INTERPOLATING AND SAMPLING SEQUENCES IN FINITE RIEMANN SURFACES

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ABSTRACT. We provide a description of the interpolating and sampling sequences on a space of holomorphic functions with a uniform growth restriction defined on finite Riemann surfaces.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let S be an open finite Riemann surface endowed with the Poincaré (hyperbolic) metric. We will study some properties of holomorphic functions in the Riemann surface with uniform growth control. Namely we will deal with the Banach space $A_\phi(S)$ of holomorphic functions in S such that $\|f\| := \sup_S |f|e^{-\phi} < \infty$ where ϕ is a given subharmonic function that controls the growth of the functions in the space.

The fact that ϕ is subharmonic is a natural assumption on the weight that limits the growth since any other growth control given by a weight ψ, $\|f\|_* = \sup_S |f|e^{-\psi}$ can be replaced by an equivalent subharmonic function because $\phi = \sup_{\|f\|_*\le 1} \log|f|$ is a subharmonic function and $A_\psi(S) = A_\phi(S)$ with equality of norms, $\sup_S |f|e^{-\psi} = \sup_S |f|e^{-\phi}$.

We have fixed a metric. It is then natural to restrict the possible weights ϕ, in a way that the functions in A_ϕ oscillate in a controlled way when the points are nearby in the Poincaré metric. This is achieved for instance by assuming that ϕ has bounded Laplacian (the Laplace-Beltrami operator with respect to the hyperbolic measure). That is, if in a local coordinate chart the Poincaré metric is of the form $ds^2 = e^{2\nu(z)}|dz|^2$, then we assume that $\Delta\phi = 4e^{-2\nu(z)}\frac{\partial^2\phi}{\partial z\partial\bar z}$ satisfies $C^{-1} \le \Delta\phi \le C$. If we want to deal with other weights then it is possible to introduce a natural metric associated to the weight as it is done in the plane in [MMOC03]. In this work we will only consider the Poincaré metric and bounded Laplacian since it already covers many interesting cases and it is technically simpler.

The problems that we will consider are the following:

(A) The description of the interpolating sequences for $A_\phi(S)$: i.e. the sequences $\Lambda \subset S$ such that it is always possible to find an $f \in A_\phi(S)$ such that $f(\lambda) = v_\lambda$ for all $\lambda \in \Lambda$ whenever the data $\{v_\lambda\}_\Lambda$, satisfies the compatibility condition $\sup_\Lambda |v_\lambda|e^{-\phi(\lambda)} < +\infty$

(B) The description of sampling sets for $A_\phi(S)$: i.e. the sets $E \subset S$ such that there is a constant $C > 0$ that satisfies

$$\sup_S |f|e^{-\phi} \le C \sup_E |f|e^{-\phi}, \quad \forall f \in A_\phi(S).$$

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In the solution of these problems the Poincaré distance and the potential theory in the surface play a key role. This has already been observed by A. Schuster and D. Varolin in [SV04], where they provide sufficient conditions for a sequence to be interpolating/sampling for functions in a slightly different context where the weighted uniform control of the growth of the functions is replaced by a weighted L^2 control. Their condition basically coincides with the description that we reach so our work can be considered as the counterpart of their theorems, although we will give a different proof of their results as well. We will rely on the well-known case of the disk and some simplifying properties of finite Riemann surfaces. Their method of proof looks more promising if one wants to extend the result to Riemann surfaces with more complicated topology.

When the surface is a disk, which will be our model situation, the corresponding problems have been solved in [BOC95], [OCS98] and in a different way in [Sei98]. Of course, the more basic problem of describing the interpolating sequences for bounded holomorphic functions in finite Riemann surfaces (in our notation $\phi \equiv 0$), has been known for a long time, see [Sto65]).

We introduce now some definitions that will be needed to state our results. For any point $z \in S$ and any $r > 0$ we denote by $D(z,r)$ the domain in the surface S that consits of points at hyperbolic distance from z less than r. They are topological disks if the center z is outside a big compact of S, or if r is small enough, as we will see in Section 2.

A sequence Λ of points in S is hyperbolically separated if there is an $\varepsilon > 0$ such that the domains $\{D(\lambda,\varepsilon)\}_{\lambda\in\Lambda}$ are pairwise disjoint.

