SOME NON-BRAIDED FUSION CATEGORIES OF RANK 3

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Abstract. We classify and explicitly describe all fusion categories with fusion rules given by $x \otimes x \cong x \oplus x \oplus y \oplus \mathbf{1}$, $x \otimes y \cong y \otimes x \cong x$ and $y \otimes y \cong \mathbf{1}$. We describe a convenient, concrete and useful variation of the graphical calculus for fusion categories, discuss pivotality and sphericity in this framework, and give a short and elementary re-proof of the fact that the quadruple dual functor is naturally isomorphic to the identity.

1. Introduction

Let k be an algebraically closed field. A *fusion category* $\mathcal{C}$ over k is a k-linear semisimple rigid monoidal category with finitely many (isomorphism classes of) simple objects, finite dimensional morphism spaces, and $End(\mathbf{1}) \cong k$. See [7] or [4] for definitions.

The *rank* r of $\mathcal{C}$ is the number of isomorphism classes of simple objects in $\mathcal{C}$. Let $\{x_i\}_{1\leq i\leq r}$ be a set of simple object representatives. The *fusion rules* for $\mathcal{C}$ are a set of $r \times r$ $\mathbb{N}$-valued matrices $N = \{N_i\}_{1\leq i\leq r}$, with $(N_i)_{j,k}$ denoted N_{ij}^k or, when convenient, $N_{x_i x_j}^{x_k}$, such that $x_i \otimes x_j \cong \bigoplus_{1\leq k\leq r} N_{ij}^k x_k$. In the sequel, assume $k = \mathbb{C}$.

Fusion categories appear in representation theory, operator algebras, conformal field theory, and in constructions of invariants of links, braids, and higher dimensional manifolds. There is currently no general classification of them. Classifications of fusion categories for various families of fusion rules have been given in work by Kerler ([3]), Tambara and Yamagami ([12]), Kazhdan and Wenzl ([5]), and Wenzl and Tuba ([13]).

For a given set of fusion rules, there are only finitely many monoidal natural equivalence classes of fusion categories. This property is called Ocneanu rigidity (see [2]). It is not known whether or not the number of fusion categories of a given rank is finite. If one assumes a modular structure, the possibilities up to rank four have been classified by Rowell, Stong and Wang in [11]. Ostrik has classified fusion categories up to rank two in [10], and constructed a finite list of realizable fusion rules for braided categories up to rank three in [9], in which the number of categories for each set of fusion rules is known. The rank two classification relies in an essential way on the theory of modular tensor categories; Ostrik shows that the quantum double of a rank two category must be modular, and uses the theory of modular tensor categories to eliminate most of the possibilities. The classification of modular tensor categories is of independent interest; in many contexts one must assume modularity.

We consider the only set of rank three fusion rules which is known to be realizable as a fusion category but which has no braided realizations. Ostrik conjectured in [9] that a classification for this rule set completes the classification of rank three fusion categories. If $\mathbf{1}$ is the trivial object, and x and y are representatives of the other two simple object types, the fusion rules are given by the following: $x \otimes x \cong x \oplus x \oplus y \oplus \mathbf{1}$, $x \otimes y \cong y \otimes x \cong x$, $y \otimes y \cong \mathbf{1}$.

The axioms for fusion categories over $\mathbb{C}$ reduce to a system of polynomial equations over $\mathbb{C}$. In this context, Ocneanu rigidity, roughly translated, says that normalization of some of the variables in the equations gives a finite solution set. In this case, one can compute a Gröbner basis for the system and obtain the solutions (see [1]). However, normalization becomes complicated when there are i, j, k such that $N_{ij}^k > 1$. The fusion rules we consider are the smallest realizable set with this property.

2. Preliminaries and Notational Conventions

This paper uses the "composition of morphisms" convention for functions as well as morphisms, and left to right matrix multiplication. For calculations of the fusion rules, our treatment is similar to [12], but the notation differs superficially for typographic reasons. The notation captures algebraic data sufficient to classify a fusion category up to monoidal natural equivalence, and is reviewed later in this section.

Given a fusion category $\mathcal{C}$, one may consider two categories $\mathcal{C}^{\mathcal{SKEL}}$ and $\mathcal{C}^{\mathcal{STR}}$, each of which is monoidally equivalent to $\mathcal{C}$. $\mathcal{C}^{\mathcal{SKEL}}$ is a full subcategory of $\mathcal{C}$ containing exactly one object from each isomorphism class of objects in $\mathcal{C}$. By semisimplicity of $\mathcal{C}$, every object in $\mathcal{C}$ is isomorphic to a direct sum of simple objects in $\mathcal{C}^{\mathcal{SKEL}}$. One may assume without loss that the objects of $\mathcal{C}^{\mathcal{SKEL}}$ consist of simple object representatives and direct sums of such.

One monoidal equivalence between $\mathcal{C}$ and $\mathcal{C}^{\mathcal{SKEL}}$ has as its functors the inclusion functor and the functor $F : \mathcal{C} \to \mathcal{C}^{\mathcal{SKEL}}$, defined as follows. For each object $x \in \mathcal{C}$, set $F(x) = x' \cong x$, and fix an isomorphism $i_x : x \to x'$. For morphisms $f : x \to y$, define $F(f) = {i_x}^{-1} \circ f \circ i_y$. The skeleton of $\mathcal{C}$ is unique up to strict natural equivalence, that is, a natural equivalence with invertible functors. Another advantage is that isomorphisms in $\mathcal{C}$ become automorphisms in $\mathcal{C}^{\mathcal{SKEL}}$.

Reassociation of tensor factors in a monoidal category is described by a natural isomorphism of trifunctors $\alpha : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$. Tensor products with $\mathbf{1}$ have natural isomorphisms $\rho : - \otimes \mathbf{1} \to -$ and $\lambda : \mathbf{1} \otimes - \to -$. These isomorphisms are subject to a coherency condition, namely that for any pair of multifunctors there is at most one natural isomorphism between them which may be constructed from λ, ρ, α and their inverses, along with Id and $\otimes$.

The second construction, strictification, turns $\mathcal{C}$ gives a strict monoidal category. In a strict monoidal category, α, λ and ρ are the identity. Strictification of a monoidal category is analogous to the construction of a tensor algebra; it gives an equivalent strict category $\mathcal{C}^{\mathcal{STR}}$ by replacing the tensor product with a strictly associative formal tensor product. The construction appears in [7].

It is not usually possible to make a fusion category strict and skeletal at the same time. However, the category $(\mathcal{C}^{\mathcal{SKEL}})^{\mathcal{STR}}$, while not a skeleton, is still unique up to strict natural equivalence. Also, it is a categorical realization of a graphical calculus, as will be seen. The next subsection describes what strictified skeleta of fusion categories look like, up to strict equivalence.

2.1. Strictified skeletal fusion categories.

A *strictified skeletal fusion category* $\mathcal{C}$ is as follows: Let N be a set of fusion rules for a set of objects S. Then the objects in $\mathcal{C}$ are multisets of finite sequences of elements of S. $\mathcal{C}$ has a tensor product $\otimes$, which is defined on objects by pairwise concatenation of sequences, distributed over elements of multisets. Direct sum of objects is given by multiset disjoint union.

A *strand* is an object which is a sequence of length one. Strands correspond to simple object types. If x, y and z are strands, define $Mor(x \otimes y, z)$ to be a k vector space isomorphic to $k^{N^z_{xy}}$. For brevity, V^y_x will denote $Mor(x, y)$, and tensor products will be omitted when the context is clear. A morphism is (n, m)-*stranded* if its source and target are sequences of length n and m, respectively. A morphism is (n)-*stranded* if it is $(m, n-m)$-stranded for some $0 \leq m \leq n$.

Semisimplicity of $\mathcal{C}$ means that for all objects w, x, y and z there are vector space isomorphisms $\sum_{v \in S} V^v_{xy} \otimes V^z_{wv} \cong V^z_{wxy} \cong \sum_{v \in S} V^v_{wx} \otimes V^z_{vy}$. The first isomorphism is given by $f \otimes g \to (Id_w \otimes f) \circ g$, the inverse of the second by $h \otimes l \to (h \otimes Id_y) \circ l$. The composition of the two isomorphisms is denoted $\alpha^z_{w,x,y}$. Additionally, each morphism space V^z_{xy} has an algebraically dual space V^{xy}_z, in the sense that there are bases $\{v_i\}_i \subset V^z_{xy}$ and $\{w_i\}_i \subset V^{xy}_z$ such that $w_i \circ v_j = \delta_{ij} Id_z$.

