EXPLICIT JACQUET-LANGLANDS FOR GENUS 2 HILBERT-SIEGEL MODULAR FORMS OVER $\mathbb{Q}(\sqrt{5})$

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Abstract. In this paper we present an algorithm for computing Hecke eigensystems of Hilbert-Siegel cusp forms over real quadratic fields of narrow class number one. We give some illustrative examples using the quadratic field $\mathbb{Q}(\sqrt{5})$. In those examples, we identify Hilbert-Siegel eigenforms that are possible lifts from Hilbert eigenforms.

Introduction

Let F be a real quadratic field of narrow class number one and let B be the unique (up to isomorphism) quaternion algebra over F which is ramified at both archimedean places of F and unramified everywhere else. Let $\mathbf{GU}_2(B)$ be the unitary similitude group of $B^{\oplus 2}$. This is the set of $\mathbb{Q}$-rational points of an algebraic group G^B defined over $\mathbb{Q}$. The group G^B is an inner form of $G := \mathrm{Res}_{F/\mathbb{Q}}(\mathbf{GSp}_4)$ such that $G^B(\mathbb{R})$ is compact modulo its centre.

In this paper we develop an algorithm which computes automorphic forms on G^B in the following sense: given an ideal N in $\mathcal{O}_F$ and an integer k greater than 3, the algorithm returns the Hecke eigensystems of all automorphic forms f of level N and parallel weight k. More precisely, given a prime $\mathfrak{p}$ in $\mathcal{O}_F$, the algorithm returns the Hecke eigenvalues of f at $\mathfrak{p}$, and hence the Euler factor $L_{\mathfrak{p}}(f, s)$, for each eigenform f of level N and parallel weight k. The algorithm is a generalization of the one developed in [D1 2005] to the genus 2 case. Although we have only described the algorithm in the case of a real quadratic field in this paper, it should be clear from our presentation that it can be adapted to any totally real number field of narrow class number one. And so, it can be seen as a slight improvement of Lansky and Pollack [LP 2002].

The Jacquet-Langlands Correspondence predicts the existence of a transfer map $JL : \Pi(G^B) \to \Pi(G)$ from automorphic representations of G^B to automorphic representations on G, which is injective, matches L-functions and enjoys other properties compatible with the principle of functoriality; in particular, the image of the Jacquet-Langlands Correspondence is to be

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contained in the space of holomorphic automorphic representations. If we admit this conjecture, then the algorithm above provides a way to produce examples of cuspidal Hilbert-Siegel modular forms of genus 2 over F and allows us to compute the L-factors of the corresponding automorphic representations for arbitrary finite primes $\mathfrak{p}$ of F.

In order to support the claim that we are in fact computing Hilbert-Siegel modular forms, we must compare the Euler factors we find with those of known Hilbert-Siegel modular forms. This we do in the final section of the paper where we observe that, in certain cases, the Euler factors we compute match those of lifts of Hilbert modular forms, for the primes we computed.

The first systematic approach to Siegel modular forms from a computational viewpoint is due to Skoruppa [S 1992] who used Jacobi symbols to generate spaces of such forms. His algorithm, which has been extensively exploited by Ryan [R 2006], works only for the full level structure. More recently, Faber and van der Geer [FvdG1 2004] and [FvdG2 2004] also produced examples of Siegel modular forms by counting points on hyperelliptic curves of genus 2, but again their results are available only in the full level structure case. The most substantial progress toward the computation of Siegel modular forms for proper level structure is by Gunnells [Gu 2000] who extended the theory of modular symbols to the symplectic group $\mathbf{Sp}_4/\mathbb{Q}$. However, this work does not see the cuspidal cohomology which is relevant to us. To the best of our knowledge, there are no numerical examples of Hilbert-Siegel modular forms for proper level structure in the literature, with the exception of those produced from liftings of Hilbert modular forms.

The outline of the paper is as follows. In Section 1 we recall the basic properties of Hilbert-Siegel modular forms and algebraic automorphic forms together with the Jacquet-Langlands Correspondence. In Section 2 we give a detailed description of our algorithm. Finally, in Section 3 we present numerical results for the quadratic field $\mathbb{Q}(\sqrt{5})$.

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1. **Hilbert-Siegel modular forms and the Jacquet-Langlands correspondence**

Throughout this paper, F denotes a real quadratic field of narrow class number one. The two archimedean places of F and the real embeddings of F will both be denoted v_0 and v_1. For every $a \in F$, we write a_0 (resp. a_1)

for the image of a under v_0 (resp. v_1). The ring of integers of F is denoted by $\mathcal{O}_F$. For every prime ideal $\mathfrak{p}$ in $\mathcal{O}_F$, the completion of F and $\mathcal{O}_F$ at $\mathfrak{p}$ will be denoted by $F_\mathfrak{p}$ and $\mathcal{O}_{F_\mathfrak{p}}$, respectively.

Let B be the unique (up to isomorphism) totally definite quaternion algebra over F which is unramified at all finite primes of F. We fix a maximal order $\mathcal{O}_B$ of B. Also, we choose a splitting field K/F of B that is Galois over $\mathbb{Q}$ and such that there exists an isomorphism $j : \mathcal{O}_B \otimes_\mathbb{Z} \mathcal{O}_K \cong \mathbf{M}_2(\mathcal{O}_K) \oplus \mathbf{M}_2(\mathcal{O}_K)$, where $\mathbf{M}_2(A)$ denotes the ring of 2×2-matrices with entries from a ring A. For every finite prime $\mathfrak{p}$ in F, we fix an isomorphism $B_\mathfrak{p} \cong \mathbf{M}_2(F_\mathfrak{p})$ which restricts to an isomorphism from $\mathcal{O}_{B,\,\mathfrak{p}}$ onto $\mathbf{M}_2(\mathcal{O}_{F_\mathfrak{p}})$.

