Proper J-holomorphic discs in Stein domains of dimension 2

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Abstract. We prove the existence of global Bishop discs in a strictly pseudoconvex Stein domain in an almost complex manifold of complex dimension 2.

1 Introduction

The problem of embedding complex discs or general Riemann surfaces into complex manifolds has been well-known for a long time. The interest to the case of almost complex manifolds has grown due to a strong link with symplectic geometry (Gromov [16]). We present the following result.

Theorem 1.1 *Let (M, J) be an almost complex manifold of complex dimension 2 admitting a strictly plurisubharmonic exhaustion function ρ. Then for every non-critical value c of ρ, every point $p \in \Omega_c = \{\rho < c\}$ and every vector $v \in T_p(M)$ there exists a J-holomorphic immersion $f : \overline{\mathbb{D}} \longrightarrow \Omega_c$, where $\mathbb{D} \subset \mathbb{C}$ is the unit disc, such that $f(b\mathbb{D}) \subset b\Omega_c$, $f(0) = p$, and $df_0 \left(\frac{\partial}{\partial \mathrm{Re}\,\zeta} \right) = \lambda v$ for some $\lambda > 0$.*

For a domain $M \subset \mathbb{C}^n$ with the standard complex structure, the result is due to Forstnerič and Globevnik [15]; there are various generalizations including embedding bordered Riemann surfaces into singular complex spaces (see [10] and references there).

Recently Biolley [7] proved a similar result for an almost complex manifold M of any dimension n, but under the additional hypothesis that the defining function ρ is subcritical. The latter means that ρ does not have critical points of the maximum Morse index n. (A plurisubharmonic function can not have critical points of index higher than n.) We don't impose such restriction. Furthemore, Biolley [7] does not prescribe the direction of the disc. Her method is based on the Floer homology and substantially uses recent work of Viterbo and Hermann [17, 26]. Our proof is self-contained; we adapt the ideas of Forstnerič and Globevnik [15] to the almost complex case using the methods of classical complex analysis and PDE.

In most work on the existence of global discs with boundaries in prescribed totally real manifolds [2, 3, 12, 13, 18, 20, 21] the authors use the continuity principle. By the implicit function theorem and the linearized equation they show that any given disc generates a family of nearby discs. Then the compactness argument allows for passing to the limit. In contrast, we construct the discs by solving the almost Cauchy-Riemann equation directly.

Following [15], we start with a small disc passing through the given point in given direction and push the boundary of the disc in the directions complex-tangent to the level sets of the defining function ρ; it results in increasing ρ due to pseudoconvexity. This plan leads to the problem of attaching J-holomorphic discs to totally real tori in a level set of ρ. The latter problem is of independent interest and may occur elsewhere. It can be viewed as a far reaching generalization of the Riemann mapping theorem. The problem in turn reduces to an existence theorem for a boundary value problem for a quasilinear elliptic system of partial differential equations in the unit disc (Theorem 4.1). We prove it by the classical methods of the theory of the Beltrami type equations and quasiconformal mappings (Ahlfors, Bers, Boyarskii, Lavrentiev, Morrey, Vekua; see [4, 5, 25] and references there).

The classical theory of the Beltrami equation in the whole complex plane $\mathbb{C}$ is largely based on the fact that the Ahlfors-Beurling transform is an isometry of $L^2(\mathbb{C})$. We introduce a new method for solving boundary value problems for the Beltrami type equations in the unit disc $\mathbb{D}$ by a suitable modification of the Ahlfors-Beurling transform, which is an isometry of $L^2(\mathbb{D})$ (Theorem 3.2). Then we reduce the PDE to an integral equation and prove the existence of solutions by L^p estimates.

Since the almost Cauchy-Riemann equation is nonlinear, one can only hope to find a solution close to a current disc f. By measuring the closeness in the L^p norm, we are able in fact to construct a disc sufficiently far from f in the sup-norm. To make sure we are looking for a disc close to f, we adapt the idea of [15] of adding to $f(\zeta)$ a term with a factor of ζ^n ($\zeta \in \mathbb{D}$) with big n. We develop a nonlinear version of this idea.

The above procedure works well in the absence of critical points of ρ. In order to push the boundary of the disc through critical level sets, we use a method by Drinovec Drnovšek and Forstnerič [10, 14], which consists of temporarily switching to another plurisubharmonic function at each critical level set. We point out that adapting this method to the almost complex case is not a major problem because the difficulties are localized near the critical points, in which the almost complex structure can be closely approximated by the standard complex structure.

Although higher dimension gives us more freedom for constructing J-holomorphic discs, we must admit that our proof of the main result goes through in dimension 2 only. The reason is that our main tool (Theorem 4.1) needs a special coordinate system in which coordinate hyperplanes $z = const$ are J-complex, which generally can be achieved only in dimension 2. For a domain in $\mathbb{C}^n$ with the standard complex structure, the result is obtained in [15] by reduction to dimension 2 using sections by 2-dimensional complex hypersurfaces. Such reduction in not possible for almost complex structures.

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2 Almost complex manifolds

Let (M, J) be an almost complex manifold. Denote by $\mathbb{D}$ the unit disc in $\mathbb{C}$ and by J_{st} the standard complex structure of $\mathbb{C}^n$; the value of n is usually clear from the context. Let f be a smooth map from $\mathbb{D}$ into M. Recall that f is called *J-holomorphic* if $df \circ J_{st} = J \circ df$. We also call such a map f a *J-holomorphic* disc or a *pseudoholomorphic* disc or just a *holomorphic* disc when a complex structure is fixed. We will often denote by ζ the standard complex coordinate on $\mathbb{C}$.

The fundamental result of the analysis and geometry of almost complex structures is *the Nijehnuis - Woolf theorem* which states that given point $p \in M$ and given tangent vector $v \in T_p M$ there exists a J-holomorphic disc $f : \mathbb{D} \longrightarrow M$ centered at p that is $f(0) = p$ and such that $df(0)(\partial / \partial \mathrm{Re}\, \zeta) = \lambda v$ for some $\lambda > 0$. This disc f depends smoothly on the initial data (p, v) and the structure J. A short complete proof of this theorem is contained in [23]. This result will be used several times in the present paper.

It is well-known that an almost complex manifold (M, J) of complex dimension n can be locally viewed as the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ equipped with an almost complex structure which is a small deformation of J_{st}. More precisely, let (M, J) be an almost complex manifold of complex dimension n. Then for every $p \in M$, $\delta_0 > 0$, and $k \geq 0$ there exist a neighborhood U of p and a smooth coordinate chart $z : U \longrightarrow \mathbb{B}$ such that $z(p) = 0$, $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$, and the direct image $z_*(J) := dz \circ J \circ dz^{-1}$ satisfies the inequality $||z_*(J) - J_{st}||_{C^k(\bar{\mathbb{B}})} \leq \delta_0$. For the proof we point out that there exists a diffeomorphism z from a neighborhood U' of $p \in M$ onto $\mathbb{B}$ such that $z(p) = 0$ and $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$. For $\delta > 0$ consider the isotropic dilation $d_\delta : t \mapsto \delta^{-1} t$ in $\mathbb{C}^n$ and the composite $z_\delta = d_\delta \circ z$. Then $\lim_{\delta \to 0} ||(z_\delta)_*(J) - J_{st}||_{C^k(\bar{\mathbb{B}})} = 0$. Setting $U = z_\delta^{-1}(\mathbb{B})$ for positive δ small enough, we obtain the desired result. As one of the consequences we obtain that for every point $p \in M$ there exists a neighborhood U of p and a diffeomorphism $z : U \to \mathbb{B}$ with center at p (in the sense that $z(p) = 0$) such that the function $|z|^2$ is J-plurisubharmonic on U and $z_*(J) = J_{st} + O(|z|)$.

Let u be a function of class C^2 on M, let $p \in M$ and $v \in T_pM$. *The Levi form* of u at p evaluated on v is defined by the equality $L^J(u)(p)(v) := -d(J^* du)(v, Jv)(p)$.

The following result is well known (see, for instance, [9]).

Proposition 2.1 *Let u be a real function of class C^2 on M, let $p \in M$ and $v \in T_pM$. Then $L^J(u)(p)(v) = \Delta(u \circ f)(0)$ where $f : r\mathbb{D} \longrightarrow M$ for some $r > 0$ is an arbitrary J-holomorphic map such that $f(0) = p$ and $df(0)(\partial/\partial \operatorname{Re} \zeta) = v$, $\zeta \in r\mathbb{D}$.*

The Levi form is invariant with respect to J-biholomorphisms. More precisely, let u be a C^2 real function on M, let $p \in M$ and $v \in T_pM$. If Φ is a (J, J')-holomorphic diffeomorphism from (M, J) into (M', J'), then $L^J(u)(p)(v) = L^{J'}(u \circ \Phi^{-1})(\Phi(p))(d\Phi(p)(v))$.

