NEW SIMPLE MODULAR LIE SUPERALGEBRAS AS GENERALIZED PROLONGS

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ABSTRACT. Over algebraically closed fields of characteristic $p > 2$, prolongations of the simple finite dimensional Lie algebras and Lie superalgebras with Cartan matrix are studied for certain simplest gradings of these algebras. Several new simple Lie superalgebras are discovered, serial and exceptional, including superBrown and superMelikyan superalgebras. Simple Lie superalgebras with Cartan matrix of rank 2 are classified.

1. Introduction

1.1. Setting. We use standard notations of [FH, S]; for the precise definition (algorithm) of generalized Cartan-Tanaka–Shchepochkina (CTS) complete and partial prolongations, and algorithms of their construction, see [Shch]. Hereafter $\mathbb{K}$ is an algebraically closed field of characteristic $p > 2$, unless specified. Let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, and $\mathfrak{c}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{center}$, where $\dim \mathfrak{center} = 1$. Let ${}^{n)}\mathfrak{g}$ denote the incarnation of the Lie (super)algebra $\mathfrak{g}$ with the n)th Cartan matrix, cf. [GL4, BGL1]. On classification of simple vectorial Lie superalgebras with polynomial coefficients (in what follows referred to as *vectorial Lie superalgebras of polynomial vector fields* over $\mathbb{C}$, see [LSh, K3]).

The works of S. Lie, Killing and È. Cartan, now classical, completed classification over $\mathbb{C}$ of

(1) **simple Lie algebras of finite dimension and of polynomial vector fields**.

Lie algebras and Lie superalgebras over fields in characteristic $p > 0$, a.k.a. *modular* Lie (super)algebras, were first recognized and defined in topology, in the 1930s. The **simple** Lie algebras drew attention (over finite fields $\mathbb{K}$) as a step towards classification of simple finite groups, cf. [St]. Lie *super*algebras, even simple ones and even over $\mathbb{C}$ or $\mathbb{R}$, did not draw much attention of mathematicians until their (outstanding) usefulness was observed by physicists in the 1970s. Meanwhile mathematicians kept discovering new and new examples of simple modular Lie algebras until Kostrikin and Shafarevich ([KS]) formulated a conjecture embracing all previously found examples for $p > 7$. Its generalization reads: *select a $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$ of every $\mathfrak{g}$ of type*[1)] *(1), take $\mathfrak{g}_{\mathbb{K}} := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K}$ and its simple finite dimensional subquotient $\mathfrak{si}(\mathfrak{g}_{\mathbb{K}})$ (there can be several such $\mathfrak{si}(\mathfrak{g}_{\mathbb{K}})$). Together with deformations*[2)] *of these examples we get in this way all simple finite dimensional Lie algebras over algebraically closed fields if $p > 5$. If $p = 5$, we should add to the above list Melikyan's examples.*

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[1)]Observe that the algebra of divided powers (the analog of the polynomial algebra for $p > 0$) *and hence all prolongs* (Lie algebras of vector fields) acquire one more — shearing — parameter: $\underline{N}$, see [S].

[2)]It is not clear, actually, if the conventional notion of deformation can always be applied if $p > 0$ (for the arguments, see [LL]; cf. [Vi]); to give the correct (better say, universal) notion is an open problem, but in some cases it is applicable, see [BGL4].

Having built upon ca 30 years of work of several teams of researchers, and having added new ideas and lots of effort, Block, Wilson, Premet and Strade proved the generalized KSh conjecture for $p > 3$, see [S]. For $p \leq 5$, the above KSh-procedure does not produce all simple finite dimensional Lie algebras; there are other examples. In [GL4], we returned to É. Cartan's description of $\mathbb{Z}$-graded Lie algebras as CTS prolongs, i.e., as subalgebras of vectorial Lie algebras preserving certain distributions; we thus interpreted the "mysterious" at that moment exceptional examples of simple Lie algebras for $p = 3$ (the Brown, Frank, Ermolaev and Skryabin algebras), further elucidated Kuznetsov's interpretation [Ku1] of Melikyan's algebras (as prolongs of the nonpositive part of the Lie algebra $\mathfrak{g}(2)$ in one of its $\mathbb{Z}$-gradings) and discovered three new series of simple Lie algebras. In [BjL], the same approach yielded $\mathfrak{bj}$, a simple super versions of $\mathfrak{g}(2)$, and $\mathfrak{Bj}(1;N|7)$, a simple $p = 3$ super Melikyan algebra. Both $\mathfrak{bj}$ and $\mathfrak{Bj}(1;N|7)$ are indigenous to $p = 3$, the case where $\mathfrak{g}(2)$ is not simple.

1.2. Classification: Conjectures and results. Recently, Elduque considered super analogs of the exceptional simple Lie algebras; his method leads to a discovery of 10 new simple (presumably, exceptional) Lie superalgebras for $p = 3$. For a description of the Elduque superalgebras, see [CE, El1, CE2, El2]; for their description in terms of Cartan matrices and analogs of Chevalley relations and notations we use in what follows, see [BGL1, BGL2].

In [L], a super analog of the KSh conjecture embracing all types of simple (finite dimensional) Lie superalgebras is formulated based on an entirely different idea in which the CTS prolongs play the main role:

For every simple finite dimensional Lie (super)algebra of the form $\mathfrak{g}(A)$, take its non-positive part with respect to a certain simplest $\mathbb{Z}$-grading, consider its complete and partial prolongs and take their simple subquotients.

The new examples of simple modular Lie superalgebras ($\mathfrak{BRJ}$, $\mathfrak{Bj}(3;\underline{N}|3)$, $\mathfrak{Bj}(3;\underline{N}|5)$) support this conjecture. (This is how Cartan got all simple $\mathbb{Z}$-graded Lie algebras of polynomial growth and finite depth — the Lie algebras of type (1) — at the time when the root technique was not discovered yet.)

1.2.1. Yamaguchi's theorem ([Y]). This theorem, reproduced in [GL4, BjL], states that *for almost all simple finite dimensional Lie algebras* $\mathfrak{g}$ *over* $\mathbb{C}$ *and their* $\mathbb{Z}$*-gradings* $\mathfrak{g} = \mathop{\oplus}\limits_{-d\leq i} \mathfrak{g}_i$ *of finite depth* d*, the CTS prolong of* $\mathfrak{g}_{\leq} = \mathop{\oplus}\limits_{-d\leq i\leq 0} \mathfrak{g}_i$ *is isomorphic to* $\mathfrak{g}$*, the rare exceptions being two of the four series of simple vectorial algebras* (the other two series being partial prolongs).

1.2.2. Conjecture. In the following theorems, we present the results of `SuperLie`-assisted ([Gr]) computations of the CTS-prolongs of the non-positive parts of the simple finite dimensional Lie algebras and Lie superalgebras $\mathfrak{g}(A)$; we have only considered $\mathbb{Z}$-grading corresponding to each (or, for larger ranks, even certain *selected*) of the **simplest** gradings $r = (r_1, \dots, r_{\mathrm{rk}\,\mathfrak{g}})$, where all but one coordinates of r are equal to 0 and only one — *selected* — is equal to 1, and where we set $\deg X_i^{\pm} = \pm r_i$ for the Chevalley generators $X_i^{\pm}$ of $\mathfrak{g}(A)$, see [BGL1].

Other gradings (as well as algebras $\mathfrak{g}(A)$ of higher ranks) do not yield new simple Lie (super)algebras as prolongs of the non-positive parts.

1.3. Theorem. *The CTS prolong of the nonpositive part of $\mathfrak{g}$ returns $\mathfrak{g}$ in the following cases: $p = 3$ and $\mathfrak{g} = \mathfrak{f}(4)$, $\mathfrak{e}(6)$, $\mathfrak{e}(7)$ and $\mathfrak{e}(8)$ considered with the $\mathbb{Z}$-grading with one selected root corresponding to the endpoint of the Dynkin diagram.*

1.3.1. Conjecture. [The computer got stuck here, after weeks of computations] To the cases of Theorem 1.3, one can add the case for $p = 5$ and $\mathfrak{g} = \mathfrak{el}(5)$ *(see* [BGL2]*)* in its $\mathbb{Z}$-grading with only one odd simple root and with one selected root corresponding to any endpoint of the Dynkin diagram.

1.4. Theorem. *Let $p = 3$. For the previously known (we found more, see Theorems 1.6, 1.7) simple finite dimensional Lie superalgebras $\mathfrak{g}$ of rank ≤ 3 with Cartan matrix and for their simplest gradings r, the CTS prolongs (of the non-positive part of $\mathfrak{g}$) different from $\mathfrak{g}$ are given in the following table elucidated below.*

1.5. Melikyan superalgebras for $p = 3$. There are known the two constructions of the Melikyan algebra $\mathfrak{Me}(5; \underline{N}) = \bigoplus\limits_{i\geq -2} \mathfrak{Me}(5; \underline{N})_i$, defined for $p = 5$:

1) as the CTS prolong of the triple $\mathfrak{Me}_0 = \mathfrak{cvect}(1; \underline{1})$, $\mathfrak{Me}_{-1} = \mathcal{O}(1; \underline{1})/\mathrm{const}$ and the trivial module $\mathfrak{Me}_{-2}$, see [S]; this construction would be a counterexample to our conjecture were there no alternative:

2) as the *complete* CTS prolong of the non-negative part of $\mathfrak{g}(2)$ in its grading $r = (01)$, with $\mathfrak{g}(2)$ obtained now as a *partial* prolong, see [Ku1, GL4].

In [BjL], we have singled out $\mathfrak{Bj}(1; \underline{N}|7)$ as a $p = 3$ simple analog of $\mathfrak{Me}(5; \underline{N})$ as a partial CTS prolongs of the pair (the negative part of $\mathfrak{k}(1; \underline{N}|7)$, $\mathfrak{Bj}(1; \underline{N}|7)_0 = \mathfrak{pgl}(3)$), and $\mathfrak{bj}$ as a $p = 3$ simple analog of $\mathfrak{g}(2)$ whose non-positive part is the same as that of $\mathfrak{Bj}(1; \underline{N}|7)$, i.e., $\mathfrak{bj}$ and $\mathfrak{Bj}(1; \underline{N}|7)$ are analogs of the construction 2).

The original Melikyan's construction 1) also has its super analog for $p = 3$ (only in the situation described in Theorem 1.6) and it yields a new series of simple Lie superalgebras as the complete prolongs, with another simple analog of $\mathfrak{g}(2)$ as a partial prolong.

Recall ([BGL1]) that we normalize the Cartan matrix A so that $A_{ii} = 1$ or 0 if the ith root is odd, whereas if the ith root is even, we set $A_{ii} = 2$ or 0 in which case we write $\bar{0}$ instead of 0 in order not to confuse with the case of odd roots.

1.6. Theorem. *A $p = 3$ analog of the construction 1) of the Melikyan algebra is given by setting $\mathfrak{g}_0 = \mathfrak{ck}(1; \underline{1}|1)$, $\mathfrak{g}_{-1} = \mathcal{O}(1; \underline{1}|1)/const$ and $\mathfrak{g}_{-2}$ being the trivial module. It yields a simple super Melikyan algebra that we denote by $\mathfrak{Me}(3; \underline{N}|3)$, non-isomorphic to a superMelikyan algebra $\mathfrak{Bj}(1; \underline{N}|7)$.*

The partial prolong of the non-positive part of $\mathfrak{Me}(3; \underline{N}|3)$ is a new (exceptional) simple Lie superalgebra that we denote by $\mathfrak{brj}(2; 3)$. This $\mathfrak{brj}(2; 3)$ has the three Cartan matrices: 1) $\begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}$ and 2) $\begin{pmatrix} 0 & -1 \\ -1 & \bar{0} \end{pmatrix}$ joined by an odd reflection, and $\begin{pmatrix} 1 & -1 \\ -1 & \bar{0} \end{pmatrix}$. It is a super analog of the Brown algebra $\mathfrak{br}(2) = \mathfrak{brj}(2; 3)_{\bar{0}}$, its even part.

The CTS prolongs for the simplest gradings r of ${}^{1)}\mathfrak{brj}(2; 3)$ returns known simple Lie superalgebras, whereas the CTS prolong for a simplest grading r of ${}^{2)}\mathfrak{brj}(2; 3)$ returns, as a partial prolong, a new simple Lie superalgebra we denote $\mathfrak{BRJ}$.

Unlike $\mathfrak{br}(2)$, the Lie superalgebra $\mathfrak{brj}(2; 3)$ has analogs for $p \neq 3$, e.g., for $p = 5$, we get a new simple Lie superalgebra $\mathfrak{brj}(2; 5)$ such that $\mathfrak{brj}(2; 5)_{\bar{0}} = \mathfrak{sp}(4)$ with the two Cartan matrices 1) $\begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}$ and 2) $\begin{pmatrix} 0 & -4 \\ -3 & 2 \end{pmatrix}$. The CTS prolongs of $\mathfrak{brj}(2; 5)$ for all its Cartan matrices and the simplest r return $\mathfrak{brj}(2; 5)$.

Having got this far, it was impossible not to try to get classification of simple $\mathfrak{g}(A)$'s. Here is its beginning part, see [BGL5].

1.7. Theorem. *If $p>5$, every finite dimensional simple Lie* super*algebra with a 2×2 Cartan matrix is isomorphic to $\mathfrak{osp}(1|4)$, $\mathfrak{osp}(3|2)$, or $\mathfrak{sl}(1|2)$. If $p=5$, we should add $\mathfrak{brj}(2;5)$. If $p=3$, we should add $\mathfrak{brj}(2;3)$.*

Remark. For details of description of the new simple Lie superalgebras of types $\mathfrak{Bj}$ and $\mathfrak{Mk}$ and their subalgebras, in particular, presentations of $\mathfrak{brj}(2;3)$ and $\mathfrak{brj}(2;5)$, and proof of Theorem 1.7 and its generalization for higher ranks, see [BGL4, BGL5].

The new simple Lie superalgebras obtained are described in the next subsections.

