mathbf{V}:=(V,\{x_i\}_{i=1,\dots,n},F,d,\Gamma)$ be a generic decorated complex. We recall once again that for brevity we write $\mathcal{K}$ instead of $\mathcal{K}_\Gamma$ wherever it is clear what Γ is taken.

An element $x\in V$ is called *normalized* if

$$x=x_p+\sum_{i\neq p}\lambda_i x_i\ ,\lambda_i\in\mathcal{K},\ F(x_p)>\max_{i\neq p}F(\lambda_i x_i)\ .$$

We shall use the notation $x=x_p+o(x_p)$. In generic complexes, every element $x\neq 0$ can be uniquely written as $x=\lambda(x_p+o(x_p))$ for some $p=1,\dots,n$ and $\lambda\in\mathcal{K}$. A system of vectors $e_1,\dots,e_m$ in V is called *normal* if every e_i has the form $e_i=x_{j_i}+o(x_{j_i})$ for $j_i\in\{1,\dots,n\}$ and the numbers j_i are pair-wise distinct.

Lemma 5.3. *Let $e_1,\dots,e_m$ be a normal system. Then*

$$F(\sum_{i=1}^{n}\lambda_i e_i)=\max_i F(\lambda_i e_i)\ .$$

Proof. We prove the result using induction in m. For $m = 1$ the statement is obvious. Let's check the induction step $m - 1 \to m$. Observe that it suffices to check that

$$F(e_1 + \sum_{i=2}^{n} \lambda_i e_i) \geq F(e_1) \ . \tag{24}$$

Then obviously

$$F(\sum_{i=1}^{n} \lambda_i e_i) \geq \max_i F(\lambda_i e_i) \ ,$$

while the reversed inequality is an immediate consequence of the definitions.

By the induction step,

$$F(\sum_{i=2}^{n} \lambda_i e_i) = \max_{i=2,\ldots,n} F(\lambda_i e_i) \ .$$

In view of the genericity, the maximum at the right hand side can be uniquely written as $F(\lambda_{i_0} x_{i_0})$. Without loss of generality we shall assume that $e_i = x_i + o(x_i)$ and $i_0 = 2$.

Put

$$v = \sum_{i \geq 2} \lambda_2^{-1} \lambda_i e_i = x_2 + o(x_2) \ .$$

Write

$$e_1 = x_1 + \alpha x_2 + X, \ v = x_2 + \beta x_1 + Y,$$

where $\alpha, \beta \in \mathcal{K}$ and $X, Y \in \mathrm{Span}_{\mathcal{K}}(x_3, \ldots, x_n)$. Note that $F(x_1) > F(\alpha x_2)$, $F(x_2) > F(\beta x_1)$, which yields

$$\nu(\alpha) < F(x_1) - F(x_2) < -\nu(\beta) = \nu(\beta^{-1}) \ . \tag{25}$$

In particular, $\nu(\alpha) < \nu(\beta^{-1})$. Note that

$$e_1 + \lambda_2 v = (1 + \lambda_2 \beta) x_1 + (\alpha + \lambda_2) x_2 + Z, \ Z \in \mathrm{Span}_{\mathcal{K}}(x_3, \ldots, x_n) \ .$$

Thus

$$F(e_1 + \lambda_2 v) \geq \max(\nu(1 + \lambda_2 \beta) + F(x_1), \nu(\alpha + \lambda_2) + F(x_2)) \ .$$

If $\nu(1 + \lambda_2 \beta) \geq 0$ we have $F(e_1 + \lambda_2 v) \geq F(x_1) = F(e_1)$ and inequality (24) follows. Assume that $\nu(1+\lambda_2\beta) < 0 = \nu(1)$. Then $\nu(\lambda_2 \beta) = 0 = \nu(\lambda_2) + \nu(\beta)$, and hence $\nu(\lambda_2) = \nu(\beta^{-1}) \neq \nu(\alpha)$. Thus

$$\nu(\alpha + \lambda_2) \geq \nu(\lambda_2) = -\nu(\beta) \ .$$

Combining this inequality with (25) we get that

$$F(e_1 + \lambda_2 v) \geq \nu(\alpha + \lambda_2) + F(x_1) + (F(x_2) - F(x_1))$$

$$\geq F(x_1) + (\nu(\alpha + \lambda_2) + \nu(\beta)) \geq F(x_1) = F(e_1) \ .$$

This completes the proof of inequality (24), and hence of the lemma. $\square$

It readily follows from the lemma that every normal system is linearly independent.

Lemma 5.4. *Every subspace $L \subset V$ has a normal basis.*

Proof. We use induction over $m = \dim_{\mathcal{K}} L$. The case $m = 1$ is obvious, so let us handle the induction step $m - 1 \to m$. It suffices to show the following: Let $e_1, \ldots, e_{m-1}$ be a normal basis in a subspace L', and let $v \notin L'$ be any vector. Put $L = \mathrm{Span}_{\mathcal{K}}(L' \cup \{v\})$. Then there exists $e_m \in L$ so that $e_1, \ldots, e_m$ is a normal basis. Indeed, assume without loss of generality that for all $i = 1, \ldots, m-1$ one has $e_i = x_i + o(x_i)$. Put $W = \mathrm{Span}_{\mathcal{K}}(x_m, \ldots, x_n)$. We claim that $L' \cap W = \{0\}$. Indeed, otherwise

$$\lambda_1 e_1 + \ldots + \lambda_{m-1} e_{m-1} = \lambda_m x_m + \ldots + \lambda_n x_n$$

where the linear combinations in the right and the left-hand sides are non-trivial. Apply F to both sides of this equality. By Lemma 5.3

$$F(\lambda_1 e_1 + \ldots + \lambda_{m-1} e_{m-1}) = F(x_p) \mod \Gamma, \text{ where } 1 \leq p \leq m-1 \ ,$$

while

$$F(\lambda_m x_m + \ldots + \lambda_n x_n) = F(x_q) \mod \Gamma, \text{ where } q \geq m \ .$$

This contradicts the genericity of our decorated complex, and the claim follows. Since $\dim L' + \dim W = \dim V$, we have that $V = L' \oplus W$. Decompose v as $u + w$ with $u \in L', w \in W$, and note that $w \in L$. Note that $e_1, \ldots, e_{m-1}, w$ are linearly independent. Furthermore, $w = \lambda(x_p + o(x_p))$ for some $p \geq m$. Put $e_m = \lambda^{-1} w$. The vectors $e_1, \ldots, e_m$ form a normal basis in L. $\square$

The same proof shows that if $L_1 \subset L_2$ are subspaces of V, every normal basis in L_1 extends to a normal basis in L_2.

Now we turn to the analysis of the differential d. Choose a normal basis $g_1, \ldots, g_q$ in $\mathrm{Im}\, d$, and extend it to a normal basis $g_1, \ldots, g_q, h_1, \ldots, h_p$ in $\mathrm{Ker}\, d$. Note that each of these $p+q$ vectors has the form $x_j + o(x_j)$ with distinct j. Let us assume without loss of generality that the remaining $n-p-q$ elements of the preferred basis in V are $x_1, \ldots, x_q$, and

$$g_i = x_{i+q} + o(x_{i+q}), h_j = x_{j+2q} + o(x_{j+2q}) \ .$$

Here we use that, by the dimension theorem, $n = p + 2q$. Note that

$$x_1, \ldots, x_q, g_1, \ldots, g_q, h_1, \ldots, h_p$$

is a normal system, and hence a basis in V. We call such a basis a *spectral basis* of the decorated complex $\mathbf{V}$.

Note that $[h_1], \ldots, [h_p]$ is a basis in the homology $H(\mathbf{V})$. Consider any homology class $a = \sum \lambda_i [h_i]$. Every element $v \in V$ with $a = [v]$ can be written as $v = \sum \lambda_i h_i + \sum \alpha_j g_j$. Thus, by Lemma 5.3, $F(v) \geq \max_i F(\lambda_i h_i)$ and hence

$$c(a) = \max_i F(\lambda_i h_i) \ . \tag{26}$$

This proves in particular that the spectral invariants are *finite* provided $a \neq 0$.

For finite sets $A = \{v_1, \ldots, v_s\}$ and $B = \{w_1, \ldots, w_s\}$ we write $A \otimes B$ for the finite set $\{v_i \otimes w_j\}$.

