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Viscosity, Black Holes, and Quantum Field Theory

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Abstract We review recent progress in applying the AdS/CFT correspondence to finite-temperature field theory. In particular, we show how the hydrodynamic behavior of field theory is reflected in the low-momentum limit of correlation functions computed through a real-time AdS/CFT prescription, which we formulate. We also show how the hydrodynamic modes in field theory correspond to the low-lying quasinormal modes of the AdS black p-brane metric. We provide a proof of the universality of the viscosity/entropy ratio within a class of theories with gravity duals and formulate a viscosity bound conjecture. Possible implications for real systems are mentioned.

CONTENTS

1 INTRODUCTION

This review is about the recently emerging connection, through the gauge/gravity correspondence, between hydrodynamics and black hole physics.

The study of quantum field theory at high temperature has a long history. It was first motivated by the Big Bang cosmology when it was hoped that early phase transitions might leave some imprints on the Universe [1]. One of those phase transitions is the QCD phase transitions (which could actually be a crossover) which happened at a temperature around $T_c \sim 200$ MeV, when matter turned from a gas of quarks and gluons (the quark-gluon plasma, or QGP) into a gas of hadrons.

An experimental program was designed to create and study the QGP by colliding two heavy atomic nuclei. Most recent experiments are conducted at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory. Although significant circumstantial evidence for the QGP was accumulated [2], a theoretical interpretation of most of the experimental data proved difficult, because the QGP created at RHIC is far from being a weakly coupled gas of quarks and gluons. Indeed, the temperature of the plasma, as inferred from the spectrum of final particles, is only approximately 170 MeV, near the confinement scale of QCD. This is deep in the nonperturbative regime of QCD, where reliable theoretical tools are lacking. Most notably, the kinetic coefficients of the QGP, which enter the hydrodynamic equations (reviewed in Sec. 2), are not theoretically computable at these temperatures.

The paucity of information about the kinetic coefficients of the QGP in particular and of strongly coupled thermal quantum field theories in general is one of the main reasons for our interest in their computation in a class of strongly coupled field theories, even though this class does not include QCD. The necessary technological tool is the anti–de Sitter–conformal field theory (AdS/CFT) correspondence [3, 4, 5], discovered in the investigation of D-branes in string theory. This correspondence allows one to describe the thermal plasmas in these theories in terms of black holes in AdS space. The AdS/CFT correspondence is reviewed in Sec. 3.

The first calculation of this type, that of the shear viscosity in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory [6], is followed by the theoretical work to establish the rules of real-time finite-temperature AdS/CFT correspondence [7, 8]. Applications of these rules to various special cases [9, 10, 11, 12] clearly show that even very exotic field theories, when heated up to finite temperature, behave hydrodynamically at large distances and time scales (provided that the number of space-time dimensions is 2+1 or higher). This development is reviewed in Sec. 4. Moreover, the way AdS/CFT works reveals very deep connections to properties of black holes in classical gravity. For example, the hydrodynamic modes of a thermal medium are mapped, through the correspondence, to the low-lying quasi-normal modes of a black-brane metric. It seems that our understanding of the connection between hydrodynamics and black hole physics is still incomplete; we may understand more about gravity by studying thermal field theories. One idea along this direction is reviewed in Sec. 5.

From the point of view of heavy-ion (QGP) physics, a particularly interesting finding has been the formulation of a conjecture on the lowest possible value of the ratio of viscosity and volume density of entropy. This conjecture was motivated by the universality of this ratio in theories with gravity duals. This is reviewed in Sec. 6.

This review is written primarily for readers with a background in QCD and QGP physics who are interested in learning about AdS/CFT correspondence and its applications to finite-temperature field theory. Some parts of this review (for example, the section about hydrodynamics) should be useful for readers with a string theory or general relativity background who are interested in the connection between string theory, gravity, and hydrodynamics. The perspectives here are shaped by our personal taste and therefore may appear narrow, but the authors believe that this review may serve as the starting point to explore the much richer original literature.

In this review we use the "mostly plus" metric signature $-+++$.

2 HYDRODYNAMICS

From the modern perspective, hydrodynamics [13] is best thought of as an effective theory, describing the dynamics at large distances and time-scales. Unlike the familiar effective field theories (for example, the chiral perturbation theory), it is normally formulated in the language of equations of motion instead of an action principle. The reason for this is the presence of dissipation in thermal media.

In the simplest case, the hydrodynamic equations are just the laws of conservation of energy and momentum,

$$\partial_\mu T^{\mu\nu} = 0\,. \tag{1}$$

To close the system of equations, we must reduce the number of independent elements of $T^{\mu\nu}$. This is done through the assumption of *local thermal equilibrium:* If perturbations have long wavelengths, the state of the system, at a given time, is determined by the temperature as a function of coordinates $T(\mathbf{x})$ and the local fluid velocity u^μ, which is also a function of coordinates $u^\mu(\mathbf{x})$. Because $u_\mu u^\mu = -1$, only three components of u^μ are independent. The number of hydrodynamic variables is four, equal to the number of equations.

In hydrodynamics we express $T^{\mu\nu}$ through $T(x)$ and $u^\mu(x)$ through the so-called constitutive equations. Following the standard procedure of effective field theories, we expand in powers of spatial derivatives. To zeroth order, $T^{\mu\nu}$ is given by the familiar formula for ideal fluids,

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}\,, \tag{2}$$

where ϵ is the energy density, and P is the pressure. Normally one would stop at this leading order, but qualitatively new effects necessitate going to the next order. Indeed, from Eq. 2 and the thermodynamic relations $d\epsilon = TdS$, $dP = sdT$, and $\epsilon + P = Ts$ (s is the entropy per unit volume), one finds that entropy is conserved [14]

$$\partial_\mu (s u^\mu) = 0\,. \tag{3}$$

Thus, to have entropy production, one needs to go to the next order in the derivative expansion.

At the next order, we write

$$T^{\mu\nu} = (\epsilon + P)g^{\mu\nu} + P u^\mu u^\nu - \sigma^{\mu\nu}\,, \tag{4}$$

where $\sigma^{\mu\nu}$ is proportional to derivatives of $T(x)$ and $u^\mu(x)$ and is termed the dissipative part of $T^{\mu\nu}$. To write these terms, let us first fix a point x and go to the local rest frame where $u^i(x) = 0$.

In this frame, in principle one can have dissipative corrections to the energy-momentum density $T^{0\mu}$. However, one recalls that the choice of T and u^μ is arbitrary, and thus one can always redefine them so that these corrections vanish, $\sigma^{00} = \sigma^{0i} = 0$, and so at a point x,

$$T^{00} = \epsilon, \qquad T^{0i} = 0\,. \tag{5}$$

The only nonzero elements of the dissipative energy-momentum tensor are σ_{ij}. To the next-to-leading order there are extra contributions whose forms are dictated by rotational symmetry:

$$\sigma_{ij} = \eta\left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\partial_k u^k\right) + \zeta\delta_{ij}\partial_k u^k\,. \tag{6}$$

Going back to the general frame, we can now write the dissipative part of the energy-momentum tensor as

$$\sigma^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\left[\eta\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}g_{\alpha\beta}\partial_\lambda u^\lambda\right) + \zeta g_{\alpha\beta}\partial_\lambda u^\lambda\right]\,, \tag{7}$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projection operator onto the directions perpendicular to u^μ.

