Some aspects of the nonperturbative renormalization of the φ^4 model

J. Kaupužs *
Institute of Mathematics and Computer Science, University of Latvia
29 Rainja Boulevard, LV–1459 Riga, Latvia

August 9, 2013

Abstract

A nonperturbative renormalization of the φ^4 model is considered. First we integrate out only a single pair of conjugated modes with wave vectors $\pm\mathbf{q}$. Then we are looking for the RG equation which would describe the transformation of the Hamiltonian under the integration over a slice $\Lambda - d\Lambda < k < \Lambda$, where $d\Lambda \to 0$. Assuming a simple superposition of the integration results for $\pm\mathbf{q}$, we arrive at the known Wegner–Houghton equation which intuitively is expected to be an exact RG equation for relevant field configurations in the thermodynamic limit. However, our main and counterintuitive result is that the simple superposition fails here, and therefore the Wegner–Houghton equation is only an approximation. We show it considering the limit of vanishing φ^4 coupling constant u in the high temperature phase, where the expansion coefficients in powers of u can be calculated exactly without assumption of the superposition. We briefly discuss also some other nonperturbative RG equations.

1 Introduction

The renormalization group (RG) approach, perhaps, is the most extensively used one in numerous studies of critical phenomena [1, 2]. Particularly, the perturbative RG approach to the φ^4 or Ginzburg–Landau model is widely known [3, 4, 5, 6]. However, as we have shown recently [7, 8], the perturbative approach suffers from serious problems. Therefore it is interesting to look for a nonperturbative approach. Historically, nonperturbative RG equations have ben developed in parallel to the perturbative ones, and there are two classes of such so called exact RG equations (ERGE) reviewed in [9] and [10], respectively (see also the original papers [11, 12] for one and [13, 14] for the other approach). The equations considered in [9] are closer in spirit to the famous Wilson's approach, where the basic idea is to integrate out the short–wave fluctuations. We have reconsidered in some detail the oldest nonperturbative equation of this type, i. e., the Wegner–Houghton equation presented originally in [11], and have revealed a new striking feature. Although it is widely believed that the Wegner–Houghton equation is exact, our results clearly show that it is approximate. We think that this finding is sufficiently important, as it changes basically the knowledge about the equations of this kind, and is relevant also for understanding of other nonperturbative RG equations discussed at the end of our paper.

*E–mail: kaupuzs@latnet.lv

2 An elementary step of renormalization

To derive a nonperturbative RG equation for the φ^4 model, we should start with some elementary steps, as explained in this section.

Consider the action $S[\varphi]$ which depends on the configuration of the order parameter field $\varphi(\mathbf{x})$ depending on coordinate $\mathbf{x}$. By definition, it is related to the Hamiltonian H of the model via $S = H/T$, where T is the temperature measured in energy units. In general, $\varphi(\mathbf{x})$ is an n–component vector with components $\varphi_j(\mathbf{x})$ given in the Fourier representation as $\varphi_j(\mathbf{x}) = V^{-1/2} \sum_{k<\Lambda} \varphi_{j,\mathbf{k}} e^{i\mathbf{kx}}$, where $V = L^d$ is the volume of the system, d is the spatial dimensionality, and Λ is the upper cutoff of the wave vectors. We consider the action of the Ginzburg–Landau form. For simplicity, we include only the φ^2 and φ^4 terms. The action of such φ^4 model is given by

$$S[\varphi] = \sum_{j,\mathbf{k}} \Theta(\mathbf{k}) \varphi_{j,\mathbf{k}} \varphi_{j,-\mathbf{k}} + uV^{-1} \sum_{j,l,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \varphi_{j,\mathbf{k}_1} \varphi_{j,\mathbf{k}_2} \varphi_{l,\mathbf{k}_3} \varphi_{l,-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3} \;, \tag{1}$$

where $\Theta(\mathbf{k})$ is some function of wave vector $\mathbf{k}$, e. g., $\Theta(\mathbf{k}) = r_0 + ck^2$ like in theories of critical phenomena [4, 5, 6, 7]. In the sums we set $\varphi_{l,\mathbf{k}} = 0$ for $k > \Lambda$.

