p-ADIC LIMIT OF WEAKLY HOLOMORPHIC MODULAR FORMS OF HALF INTEGRAL WEIGHT

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Abstract. In [18] Serre obtained the p-adic limit of the integral Fourier coefficient of modular forms on $SL_2(\mathbb{Z})$ for $p = 2, 3, 5, 7$. In this paper, we extend the result of Serre to weakly holomorphic modular forms of half integral weight on $\Gamma_0(4N)$ for $N = 1, 2, 4$. A proof is based on linear relations among Fourier coefficients of modular forms of half integral weight. As applications we obtain congruences of Borcherds exponents, congruences of quotient of Eisentein series and congruences of values of L-functions at a certain point are also studied. Furthermore, the congruences of the Fourier coefficients of Siegel modular forms on Maass Space are obtained using Ikeda lifting ([16] and [15]).

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1. Introduction and Statement of Main Results

In [18] Serre obtained the p-adic limits of the integral Fourier coefficient of modular forms on $SL_2(\mathbb{Z})$ for $p = 2, 3, 5, 7$. In this paper, we extend the result of Serre to weakly holomorphic modular forms of half integral weight on $\Gamma_0(4N)$ for $N = 1, 2, 4$. A proof is based on linear relations among Fourier coefficients of modular forms of half integral weight. As applications congruences of Borcherds exponents, congruences of values of L-functions at a certain point and congruences of quotient of Eisentein series are obtained. Furthermore, the congruences of the Fourier coefficients of Siegel modular forms on Maass Space are obtained using Ikeda lifting ([16] and [15]).

First we need some notations to state the main theorems. Let, for odd d, by

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv 3 \pmod 4, \end{cases}$$

$$\Gamma_0(4N)_t := \{\gamma \in \Gamma_0(4N) \ : \ \gamma(t) = t\}$$

and

$$\left\langle \left(\begin{smallmatrix} 1 & h_t \\ 0 & 1 \end{smallmatrix}\right) \right\rangle := \gamma \Gamma_0(4N)_t \gamma_t^{-1},$$

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where $\gamma_t = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ such that $\gamma_t(t) = \infty$. Denote the q-expansion of a modular form $f \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ at each cusp t of $\Gamma_0(4N)$ by
(1.1)
$$(f\mid_{\lambda+\frac{1}{2}} \gamma_t)(z) = \left(\frac{c}{d}\right)^{2\lambda+1} \epsilon_d^{-1-2\lambda}(cz+d)^{-\lambda-\frac{1}{2}} f\left(\frac{az+b}{cz+d}\right) = q_t^r \sum_{n=b_t}^{\infty} a_f^t(n) q_t^n,\ q_t := q^{\frac{2\pi i z}{h_t}}.$$

When $t \sim \infty$, we denote $a_f^{\infty}(n)$ by $a_f(n)$. A cusp t is called a regular cusp if $r = 0$. Let $U_{4N} := \{t_1, \cdots, t_{\nu(4N)}\}$ be the set of the inequivalent regular cusps of $\Gamma_0(4N)$. Note that the genus of $\Gamma_0(4N)$ is zero if and only if $1 \le N \le 4$. Let $\mathcal{M}_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ be the space of weakly holomorphic modular forms of weight $\lambda + \frac{1}{2}$ on $\Gamma_0(4N)$ and $\mathcal{M}^0_{\lambda+\frac{1}{2}}(\Gamma_0(N))$ denote the set of $f(z) \in \mathcal{M}_{\lambda+\frac{1}{2}}(\Gamma_0(N))$ whose the constant term of q-expansion at each cusps is zero. Define an operator U_p for $f(z) = \sum_{n=n_0}^{\infty} a_f(n)q^n$ by
$$f(z)|U_p := \sum_{n=n_0}^{\infty} a_f(pn)q^n.$$

With these notations we state the following theorem.

Theorem 1. *Suppose that p is a prime and that $N = 1, 2$ or 4. Let*
$$f(z) := \sum_{n=n_0}^{\infty} a_f(n)q^n \in \mathcal{M}^0_{\lambda+\frac{1}{2}}(\Gamma_0(4N)) \cap \mathbb{Z}_p[[q]],$$
where $\mathbb{Z}_p$ denotes the ring of p-adic integers.

(1) If $p = 2$ and $a_f(0) = 0$, then, for each $j \in \mathbb{N}$, there exists a positive integer b such that
$$f(z)|(U_p)^b \equiv 0 \pmod{p^j}.$$
(2) If $p \ge 3$ and $f(z) \in \mathcal{M}^0_{\lambda+\frac{1}{2}}(\Gamma_0(N))$ with $\lambda \equiv 2$ or $2 + \left[\frac{1}{N}\right] \pmod{\frac{p-1}{2}}$, then, for each $j \in \mathbb{N}$, there exists a positive integer b such that
$$f(z)|(U_p)^b \equiv 0 \pmod{p^j}.$$

Remark 1.1. In [11] the p-adic limit of a certain sum of Fourier coefficients is obtained when the weight of a modular form is $\frac{3}{2}$.

Let $\Delta_{4N,\lambda}$ be the unique normalized modular form of weight $\lambda + \frac{1}{2}$ on $\Gamma_0(4N)$ with the zero of the maximum order at ∞. Further let
$$R_4(z) := \frac{\eta(4z)^8}{\eta(2z)^4},\ R_8(z) := \frac{\eta(8z)^8}{\eta(4z)^4},$$
$$R_{12}(z) := \frac{\eta(12z)^{12}\eta(2z)^2}{\eta(6z)^6\eta(4z)^4} \text{ and } R_{16}(z) := \frac{\eta(16z)^8}{\eta(8z)^4},$$

where $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$. Define, for $\ell,\ n \in \mathbb{N}$,

$$\mathbf{m}(\ell : n) := \begin{cases} 0 \text{ if } \left[\frac{2n}{\ell-1}\right] \equiv 0 \pmod 2 \\ 1 \text{ if } \left[\frac{2n}{\ell-1}\right] \equiv 1 \pmod 2 \end{cases}$$

and

$$\alpha(\ell : n) := n - \frac{\ell-1}{2}\left[\frac{2n}{\ell-1}\right].$$

It turns out that the congruence properties of $f(z)$ are more complicated if $f(z) \notin \mathcal{M}^0_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$.

Theorem 2. *Suppose that $p \geq 5$ is a prime and $1 \leq N \leq 4$. Let*

$$f(z) := \sum_{n \gg -\infty} a_f(n)q^n \in \mathcal{M}_{\lambda+\frac{1}{2}}(\Gamma_0(4N)) \cap \mathbb{Z}_p[[q, q^{-1}]].$$

If $\lambda \equiv 2$ or $2 + \left[\frac{1}{N}\right] \pmod{\frac{p-1}{2}}$, then there exists a positive integer b_0 such that

$$a_f\left(p^{2b-\mathbf{m}(p:\lambda)}\right) \equiv -\sum_{t \in U_{4N}} h_t a^t_{\frac{\Delta_{4N,3-\alpha(p:\lambda)}(z)}{R_{4N}(z)^{e\cdot\omega(4N)}}}(0) a^t_f(0) \pmod p$$

for every positive integer $b > b_0$.

Example 1.2. An overpatition of n is a partition of n in which the fist occurrence of a number may be overlined. Let $\bar{P}(n)$ be the overpartition of n. It is known that

$$\sum_{n=0}^{\infty} \bar{P}(n)q^n = \frac{\eta(2z)}{\eta(z)^2}$$

and that $\frac{\eta(2z)}{\eta(z)^2} \in \mathcal{M}_{-\frac{1}{2}}(\Gamma_0(16))$. From Theorem 2 we have

$$\bar{P}(5^{2b}) \equiv 1 \pmod 5$$

for every positive integer b.

2. **Applications: More Congruences of Modular Forms of Half Integral Weight**

In this section, we study several results related to p-adic limit of Borcherds exponents, congruence relations of values of L-functions and congruences of quotients of Eisenstein series, which are derived from Theorem 1 and 2.