If the system contains a conserved current, there is an additional hydrodynamic equation related to the current conservation,

$$\partial_\mu j^\mu = 0\,. \tag{8}$$

The constitutive equation contains two terms:

$$j^\mu = \rho u^\mu - D P^{\mu\nu}\partial_\nu \alpha\,, \tag{9}$$

where ρ is the charge density in the fluid rest frame and D is some constant. The first term corresponds to convection, the second one to diffusion. In the fluid rest frame, $\mathbf{j} = -D\nabla\rho$, which is Fick's law of diffusion, with D being the diffusion constant.

2.1 Kubo's Formula For Viscosity

As mentioned above, the hydrodynamic equations can be thought of as an effective theory describing the dynamics of the system at large lengths and time scales. Therefore one should be able to use these equations to extract information about the low-momentum behavior of Green's functions in the original theory.

For example, let us recall how the two-point correlation functions can be extracted. If we couple sources $J_a(\mathbf{x})$ to a set of (bosonic) operators $O_a(x)$, so that the new action is

$$S = S_0 + \int_x J_a(x) O_a(x)\,, \tag{10}$$

then the source will introduce a perturbation of the system. In particular, the average values of O_a will differ from the equilibrium values, which we assume to be zero. If J_a are small, the perturbations are given by the linear response theory as

$$\langle O_a(x)\rangle = -\int_y G^R_{ab}(x-y) J_b(y)\,, \tag{11}$$

where G^R_{ab} is the retarded Green's function

$$iG^R_{ab}(x-y) = \theta(x^0 - y^0)\langle[O_a(x),\, O_b(y)]\rangle\,. \tag{12}$$

The fact that the linear response is determined by the retarded (and not by any other) Green's function is obvious from causality: The source can influence the system only after it has been turned on.

Thus, to determine the correlation functions of $T^{\mu\nu}$, we need to couple a weak source to $T^{\mu\nu}$ and determine the average value of $T^{\mu\nu}$ after this source is turned on. To find these correlators at low momenta, we can use the hydrodynamic theory. So far in our treatment of hydrodynamics we have included no source coupled to $T^{\mu\nu}$. This deficiency can be easily corrected, as the source of the energy-momentum tensor is the metric $g_{\mu\nu}$. One must generalize the hydrodynamic equations to curved space-time and from it determine the response of the thermal medium to a weak perturbation of the metric. This procedure is rather straightforward and in the interest of space is left as an exercise to the reader.

Here we concentrate on a particular case when the metric perturbation is homogeneous in space but time dependent:

$$g_{ij}(t,\mathbf{x}) = \delta_{ij} + h_{ij}(t), \qquad h_{ij} \ll 1 \tag{13}$$

$$g_{00}(t,\mathbf{x}) = -1, \qquad g_{0i}(t,\mathbf{x}) = 0\,. \tag{14}$$

Moreover, we assume the perturbation to be traceless, $h_{ii}=0$. Because the perturbation is spatially homogeneous, if the fluid moves, it can only move uniformly: $u^i = u^i(t)$. However, this possibility can be ruled out by parity, so the fluid must remain at rest all the time: $u^\mu = (1,0,0,0)$. We now compute the dissipative part of the stress-energy tensor. The generalization of Eq. 7 to curved space-time is

$$\sigma^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\left[\eta(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) + \left(\zeta - \frac{2}{3}\eta\right) g_{\alpha\beta}\nabla\cdot u\right]. \tag{15}$$

Substituting $u^\mu = (1,0,0,0)$ and $g_{\mu\nu}$ from Eq. 13, we find only contributions to the traceless spatial components, and these contributions come entirely from the Christoffel symbols in the covariant derivatives. For example,

$$\sigma_{xy} = 2\eta\Gamma^0_{xy} = \eta\partial_0 h_{xy}\,. \tag{16}$$

By comparison with the expectation from the linear response theory, this equation means that we have found the zero spatial momentum, low-frequency limit of the retarded Green's function of T^{xy}:

$$G^R_{xy,xy}(\omega,\mathbf{0}) = \int dt\, d\mathbf{x}\, e^{i\omega t}\theta(t)\langle[T_{xy}(t,\mathbf{x}),\, T_{xy}(0,\mathbf{0})]\rangle = -i\eta\omega + O(\omega^2) \tag{17}$$

(modulo contact terms). We have, in essence, derived the Kubo's formula relating the shear viscosity and a Green's function:

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega}\,\mathrm{Im}\, G^R_{xy,xy}(\omega,\mathbf{0})\,. \tag{18}$$

There is a similar Kubo's relation for the charge diffusion constant D.

2.2 Hydrodynamic Modes

If one is interested only in the locations of the poles of the correlators, one can simply look for the normal modes of the linearized hydrodynamic equations, that is, solutions that behave as $e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$. Owing to dissipation, the frequency $\omega(\mathbf{k})$ is complex. For example, the equation of charge diffusion,

$$\partial_t\rho - D\nabla^2\rho = 0, \tag{19}$$

corresponds to a pole in the current-current correlator at $\omega = -iDk^2$.

To find the poles in the correlators between elements of the stress-energy tensor one can, without loss of generality, choose the coordinate system so that $\mathbf{k}$ is aligned along the x^3-axis: $\mathbf{k} = (0, 0, k)$. Then one can distinguish two types of normal modes:

1. Shear modes correspond to the fluctuations of pairs of components T^{0a} and T^{3a}, where $a = 1, 2$. The constitutive equation is

$$T^{3a} = -\eta \partial_3 u^a = -\frac{\eta}{\epsilon + P} \partial_3 T^{0a} \,, \tag{20}$$

and the equation for T^{0a} is

$$\partial_t T^{0a} - \frac{\eta}{\epsilon + P} \partial_3^2 T^{0a} = 0 \,. \tag{21}$$

That is, it has the form of a diffusion equation for T^{0a}. Substituting $e^{-i\omega t + ikx^3}$ into the equation, one finds the dispersion law

$$\omega = -i \frac{\eta}{\epsilon + P} k^2 \,. \tag{22}$$

2. Sound modes are fluctuations of T^{00}, T^{03}, and T^{33}. There are now two conservation equations, and by diagonalizing them one finds the dispersion law

$$\omega = c_s k - \frac{i}{2} \left(\frac{4}{3} \eta + \zeta \right) \frac{k^2}{\epsilon + P} \,, \tag{23}$$

where $c_s = \sqrt{dP/d\epsilon}$. This is simply the sound wave, which involves the fluctuation of the energy density. It propagates with velocity c_s, and its damping is related to a linear combination of shear and bulk viscosities.

In CFTs it is possible to use conformal Ward identities to show that the bulk viscosity vanishes: $\zeta = 0$. Hence, we shall concentrate our attention on the shear viscosity η.

2.3 Viscosity In Weakly Coupled Field Theories

We now briefly consider the behavior of the shear viscosity in weakly coupled field theories, with the $\lambda\phi^4$ theory as a concrete example. At weak coupling, there is a separation between two length scales: The mean free path of particles is much larger than the distance scales over which scatterings occur. Each scattering event takes a time of order T^{-1} (which can be thought of as the time required for final particles to become on-shell). The mean free path $\ell_{\rm mfp}$ can be estimated from the formula

$$\ell_{\rm mfp} \sim \frac{1}{n\sigma v} \,, \tag{24}$$

where n is the density of particles, σ is the typical scattering cross section, and v is the typical particle velocity. Inserting the values for thermal $\lambda\phi^4$ theory, $n \sim T^3$, $\sigma \sim \lambda^2 T^{-2}$, and $v \sim 1$, one finds

$$\ell_{\rm mfp} \sim \frac{1}{\lambda^2 T} \gg \frac{1}{T} \,. \tag{25}$$

The viscosity can be estimated from kinetic theory to be

$$\eta \sim \epsilon \ell_{\rm mfp} \,, \tag{26}$$

where ϵ is the energy density. From $\epsilon \sim T^4$ and the estimate of $\ell_{\rm mft}$, one finds

$$\eta \sim \frac{T^3}{\lambda^2}\,. \tag{27}$$

In particular, the weaker the coupling λ, the larger the viscosity η. This behavior is explained by the fact that the viscosity measures the rate of momentum diffusion. The smaller λ is, the longer a particle travels before colliding with another one, and the easier the momentum transfer.

