Smooth maps with singularities of bounded $\mathcal{K}$-codimensions *†

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Abstract

Let N and P be smooth manifolds of dimensions n and p respectively such that $n \geqq p \geqq 2$ or $n < p$. Let $\mathcal{O}_\ell(N,P)$ denote a $\mathcal{K}$-invarinat open subspace of $J^\infty(N,P)$ which consists of all regular jets and singular jets z with codim$\mathcal{K}z \leqq \ell$ (including fold jets if $n \geqq p$). An $\mathcal{O}_\ell$-regular map $f : N \to P$ refers to a smooth map such that $j^\infty f(N) \subset \mathcal{O}_\ell(N,P)$. We will prove that a continuous section s of $\mathcal{O}_\ell(N,P)$ over N has an $\mathcal{O}_\ell$-regular map f such that s and $j^\infty f$ are homotopic as sections. We next study the filtration of the group of homotopy self-equivalences of a manifold P which is constructed by the sets of $\mathcal{O}_\ell$-regular homotopy self-equivalences for nonnegative integers ℓ.

1 Introduction

Let N and P be smooth (C^∞) manifolds of dimensions n and p respectively. Let $J^k(N,P)$ denote the k-jet space of the manifolds N and P with the projections π^k_N and π^k_P onto N and P mapping a jet onto its source and target respectively. The canonical fiber is the k-jet space $J^k(n,p)$ of C^∞-map germs $(\mathbb{R}^n,0) \to (\mathbb{R}^p,0)$. Let $\mathcal{K}$ denote the contact group defined in [MaIII]. Let $\mathcal{O}(n,p)$ denote a $\mathcal{K}$-invariant nonempty open subset of $J^k(n,p)$ and let $\mathcal{O}(N,P)$ denote an open subbundle of $J^k(N,P)$ associated to $\mathcal{O}(n,p)$. In this paper a smooth map $f : N \to P$ is called an *$\mathcal{O}$-regular map* if $j^k f(N) \subset \mathcal{O}(N,P)$.

We will study what is called the homotopy principle for $\mathcal{O}$-regular maps. As for the long history of the several types of homotopy principles and their applications we refer to the Smale-Hirsch Immersion Theorem ([Sm] and [H]), the Feit k-mersion Theorem ([F]), the Phillips Submersion Theorem ([P]) and the general theorems due to Gromov ([G1]) and du Plessis ([duP1], [duP2] and [duP3]). Furthermore, we should refer to the homotopy principle on the 1-jet level for fold-maps due to Èliašberg ([E1] and [E2]) (see further references in [G2]).

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Let $C^\infty_{\mathcal{O}}(N,P)$ denote the space consisting of all $\mathcal{O}$-regular maps, $N \to P$ equipped with the C^∞-topology. Let $\Gamma_{\mathcal{O}}(N,P)$ denote the space consisting of all continuous sections of the fiber bundle $\pi^k_N|\mathcal{O}(N,P) : \mathcal{O}(N,P) \to N$ equipped with the compact-open topology. Then there exists a continuous map $j_{\mathcal{O}} : C^\infty_{\mathcal{O}}(N,P) \to \Gamma_{\mathcal{O}}(N,P)$ defined by $j_{\mathcal{O}}(f) = j^k f$. Let C be a closed subset of N. Let s be a section in $\Gamma_{\mathcal{O}}(N,P)$ which has an $\mathcal{O}$-regular map g defined on a neighborhood of C to P, where $j^k g = s$. If for given C and g, any such a section s in $\Gamma_{\mathcal{O}}(N,P)$ has an $\mathcal{O}$-regular map f such that s and $j^k f$ are homotopic relative to a neighborhood of C as sections in $\Gamma_{\mathcal{O}}(N,P)$, then we say in this paper that the *relative homotopy principle on the existence level* holds for $\mathcal{O}$-regular maps.

As an important application of [A7, Theorem 0.1] we will prove the following theorem. Here, $\Sigma^{n-p+1,0}(n,p)$ refers to the space consisting of all fold jets in $J^k(n,p)$.

Theorem 1.1 *Let n and p be positive integers with $n \geqq p \geqq 2$ or $n < p$. Let k be an integer with $k \geqq n - |n-p| + 2$. Let $\mathcal{O}(n,p)$ denote a $\mathcal{K}$-invariant open subspace of $J^k(n,p)$ containing all regular jets such that if $n \geqq p \geqq 2$, then $\mathcal{O}(n,p)$ contains $\Sigma^{n-p+1,0}(n,p)$ at least. Let N and P be connected smooth manifolds of dimensions n and p respectively with $\partial N = \emptyset$. Let C be a closed subset of N. Let s be a section in $\Gamma_{\mathcal{O}}(N,P)$ which has an $\mathcal{O}$-regular map g defined on a neighborhood of C to P, where $j^k g = s$.*

Then there exists an $\mathcal{O}$-regular map $f : N \to P$ such that $j^k f$ is homotopic to s relative to a neighborhood of C by a homotopy s_λ in $\Gamma_{\mathcal{O}}(N,P)$ with $s_0 = s$ and $s_1 = j^k f$.

Let ρ be an integer with $\rho \geqq 1$. Let W^k_ρ denote the subset consisting of all $z \in J^k(n,p)$ such that the codimension of $\mathcal{K}z$ in $J^k(n,p)$ is not less than ρ (k may be ∞). Let $\mathcal{O}^k_\ell(n,p)$ denote a $\mathcal{K}$-invariant nonempty open subset of $J^k(n,p)\backslash W^k_{\ell+1}$. By applying Theorem 1.1 we will prove the following theorem.

Theorem 1.2 *Let ℓ be an integer with $\ell \geqq 1$. Let $k \geqq \max\{\ell+1, n-|n-p|+2\}$ or $k = \infty$. Let $\mathcal{O}^k_\ell(n,p)$ denote a $\mathcal{K}$-invariant open subspace of $J^k(n,p)$ containing all regular jets such that if $n \geqq p \geqq 2$, then $\mathcal{O}^k_\ell(n,p)$ contains $\Sigma^{n-p+1,0}(n,p)$ at least. Then $\mathcal{O}^k_\ell(n,p)$ satisfies the relative homotopy principle on the existence level for $\mathcal{O}^k_\ell$-regular maps.*

It is well known that any smooth map $f : N \to P$ is homotopic to a smooth map $g : N \to P$ such that $j^\infty_x g$ is of finite $\mathcal{K}$-codimension for any $x \in N$ (see, for example, [W, Theorem 5.1]).

There have been described many important applications of the homotopy principles in [G2]. We only refer to the recent applications of the relative homotopy principle on the existence level to the problems in topology such as the elimination of singularities and the existence of $\mathcal{O}^k_l$-regular maps in [A1-7] and [Sa] and the relation between the stable homotopy groups of spheres and higher singularities in [A4].

Let P be a closed manifold of dimension p. Let $\mathfrak{h}(P)$ denote the group of all homotopy classes of homotopy equivalences of P. Let $\mathfrak{h}_\ell(P)$ denote the subset of $\mathfrak{h}(P)$ which consists of all homotopy classes of maps which are homotopic to $\mathcal{O}_l^k$-regular homotopy equivalences. In particular, $\mathfrak{h}_0(P)$ is the subset of all homotopy classes of maps which are homotopic to diffeomorphisms of P. In this paper we will prove that the following filtration

$$\mathfrak{h}_0(P) \subset \mathfrak{h}_1(P) \subset \cdots \subset \mathfrak{h}_\ell(P) \subset \cdots \subset \mathfrak{h}(P). \tag{1.1}$$

is never trivial in general.

