On Equivariant Embedding of Hilbert C^* modules

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Abstract

We prove that an arbitrary (not necessarily countably generated) Hilbert G-$\mathcal{A}$ module on a $G-C^*$ algebra $\mathcal{A}$ admits an equivariant embedding into a trivial $G-\mathcal{A}$ module, provided G is a compact Lie group and its action on $\mathcal{A}$ is ergodic.

1 Introduction

Let G be a locally compact group, $\mathcal{A}$ be a C^*-algebra, and assume that there is a strongly continuous representation $\alpha : G \to Aut(\mathcal{A})$. Following the terminology of [6], we introduce the concept of a Hilbert C^* $G-\mathcal{A}$-module as follows :

Definition 1.1 A *Hilbert C^* $G-\mathcal{A}$ module* (or *$G-\mathcal{A}$ module* for short) is a pair (E, β) where E is a Hilbert C^* $\mathcal{A}$-module and β is a map from G into the set of $\mathbb{C}$-linear (caution : **not** $\mathcal{A}$-linear !) maps from E to E, such that $\beta_g \equiv \beta(g), g \in G$ satisfies the following :
(i) $\beta_{gh} = \beta_g \circ \beta_h$ for $g, h \in G$, $\beta_e = \mathrm{Id}$, where e is the identity element of G;
(ii) $\beta_g(\xi a) = \beta_g(\xi)\alpha_g(a)$ for $\xi \in E, a \in \mathcal{A}$;
(iii) $g \mapsto \beta_g(\xi)$ is continuous for each fixed $\xi \in E$;
(iv) $\langle \beta_g(\xi), \beta_g(\eta) \rangle = \alpha_g(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in E$, where $\langle \cdot, \cdot \rangle$ denotes the $\mathcal{A}$-valued inner product of E.

When β is understood from the context, we may refer to E as a $G-\mathcal{A}$ module, without explicitly mentioning the pair (E, β). Given two $G-\mathcal{A}$ modules (E_1, β) and (E_2, γ), there is a natural G-action induced on $\mathcal{L}(E_1, E_2)$, given by $\pi_g(T)(\xi) := \gamma_g(T(\beta_{g^{-1}}(\xi)))$ for $g \in G, \xi \in E_1, T \in \mathcal{L}(E_1, E_2)$. $T \in \mathcal{L}(E_1, E_2)$ is said to be *G-equivariant* if $\pi_g(T) = T$ for all $g \in G$. It is clear that for each fixed $T \in \mathcal{L}(E_1, E_2)$ and $\xi \in E_1$, $g \mapsto \pi_g(T)\xi$ is continuous. We say that T is *G-continuous* if $g \mapsto \pi_g(T)$ is continuous with respect to the norm topology on $\mathcal{L}(E_1, E_2)$. We say that E_1 and E_2 are *isomorphic as $G-\mathcal{A}$-modules*, or that they are *equivariantly isomorphic* if there is a G-equivariant unitary map $T \in \mathcal{L}(E_1, E_2)$. We call a $(G-\mathcal{A})$ module of the form $(\mathcal{A} \otimes \mathcal{H}, \alpha_g \otimes \gamma_g)$ (where $\mathcal{H}$ is some Hilbert space) a trivial $G - \mathcal{A}$ module. We say that (E, β) is *embeddable* if there is an equivariant isometry from E to $\mathcal{A} \otimes \mathcal{H}$ for some Hilbert space $\mathcal{H}$ with a G-action γ, or in other words, (E, β) is equivariantly isomorphic with a sub-$G - \mathcal{A}$ module of $(\mathcal{A} \otimes \mathcal{H}, \beta \otimes \gamma)$. Note that $\mathcal{A} \otimes \mathcal{H}$ is the closure of $\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{H}$ under the norm inherited from $\mathcal{B}(\mathcal{H}_0, \mathcal{H}_0 \otimes \mathcal{H})$ where $\mathcal{H}_0$ is any Hilbert space such that $\mathcal{A}$ is isometrically embedded into $\mathcal{B}(\mathcal{H}_0)$. The following result on the embeddabiity is due to Mingo and Phillips ([6]).

Theorem 1.2 *Let (E, β) be a Hilbert C^* $G - \mathcal{A}$ module and assume that E is countably generated as a Hilbert $\mathcal{A}$-module, that is, there is a countable set $S = \{e_1, e_2, ...\}$ of elements of E such that the right $\mathcal{A}$-linear span of S is dense in E. Assume furthermore that G is compact. Then (E, β) is embeddable.*

When G is the trivial singleton group, the above result was proved by Kasparov.

