

# CLASSIFICATION OF SUPERPOTENTIALS

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ABSTRACT. We extend our previous classification [DW4] of superpotentials of “scalar curvature type” for the cohomogeneity one Ricci-flat equations. We now consider the case not covered in [DW4], i.e., when some weight vector of the superpotential lies outside (a scaled translate of) the convex hull of the weight vectors associated with the scalar curvature function of the principal orbit. In this situation we show that either the isotropy representation has at most 3 irreducible summands or the first order subsystem associated to the superpotential is of the same form as the Calabi-Yau condition for submersion type metrics on complex line bundles over a Fano Kähler-Einstein product.

## 0. Introduction

In this paper we continue the study we began in [DW4] of superpotentials for the cohomogeneity one Einstein equations. These equations are the ODE system obtained as a reduction of the Einstein equations by requiring that the Einstein manifold admits an isometric Lie group action whose principal orbits  $G/K$  have codimension one [BB], [EW]. As discussed in [DW3], these equations can be viewed as a Hamiltonian system with constraint for a suitable Hamiltonian  $H$ , in which the potential term depends on the Einstein constant and the scalar curvature of the principal orbit, and the kinetic term is essentially the Wheeler-deWitt metric, which is of Lorentz signature.

For any Hamiltonian system with Hamiltonian  $H$  and position variable  $q$ , a *superpotential* is a *globally defined* function  $u$  on configuration space that satisfies the equation

$$(0.1) \quad H(q, du_q) = 0.$$

From the classical physics viewpoint,  $u$  is a  $C^2$  (rather than a viscosity) solution of a time-independent Hamilton-Jacobi equation. The literature for implicitly defined first order partial differential equations then suggests that such solutions are fairly rare. It is therefore not unreasonable to expect in our case that one can classify (at least under appropriate conditions) those principal orbits where the associated cohomogeneity one Einstein equations admit a superpotential.

The existence of such a superpotential  $u$  in our case leads naturally to a subsystem of equations of half the dimension of the full Einstein system. One way to see this is via generalised first integrals which are linear in momenta, described in [DW4]. Schematically, the subsystem may be written as  $\dot{q} = \mathcal{J}\nabla u$  where  $\mathcal{J}$  is an endomorphism related to the kinetic term of the Einstein Hamiltonian.

String theorists have exploited the superpotential idea in their search for explicit metrics of special holonomy (see for example [CGLP1], [CGLP2], [CGLP3], [BGGG] and references in [DW4]). The point here is that the subsystem defined by the superpotential often (though not always) represents the condition that the metric has special holonomy. Also, the subsystem can often be integrated explicitly.

In [DW4], §6, we obtained classification results for superpotentials of the cohomogeneity one Ricci-flat equations. Besides assuming that  $G$  and  $K$  are both compact, connected Lie groups such that the isotropy representation of  $G/K$  is multiplicity-free, we also mainly restricted our attention to superpotentials which are of the same form as the scalar curvature function of  $G/K$ , i.e., a finite sum with constant coefficients of exponential terms. Almost all the known superpotentials are of this kind.

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However, the above classification results were further subject to the technical assumption that the extremal weights for the superpotential did not lie in the null cone of the Wheeler-de Witt metric. In [DW4] we gave some examples of superpotentials which do not satisfy this hypothesis. These included several new examples which do not seem to be associated to special holonomy.

In this paper, therefore, we attempt to solve the classification problem without the non-null assumption on the extremal weights.

As in [DW4], we use techniques of convex geometry to analyse the two polytopes naturally associated to the classification problem. The first is (a rescaled translate of) the convex hull  $\text{conv}(\mathcal{W})$  of the weight vectors appearing in the scalar curvature function of the principal orbit. The second is the convex hull  $\text{conv}(\mathcal{C})$  of the weight vectors in the superpotential. In [DW4] we showed that the non-null assumption forces these polytopes to be equal, so we could analyse the existence of superpotentials by looking at the geometry of  $\text{conv}(\mathcal{W})$ .

In the current paper,  $\text{conv}(\mathcal{C})$  may be strictly bigger than  $\text{conv}(\mathcal{W})$  because of the existence of vertices outside  $\text{conv}(\mathcal{W})$  but lying on the null cone of the Wheeler-de Witt metric. Our strategy is to consider such a vertex  $c$  and project  $\text{conv}(\mathcal{W})$  onto an affine hyperplane separating  $c$  from  $\text{conv}(\mathcal{W})$ . We can now analyse the existence of superpotentials in terms of the projected polytope.

The analysis becomes considerably more complicated because, whereas in [DW4] we could analyse the situation by looking at the vertices and edges of  $\text{conv}(\mathcal{W})$ , now, because we have projected onto a subspace of one lower dimension, we have to consider the 2-dimensional faces of  $\text{conv}(\mathcal{W})$  also.

We find that in this situation the only polytopes  $\text{conv}(\mathcal{W})$  arising from principal orbits with more than three irreducible summands in their isotropy representations are precisely those coming from principal orbits which are circle bundles over a (homogeneous) Fano product. In the latter case, the solutions of the subsystem defined by the superpotential correspond to Calabi-Yau metrics, as discussed in [DW4].

After a review of basic material in §1, we state the main classification theorem of the paper in §2 and give an outline of the strategy of the proof there.

## 1. Review and notation

In this section we fix notation for the problem and review the set-up of [DW4].

Let  $G$  be a compact Lie group,  $K \subset G$  be a closed subgroup, and  $M$  be a cohomogeneity one  $G$ -manifold of dimension  $n + 1$  with principal orbit type  $G/K$ , which is assumed to be connected and almost effective. A  $G$ -invariant metric  $\bar{g}$  on  $M$  can be written in the form  $\bar{g} = \varepsilon dt^2 + g_t$  where  $t$  is a coordinate transverse to the principal orbits,  $\varepsilon = \pm 1$ , and  $g_t$  is a 1-parameter family of  $G$ -homogeneous Riemannian metrics on  $G/K$ . When  $\varepsilon = 1$ , the metric  $\bar{g}$  is Riemannian, and when  $\varepsilon = -1$ , the metric  $\bar{g}$  is spatially homogeneous Lorentzian, i.e., the principal orbits are space-like hypersurfaces.

We choose an  $\text{Ad}(K)$ -invariant decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  where  $\mathfrak{g}$  and  $\mathfrak{k}$  are respectively the Lie algebras of  $G$  and  $K$ , and  $\mathfrak{p}$  is identified with the isotropy representation of  $G/K$ . Let

$$(1.1) \quad \mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r$$

be a decomposition of  $\mathfrak{p} \approx T_{(K)}(G/K)$  into irreducible real  $K$ -representations. We let  $d_i$  be the real dimension of  $\mathfrak{p}_i$ , and  $n = \sum_{i=1}^r d_i$  be the dimension of  $G/K$  (so  $\dim M = n + 1$ ). We use  $d$  for the vector of dimensions  $(d_1, \dots, d_r)$ . We shall assume that the isotropy representation of  $G/K$  is *multiplicity free*, i.e., all the summands  $\mathfrak{p}_i$  in (1.1) are distinct as  $K$ -representations. In particular, if there is a trivial summand it must be 1-dimensional.

We use  $q = (q_1, \dots, q_r)$  to denote exponential coordinates on the space of  $G$ -invariant metrics on  $G/K$ . The Hamiltonian  $H$  for the cohomogeneity one Einstein equations with principal orbit  $G/K$  is now given by:

$$H = v^{-1}J + \varepsilon v((n-1)\Lambda - S),$$

where  $\Lambda$  is the Einstein constant,  $v = \frac{1}{2}e^{d \cdot q}$  is the relative volume and

$$(1.2) \quad J(p, p) = \frac{1}{n-1} \left( \sum_{i=1}^r p_i \right)^2 - \sum_{i=1}^r \frac{p_i^2}{d_i},$$

which has signature  $(1, r-1)$ . The scalar curvature  $S$  of  $G/K$  above can be written as

$$S = \sum_{w \in \mathcal{W}} A_w e^{w \cdot q},$$

where  $A_w$  are *nonzero* constants and  $\mathcal{W}$  is a finite collection of vectors  $w \in \mathbb{Z}^r \subset \mathbb{R}^r$ . The set  $\mathcal{W}$  depends only on  $G/K$  and its elements will be referred to as *weight vectors*. These are of three types

- (i) type I: one entry of  $w$  is  $-1$ , the others are zero,
- (ii) type II: one entry is  $1$ , two are  $-1$ , the rest are zero,
- (iii) type III: one entry is  $1$ , one is  $-2$ , the rest are zero.

**Notation 1.1.** As in [DW4] we use  $(-1^i, -1^j, 1^k)$  to denote the type II vector  $w \in \mathcal{W} \subset \mathbb{R}^r$  with  $-1$  in places  $i$  and  $j$ , and  $1$  in place  $k$ . Similarly,  $(-2^i, 1^j)$  will denote the type III vector with  $-2$  in place  $i$  and  $1$  in place  $j$ , and  $(-1^i)$  the type I vector with  $-1$  in place  $i$ .

**Remark 1.2.** We collect below various useful facts from [DW4] and [WZ1]. Also, we shall use standard terminology from convex geometry, as given, e.g., in [Zi]. In particular, a “face” is not necessarily 2-dimensional. However, a vertex and an edge are respectively zero and one-dimensional. The convex hull of a set  $X$  in  $\mathbb{R}^r$  will be denoted by  $\text{conv}(X)$ .

- (a) For a type I vector  $w$ , the coefficient  $A_w > 0$  while for type II and type III vectors,  $A_w < 0$ .
- (b) The type I vector with  $-1$  in the  $i$ th position is absent from  $\mathcal{W}$  iff the corresponding summand  $\mathfrak{p}_i$  is an abelian subalgebra which satisfies  $[\mathfrak{k}, \mathfrak{p}_i] = 0$  and  $[\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{p}_j$  for all  $j \neq i$ . If the isotropy group  $K$  is connected, these last conditions imply that  $\mathfrak{p}_i$  is 1-dimensional, and the  $\mathfrak{p}_j, j \neq i$ , are irreducible representations of the (compact) analytic group whose Lie algebra is  $\mathfrak{k} \oplus \mathfrak{p}_i$ .
- (c) If  $(1^i, -1^j, -1^k)$  occurs in  $\mathcal{W}$  then its permutations  $(-1^i, 1^j, -1^k)$  and  $(-1^i, -1^j, 1^k)$  do also.
- (d) If  $\dim \mathfrak{p}_i = 1$  then no type III vector with  $-2$  in place  $i$  is present in  $\mathcal{W}$ . If in addition  $K$  is connected, then no type II vector with nonzero entry in place  $i$  is present.
- (e) If  $I$  is a subset of  $\{1, \dots, r\}$ , then each of the equations  $\sum_{i \in I} x_i = 1$  and  $\sum_{i \in I} x_i = -2$  defines a face (possibly empty) of  $\text{conv}(\mathcal{W})$ . In particular, all type III vectors in  $\mathcal{W}$  are vertices and  $(-1^i, -1^k, 1^j) \in \mathcal{W}$  is a vertex unless both  $(-2^i, 1^j)$  and  $(-2^k, 1^j)$  lie in  $\mathcal{W}$ .
- (f) For  $v, w \in \mathcal{W}$  (or indeed for any  $v, w$  such that  $\sum v_i$  or  $\sum w_i = -1$ ), we have

$$(1.3) \quad J(v + d, w + d) = 1 - \sum_{i=1}^r \frac{v_i w_i}{d_i}.$$

For the remainder of the paper, we shall work in the Ricci-flat Riemannian case, that is, we take  $\varepsilon = 1$  and  $\Lambda = 0$ . As in [DW4], any argument that does not use the sign of  $A_w$  would be valid in the Lorentzian case. We shall also assume that  $\text{conv}(\mathcal{W})$  is  $r-1$  dimensional. This is certainly the case if  $G$  is semisimple, as  $\mathcal{W}$  spans  $\mathbb{R}^r$  (see the proof of Theorem 3.11 in [DW3]).

The superpotential equation (0.1) now becomes

$$(1.4) \quad J(\nabla u, \nabla u) = e^{d \cdot q} S,$$

where  $\nabla$  denotes the Euclidean gradient in  $\mathbb{R}^r$ . As in [DW4] we shall look for solutions to Eq.(1.4) of the form

$$(1.5) \quad u = \sum_{\bar{c} \in \mathcal{C}} F_{\bar{c}} e^{\bar{c} \cdot q}$$

where  $\mathcal{C}$  is a finite set in  $\mathbb{R}^r$ , and the  $F_{\bar{c}}$  are nonzero constants. Now Eq.(1.4) reduces to, for each  $\xi \in \mathbb{R}^r$ ,

$$(1.6) \quad \sum_{\bar{a}+\bar{c}=\xi} J(\bar{a}, \bar{c}) F_{\bar{a}} F_{\bar{c}} = \begin{cases} A_w & \text{if } \xi = d + w \text{ for some } w \in \mathcal{W} \\ 0 & \text{if } \xi \notin d + \mathcal{W}. \end{cases}$$

We shall assume henceforth that  $r \geq 2$  since the superpotential equation always has a solution in the  $r = 1$  case, as was noted in [DW4], and  $J$  is of Lorentz signature only when  $r \geq 2$ . The following facts were deduced in [DW4] from Eq.(1.6).

**Proposition 1.3.**  $\text{conv}(\frac{1}{2}(d + \mathcal{W})) \subset \text{conv}(\mathcal{C})$ .

*Proof.* If  $w \in \mathcal{W}$ , then Eq.(1.6) implies that  $d + w = \bar{a} + \bar{c}$  for some  $\bar{a}, \bar{c} \in \mathcal{C}$ , and hence that  $\frac{1}{2}(d + w) = \frac{1}{2}(\bar{a} + \bar{c}) \in \text{conv}(\mathcal{C})$ .  $\square$

**Proposition 1.4.** *If  $\bar{a}, \bar{c} \in \mathcal{C}$  and  $\bar{a} + \bar{c}$  cannot be written as the sum of two non-orthogonal elements of  $\mathcal{C}$  distinct from  $\bar{a}, \bar{c}$  then either  $J(\bar{a}, \bar{c}) = 0$  or  $\bar{a} + \bar{c} \in d + \mathcal{W}$ .*

*In particular, if  $\bar{c}$  is a vertex of  $\mathcal{C}$ , then either  $J(\bar{c}, \bar{c}) = 0$ , or  $2\bar{c} = d + w$  for some  $w \in \mathcal{W}$  and  $J(\bar{c}, \bar{c}) F_{\bar{c}}^2 = A_w$ . In the latter case,  $J(d + w, d + w)$  has the same sign as  $A_w$  so is  $> 0$  if  $w$  is type I and  $< 0$  if  $w$  is type II or III.*

As mentioned in the Introduction, for the classification in [DW4] we made the assumption that all vertices  $\bar{c}$  of  $\mathcal{C}$  are non-null. Under this assumption, the second assertion of Prop 1.4 implies that all vertices of  $\mathcal{C}$  lie in  $\frac{1}{2}(d + \mathcal{W})$ . Hence  $\text{conv}(\mathcal{C})$  is contained in  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ , and by Prop 1.3 they are equal. This meant that in [DW4], subject to the non-null assumption, we could study the existence of a superpotential in terms of the convex geometry of  $\mathcal{W}$ .

The aim of the current paper is to drop this assumption. We still have

$$\text{conv}(\frac{1}{2}(d + \mathcal{W})) \subset \text{conv}(\mathcal{C}),$$

but can no longer deduce that these sets are equal. The problem is that a vertex  $\bar{c}$  of  $\text{conv}(\mathcal{C})$  may lie outside  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  if it is null.

In fact, it is clear from the above discussion that  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  is strictly contained in  $\text{conv}(\mathcal{C})$  if and only if  $\mathcal{C}$  has a null vertex. For if  $\bar{c}$  is a null vertex of  $\mathcal{C}$  and  $2\bar{c} = d + w$  for some  $w \in \mathcal{W}$ , then Eq.(1.6) fails for  $\xi = d + w$ .

We conclude this section by proving an analogue of Proposition 2.5 in [DW4]. The arguments below using Prop 1.4 are ones which will recur throughout this paper. Henceforth when we use the term “orthogonal” we mean orthogonal with respect to  $J$  unless otherwise stated.

**Theorem 1.5.**  $\mathcal{C}$  lies in the hyperplane  $\{\bar{x} : \sum \bar{x}_i = \frac{1}{2}(n - 1)\}$  (possibly after subtracting a constant from the superpotential).

*Proof.* We can assume  $0 \notin \mathcal{C}$  by subtracting a constant from the superpotential. We shall also use repeatedly below the fact that as  $J$  has signature  $(1, r - 1)$  there are no null planes, only null lines.

Denote by  $H_\lambda$  the hyperplane  $\sum \bar{x}_i = \lambda$ , so  $\frac{1}{2}(d + \mathcal{W})$  lies in  $H_{\frac{1}{2}(n-1)}$ . Suppose there exist elements of  $\mathcal{C}$  with  $\sum \bar{x}_i > \frac{1}{2}(n - 1)$ . Let  $\lambda_{\max}$  denote the greatest value of  $\sum \bar{x}_i$  over  $\mathcal{C}$ . If  $\tilde{a}\tilde{c}$  is an edge of  $\text{conv}(\mathcal{C}) \cap H_{\lambda_{\max}}$ , then Prop 1.4 shows that  $\tilde{a}, \tilde{c}$  are null, and that  $\tilde{c}$  is orthogonal to the element of  $\mathcal{C}$  closest to it on the edge. Hence  $\tilde{c}$  is orthogonal to the whole edge. Now  $J$  is totally null on  $\text{Span}\{\tilde{a}, \tilde{c}\}$ , so since there are no null planes,  $\tilde{a}, \tilde{c}$  are proportional, which is impossible as they are both in  $H_{\lambda_{\max}}$ . So  $\mathcal{C} \cap H_{\lambda_{\max}}$  is a single point  $\tilde{c}_{\max}$ , which is null.

Next we claim that all elements of  $\mathcal{C}$  lying in the half-space  $\sum \bar{x}_i > \frac{1}{2}(n - 1)$  must be multiples of  $\tilde{c}_{\max}$ . If not, let  $\lambda_*$  be the greatest value such that there is an element of  $\mathcal{C}$ , not proportional to  $\tilde{c}_{\max}$ , in  $H_{\lambda_*}$ . Let  $\tilde{a}$  be a vertex of  $\text{conv}(\mathcal{C}) \cap H_{\lambda_*}$ , not proportional to  $\tilde{c}_{\max}$ . Now, by Prop 1.4,  $J(\tilde{a}, \tilde{c}_{\max}) = 0$ , and so  $\tilde{a}$  is not null. Since  $\lambda_* > \frac{1}{2}(n - 1)$ , we see  $\tilde{a} + \tilde{a}$  must be written in another

way as a sum of two non-orthogonal elements of  $\mathcal{C}$ . This sum must be of the form  $\mu\tilde{c}_{\max} + \tilde{f}$ . But  $\tilde{c}_{\max}$  is orthogonal to  $\tilde{a}$  and to itself, hence to  $\tilde{f}$ , a contradiction establishing our claim.

Similarly, all elements of  $\mathcal{C}$  lying in  $\sum \bar{x}_i < \frac{1}{2}(n-1)$  are multiples of an element  $\tilde{c}_{\min}$ , should they occur. (Note that  $J$  is negative definite on  $H_0$  and we have assumed  $0 \notin \mathcal{C}$  so  $\lambda_{\min} \neq 0$ .)

We denote the sets of elements lying in these open half-spaces by  $\mathcal{C}_+$  and  $\mathcal{C}_-$  respectively. Note that, when non-empty,  $\mathcal{C}_+$  and  $\mathcal{C}_-$  are orthogonal to all elements of  $\mathcal{C} \cap H_{\frac{1}{2}(n-1)}$ . (For if  $\tilde{a} \in \mathcal{C} \cap H_{\frac{1}{2}(n-1)}$  then  $\tilde{a} + \tilde{c}_{\max}$  cannot be written in another way as a sum of two non-orthogonal elements of  $\mathcal{C}$ .) In particular, if  $\tilde{c}_{\max}$  and  $\tilde{c}_{\min}$  are orthogonal, then  $\tilde{c}_{\max}$  is orthogonal to all of  $\text{conv}(\mathcal{C})$ , which is  $r$ -dimensional by assumption. So  $\tilde{c}_{\max}$  is zero, a contradiction. The same argument implies that  $\mathcal{C}_+$  and  $\mathcal{C}_-$  are both non-empty.

Let  $\nu\tilde{c}_{\min}$  and  $\mu\tilde{c}_{\max}$  be respectively the elements of  $\mathcal{C}_-$  and  $\mathcal{C}_+$  closest to  $H_{\frac{1}{2}(n-1)}$ . Suppose that  $\tilde{c}_{\max} + \nu\tilde{c}_{\min} = \tilde{c}^{(1)} + \tilde{c}^{(2)}$  with  $\tilde{c}^{(i)} \in \mathcal{C}$  and  $J(\tilde{c}^{(1)}, \tilde{c}^{(2)}) \neq 0$ . Non-orthogonality means the  $\tilde{c}^{(i)}$  cannot belong to the same side of  $H_{\frac{1}{2}(n-1)}$  and by the choice of  $\nu$ , they cannot belong to opposite sides of  $H_{\frac{1}{2}(n-1)}$ . Both therefore lie in  $H_{\frac{1}{2}(n-1)}$ . But by the previous paragraph,  $J(\tilde{c}_{\max} + \nu\tilde{c}_{\min}, \tilde{c}^{(1)} + \tilde{c}^{(2)}) = 0$ . This means that  $\tilde{c}_{\max} + \nu\tilde{c}_{\min}$  is null, which contradicts  $J(\tilde{c}_{\max}, \tilde{c}_{\min}) \neq 0$ . Hence  $\tilde{c}_{\max} + \nu\tilde{c}_{\min}$  lies in  $d + \mathcal{W} \subset H_{n-1}$ . Applying the same argument to  $\tilde{c}_{\min} + \mu\tilde{c}_{\max}$ , we find that in fact  $\mu = \nu = 1$ , i.e.,  $\mathcal{C}_+ = \{\tilde{c}_{\max}\}$  and  $\mathcal{C}_- = \{\tilde{c}_{\min}\}$ .

Now  $\mathcal{C} \cap H_{\frac{1}{2}(n-1)}$  (and hence its convex hull) is contained in the hyperplanes  $\tilde{c}_{\max}^\perp, \tilde{c}_{\min}^\perp$  in  $H_{\frac{1}{2}(n-1)}$ . These hyperplanes are distinct as  $\tilde{c}_{\max}$  is orthogonal to itself but not to  $\tilde{c}_{\min}$ . Hence  $d + \mathcal{W} \subset (\mathcal{C} + \mathcal{C}) \cap H_{n-1}$  is contained in the union of the point  $\tilde{c}_{\max} + \tilde{c}_{\min}$  and the codimension 2 subspace  $\tilde{c}_{\max}^\perp \cap \tilde{c}_{\min}^\perp$  of  $H_{n-1}$ . So  $\text{conv}(d + \mathcal{W})$  is contained in a codimension 1 subspace of  $H_{n-1}$ , contradicting our assumption that  $\dim \text{conv}(d + \mathcal{W}) = r - 1$ .  $\square$

**Remark 1.6.** A notational difficulty arises from the fact that, as seen above, points of  $\mathcal{C}$  are on the same footing as points in  $\frac{1}{2}(d + \mathcal{W})$  rather than points of  $\mathcal{W}$ . Accordingly, we shall use letters  $c, u, v, \dots$  to denote elements of the hyperplane  $\sum u_i = -1$  (such as elements of  $\mathcal{W}$ ), and  $\bar{c}, \bar{u}, \bar{v}, \dots$  to denote the *associated elements*  $\frac{1}{2}(d + c), \frac{1}{2}(d + u), \frac{1}{2}(d + v), \dots$  of the hyperplane  $\sum \bar{u}_i = \frac{1}{2}(n-1)$  (such as elements of  $\mathcal{C}$  or of  $\frac{1}{2}(d + \mathcal{W})$ ).

Note that for any convex or indeed affine sum  $\sum \lambda_j \xi^{(j)}$  of vectors  $\xi^{(j)}$  in  $\mathbb{R}^r$ , we have

$$\overline{\sum_j \lambda_j \xi^{(j)}} = \sum_j \lambda_j \overline{\xi^{(j)}}.$$

Since we now know that the set  $\mathcal{C}$ , like  $\frac{1}{2}(d + \mathcal{W})$ , lies in  $H_{\frac{1}{2}(n-1)} := \{\bar{x} : \sum \bar{x}_i = \frac{1}{2}(n-1)\}$ , we will adopt the convention, as in the last paragraph, that when we refer to hyperplanes such as  $\bar{c}^\perp$  in the rest of the paper, we mean “affine hyperplanes in  $H_{\frac{1}{2}(n-1)}$ ”.

## 2. The classification theorem and the strategy of its proof

We can now state the main theorem of the paper.

**Theorem 2.1.** *Let  $G$  be a compact connected Lie group and  $K$  a closed connected subgroup such that the isotropy representation of  $G/K$  is the direct sum of  $r$  pairwise inequivalent  $\mathbb{R}$ -irreducible summands. Assume that  $\dim \text{conv}(\mathcal{W}) = r - 1$ , where  $\mathcal{W}$  is the set of weights of the scalar curvature function of  $G/K$  (cf §1). (This holds, for example, if  $G$  is semisimple.)*

*If the cohomogeneity one Ricci-flat equations with  $G/K$  as principal orbit admit a superpotential of form (1.5) where  $\mathcal{C}$  contains a  $J$ -null vertex, then we are in one of the following situations (up to permutations of the irreducible summands):*

- (i)  $\mathcal{W} = \{(-1)^i, (1^1, -2^i) : 2 \leq i \leq r\}$ ,  $d_1 = 1$ ,  $\mathcal{C} = \frac{1}{2}(d + \{(-1^1), (1^1, -2^i) : 2 \leq i \leq r\})$  and  $r \geq 2$ ;

(ii)  $r \leq 3$ .

**Remark 2.2.** As mentioned before, the situation where  $\mathcal{C}$  has no null vertex was analysed in [DW4]. Hence, except for the  $r \leq 3$  case, Theorem 2.1 completes the classification of superpotentials of scalar curvature type subject to the above assumptions on  $G$  and  $K$ .

**Remark 2.3.** The first case of the above theorem is realized by certain circle bundles over a product of  $r-1$  Fano (homogeneous) Kähler-Einstein manifolds (cf. Example 8.1 in [DW4], and [BB], [WW], [CGLP3]), and the subsystem of the Ricci-flat equations singled out by the superpotential in these examples corresponds to the Calabi-Yau condition. For more on the  $r = 2$  case, see the concluding remarks in §10.

**Remark 2.4.** Theorem 2.1 remains true if we replace the connectedness of  $G$  and  $K$  by the connectedness of  $G/K$  and the extra condition on the isotropy representation given by the second statement in Remark 1.2(d), i.e., if  $\mathfrak{p}_i$  is an irreducible summand of dimension 1 in the isotropy representation of  $G/K$ , then  $[\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{k} \oplus \mathfrak{p}_j$  for all  $j \neq i$ .

This weaker property does hold in practice. For example, the exceptional Aloff-Wallach space  $N_{1,1}$  can be written as  $(SU(3) \times \Gamma)/(U_{1,1} \cdot \Delta\Gamma)$ , where  $U_{1,1}$  is the set of diagonal matrices of the form  $\text{diag}(\exp(i\theta), \exp(i\theta), \exp(-2i\theta))$  and  $\Gamma$  is the dihedral group with generators

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} e^{2\pi i/3} & 0 & 0 \\ 0 & e^{-2\pi i/3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In order to prove Theorem 2.1 we have to analyse the situation when there is a null vertex  $\bar{c} \in \mathcal{C}$ . As discussed in §1,  $\text{conv}(\mathcal{C})$  now strictly includes  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  as  $\bar{c}$  is not in  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . Our strategy is to take an affine hyperplane  $H$  separating  $\bar{c}$  from  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ , and consider the projection  $\Delta^{\bar{c}}$  of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  onto  $H$  from  $\bar{c}$ .

Roughly speaking, whereas in [DW4] we could analyse the situation by looking at the vertices and edges of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ , now, because we have projected onto a subspace of one lower dimension, we have to consider the 2-dimensional faces of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  also.

This is a natural method of dealing with the situation of a point outside a convex polytope. It has some relation to the notion of “lit set” introduced in a quite different context by Ginzburg-Guillemin-Karshon [GGK].

The analysis in the next section will show that the vertices of the projected polytope  $\Delta$  can be divided into three types (Theorem 3.8). We label these types (1A), (1B) and (2). Roughly, these correspond to vertices orthogonal to  $\bar{c}$ , vertices  $\bar{\xi}$  such that the line through  $\bar{c}$  and  $\bar{\xi}$  meets  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  at a vertex, and vertices  $\bar{\xi}$  such that this line meets  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  in an edge.

In the remainder of the paper we shall gradually narrow down the possibilities for each type. In §3 we begin a classification of type (2) vertices. In §4 we are able to deduce that  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  lies in the half space  $J(\bar{c}, \cdot) \geq 0$ . We are able to deduce an orthogonality result for vectors on edges in  $\text{conv}(\frac{1}{2}(d + \mathcal{W})) \cap \bar{c}^\perp$ . This is analogous to the key result Theorem 3.5 of [DW4] that held (in the more restrictive situation of that paper) for general edges in  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . In §5 we exploit this result and some estimates to classify the possible configurations of (1A) vertices (i.e. vertices in  $\bar{c}^\perp$ ), see Theorem 5.18.

In §6 we attack the (1B) vertices, exploiting the fact that adjacent (1B) vertices give rise to a 2-dimensional face of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . This is the most laborious part of the paper, as it involves a case-by-case analysis of such faces. We show that adjacent (1B) vertices can arise only in a very small number of situations (Theorem 6.18). In §7 we exploit the listing of 2-dim faces to show that there is at most one type (2) vertex, except in two special situations (Theorem 7.1). In §8 and 9, we eliminate more possibilities for adjacent (1B) and type (2) vertices. We find that if  $r \geq 4$  then we are either in case (i) of the Theorem or there are no type (2) vertices and no adjacent type (1B)

vertices. Using the results of §4, in the latter case we find that all vertices are (1A) except for a single (1B). Building on the results of §5 for (1A) vertices, we are able to rule out this situation in §10, see Theorem 10.15 and Corollary 10.16.

### 3. Projection onto a hyperplane

We first present some results about null vectors in  $H_{\frac{1}{2}(n-1)}$ .

**Remark 3.1.** From Eq.(1.3), the set of null vectors in the hyperplane  $H_{\frac{1}{2}(n-1)}$  form an ellipsoid  $\{\sum x_i^2/d_i = 1\}$ . If  $\bar{c}$  is null, then the hyperplane  $\bar{c}^\perp$  in  $H_{\frac{1}{2}(n-1)}$  is the tangent space to this ellipsoid. So any element  $\bar{x} \neq \bar{c}$  of  $\bar{c}^\perp$  satisfies  $J(\bar{x}, \bar{x}) < 0$ .

**Lemma 3.2.** *Let  $x, y$  satisfy  $\sum x_i = \sum y_i = -1$ .*

*Suppose that  $J(\bar{x}, \bar{x})$  and  $J(\bar{y}, \bar{y}) \geq 0$ . Then  $J(\bar{x}, \bar{y}) \geq 0$ , with equality iff  $\bar{x}$  is null and  $\bar{x} = \bar{y}$ . In particular, if  $\bar{x}, \bar{y}$  are distinct null vectors then  $J(\bar{x}, \bar{y}) > 0$ .*

*Proof.* This follows from Eq.(1.3) and Cauchy-Schwartz.  $\square$

**Proposition 3.3.** *Let  $H = \{\bar{x} : h(\bar{x}) = \lambda\}$  be an affine hyperplane, where  $h$  is a linear functional such that  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  lies in the open half-space  $\{\bar{x} : h(\bar{x}) < \lambda\}$ . Then there is at most one element of  $\mathcal{C}$  in the complementary open half-space  $\{\bar{x} : h(\bar{x}) > \lambda\}$ . Such an element is a null vertex of  $\text{conv}(\mathcal{C})$ . Hence any element of  $\mathcal{C}$  outside  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  is a null vertex of  $\mathcal{C}$ .*

*Proof.* Suppose the points of  $\mathcal{C}$  with  $h(\bar{x}) > \lambda$  are  $\bar{c}^{(1)}, \dots, \bar{c}^{(m)}$  with  $m > 1$ . Our result is stable with respect to sufficiently small perturbations of  $H$ , so we can assume that  $h(\bar{c}^{(1)}) > h(\bar{c}^{(2)}) \geq h(\bar{c}^{(3)}), \dots, h(\bar{c}^{(m)})$ .

Now  $\bar{c}^{(1)} + \bar{c}^{(1)}$  and  $\bar{c}^{(1)} + \bar{c}^{(2)}$  cannot be written in any other way as the sum of two elements of  $\mathcal{C}$ . Hence, by Prop. 1.4,  $\bar{c}^{(1)}$  is null and  $J(\bar{c}^{(1)}, \bar{c}^{(2)}) = 0$ . The only other way  $\bar{c}^{(2)} + \bar{c}^{(2)}$  can be written is as  $\bar{c}^{(1)} + \bar{c}$  for some  $\bar{c} \in \mathcal{C}$ . But then  $\bar{c} = 2\bar{c}^{(2)} - \bar{c}^{(1)}$ , so  $J(\bar{c}^{(1)}, \bar{c}) = 0$ , and such sums will not contribute. Hence  $J(\bar{c}^{(2)}, \bar{c}^{(2)}) = 0$ , contradicting Lemma 3.2.  $\square$

**Corollary 3.4.** *For distinct elements  $\bar{c}, \bar{a}$  of  $\mathcal{C}$ , the line segment  $\bar{c}\bar{a}$  meets  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ .*

This gives us some control over the extent to which  $\text{conv}(\mathcal{C})$  can be bigger than the set  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ .

**Lemma 3.5.** *Let  $A \subset H_{\frac{1}{2}(n-1)}$  be an affine subspace such that  $A \cap \text{conv}(\frac{1}{2}(d + \mathcal{W}))$  is a face of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . Suppose there exists  $\bar{c} \in \mathcal{C} \cap A$  with  $\bar{c} \notin \text{conv}(\frac{1}{2}(d + \mathcal{W}))$ .*

*Let  $\bar{x} \in A$ . If  $\bar{x} = \frac{1}{2}(\bar{a} + \bar{a}')$  with  $\bar{a}, \bar{a}' \in \mathcal{C}$ , then in fact  $\bar{a}, \bar{a}' \in A$ .*

*Proof.* If  $\bar{a}$  or  $\bar{a}'$  equals  $\bar{c}$  this is clear.

We know by Cor 3.4 that if  $\bar{a}, \bar{a}' \neq \bar{c}$  then the segments  $\bar{a}\bar{c}, \bar{a}'\bar{c}$  meet  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . So there exist  $0 < s, t \leq 1$  with  $t\bar{a} + (1-t)\bar{c}$  and  $s\bar{a}' + (1-s)\bar{c}$  in  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . Hence

$$\left(\frac{2st}{s+t}\right)\bar{x} + \left(1 - \frac{2st}{s+t}\right)\bar{c} = \frac{s}{s+t}(t\bar{a} + (1-t)\bar{c}) + \frac{t}{s+t}(s\bar{a}' + (1-s)\bar{c}) \in \text{conv}(\frac{1}{2}(d + \mathcal{W})).$$

As it is an affine combination of  $\bar{x}, \bar{c}$  this point also lies in  $A$ , so it lies in  $A \cap \text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . Also, it is a convex linear combination of the points  $t\bar{a} + (1-t)\bar{c}$  and  $s\bar{a}' + (1-s)\bar{c}$  of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . Hence, by our face assumption, both these points lie in  $A$ , so  $\bar{a}, \bar{a}'$  lie in  $A$ .  $\square$

**Remark 3.6.** The above lemma will be very useful because it means that in all our later calculations using Prop 1.4 for a face defined by an affine subspace  $A$ , we need only consider elements of  $\mathcal{C}$  lying in  $A$ .

**Proposition 3.7.** *Let  $vw$  be an edge of  $\text{conv}(\mathcal{W})$  and suppose  $\bar{v}, \bar{w} \in \mathcal{C}$ .*

- (i) *If there are no points of  $\mathcal{W}$  in the interior of  $vw$ , then  $J(\bar{v}, \bar{w}) = 0$ .*
- (ii) *If  $u = \frac{1}{2}(v + w)$  is the unique point of  $\mathcal{W}$  in the interior of  $vw$ ,  $J(\bar{v}, \bar{w}) > 0$ , and  $u$  is type II or III, then  $F_{\bar{v}}, F_{\bar{w}}$  are of opposite signs.*

*Proof.* Part(i) is a generalization of Theorem 3.5 in [DW4] and we will be able to apply the proof of that result after the following argument. Let the edge  $\bar{v}\bar{w}$  of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  be defined by equations  $\langle \bar{x}, u^{(i)} \rangle = \lambda_i : i \in \mathcal{I}$  where  $\langle \bar{x}, u^{(i)} \rangle \leq \lambda_i$  for  $i \in \mathcal{I}$  and  $\bar{x} \in \text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . (In the above,  $\langle \cdot, \cdot \rangle$  is the Euclidean inner product in  $\mathbb{R}^r$ .) Note that  $\text{Span} \{u^{(i)} : i \in \mathcal{I}\}$  is the  $\langle \cdot, \cdot \rangle$ -orthogonal complement of the direction of the edge.

Let  $H$  be a hyperplane whose intersection with  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  is the edge  $\bar{v}\bar{w}$ . We can take  $H$  to be defined by the equation  $\langle \bar{x}, \sum_{i \in \mathcal{I}} b_i u^{(i)} \rangle = \sum_{i \in \mathcal{I}} b_i \lambda_i$  where  $b_i$  are arbitrary positive numbers summing to 1.

If  $\bar{a}, \bar{a}'$  are elements of  $\mathcal{C}$  whose midpoint lies in  $\bar{v}\bar{w}$ , then either they are both in  $H$  or one of them,  $\bar{a}$  say, is on the opposite side of  $H$  from  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . In the latter case  $\bar{a}$  is null and the only element of  $\mathcal{C}$  on this side of  $H$  so, by Prop 1.4, is  $J$ -orthogonal to  $\bar{v}, \bar{w}$ . Hence, as  $\frac{1}{2}(\bar{a} + \bar{a}')$  is an affine combination of  $\bar{v}, \bar{w}$ , we see that  $J(\bar{a}, \bar{a}') = 0$ , and so such sums do not contribute in Eq.(1.6). We may therefore assume that  $\bar{a}, \bar{a}'$  are in  $H$ . But as this is true for all  $H$  of the above form, the only sums that will contribute are those where  $\bar{a}, \bar{a}'$  are collinear with  $\bar{v}\bar{w}$ .

Now if  $\bar{a}$ , say, lies outside the line segment  $\bar{v}\bar{w}$ , then it is null and  $J$ -orthogonal to  $\bar{v}$  or  $\bar{w}$ , and hence to the whole line. So the only sums which contribute are those where  $\bar{a}, \bar{a}'$  lie on the line segment  $\bar{v}\bar{w}$ . Now the proof of Theorem 3.5 in [DW4] gives (i).

Turning to (ii), note first that the above arguments and Prop 1.4 give (ii) immediately if no interior points of the edge  $\bar{v}\bar{w}$  lie in  $\mathcal{C}$ . If there are  $m$  interior points in  $\mathcal{C}$ , we again proceed as in the proof of Theorem 3.5 in [DW4] and use the notation there. We may assume that Lemma 3.2 (and hence Cor 3.3 and Lemma 3.4) of [DW4] still holds; for the only issue is the statement for  $\lambda_{m+1}$ , but if  $c^{(0)} + c^{(\lambda_{m+1})}$  cannot be written as  $c^{(\lambda_j)} + c^{(\lambda_k)}$  ( $0 < \lambda_j, \lambda_k < m + 1$ ) then what we want to prove is already true.

Now Lemma 3.4 in [DW4] and our hypothesis  $J(\bar{v}, \bar{w}) > 0$  imply that  $J_{00} < 0$  and  $J_{\lambda_i, \lambda_j} > 0$  except in the three cases listed there. The proof that the elements of  $\mathcal{C}$  are equi-distributed in  $\bar{v}\bar{w}$  carries over from [DW4] since the midpoint  $\bar{u}$  is not involved in the arguments.

Suppose next that the points in  $\mathcal{C} \cap \bar{v}\bar{w}$  are equi-distributed. In the special case where  $m = 1$ , we have  $J(\bar{v}, \bar{u}) = 0 = J(\bar{u}, \bar{w})$ , which imply  $J(\bar{u}, \bar{u}) = 0$ . So the midpoint does not contribute to the equation from  $c^{(0)} + c^{(\lambda_{m+1})}$ . If  $m > 1$ , we write down the equations arising from  $c^{(0)} + c^{(\lambda_{m+1})}$  and  $c^{(\lambda_{m-1})} + c^{(\lambda_{m+1})}$ . The formula for  $F_{\lambda_j}$  in [DW4] still holds for  $1 \leq j \leq m$ , and using this and the second equation we obtain the analogous formula for  $F_{\lambda_{m+1}}$ .

