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NILPOTENT SYMMETRY INVARIANCE IN THE SUPERFIELD FORMULATION: THE (NON-)ABELIAN 1-FORM GAUGE THEORIES

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Abstract: We capture the off-shell as well as the on-shell nilpotent Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetry invariance of the Lagrangian densities of the four (3 + 1)-dimensional (4D) (non-)Abelian 1-form gauge theories within the framework of the superfield formalism. In particular, we provide the geometrical interpretations for (i) the above nilpotent symmetry invariance, and (ii) the above Lagrangian densities, in the language of the specific quantities defined in the domain of the above superfield formalism. Some of the subtle points, connected with the 4D (non-)Abelian 1-form gauge theories, are clarified within the framework of the above superfield formalism where the 4D ordinary gauge theories are considered on the (4, 2)-dimensional supermanifold parametrized by the four spacetime coordinates x^μ (with $\mu = 0, 1, 2, 3$) and a pair of Grassmannian variables θ and $\bar{\theta}$. One of the key results of our present investigation is a great deal of simplification in the geometrical understanding of the nilpotent (anti-)BRST symmetry invariance.

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1 Introduction

The geometrical superfield approach [1-8] to Becchi-Rouet-Stora-Tyutin (BRST) formalism is one of the most attractive and intuitive approaches which enables us to gain some physical insights into the beautiful (but abstract mathematical) structures that are associated with the nilpotent (anti-)BRST symmetry transformations and their corresponding generators. The latter quantities play a very decisive role in (i) the covariant canonical quantization of the gauge theories, (ii) the proof of the unitarity of the "quantum" gauge theories at any arbitrary order of perturbative computations for a given physical process (that is allowed by the theory), (iii) the definition of the physical states of the "quantum" gauge theories in the quantum Hilbert space, and (iv) the cohomological description of the physical states of the quantum Hilbert space w.r.t. the conserved and nilpotent BRST charge.

To be specific, in the superfield formulation [1-8] of the 4D 1-form gauge theories, one defines the super curvature 2-form $\tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)} + i\, \tilde{A}^{(1)} \wedge \tilde{A}^{(1)}$ in terms of the super exterior derivative $\tilde{d} = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta}\partial_{\bar{\theta}}$ (with $\tilde{d}^2 = 0$) and the super 1-form connection $\tilde{A}^{(1)}$ on a (4, 2)-dimensional supermanifold parametrized by the usual spacetime variables x^μ (with $\mu = 0, 1, 2, 3$) and a pair of anticommuting (i.e. $\theta^2 = \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$) Grassmannian variables θ and $\bar{\theta}$. The above super 2-form is subsequently equated, due to the so-called horizontality condition [1-8], to the ordinary curvature 2-form $F^{(2)} = dA^{(1)} + iA^{(1)} \wedge A^{(1)}$ defined on the ordinary 4D flat Minkowski spacetime manifold in terms of the ordinary exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) and the 1-form connection $A^{(1)} = dx^\mu A_\mu$. The above super exterior derivative $\tilde{d}$ and super 1-form connection $\tilde{A}^{(1)}$ are the generalization of the 4D ordinary exterior derivative d and 1-form connection $A^{(1)}$ to the (4, 2)-dimensional supermanifold because $\tilde{d} \to d, \tilde{A}^{(1)} \to A^{(1)}$ in the limit $(\theta, \bar{\theta}) \to 0$.

The above horizontality condition (HC) has been referred to as the soul-flatness condition in [9] which amounts to setting equal to zero all the Grassmannian components of the (anti)symmetric second-rank super tensor that constitutes the super curvature 2-form $\tilde{F}^{(2)}$ on the (4, 2)-dimensional supermanifold. The key consequences, that emerge from the HC, are (i) the derivation of the nilpotent (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields of a given 4D 1-form gauge theory, (ii) the geometrical interpretation of the (anti-)BRST symmetry transformations for the 4D local fields as the translation of the corresponding superfields along the Grassmannian directions of the supermanifold, (iii) the geometrical interpretation of the nilpotency property as a pair of successive translations of the superfield along a particular Grassmannian direction of the supermanifold, and (iv) the geometrical interpretation of the anticommutativity property of the (anti-)BRST symmetry transformations for a 4D local field as the *sum* of (a) the translation of the corresponding superfield first along the θ-direction followed by the translation along the $\bar{\theta}$-direction, and (b) the translation of the same superfield first along the $\bar{\theta}$-direction followed by the translation along the θ-direction.

It will be noted that the above HC (i.e. $\tilde{F}^{(2)} = F^{(2)}$) is valid for the non-Abelian (i.e.

$A^{(1)(n)} \wedge A^{(1)(n)} \neq 0$) 1-form gauge theory as well as the Abelian (i.e. $A^{(1)} \wedge A^{(1)} = 0$) 1-form gauge theory. As expected, for both types of theories, the HC leads to the derivation of the nilpotent (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields of the respective theories. We lay emphasis on the fact that the HC does not shed any light on the derivation of the nilpotent (anti-)BRST symmetry transformations associated with the *matter* fields of the *interacting* 4D (non-)Abelian 1-form gauge theories.

In a recent set of papers [10-17], the above HC condition has been generalized, in a consistent manner, so as to compute the nilpotent (anti-)BRST symmetry transformations associated with the *matter* fields of a given 4D interacting 1-form gauge theory (along with the well-known nilpotent transformations for the gauge and (anti-)ghost fields) without spoiling the cute geometrical interpretations of the (anti-)BRST symmetry transformations (and their corresponding generators) that emerge from the HC *alone*. The latter approach has been christened as the augmented superfield approach to BRST formalism where the restrictions imposed on the (4, 2)-dimensional superfields are (i) the HC plus the invariance of the (super) matter Noether conserved currents [10-14], (ii) the HC plus the equality of any (super) conserved quantities [15], (iii) the HC plus a restriction that owes its origin to the gauge invariance and the (super) covariant derivatives on the matter (super)fields [16,17], and (iv) an alternative to the HC where the gauge invariance and the property of a pair of (super) covariant derivatives on the (super) matter fields (and their intimate connection with the (super) curvatures) play a crucial role [18-20].

In all the above approaches [1-20], however, the invariance of the Lagrangian densities of the 4D (non-)Abelian 1-form gauge theories, under the nilpotent (anti-)BRST symmetry transformations, has not yet been discussed at all. Some attempts in this direction have been made in our earlier works where the specific topological features [21,22] of the 2D free (non-)Abelian 1-form gauge theories have been captured in the superfield formulation [23-25]. In particular, the invariance of the Lagrangian density under the nilpotent and anticommuting (anti-)BRST and (anti-)co-BRST symmetry transformations has been expressed in terms of the superfields and the Grassmannian derivatives on them. These are, however, a bit more involved in nature because of the existence of a new set of nilpotent (anti-)co-BRST symmetries in the theory. The geometrical interpretations for the Lagrangian densities and the symmetric energy-momentum tensor (for the above topological theory) have also been provided within the framework of the superfield formulation.

The purpose of our present paper is to capture the (anti-)BRST symmetry invariance of the Lagrangian density of the 4D (non-)Abelian 1-form gauge theories within the framework of the superfield approach to BRST formalism and to demonstrate that the above symmetry invariance could be understood in a very simple manner in terms of the translational generators along the Grassmannian directions of the (4, 2)-dimensional supermanifold on which the above 4D ordinary gauge theories are considered. In addition, the reason behind the existence (or non-existence) of any specific nilpotent symmetry transformation could also be explained within the framework of the above superfield approach. We demonstrate the *uniqueness* of the existence of the nilpotent (anti-)BRST symmetry transformations for the Lagrangian density of a U(1) Abelian 1-form gauge theory. We go a step further and show the existence of the nilpotent BRST symmetry transformations for the specific Lagrangian densities (cf. (4.1) and (4.4) below) of the 4D non-Abelian 1-form gauge theory and clarify the non-existence of the anti-BRST symmetry transformations for these specific Lagrangian densities within the framework of the superfield formulation (cf. section 5 below). Finally, we provide the geometrical basis for the existence of the off-shell nilpotent and anticommuting (anti-)BRST symmetry transformations (and their corresponding generators) for the specifically defined Lagrangian densities (cf. (4.7) and/or (4.8) below) of the 4D non-Abelian 1-form gauge theory in the Feynman gauge.

The motivating factors that have propelled us to pursue our present investigation are as follows. First and foremost, to the best of our knowledge, the property of the symmetry invariance of a given Lagrangian density has not yet been captured in the language of the superfield approach to BRST formalism. Second, the above (anti-)BRST invariance of the theory has never been shown, in as simplified fashion, as we demonstrate in our present endeavour. The geometrical interpretations for (i) the existence of the above nilpotent (anti-)BRST symmetry invariance, and (ii) the on-shell conditions of the on-shell nilpotent (anti-)BRST symmetries, turn out to be quite transparent in our present work. Third, we establish the *uniqueness* of the existence of the (anti-)BRST symmetry invariance in their various forms. The non-existence of the specific symmetry transformation is also explained within the framework of the superfield approach to BRST formalism. Finally, our present investigation is the first modest step in the direction to gain some insights into the existence of the nilpotent symmetry transformations and their invariance for the higher form (e.g. 2-form, 3-form, etc.) gauge theories within the framework of the superfield formulation.

