LINKEDNESS AND ORDERED CYCLES IN DIGRAPHS

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ABSTRACT. Given a digraph D, let $\delta(D) := \min\{\delta^+(D), \delta^-(D)\}$ be the minimum degree of D. We show that every sufficiently large digraph D with $\delta(D) \geq n/2 + \ell - 1$ is ℓ-linked. The bound on the minimum degree is best possible and confirms a conjecture of Manoussakis [16]. We also determine the smallest minimum degree which ensures that a sufficiently large digraph D is k-ordered, i.e. that for every sequence $s_1, \dots, s_k$ of distinct vertices of D there is a directed cycle which encounters $s_1, \dots, s_k$ in this order.

1. Introduction

The *minimum degree* $\delta(D)$ of a digraph D is the minimum of its minimum outdegree $\delta^+(D)$ and its minimum indegree $\delta^-(D)$. When referring to paths and cycles in digraphs we always mean that these are directed without mentioning this explicitly. A digraph D is ℓ-linked if $|D| \geq 2\ell$ and if for every sequence $x_1, \dots, x_\ell, y_1, \dots, y_\ell$ of distinct vertices there are disjoint paths $P_1, \dots, P_\ell$ in D such that P_i joins x_i to y_i. Since this is a very strong and useful property to have in a digraph, the question of course arises how it can be forced by other properties.

In the case of (undirected) graphs, much progress has been made in this direction. In particular, linkedness is closely related to connectivity: Bollobás and Thomason [2] showed that every $22k$-connected graph is k-linked (this was recently improved to $10k$ by Thomas and Wollan [17]). However, for digraphs the situation is quite different: Thomassen [18] showed that for all k there are strongly k-connected digraphs which are not even 2-linked.

Our first result determines the minimum degree forcing a (large) digraph to be ℓ-linked, which confirms a conjecture of Manoussakis [16] for large digraphs.

Theorem 1. *Let $\ell \geq 2$. Every digraph D of order $n \geq 1600\ell^3$ which satisfies $\delta(D) \geq n/2 + \ell - 1$ is ℓ-linked.*

It is not hard to see that the bound on minimum degree in Theorem 1 is best possible (see Proposition 3). It is also easy to see that for $\ell = 1$ the correct bound is $\delta(D) \geq \lfloor n/2 \rfloor$. The cases $\ell = 2, 3$ of Theorem 1 were proved by Heydemann and Sotteau [9] and Manoussakis [16] respectively. Manoussakis [16] also determined the number of edges which force a digraph to be ℓ-linked. A discussion of these and related results can be found in the monograph by Bang-Jensen and Gutin [1].

Note that it does not make sense to ask for the minimum outdegree of a digraph D which ensures that D is ℓ-linked (or similarly, to ask for the minimum indegree). Indeed, the digraph obtained from a complete digraph A of order $n - 1$ by adding a new vertex x which sends an edge to every vertex in A has minimum outdegree $n - 2$ but is not even 1-linked.

A slightly weaker notion is that of a k-ordered digraph: a digraph D is *k-ordered* if $|D| \geq k$ and if for every sequence $s_1, \dots, s_k$ of distinct vertices of D there is a cycle which encounters $s_1, \dots, s_k$ in this order. It is not hard to see that every ℓ-linked digraph is also ℓ-ordered. Conversely, every 2ℓ-ordered digraph D is also ℓ-linked: if $x_1, \dots, x_\ell, y_1, \dots, y_\ell$ is a sequence of vertices as in the definition of ℓ-linkedness then a cycle which encounters $x_1, y_1, x_2, y_2, \dots, x_\ell, y_\ell$ in this order would yield the paths required for the linking. The next result says that as far as the minimum degree is concerned it is no harder to guarantee the 2ℓ paths forming such a cycle than to guarantee just the ℓ paths required for the linking. In particular, note that Theorem 2 immediately implies Theorem 1.

Theorem 2. *Let $k \geq 2$. Every digraph D of order $n \geq 200k^3$ which satisfies $\delta(D) \geq (n+k)/2 - 1$ is k-ordered.*

Again, the bound on the minimum degree is best possible (see Proposition 4). Moreover, it is easy to see that if $k = 1$ then the correct bound is $\delta(D) \geq n/2 - 1$. The proof of Theorem 2 yields paths between the k 'special' vertices whose length is at most 6 and it is also easy to translate the proof into an algorithm which finds these paths in polynomial time (see the remarks after the end of the proof).