We denote the finite adèles of $\mathbb{Q}$ (resp. $\mathbb{Z}$) by $\hat{\mathbb{Q}}$ (resp. $\hat{\mathbb{Z}}$), and for any $\mathbb{Q}$-algebra (resp. $\mathbb{Z}$-algebra) A, we set $\hat{A} := A \otimes_\mathbb{Q} \hat{\mathbb{Q}}$ (resp. $\hat{A} := A \otimes_\mathbb{Z} \hat{\mathbb{Z}}$).

Recall that the algebraic group $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathbf{GSp}_4)$ is defined as follows. For any $\mathbb{Q}$-algebra A, the set of A-rational points is given by

$$G(A) = \left\{ \gamma \in \mathbf{GL}_4(A \otimes_\mathbb{Q} F) \middle| \begin{array}{c} \gamma J_2 \gamma^t = \nu_G(\gamma) J_2 \\ \nu_G(\gamma) \in (A \otimes_\mathbb{Q} F)^\times \end{array} \right\},$$

where

$$J_2 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix}.$$

This group admits an integral model denoted the same way, and whose A-rational points for every $\mathbb{Z}$-algebra A is given by

$$G(A) = \left\{ \gamma \in \mathbf{GL}_4(A \otimes_\mathbb{Z} \mathcal{O}_F) \middle| \begin{array}{c} \gamma J_2 \gamma^t = \nu_G(\gamma) J_2 \\ \nu_G(\gamma) \in (A \otimes_\mathbb{Z} \mathcal{O}_F)^\times \end{array} \right\}.$$

Now, for any $\mathbb{Q}$-algebra A, the conjugation on B extends in a natural way to the matrix algebra $\mathbf{M}_2(B \otimes_\mathbb{Q} A)$. The algebraic group $G^B/\mathbb{Q}$ is defined by letting

$$G^B(A) = \left\{ \gamma \in \mathbf{M}_2(B \otimes_\mathbb{Q} A) \middle| \begin{array}{c} \gamma \bar{\gamma}^t = \nu_{G^B}(\gamma) \mathbf{1}_2 \\ \nu_{G^B}(\gamma) \in (A \otimes_\mathbb{Q} F)^\times \end{array} \right\}.$$

This group admits an integral model denoted the same way, and whose A-rational points for every $\mathbb{Z}$-algebra is given by

$$G^B(A) = \left\{ \gamma \in \mathbf{M}_2(\mathcal{O}_B \otimes_\mathbb{Z} A) \middle| \begin{array}{c} \gamma \bar{\gamma}^t = \nu_{G^B}(\gamma) \mathbf{1}_2 \\ \nu_{G^B}(\gamma) \in (A \otimes_\mathbb{Z} \mathcal{O}_F)^\times \end{array} \right\}.$$

The group $G^B/\mathbb{Q}$ is an inner form of $G/\mathbb{Q}$ such that $G^B(\mathbb{R})$ is compact modulo its center. By the choice of the quaternion algebra B, we have $G^B(\hat{\mathbb{Q}}) \cong G(\hat{\mathbb{Q}})$. Also, combining the isomorphism j with conjugation by a permutation matrix, we obtain an isomorphism $G^B(\mathcal{O}_K) \cong G(\mathcal{O}_K)$, which we fix from now on.

1.1. **Hilbert-Siegel modular forms.** We fix an integer $k \geq 3$ and, for simplicity, we restrict ourselves to Hilbert-Siegel modular forms of parallel weight k. The real embeddings v_0 and v_1 of F extend to $G(\mathbb{Q}) = \mathbf{GSp}_4(F)$ in a natural way. We denote by $\mathbf{GSp}_4^+(F)$ the subgroup of elements γ

with totally positive similitude factor $\nu_G(\gamma) \gg 0$. We recall that the Siegel upper-half plane of genus 2 is defined by

$$\mathfrak{H}_2 = \{\gamma \in \mathbf{GL}_2(\mathbb{C}) \mid \gamma^t = \gamma \,\text{and}\, \mathrm{Im}(\gamma) > 0\},$$

and that $\mathbf{GSp}_4^+(F)$ acts on $\mathfrak{H}_2^2$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau_0,\, \tau_1) := \left((a_0\tau_0 + b_0)(c_0\tau_0 + d_0)^{-1},\, (a_1\tau_1 + b_1)(c_1\tau_1 + d_1)^{-1}\right).$$

This induces an action on the space of functions $f : \mathfrak{H}_2^2 \to \mathbb{C}$ by

$$\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad f|_k\gamma(\tau) = \prod_{i=0}^{1} \frac{\nu_G(\gamma_i)^{k/2}}{\det(c_i\tau_i + d_i)^k} f(\tau).$$

Let N be an ideal in $\mathcal{O}_F$ and set

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GSp}_4^+(\mathcal{O}_F) \mid c \equiv 0(N) \right\}.$$

A **Hilbert-Siegel modular form** of level N and parallel weight k is a holomorphic function $f : \mathfrak{H}_2^2 \to \mathbb{C}$ such that

$$\forall \gamma \in \Gamma_0(N), \qquad f|_k\gamma = f.$$

The space of Hilbert-Siegel modular forms of parallel weight k and level N is denoted $M_k(N)$. Each $f \in M_k(N)$ admits a Fourier expansion, which by the Koecher principle takes the form

$$\forall \tau \in \mathfrak{H}_2^2, \qquad f(\tau) = \sum_{\{Q \gg 0\} \cup \{0\}} a_Q e^{2\pi i \mathrm{Tr}(Q\tau)},$$

where $Q \in \mathbf{M}_2(F)$ runs over all symmetric totally positive and semi-definite matrices. A Hilbert-Siegel modular forms f is a **cusp form** if, for all $\gamma \in \mathbf{GSp}_4^+(F)$, the first coefficient in the Fourier expansion of $f|_k\gamma$ is zero. The space of Hilbert-Siegel cusp forms is denoted $S_k(N)$.