The contents of our present paper are organized as follows. In section 2, we recapitulate some of the key points connected with the nilpotent (anti-)BRST symmetry transformations for the free 4D Abelian 1-form gauge theory (having no interaction with matter fields) in the Lagrangian formulation. The above symmetry transformations as well as the symmetry invariance of the Lagrangian densities are captured in the geometrical superfield approach to BRST formalism in section 3 where the HC on the gauge superfield plays a crucial role. Section 4 deals with the bare essentials of the nilpotent (anti-)BRST symmetry transformations for the 4D non-Abelian 1-form gauge theory in the Lagrangian formulation. The subject matter of section 5 concerns itself with the superfield formulation of the symmetry invariance of the appropriate Lagrangian densities of the above 4D non-Abelian 1-form gauge theory. Finally, in section 6, we summarize our key results, make some concluding remarks and point out a few future directions for further investigations.

2 (Anti-)BRST symmetries in Abelian theory: Lagrangian formulation

Let us begin with the following (anti-)BRST invariant Lagrangian density of the 4D Abelian

1-form gauge theory* in the Feynman gauge [26,27,9]

$$\mathcal{L}_B^{(a)} = -\frac{1}{4}\, F^{\mu\nu} F_{\mu\nu} \;+\; B\,(\partial_\mu A^\mu) \;+\; \frac{1}{2}\, B^2 - i\,\partial_\mu \bar{C}\,\partial^\mu C, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the antisymmetric ($F_{\mu\nu} = -F_{\nu\mu}$) curvature tensor that constitutes the Abelian 2-form $F^{(2)} = dA^{(1)} \equiv \frac{1}{2!}(dx^\mu \wedge dx^\nu) F_{\mu\nu}$, B is the Nakanishi-Lautrup auxiliary multiplier field and $(\bar{C})C$ are the anticommuting (i.e. $C^2 = \bar{C}^2 = 0, C\bar{C} + \bar{C}C = 0$) (anti-)ghost fields of the theory. The above Lagrangian density respects the off-shell nilpotent ($s_{(a)b}^2 = 0$) (anti-)BRST symmetry transformations $s_{(a)b}$ (with $s_b s_{ab} + s_{ab} s_b = 0$)†

$$\begin{array}{lllll} s_b A_\mu = \partial_\mu C, & s_b C = 0, & s_b \bar{C} = iB, & s_b B = 0, & s_b F_{\mu\nu} = 0, \\ s_{ab} A_\mu = \partial_\mu \bar{C}, & s_{ab} \bar{C} = 0, & s_{ab} C = -iB, & s_{ab} B = 0, & s_{ab} F_{\mu\nu} = 0. \end{array} \tag{2.2}$$

It is clear that, under the nilpotent (anti-)BRST symmetry transformations $s_{(a)b}$, the curvature tensor $F_{\mu\nu}$ is found to be invariant. In other words, the 2-form $F^{(2)}$, owing its origin to the cohomological operator $d = dx^\mu \partial_\mu$, is an (anti-)BRST invariant object for the Abelian U(1) 1-form gauge theory and is, therefore, a physically meaningful (i.e. gauge-invariant) quantity. These observations will play an important role in our discussion on the horizontality condition that would be exploited in the context of our superfield approach to (anti-)BRST invariance of the Lagrangian densities in sections 3 and 5 (see below).

A noteworthy point, at this stage, is the observation that the gauge-fixing and Faddeev-Popov ghost terms can be written, modulo a total derivative, in the following fashion

$$\begin{array}{ll} s_b\Big[-i\,\bar{C}\,\{(\partial_\mu A^\mu) + \frac{1}{2}\,B\}\Big], & s_{ab}\Big[+i\,C\,\{(\partial_\mu A^\mu) + \frac{1}{2}\,B\}\Big], \\ s_b\; s_{ab}\,\Big[\,\frac{i}{2}\,A_\mu\,A^\mu + \frac{1}{2}\,C\,\bar{C}\,\Big]. & \end{array} \tag{2.3}$$

The above equation establishes, in a very simple manner, the (anti-)BRST invariance of the 4D Lagrangian density (2.1). The simplicity ensues due to (i) the nilpotency $s_{(a)b}^2 = 0$ of the (anti-)BRST symmetry transformations, (ii) the anticommutativity property (i.e. $s_b s_{ab} + s_{ab} s_b = 0$) of $s_{(a)b}$, and (iii) the invariance of the $F_{\mu\nu}$ term under $s_{(a)b}$.

As a side remark, it is interesting to note that the following on-shell (i.e. $\Box C = \Box \bar{C} = 0$) nilpotent ($\tilde{s}_{(a)b}^2 = 0$) (anti-)BRST symmetry transformations (with $\tilde{s}_b \tilde{s}_{ab} + \tilde{s}_{ab} \tilde{s}_b = 0$)

$$\begin{array}{llll} \tilde{s}_b A_\mu = \partial_\mu C, & \tilde{s}_b C = 0, & \tilde{s}_b \bar{C} = -i(\partial_\mu A^\mu), & \tilde{s}_b F_{\mu\nu} = 0, \\ \tilde{s}_{ab} A_\mu = \partial_\mu \bar{C}, & \tilde{s}_{ab} \bar{C} = 0, & \tilde{s}_{ab} C = +i(\partial_\mu A^\mu), & \tilde{s}_{ab} F_{\mu\nu} = 0, \end{array} \tag{2.4}$$

*We adopt here the notations and conventions such that the flat Minkowski metric in 4D is $\eta_{\mu\nu} =$ diag $(+1, -1, -1, -1)$ so that $A_\mu B^\mu = \eta_{\mu\nu} A^\mu B^\nu = A_0 B_0 - A_i B_i$ for two non-null 4-vectors A_μ and B_μ. The Greek indices $\mu, \nu ...... = 0, 1, 2, 3$ and Latin indices $i, j, k .... = 1, 2, 3$ stand for the 4D spacetime and 3D space directions on the 4D Minkowski spacetime manifold, respectively, and the symbol $\Box = (\partial_0)^2 - (\partial_i)^2$.

†We follow here the notations and conventions adopted in [27]. In its full blaze of glory, the nilpotent (anti-)BRST transformations $\delta_{(A)B}$ are a product of an anticommuting spacetime independent parameter η and $s_{(a)b}$ (i.e. $\delta_{(A)B} = \eta s_{(a)b}$) where the nilpotency property is encoded in the operators $s_{(a)b}$.

are the symmetry transformations for the following Lagrangian density

$$\mathcal{L}_b^{(a)} = -\frac{1}{4}\, F^{\mu\nu} F_{\mu\nu} \;-\; \frac{1}{2}\,(\partial_\mu A^\mu)^2 \;-i\; \partial_\mu \bar{C}\; \partial^\mu C. \tag{2.5}$$

The above transformations (2.4) and the Lagrangian density (2.5) have been derived from (2.2) and (2.1) by the substitution $B = -(\partial_\mu A^\mu)$. An interesting point, connected with the on-shell nilpotent symmetry transformations, is to express the analogue of (2.3) as ‡

$$\begin{array}{ll} \tilde{s}_b \Big[+\dfrac{i}{2}\; \bar{C}\; (\partial_\mu A^\mu) + i\; A_\mu \partial^\mu \bar{C} \Big], & \tilde{s}_{ab} \Big[-\dfrac{i}{2}\; C\; (\partial_\mu A^\mu) - i\; A_\mu \partial^\mu C\; \Big], \\ \tilde{s}_b\; \tilde{s}_{ab}\; \Big[\; \dfrac{i}{2}\; A_\mu\; A^\mu + \dfrac{1}{2}\; C\; \bar{C}\; \Big]. & \end{array} \tag{2.6}$$

It should be noted that, in the above precise computation, one has to take into account the on-shell ($\Box C = \Box \bar{C} = 0$) conditions so that, for all practical purposes $\tilde{s}_{(a)b}(\partial_\mu A^\mu) = 0$.

The above nilpotent (anti-)BRST symmetry transformations (i.e. $s_r, \tilde{s}_r$ with $r = b, ab$) are connected with the conserved and nilpotent generators (i.e. $Q_r, \tilde{Q}_r$ with $r = b, ab$). This statement can be succinctly expressed, in the mathematical form, as

$$s_r\; \Omega = -i\; [\; \Omega,\; Q_r\;]_{(\pm)}, \qquad \tilde{s}_r\; \tilde{\Omega} = -i\; [\; \tilde{\Omega},\; \tilde{Q}_r\;]_{(\pm)}, \qquad\qquad r = b, ab, \tag{2.7}$$

where the subscripts (with the signatures $(\pm)$) on the square bracket stand for the bracket to be an (anti)commutator, for the generic fields $\Omega = A_\mu, C, \bar{C}, B$ and $\tilde{\Omega} = A_\mu, C, \bar{C}$ (of the Lagrangian densities (2.1) and (2.5)), being (fermionic)bosonic in nature. The above charges $Q_r, \tilde{Q}_r$ are found to be anticommuting (i.e. $Q_b Q_{ab} + Q_{ab} Q_b = 0, \tilde{Q}_b \tilde{Q}_{ab} + \tilde{Q}_{ab} \tilde{Q}_b = 0$) and off-shell as well as on-shell nilpotent ($Q^2_{(a)b} = 0, \tilde{Q}^2_{(a)b} = 0$) in nature, respectively.

