NOTES ON C-FREE PROBABILITY WITH AMALGAMATION

MIHAI POPA

Abstract. As in the cases of freeness and monotonic independence, the notion of conditional freeness is meaningful when complex-valued states are replaced by positive conditional expectations. In this framework, the paper presents several positivity results, a version of the central limit theorem and an analogue of the conditionally free R-transform constructed by means of multilinear function series.

1. Introduction

The paper addresses a topic related to conditionally free (or, shortly, using the term from [2], c-free) probability. This notion was developed in the '90's (see [1], [2]) as an extension of freeness within the framework of $*$-algebras endowed with not one, but two states. Namely, given a family of unital algebras $\{\mathfrak{A}\}_{i\in I}$, each $\mathfrak{A}_i$ endowed with two expectations $\varphi_i, \psi_i : \mathfrak{A}_i \longrightarrow \mathbb{C}$, their c-free product is the triple $(\mathfrak{A}, \varphi, \psi)$, where:

(i) $\mathfrak{A} = *_{1\in I}\mathfrak{A}_i$ is the free product of the algebras $\mathfrak{A}_i$.

(ii) $\psi = *_{i\in I}\psi_i$ and $\varphi = *_{(\psi_i, i\in I)}\varphi_i$ are conditional expectations given by the relations

(a) $\psi(a_1 \cdots a_n) = 0$

(b) $\varphi(a_1 \cdots a_n) = \varphi_{\varepsilon(1)}(a_1)\cdots\varphi_{\varepsilon(n)}(a_n)$

for all $a_j \in \mathfrak{A}_{\varepsilon(j)}, j = 1, \dots, n$ such that $\psi_{\varepsilon(j)}(a_j) = 0$ and $\varepsilon(1) \neq \cdots \neq \varepsilon(n)$.

An important result is that if the $\mathfrak{A}_i$ are $*$-algebras and φ_i, ψ_i are states, then φ and ψ are also states.

In [2] is constructed a c-free version of Voiculescu's R-transform, which we will call the ${}^cR$-transform, with the property that ${}^cR_{X+Y} = {}^cR_X + {}^cR_Y$ if X and Y are c-free elements from the algebra $\mathfrak{A}$ relative to φ and ψ (i.e. the relations (a) and (b) from the definition of the c-free product hold true for the subalgebras generated by X and Y.)

In [6], the notion of c-freeness is extended to the case when $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ and $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a conditional expectation, while ψ is still $\mathbb{C}$-valued. Also, (see Theorem 3, Section 6, from [6]) the construction is discussed in an even more general situation, when φ, ψ are operator valued function of the form $P_0\pi(a)|_{\mathcal{H}_0}$ with π a $*$-representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ and P_0 is the orthogonal projection onto the Hilbert subspace $\mathcal{H}_0$ of $\mathcal{H}$.

In [8] it was proved that for $\mathfrak{A}$ a $*$-algebra, the analogous construction with both φ and ψ valued in a C^*-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ still retains the positivity. The present paper further develops this result.

The apparatus of multilinear function series is used in recent work of K. Dykema ([3] and [4]) to construct suitable analogues for the R and S-transforms in the framework of freeness with amalgamation. We will show that this construction is also appropriate for the ${}^cR$-transform mentioned above. The techniques used differ from the ones of [3], the Fock space type construction being substituted by combinatorial techniques similar to [2] and [7].

The basic definitions and positivity results are stated in Section 2. Section 3 describes the construction and the basic property of the multilinear function series ${}^cR$-transform and Section 4 treats the central limit theorem and a related positivity result.

2. Definitions and positivity results

Definition 2.1. Let $\mathfrak{A}_i, i \in \mathfrak{I}$, a family of algebras, all containing the subalgebra $\mathfrak{B}$. Suppose $\mathfrak{D}$ is a subalgebra of $\mathfrak{B}$ and $\Psi_i : \mathfrak{A}_i \longrightarrow \mathfrak{D}$ and $\Phi_i : \mathfrak{A}_i \longrightarrow \mathfrak{B}$ are conditional expectations, $i \in \mathfrak{I}$. We say that the triple $(\mathfrak{A}, \Phi, \Psi) = \ast_{i\in\mathfrak{I}}(\mathfrak{A}_i, \Phi_i, \Psi_i)$ is the *conditionally free product* with amalgamation over $(\mathfrak{B}$,$\mathfrak{D})$, or shortly, the *c-free product*, of the triples $(\mathfrak{A}_i, \Phi_i, \Psi_i)_{i\in\mathfrak{I}}$ if

(1) $\mathfrak{A}$ is the free product with amalgamation over $\mathfrak{B}$ of the family $(\mathfrak{A}_i)_{i\in\mathfrak{I}}$
(2) $\Psi = \ast_{i\in\mathfrak{I}}\Psi_i$ and $\Phi = \ast_{(\Psi_i), i\in\mathfrak{I}}\Phi_i$ are determined by the relations

$$\begin{aligned} \Psi(a_1a_2\dots a_n) &= 0 \\ \Phi(a_1a_2\dots a_n) &= \Phi(a_1)\Phi(a_2)\dots\Phi(a_n) \end{aligned}$$

for any $a_i \in \mathfrak{A}_{\varepsilon(i)}, \varepsilon(i) \in \{1, 2\}$, such that $\varepsilon(1) \neq \varepsilon(2) \neq \cdots \neq \varepsilon(n)$ and $\Psi_{\varepsilon(i)}(a_i) = 0$.

When $\mathfrak{D} = \mathbb{C}$, this definition reduces to the one given in [6]. When both $\mathfrak{B}$ and $\mathfrak{D}$ are equal to $\mathbb{C}$, this definition was given in [2].

When discussing positivity, we need a $\ast$-structure on our algebras. We will demand that $\mathfrak{B}$ and $\mathfrak{D}$ be C*-algebras, while $\mathfrak{A}_i$ and $\mathfrak{A}$ are only required to be $\ast$-algebras.

The following results are slightly modified versions of Lemma 6.4 and Theorem 6.5 from [8].

Lemma 2.2. *Let $\mathfrak{B}$ be a C^*-algebra and $\mathfrak{A}_1$, $\mathfrak{A}_2$ be two $\ast$-algebras containing $\mathfrak{B}$ as a $\ast$-subalgebra, endowed with positive conditional expectations $\Phi_j : \mathfrak{A}_j \longrightarrow \mathfrak{B}, j = 1, 2$ If $a_1, \dots, a_n \in \mathfrak{A}_1$, $a_{n+1}, \dots, a_{n+m} \in \mathfrak{A}_2$ and $A = (A_{i,j}) \in M_{n+m}(\mathfrak{B})$ is the matrix with the entries*

$$A_{i,j} = \begin{cases} \Phi_1(a_i^*a_j) & \text{if} \quad i, j \leq n \\ \Phi_1(a_i^*)\Phi_2(a_j) & \text{if} \quad i \leq n, j > n \\ \Phi_2(a_i^*)\Phi_1(a_j) & \text{if} \quad i > n, j \leq n \\ \Phi_2(a_i^*a_j) & \text{if} \quad i, j > n \end{cases}$$

then A is positive.

