ON THE NUMBER OF TOPOLOGICAL TYPES OCCURRING IN A PARAMETRIZED FAMILY OF ARRANGEMENTS

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Abstract. Let $\mathcal{S}(\mathbb{R})$ be an o-minimal structure over $\mathbb{R}$, $T \subset \mathbb{R}^{k_1+k_2+\ell}$ a closed definable set, and

$$\pi_1 : \mathbb{R}^{k_1+k_2+\ell} \to \mathbb{R}^{k_1+k_2}, \quad \pi_2 : \mathbb{R}^{k_1+k_2+\ell} \to \mathbb{R}^{\ell}, \quad \pi_3 : \mathbb{R}^{k_1+k_2} \to \mathbb{R}^{k_2}$$

the projection maps as depicted below.

$$\begin{tikzcd}
\mathbb{R}^{k_1+k_2} \arrow[d, "\pi_3"'] & \mathbb{R}^{k_1+k_2+\ell} \arrow[l, "\pi_1"'] \arrow[d, "\pi_2"'] \\
\mathbb{R}^{k_2} & \mathbb{R}^{\ell}
\end{tikzcd}$$

For any collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ of subsets of $\mathbb{R}^{k_1+k_2}$, and $\mathbf{z} \in \mathbb{R}^{k_2}$, let $\mathcal{A}_{\mathbf{z}}$ denote the collection of subsets of $\mathbb{R}^{k_1}$, $\{A_{1,\mathbf{z}}, \ldots, A_{n,\mathbf{z}}\}$, where $A_{i,\mathbf{z}} = A_i \cap \pi_3^{-1}(\mathbf{z})$, $1 \leq i \leq n$. We prove that there exists a constant $C = C(T) > 0$, such that for any family $\mathcal{A} = \{A_1, \ldots, A_n\}$ of definable sets, where each $A_i = \pi_1(T \cap \pi_2^{-1}(\mathbf{y}_i))$, for some $\mathbf{y}_i \in \mathbb{R}^{\ell}$, the number of distinct stable homotopy types of $\mathcal{A}_{\mathbf{z}}$, $\mathbf{z} \in \mathbb{R}^{k_2}$, is bounded by $C \cdot n^{(k_1+1)k_2}$, while the number of distinct homotopy types is bounded by $C \cdot n^{(k_1+3)k_2}$. This generalizes to the general o-minimal setting, bounds of the same type proved in [4] for semi-algebraic and semi-Pfaffian families. One main technical tool used in the proof of the above results, is a topological comparison theorem which might be of independent interest in the study of arrangements.

1. Introduction

The study of arrangements is a very important subject in discrete and computational geometry, where one studies *arrangements* of n objects in $\mathbb{R}^k$ for fixed k and large values of n. The precise nature of these objects will be discussed in more details below. Common examples consist of arrangements of hyperplanes, balls or simplices in $\mathbb{R}^k$. More generally one considers arrangements of objects of "bounded description complexity" (see [11]). This means that each set in the arrangement is defined by a first order formula in the language of ordered fields involving at most a constant number polynomials whose degrees are also bounded by a constant.

In this paper we will consider parametrized families of arrangements. The question we will be interested in most, is the number of "topologically" distinct arrangements which can occur in such a family (precise definition of the topological type of an arrangement will be given later). Parametrized arrangements occur quite frequently in practice. For instance, take any arrangement $\mathcal{A}$ in $\mathbb{R}^{k_1+k_2}$ and let

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$\pi : \mathbb{R}^{k_1+k_2} \to \mathbb{R}^{k_2}$ be the projection on the last k_2 co-ordinates. Then for each $\mathbf{y} \in \mathbb{R}^{k_2}$, the intersection of the arrangement $\mathcal{A}$ with the fiber $\pi^{-1}(\mathbf{y})$, is an arrangement $\mathcal{A}_{\mathbf{y}}$ in $\mathbb{R}^{k_1}$ and the family of the arrangements $\{\mathcal{A}_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{R}^{k_2}}$ is an example of a parametrized family of arrangements. Even though the number of arrangements in the family $\{A_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{R}^{k_2}}$ is infinite, it follows from Hardt's triviality theorem (see below) that the number of "topological types" occurring amongst them is finite and can be effectively bounded in terms of the n, k_1, k_2 as well as the number and degrees of the polynomials defining the individual objects in the arrangement. If by topological type we mean homeomorphism type, then the best known upper bound on the number of types occurring is doubly exponential in k_1, k_2. However, if we consider the weaker notion of homotopy type, then we obtain a singly exponential bound, which we suspect is closer to the truth, even for homeomorphism types.

We now make precise the class of arrangements that we consider.

1.1. Combinatorial Complexity in O-minimal Geometry.

In order to put the study of the combinatorial complexity of arrangements in a more natural mathematical context, as well as to elucidate the proofs of the main results in the area, a new framework was introduced in [2] which is a significant generalization of the settings mentioned above. We recall here the basic definitions of this framework from [2], referring the reader to the same paper for further details and examples.

1.1.1. *O-minimal Structures.*

The theory of o-minimal structures developed by, van den Dries [8] and others, show that the tame topological properties exhibited by the class of semi-algebraic sets are consequences of a set of few simple axioms. An o-minimal structure on $\mathbb{R}$ is just a class of subsets of $\mathbb{R}^k, k \geq 0$, (called the *definable sets* in the structure) satisfying these axioms (see below). The class of semi-algebraic sets is one obvious example of such a structure, but in fact there are much richer classes of sets which have been proved to be o-minimal (see [6, 8]). For instance, subsets of $\mathbb{R}^k$ defined in terms of inequalities involving not just polynomials, but also trigonometric and exponential functions on a bounded domain have been proved to be o-minimal.

For the sake of completeness we include the set of axioms of an o-minimal structure (following [6]). Further details can be found in the book by van den Dries [8], as well as in [6].

Definition 1.1. An o-minimal structure on $\mathbb{R}$ is a sequence $\mathcal{S}(\mathbb{R}) = (\mathcal{S}_n)_{n\in\mathbb{N}}$, where each $\mathcal{S}_n$ is a collection of subsets of $\mathbb{R}^n$, satisfying the following axioms [6].

(1) All algebraic subsets of $\mathbb{R}^n$ are in $\mathcal{S}_n$.
(2) The class $\mathcal{S}_n$ is closed under complementation and finite unions and intersections.
(3) If $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$ then $A \times B \in \mathcal{S}_{m+n}$.
(4) If $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection map on the first n co-ordinates and $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$.
(5) The elements of $\mathcal{S}_1$ are precisely finite unions of points and intervals.

1.1.2. *Admissible Sets.*

We now recall from [2] the definition of the class of sets that will play the role of sets with bounded description complexity mentioned above.

Definition 1.2. Let $\mathcal{S}(\mathbb{R})$ be an o-minimal structure on a real closed field $\mathbb{R}$ and let $T \subset \mathbb{R}^{k+\ell}$ be a fixed definable set. Let $\pi_1 : \mathbb{R}^{k+\ell} \to \mathbb{R}^k$ (resp. $\pi_2 : \mathbb{R}^{k+\ell} \to \mathbb{R}^\ell$), be the projections onto the first k (resp. last ℓ) co-ordinates.

$$
\begin{tikzcd}
& T \subset \mathbb{R}^{k+\ell} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathbb{R}^k & & \mathbb{R}^\ell
\end{tikzcd}
$$

We will call a subset S of $\mathbb{R}^k$ to be a (T, π_1, π_2)-set if

$$S = T_{\mathbf{y}} = \pi_1(\pi_2^{-1}(\mathbf{y}) \cap T)$$

for some $\mathbf{y} \in \mathbb{R}^\ell$. In this paper, we will consider finite families of (T, π_1, π_2)-sets, where T is some fixed definable set for each such family, and we will call a family of (T, π_1, π_2)-sets to be a (T, π_1, π_2)-family. We refer to a finite (T, π_1, π_2)-family as an *arrangement* of (T, π_1, π_2)-sets.

1.2. **Diagrams and their limits.** The arrangements that we will consider are all finitely triangulable, that is they are homeomorphic to a finite simplicial complex, and each individual object in the arrangement will correspond to a sub-complex of this simplicial complex. It will be more convenient to work in the category of finite regular cell complexes, instead of just simplicial complexes.

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$, where each A_i is a sub-complex of a finite regular cell complex. We will denote by $[n]$ the set $\{1, \ldots, n\}$ and for $I \subset [n]$ we will denote by $\mathcal{A}^I$ (resp. $\mathcal{A}_I$) the regular cell complexes $\cup_{i \in I} A_i$ (resp. $\cap_{i \in I} A_i$). Notice that if $J \subset I \subset [n]$, then

$$\mathcal{A}^J \subset \mathcal{A}^I,$$
$$\mathcal{A}_I \subset \mathcal{A}_J.$$

We will call the collection of sets $\{\mathcal{A}_I\}_{I \subset [n]}$ together with the inclusion maps $i_{I,J} : \mathcal{A}_I \hookrightarrow \mathcal{A}_J$, $J \subset I$, the *diagram* of $\mathcal{A}$, which we denote by diagram$(\mathcal{A})$. Using a standard notion from homotopy theory (see for instance [5]), $\mathcal{A}^{[n]}$ is then the co-limit of diagram(A).