Let $g_r(z,w)$ be the Green function associated to the surface $D(z,r)$ with pole at the "center" z and $g(z,w) = g_\infty(z,w)$ be the Green function associated to the surface S. We define the densities

$$
\begin{aligned}
D^+_\phi(\Lambda) &:= \limsup_{r\to\infty} \sup_{z\in S} \frac{\displaystyle\sum_{\substack{1/2<d(z,\lambda)<r\\ \lambda\in\Lambda}} g_r(z,\lambda)}{\displaystyle\int_{D(z,r)} g_r(z,w) i\partial\bar\partial\phi(w)}.\\
D^-_\phi(\Lambda) &:= \liminf_{r\to\infty} \inf_{z\in S} \frac{\displaystyle\sum_{\substack{1/2<d(z,\lambda)<r\\ \lambda\in\Lambda}} g_r(z,\lambda)}{\displaystyle\int_{D(z,r)} g_r(z,w) i\partial\bar\partial\phi(w)}.
\end{aligned}
\tag{1}
$$

The main result is

Theorem 1. *Let S be a finite Riemann surface and let ϕ be a subharmonic function with bounded Laplacian.*

(A) *A sequence $\Lambda \subset S$ is an interpolating sequence for $A_\phi(S)$ if and only if it is hyperbolically separated and $D^+_\phi(\Lambda) < 1$.*

(B) *A set $E \subset S$ is a sampling set for $A_\phi(S)$ if and only if it contains an hyperbolically separated sequence $\Lambda \subset E$ such that $D^-_\phi(\Lambda) > 1$.*

In Section 2 we will prove some key properties of finite Riemann surfaces. In particular we need to study the behavior of the hyperbolic metric as we approach the boundary of the surface. We will also prove some weighted uniform estimates for the inhomogeneous Cauchy-Riemann equation in the surface, Theorem 6, that has an interest by itself.

In the next section, we use the tools and Lemmas proved in Section 2 to reduce the interpolating and sampling problem in S to a problem near the boundary that can be reduced to the known case of the disk.

Finally in Section 4 we show how our results can be extended to other Banach spaces of holomorphic functions where the uniform growth is replaced by weighted L^p spaces.

A final word on notation. By $f \lesssim g$ we mean that there is a constant C independent of the relevant variables such that $f \leq Cg$ and by $f \simeq g$ we mean that $f \lesssim g$ and $g \lesssim f$.

2. Basic properties of finite Riemann surfaces

We start by the definition and then we collect some properties of S that follow from the restrictions that we are assuming on the topology of S.

Definition 2. A finite Riemann Surface is the interior of a smooth bordered compact Riemann surface.

Our surface is an open Riemann surface and it is in fact an open subset of a compact surface (the double, see [SS54]). See Figure 1 for a typical representation. Observe that the genus is finite and the border of the surface consists of a finite number of smooth closed Jordan curves. In most of what follows the particular case of a smooth finitely connected open set in $\mathbb{C}$ has all the difficulties of the general case.

The following claim follows from instance from [Sch78, Prop 7.1-7.4]

Lemma 3. *For any $(0,1)$-form ω there is a solution u to the inhomogeneous Cauchy-Riemann equation $\bar\partial u = \omega$. Moreover since S has an essential extension to a compact Riemann surface if the data is a smooth form with compact support K in S then there is a bounded linear solution $u = T[w]$ with the bound $|u| \leq C_K\langle\omega\rangle$.*

In this statement and in the following $\langle\omega\rangle$ is the Poincaré length of the $(0,1)$-form ω.

In the disk we have Blaschke factors that are very convenient to divide out zeros of holomorphic functions without changing essentially the norm. The analogous functions that provide us with the same property in the case of finite Riemann surfaces are given by the next proposition:

Proposition 4. *There is a constant $C = C(S) > 0$ such that for any point $z \in S$ there is a function $h_z \in \mathcal{H}(S)$ with*

$$\sup_{w\in S} |\log|h_z(w)| - g(z,w)| < C.$$

In particular $h_z(w)$ is a bounded holomorphic function that vanishes only on the point z and for any $\varepsilon > 0$ $K > |h_z(w)| > C(\varepsilon)$ if $d(z,w) > \varepsilon$.

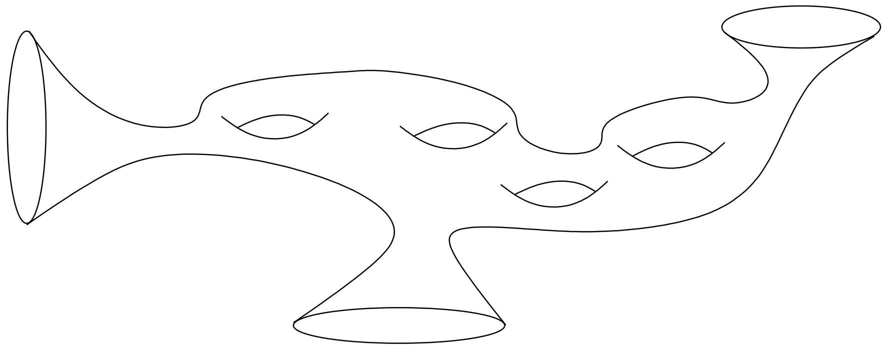

FIGURE 1. A finite Riemann surface with three funnels

Proof. The obstruction for an harmonic function u to have an harmonic conjugate is that for a set of generators $\{\gamma_i\}_{i=1}^m$ of the homology we have $\int_{\gamma_i} *du = 0$, $i = 1, \dots, m$. If we want $u = \log|f|$ for an $f \in \mathcal{H}(S)$, we just need that $\int_{\gamma_i} *du \in \mathbb{Z}$.

