A GLOBAL APPROACH TO THE THEORY OF SPECIAL FINSLER MANIFOLDS

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Dedicated to the memory of Prof. Dr. A. TAMIM

Abstract. The aim of the present paper is to provide a *global* presentation of the theory of special Finsler manifolds. We introduce and investigate *globally* (or intrinsically, free from local coordinates) many of the most important and most commonly used special Finsler manifolds : locally Minkowskian, Berwald, Landesberg, general Landesberg, P-reducible, C-reducible, semi-C-reducible, quasi-C-reducible, P^*-Finsler, C^h-recurrent, C^v-recurrent, C^0-recurrent, S^v-recurrent, S^v-recurrent of the second order, C_2-like, S_3-like, S_4-like, P_2-like, R_3-like, P-symmetric, h-isotropic, of scalar curvature, of constant curvature, of p-scalar curvature, of s-ps-curvature.

The global definitions of these special Finsler manifolds are introduced. Various relationships between the different types of the considered special Finsler manifolds are found. Many local results, known in the literature, are proved globally and several new results are obtained. As a by-product, interesting identities and properties concerning the torsion tensor fields and the curvature tensor fields are deduced.

Although our investigation is entirely global, we provide; for comparison reasons, an appendix presenting a local counterpart of our global approach and the *local* definitions of the special Finsler spaces considered. [1]

Keywords and phrases. Berwald, Landesberg, P-reducible, C-reducible, Semi-C-reducible, Quasi-C-reducible, P^*-Finsler, C^h-recurrent, C^v-recurrent, S^v-recurrent, C_2-like, S_3-like, S_4-like, P_2-like, R_3-like, P-symmetric, h-isotropic, Of scalar curvature, Of constant curvature, Of p-scalar curvature, Of s-ps-curvature.

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Introduction

In Finsler geometry all geometric objects depend not only on positional coordinates, as in Riemannian geometry, but also on directional arguments. In Riemannian geometry there is a canonical linear connection on the manifold M, while in Finsler geometry there is a corresponding canonical linear connection, due to E. Cartan, which is not a connection on M but on $\pi^{-1}(TM)$, the pullback of the tangent bundle TM by $\pi : \mathcal{T}M \longrightarrow M$ (*the pullback approach*). Moreover, in Riemannian geometry there is one curvature tensor and one torsion tensor associated with a given linear connection on the manifold M, whereas in Finsler geometry there are three curvature tensors and five torsion tensors associated with a given linear connection on $\pi^{-1}(TM)$.

Most of the special spaces in Finsler geometry are derived from the fact that the π-tensor fields (torsions and curvatures) associated with the Cartan connection satisfy special forms. Consequently, special spaces of Finsler geometry are more numerous than those of Riemannian geometry. Special Finsler spaces are investigated locally (using local coordinates) by many authors: M. Matsumoto [16], [18], [15], [14] and others [6], [19], [8], [7]. On the other hand, the global (or intrinsic, free from local coordinates) investigation of such spaces is very rare in the literature. Some considerable contributions in this direction are due to A. Tamim [24], [25].

In the present paper, we provide a *global* presentation of the theory of special Finsler manifolds. We introduce and investigate *globally* many of the most important and most commonly used special Finsler manifolds: locally Minkowskian, Berwald, Landesberg, general Landesberg, P-reducible, C-reducible, semi-C-reducible, quasi-C-reducible, P^*-Finsler, C^h-recurrent, C^v-recurrent, C^0-recurrent, S^v-recurrent, S^v-recurrent of the second order, C_2-like, S_3-like, S_4-like, P_2-like, R_3-like, P-symmetric, h-isotropic, of scalar curvature, of constant curvature, of p-scalar curvature, of *s-ps*-curvature.

The paper consists of two parts, preceded by a preliminary section (§1), which provides a brief account of the basic concepts of the pullback approach to Finsler geometry necessary to this work. For more detail, the reader is referred to [1], [3], [5] and [24].