Now let $\mathcal{A} = \{A_1, \ldots, A_n\}$, $\mathcal{B} = \{B_1, \ldots, B_n\}$ where each A_i, B_j is a sub-complex of a finite regular cell complex and $m \geq 0$.

1.2.1. *Diagram Preserving Maps.*

Definition 1.3. We call a map $f : |\mathcal{A}^{[n]}| \to |\mathcal{B}^{[n]}|$ to be *diagram preserving*, if $f(\mathcal{A}_I) \subset \mathcal{B}_I$ for every $I \subset [n]$. We say that two maps, $f, g : |\mathcal{A}^{[n]}| \to |\mathcal{B}^{[n]}|$, are *diagram homotopic*, if there exists, a homotopy $h : |\mathcal{A}^{[n]}| \times [0,1] \to |\mathcal{B}^{[n]}|$, such that $h(\cdot, 0) = f, h(\cdot, 1) = g$ and $h(\cdot, t)$ is diagram preserving for each $t \in [0,1]$.

We say that $f : |\mathcal{A}^{[n]}| \to |\mathcal{B}^{[n]}|$ is a diagram preserving homeomorphism, if there exists a diagram preserving inverse map, $g : |\mathcal{B}^{[n]}| \to |\mathcal{A}^{[n]}|$, such that the induced maps, $g \circ f : |\mathcal{A}^{[n]}| \to |\mathcal{A}^{[n]}|$ and $f \circ g : |\mathcal{B}^{[n]}| \to |\mathcal{B}^{[n]}|$ are $Id_{|\mathcal{A}^{[n]}|}$ and $Id_{|\mathcal{B}^{[n]}|}$, respectively.

We say that $f : |\mathcal{A}^{[n]}| \to |\mathcal{B}^{[n]}|$ is a diagram preserving homotopy equivalence, if there exists a diagram preserving inverse map, $g : |\mathcal{B}^{[n]}| \to |\mathcal{A}^{[n]}|$, such that the induced maps, $g \circ f : |\mathcal{A}^{[n]}| \to |\mathcal{A}^{[n]}|$ and $f \circ g : |\mathcal{B}^{[n]}| \to |\mathcal{B}^{[n]}|$ are diagram homotopic to $Id_{|\mathcal{A}^{[n]}|}$ and $Id_{|\mathcal{B}^{[n]}|}$, respectively.

We say that $f : |\mathcal{A}^{[n]}| \to |\mathcal{B}^{[n]}|$ is a diagram preserving stable homotopy equivalence, if there exists a diagram preserving inverse map, $g : |\mathcal{B}^{[n]}| \to |\mathcal{A}^{[n]}|$, such that the induced maps, $g \circ f : \Sigma\mathcal{A}^{[n]} \to \Sigma\mathcal{A}^{[n]}$ and $f \circ g : \Sigma\mathcal{B}^{[n]} \to \Sigma\mathcal{B}^{[n]}$ are diagram homotopic to $Id_{\Sigma\mathcal{A}^{[n]}}$ and $Id_{\Sigma\mathcal{B}^{[n]}}$ respectively (Σ stands for the suspension operator).

Translating these topological definitions into the language of arrangements, we will say that two arrangements $\mathcal{A}, \mathcal{B}$ are homeomorphic (resp. homotopy equivalent, stable homotopy equivalent) if there exists a diagram preserving homeomorphism (resp. homotopy equivalence, stable homotopy equivalence) between them.

The main results of this paper can now be stated.

1.3. **Main Results.** Let $\mathcal{S}(\mathbb{R})$ be an o-minimal structure over $\mathbb{R}$, and $T \subset \mathbb{R}^{k_1+k_2+\ell}$ a closed and bounded definable set, and $\pi_1 : \mathbb{R}^{k_1+k_2+\ell} \to \mathbb{R}^{k_1+k_2}$, $\pi_2 : \mathbb{R}^{k_1+k_2+\ell} \to \mathbb{R}^{\ell}$, $\pi_3 : \mathbb{R}^{k_1+k_2} \to \mathbb{R}^{k_2}$, the projection maps.

For any collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ of (T, π_1, π_2)-sets, and $\mathbf{y} \in \mathbb{R}^{k_2}$, we will denote by $\mathcal{A}_{\mathbf{y}}$ the collection of sets, $\{A_{1,\mathbf{y}}, \ldots, A_{n,\mathbf{y}}\}$, where $A_{i,\mathbf{y}} = A_i \cap \pi_3^{-1}(\mathbf{y}), 1 \leq i \leq n$.

A fundamental theorem in o-minimal geometry is Hardt's trivialization theorem, (Theorem 4.2 below) which says that there exists a definable partition of $\mathbb{R}^{k_2}$ into a finite number of definable sets $\{T_i\}_{i \in I}$ such that for each $i \in I$, all fibers $\mathcal{A}_{\mathbf{z}}$ with $\mathbf{z} \in T_i$ are definably homeomorphic. A very natural question is to ask for an upper bound on the size of this partition (which will also give an upper bound on the number of homeomorphism types amongst the $\mathcal{A}_{\mathbf{z}}$).

Hardt's theorem is a corollary of the existence of *cylindrical cell decompositions* of definable sets [6]. When $\mathcal{A}$ a (T, π_1, π_2)-family for some fixed definable set $T \subset \mathbb{R}^{k_1+k_2+\ell}$, with $\pi_1 : \mathbb{R}^{k_1+k_2+\ell} \to \mathbb{R}^{k_1+k_2}$, $\pi_2 : \mathbb{R}^{k_1+k_2+\ell} \to \mathbb{R}^{\ell}$, $\pi_2 : \mathbb{R}^{k_1+k_2} \to \mathbb{R}^{k_2}$, the usual projections, and $\#\mathcal{A} = n$, the quantitative cylindrical definable cell decomposition theorem in [2] gives a doubly exponential (in $k_1 k_2$) upper bound on the cardinality of I and hence on the number of homeomorphism types of amongst the fibers $\mathcal{A}_{\mathbf{z}}, \mathbf{z} \in \mathbb{R}^{k_2}$. A tighter (say single exponential) bound on the number of homeomorphism types of the fibers would be very interesting but is unknown at present.

In this paper we give tighter (single exponential) upper bounds on the number of homotopy types occurring amongst the fibers $\mathcal{A}_{\mathbf{z}}, \mathbf{z} \in \mathbb{R}^{k_2}$.

We have the following theorems. The first theorem gives a bound on the number of *stable* homotopy types (see Section 3.1.1 below for the precise definition of stable homotopy equivalence) of the arrangements $\mathcal{A}_{\mathbf{z}}, \mathbf{z} \in \mathbb{R}^{k_2}$, while the second theorem gives a slightly worse bound for homotopy types.

Theorem 1.4. *There exists a constant $C = C(T) > 0$, such that for any collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ of (T, π_1, π_2)-sets, there exists a finite set $A \subset \mathbb{R}^{k_2}$, with*

$$\#A \leq C \cdot n^{(k_1+1)k_2},$$

such that for every $\mathbf{y} \in \mathbb{R}^{k_2}$ there exists $\mathbf{z} \in A$ such that $\mathcal{A}_{\mathbf{y}}$ is stable homotopy equivalent to $\mathcal{A}_z$. In particular, the number of distinct stable homotopy types of $\mathcal{A}_y$ for various $\mathbf{y} \in \mathbb{R}^{k_2}$ is also bounded by

$$C \cdot n^{(k_1+1)k_2}.$$

If we replace stable homotopy type by homotopy type, we get a slightly weaker bound.

Theorem 1.5. *There exists a constant $C = C(T) > 0$, such that for any collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ of (T, π_1, π_2)-sets, there exists a finite set $A \subset \mathbb{R}^{k_2}$, with*

$$\#A \leq C \cdot n^{(k_1+3)k_2},$$

such that for every $\mathbf{y} \in \mathbb{R}^{k_2}$ there exists $\mathbf{z} \in A$ such that $\mathcal{A}_{\mathbf{y}}$ is homotopy equivalent to $\mathcal{A}_z$. In particular, the number of distinct homotopy types of $\mathcal{A}_y$ for various $\mathbf{y} \in \mathbb{R}^{k_2}$ is also bounded by

$$C \cdot n^{(k_1+3)k_2}.$$

2. Background

In this section we describe some prior work in the area of bounding the number of homotopy types of fibers of a definable map and their connections with the results presented in this paper.

We begin with a definition.

Definition 2.1. Let $\mathcal{A} = \{S_1, \ldots, S_n\}$, such that each $S_i \subset \mathbb{R}^k$ is a (T, π_1, π_2)-set. For $I \subset \{1, \ldots, n\}$, we let $\mathcal{A}(I)$ denote the set

$$\bigcap_{i \in I \subset [n]} S_i \;\cap \bigcap_{j \in [n] \setminus I} \mathbb{R}^k \setminus S_j, \tag{2.1}$$

and we will call such a set to be a basic $\mathcal{A}$-set. We will denote by, $\mathcal{C}(\mathcal{A})$, the set of non-empty connected components of all basic $\mathcal{A}$-sets.

