

# Correlation functions in the Non Perturbative Renormalization Group and field expansion

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## Abstract

In the context of a general method recently introduced to calculate  $n$ -point correlation functions at finite momenta, within the Non Perturbative Renormalization Group, expansion in powers of an external field is discussed. The study is performed using the self-energy of the tridimensional scalar model at criticality. At least in this example, the expansion around zero external field rapidly converges to the complete solution. The present study can be used to assess the quality of calculations made so far, truncating flow equations, which includes only a small number of vertices.

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## I. INTRODUCTION

In nearly all fields in physics, there are systems having a large number of strongly correlated constituents. These cannot be treated with usual perturbative methods. Phase transitions and critical phenomena, disordered systems, strongly correlated electrons, quantum chromodynamics at large distance, are just a few examples which demand a general and efficient method to treat non-perturbative situations. In problems as those just quoted, the calculation of correlation functions of the configuration variables is, in general, a very complicated task.

The non perturbative renormalization group (NPRG) [1, 2, 3] has proven to be a powerful tool to achieve this goal. It presents itself as an infinite hierarchy of flow equations relating sequentially the various  $n$ -point functions. It has been successfully applied in many different problems, either in condensed matter, particle or nuclear physics (for reviews, see e.g. [4, 5, 6]; a pedagogical introduction can be found in [7]). In most of these problems however, one is interested in observables dominated by long wavelength modes. In these cases, it is then possible to close approximately the infinity hierarchy of NPRG equations performing an expansion in the number of derivatives of the field. This approximation scheme is known as derivative expansion. The price to pay is that the  $n$ -point functions can be calculated only at small external momenta (vanishing momenta in the case of critical phenomena).

In many other physical problems however, this is not enough: the full knowledge of the momentum dependence of correlation functions is needed in order to calculate quantities of physical interest (e.g. to get the spectrum of excitations, the shape of a Fermi surface, the scattering matrix, etc.). There have been many attempts to solve the infinite system of flow equations at finite momenta; they are based on various forms of an early proposal by Weinberg [8]: Although some of these attempts [9, 10, 11] introduce sophisticated ansatz for the unknown correlation functions appearing in a given flow equation, most efforts simply ignores high order vertices. In all these works, only low order vertices are taken into account and it is not possible to measure the quality of the approximation.

Recently, an alternative general method to get  $n$ -point functions at any finite momenta within the NPRG has been proposed [12]. It has many similarities with derivative expansion. First, it is an approximation scheme that can be systematically improved. Second, the scheme yields a closed set of flow equations including simultaneously an infinite number

of vertices; one thus goes far beyond truncation schemes, as those quoted in the previous paragraph. Moreover, one can prove that, in their corresponding limits, both perturbative and derivative expansion results are recovered; this remains valid at each order of the respective expansion. Finally, in the large- $N$  limit of  $O(N)$  models, the leading-order (LO) of the approximation scheme becomes exact for all  $n$ -point functions. (The expression "leading order" means the first step in the approximation scheme; it does not refer to an expansion in a small parameter, which usually does not exist in these kind of problems.).

In [13] the method has been applied, in its leading order, to the calculation of the self-energy of the scalar model, at criticality. Already at this order, one gets a function having all the expected physical properties: It presents the correct scaling behavior in the infrared limit and the expected logarithmic shape of the perturbative regime. In order to check the quality of the solution in the intermediate momentum region, a quantity sensitive to this crossover sector has been calculated: one gets a number almost within the error bars of the best results found in the literature.

Another interesting similarity between derivative expansion and the method presented in [12] is that, as a price to pay in order to close the equations including an infinite number of vertices, one has to study the problem in an external constant field. Accordingly, one ends up with partial differential equations which may be difficult to solve. A useful approximation scheme, widely used in derivative expansion calculations, is to perform, on top of the expansion in derivatives of the field, an extra expansion in powers of the field (see e.g. [5]), in the spirit of Weinberg proposal. During the last 10 years, this strategy has been widely used [14, 15, 16, 17, 18, 19, 20, 21]; in many studied situations this expansion seems to converge (generally oscillating), while in many others it does not.

In this work we shall explore this procedure of expansion in powers of the fields, in the framework of the calculation scheme presented in [12]. More precisely, we shall compare the solution for the 2-point function found in [13] with the result obtained when making an expansion in powers of the field on top of the approximate flow equations exactly solved in [13]. In doing so, we have two goals. First, we shall study the convergence of this procedure. This comparison is essential if one hopes to apply the scheme described in [12] to situations more complicated than that considered in [13]. For example, within the derivative expansion scheme, when trying to go to higher orders, or when considering more involved models, the expansion in powers of the field, on top of the corresponding approximate flow equations,

is sometimes the only practical strategy to solve them [6, 19, 22]. The second goal is the following: as truncation in powers of the field is equivalent to ignoring high order vertices entering the flow equations (see section III below), the comparison presented here can help to estimate the quality of the calculations made so far to get  $n$ -point functions at finite momenta neglecting high order vertices.

The article is organized as follows. In the next section we describe the basics ingredients of both the NPRG and the approximation scheme introduced in [12]. We also present the results obtained in [13], when using the scheme to find the 2-point function of the scalar model. In section III, we apply the expansion in the field at various orders and compare these results with those found in [13]. Finally, we present the conclusions of the study.

