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Nilpotent Symmetry Invariance In Superfield Formalism: 1-Form (Non-)Abelian Gauge Theories

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Abstract: We capture the well-known off-shell as well as the on-shell nilpotent Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetry transformations for the Lagrangian densities of the four (3 + 1)-dimensional (4D) 1-form (non-)Abelian gauge theories within the framework of the superfield formalism. In particular, we provide the geometrical interpretations for (i) the above nilpotent symmetry invariance(s), and (ii) the Lagrangian densities of the theories, within the framework of the superfield approach to BRST formalism. Some of the subtle points, connected with the 4D (non-)Abelian 1-form gauge theories, are clarified within the above superfield approach where the 4D ordinary gauge theories are considered on the (4, 2)-dimensional supermanifold parametrized by the four spacetime coordinates x^μ (with $\mu = 0, 1, 2, 3$) and a pair of Grassmannian variables θ and $\bar{\theta}$. One of the key results of our present investigation is a great deal of simplification in the understanding of the nilpotent (anti-)BRST symmetry invariance(s) of the present 4D 1-form (non-)Abelian gauge theories.

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1 Introduction

The geometrical superfield approach [1-8] to Becchi-Rouet-Stora-Tyutin (BRST) formalism is one of the most attractive and intuitive approaches to gain some physical insights into the beautiful (but abstract mathematical) structures that are associated with the nilpotent (anti-)BRST symmetry transformations and their corresponding generators. The latter objects play a very decisive role in (i) the covariant canonical quantization of the gauge theories, (ii) the proof of the unitrarity of the quantum gauge theories at any arbitrary order of perturbative computations for a given physical process , (iii) the definition of the physical states of the quantum gauge theories in the quantum Hilbert space (that is consistent with the Dirac's method of quantization of the systems with constraints), and (iv) the cohomological description of the physical states of the quantum Hilbert space w.r.t. the conserved and nilpotent BRST charge (which generates the BRST symmetries).

To be specific, in the superfield formulation [1-8] of the 4D 1-form gauge theories, one defines the super curvature 2-form $\tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)} + i\,\tilde{A}^{(1)} \wedge \tilde{A}^{(1)}$ in terms of the super exterior derivative $\tilde{d} = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta}\partial_{\bar{\theta}}$ (with $\tilde{d}^2 = 0$) and the super 1-form connection $\tilde{A}^{(1)}$ on a (4, 2)-dimensional supermanifold parametrized by the usual spacetime variables x^μ (with $\mu = 0, 1, 2, 3$) and a pair of anticommuting (i.e. $\theta^2 = \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$) Grassmannian variables θ and $\bar{\theta}$. The above super 2-form is subsequently equated, due to the so-called horizontality condition [1-8], to the ordinary curvature 2-form $F^{(2)} = dA^{(1)} + iA^{(1)} \wedge A^{(1)}$ defined on the ordinary 4D flat Minkowski spacetime manifold in terms of the ordinary exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) and the 1-form connection $A^{(1)} = dx^\mu A_\mu$.

The above horizontality condition (HC) has been referred to as the soul-flatness condition in [9] which amounts to setting equal to zero all the Grassmannian components of the (anti)symmetric second-rank super tensor that constitutes the super curvature 2-form $\tilde{F}^{(2)}$ on the (4, 2)-dimensional supermanifold. The key consequences, that emerge from the HC, are (i) the derivation of the nilpotent (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields of a given 4D 1-form gauge theory, (ii) the geometrical interpretation of the (anti-)BRST symmetry transformations for the 4D local fields as the translation of the corresponding superfields along the Grassmannian direction(s) of the supermanifold, (iii) the geometrical interpretation of the nilpotency property as a pair of successive translations of the superfield along a particular Grassmannian direction of the supermanifold, and (iv) the geometrical interpretation of the anticommutativity property of the (anti-)BRST symmetry transformations for a 4D local field as the sum of (a) the translation of the corresponding superfield first along the θ direction followed by the translation along the $\bar{\theta}$ direction, and (b) the translation of the same superfield first along the $\bar{\theta}$ direction followed by the translation along the θ direction.

It will be noted that the above HC (i.e. $\tilde{F}^{(2)} = F^{(2)}$) is valid for the 1-form non-Abelian (i.e. $A^{(1)} \wedge A^{(1)} \neq 0$) gauge theory as well as the Abelian (i.e. $A^{(1)} \wedge A^{(1)} = 0$) gauge theory. We lay emphasis on the fact that the HC does not shed any light on the derivation of the nilpotent (anti-)BRST symmetry transformations associated with the matter fields of an *interacting* 4D 1-form (non-)Abelian gauge theory.

In a recent set of papers [10-17], the above HC condition has been generalized in a consistent manner so as to compute the nilpotent (anti-)BRST symmetry transformations associated with the matter fields of a given 4D 1-form interacting gauge theory (along with that of the gauge and (anti-)ghost fields) without spoiling the cute geometrical interpretations of the (anti-)BRST symmetry transformations and their corresponding generators that emerge from the HC *alone*. The latter approach has been christened as the augmented superfield approach to BRST formalism where the restrictions imposed on the (4, 2)-dimensional superfields are (i) the HC plus the invariance of the (super) matter Noether conserved currents [10-14], (ii) the HC plus the equality of any (super) conserved quantities [15], (iii) the HC plus a restriction that owes its origin to the (super) covariant derivatives on the matter (super) fields and its connection with the (super) curvatures [16,17], and (iv) an alternative to the HC where the gauge invariance and the property of the (super) covariant derivatives on the matter (super) fields play a crucial role [18-20].

In all the above approaches [1-20], however, the invariance of the Lagrangian densities of the 1-form (non-)Abelian gauge theories under the nilpotent (anti-)BRST symmetry transformations, has not yet been discussed at all. Some attempts in this direction have been made in our earlier works [21-25] where the topological features of the 2D free 1-form (non-)Abelian gauge theories have been captured in the superfield formulation. In particular, the invariance(s) of the Lagrangian density under the nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations have been expressed in terms of the superfields and the Grassmannian derivatives on them. These are, however, a bit more involved in nature. The geometrical interpretations for the Lagrangian densities and the symmetric energy-momentum tensor (for the above topological theory) have also been provided within the framework of the superfield approach to BRST formalism.

The simplicity is the hallmark of a beautiful theory. The purpose of our present paper is to capture the (anti-)BRST symmetry invariance(s) of the Lagrangian density of the 4D 1-form (non-)Abelian gauge theories within the framework of the superfield approach to BRST formalism and to demonstrate that the above symmetry invariance(s) could be understood in a very simple manner in terms of the translational generators along the Grassmannian direction(s) of the (4, 2)-dimensional supermanifold on which the above 4D gauge theories are considered. In addition, the reason behind the existence (or non-existence) of any specific nilpotent symmetry transformation could also be explained within the framework of the above superfield approach to BRST formalism.

In our present investigation, we demonstrate the *uniqueness* of the existence of the off-shell as well as the on-shell nilpotent (anti-)BRST symmetry transformations for the Lagrangian density of a 1-form U(1) Abelian gauge theory. We go a step further and show the existence of the on-shell as well as the off-shell nilpotent BRST symmetry transformations for the specific Lagrangian densities (cf. (4.1) and (4.4) below) of the 1-form 4D non-Abelian gauge theory and clarify the non-existence of the anti-BRST symmetry transformations for these specific Lagrangian densities within the framework of the superfield formulation (cf. section 5 below). Finally, we provide the geometrical basis for the existence of the off-shell nilpotent (anti-)BRST symmetry transformations for the specifically defined Lagrangian densities (cf. (4.7) and/or (4.8) below) of the 4D non-Abelian 1-form gauge theory in the Feynman gauge.

The motivating factors that have propelled us to pursue our present investigation are as follows. First and foremost, to the best of our knowledge, the property of the symmetry invariance of a given Lagrangian density has not yet been captured within the language of the superfield approach to BRST formalism. Second, the above (anti-)BRST invariance of the theory has never been shown, in *as* simplified fashion, as we demonstrate in our present endeavour. The underlying geometrical interpretations for the existence of the above nilpotent (anti-)BRST symmetry invariance(s) also become very transparent. Third, we establish the *uniqueness* of the existence of the (anti-)BRST symmetry invariance(s) in their various garb(s). The non-existence of the specific symmetry transformation is also shown within the framework of the superfield approach to BRST formalism. Finally, our present investigation is the first modest step in the direction to gain some decisive insights into the existence of the nilpotent symmetry transformations for the *higher spin* gauge theories within the framework of the superfield approach to BRST formalism .

The contents of our present paper are organized as follows.

In section 2, we recapitulate some of the key points connected with the on-shell as well as the off-shell nilpotent (anti-)BRST symmetry transformations for the 4D 1-form Abelian gauge theory (without matter fields) in the Lagrangian formulation.

The above symmetry transformations as well as the symmetry invariances of the Lagrangian densities are captured in the geometrical superfield approach to BRST formalism in section 3 where the horizontality condition on the gauge superfield, defined on an appropriately chosen supermanifold, plays a crucial role.

Section 4 deals with the bare essentials of (i) the off-shell as well as the on-shell nilpotent BRST symmetry transformations, and (ii) the off-shell nilpotent (anti-)BRST symmetry transformations for the 4D non-Abelian gauge theory in the Lagrangian formulation where there is no interaction between matter fields and gauge field.

The above section is followed by the subject matter of section 5 which concerns itself with the superfield formulation of the symmetry invariance(s) of the appropriate Lagrangian densities of the above 4D 1-form non-Abelian gauge theory.

Finally, in section 6, we summarize our key results, make some concluding remarks and point out a few future (promising) directions for further investigations.

