ON THE MARKOV TRACE FOR TEMPERLEY–LIEB ALGEBRAS OF TYPE E_n

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Abstract. We show that there is a unique Markov trace on the tower of Temperley–Lieb type quotients of Hecke algebras of Coxeter type E_n (for all $n \geq 6$). We explain in detail how this trace may be computed easily using tom Dieck's calculus of diagrams. As applications, we show how to use the trace to show that the diagram representation is faithful, and to compute leading coefficients of certain Kazhdan–Lusztig polynomials.

1. Introduction

In the paper [**17**], Jones introduced a certain Markov trace on the tower of Hecke algebras $\mathcal{H}(A_{n-1})$ associated to the Coxeter groups $\mathcal{S}_n = W(A_{n-1})$, which are the symmetric groups. When Jones' trace is restricted to one of the algebras $\mathcal{H} = \mathcal{H}(A_{n-1})$, it is degenerate, but its radical is an ideal, J, of $\mathcal{H}$ and so we obtain a generically nondegenerate trace on the algebra $\mathcal{H}/J$, which is the Temperley–Lieb algebra TL_n occurring in statistical mechanics [**25**] (the trace is the matrix trace of a transfer matrix algebra).

In [**19**], Kazhdan and Lusztig introduced a remarkable polynomial $P_{x,w}(q)$ for any elements x, w in a Coxeter group W. These polynomials have important applications in representation theory. Although the polynomials have an elementary

1991 *Mathematics Subject Classification.* 20C08, 20F55, 57M15.

definition, the only obvious way to compute them is using a rather complicated recurrence relation. One of the main obstructions to computing the polynomials efficiently is a fast way to compute the integer $\mu(x, w)$, which is the coefficient of $q^{(\ell(w)-\ell(x)-1)/2}$ in $P_{x,w}(q)$. In [**12**], the author showed how Jones' trace can be used to compute the leading coefficients $\mu(x, w) \in \mathbb{Z}$ in the case where x and w are fully commutative elements of W (in the sense of [**24**]). In this paper, we will investigate the analogous phenomenon in Coxeter type E_n. This includes Coxeter groups of types A and D as special cases.

The algebras TL_n may be defined in terms of generators and relations in a way that generalizes readily to Coxeter systems of other types. These generalized Temperley–Lieb algebras have been studied for Coxeter type E_n by a number of people [**2**, **3**, **7**]. Although the Coxeter groups of type E_n are infinite for $n > 8$, the Hecke algebra quotient $TL(E_n)$ in this case is still finite dimensional. In [**2**], tom Dieck constructed a diagrammatic representation of $TL(E_n)$, although the question of whether this is a realisation—a faithful representation—is not tackled. In §9, we will prove

Theorem 1.1. *The diagrammatic representation of $TL(E_n)$ given in* [**2**] *is injective.*

The closing remarks of [**2**] state without proof that this representation can be used to define a Markov trace on the tower of algebras $TL(E_n)$. In Theorem 8.11, we will prove this claim and furthermore we will show that there is a unique such Markov trace. Although this is similar to what happens in type A, the analogous claim for Coxeter type D is false.

This trace is also remarkable for other reasons: after suitable rescaling, it is a tabular trace in the sense of [**10**], and a generalized Jones trace in the sense of [**12**]. The fact that the trace is tabular implies that it is (generically) nondegenerate on the algebras $TL(E_n)$. The fact that we have a generalized Jones trace will lead to the following theorem (proved in §9) where the monomial basis elements b_w are defined in §3.

Theorem 1.2. *Let $\{b_w : w \in W_c\}$ be the monomial basis of $TL(E_n)$ indexed by the fully commutative Coxeter group elements, and let* tr *be the unique Markov trace on the tower of algebras $TL(E_n)$. If $x, y \in W_c$, then the coefficient of v^{-1} in $\mathrm{tr}(b_x b_{y^{-1}})$ (after expansion as a power series) is $\tilde{\mu}(x, y)$, where*

$$\tilde{\mu}(x,y) = \begin{cases} \mu(x,y) & \text{if } x \leq y, \\ \mu(y,x) & \text{if } x \not\leq y, \end{cases}$$

and $\mu(a, b)$ is the integer defined in [**19**].

We will also show in §9 how $\tilde{\mu}(x, y)$ may be evaluated non-recursively using the diagram calculus.

2. Traces and Markov traces

By a *trace* on an R-algebra A, we mean an R-linear map $t : A \longrightarrow R$ such that $t(ab) = t(ba)$ for all $a, b \in A$. The *radical* of the trace is the set of all $a \in A$ such that $t(ab) = 0$ for all $b \in A$. The radical is always an ideal of A, and if it is trivial, the trace is said to be *nondegenerate*. In any case, if I is the radical of t, then t induces a nondegenerate trace on the quotient algebra R/I.

The set of traces on an R-algebra A has a natural R-module structure. In the special case where ρ is a representation of an R-algebra A, then the matrix trace associated to ρ is a trace in the above sense, which means that, if A is semisimple, the Grothendieck group of A gives a $\mathbb{Z}$-lattice in the space of traces, generated by the traces of the simple modules.

We will be particularly concerned with algebras where the base ring R is obtained by extending scalars from the ring of Laurent polynomials $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ to some ring $F \otimes \mathcal{A}$. This has the effect of specializing the parameter v to an invertible element of F. In this situation, a trace is called *generically nondegenerate* if it is nondegenerate as a trace over $\mathcal{A}$, and if it also remains nondegenerate as a trace over $F \otimes \mathcal{A}$ for all but finitely many specializations of v.

Suppose now that R is an integral domain and $\{A_n : n \geq N\}$ is a family of unital R-algebras such that A_n is a subalgebra of A_{n+1} for all $n \geq N$. Let A_∞ be the associated direct limit. Suppose also that there is a set of elements $\{g_n : n \in \mathbb{N}\}$ such that $g_{n+1} \in A_{n+1} \backslash A_n$ for all n and such that $\{g_n : n \leq M\}$ is an algebra generating set for A_M. Following [**5**, §4], we may now introduce the notion of Markov trace.

Definition 2.1. Maintain the above notation, and let F be a field containing R. A *Markov trace* on A_∞ with parameter $z \in F$ is an F-linear map $\tau : A_\infty \longrightarrow F$ satisfying the following conditions:

(i) $\tau(1) = 1$;

(ii) $\tau(hb_{n+1}) = z\tau(h)$ for $n \geq N$ and $h \in A_n$;

(iii) $\tau(hh') = \tau(h'h)$ for all $h, h' \in A_\infty$.

Jones [**17**] proved that there is a unique Markov trace with parameter z on the tower of Hecke algebras of type A_n, and that the only one of these traces that passes to the Temperley–Lieb quotient is the one with parameter $z = (v + v^{-1})^{-1}$. This is an important observation in the construction of the Jones polynomial, because conditions (ii) and (iii) for the trace are what is needed to ensure that the polynomial is invariant under the two types of "Markov move".

Some other notable work on Markov traces includes that of Geck and Lambropoulou [**4**], who classified the Markov traces in Coxeter types B and D, using a suitable extension of the above definition. Lambropoulou [**20**] extended this work (in type B) to generalized and cyclotomic Hecke algebras of type B.

For the purposes of studying Temperley–Lieb type quotients of Hecke algebras, a better definition of Markov traces seems to be one that appears in work of Seifert [**22**] and recent work of Gomi [**6**, Definition 3.7]. In this case, one retains conditions (i) and (iii) of Definition 2.1 and replaces condition (ii) by the requirement that

$$\tau(aT_s) = z_s T(a)$$

whenever we have $a \in \mathcal{H}(W_I)$ for some parabolic subgroup W_I corresponding to

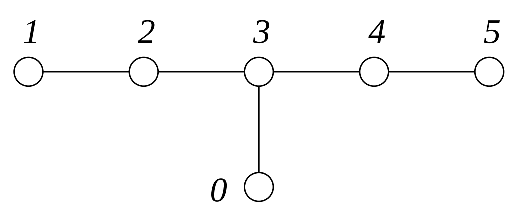
$I \subseteq S\backslash\{s\}$. (In other words, we require condition (ii) to hold for all generators of A_{n+1}, not just one particular generator.) Here, z_s is an indeterminate depending on the conjugacy class of s in W.

In this paper, we will restrict our attention to the tower of algebras $TL(E_n)$, and in this case, the above definitions happen to agree; however, they do not agree in the corresponding question for type D_n. In the latter case, it can be shown that the Seifert–Gomi formulation produces a unique Markov trace, and Definition 2.1 does not.

3. The algebras $TL(E_n)$

Let $X = X(E_n)$ be a Coxeter graph of type E_n, where $n \geq 6$. Following [**3**], we label the vertices of X by $0, 1, \ldots, n-1$ in such a way that $1, 2, 3, \ldots, n-1$ lie in a straight line, and such that 3 is the unique vertex of degree 3, which is adjacent to 2, 4 and 0. Figure 1 shows the case $n = 6$.

Figure 1. Coxeter graph of type E_6



Let $W(E_n)$ be the associated Coxeter group with distinguished set of generating involutions

$$S(E_n) = \{s_i : i \text{ is a vertex of } X(E_n)\}.$$

In other words, $W = W(E_n)$ is given by the presentation

$$W = \langle S(E_n) \mid (st)^{m(s,t)} = 1 \text{ for } m(s,t) < \infty \rangle,$$

where $m(s,s) = 1$, $m(s,t) = 2$ if s and t are not adjacent in X, and $m(s,t) = 3$ if s and t are adjacent in X. The elements of $S = S(E_n)$ are distinct as group elements, and $m(s,t)$ is the order of st. Denote by $\mathcal{H}_q = \mathcal{H}_q(E_n)$ the Hecke algebra associated to W. This is a $\mathbb{Z}[q, q^{-1}]$-algebra with a basis consisting of (invertible) elements T_w, with w ranging over W, satisfying

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ qT_{sw} + (q-1)T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

where ℓ is the length function on the Coxeter group W, $w \in W$, and $s \in S$. If $n > 8$, the group W is infinite and $\mathcal{H}_q$ has infinite rank as an $\mathcal{A}$-algebra.