If the C^* algebra $\mathcal{A}$ is replaced by a von Neumann algebra $\mathcal{B} \subseteq \mathcal{B}(h)$ for some Hilbert space h and G is a locally compact group with a strongly continuous unitary representation $g \mapsto u_g \in \mathcal{B}(h)$, one can define Hilbert von Neumann G-$\mathcal{B}$ module (E, β). The only difference is that E is now a Hilbert von Neumann $\mathcal{A}$-module equipped with the natural locally convex strong operator topology, and that we replace the norm-continuity in (iii) of the above definition by a weaker continuity : namely, the continuity of $g \mapsto \beta_g(\xi)$ (for fixed $\xi \in E$) with respect to the locally convex topology of E. In this case, we have a stronger version of the Theorem 1.2 (see [3] and [2], Theorem 4.3.5, page 99), namely without the condition of E being countably generated and without the compactness of G. It should be remarked here that the trivial Hilbert von Neumann $\mathcal{B}$ module $\mathcal{B} \otimes \mathcal{H}$ is defined to be the closure of $\mathcal{B} \otimes_{\mathrm{alg}} \mathcal{H}$ with respect to the strong-operator-topology inherited from $\mathcal{B}(h, h \otimes \mathcal{H})$.

In Theorem 1.2, the assumption that E is countably generated restricts the applicability of the result, since it is not always easy to check the property of being countably generated. However, under some special assumption on the G-C^* algebra $\mathcal{A}$, i.e. conditions on the group G, the C^* algebra $\mathcal{A}$ and also on the nature of the action, it may be possible to prove the embedability for an arbitrary Hilbert G-$\mathcal{A}$ module. The aim of the present article is to give some such sufficient conditions.

2 Ergodic action and its implication

We say that the action α of G on a unital C^*-algebra $\mathcal{A}$ is *ergodic* if $\alpha_g(a) = a$ for all $g \in G$ if and only if a is a scalar multiplie of 1. There is a considerable amount of literature on ergodic action of compact groups, and we shall quote one interesting structure theorem which will be useful for us.

Proposition 2.1 *Let G be a compact group acting ergodically on a unital C^*-algebra $\mathcal{A}$. Then there is a set of elements $t^\pi_{ij}, \pi \in \hat{G}, i = 1, ..., d_\pi, j = 1, ..., m_\pi$ of $\mathcal{A}$, where $\hat{G}$ is the set of equivalence classes of irreducible representations of G, d_π is the dimension of the irreducible representation space denoted by π, $m_\pi \leq d_\pi$ is a natural number, such that the followings hold :*
(i) There is a unique faithful G-invariant state τ on $\mathcal{A}$, which is in fact a trace,
(ii) The linear span of $\{t^\pi_{ij}\}$ is norm-dense in $\mathcal{A}$,
(iii) $\{t^\pi_{ij}\}$ is an orthonormal basis of $h = L^2(\mathcal{A}, \tau)$,
(iv) The action of u_g coincides with the πth irreducible representation of G on the vector space spanned by $t^\pi_{ij}, i = 1, ..., d_\pi$ for each fixed j and π,
(v) $\sum_{i=1,...d_\pi} (t^\pi_{ij})^* t^\pi_{ik} = \delta_{jk} d_\pi 1$, where δ_{jk} denotes the Kronecker delta symbol. Thus, in particular, $\|t^\pi_{ij}\| \leq \sqrt{d_\pi}$ for all π, i, j.

The proof can be obtained by combining the results of [7],[4] and [1].

Let now $h = L^2(\mathcal{A}, \tau)$, where τ is the unique invariant faithful trace described in Proposition 2.1. Let u_g be the unitary in h induced by the action of G, that is, on the dense set $\mathcal{A} \subseteq h$, $u_g(a) := \alpha_g(a)$, where α_g denotes the G-action on $\mathcal{A}$. Denote also by α the action $g \mapsto u_g \cdot u_g^*$ on $\hat{\mathcal{A}}$, which is the weak closure of $\mathcal{A}$ in $\mathcal{B}(h)$.