2 (Anti-)BRST Symmetries In Abelian Theory: Lagrangian Formulation

Let us begin with the following (anti-)BRST invariant Lagrangian density of the 4D 1-form

Abelian gauge theory* in the Feynman gauge [26,27,9]

$$\mathcal{L}_B^{(a)} = -\frac{1}{4}\, F^{\mu\nu} F_{\mu\nu} \;+\; B\,(\partial_\mu A^\mu) \;+\; \frac{1}{2}\, B^2 - i\,\partial_\mu \bar{C}\,\partial^\mu C, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the antisymmetric ($F_{\mu\nu} = -F_{\nu\mu}$) curvature tensor that constitutes the Abelian 2-form $F^{(2)} = dA^{(1)} \equiv \frac{1}{2!}(dx^\mu \wedge dx^\nu) F_{\mu\nu}$, B is the Nakanishi-Lautrup auxiliary (multiplier) field and $(\bar{C})C$ are the anticommuting (i.e. $C^2 = \bar{C}^2 = 0, C\bar{C} + \bar{C}C = 0$) (anti-)ghost fields of the theory. The above Lagrangian density respects the off-shell nilpotent ($s_{(a)b}^2 = 0$) (anti-)BRST symmetry transformations $s_{(a)b}$ (with $s_b s_{ab} + s_{ab} s_b = 0$)†

$$\begin{array}{lllll} s_b A_\mu = \partial_\mu C, & s_b C = 0, & s_b \bar{C} = iB, & s_b B = 0, & s_b F_{\mu\nu} = 0, \\ s_{ab} A_\mu = \partial_\mu \bar{C}, & s_{ab} \bar{C} = 0, & s_{ab} C = -iB, & s_{ab} B = 0, & s_{ab} F_{\mu\nu} = 0. \end{array} \tag{2.2}$$

It is clear that, under the nilpotent (anti-)BRST symmetry transformations $s_{(a)b}$, the curvature tensor $F_{\mu\nu}$ is found to be invariant. In other words, the 2-form $F^{(2)}$, owing its origin to the cohomological operator $d = dx^\mu \partial_\mu$, is an (anti-)BRST invariant quantity for the Abelian U(1) gauge theory and is, therefore, a physically relevant (i.e. gauge-invariant) quantity. This observation will play an important role in our discussion on the horizontality condition that would be exploited in the context of our superfield approach to (anti-)BRST invariance(s) of the Lagrangian densities in sections 3 and 5.

A note worthy point, at this stage, is the observation that the gauge-fixing and Faddeev-Popov ghost terms can be written, modulo a total derivative, in the following fashion

$$\begin{aligned} s_b\Big[-i\,\bar{C}\,\{(\partial_\mu A^\mu) + \frac{1}{2}\,B\}\Big] &= (\partial_\mu A^\mu)\,B + \frac{1}{2} B^2 - i\,\partial_\mu \bar{C}\,\partial^\mu C, \\ s_{ab}\Big[+i\,C\,\{(\partial_\mu A^\mu) + \frac{1}{2}\,B\}\,\Big] &= (\partial_\mu A^\mu)\,B + \frac{1}{2}\,B^2 - i\,\partial_\mu \bar{C}\,\partial^\mu C, \\ s_b s_{ab}\Big[\,\frac{i}{2}\,A_\mu\,A^\mu + \frac{1}{2}\,C\,\bar{C}\,\Big] &= (\partial_\mu A^\mu)\,B + \frac{1}{2}\,B^2 - i\,\partial_\mu \bar{C}\,\partial^\mu C. \end{aligned} \tag{2.3}$$

The above equation establishes, in a very simple manner, the (anti-)BRST invariance of the Lagrangian density (2.1). The simplicity is encoded, in the above equation, due to (i) the nilpotency $s_{(a)b}^2 = 0$ of the (anti-)BRST symmetry transformations, (ii) the anticommutativity property (i.e. $s_b s_{ab} + s_{ab} s_b = 0$) of $s_{(a)b}$, and (iii) the invariance of the $F_{\mu\nu}$ term under $s_{(a)b}$ (as mentioned earlier). In fact, the kinetic energy term $[(-1/4) F^{\mu\nu} F_{\mu\nu}]$ is constructed with the help of the (anti-)BRST invariant curvature term $F_{\mu\nu}$. As a consequence, the former automatically remains invariant under the (anti-)BRST symmetry transformations.

*We adopt here the notations and conventions such that the flat Minkowski metric in 4D is $\eta_{\mu\nu} =$ diag $(+1, -1, -1, -1)$ so that $A_\mu B^\mu = \eta_{\mu\nu} A^\mu B^\nu = A_0 B_0 - A_i B_i$ for two non-null 4-vectors A_μ and B_μ. The Greek indices $\mu, \nu ...... = 0, 1, 2, 3$ and Latin indices $i, j, k .... = 1, 2, 3$ stand for the spacetime and space directions on the 4D Minkowski spacetime manifold, respectively. Furthermore, in the whole body of our text, the summation convention is always followed and the d'Alembertian operator $\Box = (\partial_0)^2 - (\partial_i)^2$.

†We follow here the notations and conventions adopted in [27]. In its full blaze of glory, the nilpotent (anti-)BRST transformations $\delta_{(A)B}$ are a product of an anticommuting spacetime independent parameter η and $s_{(a)b}$ (i.e. $\delta_{(A)B} = \eta s_{(a)b}$) where the nilpotency property is carried by the operators $s_{(a)b}$.

The following on-shell (i.e. $\Box C = 0, \Box \bar{C} = 0$) nilpotent ($\tilde{s}_{(a)b}^2 = 0$) (anti-)BRST symmetry transformations (with $\tilde{s}_b \tilde{s}_{ab} + \tilde{s}_{ab} \tilde{s}_b = 0$)

$$\begin{array}{llll} \tilde{s}_b A_\mu = \partial_\mu C, & \tilde{s}_b C = 0, & \tilde{s}_b \bar{C} = -i(\partial_\mu A^\mu), & \tilde{s}_b F_{\mu\nu} = 0, \\ \tilde{s}_{ab} A_\mu = \partial_\mu \bar{C}, & \tilde{s}_{ab} \bar{C} = 0, & \tilde{s}_{ab} C = +i(\partial_\mu A^\mu), & \tilde{s}_{ab} F_{\mu\nu} = 0, \end{array} \tag{2.4}$$

can be derived from the off-shell nilpotent (anti-)BRST symmetry transformations (2.2) by the substitution $B = -(\partial_\mu A^\mu)$ which emerges, from the Lagrangian density (2.1), as an equation of motion. In an exactly similar fashion the following Lagrangian density can be found from (2.1), with the substitution $B = -(\partial_\mu A^\mu)$, namely;

$$\mathcal{L}_b^{(a)} = -\frac{1}{4}\, F^{\mu\nu}\, F_{\mu\nu} - \,\frac{1}{2}\, (\partial_\mu A^\mu)^2 - i\, \partial_\mu \bar{C}\, \partial^\mu C, \tag{2.5}$$

which remains invariant (up to a total derivative) under the above on-shell nilpotent transformations (2.4). It will be noted that, even under the transformations (2.4), the curvature tensor $F_{\mu\nu}$ remains invariant. Thus, $F_{\mu\nu}$ is a true physical quantity. An interesting point, connected with the on-shell nilpotent symmetry transformations, is to express the analogue of (2.3) in terms of these symmetry transformations. These analogues are

$$\begin{array}{l} \tilde{s}_b \Big[+\dfrac{i}{2}\, \bar{C}\, (\partial_\mu A^\mu) + i\, A_\mu\, \partial^\mu \bar{C}\, \Big] = -\dfrac{1}{2}\, (\partial_\mu A^\mu)^2 - i\, \partial_\mu \bar{C}\, \partial^\mu C, \\ \tilde{s}_{ab} \Big[-\dfrac{i}{2}\, C\, (\partial_\mu A^\mu) - i\, A_\mu\, \partial^\mu C\, \Big] = -\dfrac{1}{2}\, (\partial_\mu A^\mu)^2 - i\, \partial_\mu \bar{C}\, \partial^\mu C, \\ \tilde{s}_b \tilde{s}_{ab} \Big[\, \dfrac{i}{2}\, A_\mu\, A^\mu + \dfrac{1}{2}\, C\, \bar{C}\, \Big] = -\dfrac{1}{2}\, (\partial_\mu A^\mu)^2 - i\, \partial_\mu \bar{C}\, \partial^\mu C. \end{array} \tag{2.6}$$

It should be noted that, in the above precise proof, one has to take into account the on-shell conditions (i.e. $\Box C = \Box \bar{C} = 0$). In other words, for all practical purposes, we have to take $\tilde{s}_b (\partial_\mu A^\mu) = 0$ and $\tilde{s}_{ab} (\partial_\mu A^\mu) = 0$ in the explicit computations.