## II. GENERAL CONSIDERATIONS

Let us consider a scalar field theory with the classical action

$$S = \int d^d x \left\{ \frac{1}{2} (\partial_\mu \varphi(x))^2 + \frac{r}{2} \varphi^2(x) + \frac{u}{4!} \varphi^4(x) \right\}, \quad (1)$$

The NPRG constructs a family of effective actions,  $\Gamma_\kappa[\phi]$  (with  $\phi$  the expectation value of the field in presence of an external source), in which the magnitude of long wavelength fluctuations are controlled by an infrared regulator depending on a continuous parameter  $\kappa$ . One can write for  $\Gamma_\kappa[\phi]$  an exact flow equation [3, 11, 17, 23]:

$$\partial_\kappa \Gamma_\kappa[\phi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \partial_\kappa R_\kappa(q^2) [\Gamma_\kappa^{(2)} + R_\kappa]_{q,-q}^{-1}, \quad (2)$$

where  $\Gamma_\kappa^{(2)}$  is the second derivative of  $\Gamma_\kappa$  with respect to  $\phi$ , and  $R_\kappa$  denotes a family of “cut-off functions” depending on  $\kappa$ :  $R_\kappa(q)$  is conveniently taken so as it behaves like a mass when  $q \ll \kappa$  and it vanishes rapidly when  $q \gg \kappa$  [15, 16, 24, 25]. The effective action  $\Gamma_\kappa[\phi]$  interpolates between the classical action obtained for  $\kappa = \Lambda$  (with  $\Lambda$  the microscopic scale at which fluctuations are essentially suppressed), and the full effective action obtained when  $\kappa \rightarrow 0$ , i.e., when all fluctuations are taken into account (see e.g. [5]).

By deriving eq. (2) with respect to  $\phi(x)$ , and then letting the field be constant, one gets the flow equation for the  $n$ -point function  $\Gamma_\kappa^{(n)}$  in a constant background field  $\phi$ . For

example, for the 2-point function one gets:

$$\begin{aligned} \partial_\kappa \Gamma_\kappa^{(2)}(p; \phi) = & \int \frac{d^d q}{(2\pi)^d} \partial_\kappa R_\kappa(q) \left\{ G_\kappa(q; \phi) \Gamma_\kappa^{(3)}(p, q, -p - q; \phi) \right. \\ & \times G_\kappa(q + p; \phi) \Gamma_\kappa^{(3)}(-p, p + q, -q; \phi) G_\kappa(q; \phi) \\ & \left. - \frac{1}{2} G_\kappa(q; \phi) \Gamma_\kappa^{(4)}(p, -p, q, -q; \phi) G_\kappa(q; \phi) \right\}, \end{aligned} \quad (3)$$

where

$$G_\kappa^{-1}(q; \phi) \equiv \Gamma_\kappa^{(2)}(q, -q; \phi) + R_\kappa(q^2), \quad (4)$$

and we used the definition

$$(2\pi)^d \delta^{(d)}\left(\sum_i p_i\right) \Gamma_\kappa^{(n)}(p_1, \dots, p_n; \phi) = \int d^d x_1 \dots \int d^d x_n e^{i \sum_{j=1}^n p_j x_j} \left. \frac{\delta^n \Gamma_\kappa}{\delta \phi(x_1) \dots \delta \phi(x_n)} \right|_{\phi(x) \equiv \phi}. \quad (5)$$

The flow equation for a given  $n$ -point function involves the  $n+1$  and  $n+2$  point functions (see, e.g., eq. (3)), so that the flow equations for the  $n$ -point functions constitute an infinite hierarchy of coupled equations.

In [12] a general method was proposed to solve this infinite hierarchy. It exploits the smoothness of the regularized  $n$ -point functions, and the fact that the loop momentum  $q$  in the right hand side of the flow equations (such as eq. (2) or eq. (3)) is limited to  $q \lesssim \kappa$  by the presence of the regulator  $R_\kappa(q)$ . The leading order of the method presented in [12] thus consists in setting

$$\Gamma_\kappa^{(n)}(p_1, p_2, \dots, p_{n-1} + q, p_n - q) \sim \Gamma_\kappa^{(n)}(p_1, p_2, \dots, p_{n-1}, p_n) \quad (6)$$

in the r.h.s. of the flow equations. After making this approximation, some momenta in some of the  $n$ -point functions vanish, and the corresponding  $n$ -point functions can be obtained as the derivatives of  $m$ -point functions ( $m < n$ ) with respect to a constant background field.

Specifically, in eq. (3) for the 2-point function, the 3- and 4-point functions in the r.h.s. will contain respectively one and two vanishing momenta after we set  $q = 0$ . These can be related to the following derivatives of the 2-point function:

$$\Gamma_\kappa^{(3)}(p, -p, 0; \phi) = \frac{\partial \Gamma_\kappa^{(2)}(p, -p; \phi)}{\partial \phi}, \quad \Gamma_\kappa^{(4)}(p, -p, 0, 0; \phi) = \frac{\partial^2 \Gamma_\kappa^{(2)}(p, -p; \phi)}{\partial \phi^2}. \quad (7)$$

One then arrives at a closed equation for  $\Gamma_\kappa^{(2)}(p; \phi)$ :

$$\kappa \partial_\kappa \Gamma_\kappa^{(2)}(p^2; \phi) = J_d^{(3)}(p, \kappa; \phi) \left( \frac{\partial \Gamma_\kappa^{(2)}(p, -p; \phi)}{\partial \phi} \right)^2 - \frac{1}{2} I_d^{(2)}(\kappa; \phi) \frac{\partial^2 \Gamma_\kappa^{(2)}(p, -p; \phi)}{\partial \phi^2}, \quad (8)$$

where

$$J_d^{(n)}(p; \kappa; \phi) \equiv \int \frac{d^d q}{(2\pi)^d} \kappa \partial_\kappa R_\kappa(q^2) G_\kappa(p+q; \phi) G_\kappa^{(n-1)}(q; \phi), \quad (9)$$

and

$$I_d^{(n)}(\kappa; \phi) \equiv \int \frac{d^d q}{(2\pi)^d} \kappa \partial_\kappa R_\kappa(q^2) G_\kappa^n(q; \phi). \quad (10)$$

In fact, in order to preserve the relation

$$\Gamma_\kappa^{(2)}(p=0; \phi) = \frac{\partial^2 V_\kappa}{\partial \phi^2}, \quad (11)$$

$V_\kappa(\phi)$  being the effective potential, it is better to make the approximation (6) (followed by (7)) in the flow equation for  $\Sigma_\kappa(p; \phi)$  defined as

$$\Sigma_\kappa(p; \phi) = \Gamma_\kappa^{(2)}(p; \phi) - p^2 - \Gamma_\kappa^{(2)}(p=0; \phi). \quad (12)$$

The 2-point function is then obtained from  $\Gamma^{(2)}(p; \phi) = \partial^2 V_\kappa(\phi)/\partial \phi^2 + p^2 + \Sigma(p; \phi)$ , which demands the simultaneous solutions of the flow equations for  $V_\kappa(\phi)$  and  $\Sigma_\kappa(p; \phi)$ .