We will call definable subsets $S \subset \mathbb{R}^k$ defined by a Boolean formula whose atoms are of the form, $x \in S_i, 1 \leq i \leq n$, a $\mathcal{A}$-set. A $\mathcal{A}$-set is thus a union of basic $\mathcal{A}$-sets. If T is closed, and the Boolean formula defining S has no negations, then S is closed by definition (since each S_i is closed) and we call such a set an $\mathcal{A}$-closed set.

Moreover, if V is any closed definable subset of $\mathbb{R}^k$, and S is an $\mathcal{A}$-set (resp. $\mathcal{A}$-closed set), then we will call $S \cap V$ to be an $(\mathcal{A}, V)$-set (resp. $(\mathcal{A}, V)$-closed set).

2.0.1. *Bounds on the Betti Numbers of Admissible Sets.* The problems of bounding the Betti numbers of $\mathcal{A}$-sets is investigated in [2], where several results known in the semi-algebraic and semi-Pfaffian case are extended to this general setting. In particular, we will need the following theorem proved there.

Theorem 2.2. [2] *Let $\mathcal{S}(\mathbb{R})$ be an o-minimal structure over a $\mathbb{R}$ and let $T \subset \mathbb{R}^{k+\ell}$ be a closed definable set. Then, there exists a constant $C = C(T) > 0$ depending only on T, such that for any arrangement $\mathcal{A} = \{S_1, \ldots, S_n\}$ of (T, π_1, π_2)-sets of $\mathbb{R}^k$ the following holds.*

For every $i, 0 \leq i \leq k$,

$$\sum_{D \in \mathcal{C}(\mathcal{A})} b_i(D) \leq C \cdot n^{k-i}.$$

Remark 2.3. The intuition behind the bound in Theorem 2.2 (as well as similar results in the semi-algebraic and semi-Pfaffian settings) is that the topology (or more precisely the homotopy type) of a definable set in $\mathbb{R}^k$ defined in terms of n other definable sets, depend only on the interaction of these sets at most $k+1$ at a time. However, the proof of Theorem 2.2 in [2] (as well as the proofs of similar results in the semi-algebraic and semi-Pfaffian settings) depends on an argument involving the Mayer-Vietoris sequence for homology, and does not require more detailed information about homotopy types. However, in this paper, we have to make this intuition mathematically precise, which we do below (in Section 3). We prove a theorem (Theorem 3.4 below) which is reminiscent of Helly's theorem in convexity theory [7], but in a homotopical setting, and this auxiliary result is the key to proving the main results of this paper (Theorems 1.4 and 1.5). Moreover, the auxiliary result could also be of independent interest.

2.1. **Homotopy types of the fibers of a projection of a semi-algebraic set.** Theorem 2.2 gives tight bounds on the topological complexity of an $\mathcal{A}$-set in terms of the cardinality of $\mathcal{A}$, assuming that the sets in $\mathcal{A}$ belong to some fixed definable family. A problem closely related to the problem we consider in this paper is to bound the number of topological types of the fibers of a projection restricted to an arbitrary $\mathcal{A}$-set.

More precisely, let $S \subset \mathbb{R}^{k_1+k_2}$ be a set definable in an o-minimal structure over the reals (see [8]) and let $\pi : \mathbb{R}^{k_1+k_2} \to \mathbb{R}^{k_2}$ denote the projection map on the last k_2 co-ordinates. We consider the fibers, $S_{\mathbf{z}} = \pi^{-1}(\mathbf{z}) \cap S$ for different $\mathbf{z}$ in $\mathbb{R}^{k_2}$. Hardt's trivialization theorem, (Theorem 4.2 below) shows that there exists a definable partition of $\mathbb{R}^{k_2}$ into a finite number of definable sets $\{T_i\}_{i\in I}$ such that for each $i \in I$ and any point $\mathbf{z}_i \in T_i$, $\pi^{-1}(T_i) \cap S$ is definably homeomorphic to $S_{\mathbf{z}_i} \times T_i$ by a fiber preserving homeomorphism. In particular, for each $i \in I$, all fibers $S_{\mathbf{z}}$ with $\mathbf{z} \in T_i$ are definably homeomorphic.

Hardt's theorem is a corollary of the existence of *cylindrical cell decompositions* of definable sets [6]. In the particular case where S is an $\mathcal{A}$-set, for $\mathcal{A}$ a (T, π_1, π_2)-family for some fixed definable set $T \subset \mathbb{R}^{k_1+k_2+\ell}$, with $\pi_1 : \mathbb{R}^{k_1+k_2+\ell} \to \mathbb{R}^{k_1+k_2}$, $\pi_2 : \mathbb{R}^{k_1+k_2+\ell} \to \mathbb{R}^{\ell}$, $\pi_2 : \mathbb{R}^{k_1+k_2} \to \mathbb{R}^{k_2}$, the usual projections, and $\#\mathcal{A} = n$, the quantitative cylindrical definable cell decomposition theorem in [2] gives a doubly exponential (in k_1k_2) upper bound on the cardinality of I and hence on the number of homeomorphism types of the fibers of the map $\pi_3|_S$. A tighter (say single exponential) bound on the number of homeomorphism types of the fibers would be very interesting but is unknown at present.

Recently, the problem of obtaining a tight bound on the number of topological types of the fibers of a definable map for semi-algebraic and semi-Pfaffian sets was considered in [4], and it was shown that the number of distinct homotopy types of the fibers of such a map can be bounded (in terms of the format of the formula defining the set) by a function singly exponential in k_1k_2. In particular, the combinatorial part of the bound is also singly exponential. A more precise statement in the case of semi-algebraic sets is the following theorem which appears in [4].

Theorem 2.4. [4] *Let $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_{k_1}, Y_1, \ldots, Y_{k_2}]$, with $\deg(P) \leq d$ for each $P \in \mathcal{P}$ and cardinality $\#\mathcal{P} = n$. Then, there exists a finite set $A \subset \mathbb{R}^{k_2}$, with*

$$\#A \leq (2^{k_1} n k_2 d)^{O(k_1 k_2)},$$

such that for every $\mathbf{y} \in \mathbb{R}^{k_2}$ there exists $\mathbf{z} \in A$ such that for every $\mathcal{P}$-semi-algebraic set $S \subset \mathbb{R}^{k_1+k_2}$, the set $\pi^{-1}(\mathbf{y}) \cap S$ is semi-algebraically homotopy equivalent to $\pi^{-1}(\mathbf{z}) \cap S$. In particular, for any fixed $\mathcal{P}$-semi-algebraic set S, the number of different homotopy types of fibers $\pi^{-1}(\mathbf{y}) \cap S$ for various $\mathbf{y} \in \pi(S)$ is also bounded by

$$(2^{k_1} n k_2 d)^{O(k_1 k_2)}.$$

Remark 2.5. The proof of Theorem 2.4 however has the drawback that it relies on techniques involving perturbations of the original polynomials in order to put them in general position, as well as Thom's Isotopy Theorem, and as such does not extend easily to the o-minimal setting. The main results of this paper (see Theorem 1.4 and Theorem 1.5 below) extends the combinatorial part of Theorem 2.4 to the more general o-minimal category.

Remark 2.6. Even though the formulation of Theorem 2.4 seems a little different from the main theorems of this paper (Theorems 1.4 and 1.5), they are in fact closely related. In fact, as a consequence of Theorem 1.5 we obtain bounds on the number of homotopy types of the fibers of S for any fixed $\mathcal{A}$-set S, analogous to the one in Theorem 2.4.

More precisely,

Theorem 2.7. *Let $\mathcal{S}(\mathbb{R})$ be an o-minimal structure over $\mathbb{R}$, and $T \subset \mathbb{R}^{k_1+k_2+\ell}$ a closed and bounded definable set, and $\pi_1 : \mathbb{R}^{k_1+k_2+\ell} \to \mathbb{R}^{k_1+k_2}$, $\pi_2 : \mathbb{R}^{k_1+k_2+\ell} \to \mathbb{R}^{\ell}$, and $\pi_3 : \mathbb{R}^{k_1+k_2} \to \mathbb{R}^{k_2}$ the projection maps. Then, there exists a constant $C = C(T) > 0$, such that for any collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ of (T, π_1, π_2)-sets, there exists a finite set $A \subset \mathbb{R}^{k_2}$, with*

$$\#A \leq C \cdot n^{2(k_1+3)k_2},$$

such that for every $\mathbf{y} \in \mathbb{R}^{k_2}$ there exists $\mathbf{z} \in A$ such that for every $\mathcal{A}$ set $S \subset \mathbb{R}^{k_1+k_2}$, the set $\pi_3^{-1}(\mathbf{y}) \cap S$ is homotopy equivalent to $\pi_3^{-1}(\mathbf{z}) \cap S$. In particular, for any fixed $\mathcal{A}$-set S, the number of distinct homotopy types of fibers $\pi_3^{-1}(\mathbf{y}) \cap S$ for various $\mathbf{y} \in \pi_3(S)$ is also bounded by

$$C \cdot n^{2(k_1+3)k_2}.$$

A similar result with a bound of $C \cdot n^{2(k_1+1)k_2}$ holds for stable homotopy types as well.