Being a finite Riemann surface there are $\{h_j\}_{j=1}^n$ functions in the algebra of S without zeros such that $\int_{\gamma_i} *d\log|h_j| = \delta_{ij}$, see [Wer64, Lemma 1]. Thus the function

$$v(z) = u(z) - \sum_i \Big(\int_{\gamma_i} *du\Big) \log|h_i(z)|$$

is the logarithm of an holomorphic function $\log|f| = v$. Therefore there is a constant C such that any harmonic function u in S admits an holomorphic function f with $|u - \log|f|| < C$. Take a point $z \in S$ and any holomorphic function $k_z \in \mathcal{H}(S)$ that vanishes only on z. Then $g(z,w) - \log|k_z(w)|$ is harmonic in S and therefore there is a holomorphic function f_z such that $|g(z,w) - k_z(w) - \log|f_z|| < C$. Thus we may define $h_z(w) = f_z(w)k_z(w)$ and it has the estimate $|g(z,w) - \log|h_z|| < C$. The estimate $|g(z,w)| > C(\varepsilon)$ when $d(z,w) > \varepsilon$ holds in finite Riemann surfaces, see for instance [Dil95, Theorem 5.5]. ☐

2.1. **The hyperbolic metric in a finite Riemann surface.** The open ends of the Riemann surface can be parametrized as follows: The border of the Riemann surface S is a finite union of smooth closed curves $\tilde\gamma_i$, $i = 1, \dots, n$. Near each $\tilde\gamma_i$ there is a closed geodesic γ_i that is homotopic to $\tilde\gamma_i$. The subdomain of S bounded by γ_i and $\tilde\gamma_i$ is denoted a "funnel" following the terminology of [DPRS87] and [Dil01].

We need to be more precise about the hyperbolic metric in the funnel. There are nice coordinates in the funnel that provide good estimates. These are given by the collar theorem. Let $\mathbb{D}$ be the universal holomorphic cover of S and let $T_\gamma \in \operatorname{Aut}(\mathbb{D})$ be the deck transformation corresponding to the closed loop γ. Consider the surface $Y = \mathbb{D}/\{T_\gamma^n\}_{n\in\mathbb{Z}}$. This an annulus since $\pi_1(Y) = \mathbb{Z}$. If we quotient it by the rest of the deck transformations of the universal cover we get an holomorphic covering map π_γ from $Y \to S$ which is a local isometry (in Y and S we consider the Poincaré metric inherited from $\mathbb{D}$). In fact $Y = \{e^{-R} < |z| < e^R\}$, where $R = \pi^2/\operatorname{Length}(\gamma)$, and π_γ maps the unit circle isometrically to γ. Moreover π_γ is an isometric injection of the outer part of the annulus $\{1 < |z| < e^R\}$

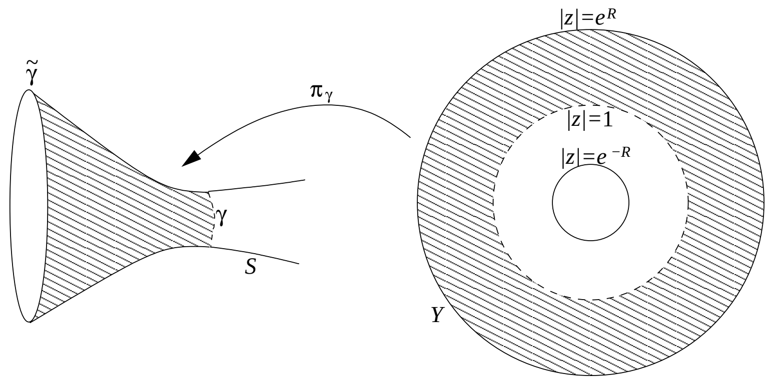

FIGURE 2. Standard coordinates on the funnel

onto the funnel. These will be called the standard coordinates of the funnel. See [Dil01] and [Bus92] for details.

The Poincare metric in the the funnel is explicit in the standard coordinates and it is comparable to the hyperbolic metric on the disk in the coordinate disk $|z| < e^R$ when restricted to $|z| > 1$.

We denote by A_i, $i = 1, \dots, n$ the funnels of S bounded by γ_i and $\tilde{\gamma}_i$.

2.2. **The inhomogeneous Cauchy-Riemann equation on the surface.** We want to solve the inhomogeneous Cauchy-Riemann equation on S with weighted uniform estimates. In order to get good estimates it is useful to find functions $f \in \mathcal{H}(S)$ with precise size control, i.e., $|f| \simeq e^\phi$ outside a neighborhood of the zero set of f. With this function we can later modify an integral formula to get a bounded solution to the $\bar{\partial}$-equation when the data has compact support. The following Lemma provides such a function that in other context has been termed a "multiplier":

Lemma 5. *Let S be a finite Riemann surface and let ϕ be a subharmonic function with bounded Laplacian. Then there is a function f with hyperbolically separated zero set Σ such that $|f| \simeq e^\phi$ whenever $d(z, \Sigma) > \varepsilon$. Moreover if we fix any compact K in S it is possible to find f with the above properties and without zeros in K.*

