Biased random walks on random combs

Tanya M Elliott and John F Wheater

Department of Physics, University of Oxford
Rudolf Peierls Centre for Theoretical Physics,
1 Keble Road,
Oxford OX1 3NP, UK

E-mail: t.elliott1@physics.ox.ac.uk, j.wheater@physics.ox.ac.uk

Abstract. We develop rigorous, analytic techniques to study the behaviour of biased random walks on combs. This enables us to calculate exactly the spectral dimension of random comb ensembles for any bias scenario in the teeth or spine. Two specific examples of random comb ensembles are discussed; the random comb with nonzero probability of an infinitely long tooth at each vertex on the spine and the random comb with a power law distribution of tooth lengths. We also analyze transport properties along the spine for these probability measures.

PACS numbers: 05.40.Fb, 04.60.Nc, 05.45.Df

1. Introduction

The behaviour of random walks on random combs is of interest from a number of points of view. Condensed matter physicists have studied such structures because they serve as a model for diffusion in more complicated fractals and percolation clusters [1, 2, 3, 4]. In the context of quantum gravity, random combs are a tractable example of a random manifold ensemble and understanding their geometric properties can provide insight into higher dimensional problems [5, 6, 7]. Most of the literature concerns approximate analytical techniques and numerical solutions, although there are exact calculations of leading order behaviour in some cases [8]. To this end, it is desirable to have rigorous methods for determining the geometric quantities of interest and that is the purpose of this paper.

One such quantity is the dimensionality of the ensemble. On a sufficiently smooth manifold all definitions of dimension will agree, but for fractal geometries like random combs this is not necessarily true. The spectral dimension is defined to be d_s provided the ensemble average probability of a random walker being back at the origin at time t, takes the asymptotic form

$$\lim_{t\to\infty} \langle p_C(t)\rangle \sim t^{-d_s/2}. \tag{1}$$

This concept of dimension does not in general agree with the Hausdorff dimension d_H, which is a measure of how the expectation value of the volume enclosed within a geodesic distance R from a marked point scales as $R \to \infty$:

$$\lim_{R\to\infty} \langle V(R)\rangle \sim R^{d_H}. \tag{2}$$

We know that for diffusion on regular structures the mean square displacement at large times is proportional to t, but for a fractal substrate there is anomalous diffusion characterised by the relation

$$\langle n^2(t)\rangle \sim t^{2/d_w}, \tag{3}$$

where d_w represents the fractal dimension of the walk and depends sensitively on the nature of the random structure.

Biased random walks on combs have also been studied in connection with disordered materials, since such a system is a paradigm for diffusion on fractal structures in the presence of an applied field [9, 10]. As we discuss later there are several different bias regimes. Topological bias, where at every vertex in the comb there is an increased probability of moving away from the origin was first studied for a random comb with a power-law distribution of tooth lengths in [11]. Other works have discussed the effects of bias away from the origin only in the teeth [12] and only in the spine [13]. The effect of going into the teeth can be viewed as creating a waiting time for the walk along the spine; the distribution of the waiting time depends on both the bias and the length of the teeth and the outcome is the result of subtle interplay between the two.

In [14] some new, rigorous techniques were developed to study random walks on combs. This enabled an exact, but very simple calculation of the spectral dimension of random combs. The principle idea is to split both random combs and random walks into subsets that give either strictly controllable or exponentially decaying contributions to the calculation of physical characteristics. These methods were later reinforced to prove that the spectral dimension of generic infinite tree ensembles is 4/3 [15]. In this paper we use and extend the techniques of [14] to deal with biased walks on combs. Some of our results are new; some qualify statements made in the literature; and some merely confirm results already derived by other, usually less rigorous, methods.

The random combs, the bias scenario, some useful generating functions and the critical exponents are defined in the next section. In Section 3 we introduce some deterministic combs, discuss general properties of the generating functions and establish bounds that will be instrumental when studying random ensembles. Section 4 looks at regions of bias where the large time behaviour is independent of the comb ensemble or simply dependent on the expectation value of the first return generating function in the teeth. In Section 5 we compute the spectral dimension in regions of bias where it is influenced by the probability measure on the teeth. Two specific cases are considered: the random comb with nonzero probability of an infinitely long tooth at each vertex on the spine and the random comb with a power law distribution of tooth lengths. Section 6 examines transport properties along the spine for these same probability measures and in the final section we review the main results, compare with the literature and discuss their significance. Some exact calculations and proofs omitted from the main text are outlined in the appendices.

2. Definitions

Wherever possible we use the definitions and notation of [14]; we repeat them here for the reader's convenience but mostly refer back to [14] for proofs and derived properties.

2.1. Random combs

Let N_∞ denote the nonnegative integers regarded as a graph so that n has the neighbours $n \pm 1$ except for 0 which only has 1 as a neighbour. Let N_ℓ be the integers $1, \ldots, \ell$ regarded as a graph so that each integer $n \in N_\ell$ has two neighbours $n \pm 1$ except for 0 and ℓ which only have one neighbour, 1 and $\ell - 1$, respectively. A comb C is an infinite

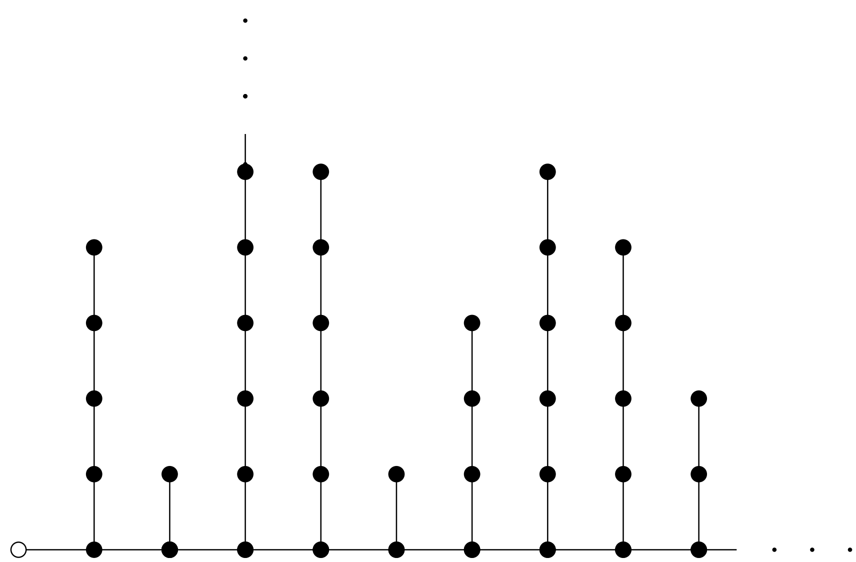

Figure 1. A comb.

rooted tree-graph with a special subgraph S called the spine which is isomorphic to N_∞ with the root, which we denote r, at $n = 0$. At each vertex of S, except the root r, there is attached by their endpoint 0 one of the graphs N_ℓ or N_∞. The linear graphs attached to the spine are called the teeth of the comb, see figure 1. We will denote by T_n the tooth attached to the vertex n on S, and by C_k the comb obtained by removing the links $(0, 1), \ldots, (k - 1, k)$, the teeth $T_1, \ldots, T_k$ and relabelling the remaining vertices on the spine in the obvious way. An arbitrary comb is specified by a list of its teeth $\{T_1, \ldots\}$ and $|T_k|$ denotes the length of the tooth. Note that we have excluded the possibility of a tooth of zero length. This is for technical convenience in what follows and can be relaxed [16].

In this paper we are interested in random combs for which the length ℓ of each tooth is identically and independently distributed with probability μ_ℓ. This induces a probability measure μ on the positive integers and expectation values with respect to this measure will be denoted $\langle\cdot\rangle_\mu$. In particular we will consider the two measures

$$\mu^A_\ell = \begin{cases} p, & \ell=\infty,\\ 1-p, & \ell=1,\\ 0, & \text{otherwise}; \end{cases}$$
$$\mu^B_\ell = \frac{C_a}{\ell^a}, \quad a>1. \tag{4}$$

However, the results proved for μ^B apply to any measure with the same behaviour at large ℓ and we note in passing that the methods used here will work for any distribution that is reasonably smooth, for example the exponential distribution. The measure μ^B has been discussed quite extensively in the literature but μ^A has not.

2.2. Biased random walks

We regard time as integer valued and consider a walker who makes one step on the graph for each unit time interval. If the walker is at the root or at the end-point of a tooth then she leaves with probability 1. If at any other vertex the probabilities are parametrized by two numbers ϵ_1 and ϵ_2 as shown in figure 2a and the allowed range of these parameters is shown in figure 2b. For walks in the teeth there is bias away from or towards the spine depending on whether ϵ_2 is positive or negative; similarly a walk on the spine is biased away from or towards the root depending on whether ϵ_1 is positive or negative. When there is no bias we say that the walk is 'critical'; the fully critical case $\epsilon_1=\epsilon_2=0$ was covered in [14]. The notation

$$\begin{aligned} b_- &= 1-\epsilon_1-\epsilon_2,\\ b_+ &= 1+\epsilon_1-\epsilon_2,\\ b_T &= 1+2\epsilon_2, \end{aligned} \tag{5}$$

will be used where applicable since these combinations appear often in our analysis. We denote by B, B', B_1, B_2 etc constants which depend on ϵ_1 and ϵ_2 and may vary from line to line but are positive and finite on the relevant range; other constants will be denoted c, c' etc.