1.2. **The Hecke algebra.** The space $S_k(N)$ comes equipped with a Hecke action, which we now recall. Take $u \in \mathbf{GSp}_4^+(F) \cap \mathbf{M}_4(\mathcal{O}_F)$, and write the finite disjoint union

$$\Gamma_0(N)u\Gamma_0(N) = \coprod_i \Gamma_0(N)u_i.$$

Then the Hecke operator $[\Gamma_0(N)u\Gamma_0(N)]$ on $S_k(N)$ is given by

$$[\Gamma_0(N)u\Gamma_0(N)]f = \sum_i f|_k u_i.$$

Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_F$ and let $\pi_\mathfrak{p}$ be a totally positive generator of $\mathfrak{p}$; let $T_1(\mathfrak{p})$ and $T_2(\mathfrak{p})$ be the Hecke operators corresponding to the double

$\Gamma_0(N)$-cosets of the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi_{\mathfrak{p}} & 0 \\ 0 & 0 & 0 & \pi_{\mathfrak{p}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi_{\mathfrak{p}} & 0 & 0 \\ 0 & 0 & \pi_{\mathfrak{p}}^2 & 0 \\ 0 & 0 & 0 & \pi_{\mathfrak{p}} \end{pmatrix},$$

respectively. The Hecke algebra $\mathbf{T}_k(N)$ is the $\mathbb{Z}$-algebra generated by the operators $T_1(\mathfrak{p})$ and $T_2(\mathfrak{p})$, where $\mathfrak{p}$ runs over all primes not dividing N.

1.3. **Algebraic Hilbert-Siegel autormorphic forms.** We only consider level structure of Siegel type. Namely, we define the compact open subgroup $U_0(N)$ of $G(\hat{\mathbb{Q}})$ by

$$U_0(N) = \prod_{\mathfrak{p}\nmid N} \mathbf{GSp}_4(\mathcal{O}_{F_{\mathfrak{p}}}) \times \prod_{\mathfrak{p}|N} U_0(\mathfrak{p}^{e_{\mathfrak{p}}}),$$

where $N = \prod_{\mathfrak{p}|N} \mathfrak{p}^{e_{\mathfrak{p}}}$ and

$$U_0(\mathfrak{p}^{e_{\mathfrak{p}}}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GSp}_4(\mathcal{O}_{F_{\mathfrak{p}}}) \mid c \equiv 0 \mod \mathfrak{p}^{e_{\mathfrak{p}}} \right\}.$$

The **weight representation** is defined as follows. Let L_k be the representation of $\mathbf{GSp}_4(\mathbb{C})$ of highest weight $(k-3,\, k-3)$. We let $V_k = L_k \otimes L_k$ and define the complex representation $(\rho_k,\, V_k)$ by

$$\rho_k :\, G^B(\mathbb{R}) \longrightarrow \mathbf{GL}(V_k),$$

where the action on the first factor is via v_0, and the action on the second one is via v_1.

The space of **algebraic Hilbert-Siegel modular forms** of weight k and level N is given by

$$M_k^B(N) := \left\{ f :\, G^B(\hat{\mathbb{Q}})/U_0(N) \to V_k \mid \forall \gamma \in G^B(\mathbb{Q}), f|_k\gamma = f \right\},$$

where $f|_k\gamma(x) = f(\gamma x)\gamma$, for all $x \in G^B(\hat{\mathbb{Q}})/U_0(N)$. When $k = 3$, we let

$$I_k^B(N) := \left\{ f :\, G^B(\mathbb{Q})\backslash G^B(\hat{\mathbb{Q}})/U_0(N) \to \mathbb{C} \mid f \text{ is constant} \right\}.$$

Then, the space of **algebraic Hilbert-Siegel cusp forms** of weight k and level N is defined by

$$S_k^B(N) := \begin{cases} M_k^B(N) & \text{if} \quad k > 3, \\ M_k^B(N)/I_k^B(N) & \text{if} \quad k = 3. \end{cases}$$

The action of the Hecke algebra on $S_k^B(N)$ is given as follows. For any $u \in G(\hat{\mathbb{Q}})$, write the finite disjoint union

$$U_0(N)uU_0(N) = \coprod_i u_iU_0(N),$$

and define

$$\begin{array}{rcl} [U_0(N)uU_0(N)] : S_k^B(N) & \to & S_k^B(N) \\ f & \mapsto & f|_k[U_0(N)uU_0(N)], \end{array}$$

by

$$f|_k[U_0(N)uU_0(N)](x) = \sum_i f(xu_i),\ x \in G(\hat{\mathbb{Q}}).$$

For any prime $\mathfrak{p} \nmid N$, let $\varpi_{\mathfrak{p}}$ be a local uniformizer at $\mathfrak{p}$. The local Hecke algebra at $\mathfrak{p}$ is generated by the Hecke operators $T_1(\mathfrak{p})$ and $T_2(\mathfrak{p})$ corresponding to the double $U_0(N)$-cosets $\Delta_1(\mathfrak{p})$ and $\Delta_2(\mathfrak{p})$ of the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \varpi_{\mathfrak{p}} & 0 \\ 0 & 0 & 0 & \varpi_{\mathfrak{p}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varpi_{\mathfrak{p}} & 0 & 0 \\ 0 & 0 & \varpi_{\mathfrak{p}}^2 & 0 \\ 0 & 0 & 0 & \varpi_{\mathfrak{p}} \end{pmatrix},$$

respectively. We let $\mathbf{T}_k^B(N)$ be the Hecke algebra generated by $T_1(\mathfrak{p})$ and $T_2(\mathfrak{p})$ for all primes $\mathfrak{p} \nmid N$.