In the first part (§2), we introduce the global definitions of the aforementioned special Finsler manifolds in such a way that, when localized, they yield the usual local definitions current in the literature (see the appendix). The definitions are arranged according to the type of the defining property of the special Finsler manifold concerned.

In the second part (§3), various relationships between the different types of the considered special Finsler manifolds are found. Many local results, known in the literature, are proved globally and several new results are obtained. As a by-product of some of the obtained results, interesting identities and properties concerning the torsion tensor fields and the curvature tensor fields are deduced, which in turn play a key role in obtaining other results.

Among the obtained results are: a characterization of Riemannian manifolds, a characterization of S^v-recurrent manifolds, a characterization of P-symmetric manifolds, a characterization of Berwald manifolds (in certain cases), the equivalence of Landsberg and general Landsberg manifolds under certain conditions, a classification of h-isotropic C^h-recurrent manifolds and a presentation of different conditions under which an R_3-like Finsler manifold becomes a Finsler manifold of *s-ps* curvature. The above results are just a non-exhaustive sample of the global results obtained in this paper.

It should finally be noted that some important results of [8], [9], [11], [13], [19], [20],...,etc. (obtained in local coordinates) are immediately derived from the obtained global results (when localized).

Although our investigation is entirely global, we conclude the paper with an appendix presenting a local counterpart of our global approach and the *local* definitions of the special Finsler spaces considered. This is done to facilitate comparison and to make the paper more self-contained.

1. Notation and Preliminaries

In this section, we give a brief account of the basic concepts of the pullback formalism of Finsler geometry necessary for this work. For more details refer to [1], [3], [5] and [24]. We make the general assumption that all geometric objects we consider are of class C^∞. The following notation will be used throughout this paper:
M: a real differentiable manifold of finite dimension n and of class C^∞,
$\mathfrak{F}(M)$: the $\mathbb{R}$-algebra of differentiable functions on M,
$\mathfrak{X}(M)$: the $\mathfrak{F}(M)$-module of vector fields on M,
$\pi_M : TM \longrightarrow M$: the tangent bundle of M,
$\pi : \mathcal{T}M \longrightarrow M$: the subbundle of nonzero vectors tangent to M,
$V(TM)$: the vertical subbundle of the bundle TTM,
$P : \pi^{-1}(TM) \longrightarrow \mathcal{T}M$: the pullback of the tangent bundle TM by π,
$P^* : \pi^{-1}(T^*M) \longrightarrow \mathcal{T}M$: the pullback of the cotangent bundle T^*M by π,
$\mathfrak{X}(\pi(M))$: the $\mathfrak{F}(M)$-module of differentiable sections of $\pi^{-1}(TM)$,

Elements of $\mathfrak{X}(\pi(M))$ will be called π-vector fields and will be denoted by barred letters $\overline{X}$. Tensor fields on $\pi^{-1}(TM)$ will be called π-tensor fields. The fundamental π-vector field is the π-vector field $\overline{\eta}$ defined by $\overline{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$. The lift to $\pi^{-1}(TM)$ of a vector field X on M is the π-vector field $\overline{X}$ defined by $\overline{X}(u) = (u, X(\pi(u)))$. The lift to $\pi^{-1}(TM)$ of a 1-form ω on M is the π-form $\overline{\omega}$ defined by $\overline{\omega}(u) = (u, \omega(\pi(u)))$.

The tangent bundle $T(\mathcal{T}M)$ is related to the pullback bundle $\pi^{-1}(TM)$ by the short exact sequence

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where the bundle morphisms ρ and γ are defined respectively by $\rho = (\pi_{\mathcal{T}M}, d\pi)$ and $\gamma(u, v) = j_u(v)$, where j_u is the natural isomorphism $j_u : T_{\pi_M(v)}M \longrightarrow T_u(T_{\pi_M(v)}M)$.