The generating function for the probability $p_C(t)$ that the walker on C is back at the root at time t having left it at $t=0$ is defined by

$$Q_C(x) = \sum_{t=0}^{\infty}(1-x)^{t/2}p_C(t). \tag{6}$$

Letting ω be a walk on C starting at r, $\omega(t)$ the vertex where the walker is to be found at time t, and $\rho_{\omega(t)}$ the probability for the walker to step from $\omega(t)$ to $\omega(t+1)$, we have

$$Q_C(x) = \sum_{\omega:r\to r}(1-x)^{\frac{1}{2}|\omega|}\prod_{t=0}^{|\omega|-1}\rho_{\omega(t)}. \tag{7}$$

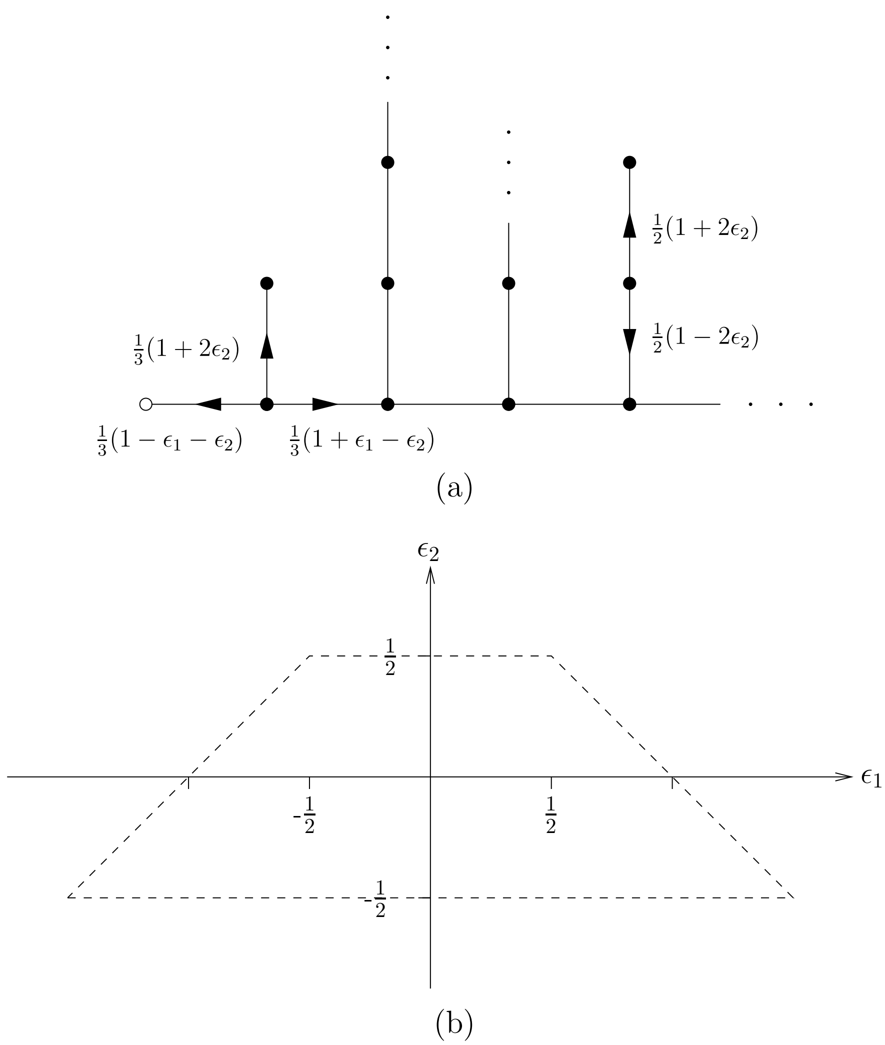

Figure 2. Bias parameterisation.

A similar relation gives the generating function for probabilities for first return to the root, $P_C(x)$, except that the trivial walk of duration 0 is excluded. The two functions are related by

$$Q_C(x) = \frac{1}{1 - P_C(x)}, \tag{8}$$

and it is straightforward to show that $P_C(x)$ satisfies the recurrence relation

$$P_C(x) = \frac{(1-x)b_-}{3 - b_+ P_{C_1}(x) - b_T P_{T_1}(x)}. \tag{9}$$

Note that $P_C(x)$ and $Q_C(x)$ depend upon ϵ_1 and ϵ_2; to avoid clutter we will normally suppress this dependence but if necessary it will appear as superscripts. It is important for what follows that Q_C is a convex function of P_C which is itself a convex function of $P_{T_1}, ...P_{T_k}, P_{C_k}$ for any $k > 0$.

The random walk is recurrent if $P_C(0) = 1$, in which case we define the exponents

β and $\tilde{\beta}$ by

$$1 - P_C(x) \sim x^{\beta} |\log x|^{-\tilde{\beta}}, \quad x \to 0 \tag{10}$$

and we expect logarithmic corrections ($\tilde{\beta} \neq 0$) only if β is an integer. Note that $u(x) \sim v(x)$ means that $u(x)/v(x) \to$ constant as $x \to 0$. Hence, for a recurrent walk $Q_C(x)$ diverges as $x \to 0$; furthermore if

$$Q_C(x) \sim x^{-\alpha} \tag{11}$$

then by a Tauberian theorem [17], $p_C(t)$ behaves asymptotically for large time as $t^{-\frac{1}{2}d_s}$ with the spectral dimension $d_s = 2-2\alpha$. It is well known that recurrence implies $d_s \leq 2$. When $d_s = 0, 2$ there may be logarithmic corrections and we define the exponent $\tilde{\alpha}$ by

$$Q_C(x) \sim x^{d_s/2-1} \, |\log x|^{\tilde{\alpha}} \tag{12}$$

and $p_C(t) \sim t^{-\frac{1}{2}d_s} |\log t|^{\tilde{\alpha}}$.

The random walk is non-recurrent, or transient, if $P_C(0) < 1$ (i.e. the walker has finite probability for never returning to the root); if then the first $k-1$ derivatives of $Q_C(x)$ are finite at $x = 0$ but

$$Q_C^{(k)}(x) \sim x^{-\alpha_k} \tag{13}$$

the large time behaviour of $p_C(t)$ is like $t^{-\frac{1}{2}d_s}$ with $d_s = 2(1 + k - \alpha_k)$; we again expect there may be logarithmic corrections when d_s is an even integer,

$$Q_C^{(k)}(x) \sim x^{d_s/2-1-k} \, |\log x|^{\tilde{\alpha}_k} \tag{14}$$

and $p_C(t) \sim t^{-\frac{1}{2}d_s} |\log t|^{\tilde{\alpha}_k}$.

For random combs we proceed analogously. The exponents β and $\tilde{\beta}$ for recurrent walks are defined by

$$\langle P_C(x) \rangle_\mu = 1 - B' x^{\beta} |\log x|^{\tilde{\beta}} (1 + o(x)) \tag{15}$$

and d_s through the first divergent derivative; namely if

$$\left\langle Q_C^{(k)}(x) \right\rangle_\mu = \frac{B}{x^{\alpha_k}} + o(x^{-\alpha_k}) \tag{16}$$

then $d_s = 2(1 + k - \alpha_k)$. Again, if d_s is an even integer the exponent $\tilde{\alpha}$ is defined as in (14). Note that for a single recurrent comb $\beta = \alpha$ but in an ensemble this is no longer necessarily the case; applying Jensen's inequality to (8) we see that $\beta \leq \alpha$.

2.3. Two-point functions

Let $p_C^1(t; n)$ denote the probability that the walker on C, having left r at $t = 0$ and not subsequently returned there, is at point n on the spine at time t. The corresponding generating function, which we will call the two-point function, is defined by

$$G_C(x; n) = \sum_{t=0}^{\infty} (1 - x)^{t/2} p_C^1(t; n). \tag{17}$$

Letting ω be a walk on C starting at r and ending at n without returning to r we have

$$G_C(x;n) = \sum_{\omega:r\to n} (1-x)^{\frac{1}{2}|\omega|} \prod_{t=0}^{|\omega|-1} \rho_{\omega(t)}. \tag{18}$$

Following the discussion in section 2.2 of [14] this leads us to the representation

$$G_C(x;n) = \frac{3}{b_+(1-x)^{n/2}} \prod_{k=0}^{n-1} \frac{b_+}{b_-} P_{C_k}(x). \tag{19}$$

2.4. The Heat kernel

Let $K_C(t;n,\ell)$ denote the probability that the walker on C, having left r at $t=0$, is at point ℓ in tooth T_n at time t. $K_C(t;n,\ell)$ satisfies the diffusion equation on C so we call it the heat kernel. The probability that the walker has travelled a distance n along the spine at time t is given by

$$K_C(t;n) = \sum_{\ell=0}^{\infty} K_C(t;n,\ell), \tag{20}$$

and has generating function

$$H_C(x;n) = \sum_{t=0}^{\infty} (1-x)^{t/2} K_C(t;n). \tag{21}$$

$H_C(x;n)$ can be written as

$$H_C(x;n) = \frac{G_C(x;n)}{1-P_C(x)} D_{|T_n|}(x), \tag{22}$$

where

$$D_\ell(x) = 1 + \frac{b_T}{3} \sum_{k=1}^{\ell} G_{N_\ell}(x;k), \tag{23}$$

and we define

$$H(x;n) = \langle H_C(x;n) \rangle_\mu . \tag{24}$$

Note that, because $K_C(t;n)$ is a probability,

$$\sum_{n=0}^{\infty} H(x;n) = \frac{1+\sqrt{1-x}}{x}. \tag{25}$$

The exponent d_k is defined through the moments in n

$$\sum_{n=0}^{\infty} n^k H(x;n) \sim x^{-1-d_k}, \quad x \to 0. \tag{26}$$

It follows that (see [17])

$$\left\langle \langle n^k \rangle_{\omega:|\omega|=t} \right\rangle_\mu \sim t^{d_k}, \quad t \to \infty, \tag{27}$$

so that $d_w = 2/d_2$. If $d_k = 0$ there may be logarithmic corrections and we define the exponent $\tilde{d}_k$ by

$$\sum_{n=0}^{\infty} n^k H(x;n) \sim x^{-1}|\log x|^{\tilde{d}_k}, \quad x \to 0, \tag{28}$$

in which case $\left\langle \langle n^k \rangle_{\omega:|\omega|=t} \right\rangle_\mu \sim |\log t|^{\tilde{d}_k}$.

3. Basic properties

3.1. Results for simple regular combs

The relation (9) can be used to compute the generating functions for a number of simple regular graphs which will be important in our subsequent analysis [14].

(i) An infinitely long tooth, N_∞:

$$P_\infty(x) = \begin{cases} 1 - x^{\frac{1}{2}} & \text{if } \epsilon_2 = 0; \\ \dfrac{1-2|\epsilon_2|}{b_T} - \dfrac{x}{4|\epsilon_2|}(1-2\epsilon_2) + O(x^2) & \text{otherwise.} \end{cases} \tag{29}$$

(ii) A tooth of length ℓ, N_ℓ:

$$P_\ell(x) = P_\infty(x) \frac{1 + XY^{1-\ell}}{1 + XY^{-\ell}} \tag{30}$$

where

$$X = \frac{b_T(1 - P_\infty(x))}{2 - b_T(1 + P_\infty(x))}, \quad Y = \frac{2 - b_T P_\infty(x)}{b_T P_\infty(x)}. \tag{31}$$

(iii) The comb $\sharp$ given by $\{T_k = N_1, \forall k\}$ has all teeth of length 1, and

$$P_\sharp(x) = \begin{cases} 1 - B_1 x^{\frac{1}{2}} + O(x) & \text{if } \epsilon_1 = 0; \\ \dfrac{1-\epsilon_2-|\epsilon_1|}{b_+} - x\dfrac{B_2}{|\epsilon_1|} + O(x^2) & \text{otherwise.} \end{cases} \tag{32}$$

Note that $\sharp$ is non-recurrent if $\epsilon_1 > 0$. It is also convenient to define $\ell\sharp$ to be $\{T_1 = N_\ell, C_1 = \sharp\}$.