As shown in [13], even if the complete solution of these equations is a priori complicated, a simple, and still accurate, way of solving them consists in assuming in the various integrals

$$G_\kappa^{-1}(q; \phi) \simeq Z_\kappa q^2 + \partial^2 V_\kappa(\phi)/\partial \phi^2 + R_\kappa(q^2), \quad (13)$$

where  $Z_\kappa \equiv Z_\kappa(\phi=0)$ , with  $Z_\kappa(\phi) \equiv 1 + \partial \Sigma_\kappa(p; \phi)/\partial p^2|_{p=0}$ . This approximation is consistent with an improved version of the Local Potential Approximation (LPA, the first order of the derivative expansion), which includes explicitly a field renormalization factor  $Z_\kappa$  [5]. Doing so, the  $p=0$  sector decouples from the  $p \neq 0$  one. Moreover, it is useful to make use of the regulator [16]

$$R_\kappa(q^2) = Z_\kappa(\kappa^2 - q^2) \Theta(\kappa^2 - q^2), \quad (14)$$

which allows the functions  $J_d^{(n)}(p; \kappa; \phi)$  and  $I_d^{(n)}(\kappa; \phi)$  to be calculated analytically. The corresponding expressions can be found in [13]. In fact, all quantities are functions of  $\rho \equiv \phi^2/2$ . The problem is then reduced to the solution of the three flow equations for  $V_\kappa(\rho)$  and  $Z_\kappa(\rho)$ , for the  $p=0$  sector, and for  $\Sigma_\kappa(p; \rho)$ , in the  $p \neq 0$  one. As only the “effective mass”

$$m_\kappa^2(\rho) \equiv \frac{\partial^2 V_\kappa(\phi)}{\partial \phi^2} = \frac{\partial^2 V_\kappa(\rho)}{\partial \rho^2} + 2\rho \frac{\partial V_\kappa(\rho)}{\partial \rho} \quad (15)$$

(and its derivatives with respect to  $\rho$ ) enters in the  $p \neq 0$  sector, it is more convenient to work with the flow equation for  $m_\kappa^2(\rho)$  unstead of that for  $V_\kappa(\rho)$  itself. The non-trivial fact is that deriving twice the flow equation for  $V_\kappa(\rho)$  one gets a closed equation for  $m_\kappa^2(\rho)$ .

In order to make explicit the fixed point in the  $\kappa \rightarrow 0$  limit, it is better to work with dimensionless variables:

$$\mu_\kappa(\tilde{\rho}) \equiv Z_\kappa^{-1} \kappa^{-2} m_\kappa^2(\rho) \quad , \quad \chi_\kappa(\tilde{\rho}) \equiv Z_\kappa^{-1} Z_\kappa(\rho) \quad , \quad \tilde{\rho} \equiv K_d^{-1} Z_\kappa \kappa^{2-d} \rho \quad , \quad (16)$$

which, in the critical case, have a finite limit when  $\kappa \rightarrow 0$ . Above,  $K_d$  is a constant conveniently taken as  $K_d^{-1} \equiv d \, 2^{d-1} \pi^{d/2} \Gamma(d/2)$  (e.g.,  $K_3 = 1/(6\pi^2)$ ). In the  $p \neq 0$  sector, the dimensionfull variable  $p$  in the self-energy flow equation makes  $\Sigma_\kappa(p; \tilde{\rho})$  reach a finite value when  $\kappa \rightarrow 0$ . As discussed in [13], the inclusion of the flow equation for the renormalization factor  $Z_\kappa(\tilde{\rho})$  is essential in order to preserve the correct scaling behavior of  $\Gamma^{(2)}(p; \tilde{\rho})$  in the infrared limit.

Putting all together, in  $d = 3$ , the three flow equations that have to be solved are then

$$\kappa \partial_\kappa \mu_\kappa(\tilde{\rho}) = -(2 - \eta_\kappa) \mu_\kappa(\tilde{\rho}) + (1 + \eta_\kappa) \tilde{\rho} \mu'_\kappa(\tilde{\rho}) - \left(1 - \frac{\eta_\kappa}{5}\right) \left( \frac{\mu'_\kappa(\tilde{\rho}) + 2\tilde{\rho} \mu''_\kappa(\tilde{\rho})}{(1 + \mu_\kappa(\tilde{\rho}))^2} - \frac{4\tilde{\rho} \mu'_\kappa(\tilde{\rho})^2}{(1 + \mu_\kappa(\tilde{\rho}))^3} \right) \quad (17)$$

and

$$\begin{aligned} \kappa \partial_\kappa \chi_\kappa &= \eta_\kappa \chi_\kappa + (1 + \eta_\kappa) \tilde{\rho} \chi'_\kappa - 2\tilde{\rho} \frac{\mu_\kappa'^2}{(1 + \mu_\kappa(\tilde{\rho}))^4} \\ &+ 8\tilde{\rho} \chi'_\kappa \left(1 - \frac{\eta_\kappa}{5}\right) \frac{\mu'_\kappa}{(1 + \mu_\kappa(\tilde{\rho}))^3} - \left(1 - \frac{\eta_\kappa}{5}\right) \frac{\chi'_\kappa + 2\tilde{\rho} \chi''_\kappa}{(1 + \mu_\kappa(\tilde{\rho}))^2} \quad , \end{aligned} \quad (18)$$

together with

$$\eta_\kappa = \frac{\chi'_\kappa}{\chi'_\kappa/5 + (1 + \mu_\kappa(0))^2} \quad , \quad \kappa \partial_\kappa Z_\kappa = -\eta_\kappa Z_\kappa \quad , \quad (19)$$