The above nilpotent (anti-)BRST symmetry transformations (i.e. $s_r, \tilde{s}_r$ with $r = b, ab$) are connected with the conserved and nilpotent generators (i.e. $Q_r, \tilde{Q}_r$ with $r = b, ab$). This statement can be succinctly expressed, in the mathematical form, as

$$s_r\, \Omega = -i\, [\, \Omega,\; Q_r\,]_{(\pm)}, \qquad \tilde{s}_r\, \tilde{\Omega} = -i\, [\, \tilde{\Omega},\; \tilde{Q}_r\,]_{(\pm)}, \qquad r = b, ab, \tag{2.7}$$

where the subscripts (with the signatures $(+)-$ on the square bracket) stand for the bracket to be an (anti)commutator, for the generic fields $\Omega = A_\mu, C, \bar{C}, B$ and $\tilde{\Omega} = A_\mu, C, \bar{C}$ of the Lagrangian densities (2.1) and (2.5), being (fermionic)bosonic in nature. Here Q_r and $\tilde{Q}_r$ are the conserved, off-shell nilpotent (i.e. $Q_r^2 = 0$ with $r = b, ab$) as well as on-shell ($\Box C = 0, \Box \bar{C} = 0$) nilpotent (i.e. $\tilde{Q}_r^2 = 0$ with $r = b, ab$) (anti-)BRST charges. These charges are found to be anticommuting (i.e. $Q_b Q_{ab} + Q_{ab} Q_b = 0$, $\tilde{Q}_b \tilde{Q}_{ab} + \tilde{Q}_{ab} \tilde{Q}_b = 0$) in nature. The nilpotency and anticommutativity properties would be useful for our present investigation. As a side remark, we would like to point out that the exact expression for these charges can be derived by exploiting the Noether theorem. For our present discussion, however, their explicit mathematical forms are not required.

3 (Anti-)BRST Invariance In Abelian Theory: Superfield Formalism

In this section, we exploit the geometrical superfield approach to BRST formalism, endowed with the theoretical arsenal of the horizontality condition, to express the (anti-)BRST symmetry transformations and the Lagrangian densities (cf. (2.1) and (2.5)) in terms of the superfields defined on the (4, 2)-dimensional supermanifold. The latter is parametrized by the spacetime coordinates x^μ (with $\mu = 0, 1, 2, 3$) and a pair of Grassmannian variables θ and $\bar{\theta}$. In particular, we wish to answer the following questions: (i) can we capture equations (2.3) and (2.6) in the language of the superfield formulation? (ii) can there be more ways to express the gauge-fixing and Faddeev-Popov ghost terms in the (anti-)BRST-exact forms than the ones expressed in (2.3) and (2.6)? (iii) what is its resolution in the language of the superfield approach to BRST formalism? (iv) how can we capture the (anti-)BRST invariance of the Lagrangian densities (2.1) and (2.5) in the language of the superfield formalism? (v) are there geometrical meanings associated with these invariance(s)?, and (vi) is there any simplicity that can be said to emerge from the superfield formalism?

To answer the above queries, let us start with the generalization of the 4D exterior derivative $d = dx^\mu \partial_\mu$, 1-form connection $A^{(1)} = dx^\mu A_\mu(x)$ and basic local fields $A_\mu(x), C(x), \bar{C}(x)$ on the (4, 2)-dimensional supermanifold. These are as follows

$$\begin{array}{l} d \rightarrow \tilde{d} = dx^\mu\, \partial_\mu \; + \; d\theta\, \partial_\theta \; + \; d\bar{\theta}\, \partial_{\bar{\theta}}, \qquad\qquad \tilde{d}^2 = 0, \\ A^{(1)} \rightarrow \tilde{A}^{(1)} = dx^\mu\, \mathcal{B}_\mu(x, \theta, \bar{\theta}) + d\theta\, \bar{\mathcal{F}}(x, \theta, \bar{\theta}) + d\bar{\theta}\, \mathcal{F}(x, \theta, \bar{\theta}), \end{array} \tag{3.1}$$

where the mapping from the 4D local fields to the superfields are: $A_\mu(x) \rightarrow \mathcal{B}_\mu(x, \theta, \bar{\theta})$, $C(x) \rightarrow \mathcal{F}(x, \theta, \bar{\theta})$ and $\bar{C}(x) \rightarrow \bar{\mathcal{F}}(x, \theta, \bar{\theta})$. The super-expansion of the superfields, in terms of the basic fields as well as the secondary fields, are:

$$\begin{array}{lcl} \mathcal{B}_\mu(x, \theta, \bar{\theta}) &=& A_\mu(x) + \theta\, \bar{R}_\mu(x) + \bar{\theta}\, R_\mu(x) + i\, \theta\, \bar{\theta}\, S_\mu(x), \\ \mathcal{F}(x, \theta, \bar{\theta}) &=& C(x) + i\, \theta\, \bar{B}_1(x) + i\, \bar{\theta}\, B_1(x) + i\, \theta\, \bar{\theta}\, s(x), \\ \bar{\mathcal{F}}(x, \theta, \bar{\theta}) &=& \bar{C}(x) + i\, \theta\, \bar{B}_2(x) + i\, \bar{\theta}\, B_2(x) + i\, \theta\, \bar{\theta}\, \bar{s}(x). \end{array} \tag{3.2}$$

It can be readily seen that, in the limiting case of $(\theta, \bar{\theta}) \rightarrow 0$, we get back our 4D basic fields $(A_\mu, C, \bar{C})$. Furthermore, on the r.h.s. of the above super expansion, the bosonic (i.e. $A_\mu, S_\mu, B_1, \bar{B}_1, B_2, \bar{B}_2$) and the fermionic $(R_\mu, \bar{R}_\mu, C, \bar{C}, s, \bar{s})$ fields do match which is the basic requirement of any arbitrary supersymmetric field theory.

At this juncture, we have to recall our observations after equation (2.2). The nilpotent (anti-)BRST symmetry transformations basically owe their origin to the cohomological operator d. This is capitalized in the horizontality condition where we impose the restriction $\tilde{d}\tilde{A}^{(1)} = dA^{(1)}$ on super 1-form connection $\tilde{A}^{(1)}$ that contains the superfields defined on the (4, 2)-dimensional supermanifold. The latter condition yields the following relationships (see, e.g., for details in our earlier works [21-25])

$$B_1 = \bar{B}_2 = s = \bar{s} = 0, \qquad \bar{B}_1 + B_2 = 0, \tag{3.3}$$

where we are free choose the secondary fields $(B_2, \bar{B}_1)$ (i.e. $B_2 = B \Rightarrow \bar{B}_1 = -B$) in terms of the Nakanishi-Lautrup auxiliary (multiplier) field B of the BRST invariant Lagrangian density (2.1). The other relations, that emerge from the above HC (i.e. $\tilde{d}\tilde{A}^{(1)} = dA^{(1)}$), are

$$R_\mu = \partial_\mu C, \qquad \bar{R}_\mu = \partial_\mu \bar{C}, \qquad S_\mu = \partial_\mu B. \tag{3.4}$$

It will be noted that all the above relations have been obtained by setting equal to zero the coefficients of the 2-form differentials $(d\theta \wedge d\theta), (d\theta \wedge d\bar{\theta}), (d\bar{\theta} \wedge d\bar{\theta}), (dx^\mu \wedge d\theta)$ and $(dx^\mu \wedge d\bar{\theta})$ that are found in the explicit computation of super 2-form $\tilde{d}\tilde{A}^{(1)}$. Finally, the comparison of the coefficients of the spacetime 2-form differentials $\frac{1}{2}\ (dx^\mu \wedge dx^\nu)$ from the l.h.s. and r.h.s. of the HC (i.e. $\tilde{d}\tilde{A}^{(1)} = dA^{(1)}$), leads to

$$\partial_\mu \mathcal{B}_\nu - \partial_\nu \mathcal{B}_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.5}$$

At this stage, the super-curvature tensor $\tilde{F}_{\mu\nu} = \partial_\mu \mathcal{B}_\nu - \partial_\nu \mathcal{B}_\mu$ is *not* equal to the ordinary curvature tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as the former contains Grassmannian dependent terms.

The substitution of the above values of the secondary fields in terms of the basic and auxiliary fields of the Lagrangian density (2.1) leads to the following expansion

$$\begin{array}{lcl} \mathcal{B}^{(h)}_\mu(x,\theta,\bar{\theta}) & = & A_\mu + \theta\ \partial_\mu \bar{C} + \bar{\theta}\ \partial_\mu C + i\ \theta\ \bar{\theta}\ \partial_\mu B, \\ \mathcal{F}^{(h)}(x,\theta,\bar{\theta}) & = & C - i\ \theta\ B, \qquad \bar{\mathcal{F}}^{(h)}(x,\theta,\bar{\theta}) = \bar{C} + i\ \bar{\theta}\ B, \end{array} \tag{3.6}$$

where the superscript (h) has been used to denote that the above expansions have been obtained after the application of the HC. It can be seen that, due to (3.6), the equation (3.5) leads to the following equality

$$\partial_\mu \mathcal{B}^{(h)}_\nu - \partial_\nu \mathcal{B}^{(h)}_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3.7}$$

where there is *no* Grassmannian θ and $\bar{\theta}$ dependence. In other words, after the application of the HC, it is quite elementary to check that the l.h.s. of the above equation is independent of the Grassmannian variables. This, in turn, implies that the super-curvature $\tilde{F}^{(h)}_{\mu\nu} = \partial_\mu \mathcal{B}^{(h)}_\nu - \partial_\nu \mathcal{B}^{(h)}_\mu$ is equal to the ordinary curvature $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

In the language of the geometry on the (4, 2)-dimensional supermanifold, the expansions (3.6) imply that the (anti-)BRST symmetry transformations $s_{(a)b}$ (and their corresponding generators $Q_{(a)b}$) for the 4D local fields (cf. (2.7)) are connected with the translational generators $(\partial/\partial\theta, \partial/\partial\bar{\theta})$ because the translation of the corresponding (4, 2)-dimensional superfields along the Grassmannian direction(s) of the supermanifold produces it. Thus, the Grassmannian independence of the super curvature tensor $\tilde{F}^{(h)}_{\mu\nu} = \partial_\mu \mathcal{B}^{(h)}_\nu - \partial_\nu \mathcal{B}^{(h)}_\mu$ implies that the 4D curvature tensor $F_{\mu\nu}$ is an (anti-)BRST (i.e. gauge) invariant physical quantity. Similarly, we shall conclude (see, (3.17), (3.18) and (3.19) below) that the Grassmannian independence of the super Lagrangian densities will automatically imply the (anti-)BRST invariance of the 4D Lagrangian densities of the 1-form (non-)Abelian gauge theories. This statement is the crux of our whole discussion on the simplification aspect of the (anti-)BRST invariance of the Lagrangian densities in the superfield approach to BRST formalism.

