Quantum Group of Isometries in Classical and Noncommutative Geometry

by

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Abstract

We formulate a quantum generalization of the notion of the group of Riemannian isometries for a compact Riemannian manifold, by introducing a natural notion of smooth and isometric action by a compact quantum group on a classical or noncommutative manifold described by spectral triples, and then proving the existence of a universal object (called the quantum isometry group) in the category of compact quantum groups acting smoothly and isometrically on a given (possibly noncommutative) manifold. Our formulation accommodates spectral triples which are not of type II. We give explicit description of quantum isometry groups of commutative and noncommutative tori, and in this context, obtain the quantum double torus defined in [6] as the universal quantum group of holomorphic isometries of the noncommutative torus.

1 Introduction

Since the formulation of quantum automorphism groups by Wang ([8], [9]), following suggestions of Alain Connes, many interesting examples of such quantum groups, particularly the quantum permutation groups of finite sets and finite graphs, have been extensively studied by a number of mathematicians (see, e.g. [1], [2], [10] and references therein), who have also found applications to and interaction with areas like free probability and subfactor theory. The underlying basic principle of defining a quantum automorphism group corresponding to some given mathematical structure (for example, a finite set, a graph, a C^* or von Neumann algebra) consists of two steps : first, to identify (if possible) the group of automorphisms of the structure as a universal object in a suitable category, and then, try to look for the universal object in a similar but bigger category by replacing groups by quantum groups of appropriate type. However, most of the work done so far concern some kind of quantum automorphism groups of a 'finite' structure, for example, of finite sets or finite dimensional matrix algebras. It is thus quite natural to try to extend these ideas to the 'infinite' or 'continuous' mathematical structures, for example classical and noncommutative manifolds. In the present article, we have made an attempt to formulate and study the quantum analogues of the groups of Riemannian isometries, which play a very important role in the classical differential geometry. The group of Riemannian isometries of a compact Riemannian manifold M can be viewed as the universal object in the category of all compact metrizable groups acting on M, with smooth and isometric action. Therefore, to define the quantum isometry group, it is reasonable to consider a category of compact quantum groups which act on the manifold (or more generally, on a noncommutative manifold given by spectral triple) in a 'nice' way, preserving the Riemannian structure in some suitable sense, to be precisely formulated. In this article, we have given a definition of such 'smooth and isometric' action by a compact quantum group on a (possibly noncommutative) manifold, extending the notion of smooth and isometric action by a group on a classical manifold. Indeed, the meaning of isometric action is nothing but that the action should commute with the 'Laplacian' coming from the spectral triple, and we should mention that this idea was already present in [2], though only in the context of a finite metric space or a finite graph. The universal object in the category of such quantum groups, if it exists, should be thought of as the quantum analogue of the group of isometries, and we have been able to prove its existence under some regularity assumptions, all of which can be verified for a general compact connected Riemannian manifold as well as the standard examples of noncommutative manifolds. We believe that a detailed study of quantum isometry groups will not only give many new and interesting examples of compact quantum groups, it will also contribute to the understanding of quantum group covariant spectral triples. In a forthcoming article [7] with J. Bhowmick; we shall provide explicit computations of quantum isometry groups of a few classical and noncommutative manifolds. However, we briefly quote some of main results of [7] in the present article. One interesting observation is that the quantum isometry group of the noncommutative two-torus $\mathcal{A}_\theta$ (with the canonical spectral triple) is (as a C^* algebra) a direct sum of two commutative and two noncommutative tori, and contains as a quantum subgroup (which is universal for certain class of isometric actions called holomorphic isometries) the 'quantum double-torus' discovered and studied by Hajac and Masuda ([6]).

2 Definition of the quantum isometry group

We begin with a well-known characterization of the isometry group of a (classical) compact Riemannian manifold. Let (M, g) be a compact Riemannian manifold and let $\Omega^1 = \Omega^1(M)$ be the space of smooth one-forms, which has a right Hilbert-$C^\infty(M)$-module structure given by the $C^\infty(M)$-valued inner product $<< \cdot, \cdot >>$ defined by

$$<< \omega, \eta >> (m) =< \omega(m), \eta(m) > |_m,$$

where $< \cdot, \cdot > |_m$ is the Riemannian metric on the cotangent space $T^*_m M$ at the point $m \in M$. The Riemannian volume form allows us to make Ω^1 a pre-Hilbert space, and we denote its completion by $\mathcal{H}_1$. Let $\mathcal{H}_0 = L^2(M, dvol)$ and consider the de-Rham differential d as an unbounded linear map from $\mathcal{H}_0$ to $\mathcal{H}_1$, with the natural domain $C^\infty(M) \subset \mathcal{H}_0$, and also denote its closure by d. Let $\mathcal{L} := -\frac{1}{2}d^*d$. The following identity can be verified by direct and easy computation using the local coordinates :

$$(\partial\mathcal{L})(\phi, \psi) \equiv \mathcal{L}(\bar{\phi}\psi) - \mathcal{L}(\bar{\phi})\psi - \bar{\phi}\mathcal{L}(\psi) =<< d\phi, d\psi >> \quad \text{for } \phi, \psi \in C^\infty(M) \quad (*).$$

Proposition 2.1 *A smooth map $\gamma : M \to M$ is a Riemannian isometry if and only if γ commutes with $\mathcal{L}$ in the sense that $\mathcal{L}(f \circ \gamma) = (\mathcal{L}(f)) \circ \gamma$ for all $f \in C^\infty(M)$.*

