On Equivariant Embedding of Hilbert C^* modules

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AMS subject classification no. : **46L08**
Key-words : Hilbert modules, Ergodic action, Equivariant triviality, Equivariant embedding.

Abstract

We prove that an arbitrary (not necessarily countably generated) Hilbert G-$\mathcal{A}$ module on a $G - C^*$ algebra $\mathcal{A}$ admits an equivariant embedding into a trivial $G - \mathcal{A}$ module, provided G is a compact Lie group and its action on $\mathcal{A}$ is ergodic.

1 Introduction

Let G be a locally compact group, $\mathcal{A}$ be a C^*-algebra, and assume that there is a strongly continuous representation $\alpha : G \to Aut(\mathcal{A})$. Following the terminology of [8], we introduce the concept of a Hilbert C^* $G - \mathcal{A}$-module as follows :

Definition 1.1 A *Hilbert C^* $G - \mathcal{A}$ module* (or *$G - \mathcal{A}$ module* for short) is a pair (E, β) where E is a Hilbert C^* $\mathcal{A}$-module and β is a map from G into the set of $\mathbb{C}$-linear (caution : **not** $\mathcal{A}$-linear !) maps from E to E, such that $\beta_g \equiv \beta(g), g \in G$ satisfies the following :
(i) $\beta_{gh} = \beta_g \circ \beta_h$ for $g, h \in G$, $\beta_e = \mathrm{Id}$, where e is the identity element of G;
(ii) $\beta_g(\xi a) = \beta_g(\xi)\alpha_g(a)$ for $\xi \in E, a \in \mathcal{A}$;
(iii) $g \mapsto \beta_g(\xi)$ is continuous for each fixed $\xi \in E$;
(iv) $\langle \beta_g(\xi), \beta_g(\eta) \rangle = \alpha_g(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in E$, where $\langle \cdot, \cdot \rangle$ denotes the $\mathcal{A}$-valued inner product of E.

When β is understood from the context, we may refer to E as a $G - \mathcal{A}$ module, without explicitly mentioning the pair (E, β). Given two $G - \mathcal{A}$ modules (E_1, β) and (E_2, γ), there is a natural G-action induced on $\mathcal{L}(E_1, E_2)$, given by $\pi_g(T)(\xi) := \gamma_g(T(\beta_{g^{-1}}(\xi)))$ for $g \in G, \xi \in E_1, T \in \mathcal{L}(E_1, E_2)$. $T \in \mathcal{L}(E_1, E_2)$ is said to be *G-equivariant* if $\pi_g(T) = T$ for all $g \in G$. It is clear that for each fixed $T \in \mathcal{L}(E_1, E_2)$ and $\xi \in E_1$, $g \mapsto \pi_g(T)\xi$ is continuous. We say that T is *G-continuous* if $g \mapsto \pi_g(T)$ is continuous with respect to the norm topology on $\mathcal{L}(E_1, E_2)$. We say that E_1 and E_2 are *isomorphic as $G - \mathcal{A}$-modules*, or that they are *equivariantly isomorphic* if there is a G-equivariant unitary map $T \in \mathcal{L}(E_1, E_2)$. We call a $(G - \mathcal{A})$ module of the form $(\mathcal{A} \otimes \mathcal{H}, \alpha_g \otimes \gamma_g)$ (where $\mathcal{H}$ is some Hilbert space) a trivial $G - \mathcal{A}$ module. We say that (E, β) is *embeddable* if there is an equivariant isometry from E to $\mathcal{A} \otimes \mathcal{H}$ for some Hilbert space $\mathcal{H}$ with a G-action γ, or in other words, (E, β) is equivariantly isomorphic with a sub-$G - \mathcal{A}$ module of $(\mathcal{A} \otimes \mathcal{H}, \beta \otimes \gamma)$. Note that $\mathcal{A} \otimes \mathcal{H}$ is the closure of $\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{H}$ under the norm inherited from $\mathcal{B}(\mathcal{H}_0, \mathcal{H}_0 \otimes \mathcal{H})$ where $\mathcal{H}_0$ is any Hilbert space such that $\mathcal{A}$ is isometrically embedded into $\mathcal{B}(\mathcal{H}_0)$. The following result on the embeddabiity is due to Mingo and Phillips ([8]).

Theorem 1.2 *Let (E, β) be a Hilbert C^* $G - \mathcal{A}$ module and assume that E is countably generated as a Hilbert $\mathcal{A}$-module, that is, there is a countable set $S = \{e_1, e_2, ...\}$ of elements of E such that the right $\mathcal{A}$-linear span of S is dense in E. Assume furthermore that G is compact. Then (E, β) is embeddable.*

When G is the trivial singleton group, the above result was proved by Kasparov.

