Differential recursion and differentially algebraic functions*

Akitoshi Kawamura
Department of Computer Science
University of Toronto

Moore introduced a class of real-valued "recursive" functions by analogy with Kleene's formulation of the standard recursive functions. While his concise definition inspired a new line of research on analog computation, it contains some technical inaccuracies. Focusing on his "primitive recursive" functions, we pin down what is problematic and discuss possible attempts to remove the ambiguity regarding the behavior of the differential recursion operator on partial functions. It turns out that in any case the purported relation to differentially algebraic functions, and hence to Shannon's model of analog computation, fails.

1. Introduction

There are several different kinds of theoretical models that talk about "computability" and "complexity" of real functions. *Computable Analysis* [19] and some other equivalent models use approximation in one way or another to bring real numbers into the framework of the standard Computability Theory that deals with discrete data in discrete time. Another well-known model is the *Blum–Shub–Smale model* [1] in which continuous quantities are treated as an entity in themselves but the machine still works with discrete clock ticks.

A third approach is *analog computation* in which not only are the data real-valued, but also the transition takes place in continuous time [15]. One of the oldest and the best-studied models of such computation is Shannon's *General Purpose Analog Computer* [18] that models the *Differential Analyzer* [3], a computing device built and put to use during the thirties through the fifties. The GPAC, after some refinements [8, 12, 16], was shown capable of generating (in a sense) all and only the *differentially algebraic* functions. We will explore this class in Section 2 and show that it can be characterized in many different ways.

Little is known about how such analog models relate to the standard (digital) computability. Moore [13] addressed this question for his new function classes that also try to express

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the power of GPAC-like computation. In imitation of Kleene's characterization of the usual recursive functions, these classes are defined as the closures under certain operators that are supposedly real-number versions of primitive recursion and minimization. He makes the following claims, among others, that relate his classes of *real primitive recursive* and *real recursive* functions to analog and digital computation, respectively [13, Propositions 9 and 13].

Claim 1. *Real primitive recursive functions are differentially algebraic.*[1]

Claim 2. *Each (partial) recursive function on the nonnegative integers (in the standard sense) is a restriction of some real recursive function.*

Despite its impact on the subsequent study on the classes and their variants [2, 4, 5, 8, 14], his work lacked formality in some ways, as already pointed out [6, 7]. In fact, the definition of the classes suffers from ambiguity. In Section 3 of this paper, we reformulate Moore's theory up to the real primitive recursive functions in a mathematically sound way. With the aid of the preparation in Section 2, we show that, even though our formulation of the class seems the most restrictive possible, Claim 1 fails. In section 4, we discuss some issues about classes other than the real primitive recursive functions, including Claim 2.

Throughout the paper, we write $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ for the sets of nonnegative integers (including 0), integers and real numbers, respectively.

Partial functions In this paper, a *function* $f :\subseteq \mathbf{R}^m \to \mathbf{R}^n$ may be *partial*, as opposed to *total*; that is, the set $\operatorname{dom} f$ of $x \in \mathbf{R}^m$ for which the value $f x \in \mathbf{R}^n$ is defined is allowed to be a proper subset of $\mathbf{R}^m$. By the *restriction* of f to a set $J \subseteq \mathbf{R}^m$ we mean the function g with $\operatorname{dom} g = J \cap \operatorname{dom} f$ such that $gx = fx$ for every $x \in \operatorname{dom} g$. When $\operatorname{dom} f$ is open, f is said to be *(real) analytic* if for every $a = (a_0, \dots, a_{m-1}) \in \operatorname{dom} f$ there are an open set $J \subseteq \operatorname{dom} f$ containing a and a family $(c_p)_{p \in \mathbf{N}^m}$ of n-tuples of real numbers such that the sum

$$\sum_{p=(p_0,\dots,p_{m-1}) \in \mathbf{N}^m} c_p \cdot (x_0 - a_0)^{p_0} \cdots (x_{m-1} - a_{m-1})^{p_{m-1}} \tag{1}$$

converges to $f x$ for each $x = (x_0, \dots, x_{m-1}) \in J$ (regardless of the ordering of summation). See Krantz and Parks [10, Chapters 1 and 2] for well-known properties of analytic functions. When f is analytic, we write $\mathrm{D}^{(a_0,\dots,a_{m-1})} f$ (and not $\partial^{a_0+\cdots+a_{m-1}} f / \partial t_0^{a_0} \cdots \partial t_{m-1}^{a_{m-1}}$) for the mixed partial derivative of f of order a_i along the i^{th} place (which is known to exist).

Moore [13] does not explicitly deal with partial functions. We believe that this is responsible for ambiguous and erroneous statements made in his seminal work as well as in some of the subsequent works by other authors. Although there are some situations in mathematical analysis where we can pretend that there are no partial functions (namely, when we are only discussing properties defined *locally*, such as continuity or analyticity), this is not the case with the notions we want to discuss here. If, say, the above Claim 2 is to make any nontrivial sense, it is clearly inappropriate to talk about "real recursiveness at x," as there is a real function which is simple locally but the restriction of which to $\mathbf{N}$ is highly complicated in the recursion-theoretic sense. We therefore emphasize that partial functions must be dealt with seriously, and devote this paper to accordingly reformulating the theory wherever possible.

[1] Moore writes M_0 for the class of real primitive recursion functions. Claim 1 was later replaced by a similar claim [6, Proposition 2] for a more "restricted" class $\mathcal{G}$ than M_0, but its definition is again unclear.

2. Differentially algebraic functions

We show some facts about single- and multi-place differentially algebraic functions.

