

Entanglement Entropy of two-dimensional anti-de Sitter black holes

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Using the AdS/CFT correspondence we derive a formula for the entanglement entropy of the anti-de Sitter black hole in two spacetime dimensions. The leading term in the large black hole mass expansion of our formula reproduces exactly the Bekenstein-Hawking entropy S_{BH} , whereas the subleading term behaves as $\ln S_{BH}$. This subleading term has the universal form typical for the entanglement entropy of physical systems described by effective conformal fields theories (e.g. one-dimensional statistical models near to the critical point). The well-known form of the entanglement entropy for a two-dimensional conformal field theory is obtained as analytic continuation of our result and is related with the entanglement entropy of a black hole with negative mass.

Quantum entanglement is a fundamental feature of quantum systems. It is related to the existence of correlations between parts of the system. The degree of entanglement of a quantum system is measured by the entanglement entropy S_{ent} . In quantum field theory (QFT), or more in general in many body systems, we can localize observable and unobservable degrees of freedom in spatially separated regions Q and R . S_{ent} is then defined as the von Neumann entropy of the system when the degrees of freedom in the region R are traced over, $S_{ent} = -\text{Tr}_Q \hat{\rho}_Q \ln \hat{\rho}_Q$, where the trace is taken over states in the observable region Q and the reduced density matrix $\hat{\rho}_Q = \text{Tr}_R \hat{\rho}$ is obtained by tracing the density matrix $\hat{\rho}$ over states in the region R .

Investigation of the entanglement entropy (EE) has become relevant in many research areas. Apart from quantum information theory, the field that gave birth to the notion of entanglement entropy, it plays a crucial role in condensed matter systems, where it helps to understand quantum phases of matter (e.g spin chains and quantum liquids)[1, 2, 3, 4, 5]. Entanglement (geometric) entropy is also an useful concept for investigating general features of QFT, in particular two-dimensional conformal field theory (CFT) and the Anti-de Sitter/conformal field theory (AdS/CFT) correspondence [6, 7, 8, 9, 10, 11, 12]. Last but not least entanglement may held the key for unraveling the mystery of black hole entropy [13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

We will be mainly concerned with the entanglement entropy of two-dimensional (2D) CFT and its relationship with the entropy of 2D black holes. It is an old idea that black hole entropy may be explained in terms of the EE of the quantum state of matter fields in the black hole geometry [13]. The main support to this conjecture comes from the fact that both the EE of matter fields and the Bekenstein-Hawking (BH) entropy depend on the area of the boundary region. On the other hand any attempt to explain the BH entropy as originating from quantum entanglement has to solve conceptual and technical difficulties.

The usual statistical paradigm explains the BH entropy in terms of a microstate gas. This is conceptually different from the EE that measures the observer's lack of information about the quantum state of the system in an inaccessible region of spacetime. Moreover, the EE depends both on the number of species n_s of the matter fields, whose entanglement should reproduce the BH entropy, and on the value of the UV cutoff δ arising owing to the presence of a sharp boundary between the accessible and inaccessible regions of the spacetime. Conversely, the BH entropy is meant to be universal, hence independent from n_s and δ . Some conceptual difficulties can be solved using Sakharov's induced gravity approach [23, 24, 25], but the problem of the dependence on n_s and δ still remains unsolved.

In this letter we will show that in the case of two-dimensional AdS black hole these difficulties can be completely solved. We will derive an expression for the black hole EE that in the large black hole mass limit reproduces exactly the BH entropy. Moreover, we will show that the subleading term has an universal behavior typical for CFTs and in particular for critical phenomena. The reason of this success is related to the peculiarities of 2D AdS gravity, namely the existence of an AdS/CFT correspondence and the fact that 2D Newton constant can be considered as wholly induced by quantum fluctuations of the dual CFT.