2.1. **p-adic Limits of Borcherds Exponents.** As the first application we obtain congruences of Borcherds exponents. Let $\mathcal{M}_H$ denote the set of integer weight meromorphic modular forms on $SL_2(\mathbb{Z})$ with a Heegner divisor, integer coefficients, and leading coefficient 1. Let

$$\mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4)) := \{f(z) = \sum_{n=m}^{\infty} a_f(n)q^n \in \mathcal{M}_{\frac{1}{2}}(\Gamma_0(4)) \mid a(n) = 0 \text{ for } n \equiv 2, 3 \pmod 4\}.$$

If $f(z) = \sum_{n=n_0}^{\infty} a_f(n)q^n \in \mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$, then define $\Psi(f(z))$ by

$$\Psi(f(z)) := q^{-h} \prod_{n=1}^{\infty} (1-q^n)^{a_f(n^2)},$$

where $h = -\frac{1}{12}a_f(0) + \sum_{1<n\equiv 0,1 \pmod 4} a_f(-n)H(-n)$ and $H(-n)$ denotes the usual Hurwitz class number of discriminant $-n$. Borcherds proved that the map Ψ is an isomorphism.

Theorem 2.1 ([4])**.** *The map Ψ is an isomorphism from $\mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$ to $\mathcal{M}_H$, and the weight of $\Psi(f(z))$ is $a_f(0)$.*

Let $j(z)$ be the usual j-invariant function. We denote the product expansion of $j(z)$ by

$$j(z) = q \prod_{n=1}^{\infty} (1-q^n)^{A(n)}.$$

Let $F(z) := q^h \prod_{n=1}^{\infty}(1-q^n)^{c(n)}$ be a meromorphic modular form of weight k in $\mathcal{M}_H$. The p-adic limit of $\sum_{d|n} d \cdot c(d)$ was studied in [5] for $p = 2, 3, 5, 7$. In the following theorem, we obtain the p-adic limit of $c(d)$ for $p = 2, 3, 5, 7$.

Theorem 3. *Let $F(z) := q^h \prod_{n=1}^{\infty}(1-q^n)^{c(n)}$ be a meromorphic modular form of weight k in $\mathcal{M}_H$.*

(1) If $p = 2$, then, for each $j \in \mathbb{N}$, there exists a positive integer b such that

$$c(mp^b) \equiv 2k \pmod{p^j}$$

for every positive integer m.

(2) If $p \in \{3, 5, 7\}$, then, for for each $j \in \mathbb{N}$, there exists a positive integer b such that

$$5c(mp^b) - \varpi(F)A(mp^b) \equiv 10k \pmod{p^j}$$

for every positive integer m. Here, $\varpi(F)$ is a constant determined by the constant term of q-expansion of $\Psi^{-1}(F)$ at 0.

2.2. **Sums of n-Squares.** As the second application we study congruences of representation numbers of the sums of n squares. Let, for $u \in \mathbb{Z}_{>0}$,

$$r_n(u) := \sharp\{(s_1, \cdots, s_n) \in \mathbb{Z}^n \ : \ s_1^2 + \cdots + s_n^2 = u\}.$$

Theorem 4. *Suppose that $p \geq 5$ is a prime. If $\lambda \equiv 2$ or $3 \pmod{\frac{p-1}{2}}$, then there exists a positive integer C_0 such that for every integer $b > C_0$*

$$r_{2\lambda+1}\left(p^{2b-\mathbf{m}(p:\lambda)}\right) \equiv -\left(14 - 4\alpha\left(p:\lambda\right)\right) + 16\left(\frac{-1}{p}\right)^{\left[\frac{\lambda}{p-1}\right]+\alpha(p:\lambda)\mathbf{m}(p:\lambda)} \pmod{p}.$$

Remark 2.2. For example, if $\lambda \equiv 2 \pmod{p-1}$ and p is a odd prime, then there exists a positive integer C_0 such that, for every integer $b > C_0$,

$$r_{2\lambda+1}\left(p^{2b}\right) \equiv 10 \pmod{p}.$$

2.3. **Values of L-Functions.** Thirdly we derive congruences among values of L-functions at a certain point. Let D be a fundamental discriminant and $\chi_D(n) := \left(\frac{D}{n}\right)$. The Dirichlet L-function of χ_D is defined by

$$L(s, \chi_D) := \sum_{n=1}^{\infty} \frac{\chi_D(n)}{n^s}.$$

In [6], Carlitz studied an analog of the Staudt-Clausen theorem for $L(1-m, \chi_D)$ modulo prime p, where $p^e(p-1)|m$ and $p \nmid D$. The value of $L(1-\frac{p+1}{2}, \chi_D)$ modulo p was also studied in [2], where $D := (-1)^{\frac{p+1}{2}}pN$ is a fundamental discriminant and $\left(\frac{-N}{p}\right) = 1$. We now give the values of $L(1-\lambda, \chi_p)$ modulo primes $p \geq 5$.

Theorem 5. *Let $p \geq 5$ be a prime and $\chi_p(n) := \left(\frac{(-1)^{\frac{p-1}{2}}p}{n}\right)$. If $\lambda \equiv 2 + \frac{p-1}{2} \pmod{p-1}$, then*

$$L(1-\lambda, \chi_p) \equiv -10 \cdot \frac{B_{2\lambda}}{2\lambda} \pmod{p}.$$

2.4. **Quotients of Eisenstein Series.** Congruences among the coefficients of quotients of Eisentein series have been studied in [3]. Let $H_{r+\frac{1}{2}}(z) := \sum_{N=0}^{\infty} H(r,N)q^n$ denote the Cohen Eisenstein series of weight $r + \frac{1}{2}$, where $r \geq 2$ is an integer. We derive congruences for the coefficients of quotients of Cohen-Eisentein series and Eisentein series.

Theorem 6. *Let*

$$F(z) := \frac{H_{\frac{5}{2}}(z)}{E_4(z)} = \sum_{n=0}^{\infty} a_F(n)q^n,$$

$$G(z) := \frac{H_{\frac{7}{2}}(z)}{E_6(z)} = \sum_{n=0}^{\infty} a_G(n)q^n$$

and

$$W(z) := \frac{H_{\frac{9}{2}}(z)}{E_6(z)} = \sum_{n=0}^{\infty} a_W(n)q^n.$$

Then there exists a positive integer C_0 *such that*

$$\begin{aligned} a_F(11^{2b+1}) &\equiv 1 \pmod{11}, \\ a_G(11^{2b+1}) &\equiv 6 \pmod{11}, \\ a_W(11^{2b+1}) &\equiv 2 \pmod{11}, \end{aligned}$$

for every integer $b > C_0$.

2.5. **Maass Space.** Next we deal with congruences of the Fourier coefficients of a Siegel modular form in Maass space. To define Maass space, we follow notations given in [15].

Let $T \in M_{2g}(\mathbb{Q})$ be a rational, half-integral, symmetric, non-degenerate matrix of size $2n$ and

$$D_T := (-1)^g \det(2T)$$

be the discriminant of T. Then $D_T \equiv 0, 1 \pmod 4$ and we write $D_T = D_{T,0}f_T^2$, where $D_{T,0}$ is the corresponding fundamental discriminant. Let

$$G_8 := \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

and G_7 be the upper $(7,7)$-submatrix of G_8. Let

$$S_0 := \begin{cases} G_8^{\bigoplus (g-1)/8} \bigoplus 2, & \text{if} \quad g \equiv 1 \pmod 8, \\ G_8^{\bigoplus (g-7)/8} \bigoplus G_7, & \text{if} \quad g \equiv -1 \pmod 8. \end{cases}$$

For $m \in \mathbb{N}$, the set of natural numbers $\mathbb{N}$, with $(-1)^g m \equiv 0, 1 \pmod 4$ define a rational, half-integral, symmetric, positive definite matrix T_m of size $2g$ by

$$T_m := \begin{cases} \begin{pmatrix} \frac{1}{2}S_s & 0 \\ 0 & m/4 \end{pmatrix}, & \text{if } m \equiv 0 \pmod 4, \\ \begin{pmatrix} \frac{1}{2}S_s & \frac{1}{2}e_{2g-1} \\ \frac{1}{2}e'_{2g-1} & [m+2+(-1)^n]/4 \end{pmatrix}, & \text{if } m \equiv (-1)^g \pmod 4, \end{cases}$$

where $e_{2g-1} = (0, \ldots, 0, 1)' \in \mathbb{Z}^{(2n-1,1)}$ is the standard column vector and e'_{2g-1} denotes the transpose of e_{2g-1}. Following Kohnen and Kojima (see [15]), we define the Maass space of weight $g + k$ and genus $2g$, which is realized as the image of Ikeda lifting.