for the  $p = 0$  sector, and

$$\begin{aligned} \kappa \partial_\kappa \Sigma_\kappa(p, \tilde{\rho}) &= (1 + \eta_\kappa) \tilde{\rho} \Sigma'_\kappa(p, \tilde{\rho}) + \frac{2\tilde{\rho} \mu_\kappa'^2(\tilde{\rho}) \kappa^2 Z_\kappa}{(1 + \mu_\kappa(\tilde{\rho}))^2} \left( f_\kappa(\tilde{p}, \tilde{\rho}) - \frac{2(1 - \eta_\kappa/5)}{(1 + \mu_\kappa(\tilde{\rho}))^2} \right) \\ &+ \frac{2\tilde{\rho} f_\kappa(\tilde{p}, \tilde{\rho})}{(1 + \mu_\kappa(\tilde{\rho}))^2} \left( 2\mu'_\kappa(\tilde{\rho}) \Sigma'_\kappa(p, \tilde{\rho}) + \frac{\Sigma_\kappa'^2(p, \tilde{\rho})}{\kappa^2 Z_\kappa} \right) - \frac{(1 - \eta_\kappa/5)}{(1 + \mu_\kappa(\tilde{\rho}))^2} (\Sigma'_\kappa(p, \tilde{\rho}) + 2\tilde{\rho} \Sigma''_\kappa(p, \tilde{\rho})) \end{aligned} \quad (20)$$

for the  $p \neq 0$  one. In these equations, the prime means  $\partial_{\tilde{\rho}}$  and we used the explicit expression for  $I_3^{(n)} = 2K_3 \kappa^{5-2n} Z_\kappa^{1-n} (1 - \eta_\kappa/5)/(1 + \mu_\kappa(\tilde{\rho}))^n$ . In eq. (20), we introduced the dimensionless expression  $f_\kappa$  defined as  $J_3^{(3)}(p; \kappa; \rho) \equiv K_3 \kappa^{-1} Z_\kappa^{-2}/(1 + \mu_\kappa(\tilde{\rho}))^2 \times f_\kappa(\tilde{p}; \tilde{\rho})$ , with  $\tilde{p} \equiv p/\kappa$ .

In [13], this strategy is used to get the 2-point function of the scalar model at criticality and zero external field (i.e.,  $\Sigma(p = 0, \rho = 0) = 0$ ), in  $d = 3$ . As remembered above, the function thus obtained has the correct shape, either in the scaling, perturbative and intermediate momenta regimes.

### III. EXPANSION IN POWERS OF THE FIELD

In this section, we shall compare the solution obtained in [13] using the procedure described above, with the solution of the same three flow equations expanded in powers of  $\tilde{\rho}$  and truncating up to a given order. Before doing so, let us first regard only the flow equation for the potential or, equivalently, that for the effective mass, i.e., eq. (17). This corresponds to the pure LPA sector and it is thus independent of the scheme presented in [12]. In  $d = 3$ , its expansion in powers of the field has been widely studied during the last ten years, using various regulators [18, 20]. In particular, it has been shown [21] that, using the regulator we consider here (see eq. (14)), the expansion seems to converge. This result follows when expanding both around finite and zero external field, although faster in the first case. However, as the latter is simpler, it is the strategy we shall try in the present work. In [21], the convergence is discussed studying the critical exponent  $\nu$ . In order to strengthen this conclusion, as a first step in our study, we have regarded the effect of the expansion on the values of the derivatives of  $\mu_\kappa(\tilde{\rho})$  with respect to  $\tilde{\rho}$ , at  $\tilde{\rho} = 0$  (that we shall call  $\mu_\kappa^{(n)}$ ) which are proportional to the couplings at zero momenta and zero external field:

$$\mu_\kappa(\tilde{\rho}) = \sum_{n=0}^{\infty} \frac{1}{n!} \mu_\kappa^{(n)} \tilde{\rho}^n. \quad (21)$$

This is motivated by the fact that the various terms  $\mu_\kappa^{(n)}$  shall appear in  $\Sigma_\kappa(p; \tilde{\rho})$  flow equation, eq. (20), when the latter shall be expanded around  $\tilde{\rho}$ . We have done this preliminary study within the LPA approximation (i.e., setting  $Z_\kappa \equiv 1$ ). Results are presented in Figure 1. The four plots present the fixed point value for the first 4 couplings,  $\mu_{\kappa=0}^{(n)}$ ,  $n = 0, \dots, 3$ . For each coupling, we present the result which follows by solving the complete LPA equation, eq. (17), together with the result obtained with the equation expanded in powers of  $\tilde{\rho}$ . For example, when going only up to the first order (i.e., neglecting all  $\mu_\kappa^{(n)}$  with  $n \geq 2$ ), the



corresponding equations for  $\mu_\kappa^{(0)}$  and  $\mu_\kappa^{(1)}$ , are:

$$\kappa \partial_\kappa \mu_\kappa^{(0)} = (\eta_\kappa - 2) \mu_\kappa^{(0)} - \frac{(1 - \eta_\kappa/5) \mu_\kappa^{(1)}}{(1 + \mu_\kappa^{(0)})^2} \quad (22)$$

and

$$\kappa \partial_\kappa \mu_\kappa^{(1)} = (2\eta_\kappa - 1) \mu_\kappa^{(1)} + \frac{6(1 - \eta_\kappa/5) (\mu_\kappa^{(1)})^2}{(1 + \mu_\kappa^{(0)})^3}. \quad (23)$$

which have to be solved simultaneously. (In fact, if solving just the LPA,  $\eta = 0$ ; nevertheless, we have kept  $\eta$  in eqs. (22)-(23) for a later use of these equations). When going to the second order, eq. (23) acquires a new term and a new flow equation, for  $\mu_\kappa^{(2)}$ , appears. And so on. According to Figure 1, an apparent convergence shows up. In all cases one observes that: 1) there seems to be an oscillating convergence, 2) the value of  $\mu^{(i)}$  is found with about 1% error truncating at order  $i + 3$ .