The renormalization group (RG) transformation implies the integration over $\varphi_{j,\mathbf{k}}$ for some set of wave vectors with $\Lambda' < k < \Lambda$, i. e., the Kadanoff's transformation, followed by certain rescaling procedure [4]. The action under the Kadanoff's transformation is changed from $S[\varphi]$ to $S_{\text{tra}}[\varphi]$ according to the equation

$$e^{-S_{\text{tra}}[\varphi]} = \int e^{-S[\varphi]} \prod_{j,\Lambda'<k<\Lambda} d\varphi_{j,\mathbf{k}} \;. \tag{2}$$

Alternatively, one often writes $-S_{\text{tra}}[\varphi] + AL^d$ instead of $-S_{\text{tra}}[\varphi]$ to separate the constant part of the action AL^d. This, however, is merely a redefinition of S_{tra}, and for our purposes it is suitable to use (2). Note that $\varphi_{j,\mathbf{k}} = \varphi'_{j,\mathbf{k}} + i\varphi''_{j,\mathbf{k}}$ is a complex number and $\varphi_{j,-\mathbf{k}} = \varphi^*_{j,\mathbf{k}}$ holds (since $\varphi_j(\mathbf{x})$ is always real), so that the integration over $\varphi_{j,\mathbf{k}}$ means in fact the integration over real and imaginary parts of $\varphi_{j,\mathbf{k}}$ for each pair of conjugated wave vectors $\mathbf{k}$ and $-\mathbf{k}$.

The Kadanoff's transformation (2) can be split in a sequence of elementary steps $S[\varphi] \to S_{\text{tra}}[\varphi]$ of the repeated integration given by

$$e^{-S_{\text{tra}}[\varphi]} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-S[\varphi]} d\varphi'_{j,\mathbf{q}} d\varphi''_{j,\mathbf{q}} \tag{3}$$

for each j and $\mathbf{q} \in \Omega$, where Ω is the subset of independent wave vectors ($\pm\mathbf{q}$ represent one independent mode) within $\Lambda' < q < \Lambda$. Thus, in the first elementary step of renormalization we have to insert the original action (1) into (3) and perform the integration for one choosen j and $\mathbf{q} \in \Omega$. In an exact treatment we must take into account that the action is already changed in the following elementary steps.

For $\Lambda' > \Lambda/3$, we can use the following exact decomposition of (1)

$$S[\varphi] = A_0 + A_1 \varphi_{j,\mathbf{q}} + A_1^* \varphi_{j,-\mathbf{q}} + A_2 \varphi_{j,\mathbf{q}} \varphi_{j,-\mathbf{q}} + B_2 \varphi^2_{j,\mathbf{q}} + B_2^* \varphi^2_{j,-\mathbf{q}} + A_4 \varphi^2_{j,\mathbf{q}} \varphi^2_{j,-\mathbf{q}} \;, \tag{4}$$

where

$$
\begin{aligned}
A_0 &= S|_{\varphi_{j,\pm\mathbf{q}}=0} \;, & (5)\\
A_1 &= \left.\frac{\partial S}{\partial \varphi_{j,\mathbf{q}}}\right|_{\varphi_{j,\pm\mathbf{q}}=0} = 4uV^{-1} \sum_{l,\mathbf{k}_1,\mathbf{k}_2}{}' \varphi_{j,\mathbf{k}_1}\varphi_{l,\mathbf{k}_2}\varphi_{l,-\mathbf{q}-\mathbf{k}_1-\mathbf{k}_2} \;, & (6)\\
A_2 &= \left.\frac{\partial^2 S}{\partial \varphi_{j,\mathbf{q}}\partial\varphi_{j,-\mathbf{q}}}\right|_{\varphi_{j,\pm\mathbf{q}}=0} = \Theta(\mathbf{q}) + \Theta(-\mathbf{q}) + 4uV^{-1}\sum_{l,\mathbf{k}}{}'(1+2\delta_{lj}) \mid \varphi_{l,\mathbf{k}} \mid^2 \;, & (7)\\
B_2 &= \frac{1}{2}\left.\frac{\partial^2 S}{\partial \varphi_{j,\mathbf{q}}^2}\right|_{\varphi_{j,\pm\mathbf{q}}=0} = 2uV^{-1}\sum_{l,\mathbf{k}}{}'(1+2\delta_{lj})\varphi_{l,\mathbf{k}}\varphi_{l,-2\mathbf{q}-\mathbf{k}} \;, & (8)\\
A_4 &= \frac{1}{4}\left.\frac{\partial^4 S}{\partial^2\varphi_{j,\mathbf{q}}\partial^2\varphi_{j,-\mathbf{q}}}\right|_{\varphi_{j,\pm\mathbf{q}}=0} = 6uV^{-1} \;. & (9)
\end{aligned}
$$