Assume now that $\mathbf{V_1}, \mathbf{V_2}$ are generic decorated complexes. We say that they are *in general position* if their tensor product $\mathbf{V} = \mathbf{V_1} \widehat{\otimes}_{\mathcal{K}} \mathbf{V_2}$ is generic. Let

$$B_i = \{x_1^{(i)}, \ldots, x_{q_i}^{(i)}, g_1^{(i)}, \ldots, g_{q_i}^{(i)}, h_1^{(i)}, \ldots, h_{p_i}^{(i)}\},\ i = 1, 2$$

be a spectral basis in $\mathbf{V_i}$. Obviously, $B_1 \otimes B_2$ is a normal basis in $V_1 \widehat{\otimes}_{\mathcal{K}} V_2$. We shall denote by d_1, d_2, d the differentials and by F_1, F_2, F the filters in $\mathbf{V_1}, \mathbf{V_2}$ and $\mathbf{V}$ respectively. Put $G_i = \{g_1^{(i)}, \ldots, g_{q_i}^{(i)}\}$, $H_i = \{h_1^{(i)}, \ldots, h_{p_i}^{(i)}\}$ and $K = G_1 \otimes B_2 \cup B_1 \otimes G_2$. Observe that

$$\mathrm{Im}\, d \subset W := \mathrm{Span}(K) \ .$$

Take any two classes

$$a_i = \sum \lambda_j^{(i)} [h_j^{(i)}] \in H(\mathbf{V}_i) \ , i = 1, 2.$$

Suppose that $a_1 \otimes a_2 = [v]$. Then v is of the form

$$v = \sum_{m,l} \lambda_m^{(1)} \lambda_l^{(2)} h_m^{(1)} \otimes h_l^{(2)} + w$$

where w must lie in W. Observe that $(H_1 \otimes H_2) \cap K = \emptyset$. By Lemma 5.3,

$$F(v) \geq \max_{m,l} F(\lambda_m^{(1)} \lambda_l^{(2)} h_m^{(1)} \otimes h_l^{(2)}) \ ,$$

and hence

$$\begin{aligned} c(a_1 \otimes a_2) &= \max_{m,l} F(\lambda_m^{(1)} \lambda_l^{(2)} h_m^{(1)} \otimes h_l^{(2)}) \\ &= \max_{m,l} \ F_1(\lambda_m^{(1)} h_m^{(1)}) + F_2(\lambda_l^{(2)} h_l^{(2)}) \\ &= \max_m F_1(\lambda_m^{(1)} h_m^{(1)}) + \max_l F_2(\lambda_l^{(2)} h_l^{(2)}) = c(a_1) + c(a_2) \ . \end{aligned}$$

In the last equality we used (26). This completes the proof of Theorem 5.2 for decorated complexes in general position.

It remains to remove the general position assumption. This will be done with the help of the following lemma. We shall work with a family of decorated complexes

$$\mathbf{V} := (V, \{x_i\}_{i=1,\ldots,n}, F, d, \Gamma)$$

which have exactly the same data (preferred basis, grading, differential and Γ) with the exception of the filter F which will be allowed to vary in the class of filters. The corresponding spectral invariants will be denoted by $c(a, F)$.

Lemma 5.5.

(i) If filters F, F' satisfy $F(x_i) \leq F'(x_i)$ for all $i = 1, \dots, n$, then $c(a, F) \leq c(a, F')$ for all non-zero classes $a \in H(\mathbf{V})$.

(ii) If F is a filter and $\theta \in \mathbb{R}$, then $F + \theta$ is again a filter and $c(a, F+\theta) = c(a, F) + \theta$ for all non-zero classes $a \in H(\mathbf{V})$.

The proof is obvious and we omit it. It follows that for any two filters F, F'

$$|c(a, F) - c(a, F')| \leq ||F - F'||_{C^0} \;\; \forall a \in H(\mathbf{V}) \setminus \{0\} \; .$$

Assume now that $\mathbf{V_1}, \mathbf{V_2}$ are decorated complexes. Denote by F_1, F_2 their filters. Fix $\epsilon > 0$. By a small perturbation of the filters we get new filters, F_1' and F_2', on our complexes so that the complexes become generic and in general position, and furthermore

$$||F_1 - F_1'||_{C^0} \leq \epsilon \; , ||F_2 - F_2'||_{C^0} \leq \epsilon \; .$$

Given homology classes $a_i \in H(\mathbf{V}_i)$ we have

$$|c(a_1, F_1) + c(a_2, F_2) - c(a_1 \otimes a_2, F_1 + F_2)| \leq$$

$$|c(a_1, F_1') + c(a_2, F_2') - c(a_1 \otimes a_2, F_1' + F_2')| + 4\epsilon = 4\epsilon \; .$$

Here we used that Theorem 5.2 is already proved for generic complexes in general position. Since $\epsilon > 0$ is arbitrary, we get that

$$c(a_1, F_1) + c(a_2, F_2) - c(a_1 \otimes a_2, F_1 + F_2) = 0 \; ,$$

which completes the proof of Theorem 5.2 in full generality. □

6 Stable non-displaceability of heavy sets

In this section we prove part (ii) of Theorem 1.2.

Proposition 6.1. *Every heavy subset is stably non-displaceable.*

For the proof we shall need the following auxiliary statement. Given $R > 0$, consider the torus $\mathbb{T}^2_R$ obtained as the quotient of the cylinder $T^*\mathbb{S}^1 = \mathbb{R}(r) \times \mathbb{S}^1$ (θ mod 1) by the shift $(r, \theta) \mapsto (r + R, \theta)$. For $\alpha > 0$ define the function $F_\alpha(r, \theta) := \alpha f(r)$ on $\mathbb{T}^2_R$, where $f(r)$ is any R-periodic function having only two non-degenerate critical points on $[0, R]$: a maximum point at $r = 0$ with $f(0) = 1$, and a minimum point at $r = R/2$, $f(R/2) =: -\beta < 0$. We denote by $[T]$ the fundamental class of $\mathbb{T}^2_R$. We work with the symplectic form $dr \wedge d\theta$ on $\mathbb{T}^2_R$.

Lemma 6.2. *$c([T], F_\alpha) = \alpha$.*

Proof. Note that the contractible closed orbits of period 1 of the Hamiltonian flow generated by F_α are fixed points forming circles $S_+ = \{r = 0\}$ and $S_- = \{r = R/2\}$. The actions of the fixed points on $S_\pm$ equal respectively to α and $-\alpha\beta$, and thus the spectral invariants of F_α lie in the set $\{\alpha, -\alpha\beta\}$. Recall from [59] that $c([T], F_\alpha) > c([\text{point}], F_\alpha)$. Thus $c([T], F_\alpha) = \alpha$. □

Lemma 6.3. *Let $H \in C^\infty(M)$ so that $H^{-1}(\max H)$ is displaceable. Then $\zeta(H) < \max H$.*

Proof. Choose $\epsilon > 0$ so that the set

$$H^{-1}((\max H - \epsilon, \max H])$$

is displaceable. Choose a real-valued cut-off function $\rho : \mathbb{R} \to [0, 1]$ which equals 1 near $\max H$ and which is supported in $(\max H - \epsilon, \max H + \epsilon)$. Thus $\rho(H)$ is supported in $H^{-1}((\max H - \epsilon; \max H])$ and $\zeta(\rho(H)) = 0$. Since H and $\rho(H)$ Poisson-commute, the vanishing and the monotonicity axioms yield

$$\zeta(H) = \zeta(\rho(H)) + \zeta(H - \rho(H)) \leq \max(H - \rho(H)) < \max H \, .$$

□

Proof of Proposition 6.1: It suffices to show that for every $R > 0$ the set

$$Y := X \times \{r = 0\} \subset M' := M \times \mathbb{T}^2_R$$

is non-displaceable. Assume on the contrary that Y is displaceable. Choose a function H on M with $H \leq 0$, $H^{-1}(0) = X$. Put

$$H' = H + F_1 = H + f(r) : M' \to \mathbb{R}.$$

Assume that the partial quasi-state ζ on M is associated to some non-zero idempotent $a \in QH_*(M)$ by means of (2). Denote by ζ' the quasi-state on M' associated to $a \otimes T$. Note that

$$Y = (H')^{-1}(\max H') \ , \quad \text{where} \quad \max H' = 1 \ ,$$

while Theorem 5.1 and Lemma 6.2 imply that

$$\zeta'(H') = \zeta(H) + 1 \ .$$

By Lemma 6.3 $\zeta'(H') < 1$ and so $\zeta(H) < 0$. In view of Proposition 4.1, we get a contradiction with the heaviness of X. □

Lemma 6.2 also yields a simple proof of the following result which also follows from Corollary 1.15:

Corollary 6.4. *Any meridian of $\mathbb{T}^2$ is heavy (with respect to the fundamental class $[T]$).*

Proof. In the notation as above identify $\mathbb{T}^2$ with $\mathbb{T}^2_1$ for $R = 1$. Since any two meridians of $\mathbb{T}^2$ can be mapped into each other by a symplectic isotopy and since such an isotopy preserves heaviness, it suffices to prove that the meridian $S := S_+ = \{r = 0\}$ (see the proof of Lemma 6.2) is heavy.

