A POVM view of the ensemble approach to polarization optics

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Statistical ensemble formalism of Kim, Mandel and Wolf (J. Opt. Soc. Am. A **4**, 433 (1987)) offers a realistic model for characterizing the effect of stochastic non-image forming optical media on the state of polarization of transmitted light. With suitable choice of the Jones ensemble, various Mueller transformations - some of which have been unknown so far - are deduced. It is observed that the ensemble approach is formally identical to the positive operator valued measures (POVM) on the quantum density matrix. This observation, in combination with the recent suggestion by Ahnert and Payne (Phys. Rev. A **71**, 012330, (2005)) - in the context of generalized quantum measurement on single photon polarization states - that linear optics elements can be employed in setting up all possible POVMs, enables us to propose a way of realizing different types of Mueller devices.

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1. Introduction

The intensity and polarization of a beam of light passing through an isolated optical device undergoes a linear transformation. But this is an ideal situation because, in general, the optical system is embedded in some media such as atmosphere or other ambient material, which further modifies the polarization properties of the light beam passing through it. A statistical ensemble model describing random linear optical media was formulated two decades ago by Kim, Mandel and Wolf [1], but is not examined in any detail in the literature, to the best of our knowledge. The purpose of the present paper is to pursue this avenue in a new way arising from the realization of a relationship, presented here, with the positive operator valued measures (POVM) of quantum measurement theory. This is because the transformation of the polarization states of a light beam propagating through an ensemble of deterministic optical devices exhibits a structural similarity with the POVM transformation of quantum density matrices. This connection motivates, in view of the recent interest in the implementations of POVMs on single photon density matrix employing linear optics elements [2], identification of experimental schemes to realize various kinds of Muller transformations. The properties of the transformation of the polarization states of light form a much studied topic in literature [3 – 17]. Thus the power of the ensemble approach becomes evident in elucidating the known optical devices as well as some hitherto unknown types [17], which had remained only a mathematical possibility.

The contents of this paper are organized as follows. In Sec. 2, a concise formulation of the Jones and Mueller matrix theory, along with a summary of main results of Gopala Rao et al. [17] is given. Based on the approach of Kim, Mandel and Wolf [1] suitable Jones ensembles, corresponding to various types of Mueller transformations are identified in Sec. 3. In Sec. 4, a structural equivalence between Jones ensemble and POVMs of quantum measurement theory is established. Following the linear optics scheme of Ahnert and Payne [2] for the implementation of POVMs on single photon density matrix, experimental setup for realizing Mueller matrices of types I and II are suggested in Sec. 5. The final section has some concluding remarks.

2. Brief summary of known results on the Jones and the Mueller formalism.

Following the standard procedure, let E_1 and E_2, defined here as a column matrix $\mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$, denote two components of the transverse electric field vector associated with a light beam. The coherency matrix (or the polarization matrix) of the light beam is a positive semidefinite 2x2 hermitian matrix defined by,

$$\mathbf{C} = \langle \mathbf{E} \otimes \mathbf{E}^\dagger \rangle. \tag{1}$$

Expressing this in terms of the standard Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the unit matrix $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$\mathbf{C} = \frac{1}{2} \sum_{i=0}^{3} s_i \sigma_i = \frac{1}{2} \begin{pmatrix} s_0 + s_3 & s_1 - is_2 \\ s_1 + is_2 & s_0 - s_3 \end{pmatrix} \tag{2}$$

The physical significance of the quantities arising here are

$$s_0 = \text{Tr}\,(\mathbf{C}\sigma_0) = \text{Intensity of the beam}$$
$$s_i = \text{Tr}\,(\mathbf{C}\sigma_i) = \text{Components of Polarization vector } \vec{\mathbf{s}} \text{ of the beam} \tag{3}$$

