

Singularity Resolution in Isotropic Loop Quantum Cosmology: Recent Developments

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Abstract

Since the past Iagrg meeting in December 2004, new developments in loop quantum cosmology have taken place, especially with regards to the resolution of the Big Bang singularity in the isotropic models. The singularity resolution issue has been discussed in terms of physical quantities (expectation values of Dirac observables) and there is also an “improved” quantization of the Hamiltonian constraint. These developments are briefly discussed.

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I. COSMOLOGY, QUANTUM COSMOLOGY, LOOP QUANTUM COSMOLOGY

Our current understanding of large scale properties of the universe is summarised by the so called Λ -CDM Big Bang model – homogeneous and isotropic, spatially flat space-time geometry with a positive cosmological constant and cold dark matter. Impressive as it is, the model is based on an Einsteinian description of space-time geometry which has the Big Bang singularity. The existence of cosmological singularities is in fact much more general. There are homogeneous but anisotropic solution space-times which are singular and even in the inhomogeneous context there is a general solution which is singular [1]. The singularity theorems give a very general argument for the existence of initial singularity for an everywhere expanding universe with normal matter content, the singularity being characterized as incompleteness of causal geodesic in the past. Secondly, in conjunction with an inflationary scenario, one imagines the origin of the smaller scale structure to be attributed to quantum mechanical fluctuations of matter and geometry. On account of both the features, a role for the quantum nature of matter *and* geometry is indicated.

Quantum mechanical models for cosmological context were in fact constructed, albeit formally. For the homogeneous and isotropic sector, the geometry is described by just the scale factor and the extrinsic curvature of the homogeneous spatial slices. In the gravitational sector, a quantum mechanical wave function is a function of the scale factor i.e. a function on the (mini-) superspace of gravity. The scale factor being positive, the minisuperspace has a boundary and wave functions need to satisfy a suitable boundary condition. Furthermore, the singularity was not resolved in that the Wheeler-De Witt equation (or the Hamiltonian constraint) which is a differential equation with respect to the scale factor, had singular coefficient due to the matter density diverging near the boundary. Thus quantization per se does not necessarily give a satisfactory replacement of the Big Bang singularity.

Meanwhile, over the past 20 years, a background independent, non-perturbative quantum theory of gravity is being constructed starting from a (gauge-) connection formulation of classical general relativity [2]. The background independence provided strong constraints on the construction of the quantum theory already at the kinematical level (i.e. before imposition of the constraints) and in particular revealed a discrete and non-commutative nature of quantum (three dimensional Riemannian) geometry. The full theory is still quite unwieldy. Martin Bojowald took to step of restricting to homogeneous geometries and quantizing such

models in a *loopy way*. Being inherited from the connection formulation, the geometry is described in terms of densitized triad which for the homogeneous and isotropic context is described by $p \sim \text{sgn}(p)a^2$ which can also take negative values (encoding the orientation of the triad). This means that the classical singularity (at $p = 0$) now lies in the interior of the superspace. Classically, the singularity prevents any relation between the two regions of positive and negative values of p . Quantum mechanically however, the wave functions in these two regions, could be related. One question that becomes relevant in a quantum theory is that if a wave function, specified for one orientation and satisfying the Hamiltonian constraint, can be unambiguously extended to the other orientation while continuing to satisfy the Hamiltonian constraint. Second main implication of loop quantization is the necessity of using holonomies – exponentials of connection variable c – as well defined operators. This makes the Hamiltonian constraint a difference equation on the one hand and also requires an indirect definition for inverse triad (and inverse volume) operators entering in the definition of the matter Hamiltonian or densities and pressures. Quite interestingly, the Hamiltonian constraint equation turns out to be non-singular (i.e. deterministic) and in the effective classical approximation suggests interesting phenomenological implication quite naturally. These two features in fact made LQC an attractive field.

We will briefly summarise the results prior to 2005 and then turn to more recent developments. An extensive review of LQC is available in [3]. For simplicity and definiteness, we will focus on the spatially flat isotropic models.

II. SUMMARY OF PRE 2005 LQC

Classical model: Using coordinates adapted to the spatially homogeneous slicing of the space-time, the metric and the extrinsic curvature are given by,

$$ds^2 := -dt^2 + a^2(t) \{ (dr^2 + r^2 d\Omega^2) \} . \quad (1)$$

Starting from the usual Einstein-Hilbert action and scalar matter for definiteness, one can get to the Hamiltonian as,

$$\begin{aligned} S &:= \int dt \int_{\text{cell}} dx^3 \sqrt{|det g_{\mu\nu}|} \left\{ \frac{R(g)}{16\pi G} + \frac{1}{2} \dot{\phi}^2 - V(\phi) \right\} \\ &= V_0 \int dt \left\{ \frac{3}{8\pi G} (-a\dot{a}^2) + \frac{1}{2} a^3 \dot{\phi}^2 - V(\phi) a^3 \right\} \end{aligned} \quad (2)$$

$$\begin{aligned}
p_a &= -\frac{3V_0}{4\pi G}a\dot{a} \quad , \quad p_\phi = V_0a^3\dot{\phi} \quad , \quad V_0 := \int_{\text{cell}} d^3x ; \\
H(a, p_a, \phi, p_\phi) &= H_{\text{grav}} + H_{\text{matter}} \\
&= \left[-\frac{2\pi G}{3} \frac{p_a^2}{V_0a} \right] + \left[\frac{1}{2} \frac{p_\phi^2}{a^3V_0} + a^3V_0V(\phi) \right] \\
&= \left(\frac{3V_0a^3}{8\pi G} \right) \left[-\frac{\dot{a}^2}{a^2} + \left(\frac{8\pi G}{3} \right) \left(\frac{H_{\text{matter}}}{V_0a^3} \right) \right]
\end{aligned} \tag{3}$$

Thus, $H = 0 \leftrightarrow$ Friedmann Equation. For the spatially flat model, one has to choose a fiducial cell whose fiducial volume is denoted by V_0 .

In the connection formulation, instead of the metric one uses the densitized triad i.e. instead of the scale factor a one has $\tilde{p}, |\tilde{p}| := a^2/4$ while the connection variable is related to the extrinsic curvature as: $\tilde{c} := \gamma\dot{a}/2$ (the spin connections is absent for the flat model). Their Poisson bracket is given by $\{\tilde{c}, \tilde{p}\} = (8\pi G\gamma)/(3V_0)$. The arbitrary fiducial volume can be absorbed away by defining $c := V_0^{1/3}\tilde{c}$, $p := V_0^{2/3}\tilde{p}$. Here, γ is the Barbero-Immirzi parameter which is dimensionless and is determined from the Black hole entropy computations to be approximately 0.23 [4]. From now on we put $8\pi G := \kappa$. The classical Hamiltonian is then given by,

$$H = \left[-\frac{3}{\kappa} \left(\gamma^{-2}c^2\sqrt{|p|} \right) \right] + \left[\frac{1}{2}|p|^{-3/2}p_\phi^2 + |p|^{3/2}V(\phi) \right] . \tag{5}$$

For future comparison, we now take the potential for the scalar field, $V(\phi)$ to be zero as well.