Definition 2.3. (Maass Space) Suppose that $g \equiv 0, 1 \pmod 4$ and let $k \in \mathbb{N}$ with $g \equiv k \pmod 2$. Let

$$S_{k+g}^{Maass}(\Gamma_{2g}) := \left\{ F(Z) = \sum_{T>0} A(T) q^{tr(TZ)} \in S_{k+g}(\Gamma_{2g}) \;\middle|\; A(T) = \sum_{a|f_T} a^{k-1}\phi(a;T)A(T_{|D_T|/a^2}) \right\}$$

(see (6.2) for more detailed notations). This space is called as a Maass space of genus $2g$ and weight $n+k$.

Recently it was proved in [15] that Maass space is the same as the image of Ikeda lifting when $g \equiv 0, 1 \pmod 4$. Using this fact with Theorem 1, we derive the following congruences of the Fourier coefficients of $F(Z)$ in $S_{k+g}^{Maass}(\Gamma_{2g})$.

Theorem 7. *Suppose that* $g \equiv 0, 1 \pmod 4$. *Let*

$$F(Z) := \sum_{T>0} A(T) q^{tr(TZ)} \in S_{k+g}^{Maass}(\Gamma_{2g})$$

with an integral $A(T)$, $T > 0$. *If* $k \equiv 2$ *or* $3 \pmod{\frac{p-1}{2}}$ *for a prime* p, *then, for each* $j \in \mathbb{N}$, *there exists a positive integer* b *for which*

$$A(T) \equiv 0 \pmod{p^j}$$

for every $T > 0$ *with* $\det(2T) \equiv 0 \pmod{p^b}$.

This paper is organized as follows: Section 3 gives a linear relation among the Fourier coefficients of modular forms of half integral weight. Using this result, in Section 4 we prove Theorem 1 and 2. In Section 5 we give a proof of Theorem 3. We prove Theorem 4, 5 6 and 7 in Section 5.

3. **Linear Relation among the Fourier Coefficients of modular forms of Half Integral Weight**

Let $V(N;k,n)$ be the subspace of $\mathbb{C}^n$ generated by the first n coefficients of q-expansion of f at ∞ for $f \in S_k(\Gamma_0(N))$, where $S_k(\Gamma_0(N))$ denotes the space of cusp forms of weight $k \in \mathbb{Z}$ on $\Gamma_0(N)$. Let $L(N;k,n)$ be the orthogonal complement of $V(N;k,n)$ in $\mathbb{C}^n$ with the usual inner product of $\mathbb{C}^n$. The vector space $L(1;k,d(k)+1)$ was studied by Siegel when the value of Dedekind zeta function at a certain point was computed, where $d(k) = \dim(S_k(\Gamma(1)))$. The vector space $L(1;k,n)$ is explicitly obtained by the principal part of negative weight modular forms in [8]. These results were extend in [7] to groups

$\Gamma_0(N)$ of genus zero. Let, for $1 \le N \le 4$,

$$EV\left(4N, \lambda+\frac{1}{2}; n\right) := \left\{\left(a_f^{t_1}(0), \cdots, a_f^{t_{\nu(4N)}}(0), a_f(1), \cdots, a_f(n)\right) \in \mathbb{C}^{n+\nu(4n)} \;\middle|\; f \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))\right\},$$

where $U_{4N} := \{t_1, \cdots, t_{\nu(4N)}\}$ is the set of the inequivalent regular cusps of $\Gamma_0(4N)$. We define $EL(4N, \lambda+\frac{1}{2}; n)$ to be the orthogonal complement of $EV(4N, \lambda+\frac{1}{2}; n)$ in $\mathbb{C}^{n+\nu(4N)}$.

Note that $R_{4N}(z) \in M_2(\Gamma_0(4N))$ has its only zero at ∞. So, the valence formula (see [17]) implies that

$$\omega(4) = 1, \omega(8) = 2, \omega(12) = 4, \omega(16) = 4. \tag{3.1}$$

For each $g \in M_{r+\frac{1}{2}}(\Gamma_0(4N))$ and $e \in \mathbb{N}$, let

$$\frac{g(z)}{R_{4N}(z)^e} = \sum_{\nu=1}^{e\cdot\omega(4N)} b(4N, e, g; \nu)q^{-\nu} + O(1) \text{ at } \infty. \tag{3.2}$$

With these notations we state the following theorem:

Theorem 3.1. *Suppose that $\lambda \ge 0$ is an integer and $1 \le N \le 4$. For each $e \in \mathbb{N}$ such that $e \ge \frac{\lambda}{2} - 1$ take $r = 2e - \lambda + 1$. The linear map $\Phi_{r,e}(4N) : M_{r+\frac{1}{2}}(\Gamma_0(4N)) \to EL(4N, \lambda+\frac{1}{2}; e\cdot\omega(4N))$, defined by*

$$\Phi_{r,e}(4N)(g) = \left(h_{t_1} a^{t_1}_{\frac{g(z)}{R_{4N}(z)^e}}(0), \cdots, h_{t_{\nu(4N)}} a^{t_{\nu(4N)}}_{\frac{g(z)}{R_{4N}(z)^e}}(0), b(4N, e, g; 1), \cdots, b(4N, e, g; e\cdot\omega(4N))\right),$$

is an isomorphism.

Proof of Theorem 3.1. Suppose that $G(z)$ is a meromorphic modular form of weight 2 on $\Gamma_0(4N)$. For $\tau \in \mathbb{H} \cup C_{4N}$, let D_τ be the image of τ under the canonical map from $\mathbb{H} \cup C_{4N}$ to a compact Riemann surface $X_0(4N)$. Here, $\mathbb{H}$ is the usual complex upper half plane, and C_{4N} denotes the set of all inequivalent cusps of $\Gamma_0(4N)$. The residue $\mathrm{Res}_{D_\tau} G dz$ of $G(z)$ at $D_\tau \in X_0(4N)$ is well-defined since we have a canonical correspondence between a meromorphic modular form of weight 2 on $\Gamma_0(4N)$ and a meromorphic 1-form of $X_0(4N)$. If $Res_\tau G$ denotes the residue of G at τ on $\mathbb{H}$, then

$$\mathrm{Res}_{D_\tau} G dz = \frac{1}{l_\tau} Res_\tau G.$$

Here, l_τ is the order of the isotropy group at τ. The residue of G at each cusp $t \in C_{4N}$ is

$$\mathrm{Res}_{D_t} G dz = h_t \cdot \frac{a_G^t(0)}{2\pi i}, \tag{3.3}$$

where $(G\mid_2 \gamma_t)(z) = (cz+d)^{-2}G\left(\frac{az+b}{cz+d}\right) = \sum_{n=m_t}^{\infty} a_G^t(n)q_t^n$ for $\gamma_t = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ such that $\gamma_t(t) = \infty$ and h_t denotes the smallest positive integer such that $(G\mid_2 \gamma_t)(z+h_t) = (G\mid_2 \gamma_t)(z)$ and $q_t := e^{\frac{2\pi i z}{h_t}}$. Now we give a proof of Theorem 3.1.

To prove Theorem 3.1, take

$$G(z) = \frac{g(z)}{R_{4N}(z)^e} f(z),$$

where $g \in M_{r+\frac{1}{2}}(\Gamma_0(4N))$ and $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$. Note that $G(z)$ is holomorphic on $\mathbb{H}$. Since $g(z)$, $R_{4N}(z)$ and $f(z)$ are holomorphic and $R_{4N}(z)$ has no zero on $\mathbb{H}$, it is enough to compute the residues of $G(z)$ only at the inequivalent cusps to apply the Residue theorem. The q-expansion of $\frac{g(z)}{R_{4N}(z)^e} f(z)$ at ∞ is

$$\frac{g(z)}{R_{4N}(z)^e} f(z) = \left(\sum_{\nu=1}^{e\cdot\omega(4N)} b(4N, e, g; \nu)q^{-\nu} + a_{\frac{g(z)}{R_{4N}(z)^e}}(0) + O(q) \right) \left(\sum_{n=0}^{\infty} a_f(n)q^n \right).$$

Since $R_{4N}(z)$ has no zero at $t \not\sim \infty$, we have

$$\left. \frac{g(z)}{R_{4N}(z)^e} f(z) \right|_2 \gamma_t = a^t_{\frac{g(z)}{R_{4N}(z)^e}}(0)a_f(0) + O(q_t).$$

Further, note that, for an irregular cusp t,

$$a^t_{\frac{g(z)}{R_{4N}(z)^e}}(0)a_f(0) = 0.$$

So, we have

$$\sum_{t\in U_{4N}} h_t a^t_{\frac{g}{R_{4N}^{e\cdot\omega(4N)}}}(0)a_f^t(0) + \sum_{\nu=1}^{e\cdot\omega(4N)} b(4N, e, g; \nu)a_f(\nu) = 0. \tag{3.4}$$

by the Residue Theorem and (3.3). This implies that $\Phi_{r,e}(4N)$ is well-defined. The linearity of the map $\Phi_{r,e}(4N)$ is clear.