3. A Topological Comparison Theorem

As noted previously, the main underlying idea behind our proof of Theorem 1.4 is that the homotopy type of a $\mathcal{A}$-set in $\mathbb{R}^k$ depend only on the interaction of sets in $\mathcal{A}$ at most $(k+1)$ at a time. In this section we make this idea precise. We show that in case $\mathcal{A} = \{A_1, \ldots, A_n\}$, with each A_i a definable, closed and bounded subset of $\mathbb{R}^k$, the homotopy type of any $\mathcal{A}$-closed set is determined by sub-complex of a complex (see Definition 3.2 below), usually called the *homotopy co-limit of the diagram associated to the partially ordered set, $\{\mathcal{A}_I \mid I \subset [n]\}$, where $\mathcal{A}_I = \cap_{i \in I} A_i$, and the set is partially ordered by inclusion.* The crucial fact here is that the sub-complex depend only on the intersections of the sets in $\mathcal{A}$ only upto $k+1$ at a time.

In order to avoid technical difficulties, we restrict ourselves to the category of finite, regular cell complexes (see [15] for the definition of a regular cell complex). The setting of finite regular, cell complexes suffices for us, since it is well known that compact definable sets in any o-minimal structure are finitely triangulable, and hence, in particular, are homeomorphic to regular cell complexes.

3.1. **Topological Preliminaries.**

3.1.1. *Stable Homotopy Equivalence.* For any CW-complex X we will denote by ΣX the suspension of X. Recall (see [13]) that two finite CW-complexes X, Y are called *stable homotopy equivalent*, if there exists a homotopy equivalence $\phi_N : \Sigma X \to \Sigma Y$.

We denote by $\mathrm{H}_i(X, \mathbb{Z})$ the i-th singular homology group of a topological space X. The following theorem characterizes stable homotopy equivalence in terms of homology.

Theorem 3.1. [13] *Let X and Y be two finite CW-complexes. Then X and Y are stable homotopy equivalent if and only if there exists a continuous map $f : X \to Y$ which induces isomorphisms $f_* : \mathrm{H}_i(X, \mathbb{Z}) \to \mathrm{H}_i(Y, \mathbb{Z})$ for all $i \geq 0$ (and we will call such a map a stable homotopy equivalence).*

3.1.2. *Homotopy co-limits.* Let $\mathcal{A} = \{A_1, \ldots, A_n\}$, where each A_i is a sub-complex of a finite regular cell complex. We now define the homotopy co-limit of diagram($\mathcal{A}$).

Let $\Delta_{[n]}$ denote the standard simplex of dimension $n-1$ with vertices in $[n]$ (identifying the ith unit vector in $\mathbb{R}^n$ with i), and for $I \subset [n]$, we denote by Δ_I the $(\#I - 1)$-dimensional face of $\Delta_{[n]}$ corresponding to I. For any simplicial (resp. cell) complex K, we will denote by $|K|$ the associated polyhedron (resp. topological space). We will denote by $\mathrm{sk}_m(K)$ the m-skeleton of the complex K.

The homotopy co-limit, $\mathcal{N}(\mathcal{A})$, is a cell complex constructed as follows.

Definition 3.2. We define,

$$\mathrm{sk}_0(\mathcal{N}(\mathcal{A})) = \{c \times |\Delta_I| \mid \#I = 1,\ c \in \mathcal{A}_I, \dim(c) = 0\}.$$

More generally, the p-dimensional cells of $\mathcal{N}(A)$ are

$$\{c \times |\Delta_I| \mid \#I = i+1,\ c \in \mathcal{A}_I, \dim(c) = j, i + j = p\}.$$

The attaching map of a p-dimensional cell, $c \times |\Delta|$ to the $(p-1)$-skeleton of $\mathrm{sk}_{p-1}\mathcal{N}(\mathcal{A})$ is defined as follows.

Notice that,

$$\partial(c \times |\Delta_I)| = \partial(c) \times |\Delta_I| \cup c \times |\partial(|\Delta_I|).$$

Note also that c is a cell of $\mathcal{A}_J$ for each $J \subset I$, and

$$\partial(|\Delta_I|) = \bigcup_{J \subset I, |J| = |I|-1} |\Delta_J|.$$

We define the attaching map,

$$\phi_{c \times |\Delta_I|} : \partial(c \times |\Delta_I|) \to |\mathrm{sk}_{p-1}(\mathcal{N}(\mathcal{A}))|$$

by defining

$$\phi_{c \times |\Delta_I|} = \phi_{c, \mathcal{A}_I} \times \mathrm{Id}$$

on $\partial(c) \times |\Delta_I|$ where $\phi_{c, \mathcal{A}_I} : \partial c \to \mathrm{sk}_{p-1}(\mathcal{A}_I)$ is the attaching map for c in $\mathcal{A}_I$, and defining

$$\phi = i_{I,J} \times \mathrm{Id}$$

on $c \times \partial(|\Delta_J|)$ for each $J \subset I, J \neq I$, where $i_{I,J} : \mathcal{A}_I \hookrightarrow \mathcal{A}_J$ is the inclusion map.

Note that there exist two natural maps

$$f_{\mathcal{A}} : |\mathcal{N}(A)| \to |\mathcal{A}^{[n]}|,$$
$$g_{\mathcal{A}} : |\mathcal{N}(A)| \to |\Delta_{[n]}|,$$

defined by,

$$f_{\mathcal{A}}(\mathbf{x},\mathbf{y}) = \mathbf{x},$$
$$g_{\mathcal{A}}(\mathbf{x},\mathbf{y}) = \mathbf{y},$$

where $(\mathbf{x},\mathbf{y}) \in c \times |\Delta_{I_c}|$ to $\mathbf{x} \in c$, and c is a cell in $\mathcal{A}^{[n]}$ and $I_c = \{i \in [n] \mid c \in A_i\}$.

For any $m, 0 \leq m \leq n$, we will denote by $\mathcal{N}_m(\mathcal{A})$ the truncated complex, $\mathcal{N}_m(\mathcal{A}) = g_{\mathcal{A}}^{-1}(\mathrm{sk}_m(\Delta_{[n]}))$.

Replacing in Definition 1.3, $|\mathcal{A}^{[n]}|$ and $|\mathcal{B}^{[n]}|$, by $|\mathcal{N}(\mathcal{A})|$ and $|\mathcal{N}(\mathcal{B})|$ respectively, as well as $\mathcal{A}_I$ and $\mathcal{B}_I$ by $f_{\mathcal{A}}^{-1}(|\mathcal{A}_I|)$ and $f_{\mathcal{B}}^{-1}(|\mathcal{B}_I|)$ respectively, we get definitions of diagram preserving, homotopy equivalences and stable homotopy equivalences between $|\mathcal{N}(\mathcal{A})|$ and $|\mathcal{N}(B)|$, and more generally between their truncations, $|\mathcal{N}_m(\mathcal{A})|$ and $|\mathcal{N}_m(\mathcal{B}))|$, for any $m \geq 0$.

Definition 3.3. We say that $\mathcal{A} \approx_m \mathcal{B}$, if there exists a diagram preserving homotopy equivalence,

$$\phi : |\mathcal{N}_m(\mathcal{A})| \to |\mathcal{N}_m(\mathcal{B})|.$$

We say that $\mathcal{A} \sim_m \mathcal{B}$, if there exists a diagram preserving stable homotopy equivalence,

$$\phi : |\mathcal{N}_m(\mathcal{A})| \to |\mathcal{N}_m(\mathcal{B})|.$$

The two following theorems are the crucial topological ingredients in the proofs of our main results.

Theorem 3.4. *Let $\mathcal{A} = \{A_1, \ldots, A_n\}, \mathcal{B} = \{B_1, \ldots, B_n\}$ be two families of subcomplexes of a finite regular cell complex, such that:*

(1) $\mathrm{H}_i(A^{[n]}, \mathbb{Z}), \mathrm{H}_i(B^{[n]}, \mathbb{Z}) = 0$, *for all $i \geq k$, and*
(2) $\mathcal{A} \sim_k \mathcal{B}$.

Then, there exists a diagram preserving stable homotopy equivalence, $\phi : |\mathcal{A}^{[n]}| \to |\mathcal{B}^{[n]}|$.

Theorem 3.5. *Let $\mathcal{A} = \{A_1, \ldots, A_n\}, \mathcal{B} = \{B_1, \ldots, B_n\}$ be two families of subcomplexes of a finite regular cell complex, such that:*

(1) $\dim(A_i), \dim(B_i) \leq k$, *for $1 \leq i \leq n$, and*
(2) $\mathcal{A} \approx_{k+2} \mathcal{B}$.