For the applications we have in mind, it is convenient to extend the scalars of $\mathcal{H}_q$ to produce an $\mathcal{A}$-algebra $\mathcal{H}$, where $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ and $v^2 = q$, and to define a scaled version of the T-basis, $\{\widetilde{T}_w : w \in W\}$, where $\widetilde{T}_w := v^{-\ell(w)} T_w$. We will write $\mathcal{A}^+$ and $\mathcal{A}^-$ for $\mathbb{Z}[v]$ and $\mathbb{Z}[v^{-1}]$, respectively.

A product $w_1 w_2 \cdots w_n$ of elements $w_i \in W$ is called *reduced* if $\ell(w_1 w_2 \cdots w_n) = \sum_i \ell(w_i)$. We reserve the terminology *reduced expression* for reduced products $w_1 w_2 \cdots w_n$ in which every $w_i \in S$. We write

$$\mathcal{L}(w) = \{s \in S : \ell(sw) < \ell(w)\}$$

and

$$\mathcal{R}(w) = \{s \in S : \ell(ws) < \ell(w)\}.$$

The set $\mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) is called the *left* (respectively, *right*) *descent set* of w.

Call an element $w \in W$ *complex* if it can be written as a reduced product $x_1 w_{ss'} x_2$, where $x_1, x_2 \in W$ and $w_{ss'}$ is the longest element of some rank 2 parabolic subgroup $\langle s, s' \rangle$ such that s and s' correspond to adjacent vertices in the Coxeter graph E_n. Denote by $W_c(E_n)$ the set of all elements of W that are not complex. The elements of $W_c = W_c(E_n)$ are the *fully commutative* elements of [**24**]; they are characterized by the property that any two of their reduced expressions may be obtained from each other by repeated commutation of adjacent generators.

Let $J(E_n)$ be the two-sided ideal of $\mathcal{H}$ generated by the elements

$$T_1 + T_s + T_t + T_{st} + T_{ts} + T_{sts},$$

where (s, t) runs over all pairs of elements of S for which $m(s, t) = 3$. Following Graham [**7**, Definition 6.1], we define the *generalized Temperley–Lieb algebra* $TL(E_n)$ to be the quotient $\mathcal{A}$-algebra $\mathcal{H}(E_n)/J(E_n)$. We denote the corresponding epimorphism of algebras by $\theta : \mathcal{H}(E_n) \longrightarrow TL(E_n)$. Let t_w (respectively, $\widetilde{t}_w$) denote the image in $TL(E_n)$ of the basis element T_w (respectively, $\widetilde{T}_w$) of $\mathcal{H}$. If $s \in S$, we define $b_s \in TL(E_n)$ by $b_s = v^{-1}1 + \widetilde{t}_s$.

A more convenient description of $TL(E_n)$ for the purposes of this paper is by generators and relations (as in [**3**, §2.2]). Since the Laurent polynomial $v + v^{-1}$ occurs frequently, we denote it by δ.

Proposition 3.1. *As a unital $\mathcal{A}$-algebra, $TL(E_n)$ is given by generators $\{b_s : s \in S\}$ and relations*

$$b_s^2 = \delta b_s,$$

$$b_s b_t = b_t b_s \quad \textit{if } m(s,t) = 2,$$

$$b_s b_t b_s = b_s \quad \textit{if } m(s,t) = 3.$$

□

The following basis theorem will be used freely in the sequel.

Theorem 3.2 [**3**, **7**].

(i) *The set $\{\widetilde{t}_w : w \in W_c\}$ is a free $\mathcal{A}$-basis for $TL(E_n)$.*

(ii) *If $w \in W_c$ and $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ is reduced, then the element*

$$b_w = b_{s_{i_1}} b_{s_{i_2}} \cdots b_{s_{i_r}}$$

is a well-defined element of $TL(E_n)$.

(iii) *The set $\{b_w : w \in W_c\}$ is a free $\mathcal{A}$-basis for $TL(E_n)$.*

Proof. Part (i) is due to Graham [**7**, Theorem 6.2]. Parts (ii) and (iii) are stated by Fan in [**3**, §2.2], and more details may be found in [**13**, Proposition 2.4]. □

Definition 3.3 [**3**, §2.3]**.** Let $P = P(n)$ denote the set of subsets of the Coxeter graph E_n that consist of non-adjacent vertices. We allow P to include the empty set,

$\emptyset$. For any $A \in P$, let $i(A)$ be the product of the elements of $S(E_n)$ corresponding to the vertices in A (with $i(\emptyset) = 1$); note that the order of the product is immaterial since the vertices in A correspond to commuting generators. Let $A, B \in P$. We say that A and B are neighbours if and only if $1 + \#(A \cap B) = \#A = \#B$, and the two vertices in $(A \cup B) \backslash (A \cap B)$ are adjacent in E_n. Define an equivalence relation on P by taking the reflexive and transitive closure of the relation $A \sim B$ if A and B are neighbours. Let $\bar{P}$ denote the set $P/\sim$.

Example 3.4. In type E_7, let $A = \{0, 2, 4, 6\}$ and $B = \{0, 1, 4, 6\}$. In this case, $i(A) = b_0 b_2 b_4 b_6$ and $i(B) = b_0 b_1 b_4 b_6$, A and B are neighbours, and the equivalence class of A is precisely $\{A, B\}$.

Definition 3.5 [**3**, §6.3]**.** Let $n \geq 6$.

If n is odd, we define $P' = P'(n)$ to be the subset of $P(n)$ consisting of the sets

$$\left\{ \{(n-1) - 2j : 0 \leq j \leq N\} : 0 \leq N \leq \frac{n-1}{2} \right\},$$

together with the set

$$\{n-1, n-3, n-5, \dots, 4\} \cup \{0\}$$

and the empty set.

If n is even, we define $P' = P'(n)$ be the subset of $P(n)$ consisting of the sets

$$\left\{ \{(n-1) - 2j : 0 \leq j \leq N\} : 0 \leq N \leq \frac{n-2}{2} \right\},$$

together with the empty set.

Example 3.6. In type E_6, we have

$$P' = \{\{5\}, \{5, 3\}, \{5, 3, 1\}, \emptyset\}.$$

In type E_7, we have

$$P' = \{\{6\}, \{6, 4\}, \{6, 4, 2\}, \{6, 4, 2, 0\}, \{6, 4, 0\}, \emptyset\}.$$

The importance of the set P' comes from the following

Proposition 3.7 (Fan, [**3**, Lemma 8.1.2]**).** *The set P' constitutes a complete set of equivalence class representatives for P with respect to $\sim$.* $\square$

4. Cells and the a-function

In §4, we recall the definitions of the **a**-function and cells arising from the monomial basis. Most of this material comes from the papers [**3**] and [**10**], or is implicit in them.

Definition 4.1 [**3**, Definition 2.3.1]**.** The **a**-*function* $\mathbf{a} : W_c \longrightarrow \mathbb{Z}^{\geq 0}$ is defined by

$$\mathbf{a}(w) := \max_{A \in P}\{\#A : w = xi(A)y \text{ is reduced}\}$$

for $w \in W_c$.

Proposition 4.2. *Let $w \in W_c$ and let $f \in \mathcal{A}$. Define the degree,* $\deg f$*, of f to be the largest integer n such that v^n occurs with nonzero coefficient in f, with the convention that* $\deg 0 = -\infty$*. Denote the structure constants with respect to the monomial basis by $g_{x,y,z} \in \mathcal{A}$, namely*

$$b_x b_y = \sum_{z \in W_c} g_{x,y,z} b_z.$$

(i) *The structure constant $g_{x,y,z}$ is either zero or a nonnegative power of δ, and, given x and y, we have $g_{x,y,z} \neq 0$ for a unique z.*
(ii) *If $s \in S$ and $g_{s,y,z} \notin \mathbb{Z}$, then $g_{s,y,z} = \delta$, $\ell(sy) < \ell(y)$ and $y = z$. Similarly, if $g_{x,s,z} \notin \mathbb{Z}$, then $g_{x,s,z} = \delta$, $\ell(xs) < \ell(x)$ and $x = z$.*
(iii) *We have* $\mathbf{a}(w) = \max_{x,y \in W_c} \deg g_{x,y,w}$.
(iv) *We have* $\mathbf{a}(w) = \max_{x,y \in W_c} \deg g_{w,x,y}$.

Proof. Parts (i) and (ii) are well known and follow easily from [**3**, Proposition 5.4.1].

Part (iii) is proved in [**10**, Proposition 4.2.3] using the results of [**3**].

The proof of [**3**, Theorem 5.5.1] shows that

$$\deg g_{w,x,y} \leq \min(\mathbf{a}(w), \mathbf{a}(x)),$$

which means that

$$\max_{x,y \in W_c} \deg g_{w,x,y} \leq \mathbf{a}(w).$$

Conversely, [**3**, Lemma 5.2.6] shows that

$$b_w b_{w^{-1}} = (v + v^{-1})^{\mathbf{a}(w)} b_d$$

for some $d \in W_c$, so taking $x = w^{-1}$ and $y = d$, we find that

$$\max_{x,y \in W_c} \deg g_{w,x,y} \geq \mathbf{a}(w),$$

which completes the proof of (iv). □

Definition 4.3 [**3**, Definition 4.1]**.**

For any $w, w' \in W_c$, we say that $w' \leq_L w$ if there exists b_x such that $g_{x,w,w'} \neq 0$, where g is as in Proposition 4.2.

For any $w, w' \in W_c$, we say that $w' \leq_R w$ if there exists b_x such that $g_{w,x,w'} \neq 0$.

For any $w, w' \in W_c$, we say that $w' \leq_{LR} w$ if there exist b_x and b_y such that $b_x b_w b_y = c b_{w'}$ for some $c \neq 0$.