Putting all the above information together in the first equation and using  $A_u < 0$ , we see that  $F_{\lambda_1}^{m+1}/F_0^{m-1}$  is positive if  $m$  is even and negative if  $m$  is odd. In either case it follows immediately that  $F_0 F_{\lambda_{m+1}} < 0$ , as required.  $\square$

We shall now set up the basic machinery of the projection of our convex hull onto an affine hyperplane.

Let  $\bar{c}$  be a null vector in  $\mathcal{C}$  and let  $H$  be an affine hyperplane separating  $\bar{c}$  from  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . Define a map  $P : \text{conv}(\frac{1}{2}(d + \mathcal{W})) \rightarrow H$  by letting  $P(\bar{z})$  be the intersection point of the ray  $\bar{c}\bar{z}$  with  $H$ . We denote by  $\Delta$  the image of  $P$  in  $H$ . ( $P$  and  $\Delta$  of course depend on  $\bar{c}$  and the choice of  $H$ . When considering projections from several null vertices, we will use the vertices as superscripts to distinguish the cases, e.g.,  $\Delta^{\bar{c}}, \Delta^{\bar{b}}$ .)

Let us now consider a vertex  $\bar{\xi}$  of  $\Delta$ . We know that  $\bar{c}$  and  $\bar{\xi}$  are collinear with a subset  $P^{-1}(\bar{\xi})$  of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . As  $\bar{\xi}$  does not lie in the interior of a positive-dimensional subset of  $\Delta$ , we see that



no point of  $P^{-1}(\bar{\xi})$  lies in the interior of a subset of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  of dimension  $> 1$ . So  $P^{-1}(\bar{\xi})$  is a vertex or an edge of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ .

If  $P^{-1}(\bar{\xi})$  is a vertex  $\bar{x}$ , then  $2\bar{x} \in d + \mathcal{W}$  and in Lemma 3.5 we can take the affine subspace  $A$  to be the line through  $\bar{c}, \bar{\xi}, \bar{x}$ . Using this lemma and also Prop 1.4 and Cor 3.4 we see that either  $\bar{x} \in \mathcal{C}$  (in which case  $J(\bar{x}, \bar{c}) = 0$ ), or  $\bar{x} \notin \mathcal{C}$  and  $\bar{x} = (\bar{a} + \bar{c})/2$  for some null element  $\bar{a} \in \mathcal{C} \cap A$ . We have therefore deduced

**Theorem 3.8.** *Let  $\bar{\xi}$  be a vertex of  $\Delta$ . Then exactly one of the following must hold:*

- (1A)  $\bar{\xi}$  (and hence  $P^{-1}(\bar{\xi})$ ) is orthogonal to  $\bar{c}$ ;
- (1B) The line through  $\bar{c}, \bar{\xi}$  meets  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  in a unique point  $\bar{x}$ , and there exists a null  $\bar{a} \in \mathcal{C}$  such that  $(\bar{a} + \bar{c})/2 = \bar{x}$ ;
- (2)  $\bar{\xi}$  is not orthogonal to  $\bar{c}$ , and  $\bar{c}$  and  $\bar{\xi}$  are collinear with an edge  $\bar{v}\bar{w}$  of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ , (and hence  $c$  and  $\xi$  are collinear with the corresponding edge  $vw$  of  $\text{conv}(\mathcal{W})$ ).  $\square$

**Remark 3.9.** If (1B) occurs, then  $\bar{a} = 2\bar{x} - \bar{c}$  being null is equivalent to  $J(\bar{x}, \bar{x}) = J(\bar{x}, \bar{c})$ , that is,

$$(3.1) \quad \sum_{i=1}^r \frac{x_i^2}{d_i} = \sum_{i=1}^r \frac{x_i c_i}{d_i}$$

In particular  $x_i$  and  $c_i$  are nonzero for some common index  $i$ . We will from now on refer to this situation by saying that the vectors  $x$  and  $c$  *overlap*.

We make a preliminary remark about (1A) vertices.

**Lemma 3.10.** *Suppose that  $u \in \mathcal{W}$  and  $\bar{u} \in \bar{c}^\perp$ .*

- (a) *If  $u = (-2^i, 1^j)$  then  $c_i \neq 0$ .*
- (b) *Suppose that  $K$  is connected. If  $u = (-1^i, -1^j, 1^k)$ , then  $c_i, c_j, c_k$  are all nonzero.*

*Proof.* After a suitable permutation, we may let  $1, \dots, s$  be the indices  $a$  with  $c_a \neq 0$ . We need

$$\sum_{a=1}^s \frac{u_a c_a}{d_a} = \sum_{a=1}^s \frac{c_a^2}{d_a} = 1.$$

In case (a) this is impossible if  $c_i = 0$  (that is,  $i \notin \{1, \dots, s\}$ ) as then we need  $d_j = 1 = c_j$  and  $c_a = 0$  for  $a \neq j$ , contradicting  $\sum_{k=1}^r c_k = -1$ .

Next, Cauchy-Schwartz on  $(\frac{u_a}{\sqrt{d_a}})_{a=1}^s, (\frac{c_a}{\sqrt{d_a}})_{a=1}^s$  shows  $\sum_{a=1}^s \frac{u_a^2}{d_a} \geq 1$ . In case (b), if, say,  $c_k = 0$ , then since  $\frac{1}{d_i} + \frac{1}{d_j} \geq 1$  and  $d_i, d_j \geq 2$  (see Remark 1.2(d)) we must have  $d_i = d_j = 2$ . The equations then imply  $c_i = c_j = -1$  and  $c_a = 0$  for  $a \neq i, j$ , also giving a contradiction. Similar arguments rule out  $c_i = 0$  or  $c_j = 0$ .  $\square$

In the next two sections we shall get stronger results on (1A) vertices. Let us now consider type (2) vertices.

**Theorem 3.11.** *Consider a type (2) vertex  $\bar{\xi}$  of  $\Delta$ . So  $c$  and  $\xi$  are collinear with an edge  $vw$  of  $\text{conv}(\mathcal{W})$ . Suppose there are no points of  $\mathcal{W}$  in the interior of  $vw$ . Then we have*

- (i)  $c = 2v - w$  or
- (ii)  $c = (4v - w)/3$ .

*In (i) the points of  $\mathcal{C}$  on the line through  $\bar{c}, \bar{\xi}$  are  $\bar{c}$  and  $\bar{w}$ . In (ii) they are  $\bar{c}, \bar{w}$  and  $\bar{c}^{(1)} = (2\bar{v} + \bar{w})/3 = (\bar{c} + \bar{w})/2$ . We need  $J(\bar{c}^{(1)}, \bar{w}) = 0$ .*

*Proof.* This is very similar to the arguments of §3 in [DW4]. We apply Lemma 3.5 to the line through  $\bar{v}, \bar{w}$ .

(A) We write the elements of  $\mathcal{C}$  on the line as  $\bar{c} = \bar{c}^{(0)}, \bar{c}^{(1)}, \dots, \bar{c}^{(m+1)}$  with  $m \geq 0$ . So  $\bar{c}^{(m+1)}$  is either null or is  $\bar{w}$ . No other  $\bar{c}^{(j)}$  can lie beyond  $\bar{w}$ , by Cor 3.4.

By assumption  $\bar{c} = \bar{c}^{(0)}$  is not orthogonal to the whole line. As  $\bar{c}$  is null, this means  $\bar{c}$  is not orthogonal to *any* other point on the line. So  $\bar{c}^{(0)} + \bar{c}^{(j)}$  is either  $2\bar{v}$ ,  $2\bar{w}$  or else is a sum of two other  $\bar{c}^{(i)}$ . In particular,  $\bar{c}^{(0)} + \bar{c}^{(1)} = 2\bar{v}$ . In fact  $\bar{c}^{(0)} + \bar{c}^{(j)}$  is never  $2\bar{w}$ ; for the only possibility is for  $\bar{c}^{(0)} + \bar{c}^{(m+1)} = 2\bar{w}$ , in which case  $\bar{c}^{(m+1)}$  is null, and so  $\bar{c}^{(m)} + \bar{c}^{(m+1)} = 2\bar{w}$ , contradicting  $\bar{v} \neq \bar{w}$ .

We deduce that for  $j > 1$ , we have  $\bar{c}^{(0)} + \bar{c}^{(j)} = \bar{c}^{(k)} + \bar{c}^{(p)}$  for some  $1 \leq k, p \leq j-1$ .

(B) Let  $\bar{c}^{(m+1)}$  be null. Since the segment  $\bar{c}^{(0)}\bar{c}^{(m+1)}$  lies in the interior of the null ellipsoid, Lemma 3.2 implies that  $J(\bar{c}^{(i)}, \bar{c}^{(j)}) > 0$  unless  $i = j = 0$  or  $m+1$ . Arguments very similar to those in §3 of [DW4] enable us to determine the signs of the  $F_{\bar{c}^{(j)}}$  in (1.5) and show that the contributions from the pairs summing to  $\bar{c}^{(1)} + \bar{c}^{(m+1)}$  cannot cancel. So we have a contradiction unless  $\bar{c}^{(1)} + \bar{c}^{(m+1)} = \bar{w}$ , which can only happen if  $m = 1$ , i.e.,  $\bar{c}^{(0)} + \bar{c}^{(1)} = 2\bar{v}$ ,  $\bar{c}^{(1)} + \bar{c}^{(2)} = 2\bar{w}$  and  $\bar{c}^{(0)} + \bar{c}^{(2)} = 2\bar{c}^{(1)}$  (otherwise  $\bar{c}^{(0)} + \bar{c}^{(2)}$  cannot cancel). Hence we have

$$c = (3v - w)/2 ; \quad c^{(1)} = (v + w)/2 ; \quad c^{(2)} = (3w - v)/2.$$

Writing  $F_j$  for  $F_{\bar{c}^{(j)}}$ , we need  $2F_0F_2J(\bar{c}, \bar{c}^{(2)}) + F_1^2J(\bar{c}^{(1)}, \bar{c}^{(1)}) = 0$  so that the contributions from  $\bar{c}^{(0)} + \bar{c}^{(2)}$  and  $\bar{c}^{(1)} + \bar{c}^{(1)}$  cancel. As  $J(\bar{c}, \bar{c}^{(2)})$  and  $J(\bar{c}^{(1)}, \bar{c}^{(1)}) > 0$ , we need  $F_0$  and  $F_2$  to have opposite signs. Now, as  $J(\bar{c}, \bar{c}^{(1)})$ ,  $J(\bar{c}^{(1)}, \bar{c}^{(2)}) > 0$ , we see that  $A_v$  and  $A_w$  have opposite signs. So we may let  $w$  be type I and  $v$  be type II or III, as long as the asymmetry between  $\bar{c}^{(0)}$  and  $\bar{c}^{(2)}$  is removed. Note that  $v, w$  cannot overlap if  $v$  is type II, as then Remark 1.2(c) means  $w$  is not a vertex. The possibilities are (up to permutation)

	$v$	$w$	$c^{(0)} = \frac{1}{2}(3v - w)$	$c^{(2)} = \frac{1}{2}(3w - v)$
(1)	$(-2, 1, 0, \dots)$	$(-1, 0, \dots)$	$(-\frac{5}{2}, \frac{3}{2}, 0, \dots)$	$(-\frac{1}{2}, -\frac{1}{2}, 0, \dots)$
(2)	$(-2, 1, 0, \dots)$	$(0, -1, 0, \dots)$	$(-3, 2, 0, \dots)$	$(1, -2, 0, \dots)$
(3)	$(-2, 1, 0, \dots)$	$(0, 0, -1, 0, \dots)$	$(-3, \frac{3}{2}, \frac{1}{2}, \dots)$	$(1, -\frac{1}{2}, -\frac{3}{2}, 0, \dots)$
(4)	$(1, -1, -1, 0, \dots)$	$(0, 0, 0, -1, \dots)$	$(\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}, \dots)$	$(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, \dots)$

Now, it is clear in (1) and (2) that  $\bar{c}^{(0)}$  and  $\bar{c}^{(2)}$  can't both give null vectors. For (3) and (4), we find that the nullity equations for  $\bar{c}^{(0)}$  and  $\bar{c}^{(2)}$  have no integral solutions in  $d_i$  (in fact  $d_3$  (resp.  $d_4$ ) must be  $5/2$ ).

Therefore in fact  $\bar{c}^{(m+1)}$  cannot be null.

(C) Now suppose that  $\bar{c}^{(m+1)} = w$  and  $m > 0$ . Since  $J(\bar{c}, \bar{v}) \neq 0$ ,  $\bar{v}$  must lie between  $\bar{c}^{(0)}$  and  $\bar{c}^{(1)}$ . So  $J(\bar{c}^{(0)}, \cdot)$  and  $J(\cdot, \bar{c}^{(m+1)})$  are affine functions on the line, vanishing at  $\bar{c}^{(0)}$  and  $\bar{c}^{(m)}$  respectively. Hence  $J(\bar{c}^{(0)}, \bar{c}^{(i)})$  ( $i \geq 1$ ) and  $J(\bar{c}^{(i)}, \bar{c}^{(m+1)})$  ( $0 \leq i \leq m-1$ ) are the same sign as  $J(\bar{c}^{(0)}, \bar{c}^{(1)})$ . It follows that  $J(\bar{c}^{(i)}, \cdot)$  is an affine function on the line, taking the same sign as  $J(\bar{c}^{(0)}, \bar{c}^{(1)})$  at  $\bar{c}^{(0)}, \bar{c}^{(m+1)}$  (for  $1 \leq i \leq m-1$ ). Thus  $J(\bar{c}^{(i)}, \bar{c}^{(j)})$  is the same sign as  $J(\bar{c}^{(0)}, \bar{c}^{(1)})$  except for the cases

$$J(\bar{c}^{(0)}, \bar{c}^{(0)}) = 0 = J(\bar{c}^{(m)}, \bar{c}^{(m+1)}) \quad : \quad \text{sign } J(\bar{c}^{(m+1)}, \bar{c}^{(m+1)}) = -\text{sign } J(\bar{c}^{(0)}, \bar{c}^{(1)}).$$

It then follows that the sign and non-cancellation arguments of (B) (taken from §3 of [DW4]) still hold, except in the case  $m = 1$ .

These give the two cases of the Theorem. If  $m = 0$ , we have  $c^{(1)} = w$  and  $c^{(0)} = 2v - w$  as  $c^{(0)} + c^{(1)} = 2v$ . If  $m = 1$ , then  $c^{(2)} = w$ ,  $c^{(0)} + c^{(1)} = 2v$  and  $c^{(0)} + c^{(2)} = 2c_1$  (for cancellation). Hence  $c^{(0)} = (4v - w)/3$ ,  $c^{(1)} = (2v + w)/3$ , as well as  $J(\bar{c}^{(1)}, \bar{c}^{(2)}) = 0$ .  $\square$

**Remark 3.12.** If there are points of  $\mathcal{W}$  in the interior of  $vw$ , we can still conclude that  $c^{(0)} + c^{(1)} = 2v$ . Hence  $c = \lambda v + (1 - \lambda)w$  for  $1 < \lambda \leq 2$ , since if  $\lambda > 2$  then  $\bar{c}^{(1)}$  is beyond  $\bar{w}$ . It must then be null, and  $m = 0$ , so there is no way of getting  $2\bar{w}$  as a sum of two elements in  $\mathcal{C}$ .

**Lemma 3.13.** *For case (i) in Theorem 3.11 (i.e.,  $c = 2v - w$ ), either  $w$  is type I, or  $w$  is type III and  $v_i = -1, w_i = -2$  for some index  $i$ .*

*Proof.* It follows from above that  $J(\bar{w}, \bar{w})F_w^2 = A_w$  so  $J(\bar{w}, \bar{w})$  is positive if  $w$  is type I and negative if  $w$  is type II or III. In the latter case,  $\sum_i \frac{w_i^2}{d_i} > 1$ , but by nullity,  $c = 2v - w$  satisfies  $\sum \frac{c_i^2}{d_i} = 1$ . Hence for some  $i$  we have  $|w_i| > |c_i| = |2v_i - w_i|$ . As  $v_i, w_i \in \{-2, -1, 0, 1\}$ , it follows that  $v_i = -1, w_i = -2$ .  $\square$

We are now able to characterise the case where  $c$  is a type I vector.

**Theorem 3.14.** *If  $c$  is a type I vector, say  $(-1, 0, \dots)$  for definiteness, then  $\mathcal{W}$  is given by  $\{(-1)^i, (1^1, -2^i) : i = 2, \dots, r\}$ .*

**Remark 3.15.** Equivalently,  $\mathcal{W}$  is as in Ex 8.1 of [DW4], where the hypersurface in the Ricci-flat manifold is a circle bundle over a product of Kähler-Einstein Fano manifolds. A superpotential was found for this example in [CGLP3].

*Proof.* Nullity of  $\bar{c}$  implies  $d_1 = 1$ , so  $(-2^1, 1^i) \notin \mathcal{W}$ . Also  $(-1^1, -1^j, 1^k) \notin \mathcal{W}$ , as then  $c$  would be in  $\text{conv}(\mathcal{W})$ . Let us consider the vertices  $\bar{\xi}$  in  $\Delta$ .  $\bar{\xi}$  cannot be of type (1A); otherwise  $\xi_1 = -1$ , which implies the existence of a type II vector in  $\mathcal{W}$  with a nonzero first component, contradicting the above. There can also be no  $\bar{\xi}$  of type (1B) since by Remark 3.9 the vector  $\bar{x}$  satisfies  $0 < -x_1$ , which we ruled out above.

Hence all vertices of  $\Delta$  are of type (2), i.e., correspond to edges  $vw$  of  $\text{conv}(\mathcal{W})$  such that  $c = \lambda v + (1 - \lambda)w$  and  $\lambda > 1$ . From this equation it follows that  $v, w$  are of the form

$$v = (-1^i), \quad w = (1^1, -2^i)$$

for some  $i > 1$ . As  $\Delta$  (being a  $(r - 2)$ -dimensional polytope in an  $(r - 2)$ -dimensional affine space) has at least  $r - 1$  vertices, such vectors occur for all  $i \neq 1$ .

Now no type II vector can be in  $\mathcal{W}$ , otherwise  $v$  would not be a vertex. Also  $(1^i, -2^j)$  with  $i, j \neq 1$  cannot be in  $\mathcal{W}$ , as then  $(-1^j)$  would not be a vertex. We have already seen  $(-2^1, 1^i)$  is not in  $\mathcal{W}$ . So  $\mathcal{W}$  is as claimed.  $\square$

We shall henceforth exclude this case, i.e. case (i) of Theorem 2.1, from our discussion.

We conclude this section by giving a preliminary listing of the possibilities for  $c$  when we have a type (2) vertex. These are given by cases (i) and (ii) of Theorem 3.11, as well as the possible cases when there is a point of  $\mathcal{W}$  in the interior of  $vw$ .

For Theorem 3.11(i) the possible  $v, w, c$  are:

	$v$	$w$	$c = 2v - w$
(1)	$(-1, 1, -1, \dots)$	$(-2, 1, \dots)$	$(0, 1, -2, \dots)$
(2)	$(-1, -1, 1, \dots)$	$(-2, 1, \dots)$	$(0, -3, 2, \dots)$
(3)	$(-1, 0, -1, 1, \dots)$	$(-2, 1, \dots)$	$(0, -1, -2, 2, \dots)$
(4)	$(-2, 1, \dots)$	$(-1, 0, \dots)$	$(-3, 2, \dots)$
(5)	$(-2, 1, \dots)$	$(0, 0, -1, \dots)$	$(-4, 2, 1, \dots)$
(6)	$(-1, 0, \dots)$	$(0, -1, \dots)$	$(-2, 1, \dots)$
(7)	$(1, -1, -1, \dots)$	$(0, 0, 0, -1, \dots)$	$(2, -2, -2, 1, \dots)$

Table 1:  $c = 2v - w$  cases

where  $\dots$  denotes zeros as usual. To arrive at this list, recall from Lemma 3.13 that  $w$  is either type I or type III with  $v_i = -1, w_i = -2$  for some  $i$ . Note also that if  $w$  is type I and  $v$  is type II then  $v$  cannot overlap with  $w$  as  $w$  cannot then be a vertex. Furthermore, the other possibility with  $w$  type I and  $v$  type III is excluded as we are assuming in Theorem 3.11 that there are no points of  $\mathcal{W}$  in the interior of  $vw$ . Finally, the case  $w = (-2, 1, \dots), v = (-1, 0, \dots)$  can be excluded as this just gives the example in Theorem 3.14.

In order to list the possibilities under Theorem 3.11(ii), recall that we need  $J(\bar{c}^{(1)}, \bar{w}) = 0$  where  $c^{(1)} = (2v + w)/3$ . Equivalently, we need

$$(3.2) \quad 2J(\bar{v}, \bar{w}) + J(\bar{w}, \bar{w}) = 0.$$

This puts constraints on the possibilities for  $v, w$ . For instance,  $w$  cannot be type I, as for such vectors  $J(\bar{v}, \bar{w}) \geq 0$  and  $J(\bar{w}, \bar{w}) > 0$ . Also, if  $w$  is type II or III, then from the superpotential equation we need  $J(\bar{w}, \bar{w}) < 0$ , so  $J(\bar{v}, \bar{w}) > 0$ . If  $w$  is type III, say  $(-2, 1, 0, \dots)$ , then since  $d_1 \geq 2$ , we have  $\frac{4}{d_1} + \frac{1}{d_2} \leq 3$ , and the above equation gives  $J(\bar{v}, \bar{w}) \leq \frac{1}{4}$  with equality iff  $d_1 = 2, d_2 = 1$ .

By the above remarks and the nullity of  $\bar{c}$ , after a moderate amount of routine computations, we arrive at the following possibilities, up to permutation of entries. In the table we have listed only the minimum number of components for each vector and all unlisted components are zero. Note that the entries (12)-(16) can occur only if  $K$  is not connected (cf. Remark 5.9).

	$v$	$w$	$c^{(1)} = (2v + w)/3$	$c = (4v - w)/3$
(1)	$(0, 0, -2, 1)$	$(-2, 1, 0, 0)$	$(-\frac{2}{3}, \frac{1}{3}, -\frac{4}{3}, \frac{2}{3})$	$(\frac{2}{3}, -\frac{1}{3}, -\frac{8}{3}, \frac{4}{3})$
(2)	$(-2, 0, 1)$	$(-2, 1, 0)$	$(-\frac{4}{3}, \frac{1}{3}, \frac{2}{3})$	$(-\frac{2}{3}, -\frac{1}{3}, \frac{4}{3})$
(3)	$(-1, 0, 0, )$	$(-2, 1, 0)$	$(-\frac{4}{3}, \frac{1}{3}, 0)$	$(-\frac{2}{3}, -\frac{1}{3}, 0)$
(4)	$(0, 0, -1)$	$(-2, 1, 0)$	$(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$	$(\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3})$
(5)	$(-1, 0, 1, -1)$	$(-2, 1, 0, 0)$	$(-\frac{4}{3}, \frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$	$(-\frac{2}{3}, -\frac{1}{3}, \frac{4}{3}, -\frac{4}{3})$
(6)	$(-1, 1, -1)$	$(-2, 1, 0)$	$(-\frac{4}{3}, 1, -\frac{2}{3})$	$(-\frac{2}{3}, 1, -\frac{4}{3})$
(7)	$(0, 0, 1, -1, -1)$	$(-2, 1, 0, 0, 0)$	$(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, -\frac{2}{3})$	$(\frac{2}{3}, -\frac{1}{3}, \frac{4}{3}, -\frac{4}{3}, -\frac{4}{3})$
(8)	$(0, 1, -1, -1)$	$(-2, 1, 0, 0)$	$(-\frac{2}{3}, 1, -\frac{2}{3}, -\frac{2}{3})$	$(\frac{2}{3}, 1, -\frac{4}{3}, -\frac{4}{3})$
(9)	$(0, -1, -1, 1)$	$(1, -1, -1, 0)$	$(\frac{1}{3}, -1, -1, \frac{2}{3})$	$(-\frac{1}{3}, -1, -1, \frac{4}{3})$
(10)	$(1, -1, 0, -1)$	$(1, -1, -1, 0)$	$(1, -1, -\frac{1}{3}, -\frac{2}{3})$	$(1, -1, \frac{1}{3}, -\frac{4}{3})$
(11)	$(1, -2, 0)$	$(1, -1, -1)$	$(1, -\frac{5}{3}, -\frac{1}{3})$	$(1, -\frac{7}{3}, \frac{1}{3})$
(12)	$(1, 0, 0, -2)$	$(1, -1, -1, 0)$	$(1, -\frac{1}{3}, -\frac{1}{3}, -\frac{4}{3})$	$(1, \frac{1}{3}, \frac{1}{3}, -\frac{8}{3})$
(13)	$(0, 0, 0, -2, 1)$	$(1, -1, -1, 0, 0)$	$(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{4}{3}, \frac{2}{3})$	$(-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{8}{3}, \frac{4}{3})$
(14)	$(0, -1, 0, 1, -1)$	$(1, -1, -1, 0, 0)$	$(\frac{1}{3}, -1, -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$	$(-\frac{1}{3}, -1, \frac{1}{3}, \frac{4}{3}, -\frac{4}{3})$
(15)	$(1, 0, 0, -1, -1)$	$(1, -1, -1, 0, 0)$	$(1, -\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3})$	$(1, \frac{1}{3}, \frac{1}{3}, -\frac{4}{3}, -\frac{4}{3})$
(16)	$(0, 0, 0, 1, -1, -1)$	$(1, -1, -1, 0, 0, 0)$	$(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, -\frac{2}{3})$	$(-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{4}{3}, -\frac{4}{3}, -\frac{4}{3})$

Table 2:  $c = \frac{1}{3}(4v - w)$  cases

We will also need a listing of those cases for which  $vw$  has interior points lying in  $\text{conv}(\mathcal{W})$ .

	$v$	$w$	$c$
(1)	$(1, -2, \dots)$	$(-2, 1, \dots)$	$(3\lambda - 2, 1 - 3\lambda, \dots)$
(2)	$(1, -2, \dots)$	$(-1, 0, \dots)$	$(2\lambda - 1, -2\lambda, \dots)$
(3)	$(-1, 0, \dots)$	$(1, -2, \dots)$	$(1 - 2\lambda, 2\lambda - 2, \dots)$
(4)	$(-2, 1, 0, \dots)$	$(0, 1, -2, \dots)$	$(-2\lambda, 1, 2\lambda - 2, \dots)$
(5)	$(1, -1, -1, \dots)$	$(-1, 1, -1, \dots)$	$(2\lambda - 1, 1 - 2\lambda, -1, \dots)$

Table 3: Cases with interior points

Recall from Remark 3.12 that  $1 < \lambda \leq 2$  and  $\dots$  denote zeros. Note that except in (4) all interior points which may lie in  $\mathcal{W}$  actually do.

#### 4. The sign of $J(\bar{c}, \bar{w})$

**Theorem 4.1.**  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  lies in the closed half-space  $J(\bar{c}, \cdot) \geq 0$ , i.e., the same closed half-space in which the null ellipsoid lies.

*Proof.* We know that if  $\bar{\xi}$  is a vertex of  $\Delta^{\bar{c}}$  then there are three possibilities, given by (1A), (1B) and (2) of Theorem 3.8. If (1A) occurs, then by definition  $J(\bar{c}, \bar{\xi}) = 0$ . If (1B) occurs, let  $\bar{a}$  be the null vector in Theorem 3.8. Then by Lemma 3.2,  $J(\bar{c}, \bar{a}) > 0$ , which in turn implies that  $J(\bar{c}, \bar{\xi}) > 0$ . It is now enough to show that  $J(\bar{c}, \bar{\xi}) \geq 0$  if  $\bar{\xi}$  is a type (2) vertex of  $\Delta^{\bar{c}}$ , since it then follows that  $\Delta^{\bar{c}}$ , and hence  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ , lies in the half-space  $J(\bar{c}, \cdot) \geq 0$ .

Suppose then that  $\bar{\xi}$  is a type (2) vertex with  $J(\bar{c}, \bar{\xi}) < 0$ . By Remark 3.12,  $c = \lambda v + (1 - \lambda)w$  for some  $v, w \in \mathcal{W}$  with  $1 < \lambda \leq 2$ , and both  $J(\bar{c}, \bar{v}), J(\bar{c}, \bar{w}) < 0$ . In particular, from Remark 3.1 and Lemma 3.2,  $J(\bar{v}, \bar{v}), J(\bar{w}, \bar{w}) < 0$  since  $\bar{v}, \bar{w}$  lie on the side of  $\bar{c}^\perp$  opposite to the null ellipsoid.

But

$$\begin{aligned} 0 = 4J(\bar{c}, \bar{c}) &= J(d + \lambda v + (1 - \lambda)w, d + \lambda v + (1 - \lambda)w) \\ &= J(\lambda(d + v) + (1 - \lambda)(d + w), \lambda(d + v) + (1 - \lambda)(d + w)) \\ &= \lambda^2 J(d + v, d + v) + 2\lambda(1 - \lambda)J(d + v, d + w) + (1 - \lambda)^2 J(d + w, d + w). \end{aligned}$$

It follows from the above remarks that  $J(d + v, d + w) < 0$ , that is

$$\sum_i \frac{v_i w_i}{d_i} > 1.$$

One then checks that this condition is only satisfied in the following cases (up to permutation of indices and interchange of  $v$  and  $w$ ):

- (a)  $v = (-2, 1, 0, \dots)$ ,  $w = (-2, 0, 1, 0, \dots)$  with  $1 < d_1 < 4$ ;
- (b)  $v = (-2, 1, 0, \dots)$ ,  $w = (-1, 1, -1, 0, \dots)$  with  $d_1 = 2$ , or  $(d_1, d_2) = (3, 2)$ , or  $d_2 = 1$ ;
- (c)  $v = (1, -1, -1, 0, \dots)$ ,  $w = (1, -1, 0, -1, 0, \dots)$  with  $d_1 = 1$  or  $d_2 = 1$ ;
- (d)  $v = (1, -1, -1, 0, \dots)$ ,  $w = (0, -1, -1, 1, 0, \dots)$  with  $d_2 = 1$  or  $d_3 = 1$ .

In case (a),  $c = (-2, \lambda, 1 - \lambda, 0, \dots)$ . The condition  $d_1 < 4$  is incompatible with the nullity of  $\bar{c}$ . Interchanging  $v$  and  $w$  reverses only the role of  $\lambda$  and  $1 - \lambda$ .

A similar argument rules out case (b) with  $v, w$  as shown, as here  $c = (-\lambda - 1, 1, \lambda - 1)$ . If we interchange  $v$  and  $w$ , then  $c = (\lambda - 2, 1, -\lambda, \dots)$ . Theorem 3.11 tells us  $\lambda = 4/3$  or  $2$ , so  $c = (-2/3, 1, -4/3, \dots)$  or  $(0, 1, -2, \dots)$ .

In the former case  $c^{(1)} := (2v + w)/3 = (-4/3, 1, -2/3, \dots)$ , so the condition  $J(\bar{w}, \bar{c}^{(1)}) = 0$  gives  $8/3d_1 + 1/d_2 = 1$ . Thus  $(d_1, d_2) = (3, 9)$  or  $(4, 3)$  but in neither case is  $\bar{c}$  null. In the latter case nullity means  $1/d_2 + 4/d_3 = 1$ , so  $J(\bar{c}, \bar{v}) = \frac{1}{4}(1 - 1/d_2 - 2/d_3) > 0$ , a contradiction.

In case (c),  $c = (1, -1, -\lambda, \lambda - 1)$  and if  $v$  and  $w$  are interchanged, the last two components of  $c$  are interchanged. But  $\bar{c}$  cannot be null if  $d_1 = 1$  or  $d_2 = 1$ . A similar argument works for case (d).  $\square$

**Corollary 4.2.**  $\text{conv}(\frac{1}{2}(d + \mathcal{W})) \cap \bar{c}^\perp$  is a (possibly empty) face of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ .  $\square$

This enables us to adapt Theorem 3.5 of [DW4] to the elements of  $\bar{c}^\perp$ .

**Corollary 4.3.** Let  $vw$  be an edge of  $\text{conv}(\mathcal{W})$  and suppose  $\bar{v}$  and  $\bar{w}$  are in  $\bar{c}^\perp$ . Suppose further that there are no elements of  $\mathcal{W}$  in the interior of  $vw$ . Then  $J(\bar{v}, \bar{w}) = 0$ .

*Proof.* This is essentially the same as the proof of Theorem 3.5 of [DW1]. As  $\text{conv}(\frac{1}{2}(d + \mathcal{W})) \cap \bar{c}^\perp$  is a face of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ , Lemma 3.5 shows that for calculations in  $\bar{c}^\perp$  we need only consider elements of  $\mathcal{C}$  in this hyperplane. Note that by Cor 3.4, no elements of  $\mathcal{C}$  lie on the opposite side of  $\bar{c}^\perp$  to  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ .

Any vertex of  $\text{conv}(\mathcal{C}) \cap \bar{c}^\perp$  outside  $\text{conv}(\frac{1}{2}(d + \mathcal{W})) \cap \bar{c}^\perp$  is, by Prop 1.4, null, so must be  $\bar{c}$  by Lemma 3.2. Now Cor 3.4 shows that  $\bar{c}$  is the only element of  $\text{conv}(\mathcal{C}) \cap \bar{c}^\perp$  outside  $\text{conv}(\frac{1}{2}(d + \mathcal{W})) \cap \bar{c}^\perp$ . But any sum  $\bar{c} + \bar{a}$  with  $\bar{a} \in \bar{c}^\perp$  does not contribute, so in fact we are in the situation of Theorem 3.5 of [DW4].  $\square$

We introduce the following sets:

$$\begin{aligned} \hat{S}_1 &= \{i \in \{1, \dots, r\} : \exists \text{ unique } w \in \mathcal{W} \text{ with } \bar{w} \in \bar{c}^\perp \text{ and } w_i = -2\} \\ \hat{S}_{\geq 2} &= \{i \in \{1, \dots, r\} : \exists \text{ more than one } w \in \mathcal{W} \text{ with } \bar{w} \in \bar{c}^\perp \text{ and } w_i = -2\} \end{aligned}$$

These are similar to the sets  $S_1, S_{\geq 2}$  of [DW4], but now we require that the vectors  $w$  to lie in  $\bar{c}^\perp$ .

It is immediate from Cor 4.3 that  $d_i = 4$  if  $i \in \hat{S}_{\geq 2}$ , (cf Prop 4.2 in [DW4]).

We next prove a useful result about which elements of  $\frac{1}{2}(d + \mathcal{W})$  can be orthogonal to  $\bar{c}$ . This will give us information about when (1A) vertices can occur.

**Lemma 4.4.** *Assume that we are not in the situation of Theorem 3.14 (i.e.,  $c$  is not of type I). Let  $u \in \mathcal{W}$  be such that  $\bar{u} \in \bar{c}^\perp$ . Then:*

- (a) *there exists  $i$  with  $c_i \neq 0$  and  $-2 < c_i < 1$ ;*
- (b) *if  $c \in \mathbb{Z}^r$  then there is at most one such  $u$ , and hence at most one (1A) vertex (wrt  $c$ ).*

*Proof.* (a) The condition  $J(\bar{u}, \bar{c}) = 0$  means  $\sum_i \frac{u_i c_i}{d_i} = 1$ , and nullity of  $\bar{c}$  means  $\sum_i \frac{c_i^2}{d_i} = 1$ . As  $u_i \in \{-2, -1, 0, 1\}$ , if the condition in (a) does not hold, then  $u_i c_i \leq c_i^2$  for all  $i$  so we must have equality for all  $i$ . Now  $c_i = u_i$  for all  $i$  with  $c_i$  nonzero. As  $\sum c_i = -1$  and  $c \neq u$  (since  $c \notin \mathcal{W}$  by definition), this means  $c$  is a type I vector and we are in the situation of Theorem 3.14.

(b) We see from the previous paragraph that we need  $u_i c_i > c_i^2$  for some  $i$ . If  $c \in \mathbb{Z}^r$  this means  $c_i = -1$  and  $u_i = -2$ . The orthogonality condition is now  $\frac{2}{d_i} + \frac{c_j}{d_j} = 1$  where  $u_j = 1$ . As  $d_i \neq 1$  we see  $c_j \geq 0$ .

If  $c_j = 0$  then  $d_i = 2$ . If  $c_j > 0$  then  $d_i \geq 3$  so  $\frac{1}{3} \leq \frac{c_j}{d_j} < \frac{1}{c_j}$ , where the second inequality is due to the nullity requirement  $\frac{1}{d_i} + \frac{c_j^2}{d_j} \leq 1$ . So  $c_j = 1$  or  $2$ . Moreover, the latter implies  $(d_i, d_j) = (3, 6)$  and  $c = (-1^i, 2^j)$ , which contradicts  $\sum c_i = -1$ .

We see that either  $c_j = 1$  and  $(d_i, d_j) = (4, 2)$  or  $(3, 3)$ , or  $c_j = 0$  and  $d_i = 2$ . Cor 4.3 implies that if there is more than one such  $u$  (say  $(-2^i, 1^j)$  and  $(-2^i, 1^k)$ ) for a given  $i$ , then  $d_i = 4$ , so  $(d_i, d_j, d_k) = (4, 2, 2)$ , and  $(c_i, c_j, c_k) = (-1, 1, 1)$ , contradicting the nullity of  $c$ .

It now readily follows that the nullity condition prevents there being more than one  $u \in \mathcal{W}$  with  $\bar{u} \in \bar{c}^\perp$  except when  $c = (-1, -1, 1, 0, \dots)$  with  $d = (4, 4, 2, \dots)$  or  $(3, 3, 3, \dots)$  and  $u = (-2, 0, 1, 0, \dots), (0, -2, 1, 0, \dots)$ . But in this case if both  $u$  occur then  $c \in \text{conv}(\mathcal{W})$ , a contradiction.  $\square$

We shall study (1A) vertices for non-integral  $c$  in the next section. The following results will be useful.

**Proposition 4.5.** *Let  $v = (-2^i, 1^j)$  and  $w = (-2^k, 1^l)$  be elements of  $\mathcal{W}$  such that  $\bar{v}, \bar{w} \in \bar{c}^\perp$ . Suppose that  $i \in \hat{S}_1$  and  $\{i, j\} \cap \{k, l\} = \emptyset$ . Then  $k \in \hat{S}_{\geq 2}$  and  $(d_i, d_k, d_l) = (2, 4, 2)$ .*

*Proof.* By Remark 1.2(e) the affine subspace  $\{\bar{x} : x_i + x_k = -2, x_j + x_l = 1\} \cap \bar{c}^\perp$  meets  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  in a face, whose possible elements are  $v, w, u = (-2^k, 1^j), y = (-1^i, 1^j, -1^k)$  and  $z = (-1^i, -1^k, 1^l)$  (since  $i \in \hat{S}_1$ ).

As  $J(\bar{v}, \bar{w}) = \frac{1}{4}$ , we see from Thm 4.3 that  $vw$  is not an edge so  $z$  is present in the face. Now Cor 4.3 on  $vz$  implies  $d_i = 2$ . Also,  $u$  must be present, otherwise  $y$  is present and Cor 4.3 on  $zw$  and  $yw$  gives a contradiction. So  $k \in \hat{S}_{\geq 2}$ , and Cor 4.3 on  $uw$  implies  $d_k = 4$ . Now considering  $zw$  implies  $d_l = 2$ .  $\square$

**Remark 4.6.** This is similar to the proof of Prop 4.6 in [DW4]. But we cannot now deduce that  $d_j = 1$  as the proof of this in [DW4] relied on the existence of  $t = (-1^i, -1^j, 1^k)$ , and although we know this is in  $\mathcal{W}$  we do not know if  $\bar{t}$  lies in  $\bar{c}^\perp$ .

**Proposition 4.7.** *If  $i \in \hat{S}_1$  and  $v = (-2^i, 1^j)$  gives an element of  $\bar{c}^\perp$  then  $w = (-1^i, -1^j, 1^k)$  cannot give an element of  $\bar{c}^\perp$ .*

*Proof.* This is similar to Prop. 4.3 in [DW4]. Since  $i \in \hat{S}_1$ , the vectors  $\bar{v}, \bar{w}$  lie on an edge in the face  $\{\bar{x} : 2x_i + x_j = -3\} \cap \bar{c}^\perp$  of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ , and  $J(\bar{v}, \bar{w}) = \frac{1}{4}(1 - \frac{2}{d_i} + \frac{1}{d_j}) \neq 0$  since  $d_i \neq 1$ .  $\square$

**Corollary 4.8.** *With  $v$  as in Prop 4.7, there are no elements  $w = (-2^j, 1^k)$  with  $\bar{w}$  in  $\bar{c}^\perp$ .*

*Proof.* This is similar to Prop 4.4 in [DW4]. If  $k = i$ , then the type I vector  $u := (-1^i) = \frac{1}{3}(2v + w)$  lies in  $\mathcal{W}$  and  $\bar{u} \in \bar{c}^\perp$ . By Lemma 5.1 below,  $u = c$ , contradicting  $c \notin \mathcal{W}$ .

We can therefore take  $k \neq i$ . Now  $\bar{v}, \bar{w}$  lie on an edge in the face  $\{\bar{x} : 3x_i + 2x_j = -4\} \cap \bar{c}^\perp$  (this is a face by Prop 4.7 and the assumption  $i \in \hat{S}_1$ ). But  $J(\bar{v}, \bar{w}) = \frac{1}{4}(1 + \frac{2}{d_j}) \neq 0$ .  $\square$

## 5. Vectors orthogonal to a null vertex

In this section we analyse the possibilities for  $\frac{1}{2}(d + \mathcal{W}) \cap \bar{c}^\perp$ . This will give us an understanding of the vertices of type (1A).