Finally, it follows from Proposition 2.1 that a C^2 function u is J-plurisubharmonic on M if and only if $L^J(u)(p)(v) \geq 0$ for all $p \in M$, $v \in T_pM$. Thus, similarly to the case of the integrable structure one arrives in a natural way to the following definition: a C^2 real valued function u on M is *strictly J-plurisubharmonic* on M if $L^J(u)(p)(v)$ is positive for every $p \in M$, $v \in T_pM \backslash \{0\}$.

Let J be a smooth almost complex structure on a neighborhood of the origin in $\mathbb{C}^n$ and $J(0) = J_{st}$. Denote by $z = (z_1, ..., z_n)$ the standard coordinates in $\mathbb{C}^n$ (in the following matrix computations we view z as a column). Then a map $z : \mathbb{D} \longrightarrow \mathbb{C}^n$ is J-holomorphic if and only if it satisfies the following system of partial differential equations

$$z_{\bar{\zeta}} - A(z)\overline{z_\zeta} = 0, \tag{1}$$

where $A(z)$ is the complex $n \times n$ matrix defined by

$$A(z)v = (J_{st} + J(z))^{-1}(J_{st} - J(z))\overline{v} \tag{2}$$

It is easy to see that right-hand side of (2) is $\mathbb{C}$-linear in $v \in \mathbb{C}^n$ with respect to the standard structure J_{st}, hence $A(z)$ is well defined. Since $J(0) = J_{st}$, we have $A(0) = 0$. Then in a sufficiently small neighborhood U of the origin the norm $\| A \|_{L^\infty(U)}$ is also small, which implies the ellipticity of the system (1).

However, we will need a more precise choice of coordinates imposing additional restrictions on the matrix function A. The proof of the following elementary statement can be found, for instance, in [9].

Lemma 2.2 *After a suitable polynomial second degree change of local coordinates near the origin*

$$z \mapsto z + \sum a_{kj} z_k \bar{z}_j$$

we can achieve

$$A(0) = 0, A_z(0) = 0$$

In these coordinates the Levi form of a given C^2 function u with respect to J at the origin coincides with its Levi form with respect to J_{st} that is

$$L^J(u)(0)(v) = L^{J_{st}}(u)(0)(v)$$

for every vector $v \in T_0\mathbb{R}^{2n}$.

3 Integral transforms in the unit disc

Let Ω be a domain in $\mathbb{C}$. Let T_Ω denote the Cauchy-Green transform

$$T_\Omega f(\zeta) = \frac{1}{2\pi i} \int \int_\Omega \frac{f(\tau) d\tau \wedge d\overline{\tau}}{\tau - \zeta}. \tag{3}$$

Let R_Ω denote the Ahlfors-Beurling transform

$$R_\Omega f(\zeta) = \frac{1}{2\pi i} \int \int_\Omega \frac{f(\tau) d\tau \wedge d\overline{\tau}}{(\tau - \zeta)^2}, \tag{4}$$

where the integral is considered in the sense of the Cauchy principal value. We omit the index Ω if it is clear form the context. Denote by B the Bergman projection for $\mathbb{D}$.

$$Bf(\zeta) = \frac{1}{2\pi i} \int \int_{\mathbb{D}} \frac{f(\tau) d\tau \wedge d\overline{\tau}}{(\overline{\tau}\zeta - 1)^2}.$$

We need the following properties of the above operators.

Proposition 3.1 *(i) Let $p > 2$ and $\alpha = (p-2)/p$. Then the linear operator $T : L^p(\mathbb{D}) \longrightarrow C^\alpha(\mathbb{C})$ is bounded, in particular, $T : L^p(\mathbb{D}) \longrightarrow L^\infty(\mathbb{D})$ is compact. If $f \in L^p(\mathbb{D})$, then $\partial_{\overline{\zeta}} T f = f$, $\zeta \in \mathbb{D}$, as a Sobolev derivative.*

(ii) Let $m \geq 0$ be integer and let $0 < \alpha < 1$. Then the linear operators $T : C^{m,\alpha}(\overline{\mathbb{D}}) \longrightarrow C^{m+1,\alpha}(\mathbb{C})$ and $R : C^{m,\alpha}(\overline{\mathbb{D}}) \longrightarrow C^{m,\alpha}(\overline{\mathbb{D}})$ are bounded. Furthermore, if $f \in C^{m,\alpha}(\overline{\mathbb{D}})$, then $\partial_{\overline{\zeta}} T f = f$ and $\partial_\zeta T f = R f$, $\zeta \in \mathbb{D}$, in the usual sense.

(iii) The operator R_Ω can be uniquely extended to a bounded linear operator $R_\Omega : L^p(\Omega) \longrightarrow L^p(\Omega)$ for every $p > 1$. If $f \in L^p(\mathbb{D})$, $p > 1$ then $\partial_\zeta T f = R f$ as a Sobolev derivative. Moreover, the operator $R_\mathbb{C}$ is an isometry of $L^2(\mathbb{C})$, therefore $\| R_\mathbb{C} \|_{L^2(\mathbb{C})} = 1$.

(iv) The Bergman projection $B : L^p(\mathbb{D}) \longrightarrow A^p(\mathbb{D})$ is bounded. Here $A^p(\mathbb{D})$ denotes the space of all holomorphic functions in $\mathbb{D}$ of class $L^p(\mathbb{D})$.

(v) The functions $p \mapsto \| T \|_{L^p(\Omega)}$ and $p \mapsto \| R \|_{L^p(\Omega)}$ are logarithmically convex and in particular, continuous for $p > 1$.

The proofs of the parts (i) - (iii) are contained in [25]. The part (iv) follows from (iii); see e.g. [11]. The part (iv) follows by the classical interpolation theorem of M.Riesz-Torin (see, e.g., [27]).

We introduce modifications of the operators T and R for solving certain boundary value problems in the unit disc $\mathbb{D}$. For $f \in L^p(\mathbb{D})$ we define

$$T_0 f(\zeta) = T f(\zeta) - \overline{T f(\overline{\zeta}^{-1})}, \quad \zeta \in \mathbb{D}. \tag{5}$$

By Proposition 3.1 for $p > 2$ and $\alpha = (p-2)/p$, the linear operator $T_0 : L^p(\mathbb{D}) \longrightarrow C^\alpha(\mathbb{D})$ is bounded, in particular, $T_0 : L^p(\mathbb{D}) \longrightarrow L^\infty(\mathbb{D})$ is compact. Since the function Tf is holomorphic and bounded in $\mathbb{C} \backslash \overline{\mathbb{D}}$, then the function $\zeta \mapsto \overline{(Tf)(\overline{\zeta}^{-1})}$ is holomorphic in $\mathbb{D}$. Hence $\partial_{\overline{\zeta}} T_0 f = \partial_{\overline{\zeta}} Tf = f$. Furthermore, for $\zeta \in b\mathbb{D}$, we have $\zeta = \overline{\zeta}^{-1}$, therefore by (5), $\operatorname{Re} T_0 f(\zeta) = 0$. Hence for $f \in L^p(\mathbb{D})$, the function $u = T_0 f$ solves the boundary value problem

$$\begin{cases} \partial_{\overline{\zeta}} u = f, \zeta \in \mathbb{D}, \\ \operatorname{Re} u|_{b\mathbb{D}} = 0 \end{cases}$$

We further define

$$R_0 f := \partial_\zeta T_0 f.$$

Since $\partial_\zeta Tf = Rf$ and $\partial_{\overline{\zeta}} Tf = f$, then

$$R_0 f(\zeta) = \partial_\zeta T_0 f(\zeta) = Rf(\zeta) - \overline{\partial_{\overline{\zeta}} Tf(\overline{\zeta}^{-1})} = Rf(\zeta) + \zeta^{-2} \overline{Rf(\overline{\zeta}^{-1})}, \tag{6}$$

and we obtain a nice formula

$$R_0 f = Rf + B\overline{f},$$

where B is the Bergman projection. By Propositions 3.1(iv) and (v), the operator $R_0 : L^p(\mathbb{D}) \longrightarrow L^p(\mathbb{D})$ is bounded, and the map $p \mapsto \| R_0 \|_{L^p(\mathbb{D})}$ is continuous for $p > 1$. By Proposition 3.1(iii), R is an isometry of $L^2(\mathbb{C})$. We obtain an analog for the operator R_0.