Let $H : \mathbb{T}^2 \to \mathbb{R}$ be a Hamiltonian and let us show that $\zeta(H) \geq \inf_S H$, where ζ is defined using $[T]$. Shifting H, if necessary, by a constant, we may assume without loss of generality that $\inf_S H = 1$. Pick $f = f(r) : \mathbb{T}^2 \to \mathbb{R}$ as in the definition of F_α so that $F_1 = f \leq H$ on $\mathbb{T}^2$ (note that f equals 1 on S). Then Lemma 6.2 yields

$$\zeta(H) \geq \zeta(F_1) = 1 = \inf_S H.$$

□

7 Analyzing stable stems

Proof of Theorem 1.6: Assume that $\mathbb{A}$ is a Poisson-commutative subspace of $C^\infty(M)$, $\Phi : M \to \mathbb{A}^*$ its moment map with the image Δ, and let $X = \Phi^{-1}(p)$ be a stable stem of $\mathbb{A}$.

Take any function $H \in C^\infty(\mathbb{A}^*)$ with $H \geq 0$ and $H(p) = 0$. We claim that $\zeta(\Phi^* H) = 0$. By an arbitrarily small C^0-perturbation of H we can assume that $H = 0$ in a small neighborhood, say U, of p. Choose an open covering $U_0, U_1, \ldots, U_N$ of Δ so that $U_0 = U$, and all $\Phi^{-1}(U_i)$ are stably displaceable for $i \geq 1$ (it exists by the definition of a stem). Let $\rho_i : \Delta \to \mathbb{R}$, $i = 0, \ldots, N$, be a partition of unity subordinated to the covering $\{U_i\}$.

Take the two-torus $\mathbb{T}^2_R$ as in Section 6. Choose $R > 0$ large enough so that $\Phi^{-1}(U_i) \times \{r = \text{const}\}$ is displaceable in $M \times \mathbb{T}^2_R$ for all $i \geq 1$. Choose now a sufficiently fine covering $V_j, j = 1, \ldots, K$, of the torus $\mathbb{T}^2_R$ by sufficiently thin annuli $\{|r - r_j| < \delta\}$ so that the sets $\Phi^{-1}(U_i) \times V_j$ are displaceable in $M \times \mathbb{T}^2_R$ for all $i \geq 1$ and all j. Let $\varrho_j = \varrho_j(r)$, $j = 1, \ldots, K$, be a partition of unity subordinated to the covering $\{V_j\}$.

Denote by ζ' the partial quasi-state corresponding to $a \otimes T$. Put $F(r, \theta) = \cos(2\pi r/R)$. Write

$$\Phi^* H + F = \sum_{i=0}^{N} \sum_{j=1}^{K} (\Phi^* H + F) \cdot \Phi^* \rho_i \cdot \varrho_j =$$

$$\Phi^*(H\rho_0) + F \cdot \Phi^* \rho_0 + \sum_{i=1}^{N} \sum_{j=1}^{K} (\Phi^* H + F) \cdot \Phi^* \rho_i \cdot \varrho_j \ .$$

Note that $H\rho_0 = 0$ and $F \cdot \Phi^* \rho_0 \leq 1$. Applying partial quasi-additivity and monotonicity we get that

$$\zeta'(\Phi^* H + F) = \zeta'(F \cdot \Phi^* \rho_0) \leq 1.$$

By Lemma 6.2 and the product formula (Theorem 5.1 above) we have

$$\zeta'(\Phi^* H + F) = \zeta(\Phi^* H) + 1 \leq 1$$

and hence $\zeta(\Phi^* H) \leq 0$. On the other hand, $\zeta(\Phi^* H) \geq 0$ since $H \geq 0$. Thus $\zeta(\Phi^* H) = 0$ and the claim follows.

Further, given any function G on M with $G \geq 0$ and $G|_X = 0$, one can find a function H on $\mathbb{A}^*$ with $H(p) = 0$ so that $G \leq \Phi^* H$. By monotonicity and the claim above

$$0 \leq \zeta(G) \leq \zeta(\Phi^* H) = 0 \ ,$$

and hence $\zeta(G) = 0$. Thus X is superheavy. $\square$

8 Monotone Lagrangian submanifolds

The main tool of proving (super)heaviness of monotone Lagrangian submanifolds satisfying the Albers condition is the spectral estimate in Proposition 8.1(iii) below, which originated in the work by Albers [2]. Later on Biran and Cornea pointed out a mistake in [2], and suggested a correction together with a far reaching generalization in [15]. Let us mention that the original Albers estimate was used in the first version of the present paper. We thank Biran and Cornea for informing us about the mistake, explaining to us their approach and helping us to correct a number of our results affected by this mistake.

The main ingredient of Biran-Cornea techniques which is needed for our purposes is the following result. Let (M,ω) be a closed monotone symplectic manifolds with $[\omega]=\kappa\cdot c_1(M)$, $\kappa>0$. Write N for the minimal Chern number of (M,ω). Let $L^n\subset M^{2n}$ be a closed monotone Lagrangian submanifold with the minimal Maslov number $N_L\geq 2$.

We shall treat slightly differently the cases when N_L is even and odd. Let us mention that for orientable L, N_L is automatically even. Thus, due to our convention, when N_L is odd we work with the basic field $\mathcal{F}=\mathbb{Z}_2$. Let $\Gamma=\kappa N\cdot\mathbb{Z}$ be the group of periods of M. Recall that the quantum ring has the form $QH_*(M)=H_*(M;\mathcal{F})\otimes_{\mathcal{F}}\Lambda$, where the Novikov ring Λ is defined as $\Lambda=\mathcal{K}_\Gamma[q,q^{-1}]$. Put $\Gamma'=(\kappa N/2)\cdot\mathbb{Z}$. Consider an extended Novikov ring $\Lambda':=\mathcal{K}_{\Gamma'}[q^{\frac{1}{2}},q^{-\frac{1}{2}}]$. Define now $QH'_*(M)$ as $QH_*(M)$ if N_L is even, and as $H_*(M,\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}\Lambda'$ if N_L is odd. In the latter case $QH'_*(M)$ is an extension of $QH_*(M)$, and we shall consider without special mentioning $QH_*(M)$, Λ, $\mathcal{K}_\Gamma$ as subrings of $QH'_*(M)$, Λ', $\mathcal{K}_{\Gamma'}$. The grading of $QH'_*(M)$ is determined by the condition $\deg q^{\frac{1}{2}}=1$. As before, we shall use notation $QH'_\bullet(M)$, where $\bullet=$ "even" when $\mathcal{F}=\mathbb{C}$ and $\bullet=*$ when $\mathcal{F}=\mathbb{Z}_2$.

Note that the spectral invariants (and hence partial symplectic quasi-states) are well-defined over the extended ring, and furthermore, their values and properties, by tautological reasons, do not alter under such an extension (cf. a discussion in [15], Section 5.4). Put $w:=s^{\kappa N_L/2}q^{N_L/2}$. Recall that j stands for the natural morphism $H_\bullet(L;\mathcal{F})\to H_\bullet(M;\mathcal{F})$.

Proposition 8.1. *Assume that $k>n+1-N_L$. If $\mathcal{F}=\mathbb{C}$ assume in addition that k is even. Then there exists a canonical homomorphism $j^q:H_k(L;\mathcal{F})\to QH'_k(M)$ with the following properties*[8]*:*

[8]The letter "q" in j^q stands for *quantum*.