Thus the coherency matrix completely specifies the physical properties of the light beam. The four-vector $\mathbf{S} = \begin{pmatrix} s_0 \\ \vec{\mathbf{s}} \end{pmatrix}$ defined by Eq. (3) is the well known Stokes vector, which represents the state of polarization of the light beam. Because $\mathbf{C}$ is hermitian, the Stokes vector is real. The positive semidefiniteness of $\mathbf{C}$ implies that the Stokes vector must satisfy the properties

$$s_0 > 0, \ s_0^2 - |\vec{\mathbf{s}}|^2 \geq 0 \tag{4}$$

A 2x2 complex matrix $\mathbf{J}$, called the Jones matrix, represents the so-called deterministic optical device [18] or medium. When a light beam represented by $\mathbf{E}$ passes through such a medium, the transformed light beam is given by $\mathbf{E}' = \mathbf{J}\mathbf{E}$. Correspondingly, the coherency matrix $\mathbf{C}$ transforms as

$$\mathbf{C}' = \mathbf{J}\mathbf{C}\mathbf{J}^\dagger \tag{5}$$

(Here $\mathbf{J}^\dagger$ is the hermitian conjugate of $\mathbf{J}$.)

Alternatively, instead of the 2×2 matrix transformation of the coherency matrix, as given by Eq. (5), a transformation

$$\mathbf{S'} = \mathbf{MS} \tag{6}$$

of the four componental Stokes column $\mathbf{S}$ through a real 4x4 matrix $\mathbf{M}$, called the Mueller matrix, is found be more useful [18].

Using Eq. (3) and Eq. (5 we have,

$$s'_i = \mathrm{Tr}(\mathbf{C'}\sigma_i) = \mathrm{Tr}(\mathbf{JCJ^\dagger}\sigma_i) = \frac{1}{2}\sum_{j=0}^{3}\mathrm{Tr}(\mathbf{J^\dagger}\sigma_\mathbf{i}\mathbf{J}\sigma_j)s_j$$

which leads to the well-known relationship [1]

$$M_{ij} = \frac{1}{2}\mathrm{Tr}(\mathbf{J^\dagger}\sigma_\mathbf{i}\mathbf{J}\sigma_j)$$

between the elements of a Jones matrix and that of corresponding Mueller matrix.

But in the case where medium cannot be represented by a Jones matrix, it is not possible to characterize the change in the state of polarization of the light beam through Eq. (5). In such a situation, Mueller formalism provides a general approach for the polarization transformation of the light beam. The Mueller matrix $\mathbf{M}$ is said to be non-deterministic when it has no corresponding Jones characterization.

Mathematically, a Mueller device can be represented by any 4×4 matrix such that the Stokes parameters of the outgoing light beam satisfy the physical constraint Eq. (4). In other words, a Mueller matrix is any 4×4 real matrix that transforms a Stokes vector into another Stokes vector. There are many aspects of the relationships between these two formulations of the polarization optics and a complete characterization of Mueller matrices has been the subject matter of Ref. [1, 3-17]. It was Gopala Rao et al. [17] who presented a complete set of necessary and sufficient conditions for any 4x4 real matrix to be a Mueller matrix. In so doing, they found that there are two algebraic types of Mueller matrices called type I and type II; and it has been shown [17] that only a subset of the type-I Mueller matrices - called deterministic or pure Mueller matrices - have corresponding Jones characterization. All the known polarizing optical devices such as retarders, polarizers, analyzers, optical rotators are pure Mueller type and are well understood. Mueller matrices of the Type II variety are yet to be physically realized and have remained as mere mathematical possibility. For the sake of completeness, we present here the characterization as well as categorization of these two

types of Mueller matrices as is given in Ref. [17]. This will enable us to show that both Type I and II Mueller devices are realizable in an unified manner in terms of the proposed ensemble approach [1].

I. A 4×4 real matrix M is called a type-I Mueller matrix iff

(i) $M_{00} \geq 0$

(ii) The G-eigenvalues ρ_0, ρ_1, ρ_2, ρ_3 of the matrix $\mathbf{N} = \widetilde{\mathbf{M}} \mathbf{G} \mathbf{M}$ are all real. (Here, $\widetilde{\mathbf{M}}$ stands for the transpose of $\mathbf{M}$; G-eigenvalues are the eigenvalues of the matrix $\mathbf{GN}$, with $\mathbf{G} = \mathrm{diag}(1, -1, -1, -1)$).

iii) The largest G-eigenvalue ρ_0 possesses a time-like G-eigenvector and the G-eigenspace of $\mathbf{N}$ contains one time-like and three space-like G-eigenvectors.

II. A 4×4 real matrix M is called a type-II Mueller matrix iff

(i) $M_{00} > 0$.