In a strictified skeleton, the trivial object $\mathbf{1}$ is the zero length sequence. There is a strand which is isomorphic to the trivial object, but not equal. This strand shall be taken to be $\mathbf{1}$ in the sequel. This choice makes the category non-strict, since λ and ρ are no longer the identity, but it is convenient for graphical calculus purposes.

(Right) rigidity in a strictified skeleton means that there is a set involution $*$ on the strands, and each strand x has morphisms $b_x : \mathbf{1} \to x \otimes x^*$ and $d_x : x^* \otimes x \to \mathbf{1}$ such that $(Id_{x^*} \otimes b_x) \circ (d_x \otimes Id_{x^*}) = Id_{x^*}$ and $(b_x \otimes Id_x) \circ (Id_x \otimes d_x) = Id_x$. This implies that $N^z_{xy} = N^{y^*}_{z^* x}$. Left rigidity is similar, and in the sequal, the right rigidity morphism for x^* will be defined to be the left rigidity morphism for x.

Define $*$, b, and d on concatenations of strands such that $d_{x \otimes y} = (Id_{y^*} \otimes d_x \otimes Id_y) \circ d_y$ and extend to direct sums. Then there is a contravariant *(right) dual* functor $*$ which sends $f \in V^y_x$ to $f^* = (Id_{y^*} \otimes b_x) \circ (Id_{y^*} \otimes f \otimes Id_{x^*}) \circ (d_y \otimes Id_{x^*}) \in V^{x^*}_{y^*}$. The definition of a left dual functor is similar, and the two duals are inverse functors by rigidity.

In a monoidal category, the coherency of α, λ and ρ is equivalent to the well known pentagon and triangle equations (see [7] for a proof). The triangle equations are given by $\rho_x \otimes y = \alpha_{x,1,y} \circ (x \otimes \lambda_y)$ for all objects x and y, where the name of an object is used as a shorthand for the identity on that object when the context is unambiguous. For objects w, x, y and z, the pentagon equations are as follows (see also Figure 1):

$$(\alpha_{x,y,z} \otimes w) \circ \alpha_{x,y \otimes z,w} \circ (x \otimes \alpha_{y,z,w}) = \alpha_{x \otimes y,z,w} \circ \alpha_{x,y,z \otimes w},$$

Monoidality for a strictified skeletal fusion category implies that the α are identity morphisms. For this to be true, it is necessary and sufficient that the following equation holds for all objects x, y, z, w, and u. Each instance will be referred to as $P^u_{x,y,z,w}$ in the sequel.

$$\bigoplus_t \alpha^t_{y,z,w} V^u_{xt} \circ \bigoplus_s V^s_{yz} \alpha^u_{x,s,w} \circ \bigoplus_t \alpha^t_{x,y,z} V^u_{tw} :$$

$$\bigoplus_{s,t} V^s_{zw} V^t_{ys} V^u_{xt} \to \bigoplus_{s,t} V^s_{yz} V^t_{sw} V^u_{xt} \to \bigoplus_{s,t} V^s_{yz} V^t_{xs} V^u_{tw} \to \bigoplus_{s,t} V^s_{xy} V^t_{sz} V^u_{tw}$$

is equal to

$$\begin{tikzcd}[column sep=small]
 & ((x\otimes y)\otimes z)\otimes w \arrow[dl, "\alpha_{x,y,z}w"'] \arrow[dr, "\alpha_{xy,z,w}"] & \\
(x\otimes(y\otimes z))\otimes w \arrow[d, "\alpha_{x,yz,w}"'] & (a) & (x\otimes y)\otimes(z\otimes w) \arrow[d, "\alpha_{x,y,zw}"] \\
x\otimes((y\otimes z)\otimes w) \arrow[rr, "x\alpha_{y,z,w}"] & & x\otimes(y\otimes(z\otimes w))
\end{tikzcd}$$

$$\begin{tikzcd}
\bigoplus_{s,t} V^s_{xy}V^t_{sz}V^u_{tw} & & \bigoplus_{s,t} V^s_{xy}V^t_{zw}V^u_{st} \arrow[ll, "\bigoplus_s V^s_{xy}\alpha^u_{s,z,w}"'] \\
\bigoplus_{s,t} V^s_{yz}V^t_{xs}V^u_{tw} \arrow[u, "\bigoplus_t \alpha^t_{x,y,z}V^u_{tw}"] & (b) & \bigoplus_{s,t} V^s_{zw}V^t_{xy}V^u_{ts} \arrow[u, "\tau"'] \\
\bigoplus_{s,t} V^s_{yz}V^t_{sw}V^u_{xt} \arrow[u, "\bigoplus_s V^s_{yz}\alpha^u_{x,s,w}"] & & \bigoplus_{s,t} V^s_{zw}V^t_{ys}V^u_{xt} \arrow[ll, "\bigoplus_t \alpha^t_{y,z,w}V^u_{xt}"'] \arrow[u, "\bigoplus_s V^s_{zw}\alpha^u_{x,y,s}"']
\end{tikzcd}$$

FIGURE 1. (a)Pentagon equality and (b)corresponding equality

$$\bigoplus_s V^s_{zw}\alpha^u_{x,y,s}\circ\tau\circ\bigoplus_s V^s_{xy}\alpha^u_{s,z,w}:$$
$$\bigoplus_{s,t} V^s_{zw}V^t_{ys}V^u_{xt}\to\bigoplus_{s,t} V^s_{zw}V^t_{xy}V^u_{ts}\to\bigoplus_{s,t} V^s_{xy}V^t_{zw}V^u_{st}\to\bigoplus_{s,t} V^s_{xy}V^t_{sz}V^u_{tw}.$$

Here $\otimes$ for vector spaces and morphisms are omitted, and τ is the isomorphism interchanging the first and the second factors of tensor products (see Figure 1).

2.2. **Remarks.**

(1) Every fusion category is monoidally naturally equivalent to a strictified skeleton. Also, two naturally equivalent strictified skeleta have an invertible equivalence functor that takes strands to strands. The self-equivalences that fix strands are given by changes of basis on the $(2,0)$ and $(2,1)$-stranded morphism spaces.

(2) The functor $**$ fixes objects. The isomorphisms $J_{x,y}: x^{**}\otimes y^{**}\to(x\otimes y)^{**}$ associated with $**$ in the definition of a monoidal functor (see [7]) may be taken to be trivial. There is an invertible scalar worth of freedom in the choice of each b_x, d_x pair.

(3) Semisimplicity allows every morphism to be built up from (3)-stranded morphisms. Choosing bases for the (3)-stranded morphisms allows morphisms in $\mathcal{C}$ to be characterized as undirected trivalent graphs with labelled edges and vertices, subject to associativity relations given by the pentagon equations. The labels for the edges are isomorphism types of simple objects; the labels for the vertices are basis vectors for the morphism spaces. This gives a categorically precise interpretation of an arrowless graphical calculus for $\mathcal{C}$.

(4) If $\mathcal{C}$ is pivotal (see section 6 for the definition), a well known construction allows one to add a second copy of each object and get a strict pivotal category. This construction gives a graphical calculus with arrows on the strands.
(5) Strictified skeleta give any categorical structure preserved under natural equivalence (and any functorial property preserved under natural isomorphism) a purely algebraic description.

3. Possible Tensor Category Structures

The fusion rules are given by $x \otimes x \cong \mathbf{1} \oplus y \oplus x \oplus x$, $x \otimes y \cong y \otimes x \cong x$, and $y \otimes y \cong \mathbf{1}$. The non-trivial vector spaces are V_{11}^1, V_{1x}^x ,V_{x1}^x, V_{1y}^y, V_{y1}^y, V_{xy}^x, V_{yx}^x, V_{yy}^1, V_{xx}^1, V_{xx}^y,and V_{xx}^x, and they are all 1-dimensional except the last space which is 2-dimensional.