Theorem 1.3 *For a given positive integer d, there exists a closed oriented p-manifold P and a sequence of positive integers $\ell_1, \ell_2, \cdots, \ell_d$ with $\ell_j < \ell_{j+1}$ for $1 \leq j < d$ such that*

$$\mathfrak{h}_0(P) \subsetneqq \mathfrak{h}_{\ell_1}(P) \subsetneqq \mathfrak{h}_{\ell_2}(P) \subsetneqq \cdots \subsetneqq \mathfrak{h}_{\ell_d}(P) \subsetneqq \mathfrak{h}(P).$$

In Section 2 we will review the results on the Boardman manifolds and the fundamental properties of $\mathcal{K}$-equivalence and $\mathcal{K}$-determinacy which are necessary in this paper. In Section 3 we will recall [A7, Theorem 0.1] and apply it in the proof of Theorem 1.1. In Section 4 we will study the nonexistence problem of $\mathcal{O}_l^k$-regular maps. In Section 5 we will study the filtration in (1.1) and prove Theorem 1.3.

2 Boardman manifolds and $\mathcal{K}$-orbits

Throughout the paper all manifolds are Hausdorff, paracompact and smooth of class C^∞. Maps are basically smooth (of class C^∞) unless otherwise stated.

For a Boardman symbol (simply symbol) $I = (i_1, \cdots, i_k)$ with $i_1 \geqq \cdots \geqq i_k \geqq 0$, let $\Sigma^I(n,p)$ denote the Boardman manifold of symbol I in $J^k(n,p)$ which has been defined in [T], [L], [Bo] and [MaTB]. Let $A_n = \mathbb{R}[[x_1, \cdots, x_n]]$ denote the formal power series of algebra on variables $x_1, \cdots, x_n$. Let $\mathfrak{m}_n$ be its maximal ideal and $A_n(k) = A_n/\mathfrak{m}_n^{k+1}$. Let $z = j_0^k f \in J^k(n,p)$ where $f = (f^1, \cdots, f^p) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$. We define $\mathcal{I}(z)$ to be the ideal in $A_n(k)$ generated by the image in $A_n(k)$ of the Taylor expansions of $f^1, \cdots, f^p$. It has been proved in [Bo] and [MaTB] that the Boardman symbol $I(z)$ of z depends only on the ideal $\mathcal{I}(z)$ by the notion of the Jacobian extension. Let $\Sigma^I(N,P)$ denote the subbundle of $J^k(N,P)$ over $N \times P$ associated to $\Sigma^I(n,p)$. Let $\Sigma^I_{x,y}(N,P)$ denote the fiber of $\Sigma^I(N,P)$ over $(x,y) \in N \times P$.

Since $\mathrm{codim}\Sigma^{i_1}(n,p) = (p-n+i_1)i_1$, the following proposition follows from [A6, Remark 2.1], which has been proved by using the results in [Bo, Section 6].

Proposition 2.1 *Let $I = (i_1, \cdots, i_\ell)$ be a symbol such that $i_1 \geqq \max\{n-p+1, 1\}$ and $\Sigma^I(n,p)$ is nonempty. Then we have*

$$\mathrm{codim}\Sigma^I(n,p) \geqq (p-n+i_1)i_1 + (1/2)\Sigma_{j=2}^\ell\ i_j(i_j+1).$$

In particular, if $i_\ell > 0$, then we have $\mathrm{codim}\Sigma^I(n,p) > |n-p| + \ell$.

Let $\Omega^I(n,p)$ denote the union of all Boardman manifolds $\Sigma^J(N,P)$ with $J \leq I$ in the lexicographic order. We have the following lemma (see [duP]).

Lemma 2.2 *The space $\Omega^I(n,p)$ is open in $J^k(n,p)$.*

Let us review the $\mathcal{K}$-equivalence of two smooth map germs $f, g : (N,x) \to (P,y)$, which has been introduced in [MaIII, (2.6)], by following [Mart, II, 1]. We say that the above two map germs f and g are $\mathcal{K}$-equivalent if there exists a smooth map germ $\phi : (N,x) \to GL(\mathbb{R}^p)$ and a local diffeomorphism $h : (N,x) \to (N,x)$ such that $f(x) = \phi(x)g(h(x))$. It is known that this $\mathcal{K}$-equivalence is nothing but the contact equivalence introduced in [MaIII]. The contact group $\mathcal{K}$ is defined as a certain subgroup of the group of germs of local diffeomorphisms $(N,x) \times (P,y)$ and acts on $J^k_{x,y}(N,P)$. For a k-jet z in $J^k_{x,y}(N,P)$ let $\mathcal{K}z$ denote the orbit of $\mathcal{K}$ through z. As is well known, $\mathcal{K}z$ is an orbit of a Lie group. Hence, $\mathcal{K}z$ is a submanifold of $J^k_{x,y}(N,P)$. This fact is also observed from the above definition. The following lemma is important in this paper.

Lemma 2.3 *The Boardman manifold $\Sigma^I_{x,y}(N,P)$ in $J^k_{x,y}(N,P)$ is invariant with respect to the action of $\mathcal{K}$.*

Proof. Let $z = j^k_x f$ and $w = j^k_x g$ be k-jets in $J^k_{x,y}(N,P)$ such that two map germs f and g are $\mathcal{K}$-equivalent as above. Let $h_* : C_x \to C_x$ be the isomorphism defined by $h_*(\phi) = \phi \circ h$. By the definition of $\mathcal{K}$-equivalence we have $h_*(\mathfrak{I}(g)) = \mathfrak{I}(f)$. The Thom-Boardman symbols of $j^k_x f$ and $j^k_x g$ are determined by $\mathfrak{I}(f)$ and $\mathfrak{I}(g)$, and are the same by [MaTB, 2, Corollary]. This proves the assertion. ∎

Let us review the results in [MaIII], [MaIV] and [MaV] which are necessary in this paper. Let $C^\infty(N,x)$ and $C^\infty(P,x)$ denote the rings of smooth function germs on (N,x) and (P,y) respectively. Let $\mathfrak{m}_x$ and $\mathfrak{m}_y$ denote their maximal ideals respectively. Let $f : (N,x) \to (P,y)$ be a germ of a smooth map. Let $f^* : C^\infty(P,x) \to C^\infty(N,x)$ denote the homomorphism defined by $f^*(a) = a \circ f$. Let $\theta(N)_x$ denote the $C^\infty(N,x)$-module of all germs at x of smooth vector fields on (N,x). We define $\theta(P)_y$ similarly for $y \in P$. Let $\theta(f)_x$ denote the $C^\infty(N,x)$-module of germs at x of smooth vector fields along f, namely which consists of all smooth germs $\varsigma : (N,x) \to TP$ such that $p_P \circ \varsigma = f$. Here, $p_P : TP \to P$ is the canonical projection. Then we have the homomorphisms

$$tf : \theta(N)_x \to \theta(f)_x \tag{2.1}$$

defined by $tf(u_N) = df \circ u_N$ for $u_N \in \theta(N)_x$. For a singular jet $z = j^k_0 f \in J^k(N,P)$ there has been defined the isomorphism

$$T_z(J^k_{x,y}(N,P)) \longrightarrow \mathfrak{m}_x\theta(f)_x/\mathfrak{m}_x^{k+1}\theta(f)_x \tag{2.2}$$

in [MaIII, (7.3)] such taht $T_z(\mathcal{K}z)$ corresponds to $tf(\mathfrak{m}_x\theta(N)_x) + f^*(\mathfrak{m}_y)(\theta(f)_x)$ modulo $\mathfrak{m}_x^{k+1}\theta(f)_x$. We do not here explain the definition. According to [MaIII] we define $d(f,\mathcal{K})$ to be

$$\dim \mathfrak{m}_x\theta(f)_x/(tf(\mathfrak{m}_x\theta(N)_x) + f^*(\mathfrak{m}_y)(\theta(f)_x)),$$

which is equal to codim$\mathcal{K}z$.