(ii) The G-eigenvalues ρ_0, ρ_1, ρ_2, ρ_3 of $\mathbf{N} = \widetilde{\mathbf{M}} \mathbf{G} \mathbf{M}$ are all real.

(iii) The largest G-eigenvalue ρ_0 possesses a null G-eigenvector and the G-eigenspace of $\mathbf{N}$ contains one null and two space-like G-eigenvectors.

(iv) If $\mathbf{X}_0 = \mathbf{e}_0 + \mathbf{e}_1$ is the null G-eigenvector of $\mathbf{N}$ such that $\mathbf{e}_0$ is a time-like vector with positive zeroth component, $\mathbf{e}_1$ is a space-like vector G-orthogonal to $\mathbf{e}_0$ then $\widetilde{\mathbf{e}}_0 \mathbf{N} \mathbf{e}_0 > 0$.

Despite the knowledge of these new category of Mueller matrices [15, 17], not much attention is paid for realizing the corresponding devices. An experimental arrangement involving a parallel combination of deterministic (pure Mueller) optical devices is proposed in Ref. [17] for realizing type-II Mueller devices. The physical situations, where the beam of light is subjected to the influence of a medium such as atmosphere was addressed in Ref. [1]. In the next section, we discuss this ensemble approach for random optical media, proposed by Kim, Mandel and Wolf [1] .

3. Mueller matrices as ensemble of Jones devices

Kim et. al. [1] associate a set of probabilities $\{p_e, \sum p_e = 1\}$ to describe the stochastic medium. Then a Jones device $\mathbf{J}_e$ associated with each element e of the ensemble gives a corresponding coherency matrix $\mathbf{C}'_e = \mathbf{J}_e \mathbf{C} \mathbf{J}_e^\dagger$. The ensemble averaged coherency matrix

$$\mathbf{C}_{av} = \sum_e p_e \mathbf{C}'_e = \sum_e p_e (\mathbf{J}_e \mathbf{C} \mathbf{J}_e^\dagger) \tag{7}$$

then describes the effects of the medium on the beam of light. In a similar fashion, the corresponding ensemble of Mueller matrices $\{\mathbf{M}_e\}$ associated with the ensemble of Jones matrices $\{\mathbf{J}_e\}$ is constructed and its ensemble averaged Mueller matrix is similarly formed as $\mathbf{M}_{av} = \sum_e p_e \mathbf{M}_e$. Since a linear combination of Mueller matrices with non-negative coefficients is also a Mueller matrix, the ensemble averaged Mueller matrix $\mathbf{M}_{av}$ is a Mueller matrix [1].

We now turn to the question of constructing an appropriate ensemble designed to describe a given physical situation. The simplest example of an ensemble is one where the elements are chosen entirely randomly, i.e., the system is described by a chaotic ensemble where the probabilities are all equal, $p_e = \frac{1}{n}$, where n denotes the number of elements in the ensemble. The coherency matrix $\mathbf{C}_{av}$ of the light beam passing through such a chaotic assembly is just an arithmetic average of the coherency matrices $\mathbf{C}'_e = \mathbf{J}_e \mathbf{C} \mathbf{J}_e^\dagger$ and hence

$$\mathbf{C}_{av} = \frac{1}{n} \sum_{e=1}^{n} \mathbf{J}_e \mathbf{C} \mathbf{J}_e^\dagger \tag{8}$$

More general models can be constructed depending on the medium for the propagation of the beam of light. For example, one may employ various types of filters or solid state systems through which the light passes; the assignment of the Jones matrices and the corresponding probabilities will then differ depending on the weights placed on these elements.

Restricting ourselves to an ensemble consisting of only two Jones devices which occur with equal probability $p_1 = 1/2$, $p_2 = 1/2$, we have found out that the resultant Mueller matrices can either be deterministic or non-deterministic. We give in the foregoing (see Table I) some examples of Mueller matrices corresponding to different choices of Jones matrices in an ensemble $\mathbf{J}_e$, $e = 1, 2$, for some representative cases. This will also serve to show the

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[1] This is because, each Mueller matrix $\mathbf{M}_e$ transforms an initial Stokes vector into a final Stokes vector and a linear combination of Stokes vectors with non-negative coefficients p_e is again a Stokes vector.

generality of the ensemble procedure in capturing the physical realizations for the Mueller devices discussed in Ref. [17].

