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Generalized Twistor Transform And Dualities With A New Description of Particles With Spin Beyond Free and Massless[1]

Itzhak Bars and Bora Orcal

Department of Physics and Astronomy,

University of Southern California, Los Angeles, CA 90089-0484, USA.

Abstract

A generalized twistor transform for spinning particles in 3+1 dimensions is constructed that beautifully unifies many types of spinning systems by mapping them to the *same twistor* $Z_A = \binom{\mu^{\dot{\alpha}}}{\lambda_\alpha}$, thus predicting an infinite set of duality relations among spinning systems with different Hamiltonians. Usual 1T-physics is not equipped to explain the duality relationships and unification between these systems. We use 2T-physics in 4+2 dimensions to uncover new properties of twistors, and expect that our approach will prove to be useful for practical applications as well as for a deeper understanding of fundamental physics. Unexpected structures for a new description of spinning particles emerge even for the massless free particle case. A new unifying symmetry $\mathrm{SU}(2,3)$ that includes conformal symmetry $\mathrm{SU}(2,2)$ =$\mathrm{SO}(4,2)$ in the massless case, turns out to be a fundamental property underlying the dualities of a large set of spinning systems, including those that occur in high spin theories. This may lead to new forms of string theory backgrounds as well as to new methods for studying various corners of M theory. In this paper we present the main concepts, and in a companion paper we give other details [1].

I. SPINNING PARTICLES IN 3+1 - BEYOND FREE AND MASSLESS

The Penrose twistor transform [2]-[5] brings to the foreground the conformal symmetry $\mathrm{SO}(4,2)$ in the dynamics of massless relativistic particles of any spin in $3+1$ dimensions. The transform relates the phase space and spin degrees of freedom $x^\mu, p_\mu, s^{\mu\nu}$ to a twistor

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$Z_A = \binom{\mu^{\dot\alpha}}{\lambda_\alpha}$ and reformulates the dynamics in terms of twistors instead of phase space. The twistor Z_A is made up of a pair of SL$(2, C)$ spinors $\mu^{\dot\alpha}, \lambda_\alpha$, $\alpha, \dot\alpha = 1, 2$, and is regarded as the 4 components $A = 1, 2, 3, 4$ of the Weyl spinor of SO$(4, 2)$ =SU$(2, 2)$.

The well known twistor transform for a spinning massless particle is [5]

$$\mu^{\dot\alpha} = -i\,(\bar x + i\bar y)^{\dot\alpha\beta}\,\lambda_\beta,\quad \lambda_\alpha\bar\lambda_{\dot\beta} = p_{\alpha\dot\beta}, \tag{1.1}$$

where $(\bar x + i\bar y)^{\dot\alpha\beta} = \frac{1}{\sqrt 2}\,(x^\mu + iy^\mu)\,(\bar\sigma_\mu)^{\dot\alpha\beta}$, and $p_{\alpha\dot\beta} = \frac{1}{\sqrt 2} p^\mu\,(\sigma_\mu)_{\alpha\dot\beta}$, while $\sigma_\mu = (1, \vec\sigma)$, $\bar\sigma_\mu = (-1, \vec\sigma)$ are Pauli matrices. $x^\mu + iy^\mu$ is a complexification of spacetime [2]. The helicity h of the particle is determined by $p\cdot y = h$. The spin tensor is given by $s^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} y_\rho p_\sigma$, and it leads to $\frac12 s^{\mu\nu}s_{\mu\nu} = h^2$. The Pauli-Lubanski vector is proportional to the momentum $W_\mu = \frac12 \varepsilon_{\mu\nu\rho\sigma}s^{\nu\rho}p^\sigma = (y\cdot p)\,p_\mu - p^2 y^\mu = hp_\mu$, appropriate for a massless particle of helicity h.

The reformulation of the dynamics in terms of twistors is manifestly SU$(2, 2)$ covariant. It was believed that twistors and the SO$(4, 2)$=SU$(2, 2)$ symmetry, interpreted as conformal symmetry, govern the dynamics of massless particles only, since the momentum p^μ of the form $p_{\alpha\dot\beta} = \lambda_\alpha\bar\lambda_{\dot\beta}$ automatically satisfies $p^\mu p_\mu = 0$.

However, recent work has shown that the *same twistor* $Z_A = \binom{\mu^{\dot\alpha}}{\lambda_\alpha}$ that describes massless *spinless* particles ($h = 0$) also describes an assortment of other spinless particle dynamical systems [6][7]. These include massive and interacting particles. The mechanism that avoids $p^\mu p_\mu = 0$ [6][7] is explained following Eq.(6.9) below. The list of systems includes the following examples worked out explicitly in previous publications and in unpublished notes.

The massless relativistic particle in $d = 4$ flat Minkowski space.

The massive relativistic particle in $d = 4$ flat Minkowski space.

The nonrelativistic free massive particle in 3 space dimensions.

The nonrelativistic hydrogen atom (i.e. $1/r$ potential) in 3 space dimensions.

The harmonic oscillator in 2 space dimensions, with its mass $\Leftrightarrow$ an extra dimension.

The particle on AdS$_4$, or on dS$_4$.

The particle on AdS$_3\times$S^1 or on $R\times S^3$.

The particle on AdS$_2\times$S^2.

The particle on the Robertson-Walker spacetime.

The particle on any maximally symmetric space of positive or negative curvature.

The particle on any of the above spaces modified by any conformal factor.

A related family of other particle systems, including some black hole backgrounds.

In this paper we will discuss these for the case of $d=4$ *with spin* $(h\neq 0)$. It must be emphasized that while the phase spaces (and therefore dynamics, Hamiltonian, etc.) in these systems are different, the twistors $\left(\mu^{\dot{\alpha}},\lambda_{\alpha}\right)$ are the same. For example, the massive particle phase space $(x^{\mu},p^{\mu})_{massive}$ and the one for the massless particle $(x^{\mu},p^{\mu})_{massless}$ are not the same (x^{μ},p^{μ}), rather they can be obtained from one another by a non-linear transformation for any value of the mass parameter m [6], and similarly, for all the other spaces mentioned above. However, under such "duality" transformations from one system to another, the twistors for all the cases are the same up to an overall phase transformation

$$\left(\mu^{\dot{\alpha}},\lambda_{\alpha}\right)_{massive}=\left(\mu^{\dot{\alpha}},\lambda_{\alpha}\right)_{massless}=\cdots=\left(\mu^{\dot{\alpha}},\lambda_{\alpha}\right). \tag{1.2}$$

This unification also shows that all of these systems share the same SO$(4,2)$=SU$(2,2)$ global symmetry of the twistors. This SU$(2,2)$ is interpreted as conformal symmetry for the massless particle phase space, but has other meanings as a hidden symmetry of all the other systems in their own phase spaces. Furthermore, in the quantum physical Hilbert space, the symmetry is realized in the *same unitary representation* of SU$(2,2)$, with the same Casimir eigenvalues (see (7.16,7.17) below), for all the systems listed above.

The underlying reason for such fantastic looking properties cannot be found in one-time physics (1T-physics) in 3+1 dimensions, but is explained in two-time physics (2T-physics) [8] as being due to a local Sp$(2,R)$ symmetry. The Sp$(2,R)$ symmetry which acts in phase space makes position and momentum indistinguishable at any instant and requires one extra space and one extra time dimensions to implement it, thus showing that the unification relies on an underlying spacetime in 4+2 dimensions. It was realized sometime ago that in 2T-physics twistors emerge as a gauge choice [9], while the other systems are also gauge choices of the same theory in 4+2 dimensions. The 4+2 phase space can be gauge fixed to many 3+1 phase spaces that are distinguishable from the point of view of 1T-physics, without any Kaluza-Klein remnants, and this accounts for the different Hamiltonians that have a duality relationship with one another. We will take advantage of the properties of 2T-physics to build the general twistor transform that relates these systems including spin.

Given that the field theoretic formulation of 2T-physics in 4+2 dimensions yields the Standard Model of Particles and Forces in 3+1 dimensions as a gauge choice [10], including spacetime supersymmetry [11], and given that twistors have simplified QCD computations [12][13], we expect that our twistor methods will find useful applications.

II. TWISTOR LAGRANGIAN

The Penrose twistor description of massless spinning particles requires that the pairs $\left(\mu^{\dot{\alpha}}, i\bar{\lambda}_{\dot{\alpha}}\right)$ or their complex conjugates $(\lambda_\alpha, i\bar{\mu}^\alpha)$ be canonical conjugates and satisfy the helicity constraint given by

$$\bar{Z}^A Z_A = \bar{\lambda}_{\dot{\alpha}}\mu^{\dot{\alpha}} + \bar{\mu}^\alpha \lambda_\alpha = 2h. \tag{2.1}$$

Indeed, Eq.(1.1) satisfies this property provided $y \cdot p = h$. Here we have defined the $\bar{\mathbf{4}}$ of SU$(2,2)$ as the contravariant twistor

$$\bar{Z}^A \equiv \left(Z^\dagger \eta_{2,2}\right)^A = \left(\bar{\lambda}_{\dot{\alpha}}\ \bar{\mu}^\alpha\right), \quad \eta_{2,2} = \begin{pmatrix} 0\,1 \\ 1\,0 \end{pmatrix} = \mathrm{SU}\,(2,2)\ \text{metric}. \tag{2.2}$$

The canonical structure, along with the constraint $\bar{Z}^A Z_A = 2h$ follows from the following worldline action for twistors

$$S_h = \int d\tau \left[i\bar{Z}_A \left(D_\tau Z^A\right) - 2h\tilde{A}\right], \quad D_\tau Z^A \equiv \frac{\partial Z^A}{\partial \tau} - i\tilde{A}Z^A. \tag{2.3}$$

In the case of $h = 0$ it was shown that this action emerges as a gauge choice of a more general action in 2T-physics [6][7]. Later in the paper, in Eq.(4.1) we give the $h \neq 0$ 2T-physics action from which (2.3) is derived as a gauge choice. The derivative part of this action gives the canonical structure $S_0 = \int d\tau i\bar{Z}_A \left(\partial_\tau Z^A\right) = i\int d\tau \left[\bar{\lambda}_{\dot{\alpha}}\partial_\tau \mu^{\dot{\alpha}} + \bar{\mu}^\alpha \partial_\tau \lambda_\alpha\right]$ that requires $\left(\mu^{\dot{\alpha}}, i\bar{\lambda}_{\dot{\alpha}}\right)$ or their complex conjugates $(\lambda_\alpha, i\bar{\mu}^\alpha)$ to be canonical conjugates. The 1-form $\tilde{A}d\tau$ is a U(1) gauge field on the worldline, $D_\tau Z^A$ is the U(1) gauge covariant derivative that satisfies $\delta_\varepsilon \left(D_\tau Z^A\right) = i\varepsilon \left(D_\tau Z^A\right)$ for $\delta_\varepsilon \tilde{A} = \partial\varepsilon/\partial\tau$ and $\delta_\varepsilon Z^A = i\varepsilon Z^A$. The term $2h\tilde{A}$ is gauge invariant since it transforms as a total derivative under the infinitesimal gauge transformation. $2h\tilde{A}$ was introduced in [6][7] as being an integral part of the twistor formulation of the spinning particle action.

Our aim is to show that this action describes not only massless spinning particles, but also all of the other particle systems listed above *with spin*. This will be done by constructing the twistor transform from Z_A to the phase space and spin degrees of freedom of these systems, and claiming the unification of dynamics via the generalized twistor transform. This generalizes the work of [6][7] which was done for the $h = 0$ case of the action in (2.3). We will use 2T-physics as a tool to construct the general twistor transform, so this unification is equivalent to the unification achieved in 2T-physics.

III. MASSLESS PARTICLE WITH ANY SPIN IN 3+1 DIMENSIONS

In our quest for the general twistor transform with spin, we first give a new alternative to the well known twistor transform of Eq.(1.1). Instead of the $y^{\mu}(\tau)$ that appears in the complexified spacetime $x^{\mu}+iy^{\mu}$ we introduce an SL$(2,C)$ *bosonic*[2] spinor $v^{\dot{\alpha}}(\tau)$ and its complex conjugate $\bar{v}^{\alpha}(\tau)$, and write the general vector y^{μ} in the matrix form as $y^{\dot{\alpha}\beta}=hv^{\dot{\alpha}}\bar{v}^{\beta}+\omega p^{\dot{\alpha}\beta}$, where $\omega(\tau)$ is an arbitrary gauge freedom that drops out. Then the helicity condition $y\cdot p=h$ takes the form $\bar{v}pv=1$. Furthermore, we can write $\lambda_{\alpha}=p_{\alpha\dot{\beta}}v^{\dot{\beta}}$ since this automatically satisfies $\lambda_{\alpha}\bar{\lambda}_{\dot{\beta}}=p_{\alpha\dot{\beta}}$ when $p^{2}=(\bar{v}pv-1)=0$ are true. With this choice of variables, the Penrose transform of Eq.(1.1) takes the new form

$$\lambda_{\alpha}=(pv)_{\alpha},\ \mu^{\dot{\alpha}}=\left[(-i\bar{x}p+h)\,v\right]^{\dot{\alpha}},\ p^{2}=(\bar{v}pv-1)=0, \tag{3.1}$$

where the last equation is a set of constraints on the degrees of freedom $x^{\mu},p_{\mu},v^{\dot{\alpha}},\bar{v}^{\alpha}$.