Proof. The vector space $\mathfrak{E} = \mathfrak{B} \oplus \ker(\Phi_1) \oplus \ker(\Phi_2)$ has a $\mathfrak{B}$-bimodule structure given by the algebraic operations on $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Consider the $\mathfrak{B}$-sesquilinear pairing

$$\langle\cdot,\cdot\rangle : \mathfrak{E} \times \mathfrak{E} \longrightarrow \mathfrak{B}$$

determined by the relations:

$$\begin{aligned} \langle b_1, b_2\rangle &= b_1^*b_2, \quad \text{for } b_1, b_2 \in \mathfrak{B} \\ \langle u_j, v_j\rangle &= \Phi_j(u_j^*v_j), \quad \text{for } u_j, v_j \in \ker(\Phi_j), j = 1, 2 \\ \langle u_1, u_2\rangle &= \langle u_2, u_1\rangle = 0 \quad \text{for } u_1 \in \ker(\Phi_1), \text{ and } u_2 \in \ker(\Phi_2). \\ \langle b, u_j\rangle &= \langle u_j, b\rangle = 0 \quad \text{for all } b \in \mathfrak{B}, u_j \in \mathfrak{A}_j \end{aligned}$$

With this notation,

$$A_{i,j} = \langle a_i, a_j \rangle,$$

hence it suffices to show that for all $a \in \mathfrak{E}$

$$\langle a, a \rangle \geq 0.$$

Indeed, for an element $a = b + u_1 + u_2$ with $b \in \mathfrak{B}, u_j \in \ker(\Phi_j), j = 1, 2$, we have:

$$\begin{aligned}
\langle a, a \rangle &= \langle b + u_1 + u_2, b + u_1 + u_2 \rangle \\
&= \langle b, b \rangle + \langle u_1, u_1 \rangle + \langle u_2, u_2 \rangle \\
&= b^* b + \Phi_1(u_1^* u_1) + \Phi_2(u_2^* u_2) \\
&\geq 0
\end{aligned}$$

□

Theorem 2.3. *Let $\mathfrak{B}$ be a C^*-algebra and $\mathfrak{D}$ a C^*-subalgebra of $\mathfrak{B}$. Suppose that $\mathfrak{A}_1$, $\mathfrak{A}_2$ are $*$-algebras containing $\mathfrak{B}$, each endowed with two positive conditional expectations $\Phi_j : \mathfrak{A}_j \longrightarrow \mathfrak{B}$, and $\Psi_j : \mathfrak{A}_j \longrightarrow \mathfrak{D}$, $j = 1, 2$. Consider the c-free product $(\mathfrak{A}, \Phi, \Psi) = \ast_{i=1,2}(\mathfrak{A}_i, \Phi_i, \Psi_i)$.*

Then Φ and Ψ are positive.

Proof. The positivity of Ψ is by now a classical result in the theory of free probability with amalgamation over a C^*-algebra (for example, see [9], Theorem 3.5.6). For the positivity of Φ we have to show that $\Phi(a^* a) \geq 0$ for any $a \in \mathfrak{A}$.

Any element of $\mathfrak{A}$ can be written as

$$a = \sum_{k=1}^{N} s_{1,k} \dots s_{n(k),k},$$

where $s_{j,k} \in \mathfrak{A}_{\varepsilon(j,k)}$ $\varepsilon(1,k) \neq \varepsilon(2,k) \neq \cdots \neq \varepsilon(n(k),k)$.
Writing

$$s_{(j,k)} = s_{(j,k)} - \Psi(s_{(j,k))} + \Psi(s_{(j,k))}$$

and expanding the product, we can consider a of the form

$$a = d + \sum_{k=1}^{N} a_{1,k} \dots a_{n(k),k}$$

where

$$\begin{aligned}
d &\in \mathfrak{D} \subset \mathfrak{B} \\
a_{j,k} &\in \mathfrak{A}_{\varepsilon(j,k)},\ \varepsilon(1,k) \neq \varepsilon(2,k) \neq \cdots \neq \varepsilon(n(k),k) \\
\Psi_{\varepsilon(j,k)}(a_{j,k}) &= 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Phi(a^* a) &= \Phi\left(d^* d + d^* \left(\sum_{k=1}^{N} a_{1,k} \dots a_{n(k),k} \right) + \left(\sum_{k=1}^{N} a_{1,k} \dots a_{n(k),k} \right)^* d + \right. \\
&\qquad \left. \left(\sum_{k=1}^{N} a_{1,k} \dots a_{n(k),k} \right)^* \left(\sum_{k=1}^{N} a_{1,k} \dots a_{n(k),k} \right) \right).
\end{aligned}$$

Since Φ is a conditional expectation and $d \in \mathfrak{D} \subset \mathfrak{B}$, the above equality becomes

$$\begin{aligned}\Phi(a^*a) &= d^*d + \sum_{k=1}^{N} d^*\Phi(a_{1,k}\dots a_{n(k),k}) + \sum_{k=1}^{N} \Phi(a^*_{n(k),k}\dots a^*_{1,k})d \\ &\quad + \sum_{k,l=1}^{N} \Phi(a^*_{n(k),k}\dots a^*_{1,k}a_{1,l}\dots a_{n(l),l}).\end{aligned}$$

Using the definition of the conditionally free product with amalgamation over $\mathfrak{B}$ and that $\Psi_{\varepsilon(j,k)}(a_{j,k}) = 0$ for all j, k, one further has

$$\begin{aligned}\Phi(a^*a) &= d^*d + \sum_{k=1}^{N} \Phi(d^*a_{1,k})\Phi(a_{2,k})\dots\Phi(a_{n(k),k}) \\ &\quad + \sum_{k=1}^{N} \Phi(a_{n(k),k})^*\dots\Phi(a^*_{2,k})\Phi(a^*_{1,k}d) \\ &\quad + \sum_{k,l=1}^{N} \left(\Phi(a_{n(k),k})^*\dots\Phi(a^*_{2,k})\right)\Phi(a^*_{1,k}a_{1,l})\Phi(a_{2,l})\dots\Phi(a_{n(l),l})\end{aligned}$$

that is

$$\begin{aligned}\Phi(a^*a) &= d^*d + \sum_{k=1}^{N} \Phi(d^*a_{1,k})\left[\Phi(a_{2,k})\dots\Phi(a_{n(k),k})\right] \\ &\quad + \sum_{k=1}^{N} \left[\Phi(a_{2,k})\dots\Phi(a_{n(k),k})\right]^*\Phi(a^*_{1,k}d) \\ &\quad + \sum_{k,l=1}^{N} \left[\Phi(a_{2,k})\dots\Phi(a_{n(k),k})\right]^*\Phi(a^*_{1,k}a_{1,l})\left[\Phi(a_{2,l})\dots\Phi(a_{n(l),l})\right]\end{aligned}$$

Denote now $a_{1,N+1} = d$ and $v_k = \Phi(a_{2,k})\dots\Phi(a_{n(k),k})$.