Theorem 3. *Let m, n and $i < m$ be nonnegative integers and $f :\subseteq \mathbf{R}^m \to \mathbf{R}^n$ be an analytic function with open domain. Let* (i)*,* (ii)*,* (iii) *and* (iv) *be the following statements:*

(i) *for any open connected set $J \subseteq \operatorname{dom} f$, there is a $\mathbf{Z}^n$-coefficient nonzero polynomial P such that*
$$P\left(f\,x, \mathrm{D}^{e_i} f\,x, \mathrm{D}^{2\cdot e_i} f\,x, \ldots, \mathrm{D}^{(\operatorname{arity} P-1)\cdot e_i} f\,x\right) = 0 \tag{2}$$
for all $x \in J$, where $e_i \in \mathbf{N}^m$ is the vector whose i^{th} component is 1 and others are 0;

(ii) *for each $x_0 \in \operatorname{dom} f$, there are a $\mathbf{Z}^n$-coefficient nonzero polynomial P and an open set J containing x_0 such that we have* (2) *for all $x \in J$;*

(iii) *for each $x_0 \in \operatorname{dom} f$, there are a $\mathbf{Z}^n$-coefficient nonzero polynomial P and an open interval J containing the i^{th} component of x_0 such that we have* (2) *for all x whose i^{th} component is in J and whose other components equal those of x_0;*

(iv) *for each $x \in \operatorname{dom} f$, there is a $\mathbf{Z}^n$-coefficient nonzero polynomial P satisfying* (2)*.*

Let ($\mathrm{i}_\mathbf{R}$)*,* ($\mathrm{ii}_\mathbf{R}$) *and* ($\mathrm{iii}_\mathbf{R}$) *be the statements obtained by replacing* $\mathbf{Z}$ *by* $\mathbf{R}$ *in* (i)*,* (ii) *and* (iii)*, respectively. Then* (i)*,* (ii)*,* (iii)*,* (iv)*,* ($\mathrm{i}_\mathbf{R}$)*,* ($\mathrm{ii}_\mathbf{R}$) *and* ($\mathrm{iii}_\mathbf{R}$) *are equivalent.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ ($\mathrm{i}_\mathbf{R}$) $\Rightarrow$ ($\mathrm{ii}_\mathbf{R}$) $\Rightarrow$ ($\mathrm{iii}_\mathbf{R}$) are obvious. It has been known that ($\mathrm{iii}_\mathbf{R}$) $\Rightarrow$ (iii), see Theorem 16. To see (iv) $\Rightarrow$ (i), consider, for each $\mathbf{Z}^n$-coefficient polynomial P, the set J_P of all $x \in J$ satisfying (2). Since by (iv) these countably many closed sets J_P cover the open set J, one of them must have nonempty interior by Baire Category Theorem 14. This J_P must then equal J by the Identity Theorem 15. ☐

Definition 4. Let m and n be nonnegative integers. An analytic function $f :\subseteq \mathbf{R}^m \to \mathbf{R}^n$ is *differentially algebraic*[2] if for each $i < m$ it satisfies one (or all) of the clauses in Theorem 3.

Note that f need not be the *unique* solution of (2). For example, every function (with open domain) that is constant on each connected component of its domain is differentially algebraic because of the single set of equations $\mathrm{D}^{e_i} f\,x = 0$.

The clauses (ii)–(iv) show that being differentially algebraic is a "local" property.

When $\operatorname{dom} f$ is connected, (i) reduces to the following statement:

(i′) there is a $\mathbf{Z}^n$-coefficient nonzero polynomial P such that we have (2) for all $x \in \operatorname{dom} f$.

Hence, the clause (iii) shows that, as long as $\operatorname{dom} f$ is connected, our definition is equivalent to that of many authors, including Moore [13], who first state the definition for $m = 1$ by (i′) and then extend it to $m > 1$ by saying that a function is differentially algebraic when it is so as a unary function of each argument when all other arguments are held fixed.[3] We proved Theorem 3 in order to use (i) to present a counterexample to Claim 1 later.

[2] Also termed *algebraic transcendental* or *hypotranscendental*. Functions *without* this property is said to be *transcendentally transcendental* or *hypertranscendental*.

[3] Their definition for the case $m = 1$ is slightly different from (i′) in that it replaces (2) by $P(x, f\,x, \mathrm{D}^{e_i} f\,x, \ldots, \mathrm{D}^{(\operatorname{arity} P-2)\cdot e_i} f\,x) = 0$. But the proof of (a) $\Rightarrow$ (b) of Lemma 5 shows that this difference is superficial.

Let us characterize differentially algebraic functions in yet another way for the case $n = 1$. For a field E, its subfield F and a set $B \subseteq E$, we write $F(B)$, agreeing tacitly on E, for the smallest subfield of E that includes F and B. We write $\overline{F}$ for the *algebraic closure* of F, that is, the set of those elements of E that annuls some F-coefficient unary nonzero polynomial.

Let $J \subseteq \mathbf{R}^m$ be an open set and consider the ring $\mathrm{C}^\omega[J]$ of analytic functions $g :\subseteq \mathbf{R}^m \to \mathbf{R}$ with $\operatorname{dom} g = J$. Note that $\mathbf{R}$ is embedded into this ring by regarding each $x \in \mathbf{R}$ as the constant function taking the value x. To assert (2) for all $x \in J$ is to say that

$$P\big(f, \mathrm{D}^{e_i} f, \mathrm{D}^{2\cdot e_i} f, \ldots, \mathrm{D}^{(\operatorname{arity} P-1)\cdot e_i} f\big) = 0 \tag{3}$$

in $\mathrm{C}^\omega[J]$. If J is connected, $\mathrm{C}^\omega[J]$ has a quotient field by the Identity Theorem 15, so the notation $\mathbf{R}(\mathbf{D}\, f)$ in the following lemma makes sense. We write $\mathbf{D}\, f = \{\, \mathrm{D}^a f \mid a \in \mathbf{N}^m \,\}$.