Let us now turn to the study of the flow equation for the 2-point function yielding from the scheme proposed in [12]. As for the effective potential (or the effective mass),  $\Gamma_\kappa^{(2)}(p; \rho)$  can be expanded in powers of the external field:

$$\Gamma_\kappa^{(2)}(p; \rho) = \sum_{n=0}^{\infty} \frac{2^n}{(2n)!} \Gamma_\kappa^{(2n+2)}(p, -p, 0, 0, \dots, 0; \rho)|_{\rho=0} \rho^n, \quad (24)$$

because

$$\Gamma_\kappa^{(m+2)}(p, -p, 0, 0, \dots, 0; \rho) = \frac{\partial^m \Gamma_\kappa^{(2)}(p; \phi)}{\partial \phi^m} \quad (25)$$

and we used that, at zero field, all odd vertex functions vanish. Equation (24) makes clear the point stated above: once approximation (6) is performed, truncating the expansion in powers of the external field is equivalent to neglecting high order vertices. Moreover, eqs. (24) and (25) show that the procedure proposed in [12] indeed includes all vertices, although approximately.

We have now all the ingredients to discuss the main goal of this paper: the analysis of the expansion of the three flow equations for  $\mu_\kappa(\tilde{\rho})$ ,  $Z_\kappa(\tilde{\rho})$  and  $\Sigma_\kappa(p; \tilde{\rho})$ , eqs. (17-20), around  $\tilde{\rho} = 0$ . In doing so, one can write:

$$\Sigma_\kappa(p, \tilde{\rho}) = \sum_{n=0}^{\infty} \frac{1}{n!} \Sigma_\kappa^{(n)}(p) \tilde{\rho}^n. \quad (26)$$

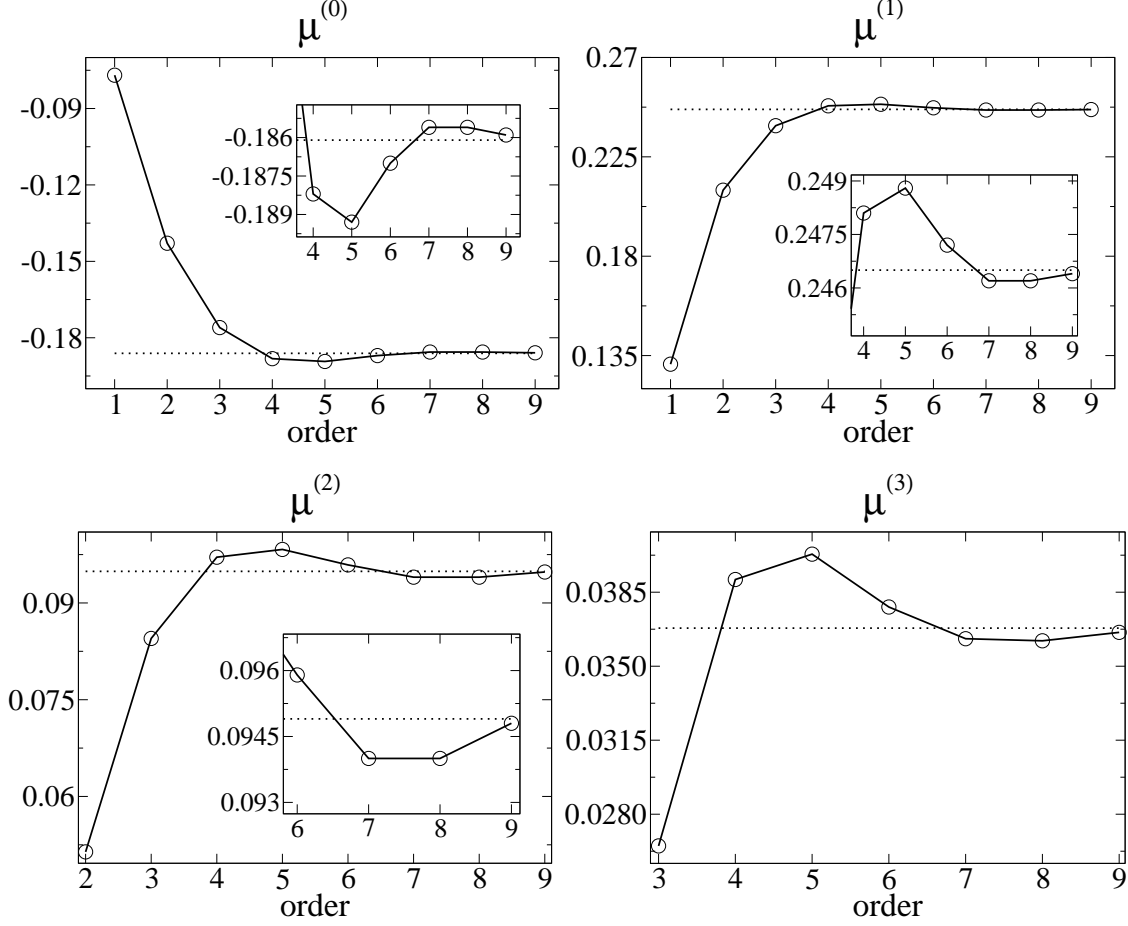


FIG. 1: First four dimensionless fixed point couplings at zero momenta and zero external field: results obtained by truncating the flow equation, as a function of the order; the corresponding value for the complete equation is represented by the dotted-line.

and

$$\chi_{\kappa}(\tilde{\rho}) = \sum_{n=0}^{\infty} \frac{1}{n!} \chi_{\kappa}^{(n)} \tilde{\rho}^n. \quad (27)$$

together with eq. (21). For example, when going to the first order, the equations that have to be solved are:

$$\kappa \partial_{\kappa} \Sigma_{\kappa}^{(0)}(p) = - \frac{(1 - \eta_{\kappa}/5) \Sigma_{\kappa}^{(1)}(p)}{(1 + \mu_{\kappa}^{(0)})^2} \quad (28)$$