We write $w \sim_L w'$ to mean that both $w' \leq_L w$ and $w \leq_L w'$. Similarly, we define $w \sim_R w'$ and $w \sim_{LR} w'$.

The relation $\sim_L$ (respectively, $\sim_R$, $\sim_{LR}$) is an equivalence relation, and the corresponding equivalence classes of W_c are called the *left* (respectively, *right*, *two-sided*) cells.

It is clear from the definitions and the fact that the identity element is a monomial basis element that two-sided cells are unions of left cells, and also unions of right cells.

Proposition 4.4.

(i) *Let $w \in W_c$. If we have $w = xi(A)y$ reduced for some A such that $\#A = \mathbf{a}(w)$, then $i(A) \sim_{LR} w$ and $w \sim_R xi(A)$.*

(ii) *The* $\mathbf{a}$*-function is constant on left, right, and two-sided cells.*

(iii) *If $w, w' \in W_c$ are such that $w' \leq_R w$ and $w' \not\sim_R w$, then $\mathbf{a}(w') > \mathbf{a}(w)$. An analogous statement holds for left cells and two-sided cells.*

(iv) *The right cell containing $i(A)$ is precisely the set*

$$\{w \in W_c : w = i(A)x \text{ reduced}, \ \mathbf{a}(w) = \#A\}.$$

(v) *A left cell and a right cell contained in the same two-sided cell intersect in a unique element.*

Proof. Statement (i) is proved during the argument establishing [**3**, Theorem 4.5.1.].

The fact that the **a**-function is constant on two-sided cells is implicit in the proof of [**3**, Theorem 4.5.1]. Since two-sided cells are unions of left (or right) cells, part (ii) follows.

Suppose now that $w, w' \in W_c$ are such that $w' \leq_R w$ and $w' \not\sim_R w$. An inductive argument using the definition of $\leq_R$ reduces the problem to the case where there is some $s \in S$ such that $b_w b_s$ is a multiple of $b_{w'}$, so let us assume that this is the situation. By [**3**, Corollary 4.2.2], the assumption that $w' \leq_R w$ implies that $\mathbf{a}(w') \geq \mathbf{a}(w)$. The statement follows unless $\mathbf{a}(w') = \mathbf{a}(w)$, so suppose we are in this case.

Let us write $w = xi(A)y$ as in statement (i). Now [**3**, Lemma 4.2.5], applied to the element $xi(A)$ and the sequence of generators corresponding to ys, shows that we have $w' = xi(A)y'$ reduced. By part (i), we find that $w' \sim_R xi(A)$, and thus that $w' \sim_R w$, a contradiction.

The statement for left cells follows by a symmetrical argument, and the statement for two-sided cells follows from the previous claims and the fact that if $w' \leq_{LR} w$, then there is a chain

$$w' = w_1, w_2, w_3, \dots, w_k = w$$

where, for each $1 \leq i < k$, we have either $w_i \leq_L w_{i+1}$ or $w_i \leq_R w_{i+1}$. This completes the proof of (iii).

Part (iv) is [**3**, Proposition 4.4.3].

Part (v) is well known and follows from the proof of [**3**, Theorem 6.1.2]. □

Remark 4.5. For finite and affine Weyl groups, the **a**-function defined above is known by [**23**, Theorem 3.1] to be the restriction of Lusztig's more general **a**-function [**21**] restricted to the subset W_c.

Although it is not true that each of the monomial cells studied above is a cell in the sense of Kazhdan–Lusztig [**19**], it can be shown fairly easily that each left (respectively, right, two-sided) monomial cell is a subset of some left (respectively, right, two-sided) Kazhdan–Lusztig cell.

5. Traces on the algebras $TL(E_n)$

In §5, we will extend scalars and deal with a K-form of $TL(E_n)$, where K is a field containing $\mathcal{A}$ and a square root of δ. (The existence of $\sqrt{\delta}$ is needed for compatibility with [**3**], but can ultimately be removed; see Remark 6.4.) We write $TL_K(E_n) := K \otimes_{\mathcal{A}} TL(E_n)$. We aim to classify the traces, $\tau : TL_K(E_n) \longrightarrow K$, that is, linear functions τ with the property that $\tau(ab) = \tau(ba)$ for all $a, b \in TL_K(E_n)$. It is clear that the set of all traces on $TL_K(E_n)$ is a K-vector space (dependent in principle on K and δ). The main result of §5 is that there is a basis for this vector space in natural bijection with the set P' of §3.

The next result shows how τ naturally induces a function $P/\sim\ \longrightarrow K$.

Lemma 5.1. *Maintain the notation of Definition 3.3. Suppose $A, B \in P$ are such that $A \sim B$, and let $\tau : TL_K(E_n) \longrightarrow K$ be a trace. Then $\tau(i(A)) = \tau(i(B))$.*

Proof. The proof immediately reduces to the case where A and B are neighbours. Let s (respectively, t) be the element of S corresponding to the unique element of $A \backslash B$ (respectively, $B \backslash A$). It is immediate from the definitions that $i(A) = b_s i(A \cap B) = i(A \cap B) b_s$ and $i(B) = b_t i(A \cap B) = i(A \cap B) b_t$. We then have

$$\begin{aligned}
\tau(i(A)) &= \tau(b_s i(A \cap B)) = \tau(b_s b_t b_s i(A \cap B)) \\
&= \tau(b_t b_s i(A \cap B) b_s) = \tau(b_t b_s b_s i(A \cap B)) \\
&= \delta \tau(b_t b_s i(A \cap B)) \\
&= \tau(b_t b_t b_s i(A \cap B)) = \tau(b_t b_s i(A \cap B) b_t) \\
&= \tau(b_t b_s b_t i(A \cap B)) = \tau(b_t i(A \cap B)) \\
&= \tau(i(B)),
\end{aligned}$$

as required. □

Lemma 5.2. *Any trace $\tau : TL_K(E_n) \longrightarrow K$ is determined by its values on the set*

$$\{i(A) : A \in P\}.$$

Proof. Suppose the values of $\tau(i(A))$ are known for each $A \in P$. We will show how to compute the value of $\tau(b_w)$, where $w \in W_c$ is arbitrary.

Let us write $w = xi(A)y$ reduced as in Proposition 4.4 (i). Using a reverse induction, we will assume that the values of $\tau(b_{w'})$ for $\mathbf{a}(w') > \mathbf{a}(w) = \#A$, if such w' exist, have been determined. By the defining relations of $TL(E_n)$, we have $b_{i(A)}b_{i(A)} = \delta^{\#A}b_{i(A)}$, and so we have

$$\begin{aligned}\tau(b_w) &= \tau(b_x b_{i(A)} b_y)\\ &= \delta^{-\#A}\tau(b_x b_{i(A)} b_{i(A)} b_y)\\ &= \delta^{-\#A}\tau(b_{i(A)} b_y b_x b_{i(A)}).\end{aligned}$$

Now $i(A)y$ and $xi(A)$ lie in W_c because w does, and Proposition 4.4 (i) and (ii) shows that $\mathbf{a}(w) = \mathbf{a}(xi(A))$. By Proposition 4.2 (i), we have

$$b_{i(A)} b_y b_x b_{i(A)} = b_{i(A)y} b_{xi(A)} = \delta^c b_z$$

for some $z \in W_c$, and it is clear from the definitions that $z \leq_L xi(A)$. By Proposition 4.4 (ii) and (iii), we see that

$$\mathbf{a}(z) \geq \mathbf{a}(xi(A)) = \mathbf{a}(w) = \#A.$$

If $\mathbf{a}(z) > \#A$ then our inductive hypothesis determines the value of $\tau(\delta^c b_z)$, which in turn determines the value of $\tau(b_w)$. We may therefore assume that $\mathbf{a}(z) = \#A$. To complete the proof, it is enough to show that $z = i(A)$, because the value of $\tau(b_z)$ will then have been determined by our assumptions.

Let $s \in A$. Since $b_s b_{i(A)} = \delta b_{i(A)}$ by the defining relations, the definition of b_z shows that $b_s b_z = \delta b_z$. By Proposition 4.2 (ii), this means that $\ell(sz) < \ell(z)$, and it follows that $A \subseteq \mathcal{L}(z)$. Because A is a set of commuting generators, standard properties of Coxeter groups show that we can write $z = i(A)z'$ reduced. Applying Proposition 4.4 (iv) to the fact that $\mathbf{a}(z) = \#A$ shows that $z \sim_R i(A)$. A symmetrical argument then shows that we have $z \sim_L i(A)$. By Proposition 4.4 (v), this can only happen if $z = i(A)$. $\square$

Theorem 5.3. *For each $\bar{A} \in \bar{P}$ (as in Definition 3.3), there is a unique trace $\tau_{\bar{A}} : TL_K(E_n) \longrightarrow K$ such that for each $B \in P$ we have*

$$\tau_{\bar{A}}(i(B)) = \begin{cases} 1 & \text{if } B \in \bar{A}, \\ 0 & \text{otherwise.} \end{cases}.$$

The set

$$\{\tau_{\bar{A}} : \bar{A} \in \bar{P}\}$$

is a K-basis for the set of all traces $\tau : TL_K(E_n) \longrightarrow K$.

Proof. It is clear from the definition of trace that the traces from $TL_K(E_n)$ to K form a K-vector space. Lemmas 5.1 and 5.2 show that this space has dimension at most the size of $\bar{P}$.

Fan [**3**, Theorem 5.6.1] shows that $TL_K(E_n)$ is semisimple and that is then a direct sum of $|\bar{P}|$ matrix rings. This proves that the dimension of the space of traces is at least the size of $\bar{P}$, and thus that the space has the claimed dimension.

A dimension count, together with another application of lemmas 5.1 and 5.2, then shows that there are unique traces $\tau_{\bar{A}}$ with the properties claimed, and that they form a basis. $\square$

We now come to the central definition of the paper.

Definition 5.4. The trace $\mathrm{tr} : TL_K(E_n) \longrightarrow K$ is defined by

$$\mathrm{tr} = \sum_{\bar{A} \in \bar{P}} \delta^{-\#A} \tau_{\bar{A}},$$

where $\tau_{\bar{A}}$ is as in Theorem 5.3.