Lemma 5. *Let $J \subseteq \mathbf{R}^m$ be open and connected. For $f \in \mathrm{C}^\omega[J]$, the following are equivalent:*

(a) *f is differentially algebraic;*

(b) *$\mathbf{D}\, f \subseteq \mathbf{R}(B)$ for some finite set $B \subseteq \mathbf{D}\, f$;*

(c) *$\mathbf{D}\, f \subseteq \overline{\mathbf{R}(B)}$ for some finite set $B \subseteq \mathbf{R}(\mathbf{D}\, f)$.*

Proof. The implication (b) $\Rightarrow$ (c) is trivial. The Transcendence Degree Theorem 17 shows (c) $\Rightarrow$ (a). For (a) $\Rightarrow$ (b), assume that for each i we have an $\mathbf{R}$-coefficient polynomial P_i with

$$P_i(f, \mathrm{D}^{e_i} f, \mathrm{D}^{2\cdot e_i} f, \ldots, \mathrm{D}^{N_i\cdot e_i} f) = 0, \tag{4}$$

where $N_i = \operatorname{arity} P_i - 1$. By choosing N_i to be smallest and then the degree of P_i in the last place to be smallest, we may assume that

$$\varXi = (\mathrm{D}^{(0,\ldots,0,1)} P_i)(f, \mathrm{D}^{e_i} f, \mathrm{D}^{2\cdot e_i} f, \ldots, \mathrm{D}^{N_i\cdot e_i} f) \tag{5}$$

is nonzero. Consider the order $\leq$ on $\mathbf{N}^m$ defined by setting $a \leq b$ when $a + c = b$ for some $c \in \mathbf{N}^m$. We will show by induction on $a \in \mathbf{N}^m$ that $\mathrm{D}^a f \in \mathbf{R}(\{\, \mathrm{D}^b f \mid b \leq (N_0, \ldots, N_{m-1}) \,\})$. The case $a \leq (N_0, \ldots, N_{m-1})$ being trivial, assume that $a \geq (N_i + 1) \cdot e_i$ for some i. Apply $\mathrm{D}^{a-N_i\cdot e_i}$ to both sides of (4) and calculate using chain rules to obtain

$$\varPsi + \varXi \cdot \mathrm{D}^a f = 0, \tag{6}$$

where $\varPsi$ can be written as a sum of products of several derivatives of f of order $\leq a$ and $\neq a$, which hence enjoy the induction hypothesis. $\square$

Apart from purely theoretical interest, the significance of differentially algebraic functions lies in their relation to the *General Purpose Analog Computer*, an analog computation model introduced by Shannon [18] and later refined by Pour-El [16]. More precisely, if a function $f :\subseteq \mathbf{R} \to \mathbf{R}$ with nonempty domain is differentially algebraic, then the restriction of f to some nonempty subset of $\operatorname{dom} f$ is GPAC generable [16, Theorem 4]; conversely, if a function $f :\subseteq \mathbf{R} \to \mathbf{R}$ with nonempty domain is GPAC generable, then the restriction of f to some nonempty subset of $\operatorname{dom} f$ is differentially algebraic [12, Theorem 2]. Graça later considered the *Polynomial GPAC*, a simpler refinement than Pour-El's, and proved analogous results [8].

3. Real primitive recursive functions

The class of real primitive recursive functions is defined [13] as the smallest class containing some basic functions and closed under the operators specified below. Unfortunately, the original definition contains ambiguity, resulting in some inconsistent claims about the class. To remedy this, we shall revisit the definitions carefully in Sections 3.1 and 3.2. Section 3.3 discusses an alternative approach by Campagnolo. Section 3.4 disproves Claim 1.

3.1. Two basic operators

The real primitive recursive functions are defined through three operators: *juxtaposition*, *composition* and *differential recursion*. The first two are very simple.

Definition 6. Given $g_0, \ldots, g_{n-1} :\subseteq \mathbf{R}^m \to \mathbf{R}$, define their juxtaposition $\textsc{jx}\,(g_0, \ldots, g_{n-1}) :\subseteq \mathbf{R}^m \to \mathbf{R}^n$ by setting $\textsc{jx}\,(g_0, \ldots, g_{n-1})x = (g_0\,x, \ldots, g_{n-1}\,x)$ whenever $x \in \operatorname{dom} g_0 \cap \cdots \cap \operatorname{dom} g_{n-1}$. Given $f :\subseteq \mathbf{R}^m \to \mathbf{R}^n$ and $g :\subseteq \mathbf{R}^l \to \mathbf{R}^m$, define their composition $\textsc{cm}\,(f, g) :\subseteq \mathbf{R}^l \to \mathbf{R}^n$ by setting $\textsc{cm}\,(f, g)x = f\,(g\,x)$ whenever $x \in \operatorname{dom} g$ and $g\,x \in \operatorname{dom} f$.

We write $f \circ g$ for $\textsc{cm}\,(f, g)$. As remarked in Section 1, it is important to define precisely what the operators do on partial functions. Note how Definition 6 specifies the domain of the functions constructed. If $g\,x$ is not defined, neither is $(f \circ g)\,x$, even if, say, f is a constant function defined everywhere. We thus work in the following (informal) general principle.

Principle 7. For the value of an expression to be defined, the value of each of its subexpressions has to be defined.

We remark that this was not explicitly intended by Moore. In fact, he presents an example to the contrary when he claims [13, Section 6] that the total function $\overline{\operatorname{inv}} :\subseteq \mathbf{R} \to \mathbf{R}$ given by

$$\overline{\operatorname{inv}}\,x = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{if } x \neq 0 \end{cases} \tag{7}$$

can be obtained by composing the binary multiplication with $\textsc{jx}\,(\text{zero?}, g)$, where zero? is from (30) and g is the restriction of $\overline{\operatorname{inv}}$ to $\mathbf{R} \setminus \{0\}$. Some authors point this out [4, p. 22] and criticize it [7, p. 47]. Without discussing which definition is more "natural," we adopt our restrictive Definition 6, simply because it is not clear how to formulate a general definition that would admit this construction of $\overline{\operatorname{inv}}$.

The operators $\textsc{jx}$ and $\textsc{cm}$ preserve analyticity [10, Proposition 2.2.8].

Theorem 8. *The property of being differentially algebraic is preserved by* $\textsc{jx}$ *and* $\textsc{cm}$.