Theorem 3.2 R_0 *is an* $\mathbb{R}$-*linear isometry of* $L^2(\mathbb{D})$, *in particular,* $\| R_0 \|_{L^2(\mathbb{D})} = 1$.

Proof : For a domain $G \subset \mathbb{C}$ we use the inner product

$$(f,g)_G = -\frac{i}{2} \int \int_G f\overline{g} d\zeta \wedge d\overline{\zeta}.$$

We put

$$\sigma f(\zeta) = \overline{\zeta}^{-2} \overline{f(\overline{\zeta}^{-1})}, \quad \psi(\zeta) = \overline{\zeta}^2 \zeta^{-2}.$$

Then $\sigma^2 = \operatorname{id}$. By substitution $\zeta \mapsto \overline{\zeta}^{-1}$ we obtain

$$(\sigma f, \sigma g)_{\mathbb{D}} = (g, f)_{\mathbb{C} \backslash \mathbb{D}}, \quad R\sigma = \psi \sigma R, \quad R = \psi \sigma R \sigma. \tag{7}$$

By (6) we have

$$R_0 f = Rf + \psi \sigma Rf.$$

Let $f \in L^2(\mathbb{D})$. Extend f to all of $\mathbb{C}$ by putting $f(\zeta) = 0$ for $|\zeta| > 1$. Then

$$\| R_0 f \|_{L^2(\mathbb{D})}^2 = (Rf + \psi \sigma Rf, Rf + \psi \sigma Rf)_{\mathbb{D}} =$$
$$(Rf, Rf)_{\mathbb{D}} + 2\operatorname{Re}(Rf, \psi \sigma Rf)_{\mathbb{D}} + (\psi \sigma Rf, \psi \sigma Rf)_{\mathbb{D}}.$$

Since $|\psi| = 1$, by (7) we obtain

$$(\psi \sigma Rf, \psi \sigma Rf)_{\mathbb{D}} = (\sigma Rf, \sigma Rf)_{\mathbb{D}} = (Rf, Rf)_{\mathbb{C} \backslash \mathbb{D}},$$

$$(Rf, \psi\sigma Rf)_{\mathbb{D}} = (\psi\sigma R\sigma f, \psi\sigma Rf)_{\mathbb{D}} = (R\sigma f, Rf)_{\mathbb{C}\setminus\mathbb{D}} = (\psi\sigma Rf, Rf)_{\mathbb{C}\setminus\mathbb{D}} = \overline{(Rf, \psi\sigma Rf)}_{\mathbb{C}\setminus\mathbb{D}}.$$

Then by the previous line and because R is an isometry

$$2\mathrm{Re}\,(Rf, \psi\sigma Rf)_{\mathbb{D}} = \mathrm{Re}\,(Rf, \psi\sigma Rf)_{\mathbb{C}} = \mathrm{Re}\,(Rf, R\sigma f)_{\mathbb{C}} = \mathrm{Re}\,(f, \sigma f)_{\mathbb{C}} = 0.$$

Hence

$$\| R_0 f \|^2_{L^2(\mathbb{D})} = (Rf, Rf)_{\mathbb{D}} + (Rf, Rf)_{\mathbb{C}\setminus\mathbb{D}} = \| Rf \|^2_{L^2(\mathbb{C})} = \| f \|^2_{L^2(\mathbb{C})} = \| f \|^2_{L^2(\mathbb{D})},$$

which proves the theorem.

4   Riemann mapping theorem for an elliptic system

The Riemann mapping theorem asserts that for every simply connected domain $G \subset \mathbb{C}$ there exists a conformal map of G onto $\mathbb{D}$. If G is smooth, then there is a diffeomorphism $f : \overline{G} \longrightarrow \overline{\mathbb{D}}$, which defines an almost complex structure $J = f_*(J_{st})$ in $\mathbb{D}$. Then the Riemann mapping theorem reduces to constructing a J-holomorphic map $z : (\mathbb{D}, J_{st}) \longrightarrow (\mathbb{D}, J)$. The latter satisfies the Beltrami type equation $\partial_{\bar\zeta} z = A(z)\partial_\zeta \bar z$, which is equivalent to the linear Beltrami equaion $\partial_{\bar z}\zeta + A(z)\partial_z\zeta = 0$. We consider the following more general system

$$\begin{cases} \partial_{\bar\zeta} z = a(z, w)\partial_\zeta \bar z, \\ \partial_{\bar\zeta} w = b(z, w)\partial_\zeta \bar z, \end{cases} \tag{8}$$

which cannot be reduced to a linear one. Here z, w are unknown functions of $\zeta \in \overline{\mathbb{D}}$ and a, b are C^∞ coefficients. By eliminating ζ, the system reduces to a nonhomogeneous quasilinear Beltrami type equation $\partial_{\bar z} w + a\partial_z w = b$, but we prefer to deal with (8) directly.

The following theorem is our main technical tool for constructing pseudoholomorphic discs with boundaries in a prescribed torus. For $r > 0$ denote $\mathbb{D}_r := r\mathbb{D}$.

Theorem 4.1 *Let $a, b : \overline{\mathbb{D}} \times \overline{\mathbb{D}}_{1+\gamma} \longrightarrow \mathbb{C}$ $(\gamma > 0)$ be smooth functions such that*

$$a(z, 0) = b(z, 0) = 0 \quad \text{and} \quad |a(z, w)| \le a_0 < 1.$$

Then there exists $C > 0$ such that for every integer $n \ge 1$ the system (8) admits a smooth solution (z_n, w_n) with the following properties:

(i) $|z_n(\zeta)| = |w_n(\zeta)| = 1$ *for* $|\zeta| = 1$.

(ii) $z_n : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$ *is a diffeomorphism with* $z_n(0) = 0$.

(iii) $|w_n(\zeta)| \le C|\zeta|^n$, $|w_n(\zeta)| < 1 + \gamma$.

Proof :   Shrinking $\gamma > 0$ if necessary, we extend the functions a and b to all of $\mathbb{C}^2$ preserving their properties. We will look for a solution of (8) in the form

$$z = \zeta e^u, w = \zeta^n e^v.$$

Then for the new unknowns u and v we have the following boundary value problem

$$\begin{cases} \partial_{\bar{\zeta}} u = A(u,v,\zeta)(1 + \overline{\zeta}\partial_{\bar{\zeta}}\overline{u}), \zeta \in \mathbb{D} \\ \partial_{\bar{\zeta}} v = B(u,v,\zeta)(1 + \overline{\zeta}\partial_{\bar{\zeta}}\overline{u}), \zeta \in \mathbb{D} \\ \operatorname{Re} u(\zeta) = \operatorname{Re} v(\zeta) = 0, |\zeta| = 1 \end{cases} \tag{9}$$

where

$$A = a\zeta^{-1}e^{\overline{u}-u},$$
$$B = b\zeta^{-n}e^{\overline{u}-v}.$$

Put $\partial_{\bar{\zeta}} u = h$ and choose u in the form $u = T_0 h$. Then $\partial_{\zeta} u = R_0 h$, which we plug into (9). We obtain the following system of singular integral equations for u, v and h:

$$\begin{cases} h = A(1 + \overline{\zeta R_0 h}), \\ u = T_0 h, \\ v = T_0(B(1 + \overline{\zeta R_0 h})) \end{cases} \tag{10}$$

We denote by $\| f \|_p$ the L^p-norm of f in $\mathbb{D}$. Since the function $p \mapsto \| R_0 \|_p$ is continuous in p and $\| R_0 \|_2 = 1$ we choose $p > 2$ such that

$$a_0 \| R_0 \|_p < 1.$$

For given $u, v \in L^\infty(\mathbb{D})$ the map $h \mapsto A(1 + \overline{\zeta R_0 h})$ is a contraction in $L^p(\mathbb{D})$ because

$$\| \overline{\zeta} A \|_\infty \| R_0 \|_p < 1.$$

Hence there exists a unique solution $h = h(u,v)$ of the first equation of (10) satisfying

$$\| h \|_p \leq \frac{\| A \|_p}{1 - a_0 \| R_0 \|_p} \tag{11}$$

Consider the map $F : L^\infty(\mathbb{D}) \times L^\infty(\mathbb{D}) \longrightarrow L^\infty(\mathbb{D}) \times L^\infty(\mathbb{D})$ defined by

$$F : (u,v) \mapsto (U,V) = (T_0 h, T_0(B(1 + \overline{\zeta R_0 h})))$$

where $h = h(u,v)$ is determined above. Then F is continuous (even Lipschitz) map. Let

$$E = \{(u,v) \in L^\infty(\mathbb{D}) \times L^\infty(\mathbb{D}) : \| u \|_\infty \leq u_0, \| v \|_\infty \leq v_0\}$$

We need the following

Lemma 4.2 *There exist $u_0 > 0$, $v_0 > 0$ such that E is invariant under F.*

Assuming the lemma, we prove the existence of the solution of (10). Indeed, since $T_0 : L^p(\mathbb{D}) \longrightarrow L^\infty(\mathbb{D})$ is compact for $p > 2$, then $F : E \longrightarrow E$ is compact. Since E is a bounded, closed and convex, then the existence of the solution of (10) follows by Schauder's principle.