Let ∇ be a linear connection (or simply a connection) in the pullback bundle $\pi^{-1}(TM)$. We associate to ∇ the map

$$K : T\mathcal{T}M \longrightarrow \pi^{-1}(TM) : X \longmapsto \nabla_X \overline{\eta},$$

called the connection (or the deflection) map of ∇. A tangent vector $X \in T_u(\mathcal{T}M)$ is said to be horizontal if $K(X) = 0$. The vector space $H_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : K(X) = 0\}$ of the horizontal vectors at $u \in \mathcal{T}M$ is called the horizontal space to M

at u . The connection ∇ is said to be regular if

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \qquad \forall u \in \mathcal{T}M.$$

If M is endowed with a regular connection, then the vector bundle maps

$$\begin{aligned} \gamma &: \pi^{-1}(TM) \longrightarrow V(\mathcal{T}M), \\ \rho|_{H(\mathcal{T}M)} &: H(\mathcal{T}M) \longrightarrow \pi^{-1}(TM), \\ K|_{V(\mathcal{T}M)} &: V(\mathcal{T}M) \longrightarrow \pi^{-1}(TM) \end{aligned}$$

are vector bundle isomorphisms. Let us denote $\beta = (\rho|_{H(\mathcal{T}M)})^{-1}$, then

$$\rho o \beta = id_{\pi^{-1}(TM)}, \qquad \beta o \rho = \begin{cases} id_{H(\mathcal{T}M)} & \text{on H(TM)} \\ 0 & \text{on V(TM)} \end{cases} \tag{1.1}$$

For a regular connection ∇ we define two covariant derivatives $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ as follows: For every vector $(1)\pi$-form A, we have

$$(\overset{1}{\nabla} A)(\overline{X}, \overline{Y}) := (\nabla_{\beta\overline{X}} A)(\overline{Y}), \qquad (\overset{2}{\nabla} A)(\overline{X}, \overline{Y}) := (\nabla_{\gamma\overline{X}} A)(\overline{Y}).$$

The classical torsion tensor $\mathbf{T}$ of the connection ∇ is defined by

$$\mathbf{T}(X, Y) = \nabla_X \rho Y - \nabla_Y \rho X - \rho[X, Y] \quad \forall X, Y \in \mathfrak{X}(\mathcal{T}M).$$

The horizontal ((h)h-) and mixed ((h)hv-) torsion tensors, denoted respectively by Q and T, are defined by

$$Q(\overline{X}, \overline{Y}) = \mathbf{T}(\beta\overline{X}\beta\overline{Y}), \;\; T(\overline{X}, \overline{Y}) = \mathbf{T}(\gamma\overline{X}, \beta\overline{Y}) \quad \forall \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)).$$

The classical curvature tensor $\mathbf{K}$ of the connection ∇ is defined by

$$\mathbf{K}(X, Y)\rho Z = -\nabla_X \nabla_Y \rho Z + \nabla_Y \nabla_X \rho Z + \nabla_{[X,Y]} \rho Z \quad \forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M).$$

The horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors, denoted respectively by R, P and S, are defined by

$$R(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X}\beta\overline{Y})\overline{Z}, \quad P(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X}, \gamma\overline{Y})\overline{Z}, \quad S(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\gamma\overline{X}, \gamma\overline{Y})\overline{Z}.$$

We also have the (v)h-, (v)hv- and (v)v-torsion tensors, denoted respectively by $\widehat{R}$, $\widehat{P}$ and $\widehat{S}$, defined by

$$\widehat{R}(\overline{X}, \overline{Y}) = R(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X}, \overline{Y}) = P(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{S}(\overline{X}, \overline{Y}) = S(\overline{X}, \overline{Y})\overline{\eta}.$$

Theorem 1.1. [25] *Let (M, L) be a Finsler manifold. There exists a unique regular connection ∇ in $\pi^{-1}(TM)$ such that*

(a) ∇ *is metric*: $\nabla g = 0$,

(b) *The horizontal torsion of* ∇ *vanishes*: $Q = 0$,

(c) *The mixed torsion T of ∇ satisfies* $\;g(T(\overline{X}, \overline{Y}), \overline{Z}) = g(T(\overline{X}, \overline{Z}), \overline{Y})$.