Here the sums are marked by a prime to indicate that terms containing $\varphi_{j,\pm\mathbf{q}}$ are omitted. This is simply a splitting of (1) into parts with all possible powers of $\varphi_{j,\pm\mathbf{q}}$. The condition $\Lambda' > \Lambda/3$, as well as the existence of the upper cutoff for the wave vectors, ensures that terms of the third power are absent in (4). Besides, the derivation is performed formally considering all $\varphi_{l,\mathbf{k}}$ as independent variables.

Taking into account (4), as well as the fact that $A_1 = A_1' + iA_1''$ and $B_2 = B_2' + iB_2''$ are complex numbers, the transformed action after the first elementary renormalization step reads

$$
\begin{aligned}
S_{\mathrm{tra}}[\varphi] = A_0 - \ln \Bigg\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} & \exp\left[-2\left(A_1'\varphi_{j,\mathbf{q}}' - A_1''\varphi_{j,\mathbf{q}}''\right)\right] \exp\left[-A_2\left({\varphi_{j,\mathbf{q}}'}^2 + {\varphi_{j,\mathbf{q}}''}^2\right)\right] \\
& \times \exp\left[-2B_2'\left({\varphi_{j,\mathbf{q}}'}^2 - {\varphi_{j,\mathbf{q}}''}^2\right) + 4B_2''\varphi_{j,\mathbf{q}}'\varphi_{j,\mathbf{q}}''\right] \\
& \times \exp\left[-A_4\left({\varphi_{j,\mathbf{q}}'}^2 + {\varphi_{j,\mathbf{q}}''}^2\right)^2\right] d\varphi_{j,\mathbf{q}}' d\varphi_{j,\mathbf{q}}'' \Bigg\} \;. \qquad (10)
\end{aligned}
$$

Considering only the field configurations which are relevant in the thermodynamic limit $V \to \infty$, Eq. (10) can be simplified, omitting the terms with B_2 and A_4. Really, using the coordinate representation $\varphi_{l,\mathbf{k}} = V^{-1/2}\int \varphi_l(\mathbf{x})\, e^{-i\mathbf{kx}} d\mathbf{x}$, we can write

$$
B_2 = 2uV^{-1}\left\{ \left[\sum_{l,j}(1+2\delta_{lj}) \int \varphi_l^2(\mathbf{x}) e^{i\mathbf{qx}}\, d\mathbf{x} \right] - 3\varphi_{j,-\mathbf{q}}^2 \right\} \;. \qquad (11)
$$

The quantity $V^{-1}\int \varphi_l^2(\mathbf{x})e^{i\mathbf{qx}}\, d\mathbf{x}$ is an average of $\varphi_l^2(\mathbf{x})$ over the volume with oscillating weight factor $e^{i\mathbf{qx}}$. This quantity vanishes for relevant configurations in the thermodynamic limit: due to the oscillations, positive and negative contributions are similar in magnitude and cancel at $V \to \infty$. Since $\langle \mid \varphi_{j,-\mathbf{q}} \mid^2\rangle = \langle \mid \varphi_{j,\mathbf{q}} \mid^2\rangle$ is the Fourier transform of the two–point correlation function, it is bounded at $V \to \infty$ and, hence, $\varphi_{j,-\mathbf{q}}^2$ also is bounded for relevant configurations giving nonvanishing contribution to the statistical averages $\langle\cdot\rangle$ in the thermodynamic limit. Consequently, for these configurations, A_2 is a quantity of order $\mathcal{O}(1)$, whereas $V^{-1}\varphi_{j,-\mathbf{q}}^2$ and B_2 vanish at $V \to \infty$. Note, however, that the term with $A_4 = \mathcal{O}\left(V^{-1}\right)$ in (10) cannot be neglected unless A_2 is positive. One can judge that the latter condition is satisfied for the relevant field configurations due to existence of the thermodynamic limit for the RG flow.