Proof of Lemma 4.2 : Since $a(z,0) = b(z,0) = 0$, we have

$$|a(z,w)| \leq C_1 |w|, \quad |b(z,w)| \leq C_1 |w|.$$

Here and below we denote by C_j constants independent of n. We have

$$|a| = |a(\zeta e^u, \zeta^n e^v)| \leq C_1 e^{\|v\|_\infty} |\zeta|^n,$$

$$\| A \|_p = \| a\zeta^{-1} \|_p \leq C_2 \| \zeta^{n-1} \|_p \, e^{\|v\|_\infty} \leq C_3 e^{\|v\|_\infty} n^{-1/p}.$$

By (11), $\| h \|_p \leq C_4 e^{\|v\|_\infty} n^{-1/p}$, hence

$$\| U \|_\infty \leq C_5 e^{\|v\|_\infty} n^{-1/p}.$$

Similarly

$$|B| = |b(\zeta e^u, \zeta^n e^v)\zeta^{-n} e^{\overline{u}-v}| \leq C_1 e^{\|u\|_\infty},$$

$$\| B \|_\infty \leq C_1 e^{\|u\|_\infty}$$

$$\| V \|_\infty \leq C_7(\| B \|_p + \| B \|_\infty \| h \|_p) \leq C_8 e^{\|u\|_\infty}.$$

Let $\delta = n^{-1/p}$. Then

$$\| U \|_\infty \leq C_9 \delta e^{\|v\|_\infty},$$
$$\| V \|_\infty \leq C_9 e^{\|u\|_\infty}$$

Consider the system

$$u_0 = C_9 \delta e^{v_0}, v_0 = C_9 e^{u_0}$$

with the unknowns u_0, v_0. Then

$$u_0 = C_9 \delta e^{C_9 e^{u_0}}$$

For small $\delta > 0$ this equation has two positive roots. Let $u_0 = u_0(\delta)$ be the smaller root and $v_0 = v_0(\delta) = C_9 e^{u_0}$. Now if $\| u \|_\infty \leq u_0$, $\| v \|_\infty \leq v_0$, then

$$\| U \|_\infty \leq C_9 \delta e^{\|v\|_\infty} \leq C_9 \delta e^{v_0} \leq u_0,$$

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$$\| V \|_\infty \leq C_9 \delta e^{\|u\|_\infty} \leq C_9 \delta e^{u_0} \leq v_0$$

Hence E is invariant under F, which proves the lemma.

Thus the solution of (10) in $L^\infty(\mathbb{D})$ exists for n big enough. Since $h \in L^p(\mathbb{D})$, $p > 2$, the second and the third equations of (10) imply that $u, v \in C^\alpha(\overline{\mathbb{D}})$, $\alpha = (p-2)/p$. Since $\partial_{\overline{\zeta}} u = h \in L^p(\mathbb{D})$ and $\partial_\zeta u = R_0 h \in L^p(\mathbb{D})$ as Sobolev's derivatives, then u and v are solutions of (9), hence $z = \zeta e^u$ and $w = \zeta^n e^v$ are solutions of (8). By the ellipticity of the system, $z, w \in C^\infty(\mathbb{D})$. The smoothness up to the boundary can be derived directly from the properties of the Beltrami equation; it also follows by the reflection principle for pseudoholomorphic discs attached to totally real manifolds (see, e.g., [22]).

Since the winding number of $z|_{b\mathbb{D}}$ about 0 equals 1 and $\left| \partial_{\overline{\zeta}} z / \partial_\zeta z \right| = |a| \leq a_0 < 1$ then $z : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$ is a homeomorphism by the classical properties of the Beltrami equation [25], and (ii) follows.

Note that $u_0 \longrightarrow 0$, $v_0 \longrightarrow C_9$ as $n \longrightarrow \infty$. Since $T_0 : L^p(\mathbb{D}) \longrightarrow C^\alpha(\overline{\mathbb{D}})$ is bounded, then we have

$$\| v \|_{C^\alpha(\overline{\mathbb{D}})} \leq C_{10}, \quad \| e^v \|_{C^\alpha(\overline{\mathbb{D}})} \leq C_{11},$$

and $|w(\zeta)| \leq C_{11} |\zeta|^n$. Furthermore, since $|e^v| = 1$ on $b\mathbb{D}$, then $|e^{v(\zeta)}| \leq 1 + C_{11}(1 - |\zeta|)^\alpha$ for $|\zeta| < 1$. Then $|w(\zeta)| \leq |\zeta|^n(1 + C_{11}(1 - |\zeta|)^\alpha)$, hence $\| w \|_\infty \longrightarrow 1$ as $n \longrightarrow \infty$. Hence $\| w \|_\infty < 1 + \gamma$ for n big enough, and (iii) follows. This completes the proof of Theorem 4.1.

5 Pseudoholomorphic discs attached to real tori

This section concerns the geometrization of Theorem 4.1. We apply Theorem 4.1 in order to obtain a crucial technical result on (approximately) attaching pseudoholomorphic discs to a given real 2-dimensional torus in (M, J). We will use this result later for pushing discs across level sets of the defining function ρ in Theorem 1.1.

The tori and the discs considered in this section are not arbitrary. We study a special case which will suffice for the proof of the main result. Given a psedoholomorphic immersed disc f, we associate with f a real 2-dimensional torus Λ formed by the boundary circles of discs h_ζ centered at the boundary points $f(\zeta), \zeta \in b\mathbb{D}$. Thus, our initial data is a pair (f, Λ). Our goal is to construct a pseudoholomorphic disc with the boundary attached to the torus Λ. First we find a suitable neighborhood of the disc f which can be parametrized by the bidisc in $\mathbb{C}^2$. We transport the structure J onto this bidisc and choose the coordinates there such that the equations for J-holomorphic discs take the form used in Theorem 4.1. The theorem will provide a pseudoholomorphic disc approximately attached to Λ.

5.1 Immersed pseudoholomorphic discs, admissible parametrizations by the bidisc and generated tori

Let $f : \mathbb{D} \longrightarrow (M, J)$ be a J-holomorphic disc of class $C^\infty(\overline{\mathbb{D}})$. Suppose f is an immersion. Let $\gamma > 0$. Given $\zeta \in \overline{D}$ consider a J-holomorphic disc

$$h_\zeta : (1 + \gamma)\mathbb{D} \longrightarrow M$$

satisfying the condition $h_\zeta(0) = f(\zeta)$ and such that the direction $dh_\zeta(0)(\frac{\partial}{\partial \mathrm{Re}\,\tau})$ is not tangent to f. Admitting some abuse of notation, we sometimes write $h_{f(\zeta)}$ for h_ζ.

This allows to define a C^∞ map

$$H : \overline{\mathbb{D}} \times (1 + \gamma)\overline{\mathbb{D}} \longrightarrow M, \qquad H(\zeta, \tau) = h_\zeta(\tau).$$

Then H has the following properties:

(i) For every $\zeta \in \mathbb{D}$ the map $h_\zeta := H(\zeta, \bullet)$ is J-holomorphic.

(ii) For every $\zeta \in \mathbb{D}$ we have $H(\zeta, 0) = f(\zeta)$.

(iii) For every $\zeta \in \mathbb{D}$ the disc h_ζ is transversal to f at the point $f(\zeta)$.

We assume in addition that

(iv) $H : \overline{\mathbb{D}} \times (1 + \gamma)\overline{\mathbb{D}} \longrightarrow M$ is locally diffeomorphic.

Then $\Lambda = H(b\mathbb{D} \times (1+\gamma)b\mathbb{D})$ is a real 2-dimensional torus immersed into M. It is formed by a family of topological circles $\gamma_\zeta = h_\zeta((1 + \gamma)b\mathbb{D})$ parametrized by $\zeta \in b\mathbb{D}$. Every such a circle bounds a J-holomorphic disc $h_\zeta : (1 + \gamma)\mathbb{D} \longrightarrow M$ centered at $f(\zeta)$. In particular the torus Λ can be continuously deformed to the circle $f(b\mathbb{D})$.