Proof :
If γ commutes with $\mathcal{L}$ then from the identity (*) we get for $m \in M$ and $\phi, \psi \in C^\infty(M)$:

$$\begin{aligned}
& < d\phi|_{\gamma(m)}, d\psi|_{\gamma(m)} > |_{\gamma(m)} \\
&= << d\phi, d\psi >> (\gamma(m)) \\
&= (\partial\mathcal{L}(\phi, \psi) \circ \gamma)(m) \\
&= \partial\mathcal{L}(\phi \circ \gamma, \psi \circ \gamma)(m) \\
&= << d(\phi \circ \gamma), d(\psi \circ \gamma) >> (m) \\
&= < d(\phi \circ \gamma)|_m, d(\psi \circ \gamma)|_m > |_m \\
&= < (d\gamma|_m)^*(d\phi|_{\gamma(m)}), (d\gamma|_m)^*(d\psi|_{\gamma(m)}) > |_m,
\end{aligned}$$

which proves that $(d\gamma|_m)^* : T^*_{\gamma(m)}M \to T^*_m M$ is an isometry. Thus, γ is a Riemannian isometry.

Conversely, if γ is an isometry, both the maps induced by γ on $\mathcal{H}_0$ and $\mathcal{H}_1$, i.e. $U^0_\gamma : \mathcal{H}_0 \to \mathcal{H}_0$ given by $U^0_\gamma(f) = f \circ \gamma$ and $U^1_\gamma : \mathcal{H}^1 \to \mathcal{H}^1$

given by $U^1_\gamma(fd\phi) = (f \circ \gamma)d(\phi \circ \gamma)$ are unitaries. Moreover, $dU^0_\gamma = U^1_\gamma d$ on $C^\infty(M) \subset \mathcal{H}_0$. From this, it follows that d^*d (and hence $\mathcal{L}$) commutes with U^0_γ. $\Box$

Now let us consider a compact metrizable (i.e. second countable) group G acting continuously on M and let $\Delta : C(M) \to C(M) \otimes C(G) \cong C(M \times G)$ be the map given by $\Delta(f)(m, g) := f(gm)$ for $g \in G$, $m \in M$ and $f \in C(M)$. For a state ϕ on $C(G)$, denote by Δ_ϕ the map $(\mathrm{id} \otimes \phi) \circ \Delta : C(M) \to C(M)$. Then we have the following

Theorem 2.2 *The G-action is smooth, i.e. $m \mapsto gm$ is C^∞ for every $g \in G$, if and only if $\Delta_\phi(C^\infty(M)) \subseteq C^\infty(M)$ for all ϕ. If the action is smooth, then it is also isometric (i.e. $m \mapsto gm$ is isometry $\forall g$) if and only if Δ_ϕ commutes with $(\mathcal{L} - \lambda)^{-1}$ for all state ϕ and all λ in the resolvent of $\mathcal{L}$ (equivalently, Δ_ϕ commutes with the heat semigroup $T_t \equiv e^{t\mathcal{L}}$ for all $t \geq 0$).*

Proof :
The 'if part' of (i) follows by considering the states corresponding to point evaluation, i.e. $C(G) \ni \xi \mapsto \xi(g)$, $g \in G$. For the converse, we note that an arbitrary state ϕ corresponds to a regular Borel measure μ on G so that $\phi(\xi) = \int \xi d\mu$, and thus, $\Delta_\phi(f)(m) = \int f(gm)d\mu(g)$ for $f \in C(M)$. From this, by interchanging differentiation and integation (which is allowed by the Dominated Convergence Theorem, since μ is a finite measure) we can prove that $\Delta_\phi(f)$ is C^∞ whenever f is so. The assertion (ii) follows from Proposition 2.1 in a straghtforward way. $\Box$

In view of the above result and the fact that the group of isometries of M, denoted by $ISO(M)$, is a compact second countable (i.e. compact metrizable) group, we see that $ISO(M)$ is the maximal compact second countable group acting on M such that the action is smooth and isometric. In other words, if we consider a catogory whose objects are compact metrizable groups acting smoothly and isometrically on M, and morphisms are the group homomorphisms commuting with the actions on M, then $ISO(M)$ (with its canonical action on M) is the initial object of this cateogory. It is now quite natural to formulate a quantum analogue of the above. In fact, we want to go beyond classical manifolds and define quantum isometry group $QISO(\mathcal{A}, \mathcal{H}, D)$ for a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we recall from [5] the construction of the space of one-forms. We have a derivation from $\mathcal{A}$ to the $\mathcal{A}$-$\mathcal{A}$ bimodule $\mathcal{B}(\mathcal{H})$ given by $a \mapsto [D, a]$. This induces a bimodule morphism π from $\Omega^1(\mathcal{A})$ (the bimodule of universal one-forms on $\mathcal{A}$) to $\mathcal{B}(\mathcal{H})$, such that