Let's choose basis vectors in each space. If we fix any non-zero vector $v_{11}^1 \in V_{11}^1$, then there are unique vectors $v_{1x}^x \in V_{1x}^x$, $v_{x1}^x \in V_{x1}^x$,$v_{1y}^y \in V_{1y}^y$, and $v_{y1}^y \in V_{y1}^y$ such that the triangle equality holds. For the other spaces, choose any non-zero vectors in each space and denote them by $v_{xy}^x \in V_{xy}^x$, $v_{yx}^x \in V_{yx}^x$, $v_{yy}^1 \in V_{yy}^1$, $v_{xx}^1 \in V_{xx}^1$, $v_{xx}^y \in V_{xx}^y$, v_1 and $v_2 \in V_{xx}^x$ where the two vectors v_1 and v_2 are linearly independent.

There are 30 associativities. It is a well known fact that if at least one of the bottom objects is $\mathbf{1}$ then the associativity is trivial. That is, with the above basis choices the matrix for $\alpha_{u,v,w}^z$ is trivial if at least one of the u, v and w is $\mathbf{1}$. Now we have ten non-trivial 1-dimensional associativities, $\alpha_{y,y,y}^y$,$\alpha_{x,y,y}^x$,$\alpha_{y,y,x}^x$,$\alpha_{x,y,x}^1$, $\alpha_{x,y,x}^y$,$\alpha_{y,x,y}^x$,$\alpha_{x,x,y}^1$,$\alpha_{x,x,y}^y$, $\alpha_{y,x,x}^1$, and $\alpha_{y,x,x}^y$, five non-trivial 2-dimensional ones, $\alpha_{x,y,x}^x$,$\alpha_{x,x,y}^x$,$\alpha_{y,x,x}^x$,$\alpha_{x,x,x}^1$, and $\alpha_{x,x,x}^y$, and one 6-dimensional one, $\alpha_{x,x,x}^x$.

With the above basis choices we obtain a basis for each tensor product of vector spaces in a canonical way and can parameterize each associativity and pentagon equation.

At this point our basis elements have not been uniquely specified, and we should expect to obtain solutions with free parameters. As the calculation progresses it will be convenient to simplify the pentagon equations by requiring certain coefficients of certain associativity matrices to be 1 or 0. These normalizations should be thought of as restrictions on the basis choices made above. Normalizations simplify the equations and have an additional advantage: once the set of possible bases is sufficiently restricted, Ocneanu rigidity [2] guarantees a finite set of possibilities for the associativity matrices[1], which can be found algorithmically by computing a Gröbner basis.

The following are 1-dimensional associativities:
$\alpha_{y,y,y}^y : v_{yy}^1 v_{y1}^y \mapsto a_{y,y,y}^y v_{yy}^1 v_{1y}^y$
$\alpha_{x,y,y}^x : v_{yy}^1 v_{x1}^x \mapsto a_{x,y,y}^x v_{xy}^x v_{xy}^x$
$\alpha_{y,y,x}^x : v_{yx}^x v_{yx}^x \mapsto a_{y,y,x}^x v_{yy}^1 v_{1x}^x$
$\alpha_{x,y,x}^1 : v_{yx}^x v_{xx}^1 \mapsto a_{x,y,x}^1 v_{xy}^x v_{xx}^1$
$\alpha_{x,y,x}^y : v_{yx}^x v_{xx}^y \mapsto a_{x,y,x}^y v_{xy}^x v_{xx}^y$
$\alpha_{y,x,y}^x : v_{xy}^x v_{yx}^x \mapsto a_{y,x,y}^x v_{yx}^x v_{xy}^x$
$\alpha_{x,x,y}^1 : v_{xy}^x v_{xx}^1 \mapsto a_{x,x,y}^1 v_{xx}^y v_{yy}^1$
$\alpha_{x,x,y}^y : v_{xy}^x v_{xx}^y \mapsto a_{x,x,y}^y v_{xx}^1 v_{1y}^y$

[1]provided that the solutions turn out to be rigid. However, section 4 will show that one can insert simple equations which enforce rigidity.

$\alpha^1_{y,x,x} : v^y_{xx}v^1_{yy} \mapsto a^1_{y,x,x}v^x_{yx}v^1_{xx}$
$\alpha^y_{y,x,x} : v^1_{xx}v^y_{y1} \mapsto a^y_{y,x,x}v^x_{yx}v^y_{xx}$
where associativity coefficients are all non-zero.

For 2-dimensional and 6-dimensional associativities we need to fix the ordering of basis elements in each Hom vector space. The orderings are as follows:
$\{v^x_{yx}v_1, v^x_{yx}v_2\}$ for $V^x_{x(yx)}$, $\{v^x_{xy}v_1, v^x_{xy}v_2\}$ for $V^x_{(xy)x}$,
$\{v^x_{xy}v_1, v^x_{xy}v_2\}$ for $V^x_{x(xy)}$, $\{v_1v^x_{xy}, v_2v^x_{xy}\}$ for $V^x_{(xx)y}$,
$\{v_1v^x_{yx}, v_2v^x_{yx}\}$ for $V^x_{y(xx)}$, $\{v^x_{yx}v_1, v^x_{yx}v_2\}$ for $V^x_{(yx)x}$,
$\{v_1v^1_{xx}, v_2v^1_{xx}\}$ for $V^1_{x(xx)}$, $\{v_1v^1_{xx}, v_2v^1_{xx}\}$ for $V^1_{(xx)x}$,
$\{v_1v^y_{xx}, v_2v^y_{xx}\}$ for $V^y_{x(xx)}$, $\{v_1v^y_{xx}, v_2v^y_{xx}\}$ for $V^y_{(xx)x}$,
$\{v^1_{xx}v^x_{x1}, v^y_{xx}v^x_{xy}, v_1v_1, v_1v_2, v_2v_1, v_2v_2\}$ for $V^x_{x(xx)}$,
and $\{v^1_{xx}v^x_{1x}, v^y_{xx}v^x_{yx}, v_1v_1, v_1v_2, v_2v_1, v_2v_2\}$ for $V^x_{(xx)x}$.

With these ordered bases, each associativity has a matrix form (here we are using the right multiplication convention). That is, $\alpha^x_{x,y,x}$ is given by the invertible 2×2 matrix $a^x_{x,y,x}$, and $\alpha^x_{x,x,y}$ is given by the invertible 2×2 matrix $a^x_{x,x,y}$,etc., and finally $\alpha^x_{x,x,x}$ is given by the invertible 6×6 matrix $a^x_{x,x,x}$.

Considering only nontrivial associativities, there are 17 1-dimensional pentagon equations, 14 2-dimensional pentagon equations, 6 6-dimensional ones, and 1 16-dimensional one. Without redundancy, the following are the 1-dimensional equations:
$P^x_{x,y,y,y} : a^y_{y,y,y}\ a^x_{x,y,y} = a^x_{x,y,y}$.
$P^1_{x,x,y,y} : a^x_{x,y,y}a^1_{x,x,y}\ a^y_{x,x,y} = 1$,
$P^1_{x,y,x,y} : a^x_{y,x,y}\ a^y_{x,y,x} = a^1_{x,y,x}$,
$P^y_{x,y,x,y} : a^x_{y,x,y}\ a^1_{x,y,x} = a^y_{x,y,x}$,
$P^1_{x,y,y,x} : a^x_{y,y,x}\ a^x_{x,y,y} = (a^1_{x,y,x})^2$,
$P^y_{x,y,y,x} : a^x_{y,y,x}\ a^x_{x,y,y} = (a^y_{x,y,x})^2$,
$P^x_{y,x,y,y} : (a^x_{y,x,y})^2 = 1$,
$P^1_{y,y,x,x} : a^y_{y,x,x}\ a^1_{y,x,x}\ a^x_{y,y,x} = 1$,
$P^1_{y,x,x,y} : a^y_{x,x,y}\ a^y_{y,x,x} = a^1_{y,x,x}\ a^1_{x,x,y}$

If we normalize the basis we may assume $a^x_{y,y,x}, a^1_{x,y,x}$ and $a^1_{x,x,y}$ to be 1 (for normalization see [12] or [6]), and we can solve the above 1-dimensional equations. Here is the solution:
$a^y_{y,y,y} = a^x_{x,y,y} = a^y_{x,x,y} = 1$, $a^x_{y,x,y} = a^y_{x,y,x} = \pm 1$, $a^1_{y,x,x} = a^y_{y,x,x} = \pm 1$.