Such a connection is called the Cartan connection associated to the Finsler manifold (M, L).

One can show that the torsion T of the Cartan connection has the property that $T(\overline{X}, \overline{\eta}) = 0$ for all $\overline{X} \in \mathfrak{X}(\pi(M))$ and associated to T we have:

Definition 1.2. [25] *Let ∇ be the Cartan connection associated to (M, L). The torsion tensor field T of the connection ∇ induces a π-tensor field of type $(0, 3)$, called the Cartan tensor and denoted again T, defined by:*

$$T(\overline{X}, \overline{Y}, \overline{Z}) = g(T(\overline{X}, \overline{Y}), \overline{Z}), \quad \textit{for all} \quad \overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\mathcal{T}M).$$

It also induces a π-form C, called the contracted torsion, defined by:

$$C(\overline{X}) := Tr\{\overline{Y} \longmapsto T(\overline{X}, \overline{Y})\}, \quad \textit{for all} \quad \overline{X} \in \mathfrak{X}(\mathcal{T}M).$$

Definition 1.3. [25] *With respect to the Cartan connection ∇ associated to (M, L), we have*

– *The horizontal and vertical Ricci tensors Ric^h and Ric^v are defined respectively by:*

$$Ric^h(\overline{X}, \overline{Y}) := Tr\{\overline{Z} \longmapsto R(\overline{X}, \overline{Z})\overline{Y}\}, \quad \textit{for all} \quad \overline{X}, \overline{Y} \in \mathfrak{X}(\mathcal{T}M),$$

$$Ric^v(\overline{X}, \overline{Y}) := Tr\{\overline{Z} \longmapsto S(\overline{X}, \overline{Z})\overline{Y}\}, \quad \textit{for all} \quad \overline{X}, \overline{Y} \in \mathfrak{X}(\mathcal{T}M).$$

– *The horizontal and vertical Ricci maps Ric_0^h and Ric_0^v are defined respectively by:*

$$g(Ric_0^h(\overline{X}), \overline{Y}) := Ric^h(\overline{X}, \overline{Y}), \quad \textit{for all} \quad \overline{X}, \overline{Y} \in \mathfrak{X}(\mathcal{T}M),$$

$$g(Ric_0^v(\overline{X}), \overline{Y}) := Ric^v(\overline{X}, \overline{Y}), \quad \textit{for all} \quad \overline{X}, \overline{Y} \in \mathfrak{X}(\mathcal{T}M).$$

– *The horizontal and vertical scalar curvatures Sc^h , Sc^v are defined respectively by:*

$$Sc^h := Tr(Ric_0^h), \qquad Sc^v := Tr(Ric_0^v),$$

where R and S are respectively the horizontal and vertical curvature tensors of ∇.

Proposition 1.4. [12] *Let (M, L) be a Finsler manifold. The vector field G determined by $i_G\Omega = -dE$ is a spray, called the canonical spray associated to the energy E, where $E := \frac{1}{2}L^2$ and $\Omega := dd_JE$.*

One can show, in this case, that $G = \beta o \overline{\eta}$, and G is thus horizontal with respect to the Cartan connection ∇.

Theorem 1.5. [26] *Let (M, L) be a Finsler manifold. There exists a unique regular connection D in $\pi^{-1}(TM)$ such that*

(a) *D is torsion free,*

(b) *The canonical spray $G = \beta o \overline{\eta}$ is horizontal with respect to D,*

(c) *The (v)hv-torsion tensor $\widehat{P}$ of D vanishes.*

Such a connection is called the Berwald connection associated to the Finsler manifold (M, L).