3 (Anti-)BRST invariance in Abelian theory: superfield formalism

In this section, we exploit the geometrical superfield approach to BRST formalism, endowed with the theoretical arsenal of the horizontality condition, to express the (anti-)BRST symmetry transformations and the Lagrangian densities (cf. (2.1) and (2.5)) in terms of the superfields defined on the (4, 2)-dimensional supermanifold. The latter is parametrized by the spacetime coordinates x^μ (with $\mu = 0, 1, 2, 3$) and a pair of Grassmannian variables θ and $\bar{\theta}$. As a consequence, the generalization of the 4D ordinary exterior derivative $d = dx^\mu \partial_\mu$ and the 1-form connection $A^{(1)} = dx^\mu A_\mu(x)$ on the (4, 2)-dimensional supermanifold, are

$$\begin{array}{ll} d \rightarrow \tilde{d} = dx^\mu\; \partial_\mu \;+\; d\theta\; \partial_\theta \;+\; d\bar{\theta}\; \partial_{\bar{\theta}}, & \tilde{d}^2 = 0, \\ A^{(1)} \rightarrow \tilde{A}^{(1)} = dx^\mu\; \mathcal{B}_\mu(x,\theta,\bar{\theta}) + d\theta\; \bar{\mathcal{F}}(x,\theta,\bar{\theta}) + d\bar{\theta}\; \mathcal{F}(x,\theta,\bar{\theta}), & \end{array} \tag{3.1}$$

where the mapping from the 4D local fields to the superfields are: $A_\mu(x) \rightarrow \mathcal{B}_\mu(x,\theta,\bar{\theta})$, $C(x) \rightarrow \mathcal{F}(x,\theta,\bar{\theta})$ and $\bar{C}(x) \rightarrow \bar{\mathcal{F}}(x,\theta,\bar{\theta})$. The super-expansion of the superfields, in terms

‡We lay emphasis on the fact that (2.6) *cannot* be derived directly from (2.3) by the simple substitution $B = -(\partial_\mu A^\mu)$. One has to be judicious to deduce the precise expression for (2.6). The logical reasons behind the derivation of (2.6) are encoded in the superfield formulation (cf. (3.9) below).

of the basic fields as well as the secondary fields, are (see, e.g., [4-7, 10-12]):

$$\begin{array}{rcl} \mathcal{B}_\mu(x,\theta,\bar\theta) &=& A_\mu(x) + \theta\, \bar R_\mu(x) + \bar\theta\, R_\mu(x) + i\,\theta\,\bar\theta\, S_\mu(x), \\ \mathcal{F}(x,\theta,\bar\theta) &=& C(x) + i\,\theta\, \bar B_1(x) + i\,\bar\theta\, B_1(x) + i\,\theta\,\bar\theta\, s(x), \\ \bar{\mathcal{F}}(x,\theta,\bar\theta) &=& \bar C(x) + i\,\theta\, \bar B_2(x) + i\,\bar\theta\, B_2(x) + i\,\theta\,\bar\theta\, \bar s(x). \end{array} \tag{3.2}$$

It can be readily seen that, in the limiting case of $(\theta, \bar\theta) \to 0$, we get back our 4D basic fields $(A_\mu, C, \bar C)$. Furthermore, on the r.h.s. of the above super expansion, the bosonic (i.e. $A_\mu, S_\mu, B_1, \bar B_1, B_2, \bar B_2$) and the fermionic $(R_\mu, \bar R_\mu, C, \bar C, s, \bar s)$ fields do match.

At this juncture, we have to recall our observations after equation (2.2). The nilpotent (anti-)BRST symmetry transformations basically owe their origin to the cohomological operator d. This is capitalized in the horizontality condition where we impose the restriction $\tilde d \tilde A^{(1)} = d A^{(1)}$ on the super 1-form connection $\tilde A^{(1)}$ that contains the superfields defined on the (4, 2)-dimensional supermanifold. The latter condition yields the following relationships (see, e.g., for details, in our earlier works [21-25]):

$$B_1 = \bar B_2 = s = \bar s = 0, \qquad \bar B_1 + B_2 = 0, \tag{3.3}$$

where we are free to choose the secondary fields $(B_2, \bar B_1)$ (i.e. $B_2 = B \Rightarrow \bar B_1 = -B$) in terms of the Nakanishi-Lautrup auxiliary field B of the BRST invariant Lagrangian density (2.1). The other relations, that emerge from the above HC (i.e. $\tilde d \tilde A^{(1)} = d A^{(1)}$), are

$$R_\mu = \partial_\mu C, \qquad \bar R_\mu = \partial_\mu \bar C, \qquad S_\mu = \partial_\mu B, \qquad \partial_\mu \mathcal{B}_\nu - \partial_\nu \mathcal{B}_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.4}$$

At this stage, the super-curvature tensor $\tilde F_{\mu\nu} = \partial_\mu \mathcal{B}_\nu - \partial_\nu \mathcal{B}_\mu$ is *not* equal to the ordinary curvature tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as the former contains Grassmannian dependent terms.

The substitution of the above values (cf. (3.3),(3.4)) of the secondary fields, in terms of the basic and auxiliary fields of the Lagrangian density (2.1), leads to

$$\begin{array}{rcl} \mathcal{B}^{(h)}_\mu(x,\theta,\bar\theta) &=& A_\mu + \theta\, \partial_\mu \bar C + \bar\theta\, \partial_\mu C + i\,\theta\,\bar\theta\, \partial_\mu B, \\ \mathcal{F}^{(h)}(x,\theta,\bar\theta) &=& C - i\,\theta\, B, \qquad \bar{\mathcal{F}}^{(h)}(x,\theta,\bar\theta) = \bar C + i\,\bar\theta\, B, \end{array} \tag{3.5}$$

where the superscript (h) has been used to denote that the above expansions have been obtained after the application of the HC. It can be seen that, due to (3.5), we get

$$\partial_\mu \mathcal{B}^{(h)}_\nu - \partial_\nu \mathcal{B}^{(h)}_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3.6}$$

where there is *no* Grassmannian θ and $\bar\theta$ dependence on the l.h.s.

In the language of the geometry on the (4, 2)-dimensional supermanifold, the expansions (3.5) imply that the (anti-)BRST symmetry transformations $s_{(a)b}$ (and their corresponding generators $Q_{(a)b}$) for the 4D local fields (cf. (2.7)) are connected with the translational generators $(\partial/\partial\theta, \partial/\partial\bar\theta)$ because the translation of the corresponding (4, 2)-dimensional superfields, along the Grassmannian directions of the supermanifold, produces it. Thus, the Grassmannian *independence* of the super curvature tensor $\tilde F^{(h)}_{\mu\nu} = \partial_\mu \mathcal{B}^{(h)}_\nu - \partial_\nu \mathcal{B}^{(h)}_\mu$ implies that the 4D curvature tensor $F_{\mu\nu}$ is an (anti-)BRST (i.e. gauge) invariant physical quantity.

In terms of the superfields, equations (2.3) can be expressed as

$$\begin{array}{l}
\mathrm{Lim}_{\theta\to 0}\; \dfrac{\partial}{\partial\bar\theta}\Big[\, -i\; \bar{\mathcal F}^{(h)}\; \{\; (\partial^\mu \mathcal B^{(h)}_\mu + \dfrac{1}{2}\; B)\; \}\; \Big],\\
\mathrm{Lim}_{\bar\theta\to 0}\; \dfrac{\partial}{\partial\theta}\Big[\, + i\mathcal F^{(h)}\; \{\; (\partial^\mu \mathcal B^{(h)}_\mu + \dfrac{1}{2}\; B)\; \}\; \Big],\\
\dfrac{\partial}{\partial\bar\theta}\; \dfrac{\partial}{\partial\theta}\; \Big[\; \dfrac{i}{2}\; \mathcal B^{\mu(h)}\mathcal B^{(h)}_\mu \;+\; \dfrac{1}{2}\; \mathcal F^{(h)}\; \bar{\mathcal F}^{(h)}\; \Big].
\end{array} \tag{3.7}$$

These equations are *unique* because there is no other way to express the above equations in terms of the derivatives w.r.t. Grassmannian variables θ and $\bar\theta$. Thus, besides (2.3), there is no other possibility to express the gauge-fixing and the Faddeev-Popov ghost terms in the language of the off-shell nilpotent (anti-)BRST symmetry transformations (2.2). The superfield approach to BRST formulation, therefore, establishes the uniqueness of (2.3).