Corollary 5.5. *Any trace* $\tau : TL_K(E_n) \longrightarrow K$ *satisfies* $\tau(b_w) = \tau(b_{w^{-1}})$ *for all* $w \in W_c$.

Proof. It follows from Proposition 3.1 that there is a unique $\mathcal{A}$-linear antiautomorphism $* : TL(E_n) \longrightarrow TL(E_n)$ fixing the generators b_s. We may extend this to a K-linear antiautomorphism $* : TL_K(E_n) \longrightarrow TL_K(E_n)$. If $a \in TL_K(E_n)$, let us write a^* for $*(a)$. Note that if $A \in P$, then $i(A)$ is invariant under $*$, because $i(A)$ is a product of commuting generators b_s.

Given a trace $\tau : TL_K(E_n) \longrightarrow K$, the K-linear map $\tau' : TL_K(E_n) \longrightarrow K$ defined by $\tau'(a) = \tau(a^*)$ is also a trace. Since τ and τ' agree on all elements $i(A)$ for $A \in P$, Lemma 5.2 shows that $\tau = \tau'$, and the assertion follows. □

Remark 5.6. The trace tr will turn out to induce the Markov trace of the title. Note that the definition makes sense because $\bar{A}, \bar{B} \in \bar{P}$ implies $\#A = \#B$.

Traces on Hecke algebras of finite Coxeter groups are known have a property similar to that given in Corollary 5.5; see [**5**, Corollary 8.2.6] for more details.

6. Cellular structure and the $\mathbf{a}$-funtion

In §6, we explain how the trace tr is particularly compatible with the structure of $TL(E_n)$ as a cellular algebra, in the sense of [**8**]. We will not recall the complete definition of a cellular algebra here, but we summarize below the properties of the cellular structure that are important for our purposes.

Definition 6.1. Let Λ be the set of two-sided cells for $TL(E_n)$, equipped with the partial order induced by $\leq_{LR}$. For each $\lambda \in \Lambda$, let $M(\lambda)$ be an indexing set for the left cells contained in λ; note that the inversion map on the Coxeter group W induces a bijection between the set of left cells in λ and the set of right cells in λ (see the remarks at the end of [**3**, §4.4]).

Proposition 6.2. *Maintain the above notation.*

(i) *Let* $T, U \in M(\lambda)$ *for some fixed* $\lambda \in \Lambda$. *Then* $T \cap U$ *contains a unique element,* w, *and we define* $C_{T,U} = b_w$.

(ii) *The $\mathcal{A}$-algebra anti-automorphism $* : TL(E_n) \longrightarrow TL(E_n)$ defined by $*(b_w) = b_{w^{-1}}$ satisfies $*(C_{T,U}) = C_{U,T}$. In particular, we have $w^2 = 1$ if and only if $b_w = C_{T,T}$ for some T.*

(iii) *Suppose that $C_{P,Q}$ and $C_{R,S}$ are arbitrary monomial basis elements, and define $C_{T,U}$ by the condition*

$$C_{P,Q}C_{R,S} = \delta^a C_{T,U}$$

(which makes sense by Proposition 4.2 (i)). If P, Q, R, S, T and U all belong to the same two-sided cell, then $P = T$ and $S = U$; if, furthermore, we have $Q = R$, then $a = \mathbf{a}(C_{T,U})$. If it is not the case that $P = T$, $S = U$ and $Q = R$, then we have $a < \mathbf{a}(C_{T,U})$.

Proof. Parts (i) and (ii), which are originally due to Graham [**7**], are proved in [**10**, Proposition 4.2.1]. Part (iii) is proved in [**10**, propositions 4.2.1 and 4.2.3] using the results of [**3**]. □

Proposition 6.3. *For all $w \in W_c$, we have $\mathrm{tr}(b_w) = \delta^a$, where $a = -\mathbf{a}(w)$ if $w^2 = 1$, and $a < -\mathbf{a}(w)$ otherwise.*

Proof. Let λ be the two-sided cell containing w. We will prove the statement by induction on the partial order on two-sided cells given in Definition 6.1. Writing $w = C_{T,U}$ for $T, U \in M(\lambda)$, as in Proposition 6.2 (i), and applying Proposition 6.2 (ii), we see that the condition $w^2 = 1$ is equivalent to $T = U$.

By Proposition 4.4, there exists a product of $\mathbf{a}(w)$ commuting generators, $i(A)$, in λ. Define $V \in M(\lambda)$ by the condition $C_{V,V} = b_{i(A)}$. Since tr is a trace, Proposition 6.2 (iii) shows that

$$\mathrm{tr}(C_{T,U}) = \delta^{-\mathbf{a}(w)}\mathrm{tr}(C_{T,V}C_{V,U}) = \delta^{-\mathbf{a}(w)}\mathrm{tr}(C_{V,U}C_{T,V}).$$

By Proposition 4.2 (i), we have

$$C_{V,U}C_{T,V} = \delta^b C_{X,Y}$$

for some $b \geq 0$ and some basis element $C_{X,Y}$. There are now two cases to consider.

The first possibility is that $C_{X,Y}$ comes from the two-sided cell λ. (If $T = U$, this case must occur by Proposition 6.2 (iii).) In this case, we have $X = Y = V$, and thus $C_{X,Y} = b_{i(A)}$. Proposition 6.2 (iii) then shows that $b = \mathbf{a}(w)$ if $T = U$, and $b < \mathbf{a}(w)$ otherwise. Since we have $\mathrm{tr}(b_{i(A)}) = \delta^{-\mathbf{a}(w)}$ by definition of tr, we have $\mathrm{tr}(C_{T,U}) = \delta^{-\mathbf{a}(w)+b-\mathbf{a}(w)}$, and the result follows.

The other possibility is that $C_{X,Y}$ comes from a two-sided cell λ' with $\lambda' < \lambda$, and $T \neq U$. In this case, Proposition 4.4 (iii) shows that $\mathbf{a}(C_{X,Y}) > \mathbf{a}(w)$. By the inductive hypothesis, we know that $\mathrm{tr}(C_{X,Y}) = \delta^{a'}$, where $a' \leq -\mathbf{a}(C_{X,Y}) < -\mathbf{a}(w)$. This means that $\mathrm{tr}(C_{T,U}) = \delta^{-\mathbf{a}(w)+b+a'}$. By propositions 4.2 (iii) and 4.4 (ii), we have $b \leq \mathbf{a}(w)$, and thus $\mathrm{tr}(C_{T,U}) = \delta^a$ for $a < -\mathbf{a}(w)$, as required. □

Remark 6.4. The above proposition shows that we do not actually need $\sqrt{\delta} \in k$ to define tr. From now on, we need only assume that K is a field containing $\mathcal{A}$.

Proposition 6.5. *If K is the field of fractions of the power series ring $\mathbb{Z}[[v^{-1}]]$, then* tr *is a nondegenerate trace on $TL_K(E_n)$, and*

$$\mathrm{tr}(C_{P,Q}C_{R,S}) - \delta_{QR}\delta_{PS} \in v^{-1}\mathbb{Q}[[v^{-1}]],$$

where δ_{QR} and δ_{PS} are the Kronecker delta.

Proof. An element x of K is uniquely representable in the form

$$x = \sum_{i=-\infty}^{N} \lambda_i v^i,$$

where $\lambda_i \in \mathbb{Q}$ for all i. If $x \neq 0$, we define $\deg x$ to be the largest integer j such that $\lambda_j \neq 0$. If $x, y \neq 0$ then $\deg(xy) = \deg x + \deg y$, so the facts that $\deg \delta = 1$ and $\deg 1 = 0$ imply that $\deg \delta^a = -a$.

The second assertion follows from the fact that $\deg \delta^a = -a$ combined with Proposition 4.4 (ii), Proposition 6.2 (iii) and Proposition 6.3.

We will now show that for any nonzero $a \in TL_K(E_n)$, we have $\mathrm{tr}(aa^*) \neq 0$, from which the assertion follows. We have

$$a = \sum_{w \in W_c} \lambda_w b_w,$$

and by clearing denominators (thus multiplying a by a nonzero scalar), we may assume that we have $\lambda_w \in \mathcal{A}$ for all $w \in W_c$. Choose w' with $\lambda_{w'} \neq 0$ and $N(w') := \deg \lambda_{w'}$ maximal, and let $c_{w'}$ be the (integer) coefficient of $v^{N(w')}$ in $\lambda_{w'}$. Setting $a_{w'} = v^{-N(w')}\lambda_{w'} b_{w'}$, we then have

$$\mathrm{tr}(a_{w'} a_{w'}^*) = c^2 \mod v^{-1}\mathbb{Q}[[v^{-1}]].$$

If $\lambda_{w''} \neq 0$ but $\deg \lambda_{w''}$ is not maximal, we may again define $a_{w''} = v^{-N(w'')}\lambda_{w''} b_{w''}$, but then

$$\mathrm{tr}(a_{w''} a_{w''}^*) \in v^{-1}\mathbb{Q}[[v^{-1}]].$$

Since the integers c^2 are strictly positive, it follows that

$$\mathrm{tr}((v^{-N(w')}a)(v^{-N(w')}a)^*) \notin v^{-1}\mathbb{Q}[[v^{-1}]],$$

which completes the proof. □

Proposition 6.6. *Let K be the field of fractions of the power series ring $\mathbb{Z}[[v^{-1}]]$, and let K' be the subfield of K consisting of the field of fractions of $\mathbb{Z}[[v^{-2}]]$.*

(i) *The field $TL_K(E_n)$ has a unique structure as a $\mathbb{Z}_2$-graded algebra over K' in which v^n has degree $n \mod 2$ and K' is precisely the set of elements of degree $0 \mod 2$.*

(ii) *The algebra $TL_K(E_n)$ has a unique structure as a $\mathbb{Z}_2$-graded algebra over K' in which v^n has degree $n \mod 2$ and the generators b_s have degree $1 \mod 2$. We denote the even subalgebra consisting of elements of degree $0 \mod 2$ by $TL_{K'}(E_n)$.*

(iii) *Let $\tau : TL_K(E_n) \longrightarrow K$ be any trace. Then there are unique K'-linear maps $\tau_{(0)}, \tau_{(1)} : TL_{K'}(E_n) \longrightarrow K'$ such that $\tau_{(0)} + v\tau_{(1)}$ is the restriction of τ to $TL_{K'}(E_n)$, and furthermore, $\tau_{(0)}$ and $\tau_{(1)}$ are themselves traces.*

Proof. Recall from the proof of Proposition 6.5 that $K = \mathbb{Q}((v^{-1})) = \mathbb{Q}[v][[v^{-1}]]$, so that each element $x \in K$ has a unique expression of the form

$$\sum_{i=N}^{\infty} q_i v^i,$$

where $q_i \in \mathbb{Q}$ and $N \in \mathbb{Z}$ depends on x. Similar reasoning shows that the subfield K' of K then consists precisely of those elements for which $q_i = 0$ whenever i is odd. Part (i) is a consequence of this construction.