3 Proof of Theorems 1.1 and 1.2.

In this section we prove Theorems 1.1 and 1.2.

Let k be a positive integer. Let $W^k_\rho = W^k_\rho(n,p)$ denote the subset consisting of all $z \in J^k(n,p)$ such that the codimension of $\mathcal{K}z$ in $J^k(n,p)$ is not less than ρ. The following lemma has been observed in [MaV, Section 7 and Proof of Theorem 8.1].

Lemma 3.1 *Let ρ be an integer with $\rho \geqq 1$. Then W^k_ρ is an algebraic subset of $J^k(n,p)$.*

The order of $\mathcal{K}$-determinacy is estimated by the codimension of a $\mathcal{K}$-orbit as follows.

Proposition 3.2 *Let k be an integer with $k > \rho$. Let $z = j^k f$ be a singular jet in $J^k(n,p)\backslash W^k_{\rho+1}$. Then there z is $\mathcal{K}$-k-determined.*

Proof. It follows from [W, Theorem 1.2 (iii)] that if d =codim$\mathcal{K}z$, then z is $\mathcal{K}$-$(d+1)$-determined. Hence, if $z \in J^k(n,p)\backslash W^k_{\rho+1}$, then $d \leq \rho$ and z is $\mathcal{K}$-k-determined. ■

We define the bundle homomorphism

$$\begin{aligned} \mathbf{d} &: (\pi^{k+1}_N)^*(TN) \longrightarrow (\pi^{k+1}_k)^*(TJ^k(N,P)), \\ \mathbf{d}_1 &: (\pi^{k+1}_N)^*(TN) \longrightarrow (\pi^{k+1}_P)^*(TP). \end{aligned} \tag{3.1}$$

Let $w = j^{k+1}_x f \in J^{k+1}_{x,y}(N,P)$ and $z = \pi^{k+1}_k(w)$. Then we have $j^k f : (N,x) \to (J^k(N,P),z)$ and $d(j^k f) : T_x N \to T_z(J^k(N,P))$. We set

$$\mathbf{d}_z(w,\mathbf{v}) = (w, d(j^k f)(\mathbf{v})) \quad \text{and} \quad (\mathbf{d}_1)_z(w,\mathbf{v}) = (w, df(\mathbf{v})).$$

Let I' be a symbol of length $k+1$. Let $\mathbf{K}(\Sigma^{I'})$ denote the kernel subbundle of $(\pi^{k+1}_N|\Sigma^{I'}(N,P))^*(TN)$ defined by

$$\mathbf{K}(\Sigma^{I'})_w = (w, \mathrm{Ker}(d_x f)).$$

The following theorem follows directly from the corresponding assertion for the case $k = \infty$ in [B, (7.7)]. This is very important in the proof of Theorem 1.1.

Theorem 3.3 *If $I' = (i_1, \cdots, i_{k-2}, 0, 0)$ and $I = (i_1, \cdots, i_{k-1}, 0)$, then we have*

$$\mathbf{d}(\mathbf{K}(\Sigma^{I'})_w) \cap (\pi^{k+1}_k|\Sigma^{I'}(N,P))^*(T(\Sigma^I(N,P))_w = \{0\}$$

for any $w \in \Sigma^{I'}(N,P)$.

Let us review a general condition on $\mathcal{O}(n,p)$ for the relative homotopy principle on the existence level in [A7]. We say that a nonempty $\mathcal{K}$-invariant open subset $\mathcal{O}(n,p)$ is *admissible* if $\mathcal{O}(n,p)$ consists of all regular jets and a finite number of disjoint $\mathcal{K}$-invariant submanifolds $V^i(n,p)$ of codimension ρ_i $(1 \leq i \leq \iota)$ such that the following properties (H-i) to (H-v) are satisfied.

(H-i) $V^i(n,p)$ consists of singular k-jets of rank r_i, namely, $V^i(n,p) \subset \Sigma^{n-r_i}(n,p)$.

(H-ii) For each i, the set $\mathcal{O}(n,p)\backslash\{\cup_{j=i}^{\iota} V^j(n,p)\}$ is an open subset.

(H-iii) For each i with $\rho_i \leq n$, there exists a $\mathcal{K}$-invariant submanifold $V^i(n,p)^{(k-1)}$ of $J^{k-1}(n,p)$ such that $V^i(n,p)$ is open in $(\pi^k_{k-1})^{-1}(V^i(n,p)^{(k-1)})$.

(H-iv) If $n \geqq p \geqq 2$, then $V^1(n,p) = \Sigma^{n-p+1,0}(n,p)$.

Here, $\Sigma^{n-p+1,0}(n,p)$ denotes the Thom-Boardman manifold in $J^k(n,p)$, which consists of $\mathcal{K}$-orbits of fold jets. Let $\mathbf{d} : (\pi^k_N)^*(TN) \longrightarrow (\pi^k_{k-1})^*(T(J^{k-1}(N,P))$ denote the bundle homomorphism defined by $\mathbf{d}(z,\mathbf{v}) = (z, d_x(j^{k-1}f)(\mathbf{v}))$ where $z = j^k_x f \in J^k(N,P)$ and $d_x(j^{k-1}f) : T_xN \rightarrow T_{\pi^k_{k-1}(z)}(J^{k-1}(N,P))$ is the differential. Let $V^i(N,P)$ denote the subbundle of $J^k(N,P)$ associated to $V^i(n,p)$. Let $\mathbf{K}(V^i)$ be the kernel bundle in $(\pi^k_N)^*(TN)|_{V^i(N,P)}$ defined by $\mathbf{K}(V^i)_z = (z, \mathrm{Ker}(d_xf))$.

(H-v) For each i with $\rho_i \leq n$ and any $z \in V^i(N,P)$, we have

$$\mathbf{d}(\mathbf{K}(V^i)_z) \cap (\pi^k_{k-1}|V^i(N,P))^*(T(V^i(N,P)^{(k-1)})_z = \{0\}. \tag{3.2}$$

Then we have proved the following theorem in [A7, Theorem 0.1].

Theorem 3.4 *Let $k \geqq n - |n-p| + 2$. Let $n \geqq p \geqq 2$ or $n < p$. Let $\mathcal{O}(n,p)$ denote a nonempty admissible open subspace of $J^k(n,p)$. Then the relative homotopy principle holds for $\mathcal{O}$-regular maps.*

We set

$$V_I(n,p) = \mathcal{O}(n,p) \cap \Sigma^I(n,p).$$

Let $J = (j_1, \cdots, j_k)$ be a symbol of a singular jet with $\mathrm{codim}\Sigma^J(n,p) \leq n$. If $k \geqq n - |n-p| + 2$, we have by Proposition 2.1 that $i_{k-1} = i_k = 0$. Indeed, if $i_{k-1} > 0$, then

$$\mathrm{codim}\Sigma^J(n,p) > |n-p| + k - 1 \geqq n + 1.$$

So we set $J = (j_1, \cdots, j_{k-2}, 0, 0)$, $J^* = (i_1, \cdots, i_{k-2}, 0)$ and

$$V_{J^*}(n,p)^{(k-1)} = \pi^k_{k-1}(\mathcal{O}(n,p)) \cap \Sigma^{J^*}(n,p).$$

Lemma 3.5 *Let $J = (j_1, \cdots, j_{k-2}, 0, 0)$ and $J^* = (j_1, \cdots, j_{k-2}, 0)$ be as above. Then $V_J(n,p)$ is open in $(\pi^k_{k-1})^{-1}(V_{J^*}(n,p)^{(k-1)})$.*

