THE EFFECT OF A FIFTH LARGE-SCALE SPACE-TIME DIMENSION ON THE CONSERVATION OF ENERGY IN A FOUR DIMENSIONAL UNIVERSE

M.B. GERRARD*AND T.J. SUMNER

Department of Physics, Imperial College London, Prince Consort Road, London. SW7 2BW, UK. E-mail: t.sumner@imperial.ac.uk

The effect of introducing a fifth large-scale space-time dimension to the equations of orbital dynamics was analysed in an earlier paper by the authors. The results showed good agreement with the observed flat rotation curves of galaxies. This analysis did not require the modification of Newtonian dynamics, but rather only their restatement in a five dimensional framework. The same analysis derived a parameter a_r, which plays an important role in the restated equations of orbital dynamics, and suggested a value for a_r of $1.2 \times 10^{-10}\,\mathrm{ms}^{-2}$ which is remarkably similar to the MOND constant a_0. In this companion paper, the principle of conservation of energy is restated within the same five-dimensional framework. The resulting analysis provides an alternative route to estimating the value of a_r, without reference to the equations of orbital dynamics, and based solely on key cosmological constants and parameters, including the gravitational constant, G. The same analysis suggests that the inverse square law of gravity may itself be due to the conservation of energy at the boundary between a four-dimensional universe and a fifth large-scale space-time dimension.

1. Introduction

Modification of Newtonian Dynamics (MOND) developed by Milgrom [1] proposes either that the Newtonian equations of gravity, or those of inertia should be modified for total accelerations below a threshold value a_0. A value for a_0 of $1.2 \times 10^{-10}\,\mathrm{ms}^{-2}$ has been proposed by Milgrom, based on observations of a range of different galaxy types, where a_0 becomes the key parameter required to account for their rotation curves, given the observed distribution of baryonic matter.

In an earlier paper [2] we introduced a fifth large-scale space-time dimension, r to Newton's Second Law, as applied to systems with angular

*corresponding author:- michael.gerrard@partnershipsuk.org.uk

velocity. The resulting analysis of the orbital motion of galaxies, which considered only the role of baryonic matter, is consistent with their observed rotation curves and the Tulley-Fisher relationship. The dimension r, is orthogonal to the three space dimensions $s(x, y, z)$ and the time dimension, t of a four-dimensional universe, but does not represent a degree of freedom of motion in this analysis. For a closed isotropic universe, r is the radius of curvature of (four-dimensional) space-time and has a value, r_u remote from gravitating matter that is estimated to be $\sim 7.5 \times 10^{26}\,\mathrm{m}$. The parameter a_r is derived from the relationship $a_r = c^2/r$. In the case of r being equal to r_u, a_r has a value of $1.2 \times 10^{-10}\,\mathrm{ms}^{-2}$, which is the same as the parameter a_0 derived by Milgrom from observing the rotation curves of more than eighty galaxies.

Using the same five-dimensional analytical framework, this paper examines the relationships between a_r, the principle of conservation of energy and gravity. The resulting derivation of a_r is, therefore, unrelated to orbital dynamics and Newton's Second Law and instead relies on key cosmological constants, such as the gravitational constant, G and parameters, such as the mass density of the universe.

2. Background Gravitational Acceleration in the Universe

The large-scale distribution of matter across the universe creates a background gravitational acceleration, g_b which is isotropic if matter itself is evenly distributed on this scale. The mutual attraction of each particle of matter towards all other matter, as represented by g_b, is similar in concept to a three dimensional "surface tension" stretching across the universe.