$\pi(\delta(a)) = [D, a]$, where $\delta : \mathcal{A} \to \Omega^1(\mathcal{A})$ denotes the universal derivation map. We set $\Omega^1_D \equiv \Omega^1_D(\mathcal{A}) := \Omega^1(\mathcal{A})/\mathrm{Ker}(\pi) \cong \pi(\Omega^1(\mathcal{A})) \subseteq \mathcal{B}(\mathcal{H})$. Assume that the spectral triple is of compact type and has a finite dimension in the sense of Connes ([4]), i.e. there is some $p > 0$ such that the operator $|D|^{-p}$ (interpreted as the inverse of the restriction of $|D|^p$ on the closure of its range, which has a finite co-dimension since D has compact resolvents) has finite nonzero Dixmier trace, denoted by Tr_ω (where ω is some suitable Banach limit, see, e.g. [4], [5]). Consider the canonical 'volume form' τ coming from the Dixmier trace, i.e. $\tau : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ defined by $\tau(A) := \frac{1}{Tr_\omega(|D|^{-p})} Tr_\omega(A|D|^{-p})$. This is a positive trace on $\mathcal{A}$, and the corresponding Hilbert space $L^2(\mathcal{A}, \tau)$ is denoted by $\mathcal{H}^0_D$. Similarly, we equip Ω^1_D with a semi-inner product given by $< \eta, \eta' > := \tau(\eta^* \eta')$, and denote the Hilbert space obtained from it by $\mathcal{H}^1_D$. The map $d_D : \mathcal{H}^0_D \to \mathcal{H}^1_D$ given by $d_D(\cdot) = [D, \cdot]$ is an unbounded linear map, which can be seen to be closable by writing down a densely defined adjoint d^*_D. We take $\mathcal{L} = -\frac{1}{2} d^*_D d_D$, $T_t := \exp(t\mathcal{L})$, and make the following assumptions:

Assumptions :
(i) $\mathcal{L}$ has compact resolvents,
(ii) $\mathcal{A} \subseteq \mathrm{Dom}(\mathcal{L})$ (where $\mathcal{A}$ is viewed as a dense subspace of $\mathcal{H}^0_D$)
(iii) $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$,
(iv) $T_t(\mathcal{H}^0_D) \subseteq \mathcal{A}$ for all $t > 0$,
(v) ('connectedness') the kernel of $\mathcal{L}$ is one-dimensional, spanned by the identity 1 of $\mathcal{A}$, viewed as a unit vector in $\mathcal{H}^0_D$.
We call $\mathcal{L}$ the noncommutative Laplacian and T_t the noncommutative heat semigroup. We summarise some simple observations in form of the following

Lemma 2.3 *(a) For $x \in \mathcal{A}$, we have $\mathcal{L}(x^*) = (\mathcal{L}(x))^*$.*
(b) Each eigenvector of $\mathcal{L}$ (which has a discrete spectrum, hence a complete set of eigenvectors) belongs to $\mathcal{A}$.

Proof :
It follows by simple calculation using the facts that τ is a trace and $d_D(x^*) = -(d_D(x))^*$ that $\tau(\mathcal{L}(x^*)^* y) = -\tau(d_D(x) d_D(y)) = \tau(\mathcal{L}(x) y)$ for all $y \in \mathcal{A}$. By density of $\mathcal{A}$ in $\mathcal{H}^0_D$ (a) follows. To prove (b), we note that if $x \in \mathcal{H}^0_D$ is an eigenvector of $\mathcal{L}$, say $\mathcal{L}(x) = \lambda x$ ($\lambda \in \mathbb{C}$), then we have $T_t(x) = e^{\lambda t} x$, hence $x = e^{-\lambda t} T_t(x) \in \mathcal{A}$ by assumption (iv). $\Box$

Since by assumption, $\mathcal{L}$ has a countable set of eigenvalues each with finite multiplicity, let us denote them by $\lambda_0 = 0, \lambda_1, \lambda_2, ...$ with $V_0 = \mathbb{C}\,1, V_1, V_2, ...$

be corresponding eigenspaces (finite dimensional), and for each i, let $\{e_{ij}, j = 1, ..., d_i\}$ be an orthonormal basis of V_i. By Lemma 2.3, $V_i \subseteq \mathcal{A}$ for each i, V_i is closed under $*$, and moreover, $\{e_{ij}^*, j = 1, ..., d_i\}$ is also an orthonormal basis for V_i, since $\tau(x^*y) = \tau(yx^*)$ for $x, y \in \mathcal{A}$. We also make the following

Assumption (vi) The complex linear span of $\{e_{ij}, i = 0, 1, ...; j = 1, ..., d_i\}$, say $\mathcal{A}_0$, is norm-dense in $\mathcal{A}$.

Lemma 2.4 *If $\Psi : \mathcal{A} \to \mathcal{A}$ is a (norm-) bounded linear map, such that $\Psi(1) = 1$, and $\Psi \circ \mathcal{L} = \mathcal{L} \circ \Psi$ on the subspace $\mathcal{A}_0$ spanned (algebraically) by $V_i, i = 1, 2, ...,$ then $\tau(\Psi(x)) = \tau(x)$ for all $x \in \mathcal{A}$.*

Proof :
Since Ψ commutes with $\mathcal{L}$ it is clear that Ψ maps each V_i into itself. By assumption (v), V_0 is spanned by 1, so Ψ maps $V_0^\perp \bigcap \mathcal{A}_0 = \text{span}\{V_i, i \geq 1\}$ into itself. But $V_0^\perp \bigcap \mathcal{A}_0 = \{x \in \mathcal{A}_0 \ : \ < 1, x > \equiv \tau(x) = 0\} = \text{Ker}(\tau) \bigcap \mathcal{A}_0$. Now, for $x \in \mathcal{A}_0$, we have $y = x - \tau(x)1 \in \text{Ker}(\tau) \bigcap \mathcal{A}_0$, so $\Psi(y) = \Psi(x) - \tau(x)1$ will belong to $\text{Ker}(\tau)$, hence $\tau(\Psi(x)) = \tau(x)$ for all $x \in \mathcal{A}_0$. By the norm-continuity of Ψ and τ it extends to the whole of $\mathcal{A}$. $\Box$