Table 1. Mueller matrices resulting from 2-element Jones ensemble.

	$\mathbf{J}_1$	$\mathbf{J}_2$	$\mathbf{M} = p_1\mathbf{M}_1 + p_2\mathbf{M}_2,$ $p_1 = p_2 = \frac{1}{2}.$	Type of $\mathbf{M}$
1.	$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix},$	$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -1 & 2 \\ 0 & 2 & 2 & -1 \end{pmatrix}$	Pure Mueller
2.	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	Type-I
3.	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	Type-I
4.	$\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$	$\frac{1}{6}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix}$	Type-I
5.	$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}$	$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$	$\frac{1}{10}\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & -1 & 2 & 4 \\ 0 & 2 & -3 & 2 \\ 0 & 4 & 2 & -1 \end{pmatrix}$	Type-I
6.	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	Type-II
7.	$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$	$\frac{1}{4}\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	Type-II

In Table I, the Jones matrices chosen are so as to give pure Mueller (deterministic) and non-deterministic type-I, type-II matrices respectively. We observe that an assembly of Jones matrices can result in a pure Mueller matrix if and only if all elements of the assembly correspond to the same optical device. This is because, with all $\mathbf{J}_e$'s are same, a transformation of the form $\mathbf{C}_{av} = \sum_e p_e(\mathbf{J}_e\mathbf{C}\mathbf{J}_e^\dagger)$ is equivalent to a transformation of the Stokes vector $\mathbf{S}$ through a Mueller matrix $\mathbf{M}_{av} = \sum p_e\mathbf{M}_e = \mathbf{M}_{\text{pure}}$. When the medium is represented by a pure Mueller matrix, the outgoing light beam will have the same degree of

polarization as the incoming light beam. In fact, pure Mueller matrix is the simplest among type-I Mueller matrices. Not all type-I Mueller matrices preserve the degree of polarization of the incident light beam. To see this, note that the type-I matrix of example 2 (see Table I) converts any incident light beam into a linearly polarized light beam; the other three type-I matrices (examples 3 to 5) transform completely polarized light beams into partially polarized light beams. Similarly, type-II Mueller matrices do not, in general, preserve the degree of polarization of the incident light beam. It may be seen that the type-II Mueller matrix of example 7 is a depolarizer matrix, since it converts any incident light beam into an unpolarized light beam.

Though one cannot a priori state which choices of Jones matrices result in type-I or type-II, it is interesting to observe that all types of Mueller matrices result - even in 2-element ensembles. It is not difficult to conclude that an ensemble, with more Jones devices and with different weight factors, can give rise to a variety of Mueller matrices of all possible algebraic types. It would certainly be interesting to physically realize such systems.

In the following section, a connection between the ensemble approach for optical devices and the POVMs of quantum measurement theory is established.

4. A connection to Positive Operator Valued Measures

We will now show that the phenomenology of the ensemble construction of Kim, Mandel and Wolf [1] described above has a fundamental theoretical underpinning, if we make a formal identification of the coherency matrix with the density matrix description of the subsystem of a composite quantum system. The coherency matrix defined by Eqs. (1) and (2) resembles a quantum density matrix in that both describe a physical system by a hermitian, trace-class, and positive semi-definite matrix. While the quantum density matrix has unit trace, the coherency matrix has intensity of the beam as the value of the trace. The Jones matrix transformation is a general transformation of the coherency matrix, which preserves its hermiticity and positive semi-definiteness - but changes the values of the elements of the coherency matrix. The most general transformation of the density matrix ρ, which preserves its hermiticity, positive semi-definiteness and also the unit trace is the positive operator valued measures (POVM) [19]:

$$\rho' = \sum_{i=1}^{n} \mathbf{V}_i \rho \mathbf{V}_i^{\dagger}; \quad \sum_{i=1}^{n} \mathbf{V}_i^{\dagger} \mathbf{V}_i = \mathbf{I} \tag{9}$$

where V_i's are general matrices and $\mathbf{I}$ is the unit element in the Hilbert space. More generally, one could relax the condition of preservation of the unit trace of the density matrix by examining the possibility of a contracting transformation, where the unit matrix condition on the POVM operators is replaced by an inequality.