Let's say $g := a^x_{y,x,y} = a^y_{x,y,x}$ and $h := a^1_{y,x,x} = a^y_{y,x,x}$ in the sequel. Also let $A := a^x_{x,y,x}$, $B := a^x_{x,x,y}$, $D := a^1_{x,x,x}$, $E := a^y_{x,x,x}$, $F := a^x_{y,x,x}$ and $\Phi := a^x_{x,x,x}$ for brevity.

Now, the following are the 2-dimensional pentagon equations using the above 1-dimensional solutions:
$P^x_{y,y,x,x} : F^2 = Id_2$
$P^x_{y,x,y,x} : gAF = FA$
$P^x_{x,y,y,x} : A^2 = Id_2$
$P^x_{y,x,x,y} : gBF = FB$
$P^x_{x,y,x,y} : gBA = AB$
$P^x_{x,x,y,y} : B^2 = Id_2$
$P^1_{y,x,x,x} : EF = D$
$P^y_{y,x,x,x} : DF = E$

$P^1_{x,y,x,x}$: $FDA = D$
$P^y_{x,y,x,x}$: $FEA = gE$
$P^1_{x,x,y,x}$: $ADB = D$
$P^y_{x,x,y,x}$: $AEB = gE$
$P^1_{x,x,x,y}$: $BE = D$
$P^y_{x,x,x,y}$: $BD = E$

It should be noted that for this particular category the large number of one dimensional morphism spaces gives us q-commutativity relations and matrices with ± 1 eigenvalues, which are of great help when simplifying the pentagon equations by hand.

To analyze 6-dimensional pentagon equations, at first let's look at the isomorphism τ interchanging the first and the second factors of tensor products. This change of basis is necessary because the image basis of the matrix for $\alpha^u_{x,y,zw}$ and the domain basis of the matrix for $\alpha^u_{xy,z,w}$ may not be the same. For $P^x_{x,y,x,x}$, τ is an isomorphism from the space $V^1_{xx}V^x_{xy}V^x_{x1} \oplus V^y_{xx}V^x_{xy}V^x_{xy} \oplus V^x_{xx}V^x_{xy}V^x_{xx}$ to $V^x_{xy}V^1_{xx}V^x_{x1} \oplus V^x_{xy}V^y_{xx}V^x_{xy} \oplus V^x_{xy}V^x_{xx}V^x_{xx}$, both of which correspond to $Hom((x \otimes y) \otimes (x \otimes x), x)$. With the canonically ordered basis $\{v^1_{xx}v^x_{xy}v^x_{x1}, v^y_{xx}v^x_{xy}v^x_{xy}, v_i v^x_{xy} v_j\}$ and $\{v^x_{xy}v^1_{xx}v^x_{x1}, v^x_{xy}v^y_{xx}v^x_{xy}, v^x_{xy}v_i v_j\}$, respectively, τ turns out to be I_6. For $P^x_{y,x,x,x}$, $P^x_{x,x,y,x}$ and $P^x_{x,x,x,y}$, τ is also I_6. But for $P^1_{x,x,x,x}$, it is τ_1, and for $P^y_{x,x,x,x}$, it is τ_2 as follows:

$$\tau_1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tau_2 := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here are the 6 6-dimensional pentagon equations:
$P^x_{y,x,x,x}$; $\Phi(I_2 \oplus I_2 \otimes F)\left(\left[\begin{smallmatrix} 0 & h \\ h & 0 \end{smallmatrix}\right] \oplus F \otimes I_2\right) = \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right] \oplus I_2 \otimes F\right)\Phi$
$P^x_{x,y,x,x}$; $\left(\left[\begin{smallmatrix} 0 & h \\ h & 0 \end{smallmatrix}\right] \oplus F \otimes I_2\right)\Phi\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right] \oplus A \otimes I_2\right) = (I_2 \oplus I_2 \otimes A)\Phi$
$P^x_{x,x,y,x}$; $\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right] \oplus A \otimes I_2\right)\Phi\left(\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \oplus B \otimes I_2\right) = \Phi(I_2 \oplus I_2 \otimes A)$
$P^x_{x,x,x,y}$; $\left(\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \oplus B \otimes I_2\right)(I_2 \oplus I_2 \otimes B)\Phi = \Phi\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right] \oplus I_2 \otimes B\right)$
$P^1_{x,x,x,x}$; $(I_2 \oplus I_2 \otimes D)\Phi = (I_2 \oplus I_2 \otimes D)\tau_1\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & h \end{smallmatrix}\right] \oplus I_2 \otimes D\right)$
$P^y_{x,x,x,x}$; $\Phi\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right] \oplus I_2 \otimes E\right)\Phi = (I_2 \oplus I_2 \otimes E)\tau_2\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & h \end{smallmatrix}\right] \oplus I_2 \otimes E\right)$.

If we normalize the basis $\{v_1, v_2\}$ of V^x_{xx}, we may assume A is of the form $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, and then get $g = -1$ and $h = 1$ using the above equations. Following is the computation for this:

At first we may assume that matrix A is of the Jordan canonical form, then $A = \pm I_2$ or $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ from $P^x_{x,y,y,x}$. We eliminate the possibility $A = \pm I_2$ from $P^x_{y,x,y,x}$, $P^1_{x,y,x,x}$ and $P^x_{y,x,x,x}$ which imply $g = 1$, $F = \pm I_2$ and then $det(\Phi) = 0$, respectively. So we conclude $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Now we eliminate the possibility $g = 1$ using $P^x_{y,x,y,x}$, $P^x_{y,y,x,x}$ and $P^x_{y,x,x,x}$, which imply F is a diagonal matrix, with entries ± 1 and then $det(\Phi) = 0$, respectively. For the case $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $g = -1$, F is of the form $\left[\begin{smallmatrix} 0 & f \\ 1/f & 0 \end{smallmatrix}\right]$ from $P^x_{y,x,y,x}$ and $P^x_{y,y,x,x}$, and B is of the form $\left[\begin{smallmatrix} 0 & b \\ 1/b & 0 \end{smallmatrix}\right]$ from $P^x_{x,y,x,y}$ and $P^x_{x,x,y,y}$. If $h = -1$, the first column of Φ has to be zero by comparing the first and the second columns of $P^x_{y,x,x,x}$, $P^x_{x,y,x,x}$, $P^x_{x,x,y,x}$, and $P^x_{x,x,x,y}$.

At this point we have fixed all 1-dimensional coefficients.

From the above equations, we get $\left[\begin{smallmatrix} 0 & f \\ 1/f & 0 \end{smallmatrix}\right]$ for F and $\left[\begin{smallmatrix} 0 & b \\ 1/b & 0 \end{smallmatrix}\right]$ for B with the relation $f^2+b^2=0$ from $P^x_{y,x,x,y}$. We note that the above normalized basis v_1, v_2 is defined up to nonzero scalar product. That means we may assume $f=1$. Then from the above 6-dimensional equations, we get the following:

$$D = d\begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix}$$

$$E = d\begin{bmatrix} b & 1 \\ -b & 1 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} \phi & \phi & -wb & w & w & -wb \\ \phi & -\phi & -wb & w & -w & wb \\ x & x & -yb & z & y & -zb \\ x & x & -zb & y & z & -yb \\ x & -x & -yb & z & -y & zb \\ -x & x & zb & -y & z & -yb \end{bmatrix}$$

Now we analyze the 16-dimensional pentagon equation $P^x_{x,x,x,x}$. It is convenient to express each Hom vector space in two different ways and put basis permutation matrices into the pentagon equation. The following are two expressions with ordered direct sum.