Somewhat surprisingly, the minimum degree in both theorems is not quite the same as in the undirected case: Kawarabayashi, Kostochka and Yu [12] proved that the smallest minimum degree which guarantees a graph on n vertices to be ℓ-linked is $\lfloor n/2 \rfloor + \ell - 1$ for large n. (Egawa et al. [4] independently determined the smallest minimum degree which guarantees the existence of ℓ disjoint cycles containing ℓ specified independent edges, which is clearly a very similar property.) Kierstead, Sarközy and Selkow [13] proved that the smallest minimum degree which guarantees a graph on n vertices to be k-ordered is $\delta(D) \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$ for large n. So in the undirected case the '2ℓ-ordered' result does not quite imply the 'ℓ-linked' result. The proofs in [4, 12, 13] do not seem to generalize to digraphs.

2. Further work and open problems

In a sequel to this paper, we hope to apply Theorem 2 to obtain the following stronger results, which would also generalize the theorem of Ghouila-Houri [6] that any digraph D on n vertices with $\delta(D) \geq n/2$ contains a Hamilton cycle: we aim to apply Theorem 2 to show that if $k \geq 2$ and D is a sufficiently large digraph whose minimum degree is as in Theorem 2 then D is even k-ordered Hamiltonian, i.e. for every sequence $s_1, \dots, s_k$ of distinct vertices of D there is a Hamilton cycle which encounters $s_1, \dots, s_k$ in this order. One can use this to prove that the minimum degree condition in Theorem 1 already implies that the digraph D is Hamiltonian ℓ-linked, i.e. the paths linking the pairs of vertices span the entire vertex set of D. Note that this in turn would immediately imply that D is ℓ-arc ordered Hamiltonian, i.e. D has a Hamilton cycle which contains any ℓ disjoint edges in a given order. Note that in each case the examples in Section 3 show that the minimum degree condition would be best possible. Undirected versions of these statements were first obtained by Kierstead, Sarközy and Selkow [13] and Egawa et al. [4] respectively (and a common generalization of these in [3]).

For graphs, the concepts 'ℓ-linked' and 'k-ordered' were generalized to 'H-linked' by Jung [11]: a graph G is *H-linked* if G contains a subdivision of H with prescribed branch vertices (so G is k-ordered if and only if it is C_k-linked). The minimum degree which forces a graph to be H-linked for an arbitrary H was determined in [5, 14, 15, 7]. Clearly, one can ask similar questions also for digraphs.

Finally, we believe that the bound on n which we require in Theorem 2 (and thus in Theorem 1) can be reduced to one which is linear in k.

3. Notation and extremal examples

Before we discuss the examples showing that the bounds on the minimum degree in Theorems 1 and 2 are best possible, we will introduce the basic notation used throughout the paper. A digraph D is *complete* if every pair of vertices of D is joined by edges in both directions. The order $|D|$ of a digraph D is the number of its vertices. We write $N^+(x)$ for the outneighbourhood of a vertex x and $d^+(x) := |N^+(x)|$ for its outdegree. Similarly, we write $N^-(x)$ for the inneighbourhood of a vertex x and $d^-(x) := |N^-(x)|$ for its indegree. We set $d(x) := \min\{d^+(x), d^-(x)\}$. Given a set A of vertices of D, we write $N^+_A(x)$ for the set of all outneighbours of x in A. $N^-_A(x)$, $d^+_A(x)$ and $d^-_A(x)$ are defined similarly. Given two vertices x, y of a digraph D, an *x-y path in D* is a directed path which joins x to y. Given two disjoint vertex sets A and B of D, an A-B edge is an edge $\overrightarrow{ab}$ where $a \in A$ and $b \in B$.

The following proposition shows that the bound on the minimum degree in Theorem 1 cannot be reduced.

Proposition 3. *For every $\ell \geq 2$ and every $n \geq 2\ell$ there exists a digraph D on n vertices with minimum degree $\lceil n/2 \rceil + \ell - 2$ which is not ℓ-linked.*

Proof. We will distinguish the following cases.

Case 1. *n is even.*

Let D be the digraph which consists of complete digraphs A and B of order $n/2 + \ell - 1$ which have precisely $2\ell - 2$ vertices in common. To see that D is not ℓ-linked let $x_1, \dots, x_{\ell-1}, y_1, \dots, y_{\ell-1}$ denote the vertices in $A \cap B$. Pick some vertex $x_\ell \in A \setminus B$ and some vertex $y_\ell \in B \setminus A$. Then D does not contain disjoint paths between x_i and y_i for all $i = 1, \dots, \ell$. The minimum degree of D is attained by the vertices in $(A \setminus B) \cup (B \setminus A)$ and thus is as desired.