From Lemma 2.2, the matrix $S = \left(\Phi(a^*_{1,i}a_{1,j})_{i,j=1}^{N+1}\right)$ is positive in $M_{N+1}(\mathfrak{B})$, therefore

$$S = T^*T, \text{ for some } T \in M_{N+1}(\mathfrak{B}).$$

The identity for $\Phi(a^*a)$ becomes:

$$\begin{aligned}\Phi(a^*a) &= (v_1, \dots, v_N, 1)^*T^*T(v_1, \dots, v_N, 1) \\ &\geq 0,\end{aligned}$$

as claimed. $\square$

Theorem 2.4. *Assume that $\mathfrak{I} = \bigcup_{j\in\mathfrak{I}} \mathfrak{I}_j$ is a partition of $\mathfrak{I}$. Then:*

$$*_{j\in\mathfrak{I}}\left(*_{i\in\mathfrak{I}_j}(\mathfrak{A}_i, \Phi_i, \Psi_i)\right) = *_{i\in\mathfrak{I}}(\mathfrak{A}_i, \Phi_i, \Psi_i)$$

Proof. The proof is identical to the proofs of similar results in [6] and [2]. Consider $a_i \in \mathfrak{A}_{\varepsilon(i)}, 1 \leq i \leq m$ such that $\varepsilon(1) \neq \varepsilon(2 \neq \cdots \neq \varepsilon(m)$ and $\Psi_{\varepsilon(i)}(a_i) = 0$. Let $1 = i_0 < i_1 < \cdots < i_k = m$ and $\mathfrak{I}_l = \{\varepsilon(i), i_{l-1} \leq i < i_l\}$.

Since

$$\left(\ast_{j\in\mathfrak{J}_l}\Psi_j\right)\left((a_{i_l-1}\cdots a_{i_l})\right)=0,$$

it suffices to show that

$$\Phi(a_1\cdots a_m)=\prod_{l=1}^{k}\left[(\ast_{(\Psi_j),j\in\mathfrak{J}_l}\Phi_j)(a_{i_l-1}\cdots a_{i_l})\right].$$

But

$$\Phi(a_1\cdots a_m)=\Phi_{\varepsilon(1)}(a_1)\cdots\Phi_{\varepsilon(m)}(a_m)$$

while, since $\Psi_{\varepsilon(i)}(a_i)=0$,

$$(\ast_{(\Psi_j),j\in\mathfrak{J}_l}\Phi_j)(a_{i_l-1}\cdots a_{i_l})=\Phi_{i_l-1}(a_{i_l-1})\cdots\Phi_{i_l}(a_{i_l})$$

and the conclusion follows. □

Definition 2.5. Let $\mathfrak{A}$ be an algebra (respectively a $*$-algebra), $\mathfrak{B}$ a subalgebra ($*$-subalgebra) of $\mathfrak{A}$ and $\mathfrak{D}$ a subalgebra ($*$-subalgebra) of $\mathfrak{B}$. Suppose $\mathfrak{A}$ is endowed with the conditional expectations $\Psi:\mathfrak{A}\longrightarrow\mathfrak{D}$ and $\Phi:\mathfrak{A}\longrightarrow\mathfrak{D}$.

(i) The subalgebras ($*$-subalgebras) $(\mathfrak{A}_i)_{i\in\mathfrak{J}}$ of $\mathfrak{A}$ are said to be *c-free* with respect to (Φ,Ψ) if
  (a) $(\mathfrak{A}_i)_{i\in\mathfrak{J}}$ are free with respect to Ψ.
  (b) if $a_i\in\mathfrak{A}_{\varepsilon(i)}, 1\leq i\leq m, \varepsilon(1)\neq\cdots\neq\varepsilon(m)$ and $\Psi(a_i)=0$, then

$$\Phi(a_1\cdots a_m)=\Phi(a_1)\cdots\Phi(a_m).$$

(ii) The elements $(X_i)_{i\in\mathfrak{J}}$ of $\mathfrak{A}$ are said to be c-free with respect to (Φ,Ψ) if the subalgebras ($*$-subalgebras) generated by $\mathfrak{B}$ and X_i are c-free with respect to (Φ,Ψ).

We will denote by $\mathfrak{B}\langle\xi\rangle$ the non-commutative algebra of polynomials in the symbol ξ and with coefficients from $\mathfrak{B}$ (the coefficients do not commute with the symbol ξ). If I is a family of indices, $\mathfrak{B}\langle\{\xi_i\}_{i\in I}\rangle$ will denote the algebra of polynomials in the non-commuting variables $\{\xi\}_{i\in I}$ and with coefficients from $\mathfrak{B}$. We will identify $\mathfrak{B}\langle\{\xi_i\}_{i\in I}\rangle$ with the free product with amalgamation over $\mathfrak{B}$ of the family $\{\mathfrak{B}\langle\xi_i\rangle\}_{i\in I}$.

If $\mathfrak{A}$ is a $*$-algebra and $\mathfrak{B}$ is with the C^*-algebra, $\mathfrak{B}\langle\xi\rangle$ will also be considered with a $*$-algebra structure, by taking $\xi^*=\xi$. If X is a selfadjoint element from $\mathfrak{A}$, we define the conditional expectations

$$\Phi_X,\Psi_X:\mathfrak{B}\langle\xi\rangle\longrightarrow\mathfrak{B}$$

given by

$$\begin{aligned}\Phi_X(f(\xi)) &= \Phi(f(X))\\ \Psi_X(f(\xi)) &= \Psi(f(X))\end{aligned}$$

for any $f(\xi)\in\mathfrak{B}\langle\xi\rangle$.

Corollary 2.6. *Suppose that $\mathfrak{A}$ is a $*$-algebra and X and Y are c-free selfadjoint elements of $\mathfrak{A}$ such that the mappings Φ_X and Φ_Y are positive. Then the mapping Φ_{X+Y} is also positive.*

Proof. Since the mappings

$$\Phi_X : \mathfrak{B}\langle\xi_1\rangle \longrightarrow \mathfrak{B}$$
$$\Phi_Y : \mathfrak{B}\langle\xi_2\rangle \longrightarrow \mathfrak{B}$$

are positive, from Proposition 2 so is

$$\Phi_x *_{(\Psi_X,\Psi_Y)} \Phi_Y : \mathfrak{B}\langle\xi_1\rangle *_{\mathfrak{B}} \mathfrak{B}\langle\xi_2\rangle = \mathfrak{B}\langle\xi_1,\xi_2\rangle \longrightarrow \mathfrak{B}$$

Remark also that

$$i_Z : \mathfrak{B}\langle\xi\rangle \ni f(\xi) \mapsto f(X+Y) \in \mathfrak{B}\langle\xi_1\rangle *_{\mathfrak{B}} \mathfrak{B}\langle\xi_2\rangle$$

is a positive $\mathfrak{B}$-functional.