To express (2.6) in terms of the superfields, one has to substitute $B = -(\partial_\mu A^\mu)$ in (3.5). Thus, the expansion (3.5), in terms of the transformations (2.4), becomes[§]

$$\begin{array}{lcl}
\mathcal B^{(h)}_{\mu(o)}(x,\theta,\bar\theta) &=& A_\mu + \theta\; \partial_\mu \bar C + \bar\theta\; \partial_\mu C - i\; \theta\; \bar\theta\; \partial_\mu(\partial^\rho A_\rho),\\
&\equiv& A_\mu + \theta\; (\tilde s_{ab} A_\mu) + \bar\theta\; (\tilde s_b A_\mu) + \theta\; \bar\theta (\tilde s_b \tilde s_{ab} A_\mu),\\
\mathcal F^{(h)}_{(o)}(x,\theta,\bar\theta) &=& C + i\; \theta\; (\partial_\mu A^\mu) \equiv C + \theta\; (\tilde s_{ab} C),\\
\bar{\mathcal F}^{(h)}_{(o)}(x,\theta,\bar\theta) &=& \bar C - i\; \bar\theta\; (\partial_\mu A^\mu) \equiv \bar C + \bar\theta\; (\tilde s_b \bar C).
\end{array} \tag{3.8}$$

We note that (3.5) and (3.8) are the super expansions (after the application of the HC) which lead to the derivation of the off-shell nilpotent (anti-)BRST symmetry transformations $s_{(a)b}$ as well as the on-shell nilpotent (anti-)BRST symmetry transformations $\tilde s_{(a)b}$, respectively, for the basic fields A_μ, C and $\bar C$ of the theory.

The gauge-fixing and Faddeev-Popov ghost terms of the Lagrangian density (2.5) can also be expressed in terms of the superfields (3.8). In other words, (*vis-à-vis* (3.7)), we have the following equations that are the analogue of (2.6), namely;

$$\begin{array}{l}
\mathrm{Lim}_{\theta\to 0}\; \dfrac{\partial}{\partial\bar\theta}\Big[\, +\dfrac{i}{2}\; \bar{\mathcal F}^{(h)}_{(o)}\; (\partial^\mu A_\mu) + i\; \mathcal B^{(h)}_{\mu(o)}\; \partial^\mu \bar{\mathcal F}^{(h)}_{(o)})\; \Big],\\
\mathrm{Lim}_{\bar\theta\to 0}\; \dfrac{\partial}{\partial\theta}\Big[\, -\; \dfrac{i}{2}\; \mathcal F^{(h)}_{(o)}\; (\partial^\mu A_\mu) - i\; \mathcal B^{(h)}_{\mu(o)}\; \partial^\mu \mathcal F^{(h)}_{(o)})\; \Big],\\
\dfrac{\partial}{\partial\bar\theta}\; \dfrac{\partial}{\partial\theta}\; \Big[\; \dfrac{i}{2}\; \mathcal B^{\mu(h)}_{(o)}\mathcal B^{(h)}_{\mu(o)} \;+\; \dfrac{1}{2}\; \mathcal F^{(h)}_{(o)}\; \bar{\mathcal F}^{(h)}_{(o)}\; \Big].
\end{array} \tag{3.9}$$

We know that, for all practical computational purposes, it is essential to take into account $\tilde s_{(a)b}(\partial_\mu A^\mu) = 0$ because of the on-shell conditions $\Box C = \Box \bar C = 0$. The logical reason behind such a restriction (i.e. $\tilde s_{(a)b}(\partial_\mu A^\mu) = 0$) in (2.6) is encoded in the superfield approach to BRST formalism as can be seen from a close look at (3.9).

The Lagrangian density (2.1) can be expressed, in terms of the (4, 2)-dimensional superfields, in the following distinct and different forms

$$\tilde{\mathcal L}^{(1)}_B = -\frac{1}{4}\tilde F^{(h)}_{\mu\nu}\tilde F^{\mu\nu(h)} + \mathrm{Lim}_{\theta\to 0}\; \frac{\partial}{\partial\bar\theta}\; \Big[-i\; \bar{\mathcal F}^{(h)}(\partial^\mu \mathcal B^{(h)}_\mu + \frac{1}{2}\; B)\Big], \tag{3.10}$$

[§]The on-shell nilpotent (anti-)BRST symmetry transformations $\tilde s_{(a)b}$ can also be obtained by invoking the (anti-)chiral superfields on the appropriately chosen supermanifolds (see, e.g. [23] for details).

$$\tilde{\mathcal{L}}_B^{(2)} = -\frac{1}{4}\tilde{F}^{(h)}_{\mu\nu}\tilde{F}^{\mu\nu(h)} + \mathrm{Lim}_{\bar\theta\to 0}\,\frac{\partial}{\partial\theta}\,\Big[+i\,\mathcal{F}^{(h)}(\partial^\mu\mathcal{B}^{(h)}_\mu + \frac{1}{2}\,B)\Big], \tag{3.11}$$

$$\tilde{\mathcal{L}}_B^{(3)} = -\frac{1}{4}\tilde{F}^{(h)}_{\mu\nu}\tilde{F}^{\mu\nu(h)} + \frac{\partial}{\partial\bar\theta}\,\frac{\partial}{\partial\theta}\,\Big[+\frac{i}{2}\mathcal{B}^{\mu(h)}\mathcal{B}^{(h)}_\mu + \frac{1}{2}\mathcal{F}^{(h)}\bar{\mathcal{F}}^{(h)}\Big]. \tag{3.12}$$

It would be noted that the kinetic energy term $-(1/4)\tilde{F}^{(h)}_{\mu\nu}\tilde{F}^{\mu\nu(h)}$ is independent of the variables θ and $\bar\theta$ because $\tilde{F}^{(h)}_{\mu\nu} = F_{\mu\nu}$. In exactly similar fashion, the Lagrangian density of (2.5) can be expressed, with the help of the super expansion (3.8), as

$$\tilde{\mathcal{L}}_b^{(1)} = -\frac{1}{4}\tilde{F}^{(h)}_{\mu\nu(o)}\tilde{F}^{\mu\nu(h)}_{(o)} + \mathrm{Lim}_{\theta\to 0}\,\frac{\partial}{\partial\bar\theta}\Big[\,+\frac{i}{2}\,\bar{\mathcal{F}}^{(h)}_{(o)}\,(\partial^\mu A_\mu) + i\,\mathcal{B}^{(h)}_{\mu(o)}\,\partial^\mu\bar{\mathcal{F}}^{(h)}_{(o)})\,\Big], \tag{3.13}$$

$$\tilde{\mathcal{L}}_b^{(2)} = -\frac{1}{4}\tilde{F}^{(h)}_{\mu\nu(o)}\tilde{F}^{\mu\nu(h)}_{(o)} + \mathrm{Lim}_{\bar\theta\to 0}\,\frac{\partial}{\partial\theta}\Big[\,-\frac{i}{2}\,\mathcal{F}^{(h)}_{(o)}\,(\partial^\mu A_\mu) - i\,\mathcal{B}^{(h)}_{\mu(0)}\,\partial^\mu\mathcal{F}^{(h)}_{(o)})\,\Big], \tag{3.14}$$

$$\tilde{\mathcal{L}}_b^{(3)} = -\frac{1}{4}\tilde{F}^{(h)}_{\mu\nu(o)}\tilde{F}^{\mu\nu(h)}_{(o)} + \frac{\partial}{\partial\bar\theta}\,\frac{\partial}{\partial\theta}\,\Big[\,+\frac{i}{2}\mathcal{B}^{\mu(h)}_{(o)}\mathcal{B}^{(h)}_{\mu(o)} + \frac{1}{2}\mathcal{F}^{(h)}_{(o)}\bar{\mathcal{F}}^{(h)}_{(o)}\,\Big]. \tag{3.15}$$

The form of the Lagrangian densities (e.g. from (3.10) to (3.15)) simplify the proof for the (anti-)BRST invariance of the Lagrangian densities in (2.1) and (2.5).

In the above forms (e.g. from (3.10) to (3.12)) of the Lagrangian density, the BRST invariance $s_b\mathcal{L}_B = 0$ and the anti-BRST invariance $s_{ab}\mathcal{L}_B = 0$ become very transparent and simple because the following equalities and mappings exist, namely;

$$s_b\mathcal{L}_B^{(a)} = 0 \Rightarrow \mathrm{Lim}_{\theta\to 0}\frac{\partial}{\partial\bar\theta}\tilde{\mathcal{L}}_B^{(1)} = 0, \qquad s_b \Leftrightarrow \mathrm{Lim}_{\theta\to 0}\frac{\partial}{\partial\bar\theta}, \qquad s_b^2 = 0 \Leftrightarrow \Big(\frac{\partial}{\partial\bar\theta}\Big)^2 = 0, \tag{3.16}$$

$$s_{ab}\mathcal{L}_B^{(a)} = 0 \Rightarrow \mathrm{Lim}_{\bar\theta\to 0}\frac{\partial}{\partial\theta}\tilde{\mathcal{L}}_B^{(2)} = 0, \quad s_{ab} \Leftrightarrow \mathrm{Lim}_{\bar\theta\to 0}\frac{\partial}{\partial\theta}, \qquad s_{ab}^2 = 0 \Leftrightarrow \Big(\frac{\partial}{\partial\theta}\Big)^2 = 0. \tag{3.17}$$

Similarly, the most beautiful relation (3.12), leads to the (anti-)BRST invariance *together*. Here one has to use the anticommutativity property $s_b s_{ab} + s_{ab} s_b = 0$ in the language of the translational generators (i.e. $(\partial/\partial\bar\theta), (\partial/\partial\theta)$) along the Grassmannian directions of the supermanifold, for its proof. This statement can be mathematically expressed as

$$s_{(a)b}\mathcal{L}_B^{(a)} = 0 \Rightarrow \Big(\frac{\partial}{\partial\theta}\Big)\frac{\partial}{\partial\bar\theta}\tilde{\mathcal{L}}_B^{(3)} = 0, \qquad s_b s_{ab} + s_{ab} s_b = 0 \Leftrightarrow \frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar\theta} + \frac{\partial}{\partial\bar\theta}\frac{\partial}{\partial\theta} = 0. \tag{3.18}$$

In exactly similar fashion, the on-shell nilpotent (anti-)BRST symmetry invariance (i.e. $\tilde{s}_{(a)b}\mathcal{L}_b^{(a)} = 0$) of the Lagrangian density (2.5) can also be captured in the language of the superfields if we exploit the expressions (3.13) to (3.15) for the Lagrangian density. In the latter case, the on-shell nilpotent (anti-)BRST invariance turns out to be like (3.16), (3.17) and (3.18) with the replacements: $s_{(a)b} \to \tilde{s}_{(a)b},\ \mathcal{L}_B^{(a)} \to \mathcal{L}_b^{(a)},\ \tilde{\mathcal{L}}_B^{(1,2,3)} \to \tilde{\mathcal{L}}_b^{(1,2,3)}$.