Case 2. *n is odd.*

In this case, we define D as follows. Let A and B be disjoint complete digraphs of order $\lceil n/2 \rceil - \ell - 1$. Add a complete digraph X of order $2\ell - 3$ and join all vertices in X to all vertices in $A \cup B$ with edges in both directions. Add a set $S := \{x_1, x_2, y_1, y_2\}$ of 4 new vertices such that each vertex in S is joined to each vertex in X with edges in both directions. Moreover, we add all the edges between different vertices in S except for $\overrightarrow{x_1y_1}$ and $\overrightarrow{x_2y_2}$. Finally, we connect the vertices in S to the vertices in $A \cup B$ as follows. Both x_1 and y_1 receive edges from every vertex in B and send edges to every vertex in A. Additionally, x_1 will receive an edge from every vertex in A and y_1 will send an edge to every vertex in B. Both x_2 and y_2 receive edges from every vertex in A and send edges to every vertex in B. Additionally, x_2 will receive an edge from every vertex in B and y_2 will send an edge to every vertex in A (see Figure 1).

To check that D has the required minimum degree, consider first any vertex $a \in A$. As a sends edges to 3 vertices in S and receives edges from 3 such vertices,

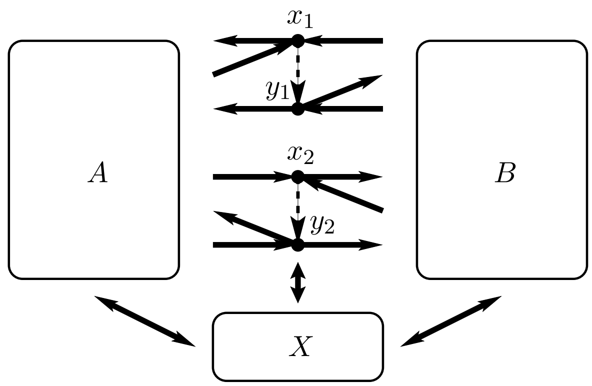

FIGURE 1. The digraph D in Case 2 of Proposition 3. The dashed arrows indicate the missing edges between x_1 and y_1 and between x_2 and y_2.

we have that $d(a) = |A| - 1 + |X| + 3 = \lceil n/2 \rceil + \ell - 2$. It follows similarly that the vertices in B have the correct degree. Thus consider any vertex $s \in S$. Then s sends edges to all vertices in A or to all vertices in B (or both) and s receives edges from all vertices in A or from all vertices in B (or both). Thus $d(s) = |A| + |X| + 2 = \lceil n/2 \rceil + \ell - 2$. It is easy to check that the vertices in X have the required degree and thus $\delta(D) = \lceil n/2 \rceil + \ell - 2$.

To see that D is not ℓ-linked, let $x, x_3, \dots, x_\ell, y_3, \dots, y_\ell$ denote the vertices in X. Then we cannot link x_i to y_i for each $i = 1, \dots, \ell$ since every x_1-y_1 path must meet $X \cup \{x_2, y_2\}$ (and thus would contain x) and the analogue is true for every x_2-y_2 path. □

We conclude this section with the examples showing that the bound on the minimum degree in Theorem 2 is best possible.

Proposition 4. *For every $k \geq 2$ and every $n \geq 2k$ there exists a digraph D on n vertices with minimum degree $\lceil (n+k)/2 \rceil - 2$ which is not k-ordered.*

Proof. We will distinguish the following cases.

Case 1. *$k \geq 3$ is odd and n is even.*

In this case, we define D as follows. Let A and B be disjoint complete digraphs of order $n/2 - k + 1$. Add a complete digraph X of order $k - 2$ and join all its vertices to all vertices in $A \cup B$ with edges in both directions. Add new vertices $s_1, \dots, s_k$ such that every s_i is joined to all vertices in X with edges in both directions. Moreover, we add all the edges $\overrightarrow{s_i s_j}$ for $j \neq i, i+1$ where $s_{k+1} := s_1$. We also add the edge $\overrightarrow{s_1 s_2}$. Finally, we connect the s_i to the vertices in $A \cup B$ as follows. Both s_1 and s_2 receive edges from every vertex in B and send edges to every vertex in B. Additionally, s_1 will send an edge to every vertex in A and s_2 will receive an edge from every vertex in A. Each of $s_3, s_5, \dots, s_k$ receives an edge from every vertex in A and sends an edge to every vertex in A. Each of $s_4, s_6, \dots, s_{k-1}$ receives an edge from every vertex in B and sends an edge to every vertex in B (see Figure 2). Let us now check that the minimum degree of the digraph D thus obtained is as required. Let $S := \{s_1, \dots, s_k\}$. Note that each