Proof. This is trivial for $\textsc{jx}$. For $\textsc{cm}$, it suffices to show that if $f :\subseteq \mathbf{R}^m \to \mathbf{R}$ and $g_0, \ldots, g_{m-1} :\subseteq \mathbf{R}^l \to \mathbf{R}$ are differentially algebraic, so is $f \circ g$, where $g = \textsc{jx}\,(g_0, \ldots, g_{m-1})$. We may assume that $\operatorname{dom} g$ and $J = \operatorname{dom}\,(f \circ g)$ are connected. We use the characterization (b) of Lemma 5. Calculate each element of $\mathbf{D}\,(f \circ g)$ by the chain rule to see that it belongs to

$$\mathbf{R}\big(\{\, d \circ g \mid d \in \mathbf{D}\,f \,\} \cup \bigcup_{i=0}^{m-1} \{\, q{\restriction_J} \mid q \in \mathbf{D}\,g_i \,\}\big), \tag{8}$$

where $q{\restriction_J}$ means the restriction of q to J. By the assumption, there are finite subsets $A \subseteq \mathbf{D}\,f$ and $B_i \subseteq \mathbf{D}\,g_i$ with $\mathbf{D}\,f \subseteq \mathbf{R}(A)$ and $\mathbf{D}\,g_i \subseteq \mathbf{R}(B_i)$ for each $i = 0, \ldots, m-1$. This implies that (8) stays unchanged by replacing $\mathbf{D}\,f$ by A and $\mathbf{D}\,g_i$ by B_i. $\square$

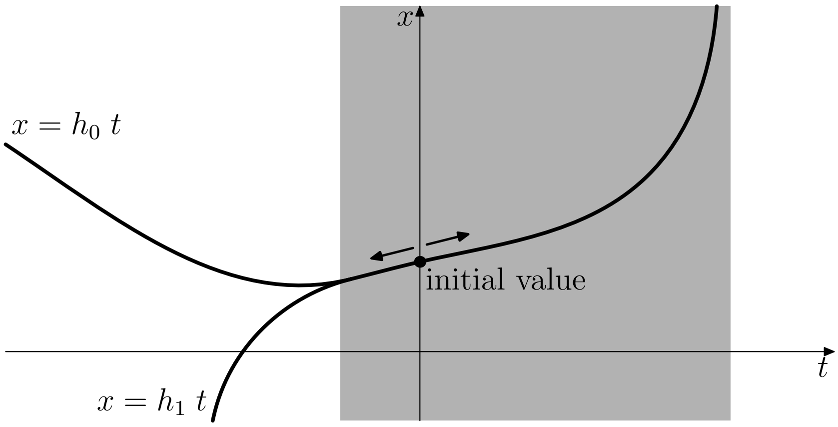

Figure 1: When the equation is satisfied by both h_0 and h_1 (as well as their restriction to each interval containing the origin), how do we say that the shaded interval is where there is a "unique solution"?

3.2. The differential recursion operator

To formulate the third operator, we need a notion of *unique solution* of an integral equation of the form (9) below, where h is the unknown. For example, it sounds natural to say that the tangent function restricted to $(-\pi/2, \pi/2)$ uniquely solves $ht = \int_0^t \bigl(1 + (h\tau)^2\bigr)\, \mathrm{d}\tau$. But as we are talking about partial functions, the word "unique" should be used carefully, because the restriction to any subinterval $J \subseteq (-\pi/2, \pi/2)$ containing 0 also satisfies the equation on J. Thus, out of the set H of all solutions, we need to pick one function that deserves to be called the unique solution defined on the largest possible interval (Figure 1). Though Moore did not discuss this, it is not hard to formulate this intuition: for a set H of functions of a type, we say that a function $h \in H$ is *unique* in H if the restriction of any function in H to $\operatorname{dom} h$ is a restriction of h.

Definition 9. Let $f :\subseteq \mathbf{R}^m \to \mathbf{R}^n$ and $g :\subseteq \mathbf{R}^{m+1+n} \to \mathbf{R}^n$. For each $v \in \mathbf{R}^m$, let H_v be the set of all functions $h :\subseteq \mathbf{R} \to \mathbf{R}^n$ such that

(a) $\operatorname{dom} h$ is either the empty set or a possibly unbounded interval containing 0,

(b) $v \in \operatorname{dom} f$ if $\operatorname{dom} h$ is nonempty,

(c) $(v, \tau, h\tau) \in \operatorname{dom} g$ for each $\tau \in \operatorname{dom} h$, and

(d) every $t \in \operatorname{dom} h$ satisfies

$$ht = fv + \int_0^t g(v, \tau, h\tau)\, \mathrm{d}\tau. \tag{9}$$

Let K_v be the set of functions unique in H_v. By Lemma 18 in the appendix, K_v has an element h_v of which all functions in K_v is a restriction. Define $\mathrm{DR}(f, g) :\subseteq \mathbf{R}^{m+1} \to \mathbf{R}^n$ by $\operatorname{dom}\bigl(\mathrm{DR}(f, g)\bigr) = \{\, (v, t) \in \mathbf{R}^{m+1} \mid t \in \operatorname{dom} h_v \,\}$ and $\mathrm{DR}(f, g)(v, t) = h_v t$.

Definition 10. The class of *real primitive recursive* functions is the smallest class containing the nullary functions $0^{0\to 1}$, $1^{0\to 1}$, $-1^{0\to 1}$ and closed under JX, CM and DR.

Lemma 11. *The following functions are real primitive recursive: for each $n \in \mathbf{N}$, the n-ary constants $0^{n\to 1}$, $1^{n\to 1}$, $-1^{n\to 1}$; for $n \in \mathbf{N}$ and $i = 0, \dots, n-1$, the n-ary projection $\mathrm{id}_i^{n\to 1}$ to the i^{th} component; binary* add *and* mul*; the functions* inv_+ *(mapping $x > 0$ to $1/x$),* sqrt_+ *(mapping $x > 0$ to $\sqrt{x}$) and* ln *(natural logarithm) defined on $(0, \infty)$; the total functions* sin, cos *and* exp*; the circle ratio π as a nullary function.*