Omitting the terms with B_2 and A_4, the integrals in (10) can be easily calculated. It yields

$$S_{\mathrm{tra}}[\varphi] = S'[\varphi] + \Delta S^{el}_{\mathrm{tra}}[\varphi] \;, \tag{12}$$

where $S'[\varphi] = A_0$ is the original action, where only the $\pm\mathbf{q}$ modes of the j-th field component are omitted, whereas $\Delta S^{el}_{\mathrm{tra}}[\varphi]$ represents the elementary variation of the action given by

$$\Delta S^{el}_{\mathrm{tra}}[\varphi] = \ln\left(\frac{A_2}{\pi}\right) - \frac{\mid A_1 \mid^2}{A_2} \;. \tag{13}$$

Acoording to the arguments provided above, this equation is exact for the relevant field configurations with $A_2 > 0$ in the thermodynamic limit.

3 Superposition hypothesis and the Wegner–Houghton equation

Intuitively, it seems very reasonable that the result of integration over Fourier modes within a slice $\Lambda - d\Lambda < k < \Lambda$ at $d\Lambda \to 0$ can be represented as a superposition of elementary contributions given by (13). (We remind, however, that strictly exact treatment requires a sequential integration of $\exp(-S[\varphi])$ over a set of $\varphi_{j,\mathbf{q}}$.) This superposition hypothesis leads to the known Wegner–Houghton equation [11]. In this case the variation of the action due to the integration over slice reads

$$\Delta S_{\mathrm{tra}}[\varphi] = \frac{1}{2} \sum_j \sum_{\Lambda - d\Lambda < q < \Lambda} \left[\ln\left(\frac{A_2(j,\mathbf{q})}{\pi}\right) - \frac{\mid A_1(j,\mathbf{q}) \mid^2}{A_2(j,\mathbf{q})} \right] \;. \tag{14}$$

The factor $1/2$ appears, since only half of the wave vectors represent independent modes. Here we have indicated that the quantities A_1 and A_2 depend on the current j and $\mathbf{q}$. They depend also on the considered field configuration $[\varphi]$. If A_1 and A_2 are represented by the derivatives of $S[\varphi]$ at $\varphi_{j,\pm\mathbf{q}} = 0$ (see (6) and (7)), then the equation has similar form as in [11]. Our derivation refers to the φ^4 model, whereas in this form it may have a more general validity, as supposed in [11]. Indeed, (13) remains correct for a generalized model provided that the third order derivatives of $S[\varphi]$ vanish for relevant field configurations in the thermodynamic limit. Based on similar arguments we have used already, the latter assumption can be justified for certain class of models, for which the action is represented by a linear combination of φ^m–kind terms with wave–vector dependent weights and vanishing sum of the wave vectors $\sum_{l=1}^m \mathbf{k}_l = \mathbf{0}$ related to the φ factors. In this case it is also possible to formulate the equation avoiding the operation of setting $\varphi_{j,\pm\mathbf{q}} = 0$, since we have

$$A_1(j,\mathbf{q}) = \frac{\partial S}{\partial \varphi_{j,\mathbf{q}}} - \varphi_{j,-\mathbf{q}} \frac{\partial^2 S}{\partial \varphi_{j,\mathbf{q}} \varphi_{j,-\mathbf{q}}} \;, \tag{15}$$

$$A_2(j,\mathbf{q}) = \frac{\partial^2 S}{\partial \varphi_{j,\mathbf{q}} \varphi_{j,-\mathbf{q}}} \tag{16}$$

for the relevant configurations at $V \to \infty$. The second term in (15) appears because the derivative $\partial S / \partial \varphi_{j,\mathbf{q}}$ contains relevant terms with $\varphi_{j,-\mathbf{q}}$, which have to be removed.