Most of the progress in understanding the EE in QFT has been achieved in the case of 2D CFT. Conformal invariance in two space-time dimension is a powerful tool that allows us to compute the EE in closed form. The entanglement entropy for the ground state of a 2D CFT originated from tracing over correlations between spacelike separated points has been calculated by Holzhey, Larsen and Wilczek [6]. Introducing an infrared cutoff Λ the spacelike coordinate of our 2D universe will belong to $\mathcal{C} = [0, \Lambda[$. The subsystem where measurements are performed is $Q = [0, \Sigma[$, whereas the outside region where the degrees of freedom are traced over is $R = [\Sigma, \Lambda[$. Because of the contribution of localized excitations arbitrarily near to the boundary the entanglement entropy diverges. Introducing

an ultraviolet cutoff δ , the regularized entanglement entropy turns out to be [6]

$$S_{ent} = \frac{c + \bar{c}}{6} \ln \left(\frac{\Lambda}{\delta\pi} \sin \frac{\pi\Sigma}{\Lambda} \right), \quad (1)$$

where c and \bar{c} are the central charges of the 2D CFT. The expression (1) emphasizes the characterizing features of the entanglement entropy, namely subadditivity and invariance under the transformation which exchanges the inside and outside regions

$$\Sigma \rightarrow \Lambda - \Sigma. \quad (2)$$

Moreover, S_{ent} is not a monotonic function of Σ , but increases and reaches its maximum for $\Sigma = \Lambda/2$ and then decreases as Σ increases further. This behavior has an obvious explanation. When the subsystem begins to fill most of the universe there is lesser information to be lost and the entanglement entropy decreases.

Let us now consider 2D AdS black holes. As classical solutions of a 2D gravity theory they are endowed with a non-constant scalar field, the dilaton Φ . In the Schwarzschild gauge the 2D AdS black hole solutions are [26],

$$ds^2 = - \left(\frac{r^2}{L^2} - a^2 \right) dt^2 + \left(\frac{r^2}{L^2} - a^2 \right)^{-1} dr^2, \quad \Phi = \Phi_0 \frac{r}{L}, \quad (3)$$

where the length L is related to cosmological constant of the AdS spacetime ($\lambda = 1/L^2$), Φ_0 is the dimensionless 2D Newton constant and a is an integration constants related to the black hole mass M and horizon radius r_h by

$$a = \frac{r_h}{L} = \sqrt{\frac{2ML}{\Phi_0}}. \quad (4)$$

The thermodynamical, Bekenstein-Hawking, entropy of the black hole is [26]

$$S_{BH} = 2\pi\Phi_0 a = 2\pi\sqrt{2\Phi_0 ML}, \quad (5)$$

whereas the black hole temperature is $T = a/2\pi L$. Setting $a = 0$ in Eq. (3) we have the AdS black hole ground state (in the following called AdS_0) with zero mass, temperature and entropy. The AdS black hole (3) can be considered as the thermalization of the AdS_0 solution at temperature $a/2\pi L$ [26].

It has been shown that the 2D black hole has a dual description in terms of a CFT with central charge [27, 28, 29, 30]

$$c = 12\Phi_0. \quad (6)$$

The dual CFT can have both the form of a 2D [29, 30] or a 1D [27, 28] conformal field theory. This $\text{AdS}_2/\text{CFT}_2$ (or $\text{AdS}_2/\text{CFT}_1$) correspondence has been used to give a microscopical meaning to the thermodynamical entropy of 2D AdS black holes. Eq. (5) has been reproduced by counting states in the dual CFT.

In Ref. [16] (see also Refs. [24, 25, 31]) it was observed that in two dimensions black hole entropy can be ascribed to quantum entanglement if 2D Newton constant is wholly induced by quantum fluctuations of matter fields. On the other hand the $\text{AdS}_2/\text{CFT}_2$ correspondence, and in particular Eq. (6), tells us that the 2D Newton constant Φ_0 is induced by quantum fluctuations of the dual CFT. It follows that the black hole entropy (5) should be explained as the entanglement of the vacuum of the 2D CFT of central charge given by Eq. (6) in the gravitational black hole background (3).

At first sight one is tempted to use Eq. (1) to calculate the entanglement entropy of the vacuum of the dual CFT. The exterior region of the 2D black hole can be easily identified with the region Q , whereas the black hole interior has to be identified with the R region where the degrees of freedom are traced over. There are two obstacles that prevents direct application of Eq. (1). First, Eq. (1) holds for a 2D flat spacetime, whereas we are dealing with a curved 2D background. Second, the calculations leading to Eq. (1) are performed for spacelike slice Q , whereas in our case the coordinate singularities at $r = r_h$ (the horizon) and $r = \infty$ (the timelike asymptotic boundary of the AdS spacetime) do not allow for a global notion of spacelike coordinate (a coordinate system covering the whole black hole spacetime in which the metric is non-singular and static). Owing to these geometrical features, in the black hole case we cannot give a direct meaning to *both* the measures Σ and $(\Lambda - \Sigma)$ of the subsystems Q, R . As a consequence of this fact invariance under the transformation (2) is meaningless in the black hole case.