Proof. The constant $0^{n\to 1}$ is built by $0^{n\to 1} = 0^{0\to 1} \circ \textsc{jx}\,(\,)$; similarly for $1^{n\to 1}$ and $-1^{n\to 1}$. Then inductively define $\mathrm{id}_i^{i+1\to 1} = \textsc{dr}\,(0^{i\to 1}, 1^{i+2\to 1})$ and $\mathrm{id}_i^{n+1\to 1} = \textsc{dr}\,(\mathrm{id}_i^{n\to 1}, 0^{n+2\to 1})$. Using these, let $\mathrm{add} = \textsc{dr}\,(\mathrm{id}_0^{1\to 1}, 1^{3\to 1})$ and $\mathrm{mul} = \textsc{dr}\,(0^{1\to 1}, \mathrm{id}_0^{3\to 1})$. For inv_+, define

$$f = \textsc{dr}\left(1^{0\to 1}, \mathrm{mul} \circ \textsc{jx}\left(-1^{1\to 1}, \mathrm{mul} \circ \textsc{jx}\,(\mathrm{id}_0^{1\to 1}, \mathrm{id}_0^{1\to 1})\right) \circ \mathrm{id}_1^{2\to 1}\right), \tag{10}$$

$$\mathrm{inv}_+ = f \circ \left(\mathrm{add} \circ \textsc{jx}\,(\mathrm{id}_0^{1\to 1}, -1^{1\to 1})\right), \tag{11}$$

or, more colloquially,

$$f\,t = 1 - \int_0^t (f\,\tau)^2 \,\mathrm{d}\tau, \qquad\qquad \mathrm{inv}_+ t = f\,(t-1). \tag{12}$$

Square root is defined analogously by

$$f\,t = 1 + \int_0^t \mathrm{inv}_+(2 \cdot f\,\tau)\,\mathrm{d}\tau, \qquad\qquad \mathrm{sqrt}_+ t = f\,(t-1). \tag{13}$$

Logarithm and exponentiation are analogous, using suitable integral equations. For the trigonometric functions, let $\sin = \mathrm{id}_0^{2\to 1} \circ \mathrm{trig}$ and $\cos = \mathrm{id}_1^{2\to 1} \circ \mathrm{trig}$, where

$$\mathrm{trig} = \textsc{dr}\left(\textsc{jx}\,(0^{0\to 1}, 1^{0\to 1}), \textsc{jx}\left(\mathrm{id}_2^{3\to 1}, \left(\mathrm{mul} \circ \textsc{jx}\,(-1^{3\to 1}, \mathrm{id}_1^{3\to 1})\right)\right)\right), \tag{14}$$

which is to say,

$$\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} \cos\tau \\ -\sin\tau \end{pmatrix} \mathrm{d}\tau. \tag{15}$$

The circle ratio is $\pi(\,) = 4 \cdot \mathrm{Arctan}\,1$, with Arctan defined by a suitable integral equation. ☐

Some authors say "the function $1/x$ is real primitive recursive" to mean that inv_+ is. It is not clear how such assertions without specification of domain can be justified.

The reader may have felt uncomfortable with the unwieldy process of Definition 9 in picking the right solution h_v out of H_v. This can be simplified if we discuss only real primitive recursive functions, because of the following facts that result from the *Uniqueness Theorem* for initial value problems [11] and the *Cauchy–Kowalewsky Theorem* [10, Section 2.4].

Theorem 12. *Let f, g, v, H_v and h_v be as in Definition 9.*

(a) *If g is an analytic[4] function with open domain, H_v is the set of all restrictions of h_v.*

(b) *If f and g are analytic functions with open domain, so is $\textsc{dr}\,(f, g)$.*

The fact (a) says that a solution of (9) may diverge to infinity at some point but can never "branch" as in Figure 1, provided g is smooth enough. We therefore could have dispensed with Lemma 18 and simply let h_v be the (graph) union of H_v, so far as real primitive recursive functions are concerned, because they are analytic by (b).

[4]This fact is often stated with a weaker assumption that g be *Lipschitz continuous*.

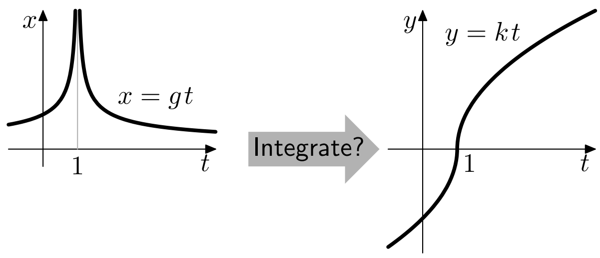

Figure 2: Integrand with a singularity.

3.3. Campagnolo's differential recursion

The clauses (a)–(c) of Definition 9 guarantee that the integral equation (9) makes sense for all $t \in \operatorname{dom} h$. The clause (c), however, could be slightly relaxed, since a small set of singularities in the integrand does not affect the integration. Define $\mathrm{DR_C}$ by replacing (c) with

(c′) $(v, \tau, h\tau) \in \operatorname{dom} g$ for any $\tau \in \operatorname{dom} h \setminus S$, where S is a countable set of isolated points.

This is due to Campagnolo [4, Definition 2.4.2], though he does not present a precise specification of the "unique" solution as we noted in the Section 3.2. The choice between (c) and (c′) is somewhat similar to the discussion regarding Principle 7. The issue is whether $g(v, \tau, h\tau)$, where $\tau \in [0, t]$, is a "subexpression" of the right-hand side of the equation (9). Without going into the philosophical discussion to ask which is "natural," we point out some differences this choice incurs.

Theorem 12 (b) fails if we replace DR by $\mathrm{DR_C}$, as the following example shows (Figure 2). The function $g :\subseteq \mathbf{R} \to \mathbf{R}$ defined by $\operatorname{dom} g = \mathbf{R} \setminus \{1\}$ and $gt = \mathrm{inv}_+\big(\mathrm{sqrt}_+\big(\mathrm{sqrt}_+(t-1)^2\big)\big) = 1/\sqrt{|t-1|}$ is real primitive recursive by Lemma 11. But $k = \mathrm{DR_C}(-2^{0\to 1}, g \circ \mathrm{id}_0^{2\to 1}) :\subseteq \mathbf{R} \to \mathbf{R}$, where $-2^{0\to 1}$ is the constant function with value -2, is the total function given by

$$kt = \begin{cases} +2 \cdot \sqrt{t-1} & \text{if } t \geqslant 1, \\ -2 \cdot \sqrt{-t+1} & \text{if } t < 1, \end{cases} \tag{16}$$

which is not differentiable at 1. Note that $\mathrm{DR}(-2^{0\to 1}, g \circ \mathrm{id}_0^{2\to 1})$ is its restriction to $(-\infty, 1)$ and thus analytic. For a subtler example, recall the equation (13) for sqrt_+; with $\mathrm{DR_C}$, the same equation produces the square root function defined on $[0, \infty)$, rather than on $(0, \infty)$.