(iv) The comb $*$ given by $\{T_k = N_\infty, \forall k\}$ has all teeth of length ∞ and is non-recurrent for $\epsilon_2 > 0$,

$$P_*(x) = \frac{1+\epsilon_2 - \sqrt{4\epsilon_2 + \epsilon_1^2}}{b_+} - x\frac{B_1}{\sqrt{4\epsilon_2 + \epsilon_1^2}} + O(x^2). \tag{33}$$

Otherwise

$$P_*(x) = \begin{cases} \dfrac{1-|\epsilon_1|}{1+\epsilon_1} - \dfrac{B_2}{|\epsilon_1|} x^{\frac{1}{2}} + O(x) & \text{if } \epsilon_2 = 0,\ \epsilon_1 \neq 0; \\ \dfrac{1-\epsilon_2-|\epsilon_1|}{b_+} - x\dfrac{B_3}{|\epsilon_1|} + O(x^2) & \text{if } \epsilon_2 < 0,\ \epsilon_1 \neq 0; \\ 1 - B_4 x^{\frac{1}{2}} + O(x) & \text{if } \epsilon_2 < 0,\ \epsilon_1 = 0. \end{cases} \tag{34}$$

(v) The comb $\flat\ell$ given by $\{T_k = N_\ell, \forall k\}$ has all teeth of length ℓ and

$$P_{\flat\ell}(x) = \begin{cases} \dfrac{1-|\epsilon_1|}{1+\epsilon_1} - \dfrac{B_1}{|\epsilon_1|}(\ell+1+|\epsilon_1|)x + O(x^2\ell^2) & \text{if } \epsilon_2 = 0,\ \epsilon_1 \neq 0; \\ \dfrac{1-\epsilon_2-|\epsilon_1|}{b_+} - x\dfrac{B_2}{|\epsilon_1\epsilon_2|} + O(xY^{-\ell}) & \text{if } \epsilon_2 < 0,\ \epsilon_1 \neq 0; \\ 1 - \dfrac{B_3}{|\epsilon_2|}x^{\frac{1}{2}} + O(x^{\frac{1}{2}}Y^{-\ell}) & \text{if } \epsilon_2 < 0,\ \epsilon_1 = 0; \end{cases} \tag{35}$$

where as $x \to 0$,

$$Y \to \frac{1+2|\epsilon_2|}{1-2|\epsilon_2|}. \tag{36}$$

When $\epsilon_2 > 0$ let $\bar{\ell} = \lfloor |\log x|/\log Y \rfloor$, where $\lfloor z \rfloor$ denotes the integer below z. For $\ell > 2\bar{\ell}$ the teeth are long enough that $P_{\flat\ell}(x)$ behaves like (33). For $\bar{\ell} < \ell \leq 2\bar{\ell}$, $P_{\flat\ell}(x)$ is non-recurrent with the leading power of x being fractional. For $\ell \leq \bar{\ell}$

$$P_{\flat\ell<\bar{\ell}}(x) = \begin{cases} \dfrac{1-\epsilon_2-|\epsilon_1|}{b_+} - x\dfrac{B_4 Y^\ell}{|\epsilon_1\epsilon_2|} + O(x) & \text{if } \epsilon_1 \neq 0; \\ 1 - \dfrac{B_5}{\sqrt{\epsilon_2}}x^{\frac{1}{2}}Y^{\frac{1}{2}\ell} + O(x^{\frac{1}{2}}Y^{-\frac{1}{2}\ell}, xY^{-\ell}) & \text{if } \epsilon_1 = 0, \end{cases} \tag{37}$$

where the notation $O(a,b)$ means $O(\max(a,b))$.

3.2. General properties of the generating functions

The generating functions for any comb satisfy three simple properties which can be derived from (9):

(i) *Monotonicity* The value of $P_C(x)$ decreases monotonically if the length of a tooth is increased.

(ii) *Rearrangement* If the comb C' is created from C by swapping the adjacent teeth T_k and T_{k+1} then $P_{C'}(x) > P_C(x)$ if $|T_{k+1}| < |T_k|$.

(iii) *Inheritance* If walks on C_k or T_k are non-recurrent for finite k then walks on C are non-recurrent.

The proof of the first two follows that given in [14] for the special case $\epsilon_2 = \epsilon_1 = 0$. The third can be shown by assuming that either $P_{C_1}(0) < 1$ or $P_{T_1}(0) < 1$; it then follows immediately from (9) that $P_C(0) < 1$ and the result follows by induction.

3.3. Useful elementary bounds

By monotonicity $G_C(x;n)$ is always bounded above by $G_\sharp(x;n)$ from which we get

$$G_C(x;n) < \frac{3}{b_+}\exp(-n\Lambda^{\epsilon_1,\epsilon_2}(x)), \tag{38}$$

where

$$\Lambda^{\epsilon_1,\epsilon_2}(x) = \begin{cases} x\,\dfrac{2+\epsilon_2}{2\epsilon_1} & \text{if } \epsilon_1 > 0, \\ x^{\frac{1}{2}}\sqrt{\dfrac{2+\epsilon_2}{1-\epsilon_2}} & \text{if } \epsilon_1 = 0, \\ \log\left(\dfrac{b_-}{b_+}\right) & \text{if } \epsilon_1 < 0. \end{cases} \tag{39}$$

Now let $P_C^{(N)}(x)$ denote the contribution to $P_C(x)$ from walks that reach beyond $n = N$ on the spine. It is straightforward to show using the arguments of section 2.5 of [14] that

$$P_C^{(N)}(x) \le \frac{1}{3} b_- G_C^{\epsilon_1,\epsilon_2}(x;N) G_C^{-\epsilon_1,\epsilon_2}(x;N). \tag{40}$$

Combining this with (38) we obtain the useful bound

$$P_C^{(N)}(x) \le \frac{3b_-}{b_+^2}\exp(-N(\Lambda^{\epsilon_1,\epsilon_2}(x) + \Lambda^{-\epsilon_1,\epsilon_2}(x))). \tag{41}$$

Now consider the ensemble μ' of combs C for which: $T_k = N_1, k = 1..K-1$, $T_K = N_\ell$; at $k > K$ teeth are short, $T_k = N_1$, with probability $1-p$ or long, $T_k = N_\ell$, with probability p; and the nth tooth is short, $T_n = N_1$. Then using the representation (19) $G_C(x;n)$ can be bounded above by noting that if $T_{k+1} = N_\ell$ then $P_{C_k} < P_{\ell\sharp}$, otherwise $P_{C_k} < P_\sharp$. This gives

$$G_C(x,n) \le \frac{3}{b_+}(1-x)^{-n/2}\left(\frac{b_+}{b_-}\right)^n P_{\ell\sharp}(x)^{n-K-k} P_\sharp(x)^{k+K}, \tag{42}$$

and hence

$$\begin{aligned} \langle G_C(x,n)\rangle_{\mu'} &= \sum_{k=0}^{n-K-1}\binom{n-K-1}{k} p^{n-K-1-k}(1-p)^k G_C(x,n) \\ &\le \frac{3}{b_+}\left(\frac{b_+}{b_-}\right)^n \frac{P_\sharp(x)^K P_{\ell\sharp}(x)}{(1-x)^{n/2}}\left((1-p)P_\sharp(x) + pP_{\ell\sharp}(x)\right)^{n-K-1}. \end{aligned} \tag{43}$$

4. Results independent of the comb ensemble μ

In this section we show that in some regions of $\epsilon_{1,2}$ the behaviour at large time is essentially independent of the comb ensemble, or else simply dependent upon $\langle P_T(x)\rangle_\mu$. The leading and, where different, the leading non-analytic, behaviour of $\langle P_T(x)\rangle_\mu$ as $x \to 0$ for the measures studied here is given in table 1. The results for μ^A are trivial, as are those for any measure when $\epsilon_2 < 0$, while the case μ^B and $\epsilon_2 = 0$ can be derived using the techniques in [14]. The calculation for μ^B and $\epsilon_2 > 0$ is somewhat subtle and is included in Appendix A.

Table 1. Leading, and leading non-analytic, behaviour of $1 - \langle P_T \rangle_\mu$ in various cases.

ensemble	$\epsilon_2 < 0$	$\epsilon_2 = 0$	$\epsilon_2 > 0$
μ^A	Bx	$Bx^{\frac{1}{2}}$	$B + B'x$
$\mu^B, a < 2$	Bx	$Bx^{a/2}$	$B(\lvert\log x\rvert^{a-1})^{-1}$
$\mu^B, a = 2k$	Bx	$Bx + \ldots B'x^k \lvert\log x\rvert$	$B(\lvert\log x\rvert^{a-1})^{-1}$
$\mu^B, a > 2, a \neq 2k$	Bx	$Bx + \ldots B'x^{a/2}$	$B(\lvert\log x\rvert^{a-1})^{-1}$

4.1. d_s when $\epsilon_2 < 0$

First we show that for any comb ensemble

$$d_s = \begin{cases} 0 & \text{if } \epsilon_1 < 0 \text{ and } \epsilon_2 < 0; \\ 1 & \text{if } \epsilon_1 = 0 \text{ and } \epsilon_2 < 0. \end{cases} \tag{44}$$

By monotonicity we have that for any comb C

$$P_*(x) \le P_C(x) \le P_\sharp(x). \tag{45}$$

Taking expectation values and using (32) and (34) it follows that for $\epsilon_2 < 0$

$$P(x) = \langle P_C(x) \rangle_\mu = \begin{cases} 1 - B_1 x^{\frac{1}{2}} + O(x) & \text{if } \epsilon_1 = 0, \\ \dfrac{1 - \epsilon_2 - |\epsilon_1|}{b_+} - x\dfrac{B_2}{|\epsilon_1|} + O(x^2) & \text{otherwise.} \end{cases} \tag{46}$$

Similarly

$$Q_*(x) \le Q_C(x) \le Q_\sharp(x) \tag{47}$$

and so

$$Q(x) = \langle Q_C(x) \rangle_\mu = \begin{cases} \dfrac{B_1}{x^{\frac{1}{2}}} + O(1) & \text{if } \epsilon_1 = 0, \\ \dfrac{B_2|\epsilon_1|}{x} + O(1) & \text{if } \epsilon_1 < 0, \end{cases} \tag{48}$$

and (44) follows.

4.2. d_s when $\epsilon_1 > 0$

When $\epsilon_1 > 0$ all combs are non-recurrent and so we must examine the derivatives of $Q(x)$. Differentiating (8) and (9) gives

$$Q_C^{(1)}(x) = Q_C(x)^2 P_C^{(1)}(x), \tag{49}$$

$$P_C^{(1)}(x) = \frac{-P_C(x)}{1-x} + \frac{P_C(x)^2}{(1-x)b_-} \times \left(b_T P_{T_1}^{(1)}(x) + b_+ P_{C_1}^{(1)}(x) \right). \tag{50}$$

By monotonicity (50) can be bounded above and below by replacing P_C with P_* and $P_\sharp$ respectively. Taking the expectation value and using translation invariance to note that $\langle P_C\rangle_\mu = \langle P_{C_1}\rangle_\mu$ shows that, if $\left\langle P_T^{(1)}(x)\right\rangle_\mu$ diverges as $x\to 0$, then

$$Q^{(1)}(x) \sim B\left\langle P_T^{(1)}(x)\right\rangle_\mu + O(\sqrt{x}\left\langle P_T^{(1)}(x)\right\rangle_\mu, 1). \tag{51}$$

As can be seen from Table 1 in some cases $\langle P_T(x)\rangle_\mu$ is analytic, or only higher derivatives diverge. For the measures considered here it can be shown that if $\langle P_T(x)\rangle_\mu$ is analytic at $x=0$ then so is $Q(x)$. If on the other hand $\langle P_T(x)\rangle_\mu$ is not analytic but the k'th derivative diverges then

$$Q^{(k)}(x) = B\left\langle P_T^{(k)}(x)\right\rangle_\mu + +O(\sqrt{x}\left\langle P_T^{(k)}(x)\right\rangle_\mu, 1). \tag{52}$$

The proof is a straightforward but tedious generalization of (49) and (50) and is relegated to Appendix B. If a derivative of $Q(x)$ diverges then d_s can be read off using (16) and (52). Otherwise if all finite order derivatives are finite then $p_C(t)$ decays at large t faster than any power and d_s is not defined.