We now introduce the quantum analogue of a smooth isometric action on the noncommutative manifold $\mathcal{A}$. We recall from [11] that a compact quantum group is a unital separable C^* algebra $\mathcal{A}$ equipped with a unital C^*-homomorphism $\Delta : \mathcal{S} \to \mathcal{S} \otimes \mathcal{S}$ (where $\otimes$ denotes the spatial tensor product) satisfying
(ai) $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ (co-associativity), and
(aii) the linear span of $\Delta(\mathcal{S})(\mathcal{S} \otimes 1)$ and $\Delta(\mathcal{S})(1 \otimes \mathcal{S})$ are norm-dense in $\mathcal{S} \otimes \mathcal{S}$.
It is well-known (see [11]) that there is a canonical dense $*$-subalgebra $\mathcal{S}^\infty$ of $\mathcal{S}$, consisting of the matrix coefficients of the finite dimensional unitary (co)-representations of $\mathcal{S}$, and maps $\epsilon : \mathcal{S}^\infty \to \mathbb{C}$ (co-unit) and $\kappa : \mathcal{S}^\infty \to \mathcal{S}^\infty$ (antipode) defined on $\mathcal{S}^\infty$ which make $\mathcal{S}^\infty$ a Hopf $*$-algebra.

We say that the compact quantum group $(\mathcal{S}, \Delta)$ (co)-acts on a unital C^* algebra $\mathcal{B}$, if there is a unital C^*-homomorphism $\alpha : \mathcal{B} \to \mathcal{B} \otimes \mathcal{S}$ satisfying the following :
(bi) $(\alpha \otimes id) \circ \alpha = (id \otimes \Delta) \circ \alpha$, and
(bii) the linear span of $\alpha(\mathcal{B})(1 \otimes \mathcal{S})$ is norm-dense in $\mathcal{B} \otimes \mathcal{S}$.
It can be proved (see [9]) that the condition (bii) is equivalent to the following :

(bii$'$) there exists of a dense unital $*$-subalgebra $\mathcal{B}^\infty$ of $\mathcal{B}$ such that the action α on $\mathcal{B}^\infty$ is algebraic in the sense that $\alpha(\mathcal{B}^\infty) \subseteq \mathcal{B}^\infty \otimes_{\rm alg} \mathcal{S}^\infty$, and
(bii$''$) $(id \otimes \epsilon)\alpha = id$ on $\mathcal{B}^\infty$.

We now formulate the notion of a smooth and isometric action of a compact quantum group on a noncommutative manifold, clearly motivated by the classical situation which we already discussed.

Definition 2.5 *A compact quantum group $(\mathcal{S}, \Delta)$ is said to act on the noncommutative manifold $\mathcal{A}$ (or, more precisely on the corresponding spectral triple) smoothly and isometrically if there is a C^*-action $\alpha : \overline{\mathcal{A}} \to \overline{\mathcal{A}} \otimes \mathcal{S}$ (where $\overline{\mathcal{A}}$ denotes the C^* algebra obtained by completing $\mathcal{A}$ in the norm of $\mathcal{B}(\mathcal{H}_D^0)$), such that $\alpha_\phi := (id \otimes \phi) \circ \alpha$ maps $\mathcal{A}$ into itself and commutes with $\mathcal{L}$ on $\mathcal{A}$, for every state ϕ on $\mathcal{S}$.*

Let us now recall the concept of universal quantum groups as in [10], [8] and references therein. We shall use most of the terminologies of [8], e.g. Woronowicz C^* -subalgebra, Woronowicz C^*-ideal etc, however with the exception that we shall call the Woronowicz C^* algebras just compact quantum groups, and not use the term compact quantum groups for the dual objects as done in [8]. For $Q \in GL_n(\mathbb{C})$, let $A_u(Q)$ denote the universal compact quantum group generated by $u_{ij}, i, j = 1, ..., n$ satisfying the relations

$$uu^* = I_n = u^*u, \quad u'Q\overline{u}Q^{-1} = I_n = Q\overline{u}Q^{-1}u',$$

where $u = ((u_{ij}))$, $u' = ((u_{ji}))$ and $\overline{u} = ((u_{ij}^*))$. We refer the reader to [10] for the definition of the coproduct and discussion on the structure and classification of such quantum groups. Let us denote by $\mathcal{U}_i$ the quantum group $A_{d_i}(I)$, where d_i is dimension of the subspace V_i. We fix a representation $\beta_i : V_i \to V_i \otimes \mathcal{U}_i$ of $\mathcal{U}_i$ on the Hilbert space V_i, given by $\beta_i(e_{ij}) = \sum_k e_{ik} \otimes u_{kj}^{(i)}$, for $j = 1, ..., d_i$, where $u_{kj}^{(i)}$ are the generators of $\mathcal{U}_i$ as discussed before. It follows from [8] that the representations β_i canonically induce a representation $\beta = *_i\beta_i$ of the free product $\mathcal{U} := *_i\mathcal{U}_i$ (which is a compact quantum group, see [8] for the details) on the Hilbert space $\mathcal{H}_D^0$, such that the restriction of β on V_i coincides with β_i for all i. We are now ready to state and prove a key lemma.