This breaks the assumption of Theorem 12 (a) and thus gives rise to incomparable functions in H_v when, say, $f = 1^{0\to 1}$ and $g = k \circ \mathrm{id}_1^{2\to 1}$, with k from (16); that is, the equation

$$ht = 1 + \int_0^t k(h\tau)\,\mathrm{d}\tau \tag{17}$$

has two solutions that take different values at a point.

Keeping the class analytic also conforms to Moore's intention [13, Definition 9] to make the equation (9) equivalent to

$$h0 = fv, \qquad \mathrm{D}^1 ht = g(v, t, ht), \tag{18}$$

which would not make sense for non-differentiable h.

3.4. A primitive recursive but not differentially algebraic function

Claim 1 would not make sense if we adopted DR_C in defining real primitive recursive functions, because there would then arise non-analytic functions, as we noted above. We now show that, even under our restrictive definition with the analyticity-preserving DR, the claim fails.

Define $\check{\Gamma} :\subseteq \mathbf{R}^2 \to \mathbf{R}$ by $\operatorname{dom} \Gamma = (0, \infty)^2$ and

$$\check{\Gamma}(R, x) = \int_{1/R}^{R} \exp\big((x-1) \cdot \ln t - t\big)\,\mathrm{d}t. \tag{19}$$

Define Euler's *gamma function* $\Gamma :\subseteq \mathbf{R} \to \mathbf{R}$ by $\operatorname{dom} \Gamma = (0, \infty)$ and

$$\Gamma\, x = \lim_{R \to \infty} \check{\Gamma}(R, x). \tag{20}$$

It can be verified that this value converges and satisfies

$$\mathrm{D}^n \Gamma\, x = \lim_{R \to \infty} \mathrm{D}^{(0,n)} \check{\Gamma}(R, x) \tag{21}$$

for each $n \in \mathbf{N}$ and $x \in (0, \infty)$. Hölder showed that Γ is not differentially algebraic [9].

We do not know if Γ is real primitive recursive, but $\check{\Gamma}$ is easily shown real primitive recursive, using Lemma 11. However, contrary to Claim 1, it is not differentially algebraic. For assume that it were. We would then have a nonzero polynomial P such that

$$P\big(\check{\Gamma}(R, x), \mathrm{D}^{(0,1)}\check{\Gamma}(R, x), \dots, \mathrm{D}^{(0, \operatorname{arity} P - 1)}\check{\Gamma}(R, x)\big) = 0 \tag{22}$$

for each $(R, x) \in (0, \infty)^2$. Note that we used the characterization (i) of Theorem 3 in order to take P independent of R. We take the limit of (22) as $R \to \infty$, which by (21) yields

$$P(\Gamma\, x, \mathrm{D}^1 \Gamma\, x, \dots, \mathrm{D}^{\operatorname{arity} P - 1} \Gamma\, x) = 0, \tag{23}$$

contradicting Hölder.

4. Other classes and related works

This section discusses some other operators introduced by Moore and other authors.

4.1. Minimization and Moore's real recursive functions

For a function $f :\subseteq \mathbf{R}^{m+1} \to \mathbf{R}$, Moore defines $\mathrm{MN}\, f :\subseteq \mathbf{R}^m \to \mathbf{R}$ by

$$\mathrm{MN}\, f\, v = \begin{cases} t^+ = \inf\{\, t \geq 0 \mid f(v, t) = 0 \,\} & \text{if } t^+ < -t^-, \\ t^- = \sup\{\, t \leq 0 \mid f(v, t) = 0 \,\} & \text{otherwise.} \end{cases} \tag{24}$$

The class of *real recursive* functions[5] is the smallest class containing all real primitive recursive functions and closed under JX, CM, DR and MN.

[5]This "recursiveness" of Moore's should not be confused with the same word also used in the context of Computable Analysis. As we see in Appendix C, Moore's real recursive functions can even be discontinuous.

Moore states the definition of MN in a way that leaves ambiguous whether (24) has a value when, say, $\operatorname{dom} f = \mathbf{R}^m \times [1, \infty)$ and $f(v, t) = 2 - t$ for all $t \geq 1$. Should it have the value 2, or be left undefined because "the zero-searching program gets stuck"?

It turns out that, whichever definition we choose, Moore's claim about iteration [13, Proposition 11] remains true, in the following modified form. Since the original proof again forgets partial functions, we present a new proof in Appendix C.

Lemma 13. *If $f :\subseteq \mathbf{R}^m \to \mathbf{R}^m$ is real recursive, there is a real recursive function $g :\subseteq \mathbf{R}^{m+1} \to \mathbf{R}^m$ that extends the function g' defined by $\operatorname{dom} g' = \{\, (v, k) \in \mathbf{R} \times (\mathbf{N} \setminus \{0\}) \mid v \in \operatorname{dom} f^k \,\}$ and $g'(v, k) = f^k v$ for all $(v, k) \in \operatorname{dom} g'$, where $f^k = \underbrace{f \circ \cdots \circ f}_{k}$.*

We have to note, however, that the class of real recursive functions is probably not well-behaved, since, with MN producing non-smooth functions, the class no longer enjoys Theorem 12. We therefore doubt the significance of Claim 2, although it could be justified by using Lemma 13 to simulate Turing machines as Moore did.

4.2. Linear differential recursion

We have seen that many of the problems in Moore's original work were caused by failure to deal with partial functions properly. Some authors avoid this trouble by studying only operators preserving totality, so that partial functions never come into discussion. Campagnolo and Moore [5] take this path by considering *linear differential recursion* in place of DR. For classes defined by this operator, some relationships with digital computation are known [2, 4].