4.3. d_k when $\epsilon_2<0$ or $\epsilon_1<0$

We show that for any comb ensemble

$$\tilde{d}_k = 0, \quad d_k = \begin{cases} 0 & \text{if } \epsilon_1<0,\\ k/2 & \text{if } \epsilon_1 = 0 \text{ and } \epsilon_2<0,\\ k & \text{if } \epsilon_1>0 \text{ and } \epsilon_2<0. \end{cases} \tag{53}$$

It is trivial to show that

$$1 \le D_\ell \le \frac{B}{|\epsilon_2|}, \quad \epsilon_2<0, \tag{54}$$

and then by monotonicity we get

$$\frac{G_*(x;n)}{1-P_*(x)} \le H(x;n) \le \frac{B}{|\epsilon_2|}\frac{G_\sharp(x;n)}{1-P_\sharp(x)}. \tag{55}$$

Combining this with (32) and (34) yields the results for $\epsilon_2<0$.

To deal with $\epsilon_1<0$ and $\epsilon_2\ge 0$ note that monotonicity gives

$$\left\langle \frac{D_{|T_n|}(x)}{1-P_C(x)}\right\rangle_\mu G_*(x;n) \le H(x;n) \le \left\langle \frac{D_{|T_n|}(x)}{1-P_C(x)}\right\rangle_\mu G_\sharp(x;n). \tag{56}$$

Using the lower bound and (19), (25) and (33) we get after summing over n

$$\left\langle \frac{D_{|T_n|}(x)}{1-P_C(x)}\right\rangle_\mu \frac{3}{b_+}\frac{b_-}{\sqrt{4\epsilon_2+\epsilon_1^2}-\epsilon_1-2\epsilon_2} \le \sum_{n=0}^{\infty} H(x;n) \le \frac{2}{x}. \tag{57}$$

Inserting this into the upper bound of (56) gives

$$H(x;n) \le \frac{2}{x}\frac{b_+}{3}\frac{\sqrt{4\epsilon_2+\epsilon_1^2}-\epsilon_1-2\epsilon_2}{b_-}G_\sharp(x;n). \tag{58}$$

It is a trivial consequence of (25) that

$$\sum_{n=0}^{\infty} n^k H(x;n) > \frac{c}{x}, \qquad k > 0, \tag{59}$$

and the results then follow by using (38).

5. The spectral dimension when $\epsilon_2 \geq 0$ and $\epsilon_1 \leq 0$

Here and in some of the sections to follow we will need to sum over the location of the first long tooth to determine the spectral dimension. Most generally we call a tooth long when it has length $\geq \ell$ and short when it has length $< \ell$. Consider combs for which the first $L-1$ teeth are short but the Lth tooth is long; the probability for this is $p(1-p)^{L-1}$, where p is the probability of a tooth being long. Denoting by ℓL a comb having the first long tooth at vertex L gives

$$Q(x) = \sum_{L=1}^{\infty} \langle Q_{\ell L}(x)\rangle_\mu \, p(1-p)^{L-1}. \tag{60}$$

$Q_{\ell L}(x)$ is bounded above by the comb in which all teeth at $n \geq L+1$ are short, and below by the comb in which all teeth at $n \geq L+1$ are infinite,

$$Q_{\{T_{n<L}=N_{\ell'},\ell'\leq\ell;T_{n\geq L}=N_\infty\}}(x) < Q_{\ell L}(x) < Q_{\{T_{n\neq L}=N_1;T_L=N_\ell\}}(x). \tag{61}$$

5.1. μ^A *– Infinite teeth at random locations*

5.1.1. $\epsilon_2 = 0$, $\epsilon_1 < 0$ We first show that the exponent $\beta = \frac{1}{2}$ – so it is unchanged from the comb $*$. This result follows from the inequalities

$$1 - \frac{pBx^{\frac{1}{2}}}{|\epsilon_1|} + O(x) \leq P(x) \leq 1 - pB'x^{\frac{1}{2}} + O(x). \tag{62}$$

The lower bound is obtained by applying Jensen's inequality to (9). To get the upper bound we average over the first tooth and then by monotonicity we obtain

$$P(x) \leq pP_{\ell\sharp}(x) + (1-p)P_\sharp(x), \tag{63}$$

with $\ell = \infty$ and using (9) and (32) gives the bound required.

The spectral dimension is given by

$$d_s = \begin{cases} 1 & \text{if } p \geq 2|\epsilon_1|(1+|\epsilon_1|)^{-1}, \\ \dfrac{\log(1-p)}{\log\left(\frac{1-|\epsilon_1|}{1+|\epsilon_1|}\right)} & \text{otherwise.} \end{cases} \tag{64}$$

This result follows from estimating the sum in (60) using the bounds in (61) with $\ell = \infty$ and short teeth being N_1. $P_C(x)$ for these bounding combs is computed in Appendix C and using (C.3) we get upper and lower bounds on $Q_{\infty L}(x)$ of the form

$$\frac{1}{Bx + B'x^{\frac{1}{2}}\left(\frac{1-|\epsilon_1|}{1+|\epsilon_1|}\right)^L}. \tag{65}$$

5.1.2. $\epsilon_2 > 0$, $\epsilon_1 < 0$ The probability that C is non-recurrent is at least p, the probability that $T_1 = N_\infty$, and hence

$$P(0) < 1. \tag{66}$$

In fact it follows from the Lemma of Appendix B that $P^{(k)}(x)$ is finite for all finite k so the exponent β is undefined.

The spectral dimension is given by

$$d_s = \frac{2\log(1-p)}{\log\left(\frac{1-|\epsilon_1|-\epsilon_2}{1+|\epsilon_1|-\epsilon_2}\right)}. \tag{67}$$

To show this we start by estimating $Q(x)$ in exactly the same way as in 5.1.1 except that the behaviour of the limiting combs is now given by (C.5) so that there are upper and lower bounds on $Q_{\infty L}(x)$ of the form

$$\frac{1}{Bx + B'\left(\frac{1-|\epsilon_1|-\epsilon_2}{1+|\epsilon_1|-\epsilon_2}\right)^L}. \tag{68}$$

When $p \le 1 - b_+/b_-$ this sum diverges at $x = 0$ and it is then straightforward to obtain (67). For larger p the sum is convergent at $x = 0$ so we next examine $Q^{(1)}(x) = \left\langle Q_C^2 P_C^{(1)}\right\rangle_\mu$. Note that $-P_C^{(1)} \ge \frac{1}{3}b_-$; then letting Z be a very large integer and using Hölder's inequality

$$b_- \left\langle Q_C(x)^2\right\rangle_\mu \le -Q^{(1)}(x) \le \left\langle Q_C(x)^{2+1/Z}\right\rangle_\mu^{\frac{2Z}{2Z+1}} \left\langle -P_C^{(1)}(x)^{2Z+1}\right\rangle_\mu^{\frac{1}{2Z+1}}. \tag{69}$$

By the lemma of Appendix B the second factor in the upper bound is finite as $x \to 0$ so we need an estimate of $\langle Q_C^2\rangle_\mu$. This is provided by (68) modified by squaring the denominator; when $p \le 1 - (b_+/b_-)^2$ this sum diverges at $x = 0$ and once again we obtain (67). For still larger p both Q and $Q^{(1)}$ are finite at $x = 0$ and we examine the second and higher derivatives. This uses (B.4), $(-1)^k P_C^{(k)} \ge b_-^k b_+^{k-1}/3^{2k-1}$, Hölder's inequality and the lemma; the term with the highest power of Q_C dominates and the result is always (67). ‡

5.1.3. $\epsilon_2 > 0$, $\epsilon_1 = 0$ By the same argument as in 5.1.2 we find $P(0) < 1$, so β is again undefined. An upper bound on $Q(x)$ may be obtained as in 5.1.1 using (C.9) to get

$$Q_{\infty L}(x) \le (L + (1-\epsilon_2)/4\epsilon_2) \tag{70}$$

which means the upper bound of (60) is finite. A proof that all derivatives of $Q(x)$ are finite is given in Appendix B.2, so $p_C(t)$ decays faster than any power at large t.

‡ Strictly speaking when $1-(b_+/b_-)^k < p \le 1-(b_+/b_-)^{k+1/Z}$ the upper bounds diverge so our proof does not work for these arbitrarily small intervals.

5.2. μ^B – Teeth of random length

In this subsection we are concerned with random combs that have a distribution of tooth lengths. The general strategy for determining quantities of interest is to identify teeth that are long enough to affect the critical behaviour of the biased random walk and consider the probability with which they occur. It will be useful to define the function

$$\lambda(\delta,\eta,\zeta) = \lfloor \frac{\delta|\log x|^{\eta} - \zeta(a-1)\log|\log x|}{\log Y} \rfloor, \tag{71}$$

which will be used to denote a tooth length, and the function

$$p_{>}(\ell) = \sum_{k=\ell}^{\infty} \frac{C_a}{k^a} = \frac{C_a}{(a-1)\ell^{a-1}}\left(1 + O(\ell^{-1})\right), \tag{72}$$

which is the probability that a tooth has length greater than $\ell - 1$.

5.2.1. $\epsilon_2 = 0$, $\epsilon_1 < 0$

We first show that

$$\beta = \begin{cases} \dfrac{a}{2} & \text{if } a < 2, \\ 1 & \text{otherwise.} \end{cases} \tag{73}$$

The proof follows the lines described in section 5.1.1 with a slight modification for the upper bound on $P(x)$. Note that, from (30), teeth of length $\ell > \lfloor x^{-\frac{1}{2}} \rfloor$ have $P_T(x) \le 1 - Bx^{\frac{1}{2}}$. We then proceed as in (63) but with $\ell = \lfloor x^{-\frac{1}{2}} \rfloor + 1$.