If space is assumed to be flat and open and matter is assumed to be evenly distributed on this large scale, with (baryonic) mass density ρ, then the background gravitational acceleration, g_b, can be derived as follows:

$$g_b = \pi G \rho H_H \tag{1}$$

where G is the gravitational constant ($6.67 \times 10^{-11}\,\mathrm{m^3Kg^{-1}s^{-2}}$), ρ for baryonic matter has a currently estimated value $\rho_u = 3 \times 10^{-28}\,\mathrm{Kg\,m^{-3}}$ and H_H is the Hubble Horizon given by $H_H = c/H$ with H being Hubble's Constant ($70\,\mathrm{Km\,s^{-1}Mpc^{-1}}$). Substitution in equation (1) gives a current value for g_b of $8.3 \times 10^{-12}\,\mathrm{ms}^{-2}$ which is noted to be two orders of magnitude less than the value of a_0.

The accuracy of equation (1) depends on three potential sources of uncertainty, namely: the value of ρ, the method of calculation of the volume

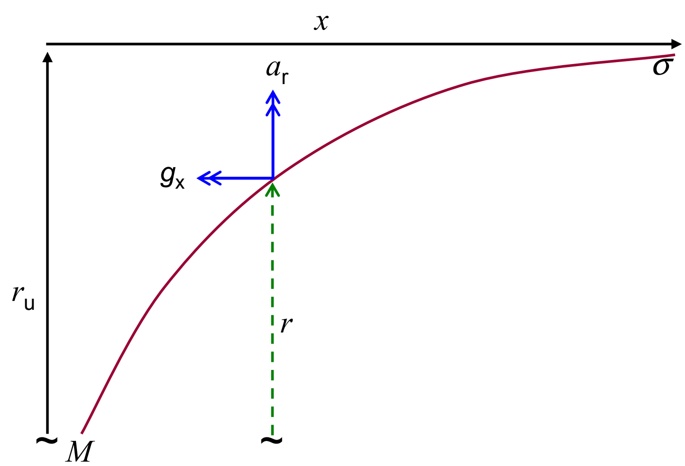

Figure 1. Locus of points $r(x)$ at which there is balance between the two accelerations g_x and a_r.

of the universe within the Hubble Horizon and the value of H itself. These will be discussed later.

3. Background Radius of Curvature of the Universe

In the earlier paper[a], an expression was derived for the locus of points $r(x)$ adjacent to a gravitating mass, M which defined the balance condition between gravitational acceleration g_x and the acceleration a_r acting everywhere in the universe in the direction of r. This locus being referred to as the brane σ[b]:

$$r\,(x) = r_u \left(1 - \frac{GM}{c^2 x}\right) \tag{2}$$

where r_u is the radius of curvature of four-dimensional space-time *remote* from gravitating matter M and x is the distance away from M as shown in figure 1.

The effect which matter has on the local radius of curvature of space-time, r is, of course, cumulative according to the amount of matter present[c]

[a]Section 3.2, ibid
[b]Figure 3, ibid
[c]The super-position of profiles in $r(x)$ is given by the approximation $\Delta r/ru = \Sigma\,\Delta r_i/r_u$, where $\Delta r = (r_u - r_i)$ and its value is small relative to r_u, as is the case here

and its proximity. Applying equation (2) to all baryonic matter contained within the Hubble Horizon (again assumed to be evenly distributed across space with density ρ and lying within a spherical volume defined by $4/3\left(\pi s^3\right)$ where s here is H_H) it is possible to calculate an overall background value of $r(x)$. This value will inevitably be somewhat less than r_u given that no point is, in practice, completely remote from all matter. This background value of r is referred to as r_b and is derived as follows[d]:

$$r_b = r_u\left(1 - \frac{2\pi G\rho H_H^2}{c^2}\right) \tag{3}$$

Substituting for known parameters and constants in equation (3), including the current value of the mass density ρ_u, gives a value for r_b equal to $0.98 \times r_u$. Substituting either value into the key relationship $a_r = c^2/r$ gives the same value to within one decimal place, namely $1.2 \times 10^{-10}\,\mathrm{ms}^{-2}$.