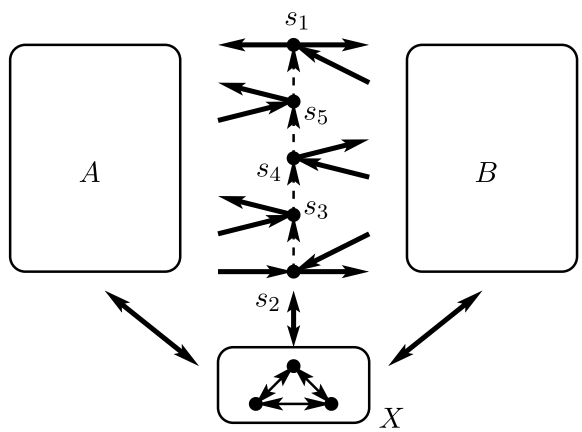

FIGURE 2. The digraph D for $k = 5$ in Case 1 of Proposition 4. The dashed arrows indicate missing edges between the vertices s_i.

vertex $v \in A \cup B$ sends edges to precisely $(k + 1)/2$ vertices in S and receives edges from precisely that many vertices. Since $|A| = |B|$, it follows that $d(v) = |A| - 1 + |X| + (k + 1)/2 = n/2 - 2 + (k + 1)/2 = \lceil (n + k)/2 \rceil - 2$. Now consider any $s_i \in S$. Then s_i receives edges from either all vertices in A or all vertices in B (or both) and s_i sends edges to either all vertices in A or all vertices in B (or both). Hence $d(s_i) \geq |A| + |X| + |S| - 2 = n/2 - 1 + k - 2 \geq \lceil (n + k)/2 \rceil - 2$. It is easy to check that the degree of the vertices in X is $> \lceil (n + k)/2 \rceil - 2$.

To see that D is not k-ordered note that every cycle in D which encounters $s_1, \dots, s_k$ in this order would use at least one vertex from X between s_i and s_{i+1} for every $i \neq 1$ (see Figure 2). But since $|X| = k - 2$ this is impossible.

Case 2. *k is even.*

Let D be the digraph which consists of a complete digraph A of order $\lceil n/2 \rceil + k/2 - 1$ and a complete digraph B of order $\lfloor n/2 \rfloor + k/2$ which has precisely $k - 1$ vertices in common with A. It is easy to check that $\delta(D) = |A| - 1 = \lceil (n + k)/2 \rceil - 2$. To see that D is not k-ordered, pick vertices $s_1, s_3, \dots, s_{k-1}$ in $A \setminus B$ and $s_2, s_4, \dots, s_k$ in $B \setminus A$. Then every cycle in D which encounters $s_1, \dots, s_k$ in this order would meet $A \cap B$ when going from s_i to s_{i+1}, i.e. it would meet $A \cap B$ k times, which is impossible.

Case 3. *$k \geq 3$ is odd and n is odd.*

This time we take D to be the digraph which consists of two complete digraphs A and B of order $(n + k)/2 - 1$ having $k - 2$ vertices in common. Then $\delta(D) = |A| - 1 = (n + k)/2 - 2$. To see that D is not k-ordered, pick vertices $s_1, s_3, \dots, s_k$ in $A \setminus B$ and $s_2, s_4, \dots, s_{k-1}$ in $B \setminus A$. $\square$

Note that in the proof of Proposition 4 we could have omitted the (easy) case when k is even as Proposition 3 already gives a digraph of the required minimum degree which is not $k/2$-linked and thus not k-ordered.

4. Proof of Theorem 2

We first prove Theorem 2 for the case when $k = 2$. So suppose that D is a digraph of minimum degree at least $\lceil n/2 \rceil$. Let s_1 and s_2 be the vertices which our cycle has to encounter. If $\overrightarrow{s_1 s_2}$ is not an edge then $s_1, s_2 \notin N^+(s_1) \cup N^-(s_2)$ and so $|N^+(s_1) \cap N^-(s_2)| \geq 2\delta(D) - (n-2) \geq 2$. Similarly, if $\overrightarrow{s_2 s_1}$ is not an edge then $|N^-(s_1) \cap N^+(s_2)| \geq 2$. Altogether this shows that there is a cycle of length at most 4 which contains both s_1 and s_2.

Thus we may assume that $k \geq 3$ and that D is a digraph of minimum degree at least $\lceil (n+k)/2 \rceil - 1$. Let $S := (s_1, \dots, s_k)$ be the given sequence of vertices of D which our cycle has to encounter. We will call these vertices *special* and will sometimes also use S for the set of these vertices. We set $s_{k+1} := s_1$. Given a set $I \subseteq [k]$ and a family $T := (t_i)_{i \in I}$ of positive integers, an (S, I, T)*-system* is a family $(\mathcal{P}_i)_{i \in I}$ where each $\mathcal{P}_i$ is a set of t_i paths joining s_i to s_{i+1} and each path in $\mathcal{P}_i$ has length at most 6 and is internally disjoint from S, from all other paths in $\mathcal{P}_i$ and from the paths in all the other $\mathcal{P}_j$. An (S, I)*-system* is an (S, I, T)*-system* where $t_i = 1$ for all $i \in I$. Thus to prove Theorem 2 we have to show that there exists an $(S, [k])$-system.