Mathematically, the (anti-)BRST invariance of the Lagrangian density (2.1) is captured in the equations (3.16) to (3.18). In the language of geometry on the (4, 2)-dimensional supermanifold, the (anti-)BRST invariance corresponds to the Grassmannian independence of the supersymmetric versions of the Lagrangian density (2.1). In other words, the translation of the super Lagrangian densities (i.e. (3.10) to (3.12)), along the $(\theta)\bar\theta$ directions of the supermanifold, is *zero*. This observation captures the (anti-)BRST invariance of (2.1).

4 (Anti-)BRST symmetries in non-Abelian theory: Lagrangian approach

We begin with the following BRST-invariant Lagrangian density, in the Feynman gauge, for the four (3 + 1)-dimensional non-Abelian 1-form gauge theory¶ (see, e.g. [26,27,9])

$$\mathcal{L}_B^{(n)} = -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} + B \cdot (\partial_\mu A^\mu) + \frac{1}{2} B \cdot B - i \partial_\mu \bar{C} \cdot D^\mu C, \tag{4.1}$$

where the curvature tensor ($F_{\mu\nu}$) is defined through the 2-form $F^{(2)(n)} = dA^{(1)(n)} + iA^{(1)(n)} \wedge A^{(1)(n)}$. Here the non-Abelian 1-form gauge connection is $A^{(1)(n)} = dx^\mu (A_\mu \cdot T)$ and the exterior derivative is $d = dx^\mu \partial_\mu$. The Nakanishi-Lautrup auxiliary field $B = B \cdot T$ is required for the linearization of the gauge-fixing term and the (anti-)ghost fields $(\bar{C})C$ are essential for the proof of the unitarity in the theory. The latter fields are fermionic (i.e. $(C^a)^2 = 0, (\bar{C}^a)^2 = 0, C^a C^b + C^b C^a = 0, C^a \bar{C}^b + \bar{C}^b C^a = 0$, etc.) in nature.

The above Lagrangian density respects the following off-shell nilpotent $((s_b^{(n)})^2 = 0)$ BRST symmetry transformations $s_b^{(n)}$, namely;

$$\begin{array}{lll} s_b^{(n)} A_\mu = D_\mu C, & s_b^{(n)} C = -\frac{i}{2} (C \times C), & s_b^{(n)} \bar{C} = iB, \\ s_b^{(n)} B = 0, & s_b^{(n)} F_{\mu\nu} = i(F_{\mu\nu} \times C). & \end{array} \tag{4.2}$$

It will be noted that (i) the curvature tensor $F_{\mu\nu} \cdot T$ transforms here under the BRST symmetry transformation. However, it can be checked explicitly that the kinetic energy term $-(1/4) F_{\mu\nu} \cdot F^{\mu\nu}$ remains invariant under the BRST symmetry transformations, (ii) the nilpotent *anti-BRST* symmetry transformations corresponding to the above BRST symmetry transformations (4.2) *cannot* be defined for the Lagrangian density (4.1), and (iii) the on-shell nilpotent version of the above BRST symmetry transformations is also possible if we substitute, in the above symmetry transformations, $B = -(\partial_\mu A^\mu)$. The ensuing on-shell (i.e. $\partial_\mu D^\mu C = 0$) nilpotent BRST symmetry transformations $\tilde{s}_b^{(n)}$ are

$$\begin{array}{ll} \tilde{s}_b^{(n)} A_\mu = D_\mu C, & \tilde{s}_b^{(n)} C = -\frac{i}{2} (C \times C), \\ \tilde{s}_b^{(n)} \bar{C} = -i(\partial_\mu A^\mu), & \tilde{s}_b^{(n)} F_{\mu\nu} = i(F_{\mu\nu} \times C). \end{array} \tag{4.3}$$

The above on-shell nilpotent transformations leave the following Lagrangian density

$$\mathcal{L}_b^{(n)} = -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu) \cdot (\partial_\rho A^\rho) - i \partial_\mu \bar{C} \cdot D^\mu C, \tag{4.4}$$

¶For the non-Abelian 1-form gauge theory, the notations used in the Lie algebraic space are: $A \cdot B = A^a B^a$, $(A \times B)^a = f^{abc} A^b B^c$, $D_\mu C^a = \partial_\mu C^a + i f^{abc} A_\mu^b C^c \equiv \partial_\mu C^a + i (A_\mu \times C)^a$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i A_\mu \times A_\nu$, $A_\mu = A_\mu \cdot T$, $[T^a, T^b] = f^{abc} T^c$ where the Latin indices $a, b, c = 1, 2, 3....N$ are in the $SU(N)$ Lie algebraic space. The structure constant f^{abc} can be chosen to be totally antisymmetric for any arbitrary semi simple Lie algebra that includes $SU(N)$, too (see, e.g., [27]).

quasi-invariant because it transforms to a total derivative.

The gauge-fixing and Faddeev-Popov ghost terms of the Lagrangian densities (4.1) and (4.4) can be written, modulo a total derivative, as a BRST-exact quantity in terms of the off-shell and on-shell nilpotent BRST symmetry transformations (4.2) and (4.3). This statement can be mathematically expressed as follows

$$s_b^{(n)} \Big[-i\, \bar{C} \cdot \{(\partial_\mu A^\mu) + \frac{1}{2} B\}\Big] = B \cdot (\partial_\mu A^\mu) + \frac{1}{2} B \cdot B - i\, \partial_\mu \bar{C} \cdot D^\mu C, \tag{4.5}$$

$$\tilde{s}_b^{(n)} \Big[+\frac{i}{2}\, \bar{C} \cdot (\partial_\mu A^\mu) + i\, A_\mu \cdot \partial^\mu \bar{C}\ \Big] = -\frac{1}{2}\, (\partial_\mu A^\mu) \cdot (\partial_\rho A^\rho) - i\, \partial_\mu \bar{C} \cdot D^\mu C. \tag{4.6}$$

It will be noted that one has to take into account $\tilde{s}_b^{(n)} (\partial_\mu A^\mu) = \partial_\mu D^\mu C = 0$ in the above proof of the exactness of the expression in (4.6).

The Lagrangian densities that respect the off-shell nilpotent (i.e. $(s_{(a)b}^{(n)})^2 = 0$) and anticommuting ($s_b^{(n)} s_{ab}^{(n)} + s_{ab}^{(n)} s_b^{(n)} = 0$) (anti-)BRST symmetry transformations are

$$\mathcal{L}_b^{(1)(n)} = -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} + B \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i \partial_\mu \bar{C} \cdot D^\mu C, \tag{4.7}$$

$$\mathcal{L}_b^{(2)(n)} = -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} - \bar{B} \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i D_\mu \bar{C} \cdot \partial^\mu C. \tag{4.8}$$

Here auxiliary fields B and $\bar{B}$ satisfy the Curci-Ferrari condition $B + \bar{B} = -(C \times \bar{C})$ [28,29]. It is also evident, from this relation, that $B \cdot (\partial_\mu A^\mu) - i \partial_\mu \bar{C} \cdot D^\mu C = -\bar{B} \cdot (\partial_\mu A^\mu) - i D_\mu \bar{C} \cdot \partial^\mu C$. Furthermore, it should be re-emphasized that the Lagrangian densities (4.1) and (4.4) do not respect the anti-BRST symmetry transformations of any kind. The BRST and anti-BRST symmetry transformations, for the above Lagrangian densities, are

$$\begin{array}{lll} s_b^{(n)} A_\mu = D_\mu C, & s_b^{(n)} C = -\dfrac{i}{2} (C \times C), & s_b^{(n)} \bar{C} = i B, \\ s_b^{(n)} B = 0, & s_b^{(n)} F_{\mu\nu} = i (F_{\mu\nu} \times C), & s_b^{(n)} \bar{B} = i (\bar{B} \times C), \end{array} \tag{4.9}$$

$$\begin{array}{lll} s_{ab}^{(n)} A_\mu = D_\mu \bar{C}, & s_{ab}^{(n)} \bar{C} = -\dfrac{i}{2} (\bar{C} \times \bar{C}), & s_{ab}^{(n)} C = i \bar{B}, \\ s_{ab}^{(n)} \bar{B} = 0, & s_{ab}^{(n)} F_{\mu\nu} = i (F_{\mu\nu} \times \bar{C}), & s_{ab}^{(n)} B = i (B \times \bar{C}). \end{array} \tag{4.10}$$

The above off-shell nilpotent (anti-)BRST symmetry transformations leave the Lagrangian densities (4.7) as well as (4.8) quasi-invariant as they transform to some total derivatives. The gauge-fixing and Faddeev-Popov ghost terms of the Lagrangian densities (4.7) and (4.8) can be written, in a symmetrical fashion with respect to $s_b^{(n)}$ and $s_{ab}^{(n)}$, as

$$\begin{aligned} s_b^{(n)} s_{ab}^{(n)} \Big[\frac{i}{2} A_\mu \cdot A^\mu + C \cdot \bar{C}\Big] &= B \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i \partial_\mu \bar{C} \cdot D^\mu C, \\ &\equiv -\bar{B} \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i D_\mu \bar{C} \cdot \partial^\mu C. \end{aligned} \tag{4.11}$$

This demonstrates the key fact that the above gauge-fixing and Faddeev-Popov ghost terms are (anti-)BRST invariant *together* because of the nilpotency and anticommutativity of the (anti-)BRST symmetry transformations $s^{(n)}_{(a)b}$ that are present in the theory.