In terms of the superfields, equations (2.3) can be expressed as

$$
\begin{array}{l}
\mathrm{Lim}_{\theta\to 0}\,\frac{\partial}{\partial\bar\theta}\Big[-i\,\bar{\mathcal F}^{(h)}\,\{\,(\partial^\mu\mathcal B^{(h)}_\mu+\frac{1}{2}\,B)\,\}\,\Big]=s_b\,[\,-i\,\bar C\{(\partial_\mu A^\mu)+\frac{1}{2}\,B\}\,],\\
\mathrm{Lim}_{\bar\theta\to 0}\,\frac{\partial}{\partial\theta}\Big[+i\mathcal F^{(h)}\,\{\,(\partial^\mu\mathcal B^{(h)}_\mu+\frac{1}{2}\,B)\,\}\,\Big]=s_{ab}\,[\,+i\,C\{(\partial_\mu A^\mu)+\frac{1}{2}\,B\}\,],\\
\frac{\partial}{\partial\bar\theta}\,\frac{\partial}{\partial\theta}\,\Big[\,\frac{i}{2}\,\mathcal B^{\mu(h)}\mathcal B^{(h)}_\mu\;+\;\frac{1}{2}\,\mathcal F^{(h)}\,\bar{\mathcal F}^{(h)}\,\Big]=s_b\,s_{ab}\,\Big(\,\frac{i}{2}\,A_\mu\,A^\mu\;+\;\frac{1}{2}\,C\,\bar C\,\Big).
\end{array}
\tag{3.8}
$$

These equations are *unique* because there is no other way to express the above equations in terms of the derivatives w.r.t. Grassmannian variables θ and $\bar\theta$. Thus, besides (2.3), there is no other possibility to express the gauge-fixing and the Faddeev-Popov ghost terms in the language of the off-shell nilpotent (anti-)BRST symmetry transformations (2.2). The superfield approach to BRST formulation, therefore, establishes the uniqueness of (2.3). To express (2.6) in terms of the superfields, one has to substitute $B=-(\partial_\mu A^\mu)$ in (3.6). Thus, the expansion (3.6), in terms of the transformations (2.4), becomes

$$
\begin{array}{rcl}
\mathcal B^{(h)}_{\mu(o)}(x,\theta,\bar\theta) & = & A_\mu+\theta\,\partial_\mu\bar C+\bar\theta\,\partial_\mu C-i\,\theta\,\bar\theta\,\partial_\mu(\partial^\rho A_\rho),\\
& \equiv & A_\mu+\theta\,(\tilde s_{ab}A_\mu)+\bar\theta\,(\tilde s_b A_\mu)+\theta\,\bar\theta(\tilde s_b\tilde s_{ab}A_\mu),\\
\mathcal F^{(h)}_{(o)}(x,\theta,\bar\theta) & = & C+i\,\theta\,(\partial_\mu A^\mu)\equiv C+\theta\,(\tilde s_{ab}C),\\
\bar{\mathcal F}^{(h)}_{(o)}(x,\theta,\bar\theta) & = & \bar C-i\,\bar\theta\,(\partial_\mu A^\mu)\equiv\bar C+\bar\theta\,(\tilde s_b\bar C).
\end{array}
\tag{3.9}
$$

We note that (3.6) and (3.9) are the super expansions (after the application of the HC) which lead to the derivation of the off-shell nilpotent (anti-)BRST symmetry transformations $s_{(a)b}$ as well as the on-shell nilpotent (anti-)BRST symmetry transformations $\tilde s_{(a)b}$, respectively, for the basic fields A_μ, C and $\bar C$ of the theory. The gauge-fixing and Faddeev-Popov terms of the Lagrangian density (2.5) can also be expressed (i) in terms of the superfields (3.9) (obtained after the application of the HC), and (ii) in terms of the on-shell nilpotent (anti-)BRST symmetry transformations (2.4). In other words, (*vis-à-vis* (3.8)), we have the following equations

$$
\begin{array}{l}
\mathrm{Lim}_{\theta\to 0}\,\frac{\partial}{\partial\bar\theta}\Big[+\frac{i}{2}\,\bar{\mathcal F}^{(h)}_{(o)}\,(\partial^\mu A_\mu)+i\,\mathcal B^{(h)}_{\mu(o)}\,\partial^\mu\bar{\mathcal F}^{(h)}_{(o)})\,\Big]=\tilde s_b\,[\,+\frac{i}{2}\bar C\,(\partial_\mu A^\mu)+i\,A_\mu\,\partial^\mu\bar C\,],\\
\mathrm{Lim}_{\bar\theta\to 0}\,\frac{\partial}{\partial\theta}\Big[-\,\frac{i}{2}\,\mathcal F^{(h)}_{(o)}\,(\partial^\mu A_\mu)-i\,\mathcal B^{(h)}_{\mu(o)}\,\partial^\mu\mathcal F^{(h)}_{(o)})\,\Big]=\tilde s_{ab}\,[\,-\frac{i}{2}\,C\,(\partial_\mu A^\mu)-i\,A_\mu\,\partial^\mu C\,],\\
\frac{\partial}{\partial\bar\theta}\,\frac{\partial}{\partial\theta}\,\Big[\,\frac{i}{2}\,\mathcal B^{\mu(h)}_{(o)}\mathcal B^{(h)}_{\mu(o)}\;+\;\frac{1}{2}\,\mathcal F^{(h)}_{(o)}\,\bar{\mathcal F}^{(h)}_{(o)}\,\Big]=\tilde s_b\,\tilde s_{ab}\,\Big(\,\frac{i}{2}\,A_\mu\,A^\mu\;+\;\frac{1}{2}\,C\,\bar C\,\Big).
\end{array}
\tag{3.10}
$$

It would be re-emphasized here that the superfields in (3.9) (obtained after the application of the HC) have to be used in the above equations for the verification of their sanctity. Furthermore, for all practical purposes of computations, it essential to take into account $\tilde s_{(a)b}(\partial_\mu A^\mu)=0$ because of the on-shell conditions $\Box C=\Box\bar C=0$.

The Lagrangian density (2.1) can be expressed, in terms of the superfields (obtained after the application of the HC) in the following distinct and different forms

$$
\tilde{\mathcal L}^{(1)}_B=-\frac{1}{4}\tilde F^{(h)}_{\mu\nu}\tilde F^{\mu\nu(h)}+\mathrm{Lim}_{\theta\to 0}\,\frac{\partial}{\partial\bar\theta}\,\Big[-i\,\bar{\mathcal F}^{(h)}(\partial^\mu\mathcal B^{(h)}_\mu+\frac{1}{2}\,B)\Big],
\tag{3.11}
$$

$$\tilde{\mathcal{L}}_B^{(2)} = -\frac{1}{4}\tilde{F}^{(h)}_{\mu\nu}\tilde{F}^{\mu\nu(h)} + \text{Lim}_{\bar\theta\to 0}\,\frac{\partial}{\partial\theta}\,\Big[+i\,\mathcal{F}^{(h)}(\partial^\mu\mathcal{B}^{(h)}_\mu + \frac{1}{2}\,B)\Big], \tag{3.12}$$

$$\tilde{\mathcal{L}}_B^{(3)} = -\frac{1}{4}\tilde{F}^{(h)}_{\mu\nu}\tilde{F}^{\mu\nu(h)} + \frac{\partial}{\partial\bar\theta}\,\frac{\partial}{\partial\theta}\,\Big[+\frac{i}{2}\mathcal{B}^{\mu(h)}\mathcal{B}^{(h)}_\mu + \frac{1}{2}\mathcal{F}^{(h)}\bar{\mathcal{F}}^{(h)}\Big]. \tag{3.13}$$

It would be noted that the kinetic energy term $-(1/4)\tilde{F}^{(h)}_{\mu\nu}\tilde{F}^{\mu\nu(h)}$ is independent of the Grassmannian variables θ and $\bar\theta$ (derived after the application of the HC) because $\tilde{F}^{(h)}_{\mu\nu} = F_{\mu\nu}$. In exactly similar fashion, the Lagrangian density of (2.5) can be expressed, with the help of the super expansion (3.9), as

$$\tilde{\mathcal{L}}_b^{(1)} = -\frac{1}{4}\tilde{F}^{(h)}_{\mu\nu}\tilde{F}^{\mu\nu(h)} + \text{Lim}_{\theta\to 0}\,\frac{\partial}{\partial\bar\theta}\Big[\,+\frac{i}{2}\,\bar{\mathcal{F}}^{(h)}_{(o)}\,(\partial^\mu A_\mu) + i\,\mathcal{B}^{(h)}_{\mu(o)}\,\partial^\mu\bar{\mathcal{F}}^{(h)}_{(o)})\,\Big], \tag{3.14}$$

$$\tilde{\mathcal{L}}_b^{(2)} = -\frac{1}{4}\tilde{F}^{(h)}_{\mu\nu}\tilde{F}^{\mu\nu(h)} + \text{Lim}_{\bar\theta\to 0}\,\frac{\partial}{\partial\theta}\Big[\,-\frac{i}{2}\,\mathcal{F}^{(h)}_{(o)}\,(\partial^\mu A_\mu) - i\,\mathcal{B}^{(h)}_{\mu(0)}\,\partial^\mu\mathcal{F}^{(h)}_{(o)})\,\Big], \tag{3.15}$$

$$\tilde{\mathcal{L}}_b^{(3)} = -\frac{1}{4}\tilde{F}^{(h)}_{\mu\nu}\tilde{F}^{\mu\nu(h)} + \frac{\partial}{\partial\bar\theta}\,\frac{\partial}{\partial\theta}\,\Big[\,+\frac{i}{2}\mathcal{B}^{\mu(h)}_{(o)}\mathcal{B}^{(h)}_{\mu(o)} + \frac{1}{2}\mathcal{F}^{(h)}_{(o)}\bar{\mathcal{F}}^{(h)}_{(o)}\,\Big]. \tag{3.16}$$

The form of the Lagrangian densities (e.g. from (3.11) to (3.16)) simplify the proof for the (anti-)BRST invariance of the Lagrangian densities in (2.1) and (2.5).