It may appear counterintuitive that viscosity tends to infinity in the limit of zero coupling $\lambda \to 0$: At zero coupling there is no dissipation, so should the viscosity be zero? The confusion arises owing to the fact that the hydrodynamic theory, and hence the notion of viscosity, makes sense only on distances much larger than the mean free path of particles. If one takes $\lambda \to 0$, then to measure the viscosity one has to do the experiment at larger and larger length scales. If one fixes the size of the experiment and takes $\lambda \to 0$, dissipation disappears, but it does not tell us anything about the viscosity.

As will become apparent below, a particularly interesting ratio to consider is the ratio of shear viscosity and entropy density s. The latter is proportional to T^3; thus

$$\frac{\eta}{s} \sim \frac{1}{\lambda^2}\,. \tag{28}$$

One has $\eta/s \gg 1$ for $\lambda \ll 1$. This is a common feature of weakly coupled field theories. Extrapolating to $\lambda \sim 1$, one finds $\eta/s \sim 1$. We shall see that theories with gravity duals are strongly coupled, and η/s is of order one. More surprisingly, this ratio is the same for all theories with gravity duals.

To compute rather than estimate the viscosity, one can use Kubo's formula. It turns out that one has to sum an infinite number of Feynman graphs to even find the viscosity to leading order. Another way that leads to the same result is to first formulate a kinetic Boltzmann equation for the quasi-particles as an intermediate effective description, and then derive hydrodynamics by taking the limit of very long lengths and time scales in the kinetic equation. Interested readers should consult Refs. [15, 16] for more details.

3 AdS/CFT CORRESPONDENCE

3.1 Review Of AdS/CFT Correspondence At Zero Temperature

This section briefly reviews the AdS/CFT correspondence at zero temperature. It contains only the minimal amount of materials required to understand the rest of the review. Further information can be found in existing reviews and lecture notes [17, 18].

The original example of AdS/CFT correspondence is between $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory and type IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ space. Let us describe the two sides of the correspondence in some more detail.

The $\mathcal{N} = 4$ SYM theory is a gauge theory with a gauge field, four Weyl fermions, and six real scalars, all in the adjoint representation of the color group. Its Lagrangian can be written down explicitly, but is not very important for our purposes. It has a vanishing beta function and is a conformal field theory (CFT) (thus the CFT in AdS/CFT). In our further discussion, we frequently use the generic terms "field theory" or CFT for the $\mathcal{N} = 4$ SYM theory.

On the string theory side, we have type IIB string theory, which contains a finite number of massless fields, including the graviton, the dilaton Φ, some other fields (forms) and their fermionic superpartners, and an infinite number of massive string excitations. It has two parameters: the string length l_s (related to the slope parameter α' by $\alpha' = l_s^2$) and the string coupling g_s. In the long-wavelength limit, when all fields vary over length scales much larger than l_s, the massive modes decouple and one is left with type IIB supergravity in 10 dimensions, which can be described by an action [19]

$$S_{\rm SUGRA} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g}\,e^{-2\Phi}\left(\mathcal{R} + 4\,\partial^\mu\Phi\partial_\mu\Phi + \cdots\right)\,, \tag{29}$$

where κ_{10} is the 10-dimensional gravitational constant,

$$\kappa_{10} = \sqrt{8\pi G} = 8\pi^{7/2} g_s l_s^4\,, \tag{30}$$

and $\cdots$ stay for the contributions from fields other than the metric and the dilaton. One of these fields is the five-form F_5, which is constrained to be self-dual. The type IIB string theory lives is a 10-dimensional space-time with the following metric:

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\mathbf{x}^2) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2\,. \tag{31}$$

The metric is a direct product of a five-dimensional sphere ($d\Omega_5^2$) and another five-dimensional space-time spanned by t, $\mathbf{x}$, and r. An alternative form of the metric is obtained from Eq. (31) by a change of variable $z = R^2/r$,

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + d\mathbf{x}^2 + dz^2) + R^2 d\Omega_5^2\,. \tag{32}$$

Both coordinates r and z are known as the radial coordinate. The limiting value $r = \infty$ (or $z = 0$) is the boundary of the AdS space.

It is a simple exercise to check that the $(t, \mathbf{x}, r)$ part of the metric is a space with constant negative curvature, or an anti de-Sitter (AdS) space. To support the metric (31) (i.e., to satisfy the Einstein equation) there must be some background matter field that gives a stress-energy tensor in the form of a negative cosmological constant in AdS$_5$ and a positive one in S^5. Such a field is the self-dual five-form field F_5 mentioned above.

Field theory has two parameters: the number of colors N and the gauge coupling g. When the number of colors is large, it is the 't Hooft coupling $\lambda = g^2N$ that controls the perturbation theory. On the string theory side, the parameters are g_s, l_s, and radius R of the AdS space. String theory and field theory each have two dimensionless parameters which map to each other through the following relations:

$$g^2 = 4\pi g_s, \tag{33}$$

$$g^2 N_c = \frac{R^4}{l_s^4}\,. \tag{34}$$

Equation (33) tells us that, if one wants to keep string theory weakly interacting, then the gauge coupling in field theory must be small. Equation (34) is particularly interesting. It says that the large 't Hooft coupling limit in field theory corresponds to the limit when the curvature radius of space-time is much larger than the string length l_s. In this limit, one can reliably decouple the massive string modes and reduce string theory to supergravity. In the limit $g_s \ll 1$ and $R \gg l_s$, one has classical supergravity instead of string theory. The practical utility of the AdS/CFT correspondence comes, in large part, from its ability to deal with the strong coupling limit in gauge theory.

One can perform a Kaluza-Klein reduction [20] by expanding all fields in S^5 harmonics. Keeping only the lowest harmonics, one finds a five-dimensional theory with the massless dilaton, SO(6) gauge bosons, and gravitons [21]:

$$S_{\rm 5D} = \frac{N^2}{8\pi^2 R^3} \int d^5x \left(\mathcal{R}_{\rm 5D} - 2\Lambda - \frac{1}{2}\partial^\mu \Phi \partial_\mu \Phi - \frac{R^2}{8} F^a_{\mu\nu} F^{a\mu\nu} + \cdots \right) . \tag{35}$$

In AdS/CFT, an operator O of field theory is put in a correspondence with a field ϕ ("bulk" field) in supergravity. We elaborate on this correspondence below; here we keep the operator and the field unspecified. In the supergravity approximation, the mathematical statement of the correspondence is

$$Z_{\rm 4D}[J] = e^{iS[\phi_{\rm cl}]} . \tag{36}$$

On the left is the partition function of a field theory, where the source J coupled to the operator O is included:

$$Z_{\rm 4D}[J] = \int D\phi \, \exp\left(iS + i \int d^4x \, JO \right) . \tag{37}$$

On the right, $S[\phi_{\rm cl}]$ is the classical action of the classical solution $\phi_{\rm cl}$ to the field equation with the boundary condition:

$$\lim_{z\to 0} \frac{\phi_{\rm cl}(z,x)}{z^\Delta} = J(x) . \tag{38}$$

Here Δ is a constant that depends on the nature of the operator O (namely, on its spin and dimension). In the simplest case, $\Delta = 0$, and the boundary condition becomes $\phi_{\rm cl}(z{=}0) = J$. Differentiating Eq. (36) with respect to J, one can find the correlation functions of O. For example, the two-point Green's function of O is obtained by differentiating $S_{\rm cl}[\phi]$ twice with respect to the boundary value of ϕ,

$$G(x-y) = -i\langle TO(x)O(y)\rangle = -\left.\frac{\delta^2 S[\phi_{\rm cl}]}{\delta J(x)\delta J(y)}\right|_{\phi(z=0)=J} . \tag{39}$$

The AdS/CFT correspondence thus maps the problem of finding quantum correlation functions in field theory to a classical problem in gravity. Moreover, to find two-point correlation functions in field theory, one can be limited to the quadratic part of the classical action on the gravity side.