Let us now specialize to the case of a compact Lie group. If G is such a group, with a basis of the Lie algebra given by $\{\chi_1, ..., \chi_N\}$, which has a strongly continuous action θ on a Banach space F, we can consider the space of 'smooth' or C^∞-elements of F, denoted by F^∞, consisting of all $\xi \in f$ such that $G \ni g \mapsto \theta_g(\eta)$ is C^∞. It is easy to prove that F^∞ is dense in F, and it is a $*$-subalgebra if F is a locally convex $*$-algebra. Moreover, we equip E^∞ with a family of seminorms $\| \cdot \|_{\infty,n}$, $n = 0, 1, ...$ given by

$$\|\xi\|_{\infty,n} := \sum_{i_1, i_2, ... i_k; k \leq n, i_t \in \{1,...,N\}} \|\partial_{i_1} \partial_{i_2} ... \partial_{i_k} \xi\|,$$

with the convention $\| \cdot \|_{\infty,0} = \| \cdot \|$ and where $\partial_j(\xi) := \frac{d}{dt}|_{t=0} \theta_{\exp(t\chi_j)}(\xi)$. The space F^∞ is complete under this family of seminorms, and thus is a

Fréchet space. When F is Hilbert space or a Hilbert module, we shall also consider a map d_j given by essentially the same expression as that of ∂_j, with χ_j replaced by $i\chi_j$, and the Hilbertian seminorms $\{\| \cdot \|_{2,n}\}$ are given by

$$\|\xi\|_{2,n}^2 := \sum_{i_1,i_2,...i_k;k\leq n,i_t\in\{1,...,N\}} \|d_{i_1}d_{i_2}...d_{i_k}\xi\|_2^2,$$

with $\| \cdot \|_2$ denoting the norm of the Hilbert space (or Hilbert module) F.

More generally, if F is a complete locally convex space given by a family of seminorms $\{\| \cdot \|^{(q)}\}$, then we can consider the smooth subspace F^∞ and the maps ∂_j as above, and make it a complete locally convex space with respect to a larger family of seminorms $\{\| \cdot \|_n^{(q)}\}$ where

$$\|\xi\|_n^{(q)} := \sum_{i_1,i_2,...i_k;k\leq n,i_t\in\{1,...,N\}} \|\partial_{i_1}\partial_{i_2}...\partial_{i_k}\xi\|^{(q)}.$$

In case F is a von Neumann algebra equipped with the locally convex strong operator topology, the locally convex space F^∞ is a topological $*$-algebra, strongly dense in F.

Lemma 2.2 *[2] Let G be a compact Lie group acting ergodically on a unital C^*-algebra $\mathcal{A}$. Then $h^\infty = \mathcal{A}^\infty$ as Fréchet spaces.*

Proof :

The fact that $\mathcal{A}^\infty = h^\infty$ as sets is contained in Lemma 8.1.20 of [2] (page 200-201). We only prove that the identity map is a topological homeomorphism.

Since the trace τ is finite, the Fréchet topology of $\mathcal{A}^\infty$ is stronger than that of h_∞. This implies that the identity map I, viewed as a linear map from the Fréchet space h^∞ to the Fréchet space $\mathcal{A}^\infty$ is closable, hence continuous. This completes the proof that the two Fréchet topologies on $\mathcal{A}^\infty = h^\infty$ are equivalent, i.e. $\mathcal{A}^\infty = h^\infty$ as topological spaces. $\Box$

Lemma 2.3 *Let $\mathcal{H}$ be a (not necessarily separable) Hilbert space with a unitary representation $w \equiv w_g$ of G, and let us consider the Fréchet modules $(\tilde{\mathcal{A}} \otimes \mathcal{H})^\infty$ and $(\mathcal{A} \otimes \mathcal{H})^\infty$ corresponding to the action $\gamma_g := \alpha_g \otimes w_g$. Let ξ be an element of $(\tilde{\mathcal{A}} \otimes \mathcal{H})^\infty$ such that $g \mapsto \gamma_g(\xi)$ is continuous in the operator-norm topology. Then ξ actually belongs to $\mathcal{A} \otimes \mathcal{H}$.*

Proof :-

We shall denote by $\| \cdot \|_p$ $(p \geq 1)$ the L^p-norm coming from the trace τ on $\mathcal{A}$. The identity 1 of $\mathcal{A}$ will also be viewed as a unit vector in $L^2(\tau)$.