One can obtain the Hamilton's equations of motion and solve them easily. On the constrained surface ($H = 0$), eliminating c in favour of p and p_ϕ , one has,

$$\begin{aligned}
c &= \pm \gamma \sqrt{\frac{\kappa}{6}} \frac{|p_\phi|}{|p|} \quad , \quad \dot{p} = \pm \sqrt{\frac{\kappa}{6}} |p_\phi| |p|^{-1/2} . \\
\dot{\phi} &= p_\phi |p|^{-3/2} \quad , \quad \dot{p}_\phi = 0 ,
\end{aligned} \tag{6}$$

$$\frac{dp}{d\phi} = \pm \sqrt{\frac{2\kappa}{3}} |p| \Rightarrow p(\phi) = p_* e^{\pm \sqrt{\frac{2\kappa}{3}}(\phi - \phi_*)} \tag{7}$$

Since ϕ is a monotonic function of the synchronous time t , it can be taken as a new “time” variable. The solution is determined by $p(\phi)$ which is (i) independent of the constant p_ϕ and (ii) passes through $p = 0$ as $\phi \rightarrow \pm\infty$ (expanding/contracting solutions). It is immediate that, along these curves, $p(\phi)$, the energy density and the extrinsic curvature diverge as $p \rightarrow 0$. Furthermore, the divergence of the density implies that $\phi(t)$ is *incomplete* i.e. t

ranges over a semi-infinite interval as ϕ ranges over the full real line¹. Thus a singularity is signalled by a solution $p(\phi)$ passing through $p = 0$ in *finite* synchronous time (or equivalently by the density diverging somewhere along the solution). A natural way to ensure that all solutions are non-singular is to ensure that either of the two terms in the Hamiltonian constraint are *bounded*. Question is: *does a quantum theory replace the Big Bang singularity by something non-singular?*

There are at least two ways to explore this question. One can imagine computing corrections to the Hamiltonian constraint such that individual terms in the effective constraint are bounded. Alternatively and more satisfactorily, one should be able to define suitable observables whose expectation values will generate the analogue of $p(\phi)$ curves along which physical quantities such as energy density, remain bounded. The former method was used pre-2005 because it could be used for more general models (non-zero potential, anisotropy etc). The latter has been elaborated in 2006, for the special case of vanishing potential. Both methods imply that classical singularity is resolved in LQC but not in Wheeler-De Witt quantum cosmology. We will first discuss the issue in terms of effective Hamiltonian because it is easier and then discuss it in terms of the expectation values.

In the standard Schrodinger quantization, one can introduce wave functions of p, ϕ and quantize the Hamiltonian operator by $c \rightarrow i\hbar\kappa\gamma/3\partial_p$, $p_\phi \rightarrow -i\hbar\partial_\phi$, in equation (5). With a choice of operator ordering, $\hat{H}\Psi(p, \phi) = 0$ leads to the Wheeler-De Witt partial differential equation which has singular coefficients.

The background independent quantization of Loop Quantum Gravity however suggest a different quantization of the isotropic model. One should look for a Hilbert space on which only exponentials of c (holonomies of the connection) are well defined operators and not \hat{c} . Such a Hilbert space consists of almost periodic functions of c which implies that the triad operator has every real number as a proper eigenvalue: $\hat{p}|\mu\rangle := \frac{1}{6}\gamma\ell_P^2\mu|\mu\rangle, \forall \mu \in \mathbb{R}$, $\ell_P^2 := \kappa\hbar$. This has a major implication: *inverses of positive powers of triad operators do not exist* [5]. These have to be defined by using alternative classical expressions and promoting them to quantum operators. This can be done with at least one parameter worth of freedom, eg.

$$|p|^{-1} = \left[\frac{3}{8\pi G\gamma l} \{c, |p|^l\} \right]^{1/(1-l)}, \quad l \in (0, 1). \quad (8)$$

¹ For the FRW metric, integral curves of ∂_t are time-like geodesics and hence incompleteness with respect to t is synonymous with geodesic incompleteness.

Only positive powers of $|p|$ appear now. However, this still cannot be used for quantization since there is no \hat{c} operator. One must use holonomies: $h_j(c) := e^{\mu_0 c \Lambda^i \tau_i}$, where τ_i are anti-hermitian generators of $SU(2)$ in the j^{th} representation satisfying $\text{Tr}_j(\tau_i \tau_j) = -\frac{1}{3}j(j+1)(2j+1)\delta_{ij}$, Λ^i is a unit vector specifying a direction in the Lie algebra of $SU(2)$ and μ_0 is the coordinate length of the loop used in defining the holonomy. Using the holonomies,

$$|p|^{-1} = (8\pi G \mu_0 \gamma l)^{\frac{1}{1-l}} \left[\frac{3}{j(j+1)(2j+1)} \text{Tr}_j \Lambda \cdot \tau h_j \{h_j^{-1}, |p|^l\} \right]^{\frac{1}{1-l}}, \quad (9)$$

which can be promoted to an operator. Two parameters, $\mu_0 \in \mathbb{R}$ and $j \in \mathbb{N}/2$, have crept in and we have a three parameter family of inverse triad operators. The definitions are:

$$\begin{aligned} \widehat{|p|_{(j)}^{-1}} |\mu\rangle &= \left(\frac{2j\mu_0}{6} \gamma \ell_P^2 \right)^{-1} (F_l(q))^{\frac{1}{1-l}} |\mu\rangle, \quad q := \frac{\mu}{2\mu_0 j} := \frac{p}{2jp_0}, \\ F_l(q) &:= \frac{3}{2l} \left[\frac{1}{l+2} \{ (q+1)^{l+2} - |q-1|^{l+2} \} \right. \\ &\quad \left. - \frac{1}{l+1} q \{ (q+1)^{l+1} - \text{sgn}(q-1) |q-1|^{l+1} \} \right] \end{aligned} \quad (10)$$

$$\begin{aligned} F_l(q \gg 1) &\approx [q^{-1}]^{1-l}, \\ F_l(q \approx 0) &\approx \left[\frac{3q}{l+1} \right]. \end{aligned} \quad (11)$$

All these operators obviously commute with \hat{p} and their eigenvalues are bounded above. This implies that the matter densities (and also intrinsic curvatures for more general homogeneous models), remain bounded at the classically singular region. Most of the phenomenological novelties are consequences of this particular feature predominantly anchored in the matter sector. We have thus two scales: $p_0 := \frac{1}{6}\mu_0 \ell_P^2$ and $2jp_0 := \frac{1}{6}\mu_0 (2j) \ell_P^2$. The regime $|p| \ll p_0$ is termed the *deep quantum regime*, $p \gg 2jp_0$ is termed the *classical regime* and $p_0 \lesssim |p| \lesssim 2jp_0$ is termed the *semiclassical regime*. The modifications due to the inverse triad defined above are strong in the semiclassical and the deep quantum regimes. For $j = 1/2$ the semiclassical regime is absent. Note that such scales are not available for the Schrodinger quantization.