This mathematical theorem has a physical basis in the Kraus operator formalism [19] when we consider the Hamiltonian description of a composite interacting system A, B described by a density matrix $\rho(A,\ B)$ and deduce the subsystem density matrix of A given by, $\rho(A) = \text{Tr}_B\ \rho(A,\ B)$. In this case, the Kraus operators are the explicit expressions of the POVM operators and contain the effects of interaction between the systems A and B in the description of the subsystem A. It is thus clear that the phenomenology of Ref. [1] has a correspondence with the Kraus formulation and the POVM theory. In order to make this association complete, we compare Eq. (9) with the expression given by Eq. (7). Apart from a phase factor, the Kraus operators $\{\mathbf{V}_i\}$, associated with POVMs, may be related to the Jones assembly $\{\mathbf{J}_i\}$, chosen in the form

$$\mathbf{V}_i = \sqrt{p_i}\mathbf{J}_i, \quad \sum_{i=1}^{n} \mathbf{V}_i^\dagger \mathbf{V}_i = \sum_{i=1}^{n} p_i \mathbf{J}_i^\dagger \mathbf{J}_i \tag{10}$$

In the construction of the Table I presented earlier, a simple model was proposed where all probabilities were chosen to be equal and the condition on the sum over the Jones matrix combinations was set equal to unit matrix. In such cases, the intensity of the beam gets reduced by 1/n and the polarization properties of the beam gets changed as was described earlier. With this identification, we have provided here an important interpretation and meaning to the phenomenology of the ensemble approach of Kim et al.[1].

Recently Ahnert and Payne [2] proposed an experimental scheme to implement all possible POVMs on single photon polarization states using linear optical elements. In view of the connection between the ensemble formalism for Jones and Mueller matrices with the POVMs, a possible experimental realization of the two types of Mueller matrices is suggested in the next section.

5.   Possible experimental realization of types I and II Mueller matrices.

We first observe that the density matrix of a single photon polarization state,

$$\rho = \rho_{HH}|H\rangle\langle H| + \rho_{HV}|H\rangle\langle V| + \rho_{HV}^*|V\rangle\langle H| + \rho_{VV}|V\rangle\langle V| \tag{11}$$

is nothing but the coherency matrix of the photon [20]

$$
\mathbf{C} = \begin{pmatrix} \langle \hat{\mathbf{a}}_H^\dagger \hat{\mathbf{a}}_H \rangle & \langle \hat{\mathbf{a}}_H^\dagger \hat{\mathbf{a}}_V \rangle \\ \langle \hat{\mathbf{a}}_V^\dagger \hat{\mathbf{a}}_H \rangle & \langle \hat{\mathbf{a}}_V^\dagger \hat{\mathbf{a}}_V \rangle \end{pmatrix}, \tag{12}
$$

where $\hat{\mathbf{a}}_H$ and $\hat{\mathbf{a}}_V$ are the creation operators of the polarization states of the single photon; $\{|H\rangle, |V\rangle\}$ denote the transverse orthogonal polarization states of photon. This is seen explicitly by noting that the average values of the Stokes operators are obtained as,

$$
\begin{aligned}
s_0 = \langle \hat{\mathbf{S}}_0 \rangle &= \langle (\hat{\mathbf{a}}_H^\dagger \hat{\mathbf{a}}_H + \hat{\mathbf{a}}_V^\dagger \hat{\mathbf{a}}_V) \rangle = \rho_{HH} + \rho_{VV} = \mathrm{Tr}(\rho), \\
s_1 = \langle \hat{\mathbf{S}}_1 \rangle &= \langle (\hat{\mathbf{a}}_H^\dagger \hat{\mathbf{a}}_V + \hat{\mathbf{a}}_V^\dagger \hat{\mathbf{a}}_H) \rangle = \rho_{HV} + \rho_{HV}^* = \mathrm{Tr}(\rho\,\sigma_1), \\
s_2 = \langle \hat{\mathbf{S}}_2 \rangle &= i\,\langle (\hat{\mathbf{a}}_V^\dagger \hat{\mathbf{a}}_H - \hat{\mathbf{a}}_H^\dagger \hat{\mathbf{a}}_V) \rangle = i\,(\rho_{HV} - \rho_{HV}^*) = \mathrm{Tr}(\rho\,\sigma_2), \\
s_3 = \langle \hat{\mathbf{S}}_3 \rangle &= \langle (\hat{\mathbf{a}}_H^\dagger \hat{\mathbf{a}}_H - \hat{\mathbf{a}}_V^\dagger \hat{\mathbf{a}}_V) \rangle = \rho_{HH} - \rho_{VV} = \mathrm{Tr}(\rho\,\sigma_3).
\end{aligned} \tag{13}
$$