We first dispose of the case of type I vectors.

**Lemma 5.1.** *If  $u$  is a type I vector and  $\bar{u} \in \bar{c}^\perp$  then  $c = u$ , so we are in the situation of Theorem 3.14.*

*Proof.* Up to a permutation we may let  $u = (-1, 0, \dots)$ . The orthogonality condition implies  $c_1 = -d_1$ . But nullity implies  $\sum c_i^2/d_i = 1$ , so  $d_1 = 1$  and  $c_i = 0$  for  $i > 1$ . (Note that in particular  $u \notin \mathcal{W}$ .)  $\square$

We shall therefore assume from now on there are no type I vectors giving points of  $\bar{c}^\perp$ .

**Lemma 5.2.** (i) *Two type II vectors whose nonzero entries lie in the same set of three indices cannot both give elements of  $\bar{c}^\perp$ .*

(ii) *Two type III vectors  $(-2^i, 1^j)$  and  $(1^i, -2^j)$  cannot both give elements in  $\bar{c}^\perp$ .*

(iii) *Three type III vectors whose nonzero entries all lie in the same set of three indices cannot all give rise to elements in  $\bar{c}^\perp$ .*

*Proof.* These all follow from Lemma 5.1 by exhibiting an affine combination of the given vectors which is of type I.  $\square$

Let  $u, v \in \mathcal{W}$  be such that  $\bar{u}$  and  $\bar{v} \in \bar{c}^\perp$ . It follows that  $\lambda\bar{u} + (1 - \lambda)\bar{v} \in \bar{c}^\perp$  for all  $\lambda$ . Hence Remark 3.1 shows that for all  $\lambda$

$$\begin{aligned} 0 &\geq J(d + \lambda u + (1 - \lambda)v, d + \lambda u + (1 - \lambda)v) \\ &= J(\lambda(d + u) + (1 - \lambda)(d + v), \lambda(d + u) + (1 - \lambda)(d + v)) \\ &= \lambda^2(J(d + u, d + u) + J(d + v, d + v) - 2J(d + u, d + v)) + \\ &\quad 2\lambda(J(d + u, d + v) - J(d + v, d + v)) + J(d + v, d + v). \end{aligned}$$

Equality occurs if and only if  $\lambda u + (1 - \lambda)v = c$ , as  $\bar{c}$  is the only null vector in  $\bar{c}^\perp$ .

Multiplying by  $-1$ , using Eq.(1.3), and recalling that the minimum value of a quadratic  $\alpha\lambda^2 + \beta\lambda + \gamma$  with  $\alpha > 0$  is  $\gamma - (\beta^2/4\alpha)$ , we deduce the following result.

**Lemma 5.3.** *If  $u, v \in \mathcal{W}$  and  $\bar{u}, \bar{v} \in \bar{c}^\perp$  then*

$$(5.1) \quad \sum_{i=1}^r \frac{u_i^2}{d_i} + \sum_{i=1}^r \frac{v_i^2}{d_i} - \left( \sum_{i=1}^r \frac{u_i^2}{d_i} \right) \left( \sum_{i=1}^r \frac{v_i^2}{d_i} \right) \leq \sum_{i=1}^r \frac{u_i v_i}{d_i} \left( 2 - \sum_{i=1}^r \frac{u_i v_i}{d_i} \right).$$

Moreover, equality occurs if and only if  $c = \lambda u + (1 - \lambda)v$  for some  $\lambda$ .  $\square$

**Remark 5.4.** By definition,  $c$  does not lie in  $\text{conv}(\mathcal{W})$ . So in the case of equality in Eq.(5.1) we cannot have  $0 \leq \lambda \leq 1$ . This observation will in many cases show that equality cannot occur.

**Remark 5.5.** The right-hand side of Eq.(5.1) is maximised when  $\sum \frac{u_i v_i}{d_i} = 1$  (i.e.,  $J(\bar{u}, \bar{v}) = 0$ ).

In this case Eq.(5.1) just follows from  $\sum \frac{u_i^2}{d_i}, \sum \frac{v_i^2}{d_i} \geq 1$ , which is true for any two vectors in  $\bar{c}^\perp$ . If  $J(\bar{u}, \bar{v}) \neq 0$ , we get sharper information.

**Corollary 5.6.** *Suppose that  $K$  is connected. If  $u, v$  are type II vectors in  $\mathcal{W}$  with  $\bar{u}, \bar{v} \in \bar{c}^\perp$  then*

$$\frac{1}{2} \leq \sum_{i=1}^r \frac{u_i v_i}{d_i} \leq \frac{3}{2},$$

*with equality if and only if  $c = \lambda u + (1 - \lambda)v$  for some  $\lambda$ , in which case all the  $d_i = 2$  whenever  $i$  is an index such that  $u_i$  or  $v_i$  is nonzero.*

*Proof.* Writing  $X = \sum \frac{u_i^2}{d_i}$  and  $Y = \sum \frac{v_i^2}{d_i}$  we see that  $1 \leq X, Y \leq \frac{3}{2}$ . The lower bound arises from  $\bar{u}, \bar{v}$  being in  $\bar{c}^\perp$ , while the upper bound follows from Remark 1.2(d) and the assumption that  $u, v$  are type II vectors.

Now  $X + Y - XY = 1 - (1 - X)(1 - Y)$  is minimised for  $X, Y$  in this range if  $X = Y = \frac{3}{2}$ , when it takes the value  $\frac{3}{4}$ . The inequality Eq.(5.1) now gives the result.  $\square$

When  $K$  is connected, it follows that any two such type II vectors must overlap. Moreover, if they have only one common index then we are in the case of equality in Cor 5.6. The nullity of  $\bar{c}$  implies that  $\lambda = \frac{1}{2}$  in this case, contradicting Remark 5.4.

Combining this remark with Cor 5.6 and Lemma 5.2 (i), we deduce the following result.

**Corollary 5.7.** *Assume  $K$  is connected. If  $u, v$  are type II vectors in  $\mathcal{W}$  with  $\bar{u}, \bar{v} \in \bar{c}^\perp$ , then either  $u = (-1^a, -1^b, 1^i), v = (-1^a, -1^b, 1^j)$  or  $u = (1^a, -1^b, -1^i), v = (1^a, -1^b, -1^j)$ .*

*Hence the collection of all such type II vectors is of the form, for some fixed  $a, b$ :*

- (i)  $(-1^a, -1^b, 1^i) : i \in I$  for some set  $I$ ; or
- (ii)  $(1^a, -1^b, -1^i) : i \in I$  for some set  $I$ ; or
- (iii)  $(1, -1, -1, 0, \dots), (1, -1, 0, -1, \dots), (1, 0, -1, -1, \dots)$ .  $\square$

We now investigate type III vectors.

**Lemma 5.8.** *Suppose  $K$  is connected. If  $u$  is a type II vector and  $v$  a type III vector in  $\mathcal{W}$  with  $\bar{u}, \bar{v} \in \bar{c}^\perp$ , then  $\sum_{i=1}^r \frac{u_i v_i}{d_i} > 0$ .*

*Proof.* With the notation of Cor 5.6 we have  $1 \leq X \leq \frac{3}{2}$  and  $1 \leq Y \leq 3$ . So  $X + Y - XY = 1 - (1 - X)(1 - Y) \geq 0$ , and Eq.(5.1) gives the desired inequality. Also, the case of equality (i.e.,  $X = \frac{3}{2}, Y = 3$ ) leads to  $\lambda = \frac{2}{3}$ , again contradicting Remark 5.4.  $\square$

**Remark 5.9.** While Cor 5.6 - Lemma 5.8 are stated under the assumption that  $K$  is connected, the actual property we used is that in Remark 2.4. By contrast, the next two results do not require this property.

**Lemma 5.10.** *Any two type III vectors  $u, v$  giving elements of  $\bar{c}^\perp$  must overlap.*

*Proof.* Write  $u = (-2^i, 1^j)$  and  $v = (-2^k, 1^l)$ . By Cor 4.3, if  $i, k \in \hat{S}_{\geq 2}$  then  $d_i = d_k = 4$ . Since  $J(\bar{c}, \bar{u}) = 0$  we have (by Cauchy-Schwartz)

$$1 = \left( -\frac{2c_i}{d_i} + \frac{c_j}{d_j} \right)^2 \leq \left( \frac{4}{d_i} + \frac{1}{d_j} \right) \left( \frac{c_i^2}{d_i} + \frac{c_j^2}{d_j} \right).$$

Hence

$$\frac{c_i^2}{d_i} + \frac{c_j^2}{d_j} \geq \frac{d_j}{1 + d_j} \geq \frac{1}{2}.$$

If  $u$  and  $v$  do not overlap, then the above and the analogous result from considering  $J(\bar{c}, \bar{v}) = 0$ , together with the nullity of  $\bar{c}$ , imply that  $d_j = 1 = d_l$  and the only nonzero components of  $c$  are  $c_i = c_k = -1, c_j = c_l = \frac{1}{2}$ . But then  $c$  is the midpoint of  $uv$ , contradicting  $c \notin \text{conv}(\mathcal{W})$ .

So if  $u$  and  $v$  do not overlap, we can take  $i \in \hat{S}_1$ . Proposition 4.5 shows that  $k \in \hat{S}_{\geq 2}$  and  $(d_i, d_k, d_l) = (2, 4, 2)$ . Hence  $2 < X \leq 3$  and  $Y = \frac{3}{2}$ , so  $X + Y - XY \geq 0$  and  $\sum \frac{u_i v_i}{d_i} \geq 0$ . Non-overlap means that equality holds. But then  $\lambda = 1/3$ , contradicting Remark 5.4.  $\square$



Lemma 5.10, together with Lemmas 5.2 and 4.8, implies the following corollary.

**Corollary 5.11.** *The type III vectors associated to elements of  $\frac{1}{2}(d+\mathcal{W}) \cap \bar{c}^\perp$  are, up to permutation of indices, either of the form*

- (a)  $(-2^1, 1^i)$ ,  $i \in I$ , (with  $d_1 = 4$  if  $|I| \geq 2$ ), or
- (b)  $(1^1, -2^i)$ ,  $i \in I$ ,

for some subset  $I \subset \{2, \dots, r\}$ .  $\square$

Having found the possible configurations for type III vectors in  $\bar{c}^\perp$ , we start to analyse the type II vectors for each such configuration. For the rest of this section we will assume that  $K$  is connected (cf Remark 5.9).

**Remark 5.12.** Lemma 5.8 now shows that in case (a) of Cor 5.11, if  $|I| \geq 2$ , then every type II vector associated to an element of  $\bar{c}^\perp$  must have “-1” in place 1. Similarly, in case (b), if  $|I| \geq 3$ , then every such type II vector has “1” in place 1. (So if a type II is present then  $d_1 \neq 1$ ). If  $|I| = 2$ , the only possible type II vectors with “0” in place 1 are  $(0^1, -1^2, -1^3, 1^i)$  where  $i \geq 4$ , and all type II vectors whose first entry is nonzero actually must have first entry equal to 1.

**Lemma 5.13.** *In case (a) of Cor. 5.11 with  $|I| \geq 2$  there are no type II vectors associated to elements of  $\bar{c}^\perp$ .*

*Proof.* Let  $v = (-2^1, 1^k)$  and  $w = (-1^1, 1^i, -1^j)$  give elements of  $\bar{c}^\perp$  with  $k \neq i, j$ . Consider the face  $\{\bar{x} : x_i + x_k = 1, x_1 + x_j = -2\} \cap \bar{c}^\perp$ . Other than  $v, w$  the possible elements in this face come from  $u = (-1^1, 1^k, -1^j)$  and  $s = (-2^1, 1^i)$ . As  $d_1 = 4$ ,  $J(\bar{v}, \bar{w}) \neq 0$ , so  $vw$  is not an edge and  $u$  must be present. But  $J(\bar{u}, \bar{w}) = \frac{1}{4}(1 - \frac{1}{d_1} - \frac{1}{d_j}) \neq 0$  since  $d_1 = 4$ , giving a contradiction. So  $k = i$  or  $j$  for every such  $v, w$ .

Hence if such a  $w$  exists there are at most two type III vectors. Now if  $|I| = 2$  and the type IIIs are  $(-2, 1, 0, \dots), (-2, 0, 1, \dots)$ , we cannot have  $w = (-1, 1, -1, \dots)$  or  $(-1, -1, 1, \dots)$  as then a suitable affine combination of the above vectors give a type I vector. (cf Lemmas 5.1, 5.2). So in fact no type II vectors give rise to elements of  $\bar{c}^\perp$ .  $\square$

**Lemma 5.14.** *The vectors  $v = (-2, 1, 0, \dots)$  and  $w = (0, 1, -1, -1, 0, \dots)$  are not both associated to elements of  $\bar{c}^\perp$ , unless  $(0, 1, -2, 0, \dots)$  or  $(0, 1, 0, -2, 0, \dots)$  is also.*

*Proof.* Suppose  $(0, 1, -2, 0, \dots), (0, 1, 0, -2, 0, \dots)$  are absent. Consider the face  $\{\bar{x} : x_2 = 1, x_1 + x_3 + x_4 = -2\} \cap \bar{c}^\perp$ . The other possible elements of this face come from  $t = (-1, 1, -1, 0, \dots)$  and  $y = (-1, 1, 0, -1, 0, \dots)$ . Both these must be present, as  $J(\bar{v}, \bar{w}) \neq 0$ . Applying Cor 4.3 to  $wt, vt$  and  $wy$  we obtain  $(d_1, d_2, d_3, d_4) = (4, 2, 2, 2)$ .

Now we have equality (for  $y, t$ ) in Eq.(5.1), as both sides equal  $15/16$ . We find that  $\lambda = 1/2$ , giving a contradiction again to Remark 5.4.  $\square$

Combining this with Lemma 5.8 (and using Lemma 4.7) yields:

**Corollary 5.15.** *If there is a unique type III vector  $u = (-2^1, 1^2)$  with  $\bar{u}$  in  $\bar{c}^\perp$ , then the type II vectors associated to elements of  $\bar{c}^\perp$  all have “-1” in place 1. Moreover  $(-1^1, -1^2, 1^i)$  cannot be present. Also, if  $(-1^1, 1^2, -1^i)$  is present for some  $i \geq 3$  then  $(d_1, d_2) = (4, 2)$  or  $(3, 3)$  and the index  $i$  is unique.*

For the last assertion, observe that  $(-1^1, 1^2, -1^i)$  and the type III vector are joined by an edge, so Cor 4.3 shows the dimensions are as stated. If we have two such type II for  $i_0$  and  $i_1$  then Eq.(5.1) implies  $d_{i_0} + d_{i_1} \leq 4$ . Hence since  $K$  is connected,  $d_{i_0} = d_{i_1} = 2$  and we have equality in Eq.(5.1) with  $\lambda = \frac{1}{2}$ , giving a contradiction.

**Lemma 5.16.** *Let the type III vectors be as in Cor 5.11(b), i.e., they are  $(1^1, -2^a)$ ,  $a \in I$ . Assume that  $|I| \geq 2$ . If we have a type II vector  $w = (1^1, -1^i, -1^j)$  with  $\bar{w}$  in  $\bar{c}^\perp$  then  $i, j \in I$*

*Proof.* Suppose for a contradiction that  $w = (1^1, -1^i, -1^j)$  is present (so  $d_1 \neq 1$ ) and  $(1^1, -2^j)$  absent (i.e.  $j \notin I$ ). Since  $|I| \geq 2$ , we can consider  $v = (1^1, -2^k)$  where  $k \in I$  (so  $k \neq j$ ) and  $k \neq i$ . Consider the face  $\{\bar{x} : x_1 = 1, x_i + x_j + x_k = -2\} \cap \bar{c}^\perp$ . As well as  $v, w$  the possible elements of  $\mathcal{W}$  in the face giving elements of  $\bar{c}^\perp$  are  $y = (1^1, -1^i, -1^k)$ ,  $t = (1^1, -1^j, -1^k)$  and  $u = (1^1, -2^i)$ . As  $d_1 \neq 1$ ,  $vw$  is not an edge so  $t$  is present. Now Cor 4.3 applied to  $vt$  and  $tw$  gives  $d_1 = d_j = 2$  and  $d_k = 4$ .

Moreover, if  $i \in I$  then  $u$  is present, so the edge  $wu$  gives  $d_i = 4$ . Thus we have shown that  $d_a = 4$  for all  $a \in I$ .

Now considering  $(1^1, -2^a)$  and  $(1^1, -2^b)$  with  $a, b \in I$ , we see that we have equality in Eq.(5.1) (both sides equal  $\frac{3}{4}$ ). In fact  $c$  is the average of these two vectors (i.e.,  $\lambda = \frac{1}{2}$ ), so as in Remark 5.4 we have a contradiction.  $\square$

**Lemma 5.17.** *Let the type III vectors be as in Cor 5.11(b), i.e., they are  $(1^1, -2^a)$ ,  $a \in I$ . Assume that  $|I| \geq 3$ . Then  $d_1 = 1$ .*

*Proof.* Each pair  $v, w$  of type III vectors gives an edge, and if  $d_1 \neq 1$ , then we have  $J(\bar{v}, \bar{w}) > 0$ . By Theorem 4.3 all the midpoint vectors  $(1^1, -1^a, -1^b)$  are present for  $a, b \in I$ . Now Prop 3.7 shows that  $F_{\bar{v}}$  and  $F_{\bar{w}}$  have opposite signs, so we have a contradiction if  $|I| \geq 3$ .  $\square$

Putting together our results so far, we obtain a description of the possibilities for  $\bar{c}^\perp \cap \frac{1}{2}(d + \mathcal{W})$ .

**Theorem 5.18.** *Assume that  $r \geq 3$  and  $K$  is connected, and that we are not in the situation of Thm 3.14. Up to permutation of the irreducible summands, the following are the possible configurations of vectors in  $\mathcal{W}$  associated to elements of  $\frac{1}{2}(d + \mathcal{W}) \cap \bar{c}^\perp$ .*

- (1)  $\{(-2^1, 1^i), 2 \leq i \leq m\}$  for fixed  $m \geq 2$ . There are no type II vectors, and  $d_1 = 4$  if  $m \geq 3$ .
- (2)  $\{(1^1, -2^i), 2 \leq i \leq m\}$  for fixed  $m \geq 3$  and  $d_1 = 1$ . There are no type II vectors.
- (3)(i)  $\{(1^1, -2^2), (1^1, -2^3), (-1^2, -1^3, 1^i), 4 \leq i \leq m\}$  with  $d_1 = 1, d_2 = d_3 = 2$ .  
 (ii)  $\{(1^1, -1^2, -1^3), (1^1, -2^2), (1^1, -2^3), (-1^2, -1^3, 1^i), 4 \leq i \leq m\}$ ,  $d_1 \neq 1, d_2 = d_3 = 2$ .
- (4)  $\{(1, -2, 0, 0, \dots), (1, 0, -2, 0, \dots), (1, -1, -1, 0, \dots)\}$  with  $d_1 \neq 1$ .
- (5) A unique type III  $(-2, 1, 0, \dots)$ . Possible type II vectors are  
 (i)  $(-1, 1, -1, 0, \dots)$  with either  $(d_1, d_2) = (4, 2)$  or  $(3, 3)$ ; or  
 (ii)  $\{(-1^1, 1^3, -1^i), 4 \leq i \leq m\}$  for fixed  $m \leq r$  and with  $d_1 = 2$ ; or  
 (iii)  $\{(-1^1, -1^3, 1^i), 4 \leq i \leq m\}$  for fixed  $m \leq r$  and with  $d_1 = 2$ .
- (6) No type III vectors. Possible type II vectors are  
 (i)  $\{(-1^1, -1^2, 1^i), 3 \leq i \leq m\}$  for fixed  $m \leq r$ , with  $d_1 = d_2 = 2$  if  $m \geq 4$ ; or  
 (ii)  $\{(1^1, -1^2, -1^i), 3 \leq i \leq m\}$  for fixed  $m \leq r$ , with  $d_1 = d_2 = 2$  if  $m \geq 4$ ; or  
 (iii)  $\{(1^1, -1^2, -1^3), (1^1, -1^2, -1^4), (1^1, -1^3, -1^4)\}$  with  $d_1 = d_2 = d_3 = d_4 = 2$ .

*Proof.* Cor 5.11 gives the possibilities for the type III vectors in  $\bar{c}^\perp$ . If there are none then Cor 5.7 gives the possibilities in (6). If there is a unique type III vector, then Cor 5.15 and Cor 5.7 give us the cases listed in (5) (or (1) with  $m = 2$  if there are no type II). If we have two or more type III vectors with  $-2$  in the same place then Lemma 5.13 shows we are in case (1).

If we have more than two type III vectors with 1 in the same place  $a$ , then  $d_a = 1$  by Lemma 5.17. Remark 5.12 then implies there are no type II vectors and we are in case (2).

If we have exactly two type III vectors with 1 in the same place, e.g.,  $(1, -2, 0, \dots)$  and  $(1, 0, -2, 0, \dots)$ , then the proof of Lemma 5.17 shows that if the type II vector  $(1, -1, -1, 0, \dots)$  is absent we must have  $d_1 = 1$ . On the other hand, if  $d_1 = 1$  we are, by Remark 5.12 and the connectedness of  $K$ , in case (2) or (3)(i). If  $d_1 \neq 1$ , then by the above, Remark 5.12, and Cor 5.7, we are in case (3)(ii) or (4).

The statements about values of the  $d_i$  follow from straightforward applications of Cor 4.3 to the obvious edges of  $\text{conv}(\frac{1}{2}(d + \mathcal{W})) \cap \bar{c}^\perp$ .  $\square$

**Remark 5.19.** The possibilities in Theorem 5.18 can be somewhat sharpened. In cases (1), (2), and (3),  $m$  cannot be  $r$ ; in other words the maximum number of vectors is not allowed. This follows easily from looking at the system of equations expressing the nullity of  $\bar{c}$ , the orthogonality of the vectors to  $\bar{c}$  and the fact that the entries of  $c$  sum up to  $-1$ . Similarly,  $r \neq 3$  in (5)(i) and  $r \neq 4$  in (6)(iii).

When  $m \geq 5$  in (5)(ii) or (5)(iii), the segment joining two type II vectors is an edge, so Cor 4.3 gives  $d_3 = 2$ .

## 6. Adjacent (1B) vertices

We now turn to (1B) vertices. Let  $\bar{\xi}, \bar{\xi}'$  be adjacent (1B) vertices of  $\Delta$ . Then there exist vertices  $\bar{x}, \bar{x}'$  of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  such that  $\bar{c}, \bar{\xi}, \bar{x}$  are collinear and  $\bar{c}, \bar{\xi}', \bar{x}'$  are collinear. Moreover, there exist null vectors  $\bar{a}, \bar{a}'$  such that  $\bar{x} = (\bar{a} + \bar{c})/2$  and  $\bar{x}' = (\bar{a}' + \bar{c})/2$ . By Cor 3.4, there must be an element  $\bar{y}$  of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  on  $\bar{a}\bar{a}'$ , so  $P^{-1}(\bar{\xi}\bar{\xi}')$  contains the convex hull of  $\bar{x}, \bar{x}', \bar{y}$  and hence is 2-dimensional. As  $\bar{\xi}\bar{\xi}'$  is by assumption an edge of  $\Delta$ ,  $P^{-1}(\bar{\xi}\bar{\xi}')$  is a 2-dimensional face of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ .

So we need to analyse the 2-dimensional faces of  $\text{conv}(\mathcal{W})$  containing vertices  $x, x'$  such that

$$(6.1) \quad x = (a + c)/2, \quad x' = (a' + c)/2, \quad \bar{a}, \bar{a}' \text{ null},$$

and such that  $c$  lies in the 2-dimensional plane defining this face. The lines through  $x, c$  (resp.  $x', c$ ) only meet  $\text{conv}(\mathcal{W})$  at  $x$  (resp.  $x'$ ).

Most 2-faces of  $\text{conv}(\mathcal{W})$  are triangular. We list below (up to permutation of components) all the possible non-triangular faces. For further details regarding how this listing is arrived at, see [DW5]. We emphasize that only the *full* faces are being listed, i.e., configurations formed by all the possible elements of  $\mathcal{W}$  in a given 2-dimensional plane. As the set of weight vectors for a given principal orbit may be a subset of the full set of possible weight vectors, these full faces may degenerate to subfaces or even lower-dimensional faces (see Remark 6.2).

*Listing convention:* In the interest of economy and clarity, we make the convention that when we list vectors in  $\mathcal{W}$  belonging to a 2-face we will use the freedom of permuting the summands to place nonzero components of the vectors first and we will only put down the minimum number of components necessary to specify the vectors.

**Hexagons:** There are 3 possibilities.

(H1) This is the face in the plane  $\{x_1 + x_2 + x_3 = -1; x_a = 0, \text{ for } a > 3\}$ . Points of  $\mathcal{W}$  are  $(-2^i, 1^j), (-1^i, 1^j, -1^k), (-1^i)$  where  $i, j, k \in \{1, 2, 3\}$ . The type III vectors form the vertices of the hexagon.

(H2) The plane here is  $\{x_1 + x_2 = -1, x_3 + x_4 = 0, x_i = 0 (i > 4)\}$ . Points of  $\mathcal{W}$  are vertices

$$u = (-2, 1, 0, 0), \quad v = (1, -2, 0, 0), \quad y = (-1, 0, 1, -1), \quad y' = (0, -1, 1, -1), \\ z = (-1, 0, -1, 1), \quad z' = (0, -1, -1, 1),$$

and the interior points

$$\alpha = (-1, 0, 0, 0), \quad \beta = (0, -1, 0, 0).$$

(H3) The plane is  $\{x_2 = -1, x_1 + x_3 + x_4 = 0, x_i = 0 (i > 4)\}$ . Points of  $\mathcal{W}$  are the vertices

$$u = (-1, -1, 1, 0), \quad v = (0, -1, 1, -1), \quad w = (1, -1, 0, -1), \\ x = (1, -1, -1, 0), \quad y = (0, -1, -1, 1), \quad z = (-1, -1, 0, 1)$$

and the centre

$$t = (0, -1, 0, 0).$$

**Square:** (S) with midpoint  $t = (0, -1, 0, 0, 0)$  and vertices

$$v = (-1, -1, 1, 0, 0), \quad u = (0, -1, 0, 1, -1), \\ s = (0, -1, 0, -1, 1), \quad w = (1, -1, -1, 0, 0).$$

**Trapezia:** We have vertices  $v, u, s, w, t$  with  $2v - s = 2u - w$  and  $t = \frac{1}{2}(s + w)$ , i.e., these are symmetric trapezia. Below we list the possible  $v, u, s, w$ .

	$v$	$u$	$s$	$w$
(T1)	$(-2, 1, 0, 0)$	$(-2, 0, 1, 0)$	$(0, 0, -2, 1)$	$(0, -2, 0, 1)$
(T2)	$(-2, 0, 1, 0)$	$(-2, 1, 0, 0)$	$(0, -1, 1, -1)$	$(0, 1, -1, -1)$
(T3)	$(-1, -1, 0, 1)$	$(0, -1, -1, 1)$	$(-2, 1, 0, 0)$	$(0, 1, -2, 0)$
(T4)	$(0, 0, 1, -1, -1)$	$(1, 0, 0, -1, -1)$	$(-2, 1, 0, 0, 0)$	$(0, 1, -2, 0, 0)$
(T5)	$(-1, 0, 0, 1, -1)$	$(0, 0, -1, 1, -1)$	$(-2, 1, 0, 0, 0)$	$(0, 1, -2, 0, 0)$
(T6)	$(1, -1, -1, 0, 0)$	$(1, -1, 0, -1, 0)$	$(0, 0, -1, 1, -1)$	$(0, 0, 1, -1, -1)$

Table 4: Possible trapezoidal faces

Note that the configuration with vertices  $(-1, -1, 1, 0, 0)$ ,  $(-1, -1, 0, 1, 0)$ ,  $(0, 0, 1, -1, -1)$ , and  $(0, 0, -1, 1, -1)$  is equivalent to (T6) under the composition of a permutation and a  $J$ -isometric involution.

**Parallelograms:** We have vertices  $v, u, s, w$  with  $v - u = s - w$ .

	$v$	$u$	$s$	$w$
(P1)	$(-2, 1, 0, 0)$	$(-1, 0, -1, 1)$	$(-2, 0, 1, 0)$	$(-1, -1, 0, 1)$
(P2)	$(-2, 1, 0, 0, 0)$	$(-2, 0, 1, 0, 0)$	$(0, 1, 0, -1, -1)$	$(0, 0, 1, -1, -1)$
(P3)	$(-2, 1, 0, 0, 0)$	$(-2, 0, 1, 0, 0)$	$(0, 0, -1, -1, 1)$	$(0, -1, 0, -1, 1)$
(P4)	$(-2, 1, 0, 0)$	$(-1, 0, 1, -1)$	$(-1, -1, 0, 1)$	$(0, -2, 1, 0)$
(P5)	$(-2, 1, 0, 0, 0)$	$(-1, 0, 0, 1, -1)$	$(-1, 0, 1, -1, 0)$	$(0, -1, 1, 0, -1)$
(P6)	$(-2, 1, 0, 0, 0)$	$(-1, 0, 0, -1, 1)$	$(-1, 0, -1, 1, 0)$	$(0, -1, -1, 0, 1)$
(P7)	$(-2, 1, 0, 0, 0)$	$(0, 1, -1, -1, 0)$	$(-1, 0, 0, 1, -1)$	$(1, 0, -1, 0, -1)$
(P8)	$(1, -1, -1, 0, 0, 0)$	$(0, 0, 0, 1, -1, -1)$	$(1, -1, 0, -1, 0, 0)$	$(0, 0, 1, 0, -1, -1)$
(P9)	$(1, -1, -1, 0, 0, 0)$	$(1, -1, 0, 0, -1, 0)$	$(0, 0, -1, 1, 0, -1)$	$(0, 0, 0, 1, -1, -1)$
(P10)	$(1, -1, -1, 0, 0, 0)$	$(1, 0, 0, 0, -1, -1)$	$(0, -1, -1, 1, 0, 0)$	$(0, 0, 0, 1, -1, -1)$
(P11)	$(0, 0, 1, -1, -1, 0)$	$(0, -1, 0, -1, 0, 1)$	$(1, 0, 0, 0, -1, -1)$	$(1, -1, -1, 0, 0, 0)$
(P12)	$(1, 0, -1, 0, -1)$	$(1, -1, -1, 0, 0)$	$(0, 0, -1, 1, -1)$	$(0, -1, -1, 1, 0)$
(P13)	$(-1, 0, -1, 0, 1)$	$(0, 0, -1, -1, 1)$	$(0, -1, -1, 1, 0)$	$(1, -1, -1, 0, 0)$
(P14)	$(-1, 0, 1, 0, -1)$	$(-1, -1, 1, 0, 0)$	$(0, 0, -1, 1, -1)$	$(0, -1, -1, 1, 0)$
(P15)	$(-1, 0, 1, 0, -1)$	$(-1, -1, 1, 0, 0)$	$(0, 0, 1, -1, -1)$	$(0, -1, 1, -1, 0)$
(P16)	$(-2, 1, 0, 0)$	$(0, 1, -2, 0)$	$(-1, 0, 1, -1)$	$(1, 0, -1, -1)$
(P17)	$(-2, 1, 0, 0)$	$(0, 1, 0, -2)$	$(-2, 0, 1, 0)$	$(0, 0, 1, -2)$

Table 5: Possible parallelogram faces

**Remark 6.1.** (P1), (P2), (P3), and (P17) are actually rectangles. (P16) also includes the midpoints  $y = (u + v)/2 = (-1, 1, -1, 0)$  and  $z = (s + w)/2 = (0, 0, 0, -1)$ . The rectangle (P17) also includes the midpoints  $y = (u + v)/2 = (-1, 1, 0, -1)$  and  $z = (s + w)/2 = (-1, 0, 1, -1)$ .

**Remark 6.2.** We must also consider subshapes of the above. Each symmetric trapezium contains two parallelograms. The two rectangles with midpoints (P17), (P16) will contain asymmetric trapezia. (P17) also contains parallelograms and squares. (For (P16), note that  $s$  is present iff  $w$  is.) Furthermore, there are numerous subshapes of the hexagons. The regular hexagon (H3) contains rectangles with midpoint (by omitting opposite pairs of vertices). Besides triangles, the hexagon (H2) contains pentagons, rectangles and squares (with midpoints), and kite-shaped quadrilaterals (e.g.  $y'uz'v$ ). For (H1) see the discussion before Theorem 6.12. Finally, the triangle with midpoints of all sides (where the vertices are the three type III vectors with 1 in the same place) contains a trapezium (by omitting one vertex) and hence parallelograms.

**Remark 6.3.** We also note for future reference that there are examples where we can have four or more coplanar elements of  $\mathcal{W}$  but the plane cannot be a face. These examples are not of course

relevant to the case of adjacent (1B) vertices, but some will be relevant when we consider multiple vertices of type (2). The examples which we will need in that context are the following three trapezia

	$v$	$u$	$s$	$w$
(T*1)	(0, 1, -1, -1)	(1, 0, -1, -1)	(-2, 1, 0, 0)	(1, -2, 0, 0)
(T*2)	(0, -1, 1, -1)	(1, -1, 0, -1)	(-2, 1, 0, 0)	(0, 1, -2, 0)
(T*3)	(1, -1, -1, 0)	(1, -1, 0, -1)	(-1, 0, -1, 1)	(-1, 0, 1, -1)

Table 6: Further trapezia

In (T\*2),(T\*3), as in (T1)-(T7), we have  $2v - s = 2u - w$ . In these examples  $t = \frac{1}{2}(s + w)$  may also be present. In (T\*1) we have  $s - w = 3(v - u)$ , and the vectors  $t = (2s + w)/3 = (-1, 0, 0, 0)$  and  $r = (s + 2w)/3 = (0, -1, 0, 0)$  will also be present.

As an example, we explain why the trapezium (T\*2) can never be a face. As  $u$  is present in  $\mathcal{W}$ , so are  $u' = (-1, 1, 0, -1)$  and  $u'' = (-1, -1, 0, 1)$ . Now  $(2u' + u'')/3 = (2s + u)/3 = (-1, \frac{1}{3}, 0, -\frac{1}{3})$  is in the plane, but  $u'$  is not, so this plane cannot give a face. Similar arguments involving  $(1, 0, -1, -1), (-1, 0, 0, 0)$  (resp.  $(-1, 0, -1, 1), (-1, 0, 1, -1)$ ) show (T\*1) (resp. (T\*3)) cannot be faces.

These arguments also show several parallelograms cannot be faces, but these will not be relevant for our purposes.

We now begin to classify the possible 2-faces which arise from adjacent (1B) vertices. We shall repeatedly use Prop 1.4, Cor 3.4, and Lemma 3.5. Let  $E$  denote the affine 2-plane determined by the 2-face being studied.

**Theorem 6.4.** *Suppose we have adjacent (1B) vertices corresponding to a parallelogram face  $vusw$  of  $\text{conv}(\mathcal{W})$ . So we have  $\bar{u} = (\bar{a} + \bar{c})/2$  and  $\bar{w} = (\bar{a}' + \bar{c})/2$  for null  $\bar{a}, \bar{a}'$ . Suppose the vertices  $v, u, s, w$  are the only elements of  $\mathcal{W}$  in the face. Then  $u, w$  are adjacent vertices of the parallelogram, and either*

- (i)  $\mathcal{C} \cap E = \{\bar{c}, \bar{a}, \bar{a}', \bar{e}\}$  where  $\bar{e}$  is null with  $v = (a + e)/2$  and  $s = (a' + e)/2$ ; or
- (ii)  $\bar{v}, \bar{s} \in \mathcal{C}$  and  $J(\bar{a}, \bar{v}) = J(\bar{a}', \bar{s}) = J(\bar{s}, \bar{v}) = 0$ .

Moreover, if none of  $v, u, s, w$  is type I, then (i) cannot occur.

*Proof.* We may introduce coordinates in the 2-plane  $E$  using the sides  $sv$  and  $sw$  to define the coordinate axes. In this way we can speak of “left” or “right”, “up” or “down”. If we extend the sides of the parallelogram to infinite lines, these lines divide the part of the plane outside the parallelogram into 8 regions, and  $\bar{c}$  must be in the interior of one such region.

We first observe that if  $\bar{c}$  is in one of the four regions which only meet the parallelogram at a vertex, then  $\bar{a}\bar{a}'$  does not meet the parallelogram, contradicting Lemma 3.4.

(A) Let  $\bar{c}$  then lie in a region which meets the parallelogram in an edge. Without loss of generality we may assume the edge is  $uw$ . By Cor 3.4, all elements of  $\mathcal{C} \cap E$  lie on or between the rays from  $\bar{c}$  through  $\bar{a}, \bar{a}'$ . Hence, by Lemma 3.2,  $J(\bar{b}, \bar{c}) > 0$  for all  $\bar{b} \in \mathcal{C} \setminus \{\bar{c}\}$ . If  $\bar{b}$  is a rightmost element of  $(\mathcal{C} \cap E) \setminus \{\bar{c}\}$ , then as  $\bar{b} + \bar{c}$  cannot be written in another way as a sum of two elements of  $\mathcal{C}$ , we deduce from Prop 1.4 that  $\bar{b} + \bar{c} \in d + \mathcal{W}$ . So  $\bar{b}$  is either  $\bar{a}$  or  $\bar{a}'$ . All other elements of  $\mathcal{C} \cap E$  lie to the left of  $\bar{a}\bar{a}'$ . Note also that a rightmost element of  $(\mathcal{C} \cap E) \setminus \{\bar{c}, \bar{a}, \bar{a}'\}$  satisfies  $b + c = a + a', 2v$  or  $2s$ .

(B) Next let  $\bar{e} = 2\bar{v} - \bar{a}$ . Observe that as well as  $\bar{v} = (\bar{a} + \bar{e})/2$ , we have  $\bar{s} = (\bar{a}' + \bar{e})/2$ , since  $2\bar{v} - \bar{a} = 2(\bar{v} - \bar{u}) + \bar{c} = 2(\bar{s} - \bar{w}) + \bar{c} = 2\bar{s} - \bar{a}'$ .

If  $\bar{e} \in \mathcal{C}$ , then it must be null, and the same argument as above shows that no elements of  $(\mathcal{C} \cap E) \setminus \{\bar{e}\}$  lie to the left of  $\bar{a}, \bar{a}'$ , so we are in case (i). Now, Lemma 3.2 shows  $J(\bar{h}, \bar{k}) > 0$  for all  $\bar{h} \neq \bar{k} \in \mathcal{C} \cap E$ . If  $v, u, s, w$  are all type II/III, we see that  $F_{\bar{c}}, F_{\bar{e}}$  are of one sign and  $F_{\bar{a}}, F_{\bar{a}'}$  the other sign. But now the contributions from  $\bar{a} + \bar{a}'$  and  $\bar{c} + \bar{e}$  in the superpotential equation cannot cancel.

If  $\bar{e} \notin \mathcal{C}$  then, as in the argument before Theorem 3.8,  $\bar{s}, \bar{v} \in \mathcal{C}$  and we are in case (ii). Prop 3.7 shows  $\bar{v}, \bar{s}$  are orthogonal. Moreover, note that the remark at the end of (A) shows that  $v + c$  or  $s + c$  is left of  $a + a'$ .  $\square$

**Lemma 6.5.** *In case (ii) of Theorem 6.4, we have  $J(\bar{v}, \bar{v}) = J(\bar{s}, \bar{s})$ .*

*Proof.* As  $\bar{c}$  and  $\bar{a} = 2\bar{u} - \bar{c}$  are both null, and similarly  $\bar{c}$  and  $\bar{a}' = 2\bar{w} - \bar{c}$  are both null, we deduce (cf Remark 3.9)

$$(6.2) \quad J(\bar{u}, \bar{u}) = J(\bar{u}, \bar{c}) \quad : \quad J(\bar{w}, \bar{w}) = J(\bar{w}, \bar{c}).$$

We also have

$$(6.3) \quad 2J(\bar{u}, \bar{v}) = J(\bar{c}, \bar{v}) \quad : \quad 2J(\bar{w}, \bar{s}) = J(\bar{c}, \bar{s})$$

from the orthogonality conditions on  $\bar{a}, \bar{v}$  and  $\bar{a}', \bar{s}$ .

Now  $J(\bar{s}, \bar{s}) - J(\bar{v}, \bar{v}) = J(\bar{s}, \bar{s}) - J(\bar{w} - \bar{u} - \bar{s}, \bar{w} - \bar{u} - \bar{s})$ , which, on expanding out and using the second relations of Eqs.(6.2),(6.3), becomes  $J(2\bar{u} - \bar{c}, \bar{w} - \bar{s}) - J(\bar{u}, \bar{u})$ . Now

$$J(2\bar{u} - \bar{c}, \bar{w} - \bar{s}) - J(\bar{u}, \bar{u}) = J(2\bar{u} - \bar{c}, \bar{u} - \bar{v}) - J(\bar{u}, \bar{u}) = J(2\bar{u} - \bar{c}, \bar{u}) - J(\bar{u}, \bar{u}) = J(\bar{u} - \bar{c}, \bar{u}) = 0.$$

We have used the first relations of Eqs.(6.3), (6.2) in the second and fourth equalities.  $\square$

**Remark 6.6.** We must also consider the case when the midpoint of one side or a pair of opposite sides of the parallelogram face is in  $\mathcal{W}$ . This can happen for (P16) and (P17). Note that  $v, u, s, w$  are type II/III in these cases.