Proof. In any of the funnels A_i we transfer the subharmonic weight ϕ to the standard coordinate chart $1 < |z| < e^{R_i}$. We define a weight ϕ_i on the disk $|z| < e^{R_i}$ in such a way that ϕ_i has bounded invariant Laplacian and moreover $|\phi - \phi_i| < C$ on the region $1 < |z| < e^{R_i}$. One way to do so is the following: we assume from the very beginning that ϕ is smooth (this is no restriction since otherwise it can be approximated by a smooth function). Define

$$\phi_i(z) = \phi(z)\chi(z) + M_i\|z\|^2, \tag{2}$$

where χ is a cutoff function such that $\chi \equiv 1$ in $e^{R_i/2} < |z| < e^{R_i}$, $\chi \equiv 0$ in $|z| < 1$ and M_i is taken big enough such that ϕ_i is subharmonic and the invariant Laplacian of ϕ_i is bounded above and below.

We are under the hypothesis of the result from [Sei95] that states that there is an holomorphic function in the disk f_i with separated zero set $Z(f_i)$ (in the hyperbolic metric of the disk) such that $|f_i| \simeq e^{\phi_i}$ whenever $d(z, Z(f_i)) > \varepsilon$. Since the hyperbolic metric of the disk is comparable to the hyperbolic metric in the funnel, we have found a function $f_i \in \mathcal{H}(A_i)$ with separated zero set such that $|f_i(z)| \simeq e^{\phi(z)}$ if $d(z, Z(f_i)) > \varepsilon$. Moreover dividing out f_i by a finite Blaschke product we can assume that f_i is zero free in any prefixed compact of the disk.

We consider the "core" of S to be $S \setminus \tilde{A}_i$, where $\widetilde{A_i}$ are the outer part of the funnels mapped by $e^{S_i} < |z| < e^{R_i}$. The values of the S_i are taken so big as to make sure that the compact K in the hypothesis of the Lemma is contained in the core of S. We adjust the f_i $i = 1, \dots, n$ as mentioned before to make sure that they are zero free in the inner part of the funnels $1 < |z| < e^{S_i}$. We finally define $f_0 \equiv 1$ in the core of S.

To patch the different f_i together we will need to solve a Cousin II problem with bounds. Our data is f_i defined on the inner parts of the funnels mapped by $1 < |z| < e^{S_i}$. The data are bounded above and below in the inner parts of the funnels (because ϕ is bounded above and below in any compact of S and f_i have no zeros there). We want to find functions $g_i \in H(A_i)$ and g_0 holomorphic on the core of S such that $f_i = g_0/g_i$ in the inner part of the funnel. If moreover g_i and g_0 are bounded (above and below) then the function f defined as $f_i g_i$ in each of the funnels A_i and g_0 on the core of S is holomorphic on S and has the desired growth properties. To find the functions g_i observe that since the intersection of the funnel A_i with the core of S strictly separates the outer part of the funnel from the inner part of the core we can reduce the Cousin II problem to solving a $\bar\partial$-equation with bounded estimates of the solution on S when the data is bounded and with compact support (the support is in the inner part of the funnels). This can be achieved by Lemma 3. $\square$

With this function we can then obtain the following result which is interesting by itself:

Theorem 6. *Let S be a finite Riemann surface and let ϕ be a subharmonic function with a bounded Laplacian. There is a constant $C > 0$ such that for any $(0,1)$-form ω on S there is a solution u to the inhomogeneous Cauchy-Riemann equation $\bar\partial u = \omega$ in S with the estimate*

$$\sup_{z\in S} |u(z)| e^{-\phi(z)} \leq C \sup_{z\in S} \langle \omega(z) \rangle e^{-\phi(z)},$$

whenever the right hand is finite.

Recall that the notation $\langle \omega(z) \rangle$ means the hyperbolic norm of ω at the point z.

Proof. Let w_i be the form w restricted to the funnel A_i. We take a standard coordinate chart and we may think of w_i as a $(0,1)$-form defined on the disk $|z| < e^{R_i}$ and with support in $1 < |z| < e^{R_i}$. Consider as in the proof of Lemma 5 a subharmonic function ϕ_i in the disk with bounded laplacian and such that $|\phi - \phi_i| < C$ if $1 < |z| < e^{R_i}$.