4.3. Open problems

Claim 1 has been the main rationale for calling variants of Moore's classes a model of analog computation. Now that we have lost it, an important challenge is the following.

Open Problem. Find a subclass of our real primitive recursive functions, preferably with an equally simple definition, that has a close relationship to the differentially algebraic functions.

Another direction would be to reformulate further the rest of Moore's work, as well as other authors' works that also suffer from the same kind of ambiguity. For example, it may be interesting to work out Mycka and Costa's class arising from the operator of taking limits [14].

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A. Old results

We list some known theorems that we used in Section 2.

The following *Baire Category Theorem* is used in the proof Theorem 3.

Theorem 14. *Let J be a subset of $\mathbf{R}^m$. The union of countably many closed subsets of J with empty interior has empty interior.*

Proof. Let J_0, J_1, … be closed subsets of J with empty interior, and U be any nonempty open subset of J. We will show that $U \setminus \bigcup_{P\in\mathbf{N}} J_P$ is nonempty. For each $P \in \mathbf{N}$, we take $x_P \in \mathbf{R}^m$ and $\varepsilon_P \in \mathbf{R}$ as follows. Write $\mathrm{B}(x, \varepsilon)$ for the open set of points in J whose distance from x is less than ε. Let $x_0 \in U$ and $\varepsilon_0 \in (0, 1)$ be such that $\mathrm{B}(x_0, \varepsilon_0) \subseteq U$. For each $P \in \mathbf{N}$, let $x_{P+1} \in U$ and $\varepsilon_{P+1} \in (0, 2^{-P-1})$ be such that $\mathrm{B}(x_{P+1}, \varepsilon_{P+1}) \subseteq \mathrm{B}(x_P, \varepsilon_P) \setminus J_P$. This is possible because $\mathrm{B}(x_P, \varepsilon_P) \setminus J_P$ is open and nonempty, since J_P is closed and has empty interior. As P tends to infinity, x_P converges to a point in $U \setminus \bigcup_{P\in\mathbf{N}} J_P$. □

The proof of Theorem 3 also uses the following *Identity Theorem* (for real analytic functions of several variables), also known as the *Principle of Analytic Continuation*. It can be proved by straightforwardly generalizing the same assertion for unary functions [10, Section 1.2].

Theorem 15. *An analytic function with open connected domain that vanishes on an open set vanishes everywhere.*

Let $J \subseteq \mathbf{R}$ be an open interval. It is well known that functions $u_0, \dots, u_{k-1} \in \mathrm{C}^\omega[J]$ are linearly dependent if and only if the determinant $\left|(\mathrm{D}^i u_j)_{i,j=0,\dots,k-1}\right|$, called their *Wronskian*, is zero. Using this fact, Ritt and Gourin [17] showed (iii$_\mathbf{R}$) ⇒ (iii) of Theorem 3.

Theorem 16. *Let $J \subseteq \mathbf{R}$ be an open interval and let $f \in \mathrm{C}^\omega[J]$. If we have*

$$P(f, \mathrm{D}^1 f, \mathrm{D}^2 f, \dots, \mathrm{D}^{\operatorname{arity} P-1} f) = 0 \tag{25}$$

for some $\mathbf{R}$-coefficient nonzero polynomial P, then we have (25) *for some $\mathbf{Z}$-coefficient nonzero polynomial P.*

Proof. By the assumption, there is a finite set $B \subseteq \mathbf{N}^{\operatorname{arity} P}$ such that the functions

$$f^{\nu_0} \cdot (\mathrm{D}f)^{\nu_1} \cdots (\mathrm{D}^{\operatorname{arity} P-1} f)^{\nu_{\operatorname{arity} P-1}}, \qquad \text{for } (\nu_0, \dots, \nu_{\operatorname{arity} P-1}) \in B, \tag{26}$$

are linearly dependent. The Wronskian of (26) thus vanishes, which is a $\mathbf{Z}$-coefficient polynomial in f, $\mathrm{D}^1 f$, …, $\mathrm{D}^{\operatorname{arity} P+|B|-1} f$. This polynomial is nonzero, since otherwise (26) would be linearly dependent for arbitrary f, which is absurd. □

One direction of Lemma 5 uses the following *Transcendence Degree Theorem*.

Theorem 17. *Let F be a subfield of a field E and D be a subset of E. If $D \subseteq \overline{F(B)}$ for some finite set $B \subseteq E$, then $D \subseteq \overline{F(C)}$ for some finite set $C \subseteq D$.*

Proof. For each $d \in D$, the assumption gives

$$d^l = \sum_{j=0}^{l-1} \beta_j \cdot d^j \tag{27}$$

for some $l \in \mathbf{N} \setminus \{0\}$ and $\beta_j \in F(B)$. Suppose that for some $d = d_0 \in D$, this equation contains some $b \in B \setminus D$, since otherwise we are done. Then we can rewrite (27) as

$$b^k = \sum_{i=0}^{k-1} \alpha_i \cdot b^i \tag{28}$$

for some $k \in \mathbf{N} \setminus \{0\}$ and $\alpha_i \in F(B')$, where $B' = (B \setminus \{b\}) \cup \{d_0\}$.

For each $d \in D$ and $t \in \mathbf{N}$, we can substitute (27) and (28) repeatedly in d^t to write

$$d^t = \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \gamma_{i,j} \cdot c^i \cdot d^j \tag{29}$$

for some $\gamma_{i,j} \in F(B')$. The $k \cdot l + 1$ elements 1, d, d^2, …, $d^{k \cdot l}$ are hence linearly dependent over $F(B')$. We have thus found a set B' with $D \subseteq \overline{F(B')}$ such that $B' \setminus D$ has strictly less elements than $B \setminus D$. Repeat. □

B. Maximal unique function

This section shows that, from a set K of functions with a certain property, we can choose a function of which all functions in K is a restriction. This was used to justify Definition 9 in the presence of non-analytic functions where Theorem 12 (a) does not apply.

We say that a set $I \subseteq \mathbf{R}$ is 0-*convex* if it is either the empty set or a possibly unbounded interval containing 0. Note that the union of 0-convex sets is 0-convex.