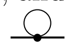
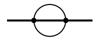
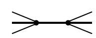
The Wegner–Houghton equation does not include the second, i. e., the rescaling step of the RG transforation. It, however, can be easily calculated for any given action, as described, e. g., in [4, 8]. It is not relevant four our further considerations.

4 The weak coupling limit

To verify the assumptions used in the derivation of the Wegner–Houghton equation, here we consider the weak coupling limit $u \to 0$ of the model with $\Theta(\mathbf{k}) = r_0 + ck^2$ at a given positive r_0, i. e., in the high temperature phase. In this case $\Delta S_{\text{tra}}[\varphi]$ can be expanded in powers of u. On the one hand, the expansion coefficients can be calculated exactly by the known perturbative methods applying the Wick's theorem [4, 5]. The Feynman diagram technique is helpful here [8]. On the other hand, the expansion can be performed in (14). Although the diagramatic perturbative renormalization can fail in vicinity of the critical point [7, 8], there are no doubts that it works in the weak coupling limit at $r_0 > 0$, which is the natural domain of validity of the perturbation theory. Hence, a comparison of the expansion coefficients is a test of (14).

Let us denote by $\Delta\tilde{S}_{\text{tra}}[\varphi]$ the variation of $S[\varphi]$ omitting the constant (independent of the field configuration) part. Then the expansion in powers of u reads

$$\Delta\tilde{S}_{\text{tra}}[\varphi] = \Delta S_1[\varphi]\, u + \left(\Delta S_2^{(a)}[\varphi] + \Delta S_2^{(b)}[\varphi] + \Delta S_2^{(c)}[\varphi]\right) u^2 + \mathcal{O}\left(u^3\right) \,, \tag{17}$$

where the expansion coefficient at u^2 is split in three parts $\Delta S_2^{(a)}[\varphi]$, $\Delta S_2^{(b)}[\varphi]$, and $\Delta S_2^{(c)}[\varphi]$ corresponding to the φ^2, φ^4, and φ^6 contributions, respectively. The contribution of order u is related to the diagram  , whereas the three second–order contributions — to the diagrams  ,  , and  . At $d\Lambda \to 0$, the diagramatic calculation (cf. [8]) for the n–component case yields

$$\Delta S_1[\varphi] = \frac{K_d \Lambda^{d-1}}{\Theta(\Lambda)} (n+2)\, d\Lambda \sum_{j,\mathbf{k}}^{\Lambda - d\Lambda} \mid \varphi_{j,\mathbf{k}} \mid^2 \tag{18}$$

$$\Delta S_2^{(b)}[\varphi] = -4V^{-1} \sum_{j,l,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}^{\Lambda - d\Lambda} \varphi_{j,\mathbf{k}_1}\varphi_{j,\mathbf{k}_2}\varphi_{l,\mathbf{k}_3}\varphi_{l,-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3} \times \left[(n+4)Q\left(\mathbf{k}_1+\mathbf{k}_2,\Lambda,d\Lambda\right) + 4Q\left(\mathbf{k}_1+\mathbf{k}_3,\Lambda,d\Lambda\right)\right] \tag{19}$$

$$\Delta S_2^{(c)}[\varphi] = -8V^{-2} \sum_{i,j,l,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_5}^{\Lambda - d\Lambda} \varphi_{i,\mathbf{k}_1}\varphi_{i,\mathbf{k}_2}\varphi_{j,\mathbf{k}_3}\varphi_{j,\mathbf{k}_4}\varphi_{l,\mathbf{k}_5}\varphi_{l,-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4-\mathbf{k}_5} \times G_0\left(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3\right) \mathcal{F}\left(\mid \mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3 \mid, \Lambda, d\Lambda\right) \,, \tag{20}$$