The complete operator to field mapping can be found in Refs. [5, 17]. For our purpose, the following is sufficient:

- The dilaton Φ corresponds to $O = -\mathcal{L} = \frac{1}{4}F^2_{\mu\nu} + \cdots$, where $\mathcal{L}$ is the Lagrangian density.
- The gauge field A^a_μ corresponds to the conserved R-charge current $J^{a\mu}$ of field theory.
- The metric tensor corresponds to the stress-energy tensor $T^{\mu\nu}$. More precisely, the partition function of the four-dimensional field theory in an external metric $g^0_{\mu\nu}$ is equal to

$$Z_{\rm 4D}[g^0_{\mu\nu}] = \exp(iS_{\rm cl}[g_{\mu\nu}]) , \tag{40}$$

where the five-dimensional metric $g_{\mu\nu}$ satisfies the Einstein's equations and has the following asymptotics at $z=0$:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{R^2}{z^2}(dz^2 + g^0_{\mu\nu}dx^\mu dx^\nu)\,. \tag{41}$$

From the point of view of hydrodynamics, the operator $\frac{1}{4}F^2$ is not very interesting because its correlator does not have a hydrodynamic pole. In contrast, we find the correlators of the R-charge current and the stress-energy tensor to contain hydrodynamic information.

We simplify the graviton part of the action further. Our two-point functions are functions of the momentum $p=(\omega,\mathbf{k})$. We can choose spatial coordinates so that $\mathbf{k}$ points along the x^3-axis. This corresponds to perturbations that propagate along the x^3 direction: $h_{\mu\nu}=h_{\mu\nu}(t,r,x^3)$. These perturbations can be classified according to the representations of the O(2) symmetry of the (x^1,x^2) plane. Owing to that symmetry, only certain components can mix; for example, h_{12} does not mix with any other components, whereas components h_{01} and h_{31} mix only with each other. We assume that only these three metric components are nonzero and introduce shorthand notations

$$\phi = h^1_2, \qquad a_0 = h^1_0, \qquad a_3 = h^1_3\,. \tag{42}$$

The quadratic part of the graviton action acquires a very simple form in terms of these fields:

$$S_{\rm quad} = \frac{N^2}{8\pi^2R^3}\int d^4x\,dr\,\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4g^2_{\rm eff}}g^{\mu\alpha}g^{\nu\beta}f_{\mu\nu}f_{\alpha\beta}\right)\,, \tag{43}$$

where $f_{\mu\nu}=\partial_\mu a_\nu - \partial_\nu a_\mu$, and $g^2_{\rm eff}=g_{xx}$. In deriving Eq. (43), our only assumption about the metric is that it has a diagonal form,

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{xx}d\mathbf{x}^2\,, \tag{44}$$

so it can also be used below for the finite-temperature metric.

As a simple example, let us compute the two-point correlation function of T^{xy}, which corresponds to ϕ in gravity. The field equation for ϕ is

$$\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi) = 0\,. \tag{45}$$

The solution to this equation, with the boundary condition $\phi(p,z=0)=\phi_0(p)$, can be written as

$$\phi(p,z) = f_p(z)\phi_0(p)\,, \tag{46}$$

where the mode function $f_p(z)$ satisfies the equation

$$\left(\frac{f_p'}{z^3}\right)' - \frac{p^2}{z^3}f_p = 0 \tag{47}$$

with the boundary condition $f_p(0)=1$. The mode equation (47) can be solved exactly. Assuming p is spacelike, $p^2>0$, the exact solution and its expansion around $z=0$ is

$$f_p(z) = \frac{1}{2}(pz)^2K_2(pz) = 1 - \frac{1}{4}(pz)^2 - \frac{1}{16}(pz)^4\ln(pz) + O((pz)^4)\,. \tag{48}$$

The second solution to Eq. (47), $(pz)^2I_2(pz)$, is ruled out because it blows up at $z\to\infty$.

We now substitute the solution into the quadratic action. Using the field equation, one can perform integration by parts and write the action as a boundary integral at $z = 0$. One finds

$$S = \frac{N^2}{16\pi^2} \int d^4x \, \frac{1}{z^3} \phi(x,z)\phi'(x,z)|_{z\to 0} = \int \frac{d^4p}{(2\pi)^4} \, \phi_0(-p)\mathcal{F}(p,z)\phi_0(p)|_{z\to 0} \,, \tag{49}$$

where

$$\mathcal{F}(p,z) = \frac{N^2}{16\pi^2} \frac{1}{z^3} f_{-p}(z) \partial_z f_p(z) \,. \tag{50}$$

Differentiating the action twice with respect to the boundary value ϕ_0 one finds

$$\langle T_{xy} T_{xy} \rangle_p = -2 \lim_{z\to 0} \mathcal{F}(p,z) = \frac{N^2}{64\pi^2} p^4 \ln(p^2) \,. \tag{51}$$

Note that we have dropped the term $\sim p^4 \ln z$, which, although singular in the limit $z \to 0$, is a contact term [i.e., a term proportional to a derivative of $\delta(x)$ after Fourier transform]. Removing such terms by adding local counter terms to the supergravity action is known as the holographic renormalization [22]. It is, in a sense, a holographic counterpart to the standard renormalization procedure in quantum field theory, here applied to composite operators.

For time-like p, $p^2 < 0$, there are two solutions to Eq. (47) which involve Hankel functions $H^{(1)}(z)$ and $H^{(2)}(z)$ instead of $K_2(z)$. Neither function blows up at $z \to \infty$, and it is not clear which should be picked. Here we encounter, for the first time, a subtlety of Minkowski-space AdS/CFT, which is discussed in great length in subsequent sections. At zero temperature this problem can be overcome by an analytic continuation from space-like p. However, this will not work at nonzero temperatures.

3.2 Black Three-Brane Metric

At nonzero temperatures, the metric dual to $\mathcal{N} = 4$ SYM theory is the black three-brane metric,

$$ds^2 = \frac{r^2}{R^2}(-f dt^2 + d\mathbf{x}^2) + \frac{R^2}{r^2 f} dr^2 + R^2 d\Omega_5^2 \,, \tag{52}$$

with $f = 1 - r_0^4/r^4$. The event horizon is located at $r = r_0$, where $f = 0$. In contrast to the usual Schwarzschild black hole, the horizon has three flat directions $\mathbf{x}$. The metric (52) is thus called a black three-brane metric.