Hence the proposed setup [2], involving only linear optics elements such as polarizing beam splitters, rotators and phase shifters, that promises to implement all possible POVMs on a single photon polarization state leads to all possible ensemble realizations for the Mueller matrices. More specifically, this provides a general experimental scheme to realize varieties of Mueller matrices - including the hitherto unreported type-II Mueller matrices. We briefly describe the scheme proposed in Ref. [2] and illustrate, by way of examples, how it leads to both type-I and type-II Mueller matrices.

In Ref. [2], a module corresponds to an arrangement having polarization beam splitters, polarization rotators, phase shifters and unitary operators. For an n element POVM, a setup involving $n-1$ modules are needed. That means, a single module is enough for a 2 element POVM; a setup involving two modules is required for a 3 element POVM and so on. We describe two, three element POVMs by specifying the optical elements in the respective modules and by specifying the corresponding Kraus operators in terms of these elements.

For any two operator POVM, the Kraus operators $\mathbf{V}_1$, $\mathbf{V}_2$ are given by $\mathbf{V}_1 = \mathbf{U}'\mathbf{D}_1\mathbf{U}$ and $\mathbf{V}_2 = \mathbf{U}''\mathbf{D}_2\mathbf{U}$. Here $\mathbf{U}$, $\mathbf{U}'$, $\mathbf{U}''$ are the three unitary operators in a single module. Denoting θ, ϕ as the angles of rotation of the two variable polarization rotators and γ, ξ, the angles of the two variable phase shifters in the module, the diagonal matrices $\mathbf{D}_1$, $\mathbf{D}_2$ are given by,

$$
\mathbf{D}_1 = \begin{pmatrix} e^{i\gamma}\cos\theta & 0 \\ 0 & \cos\phi \end{pmatrix}, \quad \mathbf{D}_2 = \begin{pmatrix} e^{i\xi}\sin\theta & 0 \\ 0 & \sin\phi \end{pmatrix} \tag{14}
$$

The POVM elements

$$\mathbf{F}_1 = \mathbf{V}_1^\dagger \mathbf{V}_1 = \mathbf{U}^\dagger \mathbf{D}_1^\dagger \mathbf{D}_1 \mathbf{U}, \quad \mathbf{F}_2 = \mathbf{V}_2^\dagger \mathbf{V}_2 = \mathbf{U}^\dagger \mathbf{D}_2^\dagger \mathbf{D}_2 \mathbf{U} \qquad (15)$$

satisfy the condition $\sum_{i=1,2} \mathbf{F}_i = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{I}$.

For any three operator POVM, the Kraus operators are given by

$$\begin{aligned}
\mathbf{V}_1 &= \mathbf{U}_\mathrm{I}' \mathbf{D}_\mathrm{I} \mathbf{U}_\mathrm{I}, \\
\mathbf{V}_2 &= \mathbf{U}_\mathrm{II}' \mathbf{D}_\mathrm{II} \mathbf{U}_{II} \mathbf{U}_\mathrm{I}'' \mathbf{D}_\mathrm{I}' \mathbf{U}_\mathrm{I}, \\
\mathbf{V}_3 &= \mathbf{U}_\mathrm{II}'' \mathbf{D}_\mathrm{II}' \mathbf{U}_\mathrm{II} \mathbf{U}_\mathrm{I}'' \mathbf{D}_\mathrm{I}' \mathbf{U}_\mathrm{I},
\end{aligned} \qquad (16)$$