It remains to show that $\Phi_{r,e}(4N)$ is an isomorphism. Since there exists no holomorphic modular form of negative weight except the constantly zero function, we obtain the injectivity of $\Phi_{r,e}(4N)$. Note that for $e \geq \frac{\lambda-1}{2}$

$$\dim_{\mathbb{C}}\left(EL\left(4N; \lambda+\frac{1}{2}, e\cdot\omega(4N)\right)\right) = e\cdot\omega(4N) + |U_{4N}| - \dim_{\mathbb{C}}\left(M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))\right).$$

However, the set C_{4N} of inequivalent cusps of $\Gamma_0(4N)$ are, for $1 \leq N \leq 4$,

$$\begin{aligned} C_4 &= \left\{\infty, 0, \tfrac{1}{2}\right\}, \\ C_8 &= \left\{\infty, 0, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}\right\}, \\ C_{12} &= \left\{\infty, 0, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{6}\right\}, \\ C_{16} &= \left\{\infty, 0, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{8}\right\}, \end{aligned}$$

and it can be checked that

$$|U_4| = 2, |U_8| = 3, |U_{12}| =, |U_{16}| = 6 \tag{3.5}$$

(for detail see Chapter 4. in [13]). We recall the dimension formula of $M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ for $1 \le N \le 4$:

TABLE 1. Dimension Formula for $M_k(\Gamma_0(4N))$

N	$k = 2n + \frac{1}{2}$	$k = 2n + \frac{3}{2}$	$k = 2n$
$N = 1$	$n + 1$	$n + 1$	$n + 1$
$N = 2$	$2n + 1$	$2n + 2$	$2n + 1$
$N = 3$	$4n + 1$	$4n + 3$	$4n + 1$
$N = 4$	$4n + 2$	$4n + 4$	$4n + 1$

So we have by (3.4), (3.5) and Table 1

$$\dim_{\mathbb{C}}\left(EL\left(4N, \lambda + \frac{1}{2}; e \cdot \omega(N)\right)\right) = \dim_{\mathbb{C}}(M_{r+\frac{1}{2}}(\Gamma_0(4N)))$$

since $r = 2e - \lambda + 1$. This implies that $\Phi_{r,e}(4N)$ is surjective since the map $\Phi_{r,e}(4N)$ is injective. This completes the proof. □

4. Proofs of Theorem 1 and 2

4.1. **Proof of Theorem 1.** First, we get linear relations among the Fourier coefficients of modular forms of half integral weight modulo primes $p \ge 5$. Let

$$\begin{aligned}\widetilde{M}_{\lambda+\frac{1}{2},\,p}(\Gamma_0(4N)) := \{H(z) = \sum_{n=0}^{\infty} a_H(n)q^n \in \mathbb{Z}_p/p\mathbb{Z}_p[[q]] \mid \\ H \equiv h \pmod{p} \text{ for some } h \in \mathbb{Z}_p[[q]] \cap M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))\}.\end{aligned}$$

and

$$\begin{aligned}\widetilde{S}_{\lambda+\frac{1}{2},\,p}(\Gamma_0(4N)) := \{H(z) = \sum_{n=1}^{\infty} a_H(n)q^n \in \mathbb{Z}_p/p\mathbb{Z}_p[[q]] \ : \\ H \equiv h \pmod{p} \text{ for some } h \in \mathbb{Z}_p[[q]] \cap S_{\lambda+\frac{1}{2}}(\Gamma_0(4N))\}.\end{aligned}$$

The following lemma gives the dimension of $\widetilde{M}_{\lambda,\,p}(\Gamma_0(4N))$.

Lemma 4.1. *Suppose that λ is a positive integer and $1 \le N \le 4$. Let p be a prime such that*

$$\begin{cases} p \ge 3 & \text{if } N = 1, 2, 4, \\ p \ge 5 & \text{if } N = 3. \end{cases}$$

Then

$$\dim \widetilde{M}_{\lambda+\frac{1}{2},\,p}(\Gamma_0(4N)) = \dim M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$$

and

$$\dim \widetilde{S}_{\lambda+\frac{1}{2},\,p}(\Gamma_0(4N)) = \dim S_{\lambda+\frac{1}{2}}(\Gamma_0(4N)).$$

Proof. Let

$$j_{4N}(z) = q^{-1} + O(q)$$

be a normalized meromorphic modular function with a pole only at ∞. Explicitly, they are

$$j_4(z) = \frac{\eta(z)^8}{\eta(4z)^8} + 8, \qquad j_8(z) = \frac{\eta(4z)^{12}}{\eta(2z)^4\eta(8z)^8},$$

$$j_{12}(z) = \frac{\eta(4z)^4\eta(6z)^2}{\eta(2z)^2\eta(12z)^4}, \qquad j_{16}(z) = \frac{\eta^2(z)\eta(8z)}{\eta(2z)\eta^2(16z)} + 2.$$

Since the Fourier coefficients of $\eta(z)$ and $\frac{1}{\eta(z)}$ are integral, the q-expansion of $j_{4N}(z)$ has integral coefficients.

Recall that $\Delta_{4N,\lambda}$ is the unique normalized modular form of weight $\lambda + \frac{1}{2}$ on $\Gamma_0(4N)$ with the zero of the maximum order at ∞. Denote the order of zero of $\Delta_{4N,\lambda}$ at ∞ by $\delta_\lambda(4N)$. Then the basis of $M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ can be chosen as

$$\left\{\Delta_{4N,\lambda}(z) j_{4N}(z)^e \ \mid\ 0 \le e \le \delta_\lambda(4N)\right\}. \tag{4.1}$$

If $\Delta_{4N,\lambda}(z)$ is p-integral, then $\{\Delta_{4N,\lambda}(z)j_{4N}(z)^e \mid 0 \le e \le \delta_\lambda(4N)\}$ forms also a basis of $\widetilde{M}_{\lambda+\frac{1}{2},p}(\Gamma_0(4N))$. Note that $\delta_\lambda(4N) = \dim M_{\lambda+\frac{1}{2}}(\Gamma_0(4N)) - 1$. So, we have, from Table 1,

$$\Delta_{4N,\lambda}(z) = \Delta_{4N,j}(z) R_{4N}(z)^{\frac{\lambda-j}{2}}, \tag{4.2}$$

where $\lambda \equiv j \pmod 2$ for $j \in \{0,1\}$. One can choose $\Delta_{4N,j}(z)$ as the following:

$$\begin{aligned}
&\Delta_{4,0}(z) = \theta(z),\ \Delta_{4,1}(z) = \theta(z)^3,\\
&\Delta_{8,0}(z) = \theta(z),\ \Delta_{8,1}(z) = \tfrac{1}{4}\left(\theta(z)^3 - \theta(z)\theta(2z)^2\right),\\
&\Delta_{12,0}(z) = \theta(z),\ \Delta_{12,1}(z) = \tfrac{1}{6}\left(\textstyle\sum_{x,y,z\in\mathbb{Z}} q^{3x^2+2\left(y^2+z^2+yz\right)} - \textstyle\sum_{x,y,z\in\mathbb{Z}} q^{3x^2+4y^2+4z^2+4yz}\right),\\
&\Delta_{16,0}(z) = \tfrac{1}{2}\left(\theta(z) - \theta(4z)\right),\ \Delta_{16,1}(z) = \tfrac{1}{8}\left(\theta(z)^3 - 3\theta(z)^2\theta(4z) + 3\theta(z)\theta(4z)^2 - \theta(4z)^3\right).
\end{aligned}$$

Since $\theta(z) = 1 + 2\sum_{n=1}^{\infty} q^n$, the coefficients of the q-expansion of $\Delta_{4N,j}(z)$, $j \in \{0,1\}$, are p-integral. This completes the proof with (4.1) and (4.2). $\square$

Remark 4.2. The proof of Lemma 4.1 implies that the spaces of $M_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ for $N = 1,2,4$ are generated by eta-quotients since $\theta(z) = \frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2}$.