Proof. It is evident that

$$\Sigma^J(n,p) = (\pi^k_{k-1})^{-1}(\Sigma^{J^*}(n,p)) \text{ and } \mathcal{O}(n,p) \subset (\pi^k_{k-1})^{-1}(\pi^k_{k-1}(\mathcal{O}(n,p)).$$

So we have $V_J(n,p) \subset (\pi^k_{k-1})^{-1}(V_{J^*}(n,p)^{(k-1)})$. Since $\pi^{k(\ell)}_{k-1}$ is an open map, we have that $V_J(n,p)$ is an open subset of $(\pi^k_{k-1})^{-1}(V_{J^*}(n,p)^{(k-1)})$. ■

Let us prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 3.4 it is enough to prove that $\mathcal{O}(n,p)$ is admissible. Let J be a symbol of length k. By Lemma 2.3, $V_J(n,p)$ is $\mathcal{K}$-invariant. We have that

(H1) $\mathcal{O}(n,p)$ is decomposed into a finite union of all $V_J(n,p)$,

(H2) For each symbol J, the set $\mathcal{O}(n,p) \cap \Omega^J(n,p)$ is an open subset of $\mathcal{O}(n,p)$,

(H3) $V_J(n,p)$ is open in $(\pi^k_{k-1})^{-1}(V_{J^*}(n,p)^{(k-1)})$ by lemma 3.5,

(H4) If $n \geqq p \geqq 2$, then $\mathcal{O}(n,p) \supset \Sigma^{n-p+1,0}(n,p)$ by the assumption,

(H5) Property (3.2) holds for $V_J(n,p)$ by Theorem 3.3 and Lemma 3.5.

Since $\mathcal{O}(n,p)$ satisfies the properties (H1) to (H5), we have proved Theorem 1.1. ■

We next prove Theorem 1.2.

Proof of Theorem 1.2. If ℓ is finite, then it follows from Lemma 3.2 that if $k > \ell$, then any k-jet z of $J^k(n,p)\backslash W^k_{\ell+1}$ is $\mathcal{K}$-k-determined and we have

$$(\pi^\infty_k)^{-1}(\mathcal{O}^k_\ell(n,p)) = \mathcal{O}^\infty_\ell(n,p).$$

Therefore, if $k \geqq \max\{\ell+1, n-|n-p|+2\}$, then the relative homotopy principle holds for $\mathcal{O}^k_\ell$-regular maps and also for $\mathcal{O}^\infty_\ell$-regular maps. ■

Corollary 3.6 *Under the same assuption of Theorem 1.2, given a map $f : N \to P$ is homotopic to an $\mathcal{O}^k_\ell$-regular map if and only if there exists a section $s \in \Gamma_{\mathcal{O}^k_\ell}(N,P)$ such that $\pi^k_P \circ s$ is homotopic to f.*

Here we give two remarks.

Remark 3.7 *Let W^∞_∞ denote the subspace of $J^\infty(n,p)$ which consists of all jets z such that any smooth map germ f with $z = j^\infty f$ is not finitely determined. Let $W^\infty_\infty(N,P)$ is the subbundle of $J^\infty(N,P)$ associated to W^∞_∞. It has been proved (see, for example, [W, Theorem 5.1]) that W^∞_∞ is not of finite codimension in $J^\infty(n,p)$. Consequently, the space of all smooth maps $f : N \to P$ with $j^\infty f(N) \subset J^\infty(N,P)\backslash W^\infty_\infty(N,P)$ is dense in $C^\infty(N,P)$. In other words if N is compact, then a smooth map $f : N \to P$ has an integer ℓ such that f is homotopic to an O^∞_ℓ-regular map.*

Remark 3.8 *It is very important to study the space $W^k_{\ell+1}(n,p)$ and obstructions for finding an O^k_ℓ-regular map. The Thom polynomials related to $W^k_{\ell+1}(n,p)$ have been studied in the dimensions $n = p \leqq 8$ in [O] and [F-R].*

4 Nonexistence theorems

In this section we will discuss the nonexistence of $\mathcal{O}^k_\ell$-regular maps $f : N \to P$. Let $W^k_{\ell+1}(N,P)$ denote the subbundle of $J^k(N,P)$ associated to $W^k_{\ell+1}(n,p)$. By the homotopy principle for $\mathcal{O}^k_\ell$-regular maps in Theorem 1.2, the existence of a section of $J^k(N,P)\backslash W^k_{\ell+1}(N,P)$ over N is equivalent to the existence of an $\mathcal{O}^k_\ell$-regular map. However, it is not so easy to find obstructions associated to

$W^k_{\ell+1}(N,P)$ such as Thom polynomials of $W^k_{\ell+1}(N,P)$, and so we will adopt a method applied in [A1], [I-K] and [duP4] in this section.

For $k \geqq p+1$, let $\Sigma(n,p;k)$ denote the algebraic subset of all C^∞-nonstable k-jets of $J^k(n,p)$ defined in [MaV]. Note that for $k' > k$, $(\pi^{k'}_k)^{-1}(\Sigma(n,p;k)) = \Sigma(n,p;k')$. We have proved the following proposition in [A1, Corollary 5.6].

Proposition 4.1 *Let $k \geqq p+1$. If*

$$(p-n+i)(\frac{1}{2}i(i+1)-p+n)-i^2 \geqq n,$$

then we have that $(\pi^k_1)^{-1}(\Sigma^i(n,p)) \subset \Sigma(n,p;k)$.

In [I-K] the following proposition has been proved, while it has not been stated explicitly and the proof has been given in the context without the details. So we give a sketchy proof.

Proposition 4.2 ([I-K]) *Let ℓ be a nonnegative integer and $k \geqq p+\ell+1$. If*

$$(p-n+i)(\frac{1}{2}i(i+1)-p+n)-i^2 \geqq n+\ell,$$

then we have that $\Sigma^i(n,p) \subset W^k_{\ell+1}(n,p)$. In particular, if $n=p$ and $\frac{1}{2}i^2(i-1) \geqq n+\ell$, then we have that $\Sigma^i(n,n) \subset W^k_{\ell+1}(n,n)$.

Proof. Take a jet z in $\Sigma^i(n,p)$ such that $z = j^k_0 f$. Suppose that $z \notin W^k_{\ell+1}$, and hence codim$\mathcal{K}z \leqq \ell$. By [MaIV] there exists a versal unfolding $F : (\mathbb{R}^n \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^p \times \mathbb{R}^\ell, 0)$ of f and $j^k_{(0,0)}F \notin \Sigma(n+\ell, p+\ell; k)$. Here, we note that $j^k_{(0,0)}F$ is of kernel rank i. By the assumption and Proposition 4.1 we have

$$\Sigma^i(n+\ell, p+\ell) \subset \Sigma(n+\ell, p+\ell; k).$$

This implies $j^k_{(0,0)}F \in \Sigma(n+\ell, p+\ell; k)$. This is a contradiction. Hence, z lies in $W^k_{\ell+1}$. ■

We show the following proposition by applying Proposition 4.2.