We frequently use an alternative radial coordinate u, defined as $u = r_0^2/r^2$. In terms of u, the boundary is at $u = 0$, the horizon at $u = 1$, and the metric is

$$ds^2 = \frac{(\pi T R)^2}{u^2}(-f(u)dt^2 + d\mathbf{x}^2) + \frac{R^2}{4u^2 f(u)} du^2 + R^2 d\Omega_5^2 \,. \tag{53}$$

The Hawking temperature is determined completely by the behavior of the metric near the horizon. Let us concentrate on the (t, r) part of the metric,

$$ds^2 = -\frac{4r_0}{R^2}(r - r_0)dt^2 + \frac{R^2}{4r_0(r - r_0)} dr^2 \,. \tag{54}$$

Changing the radial variable from r to ρ,

$$r = r_0 + \frac{\rho^2}{r_0} \,, \tag{55}$$

and the metric components become nonsingular:

$$ds^2 = \frac{R^2}{r_0^2}\left(d\rho^2 - \frac{4r_0^2}{R^2}\rho^2 dt^2\right) . \tag{56}$$

Note also that after a Wick rotation to Euclidean time τ, the metric has the form of the flat metric in cylindrical coordinates, $ds^2 \sim d\rho^2 + \rho^2 d\varphi^2$, where $\varphi = 2r_0 R^{-2}\tau$. To avoid a conical singularity at $\rho = 0$, φ must be a periodic variable with periodicity 2π. This fact matches with the periodicity of the Euclidean time in thermal field theory $\tau \sim \tau + 1/T$, from which one finds the Hawking temperature:

$$T_H = \frac{r_0}{\pi R^2} . \tag{57}$$

One of the first finite-temperature predictions of AdS/CFT correspondence is that of the thermodynamic potentials of the $\mathcal{N} = 4$ SYM theory in the strong coupling regime. The entropy is given by the Bekenstein-Hawking formula $S = A/(4G)$, where A is the area of the horizon of the metric (52); the result can then be converted to parameters of the gauge theory using Eqs. (30), (33), and (34). One obtains

$$s = \frac{S}{V} = \frac{\pi^2}{2}N^2 T^3 , \tag{58}$$

which is 3/4 of the entropy density in $\mathcal{N} = 4$ SYM theory at zero 't Hooft coupling.

We now try to generalize the AdS/CFT prescription to finite temperature. In the Euclidean formulation of finite-temperature field theory, field theory lives in a space-time with the Euclidean time direction τ compactified. The metric is regular at $r = r_0$: If one views the (τ, r) space as a cigar-shaped surface, then the horizon $r = r_0$ is the tip of the cigar. Thus, r_0 is the minimal radius where the space ends, and there is no point in space with r less than r_0. The only boundary condition at $r = r_0$ is that fields are regular at the tip of the cigar, and the AdS/CFT correspondence is formulated as

$$Z_{\text{4D}}[J] = Z_{\text{5D}}[\phi]|_{\phi(z=0)\to J} . \tag{59}$$

4 REAL-TIME AdS/CFT

In many cases we must find real-time correlation functions not given directly by the Euclidean path-integral formulation of thermal field theory. One example is the set of kinetic coefficients expressed, through Kubo's formulas, via a certain limit of real-time thermal Green's functions. Another related example appears if we want to directly find the position of the poles in the correlation functions that would correspond to the hydrodynamic modes.

In principle, some real-time Green's functions can be obtained by analytic continuation of the Euclidean ones. For example, an analytic continuation of a two-point Euclidean propagator gives a retarded or advanced Green's function, depending on the way one performs the continuation. However, it is often very difficult to directly compute a quantity of interest in that way. In particular, it is very difficult to get the information about the hydrodynamic (small ω, small $\mathbf{k}$) limit of real-time correlators from Euclidean propagators. The problem here is that we need to perform an analytic continuation from a discrete set of points in Euclidean frequencies (the Matsubara frequencies) $\omega = 2\pi i n$, where n is an integer, to the real values of ω. In the hydrodynamic limit, we are interested in real and small ω, whereas the smallest Matsubara frequency is already $2\pi T$.

Therefore, we need a real-time AdS/CFT prescription that would allow us to directly compute the real-time correlators. However, if one tries to naively generalize the AdS/CFT prescription, one immediately faces a problem. Namely, now $r = r_0$ is not the end of space but just the location of the horizon. Without specifying a boundary condition at $r = r_0$, there is an ambiguity in defining the solution to the field equations, even as the boundary condition at $r = \infty$ is set.

As an example, let us consider the equation of motion of a scalar field in the black hole background, $\partial_\mu(g^{\mu\nu}\partial_\nu\phi) = 0$. The solution to this equation with the boundary condition $\phi = \phi_0$ at $u = 0$ is $\phi(p,u) = f_p\phi_0(p)$, where $f_p(u)$ satisfies the following equation in the metric (53):

$$f_p'' - \frac{1+u^2}{uf}f_p' + \frac{w^2}{uf^2}f_p - \frac{q^2}{uf}f_p = 0\,. \tag{60}$$

Here the prime denotes differentiation with respect to u, and we have defined the dimensionless frequency and momentum:

$$w = \frac{\omega}{2\pi T}\,, \qquad q = \frac{k}{2\pi T}\,. \tag{61}$$

Near $u = 0$ the equation has two solutions, $f_1 \sim 1$ and $f_2 \sim u^2$. In the Euclidean version of thermal AdS/CFT, there is only one regular solution at the horizon $u = 1$, which corresponds to a particular linear combination of f_1 and f_2. However, in Minkowski space there are two solutions, and both are finite near the horizon. One solution termed f_p behaves as $(1-u)^{-iw/2}$, and the other is its complex conjugate $f_p^* \sim (1-u)^{iw/2}$. These two solutions oscillate rapidly as $u \to 1$, but the amplitude of the oscillations is constant. Thus, the requirement of finiteness of f_p allows for any linear combination of f_1 and f_2 near the boundary, which means that there is no unique solution to Eq. (60).

4.1 Prescription For Retarded Two-Point Functions

Physically, the two solutions f_p and f_p^* have very different behavior. Restoring the $e^{-i\omega t}$ phase in the wave function, one can write

$$e^{-i\omega t}f_p \sim e^{-i\omega(t+r_*)}\,, \tag{62}$$

$$e^{-i\omega t}f_p^* \sim e^{i\omega(t-r_*)}\,, \tag{63}$$

where the coordinate

$$r_* = \frac{\ln(1-u)}{4\pi T} \tag{64}$$

was introduced so that Eqs. (62) and (63) looked like plane waves. In fact, Eq. (62) corresponds to a wave that moves toward the horizon (incoming wave) and Eq. (63) to a wave that moves away from the horizon (outgoing wave).

The simplest idea, which is motivated by the fact that nothing should come out of a horizon, is to impose the incoming-wave boundary condition at $r = r_0$ and then proceed as instructed by the AdS/CFT correspondence. However, now we encounter another problem. If we write down the classical action for the bulk field, after integrating by parts we get contributions from both the boundary and the horizon:

$$S = \int \frac{d^4p}{(2\pi)^4}\,\phi_0(-p)\mathcal{F}(p,z)\phi_0(p)\Big|_{z=0}^{z=z_H}\,. \tag{65}$$

If one tried to differentiate the action with respect to the boundary value ϕ_0, one would find

$$G(p) = \mathcal{F}(p,z)|_0^{z_H} + \mathcal{F}(-p,z)|_0^{z_H}\,. \tag{66}$$

From the equation satisfied by f_p and from $f_p^* = f_{-p}$, it is easy to show that the imaginary part of $\mathcal{F}(p,z)$ does not depend on z; hence the quantity $G(p)$ in Eq. (66) is real. This is clearly not what we want, as the retarded Green's functions are, in general, complex. Simply throwing away the contribution from the horizon does not help because $\mathcal{F}(-p,z) = \mathcal{F}^*(p,z)$ owing to the reality of the equation satisfied by f_p.