5 (Anti-)BRST invariance in non-Abelian theory: superfield approach

To capture (i) the off-shell as well as the on-shell nilpotent (anti-)BRST symmetry transformations, and (ii) the invariance of the Lagrangian densities, in the language of the superfield approach to BRST formalism, we have to consider the 4D 1-form non-Abelian gauge theory on a (4, 2)-dimensional supermanifold. As a consequence, we have the following mappings:

$$
\begin{array}{lcl}
d \;\;\rightarrow\;\; \tilde{d} & = & dx^\mu\,\partial_\mu \;+\; d\theta\,\partial_\theta \;+\; d\bar{\theta}\,\partial_{\bar{\theta}}, \qquad\qquad \tilde{d}^2 = 0,\\
A^{(1)(n)} \rightarrow \tilde{A}^{(1)(n)} & = & dx^\mu(\mathcal{B}_\mu\cdot T)(x,\theta,\bar{\theta}) + d\theta(\bar{\mathcal{F}}\cdot T)(x,\theta,\bar{\theta}) + d\bar{\theta}(\mathcal{F}\cdot T)(x,\theta,\bar{\theta}),
\end{array}
\tag{5.1}
$$

where the (4, 2)-dimensional superfields $(\mathcal{B}_\mu\cdot T, \mathcal{F}\cdot T, \bar{\mathcal{F}}\cdot T)$ are the generalizations of the 4D basic local fields $(A_\mu\cdot T, C\cdot T, \bar{C}\cdot T)$ of the Lagrangian density (4.1), (4.7) and (4.8). These superfields can be expanded along the Grassmannian directions of the supermanifold, in terms of the basic 4D fields, auxiliary fields and secondary fields as [4,16,19]

$$
\begin{array}{lcl}
(\mathcal{B}_\mu\cdot T)(x,\theta,\bar{\theta}) & = & (A_\mu\cdot T)(x) + \theta\,(\bar{R}_\mu\cdot T)(x) + \bar{\theta}\,(R_\mu\cdot T)(x) + i\,\theta\,\bar{\theta}\,(S_\mu\cdot T)(x),\\
(\mathcal{F}\cdot T)(x,\theta,\bar{\theta}) & = & (C\cdot T)(x) + i\,\theta\,(\bar{B}_1\cdot T)(x) + i\,\bar{\theta}\,(B_1\cdot T)(x) + i\,\theta\,\bar{\theta}\,(s\cdot T)(x),\\
(\bar{\mathcal{F}}\cdot T)(x,\theta,\bar{\theta}) & = & (\bar{C}\cdot T)(x) + i\,\theta\,(\bar{B}_2\cdot T)(x) + i\,\bar{\theta}\,(B_2\cdot T)(x) + i\,\theta\,\bar{\theta}\,(\bar{s}\cdot T)(x).
\end{array}
\tag{5.2}
$$

To determine the exact expressions for the secondary fields, in terms of the basic and auxiliary fields of the theory, we have to exploit the HC. The horizontality condition, for the non-Abelian gauge theory is the requirement of the equality of the Maurer-Cartan equation on the (super) manifolds. In other words, the covariant reduction of the super 2-form curvature $\tilde{F}^{(2)(n)}$ to the ordinary 2-form curvature (i.e. $\tilde{d}\tilde{A}^{(1)(n)} + i\tilde{A}^{(1)(n)}\wedge\tilde{A}^{(1)(n)} = dA^{(1)(n)} + iA^{(1)(n)}\wedge A^{(1)(n)}$) leads to the determination of the secondary fields in terms of the basic and auxiliary fields of the theory. The ensuing expansions, in terms of the basic and auxiliary fields, lead to (i) the derivation of the (anti-)BRST symmetry transformations for the basic fields of the theory, and (ii) the geometrical interpretations of the nilpotent (anti-)BRST symmetry transformations (and their corresponding nilpotent generators) for the basic fields of the theory as the translations of the corresponding superfields along the Grassmannian directions of the (4, 2)-dimensional supermanifold (see, e.g., [16,19]).

With the identifications $B_2 = B$ and $\bar{B}_1 = \bar{B}$, the following relationships emerge after the application of the horizontality condition [‖] (see, e.g., [16]):

$$
\begin{array}{l}
R_\mu = D_\mu C, \qquad \bar{R}_\mu = D_\mu\bar{C}, \qquad B + \bar{B} = -(C\times\bar{C}), \qquad s = i(\bar{B}\times C),\\
S_\mu = D_\mu B + D_\mu C\times\bar{C} \equiv -D_\mu\bar{B} - D_\mu\bar{C}\times C,\\
\bar{s} = -i(B\times\bar{C}), \qquad B_1 = -\dfrac{1}{2}(C\times C), \qquad \bar{B}_2 = -\dfrac{1}{2}(\bar{C}\times\bar{C}).
\end{array}
\tag{5.3}
$$

[‖] In the rest of our present text, we shall be using the short-hand notations for all the fields e.g.: $A_\mu\cdot T = A_\mu,\; C\cdot T = C,\; B\cdot T = B$, etc., for the sake of brevity.

The substitution of the above expressions, which are obtained after the application of the horizontality condition, leads to the following expansions

$$\begin{aligned}
\mathcal{B}_\mu^{(h)}(x,\theta,\bar\theta) &= A_\mu + \theta\, D_\mu \bar C + \bar\theta\, D_\mu C + i\,\theta\,\bar\theta\,(D_\mu B + D_\mu C \times \bar C),\\
\mathcal{F}^{(h)}(x,\theta,\bar\theta) &= C + i\,\theta\,\bar B - \frac{i}{2}\,\bar\theta\,(C\times C) - \theta\,\bar\theta\,(\bar B \times C),\\
\bar{\mathcal{F}}^{(h)}(x,\theta,\bar\theta) &= \bar C - \frac{i}{2}\,\theta\,(\bar C\times \bar C) + i\,\bar\theta\, B + \theta\,\bar\theta\,(B\times \bar C).
\end{aligned} \tag{5.4}$$

The above expansions (see, e.g., our earlier works [16,19]) can be expressed in terms of the off-shell nilpotent (anti-)BRST symmetry transformations (4.9) and (4.10).

With the above expansion at our disposal, the gauge-fixing and Faddeev-Popov terms of the Lagrangian density (4.1) can be written, modulo a total ordinary derivative, as

$$\mathrm{Lim}_{\theta\to 0}\,\frac{\partial}{\partial\bar\theta}\Big[-i\bar{\mathcal{F}}^{(h)}\cdot\partial^\mu\mathcal{B}_\mu^{(h)} - \frac{i}{2}\,\bar{\mathcal{F}}^{(h)}\cdot B\Big] = B\cdot(\partial_\mu A^\mu) + \frac{1}{2}B\cdot B - i\,\partial_\mu\bar C\cdot D^\mu C. \tag{5.5}$$

Furthermore, it can be seen that, due to the validity and consequences of the horizontality condition, the super curvature tensor $\tilde F_{\mu\nu}$ has the following form [16,4]

$$\tilde F_{\mu\nu}^{(h)} = F_{\mu\nu} + i\theta(F_{\mu\nu}\times\bar C) + i\bar\theta(F_{\mu\nu}\times C) - \theta\,\bar\theta\,(F_{\mu\nu}\times B + F_{\mu\nu}\times C\times\bar C). \tag{5.6}$$

It is clear from the above relationship that the kinetic energy term of the present 4D non-Abelian 1-form gauge theory remains invariant, namely;

$$-\frac{1}{4}\tilde F_{\mu\nu}^{(h)}\cdot\tilde F^{\mu\nu(h)} = -\frac{1}{4}F_{\mu\nu}\cdot F^{\mu\nu}. \tag{5.7}$$

The Grassmannian independence of the l.h.s. of (5.7) has deep meaning as far as physics is concerned. It implies immediately that the kinetic energy term of the non-Abelian gauge theory is an (anti-)BRST (i.e. gauge) invariant *physical* quantity.