Proposition 4.3 *Let ℓ be a nonnegative integer and $k \geqq p+\ell+1$. If $\Sigma^i(n,p) \subset W^k_{\ell+1}(n,p)$, then we have that for any positive integer m, $\Sigma^i(m+n, m+p) \subset W^k_{\ell+1}(m+n, m+p)$.*

Proof. Let $z = j^k_0 f \in \Sigma^i(m+n, m+p)$. Setting $\alpha = j^1_0 f$, we identify α with the homomorphism $\mathbb{R}^{m+n} \to \mathbb{R}^{m+p}$. Let $\mathrm{Ker}(\alpha)^\perp$ and $\mathrm{Im}(\alpha)^\perp$ be the orthogonal complement of the kernel $\mathrm{Ker}(\alpha)$ and the image $\mathrm{Im}(\alpha)$ of α respectively. Let L and M be subspaces of $\mathrm{Ker}(\alpha)^\perp$ and $\mathrm{Im}(\alpha)$ of dimension m such that α maps L onto M isomorphically. Let $L^\perp$ and $M^\perp$ be their orthogonal complements in $\mathrm{Ker}(\alpha)^\perp$ and $\mathrm{Im}(\alpha)$ respectively. Then α is decomposed as in the following exact sequence.

$$0 \to \mathrm{Ker}(\alpha) \to L \oplus L^\perp \oplus \mathrm{Ker}(\alpha) \xrightarrow{\alpha} M \oplus M^\perp \oplus \mathrm{Im}(\alpha)^\perp \to \mathrm{Im}(\alpha)^\perp \to 0$$

Let us choose coordinates

$$(u_1, \cdots, u_m),\ (u_{m+1}, \cdots, u_{m+n-i-1}) \text{ and } (u_{m+n-i}, \cdots, u_{m+n})$$

of L, $L^\perp$ and $\mathrm{Ker}(\alpha)$, and coordinates

$$(y_1, \cdots, y_m),\ (y_{m+1}, \cdots, y_{m+n-i-1}) \text{ and } (y_{m+n-i}, \cdots, y_{m+p})$$

of M, $M^\perp$ and $\mathrm{Im}(\alpha)^\perp$ respectively. Since α maps L onto M isomorphically, there exist the new coordinates $(x_1, \cdots, x_{m+n})$ of $\mathbb{R}^{m+n}$ such that

$$x_j = x_j(u_1, \cdots, u_{m+n})\ (1 \leq j \leq m) \ \text{ and } x_j = u_j\ (m+1 \leq j \leq m+n)$$

and that

$$y_j \circ f(x_1, \cdots, x_{m+n}) = x_j\ (1 \leq j \leq m). \tag{4.1}$$

Setting $\dot{x} = (x_{m+1}, \cdots, x_{m+n})$, we define the map $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ by

$$y_j \circ g(\dot{x}) = y_j \circ f(0, \cdots, 0, \dot{x}) \quad (m+1 \leq j \leq m+p).$$

Then f is an unfolding of g by (4.1) and g is of kernel rank i at the origin. We next prove by following the argument and the notation used in [MaIV, Section 1] that $d(g, \mathcal{K})$ is equal to $d(f, \mathcal{K})$. Define $\pi : \theta(f) \to \theta(g)$ by

$$\pi\left(\sum_{j=1}^{m} a_j tf(\frac{\partial}{\partial xj}) + \sum_{j=m+1}^{m+p} a_j(\frac{\partial}{\partial yj} \circ f)\right) = \sum_{j=m+1}^{m+p} a'_j(\frac{\partial}{\partial yj} \circ g),$$

where $a_j \in C^\infty(\mathbb{R}^{m+n}, 0)$, $a'_j \in C^\infty(\mathbb{R}^n, 0)$ and $a'_j(\dot{x}) = a_j(0, \cdots, 0, \dot{x})$. We note that

$$\begin{aligned}
&tf(\partial/\partial x_j) = (\partial/\partial y_j) \circ f + \textstyle\sum_{t=m+1}^{m+p} (\partial y_t \circ f/\partial x_j)(\partial/\partial y_t) \circ f \quad (1 \leq j \leq m),\\
&tf(\partial/\partial x_j) = \textstyle\sum_{t=m+1}^{m+p} (\partial y_t \circ f/\partial x_j)(\partial/\partial y_t) \circ f \quad (m+1 \leq j \leq m+n),\\
&(\partial y_t \circ f/\partial x_j)(0, \cdots, 0, \dot{x}) = (\partial y_t \circ g/\partial x_j)(\dot{x}) \quad (m+1 \leq t \leq m+p).
\end{aligned}$$

Since

$$y_t \circ f(x_1, \cdots, x_{m+n}) - y_t \circ f(0, \cdots, 0, \dot{x}) = \sum_{u=1}^{m} x_u b_u(x_1, \cdots, x_{m+n}),$$

for some $b_j \in C^\infty(\mathbb{R}^{m+n}, 0)$, we have

$$\partial y_t \circ f/\partial x_j - \partial y_t \circ g/\partial x_j = \sum_{u=1}^{m} x_u(\partial b_u/\partial x_j) \quad (m+1 \leq j \leq m+n).$$

Hence, the assertion follows from an elementary calculation.

Since $j_0^k g \in \Sigma^i(n, p) \subset W^k_{\ell+1}(n, p)$, we have $d(g, \mathcal{K}) \geqq \ell + 1$. Hence, we have $d(f, \mathcal{K}) \geqq \ell + 1$. This shows $z \in W^k_{\ell+1}(m+n, m+p)$. This is what we want. ■

Let ξ be a stable vector bundle over a space. Let $\mathbf{c}(\Sigma^i, \xi)$ denote the determinant of the $(p-n+i)$-matrix whose (s,t)-component is the $(i+s-t)$-th Stiefel-Whitney class $W_{i+s-t}(\xi)$. If $n-p$ and i are even, say $n-p=2u$ and $i=2v$, and if ξ is orientable, then $\mathbf{c}_{\mathbb{Z}}(\Sigma^i, \xi)$ expresses the determinant of the $(v-u)$-matrix whose (s,t)-component is the $(v+s-t)$-th Pontrjagin class $P_{v+s-t}(\xi)$.

$$\begin{vmatrix} W_i & \cdots & W_{n-p+1} \\ \vdots & \ddots & \vdots \\ W_{n-p+2i-1} & \cdots & W_i \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} P_v & \cdots & P_{u+1} \\ \vdots & \ddots & \vdots \\ P_{2v-u-1} & \cdots & P_v \end{vmatrix}$$

Let τ_X denote the stable tangent bundle of a manifold X. If $f : N \to P$ is a smooth map transverse to $\Sigma^i(N, P)$ and $\xi = \tau_N - f^*(\tau_P)$, then $\mathbf{c}(\Sigma^i, \xi)$ (resp. $\mathbf{c}_{\mathbb{Z}}(\Sigma^i, \xi)$) is equal to the (resp. integer) Thom polynomial of the topological closure of $(j^k f)^{-1}(\Sigma^i(N, P))$ ([Po], [Ro] and see also [A1, Proposition 5.4]). If it does not vanish, then $(j^k f)^{-1}(\Sigma^i(N, P))$ cannot be empty by the obstruction theory in [St]. Hence, we have the following corollary of Propositions 4.2 and 4.3.

Corollary 4.4 *Let $f : N \to P$ be a smooth map. Under the same assumption of Proposition 4.2. we assume that either*
(i) $\mathbf{c}(\Sigma^i, \tau_N - f^*(\tau_P))$ does not vanish, or
(ii) N and $\tau_N - f^*(\tau_P)$ are orientable, $n-p$ and i are even and $\mathbf{c}_{\mathbb{Z}}(\Sigma^i, \tau_N - f^*(\tau_P))$ does not vanish.
Then f is not homotopic to any $\mathcal{O}^k_\ell$-regular map.

5 Homotopy equivalences

In this section we will study the filtration in (1.1) in Introduction by applying Corollaries 3.7 and 4.4 and Remark 3.8.

Let us first review what is called the Sullivan's exact sequence in the surgery theory following [M-M] (see also [K-M], [Su] and [Br]).

In what follows P is a closed and oriented n-manifold. We define the set $\mathcal{S}(P)$ to be the set of all equivalence classes of homotopy equivalences $f : N \to P$ of degree 1 under the following equivalence relation. Let N_j be closed oriented n-manifolds and let $f_j : N_j \to P$ be homotopy equivalences of degree 1 $(j = 1, 2)$. We say that f_1 and f_2 are equivalent if there exists an h-cobordism W of N_1 and N_2 and a homotopy equivalence $F : (W, N_1 \cup (-N_2)) \to (P \times [0,1], P \times 0 \cup (-P) \times 1)$ of degree 1 such that $F|N_j = f_j$ $(j = 1, 2)$.