The second difficulty can be circumvented using appropriate coordinate system and regularization procedure, the first using instead of Eq. (1) the formula derived by Fiola et al. [16], which gives the EE of the vacuum of matter fields in the case of a curved gravitational background.

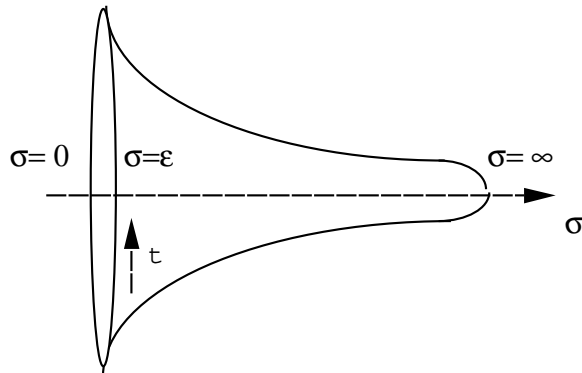


FIG. 1: Regularized euclidean instanton corresponding to the 2D AdS black hole in the coordinate system (t, σ) covering only the black hole exterior. The euclidean time is periodic. The point $\sigma = \infty$ correspond to the black hole horizon. $\sigma = 0$ corresponds to the asymptotic timelike boundary of AdS₂.

In the coordinate system used to define the vacuum of scalar fields in AdS₂, the 2D black hole metric (3) is [26]

$$ds^2 = \frac{a^2}{\sinh^2(\frac{a\sigma}{L})} (-dt^2 + d\sigma^2). \quad (7)$$

The coordinate system (t, σ) covers only the black hole exterior. The black hole horizon corresponds to $\sigma = \infty$ where the conformal factor of the metric vanishes. The asymptotic $r = \infty$ timelike conformal boundary of the AdS₂ spacetime is located at $\sigma = 0$, where the conformal factor diverges.

The entanglement entropy of the CFT vacuum in the curved background (7) can be calculated, using the formula of Ref. [16] as the half line entanglement entropy seen by an observer in the $0 < \sigma < \infty$ region. From the CFT point of view the AdS black hole has to be considered as the AdS₀ vacuum seen by the observer using the black hole coordinates (7) [26]. Moreover, this observer sees the the AdS₀ vacuum as filled with thermal radiation with *negative* flux [26]. It follows that the black hole entanglement entropy is given by the formula of Ref. [16] with reversed sign,

$$S_{ent}^{(bh)} = -\frac{c}{6} \left(\rho(\sigma = 0) - \ln \frac{\delta}{\Lambda} \right), \quad (8)$$

where ρ defines the conformal factor of the metric in the conformal gauge ($ds^2 = \exp(2\rho)(-dt^2 + d\sigma^2)$), c is the central charge given by Eq. (6) and δ, Λ are respectively UV and IR cutoffs. Notice that in Eq. (8) we have only contributions from only one sector (e.g. right movers) of the CFT. In Ref. [29, 30] it has been shown that the 2D AdS black hole is dual to an open string with appropriate boundary conditions. These boundary conditions are such that only one sector of the CFT₂ is present. The same is obviously true for the AdS₂/CFT₁ realization of the correspondence [27, 28].

The conformal factor of the metric (7), hence the entanglement entropy (8) blows up on the $\sigma = 0$ boundary of the AdS spacetime. The simplest regularization procedure that solves this problem is to consider a regularized boundary at $\sigma = \epsilon$. Notice that ϵ plays the role of a UV cutoff for the coordinate σ , which is the natural spacelike coordinate of the dual CFT. ϵ is an IR cutoff for the coordinate r , which is the natural spacelike coordinate for the AdS₂ black hole. The regularized euclidean instanton corresponding to the black hole (7) is shown in figure (1). The regularizing parameter ϵ can be set equal to the UV cutoff, $\delta = \epsilon$. Moreover, the regularized boundary is at finite proper distance from the horizon so that ϵ acts also as IR regulator, making the presence of the IR cutoff Λ in Eq. (8) redundant. It

follows that the regularized EE is given by $S_{ent}^{(bh)} = -\frac{c}{6}(\rho(\epsilon) - \ln \frac{\epsilon}{L})$, which using equations (7) and (4) becomes

$$S_{ent}^{(bh)} = \frac{c}{6} \ln \left(\frac{L^2}{r_h \epsilon} \sinh \frac{\epsilon r_h}{L^2} \right). \quad (9)$$