2. Special Finsler spaces

In this section, we introduce the global definitions of the most important and commonly used special Finsler spaces in such a way that, when localized, they yield the usual local definitions existing in the literature (see the Appendix). Here we simply set the definitions, postponing investigation of the mutual relationships between these special Finsler spaces to the next section. The definitions are arranged according to the type of defining property of the special Finsler space concerned.

Throughout the paper, g, $\widehat{g}$, ∇ and D denote respectively the Finsler metric in $\pi^{-1}(TM)$, the induced metric in $\pi^{-1}(T^*M)$, the Cartan connection and the Berwald connection associated to a given Finsler manifold (M, L). Also, T denotes the torsion tensor of the Cartan connection (or the Cartan tensor) and R, P and S denote respectively the horizontal curvature, the mixed curvature and the vertical curvature of the Cartan connection.

Definition 2.1. *A Finsler manifold (M, L) is:*

(a) *Riemannian if the metric tensor $g(x, y)$ is independent of y or, equivalently, if*

$$T(\overline{X}, \overline{Y}) = 0, \ \ \textit{for all } \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)).$$

(b) *locally Minkowskian if the metric tensor $g(x, y)$ is independent of x or, equivalently, if*

$$\nabla_{\beta\overline{X}} T = 0 \ \ \textit{and} \ \ R = 0.$$

Definition 2.2. *A Finsler manifold (M, L) is said to be:*

(a) *Berwald* [24] *if the torsion tensor T is horizontally parallel. That is,*

$$\nabla_{\beta\overline{X}} T = 0.$$

(b) *C^h-recurrent if the torsion tensor T satisfies the condition*

$$\nabla_{\beta\overline{X}} T = \lambda_o(\overline{X})\, T,$$

where λ_o is a π-form of order one.

(c) *P^*-Finsler manifold if the π-tensor field $\nabla_{\beta\overline{\eta}} T$ is expressed in the form*

$$\nabla_{\beta\overline{\eta}} T = \lambda(x, y)\, T,$$

where $\lambda(x, y) = \frac{\widehat{g}(\nabla_{\beta\overline{\eta}} C, C)}{C^2} = \frac{g(\nabla_{\beta\overline{\eta}} \overline{C}, \overline{C})}{C^2}$ and $C^2 := \widehat{g}(C, C) = C(\overline{C}) \neq 0$; $\overline{C}$ being the π-vector field defined by $g(\overline{C}, \overline{X}) = C(\overline{X})$.

Definition 2.3. *A Finsler manifold (M, L) is said to be:*

(a) *C^v-recurrent if the torsion tensor T satisfies the condition $(\nabla_{\gamma\overline{X}} T)(\overline{Y}, \overline{Z}) = \lambda_o(\overline{X}) T(\overline{Y}, \overline{Z})$.*

(b) *C^0-recurrent if the torsion tensor T satisfies the condition $(D_{\gamma\overline{X}} T)(\overline{Y}, \overline{Z}) = \lambda_o(\overline{X}) T(\overline{Y}, \overline{Z})$.*

Definition 2.4. [25] *A Finsler manifold (M, L) is said to be:*

(a) *semi-C-reducible if $dimM \geq 3$ and the Cartan tensor T has the form*

$$\begin{aligned} T(\overline{X},\overline{Y},\overline{Z}) =& \frac{\mu}{n+1}\{\hbar(\overline{X},\overline{Y})C(\overline{Z})+\hbar(\overline{Y},\overline{Z})C(\overline{X})+\hbar(\overline{Z},\overline{X})C(\overline{Y})\}+ \\ &+\frac{\tau}{C^2}C(\overline{X})C(\overline{Y})C(\overline{Z}), \end{aligned}$$

where μ and τ are scalar functions satisfying $\mu+\tau=1$, $\hbar = g-\ell\otimes\ell$ and $\ell(\overline{X}) := L^{-1}g(\overline{X},\overline{\eta})$.