If we insert our new twistor transform (3.1) into the action (2.3), the twistor action turns into the action for the phase space and spin degrees of freedom $x^{\mu},p_{\mu},v^{\dot{\alpha}},\bar{v}^{\alpha}$

$$S_{h}=\int d\tau\left\{\dot{x}^{\mu}p_{\mu}-\frac{e}{2}p^{2}+ih\left[(\bar{v}p)\,D_{\tau}v-\overline{D_{\tau}v}\,(pv)\right]-2h\tilde{A}\right\}. \tag{3.2}$$

where $D_{\tau}v=\dot{v}-i\tilde{A}v$ is the U(1) gauge covariant derivative and we have included the Lagrange multiplier e to impose $p^{2}=0$ when we don't refer to twistors. The equation of motion for $\tilde{A}$ imposes the second constraint $\bar{v}pv-1=0$ that implies U(1) gauge invariance.

From the global Lorentz symmetry of (3.2), the Lorentz generator is computed via Noether's theorem $J^{\mu\nu}=x^{\mu}p^{\nu}-x^{\nu}p^{\mu}+s^{\mu\nu}$, with $s^{\mu\nu}=\frac{i}{2}h\bar{v}\left(p\bar{\sigma}^{\mu\nu}+\sigma^{\mu\nu}p\right)v$. The helicity is determined by computing the Pauli-Lubanski vector $W^{\mu}=\frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}s_{\nu\lambda}p_{\sigma}=(h\bar{v}pv)\,p^{\mu}$. The helicity operator $h\bar{v}pv$ reduces to the constant h in the U(1) gauge invariant sector[3].

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[2] This is similar to the *fermionic* case in [5]. The bosonic spinor v can describe any spin h, which is not possible with fermions, and leads to the new features and generalizations to 4+2 dimensions.

[3] If this action is taken without the U(1) constraint $\left(\tilde{A}=0\right)$, then the excitations in the v sector describe an infinite tower of massless states with all helicities from zero to infinity (here we rescale $\sqrt{2h}v\rightarrow v$)

$$S_{all\ spins}=\int d\tau\left\{\dot{x}^{\mu}p_{\mu}-\frac{e}{2}p^{2}+\frac{i}{2}\left[\bar{v}p\dot{v}-\overline{\dot{v}}pv\right]\right\} \tag{3.3}$$

The spectrum coincides with the spectrum of the infinite slope limit of string theory with all helicities $\frac{1}{2}\bar{v}pv$. This action has a hidden SU$(2,3)$ symmetry that includes SU$(2,2)$ conformal symmetry (see footnote (6)). Along with the manifestly SU$(2,3)$ symmetric 2T-physics actions (4.1,5.4) we are proposing here a new setting for discussing high spin theories [14] including spin degrees of freedom.

The action (3.2) gives a new description of a massless particle with any helicity h in terms of the SL$(2,C)$ bosonic spinors $v,\bar{v}$. We note its similarity to the standard superparticle action [15][16] written in the first order formalism. The difference with the superparticle is that the *fermionic* spacetime spinor $\theta^{\dot{\alpha}}$ of the superparticle is replaced with the *bosonic* spacetime spinor $v^{\dot{\alpha}}$, and the gauge field $\tilde{A}$ imposes the U(1) gauge symmetry constraint $\bar{v}pv-1=0$ that restricts the system to a single, but arbitrary helicity state given by h.

Just like the superparticle case, our action has a local *kappa symmetry* with a *bosonic* local spinor parameter $\kappa_\alpha(\tau)$, namely

$$\delta_\kappa v^{\dot{\alpha}}=\bar{p}^{\dot{\alpha}\beta}\kappa_\beta,\ \delta_\kappa x_\mu=\frac{ih}{\sqrt{2}}\left(\left(\delta_\kappa\bar{v}\right)\sigma_\mu v-\bar{v}\sigma_\mu\left(\delta_\kappa v\right)\right), \tag{3.4}$$

$$\delta_\kappa p^\mu=0,\ \ \delta_\kappa e=-ih\ \left[\bar{\kappa}\left(D_\tau v\right)-\left(\overline{D_\tau v}\right)\kappa\right],\ \delta_\kappa\tilde{A}=0. \tag{3.5}$$

These kappa transformations mix the phase space degrees of freedom (x,p) with the spin degrees of freedom $v,\bar{v}$. The transformations $\delta_\kappa x_\mu,\delta_\kappa e$ are non-linear since they involve products of v with p or $\tilde{A}$.

Let us count *physical* degrees of freedom. By using the kappa and the τ-reparametrization symmetries one can choose the lightcone gauge. From phase space x^μ,p^μ there remains 3 positions and 3 momentum degrees of freedom. One of the two complex components of $v^{\dot{\alpha}}$ is set to zero by using the kappa symmetry, so $v^{\dot{\alpha}}=\binom{v}{0}$. The phase of the remaining component is eliminated by choosing the U(1) gauge, and finally its magnitude is fixed[4] by solving the constraint $\bar{v}pv-1=0$ to obtain $v^{\dot{\alpha}}=\left(p^+\right)^{-1/2}\binom{1}{0}$. Therefore, there are no *independent* physical degrees of freedom in v. The remaining degrees of freedom for the particle of any spin are just the three positions and momenta, and the constant h that appears in $s^{\mu\nu}$. This is as it should be, as seen also by counting the physical degrees of freedom from the twistor point of view. When we consider the other systems listed in the first section, we should expect that they too are described by the same number of degrees of freedom since they will be obtained from the same twistor, although they obey different dynamics (different Hamiltonians) in their respective phase spaces.

The covariant quantization of the systems (3.2,3.3) is discussed in a related paper [1].

[4] For high spin systems in Eq.(3.3), the magnitude would not be fixed.

IV. 2T-PHYSICS WITH SP$(2,R)$, SU$(2,3)$ AND KAPPA SYMMETRIES

The similarity of (3.2) to the action of the superparticle provides the hint for how to lift it to the 2T-physics formalism, as was done for the superparticle [17][9] and the twistor superstring [18][19]. This requires lifting 3+1 phase space (x^μ, p_μ) to 4+2 phase space $\left(X^M, P_M\right)$ and lifting the SL$(2,C)$ spinors $v, \bar{v}$ to the SU$(2,2)$ spinors $V_A, \bar{V}^A$. The larger set of degrees of freedom $X^M, P_M, V_A, \bar{V}^A$ that are covariant under the global symmetry SU$(2,2)$ =SO$(4,2)$, include gauge degrees of freedom, and are subject to gauge symmetries and constraints that follow from them as described below.

The point is that the SU$(2,2)$ invariant constraints on $X^M, P_M, V_A, \bar{V}^A$ have a wider set of solutions than just the 3+1 system of Eq.(3.2) we started from. This is because 3+1 dimensional spin & phase space has many different embeddings in 4+2 dimensions, and those are distinguishable from the point of view of 1T-physics because target space "time" and corresponding "Hamiltonian" are different in different embeddings, thus producing the different dynamical systems listed in section (I). The various 1T-physics solutions are reached by simply making gauge choices. One of the gauge choices for the action we give below in Eq.(4.1) is the twistor action of Eq.(2.3). Another gauge choice is the 4+2 spin & phase space action in terms of the lifted spin & phase space $X^M, P_M, V_A, \bar{V}^A$ as given in Eq.(5.4). The latter can be further gauge fixed to produce all of the systems listed in section (I) including the action (3.2) for the massless spinning particle with any spin. All solutions still remember that there is a hidden *global* symmetry SU$(2,2)$ =SO$(4,2)$, so all systems listed in section (I) are realizations of the same unitary representation of SU$(2,2)$ whose Casimir eigenvalues will be given below.

For the $4+2$ version of the superparticle [17] that is similar to the action in (5.4), this program was taken to a higher level in [9] by embedding the fermionic supercoordinates in the coset of the supergroup SU$(2,2|1)$ /SU$(2,2)$ ×U(1). We will follow the same route here, and embed the bosonic SU$(2,2)$ spinors $V_A, \bar{V}^A$ in the left coset SU$(2,3)$ /SU$(2,2)$ ×U(1). This coset will be regarded as the gauging of the group SU$(2,3)$ under the subgroup $[\mathrm{SU}(2,2) \times \mathrm{U}(1)]_L$ from the left side. Thus the most powerful version of the action that reveals the global and gauge symmetries is obtained when it is organized in terms of the

$X_i^M(\tau)$, $g(\tau)$ and $\tilde{A}(\tau)$ degrees of freedom described as

4+2 phase space $\binom{X^M(\tau)}{P_M(\tau)} \equiv X_i^M(\tau)$, $i=1,2$, *doublets of* $\mathrm{Sp}(2,R)$ *gauge symmetry,*

group element $g(\tau) \subset \mathrm{SU}(2,3)$ *subject to* $[\mathrm{SU}(2,2)\times \mathrm{U}(1)]_L \times \mathrm{U}(1)_{L+R}$ *gauge symmetry.*

We should mention that the $h=0$ version of this theory, and the corresponding twistor property, was discussed in [6], by taking $g(\tau) \subset$SU$(2,2)$ and dropping all of the U(1)'s. So, the generalized theory that includes spin has the new features that involves SU$(2,2) \to$SU$(2,3)$ and the U(1) structures. The action has the following form

$$S_h = \int d\tau \left\{ \frac{1}{2}\varepsilon^{ij}\left(D_\tau X_i^M\right) X_j^N \eta_{MN} + Tr\left(\left(iD_\tau g\right) g^{-1} \begin{pmatrix} \mathcal{L} & 0 \\ 0 & 0 \end{pmatrix} \right) - 2h\tilde{A} \right\}, \tag{4.1}$$

where $\varepsilon^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{ij}$ is the antisymmetric Sp$(2,R)$ metric, and $D_\tau X_i^M = \partial_\tau X_i^M - A_i^{\ j} X_j^M$ is the Sp$(2,R)$ gauge covariant derivative, with the 3 gauge potentials $A^{ij} = \varepsilon^{ik} A_k^{\ j} = \begin{pmatrix} A & C \\ C & B \end{pmatrix}$. For SU$(2,3)$ the group element is pseudo-unitary, $g^{-1} = (\eta_{2,3})\, g^\dagger\, (\eta_{2,3})^{-1}$, where $\eta_{2,3}$ is the SU$(2,3)$ metric $\eta_{2,3} = \begin{pmatrix} \eta_{2,2} & 0 \\ 0 & -1 \end{pmatrix}$. The covariant derivative $D_\tau g$ is given by

$$D_\tau g = \partial_\tau g - i\tilde{A}\,[q,g]\,, \quad q = \frac{1}{5}\left(\begin{array}{c|c} 1_{4\times 4} & 0 \\ \hline 0 & -4 \end{array}\right) \tag{4.2}$$

where the generator of $\mathrm{U}(1)_{L+R}$ is proportional to the 5×5 traceless matrix $q \in$ u(1) $\in$ su$(2,3)_{L+R}$. The last term of the action $-2h\tilde{A}$, which is also the last term of the action (2.3), is invariant under the $\mathrm{U}(1)_{L+R}$ since it transforms to a total derivative. Finally, the 4×4 traceless matrix $(\mathcal{L})_A^{\ B} \in$su$(2,2)$ $\in$su$(2,3)$ that appears on the *left side* of g (or right side of g^{-1}) is

$$(\mathcal{L})_A^{\ B} \equiv \left(\frac{1}{4i}\Gamma_{MN}\right)_A^{\ B} L^{MN}, \ \ L^{MN} = \varepsilon^{ij} X_i^M X_j^N = X^M P^N - X^N P^M. \tag{4.3}$$

where $\Gamma_{MN} = \frac{1}{2}\left(\Gamma_M \bar{\Gamma}_N - \Gamma_N \bar{\Gamma}_M\right)$ are the 4×4 gamma-matrix representation of the 15 generators of SU$(2,2)$. A detailed description of these gamma matrices is given in [11].

The symmetries of actions of this type for any group or supergroup g were discussed in [9][18][19][7]. The only modification of that discussion here is due to the inclusion of the U(1) gauge field $\tilde{A}$. In the absence of the $\tilde{A}$ coupling the global symmetry is given by the transformation of $g(\tau)$ from the right side $g(\tau) \to g(\tau)\, g_R$ where $g_R \subset$SU$(2,3)_R$. However, in our case, the presence of the coupling with the $\mathrm{U}(1)_{L+R}$ charge q breaks the global symmetry down to the (SU$(2,2)\times$U$(1))_R$ subgroup that acts on the right side of g.