By the results in [OC02, Thm 2] there is a solution u_i to the problem $\bar\partial u_i = w_i$ in the disk $|z| < e^{R_i}$ with the estimate

$$\sup_{|z|<e^{R_i}} |u_i| e^{-\phi_i} \leq C_i \sup_{1<|z|<e^{R_i}} \langle w_i \rangle e^{-\phi}$$

Observe that the hyperbolic metric of the disk and of the surface S in the funnel are equivalent. We consider $\tilde{u}_i = u_i\chi_i$, where χ_i is a cutoff function with support in $1 < |z| < e^{R_i}$ and such that $\chi_i \equiv 1$ if $|z| > e^{R_i/2}$. The function $\tilde{u}_i$ is extended by 0 to the remaining of S and it has the estimate $\sup_S |\tilde{u}_i|e^{-\phi} \leq C_i \sup_S \langle w\rangle e^{-\phi}$. Now $\bar\partial\tilde{u}_i$ coincides with w on the outer part of the funnel A_i. Thus the $(0,1)$-form $w_k = w - \sum_i \bar\partial\tilde{u}_i$ has compact support in S and it satisfies $\sup_S\langle w_k\rangle e^{-\phi} \leq \sup_S\langle w\rangle e^{-\phi}$. The desired solution is then $u = \sum \tilde{u}_i + v$, where v is such that $\bar\partial v = w_k$. We must then solve $\bar\partial v = w_k$ with weighted uniform estimates but with the advantage that w_k has compact support K.

Let $T(\omega_k)$ be a solution operator for $\partial\bar{u} = \omega_k$. We take the operator T given by Lemma 3 the estimate $\sup_S |T[w_k](z)| \leq C_K \sup_K\langle w_k\rangle$ holds. Take f with $|f| \simeq e^{\phi}$ and without zeros in K as given in Lemma 5. Then we define R as

$$R[\omega_k](z) = f(z)T[\omega_k/f](z), \tag{3}$$

It solves $\bar\partial R[\omega_k] = \omega_k$ with the estimate

$$\sup_S |R[\omega_k]|e^{-\phi} \leq C_K \sup_K \langle\omega_k\rangle e^{-\phi}.$$

The solution is thus $v = R[w_k]$. □

3. The main results

Proposition 7. *A separated sequence $\Lambda \subset S$ is interpolating for $A_\phi(S)$ if and only if the sequences $\Lambda_i = \Lambda \cap A_i$ are interpolating in $A_\phi(A_i)$.*

Proof. We only need to prove that we can pass from the local to the global interpolation property. We split the proof in two steps

(1) From a funnel A_i to global S: We need to prove that there are finite sets $F_i \subset \Lambda_i$ such that $\cup_{i=1}^n(\Lambda_i \setminus F_i)$ is interpolating globally.
(2) Filling up the remainder. We shall prove that by adding a finite number of points to an interpolating sequence we still get an interpolating sequence. Thus Λ is interpolating if $(\Lambda_1 \setminus F_1) \cup \cdots \cup (\Lambda_n \setminus F_n)$ is interpolating.

□

Let $\tilde\gamma$ be one of the closed curves on the boundary. Take a funnel A with outer end curve in $\tilde\gamma$ and inner end curve in γ. The constant of interpolation in the funnel A is $K > 0$. Take a cutoff function χ_ε with support in the funnel such that $\langle\bar\partial\chi_\varepsilon\rangle < \varepsilon/(KC)$ (where C is the constant in Theorem 6), the support is in a thick annulus of hyperbolic thickness $M = M(\varepsilon, K, C)$. We consider a smaller funnel where $\chi_\varepsilon \equiv 1$. The sequence Λ in this smaller funnel has still at most interpolation constant K. We can interpolate arbitrary values on Λ being small near the inner curve γ of A in the following way. Take some values v_λ with norm one. Take a function in the funnel f with norm at most K that solves the interpolation problem. We are going to approximate it by a function in A that is small near γ. Cut it off by χ_ε and correct via the following inhomogeneous Cauchy-Riemann equation:

$$\bar\partial u = f\bar\partial\chi$$

The function $h = u - f\chi$ is holomorphic. By using Theorem 6 on it is possible to solve the equation with a solution u such that $\sup|u|e^{-\phi} \leq \varepsilon$. The function h does not solve the problem directly but it almost does. We reiterate the procedure (interpolating the error $v_\lambda - h(\lambda)$ and with a convergent series we get finally a function g such that $h(\lambda) = v_\lambda$, $\sup_A |h|e^{-\phi} \leq 2$ and moreover in the inner half of the funnel that we denote by $\tilde A$, $\sup_{\tilde A} |h|e^{-\phi} \leq \varepsilon$.

Now it is easier to make it global. Take a new cutoff function χ with support in the funnel A and that is one on the outer part of (i.e. $A \setminus \tilde A$. Then we need to solve

$$\bar\partial u = h\bar\partial\chi,$$

with good global estimates in S. These are given by Theorem 6. We have solved the interpolation problem when the sequence lies in the funnels. For the general situation we only need to add a finite number of points. The existence of "Blaschke"-type factors $h_\lambda(z)$ provided by Theorem 4 shows that $\Lambda \cup \lambda$ is interpolating if Λ is interpolating (it is immediate to build functions in the space such that $f|_\Lambda \equiv 0$ and $f(\lambda) \neq 0$). □