For $1 \le N \le 4$ denote

$$\widetilde{V_S}\left(4N, \lambda + \frac{1}{2}; n\right) := \left\{(a_f(1), \cdots, a_f(n)) \in \mathbb{F}_p^n \mid f \in \widetilde{S}_{\lambda+\frac{1}{2}}(\Gamma_0(4N))\right\},$$

where $\mathbb{F}_p := \mathbb{Z}_p/p\mathbb{Z}_p$. We define $\widetilde{L_S}(4N, \lambda + \frac{1}{2}; n)$ to be the orthogonal complement of $\widetilde{V_S}(4N, \lambda + \frac{1}{2}; n)$ in $\mathbb{F}_p^n$. Using Lemma 4.1, we obtain the following proposition.

Proposition 4.3. *Suppose that λ is a positive integer and $1 \leq N \leq 4$. For each $e \in \mathbb{N}$, $e \geq \frac{\lambda}{2} - 1$, take $r = 2e - \lambda + 1$. The linear map $\widetilde{\psi_{r,e}}(4N) : \widetilde{M}_{r+\frac{1}{2}}(\Gamma_0(4N)) \to \widetilde{L_S}(4N, \lambda + \frac{1}{2}; e \cdot \omega(4N))$, defined by*

$$\widetilde{\psi_{r,e}}(4N)(g) = (b(4N, e, g; 1), \cdots, b(N, e, g; e \cdot \omega(4N))),$$

is an isomorphism. Here, $b(4N, e, g; \nu)$ is given in (3.2).

Proof. Note that $\dim S_{\frac{3}{2}}(4N) = 0$ and that

$$\dim S_{\lambda+\frac{1}{2}}(4N) + N + 1 + \left[\frac{N}{4}\right] = \dim M_{\lambda+\frac{1}{2}}(4N)$$

(see [9]). So, from Lemma 4.1 and Table 1, it is enough to show that $\psi_{r,e}(4N)$ is injective. If g is in the kernel of $\psi_{r,e}(4N)$, then $\frac{g(z)}{R_{4N}(z)^e} \cdot R_{4N}(z)^e \equiv 0 \pmod{p}$ by the Sturm's formula (see [19]). So, we have $g(z) \equiv 0 \pmod{p}$ since $R_{4N}(z)^e \not\equiv 0 \pmod{p}$. This completes our claim. □

Theorem 4.4. *Suppose that p is a prime and that $N = 1, 2$ or 4. Let*

$$f(z) := \sum_{n=1}^{\infty} a_f(n) q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(4N)) \cap \mathbb{Z}_p[[q]].$$

If $\lambda \equiv 2$ or $2 + \left[\frac{1}{N}\right] \pmod{\frac{p-1}{2}}$ or $p = 2$, then there exists a positive integer b such that, for every positive integer n,

$$a_f\left(np^b\right) \equiv 0 \pmod{p}, \ \forall n \in \mathbb{N}.$$

Proof of Theorem 4.4. **i)** First, suppose that $p \geq 3$: Take a positive integer ℓ and b such that

$$\frac{3 - 2\alpha(p:\lambda)}{2} p^{2b} + \left(\lambda + \frac{1}{2}\right) p^{\mathbf{m}(p:\lambda)} + \ell(p-1) = 2. \tag{4.3}$$

Moreover, if $b > \log_p\left(\frac{2}{3-2\alpha(p:\lambda)}\left(\lambda + \frac{1}{2}\right) p^{\mathbf{m}(p:\lambda)} - 2\right)$, then there exists a positive integer ℓ satisfying (4.3). Note that $a_f^t(0) = 0$ for every cusp t of $\Gamma_0(4N)$ since the given function $f(z)$ is a cusp form. So, if $r = 2e - \alpha(p:\lambda) + 1$, then Theorem 3.1 implies that, for every $g(z) \in \widetilde{M}_{r+\frac{1}{2}}(\Gamma_0(4N))$,

$$\sum_{\nu=1}^{e\cdot\omega(4N)} b(4N, e, g; \nu) a_f(\nu p^{2b-\mathbf{m}(p:\lambda)}) \equiv 0 \pmod{p}$$

since

$$\left(\frac{g(z)}{R_{4N}(z)^e}\right)^{p^{2b}} f(z)^{p^{\mathbf{m}(p:\lambda)}} E_{p-1}^{\ell}(z)$$
$$\equiv \left(\sum_{\nu=1}^{e\cdot\omega(4N)} b(4N,e,g;\nu)q^{-\nu p^{2b}} + a_{\frac{g(z)}{R_{4N}(z)^e}}(0) + \sum_{n=1}^{\infty} a_{\frac{g(z)}{R_{4N}(z)^e}}(n)q^{np^{2b}}\right)$$
$$\cdot \left(\sum_{n=0}^{\infty} a_f(n)q^{np^{\mathbf{m}(p:\lambda)}}\right) \pmod{p}.$$

So, we have by Proposition 4.3 that

$$\begin{aligned}&\left(a\left(p^{2b-\mathbf{m}(p:\lambda)}\right), a\left(2p^{2b-\mathbf{m}(p:\lambda)}\right), \cdots, a\left(e\cdot\omega(4N)p^{2b-\mathbf{m}(p:\lambda)}\right)\right)\\ &\in \widetilde{V_S}\left(4N, \alpha(p:\lambda)+\tfrac{1}{2}; n\right).\end{aligned}$$

If $\alpha(p:\lambda) = 2$ or $2+\left[\frac{1}{N}\right]$, then

$$\dim S_{\alpha(p:\lambda)+\frac{1}{2}}(\Gamma_0(4N)) = \widetilde{V_S}\left(4N, \alpha(p:\lambda)+\frac{1}{2}; n\right) = 0.$$

ii) We assume that $p=2$: Note that $\frac{\Delta_{4N,1}(z)}{R_{4N}(z)} = q^{-1}+O(1)$ for $N=1,2,4$. So, there exist a polynomial $F(X)\in\mathbb{Z}[X]$ such that

$$F(j_{4N}(z))\frac{\Delta_{4N,1}(z)}{R_{4N}(z)} = q^{-n}+O(1).$$

For an integer b, $\frac{2}{2^b} > \lambda+2$, let

$$G(z) := \left(F(j_{4N}(z))\frac{\Delta_{4N,1}(z)}{R_{4N}(z)}\right)^{2^b} f(z)\theta(z)^{2^{1+2b}-2\lambda+3}.$$

Since $\theta(z)\equiv 1 \pmod{2}$, Theorem 3.1 implies that $a_f(2^b\cdot n)\equiv 0 \pmod{2}$. Therefore, we completes the proof. □

Remark 4.5. In the proof of Theorem 4.4 the case when $p\geq 3$ can be also treated as the same as the case of $p=2$. However, one gets more structure information about $f(z)|U_p$ following the given proof.

To apply Theorem 4.4, we need the following two propositions.