In the above forms (e.g. from (3.11) to (3.13)) of the Lagrangian density, the BRST invariance $s_b\mathcal{L}_B = 0$ and the anti-BRST invariance $s_{ab}\mathcal{L}_B = 0$ become very transparent and simple because the following equalities and mappings exist, namely;

$$s_b\mathcal{L}_B^{(a)} = 0 \Rightarrow \text{Lim}_{\theta\to 0}\frac{\partial}{\partial\bar\theta}\tilde{\mathcal{L}}_B^{(1)} = 0, \quad s_b \Leftrightarrow \text{Lim}_{\theta\to 0}\frac{\partial}{\partial\bar\theta}, \quad s_b^2 = 0 \Leftrightarrow \Big(\frac{\partial}{\partial\bar\theta}\Big)^2 = 0, \tag{3.17}$$

$$s_{ab}\mathcal{L}_B^{(a)} = 0 \Rightarrow \text{Lim}_{\bar\theta\to 0}\frac{\partial}{\partial\theta}\tilde{\mathcal{L}}_B^{(2)} = 0, \quad s_{ab} \Leftrightarrow \text{Lim}_{\bar\theta\to 0}\frac{\partial}{\partial\theta}, \quad s_{ab}^2 = 0 \Leftrightarrow \Big(\frac{\partial}{\partial\theta}\Big)^2 = 0. \tag{3.18}$$

Similarly, the most beautiful relation (3.13), leads to the (anti-)BRST invariance *together*. Here one has to use the anticommutativity property $s_b s_{ab} + s_{ab} s_b = 0$ in the language of the translational generators (i.e. $(\partial/\partial\bar\theta), (\partial/\partial\theta)$) along the Grassmannian directions of the supermanifold, for its proof. This statement can be mathematically expressed as

$$s_b s_{ab}\mathcal{L}_B^{(a)} = 0 \Rightarrow \frac{\partial}{\partial\bar\theta}\,\frac{\partial}{\partial\theta}\tilde{\mathcal{L}}_B^{(3)} = 0, \quad s_b s_{ab} + s_{ab} s_b = 0 \Leftrightarrow \frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar\theta} + \frac{\partial}{\partial\bar\theta}\frac{\partial}{\partial\theta} = 0. \tag{3.19}$$

In exactly similar fashion, the on-shell nilpotent (anti-)BRST symmetry invariance (i.e. $\tilde{s}_{(a)b}\mathcal{L}_b^{(a)} = 0$) of the Lagrangian density (2.5) can also be captured in the language of the superfields if we exploit the expressions (3.14) to (3.16) for the Lagrangian density. In the latter case, too, the on-shell nilpotent (anti-)BRST invariance turns out to be like the above equations (3.17), (3.18) and (3.19).

4 (Anti-)BRST Symmetries In Non-Abelian Theory: Lagrangian Approach

We begin with the following BRST-invariant Lagrangian density, in the Feynman gauge, for the four (3 + 1)-dimensional non-Abelian 1-form gauge theory[‡] (see, e.g. [26,27,9])

$$\mathcal{L}_B^{(n)} = -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} + B \cdot (\partial_\mu A^\mu) + \frac{1}{2} B \cdot B - i \partial_\mu \bar{C} \cdot D^\mu C, \tag{4.1}$$

where the $SU(N)$ Lie algebraic group valued curvature tensor ($F_{\mu\nu}$) is defined through the 2-form $F^{(2)(n)} = dA^{(1)(n)} + i A^{(1)(n)} \wedge A^{(1)(n)}$. Here the 1-form non-Abelian gauge connection is $A^{(1)(n)} = dx^\mu (A_\mu \cdot T)$ and the exterior derivative $d = dx^\mu \partial_\mu$. The Nakanishi-Lautrup auxiliary field $B = B \cdot T$ is required for the linearization of the gauge-fixing term and the Lie algebraic group valued (anti-)ghost fields $(\bar{C})C$ are essential for the proof of unitarity in the theory at any arbitrary order of perturbative computation for a given physical process. The latter fields are fermionic (i.e. $(C^a)^2 = 0, (\bar{C}^a)^2 = 0, C^a C^b + C^b C^a = 0, C^a \bar{C}^b + \bar{C}^b C^a = 0$, etc.) in nature for the above 4D 1-form non-Abelian gauge theory.

The above Lagrangian density respects the following off-shell nilpotent ($s_b^2 = 0$) BRST symmetry transformations s_b

$$\begin{array}{l} s_b A_\mu = D_\mu C, \qquad s_b C = -\frac{i}{2} (C \times C), \qquad s_b \bar{C} = i B, \\ s_b B = 0, \qquad s_b F_{\mu\nu} = i (F_{\mu\nu} \times C). \end{array} \tag{4.2}$$

It will be noted that (i) the curvature tensor $F_{\mu\nu} \cdot T$ transformed here under the BRST symmetry transformation. However, it can be checked explicitly that the kinetic energy term $-(1/4) F_{\mu\nu} \cdot F^{\mu\nu}$ remains invariant under the BRST symmetry transformation because of the fact that f^{abc} is totally antisymmetric for the Lie algebra corresponding to the $SU(N)$ group, (ii) the nilpotent anti-BRST symmetry transformations corresponding to the above BRST symmetry transformations (4.2) *cannot* be defined for the Lagrangian density (4.1), and (iii) the on-shell nilpotent version of the above BRST symmetry transformations is also possible if we substitute in the above symmetry transformations $B = -(\partial_\mu A^\mu)$. The ensuing on-shell (i.e. $\partial_\mu D^\mu C = 0$ nilpotent BRST symmetry transformations $\tilde{s}_b$ are

$$\begin{array}{l} \tilde{s}_b A_\mu = D_\mu C, \qquad \tilde{s}_b C = -\frac{i}{2} (C \times C), \\ \tilde{s}_b \bar{C} = -i (\partial_\mu A^\mu), \qquad s_b F_{\mu\nu} = i (F_{\mu\nu} \times C). \end{array} \tag{4.3}$$

[‡]We follow the conventions and notations such that the Minkowskian metric is same as that used in the previous sections. For the 1-form non-Abelian gauge theory, the notations used in the Lie algebraic group space are: $A \cdot B = A^a B^a, (A \times B)^a = f^{abc} A^b B^c, D_\mu C^a = \partial_\mu C^a + i f^{abc} A_\mu^b C^c \equiv \partial_\mu C^a + i (A_\mu \times C)^a, F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i A_\mu \times A_\nu, A_\mu = A_\mu \cdot T, [T^a, T^b] = f^{abc} T^c$ and the Latin indices $a, b, c = 1, 2, 3....N$ are in the $SU(N)$ Lie group algebraic space. The structure constant f^{abc} can be chosen to be totally antisymmetric for any arbitrary semi simple Lie algebra that includes $SU(N)$, too (see, e.g., [27]).

The above transformations leave the following Lagrangian density

$$\mathcal{L}_b^{(n)} = -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu) \cdot (\partial_\mu A^\mu) - i \partial_\mu \bar{C} \cdot D^\mu C, \tag{4.4}$$

quasi-invariant because it transforms to a total derivative.

The gauge-fixing and Faddeev-Popov ghost terms of the Lagrangian density (4.1) can be written, modulo a total derivative, as a BRST-exact quantity in terms of the off-shell nilpotent BRST symmetry transformations (4.2). This statement can be mathematically expressed as follows

$$s_b \Big[-i \, \bar{C} \cdot \{ (\partial_\mu A^\mu) + \frac{1}{2} B \} \Big] = B \cdot (\partial_\mu A^\mu) + \frac{1}{2} B \cdot B - i \, \partial_\mu \bar{C} \cdot D^\mu C. \tag{4.5}$$

The above form simplifies the proof of the BRST invariance because (i) the kinetic energy term has already been shown to be BRST invariant due to the structure constants f^{abc} being totally antisymmetric in nature, and (ii) the nilpotency of the BRST transformations shows that the gauge-fixing and Faddeev-Popov ghost terms are BRST invariant, too. The on-shell version of the above expression (cf. (4.5)) can also be written. The latter form can be mathematically expressed as

$$\tilde{s}_b \Big[+ \frac{i}{2} \, \bar{C} \cdot (\partial_\mu A^\mu) + i \, A_\mu \cdot \partial^\mu \bar{C} \, \Big] = -\frac{1}{2} \, (\partial_\mu A^\mu) \cdot (\partial_\rho A^\rho) - i \, \partial_\mu \bar{C} \cdot D^\mu C. \tag{4.6}$$

It will be noted that one has to take into account $\tilde{s}_b (\partial_\mu A^\mu) = \partial_\mu D^\mu C = 0$ in the proof of the exactness of the above expression. In other words, for all practical purposes of calculations, we have to take into account $\tilde{s}_b (\partial_\mu A^\mu) = 0$.