$\mathfrak{g}$	Cartan matrix	r	prolong
$\mathfrak{osp}(3\vert 2)$	$\begin{pmatrix}0&-1\\-2&2\end{pmatrix}$	(10) (01)	$\mathfrak{k}(1\vert 3)$ $\mathfrak{k}(1\vert 3;1)$
	$\begin{pmatrix}0&-1\\-1&1\end{pmatrix}$	(10) (01)	$\mathfrak{osp}(3\vert 2)$ $\mathfrak{k}(1\vert 3;1)$
$\mathfrak{sl}(1\vert 2)$	$\begin{pmatrix}0&-1\\-1&2\end{pmatrix}$	(10) (01)	$\mathfrak{vect}(0\vert 2)$ $\mathfrak{vect}(1\vert 1)$
	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	(10) (01)	$\mathfrak{vect}(1\vert 1)$
$\mathfrak{osp}(1\vert 4)$	$\begin{pmatrix}2&-1\\-1&1\end{pmatrix}$	(10) (01)	$\mathfrak{k}(3\vert 1)$ $\mathfrak{osp}(1\vert 4)$
$\mathfrak{brj}(2;3)$	$\begin{pmatrix}0&-1\\-2&1\end{pmatrix}$	(10) (01)	$\mathfrak{Mk}(3;N\vert 3)$ $\mathfrak{Brj}(4\vert 3)$
	$\begin{pmatrix}\bar 0&-1\\-1&0\end{pmatrix}$	(10) (01)	$\mathfrak{Brj}(4;\underline{N}\vert 3)$ $\mathfrak{Brj}(3;\underline{N}\vert 4)\supset\mathfrak{BRJ}$
$\mathfrak{brj}(2;3)$	$\begin{pmatrix}\bar 0&-1\\-1&1\end{pmatrix}$	(10) (01)	$\mathfrak{Brj}(3;\underline{N}\vert 3)$ $\mathfrak{Brj}(3;\underline{N}\vert 4)\supset\mathfrak{BRJ}$
$\mathfrak{brj}(2;5)$	$\begin{pmatrix}0&-1\\-2&1\end{pmatrix}$	(10) (01)	$\mathfrak{brj}(2;5)$ $\mathfrak{brj}(2;5)$
	$\begin{pmatrix}0&1\\-3&2\end{pmatrix}$	(10) (01)	$\mathfrak{brj}(2;5)$ $\mathfrak{brj}(2;5)$

$\mathfrak{g}$	Cartan matrix	r	prolong
$\mathfrak{sl}(1\vert 3)$	$\begin{pmatrix}0&-1&0\\-1&2&-1\\0&-1&2\end{pmatrix}$	(100) (010) (001)	$\mathfrak{vect}(0\vert 3)$ $\mathfrak{sl}(1\vert 3)$ $\mathfrak{vect}(2\vert 1)$
	$\begin{pmatrix}0&-1&0\\-1&0&-2\\0&-1&2\end{pmatrix}$	(100) (010) (001)	$\mathfrak{vect}(2\vert 1)$ $\mathfrak{sl}(1\vert 3)$ $\mathfrak{vect}(2\vert 1)$
$\mathfrak{psl}(2\vert 2)$	any matrix	(100) (010) (001)	$\mathfrak{svect}(1\vert 2)$ $\mathfrak{psl}(2\vert 2)$ $\mathfrak{svect}(1\vert 2)$
$\mathfrak{osp}(1\vert 6)$	$\begin{pmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{pmatrix}$	(100) (010) (001)	$\mathfrak{k}(5\vert 1)$ $\mathfrak{osp}(1\vert 6)$ $\mathfrak{osp}(1\vert 6)$
$\mathfrak{osp}(3\vert 4)$	$\left.\begin{matrix}\begin{pmatrix}2&-1&0\\-1&0&-1\\0&-2&2\end{pmatrix}\\\begin{pmatrix}0&-1&0\\-1&0&1\\0&-1&1\end{pmatrix}\end{matrix}\right\}$	(100) (010) (001)	$\mathfrak{k}(3\vert 3)$ $\mathfrak{osp}(3\vert 4)$ $\mathfrak{osp}(3\vert 4)$
	$\begin{pmatrix}0&-1&0\\-1&2&-1\\0&-1&1\end{pmatrix}$	(100) (010) (001)	$\mathfrak{osp}(3\vert 4)$
$\mathfrak{osp}(5\vert 2)$	$\left.\begin{matrix}\begin{pmatrix}2&-1&0\\-1&0&1\\0&-1&1\end{pmatrix}\\\begin{pmatrix}0&-1&0\\-1&0&1\\0&-2&2\end{pmatrix}\end{matrix}\right\}$	(100) (010) (001)	$\mathfrak{osp}(5\vert 2)$
	$\begin{pmatrix}0&-1&0\\-1&2&-1\\0&-2&2\end{pmatrix}$	$\left.\begin{matrix}(100)\\(010)\end{matrix}\right\}$ (001)	$\mathfrak{osp}(5\vert 2)$ $\mathfrak{k}(1\vert 5)$
$\mathfrak{osp}(4\vert 2;\alpha)$ α generic	1) $\begin{pmatrix}2&-1&0\\\alpha&0&-1-\alpha\\0&-1&2\end{pmatrix}$ 2) $\begin{pmatrix}0&1&-1-\alpha\\-1&0&-\alpha\\-1-\alpha&\alpha&0\end{pmatrix}$	(100) (010) (001)	$\mathfrak{osp}(4\vert 2;\alpha)$
$\mathfrak{osp}(4\vert 2;\alpha)$ $\alpha=0,-1$	1) The simple part of ${}^{1)}\mathfrak{osp}(4\vert 2;\alpha)$ is $\mathfrak{sl}(2\vert 2)$; for the CTS of $\mathfrak{psl}(2\vert 2)$, see above 2) ${}^{2)}\mathfrak{osp}(4\vert 2;\alpha)\simeq\mathfrak{sl}(2\vert 2)$; for the CTS of $\mathfrak{sl}(2\vert 2)$, see above		

$\mathfrak{g}$	Cartan matrix	r	prolong
$\mathfrak{osp}(2\vert 4)$	1) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$	$\left.\begin{matrix}(100)\\(010)\\(001)\end{matrix}\right\}$	$\begin{cases} \mathfrak{osp}(2\vert 4) \\ \begin{cases} \mathfrak{osp}(2\vert 4) & \text{if } p>3 \\ \mathfrak{Bj}(3;N\vert 3) & \text{if } p=3 \end{cases} \\ \mathfrak{osp}(2\vert 4) \end{cases}$
	2) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 2 \end{pmatrix}$	$\left.\begin{matrix}(100)\\(010)\\(001)\end{matrix}\right\}$	$\begin{cases} \mathfrak{k}(3\vert 2) \\ \begin{cases} \mathfrak{osp}(2\vert 4) & \text{if } p>3 \\ \mathfrak{Bj}(3;N\vert 3) & \text{if } p=3 \end{cases} \\ \mathfrak{osp}(2\vert 4) \end{cases}$
	3) $\begin{pmatrix} 0 & -2 & 1 \\ -2 & 0 & 1 \\ -1 & -1 & 2 \end{pmatrix}$	$\left.\begin{matrix}(100)\\(010)\\(001)\end{matrix}\right\}$	$\begin{cases} \mathfrak{osp}(2\vert 4) \\ \mathfrak{osp}(2\vert 4) \\ \mathfrak{k}(3\vert 2) \end{cases}$
$\mathfrak{g}(2\vert 3)$	3) $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -2 \\ -1 & -2 & 2 \end{pmatrix}$	$\begin{matrix}(100)\\(010)\\(001)\end{matrix}$	$\begin{matrix}\mathfrak{Bj}(2\vert 4)\\ \mathfrak{Bj}(3\vert 5)\\ \mathfrak{bj}\end{matrix}$

1.8. A description of $\mathfrak{Bj}(3;N|3)$. For $\mathfrak{g} = {}^{1)}\mathfrak{osp}(2|4)$ and $r = (0,1,0)$, we have the following realization of the non-positive part:
(2)

$\mathfrak{g}_i$	the generators (even | odd)
$\mathfrak{g}_{-2}$	$Y_6 = \partial_1 \ \mid\ Y_8 = \partial_4$
$\mathfrak{g}_{-1}$	$Y_2 = \partial_2,\ Y_5 = x_2\partial_1 + x_5\partial_4 + \partial_3,\ \mid\ Y_4 = \partial_5,\ Y_7 = 2\,x_2\partial_4 + \partial_6,$
$\mathfrak{g}_0 \simeq$ $\mathfrak{sl}(1\vert 1)\oplus\mathfrak{sl}(2)\oplus\mathbb{K}$	$Y_3 = {x_2}^2\partial_1 + x_2x_5\partial_4 + x_2\partial_3 + 2x_5\partial_6,\ Z_3 = {x_3}^2\partial_1 + 2x_3x_6\partial_4 + x_3\partial_2 + 2x_6\partial_5$ $H_2 = 2\,x_1\partial_1 + 2\,x_2\partial_2 + x_4\partial_4 + x_5\partial_5 + 2\,x_6\partial_6,\ H_1 = [Z_1, Y_1], H_3 = [Z_3, Y_3]\ \mid$ $Y_1 = x_1\partial_4 + 2\,x_2\partial_5 + x_3\partial_6,\ Z_1 = 2\,x_4\partial_1 + 2\,x_5\partial_2 + x_6\partial_3$

The $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible, having one highest weight vector Y_2.

Let $p = 3$. The CTS prolong gives $\operatorname{sdim}(\mathfrak{g}_1) = 4|4$. The $\mathfrak{g}_0$-module $\mathfrak{g}_1$ has the following two lowest weight vectors:

V_1'	$x_1x_2\partial_4 + 2\,x_1\partial_6 + 2\,{x_2}^2\partial_5 + x_2x_3\partial_6$
V_1''	$x_1x_2\partial_1 + x_1x_5\partial_4 + 2\,x_2x_4\partial_4 + x_1\partial_3 + {x_2}^2\partial_2 + 2\,x_2x_5\partial_5 + x_2x_6\partial_6 + x_3x_5\partial_6 + x_4\partial_6$

Since $\mathfrak{g}_1$ generates the positive part of the CTS prolong, $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$, and $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-2}$, the standard criteria of simplicity ensures that the CTS prolong is simple. Since none of the $\mathbb{Z}$-graded Lie superalgebras over $\mathbb{C}$ of polynomial growth and finite depth has grading of this form (with $\mathfrak{g}_0 \simeq \mathfrak{sl}(1|1)\oplus\mathfrak{sl}(2)\oplus\mathbb{K}$), we conclude that this Lie superalgebra is new. We denote it by $\mathfrak{Bj}(3;N|3)$, where N is the shearing parameter of the even indeterminates. Our calculations show that $N_2 = N_3 = 1$ always. For $N_1 = 1$, 2, the super dimensions of the positive components of $\mathfrak{Bj}(3;N|3)$ are given in the following tables:

$N=(1,1,1)$	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$	$\mathfrak{g}_4$	$\mathfrak{g}_5$	$\mathfrak{g}_6$	$-$	$-$	$-$
sdim	$4\vert 4$	$5\vert 5$	$4\vert 4$	$4\vert 4$	$2\vert 2$	$0\vert 3$			

$N=(2,1,1)$	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$	$\mathfrak{g}_4$	$\mathfrak{g}_5$	$\mathfrak{g}_6$	$\cdots$	$\mathfrak{g}_{11}$	$\mathfrak{g}_{12}$
sdim	$4\vert 4$	$5\vert 5$	$4\vert 4$	$5\vert 5$	$4\vert 4$	$5\vert 5$	$\cdots$	$2\vert 2$	$0\vert 3$

Let V_i', V_i'' and V_i''' be the lowest height vectors of $\mathfrak{g}_i$ with respect to $\mathfrak{g}_0$. For $N=(1,1,1)$, these vectors are as follows:

$\mathfrak{g}_i$	lowest weight vectors
V_2'	$x_1{}^2\partial_4+2\,x_1x_2\partial_5+x_1x_3\partial_6+x_2x_3{}^2\partial_6$
V_2''	$x_1x_2{}^2\partial_1+x_1x_2x_5\partial_4+x_1x_2\partial_3+2\,x_1x_5\partial_6+x_2{}^2x_3\partial_3+2\,x_2{}^2x_5\partial_5+x_2x_3x_5\partial_6$
V_2'''	$x_1{}^2\partial_1+x_2{}^2x_3{}^2\partial_1+x_2{}^2x_3\partial_2+2\,x_2{}^2x_6\partial_5+2\,x_1x_2\partial_2+2\,x_1x_3\partial_3+2\,x_1x_4\partial_4$ $+x_2x_3{}^2\partial_3+x_2x_4\partial_5+2\,x_3x_4\partial_6+2\,x_2{}^2x_3x_6\partial_4+2\,x_2x_3x_6\partial_6$
V_3'	$x_1{}^2x_2\partial_4+2\,x_1{}^2\partial_6+2\,x_1x_2{}^2\partial_5+x_1x_2x_3\partial_6+x_2{}^2x_3{}^2\partial_6$
V_3''	$x_1{}^2x_2\partial_1+x_1{}^2x_5\partial_4+x_1{}^2\partial_3+x_1x_2x_3\partial_3+2\,x_1x_2x_5\partial_5+x_1x_3x_5\partial_6$ $+x_2x_3{}^2x_5\partial_6+x_1x_2x_4\partial_4+2\,x_1x_2{}^2\partial_2+x_1x_2x_3\partial_3+2\,x_1x_2x_6\partial_6$ $+2\,x_1x_4\partial_6+2\,x_2{}^2x_3{}^2\partial_3+2\,x_2{}^2x_3x_6\partial_6+2\,x_2{}^2x_4\partial_5+x_2x_3x_4\partial_6$
V_4'	$x_1{}^2x_2{}^2\partial_1+x_1{}^2x_2x_5\partial_4+x_1{}^2x_2\partial_3+2\,x_1{}^2x_5\partial_6+x_1x_2{}^2x_3\partial_3+2\,x_1x_2{}^2x_5\partial_5$ $+x_1x_2x_3x_5\partial_6+x_2{}^2x_3{}^2x_5\partial_6$
V_4''	$x_1{}^2x_4\partial_4+x_1{}^2x_5\partial_5+x_1{}^2x_6\partial_6+x_2{}^2x_3{}^2x_6\partial_6+x_1x_2{}^2x_3{}^2\partial_1+x_1x_2{}^2x_3\partial_2+2\,x_1x_2{}^2x_6\partial_5$ $+2\,x_1x_3{}^2x_5\partial_6+2\,x_1x_2x_3{}^2\partial_3+2\,x_1x_2x_4\partial_5+x_1x_3x_4\partial_6+x_2x_3{}^2x_4\partial_6+2\,x_1x_2{}^2x_3x_6\partial_4$
V_5'	$x_1{}^2x_2{}^2\partial_2+x_1{}^2x_4\partial_6+2\,x_1{}^2x_2x_3\partial_3+2\,x_1{}^2x_2x_4\partial_4+x_1{}^2x_2x_6\partial_6+2\,x_2{}^2x_3{}^2x_4\partial_6$ $+x_1x_2{}^2x_3{}^2\partial_3+x_1x_2{}^2x_4\partial_5+x_1x_2{}^2x_3x_6\partial_6+2\,x_1x_2x_3x_4\partial_6$
V_6'	$x_1{}^2x_2{}^2x_4\partial_1+x_1{}^2x_2{}^2x_5\partial_2+2\,x_1{}^2x_2{}^2x_6\partial_3+x_1{}^2x_2x_4\partial_3+2\,x_1{}^2x_4x_5\partial_6$ $+2\,x_1{}^2x_2x_3x_5\partial_3+x_1{}^2x_2x_4x_5\partial_4+x_1{}^2x_2x_5x_6\partial_6+x_2{}^2x_3{}^2x_4x_5\partial_6$ $+x_1x_2{}^2x_3{}^2x_5\partial_3+x_1x_2{}^2x_3x_4\partial_3+2\,x_1x_2{}^2x_4x_5\partial_5+x_1x_2{}^2x_3x_5x_6\partial_6+x_1x_2x_3x_4x_5\partial_6$