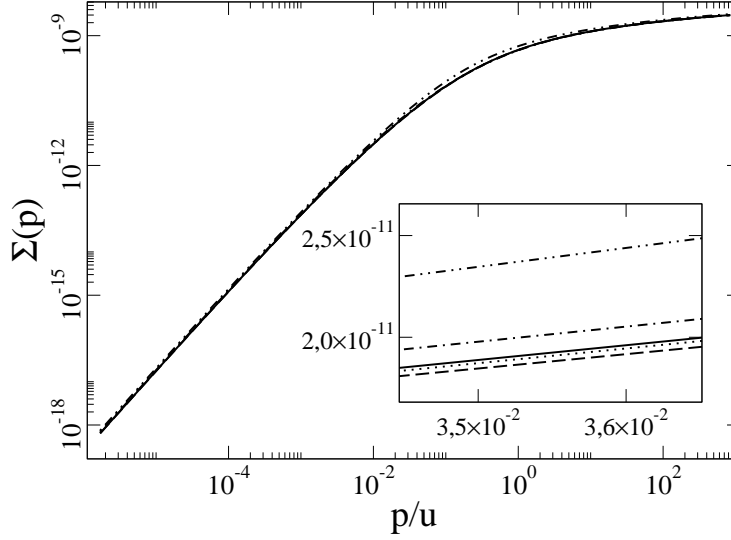


FIG. 2: Comparison of the self-energy when expanding only the flow equations for the self-energy  $\Sigma_\kappa(p; \tilde{\rho})$  and its derivative  $Z_\kappa(\tilde{\rho})$  (strategy I): truncation is made at first (double dotted-dashed), second (dotted-dashed), third (dashed) and fourth (dotted) order; the complete solution is given by the straight line.

and

$$\begin{aligned} \kappa \partial_\kappa \Sigma_\kappa^{(1)}(p) = & (1 + \eta_\kappa) \Sigma_\kappa^{(1)}(p) + \frac{2(\mu_\kappa^{(1)})^2 Z_\kappa \kappa^2}{(1 + \mu_\kappa^{(0)})^2} \left( f_\kappa(\tilde{p}, 0) - \frac{2(1 - \eta_\kappa/5)}{(1 + \mu_\kappa^{(0)})} \right) \\ & + \frac{2f_\kappa(\tilde{p}, 0)}{(1 + \mu_\kappa^{(0)})^2} \left( 2\mu_\kappa^{(1)} \Sigma_\kappa^{(1)}(p) + \frac{\Sigma_\kappa^{(1)}(p)^2}{\kappa^2 Z_\kappa} \right) + \frac{2(1 - \eta_\kappa/5) \mu_\kappa^{(0)} \Sigma_\kappa^{(1)}(p)}{(1 + \mu_\kappa^{(0)})^3} \end{aligned} \quad (29)$$

which correspond to the expansion of eq. (20),

$$\kappa \partial_\kappa \chi_\kappa^{(0)} = \eta_\kappa \chi_\kappa^{(0)} - \frac{(1 - \eta_\kappa/5) \chi_\kappa^{(1)}}{(1 + \mu_\kappa^{(0)})^2} \quad (30)$$

and

$$\kappa \partial_\kappa \chi_\kappa^{(1)} = (1 + 2\eta_\kappa) \chi_\kappa^{(1)} - \frac{2(\mu_\kappa^{(1)})^2}{(1 + \mu_\kappa^{(0)})^4} + \frac{10\mu_\kappa^{(0)} \chi_\kappa^{(1)} (1 - \eta_\kappa/5)}{(1 + \mu_\kappa^{(0)})^3} \quad (31)$$

which correspond to the expansion of eq. (18), together with eqs. (22) and (23).

In fact, it is possible to make two kinds of expansions. First, in order to isolate the effect of the field expansion just in the flow equations provided by the scheme presented in [12],

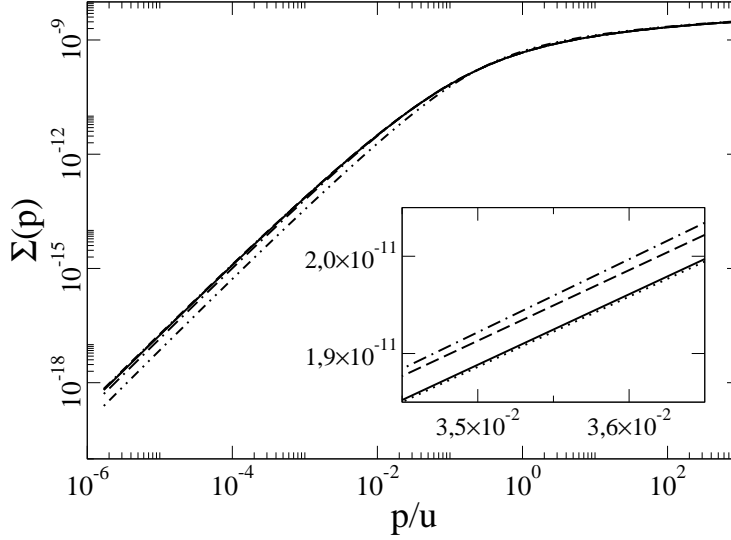


FIG. 3: Comparison of the self-energy when expanding the three flow equations (strategy II): truncation is made at first (double dotted-dashed), second (dotted-dashed), third (dashed) and fourth (dotted) order; the complete solution is given by the straight line.

we shall expand only the flow equations for  $\Sigma_\kappa(p; \tilde{\rho})$  and its derivative  $Z_\kappa(\tilde{\rho})$ , eqs. (20) and (18), solving exactly the differential flow equation for  $\mu_\kappa(\tilde{\rho})$ , eq. (17). For example, at first order, one should solve simultaneously eqs. (28-29), (30-31), and (17). This shall be called “strategy I”. Second, to consider all the effects, we shall make the expansion in the three flow equations, thus solving simultaneously eqs. (28-29), (30-31), and (22-23). We call this “strategy II”. Notice that, as explained in [13], in order to get the correct scaling behavior it is mandatory to treat the equations for  $Z_\kappa(\tilde{\rho})$  and  $\Sigma_\kappa(p; \tilde{\rho})$  with the same approximations; it is then not possible to solve one of them completely while expanding the other one.