If the above conditions (i) - (iv) hold we say that a map H is an *admissible parametrization* of a neighborhood of $f(\overline{\mathbb{D}})$ and Λ is *the torus generated by H*.

5.2 Ellipticity of admissible parametrizations

We prove the following consequence of Theorem 4.1.

Theorem 5.1 *Let $f : \overline{\mathbb{D}} \longrightarrow (M, J)$ be a C^∞ immersion J-holomorphic in $\mathbb{D}$. Suppose that there exists an admissible parametrization H of a neighborhood of $f(\mathbb{D})$ and let Λ be the generated torus. Then there exists an immersed J-holomorphic disc $\tilde{f}$ of class $C^\infty(\overline{\mathbb{D}})$ centered at $f(0)$, tangent to f at $f(0)$ and satisfying the boundary condition $\tilde{f}(b\mathbb{D}) \subset H(b\mathbb{D} \times b\mathbb{D})$.*

We stress that the boundary of $\tilde{f}$ is attached to the torus $H(b\mathbb{D} \times b\mathbb{D})$ and not to Λ. However since $\gamma > 0$ can be chosen arbitrarily close to 0, this leads to the following result sufficient for applications.

Corollary 5.2 *In the hypothesis of the former theorem for any positive integral n there exists an immersed J-holomorphic disc f^n of class $C^\infty(\overline{\mathbb{D}})$ centered at $f(0)$, tangent to f at $f(0)$ and such that $dist(f^n(b\mathbb{D}), \Lambda) \longrightarrow 0$ as $n \longrightarrow \infty$.*

Here *dist* denotes any distance compatible with the topology of M.

We begin the proof of Theorem 5.1 with the remark that the discs h_ζ, $\zeta \in \overline{D}$ fill a subset V of M containing $f(\overline{\mathbb{D}})$ which can be viewed as a fiber space with the base $f(\overline{\mathbb{D}})$ and the generic fiber $h_\zeta((1 + \gamma)\mathbb{D})$. Therefore the defined above map

$$H : \overline{\mathbb{D}} \times (1 + \gamma)\overline{\mathbb{D}} \longrightarrow V$$

gives a natural parametrization of V by the bidisc $U_\gamma := \overline{\mathbb{D}} \times (1 + \gamma)\overline{\mathbb{D}}$. Since H is locally diffeomorphic (see (iv) above) the inverse map H^{-1} is defined in a neighborhood of every point of V. This allows to define the almost complex structure $\tilde{J} = (H^{-1})_*(J) = dH^{-1} \circ J \circ dH$ on U_γ. The structure $\tilde{J}$ has a special form. Indeed, in the standard basis of $\mathbb{R}^4$ we have

$$\tilde{J} = \begin{pmatrix} \tilde{J}_{11} & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \end{pmatrix} \tag{12}$$

where $\tilde{J}_{kj}$ are real 2×2 matrices. We recall that in this basis the standard complex structure $J_{st}^{(2)}$ of $\mathbb{C}$ has the form

$$J_{st}^{(2)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

It follows by the property (i) of H that the maps $\tau \mapsto (c, \tau)$ are $\tilde{J}$-holomorphic for every fixed c. This implies that $\tilde{J}_{12} = 0$ and $\tilde{J}_{22} = J_{st}^{(2)}$. Furthermore, since the map $\zeta \mapsto (\zeta, 0)$ is $\tilde{J}$-holomorphic, we have $\tilde{J}_{11}(z, 0) = J_{st}^{(2)}$ and $\tilde{J}_{21}(z, 0) = 0$.

Let now $g : \mathbb{D} \longrightarrow U_\gamma$ be a $\tilde{J}$-holomorphic map. If we set $\zeta = \xi + i\eta$, the Cauchy-Riemann equations have expressing the $\tilde{J}$-holomorphicity of g have the form

$$\frac{\partial g}{\partial \xi} + \tilde{J}\frac{\partial g}{\partial \eta} = 0 \tag{13}$$

Suppose now that the matrix $J_{st} + J$ is invertible. Then the Cauchy - Riemann equations can be rewritten in the form

$$g_{\overline{\zeta}} + A(g)\overline{g}_{\overline{\zeta}} = 0 \tag{14}$$

where A is defined by (2). If we use the notation $g = (z, w)$ then the Cauchy-Riemann equations (14) can be written in the form

$$\begin{cases} \partial_{\overline{\zeta}}z = a(z, w)\partial_{\overline{\zeta}}\overline{z}, \\ \partial_{\overline{\zeta}}w = b(z, w)\partial_{\overline{\zeta}}\overline{z} \end{cases} \tag{15}$$

identical to (8). Furthermore, since $\tilde{J}(z, 0) = J_{st}$, the conditions $a(z, 0) = b(z, 0) = 0$ are satisfied.

Proposition 5.3 *We have $\| a \|_\infty < 1$.*

Proof :   The proof consists of two steps. First we study the matrix $\tilde{J} + J_{st}$ which determines the matrix A in the Cauchy-Riemann equations (14).

Lemma 5.4 *The matrix $\tilde{J}(z,w) + J_{st}$ is non-degenerate for any $(z,w) \in \overline{\mathbb{D}} \times (1+\gamma)\overline{\mathbb{D}}$.*

Proof :   It suffices to verify the condition $\det(\tilde{J}_{11}(z,w) + J_{st}^{(2)}) \neq 0$. For every fixed (z,w) the matrix $\tilde{J}_{11}(z,w)$ defines a complex structure on the euclidean space $\mathbb{R}^2$ so there exists a matrix $P = P(z,w)$ such that

$$\tilde{J}_{11}(z,w) = P J_{st}^{(2)} P^{-1}. \tag{16}$$

Recall that the manifold $\mathcal{J}_2$ of all complex structures on $\mathbb{R}^2$ can be identified with the quotient $GL(2,\mathbb{R})/GL(1,\mathbb{C})$ and has two connected components: $\mathcal{J}_2^+$ and $\mathcal{J}_2^-$. A structure $\tilde{J}_{11}$ belongs to $\mathcal{J}_2^+$ (resp. to $\mathcal{J}_2^-$) if in the representation (16) we have $\det P > 0$ (resp. $\det P < 0$). Suppose now that $\det(P J_{st}^{(2)} P^{-1} + J_{st}^{(2)}) = 0$ or equivalently $\det(P J_{st}^{(2)} + J_{st}^{(2)} P) = 0$ at some point (z,w). If we denote by p_{jk} the entries of the matrix P, the last equality means that $\sum_{jk=1}^2 p_{jk}^2 = 0$ which together with the non-degeneracy of P implies that $\det P < 0$ so that $\tilde{J}_{11}(z,w) \in \mathcal{J}_2^-$. On the other hand, for the point $(z,0)$ we have $\det P > 0$ since $\tilde{J}_{11}(z,0) = J_{st}^{(2)}$ so $\tilde{J}(z,0) \in \mathcal{J}_2^+$. But we can join the points $(z,0)$ and (z,w) by a real segment, so this contradiction proves lemma.

Now we can conclude the proof of Proposition 5.3. It follows by Lemma 5.4 that the Cauchy - Riemann equations (13) can be written in the form (15) on $\overline{\mathbb{D}} \times (1+\gamma)\overline{\mathbb{D}}$. The Cauchy-Riemann equations are elliptic at every point and this condition is independent of the choice of the coordinates. The system (15) is ellipitic at a point (z,w) if and only if $|a(z,w)| \neq 1$. Since $a(z,0) = 0$ we obtain by connectedness that $|a| < 1$ on $\overline{\mathbb{D}} \times (1+\gamma)\overline{\mathbb{D}}$, which concludes the proof.

Now Theorem 5.1 follows by Theorem 4.1.

5.3   Construction of an admissible parametrization with a prescribed generated torus

So far we studied a situation where an admissible parametrization of a neighborhood of an immersed J-holomorphic disc was given and proved the existence of discs with boundaries close to the generated torus. In the proof of our main result, we need an admissible parametrization of a neighborhood of a J-holomorphic disc with a given generated torus.