In fact, the argument of Theorem 6.4 is still valid if one or both of the midpoints of  $vu, sw$  is in  $\mathcal{W}$  and  $c$  lies in the region to the right of  $uw$  (or the left of  $vs$ ).

Keeping  $c$  in the region to the right of  $uw$ , we now need to consider the case where one or both of the midpoints of  $vs, uw$  is in  $\mathcal{W}$ . The conclusions (in 6.4(ii)) still hold except that we no longer have  $J(\bar{v}, \bar{s}) = 0$ .

However, we have to make slight modifications to the arguments as  $\frac{1}{2}(\bar{a} + \bar{a}')$  may be in  $\mathcal{C} \cap E$ . If  $\bar{e} \in \mathcal{C}$ , then, as  $\bar{a} + \bar{a}'$  is not in  $d + \mathcal{W}$ , the usual sign argument shows that the terms in the superpotential equation summing up to  $\bar{a} + \bar{a}'$  do not cancel, which is a contradiction. So  $\bar{e} \notin \mathcal{C}$  and our previous arguments hold except for the use of Prop 3.7.

Note that we also have to consider the possibility that  $a, a'$ , and  $e$  lie on the line through  $vs$ . But now the midpoint of  $uw$  must be present and  $\mathcal{C} \cap E = \{\bar{c}, \bar{a}, \bar{a}', \frac{1}{2}(\bar{a} + \bar{a}')\}$ , with  $v + s = a + a'$ . The usual sign argument then forces the midpoints of  $uw$  and  $vs$  to be present and of type I. Hence this special configuration cannot occur in (P16) or (P17).

Lastly, since the proof of Lemma 6.5 makes no mention of midpoints, it remains valid if midpoints are present.

The conditions of Theorem 6.4 and Lemma 6.5, together with the nullity of  $\bar{a}, \bar{a}', \bar{c}$ , put very strong constraints on  $vusw$  and the dimensions. In fact, one can check that these constraints cannot be satisfied for any of our parallelograms (including those of Remark 6.2) with one exception. This is the rectangle  $yy'z'z$  in (H2) with  $c = (-2, 1, 0, \dots)$  and  $\frac{1}{d_3} + \frac{1}{d_4} = \frac{1}{d_1}$ , which will be dealt with in Lemma 8.5. We now give an example of how to apply the above conditions in a specific case.

**Example 6.7.** Consider parallelogram (P8). The equation of the 2-plane  $E$  containing the parallelogram is

$$(6.4) \quad x_2 = -x_1, \quad x_5 = x_6, \quad x_2 + x_5 = -1, \quad x_1 + \dots + x_6 = -1$$

and  $x_i = 0$  for  $i > 6$ . As all vertices are type II/III, we must be in case (ii) of Theorem 6.4.

(A) Take  $c$  to face the side  $uw$ . Note that  $vs$  and  $uw$  have equation  $x_1 = 1, x_1 = 0$  respectively, so  $c_1 < 0$ . Also, the remarks at the end of parts (A) and (B) in the proof of Theorem 6.4 shows that  $c_1 > -\frac{1}{3}$ , as  $v + c$  or  $s + c$  is left of  $a + a'$  so  $1 + c_1 > -2c_1$

The condition  $J(\bar{v}, \bar{s}) = 0$  implies  $d_1 = d_2 = 2$  and Lemma 6.5 implies  $d_3 = d_4$ . Eqs.(6.2) and (6.3) give four linear equations in  $c_i$ . Now  $d_3 = d_4$  and Eq.(6.3) show  $c_3 = c_4$ , so the equations for the plane give  $c = (\frac{1}{2} - c_4, -\frac{1}{2} + c_4, c_4, c_4, -\frac{1}{2} - c_4, -\frac{1}{2} - c_4)$ . Next  $d_1 = d_2 = 2$  and Eq.(6.3) show  $c_4 = 3d_4/(2d_4 + 2)$  and  $c_1 = (1 - 2d_4)/(2d_4 + 2)$ .

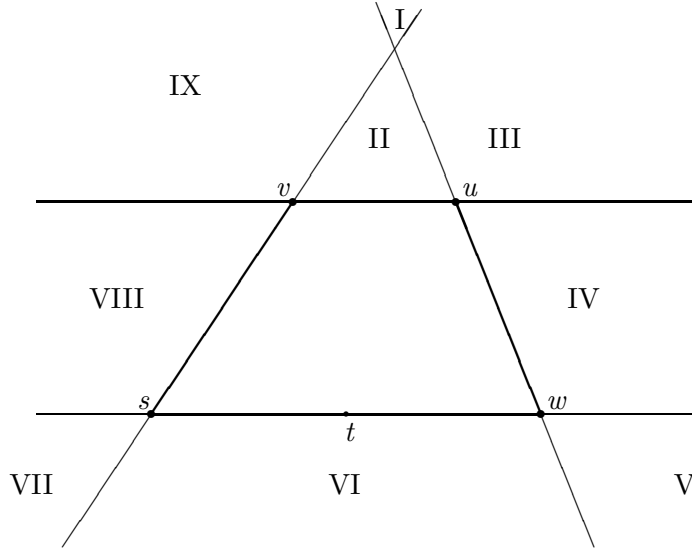
But the condition  $-\frac{1}{3} < c_1 < 0$  now implies  $d_3 = d_4 = 1$ , and it follows that  $c$  cannot be null.

(B) The argument if  $c$  faces  $vs$  is very similar. We have  $d_3 = d_4$  and  $d_5 = d_6 = 2$ , and the orthogonality equations imply  $c_3 = c_4$ . So  $c$  has the same form as in the second paragraph of (A) above. We find  $c_4 = -3d_4/(2d_4 + 2)$  and  $c_1 = (1 + 4d_4)/(2d_4 + 2)$ . But we now have the inequality  $1 < c_1 < \frac{4}{3}$ , so again  $d_3 = d_4 = 1$ , violating nullity.

(C) If  $c$  faces  $vu$  or  $sw$  then we need  $J(\bar{s}, \bar{w}) = 0$  (resp.  $J(\bar{v}, \bar{u}) = 0$ ), which is impossible.

**Example 6.8.** The example of the square (S) with midpoint can be treated in essentially the same way as the parallelograms. By symmetry, we may assume that  $c$  lies in the region that intersects  $uw$ . However, because  $\frac{1}{2}(a + a')$  may now be the midpoint and hence in  $\mathcal{W}$ , the configuration of Theorem 6.4(i) can occur, even though all vertices are type II. We have  $\mathcal{C} \cap E = \{\bar{c}, \bar{a}, \bar{a}', \bar{e}\}$  with  $a = (-1, -1, 1, 1, -1, \dots)$ ,  $a' = (1, -1, -1, -1, 1, \dots)$ ,  $c = (1, -1, -1, 1, -1, \dots)$ , and  $e = (-1, -1, 1, -1, 1, \dots)$  with nullity condition  $\sum_{i=1}^5 \frac{1}{d_i} = 1$ . We will be able to rule this case out in §7. On the other hand, the configuration of Theorem 6.4(ii) cannot occur, as one easily checks.

Next assume that adjacent (1B) vertices in  $\Delta^{\bar{c}}$  determine a trapezium  $vusw$  as shown in the diagram below:



where  $t$  is the midpoint of  $sw$  and  $vu$  is parallel to  $sw$ . We assume that  $v, u, s, w \in \mathcal{W}$  but our conclusions hold whether or not  $t$  lies in  $\mathcal{W}$ . We will now derive constraints on the 2-face and  $E \cap \mathcal{C}$  resulting from having  $c$  lie in one of the regions shown above. For theoretical considerations, we need only treat the cases where  $c$  lies in regions I to VI. In practice, for an asymmetric trapezium, we must consider  $c$  lying in the remaining regions as well. In the following we will adopt the convention that  $\bar{a}, \bar{a}'$  always denote null vectors in  $\mathcal{C}$ .

(I)  $c$  in region I: This is impossible because then  $\bar{s} = \frac{1}{2}(\bar{c} + \bar{a})$  and  $\bar{w} = \frac{1}{2}(\bar{c} + \bar{a}')$  for some  $\bar{a}, \bar{a}'$ , and so  $\bar{a}\bar{a}'$  would not intersect  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ , a contradiction to Cor 3.4.

(II)  $c$  in region II: Then  $\bar{v} = \frac{1}{2}(\bar{c} + \bar{a})$ ,  $\bar{u} = \frac{1}{2}(\bar{c} + \bar{a}')$  for some  $\bar{a}, \bar{a}'$ . We get a contradiction to Cor 3.4 if  $\bar{a}, \bar{a}'$  lie below the line  $sw$ . They also cannot lie on the line  $sw$  since the argument in (A) in the proof of Theorem 6.4 and Cor 3.4 imply that  $\mathcal{C} \cap E = \{\bar{c}, \bar{a}, \bar{a}'\}$ , and the terms corresponding to  $\bar{s}, \bar{w}$  in the superpotential equation would be unaccounted for.

Let  $e = 2s - a$ ,  $e' = 2w - a'$ . These points lie in region VI, and since we have a trapezium,  $e \neq e'$ . We may now apply Theorem 3.8 to  $\bar{a}$  and  $\bar{a}'$  to obtain the possibilities:

- (i)  $\bar{s}, \bar{w} \in \mathcal{C}$ ;  $J(\bar{a}, s) = 0 = J(\bar{a}', \bar{w})$ ,
- (ii)  $\bar{s} \in \mathcal{C}$ ,  $J(\bar{a}, \bar{s}) = 0$ ;  $w \notin \mathcal{C}$ ,  $\bar{e}' \in \mathcal{C}$  is null,  $J(\bar{e}', \bar{s}) = 0$ ,
- (iii)  $\bar{w} \in \mathcal{C}$ ,  $J(\bar{w}, \bar{a}') = 0$ ;  $\bar{s} \notin \mathcal{C}$ ,  $\bar{e} \in \mathcal{C}$  is null,  $J(\bar{e}, \bar{w}) = 0$ .

Note that the last condition in (ii) (resp. (iii)) results from applying Theorem 3.8 to  $\bar{e}'$  (resp.  $\bar{e}$ ).

(III) *c in region III*: We have  $\bar{v} = \frac{1}{2}(\bar{c} + \bar{a})$ ,  $\bar{w} = \frac{1}{2}(\bar{c} + \bar{a}')$  for some  $\bar{a}, \bar{a}'$  lying respectively in regions VIII and VI (in view of Cor 3.4). Applying Theorem 3.8 we obtain the possibilities:

- (i)  $\bar{s} \in \mathcal{C}$ ,  $J(\bar{a}, \bar{s}) = 0 = J(\bar{a}', \bar{s})$ ,
- (ii)  $\bar{s} \notin \mathcal{C}$ ,  $2\bar{s} = \bar{a} + \bar{a}'$  (which implies  $c + s = v + w$ ).

(IV) *c in region IV*: We have  $\bar{u} = \frac{1}{2}(\bar{c} + \bar{a})$ ,  $\bar{w} = \frac{1}{2}(\bar{c} + \bar{a}')$  for some  $\bar{a}, \bar{a}' \in \mathcal{C} \cap E$ .

If  $a$  lies in region IX, then Cor 3.4 implies that  $\bar{a}'$  lies in region VI. Applying Theorem 3.8 to  $\bar{a}$  and  $\bar{a}'$  we obtain the possibilities:

- (i)  $\bar{s} \in \mathcal{C}$ ,  $J(\bar{a}, \bar{s}) = 0 = J(\bar{a}', \bar{s})$ ,
- (ii)  $2s = a + a'$ , i.e.,  $c + s = u + w$ .

If  $a$  lies on the line  $sv$ , then we may apply Theorem 3.8 to  $\bar{a}'$ . We cannot have  $2\bar{s} = \bar{a}' + \bar{e}'$  with  $\bar{e}' \in \mathcal{C}$  and null, otherwise  $\bar{a}\bar{e}'$  would not intersect  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . So we have

- (iii)  $\bar{s} \in \mathcal{C}$  and  $J(\bar{s}, \bar{a}') = 0$ .

If  $a$  lies in region II, then  $\bar{a}'$  lies in region VI. Let  $\bar{e} = 2\bar{v} - \bar{a}$  and  $\bar{e}' = 2\bar{s} - \bar{a}'$ . As we have a trapezium,  $\bar{e} \neq \bar{e}'$ . Now  $e$  lies in region VII or VIII while  $e'$  lies in region VIII or IX, so by Cor 3.4  $\bar{e}$  and  $\bar{e}'$  cannot both lie in  $\mathcal{C}$  and hence be null. Theorem 3.8 now gives the possibilities:

- (iv)  $\bar{v}, \bar{s} \in \mathcal{C}$ ,  $J(\bar{a}, \bar{v}) = 0 = J(\bar{a}', \bar{s})$ , (and by Prop 3.7  $J(\bar{v}, \bar{s}) = 0$ ),
- (v)  $\bar{v} \in \mathcal{C}$ ,  $J(\bar{a}, \bar{v}) = 0$ ,  $\bar{e}' \in \mathcal{C}$  is null, and  $J(\bar{e}', \bar{v}) = 0$ ,
- (vi)  $\bar{s} \in \mathcal{C}$ ,  $J(\bar{a}', \bar{s}) = 0$ ,  $\bar{e} \in \mathcal{C}$  is null, and  $J(\bar{e}, \bar{s}) = 0$ .

(V) *c in region V*: We have  $\bar{u} = \frac{1}{2}(\bar{c} + \bar{a}')$ ,  $\bar{s} = \frac{1}{2}(\bar{c} + \bar{a})$  for some  $\bar{a}, \bar{a}'$  lying respectively in regions VIII and II (by Cor 3.4). Theorem 3.8 now gives the possibilities:

- (i)  $\bar{v} \in \mathcal{C}$ ,  $J(\bar{a}, \bar{v}) = 0 = J(\bar{a}', \bar{v})$ ,
- (ii)  $\bar{v} \notin \mathcal{C}$ ,  $2\bar{v} = \bar{a} + \bar{a}'$  (which implies  $c + v = u + s$ ).

(VI) *c in region VI*: We have  $\bar{s} = \frac{1}{2}(\bar{c} + \bar{a})$ ,  $\bar{w} = \frac{1}{2}(\bar{c} + \bar{a}')$  for some  $\bar{a}, \bar{a}'$  lying respectively in regions VIII and IV (by Cor 3.4). (To rule out  $\bar{a}, \bar{a}'$  lying in the line  $vu$ , we proceed as in case (II), except that when  $t \in \mathcal{W}$ , we conclude instead that  $\mathcal{C} \cap E = \{\bar{c}, \bar{a}, \bar{a}', \frac{1}{2}(\bar{a} + \bar{a}')\}$ . One can still check that  $\bar{v}, \bar{u}$  cannot be both accounted for.) Now let  $\bar{e} = 2\bar{v} - \bar{a}$  and  $\bar{e}' = 2\bar{u} - \bar{a}'$ . Again, having a trapezium means  $\bar{e} \neq \bar{e}'$  and Theorem 3.8 now gives the possibilities:

- (i)  $\bar{u}, \bar{v} \in \mathcal{C}$ ,  $J(\bar{a}, \bar{v}) = 0 = J(\bar{a}', \bar{u})$ , (and  $J(\bar{u}, \bar{v}) = 0$  by Prop 3.7),
- (ii)  $\bar{v} \in \mathcal{C}$ ,  $J(\bar{a}, \bar{v}) = 0$ ,  $\bar{u} \notin \mathcal{C}$ ,  $\bar{e}' \in \mathcal{C}$  is null,
- (iii)  $\bar{u} \in \mathcal{C}$ ,  $J(\bar{a}', \bar{u}) = 0$ ,  $\bar{v} \notin \mathcal{C}$ ,  $\bar{e} \in \mathcal{C}$  is null.

**Remark 6.9.** We mention a useful inequality which holds in (II) and (VI) above, as well as in parallelogram faces with the same configuration (cf Example 6.7(A)).

Let us consider (II), where we choose in  $E$  coordinates such that the first coordinate axis is parallel to  $\bar{s}\bar{w}$  (assumed to be horizontal) and the second coordinate axis is arbitrary, with the second coordinate increasing as we go up. As in (A) in the proof of Theorem 6.4, all points in  $(\mathcal{C} \cap E) \setminus \{\bar{c}, \bar{a}, \bar{a}'\}$  must lie below the line  $\bar{a}\bar{a}'$ . Let  $\bar{b}$  be a point among these with largest second coordinate. Since we have seen above that either  $\bar{s}$  or  $\bar{w}$  lies in  $\mathcal{C} \cap E$ , we have  $s_2 \leq b_2$ . Furthermore, as  $\bar{b} + \bar{c}$  cannot lie in  $d + \mathcal{W}$  it must be balanced by sums of elements in  $\mathcal{C} \cap E$ , with the limiting configuration given by  $\bar{a} + \bar{a}'$ . So we have  $\frac{1}{2}(b_2 + c_2) \leq a_2 = a'_2 = 2v_2 - c_2$ . Combining the two inequalities we get  $3c_2 \leq 4v_2 - s_2$ .

Equality in the above holds iff  $\bar{b}$  lies in  $\bar{s}\bar{w}$  and  $\bar{b} + \bar{c} = \bar{a} + \bar{a}'$ . In particular,  $\bar{b}$  is unique, so in II(i), the inequality above is strict.



Note that we only need  $\bar{v}\bar{u}$  and  $\bar{s}\bar{w}$  to be parallel and the presence or absence of  $t$  in  $\mathcal{W}$  is immaterial. Hence in Theorem 6.4(ii) we also have an analogous strict inequality, which we have already used, e.g., in (B) of Example 6.7. (For a parallelogram, there may be midpoints on the pair of non-horizontal sides lying in  $\frac{1}{2}(d + \mathcal{W})$ , but  $\frac{1}{2}(\bar{b} + \bar{c})$  can never equal these midpoints, so we still get the inequality we want.)

For the configuration in (VI), we still have an analogous inequality, but since  $\frac{1}{2}(\bar{a} + \bar{a}') \in \mathcal{C}$ , we lose uniqueness of  $\bar{b}$  and hence the strict inequality.

We will also have occasion to apply the above analysis to appropriate trapezoidal regions in hexagon (H3).

The method described above together with Remark 6.9 can now be used to rule out the trapezia (T1)-(T6) as well as those mentioned in Remark 6.2.

**Example 6.10.** For the trapezium (T3), the vectors  $v, u, s, w$  are given in Table 4, and lie in the 2-plane  $\{x_1 + x_2 + x_3 + x_4 = -1, x_2 + 2x_4 = 1\}$ .  $vu$  is given by  $x_4 = 1$  while  $sw$  is given by  $x_4 = 0$ .  $sv$  is given by  $x_3 = 0$  and  $wu$  is given by  $x_1 = 0$ . The vector  $c$  that we are looking for has the form  $(-c_3 + c_4 - 2, 1 - 2c_4, c_3, c_4)$ . Since the trapezium is symmetric, an explicit symmetry being induced by interchanging  $x_1$  and  $x_3$ , we need only consider  $c$  lying in regions II-VI.

(A) If  $c$  lies in region III, then  $c_1 > 0$ ,  $c_4 > 1$ . Since  $a = 2v - c$ , we obtain  $a = (c_3 - c_4, -3 + 2c_4, -c_3, 2 - c_4)$ . Similarly,  $a' = (c_3 - c_4 + 2, 1 + 2c_4, -4 - c_3, -c_4)$ . If we are in case (ii), then  $c = v + w - s = (1, -1, -2, 1)$ , which violates  $c_4 > 1$ . So we must be in case (i).

It follows from  $J(\bar{a}, \bar{s}) = 0 = J(\bar{a}', \bar{s})$  that  $d_1 + 3 = -2c_3 + 4c_4$  and  $J(\bar{w}, \bar{s}) = J(\bar{v}, \bar{s})$ . The second equality implies that  $d_1 = d_2$ . Using this together with the first equality and the null condition for  $\bar{a}'$  (in the form  $J(\bar{w}, \bar{w}) = J(\bar{w}, \bar{c})$ , see Remark 3.9) we get  $c_4 = d_1(d_1 - 1)/(4d_1 + 2d_3)$ . Since  $c_4 > 1$ , we have  $d_1(d_1 - 5) > 2d_3$ , so  $d_1 > 5$ . But by Remark 3.1,  $J(\bar{s}, \bar{s}) < 0$ , which gives  $d_1 < 5$  (since  $d_1 = d_2$ ), a contradiction.

(B) Let  $c$  lie in region IV, so that  $c_1 > 0$ ,  $0 < c_4 < 1$ . We obtain  $a = (2 + c_3 - c_4, 2c_4 - 3, -2 - c_3, 2 - c_4)$  and  $a' = (2 + c_3 - c_4, 1 + 2c_4, -4 - c_3, -c_4)$ . We claim that  $a_3 > 0$ , so that  $a$  lies in region IX. To see this, we solve for  $c_3, c_4$  using the null conditions  $J(\bar{u}, \bar{u}) = J(\bar{c}, \bar{u})$  and  $J(\bar{w}, \bar{w}) = J(\bar{c}, \bar{w})$  for  $\bar{a}, \bar{a}'$  respectively. We obtain  $c_4 = (d_2d_3 + 2d_3d_4 - d_2d_4)/(d_3(d_2 + 3d_4))$  and  $a_3 = -2 - c_3 = (d_2d_3 + 2d_3d_4 - d_2d_4)/(d_2(d_2 + 3d_4))$ . Since  $c_4 > 0$  we obtain our claim.

Since  $a$  lies in region IX, we first check if (ii) holds. In this case,  $c = (2, -1, -3, 1)$  which contradicts  $c_4 < 1$ . The equations in (i) together imply the contradiction  $0 = -4/d_2$ .

(C) Suppose  $c$  lies in region V, so that  $c_1 > 0$ ,  $c_4 < 0$ . We obtain  $a = (c_3 - c_4 - 2, 1 + 2c_4, -c_3, -c_4)$  and  $a' = (c_3 - c_4 + 2, 2c_4 - 3, -2 - c_3, 2 - c_4)$ . If (ii) holds then  $c = (-1, 1, -1, 0)$  and this contradicts  $c_1 > 0$ . Hence (i) must hold.

By Remark 3.9, the null condition for  $\bar{a}$  is  $J(\bar{s}, \bar{s}) = J(\bar{s}, \bar{c})$ , which is  $\frac{c_3}{d_1} - (\frac{1}{d_1} + \frac{1}{d_2})c_4 = 0$ . The two equations in (i) imply  $J(\bar{u}, \bar{v}) = J(\bar{s}, \bar{v})$  and  $J(\bar{v}, \bar{v}) < 0$ , which in turn give  $d_1 = 2$ . Using this, the null condition for  $\bar{a}$ , and  $J(\bar{a}', \bar{v}) = 0$  we obtain  $c_4 = \frac{-1}{1+d_2}$  and  $c_3 = -\frac{d_2+2}{d_2(d_2+1)}$ . But  $c_1 = c_4 - c_3 - 2 > 0$ , which simplifies to  $1 > d_2(d_2 + 1)$ , a contradiction.

(D) Let  $c$  lie now in region II. Then  $c_1 < 0$ ,  $c_3 < 0$ ,  $1 < c_4 \leq \frac{4}{3}$  where the last upper bound comes from the inequality in Remark 6.9. We obtain  $a = (c_3 - c_4, 2c_4 - 3, -c_3, 2 - c_4)$  and  $a' = (2 + c_3 - c_4, 2c_4 - 3, -2 - c_3, 2 - c_4)$ . The null conditions for  $\bar{a}, \bar{a}'$  then give

$$c_3 = -\frac{2d_1d_4 + d_1d_2 - d_2d_4}{(d_1 + d_3)(d_2 + 2d_4) - d_2d_4}, \quad c_4 = \frac{(d_1 + d_3)(d_2 + 2d_4)}{(d_1 + d_3)(d_2 + 2d_4) - d_2d_4}.$$

Suppose we are in case (i). The two equations and the above values of  $c_3, c_4$  combine to give  $(d_1 - d_3)((d_1 + d_3)(d_2 + 2d_4) - d_2d_4) = 0$ . However, the upper bound  $c_4 \leq \frac{4}{3}$  translates into  $(d_1 + d_3)(d_2 + 2d_4) \geq 4d_2d_4$ . So the second factor is positive and we have  $d_1 = d_3$ . Putting this

information into the equation  $J(\bar{a}, \bar{s}) = 0$ , we get

$$d_1 d_2 d_4 (d_2 + 15) = 2d_1^2 d_2 (d_2 + 1) - 6d_1 d_2^2 + 2d_4 (2d_1^2 d_2 + 2d_1^2 + d_2^2).$$

By Remark 3.1 we also have  $J(\bar{s}, \bar{s}) < 0$ , i.e.,  $1 < \frac{4}{d_1} + \frac{1}{d_2}$ , so either  $d_2 = 1$  or  $d_1 < 8$ . Substituting these values into the equation above and using  $c_4 \leq \frac{4}{3}$  we obtain in each instance a contradiction.

If we are in case (ii), then by adding the equations  $J(\bar{a}, \bar{s}) = 0$  and  $2J(\bar{w}, \bar{s}) - J(\bar{a}', \bar{s}) = 0$  (equivalent to  $J(\bar{e}', \bar{s}) = 0$ ), we obtain  $1 = \frac{2}{d_1} + \frac{1}{d_2}$ . Hence  $(d_1, d_2) = (4, 2)$  or  $(3, 3)$ . One then checks that these values are incompatible with the null condition for  $\bar{e}'$ ,  $J(\bar{a}, \bar{s}) = 0$ , and the bound  $c_3 < 0$ .

An analogous argument works to eliminate case (iii), where we now need the bound  $c_4 \leq \frac{4}{3}$  instead.

(E) Lastly suppose  $c$  lies in region VI, so  $c_1, c_3 < 0$  and  $-\frac{1}{3} \leq c_4 < 0$ , where the lower bound for  $c_4$  results from Remark 6.9. We have  $a = (c_3 - c_4 - 2, 1 + 2c_4, -c_3, -c_4)$  and  $a' = (2 + c_3 - c_4, 1 + 2c_4, -4 - c_3, -c_4)$ . Using the null conditions for  $\bar{a}, \bar{a}'$ , we obtain

$$c_1 = -\frac{2(d_2 + d_3)}{d_1 + d_2 + d_3}, \quad c_2 = \frac{d_1 + 5d_2 + d_3}{d_1 + d_2 + d_3}, \quad c_3 = \frac{-2(d_1 + d_2)}{d_1 + d_2 + d_3}, \quad c_4 = \frac{-2d_2}{d_1 + d_2 + d_3}.$$

If we are in case (i),  $J(\bar{u}, \bar{v}) = 0$  gives  $d_2 = d_4 = 2$ . The other two equations and the above values of  $c_3, c_4$  then give  $3(d_1 + d_3 + 2)(d_1 + d_3 - 4) = 4(3d_1 + 3d_3 - 2)$ . The lower bound  $-\frac{1}{3} \leq c_4$  becomes  $d_1 + d_2 + d_3 \geq 6d_2$ . Using this inequality in the above Diophantine relation leads to a contradiction. (Alternatively, observe the relation is a quadratic in  $d_1 + d_3$  with no rational roots).

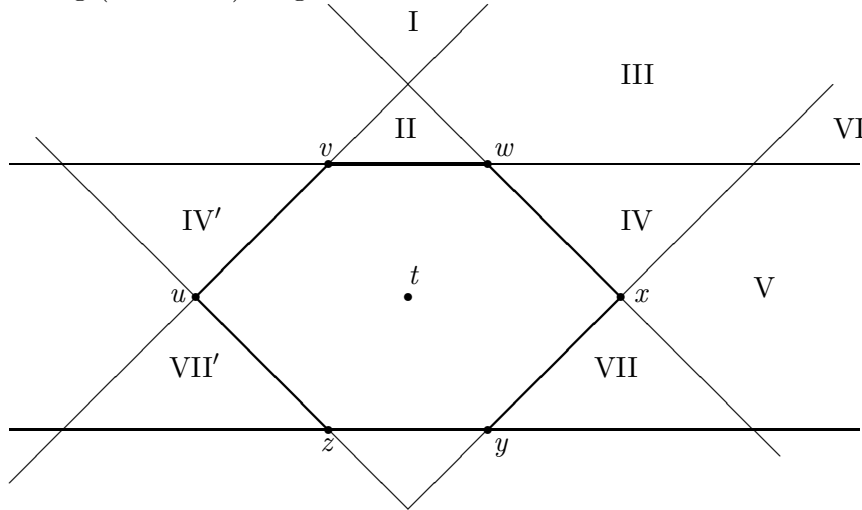
For case (ii), using the two equations and the above values for  $c_3, c_4$ , we arrive at the relation

$$(d_1 + d_2 + d_3)((d_1 - 5)d_2 d_4 + d_1 d_4 + d_2 d_3 + 2d_3 d_4) = 2d_2(d_1 d_2 + 2d_1 d_4 - d_2 d_4 + d_2 d_3 + 2d_3 d_4).$$

Using the lower bound  $-\frac{1}{3} \leq c_4$  in the above relation we see that  $d_1 \leq 3$ . By direct substitution, we further obtain  $d_1 \neq 3$ . Finally, if  $d_1 = 2$ , the null condition for  $\bar{c}$  gives  $1 > \frac{1}{2}c_1^2$  and so  $d_2 + d_3 \leq 4$ . The lower bound on  $c_4$  now implies  $d_2 = 1$ . Since  $c_2 > 1$ , the null condition for  $\bar{c}$  is violated.

Case (iii) reduces to case (ii) upon interchanging the first and third summands. Therefore, the trapezium (T3) has been eliminated.

We discuss next the hexagons (H1)-(H3). As the three cases are similar, we will focus on (H3) and refer to the following (schematic) diagram:



**Example 6.11.** The hexagon (H3) lies in the 2-plane given by  $\{x_2 = -1, x_1 + x_3 + x_4 = 0\}$ . So  $c$  has the form  $(-c_3 - c_4, -1, c_3, c_4)$ . The lines  $vw$  and  $zy$  are given respectively by  $x_1 + x_3 = 1$  and

$x_1 + x_3 = -1$ . Similarly, the lines  $uv$  and  $yx$  are given by  $x_3 = 1$  and  $x_3 = -1$  respectively. The lines  $uz$  and  $wx$  are given by  $x_1 = -1$  and  $x_1 = 1$  respectively.

Interchanging  $x_1$  and  $x_3$  induces the reflection about the perpendicular bisector of  $vw$ , while  $(x_1, x_2, x_3, x_4) \mapsto (-x_3, x_2, -x_1, -x_4)$  induces the reflection about  $ux$ . These symmetries reduce our consideration to those  $c$  lying in regions I-VI. Moreover, (H3) is actually a regular hexagon. The symmetry  $(x_1, x_2, x_3, x_4) \mapsto (-x_4, x_2, -x_3, -x_1)$  induces the reflection about  $zw$ , which swaps region II with region IV and region I with region VI. Finally, the symmetry  $(x_1, x_2, x_3, x_4) \mapsto (-x_3, x_2, -x_4, -x_1)$  induces the rotation in  $E$  about  $t$  taking  $x$  to  $w$ , and maps region V to region III. Therefore, we need only consider  $c$  lying in regions I, II, and V.

In the discussion below we again adopt the convention that  $\bar{a}, \bar{a}'$  always denote null vectors in  $\mathcal{C}$ .

If  $c$  lies in region I, then  $\bar{u} = \frac{1}{2}(\bar{c} + \bar{a})$ ,  $\bar{x} = \frac{1}{2}(\bar{c} + \bar{a}')$  for some  $\bar{a}, \bar{a}'$ , and we immediately see that  $\bar{a}\bar{a}'$  cannot meet  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ , a contradiction to Cor 3.4.

*c lying in region II:*

We have  $c_1, c_3 < 1$  and  $c_1 + c_3 > 1$ . The assumption of adjacent (1B) vertices means that  $\bar{v} = \frac{1}{2}(\bar{c} + \bar{a})$  and  $\bar{w} = \frac{1}{2}(\bar{c} + \bar{a}')$  for some  $\bar{a}, \bar{a}' \in E \cap \mathcal{C}$ . Hence  $a = (c_3 + c_4, -1, 2 - c_3, -2 - c_4)$  and  $a' = (2 + c_3 + c_4, -1, -c_3, -2 - c_4)$ . One checks easily that  $\bar{a}'$  lies in region IV and  $\bar{a}$  lies in region IV'. Moreover, the null conditions for these vectors yield

$$c_1 = \frac{d_3 + d_4}{d_1 + d_3 + d_4}, \quad c_3 = \frac{d_1 + d_4}{d_1 + d_3 + d_4}, \quad c_4 = -\frac{d_1 + d_3 + 2d_4}{d_1 + d_3 + d_4}.$$

Let  $e := 2u - a$  and  $e' := 2x - a'$ . These lie respectively in regions VII' and VII. We can now apply Theorem 3.8 to  $\bar{a}$  and  $\bar{a}'$  to obtain the following possibilities:

- (i)  $\bar{u}, \bar{x} \in \mathcal{C}$  and  $J(\bar{a}, \bar{u}) = 0 = J(\bar{a}', \bar{x})$ ;
- (ii)  $\bar{u} \in \mathcal{C}$ ,  $J(\bar{a}, \bar{u}) = 0$ ,  $\bar{x} \notin \mathcal{C}$ ,  $\bar{e}' \in \mathcal{C}$  is null;
- (iii)  $\bar{x} \in \mathcal{C}$ ,  $J(\bar{a}', \bar{x}) = 0$ ,  $\bar{u} \notin \mathcal{C}$ ,  $\bar{e} \in \mathcal{C}$  is null;
- (iv)  $\bar{u}, \bar{x} \notin \mathcal{C}$ ,  $\bar{e}, \bar{e}'$  are both null.

We can eliminate (i)-(iii) by noting that the two equations in each case together with the values of  $c_3, c_4$  above imply that  $1 = \frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3}$ . Using this relation (and the values of  $c_3, c_4$ ) in the null condition for  $\bar{c}$  then leads to a contradiction.

For case (iv) we can again apply Theorem 3.8 to the null vertices  $\bar{e}$  and  $\bar{e}'$ . The conditions  $J(\bar{e}, \bar{z}) = 0$  and  $J(\bar{e}', \bar{y}) = 0$  lead, as above, to  $1 = \frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_4}$  and  $1 = \frac{1}{d_2} + \frac{1}{d_3} + \frac{1}{d_4}$  respectively. Using this in the null condition for  $\bar{c}$  again leads to a contradiction. Hence  $\bar{z}, \bar{y} \notin \mathcal{C}$  and  $\bar{q} := 2\bar{z} - \bar{e}$  and  $\bar{q}' := 2\bar{y} - \bar{e}'$  are null vectors in  $E \cap \mathcal{C}$ . In fact we now find that  $q = q'$ , so  $caeqe'a'$  is a hexagon circumscribing (H3).

Let us consider the pair of null vertices  $\bar{c}, \bar{q}$ . We apply the argument in (A) of the proof of Theorem 6.4 to the wedge with vertex  $\bar{c}$  bounded by the rays  $\bar{c}\bar{a}$  and  $\bar{c}\bar{a}'$ . All elements of  $(\mathcal{C} \cap E) \setminus \{\bar{a}, \bar{a}', \bar{c}\}$  lie below the line  $\bar{a}\bar{a}'$ . Let  $\bar{b}$  be a highest (with respect to  $x_1 + x_3$ ) element among these. Since  $\bar{e} \in \mathcal{C}$ ,  $b_1 + b_3 > -1$  and so  $\bar{c} + \bar{b}$  cannot equal  $2\bar{u}, 2\bar{t}, 2\bar{x}$ . Hence  $\bar{c} + \bar{b} = \bar{a} + \bar{a}'$ , and we compute that  $b_1 + b_3 = \frac{d_1 + d_3 - 2d_4}{d_1 + d_3 + d_4}$ . The analogous argument applied to the wedge bounded by the rays  $\bar{q}\bar{e}$  and  $\bar{q}\bar{e}'$  gives a lowest element  $\bar{b}'$  of  $(\mathcal{C} \cap E) \setminus \{\bar{q}, \bar{e}, \bar{e}'\}$  satisfying  $\bar{b}' + \bar{q} = \bar{e} + \bar{e}'$  and  $b'_1 + b'_3 = \frac{2d_4 - d_1 - d_3}{d_1 + d_3 + d_4}$ . To avoid a contradiction, we must have  $d_1 + d_3 \geq 2d_4$ .

We can repeat the above argument with the null vertex pairs  $\{\bar{e}, \bar{a}'\}$  and  $\{\bar{e}', \bar{a}\}$ , obtaining the inequalities  $d_3 + d_4 \geq 2d_1$  and  $d_1 + d_4 \geq 2d_3$  respectively. The three inequalities then imply that in fact  $d_1 = d_3 = d_4$  and  $c = (\frac{2}{3}, -1, \frac{2}{3}, -\frac{4}{3})$ . Furthermore,  $\mathcal{C} \cap E = \{\bar{a}, \bar{a}', \bar{c}, \bar{e}, \bar{e}', \bar{t}, \bar{q}\}$  and the null condition for  $\bar{c}$  gives  $(d_1, d_2) = (3, 9)$  or  $(4, 3)$ .

By looking at the terms in the superpotential equation corresponding to the vertices (all of type II), we find that the coefficients  $F_{\bar{c}}, F_{\bar{e}}, F_{\bar{e}'}$  have the same sign, which is opposite to that of  $F_{\bar{a}}, F_{\bar{a}'}, F_{\bar{q}}$ . Next we note that the only ways to write  $d + (\frac{1}{3}, -1, \frac{1}{3}, -\frac{2}{3})$  (resp.  $d + (-\frac{1}{3}, -1, \frac{2}{3}, -\frac{1}{3})$ ) as a sum of element of  $\mathcal{C}$  are  $\bar{t} + \bar{c} = \bar{a} + \bar{a}'$  (resp.  $\bar{t} + \bar{a} = \bar{c} + \bar{e}$ ). The superpotential equation then

gives  $F_{\bar{a}}F_{\bar{a}'}J(\bar{a}, \bar{a}') + F_{\bar{t}}F_{\bar{e}}J(\bar{t}, \bar{e}) = 0$  and  $F_{\bar{e}}F_{\bar{e}'}J(\bar{e}, \bar{e}') + F_{\bar{t}}F_{\bar{a}}J(\bar{t}, \bar{a}) = 0$ . Since  $J(\bar{a}, \bar{a}')$ ,  $J(\bar{e}, \bar{e}')$ ,  $J(\bar{t}, \bar{e})$  and  $J(\bar{t}, \bar{a})$  are all positive, the above equations and facts imply that  $F_{\bar{e}}$  and  $F_{\bar{a}}$  have the same sign, a contradiction.

So  $c$  cannot lie in region II.

*c lying in region V:*

We have  $c_3 < -1 < -c_4 < 1 < c_1$ . The adjacent (1B) vertices assumption implies that  $\bar{w} = \frac{1}{2}(\bar{c} + \bar{a})$  and  $\bar{y} = \frac{1}{2}(\bar{c} + \bar{a}')$  for some  $\bar{a}, \bar{a}' \in \mathcal{C} \cap E$ . It follows that  $a = (2 + c_3 + c_4, -1, -c_3, -2 - c_4)$  and  $a' = (c_3 + c_4, -1, -2 - c_3, 2 - c_4)$ . The null conditions on these vectors give

$$c_1 = \frac{(2d_1 + d_4)(d_3 + d_4)}{d_4(d_1 + d_3 + d_4)}, \quad c_3 = -\left(1 + \frac{d_1}{d_4}\right) \left(\frac{2d_3 + d_4}{d_1 + d_3 + d_4}\right), \quad c_4 = \frac{d_3 - d_1}{d_1 + d_3 + d_4}.$$

Since  $a_3 = -c_3 > 1$ ,  $a$  lies above the line  $uv$ . Also,  $a'_1 = c_3 + c_4 = -c_1 < -1$ , so  $a'$  lies below the line  $uz$ . We can therefore apply Theorem 3.8 to  $\bar{a}$  and  $\bar{a}'$  to get the following possibilities:

- (i)  $\bar{u} \in \mathcal{C}$ ,  $J(\bar{a}, \bar{u}) = 0 = J(\bar{a}', \bar{u})$ ,
- (ii)  $\bar{u} \notin \mathcal{C}$ ,  $e := 2u - a$ ,  $e' := 2u - a'$  lie in  $\mathcal{C} \cap E$  and are null.

If (i) occurs, then the two orthogonality conditions imply that  $d_1 = d_3$ , so  $c_4 = 0$ ,  $c_1 = -c_3 = 1 + \frac{d_1}{d_4}$ . Substituting these values of  $c_i$  into  $J(\bar{a}', \bar{u}) = 0$  gives  $1 = \frac{2}{d_4} + \frac{1}{d_2}$ . But the null condition for  $\bar{e}$  is  $1 = \frac{1}{d_2} + \frac{2}{d_1}(1 + \frac{d_1}{d_4})^2 > \frac{1}{d_2} + \frac{2}{d_4} = 1$ , which is a contradiction.

Hence (ii) must occur. Note that if the above diagram is rotated so that the lines  $x_1 + x_3 = \kappa$  (for arbitrary constants  $\kappa$ ) are horizontal, then the lines  $x_1 - x_3 = \kappa$  would be vertical.  $u$  is the only point in the hexagon lying on  $x_1 - x_3 = -2$ . Observe that  $a_1 - a_3 = a'_1 - a'_3 \geq -2$ , otherwise  $\bar{a}\bar{a}'$  would not intersect  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ , which contradicts Cor 3.4. If, however,  $a_1 - a_3 > -2$ , then  $\bar{e}\bar{e}'$  would not intersect  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . So in fact  $a = e'$ ,  $e = a'$  and  $u$  all lie on  $x_1 - x_3 = -2$ . In other words, the hexagon is circumscribed by the triangle  $caa'$  with intersections at  $w, u$  and  $y$ .