1.4. **The Jacquet-Langlands Correspondence.** The Hecke modules $S_k(N)$ and $S_k^B(N)$ are related by the following conjecture known as the Jacquet-Langlands Correspondence for symplectic groups.

Conjecture 1 (Jacquet-Langlands)**.** *The Hecke algebras $\mathbf{T}_k(N)$ and $\mathbf{T}_k^B(N)$ are isomorphic and there is a compatible isomorphism of Hecke modules*

$$JL : S_k^B(N) \xrightarrow{\sim} S_k(N).$$

The Jacquet-Langlands Correspondence is a special case of the Langlands transfer map between autormorphic forms on the inner forms of a group and the group itself. To the best of our knowledge, the correspondence in this form was first discussed by Ihara [Ih 1964] in the case $F = \mathbb{Q}$, and by Ibukiyama [Ib 1984] who provided some numerical evidences. The authors intend to investigate some cases of the conjecture using theta series. The Fundamental Lemma should be the main ingredient in giving a full proof of the conjecture, although this would not be a straightforward exercise. We were told by several people that the result would be a consequence of some work in progress by James Arthur.

2. The Algorithm

In this section, we present the algorithm we used in order to compute the Hecke module of (algebraic) Hilbert-Siegel modular forms. The main assumption in this section is that the class number of the principal genus of G^B is 1. (This is not a very strong restriction and we refer to [D3 2007] to see how one could relax it). We recall that since B is totally definite, G^B satisfies Proposition 1.4 in Gross [Gr 1999]. Thus the group $G^B(\mathbb{R})$ is compact modulo its centre, and $\Gamma = G^B(\mathbb{Z})/\mathcal{O}_F^\times$ is *finite*.

For any prime $\mathfrak{p}$ in F, let $\mathbb{F}_{\mathfrak{p}} = \mathcal{O}_F/\mathfrak{p}$ be the residue field at $\mathfrak{p}$ and define the reduction map

$$\begin{array}{rcl} \mathbf{M}_2(\mathcal{O}_{B,\,\mathfrak{p}}) & \rightarrow & \mathbf{M}_4(\mathbb{F}_{\mathfrak{p}}) \\ g & \mapsto & \tilde{g}. \end{array}$$

Now, choose a totally positive generator $\pi_{\mathfrak{p}}$ of $\mathfrak{p}$ and put

$$\Theta_1(\mathfrak{p}) := \Gamma\backslash \left\{ u \in \mathbf{M}_2(\mathcal{O}_B) \;\middle|\; u\bar{u}^t = \pi_{\mathfrak{p}}\mathbf{1}_2 \text{ and } \operatorname{rank}(\tilde{g}) = 2 \right\},$$
$$\Theta_2(\mathfrak{p}) := \Gamma\backslash \left\{ u \in \mathbf{M}_2(\mathcal{O}_B) \;\middle|\; u\bar{u}^t = \pi_{\mathfrak{p}}^2\mathbf{1}_2 \text{ and } \operatorname{rank}(\tilde{g}) = 1 \right\}.$$

We let $\mathcal{H}_0^2(N) = G(\hat{\mathbb{Z}})/U_0(N)$. Then the group Γ acts on $\mathcal{H}_0^2(N)$, thus on the space of functions $f : \mathcal{H}_0^2(N) \to V_k$ by

$$\forall x \in \mathcal{H}_0^2(N), \forall \gamma \in \Gamma, \qquad f|_k\gamma(x) := f(\gamma x)\gamma.$$

The following theorem is our main result.

Theorem 2. *There is an isomorphism of Hecke modules*

$$M_k^B(N) \xrightarrow{\sim} \left\{ f : \mathcal{H}_0^2(N) \to V_k \;\middle|\; f|_k\gamma = f,\, \gamma \in \Gamma \right\},$$

where the Hecke action on the right hand side is given by

$$f|_kT_1(\mathfrak{p}) = \sum_{u \in \Theta_1(\mathfrak{p})} f|_k u,$$
$$f|_kT_2(\mathfrak{p}) = \sum_{u \in \Theta_2(\mathfrak{p})} f|_k u.$$

Proof. We first observe that the sets $\Theta_1(\mathfrak{p})$ and $\Theta_2(\mathfrak{p})$ give global cosets representatives for the double cosets $\Delta_1(\mathfrak{p})$ and $\Delta_2(\mathfrak{p})$ respectively. Since the class number of the principal genus of G^B is one, the rest of the theorem follows using a similar argument as in [D1 2005, Proposition 3.1]. (See also [D3 2007] for additional details). □

It is not hard to see that Theorem 2 provides us with a description of the Hecke action on the zero-dimensional flag scheme $\mathcal{H}_0^2(N)$ defined on $\mathcal{O}_F/N$ that lends itself better to computation than the classical approach to Brandt matrices in terms of ideal classes or lattices. In the rest of this section, we explain the main steps of our algorithm.