For $N=(2,1,1)$, the lowest hight vectors are as in the table above together with the following ones

$\mathfrak{g}_i$	lowest weight vectors
V_4'''	$x_1{}^3\partial_4+2\,x_1{}^2x_2\partial 5+x_1{}^2x_3\partial_6+x_1x_2x_3{}^2\partial_6$
...	....
V_{11}'	$x_1{}^5x_2{}^2\partial_2+x_1{}^5x_4\partial_6+x_1{}^4x_2{}^2x_3{}^2\partial_3+x_1{}^4x_2{}^2x_4\partial_5+2\,x_1{}^5x_2x_3\partial_3+2\,x_1{}^5x_2x_4\partial_4$ $+x_1{}^5x_2x_6\partial_6+2\,x_1{}^3x_2{}^2x_3{}^2x_4\partial_6+x_1{}^4x_2{}^2x_3x_6\partial_6+2\,x_1{}^4x_2x_3x_4\partial_6$
V_{12}'	$x_1{}^5x_2{}^2x_4\partial_1+x_1{}^5x_2{}^2x_5\partial_2+2\,x_1{}^5x_2{}^2x_6\partial_3+x_1{}^5x_2x_4\partial_3+2\,x_1{}^5x_4x_5\partial_6+x_1{}^4x_2{}^2x_3{}^2x_5\partial_3$ $+x_1{}^4x_2{}^2x_3x_4\partial_3+2\,x_1{}^4x_2{}^2x_4x_5\partial_5+2\,x_1{}^5x_2x_3x_5\partial_3+x_1{}^5x_2x_4x_5\partial_4+x_1{}^5x_2x_5x_6\partial_6$ $+x_1{}^3x_2{}^2x_3{}^2x_4x_5\partial_6+x_1{}^4x_2{}^2x_3x_5x_6\partial_6+x_1{}^4x_2x_3x_4x_5\partial_6$

Let us investigate if $\mathfrak{Bj}(3;N|3)$ has partial prolongs as subalgebras:

(i) Denote by $\mathfrak{g}_1'$ the $\mathfrak{g}_0$-module generated by V_1'. We have $\mathrm{sdim}(\mathfrak{g}_1')=2|2$. The CTS partial prolong $(\mathfrak{g}_-,\mathfrak{g}_0,\mathfrak{g}_1')_*$ gives a graded Lie superalgebra with the property that $[\mathfrak{g}_{-1},\mathfrak{g}_1]\simeq\{Y_1,h_1\}:=\mathfrak{aff}$. From the description of irreducible modules over solvable Lie superalgebras [Ssol], we see that the irreducible $\mathfrak{aff}$-modules are 1-dimensional. For irreducible $\mathfrak{aff}$-submodules $\mathfrak{g}_{-1}'$ in $\mathfrak{g}_{-1}$ we have two possibilities: to take $\mathfrak{g}_{-1}'=\{Y_4\}$ or $\mathfrak{g}_{-1}'=\{Y_7\}$; for both of them, $\mathfrak{g}_{-1}'$ is purely odd and we can never get a simple Cartan prolong.

(ii) Denote by $\mathfrak{g}_1''$ the $\mathfrak{g}_0$-module generated by V_1''. We have $\mathrm{sdim}(\mathfrak{g}_1'')=2|2$. The CTS partial prolong $(\mathfrak{g}_-,\mathfrak{g}_0,\mathfrak{g}_1'')_*$ returns $\mathfrak{osp}(2|4)$.

1.9. A description of $\mathfrak{Bj}(2|4)$. We consider ${}^{3)}\mathfrak{g}(2|3)$ with $r=(1,0,0)$. In this case, $\mathrm{sdim}(\mathfrak{g}(2,3)_-)=2|4$. Since the $\mathfrak{g}(2,3)_0$-module action is not faithful, we consider the quotient algebra $\mathfrak{g}_0=\mathfrak{g}(2,3)_0/\mathfrak{ann}(\mathfrak{g}_{-1})$ and embed $(\mathfrak{g}(2,3)_-,\mathfrak{g}_0)\subset\mathfrak{vect}(2|4)$. This realization is given by the following table:

$\mathfrak{g}_i$	the generators (even | odd)
$\mathfrak{g}_{-1}$	$Y_6=\partial_2,\ Y_8=\partial_1\ \mid\ Y_{11}=\partial_3,\ Y_{10}=\partial_4,\ Y_4=\partial_5,\ Y_1=\partial_6$
$\mathfrak{g}_0\simeq$ $\mathfrak{osp}(3\|2)$	$Y_3=x_2\partial_1+2\,x_4\partial_3+x_6\partial_5, Y_9=[Y_2,[Y_3,Y_5]], Z_3=x_1\partial_2+2\,x_3\partial_4+x_5\partial_6, Z_9=$ $[Z_2,[Z_3,Z_5]],\ H_2=[Z_2,Y_2], H_3=[Z_3,Y_3]\ \mid\ Y_2=x_1\partial_4+x_5\partial_2, Y_5=[Y_2,Y_3],$ $Y_7=[Y_3,[Y_2,Y_3]],\ Z_2=x_2\partial_5+2x_4\partial_1, Z_5=[Z_2,Z_3], Z_7=[Z_3,[Z_2,Z_3]]$

The $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible, having one lowest weight vector Y_{11} and one highest weight vector Y_1. The CTS prolong $(\mathfrak{g}_-,\mathfrak{g}_0)_*$ gives a Lie superalgebra of superdimension $13|14$. Indeed, $\mathrm{sdim}(\mathfrak{g}_1)=4|4$ and $\mathrm{sdim}(\mathfrak{g}_2)=1|0$. The $\mathfrak{g}_0$-module $\mathfrak{g}_1$ has one lowest vector:

$$\boxed{V_1=2\,x_1x_2\partial_3+x_1x_6\partial_1+2\,{x_2}^2\partial_4+x_2x_5\partial_1+2\,x_2x_6\partial_2+x_4x_5\partial_3+2\,x_4x_6\partial_4+x_5x_6\partial_5}$$

The $\mathfrak{g}_2$ is one-dimensional spanned by the following vector

$$\begin{array}{|l|}\hline 2\,{x_1}^2x_2\partial_1+{x_1}^2x_4\partial_3+2\,{x_1}^2x_6\partial_5+x_1{x_2}^2\partial_2+2\,x_1x_2x_3\partial_3+x_1x_2x_4\partial_4+2\,x_1x_2x_5\partial_5+x_1x_2x_6\partial_6\\ +x_1x_3x_6\partial_1+2\,x_1x_4x_5\partial_1+x_1x_4x_6\partial_2+2\,{x_2}^2x_3\partial_4+{x_2}^2x_5\partial_6+x_2x_3x_5\partial_1+2\,x_2x_3x_6\partial_2+x_2x_4x_5\partial_2\\ +x_3x_4x_5\partial_3+2\,x_3x_4x_6\partial_4+x_3x_5x_6\partial_5+2\,x_4x_5x_6\partial_6\\ \hline\end{array}$$

Besides, if $i>2$, then $\mathfrak{g}_i=0$ for all values of the sharing parameter $N=(N_1,N_2)$. A direct computation gives $[\mathfrak{g}_1,\mathfrak{g}_1]=\mathfrak{g}_2$ and $[\mathfrak{g}_{-1},\mathfrak{g}_1]=\mathfrak{g}_0$. SuperLie tells us that this Lie superalgebra has three ideals $I_1\subset I_2\subset I_3$ with the same non-positive part but different positive parts: $\mathrm{sdim}(I_1)=10|14$, $\mathrm{sdim}(I_2)=11|14$, $\mathrm{sdim}(I_3)=12|14$. The ideal I_1 is just our $\mathfrak{bj}$, see [BjL, CE]. The partial CTS prolong with I_1 returns I_1 plus an outer derivation given by the vector above (of degree 2). It is clear now that $\mathfrak{Bj}(2|4)$ is not simple.

1.10. A description of $\mathfrak{Bj}(3|5)$. We consider ${}^{3)}\mathfrak{g}(2|3)$ and $r=(0,1,0)$. In this case, $\mathrm{sdim}(\mathfrak{g}(2,3)_-)=3|5$. Since the $\mathfrak{g}(2,3)_0$-module action is again not faithful, we consider the quotient module $\mathfrak{g}_0=\mathfrak{g}(2,3)_0/\mathfrak{ann}(\mathfrak{g}_{-1})$ and embed $(\mathfrak{g}(2,3)_-,\mathfrak{g}_0)\subset\mathfrak{vect}(3;\underline{N}|5)$. This realization is given by the following table:

$\mathfrak{g}_i$	the generators (even | odd)
$\mathfrak{g}_{-2}$	$Y_9=\partial_1\ \mid\ Y_{10}=\partial_3, Y_{11}=\partial_2$
$\mathfrak{g}_{-1}$	$Y_8=\partial_4,\ Y_6=\partial_5\ \mid\ Y_5=2\,x_4\partial_2+2\,x_5\partial_3+2\,x_7\partial_1+\partial_7, Y_2=x_4\partial_3-2\,x_6\partial_1+\partial_8$ $Y_7=x_5\partial_2+\partial_6$
$\mathfrak{g}_0\simeq\mathfrak{sl}(1\|2)$	$H_1=[Z_1,Y_1], H_3=[Z_3,Y_3], Y_3=2\,x_3\partial_2+2\,x_7x_8\partial_1+x_5\partial_4+2\,x_7\partial_6+x_8\partial_7,$ $Z_3=2\,x_2\partial_3+2\,x_6x_7\partial_1+x_4\partial_5+x_6\partial_7+2\,x_7\partial_8\ \mid\ Y_4=[Y_1,Y_3],\ Z_4=[Z_1,Z_3],$ $Y_1=2\ (2\,x_1\partial_3+2\,x_6x_7\partial_2+x_6\partial_4+x_7\partial_5)$ $Z_1=2\ \left(x_3\partial_1+2\,x_4x_5\partial_2+2\,{x_5}^2\partial_3+2\,x_5x_7\partial_1+2\,x_4\partial_6+x_5\partial_7\right),$

The $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible, having one highest weight vector Y_2. We have $\mathrm{sdim}(\mathfrak{g}_1)=6|4$. The $\mathfrak{g}_0$-module $\mathfrak{g}_1$ has two lowest weight vectors given by

V_1'	$x_1x_5\partial_2+2\,x_5x_6x_8\partial_2+x_5x_7x_8\partial_3+2\,x_1\partial_6+2\,x_3\partial_4+x_5x_7\partial_4+x_5x_8\partial_5+2\,x_7x_8\partial_7$
V_1''	$x_6x_7x_8\partial_2+2\,x_1\partial_4+x_7x_8\partial_5$

Now, the $\mathfrak{g}_0$-module generated by the the vectors V_1' and V_1'' is not the whole $\mathfrak{g}_1$ but a $\mathfrak{g}_0$-module that we denote by $\mathfrak{g}_1''$, of sdim $=4|4$. The CTS prolong $(\mathfrak{g}_-,\mathfrak{g}_0,\mathfrak{g}_1)_*$ is not simple, so consider the Lie subsuperalgebra $(\mathfrak{g}_-, \mathfrak{g}_0, \mathfrak{g}_1'')_*$; the superdimensions of its positive part are

$\mathrm{ad}^i_{\mathfrak{g}_1''}(\mathfrak{g}_1'')$	$\mathfrak{g}_1''$	$\mathrm{ad}_{\mathfrak{g}_1''}(\mathfrak{g}_1'')$	$\mathrm{ad}^2_{\mathfrak{g}_1''}(\mathfrak{g}_1'')$	$\mathrm{ad}^3_{\mathfrak{g}_1''}(\mathfrak{g}_1'')$	$\mathrm{ad}^4_{\mathfrak{g}_1''}(\mathfrak{g}_1'')$
sdim	$4\vert 4$	$4\vert 4$	$4\vert 4$	$3\vert 2$	$2\vert 1$

The lowest weight vectors of the above components are precisely $\{V_2', V_2'', V_3, V_4, V_5\}$ described bellow:

$\mathrm{ad}^i_{\mathfrak{g}_1}(\mathfrak{g}_1)$	lowest weight vectors
V_2'	$x_1{}^2\partial_2 + 2\,x_1x_7\partial_4 + 2\,x_1x_8\partial_5 + x_1x_6x_8\partial_2 + 2\,x_1x_7x_8\partial_3$
V_2''	$2\,x_1{}^2\partial_1 + x_1x_2\partial_2 + x_1x_3\partial_3 + x_1x_6\partial_6 + x_1x_7\partial_7 + x_1x_8\partial_8 + 2\,x_2x_7\partial_4 + 2\,x_2x_8\partial_5$ $+2\,x_3x_6\partial_4 + 2\,x_3x_7\partial_5 + x_2x_6x_8\partial_2 + 2\,x_2x_7x_8\partial_3 + x_3x_6x_7\partial_2 + x_6x_7x_8\partial_7$
V_3	$x_1{}^2\partial_4 + 2\,x_1x_7x_8\partial_5 + 2\,x_1x_6x_7x_8\partial_2$
V_4	$x_1{}^2x_3\partial_2 + 2\,x_1{}^2x_5\partial_4 + x_1{}^2x_7\partial_6 + 2\,x_1{}^2x_8\partial_7 + x_1{}^2x_7x_8\partial_1 + 2\,x_1x_3x_7\partial_4 + 2\,x_1x_3x_8\partial_5$ $+x_1x_3x_6x_8\partial_2 + 2\,x_1x_3x_7x_8\partial_3 + x_1x_5x_7x_8\partial_5 + x_1x_5x_6x_7x_8\partial_2$
V_5	$x_1{}^2x_2\partial_4 + 2\,x_1{}^2x_3\partial_5 + 2\,x_1{}^2x_6x_7\partial_6 + x_1{}^2x_6x_8\partial_7 + 2\,x_1{}^2x_7x_8\partial_8 + 2\,x_1{}^2x_6x_7x_8\partial_1$ $+2\,x_1x_2x_7x_8\partial_5 + 2\,x_1x_3x_6x_7\partial_4 + 2\,x_1x_3x_6x_8\partial_5 + 2\,x_1x_2x_6x_7x_8\partial_2 + 2\,x_1x_3x_6x_7x_8\partial_3$

Since none of the known simple finite dimensional Lie superalgebra over (algebraically closed) fields of characteristic 0 or > 3 has such a non-positive part in any $\mathbb{Z}$-grading, it follows that $\mathfrak{Bj}(3; \underline{N}|5)$ is new.