It follows easily from the above that  $c = (2, -1, -2, 0)$ ,  $d_1 = d_3 = d_4$ , and the null condition for  $\bar{e}$  is  $1 = \frac{8}{d_1} + \frac{1}{d_2}$ . Also, we have  $\mathcal{C} \cap E = \{\bar{e}, \bar{a}, \bar{a}', \bar{t}\}$ . Since  $w, u, y$  are type II, by Lemma 3.2, we see that the signs of  $F_{\bar{a}}$ ,  $F_{\bar{e}}$ , and  $F_{\bar{a}'}$  in the superpotential equation cannot be chosen compatibly. We have thus shown that the hexagon (H3) cannot occur.

The hexagon (H2) is not regular, but has reflection symmetry about  $uv$  and the perpendicular bisector of  $yy'$ . It can be eliminated by similar arguments, but we now have to consider  $c$  lying in regions III and IV as well. The hexagon (H1) can also be eliminated by the above methods. Here the hexagon is invariant under the symmetric group permuting the coordinates  $x_1, x_2, x_3$ . Together with Cor 3.4, this fact reduces our consideration to those  $c$  lying in three of the regions formed by extending the sides of the hexagon.

As mentioned in Remark 6.2, we also need to rule out subshapes of the hexagons. For (H2) and (H3) the methods used above can also be applied to rule out all the sub-parallelograms and trapezia except the rectangle  $yy'z'z$  of (H2) (see Lemma 8.5 and the discussion immediately before Ex 6.7). All sub-triangles will be dealt with at the end of this section. (There is a triangle with midpoint in (H2) but that can be dealt with by similar methods.) For (H2) this leaves the pentagon  $yy'vz'z$  and the kite  $y'uz'v$ , both of which can still be eliminated using the above methods.

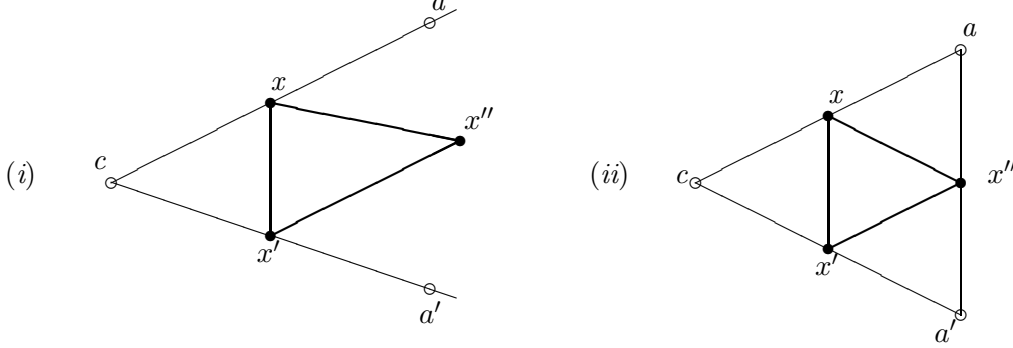
The possible subshapes of (H1) are rather numerous. However, if  $r \geq 4$  we will be able to eliminate all of them in Lemma 8.6. Without this assumption, the above methods can be used to eliminate those subshapes which do not contain all three type I vectors. Of course the following discussion will handle the sub-triangles.

Lastly, we consider triangular faces.

**Theorem 6.12.** *Suppose we have adjacent (1B) vertices in  $\Delta^{\bar{e}}$  corresponding to a triangular face  $\bar{x}\bar{x}'\bar{x}''$  of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . Let  $E$  be the affine 2-plane determined by the triangular face. So there are null vectors  $\bar{a}, \bar{a}'$  in  $\mathcal{C} \cap E$  such that  $x = \frac{1}{2}(a + c)$ ,  $x' = \frac{1}{2}(a' + c)$ .*

Suppose the vertices of the triangle are the only elements of  $\mathcal{W}$  in the face. Then we are in one of the following two situations:

- (i)  $\mathcal{C} \cap E = \{\bar{c}, \bar{a}, \bar{a}', \bar{x}''\}$ , with  $c + x'' = a + a'$  and  $J(\bar{x}'', \bar{a}) = J(\bar{x}'', \bar{a}') = 0$ ;
- (ii)  $\mathcal{C} \cap E = \{\bar{c}, \bar{a}, \bar{a}'\}$  where  $\frac{1}{2}(a + a') = x''$ , one of  $x, x', x''$  is type I, and the others are either both type I or both type II/III.



*Proof.*

(A) We may introduce coordinates in  $E$  so that  $\bar{x}\bar{x}'$  is vertical and to the right of  $\bar{c}$ . As  $\bar{a}\bar{a}'$  must meet  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ , we see  $\bar{x}''$  is on or to the right of  $\bar{a}\bar{a}'$ . Let  $\bar{b}$  be any leftmost point of  $(\mathcal{C} \cap E) \setminus \{\bar{c}\}$ . As in Theorem 6.4, we see that  $\bar{b} + \bar{c} \in d + \mathcal{W}$ , so all elements of  $\mathcal{C} \cap E$  except  $\bar{c}, \bar{a}, \bar{a}'$  are to the right of  $\bar{a}\bar{a}'$ .

(B) Considering  $\bar{a}\bar{x}''$  and  $\bar{a}'\bar{x}''$  we see (using Theorem 3.8 and Cor 3.4) that either

- (1)  $\bar{x}'' \in \mathcal{C}$  and  $J(\bar{x}'', \bar{a}) = 0 = J(\bar{x}'', \bar{a}')$ , or
- (2)  $\bar{x}'' \notin \mathcal{C}$  and  $x'' = \frac{1}{2}(a + a')$ .

In case (1),  $(\bar{x}'')^\perp \cap E$  is the line through  $\bar{a}\bar{a}'$ . By Prop. 3.3 and Cor. 3.4, observe that all elements of  $(\mathcal{C} \cap E) \setminus \{\bar{x}''\}$  are left of  $\bar{x}''$ . Let  $\bar{b}$  be a rightmost element of  $(\mathcal{C} \cap E) \setminus \{\bar{x}''\}$ . So either  $J(\bar{b}, \bar{x}'') = 0$  or  $\bar{b} + \bar{x}'' \in d + \mathcal{W}$ . Since  $\bar{b}$  is not to the left of  $\bar{a}\bar{a}'$ , the second alternative cannot hold and so  $\bar{b}$  must lie on  $\bar{a}\bar{a}'$ . Combining this with our results in (A), we see  $\mathcal{C} \cap E$  is as in (i). Also, as  $J(\bar{a}, \bar{a}') > 0$  and  $\bar{a} + \bar{a}' \notin d + \mathcal{W}$ , we see  $a + a'$  must equal  $c + x''$ .

In case (2), by Cor 3.4 there are no elements of  $\mathcal{C} \cap E$  right of  $\bar{a}\bar{a}'$ . Hence  $\mathcal{C} \cap E$  is as in (ii). Now  $J(\bar{b}, \bar{e}) > 0$  for all  $\bar{b} \neq \bar{e}$  in  $\mathcal{C} \cap E$ , so the last statement of (ii) follows.  $\square$

**Remark 6.13.** We must also consider the case when some midpoints of the sides of our triangular face lie in  $\mathcal{W}$ . (This could happen if two vertices were  $(1, -1, -1, \dots)$ ,  $(-1, 1, -1, \dots)$  or  $(1, -2, \dots)$ ,  $(1, 0, -2, \dots)$  or  $(1, -2, \dots)$ ,  $(-1, 0, \dots)$ .) Let us denote the midpoints of  $xx'$ ,  $xx''$  and  $x'x''$  respectively by  $z, y, t$ .

If  $z$  is absent, the arguments of (A) in the proof of Theorem 6.12 still hold, so we have the alternatives (1),(2) in (B). If (1) holds then, choosing  $\bar{b}$  as above, if  $b$  is right of  $aa'$ , we have  $\frac{1}{2}(b + x'') \in \mathcal{W}$ . This gives a contradiction since  $\frac{1}{2}(b + x'')$  cannot be  $y$  or  $t$  as  $b \neq x, x'$ . Now  $\mathcal{C} \cap E = \{\bar{c}, \bar{a}, \bar{a}', \bar{x}''\}$ , and as  $c + x'' \notin 2\mathcal{W}$  it must equal  $a + a'$ . It follows that the midpoints  $y, t$  cannot arise. If instead (2) holds, then  $\mathcal{C} \cap E = \{\bar{c}, \bar{a}, \bar{a}'\}$  and again no midpoints can be present.

Suppose now the midpoint  $z$  of  $xx'$  is present. The argument of (A) shows that to account for  $z$ ,  $\frac{1}{2}(\bar{a} + \bar{a}') \in \mathcal{C}$ , and all elements of  $(\mathcal{C} \cap E) \setminus \{\bar{c}, \bar{a}, \bar{a}', (\bar{a} + \bar{a}')/2\}$  are right of  $\bar{a}\bar{a}'$ . We still have the alternatives (1) and (2), but (2) immediately gives a contradiction.

In (1) we see as before there are no elements of  $\mathcal{C} \cap E$  lying to the right of  $\bar{a}\bar{a}'$ , so  $\mathcal{C} \cap E = \{\bar{c}, \bar{x}'', \bar{a}, \bar{a}', (\bar{a} + \bar{a}')/2\}$ . Note that  $J(\bar{a}, (\bar{a} + \bar{a}')/2)$  and  $J(\bar{a}', (\bar{a} + \bar{a}')/2) > 0$ .

If  $c + x'' = a + a'$ , we find after some algebra that  $\bar{a} + (\bar{a} + \bar{a}')/2 \neq 2\bar{y}$  and also cannot be written as a different sum of elements of  $\mathcal{C}$ , giving a contradiction.

If  $\bar{c} + \bar{x}'' \neq \bar{a} + \bar{a}'$  then one sees that  $\bar{c} + \bar{x}'' \notin d + \mathcal{W}$ , and by relabelling  $x$  and  $x'$ ,  $a$  and  $a'$  we may assume that  $\bar{c} + \bar{x}'' = \bar{a} + \frac{1}{2}(\bar{a} + \bar{a}')$  and also  $\bar{a} + \bar{a}' = 2\bar{y}$  and  $\bar{a}' + \frac{1}{2}(\bar{a} + \bar{a}') = 2\bar{t} = \bar{x}' + \bar{x}''$ . These relations imply  $a = x'$ , a contradiction. So no triangle with any midpoints present can arise.

**Remark 6.14.** There are also triangular faces with two points of  $\mathcal{W}$  in the interior of an edge. This can only happen if two vertices are  $(-2, 1, 0, \dots)$  and  $(1, -2, 0, \dots)$  (up to permutation). The other sides of the triangle now have no interior points in  $\mathcal{W}$  unless the triangle is contained in the hexagon (H1). We can again modify the proof of Theorem 6.12 to treat this situation.

If the interior points  $z, w$  lie on  $xx'$ , then  $(2\bar{a} + \bar{a}')/3, (\bar{a} + 2\bar{a}')/3$  must be in  $\mathcal{C}$ , and all points of  $\mathcal{C} \cap E$  except for these two and  $\bar{c}, \bar{a}, \bar{a}'$  lie to the right of  $\bar{a}\bar{a}'$ . By Prop 3.3, alternative (1) must now hold. The usual argument shows  $\bar{x}''$  is the only element of  $\mathcal{C} \cap E$  on the right of  $\bar{a}\bar{a}'$ . Now again  $J(\bar{a}, \frac{1}{3}(2\bar{a}' + \bar{a})) > 0$ ,  $J(\bar{a}', \frac{1}{3}(\bar{a} + 2\bar{a}')) > 0$ , and the sums  $a + (2a' + a)/3$  and  $a' + (a + 2a')/3$  cannot give points in  $2\mathcal{W}$ . Since they also cannot both be cancelled by  $c + x''$  in the superpotential equation, we have a contradiction.

The other possibility for two interior points is, after relabelling the vertices if necessary, when  $z = (2x + x'')/3$  and  $w = (2x'' + x)/3$ . As usual all elements of  $\mathcal{C} \cap E$  except for  $\bar{c}, \bar{a}, \bar{a}'$  are on the right of  $\bar{a}\bar{a}'$ . Alternative (1) must hold, or else we cannot account for  $z, w$ . The usual argument shows either  $\bar{x}''$  is the only element of  $\mathcal{C} \cap E$  right of  $\bar{a}\bar{a}'$ , or  $z \in \mathcal{C}$  is the rightmost element of  $(\mathcal{C} \cap E) \setminus \{\bar{x}''\}$  (so  $(z + x'')/2 = w$ ). In the former case we cannot get both  $z$  and  $w$ , as  $(c + x'')/2$  can't equal both  $z$  and  $w$ . In the latter, considering  $\bar{a}\bar{z}$  shows  $J(\bar{a}, \bar{z}) = 0$ . But as  $J(\bar{a}, \bar{x}'') = 0$ , this means  $\bar{a}$  is orthogonal to  $\bar{x}$  and hence to  $\bar{c}$ , a contradiction.

So no triangle with points of  $\mathcal{W}$  in the interior of an edge can arise (except possibly for a subtriangle of (H1)).

Nullity of  $\bar{c}, \bar{a}, \bar{a}'$  and the conditions in Theorem 6.12(i),(ii) again put severe constraints on  $x, x', x''$  and the dimensions. The possible triangles for case (i) are as follows, where (Tr11)-(Tr22) occur only if  $K$  is not connected, and we have also listed the vectors  $c, a, a'$  for future reference. Further details of how the following listing is arrived at can be found in [DW5].

	$x''$	$x$	$x'$
(Tr1)	$(-2, 1, 0, 0, 0)$	$(0, 0, -2, 1, 0)$	$(0, 0, -2, 0, 1)$
(Tr2)	$(-2, 1, 0, 0, 0)$	$(0, 1, -2, 0, 0)$	$(0, 1, -1, -1, 0)$
(Tr3)	$(0, 0, 0, -2, 1)$	$(-2, 1, 0, 0, 0)$	$(0, 1, -2, 0, 0)$
(Tr4)	$(-2, 1, 0, 0, 0, 0)$	$(0, 0, -2, 1, 0, 0)$	$(0, 0, 0, 1, -1, -1)$
(Tr5)	$(-2, 1, 0, 0, 0, 0)$	$(0, 1, -1, 0, -1, 0)$	$(0, 1, -1, -1, 0, 0)$
(Tr6)	$(-2, 1, 0, 0, 0, 0)$	$(0, 0, 1, -1, -1, 0)$	$(0, 0, 1, -1, 0, -1)$
(Tr7)	$(-2, 1, 0, 0, 0, 0)$	$(0, 0, 1, -1, -1, 0)$	$(0, 0, -1, -1, 0, 1)$
(Tr8)	$(-2, 1, 0, 0, 0, 0)$	$(0, 0, -1, -1, 1, 0)$	$(0, 0, -1, -1, 0, 1)$
(Tr9)	$(-2, 1, 0, 0, 0, 0, 0)$	$(0, 0, 1, -1, -1, 0, 0)$	$(0, 0, 1, 0, 0, -1, -1)$
(Tr10)	$(-2, 1, 0, 0, 0, 0, 0)$	$(0, 0, -1, 1, -1, 0, 0)$	$(0, 0, -1, 0, 0, 1, -1)$
(Tr11)	$(0, 0, 0, 1, -1, -1)$	$(-2, 1, 0, 0, 0, 0)$	$(-2, 0, 1, 0, 0, 0)$
(Tr12)	$(0, 1, 0, -1, -1)$	$(-2, 1, 0, 0, 0)$	$(-1, 1, -1, 0, 0)$
(Tr13)	$(0, 0, 0, -1, -1, 1)$	$(-2, 1, 0, 0, 0, 0)$	$(0, 1, -2, 0, 0, 0)$
(Tr14)	$(0, 0, 0, -1, -1, 1)$	$(0, 1, -1, -1, 0, 0, 0)$	$(-2, 1, 0, 0, 0, 0, 0)$
(Tr15)	$(0, 0, 0, 1, -1, -1, 0)$	$(1, -1, -1, 0, 0, 0, 0)$	$(1, -1, 0, 0, 0, 0, -1)$
(Tr16)	$(0, 0, 0, 1, -1, -1, 0)$	$(1, -1, -1, 0, 0, 0, 0)$	$(0, -1, 1, 0, 0, 0, 1)$
(Tr17)	$(0, 0, 0, 1, -1, -1, 0)$	$(1, -1, -1, 0, 0, 0, 0)$	$(0, -1, -1, 0, 0, 0, 1)$
(Tr18)	$(0, 0, 0, 1, -1, -1, 0)$	$(1, -1, -1, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0, -1, -1)$
(Tr19)	$(0, 0, 0, 1, -1, -1, 0)$	$(1, -1, -1, 0, 0, 0, 0, 0)$	$(0, -1, 0, 0, 0, 0, -1, 1)$
(Tr20)	$(-1, -1, 1, 0, 0, 0)$	$(0, 0, 1, -1, -1, 0)$	$(0, 0, 1, -1, 0, -1)$
(Tr21)	$(-1, 1, -1, 0, 0, 0)$	$(0, 0, -1, 1, -1, 0)$	$(0, 0, -1, 1, 0, -1)$
(Tr22)	$(-1, 1, -1, 0, 0, 0)$	$(0, 0, -1, -1, 1, 0)$	$(0, 0, -1, -1, 0, 1)$

	$3c$	$3a$	$3a'$
(Tr1)	(2, -1, -8, 2, 2)	(-2, 1, -4, 4, -2)	(-2, 1, -4, -2, 4)
(Tr2)	(2, 3, -6, -2)	(-2, 3, -6, 2)	(-2, 3, 0, -4)
(Tr3)	(-4, 4, -4, 2, -1)	(-8, 2, 4, -2, 1)	(4, 2, -8, -2, 1)
(Tr4)	(2, -1, -4, 4, -2, -2)	(-2, 1, -8, 2, 2, 2)	(-2, 1, 4, 2, -4, -4)
(Tr5)	(2, 3, -4, -2, -2)	(-2, 3, -2, 2, -4)	(-2, 3, -2, -4, 2)
(Tr6)	(2, -1, 4, -4, -2, -2)	(-2, 1, 2, -2, -4, 2)	(-2, 1, 2, -2, 2, -4)
(Tr7)	(2, -1, 0, -4, -2, 2)	(-2, 1, 6, -2, -4, -2)	(-2, 1, -6, -2, 2, 4)
(Tr8)	(2, -1, -4, -4, 2, 2)	(-2, 1, -2, -2, 4, -2)	(-2, 1, -2, -2, -2, 4)
(Tr9)	(2, -1, 4, -2, -2, -2, -2)	(-2, 1, 2, -4, -4, 2, 2)	(-2, 1, 2, 2, 2, -4, -4)
(Tr10)	(2, -1, -4, 2, -2, 2, -2)	(-2, 1, -2, 4, -4, -2, 2)	(-2, 1, -2, -2, 2, 4, -4)
(Tr11)	(-8, 2, 2, -1, 1, 1)	(-4, 4, -2, 1, -1, -1)	(-4, -2, 4, 1, -1, -1)
(Tr12)	(-6, 3, -2, 1, 1)	(-6, 3, 2, -1, -1)	(0, 3, -4, -1, -1)
(Tr13)	(-4, 4, -4, 1, 1, -1)	(-8, 2, 4, -1, -1, 1)	(4, 2, -8, -1, -1, 1)
(Tr14)	(-4, 4, -2, -2, 1, 1, -1)	(4, 2, -4, -4, -1, -1, 1)	(-8, 2, 2, 2, -1, -1, 1)
(Tr15)	(4, -4, -2, -1, 1, 1, -2)	(2, -2, -4, 1, -1, -1, 2)	(2, -2, 2, 1, -1, -1, -4)
(Tr16)	(2, -4, 0, -1, 1, 1, -2)	(4, -2, -6, 1, -1, -1, 2)	(-2, -2, 6, 1, -1, -1, -4)
(Tr17)	(2, -4, -4, -1, 1, 1, 2)	(4, -2, -2, 1, -1, -1, -2)	(-2, -2, -2, 1, -1, -1, 4)
(Tr18)	(4, -2, -2, -1, 1, 1, -2, -2)	(2, -4, -4, 1, -1, -1, 2, 2)	(2, 2, 2, 1, -1, -1, -4, -4)
(Tr19)	(2, -4, -2, -1, 1, 1, -2, 2)	(4, -2, -4, 1, -1, -1, 2, -2)	(-2, -2, 2, 1, -1, -1, -4, 4)
(Tr20)	(1, 1, 3, -4, -2, -2)	(-1, -1, 3, -2, -4, 2)	(-1, -1, 3, -2, 2, -4)
(Tr21)	(1, -1, -3, 4, -2, -2)	(-1, 1, -3, 2, -4, 2)	(-1, 1, -3, 2, 2, -4)
(Tr22)	(1, -1, -3, -4, 2, 2)	(-1, 1, -3, -2, 4, -2)	(-1, 1, -3, -2, -2, 4)

**Remark 6.15.** In making the above table, it is useful to observe from the nullity and orthogonality conditions that  $x''$  cannot be type I, and that if  $x''$  is type III, say,  $(-2^i, 1^j)$ , then  $x_i = x'_i$  iff  $x_j = x'_j$ .

The possibilities for Theorem 6.12(ii) are as follows (up to permutation of  $x, x', x''$  and the corresponding permutation of  $c, a, a'$ ):

	$x''$	$x$	$x'$
(Tr23)	(-1, 0, 0, 0, 0)	(0, -2, 1, 0, 0)	(0, 0, 0, -2, 1)
(Tr24)	(-1, 0, 0, 0, 0)	(0, 1, -2, 0)	(0, -1, -1, 1)
(Tr25)	(-1, 0, 0, 0, 0, 0)	(0, 1, -2, 0, 0, 0)	(0, 0, 0, 1, -1, -1)
(Tr26)	(-1, 0, 0, 0, 0)	(0, 1, -1, -1, 0)	(0, -1, -1, 0, 1)
(Tr27)	(-1, 0, 0, 0, 0, 0, 0)	(0, 1, -1, -1, 0, 0, 0)	(0, 0, 0, 0, 1, -1, -1)
(Tr28)	(-1, 0, 0)	(0, -1, 0)	(0, 0, -1)

	$c$	$a$	$a'$
(Tr23)	(1, -2, 1, -2, 1)	(-1, -2, 1, 2, -1)	(-1, 2, -1, -2, 1)
(Tr24)	(1, 0, -3, 1)	(-1, 2, -1, -1)	(-1, -2, 1, 1)
(Tr25)	(1, 1, -2, 1, -1, -1)	(-1, 1, -2, -1, 1, 1)	(-1, -1, 2, 1, -1, -1)
(Tr26)	(1, 0, -2, -1, 1)	(-1, 2, 0, -1, -1)	(-1, -2, 0, 1, 1)
(Tr27)	(1, 1, -1, -1, 1, -1, -1)	(-1, 1, -1, -1, -1, 1, 1)	(-1, -1, 1, 1, 1, -1, -1)
(Tr28)	(1, -1, -1)	(-1, -1, 1)	(-1, 1, -1)

**Remark 6.16.** In drawing up the above listing, recall from Theorem 6.12 that one of the vectors, without loss of generality  $x''$ , is of type I. We write  $x'' = (-1, 0, 0, \dots)$ . It now easily follows from nullity and the relations between  $x, x', x''$  and  $c, a, a'$  that  $x_1 = x'_1$ .

Also, observe that as  $x''$  is a vertex of  $\mathcal{W}$ , no type II vector may have a nonzero entry in the first position.

In contrast to the earlier listing of non-triangular faces, the above lists result from examining *all* triangular faces, including ones which arise from other faces because certain vertices are absent from  $\mathcal{W}$ .

The restrictions on the dimensions of the corresponding summands are as follows:

- (Tr1)  $(2, 1, 16, 4, 4, \dots)$
- (Tr2)  $(2, 3, 12, 4, \dots)$
- (Tr3)  $(16, 4, 16, 2, 1, \dots)$
- (Tr4)  $(2, 1, 16, 4, d_5, d_6, \dots), \quad \frac{1}{d_4} + \frac{1}{d_5} = \frac{1}{4}$
- (Tr5)  $(2, 3, 6, 6, 6, \dots)$
- (Tr6)  $(2, 1, d_3, d_4, 4, 4, \dots), \quad \frac{1}{d_3} + \frac{1}{d_4} = \frac{1}{4}$
- (Tr7)  $(2, 1, 12, 3, 12, 12, \dots)$
- (Tr8)  $(2, 1, d_3, d_4, 4, 4, \dots), \quad \frac{1}{d_3} + \frac{1}{d_4} = \frac{1}{4}$
- (Tr9)  $(2, 1, 4, d_4, d_5, d_6, d_7, \dots), \quad \frac{1}{d_4} + \frac{1}{d_5} = \frac{1}{d_6} + \frac{1}{d_7} = \frac{1}{4}$
- (Tr10)  $(2, 1, 4, d_4, d_5, d_6, d_7, \dots), \quad \frac{1}{d_4} + \frac{1}{d_5} = \frac{1}{d_6} + \frac{1}{d_7} = \frac{1}{4}$
- (Tr11)  $(16, 4, 4, 1, 1, 1, \dots)$
- (Tr12)  $(12, 3, 4, 1, 1, \dots)$
- (Tr13)  $(16, 4, 16, 1, 1, 1, \dots)$
- (Tr14)  $(16, 4, d_3, d_4, 1, 1, 1, \dots), \quad \frac{1}{d_3} + \frac{1}{d_4} = \frac{1}{4}$
- (Tr15)  $(d_1, d_2, 4, 1, 1, 1, 4, \dots), \quad \frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{4}$
- (Tr16)  $(12, 3, 12, 1, 1, 1, 12, \dots)$
- (Tr17)  $(4, d_2, d_3, 1, 1, 1, 4, \dots), \quad \frac{1}{d_2} + \frac{1}{d_3} = \frac{1}{4}$
- (Tr18)  $(4, d_2, d_3, 1, 1, 1, d_7, d_8, \dots), \quad \frac{1}{d_2} + \frac{1}{d_3} = \frac{1}{4} = \frac{1}{d_7} + \frac{1}{d_8}$
- (Tr19)  $(d_1, 4, d_3, 1, 1, 1, d_7, d_8, \dots), \quad \frac{1}{d_1} + \frac{1}{d_3} = \frac{1}{4} = \frac{1}{d_7} + \frac{1}{d_8}$
- (Tr20-22)  $(1, 1, 3, 6, 6, 6, \dots), (1, 2, 2, 8, 8, 8, \dots), \text{ or } (2, 1, 2, 8, 8, 8, \dots)$
- (Tr23)  $\frac{1}{d_1} + \frac{4}{d_2} + \frac{1}{d_3} + \frac{4}{d_4} + \frac{1}{d_5} = 1$
- (Tr24)  $d_3 = 2d_2 : \frac{1}{d_1} + \frac{9}{d_3} + \frac{1}{d_4} = 1$
- (Tr25)  $\frac{1}{d_1} + \frac{1}{d_2} + \frac{4}{d_3} + \frac{1}{d_4} + \frac{1}{d_5} + \frac{1}{d_6} = 1$
- (Tr26)  $d_2 = d_3 : \frac{1}{d_1} + \frac{4}{d_3} + \frac{1}{d_4} + \frac{1}{d_5} = 1.$
- (Tr27)  $\sum_{i=1}^7 \frac{1}{d_i} = 1$
- (Tr28)  $\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} = 1.$

Note that (Tr28) is a subtriangle of (H1), (Tr2) is a subtriangle of a triangle with midpoints of all sides in  $\mathcal{W}$ , and (Tr12) is a subtriangle of a triangle with the midpoint of one side.

Let us now illustrate by an example how one arrives at the above tables.

**Example 6.17.** One possible triangle has vertices  $V_1 = (0, 0, 0, -2, 1)$ ,  $V_2 = (-2, 1, 0, 0, 0)$ ,  $V_3 = (0, 1, -2, 0, 0)$  with the midpoint  $V_4 = (-1, 1, -1, 0, 0)$  of  $V_2V_3$  in  $\mathcal{W}$ . The triangle has a symmetry given by interchanging the first and third entries. It therefore suffices to consider  $V_1, V_2, V_4$  as possibilities for  $x''$ . Of course, by Remark 6.13 the full triangle cannot occur. The possible subtriangles  $xx'x''$  are  $V_2V_3V_1$ ,  $V_2V_4V_1$ ,  $V_4V_1V_2$ ,  $V_3V_1V_2$ , and  $V_3V_1V_4$ . Now  $3c = 2x + 2x' - x''$ ,  $3a = 4x - 2x' + x''$ , and  $3a' = -2x + 4x' + x''$  can be used to compute these vectors in each case. For  $V_2V_4V_1$  one gets  $3c = (-6, 4, -2, 2, -1)$ ,  $3a = (-6, 2, 2, -2, 1)$  and so  $\bar{c}$  and  $\bar{a}$  cannot be both null. Similarly, for the last three possibilities,  $\bar{a}'$  and  $\bar{c}$  cannot be both null. That leaves the first case, which gives (Tr3). The condition  $J(\bar{a}, \bar{x}'') = 0$  is  $3 = \frac{4}{d_4} + \frac{1}{d_5}$ , which implies  $(d_4, d_5) = (2, 1)$ . Putting this into the null conditions for  $\bar{c}, \bar{a}, \bar{a}'$  gives the equations

$$6 = \frac{16}{d_1} + \frac{16}{d_2} + \frac{16}{d_3} = \frac{64}{d_1} + \frac{4}{d_2} + \frac{16}{d_3} = \frac{16}{d_1} + \frac{4}{d_2} + \frac{64}{d_3}.$$



The last two equations imply that  $d_1 = d_3$  and the first two equations give  $d_1 = 4d_2$ . These in turn give  $(d_1, d_2, d_3) = (16, 4, 16)$ , as in the tables above.

Putting all the results in this section together we obtain

**Theorem 6.18.** *If we have two adjacent (1B) vertices, then the associated 2-face of  $\mathcal{W}$  is given by a triangle in the list (Tr1) – (Tr27), the square with midpoint (S), a proper subshape of the hexagonal face (H1) containing all three type I vectors, or the sub-rectangle  $yy'zz'$  of (H2).  $\square$*

We note for future reference the following properties of the  $c$  vector of the non-triangular faces appearing in the above theorem: for (S), all nonzero entries have the same absolute value, and there are only 3 (resp. 2) nonzero entries for the subfaces of (H1) (resp. (H2)).

## 7. More than one type (2) vertex

In this section we shall now show there is at most one type (2) vertex in  $\Delta^{\bar{c}}$ , except in the situation of Theorem 3.14 and one other possible case.

Suppose we have two type (2) vertices of  $V$ . Then we have elements  $v, w, v', w'$  of  $\mathcal{W}$  with  $c, v, w$  collinear and  $c, v', w'$  collinear. So we have four coplanar elements  $v, w, v', w'$  of  $\mathcal{W}$  where  $vw$  and  $v'w'$  are edges. Moreover, the edges  $vw$  and  $v'w'$  meet at  $c$  outside  $\text{conv}(\mathcal{W})$ . Hence  $vv'w'w'$  do not form a parallelogram or a triangle.

From our listing of polygons in §6 and considering their sub-polygons we see that the possibilities for further analysis are the following:

- Trapezia (T1)-(T6): We must have  $c = 2v - s = 2u - w$ . Also, we note for future reference that  $sw$  is always an edge of  $\text{conv}(\mathcal{W})$  in (T3) and (T5), regardless of whether or not the whole trapezium is a face, since  $sw$  can be cut out by  $\{x_2 = 1, x_1 + x_3 = -2\}$  (cf 1.2(e)).
- Hexagons (H1)-(H3)
- Rectangle with midpoints (P17): While the rectangle itself cannot occur, we need to consider the trapezia obtained by omitting one vertex, so that the edges are a side of the rectangle and the segment joining the remaining vertex to the opposite midpoint. As above, note that the longer of the two parallel sides of the trapezium is always an edge of  $\text{conv}(\mathcal{W})$ .
- Parallelogram with midpoints (P16): This case is similar to (P17). The sub-polygons to consider are the trapezia obtained by omitting one vertex of the parallelogram. By symmetry, we are reduced to omitting either  $u$  or  $w$ . But since  $s$  occurs,  $w$  cannot be omitted.
- Triangle with midpoints of all sides: We need to consider the trapezia obtained by omitting a vertex. By symmetry all three trapezia are equivalent. This triangle is always a face of  $\text{conv}(\mathcal{W})$  as it is cut out by  $\{x_2 = 1, x_1 + x_3 + x_4 = -2\}$  (cf 1.2(e)).
- Trapezia (T\*1),(T\*2), (T\*3): By Rmk 6.3 these cannot be faces of  $\text{conv}(\mathcal{W})$ , so cannot come from *adjacent* type (2) vertices of  $V$ . For (T\*1), besides the full trapezium, we need to consider the two trapezia obtained by omitting either  $s$  or  $w$ . By symmetry these are equivalent.

For (T1),(T2),(T4),(T6),(T\*2),(T\*3) we must have  $c = 2v - s = 2u - w$ , and so Lemma 3.13 applied to  $vs$  gives a contradiction. The same argument works for (P16), as up to permutations,  $c = 2v - s = 2y - w$ . For (T1\*), since Theorem 3.11 rules out  $c = (3v - s)/2 = (3u - w)/2$  (corresponding to the full trapezium), the only other possible  $c$  is  $2v - s = 2u - r$ , and again Lemma 3.13 rules this out.

For (T3),(T5) and the trapezium coming from the triangle with midpoints, we need more information from the superpotential equation. Since  $J(\bar{c}, \bar{s}), J(\bar{c}, \bar{w}) > 0$ , while  $A_v, A_u < 0$ ,  $F_{\bar{s}}, F_{\bar{w}}$  must have the same sign, which must be opposite to that of  $F_{\bar{c}}$ . Since  $stw$  is always an edge by earlier remarks, the nullity of  $\bar{c}$  implies  $J(\bar{s}, \bar{w}) > 0$ , contradicting Prop 3.7(ii).

Essentially the same argument works for (P17), as up to permutations  $c = 2s - v = 2z - u = (-2, -1, 2, 0)$ .

For (H3) most quadruples cannot give pairs of edges. For we observe that  $u$  (resp.  $v, w$ ) is present iff  $x$  (resp.  $y, z$ ) is. Thus, if  $u$  is missing, so is  $x$ , and  $v, w, y, z$  must all be present (otherwise we do not have a 2-dimensional polygon). But we now get a rectangle, which is not admissible. Hence all vertices are present and by symmetry we may assume that one of our edges is  $uz$  or  $uv$ . From this we quickly find that the two possible  $c$  (up to permutations) are  $(-1, -1, -1, 2) = 2y - x = 2z - u$  and  $(1, -1, 1, -2) = 2v - u = 2w - x$ . Both cases are ruled out by Lemma 3.13.

For (H2), observe that  $y$  (resp.  $y'$ ) is present iff  $z$  (resp.  $z'$ ) is. As these four vectors cannot all be absent (otherwise we do not have a 2-dim polygon), by the symmetries of (H2), we can assume  $y'$  is present. If  $v$  is present, then all possibilities are eliminated by Theorem 3.11. (Note that although  $2\alpha - y' = 2z - z'$  it is impossible for  $\alpha y'$  and  $zz'$  to both be edges.) On the other hand, if  $v$  is absent, then  $y'z'$  is an edge. Since the polygon cannot be a parallelogram or a triangle, it follows that  $u$  is present and the polygon is a pentagon. In this case, the only possibility compatible with Theorem 3.11 is  $c = (0, -1, 2, -2) = 2y - u = \frac{3}{2}y' - \frac{1}{2}z'$ . (This is not a priori ruled out by Theorem 3.11 as  $y'z'$  has an interior point  $\beta$ ).

To discuss (H1), we write

$$u = (-2, 1, 0), \quad p = (0, 1, -2), \quad v = (1, 0, -2), \quad w = (0, -2, 1), \quad s = (1, -2, 0), \quad q = (-2, 0, 1)$$

for the vertices,

$$x = (-1, 1, -1), \quad y = (-1, -1, 1), \quad z = (1, -1, -1),$$

for midpoints of the longer sides, and

$$\alpha = (-1, 0, 0), \quad \beta = (0, 0, -1), \quad \gamma = (0, -1, 0)$$

for the interior points, with the understanding that the rest of the components of the above vectors are zero.

As before we consider pairs of vectors which can form edges of an admissible polygon. We then compute the possibilities for  $c$  and apply Theorem 3.11. This will eliminate most possibilities. (For many quadruples of points we can see, as in (H3), that they cannot all be vertices.) So up to permutations, the remaining possibilities are as follows.

If no type II is present:

- (1)  $c = 2\alpha - u = 2\beta - p = (0, -1, 0, \dots)$
- (2)  $c = 2u - v = 2q - s = (-5, 2, 2, \dots)$
- (3)  $c = 2u - p = 2q - \gamma = (-4, 1, 2, \dots)$
- (4)  $c = 2q - u = 2\alpha - p = (-2, -1, 2, \dots)$

If all type II are present:

- (5)  $c = 2u - x = 2q - y = (-3, 1, 1, \dots)$
- (6)  $c = 2u - y = 2x - z = (-3, 3, -1, \dots)$
- (7)  $c = (3y - z)/2 = 2q - u = (-2, -1, 2, \dots)$
- (8)  $c = (3p - u)/2 = (3v - s)/2 = (1, 1, -3, \dots)$

Again, we cannot immediately rule out (7) and (8) using 3.11 because of the presence of interior points. However for (8) we easily see using the arguments of 3.11 that the elements of  $\mathcal{C}$  on the line through  $v, s$  are  $\bar{c}, \bar{c}_1 = (\bar{v} + \bar{s})/2 = \bar{z}$  and  $\bar{c}_2 = (3\bar{s} - \bar{v})/2$ . Now as  $s, v$  are type III we need  $F_{\bar{c}}$  and  $F_{\bar{c}_2}$  to have the same sign, which is the opposite sign to  $F_{\bar{c}_1}$ . But the superpotential equation now gives a contradiction to the fact that  $A_z < 0$ .

In (2)-(7) Lemma 3.13 applied respectively to  $uv, up, qu, ux, uy, qu$  gives a contradiction. Note that case (4) only occurs when  $K$  is not connected, as the vectors  $\beta, \gamma$  are absent (cf 1.2(b)). We are left with (1), which is precisely the situation of Theorem 3.14. We have therefore proved

**Theorem 7.1.** *Apart from the situation of Theorem 3.14, the only other possible case where we can have more than one type (2) vertex is, up to permutation of summands, when two type (2)*

vertices are adjacent and the 2-plane determined by them and  $\bar{c}$  intersects  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  in the pentagon with vertices  $uy'zz'$  contained in the hexagon (H2).  $\square$

We will be able to rule this case out in §8.

### 8. Adjacent (1B) vertices revisited

We now return to our classification of when adjacent (1B) vertices can occur.

The idea is as follows: each of the configurations of §6 involves, as well as the null vector  $\bar{c}$ , two new null vectors  $\bar{a}$ ,  $\bar{a}'$ . Hence the arguments of earlier sections also apply to  $\bar{a}$ ,  $\bar{a}'$ . That is, we may consider the associated polytopes  $\Delta^{\bar{a}}$  and  $\Delta^{\bar{a}'}$ .

The following lemma is useful when applied to  $\Delta^{\bar{c}}$ ,  $\Delta^{\bar{a}}$  and  $\Delta^{\bar{a}'}$ .

**Lemma 8.1.** *Suppose we have a (1B) vertex with exactly  $k$  adjacent (1B) vertices. Then*

$$r \leq \#((1A) \text{ vertices}) + \#((2) \text{ vertices}) + k + 2.$$

*Suppose we have a (1B) vertex with no adjacent (1B) vertices. Then*

$$r \leq \#((1A) \text{ vertices}) + \#((2) \text{ vertices}) + 2.$$

*If there are no (1B) vertices then*

$$r \leq \#((1A) \text{ vertices}) + \#((2) \text{ vertices}) + 1.$$

*Proof.* By our assumption that  $\dim \text{conv}(\mathcal{W}) = r - 1$ , it follows that  $\Delta^{\bar{c}}$  is a polytope of dimension  $r - 2$ . Any vertex in it has at least  $r - 2$  adjacent vertices. So for a (1B) vertex, the first two statements follow immediately. If there are no (1B) vertices the third inequality follows because  $\Delta^{\bar{c}}$  has at least  $r - 1$  vertices.  $\square$

**Lemma 8.2.** *Configurations (Tr1) - (Tr22) cannot arise from adjacent (1B) vertices.*

*Proof.* The strategy is to count the number of type (1A), (2) and adjacent (1B) vertices in  $\Delta^{\bar{a}}$  or  $\Delta^{\bar{c}}$  and apply Lemma 8.1 to get a contradiction.

(i) We first observe that for these configurations  $c$  and  $a$  have at least four nonzero entries (at least five except for (Tr2)), so they cannot be collinear with an edge  $vw$  with points of  $\mathcal{W}$  in the interior of  $vw$  (see Table 3 in §3). So if  $\Delta^{\bar{c}}$  or  $\Delta^{\bar{a}}$  has a type (2) vertex, by Theorem 3.11,  $c$  or  $a$  must equal  $2v - w$  or  $(4v - w)/3$ . It is easy to check that this is impossible except for  $c$  in (Tr3), using the forms of  $c$  in Tables 1, 2 in §3.