So the global symmetry is given by

$$\text{global:}\;\; g(\tau) \rightarrow g(\tau)\, h_R,\;\; h_R \in [\mathrm{SU}(2,2)\times \mathrm{U}(1)]_R \subset \mathrm{SU}(2,3)_R. \tag{4.4}$$

Using Noether's theorem we deduce the conserved global charges as the $[\mathrm{SU}(2,2)\times\mathrm{U}(1)]_R$ components of the the following $\mathrm{SU}(2,3)_R$ Lie algebra valued matrix $J_{(2,3)}$

$$J_{(2,3)} = g^{-1}\begin{pmatrix}\mathcal{L}\,0\\0\;0\end{pmatrix} g = \begin{pmatrix}\mathcal{J}+\frac{1}{4}J_0 & j\\ -\bar{j} & -J_0\end{pmatrix},\quad J_{2,3} = \eta_{2,3}\left(J_{2,3}\right)^{\dagger}\left(\eta_{2,3}\right)^{-1}, \tag{4.5}$$

The traceless 4×4 matrix $\left(\mathcal{J}\right)_A^{\;\;B} = \frac{1}{4i}\Gamma^{MN}J_{MN}$ is the conserved $\mathrm{SU}(2,2)$ $=\mathrm{SO}(4,2)$ charge and J_0 is the conserved U(1) charge. Namely, by using the equations of motion one can verify $\partial_\tau\left(\mathcal{J}\right)_A^{\;\;B} = 0$ and $\partial_\tau J_0 = 0$. The spinor charges $j_A, \bar{j}^A$ are not conserved[5] due to the coupling of $\tilde{A}$. As we will find out later in Eq.(6.8), j_A is proportional to the twistor

$$j_A = \sqrt{J_0}Z_A, \tag{4.6}$$

up to an irrelevant gauge transformation. It is important to note that $\mathcal{J}$ and J_0 are invariant on shell under the gauge symmetries discussed below. Therefore they generate physical symmetries $[\mathrm{SU}(2,2)\times\mathrm{U}(1)]_R$ under which all gauge invariant physical states are classified.

The local symmetries of this action are summarized as

$$\mathrm{Sp}(2,R)\times\begin{pmatrix}\mathrm{SU}(2,2) & \frac{3}{4}kappa\\ \frac{3}{4}kappa & \mathrm{U}(1)\end{pmatrix}_{left} \tag{4.7}$$

The $\mathrm{Sp}(2,R)$ is manifest in (4.1). The rest corresponds to making local SU(2,3) transformations on $g(\tau)$ from the left side $g(\tau)\rightarrow g_L(\tau)\,g(\tau)$, as well as transforming $X_i^M = \left(X^M, P^M\right)$ as vectors with the local subgroup $\mathrm{SU}(2,2)_L$ $=\mathrm{SO}(4,2)$, and A^{ij} under the kappa. The 3/4 kappa symmetry which is harder to see will be discussed in more detail below. These symmetries coincide with those given in previous discussions in [9][18][19][7] despite the presence of $\tilde{A}$. The reason is that the $\mathrm{U}(1)_{L+R}$ covariant derivative $D_\tau g$ in Eq.(4.2) can be replaced by a purely $\mathrm{U}(1)_R$ covariant derivative $D_\tau g = \partial_\tau g + igq\tilde{A}$ because the difference drops out in the trace in the action (4.1). Hence the symmetries on left side of $g(\tau)\rightarrow g_L(\tau)\,g(\tau)$ remain the same despite the coupling of $\tilde{A}$.

We outline the roles of each of these local symmetries. The $\mathrm{Sp}(2,R)$ gauge symmetry can reduce X^M, P_M to any of the phase spaces in 3+1 dimensions listed in section (I). This

[5] In the high spin version of (4.1) with $\tilde{A}=0$, the global symmetry is $\mathrm{SU}(2,3)_R$ and $j_A, \bar{j}^A$ are conserved.

is the same as the $h=0$ case discussed in [6]. The $[\mathrm{SU}(2,2)\times\mathrm{U}(1)]_L$ gauge symmetry can reduce $g(\tau)\subset\mathrm{SU}(2,3)$ to the coset $g\to t(V)\in\mathrm{SU}(2,3)/[\mathrm{SU}(2,2)\times\mathrm{U}(1)]_L$ parameterized by the $\mathrm{SU}(2,2)\times\mathrm{U}(1)$ spinors $\left(V_A,\bar{V}^A\right)$ as shown in Eq.(5.3). The remaining 3/4 kappa symmetry, whose action is shown in Eq.(5.15), can remove up to 3 out of the 4 parameters in the V_A. The $\mathrm{U}(1)_{L+R}$ symmetry can eliminate the phase of the remaining component in V. Finally the constraint due to the equation of motion of $\tilde{A}$ fixes the magnitude of V. In terms of counting, there remains only 3 position and 3 momentum *physical* degrees of freedom, plus the constant h, in agreement with the counting of physical degrees of freedom of the twistors.

It is possible to gauge fix the symmetries (4.7) partially to exhibit some intermediate covariant forms. For example, to reach the $\mathrm{SL}(2,C)$ covariant massless particle described by the action (3.2) from the 2T-physics action above, we take the massless particle gauge by using two out of the three $\mathrm{Sp}(2,R)$ gauge parameters to rotate the $M=+'$ doublet to the form $\binom{X^{+'}}{P^{+'}}(\tau)=\binom{1}{0}$, and solving explicitly two of the $\mathrm{Sp}(2,R)$ constraints $X^2=X\cdot P=0$

$$X^M=(\overset{+'}{1},\ \overset{-'}{\frac{x^2}{2}},\ \overset{\mu}{x^\mu}(\tau)),\quad P^M=(\overset{+'}{0},\ \overset{-'}{x\cdot p},\ \overset{\mu}{p^\mu}(\tau)). \tag{4.8}$$

This is the same as the $h=0$ massless case in [6]. There is a tau reparametrization gauge symmetry as a remnant of $\mathrm{Sp}(2,R)$. Next, the $[\mathrm{SU}(2,2)\times\mathrm{U}(1)]_L$ gauge symmetry reduces $g(\tau)\to t(V)$ written in terms of $\left(V_A,\bar{V}^A\right)$ as given in Eq.(5.3), and the 3/4 kappa symmetry reduces the $\mathrm{SU}(2,2)$ spinor $V_A\to\binom{v^{\dot\alpha}}{0}$ to the two components $\mathrm{SL}(2,C)$ doublet $v^{\dot\alpha}$, with a leftover kappa symmetry as discussed in Eqs.(3.4-3.5). The gauge fixed form of g is then

$$g=\exp\left(\begin{array}{cc|c}0&0&\sqrt{2h}v^{\dot\alpha}\\0&0&0\\\hline 0&\sqrt{2h}\bar{v}^{\alpha}&0\end{array}\right)=\left(\begin{array}{cc|c}1&hv^{\dot\alpha}\bar{v}^{\beta}&\sqrt{2h}v^{\dot\alpha}\\0&1&0\\\hline 0&\sqrt{2h}\bar{v}^{\alpha}&1\end{array}\right)\in\mathrm{SU}(2,3). \tag{4.9}$$

The inverse $g^{-1}=(\eta_{2,3})\,g^\dagger\,(\eta_{2,3})^{-1}$ is given by replacing $v,\bar{v}$ by $(-v),(-\bar{v})$. Inserting the gauge fixed forms of X,P,g (4.8,4.9) into the action (4.1) reduces it to the massless spinning particle action (3.2). Furthermore, inserting these X,P,g into the expression for the current in (4.5) gives the conserved $\mathrm{SU}(2,2)$ charges $\mathcal{J}$ (see Eqs.(5.9,5.20)) which have the significance of the hidden conformal symmetry of the gauge fixed action (3.2). This hidden symmetry is far from obvious in the form (3.2), but it is straightforward to derive from the 2T-physics action as we have just outlined.

Partial or full gauge fixings of (4.1) similar to (4.8,4.9) produce the actions, the hidden SU$(2,2)$ symmetry, and the twistor transforms with spin of all the systems listed in section (I). These were discussed for $h=0$ in [6], and we have now shown how they generalize to any spin $h\neq 0$, with further details below. It is revealing, for example, to realize that the *massive* spinning particle has a hidden SU$(2,2)$ "mass-deformed conformal symmetry", including spin, not known before, and that its action can be reached by gauge fixing the action (4.1), or by a twistor transform from (2.3). The same remarks applied to all the other systems listed in section (I) are equally revealing. For more information see our related paper [1].

Through the gauge (4.8,4.9), the twistor transform (3.1), and the massless particle action (3.2), we have constructed a bridge between the manifestly SU$(2,2)$ invariant twistor action (2.3) for any spin and the 2T-physics action (4.1) for any spin. This bridge will be made much more transparent in the following sections by building the general twistor transform.

V. 2T-PHYSICS ACTION WITH $X^M, P^M, V_A, \bar{V}^A$ IN 4+2 DIMENSIONS

We have hinted above that there is an intimate relation between the 2T-physics action (4.1) and the twistor action (2.3). In fact the twistor action is just a gauged fixed version of the more general 2T-physics action (4.1). Using the *local* SU$(2,2)$ =SO$(4,2)$ and local Sp$(2,R)$ symmetries of the general action (4.1) we can rotate $X^M(\tau), P^M(\tau)$ to the following form that also solves the Sp$(2,R)$ constraints $X_i\cdot X_j = X^2 = P^2 = X\cdot P = 0$ [6][7]

$$X^M = (\overset{+'}{1}, \overset{-'}{0}, \overset{+}{0}, \overset{-}{0}, \overset{i}{0}), \quad P^M = (\overset{+'}{0}, \overset{-'}{0}, \overset{+}{1}, \overset{-}{0}, \overset{i}{0}). \tag{5.1}$$

This completely eliminates all phase space degrees of freedom. We are left with the gauge fixed action $S_h = \int d\tau \left\{ Tr\left(\frac{1}{2}(D_\tau g)\, g^{-1} \left(\begin{smallmatrix}\Gamma^{-'-} & 0\\ 0 & 0\end{smallmatrix}\right)\right) - 2h\tilde{A}\right\}$, where $(i\mathcal{L}) \to \frac{1}{2}\Gamma^{-'-}L^{+'-'}$, and $L^{+'-'}=1$. Due to the many zero entries in the $4{\times}4$ matrix $\Gamma^{-'-}$ [6], only one column from g in the form $\left(\begin{smallmatrix}Z_A\\ Z_5\end{smallmatrix}\right)$and one row from g^{-1} in the form $\left(\bar{Z}^A, -\bar{Z}_5\right)$ can contribute in the trace, and therefore the action becomes $S_h = \int d\tau \left\{ i\bar{Z}^A\dot{Z}_A - i\bar{Z}_5\dot{Z}_5 + \tilde{A}\left(\bar{Z}_5 Z_5 - 2h\right)\right\}$. Here $\bar{Z}_5\dot{Z}_5$ drops out as a total derivative since the magnitude of the complex number Z_5 is a constant $\bar{Z}_5 Z_5 = 2h$. Furthermore, we must take into account $\bar{Z}^A Z_A - \bar{Z}_5 Z_5 = 0$ which is an off-diagonal entry in the matrix equation $g^{-1}g = 1$. Then we see that the 2T-physics action (4.1) reduces to the twistor action (2.3) with the gauge choice (5.1)[6].