The average mass density of the universe, ρ, decreases over time in an expanding universe. For a Euclidean (although expanding) universe, the volume of space within the Hubble Horizon is given by $(4/3)\,\pi H_H^3 \simeq (4/3)\,\pi(ct)^3$. Given that (to a first order) the total mass lying within the Hubble Horizon is constant[e], it follows that we can derive an expression for the average mass density $\rho(t)$ of the universe at any time t, in terms of the average mass density observed for the current era ρ_u (i.e. $\sim 3 \times 10^{-28}\,\mathrm{Kg\,m^{-3}}$) and the current estimated age of the universe t_u (i.e. 13.7 Bn years).

$$\rho \simeq \frac{\rho_u t_u^3}{t^3} \tag{4}$$

Given that this equation is derived (in part) from the approximation $H_H \simeq ct$, it is assumed only to be applicable in the current analysis for perturbations of time about the current era.

Substituting for ρ from equation (4) into equation (3) provides an expression for the local time-dependency of the background radius of curvature of space-time r_b in equation (5), which is similarly limited in its range of extrapolation.

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[d]By integrating the contributions towards r_b of matter lying within concentric spherical shells of space.

[e]Matter lying at a distance equal to the Hubble Horizon from a point in space is, by definition, receding form that point at the speed of light, so that matter lying beyond this horizon cannot exert any gravitational influence at that point

$$r_b = r_u \left(1 - \frac{2\pi G \rho_u t_u^3}{t}\right) \tag{5}$$

4. Conservation of Energy

In the earlier paper[f], a_r was described as a universal acceleration of expansion acting at all points in space in the direction of r. To maintain conservation of energy within four-dimensional space-time, it follows that for any mass m at a point in space P there must be an acceleration equal and opposite to a_r which prevents energy being transferred from within the four-dimensional universe to the fifth dimension r, as shown in figure 2. Accordingly, this principle may be written as:

$$a_r + \frac{d^2 r_b}{dt^2} = 0 \tag{6}$$

The second term of this equation ($\ddot{r}_b$) is identified as the acceleration acting on a mass in the direction of the dimension r (decreasing) by virtue of the expansion of the universe which causes r_b the background value of r to increase over time (but at a decelerating rate − see equation (5)). In other words, given that the universal acceleration a_r is acting everywhere along the boundary between the four-dimensional space-time and the fifth dimension r, energy can only be conserved (within four dimensional space-time) if the background radius of curvature of space-time r_b varies in time so as to satisfy equation (6). As referred above, for the current era a_r is $1.2 \times 10^{-10}\,\mathrm{ms}^{-2}$.

Using the expression for r_b from equation (5), it is straight-forward to derive an expression for $\ddot{r}_b$, namely[g]

$$\frac{d^2 r_b}{dt^2} = -4\pi\rho G r_u \tag{7}$$

Substituting values for known parameters and constants on the right-hand side (including the current mass density of the universe, ρ_u, provides the result: $\ddot{r}_b = -1.9 \times 10^{-10}\,\mathrm{ms}^{-2}$. Given the approximations used to derive equation (7), this value for $\ddot{r}_b$ appears to be in reasonably good agreement with the value expected from equation (6), namely: $-1.2 \times 10^{-10}\,\mathrm{ms}^{-2}$). The level of agreement can only be properly assessed by considering the uncertainty in the three key components to equation (7):

[f] Section 3.1, ibid

[g] Assuming all other terms other than r_b and t are constant

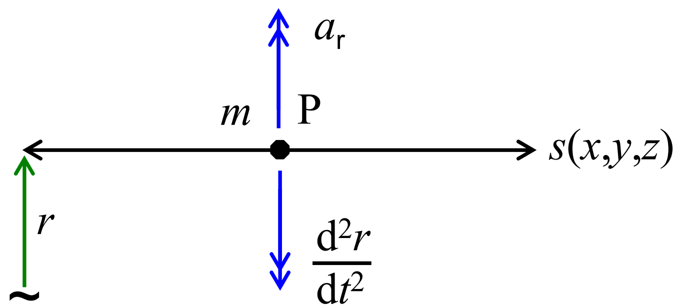

Figure 2. Conservation of energy requires the two accelerations a_r and $\ddot{r}_b$ to be equal and opposite.