2.1. **The quotient $\mathcal{H}_0^2(N)$.** Keeping the notations of the previous section, we recall that $N = \prod_{\mathfrak{p}|N} \mathfrak{p}^{e_{\mathfrak{p}}}$. Let $\mathfrak{p}$ be a prime dividing N and consider the free rank 4 $\left(\mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}^{e_{\mathfrak{p}}}\right)$-module $L = \left(\mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}^{e_{\mathfrak{p}}}\right)^4$ endowed with the symplectic pairing $\langle\, ,\, \rangle$ given by the matrix

$$J_2 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix},$$

where $\mathbf{1}_2$ is the identity matrix in $\mathbf{M}_2(\mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}^{e_{\mathfrak{p}}})$. Let M be a rank 2 $\left(\mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}^{e_{\mathfrak{p}}}\right)$-submodule which is a direct factor in L. We say that M is symplectic if $\langle u, v\rangle = 0$ for all $u, v \in M$. We recall that $\mathbf{GSp}_4(\mathcal{O}_{F_{\mathfrak{p}}})$ acts transitively on the set of symplectic $\left(\mathcal{O}_{F_\mathfrak{p}}/\mathfrak{p}^{e_\mathfrak{p}}\right)$-submodules of rank 2 and that the stabilizer of the submodule generated by $e_1 = (1,\,0,\,0,\,0)^T$ and $e_2 = (0,\,1,\,0,\,0)^T$ is $U_0(\mathfrak{p}^{e_\mathfrak{p}})$. The quotient $\mathcal{H}_0^2(\mathfrak{p}^{e_\mathfrak{p}}) = \mathbf{GSp}_4(\mathcal{O}_{F_\mathfrak{p}})/U_0(\mathfrak{p}^{e_\mathfrak{p}})$ is the set of $\left(\mathcal{O}_{F_\mathfrak{p}}/\mathfrak{p}^{e_\mathfrak{p}}\right)$-rational points of the Lagrange scheme defined over $\mathcal{O}_{F_\mathfrak{p}}/\mathfrak{p}^{e_\mathfrak{p}}$. Via the reduction map $\hat{\mathcal{O}}_F \to \mathcal{O}_F/N$, the quotient $G(\hat{\mathbb{Z}})/U_0(N)$ can be identified with the product

$$\mathcal{H}_0^2(N) = \prod_{\mathfrak{p}|N} \mathcal{H}_0^2(\mathfrak{p}^{e_\mathfrak{p}}).$$

The cardinality of $\mathcal{H}_0^2(N)$ is extremely useful and is determined using the following lemma.

Lemma 1. *Let $\mathfrak{p}$ be a prime in F and $e_\mathfrak{p} \geq 1$ an integer. Then, the cardinality of the set $\mathcal{H}_0^2(\mathfrak{p}^{e_\mathfrak{p}})$ is given by*

$$\#\mathcal{H}_0^2(\mathfrak{p}^{e_\mathfrak{p}}) = \mathbf{N}(\mathfrak{p})^{3(e_\mathfrak{p}-1)}(\mathbf{N}(\mathfrak{p})+1)(\mathbf{N}(\mathfrak{p})^2+1).$$

Proof. For $e_\mathfrak{p} = 1$, the cardinality of the Lagrange variety over the finite field $\mathbb{F}_\mathfrak{p} = \mathcal{O}_F/\mathfrak{p}$ is given by $(\mathbf{N}(\mathfrak{p})+1)(\mathbf{N}(\mathfrak{p})^2+1)$. Proceed by induction on $e_\mathfrak{p}$. □

2.2. **Brandt matrices.** Let $\mathcal{F} = \{x_1,\,\ldots,\,x_h\}$ be a fundamental domain for the action of Γ on $\mathcal{H}_0^2(N)$ and, for each i, let Γ_i be the stabilizer of x_i. It is well known that there is an isomorphism of complex spaces

$$\begin{array}{rcl} M_k^B(N) & \to & \displaystyle\bigoplus_{i=1}^{h} V_k^{\Gamma_i} \\ f & \mapsto & (f(x_i)), \end{array}$$

where $V_k^{\Gamma_i}$ is the subspace of Γ_i-invariants in V_k. For any $x,\,y \in \mathcal{H}_0^2(N)$, we let

$$\begin{aligned} \Theta_1(x,\,y,\,\mathfrak{p}) &:= \left\{u \in \Theta_1(\mathfrak{p}) \;\middle|\; \exists\gamma \in \Gamma,\ ux = \gamma y\right\}, \\ \Theta_2(x,\,y,\,\mathfrak{p}) &:= \left\{u \in \Theta_2(\mathfrak{p}) \;\middle|\; \exists\gamma \in \Gamma,\ ux = \gamma y\right\} \end{aligned}$$

Proposition 3. *The actions of the Hecke operators $T_s(\mathfrak{p})$, $s = 1,\,2$, are given by the Brandt matrices $\mathcal{B}_s(\mathfrak{p}) = (b_{ij}^s(\mathfrak{p}))$, where*

$$\begin{array}{rcl} b_{ji}^s(\mathfrak{p}) : V_k^{\Gamma_j} & \to & V_k^{\Gamma_i} \\ v & \mapsto & v \cdot \left(\displaystyle\sum_{u \in \boldsymbol{\Theta}_s(x_i,\,x_j,\,\mathfrak{p})} \gamma_u^{-1} u\right). \end{array}$$

Proof. The proof of Proposition 3 follows the lines of [D1 2005, §3]. □

2.3. **Splitting G^B at a prime $\mathfrak{p}$.** The splitting of G^B amounts to the one of the quaternion algebra B/F and we refer to [D1 2005] for further details.