Next let us gauge fix the 2T-physics action (4.1) to a manifestly $SU(2,2)=SO(4,2)$ invariant version in flat 4+2 dimensions, in terms of the phase space & spin degrees of freedom $X^M, P^M, V_A, \bar{V}^A$. For this we use the $[SU(2,2)\times U(1)]_{left}$ symmetry to gauge fix g

$$\text{gauge fix:}\ \ g \to t(V) \in \frac{SU(2,3)}{[SU(2,2)\times U(1)]_{left}} \tag{5.2}$$

The coset element $t(V)$ is parameterized by the $SU(2,2)$ spinor V and its conjugate $\bar{V} = V^\dagger \eta_{2,2}$ and given by the 5×5 $SU(2,3)$ matrix[7]

$$t(V) = \begin{pmatrix} \left(1-2hV\bar{V}\right)^{-1/2} & 0 \\ 0 & \left(1-2h\bar{V}V\right)^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2h}V \\ \sqrt{2h}\bar{V} & 1 \end{pmatrix}. \tag{5.3}$$

The factor $2h$ is inserted for a convenient normalization of V. Note that the first matrix commutes with the second one, so it can be written in either order. The inverse of the group element is $t^{-1}(V) = (\eta_{2,3})\, t^\dagger (\eta_{2,3})^{-1} = t(-V)$, as can be checked explicitly $t(V)\,t(-V) = 1$. Inserting this gauge in (4.1) the action becomes

$$S_h = \int d\tau \left\{ \dot{X}\cdot P - \frac{1}{2}A^{ij}X_i\cdot X_j - \frac{1}{2}\Omega^{MN}L_{MN} - 2h\tilde{A}\left(\frac{\bar{V}\mathcal{L}V}{1-2h\bar{V}V} - 1\right)\right\} \tag{5.4}$$

$$= \int d\tau \left\{ \frac{1}{2}\varepsilon^{ij}\left(\hat{D}_\tau X_i^M\right) X_j^N \eta_{MN} - 2h\tilde{A}\left(\frac{\bar{V}\mathcal{L}V}{1-2h\bar{V}V} - 1\right)\right\} \tag{5.5}$$

where

$$\hat{D}_\tau X_i^M = \partial_\tau X_i^M - A_i^{\ j} X_j^M - \Omega^{MN} X_{iN} \tag{5.6}$$

is a covariant derivative for local $Sp(2,R)$ as well as local $SU(2,2)=SO(4,2)$ but with a composite $SO(4,2)$ connection $\Omega^{MN}(V(\tau))$ given conveniently in the following forms

$$\frac{1}{2}\Omega^{MN}\Gamma_{MN} = \left[(i\partial_\tau t)\, t^{-1}\right]_{SU(2,2)},\ \text{or}\ \ \frac{1}{2}\Omega^{MN}L_{MN} = -Tr\left((i\partial_\tau t)\, t^{-1}\begin{pmatrix}\mathcal{L} & 0\\ 0 & 0\end{pmatrix}\right). \tag{5.7}$$

Thus, Ω is the $SU(2,2)$ projection of the $SU(2,3)$ Cartan connection and given explicitly as

$$\frac{1}{2}\Omega^{MN}\Gamma_{MN} = 2h\frac{\left(\dot{V} - V\frac{\bar{V}\dot{V}}{\bar{V}V}\right)\bar{V} - V\left(\dot{\bar{V}} - \bar{V}\frac{\dot{\bar{V}}V}{\bar{V}V}\right)}{\sqrt{1-2h\bar{V}V}\left(1+\sqrt{1-2h\bar{V}V}\right)} + h\left(\frac{V\bar{V}}{\bar{V}V} - \frac{1}{4}\right)\frac{\bar{V}\dot{V} - \dot{\bar{V}}V}{\left(1-2h\bar{V}V\right)} \tag{5.8}$$

[6] In the high spin version of (4.1) without $\tilde{A}$ (see footnote (3)), we replace $Z_5 = e^{i\phi}\sqrt{\bar{Z}^A Z_A}$ and after dropping a total derivative, the twistor equivalent becomes $S_{all\ spins} = \int d\tau \left\{ i\bar{Z}^A\dot{Z}_A + \bar{Z}Z\dot{\phi}\right\}$. For a more covariant version that displays the $SU(2,3)$ global symmetry, we introduce a new $U(1)$ gauge field for the overall phase of $\binom{Z_A}{Z_5}$ and write $S_{all\ spins} = \int d\tau \left\{ i\bar{Z}^A\dot{Z}_A - i\bar{Z}_5\dot{Z}_5 + \tilde{B}\left(\bar{Z}^A Z_A - \bar{Z}_5 Z_5\right)\right\}$.

[7] Arbitrary fractional powers of the matrix $\left(1-2hV\bar{V}\right)$ are easily computed by expanding in a series and then resuming to obtain $\left(1-2hV\bar{V}\right)^\gamma = 1 + V\bar{V}\left(\left(1-2h\bar{V}V\right)^\gamma - 1\right)/\bar{V}V$.

The action (5.4,5.5) is manifestly invariant under global SU$(2,2)$ =SO$(4,2)$ rotations, and under local $U(1)$ phase transformations applied on $V_A, \bar{V}^A$. The conserved global symmetry currents $\mathcal{J}$ and J_0 can be derived either directly from (5.4) by using Noether's theorem, or by inserting the gauge fixed form of $g \to t(V)$ into Eq.(4.5)[8] $J_{(2,3)} = t^{-1}\left(\begin{smallmatrix}\mathcal{L} & 0\\ 0 & 0\end{smallmatrix}\right) t$

$$\mathcal{J}=\frac{1}{\sqrt{1-2hV\bar{V}}}\mathcal{L}\frac{1}{\sqrt{1-2hV\bar{V}}}-\frac{1}{4}J_0,\quad J_0=\frac{2h\bar{V}\mathcal{L}V}{1-2h\bar{V}V} \tag{5.9}$$

$$j_A=\sqrt{2h}\frac{1}{\sqrt{1-2hV\bar{V}}}\mathcal{L}V\frac{1}{\sqrt{1-2h\bar{V}V}} \tag{5.10}$$

According to the equation of motion for $\tilde{A}$ that follows from the action (5.4) we must have the following constraint (this means U(1) gauge invariant physical sector)

$$\frac{\bar{V}\mathcal{L}V}{1-2h\bar{V}V}=1. \tag{5.11}$$

Therefore, in the physical sector the conserved $[\mathrm{SU}(2,2)\times\mathrm{U}(1)]_{right}$ charges take the form

$$\text{physical sector: } J_0=2h,\quad \mathcal{J}=\frac{1}{\sqrt{1-2hV\bar{V}}}\mathcal{L}\frac{1}{\sqrt{1-2hV\bar{V}}}-\frac{h}{2}. \tag{5.12}$$

Let us now explain the local kappa symmetry of the action (5.4,5.5). The action (5.4) is still invariant under the bosonic local 3/4 kappa symmetry inherited from the action (4.1). The kappa transformations of $g(\tau)$ in the general action (5.4) correspond to local coset elements $\exp\left(\begin{smallmatrix}0 & K\\ \bar{K} & 0\end{smallmatrix}\right) \in\mathrm{SU}(2,3)_{left}/[\mathrm{SU}(2,2)\times\mathrm{U}(1)]_{left}$ with a special form of the spinor K_A

$$K_A=X_i\cdot\left(\Gamma\kappa^i(\tau)\right)_A=X_M\left(\Gamma^M\kappa^1\right)_A+P_M\left(\Gamma^M\kappa^2\right)_A, \tag{5.13}$$

with $\kappa^{iA}(\tau)$ two arbitrary local spinors[9]. Now that g has been gauge fixed $g\to t(V)$, the kappa transformation must be taken as the naive kappa transformation on g followed by a $[\mathrm{SU}(2,2)\times\mathrm{U}(1)]_{left}$ gauge transformation which restores the gauge fixed form of $t(V)$

$$t(V)\to t(V')=\left[\exp\begin{pmatrix}-\omega & 0\\ 0 & Tr(\omega)\end{pmatrix}\right]\left[\exp\begin{pmatrix}0 & K\\ \bar{K} & 0\end{pmatrix}\right]t(V) \tag{5.14}$$

The SU$(2,2)$ part of the restoring gauge transformation must also be applied on X^M, P^M. Performing these steps we find the infinitesimal version of this transformation [17]

$$\delta_\kappa V=\frac{1}{\sqrt{1-2hV\bar{V}}}K\frac{1}{\sqrt{1-2h\bar{V}V}},\quad \delta_\kappa X_i^M=\ \omega^{MN}X_{iN},\quad \delta_\kappa A^{ij}=\text{see below}, \tag{5.15}$$

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[8] In the high spin version ($\tilde{A}=0$) the conserved charges include j_A as part of SU$(2,3)_R$ global symmetry. It is then also convenient to rescale $\sqrt{2h}V\to V$ in Eqs.(5.3-5.10) to eliminate an irrelevant constant.

[9] In this special form only 3 out of the 4 components of K_A are effectively independent gauge parameters. This can be seen easily in the special frame for X^M, P^M given in Eq.(5.1).

where $\omega^{MN}(K,V)$ has the same form as Ω^{MN} in Eq.(5.8) but with $\dot{V}$ replaced by the $\delta_\kappa V$ given above. The covariant derivative $\hat{D}_\tau X_i^M$ in Eq.(5.6) is covariant under the local SU$(2,2)$ transformation with parameter $\omega^{MN}(K,V)$ (this is best seen from the projected Cartan connection form $\Omega = [(i\partial_\tau t)\,t^{-1}]_{SU(2,2)}$). Therefore, the kappa transformations (5.15) inserted in (5.5) give

$$\delta_\kappa S_h = \int d\tau \left\{ -\frac{1}{2}\left(\delta_\kappa A^{ij}\right) X_i\cdot X_j + iTr\left((D_\tau t)\,t^{-1} \begin{pmatrix} 0 & \mathcal{L}K \\ -\bar{K}\mathcal{L} & 0 \end{pmatrix}\right)\right\}. \tag{5.16}$$

In computing the second term the derivative terms that contain $\partial_\tau K$ have dropped out in the trace. Using Eq.(5.13) we see that

$$\begin{aligned} \mathcal{L}K &= \frac{1}{4i}\left(\varepsilon^{li}X_l^M X_i^N\right) X_j^L \Gamma_{MN}\Gamma_L \kappa^j && (5.17)\\ &= \frac{1}{4i}\varepsilon^{li}X_l^M X_i^N X_j^L \left(\Gamma_{MNL} + \eta_{NL}\Gamma_M - \eta_{ML}\Gamma_N\right)\kappa^l && (5.18)\\ &= \frac{1}{2i}\varepsilon^{li} X_i\cdot X_j\left(X_l\cdot\Gamma\kappa^j\right) && (5.19) \end{aligned}$$

The completely antisymmetric $X_i^M X_j^N X_l^L \Gamma_{MNL}$ term in the second line vanishes since i,j,l can only take two values. The crucial observation is that the remaining term in $\mathcal{L}K$ is proportional to the dot products $X_i\cdot X_j$. Therefore the second term in (5.16) is cancelled by the first term by choosing the appropriate $\delta_\kappa A^{ij}$ in Eq.(5.16), thus establishing the kappa symmetry.

The local kappa transformations (5.15) are also a symmetry of the global SU$(2,3)_R$ charges $\delta_\kappa \mathcal{J} = \delta_\kappa J_0 = \delta_\kappa j_A = 0$ provided the constraints $X_i\cdot X_j = 0$ are used. Hence these charges are kappa invariant in the physical sector.

We have established the global SO$(4,2)$ and local Sp$(2,R)\times$(3/4 kappa)$\times$U(1) symmetries of the phase space action (5.4) in 4+2 dimensions. From it we can derive all of the phase space actions of the systems listed in section (I) by making various gauge choices for the local Sp$(2,R)\times$(3/4 kappa)$\times$U(1) symmetries. This was demonstrated for the spinless case $h=0$ in [6]. The gauge choices for X^M, P^M discussed in [6] now need to be supplemented with gauge choices for $V_A, \bar{V}^A$ by using the kappa$\times$U(1) local symmetries.

Here we demonstrate the gauge fixing described above for the massless particle of any spin h. The kappa symmetry effectively has 3 complex gauge parameters as explained in footnote (9). If the kappa gauge is fixed by using two of its parameters we reach the following forms

$$V_A \to \begin{pmatrix} v^{\dot\alpha} \\ 0 \end{pmatrix},\ \bar{V}^A \to \left(0\ \bar{v}^{\alpha}\right),\ \ \bar{V}V\to 0,\ \left(1-2hV\bar{V}\right)^{-1/2} \to \begin{pmatrix} 1 & hv\bar{v} \\ 0 & 1 \end{pmatrix}. \tag{5.20}$$

By inserting this gauge fixed form of V, and the gauge fixed form of X, P given in Eq.(4.8), into the action (5.4) we immediately recover the SL$(2, C)$ covariant action of Eq.(3.2). The U(1) gauge symmetry is intact. The kappa symmetry of the action of Eq.(3.2) discussed in Eqs.(3.4,3.5) is the residual 1/4 kappa symmetry of the more general action ((5.4).

For other examples of gauge fixing that generates some of the systems in the list of section (I) see our related paper [1].