The necessity of using holonomies also imparts a non-trivial structure for the gravitational Hamiltonian. The expression obtained is:

$$H_{\text{grav}} = -\frac{4}{8\pi G \gamma^3 \mu_0^3} \sum_{ijk} \epsilon^{ijk} \text{Tr} (h_i h_j h_i^{-1} h_j^{-1} h_k \{h_k^{-1}, V\}) \quad (12)$$

In the above, we have used $j = 1/2$ representation for the holonomies and V denotes the volume function. In the limit $\mu_0 \rightarrow 0$ one gets back the classical expression.

While promoting this expression to operators, there is a choice of factor ordering involved and many are possible. We will present two choices of ordering: the *non-symmetric* one which keeps the holonomies on the left as used in the existing choice for the full theory, and the particular *symmetric* one used in [6].

$$\hat{H}_{\text{grav}}^{\text{non-sym}} = \frac{24i}{\gamma^3 \mu_0^3 \ell_P^2} \sin^2 \mu_0 c \left(\sin \frac{\mu_0 c}{2} \hat{V} \cos \frac{\mu_0 c}{2} - \cos \frac{\mu_0 c}{2} \hat{V} \sin \frac{\mu_0 c}{2} \right) \quad (13)$$

$$\hat{H}_{\text{grav}}^{\text{sym}} = \frac{24i(\text{sgn}(p))}{\gamma^3 \mu_0^3 \ell_P^2} \sin \mu_0 c \left(\sin \frac{\mu_0 c}{2} \hat{V} \cos \frac{\mu_0 c}{2} - \cos \frac{\mu_0 c}{2} \hat{V} \sin \frac{\mu_0 c}{2} \right) \sin \mu_0 c \quad (14)$$

At the quantum level, μ_0 cannot be taken to zero since \hat{c} operator does not exist. The action of the Hamiltonian operators on $|\mu\rangle$ is obtained as,

$$\hat{H}_{\text{grav}}^{\text{non-sym}} |\mu\rangle = \frac{3}{\mu_0^3 \gamma^3 \ell_P^2} (V_{\mu+\mu_0} - V_{\mu-\mu_0}) (|\mu + 4\mu_0\rangle - 2|\mu\rangle + |\mu - 4\mu_0\rangle) \quad (15)$$

$$\begin{aligned} \hat{H}_{\text{grav}}^{\text{sym}} |\mu\rangle &= \frac{3}{\mu_0^3 \gamma^3 \ell_P^2} [|V_{\mu+3\mu_0} - V_{\mu+\mu_0}| |\mu + 4\mu_0\rangle + |V_{\mu-\mu_0} - V_{\mu-3\mu_0}| |\mu - 4\mu_0\rangle \\ &\quad - \{|V_{\mu+3\mu_0} - V_{\mu+\mu_0}| + |V_{\mu-\mu_0} - V_{\mu-3\mu_0}|\} |\mu\rangle] \end{aligned} \quad (16)$$

where $V_\mu := (\frac{1}{6}\gamma\ell_P^2|\mu|)^{3/2}$ denotes the eigenvalue of \hat{V} . Denoting quantum wave function by $\Psi(\mu, \phi)$ the Wheeler-De Witt equation now becomes a *difference* equation. For the non-symmetric one we get,

$$\begin{aligned} A(\mu + 4\mu_0)\psi(\mu + 4\mu_0, \phi) - 2A(\mu)\psi(\mu, \phi) + A(\mu - 4\mu_0)\psi(\mu - 4\mu_0, \phi) \\ = -\frac{2\kappa}{3}\mu_0^3\gamma^3\ell_P^2 H_{\text{matter}}(\mu)\psi(\mu, \phi) \end{aligned} \quad (17)$$

where, $A(\mu) := V_{\mu+\mu_0} - V_{\mu-\mu_0}$ and vanishes for $\mu = 0$.

For the symmetric operator one gets,

$$\begin{aligned} f_+(\mu)\psi(\mu + 4\mu_0, \phi) + f_0(\mu)\psi(\mu, \phi) + f_-(\mu)\psi(\mu - 4\mu_0, \phi) \\ = -\frac{2\kappa}{3}\mu_0^3\gamma^3\ell_P^2 H_{\text{matter}}(\mu)\psi(\mu, \phi) \end{aligned} \quad \text{where,} \quad (18)$$

$$f_+(\mu) := |V_{\mu+3\mu_0} - V_{\mu+\mu_0}|, \quad f_-(\mu) := f_+(\mu - 4\mu_0), \quad f_0 := -f_+(\mu) - f_-(\mu).$$

Notice that $f_+(-2\mu_0) = 0 = f_-(2\mu_0)$, but $f_0(\mu)$ is never zero. The absolute values have entered due to the $\text{sgn}(p)$ factor.

These are effectively second order difference equations and the $\Psi(\mu, \phi)$ are determined by specifying Ψ for two consecutive values of μ eg for $\mu = \hat{\mu} + 4\mu_0, \hat{\mu}$ for some $\hat{\mu}$. Since the highest (lowest) order coefficients vanishes for some μ , then the corresponding component

$\Psi(\mu, \phi)$ is undetermined by the equation. Potentially this could introduce an arbitrariness in extending the Ψ specified by data in the classical regime (eg $\mu \gg 2j$) to the negative μ .

For the non-symmetric case, the highest (lowest) A coefficients vanish for their argument equal to zero thus leaving the corresponding ψ component undetermined. However, this undetermined component is decoupled from the others. Thus apart from admitting the trivial solution $\psi(\mu, \phi) := \Phi(\phi)\delta_{\mu,0}$, $\forall \mu$, all other non-trivial solutions are completely determined by giving two consecutive components: $\psi(\hat{\mu}, \phi), \psi(\hat{\mu} + 4\mu_0, \phi)$.

For the symmetric case, due to these properties of the $f_{\pm,0}(\mu)$, it looks as if the difference equation is *non-deterministic* if $\mu = 2\mu_0 + 4\mu_0 n, n \in \mathbb{Z}$. This is because for $\mu = -2\mu_0$, $\psi(2\mu_0, \phi)$ is undetermined by the lower order ψ 's and this coefficient enters in the determination of $\psi(2\mu_0, \phi)$. However, the symmetric operator also commutes with the parity operator: $(\Pi\psi)(\mu, \phi) := \psi(-\mu, \phi)$. Consequently, $\psi(2\mu_0, \phi)$ is determined by $\psi(-2\mu_0, \phi)$. Thus, we can restrict to $\mu = 2\mu_0 + 4k\mu_0, k \geq 0$ where the equation is deterministic.

In both cases then, the space of solutions of the constraint equation, is completely determined by giving appropriate data for large $|\mu|$ i.e. in the classical regime. Such a deterministic nature of the constraint equation has been taken as a necessary condition for non-singularity at the quantum level ². As such this could be viewed as a criterion to limit the choice of factor ordering.