Then, there exists a diagram preserving homotopy equivalence, $\phi : |\mathcal{A}^{[n]}| \to |\mathcal{B}^{[n]}|$.

We now state two corollaries of Theorems 3.4 and 3.5 which might be of interest.

Given a Boolean formula $\theta(T_1, \ldots, T_n)$ with atoms $T_1, \ldots, T_n$, and a family of sets in $\mathbb{R}^k$, $\mathcal{A} = \{A_1, \ldots, A_n\}$, we will denote by $\mathcal{A}_\theta$ the set defined by the formula, $\theta_{\mathcal{A}}$, which is obtained from θ by replacing in θ the atom T_i by A_i for each $i \in [n]$ and replacing each $\wedge$ (resp. $\vee$) by $\cap$ (resp. $\cup$).

Corollary 3.6. *Let $\mathcal{A} = \{A_1, \ldots, A_n\}, \mathcal{B} = \{B_1, \ldots, B_n\}$ be two families of subcomplexes of a finite regular cell complex, satisfying the same conditions as in Theorem 3.4. Let $\theta(T_1, \ldots, T_n)$ be a Boolean formula without negations on atoms $T_1, \ldots, T_n$. Then, there exists a stable homotopy equivalence*

$$\phi : \mathcal{A}_\theta \to \mathcal{B}_\theta.$$

Corollary 3.7. *Let $\mathcal{A} = \{A_1, \ldots, A_n\}, \mathcal{B} = \{B_1, \ldots, B_n\}$ be two families of subcomplexes of a finite regular cell complex, satisfying the same conditions as in Theorem 3.5. Let $\theta(T_1, \ldots, T_n)$ be a Boolean formula without negations on atoms $T_1, \ldots, T_n$. Then, there exists a homotopy equivalence*

$$\phi : \mathcal{A}_\theta \to \mathcal{B}_\theta.$$

3.2. **Proofs of Theorems 3.4 and 3.5.** Let $\mathcal{A}$ and $\mathcal{B}$ as in Theorem 3.4.

We need a preliminary lemma.

Lemma 3.8.

$$\mathcal{A}^{[n]} \text{ is diagram preserving homotopy equivalent to } |\mathcal{N}(\mathcal{A})|.$$

Proof. Consider the map,

$$f_\mathcal{A} : |\mathcal{N}(A)| \to |\mathcal{A}^{[n]}|,$$

defined above.

Clearly, if $\mathbf{x} \in c$, $f_\mathcal{A}^{-1}(c) = |\Delta_{I_c}|$. Now applying Smale's version of the Vietoris-Beagle Theorem [12] we obtain that $f_\mathcal{A}$ is a homotopy equivalence. Clearly, $f_\mathcal{A}$ is diagram preserving. Moreover, (see for instance the proof of Theorem 6 in [12]), it is possible to choose a diagram preserving inverse,

$$h_\mathcal{A} : |\mathcal{A}^{[n]}| \to |\mathcal{N}(A)|,$$

which is a cellular map, as well as a homotopy inverse of $f_\mathcal{A}$. □

We can now prove Theorems 3.4 and 3.5 .

Proof of Theorem 3.4. Let $h_\mathcal{A} : \mathcal{A}^{[n]} \to \mathcal{N}(\mathcal{A})$ be a diagram preserving homotopy equivalence known to exist by Lemma 3.8. Since $h_\mathcal{A}$ is cellular, its image is contained in $\mathcal{N}_k(\mathcal{A})$).

We will denote by $h_{\mathcal{A},\mathcal{B}} : |\mathcal{N}_k(\mathcal{A})| \to |\mathcal{N}_k(\mathcal{B})|$ a diagram preserving stable homotopy equivalence known to exist by hypothesis (which we also assume to be cellular).

Let $i_{\mathcal{B},k} : |\mathcal{N}_k(\mathcal{B})| \hookrightarrow |\mathcal{N}(\mathcal{B})|$ denote the inclusion map. The map $i_{\mathcal{B},k}$ induces isomorphisms,

$$(i_{\mathcal{B},k})_* : \mathrm{H}_j(\mathcal{N}_k(\mathcal{B}), \mathbb{Z}) \to \mathrm{H}_j(\mathcal{N}(\mathcal{B}), \mathbb{Z}),$$

for $0 \leq j \leq k-1$.

Consequently, the map $f_\mathcal{B} \circ i_{\mathcal{B},k}$ induces isomorphisms,

$$(f_\mathcal{B} \circ i_{\mathcal{B},k})_* : \mathrm{H}_j(\mathcal{N}_k(\mathcal{B}), \mathbb{Z}) \to \mathrm{H}_j(\mathcal{B}^{[n]}, \mathbb{Z}),$$

for $0 \leq j \leq k-1$.

Composing the maps, $h_\mathcal{A}, h_{\mathcal{A}\mathcal{B}}, i_{\mathcal{B},k}, f_\mathcal{B}$ we have that the map,

$$f_\mathcal{B} \circ i_{\mathcal{B},k} \circ h_{\mathcal{A}\mathcal{B}} \circ h_\mathcal{A} : |\mathcal{A}^{[n]}| \to |\mathcal{B}^{[n]}|,$$

induces isomorphisms between

$$(f_\mathcal{B} \circ i_{\mathcal{B},k} \circ h_{\mathcal{A},\mathcal{B},k} \circ h_\mathcal{A})_* : \mathrm{H}_j(\mathcal{A}^{[n]}, \mathbb{Z}) \to \mathrm{H}_j(\mathcal{B}^{[n]}, \mathbb{Z}),$$

for all $j \geq 0$.

Moreover, the map $f_\mathcal{B} \circ i_{\mathcal{B},k} \circ h_{\mathcal{A}\mathcal{B}} \circ h_\mathcal{A}$ is diagram preserving since each constituent of the composition is diagram preserving. It now follows from Theorem 3.1, that the map

$$\phi = f_\mathcal{B} \circ i_{\mathcal{B},k} \circ h_{\mathcal{A}\mathcal{B}} \circ h_\mathcal{A} : |\mathcal{A}^{[n]}| \to |\mathcal{B}^{[n]}|,$$

is a diagram preserving stable homotopy equivalence. □

Before proving Theorem 3.5 we first need to recall a few basic facts from homotopy theory.

Definition 3.9. A map $f : X \to Y$, between two regular cell complex is called a k-equivalence if the induced morphisms,

$$f_* : \pi_i(X) \to \pi_i(Y)$$

is an isomorphism for all $0 \leq i < k$, and an epimorphism for $i = k$, and we say that X is k-equivalent to Y.

We also need the following well-known fact from algebraic topology.

Proposition 3.10. *Let X, Y be finite regular cell complexes, with*

$$\dim(X) < k, \dim(Y) \leq k,$$

and $f : X \to Y$ a k-equivalence. Then, f is a homotopy equivalence between X and Y.

Proof. See [14] (pp. 69). □

Proof of Theorem 3.5. The proof is along the same lines as that of the proof of Theorem 3.4. Let $h_{\mathcal{A}} : \mathcal{A}^{[n]} \to \mathcal{N}(\mathcal{A})$ be a diagram preserving homotopy equivalence known to exist by Lemma 3.8. By the same argment as before, its image is contained in $|\mathcal{N}_{k+2}(\mathcal{A})|$.

We will denote by $h_{\mathcal{A},\mathcal{B}} : |\mathcal{N}_{k+2}(\mathcal{A})| \to |\mathcal{N}_{k+2}(\mathcal{B})|$ a diagram preserving homotopy equivalence known to exist by hypothesis.

Let $i_{\mathcal{B},k+2} : |\mathcal{N}_{k+2}(\mathcal{B})| \hookrightarrow |\mathcal{N}(\mathcal{B})|$ denote the inclusion map. The map $i_{\mathcal{B},k+2}$ induces isomorphisms,

$$(i_{\mathcal{B},k+2})_* : \pi_j(\mathcal{N}_{k+2}(\mathcal{B})) \hookrightarrow \pi_j(\mathcal{N}(\mathcal{B})),$$

for $0 \leq j \leq k+1$. This is a consequence of the exactness of the homotopy sequence of the pair $(\mathcal{N}(\mathcal{B}), \mathcal{N}_{k+2}(\mathcal{B}))$ (see [13]).

Consequently, the map $f_{\mathcal{B}} \circ i_{\mathcal{B},k}$ induces isomorphisms,

$$(g_{\mathcal{B}} \circ i_{\mathcal{B},k})_* : \pi_j(\mathcal{N}_{k+2}(\mathcal{B})) \to \pi_j(\mathcal{B}^{[n]}),$$

for $0 \leq j \leq k+1$.

Composing the maps, $h_{\mathcal{A}}, h_{\mathcal{AB}}, i_{\mathcal{B},k+2}, f_{\mathcal{B}}$ we have that the map,

$$f_{\mathcal{B}} \circ i_{\mathcal{B},k} \circ h_{\mathcal{AB}} \circ h_{\mathcal{A}} : |\mathcal{A}^{[n]}| \to |\mathcal{B}^{[n]}|,$$

induces isomorphisms between

$$(f_{\mathcal{B}} \circ i_{\mathcal{B},k} \circ h_{\mathcal{A},\mathcal{B},k} \circ h_{\mathcal{A}})_* : \pi_j(\mathcal{A}^{[n]}) \to \pi_j(\mathcal{B}^{[n]}),$$

for all $0 \leq j \leq k+1$.