(i) $j^q(x) = j(x) + w^{-1}y$, *where* y *is a polynomial in* w^{-1} *with coefficients in* $H_\bullet(M;\mathcal{F})$*;*

(ii) $j^q([L]) = j([L])$*;*

(iii) If $j^q(x) \neq 0$ *then* $c(j^q(x), H) \leq \sup_L H$ *for every* $H \in C^\infty(M)$*.*

In particular, if S is an Albers element of L, we have $j^q(S) = j(S)+O(w^{-1}) \neq 0$.

This proposition was proved by Biran and Cornea in [15] in the case $\mathcal{F} = \mathbb{Z}_2$: The map j^q is essentially the map i_L appearing in Theorem A(iii) in [15]. Proposition 8.1(i) above is a combination of Theorem A(iii) and Proposition 4.5.1(i) in [15]. Our variable w corresponds to the variable t^{-1} in [15], while our $s^{N\kappa}q^N$ corresponds to the variable s^{-1} in Section 2.1.2 of [15]. After such an adjustment of the notation, the formula $w := s^{\kappa N_L/2}q^{N_L/2}$ above can be extracted from Section 2.1.2 of [15]. For Proposition 8.1(ii) above see Remark 5.3.2.a in [15]. Proposition 8.1(iii) above follows from Lemma 5.3.1(ii) in [15]. Finally, let us repeat the disclaimer made in Section 1.5: we take for granted that Proposition 8.1 remains valid for the case $\mathcal{F} = \mathbb{C}$.

Let us pass to the proofs of our results on (super)-heaviness of monotone Lagrangian submanifolds. We start with the following remark. Let S be an Albers element of L. The Poincaré duality property of spectral invariants (see Section 3.4 above) extends verbatim to the case of the ring $QH'(M)$ with the following modification: When N_L is odd, the pairing Π introduced in Section 3.4 extends in the obvious way to a non-degenerate $\mathcal{F}$-valued pairing on $QH'_\bullet(M)$ which we still denote by Π. Applying Poincaré duality and substituting $H := -F$ into Proposition 8.1 (iii) above we get that for every function $F \in C^\infty(M)$

$$c(T, F) \geq \inf_L F \quad \forall T \in QH'_\bullet(M) \text{ with } \Pi(T, j^q(S)) \neq 0.$$

In particular, given a non-zero idempotent $a \in QH'_\bullet(M)$ and a class $b \in QH'_\bullet(M)$, so that $\Pi(a*b, j^q(S)) \neq 0$, we get, using the normalization property of spectral invariants, that

$$c(a, F) + \nu(b) \geq c(a * b, F) \geq \inf_L F \quad \forall F \in C^\infty(M)\ . \tag{27}$$

Therefore, applying (27) to kF for $k \in \mathbb{N}$, dividing by k and passing to the limit as $k \to +\infty$, we get that for the partial quasi-state ζ, defined by a,

$$\zeta(F) \geq \inf_L F \quad \forall F \in C^\infty(M),$$

meaning that L is heavy with respect to a.

Proof of Theorem 1.15: Let S be an Albers element of L. Let $T \in H_\bullet(M;\mathcal{F})$ be any singular homology class such that $T \circ j(S) \neq 0$. Thus, applying Proposition 8.1 (i) we see that $\Pi([M]*T, j^q(S)) = \Pi(T, j^q(S)) \neq 0$, and hence inequality (27), applied to $a = [M], b = T$, yields that L is heavy with respect to $[M]$. $\square$

Let us pass to the proof of Theorem 1.25 on the effect of semi-simplicity of the quantum homology. It readily follows from the next more general statement. Let $L_1, \ldots, L_m$ be Lagrangian submanifolds satisfying the Albers condition. Let S_i be any Albers element of L_i. Denote by $Z_i = j^q(S_i) \in QH'_\bullet(M)$ its image under the inclusion morphism from Proposition 8.1 above.

Theorem 8.2. *Given such $L_1, \ldots, L_m$ and $Z_1, \ldots, Z_m$, assume, in addition, that $QH_{2n}(M)$ is semi-simple and the Lagrangian submanifolds $L_1, \ldots, L_m$ are pair-wise disjoint. Then the classes $Z_1, \ldots, Z_m$ are linearly independent over $\mathcal{K}_{\Gamma'}$.*

Proof. Arguing by contradiction, assume that

$$Z_1 = \alpha_2 Z_2 + \ldots + \alpha_m Z_m \tag{28}$$

for some $\alpha_2, \ldots, \alpha_m \in \mathcal{K}_{\Gamma'}$. Since $QH_{2n}(M)$ is semi-simple, it decomposes into a direct sum of fields with unities $e_1, \ldots, e_d$. Since the pairing Π (on $QH'_\bullet(M;\mathcal{F})$) is non-degenerate, there exists $T \in QH'_\bullet(M;\mathcal{F})$ such that

$$\Pi(T, Z_1) \neq 0. \tag{29}$$

Let us write T as

$$T = [M] * T = \sum_{i=1}^{d} e_i * T. \tag{30}$$

Equations (29), (30) imply that there exists $l \in [1, d]$ such that

$$\Pi(e_l * T, Z_1) \neq 0 \;. \tag{31}$$

Then (28) implies that there exists $r \in [2, m]$ such that

$$\Pi(e_l * T, \alpha_r Z_r) \neq 0.$$

Using (21) (for Π on $QH'_\bullet(M;\mathcal{F})$) we can rewrite the last equation as

$$\Pi(e_l * \alpha_r T, Z_r) \neq 0. \tag{32}$$

Applying now formula (27) for $S = Z_1 \in H_\bullet(L_1;\mathcal{F})$, $a = e_l$, $b = T$, and also for $S = Z_r \in H_\bullet(L_r;\mathcal{F})$, $a = e_l$, $b = \alpha_r T$, we conclude that both L_1 and L_r are heavy with respect to e_l. Thus they are superheavy with respect to e_l, because e_l is the unity in a field factor of $QH_{2n}(M)$ (see Section 1.6). Hence they must intersect – in contradiction to the assumption of the theorem. This finishes the proof of the first part of the theorem. □

Proof of Theorem 1.25(a): Assume that $L_1,\ldots,L_m$ are pair-wise disjoint Lagrangian submanifolds satisfying the condition (a) from the formulation of the theorem. Denote by N_i the minimal Maslov number of L_i. Since $N_i > n+1$, the class of a point from $H_0(L_i;\mathcal{F})$ is an Albers element for L_i. Let $Z_i \in QH'_0(M)$ be its image under the Biran-Cornea inclusion morphism associated to L_i. Note that by Proposition 8.1(i) $Z_i = p + a_i w_i^{-1}$, where $w_i = s^{\kappa N_i/2} q^{N_i/2}$, $a_i \in H_{N_i}(M;\mathcal{F})$ and $p \in H_0(M;\mathcal{F})$ is the homology class of a point. Observe that $\deg w_i = N_i > n+1$, and hence the expression for Z_i cannot contain terms in w_i^{-1} of order two and higher, since $H_{kN_i}(M;\mathcal{F}) = 0$ for $k \geq 2$.

Recall now that all N_i's lie in some set Y of positive integers. Let $W \subset QH'_0(M)$ be the span over $\mathcal{K}_{\Gamma'}$ of

$$H_0(M;\mathcal{F}) \oplus \bigoplus_{E\in Y} s^{-\kappa E/2} q^{-E/2} \cdot H_E(M;\mathcal{F}) \ .$$

Note that

$$\dim_{\mathcal{K}_{\Gamma'}} W = \beta_Y(M) + 1 < m \ .$$

Thus the elements Z_i, $i = 1,\ldots,m$, are linearly dependent over $\mathcal{K}_{\Gamma'}$. By Theorem 8.2, $QH_{2n}(M)$ is not semi-simple. □

Proof of Theorem 1.25(b): Assume that $L_1,\ldots,L_m$ are pair-wise disjoint homologically non-trivial Lagrangian submanifolds. By Proposition 8.1(ii) $j^q([L_i]) = j([L_i])$ for every $i = 1,\ldots,m$. Since the classes $j([L_i])$ are linearly dependent, Theorem 8.2 implies that $QH_{2n}(M)$ is not semi-simple. □

Proof of Theorem 1.18: Combining Proposition 8.1 (ii) and (iii) we get that for any $H \in C^\infty(M)$

$$c(j([L]),H) \leq \sup_L H \ \ \forall H \in C^\infty(M) \ .$$

By the hypothesis of the theorem, we can write $j([L]) * b = a$ for some b. Then

$$c(a, H) = c(j([L]) * b, H) \leq c(j([L]), H) + c(b, 0) \ .$$

Thus

$$c(a, H) \leq \sup_L H + c(b, 0) \ .$$

Applying this inequality to $E \cdot H$ with $E > 0$, dividing by E and passing to the limit as $E \to +\infty$ we get that $\zeta(H) \leq \sup_L H$ for all H. Thus L is superheavy. □

Remark 8.3. The results above admit the following generalizations in the framework of the Biran-Cornea theory. The main object of this theory is the quantum homology ring $QH_*(L)$ of a monotone Lagrangian submanifold, which is isomorphic to the Lagrangian Floer homology $HF_*(L, L)$ of L up to a shift of the grading.