where $K_d = S(d)/(2\pi)^d$, $S(d) = 2\pi^{d/2}/\Gamma(d/2)$ is the area of unit sphere in d dimensions, $G_0(\mathbf{k}) = 1/[2\Theta(\mathbf{k})]$ is the Fourier transformed correlation function in the Gaussian approximation, $\Theta(\Lambda)$ is the value of $\Theta(\mathbf{k})$ at $k = \Lambda$, whereas $\mathcal{F}(k,\Lambda,d\Lambda)$ is a cutoff function which has the value 1 within $\Lambda - d\Lambda < k < \Lambda$ and zero otherwise. The quantity Q is given by

$$Q(\mathbf{k},\Lambda,d\Lambda) = V^{-1} \sum_{\Lambda - d\Lambda < q < \Lambda} G_0(\mathbf{q})\, G_0(\mathbf{k}-\mathbf{q})\, \mathcal{F}(\mid \mathbf{k}-\mathbf{q} \mid, \Lambda, d\Lambda) \,. \tag{21}$$

The uper border $\Lambda - d\Lambda$ indicated in the sums implies the upper cutoff of the wave vectors according to which we set $\varphi_{l,\mathbf{k}} = 0$ for $k > \Lambda - d\Lambda$. The contribution $\Delta S_2^{(a)}[\varphi]$, not given here, is a quantity which vanishes as $\propto (d\Lambda)^3$ at $d\Lambda \to 0$ and therefore is irrelevant in this limit, where the leading terms are of order $d\Lambda$.

The expansion of (14) gives no contribution $\Delta S_2^{(a)}[\varphi]$. It is not a contradiction, since (14) is expected to be correct only asymptotically at $d\Lambda \to 0$, where $\Delta S_2^{(a)}[\varphi] \propto$

$(d\Lambda)^3$. In this limit, we can make a replacement $\sum_{l,\mathbf{k}}' \to \sum_{l,\mathbf{k}}^{\Lambda-d\Lambda}$ in (7), since $\mid \varphi_{l,\mathbf{k}} \mid^2$ is a quantity of order unity for relevant configurations. In this case the expansion of the logarithm term in (14) yields $\Delta S_1[\varphi]$ exactly consistent with (18). Similar replacement in (6) leads to $\Delta S_2^{(c)}[\varphi]$ exactly consistent with (20) and therefore, obviously, also is correct.

Note that the contributions (18) and (20) come from diagrams with only one coupled line. The term (19) is related to the diagram with two coupled lines. The expansion of (14) provides a different result for the corresponding part of $\Delta \tilde{S}_{\rm tra}[\varphi]$:

$$\Delta S_2^{(b)}[\varphi] = -\frac{K_d \, \Lambda^{d-1} \, d\Lambda}{\Theta^2(\Lambda)} \, V^{-1} \sum_{j,l,\mathbf{k}_1,\mathbf{k}_2}^{\Lambda-d\Lambda} (n+4+4\delta_{jl}) \mid \varphi_{j,\mathbf{k}_1} \mid^2 \mid \varphi_{l,\mathbf{k}_2} \mid^2 \; . \tag{22}$$

Eq. (22) is obtained if we set $Q(\mathbf{k},\Lambda,d\Lambda) \to \delta_{\mathbf{k},\mathbf{0}} \, Q(\mathbf{0},\Lambda,d\Lambda)$ in (19) (in this case only the diagonal terms $j=l$ are relevant when summing up the contributions with $Q\,(\mathbf{k}_1+\mathbf{k}_3,\Lambda,d\Lambda)$, as it can be shown by an analysis of relevant real–space configurations, since $\langle \varphi_j(\mathbf{x}) \varphi_l(\mathbf{x}) \rangle = 0$ holds for $j \neq l$). It means that a subset of terms is lost in (22). The following analysis will show that this discrepancy between (19) and (22) is important.

It is a very remarkable fact that (22) is obtained also by the diagramatic perturbation method if we first integrate out only the mode with $\varphi_{j,\pm\mathbf{q}}$ and then formally apply the superposition hypothesis, as in the derivation of the Wegner–Houghton equation. It means that the only reason for the discrepancy is the intuitively motivated assumption of a simple superposition, which does not hold exactly. Hence, the Wegner–Houghton equation is an approximation.