The conclusion follows from the fact that the c-freeness of X and Y is equivalent to

$$\Phi_{X+Y} = (\Phi_X *_{(\Psi_X,\Psi_Y)} \Phi_Y) \circ i_{X+Y}.$$

□

3. Multilinear function series and the ${}^cR$-transform

Let $\mathfrak{A}$ be a $*$-algebra containing the C^*-algebra $\mathfrak{B}$, endowed with a conditional expectation $\Psi : \mathfrak{A} \longrightarrow \mathfrak{B}$. If X is a selfadjoint element of $\mathfrak{A}$, then the moment of order n of X is the mapping

$$\begin{aligned} m_X^{(n)} &: \underbrace{\mathfrak{B} \times \cdots \times \mathfrak{B}}_{n-1 \text{ times}} \longrightarrow \mathfrak{B} \\ m_X^{(n)}(b_1,\dots,b_{n-1}) &= \Psi(Xb_1X\dots Xb_{n-1}X) \end{aligned}$$

If $\mathfrak{B} = \mathbb{C}$, then the moment-generating series of X

$$m_X(z) = \sum_{n=0}^{\infty} \Psi(X^n)z^n$$

encodes all the information about the moments of X. For $\mathfrak{B} \neq \mathbb{C}$, the straightforward generalization

$$\mathfrak{m}_X(z) = \sum_{n=0}^{\infty} \Psi(X^n)z^n$$

generally fails to keep track of all the possible moments of X. A solution to this inconvenience was proposed in [3], namely the moment-generating multilinear function series of X. Before defining this notion, we will briefly recall the construction and several results on multilinear function series.

Let $\mathfrak{B}$ be an algebra over a field K. We set $\widetilde{\mathfrak{B}}$ equal to $\mathfrak{B}$ if $\mathfrak{B}$ is unital and to the unitization of $\mathfrak{B}$ otherwise. For $n \geq 1$, we denote by $\mathcal{L}_n(\mathfrak{B})$ the set of all K-multilinear mappings

$$\omega_n : \underbrace{\mathfrak{B} \times \cdots \times \mathfrak{B}}_{n \text{ times}} \longrightarrow \mathfrak{B}$$

A *formal multilinear function series* over $\mathfrak{B}$ is a sequence $\omega = (\omega_0, \omega_1, \dots)$, where $\omega_0 \in \widetilde{\mathfrak{B}}$ and $\omega_n \in \mathcal{L}_n(\mathfrak{B})$ for $n \geq 1$. According to [3], the set of all multilinear function series over $\mathfrak{B}$ will de denoted by $Mul[[\mathfrak{B}]]$.

For $\alpha, \beta \in Mul[[\mathfrak{B}]]$, the *sum* $\alpha+\beta$ and the *formal product* $\alpha\beta$ are the elements from $Mul[[\mathfrak{B}]]$ defined by:

$$\begin{aligned}(\alpha+\beta)_n(b_1,\dots,b_n) &= \alpha_n(b_1,\dots,b_n)+\beta_n(b_1,\dots,b_n)\\ (\alpha\beta)_n(b_1,\dots,b_n) &= \sum_{k=0}^{n}\alpha_k(b_1,\dots,b_k)\beta_{n-k}(b_{k+1},\dots,b_n\end{aligned}$$

for any $b_1,\dots,b_n \in \mathfrak{B}$.

If $\beta_0 = 0$, then the *formal composition* $\alpha\circ\beta \in Mul[[\mathfrak{B}]]$ is defined by

$$(\alpha\circ\beta)_0 = \alpha_0$$

and, for $n \geq 1$, by

$$(\alpha\circ\beta)_n(b_1,\dots,b_n) = \sum_{k=1}^{n}\sum_{\substack{p_1,\dots,p_k\geq 1\\ p_1+\cdots+p_k=n}}\alpha_k(\beta_{p_1}(b_1,\dots,b_{p_1}),\dots,\beta_{p_k}(b_{q_k+1},\dots,b_{q_k+p_k}))$$

where $q_j = p_1+\cdots+p_{j-1}$.

One can work with elements of $Mul[[\mathfrak{B}]]$ as if they were formal power series. The relevant properties are described in [3], Proposition 2.3 and Proposition 2.6. As in [3], we use 1, respectively I, to denote the identity elements of $Mul[[\mathfrak{B}]]$ relative to multiplication, respectively composition. In other words, $1 = (1,0,0,\dots)$ and $I = (0, id_{\mathfrak{B}}, 0, 0, \dots)$. We will also use the fact that an element $\alpha \in Mul[[\mathfrak{B}]]$ has an inverse with respect to formal composition, denoted $\alpha^{\langle -1\rangle}$, if and only if α has the form $(0,\alpha_1,\alpha_2,\dots)$ with α_1 an invertible element of $\mathcal{L}_1(\mathfrak{B})$.

Definition 3.1. With the above notation, the moment-generating multilinear function series $\mathcal{M}_X$ of X is the element of $Mul[[\mathfrak{B}]]$ such that:

$$\begin{aligned}\mathcal{M}_{X,0} &= \Psi(X)\\ \mathcal{M}_{X,n}(b_1,\dots,b_n) &= \Psi(Xb_1X\cdots Xb_nX).\end{aligned}$$

Given an element $\alpha \in Mul[[\mathfrak{B}]]$, the multilinear function series R_α is defined by the following equation (see [3], Def 6.1):

$$R_\alpha = \Big((1+\alpha I)^{-1}\Big)\circ(I+I\alpha I)^{\langle -1\rangle}. \tag{1}$$

A key property of R is that for any $X, Y \in \mathfrak{A}$ free over $\mathfrak{B}$, we have

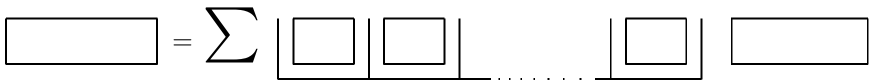
$$R_{\mathcal{M}_{X+Y}} = R_{\mathcal{M}_X} + R_{\mathcal{M}_Y}. \tag{2}$$

These relations were proved earlier in the particular case $\mathfrak{B} = \mathbb{C}$. One can also describe R_α by combinatorial means, via the recurrence relation

$$\begin{aligned}\alpha_n(b_1,\dots,b_n) = \sum_{k=0}^{n}\sum R_{\alpha,k}\Big(&[b_1\alpha_{p(1)}(b_3,\dots,b_{i_1-2})b_{i_1-1}],\dots\\ &\dots,[b_{i(k-1)}\alpha_{p(k)}(b_{i(k-1)+1},\dots,b_{i(k)-2})b_{i(k)-1}]\Big)b_{i(k)}\alpha_{n-i_k}(b_{i_{k+1}},\dots,b_n)\end{aligned}$$

where the second summation is done over all $1 = i(0) < i(1) < \cdots < i(k) \leq n$ and $p(k) = i(k) - i(k-1) - 2$.

Following an idea from [2], the above equation can be graphically illustrated by the picture:



In the case of scalar c-free probability, an analogue of the Voiculescu's R-transform is developed in [2]. In order to avoid confusions, we will denote it by ${}^cR$.