Fix an orthonormal basis $\{e_\alpha, \alpha \in T\}$ of $\mathcal{H}$ (which need not be separable), with each $e_\alpha \in \mathcal{H}^\infty$. Fix $\xi \in (\tilde{\mathcal{A}} \otimes \mathcal{H})^\infty$ satisfying the hytothesis of the lemma. Since $L^2(\tau)$ is separable, say with an orthonormal basis given by $\{x_1, x_2, ...\}$, we can find, for each i, a counteble subset T_i of T such that $< \xi 1, x_i \otimes e_\alpha >= 0$ for all $\alpha \notin T_i$. Denoting by T_∞ the countable set $\bigcup_i T_i$, we have $< \xi 1, v \otimes e_\alpha >= 0 \ \forall v \in L^2(\tau)$, for all $\alpha \notin T_\infty$. Write $T_\infty = \{e_{\alpha_1}, e_{\alpha_2}, ...\}$. Denote by ξ_n the element in $\mathcal{A}^\infty \otimes_{\rm alg} \mathcal{H}^\infty$ given by $\xi_n = (I \otimes P_n)\xi$, where P_n denotes the orthogonal projection onto the linear span of $\{e_{\alpha_1}, ..., e_{\alpha_n}\}$. It is clear that $\xi_n 1 \to \xi 1$ as $n \to \infty$. Now, for a C^∞ complex-valued function f on G and an element $\eta \in \tilde{\mathcal{A}} \otimes \mathcal{H}$, denote by $\gamma(f)(\eta)$ the element $\int_G f(g)\gamma_g(\eta)dg \ \in \tilde{\mathcal{A}} \otimes \mathcal{H} \subseteq L^2(\tau) \otimes \mathcal{H}$, where dg stands for the normalised Haar measure on G and the integral is convergent in the strong-operator topology. We claim that it is enough to prove that $\gamma(f)(\xi) \in \mathcal{A} \otimes \mathcal{H}$ for all $f \in C^\infty(G)$. Let us first prove this claim. Since $g \mapsto \gamma_g(\xi)$ is norm-continuous, given $\epsilon > 0$, we can find a nonempty open subset U of G such that $\|\gamma_g(\xi) - \xi\| \leq \epsilon$ for all $g \in G$, and then choose $f \in C^\infty(G)$ with $\mathrm{supp}(f) \subseteq U$, $f \geq 0$ and $\int_G f dg = 1$. It is easy to see that $\|\gamma(f)(\xi) - \xi\| \leq \epsilon$. Thus, ξ is the operator-norm limit of a sequence of elements of the form $\gamma(f)(\xi)$, which proves the claim.

Let us now complete the proof of the lemma by showing that $\eta := \gamma(f)(\xi)$ indeed belongs to $\mathcal{A} \otimes \mathcal{H}$ for every $f \in C^\infty(G)$. To this end, first observe that $\eta_n := \gamma(f)(\xi_n)$ belongs to $\mathcal{A}^\infty \otimes_{\rm alg} \mathcal{H}^\infty \subseteq \mathcal{A} \otimes \mathcal{H}$ for all n. Moreover, since $u_g 1 = 1$ for all g and $\gamma_g(\cdot) = \mathrm{ad}_{u_g} \otimes w_g$, it is clear that $\eta_n 1 \to \gamma(f)(\xi)1 = \eta 1$ as $n \to \infty$. Since each $\eta_{m,n}$ belongs to $\mathcal{A} \otimes \mathcal{H}$, for proving $\eta \in \mathcal{A} \otimes \mathcal{H}$ it is enough to prove that $\eta_{m,n} \to 0$ in the topology of $\mathcal{A} \otimes \mathcal{H}$, i.e. $x_{mn} :=< \eta_{m,n}, \eta_{m,n} > \to 0$ in the norm-topology of $\mathcal{A}$. We shall prove that $x_{mn} \to 0$ in the Fréchet topology of h^∞, which will prove that it converges to 0 also in the topology of $\mathcal{A}^\infty$.