Let us investigate if $\mathfrak{Bj}(3; \underline{N}|5)$ has subalgebras — partial prolongs.

(i) Denote by $\mathfrak{g}_1'$ the $\mathfrak{g}_0$-module generated by V_1'. We have $\mathrm{sdim}(\mathfrak{g}_1') = 2|3$. The CTS partial prolong $(\mathfrak{g}_{-1}, \mathfrak{g}_0, \mathfrak{g}_1')_*$ gives a graded Lie superalgebra with $\mathrm{sdim}(\mathfrak{g}_2') = 2|2$ and $\mathfrak{g}_i' = 0$ for $i > 3$. An easy computation shows that $[\mathfrak{g}_{-1}, \mathfrak{g}_1'] = \mathfrak{g}_0$ and $[\mathfrak{g}_1', \mathfrak{g}_1'] \subsetneq \mathfrak{g}_2'$. Since we are investigating simple Lie superalgebra, we take the simple part of $(\mathfrak{g}_{-1}, \mathfrak{g}_0, \mathfrak{g}_1')_*$. This simple Lie superalgebra is isomorphic to $\mathfrak{g}(2,3)/\mathfrak{c} = \mathfrak{bj}$.

(ii) Denote by $\mathfrak{g}_1''$ the $\mathfrak{g}_0$-module generated by V_1''. We just saw that $\mathrm{sdim}(\mathfrak{g}_1'') = 4|4$. The CTS partial prolong $(\mathfrak{g}_{-1}, \mathfrak{g}_0, \mathfrak{g}_1'')_*$ gives also $\mathfrak{Bj}(3|5)$.

$\underline{r = (0,0,1)}$. In this case, $\mathrm{sdim}(\mathfrak{g}(2,3)_-) = 4|5$. Since the $\mathfrak{g}(2,3)_0$-module action is not faithful, we consider the quotient algebra $\mathfrak{g}_0 = \mathfrak{g}(2,3)_0/\mathfrak{ann}(\mathfrak{g}_{-1})$ and embed $(\mathfrak{g}(2,3)_-, \mathfrak{g}_0) \subset \mathfrak{vect}(4; \underline{N}|5)$. The CTS prolong returns $\mathfrak{bj} := \mathfrak{g}(2,3)/\mathfrak{c}$.

1.11. A description of $\mathfrak{Me}(3; N|3)$. 1) Our first idea was to try to repeat the above construction with a suitable super version of $\mathfrak{g}(2)$. There is only one simple super analog of $\mathfrak{g}(2)$, namely $\mathfrak{ag}(2)$, but our attempts [BjL] to construct a super analog of Melikyan algebra in the above way as Kuznetsov suggested [Ku1] (reproduced in [GL4]) resulted in something quite distinct from the Melikyan algebra: The Lie superalgebras we obtained, an exceptional one $\mathfrak{bj}$ (cf. [CE, BGL1]) and a series $\mathfrak{Bj}$, are indeed simple but do not resemble either $\mathfrak{g}(2)$ or $\mathfrak{Me}$.

2) Our other idea is based on the following observation. The anti-symmetric form

$$(f,g) := \int f dg = \int f g' dt, \tag{3}$$

on the quotient space F/const of functions (with compact support) modulo constants on the 1-dimensional manifolds, has its counterpart in $1|1$-dimensional case in presence of a contact structure a n d o n l y i n t h i s c a s e as follows from the description of invariant bilinear differential operators, see [KLV]. Indeed, the Lie superalgebra $\mathfrak{k}(1|1)$ does not distinguish between the space of volume forms (let its generator be denoted vol) and the quotient $\Omega^1/F\alpha$, where $\alpha = dt + \theta d\theta$ is the contact form.

For any prime p therefore, on the space $\mathfrak{g}_{-1} := \mathcal{O}(1; \underline{N}|1)/\operatorname{const}$ of "functions (with compact support) in one even indeterminate u and one odd, θ modulo constants", the superantisymmetric bilinear form

$$(f, g) := \int (f dg \mod F\alpha) = \int (f_0 g_0' - f_1 g_1) dt, \tag{4}$$

where $f = f_0(t) + f_1(t)\theta$ and $g = g_0(t) + g_1(t)\theta$ and where $' := \frac{d}{dt}$, is nondegenerate.

Therefore, we may expect that, for p small and $\underline{N} = 1$, the Melikyan effect will reappear. Consider $p = 5$ as the most plausible.

We should be careful with parities. The parity of vol is a matter of agreement, let it be even. Then the integral is an odd functional but the factorization modulo $F\alpha$ makes the form (4) even. (Setting $p(\mathrm{vol}) = \bar{1}$ we make the integral an even functional and the factorization modulo $F\alpha$ makes the form (4) even again.)

Since the form (4) is even, we get the following realization of

$$\mathfrak{k}(1; \underline{1}|1) \subset \mathfrak{osp}(5|4) \simeq \mathfrak{k}(5; \underline{1, ..., 1}|5)$$

by generating functions of contact vector fields on the $5|5$-dimensional superspace with the contact form, where the coefficients are found from the explicit values of

$$d\hat{t} + \sum_i (\hat{p}_i d\hat{q}_i - \hat{q}_2 d\hat{p}_i) + \sum_j \left(\hat{\xi}_j d\eta_j + \hat{\eta}_j d\hat{\xi}_j\right) - \hat{\theta} d\hat{\theta}.$$

The coordinates on this $5|5$-dimensional superspace are hatted in order not to confuse them with generating functions of $\mathfrak{k}(1; \underline{1}|1)$:

(5)

$\mathfrak{g}_i$	basis elements
$\mathfrak{g}_{-2}$	$\hat{1}$
$\mathfrak{g}_{-1}$	$\hat{p}_1 = t,\ \hat{p}_2 = t^2,\ \hat{q}_1 = t^3,\ \hat{q}_2 = t^4,$ $\hat{\xi}_1 = \theta,\ \hat{\xi}_2 = t\theta,\ \hat{\theta} = t^2\theta,\ \hat{\eta}_2 = t^3\theta,\ \hat{\eta}_1 = t^4\theta$

We explicitly have:

$$\begin{aligned}
&(t, t^4) = \textstyle\int_{\underline{N}} t \cdot t^3 dt_{\underline{N}} = \int_{\underline{N}} 4t^4 dt_{\underline{N}} = 4 = -(t^4, t);\\
&(t^2, t^3) = \textstyle\int_{\underline{N}} t^2 \cdot t^2 dt_{\underline{N}} = \int_{\underline{N}} 6t^4 dt_{\underline{N}} = 1 = -(t^3, t^2);\\
&(t^4\theta, \theta) = -\textstyle\int_{\underline{N}} t^4 \cdot 1 dt_{\underline{N}} = -1 = (\theta, t^4\theta);\\
&(t^3\theta, t\theta) = -\textstyle\int_{\underline{N}} t^3 \cdot t dt_{\underline{N}} = -4 = (t\theta, t^3\theta);\\
&(t^2\theta, t^2\theta) = -\textstyle\int_{\underline{N}} t^2 \cdot t^2 dt_{\underline{N}} = -6 = -1.
\end{aligned} \tag{6}$$

Now, let us realize $\mathfrak{k}(1;\underline{1}|1)$ by contact fields in hatted functions:

(7)

$\mathfrak{g}_i$	basis elements
$\mathfrak{g}_{-2}$	$\hat{1}$
$\mathfrak{g}_{-1}$	$\hat{p}_1=t,\ \hat{p}_2=t^2,\ \hat{q}_2=4t^3,\ \hat{q}_1=t^4,$ $\xi_1=\theta,\ \xi_2=t\theta,\ \hat{\theta}=t^2\theta,\ \eta_2=4t^3\theta,\ \eta_1=t^4\theta$
$\mathfrak{g}_0$	$1=2\,\hat{p}_1{\cdot}\hat{q}_2+2\hat{p}_2^2+3\,\xi_1\eta_2+3\,\xi_2\hat{\theta};\quad t=2\,\hat{p}_1\hat{q}_1+4\,\hat{p}_2\hat{q}_2+4\,\xi_1\eta_1+2\,\xi_2\eta_2;$ $t^2=2\,\hat{p}_2\hat{q}_1+4\,\hat{q}_2^2+4\,\xi_2\eta_1+\hat{\theta}\eta_2;\quad t^3=3\,\hat{q}_1\hat{q}_2+4\,\hat{\theta}\eta_1;\quad t^4=\hat{q}_1^2+\eta_2\eta_1;$ $\theta=\hat{p}_1\eta_2+\hat{p}_2\hat{\theta}+\hat{q}_1\xi_1+\hat{q}_2\xi_2;\quad t\theta=\hat{p}_1\eta_1+2\,\hat{p}_2\eta_2+\hat{q}_1\xi_2+2\,\hat{q}_2\hat{\theta};$ $t^2\theta=\hat{p}_2\eta_1+\hat{q}_1\hat{\theta}+2\,\hat{q}_2\eta_2;\quad t^3\theta=4\,\hat{q}_1\eta_2+4\,\hat{q}_2\eta_1;\quad t^4\theta=\hat{q}_1\eta_1$

The CTS prolong gives that $\mathfrak{g}_1=0$.

The case where $p=3$ is more interesting because it will give us the series $\mathfrak{Me}(3;N|3)$. The non-positive part is as follows:

(8)

$\mathfrak{g}_i$	basis elements
$\mathfrak{g}_{-2}$	$\hat{1}$
$\mathfrak{g}_{-1}$	$\hat{p}_1=t,\ \hat{q}_2=t^2,\ \xi_1=\theta,\ \hat{\theta}=t\theta,\ \eta_1=t^2\theta$
$\mathfrak{g}_0$	$1=\hat{p}_1^2+2\hat{\xi}_1\hat{\eta}_1;\quad t=2\,\hat{p}_1\hat{q}_1+2\,\hat{\xi}_1\hat{\eta}_1;\ t^2=2\,\hat{q}_1^2+2\,\hat{\theta}\hat{\eta}_1;\ \theta=2\,\hat{p}_1\hat{\theta}+\hat{q}_1\hat{\xi}_1;$ $t\theta=\hat{p}_1\hat{\eta}_1+\hat{q}_1\hat{\theta};\ t^2\theta=\hat{q}_1\hat{\eta}_1$

The Lie superalgebra $\mathfrak{g}_0$ is not simple because $[\mathfrak{g}_{-1},\mathfrak{g}_1]=\mathfrak{g}_0\backslash\{t^2\theta=\hat{q}_1\hat{\eta}_1\}$. Denote $\mathfrak{g}_0':=[\mathfrak{g}_{-1},\mathfrak{g}_1]\simeq\mathfrak{osp}(1|2)$. The CTS partial prolong $(\mathfrak{g}_-,\mathfrak{g}_0')_*$ seems to be very interesting. First, our computation shows that the parameter $\underline{M}=(M_1,M_2,M_3)$ depends only on the first parameter (relative to t). Namely, $\underline{M}=(M_1,1,1)$. For $M_1=1,\,2$, the super dimensions of the positive components of $\mathfrak{Bj}(3;M|3)$ are given in the following table:

$M=(1,1,1)$	$\mathfrak{g}_1'$	$\mathfrak{g}_2'$	$\mathfrak{g}_3'$	$\mathfrak{g}_4'$	$\mathfrak{g}_5'$	$-$	$-$	$-$	$-$	$-$
sdim	$2\|4$	$4\|2$	$2\|4$	$3\|2$	$0\|1$					

$M=(2,1,1)$	$\mathfrak{g}_1'$	$\mathfrak{g}_2'$	$\mathfrak{g}_3'$	$\mathfrak{g}_4'$	$\mathfrak{g}_5'$	$\cdots$	$\mathfrak{g}_{14}'$	$\mathfrak{g}_{15}'$	$\mathfrak{g}_{16}'$	$\mathfrak{g}_{17}'$
sdim	$2\|4$	$4\|2$	$2\|4$	$4\|2$	$2\|4$	$\cdots$	$4\|2$	$2\|4$	$3\|2$	$0\|1$

Here we have that $[\mathfrak{g}_{-1},\mathfrak{g}_1]=\mathfrak{g}_0'$ and the $\mathfrak{g}_1'$ generates the positive part. The standard criteria for simplicity ensures that $\mathfrak{Me}(3;N|3)$ is simple. For $N=(1,a,b)$, the lowest weight vectors are as follows:

$\mathfrak{g}_i$	lowest weight vectors
V_1'	$2p_1{}^{(2)}\eta_1 + 2\,p_1q_1\theta + q_1{}^{(2)}\xi_1 + \xi_1\theta\eta_1$
V_1''	$2\,p_1q_1\eta_1 + q_1{}^{(2)}\theta + \eta_1 t$
V_2'	$2\,p_1q_1\theta\eta_1 + q_1{}^{(2)}\xi_1\eta_1 + tq_1{}^{(2)} + t\theta\eta_1$
V_2''	$2\,p_1{}^{(2)}q_1{}^{(2)} + p_1{}^{(2)}\theta\eta_1 + p_1q_1\xi_1\eta_1 + 2q_1{}^{(2)}\xi_1\theta + t^{(2)}$
V_3'	$2\,tp_1{}^{(2)}\eta_1 + tp_1q_1\theta + 2\,tq_1{}^{(2)}\xi_1 + t\xi_1\theta\eta_1$
V_3''	$2\,p_1{}^{(2)}q_1{}^{(2)}\eta_1 + 2\,tp_1q_1\eta_1 + 2\,q_1{}^{(2)}\xi_1\theta\eta_1 + tq_1{}^{(2)}\theta + t^{(2)}\eta_1$
V_4'	$2\,p_1{}^{(2)}q_1{}^{(2)}\theta\eta_1 + 2\,tp_1q_1\theta\eta_1 + tq_1{}^{(2)}\xi_1\eta_1 + t^{(2)}q_1{}^{(2)} + t^{(2)}\theta\eta_1$
V_5'	$p_1{}^{(2)}q_1{}^{(2)}\xi_1\theta\eta_1 + t^{(2)}p_1{}^{(2)}\eta_1 + t^{(2)}p_1q_1\theta + 2\,t^{(2)}q_1{}^{(2)}\xi_1 + 2\,t^{(2)}\xi_1\theta\eta_1$

For $N = (2, a, b)$, the lowest weight vectors are as above together with:

$\mathfrak{g}_i$	lowest weight vectors
V_4''	$2\,tp_1{}^{(2)}q_1{}^{(2)} + tp_1{}^{(2)}\theta\eta_1 + tp_1q_1\xi_1\eta_1 + 2\,tq_1{}^{(2)}\xi_1\theta + t^{(3)}$
V_5''	$2\,tp_1{}^{(2)}q_1{}^{(2)}\eta_1 + 2\,t^{(2)}p_1q_1\eta_1 + 2\,tq_1{}^{(2)}\xi_1\theta\eta_1 + t^{(2)}q_1{}^{(2)}\theta + t^{(3)}\eta_1$
...	....
V_{15}'	$2\,t^{(5)}p_1{}^{(2)}q_1{}^{(2)}\xi_1\theta\eta_1 + 2t^{(7)}p_1{}^{(2)}\eta_1 + 2\,t^{(7)}p_1q_1\theta + t^{(7)}q_1{}^{(2)}\xi_1 + t^{(7)}\xi_1\theta\eta_1$
	$2\,t^{(6)}p_1{}^{(2)}q_1{}^{(2)}\eta_1 + 2\,t^{(7)}p_1q_1\eta_1 + 2\,t^{(6)}q_1{}^{(2)}\xi_1\theta\eta_1 + t^{(7)}q_1{}^{(2)}\theta + t^{(8)}\eta_1$
V_{16}''	$2\,t^{(6)}p_1{}^{(2)}q_1{}^{(2)}\theta\eta_1 + 2\,t^{(7)}p_1q_1\theta\eta_1 + t^{(7)}q_1{}^{(2)}\xi_1\eta_1 + t^{(8)}q_1{}^{(2)} + t^{(8)}\theta\eta_1$
V_{17}''	$2t^{(6)}p_1{}^{(2)}q_1{}^{(2)}\xi_1\theta\eta_1 + 2\,t^{(8)}p_1{}^{(2)}\eta_1 + 2\,t^{(8)}p_1q_1\theta + t^{(8)}q_1{}^{(2)}\xi_1 + t^{(8)}\xi_1\theta\eta_1$

Let us investigate the subalgebras of $\mathfrak{Me}(3;N|3)$ — partial prolongs:

(i) Denote by $\mathfrak{g}_1'$ the $\mathfrak{g}_0$-module generated by V_1'. We have $\mathrm{sdim}(\mathfrak{g}_1') = 0|1$ and $\mathfrak{g}_i = 0$ for all $i > 1$. The CTS partial prolong $(\mathfrak{g}_-, \mathfrak{g}_0, \mathfrak{g}_1')_*$ gives a graded Lie superalgebra with the property that $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \simeq \mathfrak{osp}(1|2)$. The partial CTS prolong $(\mathfrak{g}_-, \mathfrak{osp}(1|2))_*$ is not simple

(ii) Denote by $\mathfrak{g}_1''$ the $\mathfrak{g}_0$-module generated by V_1''. We have $\mathrm{sdim}(\mathfrak{g}_1'') = 3|2$. The CTS partial prolong $(\mathfrak{g}_-, \mathfrak{g}_0, \mathfrak{g}_1'')_*$ returns $\mathfrak{brj}(2;3)$.

1.12. A description of $\mathfrak{Brj}(4|3)$. We have the following realization of the non-positive part inside $\mathfrak{vect}(4|3)$:

(9)

$\mathfrak{g}_i$	the generators (even | odd)
$\mathfrak{g}_{-4}$	$Y_8 = \partial_1 \ \mid\ Y_7 = \partial_5$
$\mathfrak{g}_{-3}$	$Y_6 = \partial_2 \ \mid$
$\mathfrak{g}_{-2}$	$Y_4 = \partial_3 \ \mid\ Y_5 = x_3\partial_5 + x_6\partial_1 + \partial_6$
$\mathfrak{g}_{-1}$	$Y_3 = 2\,x_2\partial_1 + 2\,x_3\partial_2 + \partial_4 \ \mid$
	$Y_2 = x_2\partial_5 + 2\,x_4{}^{(2)}x_7\partial_1 + x_4x_6\partial_1 + x_6x_7\partial_5 + x_4x_7\partial_2 + x_6\partial_2 + 2\,x_4\partial_6 + 2\,x_7\partial_3 + 1\partial_7,$
$\mathfrak{g}_0 \simeq \mathfrak{hei}(0\|2) \oplus \mathbb{K}$	$H_1 = [Z_1, Y_1],\ H_2 = 2\,x_5\partial_5 + x_2\partial_2 + 2\,x_3\partial_3 + 2\,x_4\partial_4 + x_7\partial_7,\ \mid$
	$Y_1 = 2\,x_3{}^{(2)}\partial_5 + 2\,x_3x_6\partial_1 + 2\,x_5\partial_1 + 2\,x_3\partial_6 + x_7\partial_4,$
	$Z_1 = x_4{}^{(2)}\partial_6 + 2\,x_4{}^{(2)}x_6\partial_1 + x_4{}^{(2)}x_7\partial_2 + x_4x_7\partial_3 + 2\,x_4x_6x_7\partial_5 + x_1\partial_5 + 2\,x_4\partial_7 + 2\,x_6\partial_3$

The Lie superalgebra $\mathfrak{g}_0$ is solvable, and hence the CTS prolong $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is NOT simple since $\mathfrak{g}_1$ does not generate the positive part. Our calculation shows that the prolong does not depend on N, i.e., $N = (1, 1, 1, 1)$. The simple part of this prolong is $\mathfrak{brj}(2; 3)$. The sdim of the positive parts are described as follows:

	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$	$\mathfrak{g}_4$	$\mathfrak{g}_5$	$\mathfrak{g}_6$	$\mathfrak{g}_7$	$\mathfrak{g}_8$	$\mathfrak{g}_9$	$\mathfrak{g}_{10}$
sdim	$1\vert 1$	$2\vert 2$	$1\vert 2$	$2\vert 2$	$1\vert 1$	$2\vert 2$	$1\vert 1$	$1\vert 1$	$0\vert 1$	$1\vert 1$

and the lowest weight vectors are as follows:

$\mathfrak{g}_i$	lowest weight vectors
V_1'	$2\,x_2x_3\partial_5+2\,x_2x_6\partial_1+x_3x_4\partial_6+2\,x_3x_6\partial_2+x_3x_7\partial_3+x_4x_7\partial_4+x_3{x_4}^{(2)}x_7\partial_1+2\,x_3x_4x_6\partial_1$
	$+2\,x_3x_4x_7\partial_2+2\,x_3x_6x_7\partial_5+2\,x_2\partial_6+2\,x_3\partial_7+2\,x_5\partial_2+2\,x_6\partial_4$
V_2'	${x_4}^{(2)}\partial_4+x_1x_3\partial_5+x_1x_6\partial_1+x_3{x_4}^{(2)}\partial_6+2\,x_3x_4\partial_7+2\,x_3x_6\partial_3+2\,x_4x_6\partial_4+2\,x_6x_7\partial_7$
	$+2\,x_3{x_4}^{(2)}x_6\partial_1+x_3{x_4}^{(2)}x_7\partial_2+x_3x_4x_7\partial_3+2\,x_3x_4x_6x_7\partial_5+x_1\partial_6+2\,x_5\partial_3$
V_2''	$2\,{x_2}^{(2)}\partial_1+{x_3}^{(2)}\partial_3+2\,x_2x_3\partial_2+x_3x_4\partial_4+x_3x_5\partial_5+2\,x_3x_7\partial_7+x_5x_6\partial_1+2\,x_6x_7\partial_4+x_2\partial_4+x_5\partial_6$
V_3'	$2\,x_1x_2\partial_1+2\,x_1x_3\partial_2+x_2x_3\partial_3+2\,x_2x_4\partial_4+2\,x_2x_5\partial_5+x_2x_6\partial_6+2\,x_2x_7\partial_7+2\,x_3x_6\partial_7+x_4x_5\partial_6$
	$+2\,x_5x_6\partial_2+x_5x_7\partial_3+{x_4}^{(2)}x_5x_7\partial_1+x_3x_4x_6\partial_6+x_3x_4x_7\partial_7+x_3x_6x_7\partial_3+2\,x_4x_5x_6\partial_1+2\,x_4x_5x_7\partial_2$
	$+2\,x_4x_6x_7\partial_4+2\,x_5x_6x_7\partial_5+x_3{x_4}^{(2)}x_6x_7\partial_1+2\,x_3x_4x_6x_7\partial_2+x_1\partial_4+2\,x_5\partial_7$
V_3''	$2\,{x_3}^{(2)}\partial_7+{x_3}^{(2)}x_4\partial_6+2\,{x_3}^{(2)}\partial x_6\partial_2+{x_3}^{(2)}x_7\partial_3+2\,x_2{x_3}^{(2)}\partial_5+2\,x_2x_3\partial_6+2\,x_2x_5\partial_1+x_2x_7\partial_4$
	$+2\,x_3x_5\partial_2+2\,x_3x_6\partial_4+{x_3}^{(2)}{x_4}^{(2)}x_7\partial_1+2\,{x_3}^{(2)}x_4x_6\partial_1+2\,{x_3}^{(2)}x_4x_7\partial_2+2\,{x_3}^{(2)}x_6x_7\partial_5+2\,x_2x_3x_6\partial_1$
	$+x_3x_4x_7\partial_4+x_5\partial_4$
V_4	$2\,{x_2}^{(2)}\partial_6+2\,{x_2}^{(2)}x_3\partial_5+2\,{x_2}^{(2)}x_6\partial_1+2\,{x_3}^{(2)}{x_4}^{(2)}\partial_6+{x_3}^{(2)}x_4\partial_7+{x_3}^{(2)}x_6\partial_3+2\,x_1{x_3}^{(2)}\partial_5$
	$+2\,x_1x_3\partial_6+2\,x_1x_5\partial_1+x_1x_7\partial_4+2\,x_2x_3\partial_7+2\,x_2x_5\partial_2+2\,x_2x_6\partial_4+2\,x_3x_5\partial_3+x_5x_6\partial_6+x_5x_7\partial_7$
	$+{x_3}^{(2)}{x_4}^{(2)}x_6\partial_1+2\,{x_3}^{(2)}{x_4}^{(2)}x_7\partial_2+2\,{x_3}^{(2)}x_4x_7\partial_3+2\,x_1x_3x_6\partial_1+x_2x_3x_4\partial_6+2\,x_2x_3x_6\partial_2$
	$+x_2x_3x_7\partial_3+x_2x_4x_7\partial_4+x_3{x_4}^{(2)}x_7\partial_4+2\,x_3x_4x_6\partial_4+{x_3}^{(2)}x_4x_6x_7\partial_5+x_2x_3{x_4}^{(2)}x_7\partial_1+2\,x_2x_3x_4x_6\partial_1$
	$+2\,x_2x_3x_4x_7\partial_2+2\,x_2x_3x_6x_7\partial_5$
V_4''	${x_1}^{(2)}\partial_1+{x_4}^{(2)}x_5\partial_6+2\,x_1x_3\partial_3+x_1x_4\partial_4+x_1x_5\partial_5+2\,x_1x_6\partial_6+x_1x_7\partial_7+2\,x_4x_5\partial_7+2\,x_5x_6\partial_3$
	$+2\,{x_4}^{(2)}x_5x_6\partial_1+{x_4}^{(2)}x_5x_7\partial_2+2\,{x_4}^{(2)}x_6x_7\partial_4+x_3{x_4}^{(2)}x_6\partial_6+x_3{x_4}^{(2)}x_7\partial_7+2\,x_3x_4x_6\partial_7$
	$+x_4x_5x_7\partial_3+x_3{x_4}^{(2)}x_6x_7\partial_2+x_3x_4x_6x_7\partial_3+2\,x_4x_5x_6x_7\partial_5$
V_5'	$x_1x_2\partial_6+x_1x_3\partial_7+x_1x_5\partial_2+x_1x_6\partial_4+2\,x_2x_5\partial_3+x_5x_6\partial_7+x_1x_2x_3\partial_5+x_1x_2x_6\partial_1+2\,x_1x_3x_4\partial_6$
	$+x_1x_3x_6\partial_2+2\,x_1x_3x_7\partial_3+2\,x_1x_4x_7\partial_4+x_2{x_4}^{(2)}x_7\partial_4+x_2x_3{x_4}^{(2)}\partial_6+2\,x_2x_3x_4\partial_7+2\,x_2x_3x_6\partial_3$
	$+2\,x_2x_4x_6\partial_4+2\,x_2x_6x_7\partial_7+2\,x_4x_5x_6\partial_6+2\,x_4x_5x_7\partial_7+2\,x_5x_6x_7\partial_3+2\,{x_4}^{(2)}x_5x_6x_7\partial_1$
	$+2\,x_1x_3{x_4}^{(2)}x_7\partial_1+x_1x_3x_4x_6\partial_1+x_1x_3x_4x_7\partial_2+x_1x_3x_6x_7\partial_5+2\,x_2x_3{x_4}^{(2)}x_6\partial_1+x_2x_3{x_4}^{(2)}x_7\partial_2$
	$+x_2x_3x_4x_7\partial_3+2\,x_3x_4x_6x_7\partial_7+x_4x_5x_6x_7\partial_2+2\,x_2x_3x_4x_6x_7\partial_5$
V_6'	${x_1}^{(2)}\partial_6+{x_1}^{(2)}x_3\partial_5+{x_1}^{(2)}x_6\partial_1+2\,x_1x_5\partial_3+{x_4}^{(2)}x_5x_6\partial_6+{x_4}^{(2)}x_5x_7\partial_7+x_1{x_4}^{(2)}x_7\partial_4$
	$+x_1x_3{x_4}^{(2)}\partial_6+2\,x_1x_3x_4\partial_7+2\,x_1x_3x_6\partial_3+2\,x_1x_4x_6\partial_4+2\,x_1x_6x_7\partial_7+2\,x_4x_5x_6\partial_7$
	$+{x_4}^{(2)}x_5x_6x_7\partial_2+2\,x_1x_3{x_4}^{(2)}x_6\partial_1+x_1x_3{x_4}^{(2)}x_7\partial_2+x_1x_3x_4x_7\partial_3+x_3{x_4}^{(2)}x_6x_7\partial_7$
	$+x_4x_5x_6x_7\partial_3+2\,x_1x_3x_4x_6x_7\partial_5$
V_6''	${x_2}^{(2)}x_3\partial_3+2\,{x_2}^{(2)}x_4\partial_4+2\,{x_2}^{(2)}x_5\partial_5+{x_2}^{(2)}x_6\partial_6+2\,{x_2}^{(2)}x_7\partial_7+2\,x_1{x_2}^{(2)}\partial_1+x_1{x_3}^{(2)}\partial_3$
	$+x_1x_2\partial_4+x_1x_5\partial_6+2\,x_2x_5\partial_7+2\,{x_3}^{(2)}{x_4}^{(2)}x_6\partial_6+2\,{x_3}^{(2)}{x_4}^{(2)}x_7\partial_7+{x_3}^{(2)}x_4x_6\partial_7+2\,x_1x_2x_3\partial_2$
	$+x_1x_3x_4\partial_4+x_1x_3x_5\partial_5+2\,x_1x_3x_7\partial_7+x_1x_5x_6\partial_1+2\,x_1x_6x_7\partial_4+2\,x_2x_3x_6\partial_7+x_2x_4x_5\partial_6$
	$+2\,x_2x_5x_6\partial_2+x_2x_5x_7\partial_3+x_3{x_4}^{(2)}x_5\partial_6+2\,x_3x_4x_5\partial_7+2\,x_3x_5x_6\partial_3+x_5x_6x_7\partial_7$
	$+2\,{x_3}^{(2)}{x_4}^{(2)}x_6x_7\partial_2+2\,{x_3}^{(2)}x_4x_6x_7\partial_3+x_2{x_4}^{(2)}x_5x_7\partial_1+x_2x_3x_4x_6\partial_6+x_2x_3x_4x_7\partial_7+x_2x_3x_6x_7\partial_3$
	$+2\,x_2x_4x_5x_6\partial_1+2\,x_2x_4x_5x_7\partial_2+2\,x_2x_4x_6x_7\partial_4+2\,x_2x_5x_6x_7\partial_5+2\,x_3{x_4}^{(2)}x_5x_6\partial_1+x_3{x_4}^{(2)}x_5x_7\partial_2$
	$+2\,x_3{x_4}^{(2)}x_6x_7\partial_4+x_3x_4x_5x_7\partial_3+x_2x_3{x_4}^{(2)}x_6x_7\partial_1+2\,x_2x_3x_4x_6x_7\partial_2+2\,x_3x_4x_5x_6x_7\partial_5$