For the sampling part we need the following definition

Definition 8. Given the pair (S,ϕ) of a finite Riemann surface and a subharmonic function defined on it, we associate to it the pairs: $(D_i,\phi_i)_{i=1,\dots n}$ of disks D_i and subharmonic functions ϕ_i defined on the disks as follow: If $A_i = \{1 < |z| < e^{R^i}\}$, $i = 1,\dots,n$ are the standard charts of the funnels of S we define $D_i = \{|z| < e^{R_i}\}$ and ϕ_i is any subharmonic function in D_i such that $|\phi_i - \phi| < C$ in the region $1 < |z| < e^{R_i}$, $\Delta\phi_i = \Delta\phi$ in $e^{R_i/2} < |z| < e^{R_i}$ and $\Delta\phi_i \simeq 1$ in $|z| < e^{R_i/2}$. They can be defined similarly as in (2), but to make sure $\Delta\phi_i = \Delta\phi$ we may take instead

$$\phi_i(z) = \phi(z)\chi(z) + M_i\psi(z),$$

where ψ is any bounded subharmonic function in D_i such that $\Delta\psi(z) = 1$ if $|z| < e^{R_i/2}$ and 0 elsewhere.

The funnels A_i can be considered funnels of S and they are subdomains of D_i too. We will exploit this double nature in the following theorem

Theorem 9. *Let S be a finite Riemann surface and let ϕ be a subharmonic function with bounded Laplacian. A separated sequence Λ is sampling for $A_\phi(S)$ if and only if all the sequences in the funnels $\Lambda_i = \Lambda_i \cap A_i \subset D_i$ are sampling sequences for $A_{\phi_i}(D_i)$, where (D_i,ϕ_i) are the associated pairs to S given by Definition 8.*

Thus this Theorem and Proposition 7 show that the properties of sampling and interpolation only depend on the behavior of the sequence and the weight near the boundary pieces.

To prove Theorem 9 we need some previous results

Lemma 10. *Let S be a finite Riemann surface and let ϕ be a subharmonic function with bounded Laplacian. A sequence $\Lambda \subset S$ is a uniqueness sequence for $A_\phi(S)$ if and only if all the sequences in the funnels $\Lambda_i = \Lambda_i \cap A_i \subset D_i$ are uniqueness sequences for $A_{\phi_i}(D_i)$, where (D_i, ϕ_i) are the associated pairs to S given by Definition 8.*

Proof. It is easier to deal by negation. Let Λ be contained in the zero set of a function $f \in A_\phi(S)$. Therefore Λ_i is in the zero set of $f \in A_\phi(A_i)$. We divide by a finite number of zeros E_i and we obtain a new function $g \in A_\phi(A_i)$ without zeros in $1 < |z| \le e^{R_i/2}$ and such that $\Lambda_i \setminus E_i \subset Z(g)$. Take the disk D_i and consider the cover by two open sets $|z| > 1$ and $|z| < e^{R_i/2}$. On the first set we have the function g and on the second the function 1. The quotient is bounded above and below in the intersection of the sets. This defines a bounded Cousin II in the disk D_i problem that can be solved with bounded data. We get a new function $h \in A_{\phi_i}(D_i)$ that vanishes in $Z(g)$. We can now add the finite number of zeros E_i without harm. The reciprocal implication follows with the same argument. $\square$

The next result is inspired by a result of Beurling ([Beu89, pp. 351–365]) that relates the property of sampling sequence to that of uniqueness for all weak limits of the sequence. In the context of the Bernstein space (in the original work by Beurling) the space was fixed (it was $\mathbb{C}$, the space of functions was fixed, the Bernstein class, and he considered translates and limits of it of the sampling sequence). Here we need to move and take limits of the sequence (by zooming on appropriate portions of it) but we also need to change the support space (portions of S near the funnel that look like the unit disk) and we will also move the space of functions by changing the weights. We need some definitions:

Definition 11. We consider triplets (D_n, ϕ_n, Λ_n) where D_n are disks $D_n = D(0, r_n) \subset \mathbb{D}$, ϕ_n are subharmonic functions defined in a neighborhood of D_n and Λ_n is a finite collection of points in D_n. We say that (D_n, ϕ_n, Λ_n) converges weakly to $(\mathbb{D}, \phi, \Lambda)$ (where $\mathbb{D}$ is the unit disk, ϕ a subharmonic function in $\mathbb{D}$ and Λ a discrete sequence in $\mathbb{D}$) if the following conditions are fullfilled:

- The domains D_n tend to $\mathbb{D}$, i.e.: $r_n \to 1$,
- The weights ϕ_n tend to the weight ϕ in the sense that $\Delta\phi_n$ as measures converges weakly to $\Delta\phi$.
- The sequences Λ_n converge weakly to Λ, i.e, the measure $\sum_{\lambda\in\Lambda_n} \delta_\lambda$ converges weakly to the measure $\sum_{\lambda\in\Lambda} \delta_\lambda$.