The assertion of (ii) is immediate from the observation that the defining relations of Proposition 3.1 respect the given grading.

Let $\pi : K \longrightarrow K'$ be the map

$$\pi\left(\sum_{i=N}^{\infty} q_i v^i\right) = \sum_{i=N}^{\infty} q_i' v^i,$$

where

$$q_i' = \begin{cases} q_i & \text{if } i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Our description of K' shows that π is a K'-linear map. Denoting the restriction of τ to $TL_{K'}(E_n)$ by τ', it follows that $\pi \circ \tau'$ is a trace on $TL_{K'}(E_n)$. Since $\tau_{(0)} = \pi \circ \tau'$, the maps $\tau_{(0)}$, $v\tau_{(1)} = \tau' - \tau_{(0)}$ and $\tau_{(1)}$ are also traces, completing the proof of (iii). $\square$

Note that any trace from $TL_{K'}(E_n)$ to K' extends uniquely to a trace from $TL_K(E_n)$ to K by tensoring by $K \otimes_{K'} -$.

Lemma 6.7. *The trace* $\mathrm{tr} : TL_K(E_n) \longrightarrow K$ *arises from a trace*

$$\mathrm{tr}' : TL_{K'}(E_n) \longrightarrow K'$$

by extension of scalars.

Proof. We use the notation of §5. Note that if $A \in P$, then $i(A)$ is an element of $TL_K(E_n)$ of degree $\#A \mod 2$. We also have $\mathrm{tr}(i(A)) = \delta^{\#A}$, which is an element of K of degree $\#A \mod 2$.

Recall that $TL_{K'}(E_n)$ is a K'-subalgebra of $TL_K(E_n)$ and note that if y, z are homogeneous elements of $TL_K(E_n)$, then yz and zy have the same degree. The argument of Lemma 5.2 now shows that if x is an element of $TL_{K'}(E_n)$, we have a relation

$$\mathrm{tr}(x) = \mathrm{tr}\left(\sum_{\bar{A} \in \bar{P}} \lambda_{\bar{A}} i(A)\right),$$

where for each $\bar{A} \in \bar{P}$, we have $\lambda_{\bar{A}}(i(A)) \in TL_{K'}(E_n)$. By the first paragraph of the proof, $\lambda_{\bar{A}}$ must be homogeneous of degree $\#A \mod 2$, and $\mathrm{tr}(\lambda_{\bar{A}} i(A)) \in K'$. The proof is completed by the observation that any $x \in TL_K(E_n)$ is uniquely expressible as $x_{(0)} + vx_{(1)}$ for $x_{(0)}, x_{(1)} \in TL_{K'}(E_n)$ (compare with Proposition 6.6 (iii)). $\square$

Corollary 6.8. *If $w \in W_c$ and $\mathrm{tr}(b_w) = \delta^a$ as in Proposition 6.3, then $a \equiv \lambda(w) \mod 2$.*

Proof. By Lemma 6.7, we have $\deg \mathrm{tr}(b_w) = \ell(w) \mod 2$, so the assertion follows from the fact that $\deg \delta = 1$. $\square$

§7. tom Dieck's diagram calculus

In [**2**], tom Dieck introduced a diagram calculus for the algebras $TL(E_n)$. To give a rigorous definition of tom Dieck's diagram calculus, as we do here, we first need to recall the graphical definition of the Temperley–Lieb algebra. We start by recalling Jones' formalism of k-boxes [**18**], following the approach of Martin and the author in [**15**]. For further details and references, the reader is referred to [**11**, §2].

Definition 7.1. Let k be a nonnegative integer. The *standard k-box*, $\mathcal{B}_k$, is the set $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq k+1,\ 0 \leq y \leq 1\}$, together with the $2k$ marked points

$$\begin{aligned} &1 = (1,1),\ 2 = (2,1),\ 3 = (3,1),\ \ldots,\ k = (k,1), \\ &k+1 = (k,0),\ k+2 = (k-1,0),\ \ldots,\ 2k = (1,0). \end{aligned}$$

Definition 7.2. Let X and Y be embeddings of some topological spaces (such as lines) into the standard k-box. Multiplication of such embeddings to obtain a new embedding in the standard k-box shall, where appropriate, be defined via the following procedure on k-boxes. The product XY is the embedding obtained by placing X on top of Y (that is, X is first shifted in the plane by $(0,1)$ relative to Y, so that marked point $(i,0)$ in X coincides with $(i,1)$ in Y), rescaling vertically by a scalar factor of $1/2$ and applying the appropriate translation to recover a standard k-box.

Definition 7.3. Let k be a nonnegative integer. Consider the set of smooth embeddings of a single curve (which we usually call an "edge") in the standard k-box, such that the curve is either closed (isotopic to a circle) or its endpoints coincide with two marked points of the box, with the curve meeting the boundary of the box only at such points, and there transversely.

By a smooth diffeomorphism of this curve we mean a smooth diffeomorphism of the copy of $\mathbb{R}^2$ in which it is embedded, that fixes the boundary, and in particular the marked points, of the k-box, and takes the curve to another such smooth embedding. (Thus, the orbit of smooth diffeomorphisms of one embedding contains all embeddings with the same endpoints.)

A concrete Brauer diagram is a set of such embedded curves with the property that every marked point coincides with an endpoint of precisely one curve. (In examples we can represent this set by drawing all the curves on one copy of the k-box. Examples can always be chosen in which no ambiguity arises thereby.)

Two such concrete diagrams are said to be equivalent if one may be taken into the other by applying smooth diffeomorphisms to the individual curve embeddings within it.

There is an obvious map from the set of concrete diagrams to the set of pair partitions of the $2k$ marked points. It will be evident that the image under this map is an invariant of concrete diagram equivalence.

The set $B_k(\emptyset)$ is the set of equivalence classes of concrete diagrams. Such a class (or any representative) is called a Brauer diagram.

Let D_1, D_2 be concrete diagrams. Since the k-box multiplication defined above internalises marked points in coincident pairs, corresponding curve endpoints in D_1D_2 may also be internalised seamlessly. Each chain of curves concatenated in this way may thus be put in natural correspondence with a single curve. Thus the multiplication gives rise to a closed associative binary operation on the set of concrete diagrams. It will be evident that this passes to a well defined multiplication on $B_k(\emptyset)$. Let R be a commutative ring with 1. The elements of $B_n(\emptyset)$ form the basis elements of an R-algebra $\mathcal{P}_n^B(\emptyset)$ with this multiplication.

A curve in a diagram that is not a closed loop is called *propagating* if its endpoints have different y-values, and *non-propagating* otherwise. (Some authors use the terms "through strings" and "arcs" respectively for curves of these types.)

Note that in a Brauer diagram drawn on a single copy of the k-box it is not generally possible to keep the embedded curves disjoint. Let $T_k(\emptyset) \subset B_k(\emptyset)$ denote the subset of diagrams having representative elements in which the curves are disjoint. Representatives of this kind are called Temperley–Lieb diagrams.

It will be evident that $\mathcal{P}_n^B(\emptyset)$ has a subalgebra with basis the subset $T_k(\emptyset)$. (That is to say, the disjointness property is preserved under multiplication.) We denote this subalgebra $\mathcal{P}_n(\emptyset)$

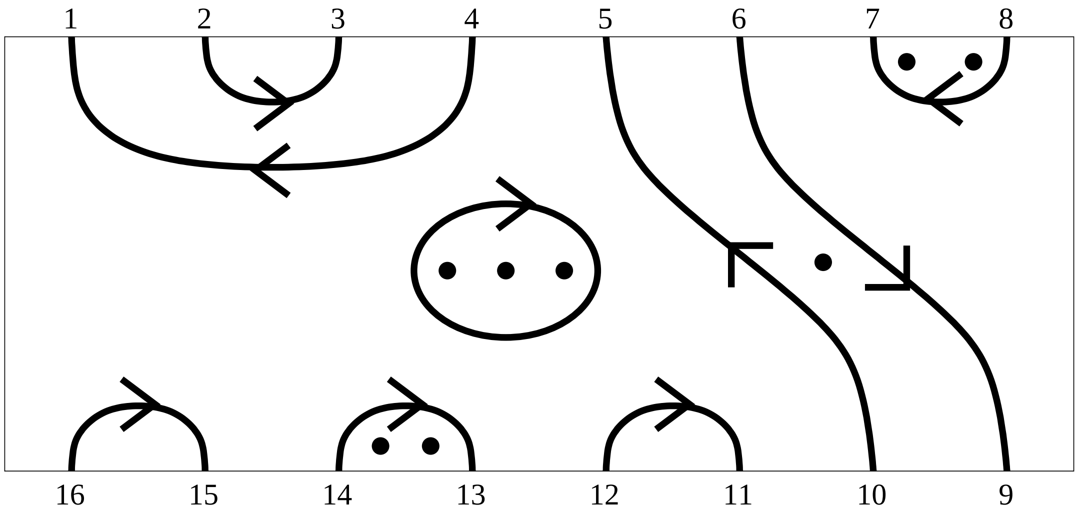
Because of the disjointness property there is, for each element of $T_k(\emptyset)$, a unique assignment of orientation to its curves that satisfies the following two conditions.