(i) If $QH_*(L)$ does not vanish then L is heavy (see Remark 1.2.9b in [15]). In fact, it follows from [15] that if L satisfies the Albers condition, $QH_*(L)$ does not vanish.

(ii) The map j^q of the Proposition 8.1 above is a footprint of the quantum inclusion map $i_L : QH_*(L) \to QH'_*(M)$ constructed in [15]. The analogue of the action estimate in item (iii) of the proposition is obtained in [15] for the classes $i_L(x)$ for elements $x \in QH_*(L)$ of a certain special form, yielding the following generalization of Theorem 1.18: for these special classes $x \in QH_*(L)$ the condition that the class $i_L(x)$ does not vanish and divides a non-trivial idempotent a implies that L is superheavy with respect to a. This enables, for instance, to generalize Example 1.19 on Lagrangian spheres in quadrics above to the case when $\dim L$ is odd.

(iii) In [15] one can find another action estimate which comes from the $QH_*(M)$-module structure on $QH_*(L)$, which yields more results on (super)heaviness of monotone Lagrangian submanifolds.

Proof of Proposition 1.4: The quantum homology $QH_{2n}(M)$ splits as an algebra over $\mathcal{K}$ into a direct sum of two algebras one of which is a field. This was proved by McDuff for the field $\mathcal{F} = \mathbb{C}$ (see [39] and [24, Section 7]), but the proof goes through for the case $\mathcal{F} = \mathbb{Z}_2$ as well. Denote the unity of the field by a. It is a non-zero idempotent in $QH_{2n}(M)$. As we already pointed out in Remark 1.21, such an idempotent a defines a genuine symplectic quasi-state and therefore the classes of heavy and superheavy sets with respect to a coincide.

By Theorem 1.2, the Lagrangian torus $L \subset M$ cannot be superheavy with respect to a, since it can be displaced from itself by a symplectic (non-Hamiltonian) isotopy. Indeed, take an obvious symplectic isotopy ϕ_t of $\mathbb{T}^{2n}$ that displaces L (a parallel shift) and compose it with a Hamiltonian isotopy ψ_t so that for every t we have that ψ_t is constant on $\phi_t(L)$ and $\psi_t\phi_t$ is identity on the ball where the blow up of $\mathbb{T}^{2n}$ was performed. Clearly, the resulting symplectic isotopy $\psi_t\phi_t$ extends to a symplectic isotopy of M that displaces L.

On the other hand, $N_L \geq 2$ because in this case $N_L = 2N$, where $N \geq 1$ is the minimal Chern number of M. Finally, note that L represents a non-trivial homology class in $H_n(M; \mathbb{Z}_2)$. Therefore we can apply Theorem 1.15 and get that L is heavy with respect to $[M]$. □

9 Rigidity of special fibers of Hamiltonian actions

In this section we prove Theorem 1.9. Denote the special fiber of Φ by $L := \Phi^{-1}(p_{spec})$.

Reduction to the case of $\mathbb{T}^1$-actions: First, we claim that it is enough to prove the theorem for Hamiltonian $\mathbb{T}^1$-actions and the general case will follow from it. Indeed, assume this is proved. The superheaviness of the special fiber immediately yields that for any function $\bar{H} : \mathbb{R} \to \mathbb{R}$

$$\zeta(\Phi^*\bar{H}) = \bar{H}(p_{spec}), \tag{33}$$

where $\Phi : M \to \mathbb{R}$ is the moment map of the $\mathbb{T}^1$-action.

Let us turn to the multi-dimensional situation and let $\Phi : M \to \mathbb{R}^k$ be the normalized moment map of a Hamiltonian $\mathbb{T}^k$-action on M. For a $\mathbf{v} \in \mathbb{R}^k$ denote by $\Phi_{\mathbf{v}}(\mathbf{x}) = \langle \mathbf{v}, \Phi(\mathbf{x})\rangle$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{R}^k$. Note that if $\mathbf{v} \in \mathbb{Z}^k$ the function $\Phi_{\mathbf{v}}$ is the normalized moment map of a Hamiltonian circle action and its special value is $\langle \mathbf{v}, p_{spec}\rangle$.

Thus by (33)

$$\zeta(\Phi_{\mathbf{v}}^* K) = K(\langle \mathbf{v}, p_{spec} \rangle)\ \forall K \in C^\infty(\mathbb{R})\ . \tag{34}$$

By homogeneity of ζ, equality (34) holds for all $\mathbf{v} \in \mathbb{Q}^k$, and by continuity for all $\mathbf{v} \in \mathbb{R}^k$.

Observe that for each pair of smooth functions $P, Q \in C^\infty(\mathbb{R})$ and for each pair of vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^k$ the functions $\Phi_{\mathbf{v}}^* P$ and $\Phi_{\mathbf{w}}^* Q$ Poisson-commute on M. Thus the triangle inequality for the spectral numbers (see Section 3.4) yields

$$\zeta(\Phi_{\mathbf{v}}^* P + \Phi_{\mathbf{w}}^* Q) \leq \zeta(\Phi_{\mathbf{v}}^* P) + \zeta(\Phi_{\mathbf{w}}^* Q)\ . \tag{35}$$

Since M is compact, it suffices to assume that the function $\bar{H} \in C^\infty(\mathbb{R}^k)$ on $\mathbb{R}^k$ is compactly supported. By the inverse Fourier transform we can write

$$\bar{H}(p) = \int_{\mathbb{R}^k} \big\{ \sin\langle \mathbf{v}, p \rangle \cdot F(\mathbf{v}) + \cos\langle \mathbf{v}, p \rangle \cdot G(\mathbf{v}) \big\} d\mathbf{v}$$

for some rapidly (say, faster than $(|p| + 1)^{-N}$ for any $N \in \mathbb{N}$) decaying functions F and G on $\mathbb{R}^k$. For every $\mathbf{v} \in \mathbb{R}^k$ define a function $K_{\mathbf{v}} \in C^\infty(\mathbb{R})$ by

$$K_{\mathbf{v}}(s) := \sin s \cdot F(\mathbf{v}) + \cos s \cdot G(\mathbf{v})\ .$$

Observe that

$$\Phi^* \bar{H} = \int_{\mathbb{R}^k} \Phi_{\mathbf{v}}^* K_{\mathbf{v}}\ d\mathbf{v}\ .$$

Denote by $B(R)$ the Euclidean ball of radius R in $\mathbb{R}^k$ with the center at the origin. Put

$$\bar{H}_R(p) = \int_{B(R)} K_{\mathbf{v}}(\langle \mathbf{v}, p \rangle)\ d\mathbf{v},\ p \in \mathbb{R}^k\ .$$

Since the functions F and G are rapidly decaying, we get that

$$||\bar{H}_R - \bar{H}||_{C^0(\mathbb{R}^k)} \to 0 \ \text{ as } \ R \to \infty\ . \tag{36}$$

We claim that for every R

$$\zeta(\Phi^* \bar{H}_R) \leq \bar{H}_R(p_{spec})\ . \tag{37}$$

Indeed, for $\epsilon > 0$ introduce the integral sum

$$\bar{H}_{R,\varepsilon}(p) = \sum_{\mathbf{v} \in\, \varepsilon \cdot \mathbb{Z}^k \cap B(R)} \varepsilon^k \cdot K_{\mathbf{v}}(\langle \mathbf{v}, p \rangle)\ .$$

Then
$$\Phi^* \bar{H}_{R,\varepsilon} = \sum_{\mathbf{v} \in \, \varepsilon \cdot \mathbb{Z}^k \cap B(R)} \varepsilon^k \cdot \Phi^*_{\mathbf{v}} K_{\mathbf{v}} \; .$$
Applying repeatedly (35) and (34) we get that
$$\zeta(\Phi^* \bar{H}_{R,\varepsilon}) \le \bar{H}_{R,\varepsilon}(p_{spec}) \; .$$
Note now that for fixed R the family $\bar{H}_{R,\epsilon}$ converges to $\bar{H}_R$ as $\varepsilon \to 0$ in the uniform norm on $C^0(\mathbb{R}^k)$. Using that ζ is Lipschitz with respect to the uniform norm on $C^0(M)$ we readily get the inequality (37).