Let us fix a point $p \in S$. If a sequence of points $z_n \in S$ goes to ∞, i.e. $d(z_n, p) \to \infty$, from a point n_0 on it will eventually belong to the union of the funnels $A_1 \cup \cdots \cup A_n$. If we take the set of points $D_n = \{z \in S;\ d(z, z_n) < d(z_n, p)/2\}$ then D_n is an hyperbolic disk contained in the funnels if n is big enough. In each of the D_n we consider the function $\phi_n = \phi|_{D_n}$ and $\Lambda_n = \Lambda \cap D_n$. Thus for any sequence of points z_n with $d(p, z_n) \to \infty$ we build a triplet (D_n, ϕ_n, Λ_n) for n big enough.

Definition 12. Let $W(S, \phi, \Lambda)$ be the set of all triplets $(\mathbb{D}, \phi^*, \Lambda^*)$ which are weak limits of triplets (D_n, ϕ_n, Λ_n) associated to any sequence z_n such that $d(p, z_n) \to \infty$.

The theorem of Beurling on our context is

Theorem 13. *Let S be Riemann surface of finite type and let ϕ be a subharmonic function with bounded Laplacian. A separated sequence Λ is sampling for $A_\phi(S)$ if and only if*

- *The sequence Λ is a uniqueness set for $A_\phi(S)$*
- *For any triplet $(\mathbb{D}, \phi^*, \Lambda^*) \in W(S, \phi, \Lambda)$, the sequence Λ^* is a uniqueness set for $A_{\phi^*}(\mathbb{D})$.*

Proof. Let us prove that the uniqueness conditions imply that Λ is a sampling sequence. If it were not, there would be a sequence of functions $f_n \in A_\phi(S)$ such that $\sup_\Lambda |f_n|e^{-\phi} \leq 1/n$ and $\sup_S |f_n|e^{-\phi} = 1$. Take a sequence of points z_n with $|f_n(z_n)|e^{-\phi} \geq 1/2$. If z_n are bounded we can take a subsequence of points that we still denote z_n convergent to $z^* \in S$ and by a normal family argument there is a partial of f_n convergent to $f \in A_\phi$, such that $f|_\Lambda \equiv 0$, $f(z^*) \neq 0$ and this is not possible. Thus z_n must be unbounded. Then we take the triplets (D_n, ϕ_n, Λ_n) associated to z_n and $D_n \to \mathbb{D}$ because $z_n \to \infty$ and the hyperbolic radius of D_n is $d(z_n, p)/2$. Since ϕ_n has bounded Laplacian, the mass of $\Delta\phi_n$ restricted to any compact K in $\mathbb{D}$ is bounded, thus we can take a subsequence that converges weakly to a positive measure μ in $\mathbb{D}$ which satisfies $(1-|z|)^2\mu \simeq 1$ because all the mesures $\Delta\phi_n$ satisfy this inequalities with uniform constants. Let ϕ be such that $\Delta\phi^* = \mu$. Since $\Lambda_n = D_n \cap \Lambda$ are all separated with uniform bound, there is a weak limit Λ^*. The functions f_n in the disks can be modified by a factor e^{g_n} in such a way that $h_n = f_n e^{g_n}$ satisfies $h_n(0) = 1$ and $|h_n| \leq e^{\phi_n + \mathrm{Re}(g_n)}$, if n big enough and $\sup_{\Lambda_n} |h_n|e^{-\phi_n + \mathrm{Re}(g_n)} \leq 1/n$. We can add an harmonic function v to ϕ^* in such a way that $\phi_n + \mathrm{Re}(g_n) \to v + \phi^*$ uniformly on compact sets. Thus h_n has a partial convergent to $h \in A_{\phi^*}$, $h(0) = 1$ and $h|_{\Lambda^*} \equiv 0$ which was not possible by assumption.

In the other direction, we assume that Λ is a sampling sequence for $A_\phi(S)$, and $(\mathbb{D}, \phi^*, \Lambda^*) \in W(S, \phi, \Lambda)$. We want to prove that any $f \in A^*_\phi(\mathbb{D})$ that vanishes in Λ^* is identically 0. Take a sequence of points z_n that escapes to infinity and (D_n, ϕ_n, Λ_n) the associated triple that converges weakly to $(\mathbb{D}, \phi^*, \Lambda^*)$. As $\phi_n \to \phi^*$ and $\Lambda_n \to \Lambda^*$ uniformly on compact sets we can take a sequence of radii s_n such that $d(\Lambda \cap D(z_n, s_n), \Lambda^* \cap D(0, s_n)) < 1/n$, $D(z_n, s_n) \subset D(z_n, r_n)$ and $|\phi_n - \phi^*| \leq 1/n$. If f vanishes in Λ^* that means that f is very small in $D(z_n, s_n) \cap \Lambda$. Assume that $f(0) = 1$. Take a cutoff function χ_n such that $\chi_n \equiv 0$ outside $D(z_n, s_n)$, $\chi(z_n) = 1$, and $\langle d\chi \rangle < \varepsilon_n$. Define $g_n = f\chi_n - u_n$, where $\bar\partial u = f\bar\partial\chi_n$ is the solution estimates by Theorem 6. Clearly g_n is small in all points of Σ and it has at least norm 1. Thus we are contradicting the fact that Λ is sampling.