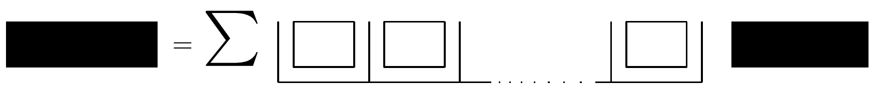
The ${}^cR$-transform has the property that it linearizes the c-free convolution of pairs of compactly supported measures. In particular, if X and Y are c-free elements from some algebra $\mathfrak{A}$, then

$$ {}^cR_{X+Y} = {}^cR_X + {}^cR_Y. $$

If the $*$-algebra $\mathfrak{A}$ is endowed with the $\mathbb{C}$-valued states φ, ψ and X is a selfadjoint element of $\mathcal{A}$, then (see [2]), the coefficients $\{{}^cR_m\}_m \geq 0$ of ${}^cR_X$ are defined by the recurrence:

$$ \varphi(X^n) = \sum_{k=1}^{n} \sum_{\substack{l(1),\dots,l(k)\geq 0 \\ l(1)+\cdots+l(k)=n-k}} {}^cR_k \cdot \psi(X^{l(1)})\cdots\psi(X^{l(k-1)})\varphi(X^{l(k)}) $$

equation that can be graphically illustrated by the picture, were the dark boxes stand for the application of φ and the light ones for the application of ψ:



The above considerations lead to the following definition:

Definition 3.2. *Let $\beta, \gamma \in Mul[[\mathfrak{B}]]$. The multilinear function series ${}^cR_{\beta,\gamma}$ is the element of $Mul[[\mathfrak{B}]]$ defined by the recurrence relation*

$$ \begin{aligned} \beta_n(b_1,\dots,b_n) = \sum_{k=0}^{n}\sum {}^cR_{\beta,\gamma,k}\Big(& [b_1\gamma_{p(1)}(b_3,\dots,b_{i_1-2})b_{i_1-1}],\dots \\ & \dots,[b_{i(k-1)}\gamma_{p(k)}(b_{i(k-1)+1},\dots,b_{i(k)-2})b_{i(k)-1}]\Big)b_{i(k)}\beta_{n-i_k}(b_{i_{k+1}},\dots,b_n) \end{aligned} $$

where the second summation is done over all $1 = i(0) < i(1) < \cdots < i(k) \leq n$ and $p(k) = i(k) - i(k-1) - 2$.

The following analytical description of ${}^cR_{\beta,\gamma}$ also shows that it is unique and well-defined:

Theorem 3.3. *For any $\beta, \gamma \in Mul[[\mathfrak{B}]]$,*

$$ {}^cR_{\beta,\gamma} = \left[\beta(1+I\beta)^{-1}\right] \circ (I + I\gamma I)^{\langle -1\rangle} \tag{3} $$

Before proving the theorem, remark that the right-hand side of (3) is well-defined and unique, since $1 + I\gamma$ is invertible with respect to the formal multiplication, $I + I\beta I$ is invertible with respect to formal composition and its inverse has 0 as first component (see [3]). We will need the following auxiliary result:

Lemma 3.4. *Let β be an element of $Mul[[\mathfrak{B}]]$ and I the identity element with respect to formal composition, $I = (0, id_{\mathfrak{B}}, 0, 0 \dots)$.*

(i) *the multilinear function series $I\beta$ is given by:*

$$\begin{aligned} (I\beta)_0 &= 0 \\ (I\beta)_n(b_1, \dots, b_n) &= b_1\beta_{n-1}(b_2, \dots, b_n) \end{aligned}$$

(ii) *the multilinear function series $I\beta I$ is given by*

$$\begin{aligned} (I\beta I)_0 &= 0 \\ (I\beta I)_1(b_1) &= 0 \\ (I\beta I)_n(b_1, \dots, b_n) &= b_1\beta_{n-2}(b_2, \dots, b_{n-1})b_n \end{aligned}$$

Proof. Since $I = (0, id_{\mathfrak{B}}, 0, \dots)$, one has:

$$(I\beta)_0 = I_0\beta_0 = 0.$$

If $n \geq 1$,

$$\begin{aligned} (I\beta)_n(b_1, \dots, b_n) &= \sum_{k=0}^{n} I_k(b_1, \dots, b_k)\beta_{n-k}(b_{k+1}, \dots, b_n) \\ &= I_1(b_1)\beta_{n-1}(b_{k+1}, \dots, b_n) \\ &= b_1\beta_{n-1}(b_{k+1}, \dots, b_n). \end{aligned}$$

For $I\beta I$, the same computations give:

$$\begin{aligned} (I\beta I)_0 &= (I\beta)_0 I_0 = 0 \\ (I\beta I)_1 &= (I\beta)_0 I_1(b_1) + (I\beta)_1(b_1)I_0 \\ &= 0. \end{aligned}$$

If $n \geq 2$, one has:

$$\begin{aligned} (I\beta I)_n(b_1, \dots, b_n) &= \sum_{k=0}^{n} (I\beta)_k(b_1, \dots, b_k)I_{n-k}(b_{k+1}, \dots, b_n) \\ &= (I\beta)_{n-1}(b_1, \dots, b_k)I_1(b_1) \\ &= b_1\beta_{n-2}(b_2, \dots, b_{n-1})b_n \end{aligned}$$

$\square$

Proof of the Theorem 3.3: Set $\sigma = I + I\beta I$. Then

$$\left({}^cR_{\beta,\gamma} \circ \sigma\right)_n(b_1, \dots, b_n) = \sum_{k=1}^{n} \sum_{\substack{p_1, \dots, p_k \geq 1 \\ p_1 + \dots + p_k = n}} {}^cR_{\beta,\gamma,k}(\sigma_{p_1}(b_1, \dots, b_{p_1}), \dots, \sigma_{p_k}(b_{q_k+1}, \dots, b_{q_k+p_k}))$$

where $q_i = p_1 + \dots + p_{i-1}$.

From Lemma (3.4)(ii),

$$\sigma_n(b_1, \dots, b_n) = (I + I\beta I)_n(b_1, \dots, b_n) =$$

therefore Definition 3.2 is equivalent to

$$\beta_n(b_1,\dots,b_n)=\sum_{k=0}^{n}\left({}^cR_{\beta,\gamma}\circ(I+I\beta I)_k(b_1,\dots,b_k)\right)b_{k+1}\beta_{n-k-2}(b_{k+2},\dots,b_n)$$

Considering now Lemma 3.4(i), the above relation becomes

$$\beta_n(b_1,\dots,b_n)=\sum_{k=0}^{n}\left({}^cR_{\beta,\gamma}\circ(I+I\beta I)_k(b_1,\dots,b_k)\right)(I+I\beta)_{n-k}(b_{k+1},\dots,b_n)$$

therefore

$$\beta=\left[{}^cR_{\beta,\gamma}\circ(I+I\gamma I)\right](1+I\beta)$$

which is equivalent to (3). □

Remark 3.5. Up to a shift in the coefficients, equation (3) is similar to the result in the case $\mathfrak{B}=\mathbb{C}$ from [2], Theorem 5.1.