The exponent β is non-trivial if $a < 2$ but, as we now show, $d_s = 0$ for all $a > 1$ so mean field theory does not apply when $a < 2$. This result follows from the inequalities

$$\frac{B'}{x|\log x|^{\frac{1}{a-1}}} \le Q(x) \le \frac{B}{x}. \tag{74}$$

The upper bound is a consequence of $Q(x) < Q_\sharp(x)$. To obtain the lower bound consider the combs for which at least the first N teeth are all shorter than ℓ_0. Then using monotonicity and (41)

$$Q(x) \ge \frac{(1 - p_{>}(\ell_0))^N}{1 - P_{\flat\ell_0}(x) + O(\exp(-N(\Lambda^{\epsilon_1,\epsilon_2}(x) + \Lambda^{-\epsilon_1,\epsilon_2}(x))))}. \tag{75}$$

Setting $\ell_0 = \lambda(1,(a-1)^{-1},0)$, $N = \lfloor 2(\Lambda^{\epsilon_1,\epsilon_2} + \Lambda^{-\epsilon_1,\epsilon_2})^{-1}|\log x| \rfloor + 1$, and using (35) the result follows for small enough x.

5.2.2. $\epsilon_2 > 0$, $\epsilon_1 < 0$

The exponent $\beta = 0$ but there are computable logarithmic corrections and we find that

$$1 - \frac{B}{|\log x|^{a-1}} \le P(x) \le 1 - \frac{B'}{|\log x|^{a-1}}. \tag{76}$$

The lower bound follows from applying Jensen's inequality to (9). For the upper bound note that teeth of length $\ell > \lambda(1,1,0)$ have $P_T < B$. Again proceed as in (63) with $\ell = \lambda(1,1,0) + 1$.

The spectral dimension is $d_s = 0$ showing again that mean field theory does not apply. For $a \geq 2$ this follows from the inequalities

$$\frac{B'}{x^{1-\delta}} \leq Q \leq \frac{B}{x}, \tag{77}$$

for small enough x, where δ is small and positive and B' depends on δ. The upper bound is a consequence of $Q(x) < Q_\sharp(x)$ and the lower bound follows from (75) by setting $\ell_0 = \lambda(\delta, 1, 0)$, $N = \lfloor 2(\Lambda^{\epsilon_1,\epsilon_2} + \Lambda^{-\epsilon_1,\epsilon_2})^{-1} |\log x|^{a-1} \rfloor + 1$ and using (37). For $a < 2$ setting $N = \lfloor 2(\Lambda^{\epsilon_1,\epsilon_2} + \Lambda^{-\epsilon_1,\epsilon_2})^{-1} |\log x| \rfloor + 1$ gives the slightly modified lower bound

$$Q \geq \frac{B' \exp\left(-B'' |\log x|^{2-a}\right)}{x^{1-\delta}}. \tag{78}$$

5.2.3. $\epsilon_2 > 0$*,* $\epsilon_1 = 0$ The exponent $\beta = 0$, but there are logarithmic corrections which follow from the inequalities

$$1 - \frac{B}{|\log x|^{(a-1)/2}} \leq P(x) \leq 1 - \frac{B'}{|\log x|^{(a-1)/2}}. \tag{79}$$

The lower bound comes from applying Jensen's inequality to the recurrence relation (9). The upper bound is obtained by requiring unitarity of the heat kernel and its proof is relegated to Appendix D.

The spectral dimension and logarithmic exponent are given by

$$\begin{aligned} &d_s = 2, \\ &\tilde{\alpha} = a - 1, \end{aligned} \tag{80}$$

which shows that mean field theory does not apply. This result follows from

$$B' \,|\log x|^{a-1} < Q(x) < B \,|\log x|^{a-1} \tag{81}$$

for small enough x which is obtained by a modified version of the argument in 5.1.1. First let $\ell_0 = \lambda(1, 1, \zeta)$, so that

$$P_{\ell_0}(x) = 1 - \frac{B}{|\log x|^{\zeta(a-1)}} + O\left(\frac{1}{|\log x|^{2\zeta(a-1)}}\right). \tag{82}$$

To obtain (81) we use (60) and (61) with $p = p_>(\ell_0)$, $\ell = \ell_0$ and for the lower bound set $T_{n<L} = N_{\ell_0}$. Then using the bounds in (C.8) with $\zeta = 1$ and (C.7) with $\zeta = 2$ and estimating the sums gives the result.

6. Heat Kernel when $\epsilon_1 \geq 0$, $\epsilon_2 \geq 0$

These calculations require $\langle D_\ell \rangle_\mu$ in the various cases which are tabulated in table 2 for convenience.

Table 2. $\langle D_\ell \rangle_\mu$ in various cases.

ensemble	$\epsilon_2 < 0$	$\epsilon_2 = 0$	$\epsilon_2 > 0$
μ^A	$B + O(x)$	$Bx^{-\frac{1}{2}} + O(1)$	$Bx^{-1} + O(1)$
$\mu^B, a \geq 2$	$B + O(x)$	$B + O(x)$	$B(x\lvert\log x\rvert^{a-1})^{-1} + O(1)$
$\mu^B, a < 2$	$B + O(x)$	$Bx^{a/2-1} + O(1)$	$B(x\lvert\log x\rvert^{a-1})^{-1} + O(1)$

6.1. μ^A – *Infinite teeth at random locations*

We show that

$$d_k = \begin{cases} 0 & \text{if } \epsilon_2 > 0 \text{ and } \epsilon_1 \geq 0, \\ k/2 & \text{if } \epsilon_2 = 0 \text{ and } \epsilon_1 > 0. \end{cases} \tag{83}$$

These results follow from (86), (87) and (88) below.

Noting that for $\epsilon_1 > 0$ all combs have $1 - B_-^{-1} - < P_C(x) < 1 - B_+^{-1}$ and using monotonicity gives

$$B_- \left\langle D_{|T_n|}(x) \right\rangle_\mu G_*(x;n) \leq H(x;n) \leq \left\langle D_{|T_n|}(x) \right\rangle_\mu \left\langle \frac{G_{C'}(x;n)}{1 - P_{C'}(x)} \right\rangle_\mu , \tag{84}$$

$$\leq B_+ \left\langle D_{|T_n|}(x) \right\rangle_\mu \left\langle G_{C'}(x;n) \right\rangle_\mu , \tag{85}$$

where C' is constructed from C by forcing $T_n = N_1$. If $\epsilon_2 > 0$ then using (43) with $K = 0, \ell = \infty$ gives the upper bound

$$H(x;n) < \frac{B}{x} \exp(-B'n). \tag{86}$$

If $\epsilon_2 = 0$ then exactly the same calculation gives

$$H(x;n) < \frac{B}{x^{\frac{1}{2}}} \exp(-B' x^{\frac{1}{2}} n) \tag{87}$$

and evaluating the left hand side of (84) gives a lower bound of the same form. If $\epsilon_2 > 0$ and $\epsilon_1 = 0$ it is necessary to sum over the location of the first infinite tooth. Using (C.9), (43) and introducing C' as in (84) gives

$$H(x;n) < \frac{B}{x} \exp(-B'n). \tag{88}$$

6.2. μ^B – *Teeth of random length*

We show that

$$d_k = \begin{cases} 0 & \text{if } \epsilon_2 > 0 \text{ and } \epsilon_1 \geq 0; \\ ka/2 & \text{if } \epsilon_2 = 0,\ \epsilon_1 > 0 \text{ and } a < 2; \\ k & \text{if } \epsilon_2 = 0,\ \epsilon_1 > 0 \text{ and } a \geq 2. \end{cases} \tag{89}$$

These results follow from

$$H(x;n) < \begin{cases} \dfrac{B}{x|\log x|^{a-1}}\exp(-B'n/|\log x|^{a-1}) & \text{if } \epsilon_2 > 0 \text{ and } \epsilon_1 \geq 0; \\ \dfrac{B}{x^{1-a/2}}\exp(-B'nx^{a/2}) & \text{if } \epsilon_2 = 0,\ \epsilon_1 > 0 \text{ and } a < 2; \\ B\exp(-nB'x) & \text{if } \epsilon_2 = 0,\ \epsilon_1 > 0 \text{ and } a \geq 2, \end{cases} \tag{90}$$

when x is small enough and lower bounds of the same form.

The upper bounds are obtained by proceeding as in subsection 6.1; for $\epsilon_1 > 0$ and $\epsilon_2 > 0$ setting $\ell = \lambda(1,1,0)+1$ and for $\epsilon_1 > 0$ and $\epsilon_2 = 0$ setting $\ell = \lfloor x^{-\frac{1}{2}} \rfloor + 1$. For $\epsilon_1 = 0$ and $\epsilon_2 > 0$ we start with the upper bound of (84); let $\ell_1 = \lambda(1,1,2)$, $p_1 = p_>(\ell_1)$ and $\ell_2 = \lambda(2,1,0)$, $p_2 = p_>(\ell_2)$. The latter shall be called long teeth and we denote by $(\ell_2 K\sharp)$ the comb with a single long tooth at vertex K. We now sum over the location of the first long tooth using (19), (43) and (C.8) and taking account of the fact that the first long tooth may be before or after the nth tooth

$$\begin{aligned} H(x;n) \leq\ & \left\langle D_{|T_n|}(x)\right\rangle_\mu \sum_{K=1}^{n-1} \frac{p_2(1-p_1)^{K-1}}{1-P_{(\ell_2 K\sharp)}(0)} \prod_{m=0}^{K-1} P_{(\ell_2 K\sharp)m} \\ & \times \left((1-p_1)P_\sharp + (p_1-p_2)P_{\ell_1\sharp} + p_2 P_{\ell_2\sharp}\right)^{n-K-1} \\ & + \left\langle \theta(\ell_2 - |T_n|) D_{|T_n|}(x)\right\rangle_\mu \sum_{K=n}^{\infty} \frac{p_2(1-p_1)^{K-1}}{1-P_{(\ell_2 K\sharp)}(0)} \prod_{m=0}^{n-1} P_{(\ell_2 K\sharp)m}. \end{aligned} \tag{91}$$

In the first sum we use the value given in Table 2 for $\left\langle D_{|T_n|}(x)\right\rangle_\mu$. In the second sum $\left\langle \theta(\ell_2 - |T_n|) D_{|T_n|}(x)\right\rangle_\mu = B(x|\log x|^{2(a-1)})^{-1} + O(1)$ for $|T_n| < \ell_2$ and the result follows.