To this end, first note that for $\beta_1, \beta_2 \in (\tilde{\mathcal{A}} \otimes \mathcal{H})$, we have

$$\| < \beta_1, \beta_2 > \|_2^2 = \tau(\beta_2^* \beta_1 \beta_1^* \beta_2) \leq \|\beta_1\|^2 \tau(\beta_2^* \beta_2),$$

hence $\| < \beta_1, \beta_2 > \|_2 \leq \|\beta_1\| \|\beta_2\|_2$. Moreover, $\| < \beta_2, \beta_1 > \|_2 = \| < \beta_1, \beta_2 >^* \|_2 = \| < \beta_1, \beta_2 > \|_2$ (since $\tau(x^* x) = \tau(x x^*)$, we have $\|x\| = \|x^*\|$). From this, we have

$$\| < \gamma(f)(\beta), \gamma(f)(\beta) > \|_2 \leq C(f)^2 \|\beta\| \|\beta\|_2,$$

where $C(f) := \int |f| dg$. Let us now fix an ordered k-tuple $I = (i_1, ..., i_k)$ (k nonnegative integer), and let C denote the maximum of $C(\chi_{j_1} ... \chi_{j_p} f)$ where $J = (j_1, ..., j_p)$ varies over all (including the empty set) ordered subsets of I.

Let us abbreviate $\partial_{j_1}...\partial_{j_p}\beta$ and $\chi_{j_1}...\chi_{j_p}f$ by $\partial_J\beta$ and f_J respectively, for $\beta \in (\mathcal{A} \otimes \mathcal{H})^\infty$. Note that

$$\partial_J\gamma(f)(\beta) = (-1)^k\gamma(f_J)(\beta).$$

Using this as well as the Leibniz formula $\partial_I < \beta, \beta >= \sum_J < \partial_J\beta, \partial_{I-J}\beta >$ (with J varying over all ordered substes of I), we have the following :

$$\begin{aligned}
&\|\partial_I x_{mn}\|_2 \\
&\leq \sum_J \| < \partial_J(\eta_m - \eta_n), \partial_{I-J}(\eta_m - \eta_n) > \|_2 \\
&\leq 2^k C^2 \|\xi_m - \xi_n\| \|\xi_m - \xi_n\|_2 \\
&\leq 2^{k+1} C^2 \|\xi\| \|\xi_m - \xi_n\|_2,
\end{aligned}$$

since the number of ordered subsets of I is 2^k and it is clear from the definition of ξ_n that $\|\xi_n\| \leq \|\xi\|$ for all n. We also have $\|\xi_m - \xi_n\|_2^2 = \tau(< \xi_m - \xi_n, \xi_m - \xi_n >) =< (\xi_m - \xi_n)(1), (\xi_m - \xi_n)(1) > \to 0$ as $m, n \to \infty$. This proves $x_{m,n} \to 0$ in the topology of h_∞, thereby completing the proof of the lemma. □

3 Main results on equivariant embedding of Hilbert modules

Let (E, β) be a $G - \mathcal{A}$ module, where $\mathcal{A}$ and G are as in the previous section, i.e. G is a compact Lie group acting ergodically on the C^* algebra $\mathcal{A}$. In this final section, we shall prove that any such (E, β) is embeddable.

Lemma 3.1 *We can find a Hilbert space $\mathcal{K}$, a strongly continuous unitary representation $g \mapsto V_g \in \mathcal{B}(\mathcal{K})$ and a $\mathcal{A}$-linear isometry $\Gamma_0 : E \to \mathcal{B}(h, \mathcal{K})$, such that $\Gamma_0\beta_g(\xi) = V_g(\Gamma_0\xi)u_g^{-1}$, and moreover, the complex linear span of elements of the form $\Gamma\xi w$ where $\xi \in E$ and $w \in h$ is dense in $\mathcal{K}$.*

Proof :
The proof of this result is adapted from [3] and [2], Theorem 4.3.5 (page 99-101). We shall give only a brief sketch of the arguments involved, omitting the details. We consider first the formal vector space (say $\mathcal{V}$) spanned by symbols (ξ, w), with $\xi \in E$ and $w \in h$, and define a semi-inner product on this formal vector space by setting

$$< (\xi, w), (\xi', w') >=< w, < \xi, \xi' > w' >,$$

where $< \xi, \xi' >$ denotes the $\mathcal{A}$-valued inner product on E. By extending this semi-inner product by linearity and then taking quotient by the subspace (say $\mathcal{V}_0$) consisting of elements of zero norm we get a pre-Hilbert space, and its completion under the pre-inner product is denoted by $\mathcal{K}$. We also define $\Gamma_0 : E \to \mathcal{B}(h, \mathcal{K})$ by setting

$$(\Gamma_0(\xi))w := [\xi, w],$$

where $[\xi, w]$ represents the equivalence class of (ξ, w) in $\mathcal{S} \equiv \mathcal{V}/\mathcal{V}_0 \subseteq \mathcal{K}$. That it is an isometry is verified by straightforward calculations. Next, we define V_g on $\mathcal{S}$ by