$Hom(x(x(xx)),x)$:

$V^x_{xx}V^1_{xx}V^x_{x1} \oplus V^x_{xx}V^y_{xx}V^x_{xy} \oplus V^1_{xx}V^x_{x1}V^x_{xx} \oplus V^y_{xx}V^x_{xy}V^x_{xx} \oplus V^x_{xx}V^x_{xx}V^x_{xx}$, and

$V^1_{xx}V^x_{x1}V^x_{xx} \oplus V^y_{xx}V^x_{xy}V^x_{xx} \oplus V^x_{xx}V^1_{xx}V^x_{x1} \oplus V^x_{xx}V^y_{xx}V^x_{xy} \oplus V^x_{xx}V^x_{xx}V^x_{xx}$

$Hom(x((xx)x)),x)$:

$V^x_{xx}V^1_{xx}V^x_{x1} \oplus V^x_{xx}V^y_{xx}V^x_{xy} \oplus V^1_{xx}V^x_{1x}V^x_{xx} \oplus V^y_{xx}V^x_{yx}V^x_{xx} \oplus V^x_{xx}V^x_{xx}V^x_{xx}$, and

$V^1_{xx}V^x_{1x}V^x_{xx} \oplus V^y_{xx}V^x_{yx}V^x_{xx} \oplus V^x_{xx}V^1_{xx}V^x_{x1} \oplus V^x_{xx}V^y_{xx}V^x_{xy} \oplus V^x_{xx}V^x_{xx}V^x_{xx}$

$Hom((x(xx))x),x)$:

$V^1_{xx}V^x_{x1}V^x_{xx} \oplus V^y_{xx}V^x_{xy}V^x_{xx} \oplus V^x_{xx}V^1_{xx}V^x_{1x} \oplus V^x_{xx}V^y_{xx}V^x_{yx} \oplus V^x_{xx}V^x_{xx}V^x_{xx}$, and

$V^x_{xx}V^1_{xx}V^x_{1x} \oplus V^x_{xx}V^y_{xx}V^x_{yx} \oplus V^1_{xx}V^x_{x1}V^x_{xx} \oplus V^y_{xx}V^x_{xy}V^x_{xx} \oplus V^x_{xx}V^x_{xx}V^x_{xx}$

$Hom((((xx)x)x),x)$:

$V^x_{xx}V^1_{xx}V^x_{1x} \oplus V^x_{xx}V^y_{xx}V^x_{yx} \oplus V^1_{xx}V^x_{1x}V^x_{xx} \oplus V^y_{xx}V^x_{yx}V^x_{xx} \oplus V^x_{xx}V^x_{xx}V^x_{xx}$, and

$V^1_{xx}V^x_{1x}V^x_{xx} \oplus V^y_{xx}V^x_{yx}V^x_{xx} \oplus V^x_{xx}V^1_{xx}V^x_{1x} \oplus V^x_{xx}V^y_{xx}V^x_{yx} \oplus V^x_{xx}V^x_{xx}V^x_{xx}$

$Hom((xx)(xx),x)$:

$V^1_{xx}V^x_{xx}V^x_{x1} \oplus V^y_{xx}V^x_{xx}V^x_{xy} \oplus V^x_{xx}V^1_{xx}V^x_{1x} \oplus V^x_{xx}V^y_{xx}V^x_{yx} \oplus V^x_{xx}V^x_{xx}V^x_{xx}$, and

$V^1_{xx}V^x_{xx}V^x_{1x} \oplus V^y_{xx}V^x_{xx}V^x_{yx} \oplus V^x_{xx}V^1_{xx}V^x_{x1} \oplus V^x_{xx}V^y_{xx}V^x_{xy} \oplus V^x_{xx}V^x_{xx}V^x_{xx}$

where each direct summand space has canonical ordered basis. For example $V^x_{xx}V^x_{xx}V^x_{xx}$ has basis $\{v_iv_jv_k\}$ where (i,j,k) range from 1 to 2 in the order $(1,1,1),(1,1,2),(1,2,1)$, etc., and $V^x_{xx}V^1_{xx}V^x_{1x}$ has $\{v_1v^1_{xx}v^x_{1x}, v_2v^1_{xx}v^x_{1x}\}$.

Let

$$\tau_3 := \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes I_4\right) \oplus I_8 \text{ and}$$

$$\tau_4 := \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes I_4\right) \oplus \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes I_2\right).$$

Then the pentagon equation $P^x_{x,x,x,x}$ is of the form:

$$(D \oplus E \oplus \Phi \otimes I_2)\tau_3(I_2 \oplus A \oplus \tilde{\Phi})\tau_3(D \oplus E \oplus \Phi \otimes I_2)\tau_3$$

$$= \tau_3(I_2 \oplus B \oplus \tilde{\Phi})\tau_4(I_2 \oplus F \oplus \tilde{\Phi})$$

where

$$\tilde{\Phi} = \begin{bmatrix} \phi & 0 & \phi & 0 & -wb & w & w & -wb & 0 & 0 & 0 & 0 \\ 0 & \phi & 0 & \phi & 0 & 0 & 0 & 0 & -wb & w & w & -wb \\ \phi & 0 & -\phi & 0 & -wb & w & -w & wb & 0 & 0 & 0 & 0 \\ 0 & \phi & 0 & -\phi & 0 & 0 & 0 & 0 & -wb & w & -w & wb \\ x & 0 & x & 0 & -yb & z & y & -zb & 0 & 0 & 0 & 0 \\ x & 0 & x & 0 & -zb & y & z & -yb & 0 & 0 & 0 & 0 \\ x & 0 & -x & 0 & -yb & z & -y & zb & 0 & 0 & 0 & 0 \\ -x & 0 & x & 0 & zb & -y & z & -yb & 0 & 0 & 0 & 0 \\ 0 & x & 0 & x & 0 & 0 & 0 & 0 & -yb & z & y & -zb \\ 0 & x & 0 & x & 0 & 0 & 0 & 0 & -zb & y & z & -yb \\ 0 & x & 0 & -x & 0 & 0 & 0 & 0 & -yb & z & -y & zb \\ 0 & -x & 0 & x & 0 & 0 & 0 & 0 & zb & -y & z & -yb \end{bmatrix}.$$

We may assume $x = 1$ once we normalize basis vector v^x_{xy}. Then from the equations, we get four explicit solution sets for the parameters b, ϕ, d, w, y, and z. We list one solution here; the others can be obtained by Galois automorphisms. The full set of associativity matrices with these basis choices is given in Appendix A.

$$b = i, \phi = \frac{-1+\sqrt{3}}{2}, d = \frac{1}{\sqrt{2}} e^{7\pi i/12}, w = \frac{1-\sqrt{3}}{4} e^{2\pi i/3},$$

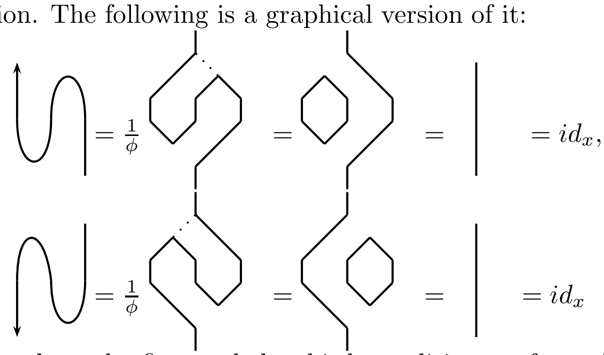
$$y = \frac{1}{2}(e^{-\pi i/3} + i), z = \frac{1}{2} e^{5\pi i/6}.$$

An examination of the normalizations made during the calculation shows that the solutions really do lie in four distinct natural equivalence classes. We omit this analysis.

4. Rigidity Sturctures

Given $v^1_{xx} \in V^1_{xx}$, choose a vector $v^{xx}_1 \in V^{xx}_1$ such that $v^{xx}_1 \circ v^1_{x,x} = id_1$(see Figure 2). Now we define right death and birth, $d_x := v^1_{xx} : x \otimes x \to \mathbf{1}$, $b_x := \frac{1}{\phi} v^{xx}_1 : \mathbf{1} \to x \otimes x$ (see Figure 3).

With these definitions, right rigidity is an easy consequence by direct computation. The following is a graphical version of it:



where the first and the third equalities are from the definitions above, and the second equalities are the associativity α^x_{xxx} and $(\alpha^x_{xxx})^{-1}$, respectively.