Let X be a selfadjoint element of $\mathfrak{A}$. If $\mathfrak{A}$ is endowed with two $\mathfrak{B}$-valued conditional expectations Φ,Ψ, the element X will have two moment-generating multilinear function series, one with respect to Ψ, that we will denote by $\mathcal{M}_X$, and one with respect to Φ, denoted $\mathfrak{M}_X$. For brevity, we will use the notation ${}^cR_X$ for the multilinear function series ${}^cR_{\mathcal{M}_X,\mathfrak{M}_X}$.

Theorem 3.6. *Let X and Y be two elements of $\mathfrak{A}$ that are c-free with respect to the pair of conditional expectations (Φ,Ψ). Then*

$${}^cR_{X+Y}={}^cR_X+{}^cR_Y$$

Proof. Let $\mathcal{A}$ be an algebra containing $\mathfrak{B}$ as a subalgebra and endowed with the conditional expectations $\Phi,\Psi:\mathcal{A}\longrightarrow\mathfrak{B}$. Consider the set $\mathcal{A}_0=\mathcal{A}\setminus\mathfrak{B}$ (set difference). For $n\geq 1$ define the mappings

$${}^cr:\underbrace{\mathcal{A}_0\times\cdots\times\mathcal{A}_0}_{n\text{ times}}\longrightarrow\mathfrak{B}$$

given by the recurrence formula:

$$\begin{aligned}\Phi(a_1\cdots a_n) \;=\; \sum_{k=1}^{n}\sum_{\substack{l(1)<\cdots<l(k)\\ 1<l(1),l(k)\leq n}} {}^cr_k\Big(&a_1[\Psi(a_2\cdots a_{l(1)-1})],\dots,\\ &\dots,a_{l(k-1)}[\Psi(a_{l(k-1)+1}\cdots a_{l(k)_1})],a_{l(k)}[\Phi(a_{l(k)+1}\cdots a_n)]\Big)\end{aligned}$$

Note that ${}^cr_n$ is well defined, and that, for any $b_1,\dots,b_n\in\mathfrak{B}$,

$$ {}^cr_{n+1}(X,b_1X,\dots,b_nX)={}^cR_{X,n}(b_1,\dots,b_n). \tag{4}$$

As in Section 2, consider $\mathfrak{B}\langle\xi_i\rangle$, the noncommutative algebras of polynomials in the symbols $\xi_i, i=1,2$ and with coefficients from $\mathfrak{B}$ and the conditional expectations

$$\Phi_X,\Psi_X:\mathfrak{B}\langle\xi_1\rangle\longrightarrow\mathfrak{B}$$

given by

$$\begin{aligned}\Phi_X(f(\xi_1)) &= \Phi(f(X))\\ \Psi_X(f(\xi_1)) &= \Psi(f(X))\end{aligned}$$

and their analogues Φ_Y,Ψ_Y for $\mathfrak{B}\langle\xi_2\rangle$.

On $\mathfrak{B}\langle\xi_1,\xi_2\rangle$, identified to $\mathfrak{B}\langle\xi_1\rangle *_{\mathfrak{B}} \mathfrak{B}\langle\xi_2\rangle$, consider the conditional expectations Ψ_0, Φ_0, φ given by:

$$\begin{aligned}
\Psi_0 &= \Psi_X * \Psi_Y \\
\Phi_0(f(\xi_1,\xi_2)) &= \Phi(f(X,Y)) \\
\varphi(a_1a_2\dots a_n) &= \sum_{k=1}^{n} \sum_{\substack{l(1)<\dots<l(k)\\ 1<l(1),l(k)\leq n}} \rho_k\Big(a_1[\Psi_0(a_2\cdots a_{l(1)-1})],\dots, \\
&\qquad \dots, a_{l(k-1)}[\Psi_0(a_{l(k-1)+1}\cdots a_{l(k)_1})], a_{l(k)}[\varphi(a_{l(k)+1}\cdots a_n)]\Big)
\end{aligned}$$

where $a_1,\dots,a_n$ are elements of the set

$$\mathfrak{B}\langle\xi_1,\xi_2\rangle_0 = \mathfrak{B}\langle\xi_1\rangle \cup \mathfrak{B}\langle\xi_2\rangle \setminus \mathfrak{B}$$

and the mappings

$$\rho_n : \underbrace{\mathfrak{B}\langle\xi_1,\xi_2\rangle_0 \times \dots \mathfrak{B}\langle\xi_1,\xi_2\rangle_0}_{n \text{ times}} \longrightarrow \mathfrak{B}$$

are given by:

$$\rho_n(a_1,\dots,a_n) = \begin{cases} {}^c r(a_1,\dots,a_n) & \text{if all } a_1,\dots,a_n \in \mathfrak{B}\langle\xi_1\rangle \\ {}^c r(a_1,\dots,a_n) & \text{if all } a_1,\dots,a_n \in \mathfrak{B}\langle\xi_2\rangle \\ 0 & \text{otherwise} \end{cases}.$$

We will show that $\varphi = \Phi_0$, in particular φ is also well-defined. Consider the element $a \in \mathfrak{B}\langle\xi_1,\xi_2\rangle$ of the form $a = a_1\cdots a_n$ with $a_j \in \mathfrak{B}\langle\xi_{\varepsilon(j)}\rangle, \varepsilon(1) \neq \varepsilon(2) \neq \dots \neq \varepsilon(n)$ and $\Psi_0(a_j) = 0$. The computation $\varphi(a_1\cdots a_n)$ is done via the recurrence relation above. Because of the definition of ρ and the fact that $\Psi_0 = \Psi_X * \Psi_Y$, only the term with $k = 1$ contribute at the sum, i.e.

$$\begin{aligned}
\varphi(a_1\cdots a_n) &= \varphi(a_1\varphi(a_2\cdots a_n)) \\
&= \varphi_{\varepsilon(1)}(a_1\varphi(a_2\cdots a_n) \\
&= \varphi_{\varepsilon(1)}(a_1)\varphi(a_2\cdots a_n)
\end{aligned}$$

and the identity between φ and Φ_0 follows by induction over n.

Since $\varphi = \Phi_0$, the mappings ρ_n and ${}^c r_n$ are satisfying the same recurrence relation, hence

$$\rho_n(a_1,\dots,a_n) = {}^c r(a_1,\dots,a_n),$$

in particular

$$\begin{aligned}
{}^c R_{X+Y,n}(b_1,\dots,b_n) &= {}^c r_{n+1}((X+Y)b_1(X+Y)\dots(X+Y)b_n(X+Y)) \\
&= \rho_{n+1}((X+Y)b_1(X+Y)\dots(X+Y)b_n(X+Y)) \\
&= \rho_{n+1}((X)b_1(X)\dots(X)b_n(X)) + \rho_{n+1}((Y)b_1(Y)\dots(Y)b_n(Y)) \\
&= {}^c R_{X,n}(b_1,\dots,b_n) + {}^c R_{Y,n}(b_1,\dots,b_n).
\end{aligned}$$

□

4. CENTRAL LIMIT THEOREM

Consider the ordered set $\langle n\rangle = \{1, 2, \dots, n\}$ and π a partition of $\langle n\rangle$ with blocks $B_1, \dots, B_m$:

$$\langle n\rangle = B_1 \sqcup B_2 \sqcup \dots \sqcup B_m.$$

The blocks B_p and B_q of π are said to be *crossing* if there exist $i < j < k < l$ in $\langle n\rangle$ such that $i, k \in B_p$ and $j, l \in B_q$.