Proposition 4.6 ([20])**.** *Suppose that p is an odd prime, k and N are integers with $(N,p)=1$, and χ is a Dirichlet character modulo $4N$. Let*

$$f(z) = \sum a(n)q^n \in \mathcal{M}_{\lambda+\frac{1}{2}}(\Gamma_0(4N)).$$

Suppose that that $\xi := \left(\begin{smallmatrix} a & b \\ cp^2 & d\end{smallmatrix}\right)$, with $ac>0$. then there exists an integer n_0, a sequence $\{a_0(n)\}_{n\geq n_0}$, a positive integer $h_0|N$, and an $r_0\in\{0,1,2,3\}$ such that for each $m\geq 1$, *we have*

$$(f(z)|U_{p^m})|_{\lambda+\frac{1}{2}}\xi = \sum_{\substack{n\geq n_0 \\ 4n+r_0\equiv 0 \pmod{p^m}}} a_0(n)q^{\frac{4n+r_0}{4h_0p^m}}.$$

Proposition 4.7 ([1])**.** *Suppose that p is an odd prime. Suppose that $p \nmid N$, that $j \geq 1$ is an integer, and that*

$$g(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(4Np^j)) \cap \mathbb{Z}[[q]].$$

Then there is a cusp form $G(z) \in S_{\lambda'+\frac{1}{2}}(\Gamma_0(4N)) \cap \mathbb{Z}[[q]]$ such that

$$G(z) \equiv g(z) \pmod{p},$$

where $\lambda' + \frac{1}{2} = (\lambda + \frac{1}{2})p^j + p^e(p-1)$ for enough large $e \in \mathbb{N}$.

Remark 4.8. In [1], Proposition 4.7 is proved for $p \geq 5$. But, one can check, by following the method given in [1], that Proposition 4.7 holds when $p = 3$.

Now, we prove Theorem 1.

Proof of Theorem 1. Let

$$G_p(z) := \begin{cases} \frac{\eta(8z)^{48}}{\eta(16z)^{24}} \in M_{12}(\Gamma_0(16)) & \text{if } p = 2, \\ \frac{\eta(z)^{27}}{\eta(9z)^3} \in M_{12}(\Gamma_0(9)) & \text{if } p = 3, \\ \frac{\eta(4z)^{p^2}}{\eta(4p^2z)} \in M_{\frac{p^2-1}{2}}(\Gamma_0(p^2)) & \text{if } p \geq 5. \end{cases}$$

By the well known properties of eta-quotients (see [10]) and Proposition 4.6, there exit positive integers ℓ, m such that

$$\begin{cases} (f(z)|U_{p^m})G_p(z)^\ell \in S_{k'+\frac{1}{2}}(\Gamma_0(16)) & \text{if } p = 2, \\ (f(z)|U_{p^m})G_p(z)^\ell \in S_{k'+\frac{1}{2}}(\Gamma_0(4p^2)) & \text{if } p \geq 3, \end{cases}$$

Using Proposition 4.7, we find

$$F(z) \in S_{k''+\frac{1}{2}}(\Gamma_0(4N))$$

such that $F(z) \equiv (f(z)|U_{p^m})G_p(z)^\ell \equiv (f(z)|U_{p^m}) \pmod{p}$. Theorem 4.4 implies that there exist a positive integer b such that $F(z)|U_{p^{2b}} \equiv 0 \pmod{p}$. So, we have that $\frac{1}{p} \cdot F(z)|U_{p^{m+2b}} \in \mathbb{Z}[q, q^{-1}]$. Repeating this method, we obtain the result. $\square$

4.2. **Proof of Theorem 2.** Using Theorem 3.1 with a special modular form, we prove Theorem 2.

Proof of Theorem 2. We take a positive integer ℓ and a positive odd integer b such that

$$\frac{3-2\alpha(p:\lambda)}{2}p^b + \left(\lambda+\frac{1}{2}\right)p^{\mathbf{m}(p:\lambda)} + \ell(p-1) = 2.$$

Let $F(z) := \left(\frac{\Delta_{4N,3-\alpha(p:\lambda)}(z)}{R_{4N}(z)}\right)^{p^b}$ and $G(z) := E_{p-1}(z)^\ell f(z)^{p^{\mathbf{m}(p:\lambda)}}$. Since that $E_{p-1} \equiv 1 \pmod{p}$, we have

$$F(z)G(z) \equiv \left(\sum_{n=-1}^{\infty} a_{\frac{\Delta_{4N,3-\alpha(p:\lambda)}(z)}{R_{4N}(z)}}(n)q^{np^b}\right)\left(\sum_{n=m_\infty}^{\infty} a_f^t(n)q^n\right) \pmod{p}.$$

We claim that the Fourier coefficients of $f(z)$ at each cusps are p-integral. Then we have

$$\begin{aligned}\left(F\cdot G|_2\gamma_t\right)(z) &\equiv \left(q_t^r\sum_{n=m_t}^{\infty} a_F^t(n)q_t^n\right)\left(q_t^r\sum_{n=0}^{\infty} a_G^t(n)q_t^n\right)\\ &\equiv \left(q_t^r\sum_{n=m_t}^{\infty} a_f^t(n)q_t^n\right)\left(q^{\frac{r}{h_t}p^u}\sum_{n=0}^{\infty} a^t_{\frac{\Delta_{4N,3-\alpha(p:\lambda)}(z)}{R_{4N}(z)}}(n)q^{\frac{n}{h_t}p^u}\right) \pmod{p}\end{aligned}$$

for $t \not\sim \infty$. If u is large enough, then

$$\begin{aligned} a_{F(z)G(z)}(0) &\equiv a_{\frac{\Delta_{4N,3-\alpha(p:\lambda)}(z)}{R_{4N}(z)}}(0)a_f(0) + a_f(p^u) && \pmod{p} \quad, \\ a^t_{F(z)G(z)}(0) &\equiv a^t_{\frac{\Delta_{4N,3-\alpha(p:\lambda)}(z)}{R_{4N}(z)}}(0)a_f^t(0) && \pmod{p} \quad \text{for } t \not\sim \infty.\end{aligned}$$

Then, using the Residue theorem again, we completes the proof.

We check the claim. Let $\Delta(z) := q\prod_{n=1}^{\infty}(1-q^n)^{24}$. Note that $\Delta(z)$ is a cusp form of weight 12 on $\Gamma(1)$. So, we may take a positive integer e such that $\Delta(z)^e f(z)$ is a holomorphic modular form. Note that the q-expansion of $j_{4N}(z)$ and $\Delta_{4N,12e+\lambda}(z)$ at each cusps are p-integral. Then we have

$$\Delta(z)^e f(z) = \sum_{n=0}^{\delta_{12e+\lambda}(4N)} c_n j_{4N}(z)^n \Delta_{4N,12e+\lambda}(z)$$

by the formula (4.1). Moreover, c_n is p-integral since

$$j_{4N}(z)^n\Delta_{4N,12e+\lambda}(z) = q^{\delta_{12e+\lambda}(4N)-n} + O\left(q^{\delta_{12e+\lambda}(4N)-n+1}\right)$$

and $f(z) \in \mathbb{Z}_p[[q,q^{-1}]]$. Note that $p \nmid 4N$ since $1 \le N \le 4$ and $p > 3$ is a prime. So, the Fourier coefficients of $j_{4N}(z)$, $\Delta_{N,12e+\lambda}(z)$ and $\frac{1}{\Delta(z)}$ at each cusps are p-integral. This completes the proof. $\square$

5. Proof of Theorem 3

Using Theorem 1 and Theorem 2.1, we prove Theorem 3.

Proof of Theorem 3. Note that $j(z) \in \mathcal{M}_H$. Let

$$g(z) := \Psi^{-1}(j(z)) \text{ and } f(z) := \Psi^{-1}(F(z)) = \sum_{n=n_0}^{\infty} a_f(n)q^n.$$

It is known (see §14 in [4]) that

$$\frac{1}{3}g(z) = \frac{\frac{d}{dz}(\theta(z))E_{10}(4z)}{4\pi i\Delta(4z)} - \frac{\theta(z)\frac{d}{dz}(E_{10}(4z))}{80\pi i\Delta(4z)} - \frac{152}{5}\theta(z).$$

Since the constant terms of q-expansions at ∞ of $f(z) - k\theta(z)$ and $g(z)$ are zero and $a^0_{\theta(z)}(0) = \frac{1-i}{2}$, we have

$$f(z) - k\theta(z) - \frac{a^0_f(0) + k(1-i)/2}{a^0_g(0)}g(z) \in \mathcal{M}^0_{\frac{1}{2}}(\Gamma_0(4)).$$

Here, $a^0_g(0)$ is defined by (1.1). By Theorem 1, we completes the proof. $\square$

6. Proofs of Theorem 4, 5 and 6

We begin by introducing the following proposition.