Moreover, the map $f_{\mathcal{B}} \circ i_{\mathcal{B},k} \circ h_{\mathcal{AB}} \circ h_{\mathcal{A}}$ is diagram preserving since each constituent of the composition is diagram preserving. It now follows from Proposition 3.10, that the map

$$\phi = f_{\mathcal{B}} \circ i_{\mathcal{B},k} \circ h_{\mathcal{AB}} \circ h_{\mathcal{A}} : |\mathcal{A}^{[n]}| \to |\mathcal{B}^{[n]}|,$$

is a diagram preserving homotopy equivalence. □

Proof of Corollary 3.7. The proof is similar to that of Corollary 3.6, using Theorem 3.5 in place of Theorem 3.4 and is omitted. □

4. Proofs of the main theorems

In order to prove Theorem 1.4 we first need to recall a few results from o-minimal geometry.

4.1. Finite unions of definable families.

We first note an elementary property of families of admissible sets (see [2] for a proof).

Observation 4.1. Suppose that $T_1, \ldots, T_m \subset \mathbb{R}^{k+\ell}$ are definable sets, $\pi_1 : \mathbb{R}^{k+\ell} \to \mathbb{R}^k$ and $\pi_2 : \mathbb{R}^{k+\ell} \to \mathbb{R}^\ell$ the two projections. Then, there exists a definable subset $T' \subset \mathbb{R}^{k+\ell+m}$ depending only on $T_1, \ldots, T_m$, such that for any collection of (T_i, π_1, π_2) families $\mathcal{A}_i$, $1 \leq i \leq m$, the union, $\bigcup_{i=1}^{m} \mathcal{A}_i$, is a (T', π_1', π_2')-family, where $\pi_1' : \mathbb{R}^{k+m+\ell} \to \mathbb{R}^k$ and $\pi_2' : \mathbb{R}^{k+\ell+m} \to \mathbb{R}^{\ell+m}$ are the projections on to the first k, and the last $\ell + m$ co-ordinates respectively.

4.2. Hardt triviality for definable sets.

One important technical tool will be the following o-minimal version of Hardt's triviality theorem.

Let $X \subset \mathbb{R}^k \times \mathbb{R}^\ell$ and $A \subset \mathbb{R}^k$ be definable subsets of $\mathbb{R}^k \times \mathbb{R}^\ell$ and $\mathbb{R}^\ell$ respectively, and let $\pi : X \to \mathbb{R}^\ell$ denote the projection map.

We say that X *is definably trivial over* A if there exists a definable set F and a definable homeomorphism
$$h : F \times A \to X \cap \pi^{-1}(A),$$
with $\pi \circ h(\mathbf{y}, \mathbf{a}) = \mathbf{a}$ and such that the following diagram commutes,
$$\begin{tikzcd} F \times A \arrow[r, "h"] \arrow[d, "\pi_2"'] & X \cap \pi^{-1}(A), \arrow[dl, "\pi"] \\ A & \end{tikzcd}$$
where $\pi_2 : F \times A \to A$ is the projection onto the second factor. We call h *a definable trivialization of* X *over* A.

If Y is a definable subset of X, we sat that the trivialization h is *compatible* with Y if there is a definable subset G of F such that $h(G \times A) = Y \cap \pi^{-1}(A)$. Clearly, the restriction of h to $G \times A$ is a trivialization of Y over A.

Theorem 4.2 (Hardt's theorem for definable families). *Let $X \subset \mathbb{R}^k \times \mathbb{R}^\ell$ be a definable set and let $Y_1, \ldots, Y_m$ be definable subsets of X. Then, there exists a finite partition of $\mathbb{R}^\ell$ into definable sets $C_1, \ldots, C_N$ such that X is definably trivial over each C_i, and moreover the trivializations over each C_i are compatible with $Y_1, \ldots, Y_m$.*

Remark 4.3. We first remark that it is straightforward to derive from the proof of Theorem 4.2 that the definable sets $C_1, \ldots, C_N$ can be chosen to be locally closed, and can be express as, $C_1 = \mathbb{R}^\ell \setminus B_1, C_2 = B_1 \setminus B_2, \ldots, C_N = B_{N-1} \setminus B_N$ for closed definable sets $B_1, \ldots, B_N$.

Remark 4.4. Note also that it follows from Theorem 4.2, that there are only a finite number of topological types amongst the fibers of any definable map $f : X \to Y$ between definable sets X and Y. This remark would be used a number of times later in the paper.

Since in what follows we will need to consider many different projections, we adopt the following convention. Given m and p, $p \leq m$, we will denote by $\pi_m^{\leq p} : \mathbb{R}^m \to \mathbb{R}^p$ (resp. $\pi_m^{>p} : \mathbb{R}^m \to \mathbb{R}^{m-p}$) the projection onto the first p (resp. the last $m-p$) coordinates.

4.3. **Definable Triangulations.** A triangulation of a compact definable set S is a simplicial complex Δ together with a definable homeomorphism from $|\Delta|$ to S. Given such a triangulation we will often identify the simplices in Δ with their images in S under the given homeomorphism, and will refer to the triangulation by Δ.

We call a triangulation $h_1 : |\Delta_1| \to S$ of a definable set S, to be a *refinement* of a triangulation $h_2 : |\Delta_2| \to S$ if for every simplex $\sigma_1 \in \Delta_1$, there exists a simplex $\sigma_2 \in \Delta_2$ such that $h_1(\sigma_1) \subset h_2(\sigma_2)$.

Let $S_1 \subset S_2$ be two compact definable subsets of $\mathbb{R}^k$. We say that a definable triangulation $h : |\Delta| \to S_2$ of S_2, *respects* S_1 if for every simplex $\sigma \in \Delta$, $h(\sigma) \cap S_1 = h(\sigma)$ or $\emptyset$. In this case, $h^{-1}(S_1)$ is identified with a sub-complex of Δ and $h|_{h^{-1}(S_1)} : h^{-1}(S_1) \to S_1$ is a definable triangulation of S_1. We will refer to this sub-complex by $\Delta|_{S_1}$.

Let $T \subset \mathbb{R}^{k_1+k+2+\ell}$ be a closed and bounded definable subset of $\mathbb{R}^{k_1+k_2+\ell}$. For each $m \geq 0$, and $(\mathbf{z}, \mathbf{y}_0, \ldots, \mathbf{y}_m) \in \mathbb{R}^{k_2+(m+1)\ell}$, we will denote by $T_{\mathbf{z},\mathbf{y}_0,\ldots,\mathbf{y}_m} \subset \mathbb{R}^{k_1}$ the definable set, $\bigcup_{1 \leq i \leq m} \{\mathbf{x} \in \mathbb{R}^{k_1} \mid (\mathbf{x}, \mathbf{z}) \in T_{\mathbf{y}_i}\}$.

For $\{j_0, \ldots, j_n\} \subset [m]$, we will denote by $\pi_{m,j_0,\ldots,j_n} : \mathbb{R}^{(m+1)\ell} \to \mathbb{R}^{(n+1)\ell}$ the projection map on the appropriate blocks of co-ordinates.

The usual proof of Hardt's triviality theorem can be extended to produce a definable triangulation and this can be done in a parametrized way. We omit the proof of the following proposition since it is a straightforward extension of the Hardt triviality theorem.

Proposition 4.5. *For each $m \geq 0$, there exists*

(1) *a definable partition $\{T_{m,\alpha}\}_{\alpha \in I_m}$ of $\mathbb{R}^{k_2+(m+1)\ell}$, and*
(2) *for each $\alpha \in I_m$, a definable continuous map,*

$$h_{m,\alpha} : |\Delta_{m,\alpha}| \times T_{m,\alpha} \to \bigcup_{(\mathbf{z},\mathbf{y}_0,\ldots,\mathbf{y}_m) \in T_{m,\alpha}} T_{\mathbf{z},\mathbf{y}_0,\ldots,\mathbf{y}_m},$$

where $\Delta_{m,\alpha}$ is a simplicial complex, and such that for each $(\mathbf{z}, \mathbf{y}_0, \ldots, \mathbf{y}_m) \in T_{m,\alpha}$, the restriction,

$$h_{m,\alpha} : |\Delta_{m,\alpha}| \times (\mathbf{z}, \mathbf{y}_0, \ldots, \mathbf{y}_m) \to T_{\mathbf{z},\mathbf{y}_0,\ldots,\mathbf{y}_m}$$

is a definable triangulation respecting the sets, $T_{\mathbf{z},\mathbf{y}_0}, \ldots, T_{\mathbf{z},\mathbf{y}_m}$, and

(3) *for each subset $\{j_0, \ldots, j_n\} \subset [m]$, $\mathrm{Id}_{k_2} \oplus \pi_{m,j_0,\ldots,j_n}(T_{m,\alpha}) \subset T_{n,\beta}$ for some $\beta \in I_n$, and for each $(\mathbf{z}, \mathbf{y}_0, \ldots, \mathbf{y}_m) \in T_{m,\alpha}$, the definable triangulation,*

$$h_{m,\alpha} : |\Delta_{m,\alpha}| \times (\mathbf{z}, \mathbf{y}_0, \ldots, \mathbf{y}_m) \to T_{\mathbf{z},\mathbf{y}_0,\ldots,\mathbf{y}_m}$$

restricted to the set $T_{\mathbf{z},\mathbf{y}_{j_0},\ldots,\mathbf{y}_{j_n}}$, is a refinement of the definable triangulation,

$$h_{n,\beta} : |\Delta_{n,\beta}| \times (\mathbf{z}, \mathbf{y}_{j_0}, \ldots, \mathbf{y}_{j_n}) \to T_{\mathbf{z},\mathbf{y}_{j_0},\ldots,\mathbf{y}_{j_n}}.$$

We will also need the following technical result.