V_7'	${x_1}^{(2)}\partial_4 + 2\,{x_1}^{(2)}x_2\partial_1 + 2\,{x_1}^{(2)}x_3\partial_2 + 2\,x_1x_5\partial_7 + x_1x_2x_3\partial_3 + 2\,x_1x_2x_4\partial_4 + 2\,x_1x_2x_5\partial_5 + x_1x_2x_6\partial_6$ $+2\,x_1x_2x_7\partial_7 + 2\,x_1x_3x_6\partial_7 + x_1x_4x_5\partial_6 + 2\,x_1x_5x_6\partial_2 + x_1x_5x_7\partial_3 + 2\,x_2{x_4}^{(2)}x_5\partial_6 + x_2x_4x_5\partial_7$ $+x_2x_5x_6\partial_3 + x_1{x_4}^{(2)}x_5x_7\partial_1 + x_1x_3x_4x_6\partial_6 + x_1x_3x_4x_7\partial_7 + x_1x_3x_6x_7\partial_3 + 2\,x_1x_4x_5x_6\partial_1 + 2\,x_1x_4x_5x_7\partial_2$ $+2\,x_1x_4x_6x_7\partial_4 + 2\,x_1x_5x_6x_7\partial_5 + x_2{x_4}^{(2)}x_5x_6\partial_1 + 2\,x_2{x_4}^{(2)}x_5x_7\partial_2 + x_2{x_4}^{(2)}x_6x_7\partial_4 + 2\,x_2x_3{x_4}^{(2)}x_6\partial_6$ $+2\,x_2x_3{x_4}^{(2)}x_7\partial_7 + x_2x_3x_4x_6\partial_7 + 2\,x_2x_4x_5x_7\partial_3 + x_4x_5x_6x_7\partial_7 + x_1x_3{x_4}^{(2)}x_6x_7\partial_1 + 2\,x_1x_3x_4x_6x_7\partial_2$ $+2\,x_2x_3{x_4}^{(2)}x_6x_7\partial_2 + 2\,x_2x_3x_4x_6x_7\partial_3 + x_2x_4x_5x_6x_7\partial_5$
V_8'	${x_1}^{(2)}{x_3}^{(2)}\partial_5 + {x_1}^{(2)}x_3\partial_6 + {x_1}^{(2)}x_5\partial_1 + 2\,{x_1}^{(2)}x_7\partial_4 + 2\,{x_2}^{(2)}x_5\partial_3 + x_1{x_2}^{(2)}\partial_6 + {x_1}^{(2)}x_3x_6\partial_1$ $+{x_2}^{(2)}{x_4}^{(2)}x_7\partial_4 + {x_2}^{(2)}x_3{x_4}^{(2)}\partial_6 + 2\,{x_2}^{(2)}x_3x_4\partial_7 + 2\,{x_2}^{(2)}x_3x_6\partial_3 + 2\,{x_2}^{(2)}x_4x_6\partial_4 + 2\,{x_2}^{(2)}x_6x_7\partial_7$ $+x_1{x_2}^{(2)}x_3\partial_5 + x_1{x_2}^{(2)}x_6\partial_1 + x_1{x_3}^{(2)}{x_4}^{(2)}\partial_6 + 2\,x_1{x_3}^{(2)}x_4\partial_7 + 2\,x_1{x_3}^{(2)}x_6\partial_3 + x_1x_2x_3\partial_7 + x_1x_2x_5\partial_2$ $+x_1x_2x_6\partial_4 + x_1x_3x_5\partial_3 + 2\,x_1x_5x_6\partial_6 + 2\,x_1x_5x_7\partial_7 + x_2x_5x_6\partial_7 + 2\,{x_2}^{(2)}x_3{x_4}^{(2)}x_6\partial_1 + {x_2}^{(2)}x_3{x_4}^{(2)}x_7\partial_2$ $+{x_2}^{(2)}x_3x_4x_7\partial_3 + {x_3}^{(2)}{x_4}^{(2)}x_6x_7\partial_7 + 2\,x_1{x_3}^{(2)}{x_4}^{(2)}x_6\partial_1 + x_1{x_3}^{(2)}{x_4}^{(2)}x_7\partial_2 + x_1{x_3}^{(2)}x_4x_7\partial_3$ $+2\,x_1x_2x_3x_4\partial_6 + x_1x_2x_3x_6\partial_2 + 2\,x_1x_2x_3x_7\partial_3 + 2\,x_1x_2x_4x_7\partial_4 + 2\,x_1x_3{x_4}^{(2)}x_7\partial_4 + x_1x_3x_4x_6\partial_4$ $+2\,x_2x_4x_5x_6\partial_6 + 2\,x_2x_4x_5x_7\partial_7 + 2\,x_2x_5x_6x_7\partial_3 + 2\,x_3{x_4}^{(2)}x_5x_6\partial_6 + 2\,x_3{x_4}^{(2)}x_5x_7\partial_7 + x_3x_4x_5x_6\partial_7$ $+2\,{x_2}^{(2)}x_3x_4x_6x_7\partial_5 + 2\,x_1{x_3}^{(2)}x_4x_6x_7\partial_5 + 2\,x_1x_2x_3{x_4}^{(2)}x_7\partial_1 + x_1x_2x_3x_4x_6\partial_1 + x_1x_2x_3x_4x_7\partial_2$ $+x_1x_2x_3x_6x_7\partial_5 + 2\,x_2{x_4}^{(2)}x_5x_6x_7\partial_1 + 2\,x_2x_3x_4x_6x_7\partial_7 + x_2x_4x_5x_6x_7\partial_2 + 2\,x_3{x_4}^{(2)}x_5x_6x_7\partial_2+$ $2\,x_3x_4x_5x_6x_7\partial_3$
V_9'	${x_1}^{(2)}x_2x_3\partial_5 + {x_1}^{(2)}x_2x_6\partial_1 + 2\,{x_1}^{(2)}x_3{x_4}^{(2)}x_7\partial_1 + {x_1}^{(2)}x_3x_4x_6\partial_1 + {x_1}^{(2)}x_3x_6x_7\partial_5 + 2\,x_1x_2x_3{x_4}^{(2)}x_6\partial_1$ $+2\,x_1x_2x_3x_4x_6x_7\partial_5 + 2\,x_1{x_4}^{(2)}x_5x_6x_7\partial_1 + {x_1}^{(2)}x_3x_4x_7\partial_2 + {x_1}^{(2)}x_3x_6\partial_2 + {x_1}^{(2)}x_5\partial_2$ $+x_1x_2x_3{x_4}^{(2)}x_7\partial_2 + x_1x_4x_5x_6x_7\partial_2 + x_2{x_4}^{(2)}x_5x_6x_7\partial_2 + {x_1}^{(2)}x_2\partial_6 + 2\,{x_1}^{(2)}x_3x_4\partial_6 + 2\,{x_1}^{(2)}x_3x_7\partial_3$ $+x_1x_2x_3{x_4}^{(2)}\partial_6 + x_1x_2x_3x_4x_7\partial_3 + 2\,x_1x_2x_3x_6\partial_3 + 2\,x_1x_2x_5\partial_3 + 2\,x_1x_4x_5x_6\partial_6 + 2\,x_1x_5x_6x_7\partial_3$ $+x_2{x_4}^{(2)}x_5x_6\partial_6 + x_2x_4x_5x_6x_7\partial_3 + {x_1}^{(2)}x_3\partial_7 + 2\,{x_1}^{(2)}x_4x_7\partial_4 + {x_1}^{(2)}x_6\partial_4 + 2\,x_1x_2x_3x_4\partial_7$ $+x_1x_2{x_4}^{(2)}x_7\partial_4 + 2\,x_1x_2x_4x_6\partial_4 + 2\,x_1x_2x_6x_7\partial_7 + 2\,x_1x_3x_4x_6x_7\partial_7 + 2\,x_1x_4x_5x_7\partial_7 + x_1x_5x_6\partial_7$ $+x_2x_3{x_4}^{(2)}x_6x_7\partial_7 + x_2{x_4}^{(2)}x_5x_7\partial_7 + 2\,x_2x_4x_5x_6\partial_7$
V_{10}'	${x_1}^{(2)}{x_2}^{(2)}\partial_1 + 2\,{x_1}^{(2)}{x_3}^{(2)}\partial_3 + 2\,{x_1}^{(2)}x_2\partial_4 + 2\,{x_1}^{(2)}x_5\partial_6 + {x_1}^{(2)}x_2x_3\partial_2 + 2\,{x_1}^{(2)}x_3x_4\partial_4$ $+2\,{x_1}^{(2)}x_3x_5\partial_5 + {x_1}^{(2)}x_3x_7\partial_7 + 2\,{x_1}^{(2)}x_5x_6\partial_1 + {x_1}^{(2)}x_6x_7\partial_4 + {x_2}^{(2)}{x_4}^{(2)}x_5\partial_6 + 2\,{x_2}^{(2)}x_4x_5\partial_7$ $+2\,{x_2}^{(2)}x_5x_6\partial_3 + 2\,x_1{x_2}^{(2)}x_3\partial_3 + x_1{x_2}^{(2)}x_4\partial_4 + x_1{x_2}^{(2)}x_5\partial_5 + 2\,x_1{x_2}^{(2)}x_6\partial_6 + x_1{x_2}^{(2)}x_7\partial_7$ $+x_1x_2x_5\partial_7 + 2\,{x_2}^{(2)}{x_4}^{(2)}x_5x_6\partial_1 + {x_2}^{(2)}{x_4}^{(2)}x_5x_7\partial_2 + 2\,{x_2}^{(2)}{x_4}^{(2)}x_6x_7\partial_4 + {x_2}^{(2)}x_3{x_4}^{(2)}x_6\partial_6$ $+{x_2}^{(2)}x_3{x_4}^{(2)}x_7\partial_7 + 2\,{x_2}^{(2)}x_3x_4x_6\partial_7 + {x_2}^{(2)}x_4x_5x_7\partial_3 + x_1{x_3}^{(2)}{x_4}^{(2)}x_6\partial_6 + x_1{x_3}^{(2)}{x_4}^{(2)}x_7\partial_7$ $+2\,x_1{x_3}^{(2)}x_4x_6\partial_7 + x_1x_2x_3x_6\partial_7 + 2\,x_1x_2x_4x_5\partial_6 + x_1x_2x_5x_6\partial_2 + 2\,x_1x_2x_5x_7\partial_3 + 2\,x_1x_3{x_4}^{(2)}x_5\partial_6$ $+x_1x_3x_4x_5\partial_7 + x_1x_3x_5x_6\partial_3 + 2\,x_1x_5x_6x_7\partial_7 + {x_2}^{(2)}x_3{x_4}^{(2)}x_6x_7\partial_2 + {x_2}^{(2)}x_3x_4x_6x_7\partial_3$ $+2\,{x_2}^{(2)}x_4x_5x_6x_7\partial_5 + x_1{x_3}^{(2)}{x_4}^{(2)}x_6x_7\partial_2 + x_1{x_3}^{(2)}x_4x_6x_7\partial_3 + 2\,x_1x_2{x_4}^{(2)}x_5x_7\partial_1 + 2\,x_1x_2x_3x_4x_6\partial_6$ $+2\,x_1x_2x_3x_4x_7\partial_7 + 2\,x_1x_2x_3x_6x_7\partial_3 + x_1x_2x_4x_5x_6\partial_1 + x_1x_2x_4x_5x_7\partial_2 + x_1x_2x_4x_6x_7\partial_4 + x_1x_2x_5x_6x_7\partial_5$ $+x_1x_3{x_4}^{(2)}x_5x_6\partial_1 + 2\,x_1x_3{x_4}^{(2)}x_5x_7\partial_2 + x_1x_3{x_4}^{(2)}x_6x_7\partial_4 + 2\,x_1x_3x_4x_5x_7\partial_3 + 2\,x_2x_4x_5x_6x_7\partial_7$ $+2\,x_3{x_4}^{(2)}x_5x_6x_7\partial_7 + 2\,x_1x_2x_3{x_4}^{(2)}x_6x_7\partial_1 + x_1x_2x_3x_4x_6x_7\partial_2 + x_1x_3x_4x_5x_6x_7\partial_5$

1.13. A description of $\mathfrak{Brj}(3;\underline{N}|4)$. We have the following realization of the non-positive part inside $\mathfrak{vect}(3;\underline{N}|4)$:

(10)

$\mathfrak{g}_i$	the generators (even | odd)
$\mathfrak{g}_{-3}$	$\mid Y_6=\partial_4$
$\mathfrak{g}_{-2}$	$Y_5=\partial_1,\ Y_6=\partial_2,\ Y_7=\partial_3\ \mid$
$\mathfrak{g}_{-1}$	$\mid Y_2=2\,x_3\partial_4+\partial_5,\ Y_3=x_2\partial_4+x_6\partial_1+\partial_6$ $Y_4=2\,x_1\partial_4+2\,x_5x_7\partial_4+x_5\partial_1+x_6\partial_2+2\,x_7\partial_3+\partial_7$
$\mathfrak{g}_0\simeq\mathfrak{hei}(2\vert 0)\Subset\mathbb{K}H_2$	$H_1=[Z_1,Y_1],\ H_2=2\,x_1\partial_1+x_3\partial_3+x_4\partial_4+x_6\partial_6+2\,x_7\partial_7$ $Y_1=2\,x_5x_6x_7\partial_4+2\,x_1\partial_2+2\,x_2\partial_3+2\,x_5x_6\partial_1+x_6x_7\partial_3+2\,x_5\partial_6+2\,x_6\partial_7,$ $Z_1=2\,x_2\partial_1+2\,x_3\partial_2+x_6x_7\partial_1+2\,x_6\partial_5+x_7\partial_6\ \mid$