2.4. **Computing the group $G^B(\mathbb{Z})$.** It is enough to compute the subgroup Γ consisting of the elements in $G^B(\mathbb{Z})$ with similitude factor 1. But it is easy to see that

$$\Gamma = \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \;\middle|\; u,\, v \in \mathcal{O}_B^1 \right\} \cup \left\{ \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} \;\middle|\; u,\, v \in \mathcal{O}_B^1 \right\},$$

where $\mathcal{O}_B^1$ is the group of norm 1 elements.

2.5. **Computing the sets $\Theta_1(\mathfrak{p})$ and $\Theta_2(\mathfrak{p})$.** Let us consider the quadratic form on the vector space $V = B^2$ given by

$$\begin{array}{rcl} V & \to & F \\ (a,\, b) & \mapsto & \|(a,\, b)\| := \mathbf{nr}(a) + \mathbf{nr}(b), \end{array}$$

where $\mathbf{nr}$ is the reduced norm on B. This determines an inner form

$$\begin{array}{rcl} V \times V & \to & F \\ (u,\, v) & \mapsto & \langle u,\, v\rangle. \end{array}$$

An element of $\Theta_1(\mathfrak{p})$ (resp. $\Theta_2(\mathfrak{p})$) is a unitary matrix $\gamma \in \mathbf{M}_2(\mathcal{O}_B)$ with respect to this inner form such that the norm of each row is $\pi_\mathfrak{p}$ (resp. $\pi_\mathfrak{p}^2$ and the rank of the reduced matrix is 1). So we first start by computing all the vectors $u = (a,\, b) \in \mathcal{O}_B^2$ such that $\|u\| = \pi_\mathfrak{p}$ (resp. $\|u\| = \pi_\mathfrak{p}^2$). And for each such vector u, we compute the vectors $v = (c,\, d) \in \mathcal{O}_B^2$ of the same norm such that $\langle u,\, v\rangle = 0$. The corresponding matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $\Theta_1(\mathfrak{p})$ (resp. $\Theta_2(\mathfrak{p})$) when its reduction mod $\mathfrak{p}$ has the appropriate rank. We list all these matrices up to equivalence and stop when we reach the right cardinality.

2.6. **The implementation of the algorithm.** The implementation of the algorithm is similar to that of [D1 2005]. However, it is important to note how we represent elements in $\mathcal{H}_0^2(N)$ so that we can retrieve them easily once stored. As in [D1 2005] we choose to work with the product

$$\mathcal{H}_0^2(N) = \prod_{\mathfrak{p}|N} \mathcal{H}_0^2(\mathfrak{p}^{e_\mathfrak{p}}).$$

Using Plucker's coordinates, we can view $\mathcal{H}_0^2(\mathfrak{p}^{e_\mathfrak{p}})$ as a closed subspace of $\mathbf{P}^5(\mathcal{O}_{F_\mathfrak{p}}/\mathfrak{p}^{e_\mathfrak{p}})$. We then represent each element in $\mathcal{H}_0^2(\mathfrak{p}^{e_\mathfrak{p}})$ by choosing a point $x = (a_0 : \cdots : a_5) = [u \wedge v] \in \mathbf{P}^5(\mathcal{O}_{F_\mathfrak{p}}/\mathfrak{p}^{e_\mathfrak{p}})$ such that the submodule M generated by u and v is a Lagrange submodule, and the first *invertible* coordinate is scaled to 1.

3. Numerical examples: $F = \mathbb{Q}(\sqrt{5})$ and $B = \left(\frac{-1,-1}{F}\right)$

In this section, we provide some numerical examples using the quadratic field $F = \mathbb{Q}(\sqrt{5})$. It is proven in K. Hashimoto and T. Ibukiyama [HI 1980] that, for the Hamilton quaternion algebra B over F, the class number of the principal genus of G^B is one. We use our algorithm to compute all the systems of Hecke eigenvalues of Hilbert-Siegel cusp forms of weight 3 and level N that are defined over real quadratic fields, where N runs over all prime ideals of norm less than 50. We then determine which of the forms we obtained are possible lifts of Hilbert cusp forms by comparing the Hecke eigenvalues for those primes.

3.1. **Tables of Hilbert-Siegel cusp forms of parallel weight 3.** In Table 1 below, we list all the systems of eigenvalues of Hilbert-Siegel cusp forms of weight 3 and level N that are defined over real quadratic fields, where N runs over all prime ideals in F of norm less than 50. Here are the conventions we use in the tables.

(1) For a quadratic field K of discriminant D, we let ω_D be a generator of the ring of integers $\mathcal{O}_K$ of K.
(2) The first row contains the level N, given in the format $(\mathrm{Norm}(N), \alpha)$ for some generator $\alpha \in F$ of N, and the dimensions of the relevant spaces.
(3) The second row lists the Hecke operators that have been computed.
(4) For each eigenform f, the Hecke eigenvalues are given in a row, and the last entry of that row indicates if the form f is a probable lift.
(5) The levels and the eigenforms are both listed up to Galois conjugation.