Proposition 6.1. *Let p be an odd prime and*

$$f(z) := \sum_{n=0}^{\infty} a_f(n)q^n \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4)) \cap \mathbb{Z}_p[[q]].$$

If $\lambda \equiv 2$ or $3 \pmod{\frac{p-1}{2}}$, then

$$\begin{aligned} &a_f\left(p^{2b-\mathbf{m}(p:\lambda)}\right) \\ &\qquad \equiv -(14 - 4\alpha(p:\lambda))a_f(0) + 2^8\left(2^{-1} - 2^{-1}i\right)^{p^b(7-2\alpha(p:\lambda))} a^0_f(0) \pmod{p} \end{aligned}$$

for every integer $b > \log_p\left(\frac{2}{2\alpha(p:\lambda)-3}\left(\lambda + \frac{1}{2}\right)p^{\mathbf{m}(p:\lambda)} + 2\right)$.

Proof of Proposition 6.1. Note that the given function $f(z) = \sum_{n=0}^{\infty} a_f(n)q^n$ is a modular form of weight $\lambda + \frac{1}{2}$, where $\lambda \equiv 2$ or $3 \pmod{\frac{p-1}{2}}$. So let

$$\left(\lambda + \frac{1}{2}\right)p^{\mathbf{m}(p:\lambda)} := \nu \cdot (p-1) + \alpha(p:\lambda) + \frac{1}{2}$$

for $\nu \in \mathbb{Z}_{\geq 0}$. For every integer b,

$$b > \frac{1}{2}\log_p\left(\frac{2}{3 - 2\alpha(p:\lambda)}\left(\left(\lambda + \frac{1}{2}\right)p^{\mathbf{m}(p:\lambda)} - 2\right)\right),$$

there exist an positive integer ℓ such that

$$\frac{3-2\alpha(p:\lambda)}{2}p^{2b}+\left(\lambda+\frac{1}{2}\right)p^{\mathbf{m}(p:\lambda)}+\ell(p-1)=2,$$

since

$$\frac{3-2\alpha(p:\lambda)}{2}p^{2b}+\left(\lambda+\frac{1}{2}\right)p^{\mathbf{m}(p:\lambda)}-2=\frac{3-2\alpha(p:\lambda)}{2}(p^{2b}-1)+\nu(p-1).$$

We have

$$F(z)\equiv \textstyle\sum_{n=0}^{\infty} a_f(n)q^{np^{\mathbf{m}(p:\lambda)}} \pmod{p},$$
$$G(z)\equiv q^{-p^b}+14-4\alpha(p:\lambda)+a_G(1)q+\cdots \pmod{p}.$$

Note $a_G(n)$ is p-integral for every integer n. Moreover, we obtain

$$F(z)G(z)|_2\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\equiv\left(a_f^0(0)+\cdots\right)\left(-2^{6p^b}\left(\frac{1}{2}-\frac{i}{2}\right)^{p^b(7-2\alpha(p:\lambda))}+\cdots\right)\pmod{p},$$

where $a_f^0(0)$ is given in (1.1). Note that $\left\{\infty,0,\frac{1}{2}\right\}$ is the set of distinct cusps of $\Gamma_0(4)$ and that $\frac{1}{2}$ is a irregular cusp. So, Theorem 2 implies that that

$$a_f(p^{2u-\mathbf{m}(p:n)})+(14-4\alpha(p:\lambda))a_f(0)-2^8a_f^0(0)\left(\frac{1}{2}-\frac{i}{2}\right)^{p^b(7-2\alpha(p:\lambda))}\equiv 0 \pmod{p}.$$

This gives a proof of Proposition 6.1. □

6.1. **Proof of Theorem 4.** Now we prove Theorem 4.

Proof of Theorem 4. Let $\theta(z):=1+2\sum_{n=1}^{\infty}q^2$. To use Theorem 6.1, take

$$f(z):=\theta^{2\lambda+1}(z)=1+\sum_{\ell=1}^{\infty}r_{2\lambda+1}(\ell)q^{\ell}=\sum_{n=0}^{\infty}a_f(n)q^n.$$

Note that $\theta(z)\in M_{\frac{1}{2}}(\Gamma_0(4))$. Since $\theta(z)|_{\frac{1}{2}}\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)=\frac{1-i}{2}+O\left(q^{\frac{1}{4}}\right)$, we obtain

$$a_f(0)=1 \text{ and } a_f^0(0)=\left(\frac{1}{2}-\frac{i}{2}\right)^{2\lambda+1}.$$

Since $\lambda\equiv 2,3 \pmod{\frac{p-1}{2}}$ and $\left(\frac{1}{2}-\frac{i}{2}\right)^8=\frac{1}{16}$, we have

$$\begin{aligned}
&\left(\frac{1}{2}-\frac{i}{2}\right)^{p^{2u}(7-2\alpha(p:\lambda))}a_f^0(0)^{p^{\mathbf{m}(p:\lambda)}}\\
&\equiv\left(\frac{1}{2}-\frac{i}{2}\right)^{p^{2u}(7-2\alpha(p:\lambda))}\left(\frac{1}{2}-\frac{i}{2}\right)^{p^{\mathbf{m}(p:\lambda)}\left(2\alpha(p:\lambda)+(p-1)\left(2\left[\frac{\lambda}{p-1}\right]+\mathbf{m}(p:\lambda)\right)+1\right)}\\
&\equiv\left(\frac{1}{2}-\frac{i}{2}\right)^{(7-2\alpha(p:\lambda))(p^{2u}-1)}\left(\frac{1}{2}-\frac{i}{2}\right)^{8+2(p-1)\left[\frac{\lambda}{p-1}\right]+\mathbf{m}(p:\lambda)p^{\mathbf{m}(p:\lambda)}(p-1)+(p^{\mathbf{m}(p:\lambda)}-1)(1+2\alpha(p:\lambda))}\\
&\equiv\left(\frac{1}{2}-\frac{i}{2}\right)^{8+2\left[\frac{\lambda}{p-1}\right](p-1)+2\alpha(p:\lambda)(p^{\mathbf{m}(p:\lambda)}-1)}\equiv\frac{1}{16}\left(\frac{-1}{p}\right)^{\left[\frac{\lambda}{p-1}\right]+\alpha(p:\lambda)\mathbf{m}(p:\lambda)} \pmod{p},
\end{aligned}$$

where u is a positive integer. Applying Theorem 6.1, we derive the result. $\square$

6.2. **Proofs of Theorem 5 and 6.** A modular form $H_{r+\frac{1}{2}}(z) := \sum_{N=0}^{\infty} H(r,N)q^n$ denotes the Cohen Eisenstein series of weight $r+\frac{1}{2}$, where $r \geq 2$ is an integer. If $(-1)^r N \equiv 0, 1 \pmod 4$, then $H(r,N) = 0$. If $N = 0$, then $H(r,0) = \frac{-B_{2r}}{2r}$. If N is a positive integer and $Df^2 = (-1)^r N$, where D is a fundamental discriminant, then

$$H(r,N) = L(1-r,\chi_D) \sum_{d|f} \mu(d)\chi_D(d)d^{r-1}\sigma_{2r-1}(f/d). \tag{6.1}$$

Here, $\mu(d)$ is a *Möbius* function. The following theorem implies that the Fourier coefficients of Cohen Eisenstein series $H_{r+\frac{1}{2}}(z)$ are p-integral if $\frac{p-1}{2} \nmid r$.

Theorem 6.2 ([6]). *Let D be a fundamental discriminant. If D is divisible by at least two different primes, then $L(1-n,\chi_D)$ is an integer for every positive integer n. If $D = p$, $p > 2$, then $L(1-n,\chi_D)$ is an integer for every positive integer n unless $\gcd(p, 1-\chi_D(g)g^n) \neq 1$, where g is a primitive root $\pmod p$.*

Using $H_{r+\frac{1}{2}}(z)$, we give proofs of Theorem 5 and 6.