Figure 2 presents the self-energy one gets truncating up to fourth order, following strategy I; it is also shown the function obtained in [13] (from now on, the latter function, obtained by solving the 3 differential equations, eqs. (17), (18) and (20), shall be called the “complete solution”). Figure 3 presents the same results when following strategy II. These Figures show that, in both strategies of expansion, truncating at first order one already gets a function with the correct shape in all momenta regimes.

In order to make a stringent evaluation of the approximate solution obtained doing the

expansion, we have calculated different numbers assessing the physical properties of the self-energy. First, as can be seen in both figures above, all solutions have, in the infrared ( $p \ll u$ ), the potential behavior characterising the scaling regime:  $\Sigma(p) + p^2 \sim p^{2-\eta}$ , where  $\eta$  is the anomalous dimension. We have checked, at each order and in both strategies, that the resulting self-energy does have scaling, and we extracted the corresponding value of  $\eta$ . In fact, this can be done in two different ways; either using the  $\kappa$ -dependence of  $Z_\kappa$  ( $\eta = -\lim_{\kappa \rightarrow 0} \kappa \partial_\kappa \log Z_\kappa$ ) or the  $p$ -dependence of  $\Sigma(p)$  stated above; we checked that those two values always coincide, within numerical uncertainties. Figure 4 presents the relative error for  $\eta$ , at each order, when compared with the value following from the complete solution. One observes: 1) in both strategies of expansions there is an apparent convergence, which is oscillatory; 2) the solution from strategy I reaches faster the correct result; 3) when following strategy I, already with a second order truncation the error is about 3% and it drops to less than 1% at the third order. Nevertheless, due to the mixed characteristic of strategy I, when using this strategy at higher orders numerical problems arise: indeed, this task demands the numerical evaluation of higher order derivatives of  $\mu_\kappa(\tilde{\rho})$ , to be used in the various flow equations obtained when expanding that of  $\Sigma_\kappa(p; \tilde{\rho})$ . If willing high precision in the result, strategy II is then numerically preferable.

A second number to assess the quality of the approximate solution is the critical exponent  $\nu$ . With the complete solution one gets  $\nu = 0.647$ , to be compared with the best accepted value [26]:  $\nu = 0.6304 \pm 0.0013$ . Figure 5 presents the relative error of the value of  $\nu$  extracted from the expansion. Once again, one observes that the convergence is much faster when following strategy I, i.e., when considering the effect of the expansion only on the self-energy equation.

The large momenta regime ( $p \gg u$ ) of the self-energy can be calculated using perturbation theory, yielding the well known logarithmic shape:  $\Sigma(p) \sim A \log(p/B)$ , where  $A$  and  $B$  are constants. For the complete solution presented in [13] one has  $A = u^2/9\pi^4$ , which is only 8% away from the exact result  $A = u^2/96\pi^2$ . We have checked that, at any order and in both strategies of expansion, the approximate solution for the self-energy has the correct shape, with  $A = u^2/9\pi^4$ . This is due to the fact that already the first order in the expansion of  $\Sigma_\kappa(p; \tilde{\rho})$  around  $\tilde{\rho} = 0$  contains the same 2-loop diagrams contributing to the complete solution.

In order to study the quality of the self-energy in the intermediate momenta regime, we

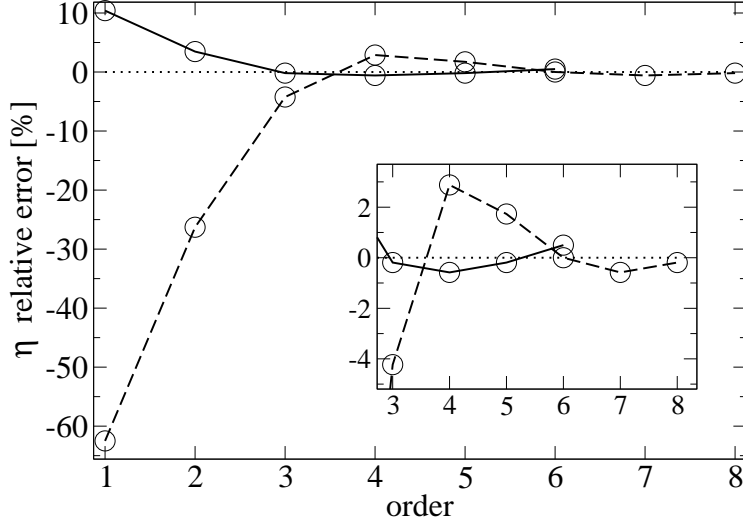


FIG. 4: Relative error (measured in percent) for the anomalous dimension, with respect to the value yielding from the complete solution, as a function of the truncation order. Full line: expanding only the flow equations for the self-energy  $\Sigma_\kappa(p; \rho)$  and its derivative  $Z_\kappa(\rho)$  (strategy I); dashed line: expanding all flow equations (strategy II).

have calculated a quantity which is very sensitive to this cross-over region:

$$\Delta\langle\phi^2\rangle = \int \frac{d^3p}{(2\pi)^3} \left( \frac{1}{p^2 + \Sigma(p)} - \frac{1}{p^2} \right). \quad (32)$$

(the integrand is non zero only in the region  $10^{-3} \lesssim p/u \lesssim 10$ , see for example [27]). This quantity received recently much attention because it has been shown [28] that for a scalar model with  $O(N)$  symmetry, in  $d = 3$  and  $N = 2$ , it determines the shift of the critical temperature of the weakly repulsive Bose gas. It has then been widely evaluated by many methods, for different values of  $N$ , in particular, for  $N = 1$ . With the numerical solution found in [13], one gets a number almost within the error bars of the best accepted results available in the literature, using lattice and 7 loops resummed perturbative calculations. In Figure 6 we plot the relative error in  $\Delta\langle\phi^2\rangle$ , at each order of the expansion, when compared with the complete solution result found in [13]. One can appreciate that 1) for both expansion strategies there is an apparent convergence, which is also oscillatory; 2) in both strategies, already with a second order truncation the error is about 1%.