To obtain the lower bounds when $\epsilon_1 > 0$ we note that

$$\begin{aligned} H(x;n) &\geq B_- \left\langle D_{|T_n|}(x) G_C(x;n)\right\rangle_\mu \\ &= B_- \left\langle D_{|T_n|}(x)\right\rangle_\mu \left\langle G_C(x;n)\right\rangle_{\overline{\mu}} \end{aligned} \tag{92}$$

where the measure $\overline{\mu}$ is defined by

$$\begin{aligned} \overline{\mu}_\ell &= \mu_\ell, && \text{for teeth } T_k,\ k \neq n, \\ \overline{\mu}_\ell &= \frac{\mu_\ell D_\ell}{\langle D_\ell\rangle_\mu}, && \text{for tooth } T_n. \end{aligned} \tag{93}$$

Using the decomposition (19) and Jensen's inequality

$$\langle G_C(x;n)\rangle_{\overline{\mu}} \geq \frac{3(1-x)^{n/2}}{b_+}\exp(-S_n), \tag{94}$$

where

$$S_n = \sum_{k=0}^{n-1} \left\langle \frac{b_-}{b_+} - P_{C_{k+1}}(x) \right\rangle_{\overline{\mu}} + \frac{b_T}{b_+}\left\langle 1 - P_{T_{k+1}}(x)\right\rangle_{\overline{\mu}}. \tag{95}$$

Now applying Jensen's inequality with the measure $\overline{\mu}$ to (9) shows that the lower bounds satisfy a recursion formula of exactly the same form as discussed in Appendix C. So from (C.1) we find that

$$\begin{aligned} S_n \le n &\left(\frac{b_-}{b_+} - P(x) + \frac{b_T}{b_+}\left\langle 1 - P_T(x)\right\rangle_\mu\right) \\ &- \frac{b_T}{b_+}(\langle P_T(x)\rangle_{\overline{\mu}} - \langle P_T(x)\rangle_\mu) + \\ &- \sum_{k=1}^{n-1} \frac{P(x)(1-A(x))}{A(x)^{k-1}(\bar{P}(x) - A(x)P(x))/(\bar{P}(x) - P(x)) - 1}, \end{aligned} \tag{96}$$

where

$$\begin{aligned} P(x) &= \frac{(1-x)b_-}{3 - b_T\left\langle P_T(x)\right\rangle_\mu - b_+ P(x)} \;, \\ \bar{P}(x) &= \frac{(1-x)b_-}{3 - b_T\left\langle P_T(x)\right\rangle_{\overline{\mu}} - b_+ P(x)} \;, \\ A(x) &= \frac{(1-x)b_-}{P(x)^2\, b_+}. \end{aligned} \tag{97}$$

For $\epsilon_1 > 0$ it is straightforward to check that $A(x) > c > 1$ and that the sum in (96) is bounded above by an n independent constant. Lower bounds of the form of (90) then follow by inserting the appropriate $\left\langle P_T\right\rangle_\mu$ in (97) and (96).

When $\epsilon_1 = 0$

$$H(x;n) \ge \left\langle D_{|T_n|}(x)\right\rangle_\mu \left\langle \frac{G_{C'}(x;n)}{1 - P_{C'}(x)}\right\rangle_\mu , \tag{98}$$

where C' is constructed from C by setting $T_{k\ge n} = N_\infty$. Choosing $\ell_0 = \lambda(1,1,2)$ and using (19) and (C.7) gives

$$\begin{aligned} H(x;n) &\ge \left\langle D_{|T_n|}(x)\right\rangle_\mu (1 - p_>(\ell_0))^{n-1} 3(1-x)^{-n/2} P_*(x)^2 \frac{\prod_{k=3}^{n-1}\left(P_{\flat\ell_0}(x) - \frac{1}{k-1}\right)}{1 - P_{\flat\ell_0}(x) + \frac{1}{n-1}} \\ &> \left\langle D_{|T_n|}(x)\right\rangle_\mu (1 - p_>(\ell_0))^{n-1} 3(1-x)^{-n/2} P_*(x)^2 \\ &\quad \times \frac{1}{n-2}\cdot\frac{1}{1 - P_{\flat\ell_0}(x) + \frac{1}{n-1}} \exp\left\{\frac{(n-3)2(1 - P_{\flat\ell_0}(x))}{2P_{\flat\ell_0}(x) - 1}\right\}, \end{aligned} \tag{99}$$

for $n \ge 4$ which gives the result.

7. Results and discussion

Figure 3 outlines the results that we have computed for μ^A. These are new and show that the most interesting regime is actually when the bias along the spine is *towards* the origin, a circumstance which has not been studied much in the literature. When $\epsilon_1 \ge 0$ and $\epsilon_2 > 0$ the walker disappears rapidly, never to return, and $p(t)$ decays faster than any power. When $\epsilon_1 < 0$ the bias along the spine is keeping the walker close to the origin *but* if there are any infinite teeth present the walker can spend a lot of time in the teeth; the conflict between these effects leads to a non-trivial d_s. The fact that $d_1 = 0$ whenever $\epsilon_2 > 0$ shows that the walker never gets far down the spine; if she disappears then it is up a tooth that she is lost. The Hausdorff dimension for μ^A is $d_H = 2$, regardless of bias and so we have here several examples of violation of the bound $2d_H/(1+d_H) \leq d_s \leq d_H$, which applies for unbiased diffusion [18].

Figure 4 shows our results for μ^B as well as the results for the unbiased case studied in [14]. This length distribution has been studied quite extensively in the literature but usually under the assumption that $\epsilon_2 \geq 0$. As can be seen the interesting behaviour displayed by μ^A when $\epsilon_1 < 0$ does not occur here – essentially because very long teeth are not common enough. We believe that with more work the bounds on $\tilde{\alpha}$ when $\epsilon_1 < 0$, $\epsilon_2 \geq 0$ can be made precise using our methods, but as this will not give further physical insight we leave the calculations to elsewhere [16]. The case $\epsilon_1 > 0$, $\epsilon_2 > 0$ (often called topological bias) was originally studied using mean field theory, which gave the mean square displacement

$$\langle n^2(t)\rangle \sim (\log t)^{2(a-1)}, \tag{100}$$

and this is in fact correct since the walker spends much of the time in the teeth. However the claim in [3] that (100) holds for $\epsilon_2 > 0$ regardless of ϵ_1 is false. The mean field method gives the correct result when $\epsilon_1 = 0$ only because the walk on the spine is ignored, which amounts to using $P_T(x)$ for $P_C(x)$ in (22) and naively applying Jensen's inequality. The case $\epsilon_1 > 0$, $\epsilon_2 = 0$ was studied by Pottier [13] who computed the leading contribution exactly, but without complete control over the sub-leading terms; she also calculated the leading behaviour $\langle n^2\rangle - \langle n\rangle^2$ which we have not. Of course our results for d_s and d_1 agree with hers. The Hausdorff dimension for μ^B is $d_H = 3 - a$ when $a < 2$ and $d_H = 1$ when $a \geq 2$ and so again we see that, as expected, a biasing field intensifies the difference between the purely geometric definition of dimension and that which is related to particle propagation.

The results for $\epsilon_2 < 0$ are intuitively obvious and, as we have proved, apply for any model with identically and independently distributed tooth lengths. The walker never gets far into the tooth and therefore combs have long time behaviour characteristic of the spine alone.

This paper has given a comprehensive treatment of biased random walks on combs using rigorous techniques – namely recursion relations for generating functions combined with unitarity and monotonicity arguments. It serves to put in context many previous results as well as present new ones. In the unbiased case [14] and in some bias regimes mean field theory is sufficient to compute the leading order behaviour because the walker either does not reach the ends of the longest teeth or does not travel far enough down the spine for variations from average to be important. But, as is illustrated in many examples here, a full treatment is needed when such fluctuations cannot be ignored. Finally, while the results are of interest in themselves, an important point of the paper was to demonstrate that rigorous analytic methods can be used to treat biased diffusion on random geometric structures and it is to be hoped that these tools can be extended to higher dimensional problems.

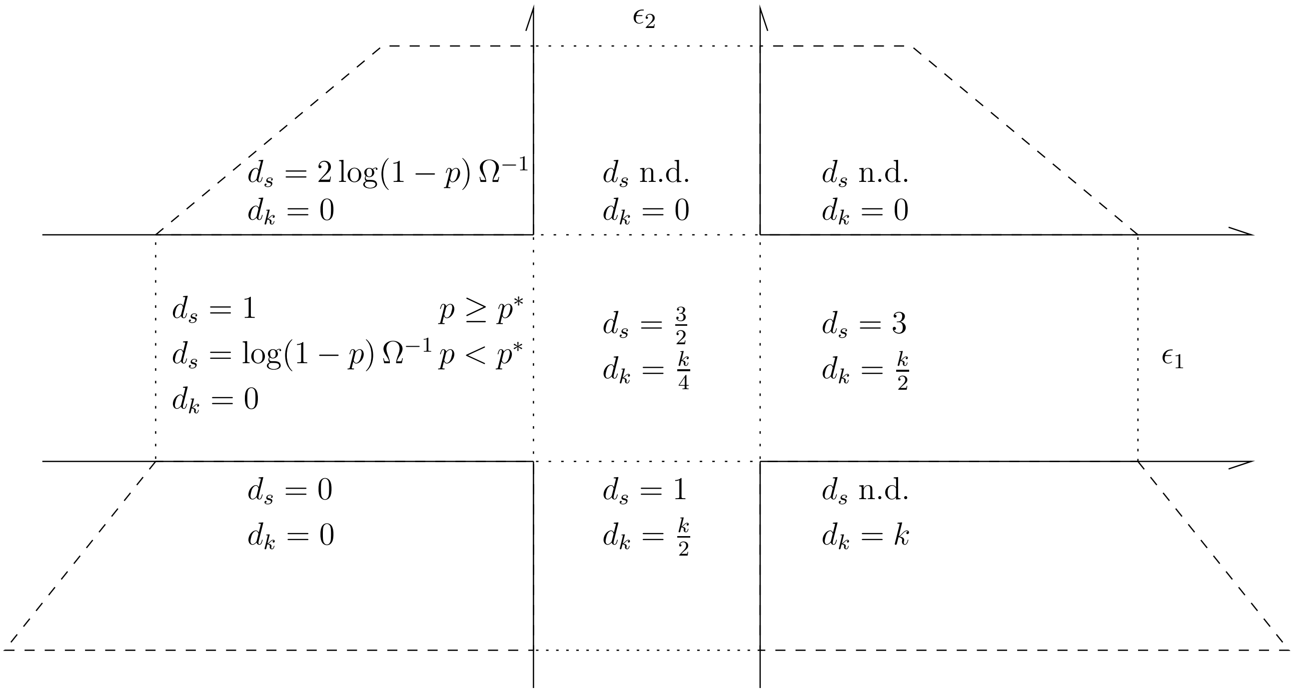

Figure 3. Results for μ^A where $\Omega = \log\left(\frac{1-|\epsilon_1|-\epsilon_2}{1+|\epsilon_1|-\epsilon_2}\right)$ and $p^* = 2|\epsilon_1|(1+|\epsilon_1|-\epsilon_2)^{-1}$. The logarithmic exponents $\tilde{\alpha}$ and $\tilde{d_k}$ are always zero for μ^A.