For an eigenform f and a given prime $\mathfrak{p} \nmid N$, let $a_1(\mathfrak{p},\, f)$ and $a_2(\mathfrak{p},\, f)$ be the eigenvalues of the Hecke operators $T_1(\mathfrak{p})$ and $T_2(\mathfrak{p})$, respectively. Then the Euler factor $L_{\mathfrak{p}}(f,\, s)$ is given by

$$L_{\mathfrak{p}}(f, s) = Q_{\mathfrak{p}}(q^{-s})^{-1},$$

where

$$\begin{aligned} Q_{\mathfrak{p}}(x) &= 1 - a_1(\mathfrak{p}, f)x + b_1(\mathfrak{p}, f)x^2 - a_1(\mathfrak{p}, f)q^{2k-3}x^3 + q^{4k-6}x^4, \\ b_1(\mathfrak{p}, f) &= a_1(\mathfrak{p}, f)^2 - a_2(\mathfrak{p}, f) - q^{2k-4}, \\ q &= \mathbf{N}(\mathfrak{p}). \end{aligned}$$

3.2. **Tables of Hilbert cusp forms of parellel weight 4.** In Table 2, we list all the Hilbert cusp forms of parallel weight 4 and level N that are defined over real quadratic fields, with N running over all prime ideals of norm less than 50. We use this data in order to determine the forms in Table 1 that are possible lifts from $\mathbf{GL}_2$.

3.3. **Lifts.** There are two types of lifts from $\mathbf{GL}_2$ to $\mathbf{GSp}_4$. The first one corresponds to the homomorphism of L-groups determined by the long root embedding into $\mathbf{GSp}_4$, and the second one by the short root embedding. (For further details, we refer to [LP 2002]). Let f be a Hilbert cusp form of parallel weight k and level N with Hecke eigenvalues $a(\mathfrak{p},\, f)$, where $\mathfrak{p}$ is a

$N=(4,2)$: $\dim M_3^B(N)=2$, $\dim S_3^B(N)=1$							
	$T_1(2)$	$T_2(2)$	$T_1(\sqrt{5})$	$T_2(\sqrt{5})$	$T_1(3)$	$T_2(3)$	Lift?
f_1	-4	0	20	-36	140	580	*yes*

$N=(5,2+\omega_5)$: $\dim M_3^B(N)=2$, $\dim S_3^B(N)=1$							
	$T_1(2)$	$T_2(2)$	$T_1(\sqrt{5})$	$T_2(\sqrt{5})$	$T_1(3)$	$T_2(3)$	Lift?
f_1	20	15	-5	0	40	-420	*yes*

$N=(9,3)$: $\dim M_3^B(N)=3$, $\dim S_3^B(N)=2$							
	$T_1(2)$	$T_2(2)$	$T_1(\sqrt{5})$	$T_2(\sqrt{5})$	$T_1(3)$	$T_2(3)$	Lift?
f_1	$25-3\omega_{41}$	$40-15\omega_{41}$	$30+6\omega_{41}$	$24+36\omega_{41}$	-9	0	*yes*

$N=(11,3+\omega_5)$: $\dim M_3^B(N)=3$, $\dim S_3^B(N)=2$							
	$T_1(2)$	$T_2(2)$	$T_1(\sqrt{5})$	$T_2(\sqrt{5})$	$T_1(3)$	$T_2(3)$	Lift?
f_1	24	35	34	48	88	60	*yes*
f_2	-20	35	-10	4	0	60	*no*

$N=(19,4+\omega_5)$: $\dim M_3^B(N)=5$, $\dim S_3^B(N)=4$							
	$T_1(2)$	$T_2(2)$	$T_1(\sqrt{5})$	$T_2(\sqrt{5})$	$T_1(3)$	$T_2(3)$	Lift?
f_1	4	11	-20	28	6	76	*no*
f_2	7	-50	15	-66	73	-90	*yes*
f_3	$24+\omega_{161}$	$35+5\omega_{161}$	$36-\omega_{161}$	$60-6\omega_{161}$	$98-3\omega_{161}$	$160-30\omega_{161}$	*yes*

$N=(29,5+\omega_5)$: $\dim M_3^B(N)=9$, $\dim S_3^B(N)=8$							
	$T_1(2)$	$T_2(2)$	$T_1(\sqrt{5})$	$T_2(\sqrt{5})$	$T_1(3)$	$T_2(3)$	Lift?
f_1	-4	11	10	20	30	60	*no*
f_2	8	-45	30	24	50	-320	*yes*
f_3	17	0	9	-102	86	40	*yes*

$N=(31,5+2\omega_5)$: $\dim M_3^B(N)=12$, $\dim S_3^B(N)=11$							
	$T_1(2)$	$T_2(2)$	$T_1(\sqrt{5})$	$T_2(\sqrt{5})$	$T_1(3)$	$T_2(3)$	Lift?
f_1	13	-20	20	-36	76	-60	*yes*

$N=(41,6+\omega_5)$: $\dim M_3^B(N)=19$, $\dim S_3^B(N)=18$							
	$T_1(2)$	$T_2(2)$	$T_1(\sqrt{5})$	$T_2(\sqrt{5})$	$T_1(3)$	$T_2(3)$	Lift?
f_1	10	20	-10	29	30	-20	*no*
f_2	-1	1	5	14	-2	-56	*no*
f_3	27	50	40	84	124	420	*yes*
f_4	-12	19	30	65	0	0	*no*
f_5	$16-2\omega_{21}$	$-5-10\omega_{21}$	$21+4\omega_{21}$	$-30+24\omega_{21}$	$72-2\omega_{21}$	$-100-20\omega_{21}$	*yes*
f_6	$2-6\omega_5$	$11-2\omega_5$	$8+4\omega_5$	$11-4\omega_5$	$-12+54\omega_5$	$160+40\omega_5$	*no*

$N=(49,7)$: $\dim M_3^B(N)=26$, $\dim S_3^B(N)=25$							
	$T_1(2)$	$T_2(2)$	$T_1(\sqrt{5})$	$T_2(\sqrt{5})$	$T_1(3)$	$T_2(3)$	Lift?
f_1	5	-60	46	120	40	-420	*yes*
f_2	$4+4\omega_{65}$	$32+3\omega_{65}$	$12-4\omega_{65}$	$44-4\omega_{65}$	$-6-12\omega_{65}$	$145+8\omega_{65}$	*no*