Combining the fact that ζ is Lipschitz with (36) and (37) we get that
$$\zeta(\Phi^* \bar{H}) = \lim_{R \to \infty} \zeta(\Phi^* \bar{H}_R) \le \lim_{R \to \infty} \bar{H}_R(p_{spec}) = \bar{H}(p_{spec}) \; .$$
Now, assume that $\bar{H} \ge 0$ and $\bar{H}(p_{spec}) = 0$. We just have proved that $\zeta(\Phi^* \bar{H}) \le 0$, and hence $\zeta(H) = 0$, which immediately yields the desired superheaviness of the special fiber. This completes the reduction of the general case to the 1-dimensional case.

From now on we will consider only the case of an effective Hamiltonian $\mathbb{T}^1$-action on M with a moment map $\Phi : M \to \mathbb{R}$. Its moment polytope Δ is a closed interval in $\mathbb{R}$ and $p_{spec} = -I(\Phi) \in \mathbb{R}$.

Reduction to the case of a strictly convex function: We claim that it is enough to show the following proposition:

Proposition 9.1. *Assume $\bar{H} : \mathbb{R} \to \mathbb{R}$ is a strictly convex smooth function reaching its minimum at p_{spec}. Set $H := \Phi^* \bar{H}$. Then $\zeta(H) = \bar{H}(p_{spec})$.*

Postponing the proof of the proposition for a moment let us show that it implies the theorem. Indeed, let $F : M \to \mathbb{R}$ be a Hamiltonian on M. In order to show the superheaviness of $L = \Phi^{-1}(p_{spec})$ we need to show that $\zeta(F) \le \sup_L F$. Pick a very steep strictly convex function $\bar{H} : \mathbb{R} \to \mathbb{R}$ with the minimum value $\sup_L F$ reached at p_{spec} and such that $\Phi^* \bar{H} =: H \ge F$ everywhere on M. Then using Proposition 9.1 and the monotonicity of ζ we get
$$\zeta(F) \le \zeta(H) = \bar{H}(p_{spec}) = \sup_L F,$$
yielding the claim.

Preparations for the proof of Proposition 9.1: Given a (time-dependent, not necessarily regular) Hamiltonian G, we associate to every pair $[\gamma, u] \in \widetilde{\mathcal{P}}_G$ a number

$$D_G([\gamma, u]) := \mathcal{A}_G([\gamma, u]) - \frac{\kappa}{2} \cdot CZ_G([\gamma, u]).$$

(Recall that we defined the Conley-Zehnder index for *all* Hamiltonians and not only the regular ones – see Section 3.3). The number $D_G([\gamma, u])$ is invariant under a change of the spanning disc u – an addition of a sphere $jS \in H_2^S(M)$ to the disc u changes both $\mathcal{A}_G([\gamma, u])$ and $\kappa/2 \cdot CZ_G([\gamma, u])$ by the same number. Thus we can write $D_G([\gamma, u]) = D_G(\gamma)$.

Given $[\gamma, u] \in \widetilde{\mathcal{P}}_G$ and $l \in \mathbb{N}$ define $\gamma^{(l)}$ and $u^{(l)}$ as the compositions of γ and u with the map $z \to z^l$ on the unit disc $\mathbb{D}^2 \subset \mathbb{C}$ (here z is a complex coordinate on $\mathbb{C}$). Denote by $t \mapsto g_t$ the time-t flow of G and by $G^{(l)} : M \times \mathbb{R} \to \mathbb{R}$ the Hamiltonian whose time-t flow is $t \mapsto (g_t)^l$ and which is defined by

$$G^{(l)} := G\sharp \ldots \sharp G \quad (l \text{ times}),$$

where $G\sharp K(x, t) := G(x, t) + K(g_t^{-1}x, t)$ for any $K : M \times \mathbb{R} \to \mathbb{R}$.

Proposition 9.2. *There exists a constant $C > 0$, depending only on n, with the following property. Given a 1-periodic orbit $\gamma \in \mathcal{P}_G$ of the flow $t \mapsto g_t$ generated by G, assume that $\gamma^{(l)}$ is a 1-periodic orbit of the flow $t \mapsto g_t^l$ generated by $G^{(l)}$, and therefore for any u such that $[\gamma, u] \in \widetilde{\mathcal{P}}_G$ we have $[\gamma^{(l)}, u^{(l)}] \in \widetilde{\mathcal{P}}_{G^{(l)}}$. Then*

$$|D_{G^{(l)}}([\gamma^{(l)}, u^{(l)}]) - lD_G([\gamma, u])| \leq l \cdot C.$$

Proof. The action term in D_G gets multiplied by l as we pass from G to $G^{(l)}$. As for the Conley-Zehnder term, the quasi-morphism property of the Conley-Zehnder index (see Proposition 3.5) implies that there exists a constant $C > 0$ (depending only on n) such that

$$|lCZ_G[\gamma, u] - CZ_{G^{(l)}}([\gamma^{(l)}, u^{(l)}])| \leq C.$$

This immediately proves the proposition. $\square$

Proposition 9.3. *Let $G : M \times [0,1] \to \mathbb{R}$ be a Hamiltonian as above. Then one can choose $\epsilon > 0$, depending on G, and a constant $C_n > 0$, depending only on $n = \dim M/2$, so that any function $F : M \times [0,1] \to \mathbb{R}$ which is ϵ-close to G in a C^∞-metric on $C^\infty(M \times [0,1])$ satisfies the following condition: for every $\gamma_0 \in \mathcal{P}_F$ there exists $\gamma \in \mathcal{P}_G$ such that the difference between $D_F(\gamma_0)$ and $D_G(\gamma)$ is bounded by C_n.*

Proof. Denote the flow of G by g_t (as before) and the flow of F by f_t. We will view time-1 periodic trajectories of these flows both as maps of $[0,1]$ to M having the same value at 0 and 1 and as maps from $\mathbb{S}^1$ to M.

First, consider the fibration $\mathbb{D}^2 \times M \to M$ and, slightly abusing notation, denote the natural pullback of ω again by ω. Second, look at the fibration $pr : \mathbb{D}^2 \times M \to \mathbb{D}^2$. Denote by $Vert$ the vertical bundle over $\mathbb{D}^2 \times M$ formed by the tangent spaces to the fibers of pr. For each loop $\sigma : \mathbb{S}^1 \to M$ define by $\widehat{\sigma} : \mathbb{S}^1 \to \mathbb{D}^2 \times M$ the map $\widehat{\sigma}(t) := (t, \gamma(t))$. The bundles $\sigma^* TM$ and $\widehat{\sigma}^* Vert$ over $\mathbb{S}^1$ coincide. Similarly for each $w : \mathbb{D}^2 \to M$ denote by $\widehat{w} : \mathbb{D}^2 \to \mathbb{D}^2 \times M$ the map $\widehat{w}(z) := (z, w(z))$.

There exists $\delta > 0$, depending on G, such that for each $\gamma \in \mathcal{P}_G$ a tubular δ-neighborhood of the image of $\widehat{\gamma}$ in $\mathbb{S}^1 \times M \subset \mathbb{D}^2 \times M$, denoted by $U_{\widehat{\gamma}}$, has the following properties:

- there exists a 1-form λ on $U_{\widehat{\gamma}}$ satisfying $d\lambda = \omega$;
- $Vert$ admits a trivialization over $U_{\widehat{\gamma}}$.

Given an $\epsilon > 0$, we can choose F sufficiently C^∞-close to G so that the paths $t \mapsto f_t$ and $t \mapsto g_t$ in $Ham(M)$ are arbitrarily C^∞-close and therefore

- for every $x \in \mathrm{Fix}\,(F)$ there exists $y \in \mathrm{Fix}\,(G)$ which is ϵ-close to x (think of the fixed points as points of intersection of the graph of a diffeomorphism with the diagonal);
- the C^∞-distance between the maps $\gamma_0 : t \mapsto f_t(x)$ and $\gamma : t \mapsto g_t(y)$ from $[0,1]$ to M is bounded by ϵ and the image of $\widehat{\gamma}_0$ lies in $U_{\widehat{\gamma}}$.