□

Observe that one particular instance of finite Riemann surface, where we can apply the result are the disks D_i associated to the funnels with the metric ϕ_i. The final piece for the proof of Theorem 9 is then

Lemma 14. *If S is a finite Riemann surface, ϕ a subharmonic function with bounded Laplacian and Λ is a uniformly separated sequence, then all possible weak limits coincide with the weak limits of the disks associated to the surface, i.e,*

$$W(S, \phi, \Lambda) = W(D_1, \phi_1, \Lambda_1) \cup \cdots \cup W(D_n, \phi_n, \Lambda_n).$$

Proof. The proof amounts to the observation that the metric in D_i converges uniformly to the metric in S as $z \to \partial\mathbb{D}_i$, and in the definition of weak limits we only consider uniform convergence over compacts. □

Theorem 9 follows now immediately from Theorem 13 and Lemmas 10 and 14. □

Now Theorem 9 and Theorem 7 show that the property of being a sampling/interpolating sequence are determined by the behavior near the boundary, more precisely in the associated disks. In these disks there is a precise description of the interpolating and sampling sequences (see [BOC95] and [OCS98]) that can be transported to the surface. If we rewrite it we get the density conditions of Theorem 1, but the disks are not hyperbolic disks on the surface, they correspond to hyperbolic disks in disks D_i, but since the condition is only relevant near the boundary, then the disks in both metrics look more an more similar. Moreover the difference between the corresponding Green functions converge to 0 uniformly as we go to the boundary. Finally, as the sequence is uniformly discrete and the Laplacian of the weight is bounded above and bellow, the small difference is absorbed by the fact that the inequalities are strict and this proves Theorem 1. In fact it is possible to replace in the definition of the density, (1) the Green function g_r of $D(z,r)$ by the Green function g of S, because as before $\sup_{w\in D(z,r)} |g_r(z,w) - g(z,w)| \to 0$ as z approaches the boundary.

4. Some L^p-variants

We have considered up to now pointwise growth restrictions. It is possible to obtain from our Theorem other results in different Banach spaces of holomorphic functions. Consider for instance the weighted Bergman spaces

$$A^p_\phi(S) = \{f \in \mathcal{H}(S);\ \int_S |f|^p e^{-\phi}\, dA < +\infty\},$$

where dA is s the hyperbolic area measure in S and $p \in [1,\infty)$. The natural problem in this context is the following:

Definition 15. Let S be a finite Riemann surface, and let ϕ be a subharmonic function with bounded Laplacian bigger than one, i.e., $1+\varepsilon < \Delta\phi < M$.

- A sequence $\Lambda \subset S$ is interpolating for $A^p_\phi(S)$ if for any values v_λ such that

  $$\sum_{\lambda\in\Lambda} |v_\lambda|^p e^{-\phi(\lambda)} < \infty$$

  there is a function $f \in A^p_\phi(S)$ such that $f(\lambda) = v_\lambda$.

The spaces A^p_ϕ can be empty if we only ask ϕ to be with positive bounded Laplacian. It is then natural to require that the Laplacian is strictly bigger than one so that the Laplacian plus the curvature of the metric in the manifold is strictly positive and there are functions in the space (consider the case of the disk $S = \mathbb{D}$ for instance).

Let ϕ_0 be a subharmonic function in S such that $\Delta\phi_0 = 1$. The corresponding theorem will be

Theorem 16. *Let S be a finite Riemann surface, and let ϕ be a subharmonic function with bounded Laplacian strictly bigger than one. Let $p \in [1, +\infty)$ and Λ be a separated sequence.*

- *The sequence Λ is interpolating for $A^p_\phi(S)$ if and only if $D^+_{(\phi-\phi_0)}(\Lambda) < 1/p$.*

In the case of the unit disk $dA(z) = (1-|z|)^{-2}$ this description is well-known, see for instance [Sei98, Thm 2,3].

Proof. The proof of the theorem is the same mutatis-mutandi as in the L^∞ setting. The basic tool that allows us to glue the pieces together is the next theorem which is the generalization of Theorem 6 and it is proved in the same way:

Theorem 17. *Let S be a finite Riemann surface, let ϕ be a subharmonic function with a bounded Laplacian strictly bigger than one and let $p \in [1, \infty)$. There is a constant $C = C(p, S) > 0$ such that for any $(0,1)$-form ω on S there is a solution u to the inhomogeneous Cauchy-Riemann equation $\bar\partial u = \omega$ in S with the estimate*

$$\int_S |u(z)|^p e^{-\phi(z)} dA(z) \le C \int_S \langle \omega(z)\rangle^p e^{-\phi(z)} dA(z),$$

whenever the right hand is finite.

The proof of this result is again the same as in Theorem 6. We can separately solve the C-R equation in each funnel using Theorem 2 from [OC02]. We glue them together with a C-R equation with data that has compact support that can be solved with the operator (3). □

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