The partition π is said to be *non-crossing* if all pairs of distinct blocks of π are not crossing. We will denote by $NC_2(n)$ the set of all non-crossing partitions of $\langle n\rangle$ whose blocks contain exactly 2 elements and by $NC_{\leq s}(n)$ the set of all non-crossing partitions of $\langle n\rangle$ whose blocks contain at most s elements.

Let now γ be a non-crossing partition of $\langle n\rangle$ and B and C be two blocks of π. We say that B is interior to C if there exist two indices $i < j$ in $\langle n\rangle$ such that $i, j \in c$ and $B \subset \{i+1, \dots, j-1\}$. The block B is said to be *outer* if it is not interior to any other block of γ. In a non-crossing partition of $\langle n\rangle$, the block containing 1 is always outer.

Consider now an element X of $\mathfrak{A}$. Let π be a partition from $NC_2(n+1)$ ($n =$ odd) and $B_1 = (1, k)$ be the block of π containing 1. We define, by recurrence, the following expressions:

$$\begin{aligned} V_\pi(X, b_1, \dots, b_n) &= \Psi(X b_1 V_{\pi|\{2,\dots,j-1\}}(X, b_2, \dots, b_{k-2}) b_{k-1} X) b_k \\ &\qquad V_{\pi|\{k+1,\dots,n+1\}}(X, b_{k+1}, \dots, b_n) \\ W_\pi(X, b_1, \dots, b_n) &= \Phi(X b_1 V_{\pi|\{2,\dots,j-1\}}(X, b_2, \dots, b_{k-2}) b_{k-1} X) b_k \\ &\qquad W_{\pi|\{k+1,\dots,n+1\}}(X, b_{k+1}, \dots, b_n) \end{aligned}$$

Theorem 4.1. *(Central Limit Theorem) Let $(X_n)_{n\geq 1}$ be a sequence of c-free elements of $\mathfrak{A}$ such that:*

1. *all X_n have the same moment-generating multilinear function series, $\mathfrak{M}$ with respect to Φ and M with respect to Ψ.*
2. $\Psi(X_n) = \Phi(X_n) = 0$.

Set

$$S_N = \frac{X_1 + \dots + X_N}{\sqrt{N}},$$

Then:

(i) $\displaystyle \lim_{N\to\infty} {}^{c}R_{S_N,n} = \begin{cases} 0 & \text{if } n \neq 1 \\ \mathfrak{M}_1(\cdot) & \text{if } n = 1 \end{cases}$

(ii) $\displaystyle \lim_{N\to\infty} R_{S_N,n} = \begin{cases} 0 & \text{if } n \neq 1 \\ M_1(\cdot) & \text{if } n = 1 \end{cases}$

(iii) *there exists two conditional expectations $\nu : \mathfrak{B}\langle\xi\rangle \longrightarrow \mathfrak{B}$, depending only on $M_1(\cdot)$, and $\mu : \mathfrak{B}\langle\xi\rangle \longrightarrow \mathfrak{B}$, depending only on $M_1(\cdot)$ and $\mathfrak{M}_1(\cdot)$, such that*

$$\begin{aligned} \lim_{N\to\infty} \Psi_{S_N} &= \nu \\ \lim_{N\to\infty} \Phi_{S_N} &= \mu; \end{aligned}$$

in the weak sense; in particular,

$$\begin{aligned}\nu(\xi b_1\xi \dots b_n\xi) &= \sum_{\pi\in NC_2(n=1)} V_\pi(X_1, b_1, \dots, b_n)\\ \mu(\xi b_1\xi \dots b_n\xi) &= \sum_{\pi\in NC_2(n=1)} W_\pi(X_1, b_1, \dots, b_n).\end{aligned}$$

Proof. Let X be an element of $\mathfrak{A}$ with the same moment generating series as $X_j, j \geq 1$. As shown in [3],

$$R_{S_N} = \sum_{k=1}^{N} R_{\frac{X_k}{\sqrt{N}}} = NR_{\frac{X}{\sqrt{N}}}.$$

Also, from Theorem 2.4 and Theorem 3.6, it follows that

$${}^cR_{S_N} = \sum_{k=1}^{N} {}^cR_{\frac{X_k}{\sqrt{N}}} = N^cR_{\frac{X}{\sqrt{N}}}.$$

Since R and ${}^cR$ are multilinear and $M_0 = \mathfrak{M}_0 = 0$, we have that

$$\begin{aligned}\lim_{N\to\infty} {}^cR_{S_N,n} &= \lim_{N\to\infty} \frac{N}{N^{\frac{n+1}{2}}} {}^cR_{X,n}\\ &= \begin{cases} 0 & \text{if } n \neq 1\\ \\ \mathfrak{M}_1(\cdot) & \text{if } n = 1\end{cases}\end{aligned}$$

and the similar relations for $R_{S_N,n}$, hence (i) and (ii) are proved.

For (iii) it suffices to check the relations for $\nu(\xi b_1\xi \dots b_n\xi)$ and $\mu(\xi b_1\xi \dots b_n\xi)$, which are a trivial corollary of (i), (ii), and the recurrence formulas that define R and ${}^cR$. □

Remark 4.2. For $\mathfrak{B} = \mathbb{C}$, the theorem is a weaker version of Theorem 4.3 from [2]. If Ψ is $\mathbb{C}$-valued, then the result is similar to Corollary 5.1 from [6]. Also, under the assumptions that for some $a, b \in \mathfrak{B}$ we have that:

$$\begin{aligned}\lim_{N\to\infty} N\Psi(X_1 \cdots X_N) &= a\\ \lim_{N\to\infty} N\Psi(X_1 \cdots X_N) &= b\end{aligned}$$

the same techniques lead to a Poisson-type limit Theorem, similar to Corollary 2, Section 5 of [6].

In the following remaining pages we will describe the positivity of the limit functionals μ and ν in terms of Φ and Ψ. The central result is Corollary 4.4.

For simplicity, suppose that $\mathfrak{B}$ is a unital $*$-algebra (otherwise, we can replace $\mathfrak{B}$ by its unitisation). Consider the symbol ξ, the $*$-algebra $\mathfrak{B}\langle\xi\rangle$ of polynomials in ξ with coefficients from $\mathfrak{B}$, as defined before, and consider also the linear space $\mathfrak{B}\xi\mathfrak{B}$ generated by the set $\{b_1\xi b_2;\ b_1, b_2 \in \mathfrak{B}\}$ with the $\mathfrak{B}$-bimodule structure given by

$$a_1b_1\xi b_2a_2 = (a_1b_1)\xi(b_2a_2)$$

for all $a_1, a_2, b_1, b_2 \in \mathfrak{B}$.