(ii) Next we consider (1A) vertices. For (Tr1)-(Tr10) we have  $|a_i/d_i| \leq \frac{1}{3}$  for all  $i$ . For Tr(1), Tr(6), Tr(8) there are three  $i$  where equality holds. In these cases one of the associated  $d_i$  equals 1. Moreover for (Tr1) and (Tr8) two of these  $a_i/d_i$  equal  $1/3$  and the third is  $-1/3$ , whereas for (Tr6) it is the other way round. For (Tr2)-(Tr5), (Tr7) and (Tr9)-(Tr10) there are only two  $i$  where equality holds. Further, for (Tr2) and (Tr5)  $|c_i/d_i| \leq \frac{1}{3}$  for all  $i$ , with equality for just two  $i$ , and here  $c_i/d_i = \frac{1}{3}$ . It follows that for (Tr1), (Tr8) there are at most two (1A) vertices in  $\Delta^{\bar{a}}$ , while for (Tr2)-(Tr5), (Tr7) and (Tr9)-(Tr10) there is at most one. In the case of (Tr6) there are at most three (1A) vertices in general but at most two if  $K$  is connected. For (Tr2) and (Tr5) there are no (1A) vertices in  $\Delta^{\bar{c}}$ .

By means of similar considerations, we find that there is one (1A) vertex (corresponding to  $\bar{x}''$ ) in  $\Delta^{\bar{a}}$  for (Tr12), (Tr13), (Tr14), (Tr16), (Tr18), (Tr19) and at most two (1A) vertices for (Tr11), (Tr17), (Tr20) and the  $d = (1, 1, 3, 6, 6, 6, \dots)$  case of (Tr21), (Tr22). For (Tr15) there are at most four (1A) vertices in  $\Delta^{\bar{a}}$ , and for the remaining cases of (Tr21) and (Tr22) there are at most  $r - 4$  (1A) vertices ( $r - 6$  of those correspond to  $(-2^3, 1^j)$  where  $j > 6$ ).

(iii) Finally, consider the (1B) vertex  $\bar{\xi}$  in  $\Delta^{\bar{a}}$  corresponding to  $\bar{x}$  in each of the triangles. In order for there to be an adjacent (1B) vertex,  $\bar{a}$  must be (up to permutation) of the form of the null vector  $\bar{c}$  in the 2-faces in Theorem 6.18. Now observe that for examples (Tr1), (Tr3), (Tr4), (Tr6)-(Tr20)

the null vector  $a$  does not appear in the list of possible  $c$ . Hence  $\bar{\xi}$  has no adjacent (1B) vertices. From above, type (2) vertices cannot occur, so combining the bounds for (1A) vertices in  $\Delta^{\bar{a}}$  with Lemma 8.1 gives an upper bound for  $r$  less than the minimum required by each configuration, a contradiction.

(iv) Let us now consider (Tr21) and (Tr22). The vector  $a$  of (Tr21) has the same form as  $c$  in (Tr22) and vice versa. An adjacent 2-face containing  $a\xi c$  of (Tr21) can only be a triangle of type (Tr22) containing  $\frac{1}{3}(1, -1, -3, -2, -2, 4, 0, \dots)$ . Thus  $\xi$  has at most one adjacent (1B) vertex, and we get the bound  $r - 2 \leq 2 + 1 + 0$  in the  $d = (1, 1, 3, 6, 6, 6, \dots)$  subcase and  $r - 2 \leq (r - 4) + 1 + 0$  in the other two subcases, a contradiction.

(v) For the remaining two triangles (Tr5) and (Tr2) we consider  $\Delta^{\bar{c}}$  instead.

For (Tr5), observe that  $c$  determines the plane  $xx'x''$  and does not occur as a possible null vector for any other configurations. So we have at most one adjacent pair of (1B) vertices in  $\Delta^{\bar{c}}$ . From above, there are no type (1A) or (2) vertices. But  $r \geq 5$ , giving a contradiction.

For (Tr2), consider the vertex  $\xi'$  of  $\Delta^{\bar{c}}$  corresponding to  $x'$ . If there is a (1B) vertex adjacent to it, we have a 2-dim face including  $c, x'$ . By Theorem 6.18 the only one is the face including  $x$ , so there is at most one (1B) vertex of  $\Delta^{\bar{c}}$  adjacent to  $\xi'$ . Also, type (1A) and (2) vertices cannot occur, so  $r \leq 3$ , a contradiction.  $\square$

**Lemma 8.3.** *Configurations (Tr23)-(Tr27) cannot arise from adjacent (1B) vertices.*

*Proof.* Note first that all entries of  $c, a, a'$  are integers, so Lemma 4.4 shows in each case there is at most one (1A) vertex, and for (Tr23),(Tr24) one checks that there are no (1A) vertices in  $\Delta^{\bar{c}}$ . Note also that for all these configurations, as  $x''$  is a vertex, there are no type II vectors with nonzero entry in place 1.

Observe as in Lemma 8.2 that there are no type (2) vertices in  $\Delta^{\bar{c}}, \Delta^{\bar{a}}$  or  $\Delta^{\bar{a}'}$ . (For (Tr24), we need to rule out the possibility that  $c$  has the form (4) in Table 3 in §3 with  $\lambda = 3/2$ . This follows since the interior point in that case would be a type II vector with nonzero entry in place 1.)

So in all cases if we have a (1B) vertex with exactly  $k$  (1B) vertices adjacent to it, then by Lemma 8.1 we have  $r \leq k + 3$ . For (Tr23), (Tr24) (using  $\Delta^{\bar{c}}$ ), we have  $r \leq k + 2$ . We will work with  $\Delta^{\bar{c}}$  below.

First consider (Tr23) and look for (1B) vertices adjacent to  $\bar{\xi}$  where  $\bar{\xi}$  corresponds to the vertex  $x$ . We need a 2-face including  $c, x$ . By Theorem 6.18 such a face must be of type (Tr23), and having fixed  $c$  and  $a$ , the only freedom lies in assigning 1 in the third null vector to the first or fifth place. So  $k \leq 2$ , which gives  $r \leq 4$ , a contradiction.

Similarly, for (Tr24), since a type (H1) face cannot contain  $c$  and  $x$ , we need only consider faces of type (Tr24), for which there are again two possibilities. However, as mentioned above, in one of these possibilities the vector “ $x'$ ” has a 1 in place 1 and hence cannot occur. So there is at most one (1B) vertex adjacent to  $\xi$ , and we deduce  $r \leq 3$ , a contradiction.

For (Tr25),(Tr27) we similarly deduce that the only 2-face containing  $x, c$  is itself because as above we cannot have any type II vectors with nonzero entry in place 1. So  $r \leq 4$ , a contradiction.

Finally, for (Tr26) the above argument still works since (Tr24) has been ruled out (the vectors  $a, a'$  of (Tr24) are of the same form as  $c, a$  of (Tr26), so a priori (Tr24) could be an adjacent 2-face).  $\square$

**Lemma 8.4.** *Configuration (S) (square with midpoint) cannot arise from adjacent (1B) vertices.*

*Proof.* We refer to §6 for the expressions for the vertices  $vusw$  of the square. The null vertex  $\bar{c}$  corresponds to  $(1, -1, -1, 1, -1, 0, \dots)$  and the 2-dimensional face is cut out by  $x_2 = -1, x_1 + x_3 = 0 = x_4 + x_5$ , and  $x_k = 0$ , for  $k > 5$ . Lemma 4.4(b) shows that there is at most one (1A) vertex in  $\Delta^{\bar{c}}$ . As  $r \geq 5$  and all the nonzero entries of  $c$  have the same absolute value, it follows that there are no type (2) vertices. Let  $\bar{\xi}$  denote the vertex of  $\Delta^{\bar{c}}$  so that  $\xi$  is collinear with  $u$  and  $a = 2u - c = (-1, -1, 1, 1, -1, 0, \dots)$ . A (1B) vertex adjacent to  $\xi$  gives a 2-dimensional face

including  $\bar{c}, \bar{u}$ . By what we have analysed so far about 2-faces given by adjacent (1B) vertices, this face must again be a face of type (S), and the only possibilities are itself or the face obtained from this by swapping indices 2 and 5. Hence there are at most two (1B) vertices adjacent to  $\bar{\xi}$ , and at vertex  $\bar{\xi}$ , we have  $3 \leq r - 2 \leq 1 + 2$ . Thus  $r = 5$  is the remaining possibility, in which case  $\bar{\xi}$  has exactly two adjacent (1B) vertices and one adjacent (1A) vertex.

Let us denote by  $\bar{\xi}'$  the (1B) vertex such that  $\bar{\xi}'$  is collinear with  $w$  and  $a' := 2w - c = (1, -1, -1, -1, 1)$ . Let  $\bar{\eta}$  denote the other (1B) vertex adjacent to  $\bar{\xi}$ . Then the 2-face determined by  $c, \bar{\xi}, \bar{\eta}$  is cut out by  $x_5 = -1, x_1 + x_3 = 0 = x_2 + x_4$ . The ray  $c\eta$  intersects  $\text{conv}(\mathcal{W})$  at  $z = (1, 0, -1, 0, -1)$  and  $b := 2z - c = (1, 1, -1, -1, -1)$  corresponds to a null vertex. Similarly, there is a (1B) vertex  $\bar{\eta}'$  (besides  $\bar{\xi}$ ) adjacent to  $\bar{\xi}'$ , and the corresponding 2-face (also of type (S)) is cut out by  $x_3 = -1, x_1 + x_2 = 0 = x_4 + x_5$ . The ray  $c\eta'$  intersects  $\text{conv}(\mathcal{W})$  at  $z' = (0, 0, -1, 1, -1)$ . The vector  $b' := 2z' - c = (-1, 1, -1, 1, -1)$  corresponds to a null vertex.

Let us examine the (1A) vertex in  $\Delta^{\bar{c}}$  more closely. Let  $y \in \mathcal{W}$  such that  $J(\bar{y}, \bar{c}) = 0$ . As  $r = 5$ , the null condition for  $\bar{c}$  implies that  $d_i \geq 2$  with at most one equal to 2. Also, for some  $j \in \{2, 3, 5\}$  (i.e.,  $j$  is an index for which the corresponding entry of  $c$  is  $-1$ ) we must have  $y_j = -2$ , so  $y$  is type III. Let  $i$  be the index such that  $y_i = 1$ . Then  $i \in \{1, 4\}$  (i.e.,  $i$  is an index for which the corresponding entry of  $c$  is 1), and the orthogonality condition implies  $(d_i, d_j) = (2, 4)$  or  $(3, 3)$ . There are thus six possibilities for  $y$ , but only one can actually occur.

With the possible exception of the existence of the (1A) vertex, the above arguments apply equally to the projected polytopes  $\Delta^{\bar{a}}, \Delta^{\bar{b}}, \Delta^{\bar{a}'}$ , and  $\Delta^{\bar{b}'}$  as the entries of  $a, b, a'$  and  $b'$  are just permutations of those of  $c$ . We claim that whichever possibility for  $y$  occurs in  $\Delta^{\bar{c}}$ , there is another projected polytope with no (1A) vertex. Applying the above arguments to this polytope would result in the contradiction  $r - 2 \leq 2$  and complete our proof.

We can use  $\Delta^{\bar{a}}$  for the contradiction if  $d_1 = 2$  or if any of  $(d_1, d_2), (d_3, d_4), (d_1, d_5) = (3, 3)$ . If  $d_4 = 2$  or if  $(d_2, d_4) = (3, 3)$  we can use  $\Delta^{\bar{a}'}$  instead. Finally, if  $(d_1, d_3) = (3, 3)$  we can use  $\Delta^{\bar{b}'}$  and if  $(d_4, d_5) = (3, 3)$  we can use  $\Delta^{\bar{b}}$ .

For example, when  $(d_1, d_3) = (3, 3)$  (so  $y = (1^1, -2^3)$ ), the null condition for  $\bar{c}$  implies that  $d_2, d_4$  in particular cannot equal 2 or 3. In order to have a (1A) vertex in  $\Delta^{\bar{b}'}$ , we must have a type III vector  $(1^i, -2^j)$  with  $i \in \{2, 4\}$ . But this requires one of  $d_2, d_4$  to be 2 or 3.

When  $(d_4, d_5) = (3, 3)$ , then  $d_1, d_2$  cannot be 2 or 3. But in  $\Delta^{\bar{b}}$  a (1A) vertex corresponds to  $(1^i, -2^j)$  with  $i \in \{1, 2\}$ , which implies that one of  $d_1, d_2$  is 2 or 3.

The remaining cases are handled similarly.  $\square$

**Lemma 8.5.** *The subrectangle  $yy'zz'$  of (H2) cannot arise from adjacent (1B) vertices.*

*Proof.* Recall  $c = (-2, 1, 0, \dots)$ , so by Lemma 4.4 there are no (1A) vertices of  $\Delta^{\bar{c}}$ . Moreover, using Tables 1-3 in §3, one may check that there are no type (2) vertices either. (Note that type II vectors other than  $y, z$  with a nonzero entry in place 1 cannot occur as then the subrectangle cannot be a face. Similarly the line through  $c, \alpha, \beta$  and  $(1, -2, 0, \dots)$  will not give a type (2) vertex as this line cannot be an edge.)

Let  $\bar{\eta}$  denote the vertex of  $\Delta^{\bar{c}}$  collinear with  $c$  and  $y$ . Any (1B) vertex adjacent to  $\bar{\eta}$  will give rise to a face containing  $c$  and  $y$ , which cannot be of type (H1), and must therefore be of type (H2), since we have eliminated all other possibilities. In fact, it must be the face we started with.

So there is just one (1B) vertex adjacent to  $\bar{\eta}$ , and from above there are no (1A) or (2) vertices. As  $r \geq 4$  for (H2), this contradicts Lemma 8.1.  $\square$

**Lemma 8.6.** *If  $r \geq 4$ , configuration (H1) or subshapes cannot arise from adjacent (1B) vertices.*

*Proof.* We first note some special properties of  $\mathcal{W}$ . Since (H1) is a face, there can be no type II vectors in  $\mathcal{W}$  with nonzero entry in a place  $\in \{1, 2, 3\}$  and in a place  $\notin \{1, 2, 3\}$ . Also, if  $(-2^i, 1^k)$  with  $i \in \{1, 2, 3\}$ ,  $k \notin \{1, 2, 3\}$ , then  $(-1^k)$  must be absent, which has strong implications, as noted in Remark 1.2(b).

Let  $\bar{\xi}$  be a (1B) vertex in the plane. We have at most one (1B) vertex adjacent to  $\bar{\xi}$ , as the associated face must again be of type (H1) and is now determined by  $\bar{c}, \bar{\xi}$ . It also readily follows that  $c$  cannot be collinear with an edge of  $\mathcal{W}$  not in the face (assuming as usual we are not in the situation of Theorem 3.14).

Now the special properties of  $\mathcal{W}$  in the first paragraph imply that (1A) vertices in  $\Delta^{\bar{c}}$  can correspond only to type III vectors in  $\mathcal{W}$  which overlap with  $c$ . A straight-forward check using the null condition for  $\bar{c}$  shows that the possible type IIIs have form  $(-2^i, 1^k)$  with  $i \in \{1, 2, 3\}$ ,  $k \notin \{1, 2, 3\}$  and  $c_i/d_i = -1/2$ . It follows that  $d_i = 2$  or  $3$  and hence, by nullity, the index  $i$  is unique. So there are at most  $r-3$  (1A) vertices. By Lemma 8.1, all  $r-3$  (1A) vertices must occur. Applying Cor 4.3 we conclude that  $r \leq 4$  (as  $d_i \neq 4$  and  $r > 4$  forces  $i \in \hat{S}_{\geq 2}$ ).

We will now improve this estimate to  $r \leq 3$ . Let the vertices of (H1) be as in §7. If  $r = 4$  then a (1A) vertex does exist and we can take it to come from  $t = (-2, 0, 0, 1)$  with  $(-1^4)$  absent. It follows that besides  $t$  the only other possible members of  $\mathcal{W}$  lying outside the 2-plane containing (H1) are  $(0, -2, 0, 1)$  and  $(0, 0, -2, 1)$ . As noted just before Theorem 6.12 we may assume the type I vectors  $\alpha, \beta, \gamma$  are all present. (If  $K$  is connected,  $d_4 = 1$  and so this last fact follows without having first to eliminate those subshapes not containing one of the type I vectors.)

As noted above,  $d_1 = 2$  or  $3$ , and  $c_1 = -1$  or  $-\frac{3}{2}$  respectively.

First consider  $c_1 = -1$ , so  $d_1 = 2$ . Now  $c = (-1, c_2, -c_2, 0)$ , and by swapping the 2, 3 coordinates if necessary, we may take  $c_2 > 0$ . Observe  $u, q$  are absent, as if  $u$  is present or if  $u$  is absent but  $q$  is present, then  $u$  (resp.  $q$ ) gives a (1B) vertex, which contradicts nullity as the associated  $a$  would have  $a_1 = -3$ . Now the type II vectors  $x, y, z$  are absent, as if one is present they all are, and we have a type (2) vertex. We deduce  $\alpha$  gives a (1B) vertex so  $a = (-1, -c_2, c_2, 0)$ . The other (1B) vertex cannot correspond to  $w$  since  $\beta, \gamma$  are present. It also cannot be given by  $v, s$  as this violates nullity, so must correspond to  $p$  or  $\beta$ .

If it is  $p$ , we have  $a' = (1, 2 - c_2, c_2 - 4, 0)$ . Now Remark 3.9 implies  $c_2 = (d_3 + 4d_2)/(d_3 + 2d_2)$  so  $1 < c_2 < 2$ . But now no entry of  $a'$  equals  $-1$  or  $-\frac{3}{2}$ . We can now check that there are no (1A) or (2) vertices with respect to  $a'$ , so there is at most one vertex of  $\Delta^{\bar{a}'}$  adjacent to  $p$ , a contradiction.

If it is  $\beta$  then  $p, v$  must be absent. Now  $a' = (1, -c_2, c_2 - 2)$  and Remark 3.9 implies  $c_2 = 1$ . Hence  $c = (-1, 1, -1)$ ,  $a = (-1, -1, 1)$ ,  $a' = (1, -1, -1)$ , and nullity implies  $\frac{1}{d_2} + \frac{1}{d_3} = \frac{1}{2}$ . It is easy to check by considering the vertices of  $\Delta^{\bar{a}'}$  that  $w, s$  must also be absent, so  $\mathcal{W}$  just contains the three type I vectors,  $t$  and possibly one or both of  $(0, -2, 0, 1), (0, 0, -2, 1)$ . But we can check that, if present, these three latter vectors give respectively vertices with respect to  $a, a'$  which cannot satisfy any of the conditions (1A), (1B) or (2). So in fact we have  $r = 3$ .

Similar arguments rule out the case  $c_1 = -\frac{3}{2}$ . □

Lemmas 8.2-8.6 give the following improvement of Theorem 6.18.

**Theorem 8.7.** *It is impossible to have adjacent (1B) vertices except possibly when  $r = 3$ , in which case  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  is a proper subface of (H1) containing all three type I vectors (e.g., the tri-warped example (Tr28)).* □

We are now in a position to strengthen Theorem 7.1 by eliminating the remaining case of the pentagon.

**Theorem 8.8.** *Let  $\bar{c}$  be a null vertex of  $\text{conv}(\mathcal{C})$  such that  $\Delta^{\bar{c}}$  contains more than one type (2) vertex. Then we are in the situation of Theorem 3.14.*

*Proof.* We just have to eliminate the case of the pentagon  $uyy'zz'$  in Theorem 7.1. Recall  $r \geq 4$  for this configuration, and  $c$  is  $(0, -1, 2, -2, \dots)$ .

Using the nullity of  $\bar{c}$  we check that the only elements of  $\mathcal{W}$  which can give an element of  $\bar{c}^\perp$  are  $(-2^2, 1^i)$  where  $i > 4$  and we have  $d_2 = 2$ . Note that  $(1^1, -2^2)$  cannot be present as then  $y'z'$  is not an edge. By Cor 4.3, at most one such vector can arise. So there is at most one (1A) vertex, which occurs only if  $r \geq 5$ .

If we can show there are no (1B) vertices, then we are done because if we look at the adjacent vertices of the type (2) vertex associated to  $y$  (in the pentagon), besides one (1A) possibility, the other possibility is the type (2) vertex associated to  $y'$  (by Theorem 7.1). As there must be at least  $r - 2$  adjacent vertices, we deduce  $r - 2 \leq 2$ , so  $r = 4$ . But now, from above there is no (1A) vertex so in fact we get  $r - 2 \leq 1$ , a contradiction.

We now use Remark 3.9 to make a list of the possible  $x \in \mathcal{W}$  associated to (1B) vertices of  $\Delta^{\bar{c}}$ . These are  $(0, 1, -1, -1, 0, \dots)$ ,  $(0, -2, 1, 0, \dots)$ ,  $(1^3, -1^i, -1^j)$ ,  $(-1^4, 1^i, -1^j)$ ,  $(1^3, -1^4, -1^i)$ ,  $(-1^3, -1^4, 1^i)$  and  $(1^3, -2^i)$  where  $i, j \neq 2, 3, 4$ . Note that type II vectors with nonzero entries in places  $2, k, m$  cannot occur except for  $y', z'$  as then  $y'z'$  is not an edge. For each  $x$  in this list, we consider the projected polytope  $\Delta^{\bar{a}}$ , where  $a = 2x - c$ . By looking at the form of  $a$ , we see from Theorem 7.1 that there is at most one type (2) vertex in  $\Delta^{\bar{a}}$ . Also, the nonzero components of  $a$  are either  $\geq 1$  or  $\leq -2$ . By Lemma 4.4(a), there are no vertices of type (1A) in  $\Delta^{\bar{a}}$ . Since  $r \geq 4$ , by Theorem 8.7, the type (1B) vertex in  $\Delta^{\bar{a}}$  corresponding to  $\bar{x}$  has no adjacent (1B) vertices. So we have a contradiction to Theorem 8.1.  $\square$

The above result together with Theorem 8.7 and Lemma 8.1 gives us lower bounds on the number of (1A) vertices.

**Theorem 8.9.** *Let  $\bar{c}$  be a null vertex of  $\text{conv}(\mathcal{C})$  and  $\Delta^{\bar{c}}$  be the corresponding projected polytope. Suppose further that  $c$  is not type I, i.e., we are not in the case of Theorem 3.14.*

- (i) *If there are no (1B) vertices in  $\Delta^{\bar{c}}$ , then there are at least  $r - 2$  type (1A) vertices.*
- (ii) *If either there is a type (2) vertex or  $r \geq 4$ , then there are at least  $r - 3$  type (1A) vertices in  $\Delta^{\bar{c}}$ . Hence there are at least  $r - 3$  elements of  $\frac{1}{2}(d + \mathcal{W})$  orthogonal to  $\bar{c}$ .  $\square$*

## 9. Type (2) vertices

In this section we consider again type (2) vertices of  $\Delta^{\bar{c}}$ . In view of Theorem 8.8, it remains to deal with the case of a unique type (2) vertex in  $\Delta^{\bar{c}}$ . By Theorem 8.7 there are no adjacent (1B) vertices in this situation.

Let  $c$  be collinear with an edge  $vw$  of  $\text{conv}(\mathcal{W})$ . We first consider the situation where there are no interior points of  $vw$  lying in  $\mathcal{W}$ . By Theorem 3.11, we have the two possibilities  $c = 2v - w$  and  $c = (4v - w)/3$ . Moreover, a preliminary listing of the cases appears in Tables 1 and 2 of §3.

**Case (i):**  $c = 2v - w$

We have to analyse cases (1)-(7) in Table 1 of §3. The idea is to determine the number of (1A) and (1B) vertices using respectively Lemma 4.4 and Remark 3.9, and then get a contradiction (sometimes using Theorem 8.9). Note that  $J(\bar{w}, \bar{w}) < 0$  for (1)-(3).

In (1), (2) and (4)-(7), Lemma 4.4 shows that there are no elements of  $\frac{1}{2}(d + \mathcal{W})$  orthogonal to  $\bar{c}$  (recall  $c \notin \mathcal{W}$ ), so Theorem 8.9 shows that  $r \leq 3$ . This already gives a contradiction in case (7). (Note that when  $r = 3$  and  $w$  is type I, since  $w$  is a vertex there are no type II vectors in  $\mathcal{W}$ .)

In (1) the only  $x \in \mathcal{W}$  that could satisfy Eq.(3.1) and give a (1B) vertex with respect to  $\bar{c}$  is  $(1, -1, -1)$ . But the associated  $a = 2x - c$  is  $(2, -3, 0)$  and it easily follows that  $\bar{a}, \bar{c}$  cannot both be null. For (2), the possible  $x \in \mathcal{W}$  which correspond to (1B) vertices are  $(1, -2, 0)$  and  $(1, -1, -1)$  respectively. In each case we find the nullity of  $\bar{c}$  and Remark 3.9 imply  $J(\bar{w}, \bar{w}) > 0$ , a contradiction to  $F_w^2 J(\bar{w}, \bar{w}) = A_w < 0$  (as  $w$  is type III).

In (5) with  $r = 3$ , one checks that the only possible  $x \in \mathcal{W}$  corresponding to a (1B) vertex is  $(0, 1, -2)$ . Let us consider the distribution of points of  $\mathcal{W}$  in the plane  $x_1 + x_2 + x_3 = -1$ . The point  $(0, 1, -2)$ , if present, would lie on one side of the line  $vw$  while the point  $(-1, 0, 0)$  lies on the other side. Now  $(-1, 0, 0)$  must lie in  $\mathcal{W}$  as otherwise  $v$  cannot be present by Remark 1.2(b). So since  $vw$  is an edge by assumption,  $(0, 1, -2)$  cannot lie in  $\mathcal{W}$ , which gives a contradiction to Theorem 8.9(i).

Hence in (1),(2),(5) Theorem 8.9 shows  $r \leq 2$ , which is a contradiction.

In case (4) the nullity of  $\bar{c}$  translates into  $1 = 9/d_1 + 4/d_2$ . Hence  $d_2 \neq 1$ , so if  $K$  is connected  $(0, -1, 0, \dots)$  is present and  $w$  is not a vertex, which is a contradiction. If  $r = 3$  and  $K$  is not connected, by Remark 1.2(b),  $(1, -2, 0)$  and  $(0, -2, 1)$  must be absent, and, from Remark 3.9, the possibilities for  $x \in \mathcal{W}$  associated to the (1B) vertex are  $x = (-2, 0, 1)$  and  $y = (0, 1, -2)$ . In the first case,  $\text{conv}(\mathcal{W})$  is the triangle with vertices  $v, w, x$  and  $a = 2x - c = (-1, -2, 2)$ . Now  $J(\bar{a}, \bar{w}) > 0$ , contradicting the superpotential equation. In the second case,  $a = (3, 0, -4)$  with  $J(\bar{a}, \bar{y}) > 0$ , and  $aw$  intersects  $\text{conv}(\mathcal{W})$  in an edge. By Theorem 3.11,  $t = (1, 0, -2) \in \mathcal{W}$  and  $\text{conv}(\mathcal{W})$  is a parallelogram with vertices  $v, y, w, t$ . Moreover, Remark 3.12 implies that  $a$  and  $w$  are the only elements of  $\mathcal{C}$  in  $aw$ . But then the midpoint  $(0, 0, -1)$  of  $wt$  is unaccounted for in the superpotential equation.

For (6) with  $r = 3$ , there should be at least two vertices in  $\Delta^{\bar{c}}$ . But we find there are no (1B) vertices, a contradiction. So  $r = 2$ , and we are in the situation of the double warped product Example 8.2 of [DW4].

In case (3), Lemma 4.4 shows  $\frac{1}{2}(d + \mathcal{W}) \cap \bar{c}^\perp$  has at most one element. Hence  $\Delta^{\bar{c}}$ , which has dimension  $\geq 2$  since  $r \geq 4$ , must contain at least one (1B) vertex. By Theorem 8.7, such a (1B) vertex has at most 2 adjacent vertices. It follows that  $r = 4$  and  $(1, -2, 0, 0)$  corresponds to the (1A) vertex; also  $d_2 = 2$ . Also, since  $(-1, 0, 0, 0) \in \mathcal{W}$ ,  $(0, -1, 0, 0)$  cannot be a vertex of  $\text{conv}(\mathcal{W})$ . But now routine computations using Eq.(3.1) show there are no (1B) vertices, a contradiction.

So the only possible case if  $K$  is connected is that giving Example 8.2 of [DW4]. If  $K$  is disconnected there is the further possibility of (4) with  $r = 2$ , i.e.,  $\mathcal{W} = \{(-2, 1), (-1, 0)\}$ . This is discussed in the third paragraph of Example 8.3 of [DW4]. An example in the inhomogeneous setting is treated there and in [DW2]. An example where the hypersurface is a homogeneous space  $G/K$  is discussed in the concluding remarks at the end of section §10.

### Case (ii): $c = (4v - w)/3$

For clarity of exposition let us assume  $K$  is connected, using the assumption as indicated in Remark 5.9. We examine the cases (1)-(11) in Table 2 of §3.

Some of these cases can be immediately eliminated. In (3), Eq.(3.2) implies  $(d_1, d_2) = (3, 3)$  or  $(4, 1)$ . In neither case is  $\bar{c}$  null. In (11) Eq.(3.2) and  $J(\bar{v}, \bar{w}) > 0$  imply  $(d_1, d_2, d_3) = (2, 4, 4), (2, 5, 2)$  or  $(3, 3, 3)$ , and again  $\bar{c}$  is not null. In (4) and (6) Eq.(3.2) implies  $(d_1, d_2) = (2, 1)$  and  $(d_1, d_2) = (3, 9)$  or  $(4, 3)$  respectively. In neither case does the nullity condition have an integral solution in  $d_3$ .

Further cases can be eliminated by finding the possible (1A) vertices (using Lemmas 3.10 and 4.4) for the given value of  $c$  and using Theorem 8.9. In particular, we get a contradiction whenever  $r \geq 4$  and there are no (1A) vertices.

In (1), Eq.(3.2) implies  $(d_1, d_2) = (2, 1)$ , and nullity of  $\bar{c}$  implies  $\frac{32}{d_3} + \frac{8}{d_4} = 3$ . But we now find that  $\frac{1}{2}(d + \mathcal{W}) \cap \bar{c}^\perp$  is empty, giving a contradiction as  $r \geq 4$ .

In (5) Eq.(3.2) and nullity imply  $(d_1, d_2) = (3, 3)$  and  $\{d_3, d_4\} = \{3, 8\}$ . One can now check that  $\frac{1}{2}(d + \mathcal{W}) \cap \bar{c}^\perp$  is empty, which is a contradiction as  $r \geq 4$ .

In (7), Eq.(3.2) implies  $(d_1, d_2) = (2, 1)$  and nullity implies  $\frac{1}{d_3} + \frac{1}{d_4} + \frac{1}{d_5} = \frac{3}{8}$ . One can now check that the only possible elements  $\bar{u}$  orthogonal to  $\bar{c}$  correspond to  $u = (1, 0, 0, -2, \dots)$  if  $d_4 = 4$  and  $(1, 0, 0, 0, -2, \dots)$  if  $d_5 = 4$ . The nullity condition means that at most one of these can occur. This is a contradiction as  $r \geq 5$ .

In (8) Eq.(3.2) gives  $(d_1, d_2) = (2, 3)$  and nullity of  $\bar{c}$  gives  $\frac{1}{d_3} + \frac{1}{d_4} = \frac{1}{4}$ . Again one can check that  $\frac{1}{2}(d + \mathcal{W}) \cap \bar{c}^\perp$  is empty, a contradiction as  $r \geq 4$ .

In (9), Eq.(3.2) and the nullity of  $\bar{c}$  give  $(d_1, d_4) = (2, 16)$  and  $\{d_2, d_3\} = \{2, 3\}$ . The only  $u$  which can give  $\bar{u} \in \bar{c}^\perp$  are  $(0, -2, 0, 0, 1^i, \dots)$  if  $d_2 = 2$  or  $(0, 0, -2, 0, 1^i, \dots)$  if  $d_3 = 2$ , where  $i \geq 5$ . In each case,  $i$  is unique since  $d_2$  (resp.  $d_3$ )  $\neq 4$ . Since  $r \geq 4$ , Theorem 8.9 now implies  $r = 4$ . But now these  $u$  are not present (as  $i \geq 5$ ). Therefore there are actually no (1A) vertices, a contradiction to  $r = 4$ .



In (10) we have  $(d_3, d_4) = (2, 16)$  and  $\{d_1, d_2\} = \{2, 3\}$ . The only  $u$  which can give an element of  $\bar{c}^\perp$  is  $(0, -2, 0, 0, 1^i, \dots)$  (for  $i$  unique and  $\geq 5$ ) if  $d_2 = 2$ . The final argument in (9) now applies equally here.

Finally, we can eliminate (2) by an analysis of both the (1A) and (1B) vertices. First, Eq.(3.2) and nullity of  $\bar{c}$  force  $(d_1, d_2, d_3) = (6, 1, 8)$ . Next we check that  $\frac{1}{2}(d + \mathcal{W}) \cap \bar{c}^\perp$  is empty, so  $r = 3$ . Using Remark 3.9 we then find there can be no (1B) vertices, giving a contradiction.

So case (ii) cannot occur if  $K$  is connected.

**Remark 9.1.** Case (ii) is the only part of this section that relies on the connectedness of  $K$ . In fact, the analysis of the cases where  $w$  is type III does not use this assumption. If  $K$  is not connected, using the same methods and with more computation we obtain the following additional possibilities (all of which are associated to a  $w$  of type II).

	$v$	$w$	$c = (4v - w)/3$	$d$	$r$
(9*)	$(0, -1, -1, 1)$	$(1, -1, -1, 0)$	$(-\frac{1}{3}, -1, -1, \frac{4}{3})$	$(1, 2, 6, 8)$	4, 5
				$(1, 6, 2, 8)$	4, 5
(10*)	$(1, -1, 0, -1)$	$(1, -1, -1, 0)$	$(1, -1, \frac{1}{3}, -\frac{4}{3})$	$(3, 3, 1, 8)$	4
				$(6, 2, 1, 8)$	4, 5
(14)	$(0, -1, 0, 1, -1)$	$(1, -1, -1, 0, 0)$	$(-\frac{1}{3}, -1, \frac{1}{3}, \frac{4}{3}, -\frac{4}{3})$	$(1, 3, 1, d_4, d_5)$	5

In (9\*) and (10\*) there is always a (1B) vertex in  $\Delta^{\bar{c}}$ , and  $r = 4$  or  $5$  according to whether the cardinality of  $\bar{c}^\perp \cap \frac{1}{2}(d + \mathcal{W})$  is 1 or 2. The dimensions  $d_4, d_5$  in (14) must satisfy  $\frac{1}{4} = \frac{1}{d_4} + \frac{1}{d_5}$  (i.e.,  $\{d_4, d_5\} = \{5, 20\}, \{6, 12\}, \{8, 8\}$ ) and again there is always a (1B) vertex in  $\Delta^{\bar{c}}$ .

### Interior points

Finally, we must consider the cases, listed in Table 3, §3, when there may be points of  $\mathcal{W}$  in the interior of  $vw$ . As in the earlier cases, we analyse the possible (1A) and (1B) vertices for these  $c$ .

For (1) and (2), as  $1 < \lambda \leq 2$ , the nonzero entries of  $c$  are either  $< -2$  or  $> 1$ . Hence by Lemma 4.4 there are no (1A) vertices. By Theorem 8.9 we have  $r = 2$  or  $3$ .

In case (3) the nullity of  $\bar{c}$  implies that a vector  $u \in \mathcal{W}$  not collinear with  $vw$  and with  $\bar{u} \in \bar{c}^\perp$  must be of the form  $(-2, 0, 1^j)$ . So  $2\lambda - 1 = \frac{d_1}{2}$  and  $c = (-\frac{d_1}{2}, -1 + \frac{d_1}{2}, 0, \dots)$ . From the range for  $\lambda$  and the nullity of  $\bar{c}$ , we have  $d_1 = 3$ ,  $d_2 = 1$ . But  $d_2 \neq 1$  since  $w \in \mathcal{W}$ . So again there are no (1A) vertices and by Theorem 8.9 we have  $r \leq 3$ .

In case (4), a straight-forward preliminary analysis reduces the possibilities of  $u \in \mathcal{W}$  such that  $\bar{u} \in \bar{c}^\perp$  to the choices  $u = (-2, 0, 1, \dots)$ ,  $(0, -2, 1, \dots)$  or  $(-1, -1, 1, \dots)$ . Note that  $c_1 < -2$  and  $c_2 = 1$ , so by Lemma 4.4(a) we see  $c_3 < 1$ , i.e.  $\lambda < \frac{3}{2}$ . Now the second vector cannot occur because the orthogonality equation and  $\lambda \leq 2$  imply that  $d_3 = 1$ , contradicting the presence of  $w$ . Since the three vectors are collinear, if two satisfy the orthogonality equation then all do. So there is at most one (1A) vertex and so  $r \leq 4$  by Theorem 8.9. This can be improved to  $r \leq 3$  as follows. If the third vector  $(-1, -1, 1, \dots)$  occurs then the orthogonality relation, the bound on  $\lambda$ , and nullity imply that  $d_1 = 5, d_3 = 2$  and  $\lambda = \frac{5}{7} \left(2 + \frac{1}{d_2}\right)$ . Now the nullity equation may be written as a quadratic in  $2 + \frac{1}{d_2}$  with no rational root. If the first vector  $(-2, 0, 1, \dots)$  occurs then orthogonality implies  $\lambda = \frac{d_1(d_3+2)}{4d_3+2d_1}$ , and the bound on  $\lambda$  gives  $\frac{6}{d_1} + \frac{1}{d_3} > 1$ . Nullity implies  $d_1 \geq 5$  and  $d_1 > d_3^2$ . We can deduce  $d_3 = 2$  and  $\lambda = \frac{2d_1}{d_1+4}$ , and one can check that nullity fails.

For case (5), again a straight-forward analysis of the orthogonality condition with the help of the nullity of  $\bar{c}$  gives the following  $u \in \mathcal{W}$  as possibilities such that  $J(\bar{c}, \bar{u}) = 0$ :

- (a)  $(-2^3, 1^i)$ ,  $i \geq 4$  and  $d_3 = 2$ ,
- (b)  $(1, -2, 0, \dots)$ ,
- (c)  $(-2^2, 1^i)$ ,  $i \geq 4$  and  $d_2 = 3$ ,
- (d)  $(0, -2, 1, 0, \dots)$ .

Note  $(1, 0, -2, \dots)$  cannot be in  $\mathcal{W}$  as then  $vw$  is not an edge.

It follows from Cor 4.3 that among (a) only one vector can occur and among (b), (c), (d) also only one vector can occur. (The orthogonality conditions of (b) and (d) are incompatible with  $1 < \lambda \leq 2$ .) So  $\bar{c}^\perp \cap \frac{1}{2}(d + \mathcal{W})$  contains at most two elements. If it has two elements, one must then come from (a) and the other from (b)-(d). Together they give an edge of  $\bar{c}^\perp \cap \text{conv}(\frac{1}{2}(d + \mathcal{W}))$  with no interior points in  $\frac{1}{2}(d + \mathcal{W})$ . Using Cor 4.3 and the null condition, we find that all these two-element cases cannot occur. Hence  $r \leq 4$ .

If  $r = 4$  then the possible  $u$  with  $J(\bar{c}, \bar{u}) = 0$  are given by (a)-(d) with  $i = 4$ . Now, we can show using techniques similar to those of Theorem 3.11 that  $c^* = (1 - 2\lambda, 2\lambda - 1, -1, 0)$  also gives a null element of  $\mathcal{C}$ . The possible vectors orthogonal to this element come from (a) and the vectors  $(b^*)$ ,  $(c^*)$ ,  $(d^*)$  obtained from (b), (c), (d) by swapping places 1 and 2.

If (a) does not give an element in  $\bar{c}^\perp \cap \frac{1}{2}(d + \mathcal{W})$ , it is straightforward to show, using the orthogonality and nullity conditions for  $\bar{c}$  and  $\bar{c}^*$  together, that the (1A) vertices for  $\bar{c}$  and  $\bar{c}^*$  are given by (b) and  $(b^*)$  respectively. Also we must have  $c = (\frac{4}{3}, -\frac{4}{3}, -1, 0)$  and  $d = (4, 4, 9, d_4)$ . We need a (1B) vertex outside  $x_4 = 0$ . From Remark 3.9, the only possible (1B) vertices for  $\bar{c}$  correspond to  $(1, 0, 0, -2)$  and  $(1, -1, 0, -1)$ . In particular there can be no vertices, and hence no elements of  $\mathcal{W}$ , with  $x_4 > 0$ . Hence  $(-1, -1, 0, 1)$  and therefore  $(1, -1, 0, -1)$  are not in  $\mathcal{W}$ . So the (1B) vertices for  $\bar{c}$  and  $\bar{c}^*$  are given by  $(1, 0, 0, -2)$  and  $(0, 1, 0, -2)$  respectively. Now the line joining the corresponding null vectors  $a, a^*$  misses  $\text{conv}(\mathcal{W})$ , a contradiction.