VI. GENERAL TWISTOR TRANSFORM (CLASSICAL)

The various formulations of spinning particles described above all contain gauge degrees of freedom of various kinds. However, they all have the global symmetry SU$(2,2)$=SO$(4,2)$ whose conserved charges $\mathcal{J}_A{}^B$ are gauge invariant in all the formulations. The most symmetric 2T-physics version gave the $\mathcal{J}_A{}^B$ as embedded in SU$(2,3)_R$ in the SU$(2,2)$ projected form in Eq.(4.5)

$$\mathcal{J} = \left[g^{-1} \begin{pmatrix} \mathcal{L} \, 0 \\ 0 \; 0 \end{pmatrix} g \right]_{SU(2,2)} . \tag{6.1}$$

Since this is gauge invariant, when gauge fixed, it must agree with the Noether charges computed in any version of the theory. So we can equate the general phase space version of Eq.(5.9) with the twistor version that follows from the Noether currents of (2.3) as follows

$$\mathcal{J} = Z^{(h)}\bar{Z}^{(h)} - \frac{1}{4} Tr\left(Z^{(h)}\bar{Z}^{(h)}\right) = \frac{1}{\sqrt{1-2hV\bar{V}}}\mathcal{L}\frac{1}{\sqrt{1-2hV\bar{V}}} - \frac{1}{4}J_0 \tag{6.2}$$

The trace corresponds to the U(1) charge $J_0 = Tr\left(Z^{(h)}\bar{Z}^{(h)}\right)$, so

$$\mathcal{J} + \frac{1}{4}J_0 = Z^{(h)}\bar{Z}^{(h)} = \frac{1}{\sqrt{1-2hV\bar{V}}}\mathcal{L}\frac{1}{\sqrt{1-2hV\bar{V}}}. \tag{6.3}$$

In the case of $h = 0$ this becomes

$$Z^{(0)}\bar{Z}^{(0)} = \mathcal{L}. \tag{6.4}$$

Therefore the equality (6.3) is solved up to an irrelevant phase by

$$Z^{(h)} = \frac{1}{\sqrt{1-2hV\bar{V}}} Z^{(0)}. \tag{6.5}$$

By inserting (6.4) into the constraint (5.11) we learn a new form of the constraints

$$\bar{V}Z^{(0)} = \sqrt{1-2h\bar{V}V}, \;\; \bar{V}Z^{(h)} = 1. \tag{6.6}$$

In turn, this implies

$$Z^{(0)} = \frac{\mathcal{L}V}{\sqrt{1-2h\bar{V}V}} \tag{6.7}$$

which is consistent[10] with $Z^{(0)}\bar{Z}^{(0)} = \mathcal{L}$, and its vanishing trace $\bar{Z}^{(0)}Z^{(0)} = 0$ since $\mathcal{L}\mathcal{L} = 0$ (due to $X^2 = P^2 = X \cdot P = 0$). Putting it all together we then have

$$Z^{(h)} = \frac{1}{\sqrt{1-2hV\bar{V}}}\mathcal{L}V\frac{1}{\sqrt{1-2h\bar{V}V}} = \left(\mathcal{J}+\frac{1}{4}J_0\right)V. \tag{6.8}$$

We note that this $Z^{(h)}$ is proportional to the non-conserved coset part of the SU$(2,3)$ charges $J_{2,3}$, that is $j_A = \sqrt{J_0}Z^{(h)}$ given in Eqs.(4.5,4.6) or (5.10), when g and $\mathcal{L}$ are replaced by their gauge fixed forms, and use the constraint[11] $J_0 = 2h$.

The key for the general twistor transform for any spin is Eq.(6.5), or equivalently (6.8). The general twistor transform between $Z^{(0)}$ and X^M, P^M which satisfies $Z^{(0)}\bar{Z}^{(0)} = \mathcal{L}$ is already given in [6] as

$$Z^{(0)} = \begin{pmatrix} \mu^{(0)} \\ \lambda^{(0)} \end{pmatrix}, \quad \left(\mu^{(0)}\right)^{\dot{\alpha}} = -i\frac{X^\mu}{X^{+'}}\left(\bar{\sigma}_\mu\lambda^{(0)}\right)^{\dot{\alpha}}, \quad \lambda_\alpha^{(0)}\bar{\lambda}_{\dot{\beta}}^{(0)} = \left(X^+P^\mu - X^\mu P^+\right)(\sigma_\mu)_{\alpha\dot{\beta}}. \tag{6.9}$$

Note that $(X^+P^\mu - X^\mu P^+)$ is compatible with the requirement that any SL$(2,C)$ vector constructed as $\lambda_\alpha^{(0)}\bar{\lambda}_{\dot{\beta}}^{(0)}$ must be lightlike. This property is satisfied thanks to the Sp$(2,R)$ constraints $X^2 = P^2 = X \cdot P = 0$ in 4+2 dimensions, thus allowing a particle of *any mass* in the $3+1$ subspace (since $P^\mu P_\mu$ is not restricted to be lightlike). Besides satisfying $Z^{(0)}\bar{Z}^{(0)} = \mathcal{L}$, this $Z^{(0)}$ also satisfies $\bar{Z}^{(0)}Z^{(0)} = 0$, as well as the canonical properties of twistors. Namely, $Z^{(0)}$ has the property [6]

$$\int d\tau\ \bar{Z}^{(0)}\partial_\tau Z^{(0)} = \int d\tau\ \dot{X}^M P_M. \tag{6.10}$$

From here, by gauge fixing the Sp$(2,R)$ gauge symmetry, we obtain the twistor transforms for all the systems listed in section (I) for $h = 0$ directly from Eq.(6.9), as demonstrated in [6]. All of that is now generalized at once to any spin h through Eq.(6.5). Hence (6.5) together with (6.9) tell us how to construct explicitly the *general* twistor $Z_A^{(h)}$ in terms

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10 To see this, we note that Eqs.(6.4,6.6) lead to $\frac{\mathcal{L}V\bar{V}\mathcal{L}}{1-2h\bar{V}V} = \frac{Z^{(0)}\bar{Z}^{(0)}V\bar{V}Z^{(0)}\bar{Z}^{(0)}}{1-2h\bar{V}V} = Z^{(0)}\bar{Z}^{(0)} = \mathcal{L}$.

11 For the high spin version ($\tilde{A} = 0$) we don't use the constraint. Instead, we use $Z^{(h)} = \frac{1}{\sqrt{1-2hV\bar{V}}}Z^{(0)}$ only in its form (6.5), and note that, after using Eq.(6.4), the j_A in Eq.(5.10) takes the form $j_A = \sqrt{J_0}Z^{(h)}$ with $\sqrt{J_0} = \frac{\bar{Z}^0 V\sqrt{2h}}{\sqrt{1-2h\bar{V}V}}$, and it is possible to rescale h away everywhere $\sqrt{2h}V \to V$.

of spin & phase space degrees of freedom $X^M, P^M, V_A, \bar{V}^A$. Then the Sp$(2,R)$ and kappa gauge symmetries that act on $X^M, P^M, V_A, \bar{V}^A$ can be gauge fixed for any spin h, to give the *specific* twistor transform for any of the systems under consideration.

We have already seen in Eq.(6.2) that the twistor transform (6.5) relates the conserved SU$(2,2)$ charges in twistor and phase space versions. Let us now verify that (6.5) provides the transformation between the twistor action (2.3) and the spin & phase space action (5.4). We compute the canonical structure as follows

$$\int d\tau\ \bar{Z}^{(h)}\partial_\tau Z^{(h)} = \int d\tau\ \bar{Z}^{(0)}\frac{1}{\sqrt{1-2hV\bar{V}}}\partial_\tau\left(\frac{1}{\sqrt{1-2hV\bar{V}}}Z^{(0)}\right) \tag{6.11}$$

$$= \int d\tau\ \left\{ \begin{array}{c} \bar{Z}^{(0)}\frac{1}{\sqrt{1-2hV\bar{V}}}\left(\partial_\tau\frac{1}{\sqrt{1-2hV\bar{V}}}\right)Z^{(0)} \\ +\bar{Z}^{(0)}\frac{1}{1-2hV\bar{V}}\partial_\tau Z^{(0)} \end{array} \right\} \tag{6.12}$$

$$= \int d\tau\ \left\{\dot{X}\cdot P + Tr\left[\left(i\partial_\tau t\right)t^{-1}\begin{pmatrix}\mathcal{L}\ 0\\ 0\ 0\end{pmatrix}\right]\right\} \tag{6.13}$$

The last form is the canonical structure of spin & phase space as given in (5.4). To prove this result we used Eq.(6.10), footnote (7), and the other properties of $Z^{(0)}$ including Eqs.(6.4-6.7), as well as the constraints $X^2 = P^2 = X\cdot P = 0$, and dropped some total derivatives. This proves that the canonical properties of $Z^{(h)}$ determine the canonical properties of spin & phase space degrees of freedom and vice versa.

Then, including the terms that impose the constraints, the twistor action (2.3) and the phase space action (5.4) are equivalent. Of course, this is expected since they are both gauge fixed versions of the master action (4.1), but is useful to establish it also directly via the general twistor transform given in Eq.(6.5).

VII. QUANTUM MASTER EQUATION, SPECTRUM, AND DUALITIES

In this section we derive the quantum algebra of the gauge invariant observables $\mathcal{J}_A{}^B$ and J_0 which are the conserved charges of $[\mathrm{SU}(2,2)\times\mathrm{U}(1)]_R$. Since these are gauge invariant symmetry currents they govern the system in any of its gauge fixed versions, including in any of its versions listed in section (I). From the quantum algebra we deduce the constraints among the physical observables $\mathcal{J}_A{}^B$,J_0 and quantize the theory covariantly. Among other things, we compute the Casimir eigenvalues of the unitary irreducible representation of SU$(2,2)$ which classifies the physical states in any of the gauge fixed version of the theory (with the different 1T-physics interpretations listed in section (I)).

The simplest way to quantize the theory is to use the twistor variables, and from them compute the gauge invariant properties that apply in any gauge fixed version. We will apply the covariant quantization approach, which means that the constraint due to the U(1) gauge symmetry will be applied on states. Since the quantum variables will generally not satisfy the constraints, we will call the quantum twistors in this section $Z_A, \bar{Z}^A$ to distinguish them from the classical $Z_A^{(h)}, \bar{Z}^{(h)A}$ of the previous sections that were constrained at the classical level. So the formalism in this section can also be applied to the high spin theories (discussed in several footnotes up to this point in the paper) by ignoring the constraint on the states.

According to the twistor action (2.3) Z_A and $i\bar{Z}^A$ (or equivalently λ_α and $i\bar{\mu}^\alpha$) are canonical conjugates. Therefore the quantum rules (equivalent to spin & phase space quantum rules) are

$$\left[Z_A, \bar{Z}^B\right] = \delta_A^{\ B}. \tag{7.1}$$

These quantum rules, as well as the action, are manifestly invariant under SU$(2,2)$. In covariant SU$(2,2)$ quantization the Hilbert space contains states which do not obey the U(1) constraint on the twistors. At the classical level the constraint was $J_0 = \bar{Z}Z = 2h$, but in covariant quantization this is obeyed only by the U(1) gauge invariant subspace of the Hilbert space which we call the physical states. The quantum version of the constraint requires $\hat{J}_0$ as a Hermitian operator applied on states (we write it as $\hat{J}_0$ to distinguish it from the classical version)

$$\hat{J}_0 = \frac{1}{2}\left(Z_A\bar{Z}^A + \bar{Z}^A Z_A\right), \quad \hat{J}_0|phys\rangle = 2h|phys\rangle. \tag{7.2}$$

The operator $\hat{J}_0$ has non-trivial commutation relations with $Z_A, \bar{Z}^{\ A}$ which follow from the basic commutation rules above

$$\left[\hat{J}_0, Z_A\right] = -Z_A, \quad \left[\hat{J}_0, \bar{Z}^{\ A}\right] = \bar{Z}^{\ A}. \tag{7.3}$$

By rearranging the orders of the quantum operators $Z_A\bar{Z}^A = \bar{Z}^A Z_A + 4$ we can extract from (7.2) the following relations

$$\bar{Z}Z = \hat{J}_0 - 2, \;\; Tr\left(Z\bar{Z}\right) = \hat{J}_0 + 2. \tag{7.4}$$

Furthermore, by using Noether's theorem for the twistor action (2.3) we can derive the 15 generators of SU$(2,2)$ in terms of the twistors and write them as a traceless 4×4 matrix

$\mathcal{J}_A{}^B$ at the quantum level as follows

$$\mathcal{J}_A{}^B = Z_A \bar{Z}^B - \frac{1}{4} Tr\left(Z\bar{Z}\right) \delta_A{}^B = \left(Z\bar{Z} - \frac{\hat{J}_0 + 2}{4}\right)_A^B . \tag{7.5}$$

In this expression the order of the quantum operators matters and gives rise to the shift $J_0 \to \hat{J}_0 + 2$ in contrast to the corresponding classical expression. The commutation rules among the generators $\mathcal{J}_A{}^B$ and the $Z_A, \bar{Z}^A$ are computed from the basic commutators (7.1),

$$\left[\mathcal{J}_A{}^B, Z_C\right] = -\delta_C{}^B Z_A + \frac{1}{4} Z_C\ \delta_A{}^B, \quad \left[\mathcal{J}_A{}^B, \bar{Z}^D\right] = \delta_A{}^D \bar{Z}^B - \frac{1}{4} \bar{Z}^D\ \delta_A{}^B \tag{7.6}$$

$$\left[\mathcal{J}_A{}^B, \mathcal{J}_C{}^D\right] = \delta_A{}^D \mathcal{J}_C{}^B - \delta_C{}^B \mathcal{J}_A{}^D, \quad \left[\hat{J}_0, \mathcal{J}_A{}^B\right] = 0. \tag{7.7}$$

We see from these that the gauge invariant observables $\mathcal{J}_A{}^B$ satisfy the SU$(2,2)$ Lie algebra, while the $Z_A, \bar{Z}^A$ transform like the quartets $4, \bar{4}$ of SU$(2,2)$. Note that the operator $\hat{J}_0$ commutes with the generators $\mathcal{J}_A{}^B$, therefore $\mathcal{J}_A{}^B$ is U(1) gauge invariant, and furthermore $\hat{J}_0$ must be a function of the Casimir *operators* of SU$(2,2)$. When $\hat{J}_0$ takes the value $2h$ on physical states, then the Casimir operators also will have eigenvalues on physical states which determine the SU$(2,2)$ representation in the physical sector.