Let I be the set of all those indices $i \in [k]$ for which D does not contain at least $6k$ internally disjoint s_i-s_{i+1} paths of length at most 6.

Claim 1. *It suffices to show that D contains an (S, I)-system.*

Indeed, suppose that $(\mathcal{P}_i)_{i \in I}$ is an (S, I)-system in D. So each $\mathcal{P}_i$ contains precisely one path P_i. We will show that for every $i \in [k] \setminus I$ we can find an s_i-s_{i+1} path P_i of length at most 6 which meets S only in s_i and s_{i+1} such that all the paths $P_1, \dots, P_k$ are internally disjoint. We will choose such a path P_i for every $i \in [k] \setminus I$ in turn. Suppose that next we want to find P_j. Recall that since $j \in [k] \setminus I$ the digraph D contains a set $\mathcal{P}$ of at least $6k$ internally disjoint s_j-s_{j+1} paths of length at most 6. Since at most $5(k-1) + k < 6k$ vertices of D lie in S or in the interior of some of the other paths P_i, one of the paths in $\mathcal{P}$ must be internally disjoint from S and all the other paths P_i, and so we can take this path to be P_j. This proves Claim 1.

In order to prove the existence of an (S, I)-system, choose an (S, J, T)-system $(\mathcal{P}_j)_{j \in J}$ in D such that $J \subseteq I$ is as large as possible and subject to this $\sum_{j \in J} t_j$ is maximal. Note that $t_j < 6k$ for all $j \in J$ since $J \subseteq I$. Assume that $|J| < |I|$. By relabelling the special vertices, we may assume that $k \in I \setminus J$. So we would like to extend $(\mathcal{P}_j)_{j \in J}$ by a suitable s_k-s_1 path. Let X' be the set of all those vertices which lie in the interior of some path belonging to $(\mathcal{P}_j)_{j \in J}$. Note that

$$|S \cup X'| < 6k \cdot 5(k-1) + |S| < 30k^2 =: k_0.$$

Let $A := N^+(s_k) \setminus (S \cup X')$ and $B := N^-(s_1) \setminus (S \cup X')$. Then

$$|A|, |B| \geq \delta(D) - |S \cup X'| \geq n/2 - k_0. \tag{1}$$

Moreover, $A \cap B = \emptyset$ as otherwise we could extend our (S, J, T)-system $(\mathcal{P}_j)_{j \in J}$ by adding the path $P_k := s_k x s_1$ where $x \in A \cap B$, a contradiction to the choice of $(\mathcal{P}_j)_{j \in J}$. In particular, this shows that the set X'' of all vertices outside $A \cup B \cup S \cup X'$ has size at most $2k_0$ and thus, setting $Y := S \cup X' \cup X''$, we have that

$$|Y| \leq 3k_0.$$

Note that D does not contain an edge $\overrightarrow{ab}$ with $a \in A$ and $b \in B$. Indeed, otherwise we could extend $(\mathcal{P}_j)_{j\in J}$ by adding the path $P_k := s_k abs_1$. We will often use the following claim.

Claim 2. *Let $a \in A$ and let $A' \subseteq A$ be a set of size at least k_0. Then $N^+(a)\cap A' \neq \emptyset$. Similarly, if $b \in B$ and $B' \subseteq B$ is a set of size at least k_0 then $N^-(b) \cap B' \neq \emptyset$.*

Suppose that $N^+(a) \cap A' = \emptyset$. Then (1) together with the fact that D does not contain an A-B edge implies that $d^+(a) \leq n - |B| - k_0 \leq n/2$, a contradiction. The proof of the second part of the claim is similar.

We say that a special vertex s_i has *out-type* A if s_i sends at least k_0 edges to A. Similarly we define when s_i has *out-type* B, *in-type* A and *in-type* B. As $|Y|+2k_0 \leq 5k_0 \leq \delta(G)$, it follows that each s_i has out-type A or out-type B (or both) and in-type A or in-type B (or both). Note that s_1 has in-type B but not in-type A whereas s_k has out-type A but not out-type B.

Claim 3. *Let $j \in J$. If s_j has out-type A then s_{j+1} has in-type B but not in-type A. Similarly, if s_j has out-type B then s_{j+1} has in-type A but not in-type B.*

Suppose that s_j has out-type A and s_{j+1} has in-type A. Let $a \in N^+_A(s_j)$. Claim 2 implies that a sends an edge to one of the at least k_0 vertices in $N^-_A(s_{j+1})$. Let $a' \in N^-_A(s_{j+1})$ be such a neighbour of a. Then we could extend our (S, J, T)-system by adding the path $s_j aa' s_{j+1}$, a contradiction. The proof of the second part of Claim 3 is similar.