the value of the Hubble Horizon, the average mass density of the universe and the estimated volume of the universe. Consistency between $\ddot{r}_b$ from equation (7) and equation (6)) lies within the uncertainty ranges of $\pm 12\%$ in each of these three components. However, the principal source of uncertainty in $\ddot{r}_b$ is expected to be the method used to calculate the volume of the universe lying within the Hubble Horizon.

The form of universe that underpins the derivation of a_r is closed (i.e. curved) and isotropic[h] and yet, so far in this paper, we have used the Euclidean derivation of a three dimension spherical volume $4/3\left(\pi s^3\right)$, where s is the radius of the volume - i.e. a derivation appropriate to a flat and open universe. A closed isotropic three dimensional space is the "surface area" of a 4-dimensional hyper-sphere, the 3-dimensional volume of which is given not by $4/3\left(\pi s^3\right)$ but by the expression $2\pi R^3$, where R is the radius of curvature of the hyper-sphere. The relevant feature of this 3-dimensional "surface area" is that at increasing distances s from a point P, the volumes of concentric spherical shells of space centred on P become progressively smaller than those derived from the (Euclidean) expression $4\pi s^2 ds$

Accordingly, failure to take account of this effect will have led to an over-estimation of the volume of the universe lying within the Hubble Horizon and so to an over-estimation of $\ddot{r}_b$ in equation (7). The value of g_b in equation (1) will have, likewise, been overestimated for this reason.

There are two important aspects of the application of ρ in the calculation of $\ddot{r}_b$ and g_b that also need to be highlighted: the first in relation to a closed universe; and the second in relation to an expanding universe.

[h] Section 3.2, ibid

4.1. *A closed universe*

The application of a single average value for ρ to a closed universe, defined by the 3-dimensional "surface area" of a 4-dimensional hyper-sphere, means that the contributions of matter lying within ever more distant volumes of space[i] to the measured values of $\ddot{r}_b$ and g_b, will ultimately diminish with distance. Consequently, inaccuracies in the value of H and, thereby, the Hubble Horizon should not be primary sources of error in $\ddot{r}_b$ and g_b. Moreover, recent observations that indicate lower values for H at the furthest distances should not, for the same reason, undermine the validity of using a single value for H in the derivation of equations (1) or (7).

4.2. *An expanding universe*

The nature of expansion of the universe (whether open or closed) that is assumed here, is one in which mass density is determined by a fixed amount of matter lying within the Hubble Horizon[j] and (to a first order) is not affected by mass flows across either the Hubble Horizon, or across regions of space lying within the Hubble Horizon[k]. Accordingly, a profile of steadily increasing mass densities at further distances from a point P, due to these further distances being observations of the universe's past, should not affect the determination of $\ddot{r}_b$ and g_b, to the extent that greater mass densities (in the past) are off-set by reductions in the volume of space (in the past).

If the same adjustment for space being closed as would be needed to bring to $\ddot{r}_b$ into equality with a_r in equation (6) is also applied to the derivation of g_b in equation (1), g_b reduces by circa 25% to $6.1 \times 10^{-12}\,\mathrm{ms}^{-2}$. Having made the same correction for volume, the relationship between the background value for the radius of curvature of space-time r_b and r_u also remain unchanged (to one decimal place), namely $r_b = 0.98 \times r_u$. Hence, the corrected calculation of the volume of space lying within the Hubble Horizon does not affect the calculated value for a_r, which remains $1.2 \times 10^{-10}\,\mathrm{ms}^{-2}$ (i.e. the same as a_0).