The same morphisms give a left rigidity structure when treated as left birth and left death. Treat the objects y and $\mathbf{1}$ analogously by replacing ϕ with 1.

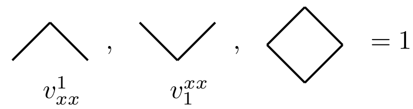


FIGURE 2. Graphical notation of v^1_{xx} and v_1^{xx} and property

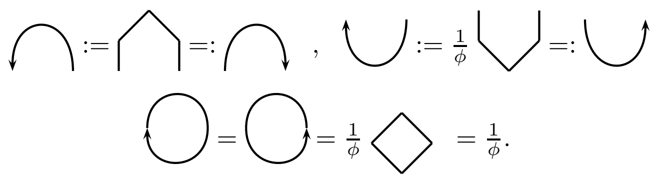


FIGURE 3. Definitions of b_x and d_x, and elementary properties

$$
\begin{tikzcd}
 & (yx)z \arrow[rr, "\alpha_{y,x,z}"] & & y(xz) \arrow[dr, "yc_{x,z}"] & \\
(xy)z \arrow[ur, "c_{x,y}z"] \arrow[dr, "\alpha_{x,y,z}"'] & & & & y(zx) \\
 & x(yz) \arrow[rr, "c_{x,yz}"] & & (yz)x \arrow[ur, "\alpha_{y,z,x}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & (yx)z \arrow[rr, "\alpha_{y,x,z}"] & & y(xz) \arrow[dr, "yc^{-1}_{z,x}"] & \\
(xy)z \arrow[ur, "c^{-1}_{y,x}z"] \arrow[dr, "\alpha_{x,y,z}"'] & & & & y(zx) \\
 & x(yz) \arrow[rr, "c^{-1}_{yz,x}"] & & (yz)x \arrow[ur, "\alpha_{y,z,x}"'] &
\end{tikzcd}
$$

FIGURE 4. Hexagon equalities

5. The Absence of Braidings

The categories under consideration are known not to be braided (see [9]). However, once associativity matrices are known it is in principle not difficult to classify braidings by direct computation. In this section we perform this computation and show that no braidings are possible.

A braiding consists of a natural family of isomorphisms $\{c_{x,y} : x \otimes y \to y \otimes x\}$ such that two hexagon equalities hold:

$(c_{x,y} \otimes z) \circ \alpha_{y,x,z} \circ (y \otimes c_{x,z}) = \alpha_{x,y,z} \circ c_{x,yz} \circ \alpha_{y,z,x}$ and

$((c_{y,x})^{-1} \otimes z) \circ \alpha_{y,x,z} \circ (y \otimes (c_{z,x})^{-1}) = \alpha_{x,y,z} \circ (c_{yz,x})^{-1} \circ \alpha_{y,z,x}$.

We define isomorphisms $R^z_{x,y} : V^z_{yx} \to V^z_{xy}$ by $f \mapsto c_{x,y} \circ f$ and $\bar{R}^z_{x,y} : V^z_{yx} \to V^z_{xy}$ by $f \mapsto (c_{x,y})^{-1} \circ f$ for any $f \in V^z_{yx}$. Figure 5 shows the 1-dimensional case where $r^z_{x,y}$ is nonzero and $\bar{r}^z_{x,y} = (r^z_{y,x})^{-1}$. For higher dimensional spaces it can

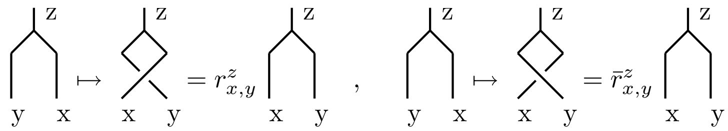


FIGURE 5. Isomorphisms $R^z_{x,y}$ and $\bar{R}^z_{x,y}$

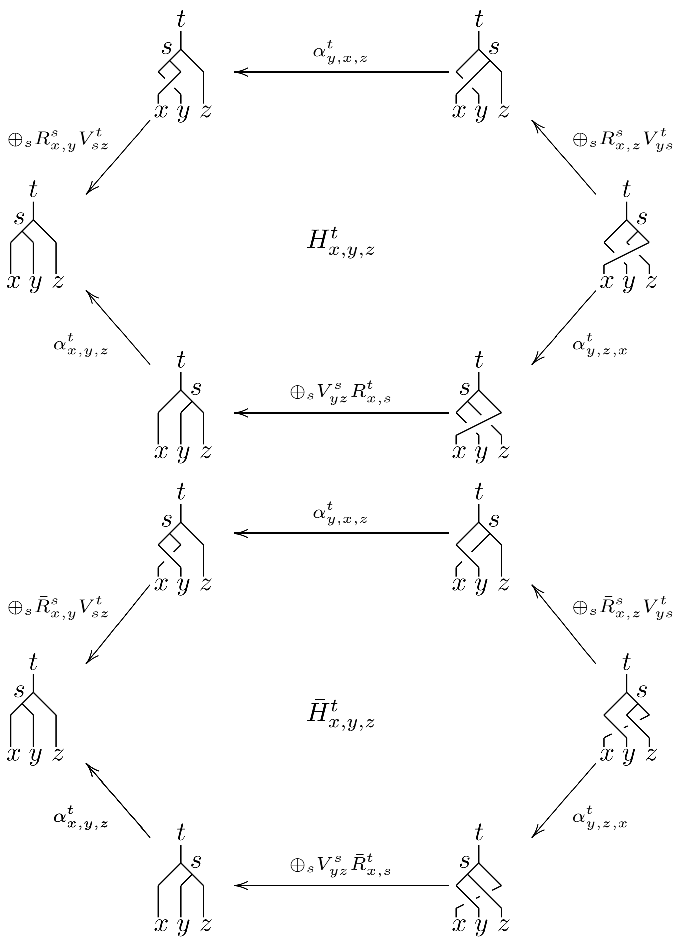


FIGURE 6. Equivalent Hexagon equalities

be expressed as an invertible matrix, also denoted $r^z_{x,y}$ on the canonically ordered basis as before.

With this linear isomorphism, the hexagon equations are equivalent to the equations

$\oplus_s R^s_{x,z} V^t_{ys} \circ \alpha^t_{y,x,z} \circ \oplus_s R^s_{x,y} V^t_{sz} = \alpha^t_{y,z,x} \circ \oplus_s V^s_{yz} R^t_{x,s} \circ \alpha^t_{x,y,z}$, and

$\oplus_s \bar{R}^s_{x,z} V^t_{ys} \circ \alpha^t_{y,x,z} \circ \oplus_s \bar{R}^s_{x,y} V^t_{sz} = \alpha^t_{y,z,x} \circ \oplus_s V^s_{yz} \bar{R}^t_{x,s} \circ \alpha^t_{x,y,z}$, which we still call hexagon equations, referred to as $H^t_{x,y,z}$ and $\bar{H}^t_{x,y,z}$, respectively. These are illustrated graphically in Figure 6).

We show the absence of a braiding by assuming the existence and deriving a contradiction.