Proposition 4.6. *Let B be a closed and bounded definable set contained in $\mathbb{R}^k$ and let $C_t \subset \mathbb{R}^k, t \geq 0$ be a definable family of closed and bounded sets, and let $C \subset \mathbb{R}^{k+1}$ be the definable set $\bigcup_{t\geq 0} C_t \times \{t\}$. If for every $0 \leq t < t'$, $C_t \subset C_{t'}$, and $B = C_0 = \pi_{k+1}^{\leq k}(\overline{C} \cap (\pi_{k+1}^{>k})^{-1}(0))$, then there exists $t_0 > 0$ such that, B has the same homotopy type as C_t for every t with $0 \leq t \leq t_0$.*

Proof. The proof given in [3] (see Lemma 16.17) for the semi-algebraic case can be easily adapted to the o-minimal setting using Hardt triviality for definable families instead of for semi-algebraic ones. □

We now introduce a notational convenience.

Definition 4.7. Let $\mathcal{F}(x)$ be a predicate defined over $\mathbb{R}_+$ and $y \in \mathbb{R}_+$. The notation $\forall(0 < x \ll y)\ \mathcal{F}(x)$ stands for the statement

$$\exists z \in (0, y)\ \forall x \in \mathbb{R}_+\ (\text{if } x < z, \text{ then } \mathcal{F}(x)),$$

and can be read "for all positive x sufficiently smaller than y, $\mathcal{F}(x)$ is true".

More generally,

Definition 4.8. for $\bar{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_s$ and a predicate $\mathcal{F}(\bar{\varepsilon})$ over $\mathbb{R}_+^n$ we say "for all sufficiently small $\bar{\varepsilon}$, $\mathcal{F}(\bar{\varepsilon})$ is true" if

$$\forall(0 < \varepsilon_0 \ll 1)\forall(0 < \varepsilon_1 \ll \varepsilon_0)\cdots\forall(0 < \varepsilon_s \ll \varepsilon_{s-1})\mathcal{F}(\bar{\varepsilon}).$$

4.4. **Infinitesimal thickenings of the faces of a standard simplex.** We will need the following construction.

Let $\bar{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_n) \in \mathbb{R}_+^{n+1}$, with $0 \leq \varepsilon_n < \cdots < \varepsilon_0 < 1$. Later we will require $\bar{\varepsilon}$ to be sufficiently small (see Definition 4.8).

For a face $\Delta_J \in \Delta_{[n]}$, we denote by $C_J(\bar{\varepsilon})$ the subset of $|\Delta_J|$ defined by,

$$C_J(\bar{\varepsilon}) = \{x \in |\Delta_J| \ \mid \ \text{ and } \operatorname{dist}(x, |\Delta_I|) \geq \varepsilon_{\#I-1} \text{ for all } I \subset J\}.$$

Now, let $I \subset J \subset [n]$. We denote by $C_{I,J}(\bar{\varepsilon})$ the subset of $|\Delta_J|$ defined by,

$$C_{I,J}(\bar{\varepsilon}) = \{x \in |\Delta_J| \ \mid \ \operatorname{dist}(x, |\Delta_I|) \leq \varepsilon_{\#I-1}, \text{ and } \operatorname{dist}(x, K) \geq \varepsilon_{\#K-1} \text{ for all } K \subset I\}.$$

Note that,

$$|\Delta_{[n]}| = \bigcup_{I\subset[n]} C_I(\bar{\varepsilon}) \cup \bigcup_{I\subset J\subset[n]} C_{I,J}(\bar{\varepsilon}).$$

Also, observe that for sufficiently small $\bar{\varepsilon}$, the various $C_J(\bar{\varepsilon})$'s and $C_{I,J}(\bar{\varepsilon})$'s are all homeomorphic to balls, and moreover all non-empty intersections between them also have the same property. Thus, the union of the $C_J(\bar{\varepsilon})$'s and $C_{I,J}(\bar{\varepsilon})$'s together with the non-empty intersections between them form a regular cell complex, $\mathcal{C}(\Delta_{[n]}, \bar{\varepsilon})$, whose underlying topological space is $|\Delta_{[n]}|$ (see Figures 1 and 2).

Proof of Theorem 1.4. For $m \geq 0$, and $(\mathbf{z}, \mathbf{y}_0, \ldots, \mathbf{y}_m) \in \mathbb{R}^{k_2+(m+1)\ell}$, we will denote by, $T_{\mathbf{z},\mathbf{y}_0,\ldots,\mathbf{y}_m}$ the definable set,

$$\bigcup_{i=1}^{m} T_{\mathbf{z},\mathbf{y}_i} \subset \mathbb{R}^\ell.$$

Now, applying Proposition 4.5 to the set T, with $m = k_1$, we obtain

(1) a definable partition $\{T_{k_1,\alpha}\}_{\alpha\in I_{k_1}}$ of $\mathbb{R}^{k_2+(m+1)\ell}$, and

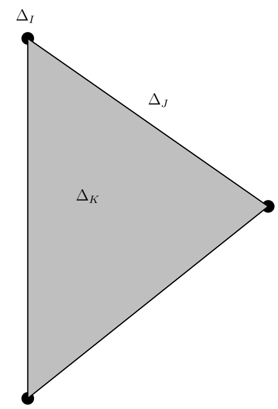

FIGURE 1. The complex $\Delta_{[n]}$.

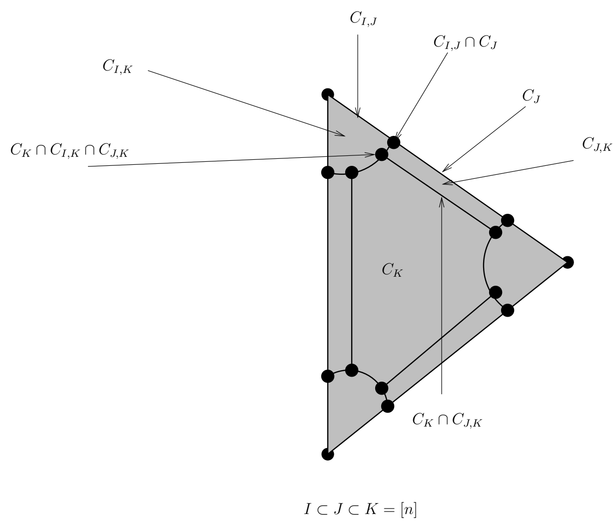

FIGURE 2. The corresponding complex $\mathcal{C}(\Delta_{[n]})$ with $I \subset J \subset K = [n]$.

(2) for each $\alpha \in I_{k_1}$, a definable continuous map,

$$h_{k_1,\alpha} : |\Delta_{k_1,\alpha}| \times T_{k_1,\alpha} \to \bigcup_{(\mathbf{z},\mathbf{y}_0,\ldots,\mathbf{y}_{k_1}) \in T_{k_1,\alpha}} T_{\mathbf{z},\mathbf{y}_0,\ldots,\mathbf{y}_{k_1}},$$

where $\Delta_{k_1,\alpha}$ is a simplicial complex, such that for each $(\mathbf{z}, \mathbf{y}_0, \ldots, \mathbf{y}_{k_1}) \in T_{k_1,\alpha}$, the restriction,

$$h_{k_1,\alpha} : |\Delta_{k_1,\alpha}| \times (\mathbf{z}, \mathbf{y}_0, \ldots, \mathbf{y}_{k_1}) \to T_{\mathbf{z},\mathbf{y}_0,\ldots,\mathbf{y}_{k_1}}$$

is a definable triangulation respecting the sets, $T_{\mathbf{z},\mathbf{y}_0}, \ldots, T_{\mathbf{z},\mathbf{y}_{k_1}}$, and