As a check of the validity of our formula we note that in the case of AdS₀ ($r_h = 0$) the entanglement entropy vanishes.

The AdS/CFT correspondence enable us to identify the cutoff ϵ as the UV cutoff of the CFT : $\epsilon \propto L$. The proportionality factor can be determined by requiring that the analytical continuation of Eq. (9) is invariant under the transformation (2) (see later). This requirement fixes $\epsilon = \pi L$. With this position we get

$$S_{ent}^{(bh)} = \frac{c}{6} \ln \left(\frac{L}{\pi r_h} \sinh \frac{\pi r_h}{L} \right). \quad (10)$$

This formula is our main result, it gives the entanglement entropy of the 2D AdS black hole. This entanglement entropy has the expected behavior as a function of the horizon radius r_h or, equivalently, of the black hole mass M . $S_{ent}^{(bh)}$ becomes zero in the AdS₀ ground state, $r_h = 0$ ($M = 0$), whereas it grows monotonically for $r_h > 0$ ($M > 0$).

In order to compare the black hole EE (10) with the BH entropy (5) let us consider the limit of macroscopic black holes, that is the limit $a \rightarrow \infty$ or equivalently $r_h \gg L$ or also $M \gg 1/L$. Expanding Eq. (10) and using Eqs. (4) and (6) we get

$$S_{ent}^{(bh)} = 2\pi\sqrt{2\Phi_0 ML} - \Phi_0 \ln LM + O(1) = S_{BH} - 2\Phi_0 \ln S_{BH} + O(1). \quad (11)$$

We have obtained the remarkable result that the leading term in the large mass expansion of the black hole entanglement entropy reproduces exactly the Bekenstein-Hawking entropy. Moreover, the subleading term behaves as the logarithm of the BH entropy and describes quantum corrections to S_{BH} . It is an universally accepted result that the quantum corrections to the BH entropy behave as $\ln S_{BH}$ [32, 33, 34, 35, 36, 37, 38, 39, 40, 41]. However, there is no general consensus about the value of the prefactor of this term. For the microcanonical ensemble this term has to be negative, whereas there are positive contributions coming from thermal fluctuation. Equation (11) fixes the prefactor of $\ln S_{BH}$ in terms of the 2D newton constant Φ_0 . This result contradicts some previous results supporting a Φ_0 -independent value of the prefactor. Our result is consistent with the approach followed in this paper, which considers 2D gravity as induced from the quantum fluctuations of a CFT with central charge $12\Phi_0$. The first (Bekenstein-Hawking) term in Eq. (11) is the induced entanglement entropy, whereas the second term, $-(c/6)\ln(r_h/L)$, is determined by the conformal symmetry. It gives the entanglement entropy (1) of a CFT in 2D flat spacetime with central charge $12\Phi_0$ and $\Sigma = r_h$ in the limit $\Sigma \ll \Lambda$ [6]. The subleading term in Eq. (11) represents therefore an universal behavior shared with other systems described by effective CFTs, such as one-dimensional statistical models near to the critical point (with the black hole radius r_h corresponding to the correlation length) or free scalars fields [7, 9].

Eq. (10) shows a close resemblance with the CFT entanglement entropy (1). Eqs. (10) and (1) differs in two main points: the absence in the black hole case of something corresponding to the measure of the whole space (the parameter Λ in Eq. (1)) and the appearance of hyperbolic instead of trigonometric functions. These are expected features for the entanglement entropy of a black hole. They solve the problems concerning the application of formula (1) to the black hole case. For a black hole one cannot define a measure of the whole space analogue to Λ . For static solutions the coordinate system covers only the black hole exterior. The appearance of hyperbolic instead of trigonometric functions allows for monotonic increasing of $S_{ent}^{(bh)}(r_h)$, eliminating the unphysical decreasing behavior of $S_{ent}(\Sigma)$ in the region $\Sigma > \Lambda/2$.