The remaining case is when (a) gives the element in  $\bar{c}^\perp \cap \frac{1}{2}(d + \mathcal{W})$  and in  $\bar{c}^{*\perp} \cap \frac{1}{2}(d + \mathcal{W})$ . Now for  $vw$  to be an edge we need  $(-1^4)$  absent, so by Remark 1.2(b) the only possible members of  $\mathcal{W}$  lying outside  $\{x_4 = 0\}$  are the three type IIIs with  $x_4 = 1$ . In particular, all vectors in  $\text{conv}(\mathcal{W})$  have  $x_4 \geq 0$ . As  $(-1, -1, 1, 0) \in \mathcal{W}$ , there must be (1B) vertices lying in  $\{x_4 = 0\}$  for both  $\bar{c}$  and  $\bar{c}^*$ . We then find that the only possibilities for such a (1B) vertex are given by (b),  $(b^*)$  respectively. It follows that  $d_1 = d_2$ , but now nullity is violated.

We conclude that there are no (1A) vertices, so  $r \leq 3$ .

**Theorem 9.2.** *Let  $\bar{c}$  be a null vertex in  $\mathcal{C}$  such that  $\Delta^{\bar{c}}$  contains a type (2) vertex corresponding to an edge  $vw$  of  $\text{conv}(\mathcal{W})$ . Suppose we are not in the situation of Theorem 3.14.*

(i) *If there are no points of  $\mathcal{W}$  in the interior of  $vw$ , then either we are in the situation of Example 8.2 of [DW4] or  $K$  is not connected and we are in one of cases in the table of Remark 9.1 or in the situation of the third paragraph of Example 8.3 of [DW4].*

(ii) *If there are interior points of  $vw$  in  $\mathcal{W}$  then  $r \leq 3$ .  $\square$*

For further remarks about the  $r = 2$  case see the concluding remarks at the end of §10.

## 10. Completing the classification

Throughout this section we will assume that  $K$  is connected (and we are not in the situation of Theorem 3.14). Theorems 8.7 and 9.2 then tell us that if  $r \geq 4$  there are no type (2) vertices and no adjacent (1B) vertices in  $\Delta^{\bar{c}}$ , for *any* null vector  $\bar{c} \in \mathcal{C}$ . Since all (1B) vertices lie in the half-space  $\{J(\bar{c}, \cdot) > 0\}$  bounded by the hyperplane  $\bar{c}^\perp$  containing the (1A) vertices, we must therefore have in *each*  $\Delta^{\bar{c}}$  exactly one (1B) vertex, with the remaining vertices all of type (1A). So if  $r \geq 4$  the only remaining task is to analyse such a situation.

As  $\dim \Delta^{\bar{c}} = r - 2$ , we see  $\dim(\Delta^{\bar{c}} \cap \bar{c}^\perp) = r - 3$ . In particular, there must be at least  $r - 2$  elements of  $\mathcal{W}$  giving elements of  $\bar{c}^\perp \cap \frac{1}{2}(d + \mathcal{W})$ .

Theorem 5.18 lists the possible configurations of such elements when  $r \geq 3$ . The above discussion, together with Remark 5.19, shows that in cases (1), (2), (3) we can take  $m = r - 1$ , in cases (5)(ii), (5)(iii), (6)(i), (6)(ii) we can take  $m = r$ , and in (6)(iii) we have  $r = 5$ .

Finally, since the vectors in (4) are collinear, so that  $\dim(\Delta \cap \bar{c}^\perp) = 1$ , we have  $r = 3$  or 4. If  $r = 3$ , it follows that  $\bar{c}$  and the edge in  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$  determined by the vectors in (4) are

collinear. This contradicts the orthogonality condition for the configuration of vectors in (4) and we conclude that  $r = 4$ .

For each configuration we can consider the possible vectors  $u \in \mathcal{W}$  giving the (1B) vertex. Besides the nullity condition on  $\bar{c}$  and the condition that  $\bar{c}$  should be orthogonal to the (1A) vectors, we have a further relation coming from the null condition in Remark 3.9. In most cases, routine (but occasionally tedious) computations show that these relations have no solution.

As a result, we obtain the following possibilities for  $u$  (up to obvious permutations):

case	(1A) vectors	possible (1B) vector
(1)	$(-2^1, 1^i), 2 \leq i \leq r-1$	$(-1^r), (1^2, -2^r), (-1^1, -1^2, 1^r)$ $(-1^1, 1^2, -1^3), (-1^1, 1^2, -1^r)$ $(-1^2), (-1^2, -1^3, 1^r)$
(2)	$(1^1, -2^i), 2 \leq i \leq r-1$	
(3)	$(1^1, -2^2); (1^1, -2^3);$ possibly $(1^1, -1^2, -1^3);$ $(-1^2, -1^3, 1^i), 4 \leq i \leq r-1$	$(1^4, -2^r), (1^1, -2^r), (-1^r)$ $(-1^4), (1^1, -2^4), (1^1, -1^2, -1^4),$ $(-1^1), (-1^4)$
(4)	$(1^1, -2^2), (1^1, -2^3), (1^1, -1^2, -1^3)$	$(1^3, -2^4), (-1^1, 1^2, -1^4), (1^3, -1^4, -1^5)$
(5)(i)	$(-2^1, 1^2), (-1^1, 1^2, -1^3)$	$(-2^2, 1^4), (-1^1, -1^2, 1^4), (-1^3, 1^4, -1^5)$
(5)(ii)	$(-2^1, 1^2); (-1^1, 1^3, -1^i), 4 \leq i \leq r$	$(-1^1, -1^3, 1^4)$
(5)(iii)	$(-2^1, 1^2); (-1^1, -1^3, 1^i), 4 \leq i \leq r$	$(1^1, -1^3, -1^4), (-1^2, 1^3, -1^4), (1^1, -2^3)$ $(-1^5)$
(6)(i)	$(-1^1, -1^2, 1^i), 3 \leq i \leq r$	
(6)(ii)	$(1^1, -1^2, -1^i), 3 \leq i \leq r$	
(6)(iii)	$(1^1, -1^2, -1^i), i = 3, 4; (1^1, -1^3, -1^4)$	

Table 7: Unique 1B cases

**Remark 10.1.** The possibilities for the (1B) vertex in cases (5)(ii) and (5)(iii) only apply to the  $r \geq 5$  situation. When  $r = 4$ , the two cases become the same if we switch the third and fourth summands and the possibilities are discussed in Lemma 10.14 below.

**Remark 10.2.** Note that  $u$  such as  $(-1^2), (-1^3)$  in (4),  $(-1^4)$  in (5)(ii) or  $(-1^i)$  with  $i > 2$  in (6)(ii) cannot arise because they will not be vertices, due to the presence of the type II vectors in  $\mathcal{W}$ .

**Remark 10.3.** The dimensions must satisfy certain constraints in each case. Some such constraints were stated in Theorem 5.18 and Remark 5.19. We also have constraints coming from the nullity conditions for  $\bar{c}$  and  $\bar{a}$ . These typically involve the requirement that some expression in the  $d_i$  is a perfect square. The following is a summary of general constraints in each case:

case (1):  $d_1 = 4$ ; case (2):  $d_1 = 1$ ; case (3):  $d_2 = d_3 = 2$ ; case (4):  $d_2 + d_3 \leq 4d_1/(d_1 - 1)$  and  $d_2, d_3, \geq 2$ ; case (5)(i):  $(d_1, d_2) = (4, 2), (3, 3)$ ; case (5)(ii,iii):  $d_1 = 2$  and if  $r \geq 5$  also  $d_3 = 2$ ; case (6)(i, ii):  $d_1 = d_2 = 2$ ; case (6)(iii):  $d_1 = d_2 = d_3 = d_4 = 2, d_5 = 25$ .

Our strategy now is reminiscent of that in §8. We have a (1B) vertex corresponding to  $\bar{u}$  and a second null vector  $\bar{a}$  satisfying  $a = 2u - c$ . Now we may apply our arguments to  $\bar{a}$ , and conclude that the vectors in  $\bar{a}^\perp \cap \frac{1}{2}(d + \mathcal{W})$  are also of the form given in the above table, up to permutation. The resulting constraints will allow us to finish our classification.

In some cases we can actually show that  $\bar{a}^\perp \cap \frac{1}{2}(d + \mathcal{W})$  is empty and we have a contradiction. A simple example when this happens is case 6(iii), where we now have  $c = (\frac{6}{5}, -\frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}, -1)$  and  $(a_i/d_i) = (-\frac{3}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, -\frac{1}{25})$ . Other cases are treated in Lemma 10.4 below.

Next we shall show that cases (6)(i)(ii) cannot arise (cf Lemma 10.6), so in all remaining cases there must be at least one type III vector  $w$  with  $\bar{w}$  in  $\bar{a}^\perp$ . We now use our explicit formulae for  $c$  and  $a$  to derive inequalities on the entries of  $a$  and find when there can be such a type III vector orthogonal to  $\bar{a}$ . For each such instance we then check whether  $\bar{a}^\perp \cap \frac{1}{2}(d + \mathcal{W})$  forms a configuration equivalent to one of those in Table 4. This turns out to be possible in only two

situations (cf Lemmas 10.7, 10.11 and Lemma 10.9). These have such distinctive features that  $\mathcal{W}$  can be completely determined and judicious applications of Prop 3.7 lead to contradictions. This yields our main classification theorem.

**Lemma 10.4.** *The following cases cannot arise:*

- case (4) with  $u = (1, -1, 0, -1)$ ,
- case (6)(ii) with  $u = (1, 0, -2, \dots)$ ,
- case (4) with  $u = (0, 0, 0, -1)$  except for the case  $c = (\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}, -1)$  with  $d = (4, 2, 2, 9)$ .

*Proof.* ( $\alpha$ ) For case (4) with  $u = (1, -1, 0, -1)$  we find that the nullity and orthogonality conditions and Remark 3.9 leave us with the following possibilities:

$d$	$c$	$(a_i/d_i)$
$(2, 5, 3, 20)$	$(1, -\frac{5}{4}, -\frac{3}{4}, 0)$	$(\frac{1}{2}, -\frac{3}{20}, \frac{1}{4}, -\frac{1}{10})$
$(2, 6, 2, 12)$	$(1, -\frac{1}{2}, -\frac{1}{2}, 0)$	$(\frac{1}{2}, -\frac{1}{12}, \frac{1}{4}, -\frac{1}{6})$
$(3, 4, 2, 12)$	$(1, -\frac{4}{3}, -\frac{2}{3}, 0)$	$(\frac{1}{3}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6})$
$(5, 3, 2, 15)$	$(1, -\frac{5}{5}, -\frac{4}{5}, 0)$	$(\frac{1}{5}, -\frac{4}{15}, \frac{2}{5}, -\frac{2}{15})$

It is easy to see that we can never have  $\sum_i \frac{w_i a_i}{d_i} = 1$  for  $w \in \mathcal{W}$ , so  $\bar{a}^\perp \cap \frac{1}{2}(d + \mathcal{W})$  is empty and we have a contradiction.

( $\beta$ ) Similarly, nullity, orthogonality and Remark 3.9 give:

$d$	$c$	$(a_i/d_i)$
$(2, 2, 225, d_4, \dots, d_r)$	$(\frac{232}{225}, -\frac{29}{30}, -\frac{1}{4}, -\frac{d_4}{900}, \dots, -\frac{d_r}{900})$	$(\frac{109}{225}, \frac{29}{60}, -\frac{1}{60}, \frac{1}{900}, \dots, \frac{1}{900})$
$(2, 2, 98, d_4, \dots, d_r)$	$(\frac{52}{49}, -\frac{13}{14}, -\frac{1}{2}, -\frac{d_4}{196}, \dots, -\frac{d_r}{196})$	$(\frac{23}{49}, \frac{13}{28}, -\frac{1}{28}, \frac{1}{196}, \dots, \frac{1}{196})$
$(2, 2, 36, d_4, \dots, d_r)$	$(\frac{10}{9}, -\frac{5}{6}, -1, -\frac{d_4}{36}, \dots, -\frac{d_r}{36})$	$(\frac{4}{9}, \frac{5}{12}, -\frac{1}{12}, \frac{1}{36}, \dots, \frac{1}{36})$

Moreover  $n$  equals 962, 226, 50 respectively. It is easy to see that we can never have  $\sum_i \frac{w_i a_i}{d_i} = 1$  for  $w \in \mathcal{W}$ , so we have a contradiction.

( $\gamma$ ) For case (4) with  $u = (0, 0, 0, -1)$  we find that the nullity and orthogonality conditions and Remark 3.9 leave us with the following possibilities, up to swapping places 2 and 3:

$d$	$c$	$(a_i/d_i)$
$(2, 2, 4, 25)$	$(\frac{6}{25}, -\frac{2}{5}, -\frac{4}{5}, -1)$	$(-\frac{3}{5}, \frac{1}{5}, \frac{1}{5}, -\frac{1}{25})$
$(2, 3, 3, 25)$	$(\frac{6}{25}, -\frac{3}{5}, -\frac{3}{5}, -1)$	$(-\frac{3}{5}, \frac{1}{5}, \frac{1}{5}, -\frac{1}{25})$
$(3, 2, 3, 121)$	$(\frac{15}{11}, -\frac{6}{11}, -\frac{9}{11}, -1)$	$(-\frac{5}{11}, \frac{3}{11}, \frac{3}{11}, -\frac{1}{121})$
$(2m-2, 2, 2, m^2)$	$(\frac{2(m-1)}{m}, \frac{1-m}{m}, \frac{1-m}{m}, -1)$	$(-\frac{1}{m}, \frac{m-1}{2m}, \frac{m-1}{2m}, -\frac{1}{m^2})$

It is now straightforward to see that we cannot have  $w \in \mathcal{W}$  with  $\sum_i \frac{w_i a_i}{d_i} = 1$ , except in two cases (both associated to the last entry of the table). One is the case stated in the Lemma. The other occurs if  $m = 2$ , so  $a = (-1, \frac{1}{2}, \frac{1}{2}, -1)$  which is orthogonal to  $(-1, 0, 1, -1), (-1, 1, 0, -1)$ . But as  $(0, 0, 0, -1)$  is a vertex, neither of these vectors can be in  $\mathcal{W}$ . So  $\bar{a}^\perp \cap \frac{1}{2}(d + \mathcal{W})$  is still empty, giving the desired contradiction.  $\square$

As discussed above, we now turn to showing that case (6) cannot occur. The following remark is useful in finding when type III vectors can give elements of  $\bar{a}^\perp \cap \frac{1}{2}(d + \mathcal{W})$ .

**Lemma 10.5.** *If  $w = (-2^i, 1^j)$  and  $\bar{w} \in \bar{a}^\perp$ , then  $\frac{a_i}{d_i} < 0$  (assuming we are not in the situation of Theorem 3.14).*

*Proof.* We need

$$(10.1) \quad \frac{a_j}{d_j} = 1 + \frac{2a_i}{d_i},$$

so if  $\frac{a_i}{d_i} \geq 0$  then  $\frac{a_j}{d_j} \geq 1$ . Hence  $\frac{a_j^2}{d_j^2} = d_j \left(\frac{a_j}{d_j}\right)^2 \geq d_j \geq 1$ . As  $\bar{a}$  is null this means  $a = (-1^j)$  and we are in the situation of Theorem 3.14.  $\square$

**Lemma 10.6.** *Configurations of type (6) cannot arise.*

*Proof.* Recall that we have dealt with (6)(iii) and we have  $d_1 = d_2 = 2$ .

Case (6)(i): We have  $u = (-1, 0, -1, 1, \dots)$ , and from the nullity and orthogonality conditions we deduce that

$$(c_i/d_i) = \left( -\frac{m+2}{2(m+1)}, -\frac{m-2}{2(m-1)}, \frac{1}{m^2-1}, \dots, \frac{1}{m^2-1} \right)$$

where  $\frac{1}{d_3} + \frac{1}{d_4} = \frac{1}{2(m+1)}$  and  $n-1 = m^2$  for some positive integer  $m$ . We have, therefore,

$$(a_i/d_i) = \left( \frac{-m}{2(m+1)}, \frac{m-2}{2(m-1)}, -\frac{2}{d_3} - \frac{1}{m^2-1}, \frac{2}{d_4} - \frac{1}{m^2-1}, \frac{-1}{m^2-1}, \dots, \frac{-1}{m^2-1} \right).$$

Let us estimate the size of the entries in  $(a_i/d_i)$ . First observe that, as  $d_3, d_4 > 2(m+1)$  from above, we have

$$4(m+1) < d_3 + d_4 \leq n - d_1 - d_2 = n - 4 = m^2 - 3$$

so we deduce  $m \geq 6$ . Hence we have  $\frac{3}{7} \leq |\frac{a_1}{d_1}| < \frac{1}{2}$ ,  $\frac{2}{5} \leq |\frac{a_2}{d_2}| < \frac{1}{2}$ ,  $|\frac{a_3}{d_3}| \leq \frac{6}{35}$ ,  $|\frac{a_i}{d_i}| \leq \frac{1}{35}$  for  $i \geq 5$ . Also note that  $\frac{a_4}{d_4} < \frac{2}{d_4} < \frac{1}{m+1} \leq \frac{1}{7}$ . Finally,  $\frac{a_4}{d_4} > 0$ , else we would have  $2n-4 = 2m^2-2 \leq d_4$ , which is impossible.

Consider now a type III vector  $w = (-2^i, 1^j)$  with  $\bar{w} \in \bar{a}^\perp$ . By Remark 10.5, we need  $i = 1, 3$  or  $\geq 5$ . If  $i = 1$  then, by Eq.(10.1), we have  $\frac{a_j}{d_j} = \frac{1}{m+1} \in (0, \frac{1}{7}]$ . So we must have  $j = 4$  and  $\frac{1}{m+1} < \frac{2}{d_4}$ , contradicting our above remarks. If  $i = 3$  then  $\frac{a_j}{d_j} \geq \frac{23}{35}$ , which is impossible. Similarly, if  $i \geq 5$  then  $\frac{a_j}{d_j} \geq \frac{33}{35}$ , which is impossible.

Hence there are no such type III vectors, so we are in case (6) with respect to  $\bar{a}$ . We cannot be in 6(ii) as then the null vector has exactly one positive entry (see below), but  $a$  has two positive entries. For 6(i), the null vector has exactly two negative entries. Now  $a$  has  $r-2$  negative entries, so  $r = 4$ . But for 6(i) the negative entries have modulus  $< 2$ , while  $a_3 = -2 - \frac{d_3}{m^2-1}$ , a contradiction.

Case 6(ii): Here there are two possibilities.

*Subcase ( $\alpha$ ):*  $u = (0, -1, 1, -1, \dots)$ . Then, as above, the null condition for  $\bar{c}$  gives

$$(c_i/d_i) = \left( \frac{(m-1)(m+2)}{2(m+1)^2}, -\frac{m+2}{2(m+1)}, \frac{-1}{(m+1)^2}, \dots, \frac{-1}{(m+1)^2} \right)$$

where  $\frac{1}{d_3} + \frac{1}{d_4} = \frac{1}{2(m+1)}$  and  $n-1 = m^2$ . So the vector  $(a_i/d_i)$  is given by

$$\left( -\frac{(m-1)(m+2)}{2(m+1)^2}, \frac{-m}{2(m+1)}, \frac{2}{d_3} + \frac{1}{(m+1)^2}, -\frac{2}{d_4} + \frac{1}{(m+1)^2}, \frac{1}{(m+1)^2}, \dots, \frac{1}{(m+1)^2} \right).$$

As before, we have  $m \geq 6$ . We deduce  $|\frac{a_1}{d_1}| \leq \frac{1}{2}$ ,  $\frac{3}{7} \leq |\frac{a_2}{d_2}| < \frac{1}{2}$ ,  $|\frac{a_3}{d_3}| \leq \frac{8}{49}$ ,  $|\frac{a_4}{d_4}| \leq \frac{1}{7}$ ,  $|\frac{a_i}{d_i}| \leq \frac{1}{49}$  for  $i \geq 5$ . Also  $\frac{a_4}{d_4} < 0$ , else  $d_4 \geq 2(m+1)^2 > 2n$ , which is impossible.

We look for vectors  $w = (-2^i, 1^j)$  with  $\bar{w} \in \bar{a}^\perp$ . By Lemma 10.5 we have  $i = 1, 2$  or  $4$ . If  $i = 1$  then  $\frac{a_j}{d_j} = \frac{m+3}{(m+1)^2}$ , so  $j = 3$ , but now Eq.(10.1) contradicts  $\frac{2}{d_3} < \frac{1}{m+1}$ . A similar argument works if  $i = 2$ , while if  $i = 4$ , Eq.(10.1) implies  $\frac{a_j}{d_j} > \frac{5}{7}$ , a contradiction.

So 6(ii) must hold for  $\bar{a}$ , as we have already ruled out 6(i). But now we need  $a$  to have exactly one positive entry, which has modulus  $< 2$ . So  $r = 4$  and this positive entry is  $a_3$ , but we have  $a_3 > 2$ , a contradiction.

Subcase ( $\beta$ ):  $u = (1, 0, -1, -1, \dots)$ . We similarly have

$$(c_i/d_i) = \left( \frac{(m-2)(m+1)}{2(m-1)^2}, -\frac{m-2}{2(m-1)}, \frac{-1}{(m-1)^2}, \dots, \frac{-1}{(m-1)^2} \right).$$

$$(a_i/d_i) = \left( \frac{m^2 - 3m + 4}{2(m-1)^2}, \frac{m-2}{2(m-1)}, \frac{1}{(m-1)^2} - \frac{2}{d_3}, \frac{1}{(m-1)^2} - \frac{2}{d_4}, \frac{1}{(m-1)^2}, \dots, \frac{1}{(m-1)^2} \right)$$

where  $n-1 = m^2$  and  $\frac{1}{d_3} + \frac{1}{d_4} = \frac{m+1}{2(m-1)^2}$ . The last two equations easily imply that  $m \geq 6$ , and so  $|\frac{a_i}{d_i}| < \frac{1}{2}$  for all  $i$ .

A type III  $(-2^i, 1^j)$  giving an element of  $\bar{a}^\perp$  must have  $i = 3$  or  $4$ , by Lemma 10.5. In both cases we find from Eq.(10.1) that  $\frac{a_j}{d_j} > \frac{1}{2}$ , which is impossible. So 6(ii) holds for  $a$ , which is impossible as  $a$  has at least two positive entries.  $\square$

**Lemma 10.7.** *The only possible example in case (4) is when  $c = (\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}, -1)$ ,  $u = (0, 0, 0, -1)$ , and  $d = (4, 2, 2, 9)$ .  $\bar{a}^\perp$  is then in case (1) with  $a = (-\frac{4}{3}, \frac{2}{3}, \frac{2}{3}, -1)$  and  $\bar{a}^\perp \cap \frac{1}{2}(d + \mathcal{W})$  consists of  $(-2, 1, 0, 0), (-2, 0, 1, 0)$ .*

*Proof.* By Lemma 10.4 we just have to eliminate the possibility  $u = (1, 0, 0, -2)$ . Now

$$(a_i/d_i) = \left( \frac{8 - 2d_1 + d_4}{d_1(4 + 2d_1 + d_4)}, \frac{(d_1 - 1)(2d_1 + d_4)}{2d_1(4 + 2d_1 + d_4)}, \frac{(d_1 - 1)(2d_1 + d_4)}{2d_1(4 + 2d_1 + d_4)}, -\frac{2}{d_4} - \frac{2(d_1 - 1)}{d_1(2d_1 + d_4 + 4)} \right).$$

The null condition is

$$-(d_1 - 1)d_4^3 - 4(d_1^2 - d_1 - 1)d_4^2 + 4d_1(3d_1^2 + d_1 + 8)d_4 + 16d_1^2(d_1 + 2)^2 = 0.$$

For  $w = (-2^i, 1^j)$  with  $\bar{w} \in \bar{a}^\perp$  we need, by Lemma 10.5,  $i = 1$  or  $4$ . If  $i = 1$ , then for  $j = 2$  or  $3$ , Eq.(10.1) can be rewritten as  $2d_1^2 + d_1d_4 + 2d_1 = -5d_4 - 32$ , which is absurd. For  $j = 4$  it can be rewritten as  $-1 - \frac{2}{d_4} = \frac{14 + 2d_4 - 2d_1}{d_1(2d_1 + d_4 + 4)}$ . So the right hand side is  $< -1$ , which on clearing denominators is easily seen to be false.

If  $i = 4$ , then for  $j = 1$  Eq.(10.1) becomes  $\frac{4}{d_4} + \frac{1}{d_1} = 1$ . So  $(d_1, d_4) = (2, 8), (3, 6)$  or  $(5, 5)$ , all of which violate the null condition. For  $j = 2, 3$  we obtain from Eq.(10.1) the equation  $4d_4(d_1 - 1) = (2d_1 + d_4 + 4)(d_1d_4 + d_4 - 8d_1)$ , which can only have solutions if  $d_4 \leq 9$ . On the other hand, the null condition has no integer solutions if  $d_4 \leq 9$ .

So no such type III exists, contradicting Lemma 10.6.  $\square$

**Lemma 10.8.** *Configurations of type (5)(i) cannot occur.*

*Proof.* It is useful to note that the null condition for  $\bar{c}$  implies that  $d_3 \leq 4$  when  $(d_1, d_2) = (4, 2)$  and  $d_3 \leq 3$  when  $(d_1, d_2) = (3, 3)$ . One further finds the following possibilities:

$u$	$d$	$c$	$(a_i/d_i)$
$(-1, 0, 0, 0)$	$(3, 3, 1, 2)$	$(-1, 1, -\frac{1}{3}, -\frac{2}{3})$	$(-\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
	$(3, 3, 2, 1)$	$(-1, 1, -\frac{2}{3}, -\frac{1}{3})$	$(-\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
	$(4, 2, 1, 3)$	$(-1, 1, -\frac{1}{4}, -\frac{3}{4})$	$(-\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
	$(4, 2, 2, 2)$	$(-1, 1, -\frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
	$(4, 2, 3, 1)$	$(-1, 1, -\frac{3}{4}, -\frac{1}{4})$	$(-\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
$(0, 0, 0, -1)$	$(3, 3, 2, 121)$	$(-\frac{9}{11}, \frac{15}{11}, -\frac{6}{11}, -1)$	$(\frac{3}{11}, -\frac{5}{11}, \frac{3}{11}, -\frac{1}{121})$
	$(4, 2, 2, 25)$	$(-\frac{4}{5}, \frac{6}{5}, -\frac{2}{5}, -1)$	$(\frac{1}{5}, -\frac{3}{5}, \frac{1}{5}, -\frac{1}{25})$

One easily checks that  $\bar{a}^\perp \cap \frac{1}{2}(d + \mathcal{W})$  is empty in the last two cases, and consists only of type II vectors in the third to fifth cases, giving a contradiction to Lemma 10.6. For the first two cases, note that  $\bar{a}^\perp \cap \frac{1}{2}(d + \mathcal{W})$  contains  $\frac{1}{2}(d + (-1, -1, 1, 0))$  since by hypothesis for 5(i)  $(-1, 1, -1, 0)$  is in  $\mathcal{W}$ . Hence (1),(2) cannot hold with respect to  $\bar{a}$ . Also the vector  $d$  of dimensions rules out (3),(4) and (5), so we have a contradiction.  $\square$

**Lemma 10.9.** *The only possible example for case (3) is when  $c = (\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, -1)$ ,  $u = (0, 0, 0, 0, -1)$ ,  $d = (2, 2, 2, 2, 9)$ . Then  $a = (-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, -\frac{2}{3}, -1)$  and  $\Delta^{\bar{a}}$  is again in case (3).*

*Proof.* (A) Let  $u = (-1^r)$ . The null condition for  $\bar{c}$  gives  $4d_r = (\delta + d_1 + 2)^2$ , where  $\delta = d_4 + \dots + d_{r-1}$ . In particular,  $d_r$  is a square. Also,  $(a_i/d_i) = (\frac{-1}{\sqrt{d_r}}, \frac{1}{2}(1 - \frac{1}{\sqrt{d_r}}), \frac{1}{2}(1 - \frac{1}{\sqrt{d_r}}), \frac{-1}{\sqrt{d_r}}, \dots, \frac{-1}{\sqrt{d_r}}, \frac{-1}{d_r})$ .

If  $d_r = 4$ , then we find there are no type III vectors in  $\bar{a}^\perp$ , a contradiction. So  $d_r \geq 9$  and we have  $|\frac{a_i}{d_i}| \leq \frac{1}{3}$  for  $i = 1, 4, \dots, r-1$ ,  $\leq \frac{1}{9}$  for  $i = r$  and  $< \frac{1}{2}$  for  $i = 2, 3$ . Lemma 10.5 shows  $i \neq 2, 3$ . From Eq.(10.1) and the above estimates, we first get  $i \neq r$ , and for the remaining values of  $i$ , we have  $\frac{a_j}{d_j} > 0$ , so that  $j = 2$  or  $3$ . Also,  $d_r = 9$  (and hence  $d_1 + d_4 + \dots + d_{r-1} = 4$ ), and  $(a_i/d_i) = (-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, \dots, -\frac{1}{3}, -\frac{1}{9})$ . Upon applying Theorem 5.18 to  $\Delta^{\bar{a}}$  together with Theorem 8.9 and the above Lemmas, we deduce that we are in case (3)(ii) with  $r = 5$  and  $d_1 = d_4 = 2$ , giving the example in the statement of the Lemma.

(B) Next let  $u = (1, 0, \dots, -2)$ . Now  $(a_i/d_i) = (\frac{2}{d_1} - \alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{2}, -\alpha, \dots, -\alpha, \frac{(n-2-d_r)\alpha-5}{d_r})$  where, as a consequence of the null condition for  $\bar{c}$ , we have

$$(10.2) \quad \alpha = \frac{c_1}{d_1} = \frac{1}{n-2-m} \quad : \quad m^2 = d_r(n-1).$$

Next we get the identity  $(n-2)^2 - m^2 = (n-1)(d_1 + 1 + \delta) + 1 = \frac{m^2(d_1+1+\delta)}{d_r} + 1$ , where  $\delta$  is as given in (A) above. We deduce  $m < \sqrt{\frac{d_r}{n-3}}(n-2)$ , and hence  $\alpha < \frac{2}{d_1+1+\delta} = \frac{2}{n-d_r-3} \leq \min(\frac{1}{2}, \frac{2}{d_1})$ . So  $\frac{a_i}{d_i}$  is positive for  $i \leq 3$  and negative for  $3 < i \leq r-1$ . Note also that  $(n-2-d_r)\alpha < \frac{2(n-2-d_r)}{n-3-d_r} = 2(1 + \frac{1}{n-3-d_r}) \leq \frac{5}{2}$ . In particular,  $\frac{a_r}{d_r} < -\frac{5}{2d_r} < 0$ .

As usual, we look for  $(-2^i, 1^j)$  giving an element of  $\bar{a}^\perp$ . By Lemma 10.5,  $i \neq 1, 2, 3$ . If  $4 \leq i \leq r-1$  then Eq.(10.1) says  $\frac{a_j}{d_j} = 1 - 2\alpha > 0$ , so  $j = 1, 2$  or  $3$ . If  $j = 1$  we obtain  $\alpha = 1 - \frac{2}{d_1}$ . Comparing this with Eq.(10.2) shows  $d_1 = 3$  and  $\alpha = \frac{1}{3}$ , but now  $\bar{a}$  is not null. If  $j = 2$  or  $3$ , we obtain  $\alpha = \frac{1}{3}$ ; we deduce from Eq.(10.2) that  $d_1 \leq 5$ , and again one can check that all possibilities violate nullity.

So all type III  $(-2^i, 1^j)$  have  $i = r$ . Therefore we must be in case (1) or (5) with respect to  $\bar{a}$ , and  $d_r = 4$  or  $2$  respectively.

If  $j = 1, 2, 3$  then  $\frac{a_j}{d_j} > 0$ . Now Eq.(10.1) combined with the estimate above for  $\frac{a_r}{d_r}$  show that  $-\frac{5}{2} > a_r > -\frac{d_r}{2}$ , so  $d_r > 5$ , a contradiction.

If  $4 \leq j \leq r-1$ , then in the case  $d_r = 4$ , we find Eq.(10.1) gives  $\alpha = \frac{3}{n-4}$ . Combining with Eq.(10.2) we get  $n = 10$ , which is incompatible with  $d_r = 4$  and  $r \geq 5$ . If  $d_r = 2$  we find similarly that  $\alpha = \frac{4}{n-3}$  and  $m$  satisfies  $3m^2 - 8m - 4 = 0$ ; but this has no integral roots.

So  $u = (1, 0, \dots, -2)$  cannot occur.

(C) For  $u = (1^4, -2^r)$ , we have  $(a_i/d_i) = (-\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{2}, \frac{2}{d_4} - \alpha, -\alpha, \dots, -\alpha, \frac{(n-2-d_r)\alpha-5}{d_r})$  and Eq.(10.2) still holds. The arguments of case (B) carry over to this case, on swapping indices 1, 4.  $\square$

**Lemma 10.10.** *Case (2) cannot occur.*

*Proof.* (A) Consider  $u = (-1^2)$ . Now  $(a_i/d_i) = (\frac{2}{d_2} - 1, \frac{-1}{d_2}, \frac{1}{d_2}, \dots, \frac{1}{d_2}, \frac{1}{d_r}(2 - \frac{n+1-d_r}{d_2}))$ .

Nullity implies  $d_2 \geq 3$ , so  $-1 < \frac{a_1}{d_1} \leq -\frac{1}{3}$  and  $|\frac{a_i}{d_i}| \leq \frac{1}{3}$  for  $2 \leq i \leq r-1$ . Also we have  $d_r(n-1) = m^2$ , and for this choice of  $u$  we have  $m = n+1-2d_2$ , so  $\frac{a_r}{d_r} = \frac{d_r-m}{d_2 d_r}$  which is positive if  $m < 0$  and negative if  $m > 0$ . By Lemma 10.5 and the fact that  $d_1 = 1$ , we only have to consider  $(-2^i, 1^j)$  with  $i = 2$  or  $r$ .

If  $i = 2$ , then Eq.(10.1) says  $\frac{a_j}{d_j} = 1 - \frac{2}{d_2} > 0$  so  $j \geq 3$ . If  $3 \leq j \leq r-1$  then Eq.(10.1) shows  $d_2 = 3$ , so  $m = n-5$  and  $d_r = \frac{(n-5)^2}{n-1} = n-9 + \frac{16}{n-1}$ . As  $n \geq 7$  we must then have  $(d_r, n) = (9, 17)$  or  $(2, 9)$ . Imposing the nullity condition on  $\bar{a}$  shows there are only three possibilities, corresponding

to  $d = (1, 3, 2, 2, 9), (1, 3, 4, 9), (1, 3, 3, 2)$ . In the first two there is only one type III in  $\bar{a}^\perp$ , as  $d_1 = 1$  and  $d_2 \neq 4$ , so we are in case (5) with respect to  $\bar{a}$ , contradicting the fact that  $d_2 \neq 2$ . In the last case we must be in case (4) with respect to  $\bar{a}$ , but now  $(0, -1, 1, -1)$  is present, so  $u$  is not a vertex. If  $j = r$  then Eq.(10.1) becomes  $d_2 = \frac{3d_r - n - 1}{d_r - 2}$  which is less than 3, a contradiction. (We cannot have  $d_r = 2$  and  $3d_r = n + 1$  as  $n \geq 7$ .)

Hence all such  $(-2^i, 1^j)$  have  $i = r$  and we are in case (1) or (5) with respect to  $\bar{a}$ . For case (1) we need  $r - 2$  of the  $\frac{a_j}{d_j}$  ( $j < r$ ) equal. This can only happen for our  $a$  if  $d_2 = 3$ , which is ruled out as in the previous paragraph. For case (5) we have  $d_r = 2$  and  $\frac{a_j}{d_j} = 3 - \frac{n-1}{d_2}$ . The possibilities on the left-hand side are  $\frac{2}{d_2} - 1, -\frac{1}{d_2}, \frac{1}{d_2}$  respectively. On using our relations for  $m, n, d_r$  we find that only the third possibility can occur, and  $d_2 = 3$ . The argument in the previous paragraph again eliminates this case.

(B) Consider  $u = (0, -1, -1, \dots, 1)$ . Now  $(a_i/d_i) = (2\beta - 1, \beta - \frac{2}{d_2}, \beta - \frac{2}{d_3}, \beta, \dots, \beta, \frac{4-(n+1-d_r)\beta}{d_r})$  where again from the null condition of  $\bar{c}$  we have

$$(10.3) \quad \beta := \frac{1}{2}(1 - c_1) = \frac{2}{n + 1 - m} = \frac{3 + d_r(\frac{1}{d_2} + \frac{1}{d_3})}{n + d_r + 1} : d_r(n - 1) = m^2 \ (m > 0).$$

Now  $0 < \beta < \frac{1}{2}$  (the case  $\beta = \frac{1}{2}$  leads to  $r = 4, d_2 = d_3 = 2$  and  $a = (0, -1, -1, 1)$  which violates nullity). So for  $\bar{w} \in \bar{a}^\perp$  we just have to consider  $w = (-2^i, 1^j)$  with  $i = 2, r$  (as  $d_1 = 1$  we can't have  $i = 1$ ; also by symmetry the case  $i = 3$  is treated the same way as  $i = 2$ ).

If  $i = 2$  then Eq.(10.1) says  $\frac{a_j}{d_j} = 2\beta + 1 - \frac{4}{d_2}$ . If  $4 \leq j \leq r - 1$  we get  $\beta = \frac{4}{d_2} - 1$ ; the only possibility consistent with our bounds on  $\beta$  is  $d_2 = 3, \beta = \frac{1}{3}$  and it is straightforward to check this is incompatible with the null condition for  $\bar{a}$ .

If  $i = 2$  and  $j = 1$  then Eq.(10.1) implies  $d_2 = 2$  and again one checks that nullity for  $\bar{c}$  fails. If  $j = 3$  Eq.(10.1) says  $\beta = \frac{4}{d_2} - \frac{2}{d_3} - 1$ , so as  $\beta > 0$  either  $d_2 = 2$  and  $\beta = 1 - \frac{2}{d_3}$  or  $d_2 = 3$  and  $\beta = \frac{1}{3} - \frac{2}{d_3}$ . In the former case the bound  $\beta < \frac{1}{2}$  shows  $d_3 = 3$  and  $\beta = \frac{1}{3}$ , and now nullity for  $\bar{a}$  fails. In the latter case the bound  $\beta > 0$  shows  $d_3 > 6$ . Substituting this into the quadratic which must vanish for nullity of  $\bar{c}$ , we see  $\delta = d_4 + \dots + d_{r-1}$  is  $< 4$ . Checking the resulting short list of cases yields no examples where nullity holds. If  $j = r$  then Eqs.(10.1) and Eq.(10.3) imply  $\frac{1}{d_r} + \frac{3}{d_2} = 1 + \frac{1}{d_3}$  and one check that the possible  $(d_2, d_r)$  yield no examples where nullity of  $\bar{c}$  holds.

So all such type III have  $i = r$ , and case (1) or (5) holds for  $\bar{a}$ . For case (1), then, as in (A),  $r - 2$  of the  $\frac{a_j}{d_j}$  ( $j \leq r - 1$ ) must be equal. So either  $r = 5$  and  $d_2 = d_3$  with  $\beta = 1 - \frac{2}{d_2}$  or  $r = 4$  and one of the preceding equalities holds. If  $\beta = 1 - \frac{2}{d_2}$  holds, then the bounds on  $\beta$  show  $d_2 = 3, \beta = \frac{1}{3}$  and as usual nullity for  $\bar{a}$  fails. If  $d_2 = d_3$  holds, then using our formulae for  $\beta$  and substituting into the null condition for  $\bar{a}$  gives a quadratic with no integer roots.

If case (5) holds, then  $d_r = 2$ . Now Eq.(10.1) gives  $\frac{a_j}{d_j} = 5 - (n - 1)\beta$ . If  $j = 1, j = 2$ , or  $4 \leq j \leq r - 1$  we get  $\beta = \frac{6}{n+1}, \frac{5+(2/d_2)}{n}, \frac{5}{n}$  respectively. (As usual, the case  $j = 3$  is treated in just the same way as  $j = 2$ .) Now using the equations in Eq.(10.3) relating  $n, m$  in each case gives a quadratic with no real roots.  $\square$

**Lemma 10.11.** *For case (1) the only possibility is when  $c = (-\frac{4}{3}, \frac{2}{3}, \frac{2}{3}, -1)$  with  $u = (0, 0, 0, -1)$  and  $d = (4, 2, 2, 9)$ .  $\Delta^{\bar{a}}$  is then in case (4).*

*Proof.* (A) Consider  $u = (0, \dots, 0, -1)$ . From the null condition for  $\bar{c}$  we see that  $d_r = k^2, n - 1 = (k + 1)^2$  for some positive integer  $k$  and  $(a_i/d_i) = (\frac{1}{2}(1 - \frac{1}{k}), -\frac{1}{k}, \dots, -\frac{1}{k}, \frac{1}{k^2})$ . Note that since  $d_1 = 4, n > 5$  and so  $k \neq 1$ .

We must consider solutions of Eq.(10.1). By Lemma 10.5,  $i \neq 1$ . If  $i = r$  we have  $\frac{a_j}{d_j} = 1 - \frac{2}{k^2}$ . The resulting equation has no solution in integer  $k > 1$  for any choice of  $j$ . If  $2 \leq i \leq r - 1$ , we need  $\frac{a_j}{d_j} = 1 - \frac{2}{k}$ . We only obtain a solution  $k > 1$  if  $j = 1$ ; in this case  $k = 3$ , so  $n = 17, d_r = 9$



and we see  $r = 4$  with  $\{d_2, d_3\} = \{2, 2\}$  or  $\{3, 1\}$ . The former case is that in the statement of the Lemma. In the latter case we can have just one type III and one type II in  $\bar{a}^\perp$  (since  $d_2$  or  $d_3$  is 1, one potential type III is missing), so we must be in case (5) with respect to  $\bar{a}$ ; but no  $d_i$  is 2, a contradiction.