(b) *C-reducible if $dimM \geq 3$ and the Cartan tensor T has the form*

$$T(\overline{X},\overline{Y},\overline{Z}) = \frac{1}{n+1}\{\hbar(\overline{X},\overline{Y})C(\overline{Z})+\hbar(\overline{Y},\overline{Z})C(\overline{X})+\hbar(\overline{Z},\overline{X})C(\overline{Y})\}.$$

(c) *C_2-like if $dimM \geq 2$ and the Cartan tensor T has the form*

$$T(\overline{X},\overline{Y},\overline{Z}) = \frac{1}{C^2}C(\overline{X})C(\overline{Y})C(\overline{Z}).$$

Definition 2.5. *A Finsler manifold (M, L), where $dimM \geq 3$, is said to be quasi-C-reducible if the Cartan tensor T is written as:*

$$T(\overline{X},\overline{Y},\overline{Z}) = A(\overline{X},\overline{Y})C(\overline{Z})+A(\overline{Y},\overline{Z})C(\overline{X})+A(\overline{Z},\overline{X})C(\overline{Y}),$$

where A is a symmetric indicatory (2) π-form ($A(\overline{X},\overline{\eta})=0$ for all $\overline{X}$).

Definition 2.6. [25] *A Finsler manifold (M, L) is said to be:*

(a) *S_3-like if $dim(M) \geq 4$ and the vertical curvature tensor $S(\overline{X},\overline{Y},\overline{Z},\overline{W}) := g(S(\overline{X},\overline{Y})\overline{Z},\overline{W})$ has the form:*

$$S(\overline{X},\overline{Y},\overline{Z},\overline{W}) = \frac{Sc^v}{(n-1)(n-2)}\{\hbar(\overline{X},\overline{Z})\hbar(\overline{Y},\overline{W})-\hbar(\overline{X},\overline{W})\hbar(\overline{Y},\overline{Z})\}.$$

(b) *S_4-like if $dim(M) \geq 5$ and the vertical curvature tensor $S(\overline{X},\overline{Y},\overline{Z},\overline{W})$ has the form:*

$$\begin{aligned} S(\overline{X},\overline{Y},\overline{Z},\overline{W}) =& \hbar(\overline{X},\overline{Z})\boldsymbol{F}(\overline{Y},\overline{W})-\hbar(\overline{Y},\overline{Z})\boldsymbol{F}(\overline{X},\overline{W})+ \\ &+\hbar(\overline{Y},\overline{W})\boldsymbol{F}(\overline{X},\overline{Z})-\hbar(\overline{X},\overline{W})\boldsymbol{F}(\overline{Y},\overline{Z}), \end{aligned} \tag{2.1}$$

where $\boldsymbol{F}$ is the (2)π-form defined by $\boldsymbol{F} = \frac{1}{n-3}\{Ric^v - \frac{Sc^v\,\hbar}{2(n-2)}\}$.

Definition 2.7. *A Finsler manifold (M, L) is said to be:*

(a) *S^v-recurrent if the v-curvature tensor S satisfies the condition*

$$(\nabla_{\gamma\overline{X}}S)(\overline{Y},\overline{Z},\overline{W}) = \lambda(\overline{X})S(\overline{Y},\overline{Z})\overline{W},$$

where λ is a π-form of order one.

(b) *S^v-recurrent of the second order if the v-curvature tensor S satisfies the condition*

$$(\overset{2}{\nabla}\overset{2}{\nabla} S)(\overline{Y},\overline{X},\overline{Z},\overline{W},\overline{U}) = \Theta(\overline{X},\overline{Y})S(\overline{Z},\overline{W})\overline{U},$$

where Θ is a π-form of order two.