The difference between (19) and (22) can be better seen in the coordinate representation. In this case (19) reads

$$\begin{aligned} \Delta S_2^{(b)}[\varphi] = & - (4n+16) \int\!\!\int \varphi^2(\mathbf{x}_1) \, R^2(\mathbf{x}_1-\mathbf{x}_2) \, \varphi^2(\mathbf{x}_2) \, d\mathbf{x}_1 d\mathbf{x}_2 \\ & - 16 \sum_{j,l} \int\!\!\int \varphi_j(\mathbf{x}_1)\varphi_l(\mathbf{x}_1) \, R^2(\mathbf{x}_1-\mathbf{x}_2) \, \varphi_j(\mathbf{x}_2)\varphi_l(\mathbf{x}_2) \, d\mathbf{x}_1 d\mathbf{x}_2 \; , \end{aligned} \tag{23}$$

where

$$R(\mathbf{x}) = V^{-1} \sum_{\mathbf{q}} G_0(\mathbf{q}) \, \mathcal{F}(q,\Lambda,d\Lambda) e^{i\mathbf{q}\mathbf{x}} \tag{24}$$

is the Fourier transform of $G_0\mathcal{F}$, and $\varphi^2(\mathbf{x}) = \sum_l \varphi_l^2(\mathbf{x})$. In three dimensions we have

$$R(\mathbf{x}) = \frac{\Lambda \, d\Lambda}{(2\pi)^2 \Theta(\Lambda)} \, \frac{1}{x} \sin(\Lambda x) \tag{25}$$

for any given $\mathbf{x}$ at $d\Lambda \to 0$ and $L \to \infty$, where L is the linear system size. The continuum approximation (25), however, is not correct for $x \sim L$ and therefore, probably, should not be used for the evaluation of (23).

The coordinate representation of (22) is

$$\begin{aligned} \Delta S_2^{(b)}[\varphi] \; = \; & -\frac{K_d \, \Lambda^{d-1} \, d\Lambda}{\Theta^2(\Lambda)} \Bigg[(n+4) \int\!\!\int \varphi^2(\mathbf{x}_1) \, V^{-1} \, \varphi^2(\mathbf{x}_2) \, d\mathbf{x}_1 d\mathbf{x}_2 \\ & + 4 \sum_j \int\!\!\int \varphi_j^2(\mathbf{x}_1) \, V^{-1} \, \varphi_j^2(\mathbf{x}_2) \, d\mathbf{x}_1 d\mathbf{x}_2 \Bigg] \; . \end{aligned} \tag{26}$$

It is obviously not consistent with (23). In fact, the term (26) represents a mean-field interaction, which is proportional to $1/V$ and independent of the distance, whereas (23) corresponds to another non-local interaction given by $R^2(\mathbf{x}_1-\mathbf{x}_2)$.

5 Discussion

We have shown in Sec. 4 that the Wegner–Houghton equation is approximate. Since the superposition fails, there is apparently no way to obtain a truly exact equation of such a relatively simple form by integrating out the fluctuation modes within $\Lambda - d\Lambda < k < \Lambda$.

It causes some questions regarding other nonperturbative RG equations, which are claimed to be exact. In particular, nonperturbative RG equations are derived in [9] which are interpreted as generalizations of the RG transformation to the case of smooth momentum (wave vector) cutoff. However, no integration over the field fluctuations is used in the derivation. To the contrary, it is based on an assumption that this renormalization step corresponds to a transformation of the field which leaves the partition function invariant (see Sec. 2.2.2 in [9]). If the obtained RG equations are generalizations of the sharp cutoff version, then the corresponding transformation step has to be consistent with the exact equation (2) in certain limit of the cutoff function. Confusing is the fact that the equations of [9] contain in a simple form only two derivatives with respect to the field, like in the approximate Wegner–Houghton equation. Hence, the equations derived in [9] and the Wegner–Houghton equation, probably, represent approximations of the same type.