(i) A curve meeting the r-th marked point of the standard k-box, where r is odd, must exit the box at that point.

(ii) Each connected component of the complement of the union of the curves in the standard k-box may be oriented in such a way that the orientation of a curve coincides with the orientation induced as part of the boundary of the connected component.

Note that the orientations match up automatically in composition. If D_1 and D_2 are equivalent concrete Temperley–Lieb diagrams, the diffeomorphisms that give rise to the equivalence set up a bijection between the connected components of D_1 and those of D_2.

FIGURE 2. A pillar diagram corresponding to an element of $T_8(\emptyset)$



Definition 7.4. A *pillar diagram* consists of a pair (D, f), where $D \in T_k(\emptyset)$ is a Temperley–Lieb diagram and f is a function from the connected components of D to $\mathbb{Z}^{\geq 0}$, such that any component with anticlockwise orientation is mapped to zero.

On the diagram D, we indicate the values of f on the clockwise connected components either by writing in the appropriate integer, or by inserting k disjoint discs (the "pillars" of [**2**]).

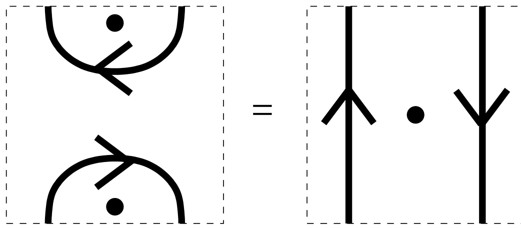
The set of pillar diagrams arising from the set $T_k(\emptyset)$ will be denoted $T_k(\bullet)$.

Example 7.5. Let $k = 8$. A pillar diagram corresponding to an element of $T_k(\bullet)$ is shown in Figure 2. Note that there are 10 connected components, precisely 7 of which inherit a clockwise orientation. The values of f on these 7 components are $3, 2, 2, 1, 0, 0, 0$.

We define an algebra $\mathcal{P}_n(\bullet)$, analogous to $\mathcal{P}_n(\emptyset)$, with the set $T_k(\bullet)$ as a basis. The multiplication is k-box multiplication with the added convention that function values on the connected components are additive. (This is natural if one represents the function values with pillars as in Figure 2.)

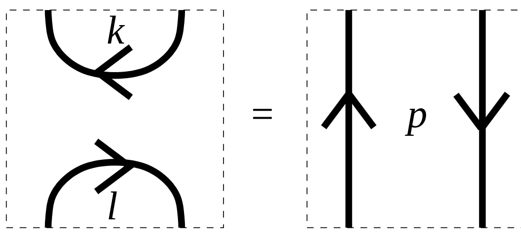
For our purposes, we need to apply an equivalence relation on the concrete diagrams of $T_k(\bullet)$. Locally, this is given by the relation shown in Figure 3.

FIGURE 3. A topological reduction rule



In the notation where clockwise regions are labelled by nonnegative integers, the relation of Figure 3 is that shown in Figure 4.

FIGURE 4. Alternative notation for the topological reduction



If the regions labelled k and l are connected to each other, Figure 3 shows that we have $k = l > 1$ and $p = k - 1$. On the other hand, if the regions labelled k and l are genuinely distinct, that is, the arcs shown on the left hand side of figure 3 are not sections of some longer arc, then we have $p = k + l - 1 \geq 1$. In the latter case, it is not possible for any regions labelled by the integer zero to be created or destroyed by the topological reduction. Note that the other partial regions shown in figures 2 and 3 have anticlockwise orientation, and as such they are labelled by the integer 0.

Definition 7.6. If L is a closed loop in a concrete diagram of $T_k(\bullet)$, we define $m(L)$ to be the integer label of the region immediately interior to L; in particular, we have $m(L) = 0$ if L has anticlockwise orientation.

Let R be a commutative ring with 1. The R-algebra $\mathcal{P}_n^E(\bullet)$ is the quotient of the R-algebra $\mathcal{P}_n(\bullet)$ obtained by applying the following three relations:

(i) for each closed loop L whose immediate interior is labelled 1 and whose immediate exterior is necessarily labelled 0, relabel the immediate interior of L by 0 and remove L;

(ii) for each closed loop L whose immediate interior is labelled 0 and whose immediate exterior is labelled k, relabel the immediate interior of L by k, remove L and multiply by δ;

(iii) for each region R labelled by $k \geq 2$ (whether or not R is a closed loop), decrease the label of R by 1 and multiply by δ.

A basis for $\mathcal{P}_n^E(\bullet)$ may be obtained by using the notion of "reduced" diagrams given in [**2**, §2] and Bergman's diamond lemma [**1**]. However, we do not pursue this because we do not need it for our purposes.

Definition 7.7. Suppose $n > 1$ and $1 \leq k < n$.

The diagram E_k^n of $\mathcal{P}_n^E(\bullet)$ is the one where each point i is connected by a propagating edge to point $2n+1-i$, unless $i \in \{k, k+1, 2n-k, 2n+1-k\}$. Points k and $k+1$ are connected by an edge, as are points $2n-k$ and $2n+1-k$. All regions are labelled by 0.

The diagram B_k^n of $\mathcal{P}_n^E(\bullet)$ is the one where each point i is connected by a propagating edge to point $2n+1-i$, and all regions are labelled by 0, except the rectangular region bounded by $k, k+1, 2n-k$ and $2n+1-k$, which is labelled by 1.

Proposition 7.8. *There is a unique homomorphism $\rho : TL(E_n) \longrightarrow \mathcal{P}_n^E(\bullet)$ of unital $\mathcal{A}$-algebras sending b_0 to B_3^n and b_s to E_s^n for $i \in \{1, 2, \ldots, n-1\}$, where the numbering of generators is as in §3.*

Proof. This is a routine (but important) exercise using the presentation of Proposition 3.1, and is essentially the same as the proof of [**2**, Theorem 2.5]. □

We shall see later that ρ is in fact a faithful representation. We will not determine the image of ρ, but this can be done by an inductive combinatorial argument similar to those in [**9**, §5].

§8. Existence and uniqueness of the Markov trace

There is a well-known embedding $\iota_n : TL(E_n) \longrightarrow TL(E_{n+1})$ sending b_s to b_s for each generator of $TL(E_n)$ (see [**3**, §6.3]). This means that the tower of algebras $TL(E_n)$, equipped with the generators b_s, fits into the framework of Markov traces defined in §2. We recall the definition in order to fix some notation.

Definition 8.1. Let K be a field containing $\mathcal{A}$. A *Markov trace* on $TL_K(E_\infty)$ with parameter $z \in K$ is a K-linear map $\tau : TL_K(E_\infty) \longrightarrow K$ satisfying the following conditions:

(i) $\tau(1) = 1$;

(ii) $\tau(hb_n) = z\tau(h)$ for $n \geq 6$ and $h \in TL_K(E_n)$;

(iii) $\tau(hh') = \tau(h'h)$ for all $n \geq 6$ and $h, h' \in TL_K(E_n)$.

Remark 8.2. Note that in condition (ii), b_n is the unique generator in $TL(E_{n+1})$ that does not lie in $TL(E_n)$. As mentioned in [**3**, §2.2], the algebras $TL(E_n)$ are quotients of the Hecke algebras of the Coxeter groups $W(E_n)$, and $b_s = q^{-1/2}(T_s + 1)$, where the T_s are the usual generators for the Hecke algebra as given in [**16**, §7]. This means that the Markov trace can also be regarded as a trace on a tower of Hecke algebras.

Proposition 8.3. *If τ is a Markov trace on $TL_K(E_\infty)$, then the parameter z must be equal to δ^{-1}, and τ is unique. Restricted to $TL(E_n)$, such a Markov trace must agree with the trace* tr.

Proof. Let $n \geq 6$. Part (ii) of Definition 8.1 shows that $\tau(b_{n-1}b_n) = z\tau(b_{n-1})$. On the other hand, the defining relations and part (iii) of the definition show that

$$\tau(b_{n-1}b_n) = \delta^{-1}\tau(b_{n-1}(b_{n-1}b_n)) = \delta^{-1}\tau(b_{n-1}b_nb_{n-1}) = \delta^{-1}\tau(b_{n-1}),$$

proving the assertion about the parameter.

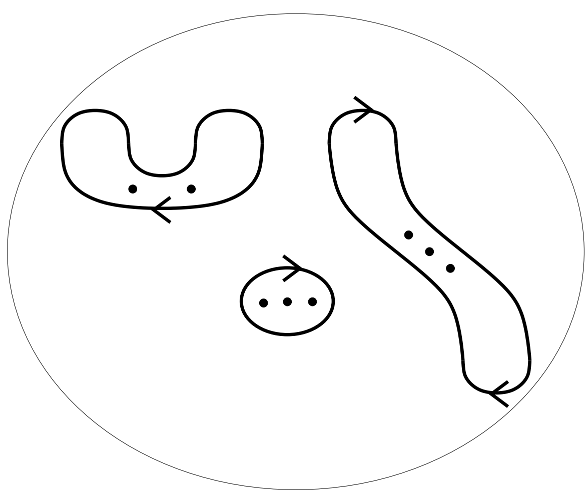
To prove the other assertions, it suffices to show that, regarding $TL_K(E_n)$ as a subalgebra of $TL_K(E_\infty)$, we have $\tau(i(A)) = \delta^{-\#A}$ for $A \in P = P(n)$. Choose such an A. It follows from Definition 3.3 that for sufficiently large $N \geq n$, and identifying A in the obvious way with an element of $P(N)$, we can find $B \in P(N)$ with $A \sim B$ and $B \cap \{b_0, b_1, \dots, b_5\} = \emptyset$. The first assertion together with repeated applications of part (ii) of Definition 8.1 (and one application of part (i)) now show that $\tau(i(B)) = \delta^{-\#B} = \delta^{-\#A}$, and Lemma 5.1 completes the proof. $\square$

To prove that the Markov trace on $TL_K(E_\infty)$ exists, we make use of the diagram calculus, as hinted in [**2**, §6].