$$V_g[\xi, w] := [\beta_g(\xi), u_g w],$$

and verify that it is indeed an isometry, and since its range clearly contains a total subset, V_g extends to a unitary on $\mathcal{K}$. Furthermore, $V_g V_h = V_{gh}$ and $V_e = \mathrm{Id}$ (where e is the identity of G) on $\mathcal{S}$,and hence on the whole of $\mathcal{K}$. The strong continuity of $g \mapsto V_g$ is also easy to see. Indeed, it is enough to prove that $g \mapsto V_g X$ is continuous for any X of the form $[\xi, v]$, $\xi \in E, v \in h$. But $\|V_g([\xi, v]) - [\xi, v]\|^2 = 2\langle [\xi, v], [\xi, v]\rangle - \langle V_g([\xi, v]), [\xi, v]\rangle - \langle [\xi, v], V_g([\xi, v])\rangle$, and we have, $\langle V_g([\xi, v]), [\xi, v]\rangle - \langle [\xi, v], [\xi, v]\rangle = \langle (u_g v - v)\langle \beta_g(\xi), \xi\rangle v\rangle + \langle v, \langle (\beta_g(\xi) - \xi), \xi\rangle v\rangle$. By assumption $\lim_{g \to e}(\beta_g(\xi) - \xi) = 0$ in the norm topology of E, so $\langle (\beta_g(\xi) - \xi)v, \xi v\rangle \to 0$ as $g \to e$. Furthermore, $g \mapsto u_g v$ is continuous. This completes the proof of strong continuity of V_g.
$\square$

In view of the above result, we assume without loss of genetrality that $E \subset \mathcal{B}(h, \mathcal{K})$ (with the natural Hilbert module structure inherited from that of $\mathcal{B}(h, \mathcal{K})$), and $\beta_g(\cdot) = V_g \cdot u_g^{-1}$. Consider the strong operator closure of $\tilde{E}$ of E in $\mathcal{B}(h, \mathcal{K})$. It is a Hilbert von Neumann $\tilde{\mathcal{A}}$ module (where $\tilde{\mathcal{A}}$ is the weak closure of $\mathcal{A}$ in h) . Moreover, the G-action $\beta_g = V_g \cdot u_g^{-1}$ can be extended to the whole of $\mathcal{B}(h, \mathcal{K})$, and denoted again by β_g. Clearly, this action leaves $\tilde{E}$ invariant, hence $(\tilde{E}, \beta)$ is a Hilbert von Neumann G-$\tilde{\mathcal{A}}$ module. Let us recall that by $\tilde{E}^\infty$ we denote the locally convex space of elements ξ in $\tilde{E}$ such that $g \mapsto \beta_g(\xi)$ is C^∞ in the strong operator topology of $\tilde{E}$.

Theorem 3.2 *There exist a Hilbert space k_0, a unitary representation w_g of G in k_0 and an isometry Σ from $\mathcal{K}$ to $h \otimes k_0$ such that*
(i) Σ is equivariant in the sense that $\Sigma V_g = (u_g \otimes w_g)\Sigma$ for all g;
(ii) $\Sigma\xi \in \mathcal{A} \otimes k_0$ for all $\xi \in E$.

Proof :
The statement (i) is contained in the Theorem 4.3.5 of [2] (page 99). For

proving (ii), we note that E^∞ (w.r.t. the action β) is mapped by Σ into $(\tilde{\mathcal{A}} \otimes k_0)^\infty$ (w.r.t. the action $\gamma_g := \mathrm{ad}_{u_g} \otimes w_g$), and moreover, for $\xi \in E$, $g \mapsto \gamma_g(\Sigma(\xi)) = \Sigma\beta_g(\xi)$ is norm-continuous since $g \mapsto \beta_g(\xi)$ is so and Σ is isometry. Thus, (ii) follows from Lemma 2.3. $\Box$

It follows from the above theorem that E can be equivariantly embedded in the trivial $G - \mathcal{A}$ module $(\mathcal{A} \otimes k_0, \alpha \otimes w)$. In particular, we have that

Theorem 3.3 *If a compact Lie group G has an ergodic action on a C^*-algebra $\mathcal{A}$, then every $G - \mathcal{A}$ module (E, β) is embeddable.*

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