If the C^* algebra $\mathcal{A}$ is replaced by a von Neumann algebra $\mathcal{B} \subseteq \mathcal{B}(h)$ for some Hilbert space h and G is a locally compact group with a strongly continuous unitary representation $g \mapsto u_g \in \mathcal{B}(h)$, one can define Hilbert von Neumann G-$\mathcal{B}$ module (E, β). The only difference is that E is now a Hilbert von Neumann $\mathcal{A}$-module equipped with the natural locally convex strong operator topology, and that we replace the norm-continuity in (iii) of the above definition by a weaker continuity : namely, the continuity of $g \mapsto \beta_g(\xi)$ (for fixed $\xi \in E$) with respect to the locally convex topology of E. We have a stronger version of the Theorem 1.2 (see [3] and [2]), namely without the condition of E being countably generated and without the compactness of G. It should be remarked here that the trivial Hilbert von Neumann $\tilde{\mathcal{A}}$ module $\tilde{\mathcal{A}} \otimes \mathcal{H}$ is defined to be the closure of $\tilde{\mathcal{A}} \otimes_{\mathrm{alg}} \mathcal{H}$ with respect to the strong-operator-topology inherited from $\mathcal{B}(h, h \otimes \mathcal{H})$.

In Theorem 1.2, the assumption that E is countably generated restricts the applicability of the result, since it is not always easy to check the property of being countably generated. However, under some special assumption on the G-C^* algebra $\mathcal{A}$, i.e. conditions on the group G, the C^* algebra $\mathcal{A}$ and also on the nature of the action, it may be possible to prove the embedability for an arbitrary Hilbert G-$\mathcal{A}$ module. The aim of the present article is to give some such sufficient conditions.

2 Ergodic action and its implication

We say that the action α of G on a unital C^*-algebra $\mathcal{A}$ is *ergodic* if $\alpha_g(a) = a$ for all $g \in G$ if and only if a is a scalar multiplie of 1. There is a considerable amount of literature on ergodic action of compact groups, and we shall quote one interesting structure theorem which will be useful for us.

Proposition 2.1 *Let G be a compact group acting ergodically on a unital C^*-algebra $\mathcal{A}$. Then there is a set of elements $t^{\pi}_{ij}, \pi \in \hat{G}, i = 1, ..., d_{\pi}, j = 1, ..., m_{\pi}$ of $\mathcal{A}$, where $\hat{G}$ is the set of equivalence classes of irreducible representations of G, d_{π} is the dimension of the irreducible representation space denoted by π, $m_{\pi} \leq d_{\pi}$ is a natural number, such that the followings hold :*
(i) There is a unique faithful G-invariant state τ on $\mathcal{A}$, which is in fact a trace,
(ii) The linear span of $\{t^{\pi}_{ij}\}$ is norm-dense in $\mathcal{A}$,
(iii) $\{t^{\pi}_{ij}\}$ is an orthonormal basis of $h = L^2(\mathcal{A}, \tau)$,
(iv) The action of u_g coincides with the πth irreducible representation of G on the vector space spanned by $t^{\pi}_{ij}, i = 1, ..., d_{\pi}$ for each fixed j and π,
(v) $\sum_{i=1,...d_{\pi}} (t^{\pi}_{ij})^* t^{\pi}_{ik} = \delta_{jk} d_{\pi} 1$, where δ_{jk} denotes the Kronecker delta symbol. Thus, in particular, $\|t^{\pi}_{ij}\| \leq \sqrt{d_{\pi}}$ for all π, i, j.

The proof can be obtained by combining the results of [10],[5] and [1].

Let now $h = L^2(\mathcal{A}, \tau)$, where τ is the unique invariant faithful trace described in Proposition 2.1. Denoting by $\tilde{\mathcal{A}}$ the weak closure of $\mathcal{A}$ in $\mathcal{B}(h)$, we have the following result due to Goswami and Sinha ([2]). However, we give a proof here for readers' convenience.