By introducing an interpolating, slowly varying smooth function, $\Psi(p(\mu) := \frac{1}{6}\gamma\ell_P^2)$, and keeping only the first non-vanishing terms, one deduces the Wheeler-De Witt differential equation (with a modified matter Hamiltonian) from the above difference equation. Making a WKB approximation, one infers an effective Hamiltonian which matches with the classical Hamiltonian for large volume ($\mu \gg \mu_0$) and small extrinsic curvature (derivative of the WKB phase is small). There are terms of $o(\hbar^0)$ which contain arbitrary powers of the first derivative of the phase which can all be summed up. The resulting effective Hamiltonian now contains modifications of the classical gravitational Hamiltonian, apart from the modifications in the matter Hamiltonian due to the inverse powers of the triad. The largest possible domain of validity of effective Hamiltonian so deduced must have $|p| \gtrsim p_0$ [7, 8].

An effective Hamiltonian can alternatively be obtained by computing expectation values of

² For contrast, if one just symmetrizes the non-symmetric operator (without the sgn factor), one gets a difference equation which is *non-deterministic*.

the Hamiltonian operator in semiclassical states peaked in classical regimes [9]. The leading order effective Hamiltonian that one obtains is (spatially flat case):

$$\begin{aligned}
H_{\text{eff}}^{\text{non-sym}} &= -\frac{1}{16\pi G} \left(\frac{6}{\mu_0^3 \gamma^3 \ell_{\text{P}}^2} \right) \left[B_+(p) \sin^2(\mu_0 c) + \left(A(p) - \frac{1}{2} B_+(p) \right) \right] + H_{\text{matter}} ; \\
B_+(p) &:= A(p + 4p_0) + A(p - 4p_0) , \quad A(p) := (|p + p_0|^{3/2} - |p - p_0|^{3/2}) , \\
p &:= \frac{1}{6} \gamma \ell_{\text{P}}^2 \mu , \quad p_0 := \frac{1}{6} \gamma \ell_{\text{P}}^2 \mu_0 .
\end{aligned} \tag{19}$$

For the symmetric operator, the effective Hamiltonian is the same as above except that $B_+(p) \rightarrow f_+(p) + f_-(p)$ and $2A(p) \rightarrow f_+(p) + f_-(p)$.

The second bracket in the square bracket, is the quantum geometry potential which is negative and higher order in ℓ_{P} but is important in the small volume regime and plays a role in the genericness of bounce deduced from the effective Hamiltonian [10]. This term is absent in effective Hamiltonian deduced from the symmetric constraint. The matter Hamiltonian will typically have the eigenvalues of powers of inverse triad operator which depend on the ambiguity parameters j, l .

We already see that the quantum modifications are such that both the matter and the gravitational parts in the effective Hamiltonian, are rendered bounded and effective dynamics must be non-singular.

For large values of the triad, $p \gg p_0$, $B_+(p) \sim 6p_0 \sqrt{p} - o(p^{-3/2})$ while $A(p) \sim 3p_0 \sqrt{p} - o(p^{-3/2})$. In this regime, the effective Hamiltonians deduced from both symmetric and non-symmetric ordering are the same. The classical Hamiltonian is obtained for $\mu_0 \rightarrow 0$. From this, one can obtain the equations of motion and by computing the left hand side of the Friedmann equation, infer the effective energy density. For $p \gg p_0$ one obtains ³,

$$\frac{3}{8\pi G} \left(\frac{\dot{a}^2}{a^2} \right) := \rho_{\text{eff}} = \left(\frac{H_{\text{matter}}}{p^{3/2}} \right) \left\{ 1 - \frac{8\pi G \mu_0^2 \gamma^2}{3} p \left(\frac{H_{\text{matter}}}{p^{3/2}} \right) \right\} , \quad p := a^2/4 . \tag{20}$$

The effective density is quadratic in the classical density, $\rho_{\text{cl}} := H_{\text{matter}} p^{-3/2}$. This modification is due to the quantum correction in the gravitational Hamiltonian (due to the \sin^2 feature). This is over and above the corrections hidden in the matter Hamiltonian (due to the “inverse volume” modifications). As noted before, we have two scales: p_0 controlled by μ_0 in the gravitational part and $2p_0 j$ in the matter part. For large j it is possible that we

³ For p in the semiclassical regime, one should include the contribution of the quantum geometry potential present in the non-symmetric ordering, especially for examining the bounce possibility [8].

can have $p_0 \ll p \ll 2p_0j$ in which case the above expressions will hold with j dependent corrections in the matter Hamiltonian. In this semiclassical regime, the corrections from \sin^2 term are smaller in comparison to those from inverse volume. If $p \gg 2p_0j$ then the matter Hamiltonian is also the classical expression. For $j = 1/2$, there is only the $p \gg p_0$ regime and ρ_{cl} is genuinely the classical density.

Let us quickly note a comparison of the two quantizations as reflected in the corresponding effective Hamiltonians, particularly with regards to the extrema of $p(t)$. For this, we will assume same ambiguity parameters (j, l) in the matter Hamiltonian, $(1/2)p_\phi^2 F_{jl}^{3/2}$ and explore the regime $p \gtrsim p_0$. The two effective Hamiltonians differ significantly in the semiclassical regime due to the quantum geometry potential.

The equations of motion imply that p_ϕ is a constant of motion, ϕ varies monotonically with t and on the constraint surface, we can eliminate c in favour of p and p_ϕ . Let us focus on $p(t)$ and in particular consider its possible extrema. It is immediate that $\dot{p} = 0$ implies $\sin(2\mu_0 c) = 0$. This leads to two possibilities: (A) $\sin(\mu_0 c) = 0$ or (B) $\cos(\mu_0 c) = 0$. A local *minimum* signifies a *bounce* while a local *maximum* signifies a *re-collapse*. The value of p at an extremum, p_* , gets determined in terms of the constant p_ϕ . The bounce/re-collapse nature of an extremum depends upon whether p_* is in the classical regime or in the semiclassical regime and also on the case (A) or (B). Note that for the case (A) to hold, it is necessary that the quantum geometry potential is present. Thus, for the symmetric ordering, case (A) cannot be realised – it will imply $p_\phi = 0$.

An extremum determined by case (A): It is a bounce if p_* is in the *semiclassical* regime; p_* varies inversely with p_ϕ while the corresponding density varies directly. p_* being limited to the semiclassical regime implies that p_ϕ is also bounded both above and below, for such an extremum to occur. It turns out that p_* can be in the *classical* regime, provided $p_\phi \sim \ell_P^2$. Thus, the non-symmetric constraint, at the effective level, can accommodate a bounce only in the semiclassical regime and with large densities.

An extremum determined by case (B): It is bounce if p_* is in the *classical* regime; p_* varies directly with p_ϕ and the corresponding density varies inversely. p_* being limited to the classical regime implies that p_ϕ must be bounded below but can be arbitrarily large and thus the density can be arbitrarily small. This is quite unreasonable and has been cited as one of the reasons for considering the “improved” quantization (more on this later). If p_* is in the *semiclassical* regime, it has to be a re-collapse with $p_\phi \sim \ell_P^2$.