There is another class of nonperturbative RG equations reviewed in [10], where a smooth small wave vector $\mathbf{q}$ (or infrared) cutoff is used instead of the more conventional for critical theories large $\mathbf{q}$ (or ultraviolet) cutoff. Such RG equations describe the variation of an average effective action $\Gamma_k[\phi]$ depending on the running cutoff scale k. Here $\phi(\mathbf{x}) = \langle \varphi(\mathbf{x}) \rangle$ is the averaged field imposed by sources $J(\mathbf{x})$. For simplicity, here we refer to the case of scalar field. Following [10, 14], we denote the Fourier amplitudes by $\phi(\mathbf{q})$.

The approah apparently is based on some assumptions. In particular, assuming that an exact relation is obtained when the Gaussian propogator is replaced with the full propogator in a one–loop perturbative equation, it is stated that the inverse average propogator $1/\langle \phi(\mathbf{q})\phi(-\mathbf{q}) \rangle$ is exactly the second functional derivative $\delta^2 \Gamma_k[\phi]/(\delta\phi(\mathbf{q})\delta\phi(-\mathbf{q}))$ of the action with respect to the field. The second derivative of $\Gamma_k[\phi]$ depends on the set of Fourier amplitudes $\phi(\mathbf{q})$ in general, when the action contains ϕ^m–kind terms with $m > 2$, e. g., a ϕ^4 term which is similar to that in (1). A paradox (contradiction) here is that the averaged over the field configurations propogator, by definition, is indepent of the field.

A derivation of the basic RG equation of this approach has been proposed in [10] avoiding a reference to the perturbation theory, however, assuming without proof that $\delta J(\mathbf{x})/\delta\phi(\mathbf{y}) = [\delta\phi(\mathbf{x})/\delta J(\mathbf{y})]^{-1}$ holds (see the text between (2.28) and (2.29) in [10]).

6 Conclusions

1. The nonperturbative Wegner–Houghton RG equation has been rederived (Secs. 2 and 3), discussing explicitly some assumptions which are used here. In particular, this approach assumes the superposition of small contributions provided by elementary integration steps over the short–wave fluctuation modes. Apparently, it has been accepted in the previous derivations as an intuitively self understandable fact. Our main (counterintuitive) result, however, is that this superposition hypothesis is false, as we have shown it by an exact calculation of the expansion coefficients in the weak coupling limit at $r_0 > 0$ (Sec. 4).

2. We have demonstrated that the exact renormalization of the φ^4 model is tricky, and therefore one has to be very careful with claims about exact RG equations. Our

results imply that the Wegner–Houghton equation is approximate and, according to the arguments provided in Sec. 5, the other nonperturbative RG equations also should be carefully reconsidered to clarify how accurate they are.

References

[1] D. J. Amit, Field theory, the renormalization group, and critical phenomena, World Scientific, Singapore, 1984

[2] D. Sornette, Critical Phenomena in Natural Sciences, Springer, Berlin, 2000

[3] K. G. Wilson, M. E. Fisher, Phys. Rev. Lett. **28**, 240 (1972)

[4] Shang–Keng Ma, Modern Theory of Critical Phenomena, W.A. Benjamin, Inc., New York, 1976

[5] J. Zinn–Justin, Quantum Field Theory and Critical Phenomena, Clarendon Press, Oxford, 1996

[6] H. Kleinert, V. Schulte–Frohlinde, Critical properties of ϕ^4 theories, World Scientific, 2001

[7] J. Kaupužs, Ann. Phys. (Leipzig) **10**, 299 (2001)

[8] J. Kaupužs, e–print cond-mat/0610015 (2006)

[9] C. Bagnuls, C. Bervillier, Phys. Rep. **348**, 91 (2001)

[10] J. Berges, N. Tetradis, C. Wetterich, Phys. Rep. **363**, 223 (2002)

[11] F. Wegner, A. Houghton, Phys. Rev. A **8**, 401 (1973)

[12] J. Polchinski, Nucl. Phys. B **231**, 269 (1984)

[13] C. Wetterich, Phys. Lett. B **301**, 90 (1993)

[14] L. Canet, B. Delamotte, D. Mouhanna, J. Vidal, Phys. Rev. B **68**, 064421 (2003)