Proposition 2.2 *Let u_g be the unitary in h induced by the action of G, that is, on the dense set $\mathcal{A} \subseteq h$, $u_g(a) := \alpha_g(a)$, where α_g denotes the G-action on $\mathcal{A}$. Denoting also by α the action $g \mapsto u_g . u_g^*$ on $\bar{\mathcal{A}}$, we have that $\tilde{\mathcal{A}} > \!\!\! \lhd_{\alpha} G$ is isomorphic with the von Neumann algebra generated by $\bar{\mathcal{A}}$ and $u_g, g \in G$.*

Proof :
We shall use the notation of Proposition 2.1. The crossed product von Neumann algebra $\mathcal{C} := \tilde{\mathcal{A}} > \!\!\! \lhd G$ is by definition the von Neumann algebra generated by $\{(t^{\pi}_{ij} \otimes 1), \pi, i, j; (u_g \otimes L_g), g \in G\}$ in $L^2(\tau) \otimes L^2(G)$, where L_g is the regular representation of G in $L^2(G)$. Let ρ be the normal $*$-homomorphism from $\mathcal{C}$ onto $\{\tilde{\mathcal{A}}, u_g, g \in G\}'' \subseteq \mathcal{B}(L^2(\tau))$ which satisfies $\rho(t^{\pi}_{ij} \otimes 1) = t^{\pi}_{ij}$ and $\rho(u_g \otimes L_g) = u_g$. We have to show that this is an

isomorphism, that is, the kernel of ρ is trivial. Clearly, the set of elements of the form $\sum c_{\pi ij} t^{\pi}_{ij} u_{g_{\pi ij}}$ (finitely many terms), with $c_{\pi ij} \in \mathbb{C}; g_{\pi ij} \in G$ is dense with respect to the strong-operator topology in $\{\tilde{\mathcal{A}}, u_g, g \in G\}''$. Similarly, the set of elements of the form $\sum c_{\pi ij}(t^{\pi}_{ij} \otimes 1)(u_{g_{\pi ij}} \otimes L g_{\pi ij})$ (finitely many terms) will be strongly dense in $\mathcal{C}$. Now, let $\mathcal{I} \equiv \{X \in \mathcal{C} : \rho(X) = 0\}$. We need to show that $\mathcal{I} = \{0\}$. Let $X \in \mathcal{I}$ and let $X_p = \sum c^{(p)}_{\pi ij}(t^{\pi}_{ij} \otimes 1)(u_{g^{(p)}_{\pi ij}} \otimes L g^{(p)}_{\pi ij})$ be a net (indexed by p) of elements from the above dense algebra such that X_p converges strongly to X. Hence we have, $\sum |c^{(p)}_{\pi ij}|^2 = \|\rho(X_p)(1)\|^2 \to 0$. This implies that for any $\phi \in L^2(G)$, $\|X_p(1 \otimes \phi)\|^2 = \sum |c^{(p)}_{\pi ij}|^2 \|L_{g^{(p)}_{\pi ij}} \phi\|^2 \leq \|\phi\|^2 \sum |c^{(p)}_{\pi ij}|^2 \to 0$, which proves that $X(1 \otimes \phi) = 0$ for every $X \in \mathcal{I}$. But since $\mathcal{I}$ is an ideal in $\mathcal{C}$, this shows that for $a \in \mathcal{A}$, $X(a \otimes \phi) = (X(a \otimes 1))(1 \otimes \phi) = 0$, and by the fact that $\{a \otimes \phi : a \in \mathcal{A}, \phi \in L^2(G)\}$ is total in $h \otimes L^2(G)$ we conclude that $\mathcal{I} = \{0\}$. $\square$

Considering the C^*-algebra $\mathcal{A}^o := J\mathcal{A}J \subseteq \mathcal{A}' \subseteq \mathcal{B}(h)$, where J is the antilinear isometry on h given by $x \mapsto x^*$, and noting that $(\mathcal{A}^o)'' = \mathcal{A}'$, and that the action $\alpha_g(\cdot) = u_g \cdot u_g^*$ is ergogic on $\mathcal{A}^o$, we conclude from the Proposotion 2.2 the following :

Corollary 2.3 *The crossed product $\mathcal{A}' >\!\!\lhd_{\alpha} G$ is isomorphic with the von Neumann algebra generated by $\mathcal{A}'$ and $\{u_g, g \in G\}$ in $\mathcal{B}(h)$.*

Let us now specialize to the case of a compact Lie group. If G is such a group, with a basis of the Lie algebra given by $\{\chi_1, ..., \chi_N\}$, which has a strongly continuous action θ on a Banach space F, we can consider the space of 'smooth' or C^∞-elements of F, denoted by F^∞, consisting of all $\xi \in f$ such that $G \ni g \mapsto \theta_g(\eta)$ is C^∞. It is easy to prove that (see [2] and references therein) F^∞ is dense in F, and it is a $*$-subalgebra if F is a locally convex $*$-algebra. Moreover, we equip E^∞ with a family of seminorms $\|\cdot\|_{\infty,n}$, $n = 0, 1, ...$ given by

$$\|\xi\|_{\infty,n} := \sum_{i_1, i_2, ... i_k; k \leq n, i_t \in \{1,...,N\}} \|\partial_{i_1} \partial_{i_2} ... \partial_{i_k} \xi\|,$$