TABLE 1. **Hilbert-Siegel eigenforms of weight 3**

N		$(4,2)$	$(5,2+\omega_5)$	$(9,3)$	$(11,3+\omega_5)$
$\mathbf{N}(\mathfrak{p})$	$\mathfrak{p}$	$a(\mathfrak{p},\ f_1)$	$a(\mathfrak{p},\ f_1)$	$a(\mathfrak{p},\ f_1)$	$a(\mathfrak{p},\ f_1)$
4	2	-4	0	$5-3\omega_{41}$	4
5	$2+\omega_5$	-10	-5	$6\omega_{41}$	4
9	3	50	-50	-9	-2
11	$3+2\omega_5$	-28	32	$-18-6\omega_{41}$	-10
11	$3+\omega_5$	-28	32	$-18-6\omega_{41}$	-11
19	$4+3\omega_5$	60	100	$-40+24\omega_{41}$	-94
19	$4+\omega_5$	60	100	$-40+24\omega_{41}$	28

N		$(19,4+\omega_5)$		$(29,5+\omega_5)$	
$\mathbf{N}(\mathfrak{p})$	$\mathfrak{p}$	$a(\mathfrak{p},\ f_1)$	$a(\mathfrak{p},\ f_2)$	$a(\mathfrak{p},\ f_1)$	$a(\mathfrak{p},\ f_2)$
4	2	-13	$5-\omega_{161}$	-12	-3
5	$2+\omega_5$	-15	$5+\omega_{161}$	0	-21
9	3	-17	$5+3\omega_{161}$	-40	-4
11	$3+2\omega_5$	-6	$2+8\omega_{161}$	-68	37
11	$3+\omega_5$	33	$7-7\omega_{161}$	30	-66
19	$4+3\omega_5$	-139	$-15-9\omega_{161}$	-28	-40
19	$4+\omega_5$	19	-19	84	-9

N		$(31,5+2\omega_5)$	$(41,6+\omega_5)$	
$\mathbf{N}(\mathfrak{p})$	$\mathfrak{p}$	$a(\mathfrak{p},\ f_1)$	$a(\mathfrak{p},\ f_1)$	$a(\mathfrak{p},\ f_2)$
4	2	-7	7	$-4-2\omega_{21}$
5	$2+\omega_5$	-10	10	$-9+4\omega_{21}$
9	3	-14	34	$-18-2\omega_{21}$
11	$3+2\omega_5$	-20	-60	-19
11	$3+\omega_5$	-28	-2	$-24-4\omega_{21}$
19	$4+3\omega_5$	-12	74	$4-50\omega_{21}$
19	$4+\omega_5$	28	16	$-29+44\omega_{21}$

N		$(49,7)$	
$\mathbf{N}(\mathfrak{p})$	$\mathfrak{p}$	$a(\mathfrak{p},\ f_1)$	$a(\mathfrak{p},\ f_2)$
4	2	-15	-2
5	$2+\omega_5$	16	-10
9	3	-50	-11
11	$3+2\omega_5$	-8	$-7-28\omega_{13}$
11	$3+\omega_5$	-8	$-35+28\omega_{13}$
19	$4+3\omega_5$	-110	$-26+14\omega_{13}$
19	$4+\omega_5$	-110	$-12-14\omega_{13}$

TABLE 2. **Hilbert eigenforms of weight 4**

prime not dividing N. Let ϕ be the lift of f to $\mathbf{GSp}_4$ via the long root, and ψ the one via the short root. Then the Hecke eigenvalues of ϕ are given by

$$a_1(\mathfrak{p},\ \phi) = a(\mathfrak{p},\ f)\ \mathbf{N}(\mathfrak{p})^{\frac{4-k}{2}} + \mathbf{N}(\mathfrak{p})^2 + \mathbf{N}(\mathfrak{p})$$
$$a_2(\mathfrak{p},\ \phi) = a(\mathfrak{p},\ f)\ \mathbf{N}(\mathfrak{p})^{\frac{4-k}{2}}(\mathbf{N}(\mathfrak{p})+1) + \mathbf{N}(\mathfrak{p})^2 - 1,$$

and the Hecke eigenvalues of ψ are given by

$$a_1(\mathfrak{p},\,\psi) = a(\mathfrak{p},\,f)^3\ \mathbf{N}(\mathfrak{p})^{\frac{6-3k}{2}} - 2\ a(\mathfrak{p},\,f)\ \mathbf{N}(\mathfrak{p})^{\frac{4-k}{2}}$$
$$a_2(\mathfrak{p},\,\psi) = a(\mathfrak{p},\,f)^4\ \mathbf{N}(\mathfrak{p})^{4-2k} - 3\ a(\mathfrak{p},\,f)^2\ \mathbf{N}(\mathfrak{p})^{3-k} + \mathbf{N}(\mathfrak{p})^2 - 1.$$

The second lift ψ is the so-called symmetric cube lifting.

Recently, Ramakrishnan and Shahidi [RS 2007] showed the existence of symmetric cube liftings for non-CM elliptic curves $E/\mathbb{Q}$ to $\mathbf{GSp}_4/\mathbb{Q}$. This result should hold for other totally real number fields, and we expect the levels of those lifts to be of Klingen type. But all the lifts in Table 1 are via the long root embedding. Since there are modular elliptic curves over $\mathbb{Q}(\sqrt{5})$ whose conductors have norm 31, 41 and 49, there are probably more lifts than appear in Table 1.

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