From the quantum rules (7.3), it is evident that the U(1) generator $\hat{J}_0$ can only have integer eigenvalues since it acts like a number of operator. More directly, through Eq.(7.4) it is related to the number operator $\bar{Z}Z$. Therefore the theory is consistent at the quantum level (7.2) provided $2h$ is an integer.

Let us now compute the square of the matrix $\mathcal{J}_A{}^B$. By using the form (7.5) we have $(\mathcal{J}\mathcal{J}) = \left(Z\bar{Z} - \frac{\hat{J}_0+2}{4}\right)\left(Z\bar{Z} - \frac{\hat{J}_0+2}{4}\right) = Z\bar{Z}Z\bar{Z} - 2\frac{\hat{J}_0+2}{4} Z\bar{Z} + \left(\frac{\hat{J}_0+2}{4}\right)^2$ where we have used $\left[\hat{J}_0, Z_A\bar{Z}^B\right] = 0$. Now we elaborate $\left(Z\bar{Z}Z\bar{Z}\right)_A^B = Z_A\left(\hat{J}_0 - 2\right)\bar{Z}^B = \left(\hat{J}_0 - 1\right) Z_A\bar{Z}^B$ where we first used (7.4) and then (7.3). Finally we note from (7.5) that $Z_A\bar{Z}^B = \mathcal{J}_A{}^B + \frac{\hat{J}_0+2}{4}\delta_A{}^B$. Putting these observations together we can rewrite the right hand side of $(\mathcal{J}\mathcal{J})$ in terms of $\mathcal{J}$ and $\hat{J}_0$ as follows[12]

$$(\mathcal{J}\mathcal{J}) = \left(\frac{\hat{J}_0}{2} - 2\right)\mathcal{J} + \frac{3}{16}\left(\hat{J}_0^2 - 4\right). \tag{7.8}$$

[12] A similar structure at the classical level can be easily computed by squaring the expression for $\mathcal{J}$ in Eq.(6.2) and applying the classical constraint $J_0 = \bar{Z}^A Z_A = 2h$. This yields the classical version $\mathcal{J}_A{}^C \mathcal{J}_C{}^B = \frac{J_0}{2}\mathcal{J}_A{}^B + \frac{3}{16} J_0^2 \delta_A{}^B = h\mathcal{J}_A{}^B + \frac{3}{4}h^2\delta_A{}^B$, which is different than the quantum equation (7.8). Thus, the quadratic Casimir at the *classical* level is computed as $C_2 = \frac{3}{4}J_0^2 = 3h^2$ which is different than the quantum value in (7.16).

This equation is a constraint satisfied by the global $[\mathrm{SU}(2,2)\times\mathrm{U}(1)]_R$ charges $\mathcal{J}_A{}^B, \hat{J}_0$ which are gauge invariant physical observables. It is a correct equation for all the states in the theory, including those that do not satisfy the U(1) constraint (7.2). We call this the *quantum master equation* because it will determine completely all the $\mathrm{SU}(2,2)$ properties of the physical states for all the systems listed in section (I) for any spin.

By multiplying the master equation with $\mathcal{J}$ and using (7.8) again we can compute $\mathcal{J}\mathcal{J}\mathcal{J}$. Using this process repeatedly we find all the powers of the matrix $\mathcal{J}$

$$(\mathcal{J})^n = \alpha_n \mathcal{J} + \beta_n, \tag{7.9}$$

where

$$\alpha_n(\hat{J}_0) = \frac{1}{\hat{J}_0 - 1}\left[\left(\frac{3}{4}\left(\hat{J}_0 - 2\right)\right)^n - \left(\frac{-1}{4}\left(\hat{J}_0 + 2\right)\right)^n\right], \tag{7.10}$$

$$\beta_n(\hat{J}_0) = \frac{3}{16}\left(\hat{J}_0^2 - 4\right)\ \alpha_{n-1}(\hat{J}_0). \tag{7.11}$$

Remarkably, these formulae apply to all powers, including negative powers of the matrix $\mathcal{J}$. Using this result, any function of the matrix $\mathcal{J}$ constructed as a Taylor series takes the form

$$f\left(\mathcal{J}\right) = \alpha\left(\hat{J}_0\right)\mathcal{J} + \beta\left(\hat{J}_0\right) \tag{7.12}$$

where

$$\alpha\left(\hat{J}_0\right) = \frac{1}{\hat{J}_0 - 1}\left[f\left(\frac{3}{4}\left(\hat{J}_0 - 2\right)\right) - f\left(\frac{-1}{4}\left(\hat{J}_0 + 2\right)\right)\right], \tag{7.13}$$

$$\beta\left(\hat{J}_0\right) = \frac{1}{\hat{J}_0 - 1}\left[\frac{\left(\hat{J}_0 + 2\right)}{4} f\left(\frac{3}{4}\left(\hat{J}_0 - 2\right)\right) + \frac{3\left(\hat{J}_0 - 2\right)}{4} f\left(\frac{-1}{4}\left(\hat{J}_0 + 2\right)\right)\right]. \tag{7.14}$$

We can compute all the Casimir operators by taking the trace of $\mathcal{J}^n$ in Eq.(7.9), so we find[13]

$$C_n(\hat{J}_0) \equiv Tr\left(\mathcal{J}\right)^n = 4\beta_n(\hat{J}_0) = \frac{3}{4}\left(\hat{J}_0^2 - 4\right)\ \alpha_{n-1}(\hat{J}_0). \tag{7.15}$$

In particular the quadratic, cubic and quartic Casimir operators of $\mathrm{SU}(2,2) = \mathrm{SO}(6,2)$ are computed at the quantum level as

$$C_2(\hat{J}_0) = \frac{3}{4}\left(\hat{J}_0^2 - 4\right), \quad C_3(\hat{J}_0) = \frac{3}{8}\left(\hat{J}_0^2 - 4\right)\left(\hat{J}_0 - 4\right), \tag{7.16}$$

$$C_4(\hat{J}_0) = \frac{3}{64}\left(\hat{J}_0^2 - 4\right)\left(7\hat{J}_0^2 - 32\hat{J}_0 + 52\right). \tag{7.17}$$

[13] Other definitions of $C_n$could differ from ours by normalization or linear combinations of the $Tr\left(\mathcal{J}^n\right)$.

The eigenvalue of the operator $\hat{J}_0$ on physical states $\hat{J}_0|phys\rangle = 2h|phys\rangle$ completely fixes the unitary $\mathrm{SU}(2,2)$ representation that classifies the physical states, since the most general representation of $\mathrm{SO}(4,2)$ is labeled by the three independent eigenvalues of C_2, C_3 and C_4. Obviously, this result is a special representation of $\mathrm{SU}(2,2)$ since all the Casimir eigenvalues are determined in terms of a single half integer number h. Therefore we conclude that all of the systems listed in section (I) share the very same unitary representation of $\mathrm{SU}(2,2)$ with the same Casimir eigenvalues given above.

In particular, for spinless particles $\left(\hat{J}_0 \to h = 0\right)$ we obtain $C_2 = -3$, $C_3 = 6$, $C_4 = -\frac{39}{4}$, which is the unitary *singleton* representation of $\mathrm{SO}(4,2)$ $=\mathrm{SU}(2,2)$. This is in agreement with previous covariant quantization of the spinless particle in any dimension directly in phase space in $d+2$ dimensions, which gave for the $\mathrm{SO}(d,2)$ Casimir the eigenvalue as $C_2 = \frac{1}{2}L_{MN}L^{MN} \to 1 - d^2/4$ on physical states that satisfy $X^2 = P^2 = X \cdot P = 0$ [8]. So, for $d = 4$ we get $C_2 = -3$ in agreement with the quantum twistor computation above. Note that the classical computation either in phase space or twistor space would give the wrong answer $C_2 = 0$ when orders of canonical conjugates are ignored and constraints used classically.

Of course, having the same $\mathrm{SU}(2,2)$ Casimir eigenvalue is one of the *infinite number of duality relations* among these systems that follow from the more general twistor transform or the master 2T-physics theory (4.1). All dualities of these systems amount to all quantum functions of the gauge invariants $\mathcal{J}_A{}^B$ that take the same gauge invariant values in any of the physical Hilbert spaces of the systems listed in section (I).

All the physical information on the relations among the physical observables is already captured by the quantum master equation (7.8), so it is sufficient to concentrate on it. The predicted duality, including these relations, can be tested at the quantum level by computing and verifying the equality of an infinite number of matrix elements of the master equation between the dually related quantum states for the systems listed in section (I). In the case of the Casimir operators C_n the details of the individual states within a representation is not relevant, so that computation whose result is given above is among the simplest computations that can be performed on the systems listed in section (I) to test our duality predictions. This test was performed successfully for $h = 0$ at the quantum level for some of these systems directly in their own phase spaces [21], verifying for example, that the free massless particle, the hydrogen atom, the harmonic oscillator, the particle on AdS spaces, all have the same Casimir eigenvalues $C_2 = -3$, $C_3 = 6$, $C_4 = -\frac{39}{4}$ at the quantum level.

Much more elaborate tests of the dualities can be performed both at the classical and quantum levels by computing any function of the gauge invariant $\mathcal{J}_A{}^B$ and checking that it has the same value when computed in terms of the spin & phase space of any of the systems listed in section (I). At the quantum level all of these systems have the same Casimir eigenvalues of the C_n for a given h. So their spectra must correspond to the same unitary irreducible representation of $\mathrm{SU}(2,2)$ as seen above. But the rest of the labels of the representation correspond to simultaneously commuting operators that include the Hamiltonian. The Hamiltonian of each system is some operator constructed from the observables $\mathcal{J}_A{}^B$, and so are the other simultaneously diagonalizable observables. Therefore, the different systems are related to one another by unitary transformations that sends one Hamiltonian to another, but staying within the same representation. These unitary transformations are the quantum versions of the gauge transformations of Eq.(4.7), and so they are the duality transformations at the quantum level. In particular the twistor transform applied to any of the systems is one of those duality transformations. By applying the twistor transforms we can map the Hilbert space of one system to another, and then compute any function of the gauge invariant $\mathcal{J}_A{}^B$ between dually related states of different systems. The prediction is that all such computations within different systems must give the same result.

Given that $\mathcal{J}_A{}^B$ is expressed in terms of rather different phase space and spin degrees of freedom in each dynamical system with a different Hamiltonian, this predicted duality is remarkable. 1T-physics simply is not equipped to explain why or for which systems there are such dualities, although it can be used to check it. The origin as well as the proof of the duality is the unification of the systems in the form of the 2T-physics master action of Eq.(4.1) in 4+2 dimensions. The existence of the dualities, which can laboriously be checked using 1T-physics, is the evidence that the underlying spacetime is more beneficially understood as being a spacetime in 4+2 dimensions.

VIII. QUANTUM TWISTOR TRANSFORM

We have established a master equation for physical observables $\mathcal{J}$ at the quantum level. Now, we also want to establish the twistor transform at the quantum level expressed as much as possible in terms of the gauge invariant physical quantum observables $\mathcal{J}$. To this end we write the master equation (7.8) in the form

$$\left(\mathcal{J}-\frac{3}{4}\left(\hat{J}_0-2\right)\right)\left(\mathcal{J}+\frac{1}{4}\left(\hat{J}_0+2\right)\right)=0. \tag{8.1}$$

Recall the quantum equation (7.5) $\mathcal{J}+\frac{\hat{J}_0+2}{4}=Z\bar{Z}$, so the equation above is equivalent to

$$\left(\mathcal{J}-\frac{3}{4}\left(\hat{J}_0-2\right)\right)Z=0. \tag{8.2}$$

This is a 4×4 matrix eigenvalue equation with operator entries. The general solution is

$$Z=\left(\mathcal{J}+\frac{1}{4}\left(\hat{J}_0+2\right)\right)\hat{V} \tag{8.3}$$

where $\hat{V}_A$ is any spinor up to a normalization. This is verified by using the master equation (8.1) which gives $\left(\mathcal{J}-\frac{3}{4}\left(\hat{J}_0-2\right)\right)Z=\left(\mathcal{J}-\frac{3}{4}\left(\hat{J}_0-2\right)\right)\left(\mathcal{J}+\frac{1}{4}\left(\hat{J}_0+2\right)\right)\hat{V}=0$. Noting that the solution (8.3) has the same form as the classical version of the twistor transform in Eq.(6.8), except for the quantum shift $J_0\rightarrow\hat{J}_0+2$, we conclude that the $\hat{V}_A$ introduced above is the quantum version of the V_A discussed earlier (up to a possible renormalization[14]), as belonging to the coset SU$(2,3)$ /[SU$(2,2)$ ×U(1)].