Lemma 4.3. *For any positive $\mathfrak{B}$-sesquilinear pairing $\langle,\rangle$ on $\mathfrak{B}\xi\mathfrak{B}$ there exists a positive conditional expectation*

$$\varphi : \mathfrak{B}\langle\xi\rangle \longrightarrow \mathfrak{B}$$

such that for any $b_1, b_2 \in \mathfrak{B}$

$$\varphi(\xi b_1^* b_2 \xi) = \langle b_1 \xi, b_2 \xi \rangle$$

Proof. Without loss of generality, we can suppose that $\mathfrak{B}$ is unital (otherwise we can replace $\mathfrak{B}$ by its unitization).

Consider the Full Fock bimodule over $\mathfrak{B}\xi\mathfrak{B}$

$$\mathcal{F}\langle\xi\rangle = \mathfrak{B} \oplus \left(\bigoplus_{n\geq 1} \underbrace{\mathfrak{B}\xi\mathfrak{B} \otimes_{\mathfrak{B}} \cdots \otimes_{\mathfrak{B}} \mathfrak{B}\xi\mathfrak{B}}_{n \text{ times}} \right)$$

with the pairing given by

$$\begin{aligned} \langle a, b \rangle &= a^* b \\ \langle a_1\xi \otimes \cdots \otimes a_n\xi, b_1\xi \otimes \cdots \otimes b_m\xi \rangle &= \delta_{m,n} \langle a_n\xi, \langle \ldots, \langle a_1\xi, b_1\xi \rangle b_2\xi \rangle, \ldots b_n\xi \rangle. \end{aligned}$$

$(a, a_j, b, b_j \in \mathfrak{B}, j = 1, \ldots, n)$

Note that the $\mathfrak{B}$-linear operators $A_1, A_2 : \mathcal{F}\langle\xi\rangle \longrightarrow \mathcal{F}\langle\xi\rangle$ described by the relations

$$\begin{aligned} A_1 b &= \xi b \\ A_1(a_1\xi \otimes \cdots \otimes a_n\xi b) &= \xi \otimes a_1\xi \otimes \cdots \otimes a_n\xi b \\ A_2 b &= 0 \\ A_2(a_1\xi \otimes \cdots \otimes a_n\xi b) &= \langle \xi, a_1\xi \rangle a_2\xi \otimes \cdots \otimes a_n\xi b \end{aligned}$$

are self-adjoint to each other, in the sense that

$$\langle A_1\widetilde{\zeta_1}, \widetilde{\zeta_2} \rangle = \langle \widetilde{\zeta_1}, A_2\widetilde{\zeta_2} \rangle$$

for any $\widetilde{\zeta_1}, \widetilde{\zeta_2} \in \mathcal{F}\langle\xi\rangle$, therefore $S = A_1 + A_2$ is selfadjoint.

Moreover, for any $a, b \in \mathfrak{B}$,

$$\begin{aligned} \langle 1, Sa^*bS1 \rangle &= \langle aS1, bS1 \rangle \\ &= \langle a(A_1 + A_2)1, b(A_1 + A_2)1 \rangle \\ &= \langle a\xi, b\xi \rangle \end{aligned}$$

and the conclusion follows by setting $\varphi(p(\xi)) = \langle 1, p(S)1 \rangle$ for all $p \in \mathfrak{B}\langle\xi\rangle$. $\square$

Corollary 4.4. *The mappings* μ *and* ν *from Theorem 4.1 are positive if and only if for any* $b \in \mathfrak{B}$ *one has that* $\Phi(Xb^*bX) \geq 0$ *and* $\Psi(Xb^*bX) \geq 0$.

Proof. One implication is trivial, since, if ν and μ are positive, then

$$\Psi(Xb^*bX) = \nu(Xb^*bX) = \nu((bX)^*bX) \geq 0$$

and

$$\Phi(Xb^*bX) = \mu(Xb^*bX) = \mu((bX)^*bX) \geq 0.$$

Suppose now that $\Phi(Xb^*bX) \geq 0$ and $\Psi(Xb^*bX) \geq 0$ for all $b \in \mathfrak{B}$. We will use the same argument as in [9] and [8].

Consider the set of selfadjoint symbols $\{\xi_i\}_{i\geq 1}$. On each $\mathfrak{B}$-bimodule $\mathfrak{B}\xi_i\mathfrak{B}$ we have the positive $\mathfrak{B}$-sesquilinear pairings $\langle\cdot,\cdot\rangle_\Phi$ and $\langle\cdot,\cdot\rangle_\Psi$ determined by

$$\begin{aligned} \langle a\xi_i, b\xi_i \rangle_\Phi &= \Phi(Xa^*bX) \\ \langle a\xi_i, b\xi_i \rangle_\Psi &= \Psi(Xa^*bX). \end{aligned}$$

As shown in Lemma 4.3, the above $\mathfrak{B}$-sesquilinear pairings determine positive conditional expectations $\varphi_1, \psi_i : \mathfrak{A}_i \longrightarrow \mathfrak{B}$, where $\mathfrak{A}_i = \mathfrak{B}\langle\xi_i\rangle$ be the $*$-algebras of polynomials in ξ with coefficients from $\mathfrak{B}$, $i \geq 1$.

For $\tau : \mathfrak{B}\langle\xi\rangle \longrightarrow \mathfrak{B}$ a conditional expectation, and $\lambda \geq 0$, note with $D_\lambda\tau$ the dilation with λ of τ, i.e.

$$D_\lambda\tau(\xi b_1\xi\cdots b_n\xi) = \lambda^{n+1}\tau(\xi b_1\xi\cdots b_n\xi)$$

Remark that if τ is positive, then $D_\lambda\tau$ is also positive.

With the notations above, consider, as in Definition 2.1, the conditionally free product $(\mathfrak{A}, \Phi, \Psi) = \ast_{i\in\mathfrak{I}}(\mathfrak{A}_i, \Phi_i, \Psi_i)$. The elements $\{\xi_i\}_{i\geq 1}$ are conditionally free in $\mathfrak{A}$, so Theorem 4.1 implies that:

$$\begin{aligned}
\mu &= \lim_{N\longrightarrow\infty} \Phi_{\frac{\xi_1+\cdots+\xi_N}{\sqrt{N}}} = D_{\frac{1}{\sqrt{N}}}\Phi_{\xi_1+\cdots+\xi_N}\\
\nu &= \lim_{N\longrightarrow\infty} \Psi_{\frac{\xi_1+\cdots+\xi_N}{\sqrt{N}}} = D_{\frac{1}{\sqrt{N}}}\Psi_{\xi_1+\cdots+\xi_N}\\
&= D_{\frac{1}{\sqrt{N}}}\left(\ast_{i=1}^{N}\Psi_{\xi_i}\right).
\end{aligned}$$

We have that $\ast_{i=1}^{N}\Psi_{\xi_i} \geq 0$ since it is the free product of states (see, for example [9]), hence the positivity of ν.

Also, Theorem 2.4 and Corollary 2.6 imply that

$$\Phi_{\xi_1+\cdots+\xi_N} \geq 0$$

therefore $\mu \geq 0$.

□

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Indiana University at Bloomington, Department of Mathematics, Rawles Hall, 931 E 3rd St, Bloomington, IN 47405
E-mail address: mipopa@indiana.edu