Let $f : \mathbb{D} \longrightarrow M$ be an immersed J-holomorphic disc of class $C^\infty(\overline{\mathbb{D}})$. We extend f smoothly to a neighborhood of $\overline{\mathbb{D}}$. Let U be a small neighborhood of $b\mathbb{D}$. For every point $f(\zeta)$, $\zeta \in U$, consider a J-holomorphic disc $h_\zeta : 2\mathbb{D} \longrightarrow M$. Suppose that the map h_ζ smoothly depends on $\zeta \in U$. Thus we obtain a smooth map

$$H : b\mathbb{D} \times \overline{\mathbb{D}} \longrightarrow M, \quad H : (\zeta, \tau) \mapsto h_\zeta(\tau).$$

Then $\Lambda := H(b\mathbb{D} \times b\mathbb{D})$ is a real 2-dimensional torus. In order to construct an admissible parametrization with the generated torus Λ we need to extend the map H from the cylinder $b\mathbb{D} \times \overline{\mathbb{D}}$ to the bidisc $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$.

Definition 5.5 *We call the described above torus Λ admissible. We further put $X_\zeta := X_{f(\zeta)} = dh_\zeta(0)(\frac{\partial}{\partial \mathrm{Re}\,\tau})$ for every $\zeta \in U$.*

Theorem 5.6 *Let $f : \mathbb{D} \longrightarrow (M, J)$ be an immersed J-holomorphic disc of class $C^\infty(\overline{\mathbb{D}})$. Let Λ be an admissible torus. Then there is a sequence of admissible tori Λ_n converging to Λ such that for every n there exists an immersed J-holomorphic disc f^n of class $C^\infty(\overline{\mathbb{D}})$ centered at $f(0)$, tangent to f at $f(0)$ and satisfying the boundary condition $f^n(b\mathbb{D}) \subset \Lambda_n$.*

Proof :   Let Λ be an admissible torus and let X be the vector field given by Definition 5.5. In general it is impossible to extend X as a non-vanishing vector field transversal to $f(\mathbb{D})$ at every point. However, for any integral (not necessarily positive) n we can consider the discs $h_\zeta^n : \tau \mapsto h_\zeta(\zeta^n \tau)$, where $\zeta \in b\mathbb{D}$. Their tangent vectors at the points $f(\zeta)$ are equal to $X_\zeta^n := \zeta^n X_\zeta$, where by multiplying a vector by a complex number ζ^n we mean applying the operator $(\mathrm{Re}\,\zeta + (\mathrm{Im}\,\zeta)J)^n$. We need the following

Lemma 5.7 *After a suitable choice of n the vector field X_ζ^n can be extended on the disc as a nonvanishing field transversal to f at every point.*

Proof :   First we look for a global parametrization of a neighborhood of $f(\overline{\mathbb{D}})$. Fix an arbitrary vector field Y transversal to $f(\overline{\mathbb{D}})$ at every point. By Nijenhuis - Woolf theorem we obtain a family of J-holomorphic discs $g_z : w \mapsto g_z(w)$, $z \in \overline{\mathbb{D}}$ so that $g_z(0) = f(z)$ and $X_{f(\zeta)}$ is tangent to g_z. Then the map $G : (z, w) \mapsto g_z(w)$ is a local diffeomorphism from a neighborhood of $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$ onto a neighborhood of $f(\overline{\mathbb{D}})$ and $G(z, 0) = f(z)$ so we can use the coordinates (z, w). We pull back the vector field X by G^{-1} and consider the vector field $(G^{-1})_*(X) : \zeta \mapsto (G^{-1})_*(X_\zeta)$. Let m be the winding number of the w-component of the vector field $(G^{-1})_*(X)$ when ζ runs along the circle $b\mathbb{D}$. We set $n = -m$. Then the field $(G^{-1})_*(X^n)$ extends on the disc $(\zeta, 0)$ as a smooth vector field Z transversal to this disc at every point. Then the map $G_*(Z)$ associates to every point of $\mathbb{D}$ a vector transversal to $f(\mathbb{D})$ and so defines the desired extension $\tilde{X}^n$ of the vector field X^n. This proves the lemma.

Now by the Nijenhuis - Woolf theorem there exists a map $\tilde{h}_\zeta : \overline{\mathbb{D}} \longrightarrow M$ which is J-holomorphic on $\mathbb{D}$ such that $\tilde{h}_\zeta = h_\zeta$ for every ζ in a neighborhood of $b\mathbb{D}$ and the vector $\tilde{X}^n_{f(\zeta)}$ is tangent to h_ζ at the origin. Thus we can extend H to a function defined on $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$ such that the map $H(\zeta, \bullet)$ is J-holomorphic for any $\zeta \in \overline{\mathbb{D}}$. This map H is a local diffeomorphism and so determines an admissible parametrization of a neighborhood of $f(\overline{\mathbb{D}})$ such that the generated torus coincides with Λ. Theorem 5.6 now follows by Theorem 5.1.

6   Pushing discs through non-critical levels

In this section we explain how to push a given disc through non-critical level sets of a strictly plurisubharmonic function.

Proposition 6.1 *Suppose that ρ does not have critical values in the closed interval $[c_1, c_2]$. Let $f : \mathbb{D} \longrightarrow \Omega_{c_1}$ be an immersed J-holomorphic disc such that $f(b\mathbb{D}) \subset b\Omega_{c_1}$. Then there exists an immersed J-holomorphic disc $\tilde{f} : \mathbb{D} \longrightarrow \Omega_{c_2}$ such that $\tilde{f}(0) = f(0)$, $d\tilde{f}(0) = \lambda df(0)$ for some $\lambda > 0$ and $\tilde{f}(b\mathbb{D}) \subset b\Omega_{c_2}$.*

For the proof we need some preparations. Let ρ be a strictly plurisubharmonic function on an almost complex manifold (M, J). For real c consider the domain $\Omega_c = \{\rho < c\}$. Suppose that its boundary has no critical points. Let $f : \mathbb{D} \longrightarrow \Omega_c$ be a J-holomorphic disc of class $C^\infty(\overline{\mathbb{D}})$ and such that $f(b\mathbb{D}) \subset b\Omega_c$. For every point $p \in f(b\mathbb{D})$ consider a J-holomorphic disc $h_p : 2\mathbb{D} \longrightarrow M$ touching $b\Omega_c$ from outside such that $\rho \circ h_p|_{2\mathbb{D}\setminus\{0\}} > c$. We call the discs h_p the *Levi discs*. The map h_p can be chosen smoothly depending on $p \in f(b\mathbb{D})$.

An explicit construction of the Levi discs is given in [15]. In the almost complex case the proof is similar; the only thing which has to be justified is the existence of discs h_p touching a strictly pseudoconvex level set from outside. This was recently proved by Mazzilli-Barraud and Ivashkovich-Rosay [1, 19]. In [9] the result is obtained in any dimension. For reader's convenience we include a simple proof (see [9]).

Lemma 6.2 *For a point $p \in b\Omega_c$ there exists a J-holomorphic disc h_p such that $h_p(0) = p$ and $h_p(\mathbb{D}\setminus\{0\})$ is contained in $M\setminus\overline{\Omega}_c$.*

Proof :   We fix local coordinates $z = (z_1, z_2)$ near p such that $p = 0$ and $J(0) = J_{st}$. Denote by $e_j, j = 1, 2$ the vectors of the standard basis of $\mathbb{C}^2$. By an additional change of coordinates we may achieve that the map $h : \zeta \mapsto \zeta e_1$ is J-holomorphic on $\mathbb{D}$. We can assume that the Levi form $L_r^J(0, e_1) = 1$ so that

$$r(z) = 2\operatorname{Re} z_2 + 2\operatorname{Re} \sum a_{jk} z_j z_k + \sum \alpha_{jk} z_j z_k + o(|z|^2)$$

with

$$\alpha_{11} = \Delta(r \circ h)(0) = 1.$$

Now for every $\delta > 0$ consider the non-isotropic dilation $\Lambda_\delta : (z_1, z_2) \mapsto (\delta^{-1/2} z_1, \delta^{-1} z_2)$. The J-holomorphicity of the map h implies that the direct images $J_\delta := (\Lambda_\delta)_*(J)$ converge to J_{st} as $\delta \longrightarrow 0$ in the C^k norm for every positive integral k on any compact subset of $\mathbb{C}^2$. Similarly, the functions $r_\delta := \delta^{-1} r \circ \Lambda^{-1}$ converge to the function $r_0 := 2\operatorname{Re} z_2 + |z_1|^2 + 2\operatorname{Re}\beta z_1^2$ (for some $\beta \in \mathbb{C}$).