(3) for each subset $\{j_0, \ldots, j_p\} \subset [k_1]$, $(\mathrm{Id}_{k_2} \oplus \pi_{k_1,j_0,\ldots,j_p})(T_{k_1,\alpha}) \subset T_{p,\beta}$ for some $\beta \in I_p$, and for each $(\mathbf{z}, \mathbf{y}_0, \ldots, \mathbf{y}_{k_1}) \in T_{k_1,\alpha}$, the definable triangulation,

$$h_{k_1,\alpha} : |\Delta_{k_1,\alpha}| \times (\mathbf{z}, \mathbf{y}_0, \ldots, \mathbf{y}_{k_1}) \to T_{\mathbf{z},\mathbf{y}_0,\ldots,\mathbf{y}_{k_1}}$$

restricted to the set $T_{\mathbf{z},\mathbf{y}_{j_0},\ldots,\mathbf{y}_{j_p}}$, is a refinement of the definable triangulation,

$$h_{p,\beta} : |\Delta_{p,\beta}| \times (\mathbf{z}, \mathbf{y}_{j_0}, \ldots, \mathbf{y}_{j_p}) \to T_{\mathbf{z},\mathbf{y}_{j_0},\ldots,\mathbf{y}_{j_p}}.$$

We now fix $\{\mathbf{y}_1, \ldots, \mathbf{y}_n\} \subset \mathbb{R}^\ell$ and let $\mathcal{A} = \{A_1, \ldots, A_n\}$ with $A_i = T_{\mathbf{y}_i} \subset \mathbb{R}^{k_1+k_2}$. For each $\mathbf{z} \in \mathbb{R}^{k_2}$, we will denote by $\mathcal{A}_z = \{A_{1,\mathbf{z}}, \ldots, A_{n,\mathbf{z}}\}$ where $A_{i,\mathbf{z}} = \{\mathbf{x} \in \mathbb{R}^{k_1} \mid (\mathbf{x}, \mathbf{z}) \in A_i\}$.

For $\alpha \in I_{k_1}$, and $1 \le i_0 < \cdots < i_{k_1} \le n$, we will denote by $T_{k_1,\alpha,i_0,\ldots,i_{k_1}} \subset \mathbb{R}^\ell$ the definable set,

$$T_{k_1,\alpha,i_0,\ldots,i_{k_1}} = \{\mathbf{z} \in \mathbb{R}^\ell \mid (\mathbf{z}, \mathbf{y}_0, \ldots, \mathbf{y}_{k_1}) \in T_{k_1,\alpha}\}.$$

Let

$$\mathcal{T} = \bigcup_{\alpha \in I_{k_1}} \{T_{k_1,\alpha,i_0,\ldots,i_{k_1}} \mid 1 \le i_0 < i_1 < \cdots < i_{k_1} \le n\},$$

and let $D \in \mathcal{C}(\mathcal{T})$. Theorem 1.4 will follow from the following two lemmas.

Lemma 4.9. *For any $\mathbf{z}_1, \mathbf{z}_2 \in D$, $\mathcal{A}_{\mathbf{z}_1}$ is diagram preserving stable homotopy equivalent to $\mathcal{A}_{\mathbf{z}_2}$.*

Proof. Clearly, by Theorem 3.4 it suffices to prove that $|\mathcal{N}_{k_1}(\mathcal{A}_{\mathbf{z}_1})|$ is diagram preserving homotopy equivalent to $|\mathcal{N}_{k_1}(\mathcal{A}_{\mathbf{z}_2})|$.

Let $\bar{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_n) \in \mathbb{R}_+^{n+1}$. We first construct regular cell complexes, $\mathcal{N}'(\mathcal{A}_{\mathbf{z}_i}, \bar{\varepsilon}), i = 1, 2$, and show that $|\mathcal{N}'_{k-1}(\mathcal{A}_{\mathbf{z}_1}, \bar{\varepsilon})|$ is homeomorphic to $|\mathcal{N}'_{k_1}(\mathcal{A}_{\mathbf{z}_2}, \bar{\varepsilon})|$, and the homeomorphism is diagram preserving.

Finally, we prove that for all sufficiently small $\bar{\varepsilon}$, $|\mathcal{N}'_{k_1}(\mathcal{A}_{\mathbf{z}_i}, \bar{\varepsilon})|$ is homotopy equivalent to $|\mathcal{N}_{k_1}(\mathcal{A}_{\mathbf{z}_i})|, i = 1, 2$, which will prove the lemma.

We now define $\mathcal{N}'(\mathcal{A}_{\mathbf{z}_i}, \bar{\varepsilon}), i = 1, 2$. The cells of $\mathcal{N}'(\mathcal{A}_{\mathbf{z}_i}, \bar{\varepsilon})$ are,

$$\bigcup_{I \subset [n], c \in \Delta(\mathcal{A}_{I,\mathbf{z}_i})} C_I(\bar{\varepsilon}) \times c,$$

where $\Delta(\mathcal{A}_{I,\mathbf{z}_i})$ is the set of cells of $\mathcal{A}_{I,\mathbf{z}_i}$ occurring in the triangulation

$$h_{k_1,\alpha} : |\Delta_{k_1,\alpha}| \times (\mathbf{z}, \mathbf{y}_{i_0}, \ldots, \mathbf{y}_{i_{k_1}}) \to T_{\mathbf{z},\mathbf{y}_{i_0},\ldots,\mathbf{y}_{i_{k_1}}},$$

where $I = \{i_0, \ldots, i_{k_1}\}$. The compatibility properties of the triangulations ensure that $\mathcal{N}'_{k_1}(\mathcal{A}_{\mathbf{z}_i}, \bar{\varepsilon})$ is a regular cell complex for $i = 1, 2$, and also that the complex $\mathcal{N}'_{k_1}(\mathcal{A}_{\mathbf{z}_1}, \bar{\varepsilon})$ is isomorphic to $\mathcal{N}'_{k_1}(\mathcal{A}_{\mathbf{z}_2}, \bar{\varepsilon})$.

We claim that for all sufficiently small $\bar{\varepsilon} > 0$, $|\mathcal{N}'_{k_1}(\mathcal{A}_{\mathbf{z}_i}, \bar{\varepsilon})|$ is homotopy equivalent to $|\mathcal{N}_{k_1}(\mathcal{A}_{\mathbf{z}_i})|$. In order to see this, let $N_i = |\mathcal{N}'_{k_1}(\mathcal{A}_{\mathbf{z}_i}, \bar{\varepsilon})|$. First replace ε_n by a variable t in the definition of N_i to obtain a space, $N^n_{i,t}$, and observe that $N^n_{i,t} \subset N^n_{i,t'}$ for all $0, t < t' \ll 1$. Now apply Proposition 4.6 to obtain that N_i is homotopy equivalent to $N^n_{i,0}$. Now, replace ε_{n-1} by t in the definition of $N^n_{i,0}$ to obtain $N^{n-1}_{i,t}$, and applying Proposition 4.6 obtain that $N^n_{i,0}$ is homotopy equivalent to $N^{n-1}_{i,0}$. Continuing in this way we finally obtain that, N_i is homotopy equivalent to $N^0_{i,0} = |\mathcal{N}_{k_1}(\mathcal{A}_{z_i})|$, for $i = 1, 2$.

Thus, we get a diagram preserving homotopy equivalence,

$$\phi : |\mathcal{N}_{k_1}(\mathcal{A}_{\mathbf{z}_1}))| \to |\mathcal{N}_{k_1}(\mathcal{A}_{\mathbf{z}_2}))|.$$

It now follows from Theorem 3.4 that there exists a stable homotopy equivalence $\phi : \mathcal{A}_{\mathbf{z}_1} \to \mathcal{A}_{\mathbf{z}_2}$. □

Lemma 4.10. *There exists a constant $C(T)$ such that the cardinality of $\mathcal{C}(\mathcal{T})$ is bounded by $C \cdot n^{(k_1+1)k_2}$.*

Proof. Notice that each $T_{k_1,\alpha}, \alpha \in I_{k_1}$ is a definable subset of $\mathbb{R}^{k_2+(k_1+1)\ell}$ depending only on T. Also, the cardinality of the index set I_{k_1} is determined by T.

Hence, the set $\mathcal{T}$ consists of $\binom{n}{k_1+1}$ definable sets, each one of them is a

$$(T_{k_1,\alpha}, \pi^{\leq k_2}_{k_2+(k_1+1)\ell}, \pi^{>k_2}_{k_2+(k_1+1)\ell})$$

for some $\alpha \in I_{k_1}$. Using Observation 4.1, we have that $\mathcal{T}$ is a (T', π'_1, π'_2)-set for some T' determined only by T. Now apply Theorem 2.2. □

The theorem now follows from Lemmas 4.9 and 4.10 proved above. □

Proof of Theorem 1.5. The proof is similar to that of Theorem 1.4 given above, except we use Theorem 3.5 instead of Theorem 3.4, and this accounts for the slight worsening of the exponent in the bound. □

Proof of Theorem 2.7. Using a construction due to Gabrielov and Vorobjov [9] (see also [2]) by which one can replace any given $\mathcal{A}$-set by a closed bounded $\mathcal{A}'$-set (where $\mathcal{A}'$ is a new family of definable closely related to $\mathcal{A}$ with $\#\mathcal{A}' = (\#\mathcal{A})^2)$,) such that the new set has the same homotopy type as the original one. Using this construction one can directly deduce Theorem 2.7 from Theorem 1.5. We omit the details. □

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