with the convention $\|\cdot\|_{\infty,0} = \|\cdot\|$ and where $\partial_j(\xi) := \frac{d}{dt}|_{t=0} \theta_{\exp(t\chi_j)}(\xi)$. The space F^∞ is complete under this family of seminorms, and thus is a Frechet space. When F is Hilbert space or a Hilbert module, we shall also consider a map d_j given by essentially the same expression as that of ∂_j, with χ_j replaced by $i\chi_j$, and the Hilbertian seminorms $\{\|\cdot\|_{2,n}\}$ are given by

$$\|\xi\|_{2,n}^2 := \sum_{i_1,i_2,\dots i_k; k \leq n, i_t \in \{1,\dots,N\}} \|d_{i_1} d_{i_2} \dots d_{i_k} \xi\|_2^2,$$

with $\|\cdot\|_2$ denoting the norm of the Hilbert space (or Hilbert module) F.

More generally, if F is a complete locally convex space given by a family of seminorms $\{\|\cdot\|^{(q)}\}$, then we can consider the smooth subspace F^∞ and the maps ∂_j as above, and make it a complete locally convex space with respect to a larger family of seminorms $\{\|\cdot\|_n^{(q)}\}$ where

$$\|\xi\|_n^{(q)} := \sum_{i_1,i_2,\dots i_k; k \leq n, i_t \in \{1,\dots,N\}} \|\partial_{i_1} \partial_{i_2} \dots \partial_{i_k} \xi\|^{(q)}.$$

In case F is a von Neumann algebra equipped with the locally convex strong operator topology, the locally convex space F^∞ is a topological $*$-algebra, strongly dense in F.

Lemma 2.4 *[2] Let G be a compact Lie group acting ergodically on a unital C^*-algebra $\mathcal{A}$. Then $h^\infty = \mathcal{A}^\infty$ as Frechet spaces.*

Proof :
We give a detailed proof taken essentially from [2]. Let us recall from the Proposition 2.1 that for π in $\hat{G}$, the set of irreducible representations of G (which is countable since G is a compact Lie group) with n_π the dimension of the space (also denoted by π) of the representation π, $\{t_{ij}^\pi \in \mathcal{A},\ i = 1, 2, \dots n_\pi; j = 1, 2, ..m_\pi \leq n_\pi;\ \pi \in \hat{G}\}$ is an orthonormal basis of h. Moreover, let $\{\lambda_\pi, \pi \in \hat{G}\}$ be the set of the eigenvalues, of multiplicity n_π^2, of the bi-invariant Laplacian (say Δ_G) on the compact manifold G (compact Lie group) of dimension N. Then by the Weyl asymptotics of the eigenvalues of the Laplacian (see [9]) we have that

$$\lambda_\pi = O(n_\pi^{\frac{2}{N}}),$$

so that for sufficiently large n,

$$\sum_\pi \frac{n_\pi^3}{\lambda_\pi^{2n}} \leq C \sum_\pi n_\pi^{(3-\frac{4n}{N})} < \infty(\quad \text{where } C \text{ is a positive constant}).$$

It is clear from the Proposition 2.1 that $\partial_k = id_k$ on each of the finite dimensional spaces $\mathcal{H}_l^\pi$ spanned by $\{t_{il}^\pi,\ i = 1, \cdots, n_\pi\}$. We also note that the bi-invariant Lapalacian Δ_G can be expressed as a linear combination of the form $\sum_{j,k=1}^{N} p_{jk}\ \chi_j\chi_k$ (where p_{jk} are real constants). Let us denote by $\mathcal{L}_0$ the unbounded operator $\sum_{j,k=1}^{N} p_{jk}\ d_j d_k$ on h, which coincides with $-\sum_{j,k=1}^{N} p_{jk}\ \partial_j\partial_k$ on each of the finite dimensional subspaces $\mathcal{H}_l^\pi$. It is also clear that $h^\infty \subseteq \mathrm{Dom}(\mathcal{L}_0^n)$ for all nonnegative integer n. The G-action on the $\mathcal{H}_l^\pi$ induces a homomorphism (say Ψ_l^π) of the universal enveloping algebra of the Lie algebra (see [6]), which sends the generator χ_j to id_j. In particular, $\Psi_l^\pi(\Delta_G) = -\mathcal{L}_0|_{\mathcal{H}_l^\pi}$. However, since the G-action on $\mathcal{H}_l^\pi$ is the irreducible representation π, $\Psi_l^\pi(\Delta_G)$ is nothing but $\lambda_\pi I_{\mathcal{H}_l^\pi}$. Thus we have that

$$\mathcal{L}_0|_{\mathcal{H}_l^\pi} = -\lambda_\pi I_{\mathcal{H}_l^\pi}. \tag{1}$$