In the early works, one worked with the non-symmetric constraint operator and the \sin^2 corrections were not incorporated (i.e. $\mu_0 c \ll \pi/2$ was assumed) and the phenomenological implications were entirely due to the modified matter Hamiltonian. These already implied genericness of inflation and genericness of bounce. These results were discussed at the previous IAGRG meeting in Jaipur.

To summarize: LQC differs from the earlier quantum cosmology in three basic ways (a) the basic variables are different and in particular the classical singularity is in the interior of the mini-superspace; (b) the quantization is very different, being motivated by the background independent quantization employed in LQG; (c) there is a “parent” quantum theory (LQG) which is pretty much well defined at the kinematical level, unlike the metric variables based Wheeler-De Witt theory. The loop quantization has fundamentally distinct implications: its discrete nature of quantum geometry leads to bounded energy densities and bounded extrinsic and intrinsic curvatures (for the anisotropic models). These two features are construed as “resolving the classical singularity”. Quite un-expectedly, the effective dynamics incorporating quantum corrections is also singularity-free (via a bounce), accommodates an inflationary phase rather naturally and is well behaved with regards to perturbations. Although there are many ambiguity parameters, these results are robust with respect to their values.

III. POST 2004 ISOTROPIC LQC

Despite many attractive features of LQC, many points need to be addressed further:

- LQC being a constrained theory, it would be more appropriate if singularity resolution is formulated and demonstrated in terms of physical expectation values of physical (Dirac) operators i.e. in terms of “gauge invariant quantities”. This can be done at present with self-adjoint constraint i.e. a symmetric ordering and for free, massless scalar matter.
- There are at least three distinct ambiguity parameters: μ_0 related to the fiducial length of the loop used in writing the holonomies; j entering in the choice of $SU(2)$ representation which is chosen to be $1/2$ in the gravitational sector and some large value in the matter sector; l entering in writing the inverse powers in terms of Poisson

brackets. The first one was thought to be determined by the area gap from the full theory. The $j = 1/2$ in the gravitational Hamiltonian seems needed to avoid high order difference equation and larger j values are hinted to be problematic in the study of a three dimensional model [11]. Given this, the choice of a high value of j in the matter Hamiltonian seems unnatural⁴. For phenomenology however the higher values allowing for a larger semiclassical regime are preferred. The l does not play as significant a role.

- The bounce scale and density at the bounce, implied by the effective Hamiltonian (from symmetric ordering), is dependent on the parameters of the matter Hamiltonian and can be arranged such that the bounce density is arbitrarily small. This is a highly undesirable feature. Furthermore, the largest possible domain of validity of WKB approximation is given by the turning points (eg the bounce scale). However, the approximation could break down even before reaching the turning point. An independent check on the domain of validity of effective Hamiltonian is thus desirable.
- A systematic derivation of LQC from LQG is expected to tighten the ambiguity parameters. However, such a derivation is not yet available.

A. Physical quantities and Singularity Resolution

When the Hamiltonian is a constraint, at the classical level itself, the notion of dynamics in terms of the ‘time translations’ generated by the Hamiltonian is devoid of any physical meaning. Furthermore, at the quantum level when one attempts to impose the constraint as $\hat{H}|\Psi\rangle = 0$, typically one finds that there are no solutions in the Hilbert space on which \hat{H} is defined - the solutions are generically distributional. One then has to consider the space of all distributional solutions, define a new physical inner product to turn it into a Hilbert space (the physical Hilbert space), define operators on the space of solutions (which must thus act invariantly) which are self-adjoint (physical operators) and compute expectation values, uncertainties etc of these operators to make physical predictions. Clearly, the space of solutions depends on the quantization of the constraint and there is an arbitrariness in the choice of physical inner product. This is usually chosen so that a complete set of Dirac

⁴ For an alternative view on using large values of j , see reference [12].

observables (as deduced from the classical theory) are self-adjoint. This is greatly simplified if the constraint has a *separable* form with respect to some degree of freedom ⁵. For LQC (and also for the Wheeler-De Witt quantum cosmology), such a simplification is available for a free, massless scalar matter: $H_{\text{matter}}(\phi, p_\phi) := \frac{1}{2}p_\phi^2|p|^{-3/2}$. Let us sketch the steps schematically, focusing on the spatially flat model for simplicity [6, 13].

1. *Fundamental constraint equation:*

The classical constraint equations is:

$$-\frac{6}{\gamma^2}c^2\sqrt{|p|} + 8\pi G p_\phi^2 |p|^{-3/2} = 0 = C_{\text{grav}} + C_{\text{matter}} ; \quad (21)$$

The corresponding quantum equation for the wave function, $\Psi(p, \phi)$ is:

$$8\pi G \hat{p}_\phi^2 \Psi(p, \phi) = [\tilde{B}(p)]^{-1} \hat{C}_{\text{grav}} \Psi(p, \phi) , \quad [\tilde{B}(p)] \text{ is eigenvalue of } \widehat{|p|^{-3/2}} ; \quad (22)$$

Putting $\hat{p}_\phi = -i\hbar\partial_\phi$, $p := \frac{\gamma\ell_P^2}{6}\mu$ and $\tilde{B}(p) := (\frac{\gamma\ell_P^2}{6})^{-3/2}B(\mu)$, the equation can be written in a separated form as,

$$\frac{\partial^2 \Psi(\mu, \phi)}{\partial \phi^2} = [B(\mu)]^{-1} \left[8\pi G \left(\frac{\gamma}{6}\right)^{3/2} \ell_P^{-1} \hat{C}_{\text{grav}} \right] \Psi(\mu, \phi) := -\hat{\Theta}(\mu) \Psi(\mu, \phi). \quad (23)$$

The $\hat{\Theta}$ operator for different quantizations is different. For Schrodinger quantization (Wheeler-De Witt), with a particular factor ordering suggested by the continuum limit of the difference equation, the operator $\hat{\Theta}(\mu)$ is given by,

$$\hat{\Theta}_{\text{Sch}}(\mu) \Psi(\mu, \phi) = -\frac{16\pi G}{3} |\mu|^{3/2} \partial_\mu \sqrt{\mu} \partial_\mu \Psi(\mu, \phi) \quad (24)$$

while for LQC, with symmetric ordering, it is given by,

$$\begin{aligned} \hat{\Theta}_{\text{LQC}}(\mu) \Psi(\mu, \phi) &= -[B(\mu)]^{-1} \left\{ C^+(\mu) \Psi(\mu + 4\mu_0, \phi) + C^0(\mu) \Psi(\mu, \phi) + \right. \\ &\quad \left. C^-(\mu) \Psi(\mu - 4\mu_0, \phi) \right\} , \\ C^+(\mu) &:= \frac{\pi G}{9\mu_0^3} \left| |\mu + 3\mu_0|^{3/2} - |\mu + \mu_0|^{3/2} \right| , \\ C^-(\mu) &:= C^+(\mu - 4\mu_0) , \quad C^0(\mu) := -C^+(\mu) - C^-(\mu) . \end{aligned} \quad (25)$$

Note that in the Schrodinger quantization, the $B_{\text{Sch}}(\mu) = |\mu|^{-3/2}$ diverges at $\mu = 0$ while in LQC, $B_{\text{LQC}}(\mu)$ vanishes for all allowed choices of ambiguity parameters. In both cases, $B(\mu) \sim |\mu|^{-3/2}$ as $|\mu| \rightarrow \infty$.