The Lagrangian densities that respect off-shell nilpotent (i.e. $s_{(a)b}^2 = 0$) BRST as well as anti-BRST symmetry transformations are

$$\mathcal{L}_b^{(1)} = -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} + B \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i \partial_\mu \bar{C} \cdot D^\mu C, \tag{4.7}$$

$$\mathcal{L}_b^{(2)} = -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} - \bar{B} \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i D_\mu \bar{C} \cdot \partial^\mu \bar{C}. \tag{4.8}$$

Here auxiliary fields B and $\bar{B}$ satisfy the Curci-Ferrari condition $B + \bar{B} = -(C \times \bar{C})$ [28,29]. It is also evident from this relation that $B \cdot (\partial_\mu A^\mu) - i \partial_\mu \bar{C} \cdot D^\mu C = -\bar{B} \cdot (\partial_\mu A^\mu) - i D_\mu \bar{C} \cdot \partial^\mu C$. Furthermore, it should be re-emphasized that the Lagrangian densities (4.1) and (4.4) do not respect the anti-BRST symmetry transformations of any kind. The BRST and anti-BRST symmetry transformations, for the above Lagrangian densities, are [§]

$$\begin{aligned} &s_b A_\mu = D_\mu C, \qquad s_b C = -\frac{i}{2} (C \times C), \qquad s_b \bar{C} = i B, \\ &s_b B = 0, \qquad s_b F_{\mu\nu} = i (F_{\mu\nu} \times C), \qquad s_b \bar{B} = i (\bar{B} \times C), \end{aligned} \tag{4.9}$$

[§]It should be noted that we have continued with the same notations for (anti-)BRST symmetry transformations operators (i.e. $s_{(a)b}$) for the 1-form Abelian as well as the non-Abelian gauge theories because these are the well-accepted (universal) notations for any kind of BRST invariant (physical) gauge theories.

$$
\begin{array}{lll}
s_{ab} A_\mu = D_\mu \bar{C}, & s_{ab} \bar{C} = -\dfrac{i}{2}(\bar{C} \times \bar{C}), & s_{ab} C = i\bar{B}, \\
s_{ab} \bar{B} = 0, & s_{ab} F_{\mu\nu} = i(F_{\mu\nu} \times \bar{C}), & s_{ab} B = i(B \times \bar{C}).
\end{array} \tag{4.10}
$$

The above off-shell nilpotent (anti-)BRST symmetry transformations leave the Lagrangian densities (4.7) as well as (4.8) quasi-invariant as they transform to some total derivatives. The gauge-fixing and Faddeev-Popov ghost terms of the Lagrangian densities (4.7) and (4.8) can be written, in a symmetrical fashion with respect to s_b and s_{ab}, as

$$
\begin{array}{lll}
s_b s_{ab} \Big[\dfrac{i}{2} A_\mu \cdot A^\mu + C \cdot \bar{C} \Big] & = & B \cdot (\partial_\mu A^\mu) + \dfrac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B}) - i \partial_\mu \bar{C} \cdot D^\mu C, \\
& \equiv & -\bar{B} \cdot (\partial_\mu A^\mu) + \dfrac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B}) - i D_\mu \bar{C} \cdot \partial^\mu C.
\end{array} \tag{4.11}
$$

This demonstrates the key fact that the above gauge-fixing and Faddeev-Popov ghost terms are (anti-)BRST invariant *together* because of the nilpotency of the (anti-)BRST symmetry transformations (i.e. $s_{(a)b}^2 = 0$) that are present in the theory.

5 (Anti-)BRST Invariance In Non-Abelian Theory: Superfield Approach

To capture (i) the off-shell as well as the on-shell nilpotent (anti-)BRST symmetry transformations, and (ii) the invariance of the Lagrangian densities, in the language of the superfield approach to BRST formalism, we have to consider the 4D 1-form non-Abelian gauge theory on a (4, 2)-dimensional supermanifold (as discussed for the 1-form Abelian case). The four basic questions we address here are: (i) how to capture the equations (4.5), (4.6) and (4.11) in the language of the superfield approach to BRST formalism?, (ii) what is the reason, within the framework of the superfield approach to BRST formalism, that the Lagrangian density (4.1) is *not* endowed with the on-shell or off-shell nilpotent anti-BRST symmetry transformations but it respects the on-shell (cf. (4.2)) as well as the off-shell (cf. (4.3)) nilpotent BRST symmetry transformations? (iii) why is it that *only* the Lagrangian densities (4.7) and (4.8) respect off-shell nilpotent (anti-)BRST symmetry transformations, and (iv) is there any simplicity (or beauty) that arises from the above exercise?

To answer the above queries, we start off with the analogues of (3.1) and (3.2) as

$$
\begin{array}{l}
d \to \tilde{d} = dx^\mu \, \partial_\mu \; + \; d\theta \, \partial_\theta \; + \; d\bar{\theta} \, \partial_{\bar{\theta}}, \qquad \tilde{d}^2 = 0, \\
A^{(1)(n)} \to \tilde{A}^{(1)(n)} = dx^\mu (\mathcal{B}_\mu \cdot T)(x, \theta, \bar{\theta}) + d\theta (\bar{\mathcal{F}} \cdot T)(x, \theta, \bar{\theta}) + d\bar{\theta} (\mathcal{F} \cdot T)(x, \theta, \bar{\theta}).
\end{array} \tag{5.1}
$$

where the (4, 2)-dimensional superfields $(\mathcal{B}_\mu \cdot T, \mathcal{F} \cdot T, \bar{\mathcal{F}} \cdot T)$ are the generalizations of the 4D basic local fields $(A_\mu \cdot T, C \cdot T, \bar{C} \cdot T)$ of the Lagrangian density (4.1), (4.7) and (4.8). These superfields can be expanded along the Grassmannian directions of the supermanifold, in terms of the basic 4D fields, auxiliary fields and secondary fields as [4,16,19]

$$
\begin{array}{lll}
(\mathcal{B}_\mu \cdot T)(x, \theta, \bar{\theta}) & = & (A_\mu \cdot T)(x) + \theta \, (\bar{R}_\mu \cdot T)(x) + \bar{\theta} \, (R_\mu \cdot T)(x) + i \, \theta \, \bar{\theta} \, (S_\mu \cdot T)(x), \\
(\mathcal{F} \cdot T)(x, \theta, \bar{\theta}) & = & (C \cdot T)(x) + i \, \theta \, (\bar{B}_1 \cdot T)(x) + i \, \bar{\theta} \, (B_1 \cdot T)(x) + i \, \theta \, \bar{\theta} \, (s \cdot T)(x), \\
(\bar{\mathcal{F}} \cdot T)(x, \theta, \bar{\theta}) & = & (\bar{C} \cdot T)(x) + i \, \theta \, (\bar{B}_2 \cdot T)(x) + i \, \bar{\theta} \, (B_2 \cdot T)(x) + i \, \theta \, \bar{\theta} \, (\bar{s} \cdot T)(x).
\end{array} \tag{5.2}
$$

To determine the exact expressions for the secondary fields, in terms of the basic fields of the theory, we have to exploit the HC. The horizontality condition, for the non-Abelian gauge theory is the requirement of the equality of the Maurer-Cartan equation on the (super) manifolds. In other words, the covariant reduction of the super 2-form curvature $\tilde{F}^{(2)(n)}$ to the ordinary 2-form curvature (i.e. $\tilde{d}\tilde{A}^{(1)(n)} + i\tilde{A}^{(1)(n)} \wedge \tilde{A}^{(1)(n)} = dA^{(1)(n)} + iA^{(1)(n)} \wedge A^{(1)(n)}$ where $d = dx^\mu \partial_\mu$ and $A^{(1)(n)} = dx^\mu (A_\mu \cdot T)$) leads to the determination of the secondary fields in terms of the basic fields of the theory. The ensuing expansions, in terms of the basic fields, lead to (i) the derivation of the (anti-)BRST symmetry transformations for the basic fields of the theory, and (ii) the geometrical interpretations of the (anti-)BRST symmetry transformations (and their corresponding nilpotent generators) for the basic fields as the translations of the corresponding superfields along the Grassmannian directions of the (4, 2)-dimensional supermanifold (see, e.g., [16,19] for details).

With the identifications $B_2 = B$ and $\bar{B}_1 = \bar{B}$, the following relationships emerge after the application of the horizontality condition¶ (see, e.g., [16]):

$$
\begin{aligned}
&R_\mu = D_\mu C, \qquad \bar{R}_\mu = D_\mu \bar{C}, \qquad B + \bar{B} = -(C \times \bar{C}), \qquad s = i(\bar{B} \times C), \\
&S_\mu = D_\mu B + D_\mu C \times \bar{C} \equiv -D_\mu \bar{B} - D_\mu \bar{C} \times C, \\
&\bar{s} = -i(B \times \bar{C}), \qquad B_1 = -\frac{1}{2}(C \times C), \qquad \bar{B}_2 = -\frac{1}{2}(\bar{C} \times \bar{C}).
\end{aligned} \tag{5.3}
$$

The substitution of the above expressions, which are obtained after the application of the horizontality condition, leads to the following expansions

$$
\begin{aligned}
\mathcal{B}_\mu^{(h)}(x, \theta, \bar{\theta}) &= A_\mu + \theta\, D_\mu \bar{C} + \bar{\theta}\, D_\mu C + i\,\theta\,\bar{\theta}\,(D_\mu B + D_\mu C \times \bar{C}), \\
\mathcal{F}^{(h)}(x, \theta, \bar{\theta}) &= C + i\,\theta\,\bar{B} - \frac{i}{2}\,\bar{\theta}\,(C \times C) - \theta\,\bar{\theta}\,(\bar{B} \times C), \\
\bar{\mathcal{F}}^{(h)}(x, \theta, \bar{\theta}) &= \bar{C} - \frac{i}{2}\,\theta\,(\bar{C} \times \bar{C}) + i\,\bar{\theta}\,B + \theta\,\bar{\theta}\,(B \times \bar{C}).
\end{aligned} \tag{5.4}
$$

The above expansions (see, e.g., our earlier works [16,19]) can be expressed in terms of the off-shell nilpotent (anti-)BRST symmetry transformations (4.9) and (4.10).