Let $O(k)$ denote the rotation group of $\mathbb{R}^k$ and Let G_k denote the space of all homotopy equivalence of the $(k-1)$-sphere S^{k-1} equipped with the compact-open topology. By considering the canonical inclusions $O(k) \to O(k+1)$ and $G_k \to G_{k+1}$, we set $O = \lim_{k\to\infty} O(k)$ and $G = \lim_{k\to\infty} G_k$. Let BO and BG denote the classifying spaces for O and G. Then the canonical inclusion $O \to G$ induces a map $\pi : BO \to BG$. Regarding $\pi : BO \to BG$ as a fibration, we let

G/O denote its fiber. Let m be a sufficiently large number. Let $\eta_{O(m)}$ denote the universal vector bundle over $BO(m)$. Let $i_{G/O} : G(m)/O(m) \to BO(m)$ be the inclusion of a fiber and set $\eta_{G/O} = (i_{G/O})^*\eta_{O(m)}$. Then $\eta_{G/O}$ has a trivialization $t_{G/O} : \eta_{G/O} \to \mathbb{R}^m$ as a spherical fibration.

We next recall the surgery obstruction $\mathfrak{s}^P_{4k} : [P, G/O] \to \mathbb{Z}$ only in the case of $n = 4q$. For $[\alpha] \in [P, G/O]$ let $\eta = \alpha^*(\eta_{O(m)})$ with the canonical bundle map $\overline{\alpha} : \eta \to \eta_{O(m)}$ covering α and the projection π_η onto P. We deform $t_{G/O} \circ \overline{\alpha}$ to a map transverse to $0 \in \mathbb{R}^m$ and let M be the inverse image of 0 with a map $\pi_\eta|M : M \to P$ of degree 1. We define $\mathfrak{s}^P_{4q}([\alpha]) = (1/8)(\sigma(N) - \sigma(P))$. If P is simply connected in addition, then there have been defined an injection $j^P : \mathcal{S}(P) \to [P, G/O]$ such that if $\mathfrak{s}^P_{4q}([\alpha]) = 0$, $\pi_\eta|M$ is deformed to a homotopy equivalence $f : N \to P$ under a certain cobordism. The following is the Sullivan's exact sequence.

$$0 \longrightarrow \mathcal{S}(P) \xrightarrow{j^P} [P, G/O] \xrightarrow{\mathfrak{s}^P_{4k}} \mathbb{Z}$$

Let us now define the *cobordism group* Ω^{h-eq}_n of homotopy equivalences of degree 1. Let N_j and P_j be oriented closed n-manifolds and let $f_j : N_j \to P_j$ be homotopy equivalences of degree 1 ($j = 1, 2$). We say that f_1 and f_2 are cobordant if there exists an oriented $(n+1)$-manifold W, V and a homotopy equivalence $F : (W, \partial W) \to (V, \partial V)$ of degree 1 such that $\partial W = N_1 \cup (-N_2)$, $\partial V = P_1 \cup (-P_2)$ and $F|N_j = f_j$. The cobordism class of $f : N \to P$ is denoted by $[f : N \to P]$. Let Ω^{h-eq}_n denote the set which consists of all cobordism classes of homotopy equivalences of degree 1. We provide Ω^{h-eq}_n with a module structure by setting

- $[f_1 : N_1 \to P_1] + [f_2 : N_2 \to P_2] = [f_1 \cup f_2 : N_1 \cup N_2 \to P_1 \cup P_2]$,
- $-[f : N \to P] = [f : (-N) \to (-P)]$.

The null element is defined to be $[f : N \to P]$ which bound a homotopy equivalence $F : (W, \partial W) \to (V, \partial V)$ of degree 1 such that $\partial W = N$, $\partial V = P$ and $F|N = f$. Even if P is not simply connected, we can easily find $f_1 : N_1 \to P_1$ with P_1 being simply connected in the same cobordism class by killing $\pi_1(N) \approx \pi_1(P)$ by usual surgery.

Let $\mathbf{c}_{\mathbb{Q}}(\Sigma^{2i}, \eta_{G/O})$ denote the image of $\mathbf{c}_{\mathbb{Z}}(\Sigma^{2i}, \eta_{G/O})$ in $H_{4q-4i^2}(G/O; \mathbb{Q})$. Let $\alpha = j^P([f : N \to P])$. Then it induces the homomorphism $\mathcal{C}_{2i} : \Omega^{h-eq}_{4q} \to H_{4q-4i^2}(G/O; \mathbb{Q})$ defined by

$$\begin{aligned}\mathcal{C}_{2i}([f : N \to P]) &= \mathbf{c}_{\mathbb{Q}}(\Sigma^{2i}, \eta_{G/O}) \cap \alpha([P]) \\ &= \mathbf{c}_{\mathbb{Q}}(\Sigma^{2i}, \eta_{G/O}) \otimes 1 \cap (\alpha \times c_P)_*([P]),\end{aligned}$$

under the identification

$$H_{4q-4i^2}(G/O; \mathbb{Q}) = H_{4q-4i^2}(G/O; \mathbb{Q}) \otimes 1$$

in $\Sigma_{j=0}^{q-i^2} H_{4j}(G/O;\mathbb{Q}) \otimes H_{4q-4i^2-4j}(BSO;\mathbb{Q})$. We have that

$$\begin{aligned}
\mathcal{C}_{2i}(\alpha) &= \mathbf{c}_{\mathbb{Q}}(\Sigma^{2i}, \eta_{G/O}) \cap (\alpha)_*([P]) \\
&= \mathbf{c}_{\mathbb{Q}}(\Sigma^{2i}, \eta_{G/O}) \cap (\alpha \circ f)_*([N]) \\
&= (\alpha \circ f)_*((\alpha \circ f)^*(\mathbf{c}_{\mathbb{Q}}(\Sigma^{2i}, \eta_{G/O})) \cap [N]) \\
&= (\alpha \circ f)_*(\mathbf{c}_{\mathbb{Z}}(\Sigma^{2i}, \tau_N - f^*(\tau_P)) \cap [N]).
\end{aligned}$$

Furthermore, we have proved in [A5, Theorems 3.2 and 4.1] that for integers q and i with $q \geqq i \geqq 1$,

$$\dim \Omega_{4q}^{h-eq}/(\Omega_{4q}^{h-eq} \cap \mathrm{Ker}(\mathcal{C}_{2i})) \otimes \mathbb{Q} = \dim H_{4q-4i^2}(BSO;\mathbb{Q}). \tag{5.1}$$

The following theorem follows from (5.1) and Proposition 4.2.

Theorem 5.1 *Let ℓ, q and i be given integers with $\ell \geqq 0$ and $q \geqq i$. Let $k \geqq 4q + \ell + 1$. There exists a cobordism class $[f : N \to P] \in \Omega_{4q}^{h-eq}$ such that $\mathbf{c}_{\mathbb{Z}}(\Sigma^{2i}, \tau_N - f^*(\tau_P))$ is not a torsion element and that if $4i^3 - 2i^2 \geqq 4q + \ell \geqq 4i^2 + \ell$, then f is not cobordant in Ω_{4q}^{h-eq} to any $\mathcal{O}_\ell^k$-regular map.*

We can prove the following theorem using Theorem 5.1 by applying the same argument in the proof of [A5, Theorem 0.2]. However, it is very important to prove Theorem 1.2 and the situation is rather different. Therefore, we give its proof.