The Lie superalgebra $\mathfrak{g}_0$ is solvable with the property that $[\mathfrak{g}_0,\mathfrak{g}_0]=\mathfrak{hei}(2|0)$. The CTS prolong $(\mathfrak{g}_-,\mathfrak{g}_0)_*$ is NOT simple since $\mathfrak{g}_1$ does not generate the positive part. Our calculation shows that the prolong does not depend on N, i.e., $N=(1,1,1,1)$. The simple part of this prolong is $\mathfrak{brj}$. The sdim of the positive parts are described as follows:

	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$
sdim	$0\vert 3$	$3\vert 0$	$0\vert 2$

and the lowest weight vectors are

V_1'	$2\,x_1{}^{(2)}\partial_4+2\,x_1x_5x_7\partial_4+x_1x_5\partial_1+x_1x_6\partial_2+2\,x_1x_7\partial_3+x_4\partial_3+x_1\partial_7+2\,x_5x_6\partial_6+x_5x_7\partial_7$
V_2'	$2\,x_1x_4\partial_4+x_2x_5x_6x_7\partial_4+2\,x_4x_5x_7\partial_4+2\,x_1{}^{(2)}\partial_1+x_1x_2\partial_2+x_2{}^{(2)}\partial_3+x_2x_5x_6\partial_1+2\,x_2x_6x_7\partial_3$ $+x_4x_5\partial_1+x_4x_6\partial_2+2\,x_4x_7\partial_3+2\,x_1x_5\partial_5+x_1x_6\partial_6+x_2x_5\partial_6+x_2x_6\partial_7+x_4\partial_7+x_5x_6x_7\partial_6$
V_3' V_3''	$x_1{}^{(2)}x_2\partial_4+x_1x_2x_5x_7\partial_4+2\,x_4x_5x_6x_7\partial_4+x_1{}^{(2)}x_6\partial_1+2\,x_1x_2x_5\partial_1+2\,x_1x_2x_6\partial_2+x_1x_2x_7\partial_3$ $+2\,x_1x_4\partial_2+2\,x_1x_5x_6x_7\partial_1+2\,x_2x_4\partial_3+2\,x_4x_5x_6\partial_1+x_4x_6x_7\partial_3+x_1{}^{(2)}\partial_6+2\,x_1x_2\partial_7$ $+x_1x_5x_6\partial_5+2\,x_1x_5x_7\partial_6+x_2x_5x_6\partial_6+2\,x_2x_5x_7\partial_7+2\,x_4x_5\partial_6+2\,x_4x_6\partial_7$ $x_1{}^{(2)}x_3\partial_4+x_1x_2{}^{(2)}\partial_4+x_1x_3x_5x_7\partial_4+x_2{}^{(2)}x_5x_7\partial_4+x_1x_2x_6\partial_1+2\,x_1x_3x_5\partial_1+2\,x_1x_3x_6\partial_2$ $+x_1x_3x_7\partial_3+2\,x_1x_4\partial_1+2\,x_2{}^{(2)}x_5\partial_1+2\,x_2{}^{(2)}x_6\partial_2+x_2{}^{(2)}x_7\partial_3+2\,x_2x_4\partial_2+2\,x_2x_5x_6x_7\partial_1$ $+2\,x_3x_4\partial_3+2\,x_1{}^{(2)}\partial_5+x_1x_2\partial_6+2\,x_1x_3\partial_7+x_1x_6x_7\partial_6+2\,x_2{}^{(2)}\partial_7+x_2x_5x_6\partial_5+2\,x_2x_5x_7\partial_6$ $+x_3x_5x_6\partial_6+2\,x_3x_5x_7\partial_7+x_4x_5\partial_5+x_4x_6\partial_6+x_4x_7\partial_7$

Let us study now the case where $\mathfrak{g}_0'=\mathfrak{der}_0(\mathfrak{g}_-)$. Our calculation shows that $\mathfrak{g}_0'$ is generated by the vectors Y_1,Z_1,H_1,H_2 above together with $V=2x_3\partial_1+x_7\partial_5$. The Lie algebra $\mathfrak{g}_0'$ is solvable of sdim $=5|0$. The CTS prolong $(\mathfrak{g}_-,\mathfrak{g}_0')_*$ gives a Lie superalgebra that is not simple because $\mathfrak{g}_1'$ does not generate the positive part. Its simple part is a new Lie superalgebra that we denote by $\mathfrak{BRJ}$, described as follows (here also $N=(1,1,1)$:

	$\mathfrak{g}_1'$	$\mathrm{ad}_{\mathfrak{g}_1'}(\mathfrak{g}_1')$	$\mathrm{ad}^2_{\mathfrak{g}_1'}(\mathfrak{g}_1')$	$\mathrm{ad}^3_{\mathfrak{g}_1'}(\mathfrak{g}_1')$	$\mathrm{ad}^4_{\mathfrak{g}_1'}(\mathfrak{g}_1')$	$\mathrm{ad}^5_{\mathfrak{g}_1'}(\mathfrak{g}_1')$
sdim	$0\vert 6$	$6\vert 0$	$0\vert 5$	$3\vert 0$	$0\vert 3$	$1\vert 0$

1.14. A description of $\mathfrak{Brj}(3;\underline{N}|3)$. We have the following realization of the non-positive part inside $\mathfrak{vect}(3;\underline{N}|3)$:

(11)

$\mathfrak{g}_i$	the generators (even | odd)
$\mathfrak{g}_{-2}$	$Y_7=\partial_1 \mid Y_5=\partial_4,\ Y_8=\partial_5$
$\mathfrak{g}_{-1}$	$Y_1=\partial_2,\ Y_6=2\,x_2\partial_1+\partial_3 \mid Y_3=x_2\partial_4+x_3\partial_5+2\,x_6\partial_1+\partial_6$
$\mathfrak{g}_0$	$H_2=[X_2,Y_2],\ H_1=x_1\partial_1+x_3\partial_3+2\,x_4\partial_4+2\,x_6\partial_6,\ X_4=[X_2,X_2],\ Y_4=[Y_2,Y_2] \mid$ $Y_2=x_2^{(2)}\partial_4+x_2\,x_3\partial_5+2\,x_2\,x_6\partial_1+x_1\partial_5+x_2\partial_6+x_4\partial_1+x_6\partial_3$ $X_2=x_3^{(2)}\partial_5+2\,x_1\partial_4+x_3\partial_6+x_5\partial_1+2\,x_6\partial_2$

The Lie superalgebra $\mathfrak{g}_0$ is isomorphic to $\mathfrak{osp}(1|2)\oplus\mathbb{K}$. The CTS prolong $(\mathfrak{g}_-,\mathfrak{g}_0)_*$ is NOT simple since it gives back $\mathfrak{brj}(2;3)$ + an outer derivation. The sdim of the positive parts are described as follows:

	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$
sdim	$2\|1$	$1\|2$	$0\|1$

and the lowest weight vectors are

V_1'	$2\,x_1\,x_2\partial_1+x_2\,x_4\partial_4+x_3\,x_4\partial_5+2\,x_4\,x_6\partial_1+x_1\partial_3+2\,x_2\,x_3\partial_3+x_2\,x_6\partial_6+x_4\partial_6$
V_2'	$x_1^{(2)}\partial_5+x_1\,x_2^{(2)}\partial_4+x_1\,x_2\,x_3\partial_5+2\,x_1\,x_2\,x_6\partial_1+x_1\,x_4\partial_1+2\,x_2^{(2)}\,x_3^{(2)}\partial_5+2\,x_2^{(2)}\,x_5\partial_1+2\,x_4\,x_5\partial_5+x_1\,x_2\partial_6$ $+x_1\,x_6\partial_3+2\,x_2^{(2)}\,x_3\partial_6+x_2^{(2)}\,x_6\partial_2+x_2\,x_3\,x_6\partial_3+x_2\,x_4\partial_2+x_2\,x_5\partial_3+2\,x_4\,x_6\partial_6$
V_3'	$x_1^{(2)}\,x_2\partial_4+x_1^{(2)}\,x_3\partial_5+2\,x_1^{(2)}\,x_6\partial_1+2\,x_1\,x_2\,x_3^{(2)}\partial_5+2\,x_1\,x_2\,x_5\partial_1+x_2\,x_3^{(2)}\,x_4\partial_1+2\,x_2\,x_4\,x_5\partial_4+2\,x_3\,x_4\,x_5\partial_5$ $+x_4\,x_5\,x_6\partial_1+x_1^{(2)}\partial_6+2\,x_1\,x_2\,x_3\partial_6+x_1\,x_2\,x_6\partial_2+x_1\,x_3\,x_6\partial_3+x_1\,x_4\partial_2+x_1\,x_5\partial_3+2\,x_2\,x_3^{(2)}\,x_6\partial_3+2\,x_2\,x_3\,x_4\partial_2$ $+2\,x_2\,x_3\,x_5\partial_3+x_2\,x_5\,x_6\partial_6+2\,x_3\,x_4\,x_6\partial_6+2\,x_4\,x_5\partial_6$

1.15. A description of $\mathfrak{Brj}(3;\underline{N}|4)$. We have the following realization of the non-positive part inside $\mathfrak{vect}(3;\underline{N}|4)$:
(12)

$\mathfrak{g}_i$	the generators (even | odd)
$\mathfrak{g}_{-3}$	$\mid Y_8=\partial_4$
$\mathfrak{g}_{-2}$	$Y_4=\partial_1,\ Y_6=\partial_2,\ Y_7=\partial_3\ \mid$
$\mathfrak{g}_{-1}$	$\mid Y_2=x_3\partial_4+2x_5\partial_1+\partial_5,\ Y_3=\partial_6+x_5\,x_6\partial_4+x_2\partial_4+x_5\partial_2+2\,x_6\partial_3$ $Y_5=x_1\partial_4+x_5\partial_3+\partial_7$
$\mathfrak{g}_0\simeq\mathfrak{hei}(2\|0)\ltimes\mathbb{K}H_2$	$H_1=[Z_1,Y_1],\ H_2=2\,x_1\partial_1+x_2\partial_2+x_4\partial_4+x_5\partial_5+2\,x_7\partial_7$ $Y_1=x_1\partial_2+x_2\partial_3+x_5\,x_6\partial_3+2\,x_5\partial_6+2\,x_6\partial_7$ $Z_1=2\,x_5\,x_6\,x_7\partial_4+2\,x_2\partial_1+x_3\partial_2+x_5\,x_6\partial_1+2\,x_6\,x_7\partial_3+2\,x_6\partial_5+x_7\partial_6\ \mid$

The Lie superalgebra $\mathfrak{g}_0$ is solvable with the property that $[\mathfrak{g}_0,\mathfrak{g}_0]=\mathfrak{hei}(2|0)$. The CTS prolong $(\mathfrak{g}_-,\mathfrak{g}_0)_*$ is NOT simple since $\mathfrak{g}_1$ does not generate the positive part. Our calculation shows that the prolong does not depend on N, i.e., $N=(1,1,1,1)$. The simple part of this prolong is ${}^{3)}\mathfrak{brj}(2;3)$. The sdim of the positive parts are described as follows:

	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$
sdim	$0\|3$	$3\|0$	$0\|2$

and the lowest weight vectors are

V_1'	$2x_1 x_3\partial_4 + x_2^{(2)}\partial_4 + x_2 x_5 x_6\partial_4 + x_1 x_5\partial_1 + x_2 x_5\partial_2 + 2 x_2 x_6\partial_3 + 2 x_3 x_5\partial_3 + x_4\partial_3 + 2 x_1\partial_5 + x_2\partial_6 + 2 x_3\partial_7 + 2 x_5 x_7\partial_7$
V_2'	$2 x_1 x_4\partial_4 + x_1^{(2)}\partial_1 + 2 x_1 x_2\partial_2 + 2 x_2^{(2)}\partial_3 + 2 x_2 x_5 x_6\partial_3 + 2 x_4 x_5\partial_3 + 2 x_1 x_5\partial_5 + x_1 x_7\partial_7 + x_2 x_5\partial_6 + x_2 x_6\partial_7 + 2 x_4\partial_7$
V_3'	$x_1 x_3^{(2)}\partial_4 + 2 x_2^{(2)} x_3\partial_4 + 2 x_2 x_3 x_5 x_6\partial_4 + 2 x_1 x_3 x_5\partial_1 + 2 x_1 x_4\partial_1 + x_2^{(2)} x_5\partial_1 + 2 x_2 x_3 x_5\partial_2 + x_2 x_3 x_6\partial_3 + 2 x_2 x_4\partial_2$ $+2 x_2 x_5 x_6 x_7\partial_3 + x_3^{(2)} x_5\partial_3 + 2 x_3 x_4\partial_3 + x_1 x_3\partial_5 + x_1 x_6 x_7\partial_6 + 2 x_2^{(2)}\partial_5 + 2 x_2 x_3\partial_6 + 2 x_2 x_5 x_6\partial_5 + x_2 x_5 x_7\partial_6$ $+2 x_2 x_6 x_7\partial_7 + x_3^{(2)}\partial_7 + x_3 x_5 x_7\partial_7 + x_4 x_5\partial_5 + x_4 x_6\partial_6 + x_4 x_7\partial_7$

Let us study now the case where $\mathfrak{g}_0' = \mathfrak{der}_0(\mathfrak{g}_-)$. The Lie algebra $\mathfrak{g}_0'$ is solvable of sdim $= 5|0$. The CTS prolong $(\mathfrak{g}_-, \mathfrak{g}_0')_*$ gives a Lie superalgebra that is not simple because $\mathfrak{g}_1'$ does not generate the positive part. Its simple part is a new Lie superalgebra that we had denoted by $\mathfrak{BRJ}$, described as follows (here also $N = (1, 1, 1)$:

	$\mathfrak{g}_1'$	$\mathrm{ad}_{\mathfrak{g}_1'}(\mathfrak{g}_1')$	$\mathrm{ad}^2_{\mathfrak{g}_1'}(\mathfrak{g}_1')$	$\mathrm{ad}^3_{\mathfrak{g}_1'}(\mathfrak{g}_1')$	$\mathrm{ad}^4_{\mathfrak{g}_1'}(\mathfrak{g}_1')$	$\mathrm{ad}^5_{\mathfrak{g}_1'}(\mathfrak{g}_1')$
sdim	$0\|6$	$6\|0$	$0\|5$	$3\|0$	$0\|3$	$1\|0$

1.16. Constructing Melikyan superalgebras. Denote by $\mathcal{F}_{1/2} := \mathcal{O}(1;1)\sqrt{dx}$ the space of semi-densities (weighted densities of weight $\frac{1}{2}$). For $p = 3$, the CTS prolong of the triple $(\mathbb{K}, \Pi(\mathcal{F}_{1/2}), \mathfrak{cvect}(1;1))_*$ gives the whole $\mathfrak{k}(1;\underline{N}|3)$. For $p = 5$, let us realize the non-positive part in $\mathfrak{k}(1;\underline{N}|5)$:

(13)

$\mathfrak{g}_i$	the generators
$\mathfrak{g}_{-2}$	1
$\mathfrak{g}_{-1}$	$\Pi(\mathcal{F}_{1/2})$
$\mathfrak{g}_0$	$\partial_1 \longleftrightarrow 4\,\xi_1\eta_2 + \xi_2\theta,\ x_1\partial_1 \longleftrightarrow 2\,\xi_1\eta_1 + \xi_2\eta_2,\ x_1^{(2)}\partial_1 \longleftrightarrow 2\,\xi_2\eta_1 + 3\,\theta\eta_2,\ x_1^{(3)}\partial_1 \longleftrightarrow 2\,\theta\eta_1$ $x_1^{(4)}\partial_1 \longleftrightarrow 2\,\eta_2\eta_1, \qquad t$

The CTS prolong gives that $\mathfrak{g}_i$=0 for all $i > 0$.