With the above expansion at our disposal, the gauge-fixing and Faddeev-Popov terms of the Lagrangian density (4.1) can be written, modulo a total derivative, as

$$
\mathrm{Lim}_{\theta \to 0} \frac{\partial}{\partial \bar{\theta}} \Big[-i \bar{\mathcal{F}}^{(h)} \cdot \partial^\mu \mathcal{B}_\mu^{(h)} - \frac{i}{2}\, \bar{\mathcal{F}}^{(h)} \cdot B \Big] = B \cdot (\partial_\mu A^\mu) + \frac{1}{2} B \cdot B - i\, \partial_\mu \bar{C} \cdot D^\mu C. \tag{5.5}
$$

Furthermore, it can be seen that, due to the validity and consequences of the horizontality condition, the super curvature tensor $\tilde{F}_{\mu\nu}$ has the following form [16,4]

$$
\tilde{F}_{\mu\nu}^{(h)} = F_{\mu\nu} + i\theta(F_{\mu\nu} \times \bar{C}) + i\bar{\theta}(F_{\mu\nu} \times C) - \theta\,\bar{\theta}\,(F_{\mu\nu} \times B + F_{\mu\nu} \times C \times \bar{C}). \tag{5.6}
$$

It is clear from the above relationship that the kinetic energy term of the present 1-form non-Abelian gauge theory remains invariant, namely;

$$
-\frac{1}{4} \tilde{F}_{\mu\nu}^{(h)} \cdot \tilde{F}^{\mu\nu(h)} = -\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu}. \tag{5.7}
$$

¶In the rest of our present text, we shall be using the short-hand notations for all the fields e. g.: $A_\mu \cdot T = A_\mu, C \cdot T = C, B \cdot T = B$, etc., for the sake of brevity.

The above invariance is basically due to the fact that, for the semi-simple Lie algebra corresponding to $SU(N)$, the structure constants f^{abc} (that also appears in the cross product), can be chosen to be totally antisymmetric in all the Lie algebraic indices a, b, c. In other words, the total kinetic energy term remains independent of the Grassmannian variables θ and $\bar{\theta}$ because all the θ and $\bar{\theta}$ dependent terms of the l.h.s. turn out to be zero if we exploit the totally antisymmetric property of the indices a, b, c in f^{abc}. The Grassmannian independence of the l.h.s. of (5.7) has deep meaning as far as physics is concerned. It implies immediately that the kinetic energy term of the non-Abelian gauge theory is an (anti-)BRST (i.e. gauge) invariant physical quantity.

At this juncture, it is worthwhile to point out that one can also capture the equation (4.6) in the superfield approach to BRST formalism where the on-shell nilpotent version of the BRST symmetry transformations (i.e. $\tilde{s}_b$) plays an important role. For this purpose, we have to express the superfield expansion (5.4) for the on-shell nilpotent BRST symmetry transformation where one has to replace $B = -(\partial_\mu A^\mu)$. With this substitution, the equation (5.4) for the superfield expansion becomes

$$
\begin{aligned}
\mathcal{B}^{(h)}_{\mu(o)}(x,\theta,\bar{\theta}) &= A_\mu + \theta\; D_\mu \bar{C} + \bar{\theta}\; D_\mu C + i\;\theta\;\bar{\theta}\;[-D_\mu(\partial^\rho A_\rho) + D_\mu C \times \bar{C}],\\
\mathcal{F}^{(h)}_{(o)}(x,\theta,\bar{\theta}) &= C + i\;\theta\;\bar{B} - \frac{i}{2}\;\bar{\theta}\;(C \times C) - \theta\;\bar{\theta}\;(\bar{B} \times C),\\
\bar{\mathcal{F}}^{(h)}_{(o)}(x,\theta,\bar{\theta}) &= \bar{C} - \frac{i}{2}\;\theta\;(\bar{C} \times \bar{C}) - i\;\bar{\theta}\;(\partial_\mu A^\mu) - \theta\;\bar{\theta}\;[(\partial_\mu A^\mu) \times \bar{C})].
\end{aligned} \tag{5.8}
$$

Now, the equation (4.6) can be expressed in terms of the above superfields as

$$
\mathrm{Lim}_{\theta\to 0}\;\frac{\partial}{\partial\bar{\theta}}\Big[\frac{i}{2}\bar{\mathcal{F}}^{(h)}_{(o)} \cdot (\partial^\mu A_\mu) + i\;\bar{\mathcal{F}}^{(h)}_{(o)} \cdot \partial^\mu \mathcal{B}^{(h)}_{\mu(o)}\Big] = -\frac{1}{2}\;(\partial_\mu A^\mu) \cdot (\partial_\rho A^\rho) - i\;\partial_\mu \bar{C} \cdot D^\mu C. \tag{5.9}
$$

Furthermore, it can be seen that the analogue of the transformation (5.6), for the on-shell nilpotent BRST symmetry transformation, can be obtained by the replacement $B = -(\partial_\mu A^\mu)$. Once again, the equation (5.7) would remain intact even if we take into account the on-shell nilpotent BRST symmetry transformations. Thus, we note that the kinetic energy term (i.e. $(-1/4) F^{\mu\nu} \cdot F_{\mu\nu}$) of the non-Abelian gauge theory remains independent of the Grassmannian variables θ and $\bar{\theta}$ after the application of the HC. This statement is true for the off-shell as well as on-shell nilpotent (anti-)BRST symmetry transformations. Physically, it implies that the kinetic energy term of the gauge field of the non-Abelian theory is an (anti-)BRST (i.e. gauge) invariant quantity.

The above key observation helps in expressing the Lagrangian density (4.1) and (4.4) in terms of the superfields (obtained after the application of HC), as

$$
\begin{aligned}
\tilde{\mathcal{L}}^{(n)}_B &= -\frac{1}{4}\;\tilde{F}^{(h)}_{\mu\nu} \cdot \tilde{F}^{\mu\nu(h)} + \mathrm{Lim}_{\theta\to 0}\;\frac{\partial}{\partial\bar{\theta}}\Big[-i\bar{\mathcal{F}}^{(h)} \cdot \partial^\mu \mathcal{B}^{(h)}_\mu - \frac{i}{2}\;\bar{\mathcal{F}}^{(h)} \cdot B\Big],\\
\tilde{\mathcal{L}}^{(n)}_b &= -\frac{1}{4}\;\tilde{F}^{(h)}_{\mu\nu} \cdot \tilde{F}^{\mu\nu(h)} + \mathrm{Lim}_{\theta\to 0}\;\frac{\partial}{\partial\bar{\theta}}\Big[\frac{i}{2}\bar{\mathcal{F}}^{(h)}_{(o)} \cdot (\partial^\mu A_\mu) + i\;\bar{\mathcal{F}}^{(h)}_{(o)} \cdot \partial^\mu \mathcal{B}^{(h)}_{\mu(o)}\;\Big].
\end{aligned} \tag{5.10}
$$

This result, in turn, simplifies the BRST invariance of the above Lagrangian density (4.1) and (4.4), describing the 4D 1-form non-Abelian gauge theory, as follows

$$\text{Lim}_{\theta \to 0} \frac{\partial}{\partial \bar{\theta}} \tilde{\mathcal{L}}_B^{(n)} = 0 \Rightarrow s_b \mathcal{L}_B^{(n)} = 0, \qquad \text{Lim}_{\theta \to 0} \frac{\partial}{\partial \bar{\theta}} \tilde{\mathcal{L}}_b^{(n)} = 0 \Rightarrow \tilde{s}_b \mathcal{L}_b^{(n)} = 0. \tag{5.11}$$

This is a great simplification because of the fact that the total Lagrangian remains independent of the Grassmannian variable $\bar{\theta}$. This key result captures the BRST invariance of the Lagrangian densities (4.1) and (4.4) because of the fact that $(s_b, \tilde{s}_b) \Leftrightarrow \text{Lim}_{\theta \to 0}(\partial/\partial\bar{\theta})$.

It can be readily checked that the analogues of (5.5) and (5.9) (i.e. the gauge-fixing and Faddeev-Popov ghost terms of the Lagrangian densities (4.1) and (4.4)) cannot be expressed as the derivative w.r.t. the Grassmannian variable θ. To check this, one has to exploit the super expansions (5.4) and (5.8) obtained after the application of the HC (in the context of the derivation of the off-shell as well as on-shell nilpotent BRST symmetry transformations s_b and $\tilde{s}_b$). It can be checked that the derivative w.r.t. the Grassmannian variables θ on *any* combination of the superfields of (5.4) and (5.8) does *not* lead to the derivation of the r.h.s. of the equations (5.5) and (5.9). In the language of the superfield approach to BRST formalism, this is the reason behind the non-existence of the anti-BRST symmetry transformations for the Lagrangian densities (4.1) and (4.4). This is clear because of the fact that, in the framework of the superfield approach to BRST formalism, the identification of the nilpotent anti-BRST symmetry transformation to the translational generator, on the appropriately chosen supermanifold, is: $s_{ab} \Leftrightarrow \text{Lim}_{\bar{\theta} \to 0}(\partial/\partial\theta)$.