Theorem 5.2 *Let ℓ, q and i be given integers with $\ell \geqq 0$ and $q \geqq i$. Let $k \geqq 8q + \ell + 1$. If $4i^3 - 2i^2 \geqq 4q + \ell \geqq 4i^2 + \ell$, then there exists a closed connected oriented $8q$-manifold P and a homotopy equivalence $f : P \to P$ of degree 1 such that $\mathbf{c}_{\mathbb{Z}}(\Sigma^{2i}, \tau_P - f^*(\tau_P)) \neq 0$ and that f is not cobordant in Ω_{8q}^{h-eq} to any $\mathcal{O}_\ell^k$-regular map.*

Proof. It follows from Theorem 5.1 that there exists a homotopy equivalence $f : N \to P$ of degree 1 between $4q$-manifolds such that $\mathbf{c}_{\mathbb{Z}}(\Sigma^{2i}, \tau_N - f^*(\tau_P))$ is not a torsion element. Let $f^{-1} : P \to N$ be a homotopy inverse of f. Define $g : N \times P \to N \times P$ by $g(x, y) = (f^{-1}(y), f(x))$. We have $k \geq \dim N \times P + \ell + 1$. If we prove that $\mathbf{c}_{\mathbb{Z}}(\Sigma^{2i}, \tau_{N\times P} - g^*(\tau_{N\times P}))$ does not vanish, then, by Proposition 4.2, g is not homotopic to any $\mathcal{O}_\ell^k$-regular map. We set $\xi = \tau_{N\times P} - g^*(\tau_{N\times P}) = \tau_N \times \tau_P - f^*(\tau_P) \times (f^{-1})^*(\tau_N)$. Then

$$\begin{aligned}
p_j(\xi) &= \sum_{s+t=j} p_s(\tau_N \times \tau_P)\overline{p}_t(f^*(\tau_P) \times (f^{-1})^*(\tau_N)) \\
&= \sum_{s+t=j} \sum_{\substack{s_1+s_2=s \\ t_1+t_2=t}} p_{s_1}(\tau_N)\overline{p}_{t_1}(f^*(\tau_P)) \otimes p_{s_2}(\tau_P)\overline{p}_{t_2}((f^{-1})^*(\tau_N))
\end{aligned}$$

modulo torsion in $H^*(N;\mathbb{Z}) \otimes H^*(P;\mathbb{Z})$. The term of $p_j(\xi)$ which lies in $H^{4j}(N;\mathbb{Z}) \otimes H^0(P;\mathbb{Z})$ is equal modulo torsion to

$$\sum_{s+t=j} p_s(\tau_N)\overline{p}_t(f^*(\tau_P)) \otimes 1 = p_j(\tau_N - f^*(\tau_P)) \otimes 1.$$

Hence, we have that $\mathbf{c}_{\mathbb{Z}}(\Sigma^{2i}, \tau_{N\times P} - g^*(\tau_{N\times P})$ is equal to the sum of $\mathbf{c}_{\mathbb{Z}}(\Sigma^{2i}, \tau_N - f^*(\tau_P)) \otimes 1$ and the other term which lies in $\Sigma_{j=1}^{i^2} H^{4i^2-4j}(N;\mathbb{Z}) \otimes H^{4j}(P;\mathbb{Z})$ modulo torsion. Since $\mathbf{c}_{\mathbb{Z}}(\Sigma^{2i}, \tau_N - f^*(\tau_P))$ does not vanish, it follows that $\mathbf{c}_{\mathbb{Z}}(\Sigma^{2i}, \tau_{N\times P} - g^*(\tau_{N\times P}))$ does not vanish. This completes the proof. ■

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. In the proof k refers to a sufficiently large integer. Let $i_0 = 2$, which is the smallest integer such that $4i^3 - 2i^2 \geqq 4i^2$. Then we have, by Theorem 5.2, a closed connected oriented $8 \cdot 4$-manifold P_0 and a homotopy equivalence $f_0 : P_0 \to P_0$ of degree 1 such that $\mathbf{c}_{\mathbb{Z}}(\Sigma^4, \tau_{P_0} - f_0^*(\tau_{P_0})) \neq 0$ and that f_0 is not homotopic to any $\mathcal{O}_0^k$-regular map. By Remark 3.8 there exists an integer ℓ such that f_t is homotopic to an $\mathcal{O}_\ell^k$-regular map. Let ℓ_1 be such a smallest integer.

We assume the following (A-t) for an integer $t \geqq 0$, where $\ell_0 = 0$.

(A-t) We have constructed integers ℓ_t, ℓ_{t+1}, i_t, a closed oriented $8 \cdot i_t^2$-manifold P_t and an $\mathcal{O}_{\ell_{t+1}}^k$-regular homotopy equivalence $f_t : P_t \to P_t$ of degree 1 such that $4i_t^3 - 2i_t^2 \geqq 4i_t^2 + \ell_t$, $\ell_{t+1} > \ell_t$, $\mathbf{c}_{\mathbb{Z}}(\Sigma^{2i_t}, \tau_{P_t} - f_t^*(\tau_{P_t})) \neq 0$ and that f_t is not homotopic to any $\mathcal{O}_{\ell_t}^k$-regular map.

Under the assumption (A-t) we prove (A-$(t+1)$) with $\ell_{t+1} < \ell_{t+2}$. Let i_{t+1} be the smallest integer among the integers $i > 0$ with $4i^3 - 2i^2 \geqq 4i^2 + \ell_{t+1}$. Then it follows from Theorem 5.2 that there exist a closed connected oriented $8 \cdot i_{t+1}^2$-manifold P_{t+1} and a homotopy equivalence $f_{t+1} : P_{t+1} \to P_{t+1}$ of degree 1 such that $\mathbf{c}_{\mathbb{Z}}(\Sigma^{2i_t}, \tau_{P_{t+1}} - f_{t+1}^*(\tau_{P_{t+1}})) \neq 0$ and that f_{t+1} is not homotopic to any $\mathcal{O}_{\ell_{t+1}}^k$-regular map. It follows Remark 3.8 that there exists an integer ℓ such that f_{t+1} is homotopic to an $\mathcal{O}_\ell^k$-regular map. Let ℓ_{t+2} be the smallest integer among those integers ℓ. Hence, we have $\ell_{t+2} > \ell_{t+1}$. This proves (A-$(t+1)$).

Thus we have defined the sequences $\{i_t\}$, $\{\ell_t\}$, closed connected oriented manifolds $\{P_t\}$ of dimensions $\{8 \cdot i_t^2\}$ and homotopy equivalences $\{f_t\}$ of degree 1 which satisfy the above properties.

Given a positive integer d, let

$$\begin{aligned} P &= P_0 \times P_1 \times P_2 \times \cdots \times P_d, \\ F_t &= id_{P_0} \times \cdots \times id_{P_{t-1}} \times f_t \times id_{P_{t+1}} \times \cdots \times id_{P_d} \quad (0 \leqq t \leqq d), \end{aligned}$$

and $p = \sum_{t=0}^d 8 \cdot i_t^2$. We show that $F_t \notin \mathfrak{h}_{\ell_t}(P)$ and $F_t \in \mathfrak{h}_{\ell_{t+1}}(P)$. Let $q_t : P \to P_t$ be the canonical projection. Then the stable tangent bundle τ_P is isomorphic to $q_1^*(\tau_{P_1}) \oplus q_2^*(\tau_{P_2}) \oplus \cdots \oplus q_d^*(\tau_{P_d})$. Hence, $\tau_P - F_t^*(\tau_P)$ is equal to

$$
\begin{aligned}
& q_0^*(\tau_{P_0}) \oplus q_1^*(\tau_{P_1}) \oplus \cdots \oplus q_d^*(\tau_{P_d}) \\
& - ((q_0 \circ F_t)^*(\tau_{P_0}) \oplus (q_1 \circ F_t)^*(\tau_{P_1}) \oplus \cdots \oplus (q_d \circ F_t)^*(\tau_{P_d})) \\
& = q_0^*(\tau_{P_0}) \oplus q_1^*(\tau_{P_1}) \oplus \cdots \oplus q_d^*(\tau_{P_d}) \\
& - (q_0^*(\tau_{P_0}) \oplus \cdots \oplus q_{t-1}^*(\tau_{P_{t-1}}) \oplus (f_t \circ q_t)^*(\tau_{P_t}) \oplus \cdots \oplus q_d^*(\tau_{P_d})) \\
& = q_t^*(\tau_{P_t}) - (f_t \circ q_t)^*(\tau_{P_t}) \\
& = q_t^*((\tau_{P_t}) - f_t^*(\tau_{P_t})).
\end{aligned}
$$