Now $\hat{V}_A$ is a quantum operator whose commutation rules must be compatible with those of $Z_A,\bar{Z}^A,\hat{J}_0$ and $\mathcal{J}_A{}^B$. Its commutation rules with $\mathcal{J}_A{}^B,\hat{J}_0$ are straightforward and fixed uniquely by the SU$(2,2)$ ×U(1) covariance

$$\left[\hat{J}_0,\hat{V}_A\right]=-\hat{V}_A,\quad\left[\hat{J}_0,\overline{\hat{V}}^A\right]=\overline{\hat{V}}^A, \tag{8.4}$$

$$\left[\mathcal{J}_A{}^B,\hat{V}_C\right]=-\delta_C^B\hat{V}_A+\frac{1}{4}\hat{V}_C\,\delta_A^B\ ,\quad\left[\mathcal{J}_A{}^B,\overline{\hat{V}}^D\right]=\delta_A^D\overline{\hat{V}}^B-\frac{1}{4}\overline{\hat{V}}^D\delta_A^B. \tag{8.5}$$

Other quantum properties of $\hat{V}_A$ follow from imposing the quantum property $\bar{Z}Z=\hat{J}_0-2$ in (7.4). Inserting Z of the form (8.3), using the master equation, and observing the commutation rules (8.4), we obtain

$$\overline{\hat{V}}\left(\mathcal{J}+\frac{\hat{J}_0+2}{4}\right)\hat{V}=1. \tag{8.6}$$

[14] The quantum version of $\hat{V}$ is valid in the whole Hilbert space, not only in the subspace that satisfies the U(1) constraint $\hat{J}_0\rightarrow 2h$. In particular, in the high spin version, already at the classical level we must take $\hat{V}=V(\sqrt{2h}/\sqrt{J_0})$ and then rescale it $V\sqrt{2h}\rightarrow V$ as described in previous footnotes. So in the full quantum Hilbert space we must take $\hat{V}=\sqrt{2h}V(\hat{J}_0+\gamma)^{-1/2}$ (or the rescaled version $V\sqrt{2h}\rightarrow V$) with the possibly quantum shifted operator $(\hat{J}_0+\gamma)^{-1/2}$.

This is related to (5.11) if we take (5.9) into account by including the quantum shift $J_0 \to \hat{J}_0 + 2$. Considering (8.3) this equation may also be written as

$$\overline{\hat{V}} Z = \bar{Z}\hat{V} = 1. \tag{8.7}$$

Next we impose $\left[Z_A, \bar{Z}^B\right] = \delta_A^{\ B}$ to deduce the quantum rules for $[\hat{V}_A, \overline{\hat{V}}^B]$. After some algebra we learn that the most general form compatible with $\left[Z_A, \bar{Z}^B\right] = \delta_A^{\ B}$ is

$$\left[\hat{V}_A, \overline{\hat{V}}^B\right] = -\frac{\overline{\hat{V}}\hat{V}}{\hat{J}_0 - 1}\delta_A^{\ B} + \left(M(\mathcal{J} - 3\frac{\hat{J}_0 - 2}{4}) + (\mathcal{J} - 3\frac{\hat{J}_0 - 2}{4})\bar{M}\right)_A^{\ \ B}, \tag{8.8}$$

where $M_A^{\ B}$ is some complex matrix and $\bar{M} = (\eta_{2,2})\, M^\dagger\, (\eta_{2,2})^{-1}$. The matrix $M_A^{\ B}$ could not be determined uniquely because of the 3/4 kappa gauge freedom in the choice of $\hat{V}_A$ itself.

A maximally gauge fixed version of $\hat{V}_A$ corresponds to eliminating 3 of its components $\hat{V}_{2,3,4} = 0$ by using the 3/4 kappa symmetry, leaving only $A \equiv \hat{V}_1 \neq 0$. Then we find $\overline{\hat{V}}^{1,2,4} = 0$ and $\bar{V}^3 = A^\dagger$. Let us analyze the quantum properties of this gauge in the context of the formalism above. From Eq.(8.6) we determine $A = (\mathcal{J}_3^{\ 1})^{-1/2} e^{-i\phi}$, where ϕ is a phase, and then from Eq.(8.3) we find Z_A.

$$Z_A = \left(\mathcal{J}_A^{\ \ 1} + \frac{\hat{J}_0 + 2}{4}\delta_A^{\ \ 1}\right) (\mathcal{J}_3^{\ 1})^{-1/2} e^{-i\phi}, \quad \bar{Z}^A = e^{i\phi} (\mathcal{J}_3^{\ 1})^{-1/2} \left(\mathcal{J}_3^{\ \ A} + \frac{\hat{J}_0 + 2}{4}\delta_3^{\ \ A}\right). \tag{8.9}$$

We see that, except for the overall phase, Z_A is completely determined in terms of the gauge invariant $\mathcal{J}_A^{\ B}$. We use a set of gamma matrices Γ^M given in ([6],[11]) to write $\mathcal{J}_A^{\ B} = \frac{1}{4i} J^{MN} \left(\Gamma_{MN}\right)_A^{\ B}$ as an explicit matrix so that Z_A can be written in terms of the 15 SO$(4,2)$ =SU$(2,2)$ generators J^{MN}. We find

$$Z_A = \begin{pmatrix} \frac{1}{2}J^{12} + \frac{1}{2i}J^{+-} + \frac{1}{2i}J^{+'-'} + \frac{\hat{J}_0+2}{4} \\ \frac{i}{\sqrt{2}}\left(J^{+1} + iJ^{+2}\right) \\ J^{+'+} \\ \frac{i}{\sqrt{2}}\left(J^{+'1} + iJ^{+'2}\right) \end{pmatrix} \frac{e^{-i\phi}}{\sqrt{J^{+'+}}}, \tag{8.10}$$

and $\bar{Z}^A = \left(Z^\dagger \eta_{2,2}\right)^A$. The orders of the operators here are important. The basis $M = \pm', \pm, i$ with $i = 1, 2$ corresponds to using the lightcone combinations $X^{\pm'} = \frac{1}{\sqrt{2}}\left(X^{0'} \pm X^{1'}\right)$, $X^{\pm} = \frac{1}{\sqrt{2}}\left(X^0 \pm X^1\right)$.

From our setup above, the $Z_A, \bar{Z}^A$ in (8.10) are guaranteed to satisfy the twistor commutation rules $\left[Z_A, \bar{Z}^B\right] = \delta_A^{\ B}$ provided we insure that the $\hat{V}_A, \overline{\hat{V}}^B$ have the quantum properties given in Eqs.(8.4,8.5,8.8). These are satisfied provided we take the following non-trivial commutation rules for ϕ

$$\left[\phi, \hat{J}_0\right] = i,\ [\phi, J_{12}] = \frac{i}{2}\ \Rightarrow \left[\hat{J}_0, e^{\pm i\phi}\right] = \pm e^{\pm i\phi},\ [J^{12}, e^{\pm i\phi}] = \pm\frac{1}{2}e^{\pm i\phi} \tag{8.11}$$

while all other commutators between ϕ and J^{MN} vanish. Then (8.8) becomes $[\hat{V}_A, \overline{\hat{V}}^B] = 0$, so $M_A^{\ B}$ vanishes in this gauge. Indeed one can check directly that only by using the Lie algebra for the J^{MN}, $\hat{J}_0$ and the commutation rules for ϕ in (8.11), we obtain $\left[Z_A, \bar{Z}^B\right] = \delta_A^{\ B}$, which a remarkable form of the twistor transform at the quantum level.

The expression (8.10) for the twistor is not $\mathrm{SU}(2,2)$ covariant. Of course, this is because we chose a non-covariant gauge for $\hat{V}_A$. However, the global symmetry $\mathrm{SU}(2,2)$ is still intact since the correct commutation rules between the twistors and J^{MN} or the $\mathcal{J}_A^{\ \ B}$ as given in (7.6,7.7) are built in, and are automatically satisfied. Therefore, despite the lack of manifest covariance, the expression for Z_A in (8.10) transforms covariantly as the spinor of $\mathrm{SU}(2,2)$.

It is now evident that one has many choices of gauges for $\hat{V}_A$. Once a gauge is picked the procedure outlined above will automatically produce the *quantum* twistor transform in that gauge, and it will have the correct commutation rules and $\mathrm{SU}(2,2)$ properties at the quantum level. For example, in the $\mathrm{SL}(2,C)$ covariant gauge of Eq.(5.20), the quantum twistor transform in terms of J^{MN} is

$$\mu^{\dot{\alpha}} = \frac{1}{4i}J_{\mu\nu}\left(\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}_{\ \dot{\beta}}\, v^{\dot{\beta}} + \frac{1}{2i}J^{+'-'}v^{\dot{\alpha}},\ \ \lambda_\alpha = \frac{1}{\sqrt{2}}J^{+'\mu}\left(\sigma_\mu\right)_{\alpha\dot{\beta}}\, v^{\dot{\beta}}. \tag{8.12}$$

with the constraint

$$\frac{1}{\sqrt{2}}\bar{v}\sigma_\mu v J^{+'\mu} = 1. \tag{8.13}$$

This gauge for $\hat{V}_M$ covers several of the systems listed in section (I). The spinless case was discussed at the classical level in ([6]). The quantum properties of this gauge are discussed in more detail in ([1]).

The result for Z_A in (8.10) is a quantum twistor transform that relies *only on the gauge invariants* $\mathcal{J}_A^{\ B}$ or equivalently J^{MN}. It generalizes a similar result in [6] that was given at the classical level. In the present case it is quantum and with spin. All the information on spin is included in the generators $J^{MN} = L^{MN} + S^{MN}$. There are other ways of describing spinning particles. For example, one can start with a 2T-physics action that uses fermions $\psi^M(\tau)$ [22] instead of our bosonic variables $V_A(\tau)$. Since we only use the gauge invariant J^{MN}, our quantum twistor transform (8.3) applies to all such descriptions of spinning particles, with an appropriate relation between $\hat{V}$ and the new spin degrees of freedom. In particular in the gauge fixed form of $\hat{V}$ that yields (8.10) there is no need to seek a relation between $\hat{V}$ and the other spin degrees of freedom. Therefore, in the form (8.10), if the J^{MN} are produced with the correct quantum algebra SU$(2,2)$ =SO$(4,2)$ in *any theory*, (for example bosonic spinors, or fermions ψ^M, or the list of systems in section (I), or any other) then our formula (8.3) gives the twistor transform for the corresponding degrees of freedom of that theory. Those degrees of freedom appear as the building blocks of J^{MN}. So, the machinery proposed in this section contains some very powerful tools.

IX. THE UNIFYING SU$(2,3)$ LIE ALGEBRA

The 2T-physics action (4.1) offered the group SU$(2,3)$ as the most symmetric unifying property of the spinning particles for all the systems listed in section (I), including twistors. Here we discuss how this fundamental underlying structure governs and simplifies the quantum theory.

We examine the SU$(2,3)$ charges $\mathcal{J}_A{}^B, \hat{J}_0, j_A, \bar{j}^A$ given in (4.5,5.9,5.10). Since these are gauge invariant under all the gauge symmetries (4.7) they are physical quantities that should have the properties of the Lie algebra[15] of SU$(2,3)$ in all the systems listed in section (I). Using covariant quantization we construct the quantum version of all these charges in terms of twistors. By using the general quantum twistor transform of the previous section, these charges can also be written in terms of the quantized spin and phase space degrees of freedom of any of the relevant systems.