At this juncture, it is worthwhile to point out that one can also capture the equation (4.6) in the superfield approach to BRST formalism where the on-shell nilpotent version of the BRST symmetry transformations (i.e. $\tilde s_b^{(n)}$) plays an important role. For this purpose, we have to express the superfield expansion (5.4) for the on-shell nilpotent BRST symmetry transformation where one has to exploit the replacement $B = -(\partial_\mu A^\mu)$. With this substitution, the equation (5.4) for the superfield expansion becomes

$$\begin{aligned}
\mathcal{B}_{\mu(o)}^{(h)}(x,\theta,\bar\theta) &= A_\mu + \theta\, D_\mu\bar C + \bar\theta\, D_\mu C + i\,\theta\,\bar\theta\,[-D_\mu(\partial^\rho A_\rho) + D_\mu C\times\bar C],\\
\mathcal{F}_{(o)}^{(h)}(x,\theta,\bar\theta) &= C + i\,\theta\,\bar B - \frac{i}{2}\,\bar\theta\,(C\times C) - \theta\,\bar\theta\,(\bar B\times C),\\
\bar{\mathcal{F}}_{(o)}^{(h)}(x,\theta,\bar\theta) &= \bar C - \frac{i}{2}\,\theta\,(\bar C\times\bar C) - i\,\bar\theta\,(\partial_\mu A^\mu) - \theta\,\bar\theta\,[(\partial_\mu A^\mu)\times\bar C)].
\end{aligned} \tag{5.8}$$

Now, the equation (4.6) can be expressed in terms of the above superfields, as:

$$\mathrm{Lim}_{\theta\to 0}\,\frac{\partial}{\partial\bar\theta}\Big[\frac{i}{2}\bar{\mathcal{F}}_{(o)}^{(h)}\cdot(\partial^\mu A_\mu) + i\,\mathcal{B}_{\mu(o)}^{(h)}\cdot\partial^\mu\bar{\mathcal{F}}_{(o)}^{(h)}\Big] = -\frac{1}{2}\,(\partial_\mu A^\mu)\cdot(\partial_\rho A^\rho) - i\,\partial_\mu\bar C\cdot D^\mu C. \tag{5.9}$$

Furthermore, it will be noted that the analogue of (5.6), for the on-shell nilpotent BRST symmetry transformation (i.e. $\tilde{F}^{(h)}_{\mu\nu(o)}$), can be obtained by the replacement $B = -(\partial_\mu A^\mu)$. Once again, the equality (5.7) would remain intact even if we take into account the on-shell nilpotent BRST symmetry transformations. Thus, we note that the kinetic energy term (i.e. $(-(1/4)F^{\mu\nu} \cdot F_{\mu\nu} = -(1/4)\tilde{F}^{\mu\nu(h)}_{(o)} \cdot \tilde{F}^{(h)}_{\mu\nu(o)})$ of the non-Abelian gauge theory remains independent of the Grassmannian variables θ and $\bar{\theta}$ after the application of the HC. This statement is true for the off-shell as well as the on-shell nilpotent (anti-)BRST symmetry transformations. Physically, it implies that the kinetic energy term for the gauge field of the non-Abelian theory is an (anti-)BRST (i.e. gauge) invariant quantity.

The above key observation helps in expressing the Lagrangian density (4.1) and (4.4) in terms of the superfields (obtained after the application of HC), as

$$\begin{aligned}
\tilde{\mathcal{L}}^{(n)}_B &= -\frac{1}{4}\; \tilde{F}^{(h)}_{\mu\nu} \cdot \tilde{F}^{\mu\nu(h)} + \mathrm{Lim}_{\theta\to 0}\, \frac{\partial}{\partial\bar{\theta}}\Big[-i\bar{\mathcal{F}}^{(h)} \cdot \partial^\mu \mathcal{B}^{(h)}_\mu - \frac{i}{2}\; \bar{\mathcal{F}}^{(h)} \cdot B\Big], \\
\tilde{\mathcal{L}}^{(n)}_b &= -\frac{1}{4}\; \tilde{F}^{(h)}_{\mu\nu(o)} \cdot \tilde{F}^{\mu\nu(h)}_{(o)} + \mathrm{Lim}_{\theta\to 0}\, \frac{\partial}{\partial\bar{\theta}}\Big[\frac{i}{2}\bar{\mathcal{F}}^{(h)}_{(o)} \cdot (\partial^\mu A_\mu) + i\; \mathcal{B}^{(h)}_{\mu(o)} \cdot \partial^\mu \bar{\mathcal{F}}^{(h)}_{(o)}\;\Big].
\end{aligned} \tag{5.10}$$

This result, in turn, simplifies the BRST invariance of the above Lagrangian density (4.1) and (4.4) (describing the 4D 1-form non-Abelian gauge theory) as follows

$$\mathrm{Lim}_{\theta\to 0}\frac{\partial}{\partial\bar{\theta}}\tilde{\mathcal{L}}^{(n)}_B = 0 \Rightarrow s^{(n)}_b \mathcal{L}^{(n)}_B = 0, \qquad \mathrm{Lim}_{\theta\to 0}\frac{\partial}{\partial\bar{\theta}}\tilde{\mathcal{L}}^{(n)}_b = 0 \Rightarrow \tilde{s}^{(n)}_b \mathcal{L}^{(n)}_b = 0. \tag{5.11}$$

This is a great simplification because the total super Lagrangian densities (5.10) remain independent of the Grassmannian variable $\bar{\theta}$. This key result is encoded in the mapping $(s^{(n)}_b, \tilde{s}^{(n)}_b) \Leftrightarrow \mathrm{Lim}_{\theta\to 0}(\partial/\partial\bar{\theta})$ and the nilpotency $(s^{(n)}_b)^2 = 0, (\tilde{s}^{(n)}_b)^2 = 0, (\partial/\partial\bar{\theta})^2 = 0$.

It can be readily checked that the analogues of (5.5) and (5.9) cannot be expressed as the derivative w.r.t. the Grassmannian variable θ. To check this, one has to exploit the super expansions (5.4) and (5.8) obtained after the application of the HC (in the context of the derivation of the off-shell as well as the on-shell nilpotent BRST symmetry transformations $s^{(n)}_b$ and $\tilde{s}^{(n)}_b$). It can be clearly seen that the operation of the derivative w.r.t. the Grassmannian variable θ, on *any* combination of the superfields from the expansions (5.4) and (5.8), does *not* lead to the derivation of the r.h.s. of (5.5) and (5.9). In the language of the superfield approach to BRST formalism, this is the reason behind the non-existence of the anti-BRST symmetry transformations for the Lagrangian densities (4.1) and (4.4).

The form of the gauge-fixing and Faddeev-Popov terms (4.11), expressed in terms of the BRST and anti-BRST symmetry transformations *together*, can be represented in the language of the superfields (obtained after the application of HC), as

$$\frac{\partial}{\partial\bar{\theta}}\,\frac{\partial}{\partial\theta}\Big[\;\frac{i}{2}\mathcal{B}^{(h)}_\mu \cdot \mathcal{B}^{\mu(h)} + \mathcal{F}^{(h)} \cdot \bar{\mathcal{F}}^{(h)}\;\Big] \;\;=\;\; B \cdot (\partial_\mu A^\mu) + \frac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B}) - i\partial_\mu \bar{C} \cdot D^\mu C. \tag{5.12}$$

As a consequence of the above expression, the Lagrangian densities (4.7) (as well as (4.8)) can be presented, in terms of the superfields, as

$$\tilde{\mathcal{L}}^{(1,2)(n)}_b = -\frac{1}{4}\tilde{F}^{\mu\nu(h)} \cdot \tilde{F}^{(h)}_{\mu\nu} + \frac{\partial}{\partial\bar{\theta}}\,\frac{\partial}{\partial\theta}\,\Big[\frac{i}{2}\mathcal{B}^{(h)}_\mu \cdot \mathcal{B}^{\mu(h)} + \mathcal{F}^{(h)} \cdot \bar{\mathcal{F}}^{(h)}\Big]. \tag{5.13}$$

The BRST and anti-BRST invariance of the above super Lagrangian density (and that of the ordinary 4D Lagrangian densities (4.7) and (4.8)) is encoded in the following simple equations that are expressed in terms of the translational generators along the Grassmannian directions of the (4, 2)-dimensional supermanifold, namely;

$$\mathrm{Lim}_{\theta \to 0} \frac{\partial}{\partial \bar{\theta}} \tilde{\mathcal{L}}_b^{(1,2)(n)} = 0 \Rightarrow s_b^{(n)} \mathcal{L}_b^{(1)(n)} = 0, \; \mathrm{Lim}_{\bar{\theta} \to 0} \frac{\partial}{\partial \theta} \tilde{\mathcal{L}}_b^{(1,2)(n)} = 0 \Rightarrow s_{ab}^{(n)} \mathcal{L}_b^{(2)(n)} = 0. \tag{5.14}$$

This is a tremendous simplification of the (anti-)BRST invariance of the Lagrangian densities (4.7) and (4.8) in the language of the superfield approach to BRST formalism. In other words, if one is able to show the Grassmannian independence of the super Lagrangian densities of the theory, the (anti-)BRST invariance of the 4D theory follows automatically.

In the language of the geometry on the supermanifold, the (anti-)BRST invariance of a 4D Lagrangian density is equivalent to the statement that the translation of the *super* version of the above Lagrangian density, along the Grassmannian directions of the (4, 2)-dimensional supermanifold, is *zero*. Thus, the super Lagrangian density of an (anti-)BRST invariant 4D theory is a Lorentz scalar, constructed with the help of (4, 2)-dimensional superfields (obtained after the application of HC), such that, when the partial derivatives w.r.t. the Grassmannian variables (θ and $\bar{\theta}$) operate on it, the result is zero.