Claim 4. *No vertex in B sends an edge to A.*

Suppose that $\overrightarrow{b^*a^*}$ is an edge of D, where $a^* \in A$ and $b^* \in B$. Given vertices $a \in A$ and $b \in B$, put $N_{ab} := N^+(a) \cap N^-(b)$. Note that $N_{ab} \subseteq Y$ and $a, b \notin N^+(a) \cup N^-(b)$ as D does not contain an A-B edge. Thus

$$|N_{ab}| \geq 2\delta(D) - (n-2) = \left(2\left\lceil\frac{n+k}{2}\right\rceil - 2\right) - (n-2) \geq k. \tag{2}$$

Let us now show that no special vertex s_i with $i \in J$ has out-type B. So suppose $i \in J$ and s_i has out-type B. Then Claim 3 implies that s_{i+1} has in-type A. By Claim 2 some of the at least k_0 vertices in $N^-_A(s_{i+1})$ receives an edge from a^*. Let a' be such a vertex. Similarly, some of the vertices in $N^+_B(s_i)$ sends an edge to b^*. Let b' be such a vertex. Then we could extend our (S, J, T)-system by adding the path $s_i b' b^* a^* a' s_{i+1}$, a contradiction. This shows that whenever s_i is a special vertex of out-type B then $i \notin J$. Let Q denote the set of such vertices s_i. Note that $s_k \notin Q$ as s_k does not have out-type B. Thus each special vertex in Q forbids one index in J. Altogether this shows that

$$|J| \leq k - 1 - |Q|. \tag{3}$$

Let S_A be the set of all those special vertices s_i with $1 \leq N^-_A(s_i) < k_0$. Let S_B be the set of all those special vertices s_i with $1 \leq N^+_B(s_i) < k_0$. Let A^* be the set of all those vertices in A which do not send an edge to some vertex in S_A. Then $|A^*| > |A| - kk_0$. Similarly, let B^* be the set of all those vertices in B which do not receive an edge from some vertex in S_B. Then $|B^*| > |B| - kk_0$.

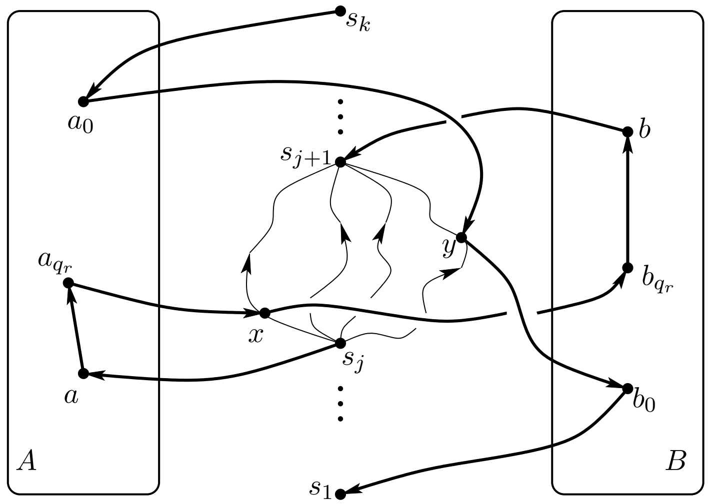

FIGURE 3. Modifying our (S,J,T)-system in the proof of Claim 4.

Consider any pair a,b with $a\in A^*$ and $b\in B^*$. As each special vertex in N_{ab} belongs to Q, it follows that

$$|N_{ab}\setminus S|\ge |N_{ab}|-|Q|\overset{(2)}{\ge} k-|Q|\overset{(3)}{>}|J|. \tag{4}$$

Suppose first that $J\neq\emptyset$. Given $j\in J$, let X''_j be the union of X'' with the set of all vertices lying in the interior of paths in $\mathcal{P}_j$. As $N_{ab}\subseteq Y$ there must be an index $j_{ab}\in J$ such that N_{ab} contains at least two vertices in $X''_{j_{ab}}$. Note that $|A^*|,|B^*|>2k_0k$. Thus there are $2k_0+1$ disjoint pairs a,b for which this index j_{ab} must be the same. Let a_q,b_q ($q=0,\dots,2k_0$) denote these pairs and let $j\in J$ denote the common index.