For proving that $h^\infty = \mathcal{A}^\infty$ as sets, it is sufficient to show that for every $k \in \{1, \cdots, N\}$, one has $h^\infty \subseteq \mathrm{Dom}(\partial_k)$ and $\partial_k(h^\infty) \subseteq h^\infty$. To this end let us consider an arbitrary element $v \in h^\infty$ given by an L^2-convergent series $\sum_{\pi,i,j} c_{\pi,i,j} t_{ij}^\pi$. The fact that v is in $h^\infty \subseteq \mathrm{Dom}(\mathcal{L}_0^n)$ implies that (by (1))

$$\sum_{\pi,i,j} \lambda_\pi^{2n} |c_{\pi,i,j}|^2 < \infty$$

for every positive integer n. Since by the Proposition 2.1, $\|t_{ij}^\pi\|_{\infty,0} \leq \sqrt{n_\pi}$, it follows that

$$\sum_{\pi,i,j} |c_{\pi,i,j}| \|t_{ij}^\pi\|_{\infty,0} \leq (\sum_{\pi,i,j} |c_{\pi,i,j}|^2 \lambda_\pi^{2n})^{\frac{1}{2}} (\sum_\pi \frac{n_\pi^3}{\lambda_\pi^{2n}})^{\frac{1}{2}} < \infty.$$

This proves that the series $\sum_{\pi,i,j} c_{\pi,i,j} t_{ij}^\pi$ converges in the norm of $\mathcal{A}$, hence $v \in \mathcal{A}$. Since $\hat{G}$ is countable, let us identify $\hat{G}$ with $I\!N$ without loss of generality for the rest of the proof. Denoting by v_n the element $\sum_{l=1}^{n} \sum_{i,j} c_{l,i,j} t_{ij}^l$, we observe that v_n converges to v in the norm of $\mathcal{A}$. Moreover, since the finite dimensional space h_n spanned by $\{t_{ij}^l, i = 1, \cdots, n_l,\ j = 1, \cdots, m_l; l = 1, \cdots, n\}$, is invariant under d_k, d_k commutes with the projection P_n on h_n. Thus, $d_k v = \lim_{n\to\infty} P_n d_k v = \lim_{n\to\infty} d_k v_n$. If we write $d_k v_n$ as a sum of the form $\sum_{l=1}^{n} \sum_{i,j} b_{l,i,j} t_{ij}^l$, then it follows similarly that the (a-priori L^2-convergent) series $d_k v = \sum_{l=1}^{\infty} \sum_{i,j} b_{l,i,j} t_{ij}^l$ converges in the norm of $\mathcal{A}$. Therefore, $d_k v_n$ converges to $d_k v$ in the norm of $\mathcal{A}$. But d_k coincides with $-i\partial_k$ on h_n, hence $\partial_k v_n = id_k v_n$, which converges in the norm of $\mathcal{A}$ to $id_k v$. Thus, both v_n and $\partial_k v_n$ converge in the $\mathcal{A}$-norm, from which it follows by using the fact that ∂_k is closed that v is in the domain of ∂_k and $\partial_k v = id_k v \in h^\infty$. Moreover, since the trace τ is finite, the Fréchet topology of $\mathcal{A}^\infty$ is stronger than that of h_∞. This implies that the identity map I, viewed as a linear map from the Fréchet space h^∞ to the Fréchet space $\mathcal{A}^\infty$ is closable, hence continuous. This completes the proof that the two Fréchet topologies on $\mathcal{A}^\infty = h^\infty$ are equivalent, i.e. $\mathcal{A}^\infty = h^\infty$ as topological spaces. $\Box$

Lemma 2.5 *We have* $(\tilde{\mathcal{A}} \otimes \mathcal{H})^\infty = (\mathcal{A} \otimes \mathcal{H})^\infty$ *for every Hilbert space* $\mathcal{H}$ *with a trivial* G *representation.*