Definition 8.4. Let k be a nonnegative integer. The *standard k-cone* is obtained from the standard k-box by identifying each pair of points $\{(x, 0), (x, 1)\}$ for each $0 \leq x \leq k+1$, and identifying all the points in the set $\{(k+1, y) : 0 \leq y \leq 1\}$. The standard k-cone is homeomorphic to a closed disc.

Let D be a diagram in $\mathcal{P}_k^E(\bullet)$. The *trace diagram*, $\overline{D}$, of D is obtained by identifying the boundary points of the k-box bounding D to form the standard k-cone.

FIGURE 5. The trace diagram of the pillar diagram in Figure 2



Example 8.5. The trace diagram $\overline{D}$ corresponding to the diagram D of Figure 2 is shown in Figure 5.

Notice that the outer part of the trace diagram (regarded as a disc) will always have an anticlockwise orientation and thus be labelled by 0. Consequently, any regions in the trace diagram not labelled by zero must be bounded by at least one closed loop. (It is possible for the closed loops to be nested.)

Definition 8.6. Let $g : \mathbb{Z}^{\geq 0} \longrightarrow \mathbb{Z}^{\geq 0}$ be given by

$$g(c) = \begin{cases} 1 & \text{if } c = 0, \\ c-1 & \text{if } c \geq 1. \end{cases}$$

If $\overline{D}$ is a trace diagram for $TL(E_n)$, we define the *content*, $c(\overline{D})$, of $\overline{D}$ to be the integer

$$\sum_L g(f(L)),$$

where the sum is over all the connected components L of $\overline{D}$ that are interior to at least one closed loop, and where $f(L)$ is the integer assigned to L as in Definition 7.4.

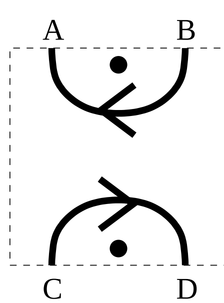
Example 8.7. The content of the trace diagram in Figure 5 is

$$g(2) + g(3) + g(3) = 5.$$

Lemma 8.8. *The content of a trace diagram $\overline{D}$ is invariant under the topological reduction rule shown in Figure 3.*

Proof. Consider the application of the topological reduction rule to a diagram that looks locally like the situation in Figure 6.

FIGURE 6. Labelling of points involved in the topological relation



As in the discussion following Figure 4, there are two cases to consider, according as the two pillar regions are connected or not in $\overline{D}$.

There are four cases to consider, according as there is an oriented curve in $\overline{D}$ from point A to point C, and (independently) according as there is an oriented curve in $\overline{D}$ from point D to point B.

Suppose first that there is no oriented curve in $\overline{D}$ from point A to point C, and also that there is no oriented curve in $\overline{D}$ from point D to point B. In this case, the two pillar regions are genuinely distinct, and applying the topological relation does not produce any new closed loops. We are then in the case $p = k + l - 1 \geq 1$ of Figure 4, so the summands $(k-1)$ and $(l-1)$ appearing in Definition 8.6 are replaced by a single $((k+l-1)-1)$, leaving the content unchanged.

We next deal with the case where there is an oriented curve from point A to point C, but no oriented curve from point D to point B. In this case, the two pillar regions are connected to each other, and the application of the topological rule produces a new closed loop (labelled zero) from the curve originally connecting point A to point C. We are now in the case $k = l > 1$ of Figure 4. This will change one of the summands $(k-1)$ of Definition 8.6 to $(k-2)$, and a new summand of 1 will be produced, corresponding to the new closed loop. The content thus remains unchanged.

Consideration of the case where there is an oriented curve from point D to point B, but not from point A to point C, proceeds in exactly the same way. The last case, in which both oriented curves exist, also works similarly, except that the oriented curves shown in Figure 6 are already part of a closed loop. Application of the topological relation splits this closed loop into two closed loops, again producing an extra summand of 1 and changing a summand $(k-1)$ to $(k-2)$, leaving the content unchanged. □

Lemma 8.9. *There is a well-defined K-linear map*

$$\tau_n^\bullet : \mathcal{P}_n^E(\bullet) \longrightarrow K$$

such that for each pillar diagram D, $\tau_n^\bullet(D) = \delta^{c(\overline{D})}$. If $x, y \in \mathcal{P}_n^E(\bullet)$, we have $\tau_n^\bullet(xy) = \tau_n^\bullet(yx)$.

Proof. For the first assertion, we need to check relations (a)–(c) of Definition 7.6. Relation (iii) holds by Lemma 8.8.

In relation (i), we have $D = D_1$, where D_1 is the result of removing a loop labelled 1 from D. Since $c(\overline{D}) = c(\overline{D_1})$, we have $\tau_n^\bullet(D) = \tau_n^\bullet(D_1)$.

In relation (ii), we have $D = \delta D_2$, where D_2 is the result of removing a loop labelled 0 from D. Since $c(\overline{D}) = c(\overline{D_2}) + 1$, we have $\tau_n^\bullet(D) = \tau_n^\bullet(D_2)$.

By linearity, we only need check the second assertion in the case where x and y are pillar diagrams, and this is immediate from the construction of trace diagrams from pillar diagrams. □

It is not hard to see that there is an algebra embedding $\iota_n^\bullet : \mathcal{P}_n^E(\bullet) \longrightarrow \mathcal{P}_{n+1}^E(\bullet)$ analogous to the map ι_n. Given a pillar diagram D of $\mathcal{P}_n^E(\bullet)$, $\iota^\bullet(D)$ is the diagram obtained by adding a vertical line on the right of the diagram.

Lemma 8.10. *Let D be a pillar diagram of $\mathcal{P}_n^E(\bullet)$.*

(i) *We have $\tau_{n+1}^\bullet(\iota_n^\bullet(D)) = \delta\tau_n^\bullet(D)$.*

(ii) *Let E_n^{n+1} be as in Definition 7.7. Then we have $\tau_n^\bullet(D) = \tau_{n+1}^\bullet(\iota_n(D)E_n)$.*

Proof. Part (i) follows from the observation that the trace diagram $\overline{\iota_n^\bullet(D)}$ differs from the trace diagram $\overline{D}$ only in having a single extra closed loop, labelled 0.

A short calculation involving diagrams shows that the trace diagrams $\overline{D}$ and $\overline{\iota_n(D)E_n}$ are equivalent, from which part (ii) follows. □

Theorem 8.11. *Let $\tau_n : TL_K(E_n) \longrightarrow K$ be the trace defined by*

$$\tau_n(x) = \delta^{-n}\tau_n^\bullet(\rho(x)).$$

The family of traces $\{\tau_n : n \geq 6\}$ is compatible with the direct limit of algebras $TL_K(E_n)$ and gives the unique Markov trace on $TL_K(E_\infty)$. Furthermore, the Markov trace agrees with the traces tr *of Definition 5.4.*

Proof. The maps τ_n are traces by Proposition 7.8 and Lemma 8.9. They are compatible with the direct limit by Lemma 8.10 (i). Since $\tau_n^\bullet(1) = \delta^n$, we have

$\tau_n(1) = 1$. Condition (ii) of Definition 8.1 follows from part (ii) of Lemma 8.10. Uniqueness of the Markov trace, and agreement with the traces tr, is given by Proposition 8.3. □

9. Proofs and applications

Proof of Theorem 1.1. We need to show that the homomorphism ρ of Proposition 7.8 is injective, and there is no loss in passing to the field of fractions K of $\mathbb{Z}[[v^{-1}]]$. In this case, Proposition 6.5 and Theorem 8.11 show that the unique Markov trace on $TL_K(E_n)$, which can be defined on $\mathrm{Im}(\rho)$, is nondegenerate on $TL_K(E_n)$. The conclusion follows. □

Proposition 9.1. *The linear map*

$$(1 + v^{-2})^n \tau_n = v^{-n} \tau_n^{\bullet} \circ \rho$$

restricted to $TL(E_n)$ takes values in $\mathcal{A}$. It is a tabular trace in the sense of [**10**], *and a positive generalized Jones trace in the sense of* [**12**].

Proof. The first assertion comes from the fact that $\tau_n^{\bullet}$ evaluated on a diagram (such as an element of the form $\rho(b_w)$ for $w \in W_c$) yields a nonnegative integer power of δ.

To check that $(1+v^{-2})^n \tau_n$ is a tabular trace, we need to check that axiom (A5) of [**10**, Definition 1.3.4] is satisfied. We have just shown that $(1+v^{-2})^n \tau_n$ takes values in $\mathcal{A}$, and it is clear from Theorem 8.11 that $(1+v^{-2})^n \tau_n$ is a trace. We have seen in Corollary 5.5 and Proposition 6.2 (ii) that $(1+v^{-2})^n \tau_n(x) = (1+v^{-2})^n \tau_n(x^*)$ for all $x \in TL(E_n)$. All that remains to check is that

$$\tau(v^{\mathbf{a}(C_{S,T})} C_{S,T}) = \delta_{S,T} \mod v^{-1}\mathcal{A}^-.$$

This follows from propositions 6.2 (ii) and 6.3 once we observe that we have

$$(1 + v^{-2})^n = 1 \mod v^{-2}\mathbb{Q}[[v^{-1}]],$$

regarded as power series in $\mathbb{Q}[v][[v^{-1}]]$.