A partial solution to this problem was suggested in Ref. [7]. It was postulated that the retarded Green's function is related to the function $\mathcal{F}$ by the same formula that was found at zero temperature:

$$G^R(p) = -2\lim_{z\to 0}\mathcal{F}(p,z)\,. \tag{67}$$

In particular, we throw away all contributions from the horizon. This prescription was established more rigorously in Ref. [8] (following an earlier suggestion in Ref. [23]) as a particular case of a general real-time AdS/CFT formulation, which establishes the connection between the close-time-path formulation of real-time quantum field theory with the dynamics of fields in the whole Penrose diagram of the AdS black brane. Here we accept Eq. (67) as a postulate and proceed to extract physical results from it.

It is also easy to generalize this prescription to the case when we have more than one field. In that case, the quantity $\mathcal{F}$ becomes a matrix $\mathcal{F}_{ab}$, whose elements are proportional to the retarded Green's function G_{ab}.

4.2 Calculating Hydrodynamic Quantities

As an illustration of the real-time AdS/CFT correspondence, we compute the correlator of T_{xy}. First we write down the equation of motion for $\phi = h^x_y$:

$$\phi_p'' - \frac{1+u^2}{uf}\phi_p' + \frac{w^2 - q^2 f}{uf^2}\phi_p = 0\,. \tag{68}$$

In contrast to the zero-temperature equation, now ω and k enter the equation separately rather than through the combination $\omega^2 - k^2$. Thus the Green's function will have no Lorentz invariance. The equation cannot be solved exactly for all ω and k. However, when ω and k are both much smaller than T, or $w, q \ll 1$, one can develop series expansion in powers of w and q. There are two solutions that are complex conjugates of each other. The solution that is an incoming wave at $u = 1$ and normalized to 1 at $u = 0$ is

$$f_p(z) = (1-u^2)^{-iw/2} + O(w^2, q^2)\,. \tag{69}$$

The kinetic term in the action for ϕ is

$$S = -\frac{\pi^2 N^2 T^4}{8}\int du\, \frac{f}{u}\phi'^2\,. \tag{70}$$

Applying the general formula (67), one finds the retarded Green's function of T_{xy},

$$G^R_{xy,xy}(\omega, k) = -\frac{\pi^2 N^2 T^4}{4} i w\,, \tag{71}$$

and, using Kubo's formula for η, the viscosity,

$$\eta = \frac{\pi}{8} N^2 T^3 \,. \tag{72}$$

It is instructive to compute other correlators that have poles corresponding to hydrodynamic modes. As a warm-up, let us compute the two-point correlators of the R-charge currents, which should have a pole at $\omega = -iDk^2$, where D is the diffusion constant. We first write down Maxwell's equations for the bulk gauge field. Let the spatial momentum be aligned along the x^3-axis: $p = (\omega, 0, 0, k)$. Then the equations for A_0 and A_3 are coupled:

$$\begin{aligned} wA_0' + qfA_3' &= 0\,, & (73)\\ A_0'' - \frac{1}{uf}(q^2 A_0 + wqA_3) &= 0\,, & (74)\\ A_3'' + \frac{f'}{f}A_3' + \frac{1}{uf^2}(w^2 A_3 + wqA_0) &= 0\,. & (75)\end{aligned}$$

One can eliminate A_3 and write down a third-order equation for A_0,

$$A_0''' + \frac{(uf)'}{uf} A_0'' + \frac{w^2 - q^2 f}{uf^2} A_0' = 0\,. \tag{76}$$

Near u=1 we find two independent solutions, $A_0' \sim (1-u)^{\pm iw/2}$, and the incoming-wave boundary condition singles out $(1-u)^{-iw/2}$. One can substitute $A_0' = (1-u)^{-iw/2} F(u)$ into Eq. (76). The resulting equation can be solved perturbatively in w and q^2. We find

$$A_0' = C(1-u)^{-iw/2} \left(1 + \frac{iw}{2} \ln \frac{2u^2}{1+u} + q^2 \ln \frac{1+u}{2u} \right) . \tag{77}$$

Using Eq. (74) one can express C through the boundary values of A_0 and A_3 at $u = 0$:

$$C = \left. \frac{q^2 A_0 + wqA_3}{iw - q^2} \right|_{u=0} . \tag{78}$$

Differentiating the action with respect to the boundary values, we find, in particular,

$$\langle J_0 J_0 \rangle_p = \frac{N^2 T}{16\pi} \frac{k^2}{i\omega - Dk^2} \,, \tag{79}$$

where

$$D = \frac{1}{2\pi T} \,. \tag{80}$$

The correlator given by Eq. (79) has the expected hydrodynamic diffusive pole, and D is the R-charge diffusion constant.

Similarly, one can observe the appearance of the shear mode in the correlators of the metric tensor. We note that the shear flow along the x^1 direction with velocity gradient along the x^3 direction involves T_{01} and T_{31}, hence the interesting metric components are $a_0 = h_0^1$ and $a_3 = h_3^1$. Two of the field equations are

$$a_0' - \frac{qf}{w} a_3' = 0\,, \tag{81}$$

$$a_3'' - \frac{1+u^2}{uf} a_3' + \frac{1}{uf^2}(w^2 a_3 + wqa_0) = 0\,. \tag{82}$$

They can be combined into a single equation:

$$a_0''' - \frac{2u}{f} a_0'' + \frac{2uf - q^2 f + w^2}{uf^2} a_0' = 0\,. \tag{83}$$

Again, the solution can be found perturbatively in w and q:

$$a_0' = C(1-u)^{-iw/2} \left[u - iw \left(1 - u - \frac{u}{2} \ln \frac{1+u}{2} \right) + \frac{q^2}{2}(1-u) \right]. \tag{84}$$

Applying the prescription, one finds the retarded Green's functions. For example,

$$G_{tx,tx}(\omega, k) = \frac{\xi k^2}{i\omega - \mathcal{D}k^2}\,, \tag{85}$$

where

$$\xi = \frac{\pi}{8} N^2 T^3, \qquad \mathcal{D} = \frac{1}{4\pi T}\,. \tag{86}$$

Thus, we found that the correlator contains a diffusive pole $\omega = -i\mathcal{D}k^2$, just as anticipated from hydrodynamics. Furthermore, the magnitude of the momentum diffusion constant $\mathcal{D}$ also matched our expectation. Indeed, if one recalls the value of η from Eq. (72) and the entropy density from Eq. (58), one can check that

$$\mathcal{D} = \frac{\eta}{\epsilon + P}\,. \tag{87}$$

5 THE MEMBRANE PARADIGM

Let us now look at the problem from a different perspective. The existence of hydrodynamic modes in thermal field theory is reflected by the existence of the poles of the retarded correlators computed from gravity. Are there direct gravity counterparts of the hydrodynamic normal modes?

If the answer to this question is yes, then there must exist linear gravitational perturbations of the metric that have the dispersion relation identical to that of the shear hydrodynamic mode, $\omega \sim -iq^2$, and of the sound mode, $\omega = c_s q - i\gamma q^2$. It turns out that one can explicitly construct the gravitational counterpart of the shear mode. (It should be possible to do the same construction for the sound mode, but it has not been done in the literature.) Our discussion is physical but somewhat sketchy; for more details see Ref. [24].