Proof :-
We shall denote by $\| \cdot \|_p$ $(p \geq 1)$ the L^p-norm coming from the trace τ on $\mathcal{A}$. It is enough to prove the nontrivial inclusion $(\tilde{\mathcal{A}} \otimes \mathcal{H})^\infty \subseteq (\mathcal{A} \otimes \mathcal{H})^\infty$ Fix an orthonormal basis $\{e_\alpha, \alpha \in T\}$ of $\mathcal{H}$ (which need not be separable). Fix $\eta \in (\tilde{\mathcal{A}} \otimes \mathcal{H})^\infty$. Clearly, each η_α is in $\tilde{\mathcal{A}}^\infty = \mathcal{A}^\infty \subset \mathcal{A}$. For $k \in \{1, ..., N\}$, let $\eta' = \partial_k \eta = \sum_\alpha \eta'_\alpha \otimes e_\alpha$, say. It is easy to argue that $\eta'_\alpha = \partial_k(\eta_\alpha)$ for all α. There will be at most countable sets I, J of T such that $< 1 \otimes e_\alpha, \eta >=< 1 \otimes e_\alpha, \eta' >= 0$ for all $\alpha \in T - I \bigcup J$. So, we can choose a countable set $\{e_i\}$ such that $\eta = \sum_i \eta_i \otimes e_i$, $\eta' = \sum_i \eta'_i \otimes e_i$, where both the series are convergent in the strong operator topology. Let $\eta_{m,n} := \sum_{j=m}^n \eta_j \otimes e_j$, for $m \leq n$. Since each $\eta_{m,n}$ belongs to $\mathcal{A} \otimes \mathcal{H}$, for proving $\eta \in \mathcal{A} \otimes \mathcal{H}$ it is enough to prove that $\eta_{m,n} \to 0$ in the topology of $\mathcal{A} \otimes \mathcal{H}$, i.e. $x_{mn} :=< \eta_{m,n}, \eta_{m,n} >= \sum_{j=m}^n \eta_j^* \eta_j \to 0$ in the norm-topology of $\mathcal{A}$. We shall prove that $x_{mn} \to 0$ in the Frechet topology of h^∞, which will prove that it converges to 0 also in the topology of $\mathcal{A}^\infty$.

To this end, first note that $< \eta, \eta >= \sum_j \eta_j^* \eta_j \in \tilde{\mathcal{A}}$, so in particular, $\|x_{mn}\| \leq \|\eta\|^2$ for all m, n. Moreover, since $\eta_{mn} \to 0$ in the strong operator topology, we have $\sum_{j=m}^n \|\eta_j\|_2^2 = \|\eta_{m,n} 1\|_2^2 \to 0$, and since x_{mn} are nonnegative elements, it follows that $\|x_{mn}\|_1 = \tau(x_{mn}) = \sum_{j=m}^n \tau(\eta_j^* \eta_j) \to 0$ as $m, n \to \infty$. Thus $\|x_{mn}\|_2^2 \leq \|x_{mn}\| \|x_{mn}\|_1 \leq \|\eta\|^2 \|x_{mn}\|_1 \to 0$. Note that for $\xi_1, \xi_2 \in (\tilde{\mathcal{A}} \otimes \mathcal{H})^\infty$, we have

$$\| < \xi_1, \xi_2 > \|_2^2 = \tau(\xi_2^* \xi_1 \xi_1^* \xi_2) \leq \|\xi_1\|^2 \tau(\xi_2^* \xi_2),$$

hence $\| < \xi_1, \xi_2 > \|_2 \leq \|\xi_1\| \| < \xi_2, \xi_2 > \|_1^{\frac{1}{2}}$. Moreover, $\| < \xi_2, \xi_1 > \|_2 = \| < \xi_1, \xi_2 >^* \|_2 = \| < \xi_1, \xi_2 > \|_2$ (since $\tau(x^* x) = \tau(x x^*)$, we have $\|x\| = \|x^*\|$). Using this, we have

$$\begin{aligned} & \|\partial_k x_{mn}\|_2 \\ = & \ \| < \partial_k \eta_{m,n}, \eta_{m,n} > + < \eta_{m,n}, \partial_k \eta_{m,n} > \|_2 \\ \leq & \ 2 \| < \partial_k \eta_{m,n}, \eta_{m,n} > \|_2 \\ \leq & \ 2 \|\partial_k \eta_{m,n}\| \|x_{mn}\|_1^{\frac{1}{2}}, \end{aligned}$$

which goes to 0 as $m, n \to \infty$ since $\lim_{m,n\to\infty} \|x_{mn}\|_1 = 0$ and for every (m,n), $\|\partial_k \eta_{m,n}\| = \| \sum_{j=m}^{n} {\eta'_j}^* \eta'_j \|^{\frac{1}{2}} \leq \|\partial_k \eta\|$. Proceeding similarly, we can prove that $\partial_{k_1}...\partial_{k_l} x_{mn} \to 0$ as $m, n \to \infty$ in the L^2-norm, for every fixed $(k_1, ..., k_l)$, i.e. x_{mn} is Cauchy in the Frechet topology of h^∞, hence in that of $\mathcal{A}^\infty$. □

3 Main results on equivariant embedding of Hilbert modules

Let (E, β) be a $G-\mathcal{A}$ module, where $\mathcal{A}$ and G are as in the previous section, i.e. G is a compact Lie group acting ergodically on the C^* algebra $\mathcal{A}$. In this final section, we shall prove that any such (E, β) is embeddable.