Consider a J_{st}-holomorphic disc $\hat{h} : \zeta \mapsto \zeta e_1 - \beta \zeta^2 e_2$. According to the Nijenhuis-Woolf theorem for every $\delta \geq 0$ small enough there exists a J_δ-holomorphic discs h^δ such that the family $(h^\delta)_{\delta \geq 0}$ depends smoothly on the parameter δ and for every $\delta \geq 0$ we have

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$h^\delta(\zeta) = \zeta e_1 + o(|\zeta|)$ and $h^0 = \hat{h}$. Since $(r_0 \circ h^0)(\zeta) = ||\zeta||^2$, we obtain that for $\delta > 0$ small enough that $(r_\delta \circ h^\delta)(\zeta) = A_\delta(\zeta) + o(|\zeta|^2)$, where A_δ is a positive definite quadratic form on $\mathbb{R}^2$. Since the structures J_δ and J are biholomorphic, then the lemma follows.

Thus we obtain a smooth map

$$H : b\mathbb{D} \times \overline{\mathbb{D}} \longrightarrow M, \qquad H : (\zeta, \tau) \mapsto h_{f(\zeta)}(\tau) =: h_\zeta(\tau).$$

We can assume that $\Lambda := H(b\mathbb{D} \times b\mathbb{D})$ is an admissible torus and $\rho|_\Lambda \geq c + \varepsilon$ for some $\varepsilon > 0$. We stress that here ε depends only on ρ (more precisely on a constant separating the norm of the gradient of ρ from zero) and the C^2-norm of J.

Now Theorem 5.6 implies that there exists a disc $\tilde{f}$ with the same direction as f at the center and with the boundary attached to a torus arbitrarily close to Λ. Now we cut off the discs h_ζ by the level set $\{\rho = c + \varepsilon/2\}$ and obtain a disc with boundary attached to this level set. Indeed, we have the following

Lemma 6.3 *Suppose that $\rho \circ f|_{b\mathbb{D}} \geq c_0$ and c_0 is a non-critical value of ρ. Then there exists a J-holomorphic disc $\tilde{f}$ centered at $f(0)$ and tangent to f at the center with boundary attached to the level set $\{\rho = c_0\}$.*

Proof : By the Hopf lemma the disc f intersects the level set $\{\rho = c_0\}$ transversally at every point. Therefore the open set $\Omega = \{\zeta \in \mathbb{D} : \rho \circ f(\zeta) < c_0\}$ has a smooth boundary. The set Ω may be disconnected, but the connected component of $0 \in \Omega$ is simply connected by the maximum principle applied to the function $\rho \circ f$. Now the lemma follows via reparametrization by the Riemann mapping theorem.

Then we again consider the Levi discs for this level set etc. By iterating this argument a finite number of times we obtain Proposition 6.1.

7 Pushing discs through a critical level

In order to push the boundary of the disc f through critical level sets of ρ, we use a method of [14, 10], which consists of temporarily switching to another plurisubharmonic function at each critical level set. We need a version of the Morse lemma for almost complex manifolds.

Proposition 7.1 *Let (M, J) be an almost complex manifold of complex dimension 2. Let ρ be a strictly plurisubharmonic Morse function on M. Then there exists another strictly plurisubharmonic Morse function $\tilde{\rho}$ close to ρ with the same critical points, such that at each critical point of Morse index k in local coordinates given by Lemma 2.2 one has*

$$\tilde{\rho}(z) = \tilde{\rho}(0) + |z_1|^2 + |z_2|^2 - a_1 \operatorname{Re} z_1^2 - a_2 \operatorname{Re} z_2^2 \tag{17}$$

where

(i) $a_1 = a_2 = 0$ if $k = 0$,

(ii) $a_1 = 2$ and $a_2 = 0$ if $k = 1$,

(iii) $a_1 = a_2 = 2$ if $k = 2$.

Remark. This is a weak version of the Morse lemma because we change the given function ρ instead of reducing it to a normal form.

The following result must be well known. For convenience we include a proof.

Lemma 7.2 *Let B be a complex symmetric $n \times n$ matrix. Then there exists a unitary matrix U such that $U^t B U$ is diagonal with nonnegative elements.*

Proof : Using coordinate-free language, given a hermitian positive definite form H and a complex symmetric bilinear form B on a vector space V, $\dim_{\mathbb{C}} V = n$, we need $u_1, ..., u_n \in V$ such that
$$H(u_i, u_j) = \delta_{ij}, \qquad B(u_i, u_j) = c_i \delta_{ij}, \quad c_i \geq 0.$$
If the above holds with just $c_i \in \mathbb{C}$, then by rotation $u_i \mapsto \sigma_i u_i$, $|\sigma_i| = 1$, we obtain $c_i \geq 0$. It suffices to find $u_1 \in V$, $H(u_1, u_1) = 1$, such that for every $x \in V$,

$$H(x, u_1) = 0 \quad \text{implies} \quad B(x, u_1) = 0. \tag{18}$$

Then the rest of u_i in the H-orthogonal complement of u_1 are found by induction. Given $u \in V$, by duality, there is a unique vector $L(u) \in V$ such that for every $x \in V$,

$$H(x, L(u)) = B(x, u). \tag{19}$$

Then $L : V \to V$ is a $\mathbb{R}$-linear ($\mathbb{C}$-antilinear) transformation. Since B is symmetric, then by (19), L is real symmetric (self-adjoint) with respect to the form $\operatorname{Re} H$. Then the eigenvalues of L are real and the eigenvectors are in V (generally they are in $V \otimes_{\mathbb{R}} \mathbb{C}$). Let $u_1 \in V$ be an eigenvector of L, that is $L(u_1) = \lambda u_1$, for some $\lambda \in \mathbb{R}$. We normalize u_1 so that $H(u_1, u_1) = 1$. Then for $u = u_1$, (19) implies (18), and the lemma follows.

Proof of Proposition 7.1 : Let p be a critical point of ρ. Introduce a coordinate system with the origin at p given by Lemma 2.2. In these coordinates the function ρ is strictly plurisubharmonic at the origin with respect to J_{st}. Then

$$\rho(z) = \rho(0) + \sum a_{ij} z_i \bar{z}_j + \operatorname{Re} \sum b_{ij} z_i z_j + O(|z|^3),$$

where $a_{ij} = \bar{a}_{ji}$ and $b_{ij} = b_{ji}$. By a linear transformation we can reduce to the form $a_{ij} = \delta_{ij}$. If we now make a unitary transformation $z \mapsto Uz$ preserving $|z_1|^2 + |z_2|^2$, then the matrix $B = (b_{ij})$ changes to $U^t B U$. By Lemma 7.2 the expression of ρ reduces to

$$\rho(z) = \rho(0) + |z_1|^2 + |z_2|^2 - \operatorname{Re}(a_1 z_1^2 + a_2 z_2^2) + O(|z|^3),$$

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where $a_j \geq 0$, $j = 1, 2$. The remainder $\varphi = O(|z|^3)$ can be removed by changing ρ to $\tilde{\rho} = \rho - \varphi\lambda$, where $\lambda(z) = \lambda_0(z/\varepsilon)$ is a smooth cut-off function with $\lambda_0 \equiv 1$ in a neighborhood of the origin and $\lambda_0(z) = 0$ for $|z| \geq 1$, $\varepsilon > 0$ small enough.

Since $\varphi(z) = O(|z|^3)$, then $|d(\varphi\lambda)| \leq C|z|^2$, $\| \varphi\lambda \|_{C^2(\mathbb{C}^2)} \leq C\varepsilon$ where $C > 0$ is independent of ε. Since $|d\rho| \geq C|z|$ in a neighborhood of 0 for some $C > 0$, then for small $\varepsilon > 0$ the function $\tilde{\rho}$ has only one critical point at the origin, is strictly plurisubharmonic and matches with ρ for $|z| > \varepsilon$.

The coefficients a_j can be reduced to the standard values 0 and 2 depending on the index k of the critical point. We need a cut-off function that falls down from 1 to 0 sufficiently slowly.

Lemma 7.3 *Given $\delta > 0$ there exists a smooth non-increasing function ϕ with a compact support on $\mathbb{R}_+$ such that*

(i) *$\phi = 1$ near the origin.*

(ii) *$|t\phi'(t)| \leq \delta$.*

(iii) *$|t^2\phi''(t)| \leq \delta$*

The lemma follows because $\int_1^\infty \frac{dt}{t} = \infty$.

Let $b_j = 0$ (resp. 2) if $0 \leq a_j < 1$ (resp. $a_j > 1$). Let $\lambda(z) = \phi(|z|/\varepsilon)$, where ϕ is provided by Lemma 7.3 for sufficiently small δ. Then the function

$$\tilde{\rho}(z) = \rho(z) + \lambda[(a_1 - b_1)\mathrm{Re}\, z_1^2 + (a_2 - b_2)\mathrm{Re}\, z_2^2]$$

for sufficiently small ε has all the desired properties. Proposition 7.1 is proved.