Note that s_j has out-type A since we have seen before that no special vertex s_i with $i\in J$ has out-type B. Claim 3 now implies that s_{j+1} has in-type B. Pick vertices $a\in N^+_A(s_j)$ and $b\in N^-_B(s_{j+1})$ such that $a\neq a_0$ and $b\neq b_0$. Claim 2 implies that there are indices $q_1,\dots,q_{k_0}$ such that a sends an edge to each a_{q_r}. Apply Claim 2 again to find an index $r\le k_0$ such that b receives an edge from b_{q_r}. Let $x\in N_{a_{q_r}b_{q_r}}$ and $y\in N_{a_0b_0}$ be distinct vertices such that $x,y\in X''_j$. We can now modify our (S,J,T)-system to obtain an $(S,J\cup\{k\},T')$-system in D by replacing $\mathcal{P}_j$ with the single path $s_jaa_{q_r}xb_{q_r}bs_{j+1}$ and adding the s_k-s_1 path $s_ka_0yb_0s_1$ (see Figure 3). If $J=\emptyset$ then we just add the s_k-s_1 path (which is still guaranteed by (4)). In both cases this contradicts the choice of our (S,J,T)-system and completes the proof of Claim 4.

Claim 5. *Let $a\in A$ and let $A'\subseteq A$ be a set of size at least k_0. Then $N^-(a)\cap A'\neq\emptyset$. Similarly, if $b\in B$ and $B'\subseteq B$ is a set of size at least k_0 then $N^+(b)\cap B'\neq\emptyset$.*

Using Claim 4, this can be shown similarly as Claim 2.

Let S^+_A be the set of all those special vertices which send an edge to A and let S^-_A be the set of all those special vertices which receive an edge from A. Define S^+_B and S^-_B similarly. Note that these sets are not disjoint. The proof of the next claim is similar to that of Claim 3. (To prove the second and third part of Claim 6 we use Claim 5 instead of Claim 2.)

Claim 6. *If $j \in J$ and $s_j \in S_A^+$ then s_{j+1} cannot have in-type A. If $j-1 \in J$ and $s_j \in S_A^-$ then s_{j-1} cannot have out-type A. If $j \in J$ and $s_j \in S_B^+$ then s_{j+1} cannot have in-type B. Finally, if $j-1 \in J$ and $s_j \in S_B^-$ then s_{j-1} cannot have out-type B.*

Let $q_A^+ := |S_A^+|$ and define q_A^-, q_B^+ and q_B^- similarly. Let $\bar{Y} := V(D) \setminus Y = A \cup B$ and $X := X' \cup X'' = Y \setminus S$. Consider any pair a, b with $a \in A$ and $b \in B$. Then

$$\left\lceil \frac{n+k}{2} \right\rceil - 1 \le |N^+(a)| \le q_A^- + |N_X^+(a)| + |N_{\bar{Y}}^+(a)| \tag{5}$$

and

$$\left\lceil \frac{n+k}{2} \right\rceil - 1 \le |N^-(b)| \le q_B^+ + |N_X^-(b)| + |N_{\bar{Y}}^-(b)|. \tag{6}$$

Since $N_{\bar{Y}}^+(a) \cap N_{\bar{Y}}^-(b) = \emptyset$ (as D does not contain an A-B edge) and $a, b \notin N_{\bar{Y}}^+(a) \cup N_{\bar{Y}}^-(b)$ we have

$$|N_{\bar{Y}}^+(a)| + |N_{\bar{Y}}^-(b)| \le |\bar{Y}| - 2 = n - |X| - k - 2.$$

Adding (5) and (6) together now gives

$$2\left\lceil \frac{n+k}{2} \right\rceil - 2 \le q_A^- + q_B^+ + |N_X^+(a)| + |N_X^-(b)| + n - |X| - k - 2.$$

Hence

$$|N_X^+(a) \cap N_X^-(b)| \ge |N_X^+(a)| + |N_X^-(b)| - |X| \ge 2\left\lceil \frac{n+k}{2} \right\rceil - n + k - q_A^- - q_B^+ \ge 2k - q_A^- - q_B^+. \tag{7}$$

Similarly, using Claim 4, one can show that

$$|N_X^-(a) \cap N_X^+(b)| \ge 2k - q_A^+ - q_B^-. \tag{8}$$

Consider any $j \in J$. Recall that by Claim 3 we have that either s_j has out-type A and s_{j+1} has in-type B or s_j has out-type B and s_{j+1} has in-type A. Let J_{AB} denote the set of all those indices $j \in J$ for which the former holds and let J_{BA} be the set of all those $j \in J$ for which the latter holds. Our next aim is to estimate $j_{AB} := |J_{AB}|$ and $j_{BA} := |J_{BA}|$. Note that Claim 6 implies that if $s_j \in S_B^+$ then $j \notin J_{AB}$. As $s_k \notin S_B^+$ and $k \notin J$, this shows that

$$j_{AB} \le k - 1 - |S_B^+ \setminus \{s_k\}| = k - 1 - |S_B^+| = k - 1 - q_B^+.$$