Theorem 3.1 *We can find a Hilbert space $\mathcal{K}$, a strongly continuous unitary representation $g \mapsto V_g \in \mathcal{B}(\mathcal{K})$ and a $\mathcal{A}$-linear isometry $\Gamma_0 : E \to \mathcal{B}(h, \mathcal{K})$, such that $\Gamma_0 \beta_g(\xi) = V_g(\Gamma_0 \xi) u_g^{-1}$, and moreover, the complex linear span of elements of the form $\Gamma \xi w$ where $\xi \in E$ and $w \in h$ is dense in $\mathcal{K}$.*

Proof :
The proof of this theorem is adapted from [3] and [2]. We shall give only a brief sketch of the arguments involved, omitting the details. We consider first the formal vector space (say $\mathcal{V}$) spanned by symbols (ξ, w), with $\xi \in E$ and $w \in h$, and define a semi-inner product on this formal vector space by setting

$$< (\xi, w), (\xi', w') >=< w, < \xi, \xi' > w' >,$$

where $< \xi, \xi' >$ denotes the $\mathcal{A}$-valued inner product on E. By extending this semi-inner product by linearity and then taking quotient by the subspace (say $\mathcal{V}_0$) consisting of elements of zero norm we get a pre-Hilbert space, and its completion under the pre-inner product is denoted by $\mathcal{K}$. We also define $\Gamma_0 : E \to \mathcal{B}(h, \mathcal{K})$ by setting

$$(\Gamma_0(\xi))w := [\xi, w],$$

where $[\xi, w]$ represents the equivalence class of (ξ, w) in $\mathcal{S} \equiv \mathcal{V}/\mathcal{V}_0 \subseteq \mathcal{K}$. That it is an isometry is verified by straightforward calculations. Next, we define V_g on $\mathcal{S}$ by

$$V_g[\xi, w] := [\beta_g(\xi), u_g w],$$

and verify that it is indeed an isometry, and since its range clearly contains a total subset, V_g extends to a unitary on $\mathcal{K}$. Furthermore, $V_g V_h = V_{gh}$

and $V_e = \mathrm{Id}$ (where e is the identity of G) on $\mathcal{S}$,and hence on the whole of $\mathcal{K}$. The strong continuity of $g \mapsto V_g$ is also easy to see. Indeed, it is enough to prove that $g \mapsto V_g X$ is continuous for any X of the form $[\xi, v]$, $\xi \in E, v \in h$. But $\|V_g([\xi, v]) - [\xi, v]\|^2 = 2\langle[\xi, v], [\xi, v]\rangle - \langle V_g([\xi, v]), [\xi, v]\rangle - \langle[\xi, v], V_g([\xi, v])\rangle$, and we have, $\langle V_g([\xi, v]), [\xi, v]\rangle - \langle[\xi, v], [\xi, v]\rangle = \langle(u_g v - v)\langle\beta_g(\xi), \xi\rangle v\rangle + \langle v, \langle(\beta_g(\xi) - \xi), \xi\rangle v\rangle$. By assumption $\lim_{g \to e}(\beta_g(\xi) - \xi) = 0$ in the norm topology of E, so $\langle(\beta_g(\xi) - \xi)v, \xi v\rangle \to 0$ as $g \to e$. Furthermore, $g \mapsto u_g v$ is continuous. This completes the proof of strong continuity of V_g. $\square$

In view of the above result, we assume without loss of genetrality that $E \subset \mathcal{B}(h, \mathcal{K})$ (with the natural Hilbert module structure inherited from that of $\mathcal{B}(h, \mathcal{K})$), and $\beta_g(\cdot) = V_g \cdot u_g^{-1}$. Consider the strong operator closure of $\tilde{E}$ of E in $\mathcal{B}(h, \mathcal{K})$. It is a Hilbert von Neumann $\mathcal{A}''$ module (where $\mathcal{A}''$ is the weak closure of $\mathcal{A}$ in h) . Moreover, the G-action $\beta_g = V_g \cdot u_g^{-1}$ can be extended to the whole of $\mathcal{B}(h, \mathcal{K})$, and denoted again by β_g. Clearly, this action leaves $\tilde{E}$ invariant, hence $(\tilde{E}, \beta)$ is a Hilbert von Neumann G-$\mathcal{A}''$ module. Let us recall that by $\tilde{E}^\infty$ we denote the locally convex space of elements ξ in $\tilde{E}$ such that $g \mapsto \beta_g(\xi)$ is C^∞ in the strong operator topology of $\tilde{E}$.