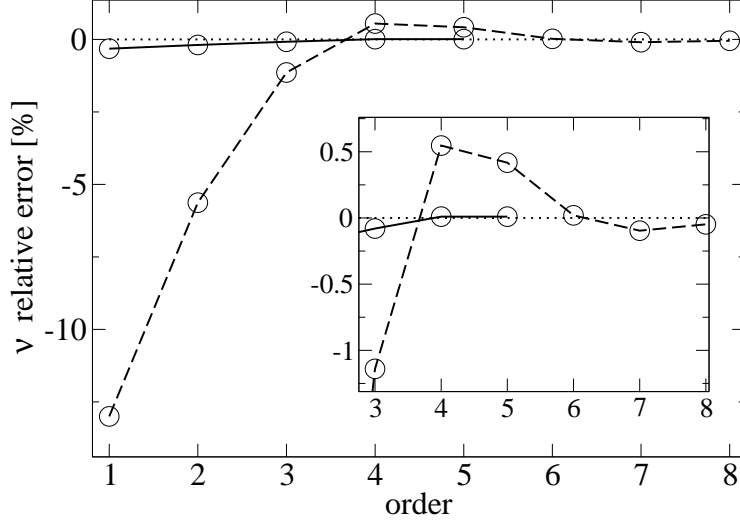


FIG. 5: Relative error (measured in percent) for the critical exponent  $\nu$ , with respect to the value yielding from the complete solution, as a function of the truncation order. Full line: expanding only the flow equations for the self-energy  $\Sigma_\kappa(p; \rho)$  and its derivative  $Z_\kappa(\rho)$  (strategy I); dashed line: expanding all flow equations (strategy II).

#### IV. SUMMARY AND CONCLUSIONS

In the framework of the calculation procedure introduced in [12] to get  $n$ -point functions at finite momenta, we have studied the effect of an expansion in powers of the field. This strategy is usually employed in derivative expansion. We considered here the calculation of the 2-point function of the scalar field, at criticality, in  $d = 3$ .

The calculation of the 2-point function demands the study of both the  $p = 0$  and the  $p \neq 0$  sectors. While the first one is given by the well studied derivative expansion flow equation for the effective potential, the latter follows from the approximation scheme introduced in [12] to calculate the flow of  $\Sigma_\kappa(p; \rho)$ . We used two different strategies to perform the expansion, both of them around zero external field: either expanding only the flow equation for the self-energy (and its derivative) (strategy I), or both the effective potential and the self-energy (and its derivative) flow equations (strategy II). We have studied convergence properties of various quantities measuring physical properties of the self-energy in all momenta regimes: the critical exponents  $\eta$  and  $\nu$  of the infrared regime, the coefficient of the ultraviolet logarithm, and  $\Delta\langle\phi^2\rangle$  which is dominated by the crossover momenta regime. We found that, within both

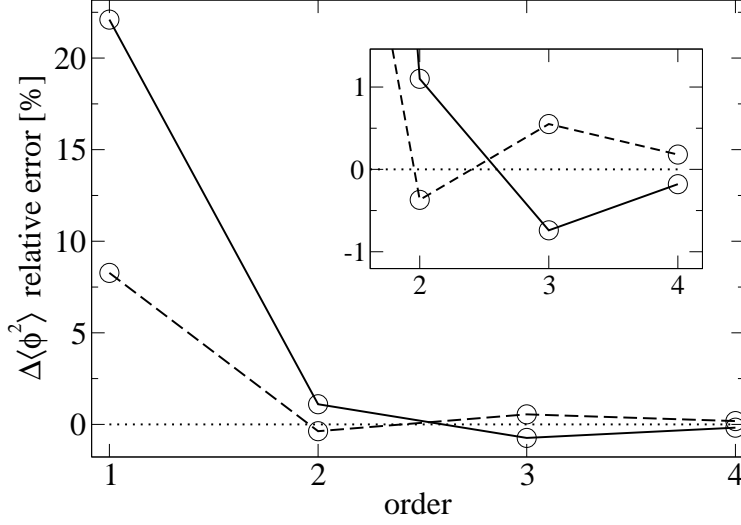


FIG. 6: Relative error (measured in percent) for  $\Delta\langle\phi^2\rangle$ , with respect to the value yielding from the complete solution, as a function of the truncation order. Full line: expanding only the flow equations for the self-energy  $\Sigma_\kappa(p; \rho)$  and its derivative  $Z_\kappa(\rho)$  (strategy I); dashed line: expanding all flow equations (strategy II).

strategies, the series for these four physical quantities rapidly converge. The convergence is faster when using strategy I, i.e., when making the expansion only for the approximate flow equation resulting from the method presented in [12]. Nevertheless, due to numerical difficulties, if trying to go to high order expansion, it is preferable to use strategy II, i.e., expanding also the effective potential flow equation, resulting from the derivative expansion scheme.

Although results presented in this article could apply only for the scalar model studied here, or even only for  $d = 3$ , two preliminary conclusions arise. First, when considering more involved models, it is possible to use the field expansion on top of the strategy proposed in [12], in order to get, at least, the shape of correlation functions at finite momenta. Second, the present work gives a first insight on the quality of the usual strategy of ignoring high order vertices in the flow equations for the  $n$ -point functions: According to this example, even including completely the 4-point vertex (i.e., in the language of field expansion, going only up to the first order of the expansion), when describing the deep infrared regime one could make errors as big as 60% (see the value of  $\eta$  in figure 4). If one wants results with less than 5% error, the inclusion of up to 8-point vertices (i.e., going up to order third)



is necessary. Finally, in situations where truncation is not a priori convenient, e.g., with non-zero external field or in  $d = 2$ , the only way out is the complete solution of approximate flow equations as, for example, those following from the procedure presented in [12].

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