Hence, if account is taken of a closed and isotropic nature of space in applying the current value for the mass density of the universe ρ_u, then the principle of conservation of energy appears to offer an alternative approach

[i] i.e. the volume of concentric shells of space centred on point P and lying at distance s from P depart increasingly from $4\pi s^2 ds$ as s increases

[j] Receding at the speed of light

[k] Nor by the inter-change between matter and energy

to the valuation of a_r and, moreover, an approach that is based on key cosmological parameters and the gravitational constant G and that is independent of orbital dynamics and Newton's Second Law used in the earlier paper.

5. Discussion

A number of simplifying assumptions have been made in this paper. These include assumptions about the Hubble Horizon, the mass density of the universe and the calculation of volumes of space over large distances. Nonetheless, the value for a_r derived from the principle of conservation of energy is in good agreement with that expected[1].

The relative dominance of proximate matter over very distant matter in the determination of the background universal gravitational acceleration g_b and in the background value for the radius of curvature of space-time r_b (assuming matter is evenly distributed on a very-large scale and the universe is closed), should make the calculations used in this paper reasonably robust to inaccuracies in the estimation of the Hubble Horizon and of volumes of space at greater distances.

The time dependencies of r_b evident in equation (5) (i.e. increasing with age of the universe) and of $|\ddot{r}_b|$ evident in equation (7) (i.e. decreasing with age of the universe) imply that we should modify the central equation for a_r proposed in the earlier paper and write it as:

$$a_r = \frac{c^2}{r_b} \tag{8}$$

For a value of $r_b = 0.98 \times r_u$, the value of a_r derived from equation (8) remains the same as a_0 (the MOND constant) to one decimal place (i.e. $1.2 \times 10^{-10}\,\mathrm{ms}^{-2}$) for the current era. The substitution of r_b for r_u in the equation for a_r and the principle of conservation of energy (i.e. equation (6)) are consistent with higher values for ρ, $|\ddot{r}_b|$ and a_r in earlier ages of the universe. The observations of rotation curves of galaxies which support the MOND constant a_0 proposed by Milgrom have, so far, mostly covered galaxies out as far as $\sim 100\,\mathrm{Mpc}$ from earth. To one decimal place, there is no change to r_b from equation (5) over these distances and so no corresponding departure from the MOND value for a_0 would be expected from equation (8).

[1]From MOND observations and from the derivation based on Hubble Constant, section 4.1, ibid

The analysis in sections 3 and 4 can, of course, be reversed and the principle of conservation of energy as expressed by equation (6) can be used as the starting point to derive the underlying relationship between matter and the radius of curvature of 4-dimensional space-time in an expanding universe, namely equation (2). If this approach is adopted, then the inverse square law of gravity (which is a derivative of equation (2)[m]) may be inferred as a consequence of the conservation of energy at the boundary between a (closed) expanding four-dimensional universe and a fifth large-scale dimension of space-time. Accordingly, a description of gravity based upon this principle of conservation of energy would appear to offer a derivation based on thermodynamics for the key dimensionless term of General Relativity (GM/c^2x). Finally, equations (7) and (8) may be substituted in equation (6) to provide an expression for the gravitational constant (G), of the following form:

$$G = \frac{kc^2 r_u}{M_{universe}} \tag{9}$$

where $M_{universe}$ is the mass of the universe and k is a dimensionless constant determined by the correct approach to calculating the volume of the universe. This equation suggests a link between G and the key fifth dimensional parameter r_u, which is identified in this and the earlier paper as the radius of curvature of space-time remote from gravitating matter. All the terms on the right-hand side of equation (9) are, as expected, constant.

References

1. Milgrom, M. (1983) *Ap. J.*, **270**, 365.
2. Gerrard, M.B. and Sumner, T.J. (2006) arXiv.gr-qc/0605080

[m] For the relationship between equation (2) and the inverse square law of gravity, see section 3.2, ibid