The twistor expressions for $\hat{J}_0, \mathcal{J}_A{}^B$ are already given in Eqs.(7.2,7.5)

$$\hat{J}_0 = \frac{1}{2}\left(Z_A\bar{Z}^A + \bar{Z}^A Z_A\right), \quad \mathcal{J}_A{}^B = Z_A\bar{Z}^B - \frac{\hat{J}_0 + 2}{4}\delta_A^{\ B}. \tag{9.1}$$

We have seen that at the classical level $(j_A)_{classical} = \sqrt{J_0}Z_A$ and now we must figure out

[15] Even when j_A is not a conserved charge when the U(1) constraint is imposed, its commutation rules are still the same in the covariant quantization approach, independently than the constraint.

the quantum version $j_A = \sqrt{\hat{J}_0 + \alpha} Z_A$ that gives the correct $\mathrm{SU}(2,3)$ closure property

$$\left[j_A, \bar{j}^B\right] = \mathcal{J}_A{}^B + \frac{5}{4}\hat{J}_0 \delta_A{}^B. \tag{9.2}$$

The coefficient $\frac{5}{4}$ is determined by consistency with the Jacobi identity $\left[\left[j_A, \bar{j}^B\right], j_C\right] + \left[\left[\bar{j}^B, j_C\right], j_A\right] + \left[\left[j_C, j_A,\right], \bar{j}^B\right] = 0$, and the requirement that the commutators of j_A with $\mathcal{J}_A{}^B, \hat{J}_0$ be just like those of Z_A given in Eqs.(7.6,7.7), as part of the $\mathrm{SU}(2,3)$ Lie algebra. So we carry out the computation in Eq.(9.2) as follows

$$\begin{aligned}
\left[j_A, \bar{j}^B\right] &= \sqrt{\hat{J}_0 + \alpha} Z_A \bar{Z}^B \sqrt{\hat{J}_0 + \alpha} - \bar{Z}^B \sqrt{\hat{J}_0 + \alpha} \sqrt{\hat{J}_0 + \alpha} Z_A && (9.3)\\
&= \left(\hat{J}_0 + \alpha\right) Z_A \bar{Z}^B - \left(\hat{J}_0 + \alpha - 1\right) \bar{Z}^B Z_A && (9.4)\\
&= \left(\hat{J}_0 + \alpha - 1\right) \left[Z_A, \bar{Z}^B\right] + Z_A \bar{Z}^B && (9.5)\\
&= \delta_A{}^B \left(\hat{J}_0 + \alpha - 1 + \frac{\hat{J}_0 + 2}{4}\right) + \mathcal{J}_A{}^B && (9.6)
\end{aligned}$$

To get (9.4) we have used the properties $Z_A f\left(\hat{J}_0\right) = f\left(\hat{J}_0 + 1\right) Z_A$ and $\bar{Z}^B f\left(\hat{J}_0\right) = f\left(\hat{J}_0 - 1\right) \bar{Z}^B$ for any function $f\left(\hat{J}_0\right)$. These follow from the commutator $\left[\hat{J}_0, Z_A\right] = -Z_A$ written in the form $Z_A \hat{J}_0 = \left(\hat{J}_0 + 1\right) Z_A$ which is used repeatedly, and similarly for $\bar{Z}^B$. To get (9.6) we have used $\left[Z_A, \bar{Z}^B\right] = \delta_A{}^B$ and then used the definitions (9.1). By comparing (9.6) and (9.2) we fix $\alpha = 1/2$. Hence the correct quantum version of j_A is

$$j_A = \sqrt{\hat{J}_0 + \frac{1}{2}} Z_A = Z_A \sqrt{\hat{J}_0 - \frac{1}{2}}\ . \tag{9.7}$$

The second form is obtained by using $Z_A f\left(\hat{J}_0\right) = f\left(\hat{J}_0 + 1\right) Z_A$.

Note the following properties of the $j_A, \bar{j}^A$

$$\begin{aligned}
\bar{j}^A j_A &= \sqrt{\hat{J}_0 - \frac{1}{2}} \bar{Z} Z \sqrt{\hat{J}_0 - \frac{1}{2}} = \left(\hat{J}_0 - \frac{1}{2}\right)\left(\hat{J}_0 - 2\right) && (9.8)\\
j_A \bar{j}^B &= \sqrt{\hat{J}_0 + \frac{1}{2}} Z_A \bar{Z}^B \sqrt{\hat{J}_0 + \frac{1}{2}} = \left(\hat{J}_0 + \frac{1}{2}\right)\left(\mathcal{J} + \frac{1}{4}\left(\hat{J}_0 + 2\right)\right) && (9.9)
\end{aligned}$$

which will be used below.

With the above arguments we have now constructed the quantum version of the $\mathrm{SU}(2,3)$ charges written as a 5×5 traceless matrix

$$\begin{aligned}
\hat{J}_{2,3} &= \left(g^{-1} \begin{pmatrix} \mathcal{L} & 0 \\ 0 & 0 \end{pmatrix} g\right)_{\text{quantum}} = \begin{pmatrix} \mathcal{J} + \frac{1}{4}\hat{J}_0 & j \\ -\bar{j} & -\hat{J}_0 \end{pmatrix} && (9.10)\\
&= \begin{pmatrix} Z_A \bar{Z}^B - \frac{1}{2}\delta_A{}^B & \sqrt{\hat{J}_0 + \frac{1}{2}} Z_A \\ -\bar{Z}^B \sqrt{\hat{J}_0 + \frac{1}{2}} & -\hat{J}_0 \end{pmatrix}, && (9.11)
\end{aligned}$$

with $\hat{J}_0, \mathcal{J}$ given in Eq.(9.1).

At the classical level, the square of the matrix $J_{2,3}$ vanishes since $\mathcal{L}^2 = 0$ as follows

$$\left[(J_{2,3})^2\right]_{classical} = \left(g^{-1}\begin{pmatrix}\mathcal{L} & 0\\ 0 & 0\end{pmatrix}g\right)\left(g^{-1}\begin{pmatrix}\mathcal{L} & 0\\ 0 & 0\end{pmatrix}g\right) = g^{-1}\begin{pmatrix}\mathcal{L}^2 & 0\\ 0 & 0\end{pmatrix}g = 0. \tag{9.12}$$

At the quantum level we find the following non-zero result which is SU$(2,3)$ covariant

$$\left(\hat{J}_{2,3}\right)^2 = \begin{pmatrix} Z\bar{Z} - \frac{1}{2} & \sqrt{\hat{J}_0 + \frac{1}{2}}Z \\ -\bar{Z}\sqrt{\hat{J}_0 + \frac{1}{2}} & -\hat{J}_0 \end{pmatrix}^2 \tag{9.13}$$

$$= -\frac{5}{2}\left(\hat{J}_{2,3}\right) - 1. \tag{9.14}$$

By repeatedly using the same equation we can compute all powers $\left(\hat{J}_{2,3}\right)^n$, and by taking traces we obtain the Casimir eigenvalues of the SU$(2,3)$ representation. For example the quadratic Casimir is

$$Tr\left(\left(\hat{J}_{2,3}\right)^2\right) = -5. \tag{9.15}$$

Written out in terms of the charges, Eq.(9.14) becomes

$$\begin{pmatrix} \mathcal{J}+\frac{1}{4}\hat{J}_0 & j \\ -\bar{j} & -\hat{J}_0 \end{pmatrix}^2 = -\frac{5}{2}\begin{pmatrix} \mathcal{J}+\frac{1}{4}\hat{J}_0 & j \\ -\bar{j} & -\hat{J}_0 \end{pmatrix} - 1. \tag{9.16}$$

Collecting terms in each block we obtain the following relations among the gauge invariant charges $\mathcal{J}, \hat{J}_0, j, \bar{j}$

$$\left(\mathcal{J}+\frac{1}{4}\hat{J}_0\right)^2 - j\bar{j} + \frac{5}{2}\left(\mathcal{J}+\frac{1}{4}\hat{J}_0\right) + 1 = 0, \tag{9.17}$$

$$\left(\mathcal{J}+\frac{1}{4}\hat{J}_0\right)j - j\hat{J}_0 + \frac{5}{2}j = 0, \tag{9.18}$$

$$-\bar{j}j + \left(\hat{J}_0\right)^2 - \frac{5}{2}\hat{J}_0 + 1 = 0. \tag{9.19}$$

Combined with the information in Eq.(9.9) the first equation is equivalent to the master quantum equation (7.8). After using $j\hat{J}_0 = \hat{J}_0 j + j$, the second equation is equivalent to the eigenvalue equation (8.2) whose solution is the quantum twistor transform (8.3). The third equation is identical to (9.8).

Hence the SU$(2,3)$ quantum property $\left(\hat{J}_{2,3}\right)^2 = -\frac{5}{2}\left(\hat{J}_{2,3}\right) - 1$, or equivalently $\left(\hat{J}_{2,3} + 2\right)\left(\hat{J}_{2,3} + \frac{1}{2}\right) = 0$, governs the quantum dynamics of all the sytems listed in section (I) and captures all of the physical information, twistor transform, and dualities as a property of a fixed $\mathrm{SU}(2,3)$ representation whose generators satisfy the given constraint. This is a remarkable simple unifying description of a diverse set of spinning systems, that shows the existence of the sophisticated higher structure $\mathrm{SU}(2,3)$ for which there was no clue whatsoever from the point of view of 1T-physics.

X. FUTURE DIRECTIONS

One can consider several paths that generalizes our discussion, including the following.

- It is straightforward to generalize our theory by replacing $\mathrm{SU}(2,3)$ with the supergroup $\mathrm{SU}(2,(2+n)\,|N)$. This generalizes the spinor V_A to V_A^a where a labels the fundamental representation of the supergroup $\mathrm{SU}(n|N)$. The case of $N=0$ and $n=1$ is what we discussed in this paper. The case of $n=0$ and any N is the superparticle with N supersymmetries discussed in [17] and in [6][7]. The cases with $n>1$ and $N=0$, or both n and N non-zero, have not been discussed so far in any form in the literature. This model has global symmetry $\mathrm{SU}(2,2)\times\mathrm{SU}(n|N)\times\mathrm{U}(1)\subset[\mathrm{SU}(2,(2+n)\,|N)]_R$ if a U(1) gauging is included, or the full global symmetry $[\mathrm{SU}(2,(2+n)\,|N)]_R$ in its high spin version. It also has local gauge symmetries that include bosonic & fermionic kappa symmetries embedded in $[\mathrm{SU}(2,(2+n)\,|N)]_L$ as well as the basic $\mathrm{Sp}(2,R)$ gauge symmetry. The gauge symmetries insure that the theory has no negative norm states. In the massless particle gauge, this model corresponds to supersymmetrizing spinning particles rather than supersymmetrizing the zero spin particle. The usual R-symmetry group in SUSY is replaced here by $\mathrm{SU}(n|N)\times\mathrm{U}(1)$. For all these cases with non-zero n,N, the 2T-physics and twistor formalisms unify a large class of new 1T-physics systems and establishes dualities among them.

- One can generalize our discussion in 4+2 dimensions, including the previous paragraph, to higher dimensions. The starting point in 4+2 dimensions was $\mathrm{SU}(2,2)=\mathrm{SO}(4,2)$ embedded in $g=\mathrm{SU}(2,3)$. For higher dimensions we start from $\mathrm{SO}(d,2)$ and seek a group or supergroup that contains $\mathrm{SO}(d,2)$ in the spinor representation. For example for 6+2 dimensions, the starting point is the 8×8 spinor version of $\mathrm{SO}(8^*)=\mathrm{SO}(6,2)$ embedded in $g=\mathrm{SO}(9^*)=\mathrm{SO}(6,3)$ or $g=\mathrm{SO}(10^*)=\mathrm{SO}(6,4)$. The spinor variables in 6+2 dimensions V_A will then be the spinor of $\mathrm{SO}(8^*)=\mathrm{SO}(6,2)$ parametrizing the coset $\mathrm{SO}(9^*)/\mathrm{SO}(8^*)$ (real spinor) or $\mathrm{SO}(10^*)/\mathrm{SO}(8^*)\times\mathrm{SO}(2)$ (complex spinor). This can be supersymmetrized. The pure superparticle version of this program for various dimensions is discussed in [6][7], where all the relevant supergroups are classified. That discussion can now be taken further by including bosonic variables embedded in a supergroup as just outlined in the previous item. As explained before [6][7], it must be mentioned that when $d+2$ exceeds $6+2$ it seems that we need to include also brane degrees of freedom in addition to particle degrees of freedom.

- The methods in this paper overlap with those in [23] where a similar master quantum equation technique for the supergroup $SU(2,2|4)$ was used to describe the spectrum of type-IIB supergravity compactified on $AdS_5\times S^5$. So our methods have a direct bearing on M theory. In the case of [23] the matrix insertion $\left(\begin{smallmatrix}\mathcal{L} & 0\\ 0 & 0\end{smallmatrix}\right)$ in the 2T-physics action was generalized to $\begin{pmatrix}\mathcal{L}_{(4,2)} & 0\\ 0 & \mathcal{L}_{(6,0)}\end{pmatrix}$ to describe a theory in 10+2 dimensions. This approach to higher dimensions can avoid the brane degrees of freedom and concentrate only on the particle limit. Similar generalizations can be used with our present better develped methods and richer set of groups mentioned above to explore various corners of M theory.

- One of the projects in 2T-physics is to take advantage of its flexible gauge fixing mechanisms in the context of 2T-physics field theory. Applying this concept to the 2T-physics version of the Standard Model [10] will generate duals to the Standard Model in 3+1 dimensions. The study of the duals could provide some non-perturbative or other physical information on the usual Standard Model. This program is about to be launched in the near future [24]. Applying the twistor techniques developed here to 2T-physics field theory should shed light on how to connect the Standard Model with a twistor version. This could lead to further insight and to new computational techniques for the types of twistor computations that proved to be useful in QCD [12][13].

- Our new models and methods can also be applied to the study of high spin theories by generalizing the techniques in [14] which are closely related to 2T-physics. The high spin version of our model has been discussed in many of the footnotes, and can be supersymmetrized and written in higher dimensions as outlined above in this section. The new ingredient is the bosonic spinor V_A and the higher symmetry, such as $SU(2,3)$

and its generalizations in higher dimensions or with supersymmetry. The last three sections on the quantum theory discussed in this paper would apply also in the high spin version of our theory.

- One can consider applying the bosonic spinor that worked well in the particle case to strings and branes. This may provide new string backgrounds with spin degrees of freedom other than the familiar Neveu-Schwarz or Green-Schwarz formulations that involve fermions.

More details and applications of our theory will be presented in a companion paper [1].

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