To show that $(1+v^{-2})^n\tau_n$ is a generalized Jones trace (see [**12**, Definition 2.9]), two further conditions must be checked. One of these is precisely that established by Lemma 6.7; the other is that, for $x, y \in W_c$, we should have

$$(1+v^{-2})^n\tau_n(c_x c_{y^{-1}}) = \begin{cases} 1 \mod v^{-1}\mathcal{A}^- & \text{if } x = y, \\ 0 \mod v^{-1}\mathcal{A}^- & \text{otherwise,} \end{cases}$$

where $\{c_w : w \in W_c\}$ is the canonical basis of $TL(E_n)$ defined by J. Losonczy and the author in [**14**]. By [**14**, Theorem 3.6], this is nothing other than the basis $\{b_w : w \in W_c\}$ in this case. The corresponding property for tr (instead of $(1+v^{-2})^n\tau_n$) follows from Proposition 6.5, and the assertion for $(1+v^{-2})^n\tau_n$ follows from the fact that $(1+v^{-2})^n = 1 \mod v^{-2}\mathcal{A}^-$.

A generalized Jones trace is positive if it sends canonical basis elements to elements of $\mathbb{N}[v, v^{-1}]$. This holds for $(1+v^{-2})^n\tau_n$ by Proposition 6.3: in this case, $(1+v^{-2})^n\tau_n(b_w) = \delta^b$ for some $b \geq 0$, so that $(1+v^{-2})^n \in \mathbb{N}[v, v^{-1}]$. □

Remark 9.2. Proposition 9.1 corrects the proof of [**10**, Theorem 4.3.5], where the proof that the tabular trace takes the same values on x and x^* contains a gap.

Proof of Theorem 1.2. By [**12**, Theorem 7.10], the conclusion of Theorem 1.2 holds for a generalized Jones trace if the underlying Coxeter group has "Property F" and a bipartite Coxeter graph. Clearly the graphs E_n are bipartite, because they contain no circuits. Property F holds by [**12**, Remark 3.5]; see [**13**, Lemma 5.6] for a fuller explanation.

To complete the proof, we simply have to transfer the result from $(1+v^{-2})^n\tau_n$ to the Markov trace, which follows from the fact that $(1+v^{-2})^n = 1 \mod v^{-2}\mathcal{A}^-$. □

The next result is an easier to use version of Theorem 1.2.

Corollary 9.3. *Let $x, y \in W_c(E_n)$. Then we have*

$$\tilde{\mu}(x, y) = \begin{cases} 1 & \textit{if } \tau_n^{\bullet} \circ \rho(b_x b_{y^{-1}}) = \delta^{n-1}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This follows from Theorem 1.2 together with the observation that $b_x b_{y^{-1}} = \delta^b b_w$ for some $b \geq 0$ and $w \in W_c$, and the fact that $\tau_n^\bullet$ sends diagrams to positive powers of δ. $\square$

Remark 9.4. It follows from [**13**, Theorem 4.6 (iv)] and [**14**, Theorem 3.6] that the monomial basis element b_x is the projection of the Kazhdan–Lusztig basis element $C'_x \in \mathcal{H}(E_n)$. Regarding tr and $\tau_n^\bullet \circ \rho$ as traces on the Hecke algebra, Theorem 1.2 and Corollary 9.3 can be used to evaluate the trace on products of certain Kazhdan–Lusztig basis elements, without evaluating the product (which would be difficult). Another noteworthy property of these results is that they give non-recursive formulae for certain of the integers $\mu(x, y)$.

Remark 9.5. In [**7**, §9], Graham showed that if $x, w \in W_c$ for $TL(E_n)$ then $\mu(x, y) \in \{0, 1\}$, and also produced a nonrecursive method of finding all the x with $\mu(x, y) = 1$ for a fixed y. (In [**7**], x and y are said to be "close" if $\tilde{\mu}(x, y) = 1$.) However, unlike the results above, this does not give an efficient way to compute $\mu(x, y)$ when both of x and y are specified. Corollary 9.3 can therefore be regarded as a quick way to tell if two elements are close or not.

Remark 9.6. It is possible to modify Theorem 1.2 and Corollary 9.3 so that they provide a nonrecursive way to test whether two diagrams represent the same algebra element. However, we do not pursue this here for reasons of space.

Example 9.7. Consider the Coxeter system of type E_n with $n = 6$, and generators $s_0, \ldots, s_5$ as numbered in Figure 1. Define $y = s_1 s_2 s_4 s_0 s_5$ and

$$w = s_1 s_2 s_3 s_4 s_0 s_3 s_5 s_2 s_4 s_1 s_3 s_2 s_0 s_3 s_4 s_5;$$

these are both reduced expressions for fully commutative elements. The diagrams $\rho(b_y)$ and $\rho(b_w)$ are shown in figures 7 and 8 respectively. To evaluate $\tau_n^\bullet(b_y b_{w^{-1}})$, we invert the diagram for b_w, compose it with b_y and identify boundary points to produce a trace diagram. The trace diagram so obtained is shown in Figure 9 (up to equivalence), and by inspection, it has content $1+1+1+(3-1) = 5 = n-1$. It follows from Corollary 9.3 that $\mu(y, w) = 1$.

FIGURE 7. The diagram $\rho(b_y)$ of Example 9.7

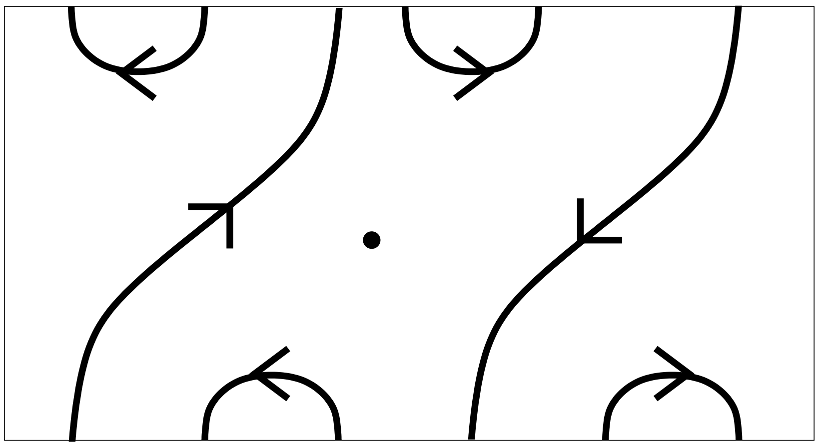


FIGURE 8. The diagram $\rho(b_w)$ of Example 9.7

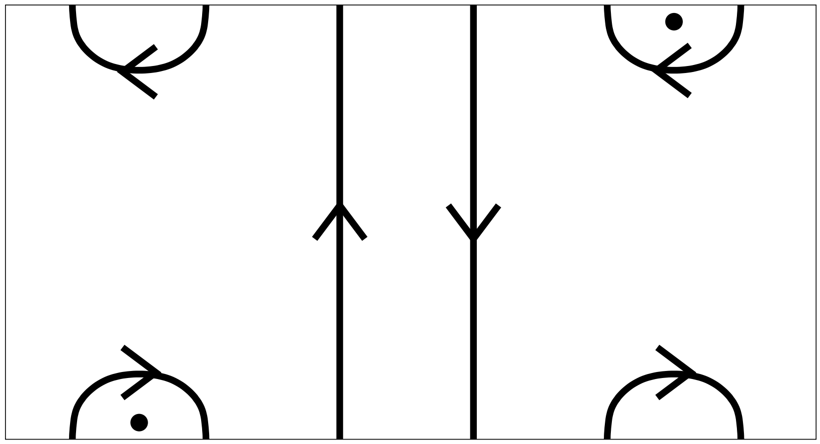
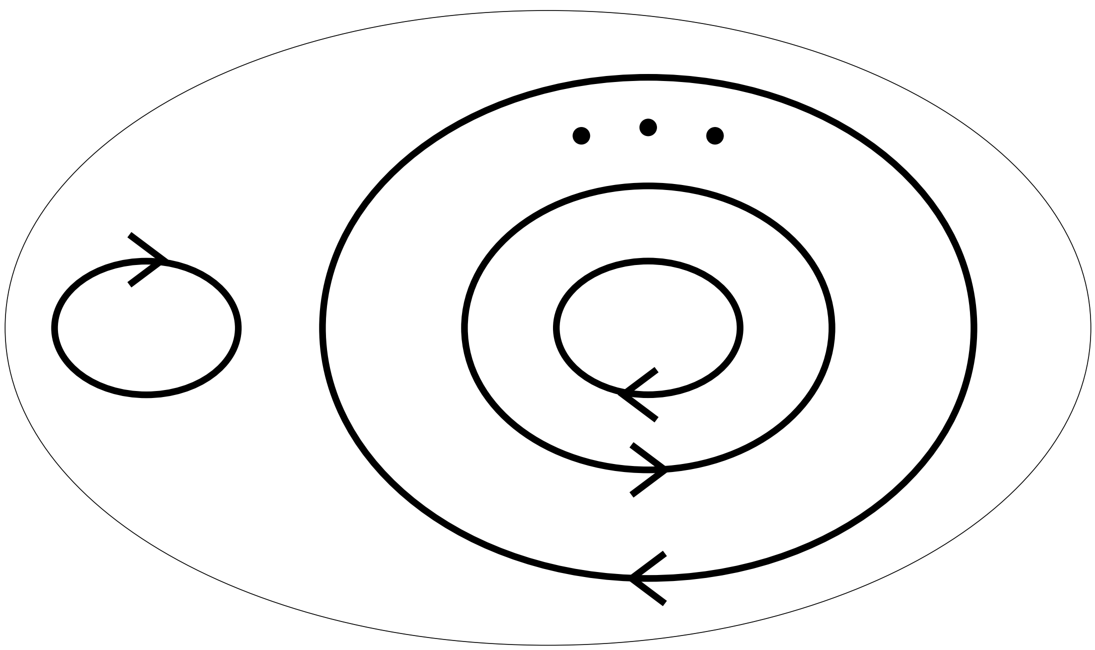


FIGURE 9. The trace diagram corresponding to $\tau_6^\bullet \circ \rho(b_y b_{w^{-1}})$ of Example 9.7



Acknowledgement

I am grateful to P.P. Martin for helpful comments on an early version of this paper.

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