Proof of Theorem 5. We take $f(z) := H_{\lambda+\frac{1}{2}}(z) = \sum_{n=0}^{\infty} a_f(n)q^n$. Then the Fourier coefficients of $f(z)$ are p-integral by Theorem 6.2 and by the assumption that

$$\lambda \equiv 2 + \frac{p-1}{2}.$$

Note that

$$a_f(p^{2n+1}) = L(1-\lambda,\chi_p) \sum_{d|p^n} \mu(d)\chi_p(n)d^{\lambda-1}\sigma_{2\lambda-1}(p^n/d),$$

$$a_f(0) = \frac{B_{2\lambda}}{2\lambda} \text{ and } a_f^0(0) = \frac{B_{2\lambda}}{2\lambda}\left(1+i^{1+2\lambda}\right)(2i)^{-1-2\lambda}.$$

We have

$$\begin{aligned} 2^8\left(2^{-1}-2^{-1}i\right)^{p(7-2\alpha(p:\lambda))} a_f^0(0) &\equiv 2^8\left(2^{-1}-2^{-1}i\right)^{3p}\frac{B_{2\lambda}}{2\lambda}\left(1+i^{1+2\lambda}\right)(2i)^{-1-2\lambda} \\ &\equiv \left(1-i\left(\frac{-1}{p}\right)\right)^3 \frac{B_{2\lambda}}{2\lambda}\left(1+i\left(\frac{-1}{p}\right)\right)i^{-1}\left(\frac{-1}{p}\right) \\ &\equiv -4\frac{B_{2\lambda}}{2\lambda} \pmod p. \end{aligned}$$

So, we obtain

$$L(1-\lambda,\chi_p) \equiv -10 \cdot \frac{B_{2\lambda}}{2\lambda} \pmod p.$$

Since $L(1-\lambda,\chi_p)$ and $\frac{B_{2\lambda}}{2\lambda}$ are a p-integral rational number, this completes the proof. $\square$

Proof of Theorem 6. Note that $E_{10}(z) = E_4(z)E_6(z)$. So, the functions $E_{10}(z)F(z)$, $E_{10}(z)G(z)$ and $E_{10}(z)W(z)$ are a modular form of weight $8 \cdot \frac{1}{2}$, $7 \cdot \frac{1}{2}$ and $8 \cdot \frac{1}{2}$ respectively. Moreover, the Fourier coefficients of those are 11-integral, since the Fourier coefficients of $H_{\frac{5}{2}}(z)$, $H_{\frac{7}{2}}(z)$ and $H_{\frac{9}{2}}(z)$ are 11-integral by Theorem 6.2. We have

$$\begin{aligned}
&E_{10}(z)F(z) = \tfrac{B_4}{4} + O(q),\\
&E_{10}(z)F(z)|_{\frac{17}{2}} \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \tfrac{B_4}{4}(1+i)(2i)^{-5} + O\left(q^{\frac{1}{4}}\right),\\
&E_{10}(z)G(z) = \tfrac{B_6}{6} + O(q),\\
&E_{10}(z)G(z)|_{\frac{15}{2}} \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \tfrac{B_6}{6}(1-i)(2i)^{-7} + O\left(q^{\frac{1}{4}}\right),\\
&E_{10}(z)W(z) = \tfrac{B_8}{8} + O(q),\\
&E_{10}(z)W(z)|_{\frac{17}{2}} \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \tfrac{B_8}{8}(1+i)(2i)^{-9} + O\left(q^{\frac{1}{4}}\right),
\end{aligned}$$

where B_{2r} is the $2r$th Bernoulli number. The conclusion follows by Theorem 6.1. $\square$

6.3. **Proofs of Theorem 7.** We begin by introducing some notations. Let $V := (\mathbb{F}_p^{2n}, Q)$ be the quadratic space over $\mathbb{F}_p$, where Q is a quadratic form obtained from a quadratic form $x \mapsto T[x] (x \in \mathbb{Z}_p^{2n})$ by reducing modulo p. We denote by $< x, y >:= Q(x,y) - Q(x) - Q(y) (x, y \in \mathbb{F}_p^{2n})$ the associated bilinear form and let

$$R(V) := \{x \in \mathbb{F}_p^{2n} \ : < x, y >= 0,\ \forall y \in \mathbb{F}_p^{2n},\ Q(x) = 0\}$$

be the radical of $R(V)$. Following [12], define a polynomial

$$H_{n,p}(T;X) := \begin{cases} 1 & \text{if } s_p = 0, \\ \prod_{j=1}^{[(s_p-1)/2]}(1-p^{2j-1}X^2) & \text{if } s_p > 0,\ s_p \text{ odd}, \\ (1+\lambda_p(T)p^{(s_p-1)/2}X)\prod_{j=1}^{[(s_p-1)/2]}(1-p^{2j-1}X^2) & \text{if } s_p > 0,\ s_p \text{ even}, \end{cases}$$

where for even s_p we denote

$$\lambda_p(T) := \begin{cases} 1 & \text{if } W \text{ is a hyperbolic space or } s_p = 2n, \\ -1 & \text{otherwise.} \end{cases}$$

Following [14], for a nonnegative integer μ, define $\rho_T(p^\mu)$ by

$$\sum_{\mu \geq 0} \rho_T(p^\mu)X^\mu := \begin{cases} (1-X^2)H_{n,p}(T;X), & \text{if } p|f_T, \\ 1 & \text{otherwise.} \end{cases}$$

We extend the functions ρ_T multiplicatively to natural numbers $\mathbb{N}$ by defining

$$\sum_{\mu \geq 0} \rho_T(p^\mu)X^{-\mu} := \prod_{p|f_p}((1-X^2)H_{n,p}(T;X)).$$

Let

$$\mathcal{D}(T) := GL_{2n}(\mathbb{Z}) \setminus \{G \in M_{2n}(\mathbb{Z}) \cap GL_{2n}(\mathbb{Q}) \ : \ T[G^{-1}] \text{ half-integral}\},$$

where $GL_{2n}(\mathbb{Z})$ operates by left-multiplication and $T[G^{-1}] = T'G^{-1}T$. Then $\mathcal{D}(T)$ is finite. For $a \in \mathbb{N}$ with $a|f_T$ let

$$\phi(a;T) := \sqrt{a} \sum_{d^2|a} \sum_{G \in \mathcal{D}(T), |\det(G)|=d} \rho_{T[G^{-1}]}(a/d^2). \tag{6.2}$$

Note that $\phi(a;T) \in \mathbb{Z}$ for all a. With these notations we state the following theorem:

Theorem 6.3 ([15]). *Suppose that $g \equiv 0, 1 \pmod 4$ and let $k \in \mathbb{N}$ with $g \equiv k \pmod 2$. A Siegel modular form F is in $S_{k+n}^{Maass}(\Gamma_{2g})$ if and only if there exists*

$$f(z) = \sum_{n=1}^{\infty} c(n)q^n \in S_{k+\frac{1}{2}}(\Gamma_0(4))$$

such that $A(T) = \sum_{a|f_T} a^{k-1}\phi(a;T)c\left(\frac{|D_T|}{a^2}\right)$ for all T. Here,

$$D_T := (-1)^g \cdot \det(2T)$$

and $D_T = D_{T,0}f_T^2$ with $D_{T,0}$ the corresponding fundamental discriminant and $f_T \in \mathbb{N}$.

Remark 6.4. The proof of Theorem 6.3 in [15] implies that if $A(T) \in \mathbb{Z}$ for all T, then $c(m) \in \mathbb{Z}$ for all $m \in \mathbb{N}$.

Proof of Theorem 7. From Theorem 6.3 we can take

$$f(z) = \sum_{n=1}^{\infty} c(n)q^n \in S_{k+\frac{1}{2}}(\Gamma_0(4)) \cap \mathbb{Z}_p[[q]]$$

such that

$$F(Z) = \sum_{T>0} A(T)q^{tr(TZ)} = \sum_{T>0} \sum_{a|f_T} a^{k-1}\phi(a;T)c\left(\frac{|D_T|}{a^2}\right) q^{tr(TZ)}.$$

From Theorem 1 there exists a positive integer b such that, for every positive integer m,

$$c(p^b m) \equiv 0 \pmod{p^j}$$

since $k \equiv 2$ or $3 \pmod{\frac{p-1}{2}}$. Suppose that $p^{b+2j} || D_T|$. If $p^j|a$ and $a|f_T$, then

$$a^{k-1}\phi(a;T)c\left(\frac{|D_T|}{a^2}\right) \equiv 0 \pmod{p^j}.$$

If $p^j \nmid a$ and $a|f_T$, then $p^b \left| \frac{|D_T|}{a^2} \right.$ and $a^{k-1}\phi(a;T)c\left(\frac{|D_T|}{a^2}\right) \equiv 0 \pmod{p^j}$. This completes the proof. $\square$

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