Lemma 2.6 *Let $(\mathcal{S}, \Delta)$ be a compact quantum group acting on $\mathcal{A}$ smoothly and isometrically. Moreover, assume that the action (say α) is faithful in the sense that there is no proper Woronowicz C^*-subalgebra $\mathcal{S}_1$ of $\mathcal{S}$ such that*

$\alpha(\mathcal{A}) \subseteq \mathcal{A} \otimes \mathcal{S}_1$. *Then* $\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{S}$ *extends to a unitary representation (denoted again by* α*) of* $\mathcal{S}$ *on* $\mathcal{H}_D^0$*. Moreover, we can find an isomorphism (of compact quantum groups)* $\phi : \mathcal{U}/\mathcal{I} \to \mathcal{S}$ *between* $\mathcal{S}$ *and a quotient of* $\mathcal{U}$ *by a Woronowicz* C^**-ideal* $\mathcal{I}$ *of* $\mathcal{U}$*, such that* $\alpha = \phi \circ \Pi_{\mathcal{I}} \circ \beta$ *on* $\mathcal{A} \subseteq \mathcal{H}_D^0$*, where* $\Pi_{\mathcal{I}}$ *denotes the quotient map from* $\mathcal{U}$ *to* $\mathcal{U}/\mathcal{I}$*.*

Proof :
Let ω be any state on $\mathcal{S}$. Since the action $\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{S}$ is smooth and isometric, we conclude by Lemma 2.4 that $\tau(\alpha_\omega(x)) = \tau(x)\omega(1)$ for all $x \in \overline{\mathcal{A}}$. Since ω is arbitrary, we have $(\tau \otimes id)\alpha(x) = \tau(x)1_{\mathcal{S}}$ for all $x \in \overline{\mathcal{A}}$. So, $< \alpha(x), \alpha(y) >=< x, y > 1_{\mathcal{S}}$, which proves that $\tilde{\alpha}$ defined by $\tilde{\alpha}(x \otimes b) := \alpha(x)(1 \otimes b)$ $(x \in \mathcal{A}, b \in \mathcal{S})$ extends to an $\mathcal{S}$-linear isometry on the Hilbert $\mathcal{S}$-module $\mathcal{H}_D^0 \otimes \mathcal{S}$. Moreover, since $\alpha(\mathcal{A})(1 \otimes \mathcal{S})$ is norm-dense in $\bar{\mathcal{A}} \otimes \mathcal{S}$, it is clear that the $\mathcal{S}$-linear span of the range of $\alpha(\mathcal{A})$ is dense in the Hilbert module $\mathcal{H}_D^0 \otimes \mathcal{S}$, or in other words, the isometry $\tilde{\alpha}$ has a dense range, so it is a unitary. Since α_ω leaves each V_i invariant, it is clear that α maps V_i into $V_i \otimes \mathcal{S}$ for each i. Let $v_{kj}^{(i)}$ $(j, k = 1, ..., d_i)$ be the elements of $\mathcal{S}$ such that $\alpha(e_{ij}) = \sum_k e_{ik} \otimes v_{kj}^{(i)}$. The algebra generated by $v_{kj}^{(i)}$ is a Hopf algebra, and $v_i := ((v_{kj}^{(i)}))$ is a unitary in $M_{d_i}(\mathbb{C}) \otimes \mathcal{S}$. Moreover, the $*$-subalgebra generated by all $v_{kj}^{(i)}$'s , with i, j, k varying, must be dense in $\mathcal{S}$ by the assumption of faithfulness.

Now, we have already remarked that $\{e_{ij}^*\}$ is also an orthonormal basis of V_i, and since α, being a C^*-action on $\overline{\mathcal{A}}$, is $*$-preserving, we have $\alpha(e_{ij}^*) = (\alpha(e_{ij}))^* = \sum_k e_{ik}^* \otimes v_{kj}^{(i)^*}$, and therefore $((v_{kj}^{(i)^*}))$ is also unitary. By universality of $\mathcal{U}_i$, there is a C^*-homomorphism from $\mathcal{U}_i$ to $\mathcal{S}$ sending $u_{kj}^{(i)}$ to $v_{kj}^{(i)}$, and by definition of the free product, this induces a C^*-homomorphism, say Π, from $\mathcal{U}$ onto $\mathcal{S}$, which is easily seen to be a surjective morphism of compact quantum groups. Thus, we obtain the Woronowicz C^*-ideal $\mathcal{I} := \mathrm{Ker}(\Pi)$, so that $\mathcal{U}/\mathcal{I} \cong \mathcal{S}$. $\Box$

Now we can give the definition of the quantum group $QISO(\mathcal{A}, \mathcal{H}, D)$. Let us consider a category whose objects are the pairs $(\mathcal{S}, \alpha)$ where $\mathcal{S}$ is a compact quantum group and α is a smooth isometric (not necessarily faithful) action of $\mathcal{S}$ on $\mathcal{A}$. The set of morphisms from $(\mathcal{S}, \alpha)$ to $(\mathcal{S}', \alpha')$ are the morphisms $\psi : \mathcal{S} \to \mathcal{S}'$ of compact quantum groups, satisfying $\psi \circ \alpha = \alpha'$. We prove below that this category has a (unique upto isomorhism) universal (initial) object. We shall need an elementary fact, which is stated as a lemma.

Lemma 2.7 *Let $\mathcal{C}$ be a C^* algebra and $\mathcal{F}$ be a collection of closed ideals of $\mathcal{C}$. Then for any $x \in \mathcal{C}$, we have*

$$\sup_{I \in \mathcal{F}} \|x + I\| = \|x + I_0\|,$$

where I_0 denotes the intersection of all I in $\mathcal{F}$ and $\|x + I\| = inf\{\|x - y\| \ : \ y \in I\}$ denotes the norm in $\mathcal{C}/I$.