⁵ A general abstract procedure using group averaging is also available.

2. Inner product and General solution:

The operator $\hat{\Theta}$ turns out to be a self-adjoint, positive definite operator on the space of functions $\Psi(\mu, \phi)$ for each fixed ϕ with an inner product scaled by $B(\mu)$. That is, for the Schrodinger quantization, it is an operator on $L^2(\mathbb{R}, B_{\text{Sch}}(\mu)d\mu)$ while for LQC it is an operator on $L^2(\mathbb{R}_{\text{Bohr}}, B_{\text{Bohr}}(\mu)d\mu_{\text{Bohr}})$. Because of this, the operator has a complete set of eigenvectors: $\hat{\Theta}e_k(\mu) = \omega^2(k)e_k(\mu)$, $k \in \mathbb{R}$, $\langle e_k | e_{k'} \rangle = \delta(k, k')$, and the general solution of the fundamental constraint equation can be expressed as

$$\Psi(\mu, \phi) = \int dk \tilde{\Psi}_+(k) e_k(\mu) e^{i\omega\phi} + \tilde{\Psi}_-(k) \bar{e}_k(\mu) e^{-i\omega\phi} . \quad (26)$$

The orthonormality relations among the $e_k(\mu)$ are in the corresponding Hilbert spaces. Different quantizations differ in the form of the eigenfunctions, possibly the spectrum itself and of course $\omega(k)$. In general, these solutions are *not* normalizable in $L^2(\mathbb{R}_{\text{Bohr}} \times \mathbb{R}, d\mu_{\text{Bohr}} \times d\mu)$, i.e. these are distributional.

3. Choice of Dirac observables:

Since the classical kinematical phase space is 4 dimensional and we have a single first class constraint, the phase space of physical states (reduced phase space) is two dimensional and we need two functions to coordinatize this space. We should thus look for two (classical) Dirac observables: functions on the kinematical phase space whose Poisson bracket with the Hamiltonian constraint vanishes on the constraint surface.

It is easy to see that p_ϕ is a Dirac observable. For the second one, we choose a one parameter family of functions $\mu(\phi)$ satisfying $\{\mu(\phi), C(\mu, c, \phi, p_\phi)\} \approx 0$. The corresponding quantum definitions, with the operators acting on the solutions, are:

$$\hat{p}_\phi \Psi(\mu, \phi) := -i\hbar \partial_\phi \Psi(\mu, \phi) , \quad (27)$$

$$\widehat{|\mu|_{\phi_0}} \Psi(\mu, \phi) := e^{i\sqrt{\hat{\Theta}}(\phi-\phi_0)} |\mu| \Psi_+(\mu, \phi_0) + e^{-i\sqrt{\hat{\Theta}}(\phi-\phi_0)} |\mu| \Psi_-(\mu, \phi_0) \quad (28)$$

On an initial datum, $\Psi(\mu, \phi_0)$, these operators act as,

$$\widehat{|\mu|_{\phi_0}} \Psi(\mu, \phi_0) = |\mu| \Psi(\mu, \phi_0) , \quad \hat{p}_\phi \Psi(\mu, \phi_0) = \hbar \sqrt{\hat{\Theta}} \Psi(\mu, \phi_0) . \quad (29)$$

4. Physical inner product:

It follows that the Dirac operators defined on the space of solutions are self-adjoint if we define a *physical inner product* on the space of solutions as:

$$\langle \Psi | \Psi' \rangle_{\text{phys}} := \left(\int_{\phi=\phi_0} d\mu B(\mu) \right) \bar{\Psi}(\mu, \phi) \Psi'(\mu, \phi) . \quad (30)$$

Thus the eigenvalues of the inverse volume operator crucially enter the definition of the physical inner product. For Schrodinger quantization, the integral is really an integral while for LQC it is actually a sum over μ taking values in a lattice. The inner product is independent of the choice of ϕ_0 .

A complete set of physical operators and physical inner product has now been specified and physical questions can be phrased in terms of (physical) expectation values of functions of these operators.

5. *Semiclassical states:*

To discuss semiclassical regime, typically one *defines* semiclassical states: physical states such that a chosen set of self-adjoint operators have specified expectation values with uncertainties bounded by specified tolerances. A natural choice of operators for us are the two Dirac operators defined above. It is easy to construct semiclassical states with respect to these operators. For example, a state peaked around, $p_\phi = p_\phi^*$ and $|\mu|_{\phi_0} = \mu^*$ is given by (in Schrodinger quantization for instance),

$$\Psi_{\text{semi}}(\mu, \phi_0) := \int dk e^{-\frac{(k-k^*)^2}{2\sigma^2}} e_k(\mu) e^{i\omega(\phi_0 - \phi^*)} \quad (31)$$

$$k^* = -\sqrt{3/2\kappa\hbar}^{-1} p_\phi^* , \quad \phi^* = \phi_0 + -\sqrt{3/2\kappa\ell n} |\mu^*| . \quad (32)$$

For LQC, the $e_k(\mu)$ functions are different and the physical expectation values are to be evaluated using the physical inner product defined in the LQC context.

6. *Evolution of physical quantities:*

Since one knows the general solution of the constraint equation, $\Psi(\mu, \phi)$, given $\Psi(\mu, \phi_0)$, one can compute the physical expectation values in the semiclassical solution, $\Psi_{\text{semi}}(\mu, \phi)$ and track the position of the peak as a function of ϕ as well as the uncertainties as a function of ϕ .

7. *Resolution of Big Bang Singularity:*

A classical solution is obtained as a curve in (μ, ϕ) plane, different curves being labelled by the points (μ^*, ϕ^*) in the plane. The curves are independent of the constant value of p_ϕ^* . These curves are already given in (7).

Quantum mechanically, we first select a semiclassical solution, $\Psi_{\text{semi}}(p_\phi^*, \mu^* : \phi)$ in which the expectation values of the Dirac operators, at $\phi = \phi_0$, are p_ϕ^* and μ^* respectively. These values serve as labels for the semiclassical solution. The former one continues to be p_ϕ^* for all ϕ whereas $\langle \widehat{|\mu|_{\phi_0}} \rangle(\phi) =: |\mu|_{p_\phi^*, \mu^*}(\phi)$, determines a curve in the (μ, ϕ) plane. In general one expects this curve to be different from the classical curve in the region of small μ (small volume).