We need five 2-dimensional hexagon equations as follows:
$H^x_{y,x,x}: R^x_{y,x}\otimes I_2\circ\alpha^x_{x,y,x}\circ R^x_{y,x}\otimes I_2=\alpha^x_{x,x,y}\circ I_2\otimes R^x_{y,x}\circ\alpha^x_{y,x,x}$
$\bar{H}^x_{y,x,x}: \bar{R}^x_{y,x}\otimes I_2\circ\alpha^x_{x,y,x}\circ \bar{R}^x_{y,x}\otimes I_2=\alpha^x_{x,x,y}\circ I_2\otimes \bar{R}^x_{y,x}\circ\alpha^x_{y,x,x}$
$H^x_{x,y,x}: R^x_{x,x}\otimes 1\circ\alpha^x_{y,x,x}\circ R^x_{x,y}\otimes I_2=\alpha^x_{y,x,x}\circ 1\otimes R^x_{x,x}\circ\alpha^x_{x,y,x}$
$\bar{H}^x_{x,y,x}: \bar{R}^x_{x,x}\otimes 1\circ\alpha^x_{y,x,x}\circ \bar{R}^x_{x,y}\otimes I_2=\alpha^x_{y,x,x}\circ 1\otimes \bar{R}^x_{x,x}\circ\alpha^x_{x,y,x}$
$H^1_{x,x,x}: R^x_{x,x}\otimes 1\circ\alpha^1_{x,x,x}\circ R^x_{x,x}\otimes 1=\alpha^1_{x,x,x}\circ I_2\otimes R^1_{x,x}\circ\alpha^1_{x,x,x}$
these are of the following forms, respectively:
$(r^x_{y,x})^2\begin{bmatrix}1&0\\0&-1\end{bmatrix}=r^x_{y,x}\begin{bmatrix}0&b\\1/b&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}$
$(r^x_{x,y})^{-2}\begin{bmatrix}1&0\\0&-1\end{bmatrix}=(r^x_{x,y})^{-1}\begin{bmatrix}0&b\\1/b&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}$
$r^x_{x,y}\begin{bmatrix}k&l\\m&n\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}k&l\\m&n\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$
$(r^x_{y,x})^{-1}\begin{bmatrix}k&l\\m&n\end{bmatrix}^{-1}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}k&l\\m&n\end{bmatrix}^{-1}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$
$d\begin{bmatrix}k&l\\m&n\end{bmatrix}\begin{bmatrix}1&b\\1&-b\end{bmatrix}\begin{bmatrix}k&l\\m&n\end{bmatrix}=d^2r^1_{x,x}\begin{bmatrix}1&b\\1&-b\end{bmatrix}^2$
where $\begin{bmatrix}k&l\\m&n\end{bmatrix}$ represents the matrix $r^x_{x,x}$.

From the first four equations, we get $r^x_{y,x}=b$, $r^x_{x,y}=1/b$, $-n=r^x_{x,y}k$, $m=r^x_{x,y}l$, $-n=r^x_{y,x}k$, which imply $k=n=0$ since $r^x_{x,y}\neq r^x_{y,x}$ as above. Now from the final one we get $l^2=dr^1_{x,x}(b+1)$ and $-blm=dr^1_{x,x}(1+b)$, and the later equality means $l^2=-dr^1_{x,x}(1+b)$ by substituting $m=r^x_{x,y}l$. We get easily a contradiction for either case $b=\pm i$.

6. The Pivotal Structure and Sphericity

Let $\mathcal{C}$ be a rigid monoidal category. A *pivotal structure* for $\mathcal{C}$ is a monoidal natural isomorphism π from $**$ to Id. A *strict pivotal structure* is a pivotal structure which is the identity. In a pivotal monoidal category, the *right trace* tr_r of an endomorphism $f:x\to x$ is given by $tr_r(f)=b_x\circ(f\otimes Id_{x^*})\circ(\pi_x^{-1}\otimes Id_{x^*})\circ d_{x^*}\in End(\mathbf{1})\cong\mathbb{C}$. The *left trace* tr_l is given by $b_{x^*}\circ(f^*\otimes Id_{x^{**}})\circ((\pi_x)^*\otimes Id_{x^{**}})\circ d_{x^{**}}$. A pivotal monoidal category is *spherical* if $tr_r=tr_l$.

Pivotal structures may not be unique. For example, in a fusion category with object types given by a finite group G, group multiplication as tensor product and trivial associativity matrices, any group homomorphism $G\to\mathbb{C}$ gives a pivotal structure. However, only the trivial homomorphism gives a spherical category with positive dimensions.

Pivotal structures depend on choices of rigidity. However, if one chooses a new rigidity structure with $b'_x=cb_x$ and $d'_x=c^{-1}d_x$, then $\pi'_x=c^{-1}\pi_x$ gives a new pivotal structure π' inducing the same traces as π.

In a pivotal strictified skeletal fusion category, $**$ is an object fixing strong monoidal endofunctor with trivial isomorphisms J. Then π is as follows: for each strand (x), there is a scalar t_x such that $\pi_x=t_xId_x$, and $\pi_{(x_1,\ldots,x_n)}=t_{x_1}\ldots t_{x_n}Id_{(x_1,\ldots x_n)}$. Then for all sequences $(x_1,\ldots,x_m)$ and $(y_1,\ldots,y_n)$, and all $f:(x_1,\ldots,x_m)\to(y_1,\ldots,y_n)$, $f=t^{-1}_{x_1}\ldots t^{-1}_{x_m}t_{y_1}\ldots t_{y_n}f^{**}$.

Since the morphisms b_x and d_x are self double dual by rigidity, $t_xt_{x^*}=1$. Then for self-dual objects, $t_x=\pm1$; in this case t_x is called the *Frobenius-Schur indicator* for x. Since the right and left births (and deaths) have been chosen to be equal, this implies the following:

Lemma 1. Every pivotal fusion category with self-dual objects is spherical.

A proof that the categories found in Section 3 have pivotal structures appears in Section 7. Therefore, these structures are spherical. One could also prove sphericity by showing that these categories are pseudo-unitary; see [2] for definitions and details.

It is not known whether every fusion category admits a pivotal structure. It is known that $****\cong Id$. This was shown in [2], using an analog of Radford's formula for S^4 for representation categories of weak Hopf algebras, which was developed in [8]. The following theorem shows that, in a strictified skeletal fusion category, a convenient choice of rigidity makes $****$ the identity on the nose. Extending the result to general fusion categories via natural equivalences gives an elementary proof that $****\cong Id$.

Theorem 2. In a strictified skeletal fusion category, there is a choice of rigidity structures such that $**** = Id$.

Proof. The functor $****$ is the identity on (2)-stranded morphisms by rigidity; it suffices to prove the result for $(2,1)$ stranded morphisms. Let $V = V^z_{xy}$ be a $(2,1)$ stranded morphism space with a basis $\{v_i\}$, and let $\{w_i\}$ be an algebraically dual basis for the space $W = V^{xy}_z$, in the sense that $w_i \circ v_i = Id_z$. For any simple object z, define the *right pseudotrace* ptr_r of an endomorphism $f : z \to z$ by $ptr_r(f) = b_z \circ (f \otimes Id_{z^*}) \circ d_{z^*}$, and the *left pseudotrace* ptr_l by $ptr_l(f) = b_{z^*} \circ (Id_{z^*} \otimes f) \circ d_z$. This definition is possible because $**$ is the identity on objects. Scale rigidity morphisms if necessary so that for any strand z, $ptr_r(Id_z) = ptr_l(Id_z)$. Because d_z and b_{z^*} are nonzero elements of one dimensional algebraically dual morphism spaces, $ptr_r(Id_z) \neq 0$. One may now exchange left pseudotraces for right pseudotraces, just like with traces in a graphical calculus for a spherical category.

Figure 7 gives the proof. On the left side, bending arms and pseudosphericity implies that the algebraic dual basis of the basis $\{w_i^{**}\}$ is $\{{}^{**}v_i\}$. However, on the right side the functoriality of the double dual implies that the algebraic dual basis of $\{w_i^{**}\}$ is $\{v_i^{**}\}$. Since the left and right double dual are inverse functors, $****$ is the identity. □

Even if a category admits a pivotal structure it is not known whether it admits a spherical pivotal structure. Pictorial considerations do not readily provide an answer. It is possible, however, to partially describe what a pivotal strictified skeleton which did not admit a spherical structure would look like.

Let $\mathcal{C}$ be a pivotal strictified skeletal fusion category which does not admit a spherical structure. Choose rigidity morphisms which give a pseudo-spherical structure as above. Then $\mathcal{C}$ is spherical iff all of the t are ± 1. For simple objects u and v, $u \otimes v$ has a nontrivial morphism to some object w, and $t_u t_v (t_w)^{-1} = \pm 1$. At least one triplet must give -1, or else $\mathcal{C}$ would have a strict pivotal structure, which would be spherical. Thus the set of scalars t and their additive inverses forms a finite subgroup G of $\mathbb{C}$.