Proof :
It is clear that $\sup_{I \in \mathcal{F}} \|x + I\|$ defines a norm on $\mathcal{C}/I_0$, which is in fact a C^*-norm since each of the quotient norms $\| \cdot + I\|$ is so. Thus the lemma follows from the uniqueness of C^* norm on the C^* algebra $\mathcal{C}/I_0$. $\square$

Theorem 2.8 *There exists a (unique upto isomorphism) compact quantum group $(\mathcal{S}_0, \Delta_0)$ which is universal in the category $\mathcal{C}$ of all compact quantum groups acting smoothly isometrically on $\mathcal{A}$.*

Proof :
Recall the quantum group $\mathcal{U}$ considered before, and its unitary representation β on $\mathcal{A} \subseteq \mathcal{H}^0_D$. By our definition of β, it is clear that $\beta(\mathcal{A}_0) \subseteq \mathcal{A}_0 \otimes_{\rm alg} \mathcal{U}$. However, β is only a linear map (unitary) but not necessarily a $*$-homomorphism. We shall construct the universal object as a suitable quotient of $\mathcal{U}$. Let $\mathcal{F}$ be the collection of all those Woronowicz C^*-ideals $\mathcal{I}$ of $\mathcal{U}$ such that the composition $\Gamma_{\mathcal{I}} := (id \otimes \Pi_{\mathcal{I}}) \circ \beta : \mathcal{A}_0 \to \mathcal{A}_0 \otimes_{\rm alg} (\mathcal{U}/\mathcal{I})$ extends to a C^*-homomorphsim from $\bar{\mathcal{A}}$ to $\bar{\mathcal{A}} \otimes (\mathcal{S}/\mathcal{I})$, where $\Pi_{\mathcal{I}}$ denotes the quotient map from $\mathcal{U}$ onto $\mathcal{U}/\mathcal{I}$. This collection is nonempty, since the trivial group, viewed as a quantum group, acts faithfully, smoothly and isometrically on $\mathcal{A}$, and by Lemma 2.6 we do get a member of $\mathcal{F}$. Now, let $\mathcal{I}_0$ be the intersection of all ideals in $\mathcal{F}$. We claim that $\mathcal{I}_0$ is again a member of $\mathcal{F}$. Since any C^*-homomorphism is contractive, we have $\|\Gamma_{\mathcal{I}}(a)\| \equiv \|\beta(a) + \bar{\mathcal{A}} \otimes \mathcal{I}\| \leq \|a\|$ for all $a \in \mathcal{A}_0$ and $\mathcal{I} \in \mathcal{F}$. By Lemma 2.7, we see that $\|\Gamma_{\mathcal{I}_0}(a)\| \leq \|a\|$ for $a \in \mathcal{A}_0$, so $\Gamma_{\mathcal{I}_0}$ extends to a norm-contractive map on $\bar{\mathcal{A}}$ by the density of $\mathcal{A}_0$ in $\bar{\mathcal{A}}$. Moreover, For $a, b \in \bar{\mathcal{A}}$ and for $\mathcal{I} \in \mathcal{F}$, we have $\Gamma_{\mathcal{I}}(ab) = \Gamma_{\mathcal{I}}(a)\Gamma_{\mathcal{I}}(b)$. Since $\Pi_{\mathcal{I}} = \Pi_{\mathcal{I}} \circ \Pi_{\mathcal{I}_0}$, we can rewrite the homomorphic property of $\Gamma_{\mathcal{I}}$ as

$$\Gamma_{\mathcal{I}_0}(ab) - \Gamma_{\mathcal{I}_0}(a)\Gamma_{\mathcal{I}_0}(b) \in \bar{\mathcal{A}} \otimes (\mathcal{I}/\mathcal{I}_0).$$

Since this holds for every $\mathcal{I} \in \mathcal{F}$, we conclude that $\Gamma_{\mathcal{I}_0}(ab) - \Gamma_{\mathcal{I}_0}(a)\Gamma_{\mathcal{I}_0}(b) \in \bigcap_{\mathcal{I} \in \mathcal{F}} \bar{\mathcal{A}} \otimes (\mathcal{I}/\mathcal{I}_0) = (0)$, i.e. $\Gamma_{\mathcal{I}_0}$ is a homomorphism. In a similar way, we can show that it is a $*$-homomorphism. To see that $\Gamma_{\mathcal{I}_0}$ is indeed an action of the compact quantum group $\mathcal{U}/\mathcal{I}_0$ on $\bar{\mathcal{A}}$, we observe that the conditions bii$'$

and bii$''$ are satisfied by taking for $\mathcal{B}^\infty$ the $*$-subalgebra generated by $\mathcal{A}_0$, since $\Pi_{\mathcal{I}_0}(\mathcal{U}^\infty) \subseteq (\mathcal{U}/\mathcal{I}_0)^\infty$. Indeed, the counit of $\mathcal{U}/\mathcal{I}_0$, say ϵ_0, is nothing but the composite map $\epsilon \circ \Pi_{\mathcal{I}_0}$ where ϵ denotes the counit of $\mathcal{U}$, defined by $\epsilon(u^{(i)}_{kj}) = \delta_{kj}$ (δ_{kj} denotes the Kronecker delta), and it is clear from the construction of β that $(id \otimes \epsilon)(\beta(a)) = a$ for $a \in \mathcal{A}_0$. It follows from this that $(id \otimes \epsilon_0) \circ \Gamma_{\mathcal{I}_0} = id$ on $\mathcal{A}_0$, hence on the $*$-algebra generated by $\mathcal{A}_0$.