Also, if $s_j \in S_A^-$ then $j - 1 \notin J_{AB}$ by Claim 6. As $s_1 \notin S_A^-$, this shows that

$$j_{AB} \le k - 1 - |S_A^- \setminus \{s_1\}| = k - 1 - |S_A^-| = k - 1 - q_A^-.$$

Adding these two inequalites gives

$$j_{AB} \le k - 1 - \frac{q_A^- + q_B^+}{2}. \tag{9}$$

In order to give an upper bound for j_{BA}, note that if $s_j \in S_A^+$ then $j \notin J_{BA}$ by Claim 6. Thus

$$j_{BA} \le k - 1 - |S_A^+ \setminus \{s_k\}| \le k - q_A^+.$$

Also, if $s_j \in S_B^-$ then $j-1 \notin J_{BA}$ by Claim 6. Thus

$$j_{BA} \le k - 1 - |S_B^- \setminus \{s_1\}| \le k - q_B^-.$$

Adding these two inequalites gives

$$j_{BA} \le k - \frac{q_A^+ + q_B^-}{2}. \tag{10}$$

Our next aim is to show that D contains a $(S, J \cup \{k\})$-system. This will complete the proof of Theorem 2 since it contradicts the choice of our (S, J, T)-system. Pick distinct vertices $a_0 \in A$, $a_j \in N_A^+(s_j)$ for all $j \in J_{AB}$, $a'_j \in N_A^-(s_{j+1})$ for all $j \in J_{BA}$, $b_0 \in B$, $b_j \in N_B^-(s_{j+1})$ for all $j \in J_{AB}$ and $b'_j \in N_B^+(s_j)$ for all $j \in J_{BA}$. Choose a vertex $x_0 \in N_X^+(a_0) \cap N_X^-(b_0)$ and link s_k to s_1 by the path $Q_k := s_k a_0 x_0 b_0 s_1$. (This can be done since the right hand side of (7) is at least 2.) To find the other paths, we distinguish two cases.

Case 1. $j_{BA} \le j_{AB}$

For all $j \in J_{BA}$ we pick a vertex $x_j \in N_X^-(a'_j) \cap N_X^+(b'_j)$ such that all these x_j are pairwise distinct and distinct from x_0. Inequalities (8) and (10) together imply that this can be done. Inequality (7) together with the fact that

$$2k - q_A^- - q_B^+ - 1 - j_{BA} \overset{(9)}{\ge} 2j_{AB} + 1 - j_{BA} \ge j_{AB} + 1,$$

implies that for all $j \in J_{AB}$ we can now pick a vertex $x_j \in N_X^+(a_j) \cap N_X^-(b_j)$ such that x_0 and all the x_j ($j \in J$) are pairwise distinct. If $j \in J_{AB}$ we link s_j to s_{j+1} by the path $Q_j := s_j a_j x_j b_j s_{j+1}$. If $j \in J_{BA}$ we link s_j to s_{j+1} by the path $Q_j := s_j b'_j x_j a'_j s_{j+1}$. The paths Q_j ($j \in J$) and Q_k are internally disjoint and have length 4, so they form an $(S, J \cup \{k\})$-system, as required.

Case 2. $j_{BA} > j_{AB}$

We proceed similiarly as in Case 1, but this time we choose the vertices $x_j \in N_X^+(a_j) \cap N_X^-(b_j)$ for all $j \in J_{AB}$ first. As

$$2k - q_A^+ - q_B^- - 1 - j_{AB} \overset{(10)}{\ge} 2j_{BA} - 1 - j_{AB} > j_{BA} - 1,$$

inequality (8) implies that we can then pick the vertices $x_j \in N_X^-(a'_j) \cap N_X^+(b'_j)$ for all $j \in J_{BA}$. The paths Q_j ($j \in J$) and Q_k are then defined as before. This completes the proof of Theorem 2.

Note that throughout the proof, the paths we constructed always had length at most 6 (the only case where they had length exactly 6 was in the proof of Claim 4). This means that the proof can easily be translated into polynomial algorithm so that the exponent of the running time does not depend on k: We simply start with any (S, J, T)-system with $J \subseteq I$. Now we go through the steps of the proof and find a 'better' (S, J', T')-system with $J' \subseteq I$. Claim 1 implies that for fixed k we only need to do this a bounded number of times. Since the paths we need have length at most 6 and there are only a bounded number of cases to consider in the proof, it is clear that one can find the better system in polynomial time with exponent independent of k. Altogether this means that the problem of finding a cycle encountering a given sequence of k vertices is fixed parameter tractable for digraphs whose minimum degree satisfies the conditon in Theorem 2 (where k is the fixed parameter). The same applies to the problem of linking ℓ given pairs of vertices. In general, even the problem of deciding whether a digraph is 2-linked is already NP-complete [10]. For a survey on fixed parameter tractable digraph problems, see [8].

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