First, let us construct a gravity perturbation that corresponds to a diffusion of a conserved charge (e.g., the R-charge in $\mathcal{N} = 4$ SYM theory). To keep the discussion general, we use the form of the metric (44), with the metric components unspecified. Our only assumptions are that the metric is diagonal and has a horizon at $r = r_0$, near which

$$g_{00} = -\gamma_0 (r - r_0), \qquad g_{rr} = \frac{\gamma_r}{r - r_0}\,. \tag{88}$$

The Hawking temperature can be computed by the method used to arrive at Eq. (57), and one finds $T = (4\pi)^{-1}(\gamma_0/\gamma_r)^{1/2}$.

We also assume that the action of the gauge field dual to the conserved current is

$$S_{\text{gauge}} = \int dx\, \sqrt{-g} \left(-\frac{1}{4g_{\text{eff}}^2} F^{\mu\nu} F_{\mu\nu} \right), \tag{89}$$

where g_{eff} is an effective gauge coupling that can be a function of the radial coordinate r. For simplicity we set g_{eff} to a constant in our derivation of the formula for D; it can be restored by replacing $\sqrt{-g} \to \sqrt{-g}/g_{\text{eff}}^2$ in the final answer.

The field equations are

$$\partial_\mu \left(\frac{1}{g_{\text{eff}}^2} \sqrt{-g}\, F^{\mu\nu} \right) = 0 \,. \tag{90}$$

We search for a solution to this equation that vanishes at the boundary and satisfies the incoming-wave boundary condition at the horizon.

The first indication that one can have a hydrodynamic behavior on the gravity side is that Eq. (90) implies a conservation law on a four-dimensional surface. We define the stretched horizon as a surface with constant r just outside the horizon,

$$r = r_h = r_0 + \varepsilon, \qquad \varepsilon \ll r_0 \,, \tag{91}$$

and the normal vector n_μ directed along the r direction (i.e., perpendicularly to the stretched horizon). Then with any solution to Eq. (90), one can associate a current on the stretched horizon:

$$j^\mu = n_\nu F^{\mu\nu} \Big|_{r_h} \,. \tag{92}$$

The antisymmetry of $F^{\mu\nu}$ implies that j^μ has no radial component, $j^r = 0$. The field equation (90) and the constancy of n_ν on the stretched horizon imply that this current is conserved: $\partial_\mu j^\mu = 0$. To establish the diffusive nature of the solution, we must show the validity of the constitutive equation $j^i = -D\partial_i j^0$.

Such constitutive equation breaks time reversal and obviously must come from the absorptive boundary condition on the horizon. The situation is analogous to the propagation of plane waves to a non-reflecting surface in classical electrodynamics. In this case, we have the relation $\mathbf{B} = -\mathbf{n} \times \mathbf{E}$ between electric and magnetic fields. In our case, the corresponding relation is

$$F_{ir} = -\sqrt{\frac{\gamma_r}{\gamma_0}}\, \frac{F_{0i}}{r - r_0} \,, \tag{93}$$

valid when r is close to r_0. This relates $j_i \sim F_{ir}$ to the parallel to the horizon component of the electric field F_{0i}, which is one of the main points of the "membrane paradigm" approach to black hole physics [25]. We have yet to relate j_i to $j_0 \sim F_{0r}$, which is the component of the electric field normal to the horizon. To make the connection to F_{0r}, we use the radial gauge $A_r = 0$, in which

$$F_{0i} \approx -\partial_i A_0 \,. \tag{94}$$

Moreover, when k is small the fields change very slowly along the horizon. Therefore, at each point on the horizon the radial dependence of the scalar potential A_0 is determined by the Poisson equation,

$$\partial_r (\sqrt{-g}\, g^{rr} g^{00} \partial_r A_0) = 0 \,, \tag{95}$$

whose solution, which satisfies $A_0(r=\infty) = 0$, is

$$A_0(r) = C_0 \int_r^\infty dr' \, \frac{g_{00}(r') g_{rr}(r')}{\sqrt{-g(r')}} \,. \tag{96}$$

This means that the ratio of the scalar potential A_0 and electric field F_{0r} approaches a constant near the horizon:

$$\left.\frac{A_0}{F_{0r}}\right|_{r=r_0} = \frac{\sqrt{-g}}{g_{00}g_{rr}}(r_0)\int\limits_{r_0}^{\infty} dr\,\frac{g_{00}g_{rr}}{\sqrt{-g}}(r)\,. \tag{97}$$

Combining the formulas $j^i \sim F_{0i} \sim \partial_i A_0$, and $A_0 \sim F_{0r} \sim j^0$, we find Fick's law $j^i = -D\partial_i j^0$, with the diffusion constant

$$D = \frac{\sqrt{-g}}{g_{xx}g_{\mathrm{eff}}^2\sqrt{-g_{00}g_{rr}}}(r_0)\int\limits_{r_0}^{\infty} dr\,\frac{-g_{00}g_{rr}g_{\mathrm{eff}}^2}{\sqrt{-g}}(r)\,. \tag{98}$$

Thus, we found that for a slowly varying solution to Maxwell's equations, the corresponding charge on the stretched horizon evolves according to the diffusion equation. Therefore, the gravity solution must be an overdamped one, with $\omega = -iDk^2$. This is an example of a quasi-normal mode. We also found the diffusion constant D directly in terms of the metric and the gauge coupling g_{eff}.

The reader may notice that our quasinormal modes satisfy a vanishing Dirichlet condition at the boundary $r{=}\infty$. This is different from the boundary condition one uses to find the retarded propagators in AdS/CFT, so the relation of the quasinormal modes to AdS/CFT correspondence may be not clear. It can be shown, however, that the quasi-normal frequencies coincide with the poles of the retarded correlators [26, 27].

We can now apply our general formulas to the case of $\mathcal{N} = 4$ SYM theory. The metric components are given by Eq. (52). For the R-charge current g_{eff} = const, Eq. (98) gives $D = 1/(2\pi T)$, in agreement with our AdS/CFT computation. For the shear mode of the stress-energy tensor we have effectively $g_{\mathrm{eff}}^2 = g_{xx}$, so $\mathcal{D} = 1/(4\pi T)$, which also coincides with our previous result. In both cases, the computation is much simpler than the AdS/CFT calculation.

6 THE VISCOSITY/ENTROPY RATIO

6.1 Universality

In all thermal field theories in the regime described by gravity duals the ratio of shear viscosity η to (volume) density of entropy s is a universal constant equal to $1/(4\pi)$ [$\hbar/(4\pi k_B)$, if one restores $\hbar$, c and the Boltzmann constant k_B].

One proof of the universality is based on the relationship between graviton's absorption cross section and the imaginary part of the retarded Green's function for T_{xy} [29]. Another way to prove the universality [30] is via the direct AdS/CFT calculation of the correlation function in Kubo's formula (18).

We, however, follow a different method. It is based on the formula for the viscosity derived from the membrane paradigm. A similar proof was given by Buchel & Liu [28].