Pick a map $u_0 : \mathbb{D}^2 \to M$, $u|_{\partial \mathbb{D}^2} = \gamma_0$. Since γ_0 and γ are C^∞-close one can enlarge $\mathbb{D}^2$ to a bigger disc $\mathbb{D}^2_1 \supset \mathbb{D}^2$ and find a smooth map $u : \mathbb{D}^2_1 \to M$ so that

- $u|_{\partial \mathbb{D}^2_1} = \gamma$;

- $u|_{\mathbb{D}^2} = u_0$;
- $u(\mathbb{D}^2_1 \setminus \mathbb{D}^2) \subset U_{\widehat{\gamma}}$.

Rescaling $\mathbb{D}^2_1$ we may assume without loss of generality that $[\gamma, u] \in \mathcal{P}_G$.

Trivialize the vector bundles $\gamma_0^* TM$ and $\gamma^* TM$ so that the trivializations extend to a trivialization of $u^* TM$ over $\mathbb{D}^2_1$ (and hence of $u_0^* TM$ over $\mathbb{D}^2$). Using the trivializations we can identify the paths $t \mapsto d_{\gamma_0(0)} f_t$ and $t \mapsto d_{\gamma(0)} g_t$ with some identity-based paths of symplectic matrices $A(t)$, $B(t)$. Fixing a small ϵ as above, we can also assume that F is chosen so C^∞-close to G that, in addition to all of the above, the C^∞-distance between the paths $t \mapsto A(t)$ and $t \mapsto B(t)$ in $Sp\,(2n)$ is bounded by ϵ (for instance, make sure first that the matrix paths obtained by writing the paths $t \mapsto d_{\gamma_0(0)} f_t$ and $t \mapsto d_{\gamma(0)} g_t$ using some trivialization of $Vert$ over $U_{\widehat{\gamma}}$ are close enough – then the matrix paths $t \mapsto A(t)$ and $t \mapsto B(t)$ will also be close enough).

We claim that by choosing ϵ sufficiently small in the construction above we can bound the difference between $D_F([\gamma_0, u_0])$ and $D_G([\gamma, u])$ by a quantity depending only on $\dim M$.

Indeed, the difference $|\int_0^1 F(\gamma_0(t), t)dt - \int_0^1 G(\gamma(t))dt|$ is bounded by a quantity depending only on some universal constants and ϵ, because γ_0 is ϵ-close to γ and F is ϵ-close to G with respect to the C^∞-metrics. It can be made arbitrarily small by choosing a sufficiently small ϵ. The difference

$$|\int_{\mathbb{D}^2} u_0^*\omega - \int_{\mathbb{D}^2} u^*\omega| = |\int_{\mathbb{D}^2} \widehat{u}_0^*\omega - \int_{\mathbb{D}^2} \widehat{u}^*\omega|$$

is bounded by the difference $|\int_0^1 \widehat{\gamma}_0^*\lambda - \int_0^1 \widehat{\gamma}^*\lambda|$. Since, γ_0 and γ are ϵ-close in the C^∞-metric the later difference can be made less than 1 if we choose a sufficiently small ϵ. Thus we have shown that by choosing a sufficiently small ϵ we can bound $|\mathcal{A}_F([\gamma_0, u_0]) - \mathcal{A}_G([\gamma, u])|$ by 1.

Now, as far as the Conley-Zehnder indices are concerned, our choice of the trivializations means that the difference between $CZ_F([\gamma_0, u_0])$ and $CZ_G([\gamma, u])$ is just the difference between the Conley-Zehnder indices for the matrix paths $t \mapsto A(t)$ and $t \mapsto B(t)$. But the latter paths in $Sp\,(2n)$ are ϵ-close in the C^∞-sense, hence represent close elements of $\widetilde{Sp}\,(2n)$ and if ϵ was chosen sufficiently small, then, as we mentioned in Section 3.3, their Conley-Zehnder indices differ at most by a constant depending only on n.

This finishes the proof of the claim and the proposition. $\square$

Plan of the proof of Proposition 9.1: We assume now that $\bar{H}$ is a fixed strictly convex function on $\mathbb{R}$. Our calculations will feature E as a large parameter. For quantities α, β depending on E we will write $\alpha \preceq \beta$ if $\alpha \leq \beta +$ *const* holds for large enough E, where *const* depends only on (M, ω), Φ and $\bar{H}$, and in particular does not depend on E. We will write $\alpha \approx \beta$ if $\alpha \preceq \beta$ and $\beta \preceq \alpha$. Using this language the proposition can be restated as

$$c(a, EH) \approx E\bar{H}(p_{spec}). \tag{38}$$

In general, 1-periodic orbits of the flow of EH are not isolated and therefore the Hamiltonian is not regular. Let F be a regular (time-periodic) perturbation of EH.

By the spectrality axiom, the spectral number $c(a, F)$ for $a \in QH_{2n}(M)$ equals $\mathcal{A}_F([\gamma_0, u_0])$ for some pair $[\gamma_0, u_0] \in \widetilde{\mathcal{P}}_F$ with $CZ_F([\gamma_0, u_0]) = 2n$. Thus $c(a, F) \approx D_F(\gamma_0)$. Combining this with Proposition 9.3 we get that for *some* $\gamma \in \mathcal{P}_{EH}$

$$E\bar{H}(p_{spec}) \preceq c(a, EH) \approx c(a, F) \approx D_F(\gamma_0) \approx D_{EH}(\gamma)\ . \tag{39}$$

Thus it would be enough to show that

$$D_{EH}(\gamma) \preceq E\bar{H}(p_{spec}) \ \text{ for } all\ \gamma \in \mathcal{P}_{EH}\ , \tag{40}$$

which together with (39) would imply (38).

Inequality (40) will be proved in the following way. Note that each $\gamma \in \mathcal{P}_{EH}$ lies in $\Phi^{-1}(p)$ for some $p \in \Delta$. We will show that

$$D_{EH}(\gamma) \approx E\bar{H}(p) + E\bar{H}'(p)(p_{spec} - p). \tag{41}$$

Note that (41) implies (40). Indeed, since $\bar{H}$ is strictly convex and reaches its minimum at p_{spec}, it follows from (41) that

$$D_{EH}(\gamma) \approx E\bar{H}(p) + E\bar{H}'(p)(p_{spec} - p) \leq E\bar{H}(p_{spec}),$$

which is true for any $\gamma \in \mathcal{P}_{EH}$ thus yielding (40).

Proof of (41): Let the $\mathbb{T}^1$-action on M be given by a loop of symplectomorphisms $\{\phi_t\}$, $t \in \mathbb{R}$, $\phi_t = \phi_{t+1}$. The flow of EH has the form $h_t x = \phi_{E\bar{H}'(\Phi(x))t}x$.

We view γ as a map $\gamma : [0, 1] \to M$ satisfying $\gamma(0) = \gamma(1)$. Denote $x := \gamma(0)$. The curve γ lies in $\Phi^{-1}(p)$.

Denote $N := \gamma([0,1])$. This is the $\mathbb{T}^1$-orbit of x and it is either a point or a circle.

In the first case γ is a constant trajectory concentrated at a fixed point $N \in M$ of the action. Using this constant curve γ together with the constant disc u spanning for the definitions of $I(\Phi)$ and $D_{EH}(\gamma)$ one gets

$$p_{spec} - p = m_\Phi(\gamma, u) \cdot \kappa/2,$$

and

$$D_{EH}(\gamma) = E\bar{H}(p) - \kappa/2 \cdot CZ_{EH}([\gamma, u]).$$

Thus proving (41) reduces in this case to proving

$$-CZ_{EH}([\gamma, u]) \approx E\bar{H}'(p) \cdot m_\Phi(\gamma, u).$$

Let us fix a symplectic basis of $T_N M$ and view each differential $d_N \phi_t$ as a symplectic matrix $A(t)$, so that $\{A(t)\}$ is an identity-based loop in $Sp\,(2n)$. Then

$$-CZ_{EH}([\gamma, u]) \approx CZ_{matr}(\{A(E\bar{H}'(p)t)\}),$$

while

$$E\bar{H}'(p) \cdot m_\Phi(\gamma, u) \approx E\bar{H}'(p) Maslov(\{A(t)\}).$$

Thus we need to prove

$$CZ_{matr}(\{A(E\bar{H}'(p)t)\}) \approx E\bar{H}'(p) Maslov(\{A(t)\}),$$

which follows easily from the definitions of the Conley-Zehnder index and the Maslov class.