Thus in what follows we assume that ρ has the properties given by Proposition 7.1. Let p be a critical point of ρ and $\rho(p) = 0$. Without loss of generality assume that the index k of p is equal to 1 or 2 since the disc obtained by Proposition 6.1 cannot approach a minimum of ρ. Choose a small neighborhood U of p. By (17) ρ is strictly plurisubharmonic with respect to J_{st}.

We apply the construction of Lemma 6.7 of [14]. Consider $c_0 > 0$ small enough such that 0 is the only critical value of ρ in the interval $[-c_0, 3c_0]$. We can assume that c_0 is small enough so that the set $K(c_0) := \{z : \rho(z) \leq 3c_0, |x'|^2 \leq c_0\}$ is compactly contained in a neighborhood of the origin corresponding to U. Here we use the notation $x' = x_1$, $x'' = x_2$ and $|x'|^2 = x_1^2$ (resp. $x' = (x_1, x_2)$ and $|x'|^2 = x_1^2 + x_2^2$) if $k = 1$ (resp. $k = 2$). We will use similar notations for the coordinates x, y and the coordinates u, v introduced below. Let

$$E = \{y' = 0, z'' = 0, |x'|^2 \leq c_0\}. \tag{20}$$

Then E is a totally real submanifold with boundary and $\dim E = k$. Consider the isotropic dilations of coordinates

$$d_{c_0} : z \mapsto w = u + iv = c_0^{-1/2}z.$$

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Set $J_{c_0} = (d_{c_0})_*(J)$. The structures J_{c_0} converge to J_{st} in any C^m norm on compact subsets of $\mathbb{C}^2$ as $c_0 \longrightarrow 0$. Consider the function $\hat{\rho}(w) := c_0^{-1}\rho(c_0^{1/2}w)$. This function has no critical values in $[-1, 3]$ and its expression in the coordinates $w = u+iv$ is the same as the expression (17) of ρ that is

$$\hat{\rho}(w) = 3v_1^2 + v_2^2 - u_1^2 + u_2^2$$

if $k = 1$ and

$$\hat{\rho}(w) = 3v_1^2 + 3v_2^2 - u_1^2 - u_2^2$$

if $k = 2$. In particular the set $K = d_{c_0}(K(c_0))$ is given by $\{w : \hat{\rho}(w) \leq 3, |u'|^2 \leq 1\}$ and is a fixed compact independent of c_0.

It is important that the origin is a critical point of the function ρ and the local coordinates and the function ρ are given by Proposition 7.1. This allows to use the isotropic dilations in contrast with Lemma 6.2.

Since the function $\hat{\rho}$ is strictly plurisubharmonic with respect to J_{st}, we can apply the construction of [14] (Lemma 6.7 and section 6.4). We replace the function $\hat{\rho}$ by a new function φ defined by

$$\varphi(w) = 3v_1^2 + v_2^2 - h(u_1^2) + u_2^2$$

if $k = 1$ and

$$\varphi(w) = 3v_1^2 + 3v_2^2 - h(u_1^2 + u_2^2)$$

if $k = 2$, where $h \geq 0$ is a suitable function. The construction of h depends on the parameter c_0 only. In our "delated" coordinates w we apply this construction taking $c_0 = 1$. Namely, according to [14] there exist constants $0 < \tau_0 < \tau_1 < 1$ depending on the eigenvalues of $\hat{\rho}$ and a function φ strictly plurisubharmonic on $\mathbb{C}^2$ with respect to J_{st} satisfying the following properties:

(i) $\hat{\rho} \leq \varphi \leq \hat{\rho} + \tau_1$,

(ii) $\hat{\rho} + \tau_0 \leq \varphi$ on the set $\{|u'|^2 \geq \tau_0\}$

(iii) $\varphi = \hat{\rho} + \tau_1$ on $\{|u'|^2 \geq 1\}$

Since $\hat{\rho}$ is strictly plurisubharmonic with respect to the structure J_{c_0}, the function φ also is strictly J_{c_0}-plurisubharmonic on $\{|u'|^2 \geq 1\}$ in view of (iii). On the other hand the structures J_{c_0} converge to J_{st} in any C^m norm on compact subsets of $\mathbb{C}^2$ as $c_0 \longrightarrow 0$. Therefore, since φ is strictly J_{st}-plurisubharmonic, it also is strictly J_{c_0}-plurisubharmonic on K if c_0 is small enough. Thus, φ is strictly J_{c_0}-plurisubharmonic on $\{\hat{\rho} \leq 3\}$.

Now consider the function $\tilde{\rho}(z) = c_0\varphi(c_0^{-1/2}z)$ and set $t_0 = \tau_0 c_0$.
The function $\tilde{\rho}$ satisfies the following properties:

(i) $\tilde{\rho}$ is strictly plurisubharmonic (with respect to J_{st}) in a neighborhood $V \subset U$ of 0 and $\tilde{\rho} = \rho + t_1$ on the complement of V. Here $t_1 > 0$ is a constant.

(ii) $\tilde{\rho}$ has no critical values on $(0, 3c_0)$

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(iii) There exists $t_0 \in (0, c_0)$ such that

$$\{\rho \le -c_0\} \cup E \subset \{\tilde\rho \le 0\} \subset \{\rho \le -t_0\} \cup E, \qquad (21)$$

where E is defined above by (20).

(iv) We have

$$\{\rho \le c_0\} \subset \{\tilde\rho \le 2c_0\} \subset \{\rho < 3c_0\} \qquad (22)$$

By Proposition 6.1 we construct an immersed J-holomorphic disc f such that $-t_0 < \rho \circ f|_{b\mathbb{D}} < 0$. The boundary of f is contained in a torus Λ formed by discs complex tangent to a level set of ρ. We will perturb the disc f slightly in order to avoid the intersection of its boundary with E.

Proposition 7.4 *Let $f : \overline{\mathbb{D}} \longrightarrow M$ be an immersed J-holomorphic disc in (M, J), where $\dim_{\mathbb{C}} M = 2$. Let E be a smooth submanifold in M. Then for every $m \ge 2$ there exists a J-holomorphic disc $\tilde f$ arbitrarily close to f in $C^m(\overline{\mathbb{D}})$ such that $\tilde f(0) = f(0)$, $d\tilde f(0) = df(0)$, and $\tilde f|_{b\mathbb{D}}$ is transverse to E. In particular, if $\dim_{\mathbb{R}} E \le 2$, then $\tilde f(b\mathbb{D}) \cap E = \emptyset$.*

Proof : By the implicit function theorem, the restriction $f|_{bD}$ admits infinitesimal perturbations in all directions. Then the proposition follows by the proof of Thom's transversality theorem.

We now assume $f(b\mathbb{D}) \cap E = \emptyset$. In view of the inclusion (21) we conclude that $\tilde\rho > 0$ on $f(b\mathbb{D})$. By Lemma 6.3 we cut off the disc f by a level set $\{\tilde\rho = c\}$ for some $c > 0$ to assume that now $f(b\mathbb{D})$ is contained in this level set. The function $\tilde\rho$ has no critical values in $(0, 3c_0)$. By Proposition 6.1 applied to the disc f and the function $\tilde\rho$ there exists a new disc $\tilde f$ with the boundary contained in $\{\tilde\rho > 2c_0\}$. In view of (22) we have the inclusion $\{\tilde\rho > 2c_0\} \subset \{\rho > c_0\}$. Now the boundary of $\tilde f$ is outside the critical level $\{\rho = 0\}$ as desired, and we switch back to the original function ρ.

8   Proof of Theorem 1.1

Since the function ρ is strictly plurisubharmonic, then after a generic perturbation of ρ which does not change the given level set, we can assume that ρ is a Morse function. Let p be the given point in D. If p is not a point of minimum of ρ, we proceed as follows. Consider a small J-holomorphic disc f centered at p with the given direction v. Consider a non-critical level set $\rho = c$ such that $\rho(p) < c$. Consider a foliation of a neighborhood of f by a complex one-parameter family of J-holomorphic discs h_q, $q \in f(\mathbb{D})$ such that the boundaries of these discs are outside the sublevel set $\rho < c$. When q runs over the circle $f(b\mathbb{D})$ these boundaries form a torus. Applying Proposition 5.1 we obtain a new disc $\tilde f$ centered at p and still in the same direction at p but with $\rho \circ \tilde f|_{b\mathbb{D}} > 0$.

If p is a point of minimum for ρ, we drop this first step and directly have this situation with $\tilde{f} = f$. Now the desired results follow by Proposition 6.1 combined with the above argument allowing to push boundaries of discs through critical levels.

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