The observation is that the shear gravitational perturbation with $k = 0$ can be found exactly by performing a Lorentz boost of the black-brane metric (52). Consider the coordinate transformations $r, t, x_i \to r', t', x_i'$ of the form

$$\begin{aligned} r &= r'\,, \\ t &= \frac{t' + vy'}{\sqrt{1-v^2}} \approx t' + vy'\,, \end{aligned}$$

$$
\begin{aligned}
y &= \frac{y' + vt'}{\sqrt{1-v^2}} \approx y' + vt' \,, \\
x_i &= x'_i \,,
\end{aligned} \tag{99}
$$

where $v < 1$ is a constant parameter and the expansion on the right corresponds to $v \ll 1$. In the new coordinates, the metric becomes

$$
ds^2 = g_{00} dt'^{\,2} + g_{rr} dr'^{\,2} + g_{xx}(r) \sum_{i=1}^{p} (dx'^{\,i})^2 + 2v(g_{00} + g_{xx}) dt' dy' \,. \tag{100}
$$

This is simply a shear fluctuation at $k = 0$. In our language, the corresponding gauge potential is

$$
a_0 = v g^{xx} (g_{00} + g_{xx}) \,. \tag{101}
$$

This field satisfies the vanishing boundary condition $a_0(r = \infty) = 0$ owing to the restoration of Poincaré invariance at the boundary: $g_{00}/g_{xx} \to -1$ when $r \to \infty$. This clearly has a much simpler form than Eq. (96) for the solution to the generic Poisson equation. The simple form of solution (101) is valid only for the specific case of the shear gravitational mode with $g^2_{\text{eff}} = g_{xx}$. We have also implicitly used the fact that the metric satisfies the Einstein equations, with the stress-energy tensor on the right being invariant under a Lorentz boost.

Equation (97) now becomes

$$
\left.\frac{a_0}{f_{0r}}\right|_{r \to r_0} = -\left.\frac{1 + g^{xx} g_{00}}{\partial_r (g^{xx} g_{00})}\right|_{r \to r_0} = \frac{g_{xx}(r_0)}{\gamma_0} \,. \tag{102}
$$

The shear mode diffusion constant is

$$
\mathcal{D} = \left.\frac{a_0}{f_{0r}}\right|_{r \to r_0} \frac{\sqrt{\gamma_0 \gamma_r}}{g_{xx}(r_0)} = \sqrt{\frac{\gamma_r}{\gamma_0}} = \frac{1}{4\pi T} \,. \tag{103}
$$

Because $\mathcal{D} = \eta/(\epsilon + P)$, and $\epsilon + P = Ts$ in the absence of chemical potentials, we find that

$$
\frac{\eta}{s} = \frac{1}{4\pi} \,. \tag{104}
$$

In fact, the constancy of this ratio has been checked directly for theories dual to Dp-brane [24], M-brane [11], Klebanov-Tseytlin and Maldacena-Nunez backgrounds [28], $\mathcal{N} = 2^*$ SYM theory [31] and others.

As remarked in Sec. 2, the ratio η/s is much larger than the one for weakly coupled theories. The fact that we found the ratio to be parametrically of order one implies that all theories with gravity duals are strongly coupled.

In $\mathcal{N} = 4$ SYM theory, the ratio η/s has been computed to the next order in the inverse 't Hooft coupling expansion [32]

$$
\frac{\eta}{s} = \frac{1}{4\pi} \left(1 + \frac{135\zeta(3)}{8(g^2 N)^{3/2}} \right) \,. \tag{105}
$$

The sign of the correction can be guessed from the fact that in the limit of zero 't Hooft coupling $g^2 N \to 0$, the ratio diverges, $\eta/s \to \infty$.

6.2 The Viscosity Bound Conjecture

From our discussion above, one can argue that

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi} \tag{106}$$

in all systems that can be obtained from a sensible relativistic quantum field theory by turning on temperatures and chemical potentials.

The bound, if correct, implies that a liquid with a given volume density of entropy cannot be arbitrarily close to being a perfect fluid (which has zero viscosity). As such, it implies a lower bound on the viscosity of the QGP one may be creating at RHIC. Interestingly, some model calculations suggest that the viscosity at RHIC may be not too far away from the lower bound [33, 34].

One place where one may think that the bound should break down is superfluids. The ability of a superfluid to flow without dissipation in a channel is sometimes described as "zero viscosity". However, within the Landau's two-fluid model, any superfluid has a measurable shear viscosity (together with three bulk viscosities). For superfluid helium, the shear viscosity has been measured in a torsion-pendulum experiment by Andronikashvili [35]. If one substitutes the experimental values, the ratio η/s for helium remains larger than $\hbar/4\pi k_B \approx 6.08 \times 10^{-13}$ K s for all ranges of temperatures and pressures, by a factor of at least 8.8.

As discussed in Sec. 2.3, the ratio η/s is proportional to the ratio of the mean free path and the de Broglie wavelength of particles,

$$\frac{\eta}{s} \sim \frac{\ell_{\rm mfp}}{\lambda}\,. \tag{107}$$

For the quasi-particle picture to be valid, the mean free path must be much larger than the de Broglie wavelength. Therefore, if the coupling is weak and the system can be described as a collection of quasi-particles, the ratio η/s is larger than 1.

We have found is that, within the $\mathcal{N}=4$ SYM theory and, more generally, theories with gravity duals, even in the limit of infinite coupling the ratio η/s cannot be made smaller than $1/(4\pi)$.

7 CONCLUSION

In this review, we covered only a small part of the applications of AdS/CFT correspondence to finite-temperature quantum field theory. Here we briefly mention further developments and refer the reader to the original literature for more details.

In addition to $\mathcal{N}=4$ SYM theory, there exists a large number of other theories whose hydrodynamic behavior has been studied using the AdS/CFT correspondence, including the worldvolume theories on M2- and M5-branes [11], theories on Dp branes [24], and little string theory [36]. In all examples the ratio η/s is equal to $1/(4\pi)$, which is not surprising because the general proofs of Sec. 6 apply in these cases.

We have concentrated on the shear hydrodynamic mode, which has a diffusive pole ($\omega \sim -ik^2$). One can also compute correlators which have a sound-wave pole from the AdS/CFT prescription [10]. One such correlator is between the energy density T^{00} at two different points in space-time. The result confirms the existence of such a pole, with both the real part and imaginary part having exactly the values predicted by hydrodynamics (recall that in conformal field theories the bulk viscosity is zero and the sound attenuation rate is determined completely by the shear viscosity).

Some of the theories listed above are conformal field theories, but many are not (e.g., the Dp-brane worldvolume theories with $p \neq 3$). The fact that $\eta/s = 1/(4\pi)$ also in those theories implies that the constancy of this ratio is not a consequence of conformal symmetry. Theories with less than maximal number of supersymmetries have been found to have the universal value of η/s, for example, the $\mathcal{N} = 2^*$ theory [37], theories described by Klebanov-Tseytlin, and Maldacena-Nunez backgrounds [28]. A common feature of these theories is that they all have a gravitational dual description. The bulk viscosity has been computed for some of these theories [38, 36].

Besides viscosity, one can also compute diffusion constants of conserved charges by using the AdS/CFT correspondence. Above we presented the computation of the R-charge diffusion constant in $\mathcal{N} = 4$ SYM theory; for similar calculations in some other theories see Ref. [11, 24].

Recently, the AdS/CFT correspondence was used to compute the energy loss rate of a quark in the fundamental representation moving in a finite-temperature plasma [39,40,41,42]. This quantity is of importance to the phenomenon of "jet quenching" in heavy-ion collisions.

So far, the only quantity that shows a universal behavior at the quantitative level, across all theories with gravitational duals, is the ratio of the shear viscosity and entropy density. Recently, it was found that this ratio remains constant even at nonzero chemical potentials [43,44,45,46,47].

What have we learned from the application of AdS/CFT correspondence to thermal field theory? Although, at least at this moment, we cannot use the AdS/CFT approach to study QCD directly, we have found quite interesting facts about strongly coupled field theories. We have also learned new facts about quasi-normal modes of black branes. However, we have also found a set of puzzles: Why is the ratio of the viscosity and entropy density constant in a wide class of theories? Is there a lower bound on this ratio for all quantum field theories? Can this be understood without any reference to gravity duals? With these open questions, we conclude this review.

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