We say that a set K of functions from $\mathbf{R}$ is *consistent* if for any $t \in \mathbf{R}$, the set $\{\, gt \mid g \in K \,\}$ has at most one element. In this case, the *union* of K means the unique function k such that $\operatorname{dom} h = \bigcup_{g \in K} \operatorname{dom} g$ and for each $t \in \operatorname{dom} h$, there is some $g \in K$ with $ht = gt$.

Lemma 18. *Let H be a set of functions from $\mathbf{R}$ with 0-convex domain. Then the set K of functions unique in H is consistent. Moreover, if its union belongs to H, it belongs to K.*

Proof. For the first claim, suppose otherwise. Then there are functions k_0, $k_1 \in K$ and $t \in \operatorname{dom} k_0 \cap \operatorname{dom} k_1$ such that $k_0 t \neq k_1 t$. This contradicts the fact that k_0 is unique in H.

For the second claim, suppose that the union k of K is not unique in H. That is, there are a function $g \in H$ and $t \in \operatorname{dom} k \cap \operatorname{dom} g$ such that $gt \neq kt$. There is $k_0 \in K$ for which $t \in \operatorname{dom} k_0$. We have $k_0 t = kt \neq gt$, contradicting the fact that k_0 is unique in H. □

This lemma can be applied to $H = H_v$ in the situation of Definition 9, because there the union of any consistent subset of H_v belongs to H_v.

C. Iteration

As we noted, the definition (24) of the operator MN is ambiguous, as it contains a subexpression $f(v, t)$ that may be undefined for some (v, t). So when is MN $f\,v$ defined? Possible answers include:

(a) When t^+ and t^- are defined.

(b) When at least either t^+ or t^- is defined; the condition $t^+ < -t^-$ will be used only when both are defined.

And when is t^+ (resp. t^-) defined? Possible answers include:

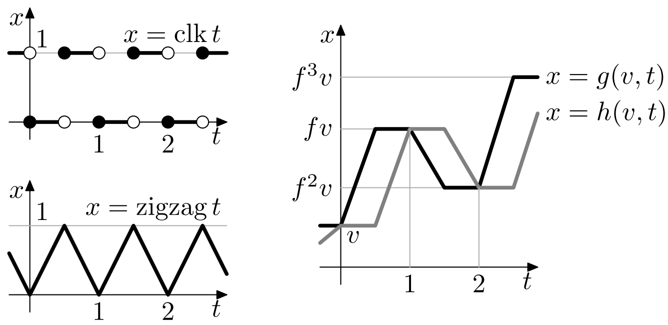

Figure 3: Simulating iteration $f^k v$ by real recursive functions.

(i) When there is $t \geq 0$ (resp. ≤ 0) such that $f(v,t) = 0$ and $(v,\tau) \in \operatorname{dom} f$ for all $\tau \in \mathbf{R}$.

(ii) When there is $t \geq 0$ (resp. ≤ 0) such that $f(v,t) = 0$ and $(v,\tau) \in \operatorname{dom} f$ for all $\tau \in [-t,t]$ (resp. $[t,-t]$).

(iii) When there is $t \geq 0$ (resp. ≤ 0) such that $f(v,t) = 0$ and $(v,\tau) \in \operatorname{dom} f$ for all $\tau \in [0,t]$ (resp. $[t,0]$).

(iv) When there is $t \geq 0$ (resp. ≤ 0) such that $f(v,t) = 0$.

For (i), (ii) and (iii), we may also consider adding the phrase "except for some countably many isolated τ" (compare (c′) in Section 3.3).

Moore's informal explanation by a programming language [13, Section 7] seems to suggest (b) and (ii). However, without discussing which is the "right" definition of MN, we show that, whichever we choose, Lemma 13 holds. The following proof is consistent with any of the above 2×7 possible definitions.

Proof of Lemma 13. Denote $\text{MN}\, f\, v$ by $\mu t.\, f(v,t)$. Let

$$\text{zero?}\, x = \mu y.\, (x^2 + y^2) \cdot (1 - y), \tag{30}$$

$$\text{integer?}\, x = \text{zero?}\big(\sin(\pi \cdot x)\big), \tag{31}$$

$$\text{round}\, x = x - \mu r.\, \text{integer?}(x - r), \tag{32}$$

so that (32) is the unique integer in $(x - 1/2, x + 1/2]$. We get $\overline{\text{inv}}$ of (7) by

$$\overline{\text{inv}}\, x = \mu t.\, x \cdot (x \cdot t - 1). \tag{33}$$

The above four functions are total. Let

$$\text{digit}(x,b,i) = \text{round}\left(\frac{x}{b^i} - \frac{1}{2}\right) - b \cdot \text{round}\left(\frac{x}{b^{i+1}} - \frac{1}{2}\right) \tag{34}$$

for $b > 0$, where $b^i = \exp(i \cdot \ln b)$. When $b > 1$ and i are integers, $\text{digit}(x,b,i)$ is the digit in b^i's place when x is written in base-b notation. Define

$$\text{clk}\, t = \text{digit}(t, 2, -1), \qquad \text{zigzag}\, t = 0 + \int_0^t (2 - 4 \cdot \text{clk}\, \tau)\, \mathrm{d}\tau, \tag{35}$$

$$\begin{pmatrix} g(v,t) \\ h(v,t) \end{pmatrix} = \begin{pmatrix} v \\ v \end{pmatrix} + \int_0^t \begin{pmatrix} 2 \cdot (1 - \text{clk}\, \tau) \cdot \big(f\big(h(v,\tau) - \text{clk}\, \tau \cdot (h(v,\tau) - v)\big) - h(v,\tau)\big) \\ 2 \cdot \text{clk}\, \tau \cdot \big(h(v,\tau) - g(v,\tau)\big) \cdot \overline{\text{inv}}(\text{zigzag}\, \tau) \end{pmatrix} \mathrm{d}\tau, \tag{36}$$

as in Figure 3. We have $f^k v = g(v, k - 1/2)$ for $k \in \mathbf{N} \setminus \{0\}$. $\square$

Note that $\text{clk}\, \tau \cdot (h(v,\tau) - v)$ in (36) cannot be dropped, because of Principle 7.