The result of the computations is that Schrodinger quantization, the curve $|\mu|_{p_\phi^*, \mu^*}(\phi)$, does approach the $\mu = 0$ axis asymptotically. However for LQC, the curve *bounces away* from the $\mu = 0$ axis. In this sense – and now inferred in terms of physical quantities – the Big Bang singularity is resolved in LQC. It also turns out that for large enough values of p_ϕ^* , the quantum trajectories constructed by the above procedure are well approximated by the trajectories by the effective Hamiltonian. All these statements are for semiclassical solutions which are peaked at large μ_* at late times.

Two further features are noteworthy as they corroborate the suggestions from the effective Hamiltonian analysis.

First one is revealed by computing expectation value of the matter density operator, $\rho_{\text{matter}} := \frac{1}{2}(\widehat{(p_\phi^*)^2 |p|})^{-3}$, at the bounce value of $|p|$. It turns out that this value is sensitive to the value of p_ϕ^* and can be made arbitrarily *small* by choosing p_ϕ^* to be *large*. Physically this is unsatisfactory as quantum effects are *not* expected to be significant for matter density very small compared to the Planck density. This is traced to the quantization of the gravitational Hamiltonian, in particular to the step which introduces the ambiguity parameter μ_0 . A novel solution proposed in the “improved quantization”, removes this undesirable feature.

The second one refers to the role of quantum modifications in the gravitational Hamiltonian compared to those in the matter Hamiltonian (the inverse volume modification or $B(\mu)$). The former is much more significant than the latter. So much so, that even if one uses the $B(\mu)$ from the Schrodinger quantization (i.e. switch-off the inverse volume modifications), one still obtains the bounce. So bounce is seen as the consequence of $\hat{\Theta}$ being different and as far as qualitative singularity resolution is concerned, the inverse volume

modifications are *un-important*. As the effective picture (for symmetric constraint) showed, the bounce occurs in the classical region (for $j = 1/2$) where the inverse volume corrections can be neglected. For an exact model which seeks to understand as to why the bounces are seen, please see [14].

B. Improved Quantization

The undesirable features of the bounce coming from the classical region, can be seen readily using the effective Hamiltonian, as remarked earlier. To see the effects of modifications from the gravitational Hamiltonian, choose $j = 1/2$ and consider the Friedmann equation derived from the effective Hamiltonian (20), with matter Hamiltonian given by $H_{\text{matter}} = \frac{1}{2}p_\phi^2|p|^{-3/2}$. The positivity of the effective density implies that $p \geq p_*$ with p_* determined by vanishing of the effective energy density: $\rho_* := \rho_{cl}(p_*) = (\frac{8\pi G\mu_0^2\gamma^2}{3}p_*)^{-1}$. This leads to $|p_*| = \sqrt{\frac{4\pi G\mu_0^2\gamma^2}{3}}|p_\phi|$ and $\rho_* = \sqrt{2}(\frac{8\pi G\mu_0^2\gamma^2}{3})^{-3/2}|p_\phi|^{-1}$. One sees that for large $|p_\phi|$, the bounce scale $|p_*|$ can be large and the maximum density – density at bounce – could be small. Thus, *within the model*, there exist a possibility of seeing quantum effects (bounce) even when neither the energy density nor the bounce scale are comparable to the corresponding Planck quantities and this is an undesirable feature of the model. This feature is independent of factor ordering as long as the bounce occurs in the classical regime.

One may notice that if we replace $\mu_0 \rightarrow \bar{\mu}(p) := \sqrt{\Delta/|p|}$ where Δ is a constant, then the effective density vanishes when ρ_{cl} equals the critical value $\rho_{\text{crit}} := (\frac{8\pi G\Delta\gamma^2}{3})^{-1}$, which is independent of matter Hamiltonian. The bounce scale p_* is determined by $\rho_* = \rho_{\text{crit}}$ which gives $|p_*| = (\frac{p_\phi^2}{2\rho_{\text{crit}}})^{1/3}$. Now although the bounce scale can again be large depending upon p_ϕ , the density at bounce is always the universal value determined by Δ . This is a rather nice feature in that quantum geometry effects are revealed when matter density (which couples to gravity) reaches a universal, critical value regardless of the dynamical variables describing matter. For a suitable choice of Δ one can ensure that a bounce always happen when *the energy density* becomes comparable to the Planck density. In this manner, one can retain the good feature (bounce) even for $j = 1/2$ thus “effectively fixing” an ambiguity parameter and also trade another ambiguity parameter μ_0 for Δ . This is precisely what is achieved by the “improved quantization” of the gravitational Hamiltonian [15].

The place where the quantization procedure is modified is when one expresses the cur-

vature in terms of the holonomies along a loop around a “plaquette”. One shrinks the plaquette in the limiting procedure. One now makes an important departure: the plaquette should be shrunk only till the physical area (as distinct from a fiducial one) reaches its minimum possible value which is given by the area gap in the known spectrum of area operator in quantum geometry: $\Delta = 2\sqrt{3}\pi\gamma G\hbar$. Since the plaquette is a square of fiducial length μ_0 , its physical area is $\mu_0^2|p|$ and this should set be to Δ . Since $|p|$ is a dynamical variable, μ_0 cannot be a constant and is to be thought of a function on the phase space, $\bar{\mu}(p) := \sqrt{\Delta/|p|}$. It turns out that even with such a change which makes the curvature to be a function of both connection and triad, the form of *both* the gravitational constraint and inverse volume operator appearing in the matter Hamiltonian, remains the same with just doing the replacement, $\mu_0 \rightarrow \bar{\mu}$ defined above, *in the holonomies*. The expressions simplify by using eigenfunctions of the volume operator $\hat{V} := |\hat{p}|^{3/2}$, instead of those of the triad. The relevant expressions are:

$$v := K \text{sgn}(\mu) |\mu|^{3/2} \quad , \quad K := \frac{2\sqrt{2}}{3\sqrt{3\sqrt{3}}} ; \quad (33)$$

$$\hat{V}|v\rangle = \left(\frac{\gamma}{6}\right)^{3/2} \frac{\ell_P^3}{K} |v||v\rangle , \quad (34)$$

$$\widehat{e^{ik\frac{\bar{\mu}}{2}c}}\Psi(v) := \Psi(v+k) , \quad (35)$$

$$\widehat{|p|^{-1/2}} \Big|_{j=1/2, l=3/4} \Psi(v) = \frac{3}{2} \left(\frac{\gamma\ell_P^2}{6}\right)^{-1/2} K^{1/3} |v|^{1/3} \left| |v+1|^{1/3} - |v-1|^{1/3} \right| \Psi(v) \quad (36)$$

$$B(v) = \left(\frac{3}{2}\right)^{3/2} K |v| \left| |v+1|^{1/3} - |v-1|^{1/3} \right|^3 \quad (37)$$

$$\hat{\Theta}_{\text{Improved}} \Psi(v, \phi) = -[B(v)]^{-1} \left\{ C^+(v) \Psi(v+4, \phi) + C^0(v) \Psi(v, \phi) + C^-(v) \Psi(v-4, \phi) \right\} , \quad (38)$$