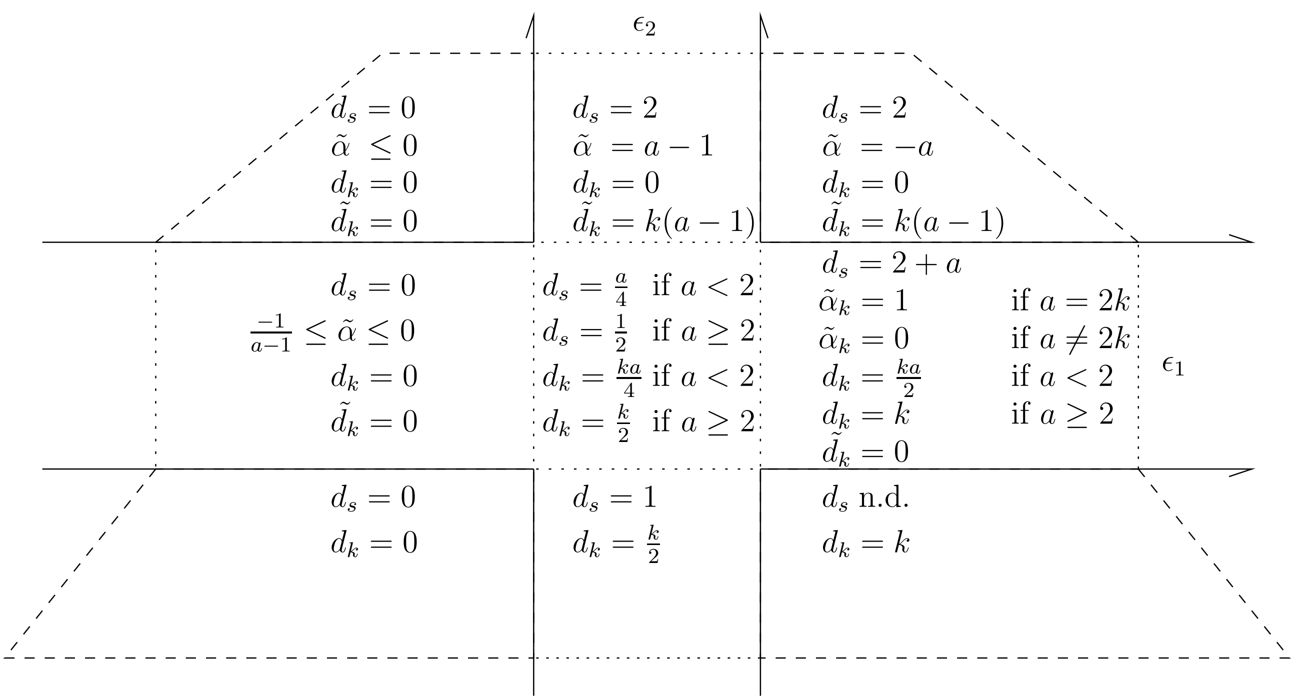

Figure 4. Results for μ^B. When $\epsilon_2 < 0$ the logarithmic exponents $\tilde{\alpha}$ and $\tilde{d_k}$ are always zero.

Acknowledgments

We would like to thank Bergfinnur Durhuus and Thordur Jonsson for valuable discussions. This work is supported in part by Marie Curie grant MRTN-CT-2004-005616 and by UK PPARC grant PP/D00036X/1. T.E. would like to acknowledge an ORS award and a Julia Mann Graduate Scholarship from St Hilda's College, Oxford.

Appendix A. Calculation of $\langle P_T(x)\rangle_{\mu^B}$ for $\epsilon_2 > 0$

First we rewrite (30) as

$$P_\ell(x) = P_\infty(x)Y - (Y-1)P_\infty(x)X^{-1}\frac{1}{X^{-1}+Y^{-\ell}}, \tag{A.1}$$

so that

$$\langle P_T(x)\rangle_{\mu^B} = P_\infty(x)Y - (Y-1)P_\infty(x)X^{-1}\left\langle\frac{1}{X^{-1}+Y^{-\ell}}\right\rangle_{\mu^B} \tag{A.2}$$

with

$$\left\langle\frac{1}{X^{-1}+Y^{-\ell}}\right\rangle_{\mu^B} = \sum_{\ell=1}^{\infty}\frac{C_a\ell^{-a}}{X^{-1}+Y^{-\ell}} \equiv S. \tag{A.3}$$

Since for $\epsilon_2 > 0$, $Y > 1$ we let $\log Y = \rho$ and write

$$S = \sum_{\ell=1}^{\lfloor\sigma\frac{|\log x|}{\rho}\rfloor}\frac{C_a\ell^{-a}}{X^{-1}+e^{-\rho\ell}} + \sum_{\lfloor\sigma\frac{|\log x|}{\rho}\rfloor+1}^{\infty}\frac{C_a\ell^{-a}}{X^{-1}+e^{-\rho\ell}}, \tag{A.4}$$

where σ is an arbitrary constant < 1. This is bounded above by taking ℓ in the exponential to be its value at the top of each sum to give

$$S \le \frac{C_a}{X^{-1}+x^\sigma}\sum_{\ell=1}^{\lfloor\sigma\frac{|\log x|}{\rho}\rfloor}\ell^{-a} + \frac{C_a}{X^{-1}}\sum_{\lfloor\sigma\frac{|\log x|}{\rho}\rfloor+1}^{\infty}\ell^{-a}$$

$$S \le \frac{1}{x^\sigma} + \frac{c_0 X}{|\log x|^{a-1}}. \tag{A.5}$$

Noting that as $x \to 0$, $X^{-1} \to Bx$ we get a lower bound on $\langle P_\ell(x)\rangle_{\mu^B}$ of

$$\langle P_T(x)\rangle_{\mu^B} \ge 1 - \frac{B_1}{|\log x|^{a-1}}, \tag{A.6}$$

for small enough x. An equivalent upper bound is calculated in the same manner by ignoring the first term in (A.4) and setting $\sigma = 1$, which leads to the result quoted in table 1. A similar procedure leads to bounds of the form $B/x|\log x|^a$ on $\left\langle P_T^{(1)}(x)\right\rangle_{\mu^B}$, which we also need, at small enough x.

Appendix B. Proof of results for non-recurrent regime

First we define a structure of ordered lists of ordered integers. Let S denote an ordered list of h_S integers

$$S = \begin{cases} [n_1, n_2, \dots n_{h_S}], & n_1 \geq n_2 \geq \dots \geq n_{h_S} \geq 1, \quad h_S \geq 1, \\ [\,], \quad h_S = 0. \end{cases} \tag{B.1}$$

Define

$$|S| = \begin{cases} \displaystyle\sum_{i=1}^{h_S} n_i, & h_S \geq 1, \\ 0, & h_S = 0, \end{cases} \tag{B.2}$$

and let $\mathcal{S}^N$ denote the set of all distinct lists S with $|S| = N$. Within $\mathcal{S}^N$ the lists S and S' are ordered by letting $j = \min(i : n_i \neq n'_i)$ and then setting $S > S'$ if $n_j > n'_j$. Finally if $S \in \mathcal{S}^N$ and $S' \in \mathcal{S}^{N'}$ with $N > N'$ then $S > S'$. It is convenient to denote by $S + 1$ the lowest list above S, and by $S \cup S'$ the list obtained by concatenating S and S' and then ordering as above.

Now define

$$H(S; f(x)) = (-1)^{|S|} \prod_{i=1}^{h_S} f^{(n_i)}(x), \tag{B.3}$$

and for the empty list $H([\,]; f(x)) = 1$. We need the following lemma, which is proved in Appendix B.1:

Lemma

(i) If $\langle H(S; P_T(x)) \rangle_\mu$ is finite as $x \to 0$ for all $S \leq \bar{S}$ then $\langle H(S; P_C(x)) \rangle_\mu$ is finite as $x \to 0$ for all $S \leq \bar{S}$ and $\epsilon_1 \neq 0$.

(ii) If the conditions of part (i) apply and, as $x \to 0$, $\left\langle H(\bar{S}+1, P_T(x)) \right\rangle_\mu$ diverges as $x^{-\gamma}, \gamma > 0$, then $\left\langle H(\bar{S}+1, P_C(x)) \right\rangle_\mu$ also diverges as $x^{-\gamma}$.

Differentiating (8) k times gives

$$Q_C^{(k)}(x) = \frac{P_C^{(k)}(x)}{(1 - P_C(x))^2} + (-1)^k \sum_{S \in \mathcal{S}^k / [k]} \frac{C(S) H(S; P_C(x))}{(1 - P_C(x))^{h_S + 1}} \tag{B.4}$$

where $C(S)$ is a combinatorial coefficient. It is straightforward to check for any S that $\langle H(S; P_T(x)) \rangle_{\mu^A}$ is analytic for $\epsilon_2 \neq 0$, and that $\langle H(S; P_T(x)) \rangle_{\mu^B}$ is analytic when $\epsilon_2 < 0$. When $\epsilon_2 = 0$

$$H(S; P_\ell(x))|_{x=0} = c_S \ell^{2|S| - h_S} \left(1 + O(l^{-2})\right) \tag{B.5}$$

from which $\langle H(S; P_\ell(x)) \rangle_{\mu^B}$ is divergent for $S = [\lceil a/2 \rceil]$, and with smaller degree for $[\lceil a/2 \rceil - 1, 1]$ if $2k < a \leq 2k + 1$, $k \in \mathbb{Z}$, but always convergent for any inferior S. The results given in section 4.2 then follow from noting that $P_*(x) < P_C(x) < P_\sharp(0) < 1$ and using the lemma.

Appendix B.1. Proof of lemma

To prove the lemma note that

$$H(S; f+g) = \sum_{S' \cup S'' = S} H(S'; f) H(S''; g) \tag{B.6}$$

and differentiate (9) n times to get

$$(-1)^n P_C^{(n)}(x) = (1-x) F_C^{(n)}(x) + n F_C^{(n-1)}(x) \tag{B.7}$$

where

$$F_C^{(n)}(x) = \frac{P_C(x)}{1-x} \sum_{S \in \mathcal{S}^n} C(S) \left(\frac{P_C(x) b_+}{(1-x) b_-} \right)^{h_S} \times \\ \sum_{S' \cup S'' = S} \left(\frac{b_T}{b_+} \right)^{h_{S''}} H(S'; P_{C_1}(x)) H(S''; P_{T_1}(x)). \tag{B.8}$$

It is then straightforward to generalise this formula to

$$H(S, P_C(x)) = \mathcal{R} + (P_C(x))^{h_S} \sum_{\substack{S' \in \mathcal{S}^{|S|} \\ S' \leq S}} C(S, S') \left(\frac{P_C(x) b_+}{(1-x) b_-} \right)^{h_{S'}} \times \\ \sum_{S'' \cup S''' = S'} \left(\frac{b_T}{b_+} \right)^{h_{S'''}} H(S''; P_{C_1}(x)) H(S'''; P_{T_1}(x)), \tag{B.9}$$

where the leading terms are written out explicitly and $\mathcal{R}$ contains contributions depending only on lists inferior to $\mathcal{S}^{|S|}$. Every term on the right hand side is positive so it can be bounded above by using $P_C(x) < P_\sharp(0)$ and then the expectation value taken; moving the $S'' = S$ term to the left hand side gives

$$\langle H(S, P_C(0)) \rangle_\mu \left(1 - \left(\frac{P_\sharp(0)^2 b_+}{b_-} \right)^{h_S} \right) \leq \\ \mathcal{R} + (P_\sharp(0))^{h_S} \sum_{\substack{S' \in \mathcal{S}^{|S|} \\ S' \leq S}} C(S, S') \left(\frac{P_\sharp(0) b_+}{b_-} \right)^{h_{S'}} \times \\ \sum_{\substack{S'' \cup S''' = S' \\ S'' \neq S}} \left(\frac{b_T}{b_+} \right)^{h_{S'''}} \langle H(S''; P_{C_1}(0)) \rangle_\mu \langle H(S'''; P_T(0)) \rangle_\mu . \tag{B.10}$$

Part (i) is true for $\bar{S} = [1]$ so the lemma then follows immediately by induction on S. To prove part (ii) use part (i) to isolate the potentially divergent terms in (B.9) leaving

$$H(\bar{S}, P_C(x)) = \left(\frac{P_C(x)^2 b_+}{(1-x) b_-} \right)^{h_{\bar{S}}} \left(H(\bar{S}; P_{C_1}(x)) + \left(\frac{b_T}{b_+} \right)^{h_{\bar{S}}} H(\bar{S}; P_{T_1}(x)) \right) \\ + \text{finite terms.} \tag{B.11}$$

For small enough x,

$$0 < \left(\frac{P_C(x)^2 b_+}{(1-x)b_-} \right) < 1, \quad \forall C \tag{B.12}$$

and part (ii) follows upon taking expectation values.