Here, the diagonal $\mathbf{D}$ matrices are

$$\mathbf{D}_\mathrm{I} = \begin{pmatrix} e^{i\gamma_\mathrm{I}} \cos\theta_\mathrm{I} & 0 \\ 0 & \cos\phi_\mathrm{I} \end{pmatrix}, \quad \mathbf{D}_\mathrm{I}' = \begin{pmatrix} e^{i\xi_\mathrm{I}} \sin\theta_\mathrm{I} & 0 \\ 0 & \sin\phi_\mathrm{I} \end{pmatrix} \qquad (17)$$

and

$$\mathbf{D}_\mathrm{II} = \begin{pmatrix} e^{i\gamma_\mathrm{II}} \cos\theta_\mathrm{II} & 0 \\ 0 & \cos\phi_\mathrm{II} \end{pmatrix}, \quad \mathbf{D}_\mathrm{II}' = \begin{pmatrix} e^{i\xi_\mathrm{II}} \sin\theta_\mathrm{II} & 0 \\ 0 & \sin\phi_\mathrm{II} \end{pmatrix} \qquad (18)$$

$(\theta_\mathrm{I},\ \phi_\mathrm{I})$, $(\gamma_\mathrm{I},\ \xi_\mathrm{I})$ are respectively the pair of angles corresponding to variable polarization rotators and variable phase shifters in the first module. Similarly, $(\theta_\mathrm{II},\ \phi_\mathrm{II})$, $(\gamma_\mathrm{II},\ \xi_\mathrm{II})$ are the pairs of angles corresponding to variable polarization rotators and variable phase shifters respectively in the second module. $\mathbf{U}_\mathrm{I}$, $\mathbf{U}_\mathrm{I}'$, $\mathbf{U}_\mathrm{I}''$ are the unitary operators used in the first module and $\mathbf{U}_\mathrm{II}$, $\mathbf{U}_\mathrm{II}'$, $\mathbf{U}_\mathrm{II}''$ are the unitary operators used in the second module. (Notice that all the unitary operators in the above schemes are arbitrary and a particular choice of the associated unitary operators gives rise to a different experimental arrangement). The extension of this scheme to n operator POVM involving n-1 modules is quite similar and is given in [2].

We had identified, in Sec. 3, that an ensemble average of Jones devices will lead to all possible types of Mueller matrices, some examples of which are given in Table 1. We now show that the experimental set up proposed in Ref. [2] can also be used to realize varieties of Mueller devices. To substantiate our claim, we identify here the linear optical elements needed in the single module set up of Ahnert and Payne [2], which lead to the physical realization of two typical Mueller matrices given in Table 1.

To obtain the type-I Mueller matrix $\mathbf{M} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ of example 3 (see Table I), we

use $\mathbf{U} = \mathbf{I}$, $\mathbf{U}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{U}'' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as the required unitary Jones devices and

both the variable polarization rotators are set with their rotation angles $\theta = \phi = \pi/4$. There

is no need of phase shifter devices in this case i.e, $\gamma = \xi = 0$.

Similarly for the type-II Mueller matrix $\mathbf{M} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ of example 7, we find that

$\mathbf{U} = \mathbf{I}$, $\mathbf{U}' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\mathbf{U}'' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ are the required unitary Jones devices. The

rotation angles of the variable polarization rotators are, as in the earlier case, $\theta = \phi = \pi/4$

and there is no need of phase shifter devices i.e., $\gamma = \xi = 0$. Notice that in both the above

examples the unitary operators $\mathbf{U}'$, $\mathbf{U}''$ correspond to linear and circular retarders [18].

These two examples illustrate that the experimental set up given in Ref. [2] may be

utilized to realize the required non-determinisitc Mueller devices. In fact Mueller matri-

ces corresponding to an ensemble with more than two Jones devices may also be realized

by employing larger number of modules as given in the experimental scheme proposed by

Ref. [2].

6. Conclusion

We have established here a connection between the phenomenological ensemble approach [1]

for the coherency matrix and the POVM transformation of quantum density matrix. This

opens up a fresh avenue to physically realize types I and II of the Mueller matrix classification

of Ref. [17]. We have also given experimental setup to implement Mueller transformations

corresponding to ensemble average of Jones devices by employing the POVM scheme on the

single photon density matrix suggested in Ref. [2], in the context of quantum measurement

theory. It is gratifying to note that two decades after the introduction of the ensemble

approach, which had remained obscure and only received passing reference in textbooks such

as [20], its value is revealed in this paper through its connection with the new developments

in quantum measurement theory. We plan on exploring further the POVM transformation in the description of quantum polarization optics.

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