The form of the gauge-fixing and Faddeev-Popov terms(4.11), expressed in terms of the BRST and anti-BRST symmetry transformations, can be represented in the language of the superfields (obtained after the application of HC), as

$$\frac{\partial}{\partial \bar{\theta}} \frac{\partial}{\partial \theta} \Big[\frac{i}{2} \mathcal{B}_\mu^{(h)} \cdot \mathcal{B}^{\mu(h)} + \mathcal{F}^{(h)} \cdot \bar{\mathcal{F}}^{(h)} \Big] = B \cdot (\partial_\mu A^\mu) + \frac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B}) - i \partial_\mu \bar{C} \cdot D^\mu C. \tag{5.12}$$

As a consequence of the above expression, the Lagrangian densities (4.7) (as well as (4.8)) can be presented, in terms of the superfields (obtained after the application of HC), as given below:

$$\tilde{\mathcal{L}}_B^{(1)(2)} = -\frac{1}{4} \tilde{F}^{\mu\nu(h)} \cdot \tilde{F}_{\mu\nu}^{(h)} + \frac{\partial}{\partial \bar{\theta}} \frac{\partial}{\partial \theta} \Big[\frac{i}{2} \mathcal{B}_\mu^{(h)} \cdot \mathcal{B}^{\mu(h)} + \mathcal{F}^{(h)} \cdot \bar{\mathcal{F}}^{(h)} \Big]. \tag{5.13}$$

The BRST and anti-BRST invariance of the above Lagrangian density (and that of the ordinary 4D Lagrangian densities (4.7) and (4.8)) is encoded in the following simple equations that are expressed in terms of the translational generators along the Grassmannian directions of the (4, 2)-dimensional supermanifold, namely;

$$\text{Lim}_{\theta \to 0} \frac{\partial}{\partial \bar{\theta}} \tilde{\mathcal{L}}_B^{(1)} = 0 \Rightarrow s_b \mathcal{L}_b^{(1)} = 0, \qquad \text{Lim}_{\bar{\theta} \to 0} \frac{\partial}{\partial \theta} \tilde{\mathcal{L}}_B^{(1)} = 0 \Rightarrow s_{ab} \mathcal{L}_b^{(2)} = 0. \tag{5.14}$$

This is a tremendous simplification, in the language of the superfield formalism, for the (anti-)BRST symmetry invariance of the Lagrangian densities (4.7) and (4.8) of the 4D

1-form non-Abelian gauge theory. In other words, if one is able to show the Grassmannian independence of the super Lagrangian densities of the theory, the (anti-)BRST invariance is implied automatically. In the language of the geometry on the supermanifold, the (anti-)BRST invariance of a 4D Lagrangian density is equivalent to the statement that the translation of the *super* version of the above Lagrangian density, along the Grassmannian direction(s) of the (4, 2)-dimensional supermanifold, is *zero*. Furthermore, the nilpotency and anticommutativity properties (that are associated with the (anti-)BRST symmetry transformations) are found to be captured (cf. (3.17)-(3.19)) very naturally when we consider the superfield formulation of a given gauge theory. In fact, one can derive the analogue of the equations (3.17), (3.18) and (3.19) for the non-Abelian gauge theory as well.

6 Conclusions

In our present investigation, we have concentrated mainly on the (anti-)BRST invariance(s) of the Lagrangian densities of the 4D one-form (non-)Abelian gauge theories (without any interaction with matter fields) within the framework of the superfield approach to BRST formalism. One of the decisive roles, played by the above approach, is to provide the geometrical origin for the existence of the (anti-)BRST invariances. As it turns out, we have been able to show that the Grassmannian independence of the super Lagrangian density is an indication of the (anti-)BRST invariance in the theory (cf. (3.17), (3.18), (3.19), (5.11), (5.14)). In other words, if the super Lagrangian density could be expressed as a sum of (i) a Grassmannian independent term, and (ii) a derivative w.r.t. the Grassmannian variable(s), then, the corresponding 4D Lagrangian density will automatically respect BRST and/or anti-BRST invariance. In the latter piece of the super Lagrangian density, the derivative could be *either* w.r.t. θ and $\bar{\theta}$ *or* w.r.t. both the derivatives *together*. More specifically, (i) if the derivative is w.r.t. $\bar{\theta}$, the nilpotent symmetry would correspond to BRST, (ii) if the derivative is w.r.t. θ, the nilpotent symmetry will be anti-BRST, and (iii) if both the derivatives are present together, both the nilpotent (anti-)BRST symmetries would be respected together by the 4D Lagrangian density.

For the 4D 1-form (non-)Abelian gauge theories, which are considered on the (4, 2)-dimensional supermanifold, it is the horizontality condition (HC) on the gauge superfield (i.e. 1-form super connection $\tilde{A}^{(1)}$) of the above supermanifold that plays a very important role in the derivation of the (anti-)BRST symmetry transformations. The cohomological origin for the above HC lies in the (super) exterior derivatives $(\tilde{d})d$. This point has been made very lucid and clear after the off-shell as well as on-shell nilpotent (anti-)BRST symmetry transformations (2.2), (2.4), (4.2), (4.3), (4.9) and (4.10). In fact, it is the kinetic energy term, owing its origin to the cohomological operator $d = dx^\mu \partial_\mu$, that remains invariant under the nilpotent (anti-)BRST symmetry transformations. This is precisely the reason that the celebrated HC yields, in one stroke, the (anti-)BRST symmetry transformations *together* for the appropriate fields of the theory. Furthermore, it produces *specifically* the nilpotent (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields because of the fact that the super 1-form connection $\tilde{A}^{(1)}/\tilde{A}^{(1)(n)}$ (cf. (3.1) and (5.1)) is constructed with the super vector multiplets $(\mathcal{B}_\mu, \mathcal{F}, \bar{\mathcal{F}})$ which are the generalizations of the gauge field A_μ and the (anti-)ghost fields $(\bar{C})C$ of the ordinary 4D 1-form (non-)Abelian gauge theory. As a consequence, the (anti-)BRST symmetry transformations are obtained for the 4D local fields A_μ, C and $\bar{C}$ when the strength of the HC is exploited.

One of the key observations of our present endeavour is to note that, the superfield version of the kinetic energy terms of the 1-form (non-)Abelian gauge theories, remain independent of the Grassmannian variables θ and $\bar{\theta}$ after the application of the HC (cf., e.g., (5.7)). Physically, this implies that the kinetic energy term is an (anti-)BRST (i.e. gauge) invariant quantity. On top of it, as it turns out, the remaining gauge-fixing and Faddeev-Popov ghost terms of the theory are found to be (i) a total derivative w.r.t. the Grassmannian variable θ, or (ii) a total derivative w.r.t. the Grassmannian variable $\bar{\theta}$, or (iii) a total derivative w.r.t. the Grassmannian variables θ and $\bar{\theta}$ *together*. This observation automatically implies, due to the nilpotency and anticommutativity of the (anti-)BRST symmetry transformations and their intimate connections with the translational generators on the supermanifold, that the super Lagrangian densities (3.14), (3.15) and (3.16) as well as (5.10) and (5.13) are independent of the Grassmannian variables θ and $\bar{\theta}$. These, in turn, imply the nilpotent (anti-)BRST symmetry invariance of the 4D Lagrangian densities of the 1-form (non-)Abelian gauge theories (without any interaction with the matter fields) .

It is worthwhile to point out that *geometrically* the super Lagrangian densities are equivalent to the sum of the kinetic energy term and the translations of some composite superfields (obtained after the application of the HC) along the Grassmannian directions θ and/or $\bar{\theta}$ directions of the (4, 2)-dimensional supermanifold. This observation is distinctly different from our earlier works on superfield approach to 2D 1-form (non-)Abelian gauge theories [24,25] which do correspond to the topological field theories. In fact, for the latter theories, the total super Lagrangian density turns out to be a total derivative w.r.t. the Grassmannian variables θ and $\bar{\theta}$. That is to say, even the kinetic energy term of the latter theory, is able to be expressed as the total derivative w.r.t. the Grassmannian variables θ and/or $\bar{\theta}$. This is due to the fact that the 2D 1-form (non-)Abelian gauge theories are endowed with the nilpotent, local, covariant and continuous (anti-)BRST as well as its dual (anti-)co-BRST symmetry transformations. In contrast, this is not the case with the 4D theories. Moreover, within the framework of the superfield approach to the former theories, it has been shown that $\mathrm{Lim}_{\theta \to 0}(\partial/\partial\bar{\theta})$ corresponds to BRST and co-BRST symmetry transformations. On the other hand, the translational generators $\mathrm{Lim}_{\bar{\theta} \to 0}(\partial/\partial\theta)$ corresponds to anti-BRST as well as anti-co-BRST symmetry transformations. This is precisely the reason that the Lagrangian densities of the 2D topological field theories are able to be expressed as the total derivative w.r.t. the Grassmannian variables θ and $\bar{\theta}$.

One of the central results of our present investigation is the simplicity and beauty that have been achieved (within the framework of the superfield approach to BRST formalism) for the (anti-)BRST symmetry invariance(s) of the Lagrangian densities of the 4D non-interacting 1-form (non-)Abelian gauge theories. The simple geometrical interpretation for the above invariance(s) is another key result of our present investigation. Our present work can be generalized to the case of interacting 4D 1-form (non-)Abelian gauge theories where there is an explicit coupling between the gauge field and matter fields. In fact, our earlier works [14-18] might come out quite handy in attempting the above problem. It seems to us that it is the combination of the HC and the restrictions, owing their origin to the (super) covariant derivative on the matter (super) fields, would play very decisive roles in proving the (anti-)BRST invariance of the Lagrangian densities of the above interacting theories. At present, this problem is under intensive investigation and our results would be reported in our forthcoming future publications [30].

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