Finally, we claim that $\mathcal{S}_0 := \mathcal{U}/\mathcal{I}_0$ is the desired universal object. To see this, consider any compact quantum group $\mathcal{S}$ acting smoothly and isometrically on $\mathcal{A}$. Without loss of generality we can assume the action to be faithful, since otherwise we can replace $\mathcal{S}$ by the Woronowicz C^*-subalgebra generated by the matrix elements of the action on $\mathcal{A}$. But by Lemma 2.6 we can further assume that $\mathcal{S}$ is isomorphic with $\mathcal{U}/\mathcal{I}$ for some $\mathcal{I} \in \mathcal{F}$. Since $\mathcal{I}_0 \subseteq \mathcal{I}$, we have a natural morphism of quantum groups from $\mathcal{U}/\mathcal{I}_0$ onto $\mathcal{U}/\mathcal{I}$, sending $x + \mathcal{I}_0$ to $x + \mathcal{I}$. $\Box$

Definition 2.9 *We shall call the universal object* $(\mathcal{S}_0, \Delta_0)$ *obtained in the theorem above the quantum isometry group of* $(\mathcal{A}, \mathcal{H}, D)$ *and denote it by* $QISO(\mathcal{A}, \mathcal{H}, D)$*, or just* $QISO(\mathcal{A})$ *(or sometimes* $QISO(\bar{\mathcal{A}})$*) if the spectral triple is understood from the context.*

Remark 2.10 *It is easy to see how to extend our formulation and results to spectral triples which are not necessarily of type II, i.e. when the trace* τ *is replaced by some non-tracial positive functional. Indeed, our construction will go through in such a situation more or less verbatim, by replacing the universal quantum groups* $A_{d_i}(I)$ *by* $A_{d_i}(Q_i)$ *for some suitable choice of matrices* Q_i *coming from the modularity property of* τ*.*

3 Examples and computations

We give some simple yet interesting explicit examples of quantum isometry groups here. However, we give only some computational details for the first example, and for the rest, the reader is referred to a forthcoming article ([7]).

Example 1 : commutative tori
Consider $M = \mathbb{T}$, the one-torus, with the usual Riemannian structure. The $*$-algebra $\mathcal{A} = C^\infty(M)$ is generated by one unitary U, which is the multiplication operator by z in $L^2(\mathbb{T})$. The Laplacian is given by $\mathcal{L}(U^n) = -\frac{1}{2}n^2 U^n$. If a compact quantum group $(\mathcal{S}, \Delta_{\mathcal{S}})$ acts on $\mathcal{A}$ smoothly, let $A_n, n \in \mathbb{Z}$ be

elements of $\mathcal{S}$ such that $\Delta(U) = \sum_n U^n \otimes A_n$ (here $\Delta : \mathcal{A} \to \mathcal{A} \otimes_{\rm alg} \mathcal{S}$ is the $\mathcal{S}$-action on $\mathcal{A}$). Note that this infinite sum converges at least in the topology of the Hilbert space $L^2(\mathbb{T}) \otimes L^2(\mathcal{S})$, where $L^2(\mathcal{S})$ denotes the GNS space for the Haar state of $\mathcal{S}$. It is clear that the condition $(\mathcal{L} \otimes id) \circ \Delta = \Delta \circ \mathcal{L}$ forces to have $A_n = 0$ for all but $n = \pm 1$. The conditions $\Delta(U)\Delta(U)^* = \Delta(U)^*\Delta(U) = 1 \otimes 1$ further imply the following:

$$A_1^* A_1 + A_{-1}^* A_{-1} = 1 = A_1 A_1^* + A_{-1} A_{-1}^*,$$

$$A_1^* A_{-1} = A_{-1}^* A_1 = A_1 A_{-1}^* = A_{-1} A_1^* = 0.$$

It follows that $A_{\pm 1}$ are partial isometries with orthogonal domains and ranges. Say, A_1 has domain P and range Q. Hence the domain and range of A_{-1} are respectively $1 - P$ and $1 - Q$. Consider the unitary $V = A + B$, so that $VP = A$, $V(1 - P) = B$. Now, from the fact that $(\mathcal{L} \otimes id)(\Delta(U^2)) = \Delta(\mathcal{L}(U^2))$ it is easy to see that the coefficient of $1 \otimes 1$ in the expression of $\Delta(U)^2$ must be 0, i.e. $AB + BA = 0$. From this, it follows that V and P commute and therefore $P = Q$. By straightforward calculation using the facts that V is unitary, P is a projection and V and P commute, we can verify that Δ given by $\Delta(U) = U \otimes VP + U^{-1} \otimes V(1 - P)$ extends to a $*$-homomorphsim from $\mathcal{A}$ to $\mathcal{A} \otimes C^*(V, P)$ satisfying $(\mathcal{L} \otimes id) \circ \Delta = \Delta \circ \mathcal{L}$. It follows that the C^* algebra $QISO(\mathbb{T})$ is commutative and generated by a unitary V and a projection P, or equivalently by two partial isometries A, B such that $A^*A = AA^*, B^*B = BB^*, AB = BA = 0$. So, as a C^* algebra it is isomorphic with $C(\mathbb{T}) \oplus C(\mathbb{T}) \cong C(\mathbb{T} \times \mathbb{Z}_2)$. The coproduct (say Δ_0) can easily be calculated from the requirement of co-associativity, and the Hopf algebra structure of $QISO(\mathbb{T})$ can be seen to coincide with that of the semi-direct product of $\mathbb{T}$ by $\mathbb{Z}_2$, where the generator of $\mathbb{Z}_2$ acts on $\mathbb{T}$ by sending $z \mapsto \bar{z}$.