Appendix B.2. $\epsilon_1 = 0$, $\epsilon_2 > 0$

We will show that

$$F_S = \left\langle \frac{H(S, P_C(x))}{(1 - P_C(x))^{h_S+1}} \right\rangle_{\mu^A} \tag{B.13}$$

is finite at $x = 0$ which together with (B.4) gives the result. Using (61) and (C.9) gives

$$F_S < \left\langle n_C^{h_S+1} H(S, P_C(x)) \right\rangle_{\mu^A}, \tag{B.14}$$

where n_C is the location of the first infinite tooth of C. Applying (B.9) iteratively we find that the right hand side is bounded above by terms of the form

$$\left\langle n_C^K \right\rangle_{\mu^A} \left\langle H(S', P_T(x)) \right\rangle_{\mu^A}. \tag{B.15}$$

The maximum value of K occurring is $h_S + 1 + \Phi_S$ where Φ_S is the number of strings inferior to S. As remarked before $\langle H(S', P_T(x)) \rangle_{\mu^A}$ is analytic and $\left\langle n_C^K \right\rangle_{\mu^A}$ is trivially finite which completes the proof.

Appendix C. Calculation of $P_C(x)$ for some useful combs

Let the comb C have $T_k = N_\ell, k < L$ and arbitrary T_L and C_L. Then following the method of Appendix A of [14] we find

$$P_C^{\epsilon_1\epsilon_2}(x) = P_{\flat\ell}^{\epsilon_1\epsilon_2}(x) \left(1 + \frac{(1-A)(P_{C_{L-1}}^{\epsilon_1\epsilon_2}(x) - P_{\flat\ell}^{\epsilon_1\epsilon_2}(x))}{A^{L-1}(P_{C_{L-1}}^{\epsilon_1\epsilon_2}(x) - AP_{\flat\ell}^{\epsilon_1\epsilon_2}(x)) - (P_{C_{L-1}}^{\epsilon_1\epsilon_2}(x) - P_{\flat\ell}^{\epsilon_1\epsilon_2}(x))} \right) \tag{C.1}$$

where

$$A = \frac{(1-x)b_-}{(P_{\flat\ell}^{\epsilon_1\epsilon_2}(x))^2 b_+}. \tag{C.2}$$

Setting $\epsilon_2 = 0, \epsilon_1 < 0$, $\ell = 1$, $T_L = N_\infty$ and $C_L = \sharp$ we find after some algebra that

$$P_C^{\epsilon_1 0}(x) = P_\sharp^{\epsilon_1 0}(x) \left(1 + A^{-L} \left(x^{\frac{1}{2}} \frac{A-1}{2\epsilon_1} + O(x) \right) \right) \tag{C.3}$$

and, as $x \to 0$,

$$A \to \frac{1 + |\epsilon_1|}{1 - |\epsilon_1|}. \tag{C.4}$$

Repeating the exercise but with $C_L = *$ yields a similar result.

If instead we set $\epsilon_2 > 0, \epsilon_1 < 0$, $\ell = 1$, $T_L = N_\infty$ and $C_L = \sharp$ we find

$$P_C^{\epsilon_1\epsilon_2}(x) = P_\sharp^{\epsilon_1\epsilon_2}(x) \left(1 + \frac{2\epsilon_2(A-1)A^{-L}}{\epsilon_1 - 2\epsilon_2(1 - A^{-L})} (1 + O(x)) \right) \tag{C.5}$$

and, as $x \to 0$,

$$A \to \frac{1+|\epsilon_1|-\epsilon_2}{1-|\epsilon_1|-\epsilon_2}. \tag{C.6}$$

Again, repeating the exercise but with $C_L = *$ yields a similar result.

With $\epsilon_2 > 0, \epsilon_1 = 0$, and $C = \{T_{k<L} = N_\ell, T_{k\geq L} = N_\infty\}$ we find that

$$P_C^{0\epsilon_2}(x) > P_{\flat\ell}^{\epsilon_1\epsilon_2}(x) - \frac{1}{L-1}, \qquad L > 2, \tag{C.7}$$

(it is good enough to use $P_*(x)$ for $k=2$); and for $C = \{T_{k\neq L} = N_1, T_L = N_\ell\}$, $x < x_0$,

$$P_C^{0\epsilon_2}(x) < P_\sharp^{0\epsilon_2}(x)\left(1 - \frac{1}{\frac{A^{L-1}-1}{A-1} + \frac{BA^{L-1}}{1-P_{N_\ell}^{\epsilon_2}(x)}}\right) \tag{C.8}$$

where $A = (1-x)(P_\sharp^{0\epsilon_2}(x))^{-2}$ and B is a positive constant depending on x_0, A and ϵ_2.

Finally for $\epsilon_2 > 0, \epsilon_1 = 0$, and $C = \{T_{k\neq L} = N_1, T_L = N_\infty\}$ we find that

$$P_C^{0\epsilon_2}(0) = 1 - \frac{1}{L + (1-\epsilon_2)/4\epsilon_2}. \tag{C.9}$$

Appendix D. Upper bound on $P(x)$ when $\epsilon_1 = 0$, $\epsilon_2 > 0$

We start by writing

$$H(x;n) = \left\langle D_{|T_n|}(x)\right\rangle_\mu \left\langle \frac{G_C(x;n)}{1-P_C(x)}\right\rangle_{\overline{\mu}}, \tag{D.1}$$

where the measure $\bar{\mu}$ is defined in (93). Applying Jensen's inequality with this measure to (9) results in a recursion formula of the same form as discussed in Appendix C and it is easy to verify that $\langle P_{C_k}(x)\rangle_{\overline{\mu}} \geq \langle P_{C_k}(x)\rangle_\mu$ to give

$$\begin{aligned}\left\langle \frac{G_C(x;n)}{1-P_C(x)}\right\rangle_{\overline{\mu}} &\geq \left\langle \frac{G_C(x;n)}{1-P_C(x)}\right\rangle_\mu \\ &\geq \frac{3}{b_+(1-x)^{n/2}} \frac{\exp\left(-n\left\langle 1-P_C(x)\right\rangle_\mu\right)}{\left\langle 1-P_C(x)\right\rangle_\mu},\end{aligned} \tag{D.2}$$

where in the last line we have again used Jensen's inequality when averaging over the ensemble. Applying this result to (D.1), summing over n, and using (25) we obtain the inequality

$$\frac{2}{x} \geq \left\langle D_{|T_n|}(x)\right\rangle_\mu \frac{B}{\left\langle 1-P_C(x)\right\rangle_\mu^2}. \tag{D.3}$$

Using the value for $\langle D_\ell(x)\rangle_\mu$ given in table 2 and rearranging gives the upper bound on $P(x)$ quoted in 5.2.3.

References

[1] G. H. Weiss and S. Havlin, *Some properties of a random walk on a comb structure*, Physica **134A** (1986) 474-484

[2] S. Revathi, V. Balakrishnan, S. Lakshmibala and K. P. N. Murthy, *Validity of the mean-field approximation for diffusion on a random comb*, Phys. Rev. E **54** (1996) 2298-2302

[3] D. ben-Avraham and S. Havlin, *Diffusion and reactions in fractals and disordered systems*, Cambridge University Press, Cambridge (2000)

[4] S. Havlin, J. E. Kiefer and G. H. Weiss, *Anomalous diffusion on a random comblike structure*, Phys. Rev. A **36** (1987) 1403-1408

[5] J. Ambjørn, B. Durhuus and T. Jonsson, *Quantum geometry: a statistical field theory approach*, Cambridge University Press, Cambridge (1997)

[6] J. Ambjørn and Y. Watabiki, *Scaling in quantum gravity,* Nucl. Phys. **B445** (1995) 129-144, hep-th/9501049

[7] J. Ambjørn, J. Jurkiewicz and R. Loll, *Spectral dimension of the universe,* Phys. Rev. Lett. **95** (2005) 171301, hep-th/0505113

[8] C. Aslangul, P. Chvosta and N. Pottier, *Analytic study of a model of diffusion on a random comblike structure*, Physica A **203** (1994) 533-565

[9] S. Havlin, A. Bunde, H. E. Stanley and D. Movsholvitz, *Diffusion on percolation clusters with a bias in topological space: non-universal behaviour*, J. Phys. A **19** (1986) L693-L698

[10] V. Balakrishnan and C. Van den Broeck, *Transport properties on a random comb*, Physica A **217** (1995) 1-21

[11] S. Havlin, A. Bunde, Y. Glaser and H. E. Stanley, *Diffusion with a topological bias on random structures with a power-law distribution of dangling ends*, Phys. Rev. A **34** (1986) 3492-3495

[12] N. Pottier, *Diffusion on random comblike structures: field-induced trapping effects*, Physica A **216** (1995) 1-19

[13] N. Pottier, *Analytic study of a model of biased diffusion on a random comblike structure*, Physica A **208** (1994) 91-123

[14] B. Durhuus, T. Jonsson and J. F. Wheater, *Random walks on combs*, J. Phys. A **39** (2006) 1009-1038, hep-th/0509191

[15] B. Durhuus, T. Jonsson and J. F. Wheater, *The spectral dimension of generic trees*, math-ph/0607020

[16] T.M. Elliott, Oxford University D.Phil Thesis, in preparation.

[17] W. Feller, *An introduction to probability theory and its applications*, Vol.2, Wiley, London (1968)

[18] A. Grigoryan and T. Coulhon, *Pointwise estimates for transition probabilities of random walks in infinite graphs*, in: Trends in mathematics: Fractals in Graz 2001, Ed. P. Grabner and W. Woess. Birkhäueser (2002)