The nilpotency and anticommutativity properties (that are associated with the conserved (anti-)BRST charges and (anti-)BRST symmetry transformations) are found to be captured very naturally (cf. (3.16)-(3.18)) when we consider the superfield formulation of the (anti-)BRST invariance of the Lagrangian density of a given 1-form gauge theory. We mention, in passing, that one could also derive the analogue of the equations (3.16), (3.17) and (3.18) for the 4D non-Abelian 1-form gauge theory in a straightforward manner.

6 Conclusions

In our present investigation, we have concentrated mainly on the (anti-)BRST invariance of the Lagrangian densities of the free 4D (non-)Abelian 1-form gauge theories (having no interaction with matter fields) within the framework of the superfield approach to BRST formalism. We have been able to provide the geometrical basis for the existence of the (anti-)BRST invariance in the above 4D theories. To be more specific, we have been able to show that the Grassmannian independence of the (4, 2)-dimensional super Lagrangian density, expressed in terms of the appropriate superfields, is a clear-cut proof that there is an (anti-)BRST invariance (cf. (3.16), (3.17), (3.18), (5.11), (5.14)) in the 4D theory.

If the super Lagrangian density could be expressed as a sum of (i) a Grassmannian independent term, and (ii) a derivative w.r.t. the Grassmannian variable, then, the corresponding 4D Lagrangian density will automatically respect BRST and/or anti-BRST invariance. In the latter piece of the above super Lagrangian density, the derivative could be *either* w.r.t. θ *or* w.r.t. $\bar{\theta}$ *or* w.r.t. both of them put *together*. More specifically, (i) if the derivative is w.r.t. $\bar{\theta}$, the nilpotent symmetry would correspond to the BRST,

(ii) if the derivative is w.r.t. θ, the nilpotent symmetry would be that of the anti-BRST type, and (iii) if both the derivatives are present together, both the nilpotent (anti-)BRST symmetries would be present together (and they would turn out to be anticommuting).

For the 4D (non-)Abelian 1-form gauge theories, that are considered on the (4, 2)-dimensional supermanifold, it is the HC on the 1-form super connection $\tilde{A}^{(1)}$ that plays a very important role in the derivation of the (anti-)BRST symmetry transformations. The cohomological origin for the above HC lies in the (super) exterior derivatives $(\tilde{d})d$. This point has been made quite clear in our discussions after the off-shell as well as the on-shell nilpotent (anti-)BRST symmetry transformations (2.2), (2.4), (4.2), (4.3), (4.9) and (4.10). In fact, it is the full kinetic energy term of the above theories (owing its origin to the cohomological operator $d = dx^\mu \partial_\mu$) that remains invariant under the above on-shell as well the off-shell nilpotent (anti-)BRST symmetry transformations.

The HC produces *specifically* the nilpotent (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields because of the fact that the super 1-form connection $\tilde{A}^{(1)}/\tilde{A}^{(1)(n)}$ (cf. (3.1) and (5.1)) is constructed with a super vector multiplet $(\mathcal{B}_\mu, \mathcal{F}, \bar{\mathcal{F}})$ which is the generalization of the gauge field A_μ and the (anti-)ghost fields $(\bar{C})C$ (of the ordinary 4D (non-)Abelian 1-form gauge theories) to the (4, 2)-dimensional supermanifold. As a consequence, *only* the nilpotent and anticommuting (anti-)BRST symmetry transformations for the 4D local fields A_μ, C and $\bar{C}$ are obtained when the full potential of the HC is exploited within the framework of the above superfield formulation.

It is worthwhile to point out that *geometrically* the super Lagrangian densities, expressed in terms of the (4, 2)-dimensional superfields, are equivalent to the sum of the kinetic energy term and the translations of some composite superfields (obtained after the application of the HC) along the Grassmannian directions (i.e. θ and/or $\bar{\theta}$) of the (4, 2)-dimensional supermanifold. This observation is distinctly different from our earlier works on the superfield approach to 2D (non-)Abelian 1-form gauge theories [24,25,23] which are found to correspond to the topological field theories. In fact, for the latter theories, the total super Lagrangian density turns out to be a total derivative w.r.t. the Grassmannian variables (θ and/or $\bar{\theta}$). That is to say, even the kinetic energy term of the latter theories, is able to be expressed as the total derivative w.r.t. the variables θ and/or $\bar{\theta}$.

In our present endeavour, within the framework of the superfield approach to BRST formalism, we have been able to provide (i) the logical reason behind the non-existence of the anti-BRST symmetry transformations for the Lagrangian densities (4.1) and (4.4) for the 4D non-Abelian 1-form gauge theory, (ii) the explicit explanation for the uniqueness of the equations (2.3) and (2.6) for the 4D Abelian 1-form gauge theory, (iii) the convincing proof for the on-shell nilpotent (anti-)BRST invariance of the gauge-fixing term (i.e. $\tilde{s}_{(a)b}(\partial_\mu A^\mu) = 0, \tilde{s}^{(n)}_{(a)b}(\partial_\mu A^\mu) = 0$) for the (non-)Abelian 1-form gauge theories, and (iv) the compelling arguments for the non-existence of the exact analogue(s) of (2.3) and (2.6) for the non-Abelian 1-form gauge theory. To the best of our knowledge, the logical explanations for the above subtle points (connected with the 1-form gauge theories) are completely new. Thus, the results of our present work are simple, beautiful and original.

It is worthwhile to mention that our superfield construction and its ensuing geometrical interpretations are not specific to the Feynman gauge (which has been taken into account in our present endeavor). To corroborate this assertion, we take the simple case of the 4D Abelian 1-form gauge theory and write the Lagrangian density (2.1) in the arbitrary gauge

$$\mathcal{L}_B^{(a,\xi)} = -\frac{1}{4}\, F^{\mu\nu} F_{\mu\nu} \; + \; B\, (\partial_\mu A^\mu) \; + \; \frac{\xi}{2}\, B^2 - i\, \partial_\mu \bar{C}\, \partial^\mu C, \tag{6.1}$$

where ξ is the gauge parameter. It is elementary to check that, in the limit $\xi \to 1$, we get back our Lagrangian density (2.1) for the Abelian theory in the Feynman gauge.

The analogue of the equation (2.3) (for the gauge-fixing and Faddeev-Popov ghost terms in the case of the arbitrary gauge) can be expressed as

$$\begin{array}{ll} s_b \Big[-i\, \bar{C}\, \{(\partial_\mu A^\mu) + \frac{\xi}{2}\, B\}\Big], & s_{ab} \Big[+i\, C\, \{(\partial_\mu A^\mu) + \frac{\xi}{2}\, B\}\, \Big], \\ s_b\, s_{ab}\, \Big[\, \frac{i}{2}\, A_\mu\, A^\mu + \frac{\xi}{2}\, C\, \bar{C}\, \Big]. & \end{array} \tag{6.2}$$

The above expression can be easily generalized to the analogues of the equations (3.10)—(3.12) in terms of the superfields by taking the help of (3.8). Thus, the geometrical interpretations remain intact even in the case of the arbitrary gauge.

In a similar fashion, for the 4D non-Abelian 1-form gauge theory, the equations (4.5), (4.6) and (4.11) can be generalized to the case of arbitrary gauge and, subsequently, can be expressed in terms of superfields as the analogues of (5.5), (5.9) and (5.12). Finally, we can obtain the analogues of (5.7), (5.10) and (5.13) which will lead to the derivation of the analogues of (5.11) and (5.14). Thus, we note that geometrical interpretations, in the arbitrary gauge, remain the same for the 4D (non-)Abelian 1-form gauge theory within the framework of our superfield approach to BRST formalism.

Our present work can be generalized to the case of the interacting 4D (non-)Abelian 1-form gauge theories where there exists an explicit coupling between the gauge field and the matter fields. In fact, our earlier works [14-18] might turn out to be quite handy in attempting the above problems. It seems to us that it is the combination of the HC and the restrictions, owing their origin to the (super) covariant derivative on the matter (super) fields and their intimate connection with the (super) curvatures, that would play a decisive role in proving the existence of the (anti-)BRST invariance for the above gauge theories.

It is gratifying to state that we have accomplished the above goals in our very recent endeavours [30-32]. In fact, we have been able to provide the geometrical basis for the existence of the (anti-)BRST invariance, in the context of the interacting (non-)Abelian 1-form gauge theories with Dirac as well as complex scalar fields, within the framework of the augmented superfield approach to BRST formalism. As it turns out, here too, the super Lagrangian density is found to be independent of the Grassmannian variables.

In our earlier works [33-35], we have been able to show the existence of the nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations for the 4D free Abelian 2-form gauge theory. We have also established the quasi-topological nature of it in [35]. In a recent work [36], the nilpotent (anti-)BRST symmetry transformations have been captured in the framework of the superfield formulation. It would be a very nice endeavour to study the (anti-)BRST and (anti-)co-BRST invariance of the 4D Abelian 2-form gauge theory within the framework of superfield formulation. At present, this issue and connected problems in the context of the 4D free Abelian 2-form gauge theory are under intensive investigation and our results would be reported in our forthcoming future publications [37].

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