$$C^+(v) := \frac{3\pi KG}{8} |v+2| \left| |v+1| - |v+3| \right| , \quad (39)$$

$$C^-(v) := C^+(v-4) \quad , \quad C^0(v) := -C^+(v) - C^-(v) . \quad (40)$$

Thus the main changes in the quantization of the Hamiltonian constraint are: (1) replace $\mu_0 \rightarrow \bar{\mu} := \sqrt{\Delta/|p|}$ in the holonomies; (2) *choose* symmetric ordering for the gravitational constraint; and (3) *choose* $j = 1/2$ in both gravitational Hamiltonian and the matter Hamiltonian (in the definition of inverse powers of triad operator). The “improvement” refers to the first point. This model is singularity free at the level of the fundamental constraint equation (even though the leading coefficients of the difference equation do vanish, because

the the parity symmetry again saves the day); the densities continue to be bounded above – and now with a bound independent of matter parameters; the effective picture continues to be singularity free and with undesirable features removed and the classical Big Bang being replaced by a quantum bounce is established in terms of *physical* quantities.

C. Close Isotropic Model

While close model seems phenomenologically disfavoured, it provides further testing ground for quantization of the Hamiltonian constraint. Because of the intrinsic (spatial) curvature, the plaquettes used in expressing the F_{ij} in terms of holonomies, are not bounded by just four edges – a fifth one is necessary. This was attempted and was found to lead to an “unstable” quantization. This difficulty was bypassed by using the holonomies of the extrinsic curvature instead of the gauge connection which is permissible in the homogeneous context. The corresponding, non-symmetric constraint and its difference equation was analysed for the massless scalar matter. Green and Unruh, found that solutions of the difference equation was always diverging (at least for one orientation) for large volumes. Further, the divergence seemed to set in just where one expected a re-collapse from the classical theory. In the absence of physical inner product and physical interpretation of the solutions, it was concluded that this version of LQC for close model is unlikely to accommodate classical re-collapse even though it avoided the Big Bang/Big Crunch singularities.

Recently, this model has been revisited [16]. One went back to using the gauge connection and the fifth edge difficulty was circumvented by using both the left-invariant and the right-invariant vector fields to define the plaquette. In addition, the symmetric ordering was chosen and finally the $\mu_0 \rightarrow \bar{\mu}$ improvement was also incorporated. Without the improvement, there were still the problems of getting bounce for low energy density and also not getting a reasonable re-collapse (either re-collapse is absent or the scale is marginally larger than the bounce scale). With the improvement, the bounces and re-collapses are neatly accommodated and one gets a cyclic evolution. In this case also, the scalar field serves as a good clock variable as it continues to be monotonic with the synchronous time.

I have focussed on the singularity resolution issue in this talk. Other developments have also taken place in the past couple of years. I will just list these giving references.

1. *Effective models and their properties:* The effective picture was shown to be non-

singular and since this is based on the usual framework of GR, it follows that energy conditions must be violated (and indeed they are thanks to the inverse volume modifications). This raised questions regarding stability of matter and causal propagation of perturbations. Golam Hossain showed that despite the energy conditions violations, neither of the above pathologies result [17].

Minimally coupled scalar has been used in elaborating inflationary scenarios. However non-minimally coupled scalars are also conceivable models. The singularity resolution and inflationary scenarios continue to hold also in this case. Furthermore sufficient e-foldings are also admissible [18].

In the improved quantization, one sets the ambiguity parameter $j = 1/2$ and shifts the dominant effects to the the gravitational Hamiltonian. All the previous phenomenological implications however were driven by the inverse volume modifications in the matter sector. Consequently, it is necessary to check if and how the phenomenology works with the improved quantization. This has been explored in [19].

Using the effective dynamics for the homogeneous mode, density perturbations were explored and power spectra were computed with the required small amplitude [20, 21].

As many of the phenomenology oriented questions have been explored using effective Hamiltonian which incorporate quantum corrections from various sources (gravity, matter etc). This motivates a some what systematic approach to constructing effective approximations. This has been initiated in [22].

2. *Anisotropic models:* The anisotropic models provide further testing grounds for loop quantization. At the difference equation level, the non-singularity has been checked also for these models in the non-symmetric scheme. For the vacuum Bianchi I model, there is no place for the inverse volume type corrections to appear at an effective Hamiltonian level and the effective dynamics would continue to be singular. However, once the gravitational corrections (\sin^2) are incorporated, the effective dynamics again is non-singular and one can obtain the non-singular version of the (singular) Kasner solution [23]. More recently, the Bianchi I model with a free, massless scalar is also analysed in the improved quantization [24]. A perturbative treatment of anisotropies has been explored in [25].

3. *Inhomogeneities:* Inhomogeneities are a fact of nature although these are small in the early universe. This suggests a perturbative approach to incorporate inhomogeneities. On the one hand one can study their evolution in the homogeneous, isotropic background (cosmological perturbation theory). One can also begin with a (simplified) inhomogeneous model and try to see how a homogeneous approximation can become viable. The work on the former has already begun. For the latter part, Bojowald has discussed a simplified lattice model to draw some lessons for the homogeneous models. In particular he has given an alternative argument for the $\mu_0 \rightarrow \bar{\mu}$ modification which does not appeal to the area operator [12].

IV. OPEN ISSUES AND OUT LOOK

In summary, over the past two years, we have seen how to phrase and understand the fate of Big Bang singularity in a quantum framework.

Firstly, with the help of a minimally coupled, free, massless scalar which serves as a good clock variable in the isotropic context, one can define physical inner product, a complete set of Dirac observables and their physical matrix elements. At present this can be done only for self-adjoint Hamiltonian constraint. Using these, one can construct trajectories in the (p, p_ϕ) plane which are followed by the peak of a semiclassical state as well as the uncertainties in the Dirac observables. It so happens that these trajectories do not pass through the zero volume – Big Bang is replaced by a Bounce. For close isotropic model, the Big Crunch is also replaced by a bounce while retaining classically understood re-collapse. In conjunction with the $\bar{\mu}$ improvement, the gravitational Hamiltonian can be given the the main role in generating the bounce. A corresponding treatment in Schrodinger quantization (Wheeler-De Witt theory), *does not* generate a bounce nor does it render the density, curvatures bounded. Thus, quantum representation plays a significant role in the singularity resolution.

Secondly, the improved quantization motivated by the regulation of the F_{ij} invoking the area operator from the full theory (or by the argument from the inhomogeneous lattice model), also leads the bounce to be “triggered” when the energy density reaches a critical value ($\sim 0.82\rho_{\text{Planck}}$) which is *independent* of the values of the dynamical variables. Close model also gives the same critical value.

While the improvement is demonstrated to be viable in the isotropic context, the proce-

ture differs from that followed in the full theory. One may either view this as something special to the mini-superspace model(s) or view it as providing hints for newer approaches in the full theory.

A general criteria for “non-singularity” is not in sight yet and so also a systematic derivation of the mini-superspace model(s) from a larger, full theory.

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