Every finite subgroup of $\mathbb{C}$ is a cyclic group of roots of unity. We have $|G| = 2k$ for some k, and since $\mathcal{C}$ is not spherical, $|G| \geq 4$. Using a homomorphism which preserves -1 we may switch to a new pivotal structure which gives $|G| = 2^k$ for

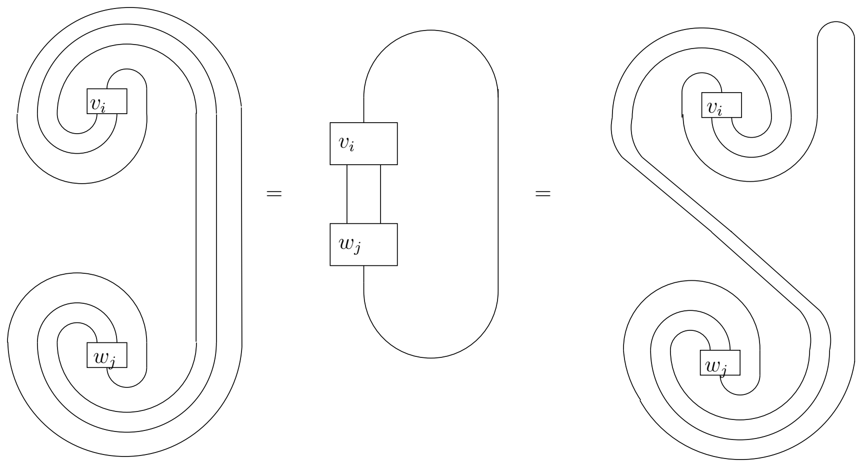


FIGURE 7. In a strictified skeletal fusion category, with the right choice of rigidity structures the quadruple dual is the identity.

some k, where $k \geq 2$ to contradict sphericity. Pick an object v with $t_v^2 = -1$. Then v is not self dual, and for a simple summand w in $v \otimes v$, one has $w \neq \mathbf{1}$ and $t_w^2 = 1$. Therefore, $\mathcal{C}$ has at least four objects, v, v^*, w and $\mathbf{1}$. The set of objects u such that $t_u^2 = 1$ generates a spherical subcategory with at least two simple objects, and missing at least two.

For the case of four objects exactly, the classification of fusion categories with two objects and the requirement that associativity makes sense at the level of fusion rules admit only one fusion ring, in which objects and tensor products are given by the group $\mathbb{Z}_4$. Any such category is pseudo-unitary and therefore spherical, as described in [2]. Therefore, a pivotal fusion category which can't be made spherical must have at least five objects.

7. Pivotal Structure Calculations

It is easy to determine whether or not a fusion category is pivotal once a set of associativity matrices is known. One way is to perform the calculations directly using the associativity matrices, but there is an easier calculation. In order to explain this calculation, it is convenient to extend the definition of composition of morphisms over extracategorical direct sums of morphism spaces. Suppose $f \in Mor(a, b)$ and $g \in Mor(c, d)$. Define $f \circ g$ as usual if $b = c$, and $f \circ g = 0 \in Mor(a, d)$ otherwise. Extend this definition over direct sums of morphism spaces, distributing composition over direct sum.

Given a strictified skeletal fusion category C and a set of associativity matrices, choose bases for the $(2, 0)$ and $(2, 1)$-stranded morphism spaces compatible with the associativity matrices and choose rigidity so that for each strand x, the basis element for $V_{x^*x}^1$ is d_x. Define morphisms $b = \oplus_x b_x$, $d = \oplus_x d_x$, and $I = \oplus_x Id_x$, taking sums over the strands.

Then B acts on $\bigoplus_{x,y,z} V_{xy}^z$ as follows:

$$B(f) = (I \otimes I \otimes b) \circ (I \otimes f \otimes I) \circ (d \otimes I)$$

For a single $(2,1)$-stranded morphism space, this action amounts to "bending arms". The cube of B is the double dual. The action of B on a morphism $f \in V^z_{xy}$ is given by the associativity matrix $a^{\mathbf{1}}_{z^*,x,y}$, since $(Id_{z^*} \otimes f) \circ d_z = (g \otimes id_y) \circ d_y$ for some $g \in V^{y^*}_{z^*x}$ implies that $B(f) = (Id_{z^*} \otimes Id_x \otimes b_y) \circ (Id_{z^*} \otimes f \otimes Id_{y^*}) \circ (d_z \otimes Id_{y^*}) = (Id_{z^*} \otimes Id_x \otimes b_y) \otimes (g \otimes Id_y \otimes Id_{y^*}) \circ (d_y \otimes Id_{y^*}) = g$ by rigidity. For the fusion rules at hand, the matrix for B is as follows:

$$
\begin{array}{c|ccccc}
 & v_1 & v_2 & v^y_{xx} & v^x_{yx} & v^x_{xy} \\
\hline
v_1 & (a^{\mathbf{1}}_{xxx})_{1,1} & (a^{\mathbf{1}}_{xxx})_{1,2} & 0 & 0 & 0 \\
v_2 & (a^{\mathbf{1}}_{xxx})_{2,1} & (a^{\mathbf{1}}_{xxx})_{2,2} & 0 & 0 & 0 \\
v^y_{xx} & 0 & 0 & 0 & a^{\mathbf{1}}_{yxx} & 0 \\
v^x_{yx} & 0 & 0 & 0 & 0 & a^{\mathbf{1}}_{xyx} \\
v^x_{xy} & 0 & 0 & a^{\mathbf{1}}_{xxy} & 0 & 0
\end{array}
$$

For all of the solutions given in section 3, B^3 is the identity matrix, so the corresponding strictified categories have a strict pivotal structure. Non-strict pivotality would mean that B^3 is a diagonal matrix with eigenvalues determined by a family of invertible scalars t, coherent as described in section 6.

Appendix A. Associativity Matrices

In this section, we give explicit associativity matrices for the categorical realization given in Section 3.

$$a^y_{y,y,y} = a^x_{x,y,y} = a^x_{y,y,x} = a^1_{x,y,x} = a^1_{x,x,y} = a^y_{x,x,y} = a^1_{y,x,x} = a^y_{y,x,x} = 1,$$
$$a^y_{x,y,x} = a^x_{y,x,y} = -1,$$
$$a^x_{x,y,x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$a^x_{x,x,y} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$
$$a^x_{y,x,x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$a^1_{x,x,x} = \frac{1}{\sqrt{2}} e^{7\pi i/12} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$
$$a^y_{x,x,x} = \frac{1}{\sqrt{2}} e^{7\pi i/12} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}$$
$$a^x_{x,x,x} = \begin{bmatrix}
\frac{-1+\sqrt{3}}{2} & \frac{-1+\sqrt{3}}{2} & \frac{1-\sqrt{3}}{4}e^{\pi i/6} & \frac{1-\sqrt{3}}{4}e^{2\pi i/3} & \frac{1-\sqrt{3}}{4}e^{2\pi i/3} & \frac{1-\sqrt{3}}{4}e^{\pi i/6} \\
\frac{-1+\sqrt{3}}{2} & \frac{1-\sqrt{3}}{2} & \frac{1-\sqrt{3}}{4}e^{\pi i/6} & \frac{1-\sqrt{3}}{4}e^{2\pi i/3} & -\frac{1-\sqrt{3}}{4}e^{2\pi i/3} & -\frac{1-\sqrt{3}}{4}e^{\pi i/6} \\
1 & 1 & -\frac{1}{2}(e^{\pi i/6}-1) & \frac{1}{2}e^{5\pi i/6} & \frac{1}{2}(e^{-\pi i/3}+i) & \frac{1}{2}e^{\pi i/3} \\
1 & 1 & \frac{1}{2}e^{\pi i/3} & \frac{1}{2}(e^{-\pi i/3}+i) & \frac{1}{2}e^{5\pi i/6} & -\frac{1}{2}(e^{\pi i/6}-1) \\
1 & -1 & -\frac{1}{2}(e^{\pi i/6}-1) & \frac{1}{2}e^{5\pi i/6} & -\frac{1}{2}(e^{-\pi i/3}+i) & -\frac{1}{2}e^{\pi i/3} \\
-1 & 1 & -\frac{1}{2}e^{\pi i/3} & -\frac{1}{2}(e^{-\pi i/3}+i) & \frac{1}{2}e^{5\pi i/6} & -\frac{1}{2}(e^{\pi i/6}-1)
\end{bmatrix}$$

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