Thus from now on we will assume that N is a circle. Take any point $x \in N$. The stabilizer of x under the $\mathbb{T}^1$-action is a finite cyclic group of order $k \in \mathbb{N}$. Thus the orbit of the $\mathbb{T}^1$-action turns k times along N. Since γ is a non-constant closed orbit of the Hamiltonian flow generated by $E\Phi^*\bar{H}$, it turns r times along N with $r \in \mathbb{Z} \setminus \{0\}$. This implies that $E\bar{H}'(p) = r/k$. We claim that without loss of generality we may assume that $l := r/k$ is an integer.

Indeed, we can always pass to $\gamma^{(k)} \in \mathcal{P}_{kEH}$, so that $(kE\bar{H})'(p) \in \mathbb{Z}$, and if we can prove the proposition for $\gamma^{(k)}$, then

$$D_{kEH}(\gamma^{(k)}) \approx kE\bar{H}(p) + kE\bar{H}'(p)(p_{spec} - p).$$

Applying Proposition 9.2 we get

$$kD_{EH}(\gamma) \approx kE\bar{H}(p) + kE\bar{H}'(p)(p_{spec} - p) + k \cdot const,$$

and hence

$$D_{EH}(\gamma) \approx E\bar{H}(p) + E\bar{H}'(p)(p_{spec} - p),$$

proving the claim for the original γ.

From now on we assume that $l := E\bar{H}'(p) \in \mathbb{Z} \setminus \{0\}$ and that $[\gamma, u] \in \widetilde{\mathcal{P}}_{l\Phi}$. Consider the Hamiltonian vector field $X := \operatorname{sgrad} \Phi$ at a point $x \in N$. Since N is a non-constant orbit we get $X \neq 0$. Then $V = T_x(\Phi^{-1}(p))$ is the skew-orthogonal complement to X. Choose a $\mathbb{T}^1$-invariant ω-compatible almost complex structure J in a neighborhood of N. Together ω and J define a $\mathbb{T}^1$-invariant Riemannian metric g. Decompose the tangent bundle TM along N as follows. Put $Z = \operatorname{Span}(JX, X)$ and set W to be the g-orthogonal complement to X in V. Thus we have a $\mathbb{T}^1$-invariant decomposition

$$T_x M = W \oplus Z \ , x \in N \ . \tag{42}$$

Furthermore, W and Z carry canonical symplectic forms. Thus W and Z define symplectic (and hence trivial) subbundles of TM over N. They induce trivial subbundles of the bundle $\gamma^* TM$ over $\mathbb{S}^1$.

We calculate

$$dh_t(x)\xi = d\phi_{EH'(\Phi(x))t}(x)\xi + EH''(\Phi(x)) \cdot d\Phi(\xi) \cdot X \ . \tag{43}$$

We consider two trivializations of the bundle $\gamma^* TM$ over $\mathbb{S}^1$. The first trivialization is defined by means of sections invariant under the $\mathbb{T}^1$-action. The second one is chosen in such a way that it extends to a trivialization of $u^* TM$ over $\mathbb{D}^2$. Using these trivializations we can identify $dh_t(x)$, respectively, with two identity-based paths $\{C_t\}$, $\{C'_t\}$ of symplectic matrices. The decomposition (42) induces a split

$$C_t = \mathbf{1} \oplus B_t \ .$$

We claim that $|CZ_{matr}(\{B_t\})|$ is bounded by a constant independent of E. Indeed, observe that in the basis (X, JX) of Z

$$B_t = \begin{pmatrix} 1 & b_{12}(t) \\ 0 & 1 \end{pmatrix} .$$

Denote by L the line spanned by $X = (1, 0)$. Perturb $\{B_t\}$ to a path $\{B'_t = R_{\delta t} B_t\}$, where R_t is the rotation by angle t, and $\delta > 0$ is small enough.

Observe that $B'(t)L \cap L = \{0\}$ for $t > 0$. It follows readily from the definitions that $|CZ_{matr}(B'_t)|$ and $|CZ_{matr}(R_{\delta t})|$ do not exceed 2. Thus by the quasi-morphism property of the Conley-Zehnder index (see Proposition 3.5) we have that $|CZ_{matr}(\{B_t\})|$ is bounded by a constant independent of E, which yields the claim. Therefore

$$CZ_{matr}\left(\{C_t\}\right) \approx 0 \ .$$

On the other hand, by formula (18)

$$CZ_{matr}\left(\{C'_t\}\right) = CZ_{matr}\left(\{C_t\}\right) + m_{l\Phi}([\gamma, u]) \ .$$

Thus

$$CZ_{EH}([\gamma, u]) := n - CZ_{matr}\left(\{C'_t\}\right) \approx -m_{l\Phi}([\gamma, u]). \tag{44}$$

Since the periodic trajectory γ lies inside $\Phi^{-1}(p)$, we get

$$\mathcal{A}_{EH}([\gamma, u]) = \int_0^1 EH(\gamma(t))dt - \int_{\mathbb{D}^2} u^*\omega = E\bar{H}(p) - \int_{\mathbb{D}^2} u^*\omega. \tag{45}$$

Using (45) and (44) the precise equality

$$D_{EH}([\gamma, u]) = \mathcal{A}_{EH}([\gamma, u]) - \frac{\kappa}{2} \cdot CZ_{EH}([\gamma, u])$$

can be turned into an asymptotic inequality

$$D_{EH}([\gamma, u]) \approx E\bar{H}(p) - \int_{\mathbb{D}^2} u^*\omega + \frac{\kappa}{2} m_{l\Phi}([\gamma, u]). \tag{46}$$

Since the periodic trajectory γ lies inside $\Phi^{-1}(p)$, we have

$$\mathcal{A}_{l\Phi}([\gamma, u]) = \int_0^1 l\Phi(\gamma(t))dt - \int_{\mathbb{D}^2} u^*\omega = lp - \int_{\mathbb{D}^2} u^*\omega. \tag{47}$$

Adding and subtracting lp from the right-hand side of (46) and using (47) we get

$$D_{EH}(\gamma) = D_{EH}([\gamma, u]) \approx \Big(E\bar{H}(p) - lp\Big) + \Big(lp - \int_{\mathbb{D}^2} u^*\omega + \frac{\kappa}{2} m_{l\Phi}([\gamma, u])\Big) =$$

$$= \Big(E\bar{H}(p) - lp\Big) + \Big(\mathcal{A}_{l\Phi}([\gamma, u]) + \frac{\kappa}{2} m_{l\Phi}([\gamma, u])\Big) = \Big(E\bar{H}(p) - lp\Big) - I(l\Phi) =$$

$$= E\bar{H}(p) + l(-I(\Phi) - p) = E\bar{H}(p) + l(p_{spec} - p) \; .$$

Recalling that $l = EH'(p)$, we finally obtain that

$$D_{EH}(\gamma) = E\bar{H}(p) + EH'(p)(p_{spec} - p),$$

which is precisely the equation (41) that we wanted to get. This finishes the proof of Proposition 9.1 and Theorem 1.9. $\square$

9.1 Calabi and mixed action-Maslov

Proof of Theorem 1.13.

Assume $H : M \times [0, 1] \to \mathbb{R}$ is a normalized Hamiltonian which generates a loop in $Ham(M)$ representing a class $\alpha \in \pi_1(Ham(M)) \subset \widetilde{Ham}(M)$. Then $H^{(l)}$ is also normalized and generates a loop representing α^l. Let us compute $\mu(\alpha) = -\mathrm{vol}\,(M) \cdot \lim_{l\to+\infty} c(a, H^{(l)})/l$.

Arguing as in the proof of (39) we get that there exists a constant $C > 0$ such that for each $l \in \mathbb{N}$ there exists $\gamma \in \mathcal{P}_{H^{(l)}}$ for which $|c(a, H^{(l)}) - D_{H^{(l)}}(\gamma)| \leq C$. But, as it follows from the definitions and from the fact that I is a homomorphism, $D_{H^{(l)}}(\gamma)$ does not depend on γ and equals $-I(\alpha^l) = -lI(\alpha)$. This immediately implies that $\mu(\alpha) = \mathrm{vol}\,(M) \cdot I(\alpha)$. $\square$

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