(B) Consider  $u = (0, 1, 0, \dots, 0, -2)$ . Now  $(a_i/d_i) = (\frac{1}{2}(1-\beta), \frac{2}{d_2} - \beta, -\beta, \dots, -\beta, \frac{(n-d_r-2)\beta-5}{d_r})$  where  $\beta = \frac{d_r+6d_2}{d_2(2n-d_r-4)}$ . The nullity condition for  $\bar{c}$  implies  $d_2 \geq 3$ ,  $d_2 > \delta$  and  $d_r > 2d_2 + 4$ , where  $\delta$  now denotes  $d_3 + \dots + d_{r-1}$ . We can then deduce that  $\beta < 1$ ,  $0 < \frac{a_1}{d_1} < \frac{1}{2}$ ,  $0 < \frac{a_2}{d_2} < \frac{1}{3}$ . In particular  $\beta < \frac{2}{d_2}$ . By Lemma 10.5, we must consider elements of  $\bar{a}^\perp$  coming from vectors  $(-2^i, 1^j)$  with  $i \geq 3$ .

If  $3 \leq i \leq r-1$ , Eq.(10.1) says  $\frac{a_j}{d_j} = 1 - 2\beta$ . As  $\beta < 1$ , this immediately rules out  $3 \leq j \leq r-1$ . If  $j = 1$  we get  $\beta = \frac{1}{3}$ . Combining this with our formula above for  $\beta$  we get  $2d_2(d_2 + \delta - 7) + (d_2 - 3)d_r = 0$ . The only possibilities are  $d_2 = 3, \delta = 4$  which violates the null condition, or  $d = (4, 4, 2, 8)$  which violates the condition that  $d_r(n-1)$  should be a square. If  $j = 2$ , we get  $\beta = 1 - \frac{2}{d_2}$ . Since we saw above that  $\beta < \frac{2}{d_2}$  we get  $d_2 = 3$  and  $\beta = \frac{1}{3}$ , which is ruled out as above. If  $j = r$ , Eq.(10.1) implies  $\beta = \frac{d_r+5}{2+d_2+\delta+2d_r}$ . Comparing this with the formula for  $\beta$  above leads to a contradiction.

The remaining possibility is for  $i = r$ . So we are in case (1) or (5) with respect to  $\bar{a}$ , and  $d_r = 4$  or 2 respectively. But  $d_r > 2d_2 + 4$ , so this is impossible.

(C) Let  $u = (-1, -1, 0, \dots, 0, 1)$ . Now  $(a_i/d_i) = (-\frac{1}{2}\beta, -\frac{2}{d_2} - \beta, -\beta, \dots, -\beta, \frac{1+(n-d_r-2)\beta}{d_r})$  where  $\beta = \frac{d_r(d_2-4)}{2d_2(2n+d_r-4)}$ , so  $0 < \beta < \frac{1}{6}$  (noting that the nullity condition for  $\bar{c}$  implies  $d_2 \geq 5$ ).

We look for vectors  $(-2^i, 1^j)$  giving elements of  $\bar{a}^\perp$ . Now Lemma 10.5 rules out  $i = r$ , while if  $3 \leq i \leq r-1$  we need  $\frac{a_j}{d_j} = 1 - 2\beta > \frac{2}{3}$ . So  $j = r$ , and Eq.(10.1) yields  $\beta = \frac{d_r-1}{n+d_r-2}$ . Equating this to the expression above for  $\beta$  gives an equation which may be rearranged so that it says a sum of positive terms is zero, which is absurd.

If  $i = 1$  then Eq.(10.1) says  $\frac{a_j}{d_j} = 1 - \beta$ . Clearly this can only possibly hold if  $j = r$ . The equation then gives  $\beta = \frac{d_r-1}{n-2}$ , and equating this with the earlier expression for  $\beta$  leads, as in the previous paragraph, to a contradiction.

So the only possibility is  $i = 2$ , and we are therefore in case (1) or (5) with respect to  $\bar{a}$ . But  $d_2 \geq 5$  so this is impossible.

(D) Let  $u = (-1, 1, -1, 0, \dots)$ . Now  $(a_i/d_i) = (-\frac{1}{2}\beta, \frac{2}{d_2} - \beta, -\frac{2}{d_3} - \beta, -\beta, \dots, -\beta, \frac{(n-d_r-2)\beta-1}{d_r})$ . and  $\beta = \frac{1}{2} - 2(\frac{1}{d_2} + \frac{1}{d_3})$ . An analysis of the nullity condition for  $\bar{c}$  shows that it can only be satisfied if  $\frac{1}{5} < \frac{1}{d_2} + \frac{1}{d_3} < \frac{1}{4}$ , so  $d_2, d_3 \geq 5$  and  $0 < \beta < \frac{1}{10}$ .

Let us now consider solutions to Eq.(10.1). If  $i = r$ , we have  $\frac{a_j}{d_j} = 1 - \frac{2}{d_r} + \frac{2(n-d_r-2)\beta}{d_r}$ . If  $j \neq 2$ , this equation implies that the positive quantity  $(1 + \frac{2(n-d_r-2)}{d_r})\beta$  (or  $(\frac{1}{2} + \frac{2(n-d_r-2)}{d_r})\beta$  if  $j = 1$ ) equals a nonpositive quantity (recall  $d_r > 1$  as  $i = r$ ). If  $j = 2$ , we get that it equals  $\frac{2}{d_2} + \frac{2}{d_r} - 1$ . But  $d_2 \geq 5$  so  $d_r = 2$  or 3, and in each case we find the nullity condition for  $\bar{c}$  is violated.

If  $i = 1$ , Eq.(10.1) says  $\frac{a_j}{d_j} = 1 - \beta > \frac{9}{10}$ , so  $j = 2$  or  $r$ . But for  $j = 2$  we get  $d_2 = 2$ , which is impossible as we know  $d_2 \geq 5$ , so in fact  $j = r$ .

If  $i = 2$ , Eq.(10.1) is  $\frac{a_j}{d_j} = 1 + \frac{4}{d_2} - 2\beta$ . We cannot then have  $j = 1, 3$  or  $4 \leq j \leq r-1$  as they lead to  $\beta > \frac{2}{3}, > 1, > 1$  respectively. So we must have  $j = r$ .

If  $i = 3$ , we see  $\frac{a_j}{d_j} = 1 - \frac{4}{d_3} - 2\beta$ . If  $j = 1, 2$  or  $4 \leq j \leq r-1$ , we see in all cases (using our bounds on  $d_2, d_3$ ) that  $\beta > \frac{1}{5}$ , a contradiction. Hence again  $j = r$ .

If  $4 \leq i \leq r-1$ , then  $\frac{a_j}{d_j} = 1 - 2\beta > \frac{4}{5}$  so  $j = 2$  or  $r$ . If  $j = 2$  we obtain  $\beta = 1 - \frac{2}{d_2} \geq \frac{3}{5}$ , contradicting our earlier inequality for  $\beta$ ; so again we have  $j = r$ .

We have shown that any  $(-2^i, 1^j)$  giving an element of  $\bar{a}^\perp$  has  $j = r$ , so we are in case (3), (4) or (5) with respect to  $\bar{a}$ . It cannot be case (3) as we know from Lemma 10.9 that then each  $d_i$  is

2 or 9, and we have  $d_1 = 4$ . If we are in case (4), then Lemma 10.7 tells us that  $d = (4, 2, 2, 9)$ . Moreover, as  $(-2, 0, 0, 1), (0, -2, 0, 1)$  are the elements of  $\bar{a}^\perp \cap \frac{1}{2}(d + \mathcal{W})$ , we must have  $\beta = \frac{4}{d_2}$ ; but now  $\beta > \frac{1}{10}$ , a contradiction. If it is case (5), then we have  $d_i = 2$  for some  $i$ , which we can take to be 4. Now  $\bar{a}$  must be orthogonal to vectors associated to  $(-1^4, 1^5, -1^k)$  or  $(-1^4, -1^5, 1^k)$ , and either case is incompatible with our expressions for  $a_i/d_i$ .

(E) Consider  $u = (-1, 1, 0, \dots, 0, -1)$ . Now  $(a_i/d_i) = (-\frac{1}{2}\beta, \frac{2}{d_2} - \beta, -\beta, \dots, -\beta, \frac{(n-d_r-2)\beta-3}{d_r})$  and  $\beta = \frac{8d_2-d_r(d_2-4)}{2d_2(2n-d_r-4)}$ . It is easy to check that  $\beta < \frac{2}{d_2}$ . Also, the nullity condition for  $\bar{c}$  implies  $d_2 \geq 3$  and  $(\frac{4}{d_2} - 1)d_r + 8 > 0$ ; hence  $\beta > 0$ .

The analysis is similar to that in (D). If  $i = r$  then Eq.(10.1) implies that a positive quantity times  $\beta$  equals a positive linear combination of reciprocals of  $d_i$ , minus 1. This sum of reciprocals is therefore  $> 1$ , which gives us upper bounds on  $d_r$ . The only case where Eq.(10.1) and the null condition can hold is if  $j = 2$  and  $d_2 = 7, d_r = 4, d_3 + \dots + d_{r-1} = 11$ .

If  $i = 1$  then Eq.(10.1) says  $\frac{a_i}{d_i} = 1 - \beta > 0$ , so  $j = 2$  or  $r$ . But  $j = 2$  implies  $d_2 = 2$ , which from above cannot hold, so  $j = r$ .

Now Lemma 10.5 rules out  $i = 2$ . If  $3 \leq i \leq r-1$ , we have  $\frac{a_i}{d_i} = 1 - 2\beta$ . If  $j = 1$  then we get  $\beta = \frac{2}{3}$ , which cannot hold. If  $j = 2$  then  $\beta = 1 - \frac{2}{d_2}$ , and as  $\beta < \frac{2}{d_2}$  we deduce  $d_2 = 3$  and  $\beta = \frac{1}{3}$ , which violates the null condition for  $\bar{c}$ . If  $3 \leq j \leq r-1$ , then  $\beta = 1$ , which is impossible. So we have  $j = r$ .

So in all cases we have  $j = r$ , except in the exceptional case discussed above where we can have  $i = r$  and  $j = 2$ . But our list (1)-(6) of possible configurations in  $\bar{a}^\perp$  shows that if the  $(i, j) = (r, 2)$  case occurs then no other type III can be in  $\bar{a}^\perp$ . So we are in case (5), which is impossible as  $d_r = 4 \neq 2$  for this example. Hence the exceptional case cannot arise.

We see therefore that  $j = r$  in all cases. So, as in (D), we must be in case (3), (4) or (5) with respect to  $\bar{a}$ . As before, the fact that  $d_1 = 4$  rules out case (3). For case (4) we need the  $d_k$  to be 4, 2, 2, 9 and  $d_r$  to be 4 (as the  $(-2^i, 1^j)$  in  $\bar{a}^\perp$  have  $j = r$ ), but this contradicts  $d_1 = 4$ .

So we are in case (5). Now the orthogonality condition for the family of type II vectors leads to  $\beta > \frac{2}{3}$ , which is impossible.  $\square$

**Lemma 10.12.** *Case (5)(ii) cannot occur if  $r \geq 5$ .*

*Proof.* (A) Consider  $u = (0, 0, 1, -2, 0, \dots)$ . We have

$(a_i/d_i) = (1 - \beta, 1 - 2\beta, \frac{n-2d_2-3}{n-d_2-2} - (\frac{n-6-3d_2}{n-d_2-2})\beta, \frac{2(d_2+2)\beta}{n-d_2-2} - \frac{d_2+1}{n-d_2-2} - \frac{4}{d_4}, \frac{2(d_2+2)\beta}{n-d_2-2} - \frac{d_2+1}{n-d_2-2}, \dots)$  where all terms from the fifth onwards are equal and where

$$\beta := 1 + \frac{c_1}{2} = \frac{8(n-d_2-2)+d_4(n+d_2)}{2d_4(n+d_2+2)}.$$

The nullity condition for  $\bar{c}$  implies  $d_4 > 52$ , so  $n > 56$  and we deduce  $0 < \beta < \frac{15}{26} < \frac{3}{5}$ . Hence  $\frac{a_1}{d_1} > 0$ . It is also easy to show that  $\frac{a_i}{d_i} > 0$  for  $i \geq 5$  and  $\frac{a_3}{d_3} > 0$ .

So if  $(-2^i, 1^j)$  gives an element of  $\bar{a}^\perp$  we need  $i = 2$  or 4. As  $d_4 > 52$ , Lemmas 10.6 - 10.11 show that case (5) must hold with respect to  $\bar{a}$ . In particular  $d_i = 2$ , so we cannot have  $i = 4$ . Hence  $i = 2$  and  $d_2 = 2$ . Now Eq.(10.1) implies  $\beta = \frac{2}{3}, \frac{2n-5}{3n-4}, \frac{3n-9}{4(n-2)} + \frac{n-4}{d_4(n-2)}$  or  $\frac{3n-9}{4(n-2)}$ , depending on whether  $j = 1, 3, 4$  or  $\geq 5$ . In all cases this contradicts the bound  $\beta < \frac{3}{5}$  and  $n > 56$ .

(B) Consider  $u = (-1, 1, 0, -1, 0, \dots)$ . We have  $(a_i/d_i) = (-\beta, \frac{2}{d_2} + 1 - 2\beta, -\frac{d_2+1}{n-d_2-2} - (\frac{n-6-3d_2}{n-d_2-2})\beta, \frac{2(d_2+2)\beta}{n-d_2-2} - \frac{d_2+1}{n-d_2-2} - \frac{2}{d_4}, \frac{2(d_2+2)\beta}{n-d_2-2} - \frac{d_2+1}{n-d_2-2}, \dots)$  where all terms from the fifth onwards are equal and

$$\beta := 1 + \frac{c_1}{2} = \frac{n+d_2-2}{2(n+d_2+2)} + \frac{n-2}{d_2(n+d_2+2)} + \frac{n-d_2-2}{d_4(n+d_2+2)}.$$

The nullity condition for  $\bar{c}$  implies  $d_2 \geq 9$  and  $d_4 \geq 4$ . It is now easy to check that  $\frac{3}{7} < \beta < \frac{31}{36}$ , and that  $\frac{a_i}{d_i} > 0$  for  $i \geq 5$ .

As in (A), case (5) must hold with respect to  $\bar{a}$ . Now if  $(-2^i, 1^j)$  gives an element of  $\bar{a}^\perp$  we need  $d_i = 2$ . This, combined with Lemma 10.5, means  $i = 1$  or 3.

If  $i = 1$ , Eq.(10.1) immediately shows  $j$  cannot be 2. Moreover, if  $j = 3$  or  $\geq 5$ , Eq.(10.1) yields a value for  $\beta$  that violates the nullity condition for  $\bar{c}$ . If  $j = 4$  we obtain  $\beta = \frac{n-1}{2n} + \frac{n-d_2-2}{d_4n}$ . As we are in case (5) with respect to  $\bar{a}$ , we need to consider the elements in  $\bar{a}^\perp \cap \frac{1}{2}(d+\mathcal{W})$  corresponding to type II vectors. Their number and pattern, as stipulated by Theorem 5.18, together with orthogonality to  $\bar{a}$ , imply further linear relations among the components of  $(a_i/d_i)$  and small upper bounds for  $r$  (usually of the form  $r = 5, 6$ ). In all cases these additional constraints can be shown to be incompatible with the above values of  $\beta$ .

As an illustration of the above method, note that our type III vector is  $(-2, 0, 0, 1, 0, \dots)$ . If  $\Delta^{\bar{a}}$  is in case (5)(iii), Theorem 5.18 says that the possible type IIs must have a  $-1$  in place 1 and a 0 in place 4. Since  $r \geq 5$ , the remaining  $-1$  must be in a place whose corresponding dimension is 2. As  $d_3 = 2$  we can have  $(-1, *, -1, 0, *, \dots)$  where  $*$  indicates a possible location of the 1 in the type II. The other possibility is for  $-1$  to be in place  $k$  for some  $k \geq 5$ . After a permutation we can assume  $k = 5$ , and  $d_5 = 2$  must hold. The type II is then of the form  $(-1, *, *, 0, -1, *, \dots)$  where  $*$  again indicates possible positions for the 1. In the first case, the orthogonality conditions imply  $\frac{a_2}{d_2} = \frac{a_5}{d_5}$ , which gives  $\beta = \frac{n-1}{2n} + \frac{n-d_2-2}{d_2n}$ . Comparing with the value of  $\beta$  from Eq.(10.1), we get  $d_2 = d_4$ . Using this in the first value of  $\beta$  in (B) gives a contradiction after some manipulation. In the second case, the argument we just gave implies that we can only have  $r = 5$  and the orthogonality condition implies  $\frac{a_2}{d_2} = \frac{a_3}{d_3}$ , which gives  $\beta = \frac{n-1}{n+d_2+2} + \frac{2(n-d_2-2)}{d_2(n+d_2+2)}$ . After a short computation, one sees that the two values of  $\beta$  are again incompatible. If  $\Delta^{\bar{a}}$  is in case (5)(ii), the argument is essentially the same, as we only have to switch the places of the second  $-1$  and the 1 in the type IIs.

Let us now take  $i = 3$ . If  $j = 1$ , Eq.(10.1) implies  $\beta = \frac{3d_2+4-n}{5d_2+10-n}$ . If the denominator is negative, then  $\beta > 1$ , which is a contradiction. If it is positive we find that this is incompatible with the inequality  $\beta > \frac{n+d_2-2}{2(n+d_2+2)}$  which comes from the displayed expression for  $\beta$  above.

If  $j = 2$ , we get  $\beta = \frac{d_2+1}{2(d_2+2)} + \frac{n-d_2-2}{2d_2(d_2+2)}$ . As above, we can rule this out by considering the vectors in  $\bar{a}^\perp \cap \frac{1}{2}(d+\mathcal{W})$  associated to type II vectors. A similar argument works for  $j \geq 5$ , where we find  $\beta = \frac{n-2d_2-3}{2(n-2d_2-4)}$ , and for  $j = 4$ , where we have  $\beta = \frac{n-d_2-2}{d_4(n-2d_2-4)} + \frac{n-2d_2-3}{2(n-2d_2-4)}$ .

(C) Next let  $u = (0, 0, 1, -1, -1, 0, \dots)$ . We have  $(a_i/d_i) = (1 - \beta, 1 - 2\beta, \frac{n-2d_2-3}{n-d_2-2} - (\frac{n-6-3d_2}{n-d_2-2})\beta, \frac{2(d_2+2)}{n-d_2-2}\beta - \frac{d_2+1}{n-d_2-2} - \frac{2}{d_4}, \frac{2(d_2+2)}{n-d_2-2}\beta - \frac{d_2+1}{n-d_2-2} - \frac{2}{d_5}, \frac{2(d_2+2)}{n-d_2-2}\beta - \frac{d_2+1}{n-d_2-2}, \dots)$  where all terms from the sixth on are equal and

$$\beta := 1 + \frac{c_1}{2} = \frac{n+d_2}{2(n+d_2+2)} + \left(\frac{1}{d_4} + \frac{1}{d_5}\right) \frac{n-d_2-2}{(n+d_2+2)}.$$

The nullity condition for  $\bar{c}$  implies  $d_4$  and  $d_5 \geq 13$ . It is now easy to check that  $0 < \beta < \frac{2}{3}$ , so  $\frac{a_1}{d_1} > 0$ . As in (A), we find also that  $\frac{a_i}{d_i} > 0$  for  $i \geq 6$ , and that  $\frac{a_3}{d_3} > 0$ .

As in (A) again, case (5) must hold with respect to  $\bar{a}$ , so if  $(-2^i, 1^j)$  gives an element of  $\bar{a}^\perp$  we need  $d_i = 2$ . This, combined with Lemma 10.5, means  $i = 2$ . In this situation in all cases Eq.(10.1) gives a value of  $\beta$  incompatible with our bounds on  $n$  and  $\beta$ .  $\square$

**Lemma 10.13.** *Case (5)(iii) cannot arise if  $r \geq 5$ .*

*Proof.* This is similar to the proof of the previous Lemma so we will be brief.

(A) Consider  $u = (0, -2, 0, 1, 0, \dots)$ . Now  $(a_i/d_i)$  is given by  $(1 - \beta, -\frac{4}{d_2} + 1 - 2\beta, \frac{(n-3)+(n+d_2-2)(\beta-1)}{n-d_2-2}, \frac{2}{d_4} + \frac{2d_2\beta-(d_2+1)}{n-d_2-2}, \frac{2d_2\beta-(d_2+1)}{n-d_2-2}, \dots, \frac{2d_2\beta-(d_2+1)}{n-d_2-2})$  where

$$\beta := 1 + \frac{c_1}{2} = \frac{1}{2} - \frac{(d_2+4d_4)(n-2-d_2-d_4)+4d_4^2}{2d_2d_4(2n-d_2-4)}.$$

The nullity condition for  $\bar{c}$  implies  $d_4 \geq 8, d_2 \geq 27$  and  $d_2 > 2d_4$ , and it readily follows that  $\frac{1}{4} < \beta < \frac{1}{2}$ . In particular,  $\frac{a_1}{d_1} > 0$ .

As before, we see that case (5) holds with respect to  $\bar{a}$ , so for  $\bar{a}^\perp$  we have to consider type III vectors  $(-2^i, 1^j)$  where  $d_i = 2$ . So we need only consider  $i = 3$  or  $i \geq 5$ .

In either situation, we proceed as in part (B) of the proof of Lemma 10.12, and obtain inconsistencies in the equations involving  $\beta$  or contradiction to the bounds on  $\beta$  or the dimensions.

(B) Let  $u = (0, 0, -1, 1, -1, 0, \dots)$ . The nullity condition on  $\bar{c}$ , which has a symmetry in  $d_4$  and  $d_5$ , now implies  $d_4, d_5 \geq 46$  and  $> 28d_2$ . Now  $(a_i/d_i)$  is given by

$$(1 - \beta, 1 - 2\beta, (\frac{n+d_2-2}{n-d_2-2})\beta - \frac{n-1}{n-d_2-2}, \frac{2}{d_4} + \frac{2d_2\beta-(d_2+1)}{n-d_2-2}, \frac{-2}{d_5} + \frac{2d_2\beta-(d_2+1)}{n-d_2-2}, \dots)$$

where all terms from the sixth on are equal and

$$\beta := 1 + \frac{c_1}{2} = \frac{1}{2} + \frac{1}{n+d_2-2} + (\frac{1}{d_4} + \frac{1}{d_5})(\frac{n-d_2-2}{n+d_2-2}).$$

We deduce that  $\frac{1}{2} < \beta < \frac{3}{5}$  and so  $\frac{1}{2} > \frac{a_1}{d_1} > 0$ .  $\Delta^{\bar{a}}$  must be in case (5), and for  $(-2^i, 1^j)$  associated to elements of  $\bar{a}^\perp \cap \frac{1}{2}(d + \mathcal{W})$ , we have  $d_i = 2$  and we need only consider  $i = 2, 3$  or  $\geq 6$ .

If  $i = 2$ , Eq.(10.1) and the upper bound on  $\beta$  imply  $\frac{a_j}{d_j} > \frac{3}{5}$ . As  $d_2 = 2$ , one checks that this never holds.

If  $i \geq 6$ , Eq.(10.1) implies  $\frac{a_j}{d_j} > \frac{47}{48}$ . The bound  $d_4, d_5 > 28d_2$  can be used to show that this never happens.

If  $i = 3$ , Eq.(10.1) and the expression for  $\beta$  above are seen to be incompatible if we use the bounds on  $d_4, d_5$  and  $\beta$ .

(C) Consider  $u = (-1, -1, 0, 1, 0, \dots)$ . The nullity condition on  $\bar{c}$  gives  $d_4 \geq 4, d_2 \geq 5$ . The vector  $(a_i/d_i)$  is

$$(-\beta, 1 - 2\beta - \frac{2}{d_2}, \frac{(n+d_2-2)\beta-(d_2+1)}{n-d_2-2}, \frac{2}{d_4} + \frac{2d_2\beta-(d_2+1)}{n-d_2-2}, \frac{2d_2\beta-(d_2+1)}{n-d_2-2}, \dots)$$

where all terms from the fifth on are equal and

$$\beta := 1 + \frac{c_1}{2} = \frac{1}{2} - \frac{d_4^2 + (d_2 + d_4)(n - d_2 - d_4 - 2)}{d_2 d_4 (3n - d_2 - 6)}.$$

Now as  $\frac{1}{d_2} + \frac{1}{d_4} < \frac{1}{2}$ , we see that  $\frac{1}{3} < \beta < \frac{1}{2}$ .

Again  $\Delta^{\bar{a}}$  is in case (5) and we consider vectors  $(-2^i, 1^j)$  associated to elements of  $\bar{a}^\perp \cap \frac{1}{2}(d + \mathcal{W})$ , where we must have  $d_i = 2$ , so  $i \neq 2, 4$ .

If  $i = 1$ , Eq.(10.1) becomes  $\frac{a_j}{d_j} = 1 - 2\beta > 0$ . This immediately means  $j \neq 2, 5, \dots, r$ . If  $j = 3$ , the value of  $\beta$  from Eq.(10.1) and the above expression for  $\beta$  lead to  $d_4 \leq 10/3$ . For  $j = 4$ , we obtain a contradiction by the method of part (B) in the proof of Lemma 10.12.

If  $i \geq 5$ , Eq.(10.1) says  $\frac{a_j}{d_j} = \frac{4d_2\beta + (n-3d_2-4)}{n-d_2-2}$ . If  $i = 3$ , Eq.(10.1) says  $\frac{a_j}{d_j} = 1 - \frac{2(d_2+1)}{n-d_2-2} + \frac{2(n+d_2-2)\beta}{n-d_2-2}$ . In both situations, we can again apply the method of part (B) in the proof of Lemma 10.12 to obtain contradictions.  $\square$

The last case to consider is case (5)(ii) with  $r = 4$ , which is the same as case (5)(iii) with  $r = 4$  if we interchange the third and fourth summands.

**Lemma 10.14.** *No configurations for case (5)(ii) with  $r = 4$  can occur.*

*Proof.* When  $r = 4$  we no longer have  $d_3 = 2$ , but the nullity condition for  $\bar{c}$  implies that  $\frac{1}{d_3} + \frac{1}{d_4} \geq \frac{1}{2}$ . Hence either  $\{d_3, d_4\}$  is one of  $\{3, 3\}, \{3, 4\}, \{3, 5\}, \{3, 6\}, \{4, 4\}$  or one of  $d_3$  or  $d_4$  is 2. Using this together with the nullity conditions for  $\bar{a}, \bar{c}$  and the orthogonality conditions, we see that we only need to consider  $u = (0, -2, 1, 0), (0, 0, 1, -2), (-1, 1, 0, -1)$  and  $(-1, -1, 1, 0)$ .

(A) Let  $u = (0, -2, 1, 0)$ . From the nullity condition for  $\bar{c}$  we deduce that  $d_4 = 2, d_2 \geq 13$ , and  $d_3 \geq 3$ . Now

$$(a_i/d_i) = (1 - \beta, 2\beta - 1 - \frac{4}{d_2}, \frac{2}{d_3} + \frac{2d_2\beta-(d_2+1)}{d_3+2}, \frac{(2d_2+d_3+2)\beta-(d_2+1)}{n-d_2-2})$$

where  $\beta := 1 + \frac{c_1}{2} = \frac{1}{2} - \frac{d_2+4d_3+2d_3^2}{d_2d_3(d_2+2d_3+4)}$ . We find that  $\frac{11}{26} < \beta < \frac{1}{2}$  and so  $\frac{a_1}{d_1} > 0$ .

The above facts imply that  $\Delta^{\bar{a}}$  is again in case (5)(ii), and for  $(-2^i, 1^j)$  associated to an element of  $\bar{a}^\perp$  we must have  $i = 4$ . If  $j = 1, 2$ , Eq.(10.1) leads to a contradiction to the above dimension restrictions. The case  $j = 3$  can be eliminated using the method of part (B) in the proof of Lemma 10.12.

(B) Next let  $u = (0, 0, 1, -2)$ . The nullity condition for  $\bar{c}$  implies that  $d_3 = 2$  and  $d_4 > 32d_2 + 14 \geq 46$ . Now  $(a_i/d_i)$  is given by

$$(1 - \beta, 1 - 2\beta, \frac{d_4 - d_2 + 1}{d_4 + 2} + (\frac{2d_2 + 2 - d_4}{d_4 + 2})\beta, -\frac{4}{d_4} + \frac{2(d_2 + 2)\beta - (d_2 + 1)}{d_4 + 2})$$

where  $\beta := 1 + \frac{c_1}{2} = 1 - \frac{d_4^2 + 2d_2d_4 - 16}{2d_4(2d_2 + d_4 + 6)}$ . One easily sees that  $\frac{1}{2} < \beta < 1$ , so that  $0 < \frac{a_1}{d_1} < \frac{1}{2}$ . Since  $d_4 > 2d_2 + 2$  we obtain  $0 < \frac{a_3}{d_3} < \frac{3}{5}$ .

Therefore,  $\Delta^{\bar{a}}$  is in case (5)(ii) and for  $(-2^i, 1^j)$  associated to an element of  $\bar{a}^\perp$  we must have (by Lemma 10.5)  $i = 2$  and so  $d_2 = 2$ . Putting this value of  $d_2$  into the nullity condition for  $\bar{c}$  gives a cubic equation in  $d_4$  with no integral roots, a contradiction.

(C) Consider now  $u = (-1, 1, 0, -1)$ . From the nullity condition for  $\bar{c}$  we deduce that  $d_2 \geq 4, d_3 = 2, d_4 \geq 3$  and  $4d_2 > 3d_4$ . Also,

$$(a_i/d_i) = (-\beta, \frac{2}{d_2} + 1 - 2\beta, \frac{(2d_2 + 2 - d_4)\beta - (d_2 + 1)}{d_4 + 2}, -\frac{2}{d_4} + \frac{2(d_2 + 2)\beta - (d_2 + 1)}{d_4 + 2})$$

where  $\beta := 1 + \frac{c_1}{2} = \frac{1}{2} + \frac{2d_2 + 2d_4 + d_4^2}{d_2d_4(2d_2 + d_4 + 6)}$ . It follows that  $\frac{1}{2} < \beta < \frac{3}{4}$  and  $\frac{a_2}{d_2} > 0$ .

Now we see that  $\Delta^{\bar{a}}$  is either in case (1) or (4) or (5)(ii). In the first two instances, by Lemmas 10.11, 10.7  $d$  is a permutation of  $(4, 2, 2, 9)$ . Since  $3d_4 < 4d_2$  we have  $d_2 = 9, d_4 = 4$ . But then the null condition for  $\bar{c}$  is violated. So we are in case (5)(ii). For  $(-2^i, 1^j)$  associated to an element of  $\bar{a}^\perp$ , as  $d_i = 2$ , we have  $i = 1$  or  $3$ .

If  $i = 1$ , then Eq.(10.1) becomes  $\frac{a_j}{d_j} = 1 - 2\beta < 0$ , so  $j = 3, 4$ . When  $j = 3$  the value of  $\beta$  given above together with Eq.(10.1) imply that  $d_4 = 3$  or  $4$ . But then the null condition for  $\bar{c}$  is violated. For  $j = 4$  we may use the argument of part (B) of the proof of Lemma 10.12.

If  $i = 3$ , using  $\beta > \frac{1}{2}$  in Eq.(10.1), we see that  $j = 2, 4$ . In either case, applying our bounds for the dimensions in Eq.(10.1) lead to contradictions.

(D) Let  $u = (-1, -1, 1, 0)$ . The null condition for  $\bar{c}$  implies that  $d_4 = 2, d_3 \geq 3, d_2 \geq 5$ . With  $\beta := 1 + \frac{c_1}{2}$ , we have

$$(a_i/d_i) = (-\beta, 1 - 2\beta - \frac{2}{d_2}, \frac{2}{d_3} + \frac{2d_2\beta - (d_2 + 1)}{d_3 + 2}, \frac{(2d_2 + d_3 + 2)\beta - (d_2 + 1)}{d_3 + 2}).$$

One computes that  $\beta = \frac{1}{2} - \frac{2d_2 + d_3^2 + 2d_3}{d_2(3d_3^2 + 2d_2d_3 + 6d_3)}$ , and from the dimension bounds one gets  $\frac{5}{12} \leq \beta < \frac{1}{2}$ .  $\Delta^{\bar{a}}$  cannot be in case (1) or (4), otherwise as  $d_2 \geq 5$ , we must have  $d_2 = 9, d_3 = 4$ , and the null condition for  $\bar{c}$  is violated. So  $\Delta^{\bar{a}}$  is in case (5)(ii). For  $(-2^i, 1^j)$  associated to an element of  $\bar{a}^\perp$ , we must then have  $i = 1, 4$ .

If  $i = 1$ , then Eq.(10.1) is  $\frac{a_j}{d_j} = 1 - 2\beta > 0$ , so  $j = 3$  or  $4$ . In either situation, we may apply the argument of part (B) of the proof of Lemma 10.12 to get a contradiction. If  $i = 4$ , Eq.(10.1) together with the dimension bounds above show first that we can only have  $j = 3$ . In that case, a more detailed look at Eq.(10.1) leads to a contradiction.  $\square$

We can summarise our discussions thus far by

**Theorem 10.15.** *Let  $r \geq 4$  and  $K$  be connected. Suppose that we are not in the situation of Theorem 3.14. Assume that  $\bar{c} \in \mathcal{C}$  is a null vector such that  $\Delta^{\bar{c}}$  has the property that there is a unique vertex of type (1B) and all other vertices are of type (1A). Then the only possibilities are given by Lemmas 10.7/10.11 and 10.9, up to interchanging  $\bar{a}$  and  $\bar{c}$  and a permutation of the irreducible summands.  $\square$*

We will now sharpen the above Theorem using Proposition 3.7.

**Corollary 10.16.** *Let  $r \geq 4$ . Assume that  $K$  is connected and we are not in the situation of Theorem 3.14. Then the possibilities given by Lemmas 10.7 and 10.9 (and hence Lemma 10.11) cannot occur.*

*Proof.* We will discuss the  $r = 4$  case (i.e. that in Lemmas 10.7, 10.11) in detail and leave the details of the  $r = 5$  case (from Lemma 10.9) to the reader, as the arguments are very similar.

In the  $r = 4$  case, first observe that  $\mathcal{C}$  has exactly two null vectors,  $\bar{c}$  and  $\bar{a}$  in the notation of Lemma 10.7, as the entries of  $a, c$  are determined by the vector  $d$  of dimensions. Hence, by Prop. 3.3, these are the only elements of  $\mathcal{C}$  outside  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ .

Since  $(-1^4)$  is a vertex of  $\mathcal{W}$ , all type II vectors in  $\mathcal{W}$  must be zero in place 4. As  $(1, -1, -1, 0)$  is associated to an element of  $\bar{c}^\perp$ , it, together with  $(-1, 1, -1, 0), (-1, -1, 1, 0)$  are the only type II vectors in  $\mathcal{W}$ .

Next we analyse vectors in  $\mathcal{W}$  and see if they are associated to elements of  $\mathcal{C}$ , this last property being important for applying Prop 3.7. Recall that  $(1, -2, 0, 0), (1, 0, -2, 0), (-2, 1, 0, 0), (-2, 0, 1, 0)$  must be in  $\mathcal{W}$ . The first two give elements of  $\bar{c}^\perp$ , the last two give elements of  $\bar{a}^\perp$ .

First consider  $v = (1, -2, 0, 0)$ . Now  $\bar{v}$  is a vertex of  $\text{conv}(\frac{1}{2}(d + \mathcal{W}))$ . By the superpotential equation,  $d + v = 2\bar{v}$  can be written as  $\bar{c}^{(\alpha)} + \bar{c}^{(\beta)}$ , with  $\bar{c}^{(\alpha)}, \bar{c}^{(\beta)} \in \mathcal{C}$ . Since  $\bar{v}$  is a vertex, every such expression must involve  $\bar{a}$  or  $\bar{c}$ , unless it is the trivial expression  $\bar{v} + \bar{v}$  and  $\bar{v} \in \mathcal{C}$ . By computing  $2v - a, 2v - c$  we find that these cannot lie in  $\text{conv}(\mathcal{W})$  (it is enough to exhibit one component  $< -2$  or  $> 1$ ). Thus  $\bar{v} \in \mathcal{C}$ . Now an analogous argument shows that if  $w = (0, -2, 1, 0)$  is in  $\mathcal{W}$  then  $\bar{w}$  also lies in  $\mathcal{C}$ . But  $vw$  is an edge of  $\text{conv}(\mathcal{W})$  with no interior points in  $\mathcal{W}$ . So Prop 3.7 gives  $1 - \frac{4}{d_2} = 4J(\bar{v}, \bar{w}) = 0$ , a contradiction to  $d_2 = 2$ . Hence  $w \notin \mathcal{W}$ . Similarly we see  $(0, -2, 0, 1) \notin \mathcal{W}$ .

Next consider  $z = (-1, -1, 1, 0)$ . By Remark 1.2(e) and the above,  $z$  is a vertex of  $\text{conv}(\mathcal{W})$ . As above we can show that  $\bar{z} \in \mathcal{C}$ . Now  $v, z$  are the only elements of the face  $\{x_1 + 2x_2 = -3\} \cap \text{conv}(\mathcal{W})$  (cf. proof of Prop. 4.3 in [DW4]). So applying Prop 3.7 to  $vz$  we obtain  $0 = 4J(\bar{v}, \bar{z}) = \frac{1}{4}$ , a contradiction.

To handle the  $r = 5$  case (from Lemma 10.9), first observe that null elements of  $\mathcal{C}$  must have entries  $\frac{2}{3}$  in two of the places  $1, \dots, 4$  and  $-\frac{2}{3}$  in the other two places. For  $a, c$  as in Lemma 10.9, we can take  $(1, -2, 0, 0, 0), (1, 0, -2, 0, 0), (0, 1, 0, -2, 0), (-2, 1, 0, 0, 0)$  to lie in  $\mathcal{W}$ . The argument above to show that  $\bar{v}$  is in  $\mathcal{C}$  still works for such type III vectors  $v$ . As above, we can use Prop 3.7 to show the other type III vectors  $(-2^i, 1^j)$  ( $i \leq 4$ ) do not lie in  $\mathcal{W}$ ; hence the  $\bar{a}, \bar{c}$  of Lemma 10.9 are the only null elements of  $\mathcal{C}$ .

Let  $z = (-1, 1, -1, 0, 0)$  (it lies in  $\mathcal{W}$  since  $(1, -1, -1, 0, 0)$  is associated to an element of  $\bar{c}^\perp$ ) and  $v = (1, 0, -2, 0, 0)$ . As above we find  $\bar{z}$  is in  $\mathcal{C}$ , and the arguments of Prop 4.3 in [DW4] show  $vz$  is an edge of  $\text{conv}(\mathcal{W})$ . A contradiction results as above from applying Prop 3.7 to  $vz$ .  $\square$

The discussion at the beginning of this section now tells us that if  $K$  is connected and  $r \geq 4$  the only case when we have a superpotential of the kind under discussion is that of Theorem 3.14. The proof of Theorem 2.1 is now complete.

### Concluding remarks.

1. When  $r = 2$ , then  $c$  is collinear with the elements of  $\mathcal{W}$ . In other words, the projected polytope  $\Delta^{\bar{c}}$  reduces to a single vertex, which must be of type (2). The possible elements of  $\mathcal{W}$  are  $(-2, 1), (-1, 0), (0, -1), (1, -2)$ . If  $\mathcal{W}$  has just two elements then Theorem 9.2 tells us we are either in the situation of Theorem 3.14 (the Bérard Bergery examples), or in Example 8.2 or the third case of Example 8.3 in [DW4]. In fact one can show that this last possibility can be realised in the class of homogeneous hypersurfaces exactly when  $(d_1, d_2) = (8, 18)$ . An example for these dimensions is provided by  $G = SU(2)^9 \ltimes \text{Sym}(9)$  (where  $\text{Sym}(9)$  acts on  $SU(2)^9$  by permutation) and  $K$  is the product of the diagonal  $U(1)$  in  $SU(2)^9$  with  $\text{Sym}(9)$ . The arguments of [DW2] show that this in fact gives an example where the cohomogeneity one Ricci-flat equations are fully integrable.

If  $\mathcal{W}$  has three elements, we may adapt the proof of Theorem 3.11 to derive a contradiction. Here the essential point is that whenever we had to check that a sum of two elements of  $\mathcal{C}$  does not lie in  $d + \mathcal{W}$ , such a fact remains true because the interior point of  $vw$  is the midpoint.

If  $\mathcal{W}$  contains all four possible elements, then  $\mathfrak{k} \subset \mathfrak{g}$  is a maximal subalgebra (with respect to inclusion). We suspect that this case also does not occur. In any event, it is of less interest because the only way to obtain a complete cohomogeneity one example is by adding a  $\mathbb{Z}/2$ -quotient of the principal orbit as special orbit.

2. The only parts of this paper which depend on  $K$  being connected (or slightly more generally, on the condition in Remark 2.4) are parts of §5, Case (ii) of §9, and all of §10. To remove this

condition, the main task would be generalizing Theorem 5.18 by getting a better handle on the type II vectors associated to (1A) vertices (cf Lemmas (5.6)-(5.8)).

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