This shows that

$$
\begin{aligned}
\mathbf{c}_{\mathbb{Z}}(\Sigma^{2i_t}, \tau_P - F_t^*(\tau_P)) &= \mathbf{c}_{\mathbb{Z}}(\Sigma^{2i_t}, q_t^*((\tau_{P_t}) - f_t^*(\tau_{P_t})) \\
&= q_t^*(\mathbf{c}_{\mathbb{Z}}(\Sigma^{2i_t}, \tau_{P_t} - f_t^*(\tau_{P_t})),
\end{aligned}
$$

which does not vanish in $H^{2i_t^2}(P;\mathbb{Z})$ since $\mathbf{c}_{\mathbb{Z}}(\Sigma^{2i_t}, \tau_{P_t} - f_t^*(\tau_{P_t})) \neq 0$ and since $q_t^* : H^{2i_t^2}(P_t;\mathbb{Z}) \to H^{2i_t^2}(P;\mathbb{Z})$ is injective. Furthermore, it follows from Proposition 4.3 that

$$\Sigma^{2i_t}(p,p) \subset W_{\ell+1}^k(p,p).$$

Therefore, F_t is not homotopic to to any $\mathcal{O}_{\ell_t}^k$-regular map. However, since f_t is homotopic to an $\mathcal{O}_{\ell_{t+1}}^k$-regular map, F_t is also homotopic to an $\mathcal{O}_{\ell_{t+1}}^k$-regular map. This proves the theorem. ■

We prepare the results which are necessary to study the filtration in (1.1). The assertions (i) and (ii) in the following theorem have been proved in [A2, Theorem 4.8] and [A4, Theorem 4.1] respectively, which are applications of the homotopy principle.

Theorem 5.3 *Let P be orientable and let $f : P \to P$.*

(i) A map f is homotopic to a fold-map if and only if τ_P is isomorphic to $f^*(\tau_P)$.

(ii) If a map f is Ω^1-regular, then f is homotopic to an $\Omega^{(1,1,0)}$-regular map.

Let $V(n,p)$ be an algebraic set of $J^k(n,p)$ which is invariant with respect to the actions of local diffeomorphisms of $(\mathbb{R}^n,0)$ and $(\mathbb{R}^n,0)$ and Let $V(N,P)$ be the sububndle of $J^k(N,P)$ associated to $V(n,p)$. By [B-H] we have the fundamental class of $V(N,P)$ under the coefficient group $\mathbb{Z}/2$, and have the Thom polynomial $\mathbf{c}(V(n,p), \tau_N - f^*(\tau_P))$ of $V(N,P)$.

Theorem 5.4 *Let $V(p,p)$ be as above and let $f : P \to P$.*

(i) If f is a homotopy equivalence, then $\mathbf{c}(V(p,p), \tau_P - f^*(\tau_P))$ vanishes.

(ii) $\mathbf{c}_{\mathbb{Z}}(W_p^k(p,p), \tau_P - f^*(\tau_P)) = 0$ for $p = 5,6,7$ and

$$\mathbf{c}_{\mathbb{Z}}(W_8^k(8,8), \tau_P - f^*(\tau_P)) = 9P_2(\tau_P - f^*(\tau_P)) + 3P_1^2(\tau_P - f^*(\tau_P))$$

for $p = 8$.

(iii) Let $p \leqq 8$. Then there exists a section s of $\mathcal{O}_{p-1}^k(P,P)$ over P with $\pi_P^k \circ s = f$ if and only if $\mathbf{c}_{\mathbb{Z}}(W_p^k(p,p), \tau_P - f^*(\tau_P)) = 0$.

Proof. (i) Let $S(\nu_P)$ denote the spherical normal fiber space of P. It follows from [Sp] that $S(\nu_P)$ is equivalent to $f^*(S(\nu_P))$. Hence, the associated spherical spaces of τ_P and $f^*(\tau_P)$ are equivalent. In particular, the Stiefel-Whitney classes of $\tau_P - f^*(\tau_P)$ vanish.

(ii) If $p \leqq 8$, then a map $f : P \to P$ is homotopic to a smooth map with only $\mathcal{K}$-simple singularities by [MaVI]. According to [F-R], the integer Thom polynomial of $W_p^k(p,p)$ is equal to the formula for $p = 8$ and vanish for $p = 5, 6, 7$ under $\mathbb{Z}$.

(iii) The primary obstruction in $H^p(P; \pi_{p-1}(\mathcal{O}_{p-1}^k(p,p))$ is the unique obstruction for finding the required section. By an elementary argument we have

$$\pi_{p-1}(\mathcal{O}_{p-1}^k(p,p)) \approx H_{p-1}(\mathcal{O}_{p-1}^k(p,p); \mathbb{Z}) \approx H^{\dim W_p^k(p,p)}(W_p^k(p,p); \mathbb{Z}).$$

This shows the assertion. ■

Finally we study the filtration in (1.1) in the case of P being orientable and $p \leqq 8$ by applying the homotopy principles in Theorems 1.2 and 5.3. We have $\mathfrak{h}_p(P) = \mathfrak{h}(P)$.

Examples.

Case: $p \leqq 3$; $\mathfrak{h}_0(P) \subset \mathfrak{h}_1(P) = \mathfrak{h}(P)$.

Since P is parallelizable, TP and $f^*(TP)$ are trivial. So a map $f : P \to P$ is homotopic to a fold-map.

Case: $p = 4$; $\mathfrak{h}_0(P) \subset \mathfrak{h}_1(P) \subset \mathfrak{h}_2(P) \subset \mathfrak{h}_4(P)$.

It is known that $\mathbf{c}_{\mathbb{Z}}(\Sigma^4; \tau_P - f^*(\tau_P)) = P_2(\tau_P - f^*(\tau_P))$. If this class vanish, then there exists a section $P \to \Omega^1(P,P)$ covering f, and hence an Ω^1-regular map by [F]. By Theorems 5.3 and 5.4 we obtain an $\Omega^{(1,1,0)}$-regular map homotopic to f.

Case: $5 \leqq p \leqq 7$; $\mathfrak{h}_0(P) \subset \mathfrak{h}_1(P) \subset \mathfrak{h}_2(P) \subset \mathfrak{h}_{p-1}(P) = \mathfrak{h}_p(P)$.

This follows from Theorems 1.2 and 5.4.

Case: $p = 8$; $\mathfrak{h}_0(P) \subset \mathfrak{h}_1(P) \subset \mathfrak{h}_2(P) \subset \mathfrak{h}_7(P) \subset \mathfrak{h}_8(P)$.

If $9P_2(\tau_P - f^*(\tau_P)) + 3P_1^2(\tau_P - f^*(\tau_P)) = 0$, then the homotopy class of f lies in $\mathfrak{h}_7(P)$ by Theorems 1.2 and 5.4.

For more precise information we must investigate the obstructions for finding sections in $\Gamma_{\mathcal{O}_\ell^k}(P,P)$ related to $W_{\ell+1}^k(p,p)$.

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