It is interesting to see how Eq. (1) can be obtained as the analytic continuation $r_h \rightarrow ir_h$ of our formula (10), i.e by considering an AdS black hole with negative mass. The analytically continued black hole solution is given by Eq. (3) with $a^2 < 0$. In the conformal gauge the solution reads now $ds^2 = [a^2/\sin^2(a\sigma/L)](-dt^2 + d\sigma^2)$. The range of the spacelike coordinate, corresponding to $0 < r < \infty$, is now $0 < \sigma < \pi L/2a$. Regularizing the solution at $\sigma = 0$ by introducing the cutoff ϵ we get the euclidean instanton shown in Fig. (2). In terms of the 2D CFT we have to trace over the degrees of freedom outside the spacelike slice $\epsilon < \sigma < \pi L/2a$. The related entanglement entropy can be calculated using the formula of Ref. [16] in the case of a spacelike slice with two boundary points: $S_{ent} = -c/6[\rho(\epsilon) + \rho(\pi L/2a) - \ln(\delta/\Lambda)]$. Applying this formula to the case of the black hole solution of negative mass, identifying ϵ in terms of the IR cutoff Λ , $\epsilon = \pi L^2/\Lambda$, and redefining appropriately the UV cutoff δ , we get

$$S_{ent} = \frac{c}{6} \ln \left(\frac{\Lambda}{\pi \delta} \sin \frac{\pi r_h}{\Lambda} \right). \quad (12)$$

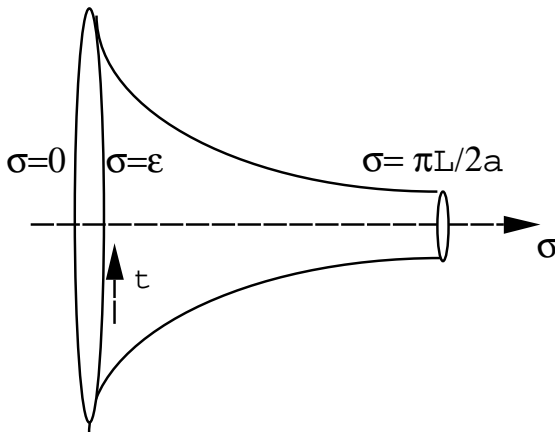


FIG. 2: Regularized euclidean instanton corresponding to the 2D AdS black hole with negative mass. The euclidean time is periodic. The point $\sigma = \pi L/2a$ corresponds to the black hole singularity at $r = 0$. $\sigma = 0$ corresponds to the asymptotic timelike boundary of AdS₂.

Thus, the entanglement entropy of the 2D CFT in the curved background given by the AdS black hole of negative mass has exactly the form given by Eq. (1) with the horizon radius r_h playing the role of Σ . Notice that the presence of the factor π in the argument of the sin-function is necessary if one wants invariance under the transformation (2). The requirement that equation (12) is the analytic continuation of Eq. (10) fixes, as previously anticipated, the proportionality factor between ϵ and L in the calculations leading to Eq. (10).

In this letter we have derived a formula for the entanglement entropy of 2D AdS black holes that has nice striking features. The leading term in the large black hole mass expansion reproduces exactly the BH entropy. The subleading term has the right $\ln S_{BH}$ behavior of the quantum corrections to the BH formula and represents an universal term typical of CFTs. Analytic continuation to negative black hole masses give exactly the entanglement entropy of 2D CFT with the black hole radius playing the role of the measure of the observable spacelike slice in the CFT. Our results rely heavily on peculiarities of 2D AdS gravity, namely the existence of an AdS/CFT correspondence and on the fact that 2D Newton constant arises from quantum fluctuation of the dual CFT. The generalization of our approach to higher dimensional gravity theories is therefore far from being trivial. A related problem is the form of the coefficient of the $\ln S_{BH}$ term. In the 2D context our result, stating that this coefficient is given in terms of the 2D Newton constant (or equivalently the central charge of the dual CFT) is rather natural. For higher dimensional gravity theories this is again a rather subtle point.

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