We summarize this in form of the following.

Theorem 3.1 *The universal quantum group of isometries $QISO(\mathbb{T})$ of the one-torus $\mathbb{T}$ is isomorphic (as a quantum group) with $C(\mathbb{T} \rtimes \mathbb{Z}_2) = C(ISO(\mathbb{T}))$.*

We can easily extend this result to higher dimensional commutative tori, and can prove that the quantum isometry group coincides with the classical isometry group. This is some kind of rigidity result, and it will be interesting to investigate the nature of quantum isometry groups of more general classical manifolds.

Example 2 : Noncommutative torus

Next we consider the simplest and well-known example of noncommutative manifold, namely the noncommutative two-torus $\mathcal{A}_\theta$, where θ is a fixed irrational number (see [4]). It is the universal C^* algebra generated by two unitaries U and V satisfying the commutation relation $UV = \lambda VU$, where $\lambda = e^{2\pi i\theta}$. There is a canonical faithful trace τ on $\mathcal{A}_\theta$ given by $\tau(U^m V^n) = \delta_{mn}$. We consider the canonical spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is the unital $*$-algebra spanned by U, V, $\mathcal{H} = L^2(\tau) \oplus L^2(\tau)$ and D is given by

$$D = \begin{pmatrix} 0 & d_1 + id_2 \\ d_1 - id_2 & 0 \end{pmatrix},$$

where d_1 and d_2 are closed unbounded linear maps on $L^2(\tau)$ given by $d_1(U^m V^n) = mU^m V^n$, $d_2(U^m V^n) = nU^m V^n$. It is easy to compute the space of one-forms Ω^1_D (see [3], [5], [4]) and the Laplacian $\mathcal{L} = -\frac{1}{2}d^* d$ is given by $\mathcal{L}(U^m V^n) = -\frac{1}{2}(m^2 + n^2)U^m V^n$. For simplicity of computation, instead of the full quantum isometry group we at first concentrate on an interesting quantum subgroup $\mathcal{G} = QISO^{\text{hol}}(\mathcal{A}, \mathcal{H}, D)$, which is the universal quantum group which leaves invariant the subalgebra of $\mathcal{A}$ consisting of polynomials in U, V and 1, i.e. span of $U^m V^n$ with $m, n \geq 0$. The proof of existence and uniqueness of such a universal quantum group is more or less identical to the proof of existence and uniqueness of QISO. We call $\mathcal{G}$ the quantum group of "holomorphic" isometries, and observe in the theorem stated below without proof (see [7]) that this quantum group is nothing but the quantum double torus studied in [6].

Theorem 3.2 *Consider the following co-product $\Delta_{\mathcal{B}}$ on the C^* algebra $\mathcal{B} = C(\mathbb{T}^2) \oplus \mathcal{A}_{2\theta}$, given on the generators A_0, B_0, C_0, D_0 as follows (where A_0, D_0 correspond to $C(\mathbb{T}^2)$ and B_0, C_0 correspond to $\mathcal{A}_{2\theta}$)*

$$\Delta_{\mathcal{B}}(A_0) = A_0 \otimes A_0 + C_0 \otimes B_0, \quad \Delta_{\mathcal{B}}(B_0) = B_0 \otimes A_0 + D_0 \otimes B_0,$$

$$\Delta_{\mathcal{B}}(C_0) = A_0 \otimes C_0 + C_0 \otimes D_0, \quad \Delta_{\mathcal{B}}(D_0) = B_0 \otimes C_0 + D_0 \otimes D_0.$$

Then $(\mathcal{B}, \Delta_0)$ is a compact quantum group and it has an action Δ_0 on $\mathcal{A}_\theta$ given by

$$\Delta_0(U) = U \otimes A_0 + V \otimes B_0, \quad \Delta_0(V) = U \otimes C_0 + V \otimes D_0.$$

Moreover, $(\mathcal{B}, \Delta_{\mathcal{B}})$ is isomorphic (as quantum group) with $\mathcal{G} = QISO^{\text{hol}}(\mathcal{A}, \mathcal{H}, D)$.

We refer to [7] for a proof of the above result, and to [6] for the computation of the Haar stat and representation theory of the compact quantum group $\mathcal{G}$.

By similar but somewhat tedious calculations (see [7]) one can also describe explicitly the full quantum isometry group $QISO(\mathcal{A}, \mathcal{H}, D)$. It is as a C^* algebra has four direct summands, two of which are isomorphic with the commutative algebra $C(\mathbb{T}^2)$, and the other two are irrational rotation algebras. For the description of the coproduct, counit, antipode and detailed study of the representation theory, the reader is again referred to [7]. It is interesting to mention here that the quantum isometry group of $\mathcal{A}_\theta$ is a Rieffel type deformation of the isometry group (which is same as the quantum ismetry group) of the commutative two-torus. The commutative two-torus is a subgroup of its isometry group, but when the isometry group is deformed into $QISO(\mathcal{A}_\theta)$, the subgroup relation is not respected, and the deformation of the commutative torus, which is $\mathcal{A}_{2\theta}$, sits in $QISO(\mathcal{A}_\theta)$ just as a C^* subalgebra (in fact a direct summand) but not as a quantum subgroup any more. This perhaps provides some explanation of the non-existence of any Hopf algebra structure on the noncommutative torus.

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