Consider now the case of $(\mathbb{K}, \Pi(\mathcal{F}_{1/2}), \mathfrak{cvect}(2;1))_*$, where $p = 3$. The non-positive part is realized in $\mathfrak{k}(1;\underline{N}|9)$ as follows:

(14)

$\mathfrak{g}_i$	the generators
$\mathfrak{g}_{-2}$	1
$\mathfrak{g}_{-1}$	$\Pi(\mathcal{F}_{1/2})$
$\mathfrak{g}_0$	$\partial_1 \longleftrightarrow 2\,\xi_1\eta_3 + x_2\theta + 2\,\xi_3\eta_4,\ x_1\partial_1 \longleftrightarrow \xi_1\eta_1 + \xi_2\eta_2 + 2\,\xi_4\eta_4,\ x_1^2\partial_1 \longleftrightarrow \xi_3\eta_1 + \xi_4\eta_3 + \theta\eta_2,$ $\partial_2 \longleftrightarrow 2\,\xi_1\eta_2 + \xi_2\xi_4 + \xi_3\theta,\ x_2\partial_2 \longleftrightarrow \xi_1\eta_1 + \xi_3\eta_3 + \xi_4\eta_4,\ x_2^2\partial_2 \longleftrightarrow \xi_2\eta_1 + \theta\eta_3 + 2\,\eta_4\eta_2,$ $x_1x_2\partial_1 \longleftrightarrow \xi_2\eta_1 + \eta_4\eta_2,\ x_1x_2\partial_2 \longleftrightarrow \xi_3\eta_1 + 2\,\xi_4\eta_3,\ x_1^2x_2\partial_1 \longleftrightarrow \theta\eta_1 + 2\,\eta_3\eta_2,$ $x_1^2x_2\partial_2 \longleftrightarrow \xi_4\eta_1,\ x_1x_2^2\partial_1 \longleftrightarrow 2\,\eta_4\eta_1,\ x_1x_2^2\partial_2 \longleftrightarrow \theta\eta_1 + \eta_3\eta_2,\ x_1^2x_2^2\partial_1 \longleftrightarrow \eta_3\eta_1,$ $x_1^2x_2^2\partial_2 \longleftrightarrow \eta_2\eta_1, \qquad t$

The CTS prolong $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ gives a Lie superalgebra that is not simple with the property that $\mathrm{sdim}(\mathfrak{g}_1) = 0|4$ and $\mathfrak{g}_i = 0$ for all $i > 1$. The generating functions of $\mathfrak{g}_1$ are

$$\xi_2\eta_2\eta_1 + 2\,\xi_3\eta_3\eta_1 + \xi_4\eta_4\eta_1 + \theta\eta_3\eta_2, \quad 2\,\xi_4\eta_3\eta_1 + \theta\eta_2\eta_1, \quad \theta\eta_3\eta_1 + \eta_4\eta_2\eta_1, \quad \eta_3\eta_2\eta_1.$$

1.17. Defining relations of the positive parts of $\mathfrak{brj}(2;3)$ and $\mathfrak{brj}(2;5)$. For the presentations of the Lie superalgebras with Cartan matrix, see [GL1, BGL1]. The only non-trivial part of these relations are analogs of the Serre relations (both the straightforward ones and the ones different in shape). Here they are:

$\underline{\mathfrak{brj}(2;3)}$; sdim $\mathfrak{brj}(2;3) = 10|8$.

1) $[[x_1, x_2], [x_2, [x_1, x_2]]] = 0,$
$[[x_2, x_2], [[x_1, x_2], [x_2, x_2]]] = 0.$

2) $\mathrm{ad}^3_{x_2}(x_1) = 0,$
$[[x_1, x_2], [[x_1, x_2], [x_1, x_2]]] = 0,$
$[[x_2, [x_1, x_2]], [[x_1, x_2], [x_2, [x_1, x_2]]]] = 0.$

3) $\mathrm{ad}^3_{x_1}(x_2) = 0,$
$[x_2, [x_1, [x_1, x_2]]] - [[x_1, x_2], [x_1, x_2]] = 0,$
$[[x_1, x_2], [x_2, x_2]] = 0.$

$\underline{\mathfrak{brj}(2;5)}$; sdim $\mathfrak{brj}(2;5) = 10|12$.

1) $[[x_2, [x_1, x_2]], [x_2, [x_1, x_2]]] = 2\,[[x_1, x_2], [[x_1, x_2], [x_2, x_2]]],$
$[[x_2, x_2], [[x_1, x_2], [x_2, x_2]]] = 0,$
$[[x_2, [x_1, x_2]], [[x_1, x_2], [x_2, [x_1, x_2]]]] = 0.$

2)$\mathrm{ad}^4_{x_2}(x_1) = 0,$
$[[x_2, [x_1, x_2]], [x_2, [x_2, [x_1, x_2]]]] = 0,$
$[[[x_1, x_2], [x_1, x_2]], [[x_1, x_2], [x_2, [x_1, x_2]]]] = 0.$

References

[BKK] Benkart, G.; Kostrikin, A. I.; Kuznetsov, M. I. The simple graded Lie algebras of characteristic three with classical reductive component L_0. Comm. Algebra 24 (1996), no. 1, 223–234.

[BL] Bernstein J., Leites D., Invariant differential operators and irreducible representations of Lie superalgebras of vector fields. Selecta Math. Sov., v. 1, 1981, no. 2, 143–160

[BjL] Bouarroudj S., Leites D., Simple Lie superalgebras and non-integrable distributions in characteristic p Zapiski nauchnyh seminarov POMI, t. 331 (2006), 15–29; Reprinted in J. Math. Sci. (NY), 141 (2007) no.4, 1390–98; math.RT/0606682

[BGL1] Bouarroudj S., Grozman P., Leites D., Cartan matrices and presentations of Elduque and Cunha simple Lie superalgebras; MPIMiS preprint 124/2006 (www.mis.mpg.de)

[BGL2] Bouarroudj S., Grozman P., Leites D., Cartan matrices and presentations of the exceptional simple Elduque Lie superalgebra; MPIMiS preprint 125/2006 (www.mis.mpg.de)

[BGL4] Bouarroudj S., Grozman P., Leites D., Infinitesimal deformations of the simple modular Lie superalgebras with Cartan matrices for $p = 3$. IN PREPARATION

[BGL5] Bouarroudj S., Grozman P., Leites D., Simple modular Lie superalgebras with Cartan matrices. IN PREPARATION

[C] Cartan É., Über die einfachen Transformationsgrouppen, Leipziger Berichte (1893), 395–420. Reprinted in: *Œuvres complètes.* Partie II. (French) [Complete works. Part II] Algèbre, systèmes différentiels et problèmes d'équivalence. [Algebra, differential systems and problems of equivalence] Second edition. Éditions du Centre National de la Recherche Scientifique (CNRS), Paris, 1984.

[Cla] Clarke B., Decomposition of the tensor product of two irreducible $\mathfrak{sl}(2)$-modules in characteristic 3, MPIMiS preprint 145/2006; for calculations, see
http://personal-homepages.mis.mpg.de/clarke/Tensor-Calculations.tar.gz

[CE] Cunha I., Elduque A., An extended Freudenthal magic square in characteristic 3; math.RA/0605379

[CE2] Cunha I., Elduque, A., The extended Freudenthal Magic Square and Jordan algebras; math.RA/0608191

[El1] Elduque, A. New simple Lie superalgebras in characteristic 3. J. Algebra 296 (2006), no. 1, 196–233

[El2] Elduque, A. Some new simple modular Lie superalgebras. math.RA/0512654

[Er] Ermolaev, Yu. B. Integral bases of classical Lie algebras. (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 2004, , no. 3, 16–25; translation in Russian Math. (Iz. VUZ) 48 (2004), no. 3, 13–22.

[FH] Fulton, W., Harris, J., *Representation theory. A first course*. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991. xvi+551 pp

[GK] Gregory, T.; Kuznetsov, M. On depth-three graded Lie algebras of characteristic three with classical reductive null component. Comm. Algebra 32 (2004), no. 9, 3339–3371

[Gr] Grozman P., **SuperLie**, `http://www.equaonline.com/math/SuperLie`

[GL1] Grozman P., Leites D., Defining relations for classical Lie superalgebras with Cartan matrix, Czech. J. Phys., Vol. 51, 2001, no. 1, 1–22; arXiv: `hep-th/9702073`

[GL2] Grozman P., Leites D., **SuperLie** and problems (to be) solved with it. Preprint MPIM-Bonn, 2003-39 (`http://www.mpim-bonn.mpg.de`)

[GL4] Grozman P., Leites D., Structures of $G(2)$ type and nonintegrable distributions in characteristic p. Lett. Math. Phys. 74 (2005), no. 3, 229–262; arXiv: `math.RT/0509400`

[GLS] Grozman P., Leites D., Shchepochkina I., Invariant operators on supermanifolds and standard models. In: In: M. Olshanetsky, A. Vainstein (eds.) *Multiple facets of quantization and supersymmetry. Michael Marinov Memorial Volume*, World Sci. Publishing, River Edge, NJ, 2002, 508–555. [math.RT/0202193; ESI preprint 1111 (2001)].

[KWK] Kac, V. G. Corrections to: "Exponentials in Lie algebras of characteristic p" [Izv. Akad. Nauk SSSR 35 (1971), no. 4, 762–788; MR0306282 (46 #5408)] by B. Yu. Veisfeiler and Kac. (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 58 (1994), no. 4, 224; translation in Russian Acad. Sci. Izv. Math. 45 (1995), no. 1, 229

[K2] Kac V., Lie superagebras, Adv. Math. v. 26, 1977, 8–96

[K3] Kac, V. Classification of supersymmetries. Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), Higher Ed. Press, Beijing, 2002, 319–344
Cheng, Shun-Jen; Kac, V., Addendum: "Generalized Spencer cohomology and filtered deformations of $\mathbb{Z}$-graded Lie superalgebras" [Adv. Theor. Math. Phys. 2 (1998), no. 5, 1141–1182; MR1688484 (2000d:17025)]. Adv. Theor. Math. Phys. 8 (2004), no. 4, 697–709.
Cantarini, N.; Cheng, S.-J.; Kac, V. Errata to: "Structure of some $\mathbb{Z}$-graded Lie superalgebras of vector fields" [Transform. Groups 4 (1999), no. 2-3, 219–272; MR1712863 (2001b:17037)] by Cheng and Kac. Transform. Groups 9 (2004), no. 4, 399–400

[KKCh] Kirillov, S. A.; Kuznetsov, M. I.; Chebochko, N. G. Deformations of a Lie algebra of type G_2 of characteristic three. (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 2000, , no. 3, 33–38; translation in Russian Math. (Iz. VUZ) 44 (2000), no. 3, 31–36

[KLV] Kochetkov Yu., Leites D., Vaintrob A. New invariant differential operators and pseudo-(co)homology of supermanifolds and Lie superalgebras. In: S. Andima et. al. (eds.) *General Topology and its Appl., June 1989*, Marcel Dekker, NY, 1991, 217–238

[KS] Kostrikin, A. I., Shafarevich, I.R., Graded Lie algebras of finite characteristic, Izv. Akad. Nauk. SSSR Ser. Mat. 33 (1969) 251–322 (in Russian); transl.: Math. USSR Izv. 3 (1969) 237–304

[Ku1] Kuznetsov, M. I. The Melikyan algebras as Lie algebras of the type G_2. Comm. Algebra 19 (1991), no. 4, 1281–1312.

[Ku2] Kuznetsov, M. I. Graded Lie algebras with the almost simple component L_0. Pontryagin Conference, 8, Algebra (Moscow, 1998). J. Math. Sci. (New York) 106 (2001), no. 4, 3187–3211.

[LL] Lebedev A., Leites D., (with Appendix by P. Deligne) On realizations of the Steenrod algebras. J. Prime Res. Math., v. 2, 2006,

[L] Leites D., Towards classification of simple finite dimensional modular Lie superalgebras in characteristic p. IN PREPARATION

[LSh] Leites D., Shchepochkina I., Classification of the simple Lie superalgebras of vector fields, preprint MPIM-2003-28 (`http://www.mpim-bonn.mpg.de`)

[Ssol] Sergeev, A. Irreducible representations of solvable Lie superalgebras. Represent. Theory 3 (1999), 435–443; math.RT/9810109

[Shch] Shchepochkina I., How to realize Lie algebras by vector fields. Theor. Mat. Fiz. 147 (2006) no. 3, 821–838; arXiv: `math.RT/0509472`

[Sk] Skryabin, S. M. New series of simple Lie algebras of characteristic 3. (Russian. Russian summary) Mat. Sb. 183 (1992), no. 8, 3–22; translation in Russian Acad. Sci. Sb. Math. 76 (1993), no. 2, 389–406

[S] Strade, H. *Simple Lie algebras over fields of positive characteristic. I. Structure theory.* de Gruyter Expositions in Mathematics, 38. Walter de Gruyter & Co., Berlin, 2004. viii+540 pp.

[St] Steinberg, R. *Lectures on Chevalley groups.* Notes prepared by John Faulkner and Robert Wilson. Yale University, New Haven, Conn., 1968. iii+277 pp.

[Vi] Viviani F., Deformations of Simple Restricted Lie Algebras I, II. math.RA/0612861, math.RA/0702499;
Deformations of the restricted Melikian Lie algebra,math.RA/0702594;
Restricted simple Lie algebras and their infinitesimal deformations, math.RA/0702755

[WK] Weisfeiler, B. Ju.; Kac, V. G. Exponentials in Lie algebras of characteristic p. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 762–788.

[Y] Yamaguchi K., Differential systems associated with simple graded Lie algebras. Progress in differential geometry, Adv. Stud. Pure Math., 22, Math. Soc. Japan, Tokyo, 1993, 413–494

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