Theorem 3.2 *There exist a Hilbert space k_0 and an isometry Σ from $\mathcal{K}$ to $h \otimes k_0$ such that*
(i) Σ is equivariant in the sense that $\Sigma V_g = (u_g \otimes I)\Sigma$ for all g;
(ii) $\Sigma\xi \in \mathcal{A} \otimes k_0$ for all $\xi \in E$.

Proof :

We shall appeal to the theory of crossed product von Neumann algebras. Construct a representation $\rho : \mathcal{A}' \to \mathcal{B}(\mathcal{K})$ by setting $\rho(a')[\xi, v] := [\xi, a'v]$, for $a' \in \mathcal{A}' \subseteq \mathcal{B}(h)$. It is easy to verify that

$$\begin{aligned}
&\|a'\|^2 \|\sum_{i=1}^n [\xi_i, v_i]\|^2 - \|\rho(a')(\sum_{i=1}^n [\xi_i, v_i])\|^2 \\
&= \sum_{i,j} < v_i, (\|a'\|^2 < \xi_i, \xi_j > - a'^* < \xi_i, \xi_j > a')v_j > \\
&= \sum_{i,j} < v_i, (\|a'\|^2 1 - a'^* a') < \xi_i, \xi_j > v_j > \\
&= \sum_{i,j} < v_i, c^* < \xi_i, \xi_j > c v_j > \qquad (\text{ where } c = (\|a'\|^2 1 - a'^* a')^{\frac{1}{2}})
\end{aligned}$$

$$\geq \quad 0,$$

which proves that $\rho(a')$ admits a bounded (in fact contractive) extension. It is also easy to prove that that ρ thus defined is a normal representation. By Corollary 2.3, we have that $\mathcal{A}'_G \equiv \mathcal{A}' >\!\!\lhd_\alpha G$ is the weak closure of the set $\{u_g, b, g \in G, b \in \mathcal{A}'\} \subseteq \mathcal{B}(h)$, and thus, there is an extension of ρ, which is a normal $*$-representation $\tilde{\rho} : \mathcal{A}'_G \to \mathcal{B}(\mathcal{K})$ satisfying $\tilde{\rho}(b) = \rho(b)$ and $\tilde{\rho}(u_g) = V_g$. By the structure theorem for normal $*$-homomorphism of von Neumann algebras (see [4]), we can find some Hilbert space k_0 and an isometry (as $\tilde{\rho}$ is clearly unital) $\Sigma : \mathcal{K} \to h \otimes k_0$ such that $\Sigma^*(X \otimes 1_{k_0})\Sigma = \tilde{\rho}(X)$, for all $X \in \mathcal{A}'_G$. We claim that this Σ satisfies the properties mentioned in the statement of the Theorem, thereby completing the proof.

Note that $P := \Sigma\Sigma^*$ commutes with $X \otimes 1_{k_0}$ for all $X \in \mathcal{A}'_G$, so in particular with $(u_g \otimes 1), g \in G$. So, $\Sigma V_g \Sigma^* = P(u_g \otimes 1)P = (u_g \otimes 1)\Sigma\Sigma^*$. So, $\Sigma V_g = (u_g \otimes 1)\Sigma$. Thus, $\Sigma\beta_g(\xi) = \Sigma V_g \xi u_g^* = (u_g \otimes 1)\Sigma\xi u_g^* = (\alpha_g \otimes 1)(\Sigma(\xi))$, hence $\Sigma\beta_g\Sigma^* = (\alpha_g \otimes 1)\Sigma\Sigma^* = (\alpha_g \otimes 1)P$. Furthermore, it is easy to see that for $b \in \mathcal{A}'$ and $\xi \in E$, we have

$$(b \otimes 1)\Sigma\xi = \Sigma\rho(b)\xi = (\Sigma\xi)b.$$

This proves that $\Sigma\xi \in (\mathcal{A}')' \otimes k_0 = \tilde{\mathcal{A}} \otimes k_0$. Since Σ is a bounded equivaraint operator from $\tilde{E}$ to $\tilde{\mathcal{A}} \otimes k_0$, it must map $\tilde{E}^\infty$ to $(\tilde{\mathcal{A}} \otimes k_0)^\infty = (\mathcal{A} \otimes k_0)^\infty \subset \mathcal{A} \otimes k_0$. $\Box$

It follows from the above theorem that E can be equivariantly embedded in the trivial $G - \mathcal{A}$ module $(\mathcal{A} \otimes k_0, \alpha \otimes \mathrm{id})$. In particular, we have that

Theorem 3.3 *If a compact Lie group G has an ergodic action on a C^*-algebra $\mathcal{A}$, then every $G - \mathcal{A}$ module (E, β) is embeddable.*

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