Definition 2.8. [24] *A Finsler manifold (M, L) is said to be*:

(a) *a Landsberg manifold if*

$$\widehat{P}(\overline{X}, \overline{Y}) = P(\overline{X}, \overline{Y})\overline{\eta} = 0 \;\; \forall \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)), \;\; \textit{or equivalently} \;\; \nabla_{\beta\overline{\eta}} T = 0.$$

(b) *a general Landsberg manifold if*

$$Tr\{\overline{Y} \longrightarrow \widehat{P}(\overline{X}, \overline{Y})\} = 0 \;\; \forall \overline{X}, \in \mathfrak{X}(\pi(M)), \;\; \textit{or equivalently} \;\; \nabla_{\beta\overline{\eta}}\, C = 0.$$

Definition 2.9. *A Finsler manifold (M, L) is said to be P-symmetric if the mixed curvature tensor P satisfies*

$$P(\overline{X}, \overline{Y})\overline{Z} = P(\overline{Y}, \overline{X})\overline{Z}, \;\; \forall\, \overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\pi(M)).$$

Definition 2.10. *A Finsler manifold (M, L), where $dimM \geq 3$, is said to be P_2-like if the mixed curvature tensor P has the form*:

$$P(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \alpha(\overline{Z})T(\overline{X}, \overline{Y}, \overline{W}) - \alpha(\overline{W})\,T(\overline{X}, \overline{Y}, \overline{Z}),$$

where α is a (1) *π-form* (positively homogeneous of degree 0).

Definition 2.11. [25] *A Finsler manifold (M, L), where $dimM \geq 3$, is said to be P-reducible if the π-tensor field $P(\overline{X}, \overline{Y}, \overline{Z}) := g(P(\overline{X}, \overline{Y})\overline{\eta}, \overline{Z})$ can be expressed in the form*:

$$P(\overline{X}, \overline{Y}, \overline{Z}) = \delta(\overline{X})\hbar(\overline{Y}, \overline{Z}) + \delta(\overline{Y})\hbar(\overline{Z}, \overline{X}) + \delta(\overline{Z})\hbar(\overline{X}, \overline{Y}),$$

where δ is a (1) *π-form satisfying $\delta(\overline{\eta}) = 0$.*

Definition 2.12. [2] *A Finsler manifold (M, L), where $dimM \geq 3$, is said to be h-isotropic if there exists a scalar k_o such that the horizontal curvature tensor R has the form*

$$R(\overline{X}, \overline{Y})\overline{Z} = k_o\{g(\overline{Y}, \overline{Z})\overline{X} - g(\overline{X}, \overline{Z})\overline{Y}\}.$$

Definition 2.13. [2] *A Finsler manifold (M, L), where $dimM \geq 3$, is said to be*:

(a) *of scalar curvature if there exists a scalar function $k : \mathcal{T}M \longrightarrow \mathbb{R}$ such that the horizontal curvature tensor $R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) := g(R(\overline{X}, \overline{Y})\overline{Z}, \overline{W})$ satisfies the relation*

$$R(\overline{\eta}, \overline{X}, \overline{\eta}, \overline{Y}) = kL^2\hbar(\overline{X}, \overline{Y}).$$

(b) *of constant curvature if the function k in* (a) *is constant.*

Definition 2.14. *A Finsler manifold (M, L) is said to be R_3-like if $dimM \geq 4$ and the horizontal curvature tensor $R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W})$ is expressed in the form*

$$\begin{aligned} R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) =& g(\overline{X}, \overline{Z})F(\overline{Y}, \overline{W}) - g(\overline{Y}, \overline{Z})F(\overline{X}, \overline{W}) + \\ & + g(\overline{Y}, \overline{W})F(\overline{X}, \overline{Z}) - g(\overline{X}, \overline{W})F(\overline{Y}, \overline{Z}), \end{aligned} \tag{2.2}$$

where F is the $(2)\pi$-form defined by $F = \frac{1}{n-2}\{Ric^h - \frac{Sc^h\, g}{2(n-1)}\}$.

3. Relationships between different types of special Finsler spaces

This section is devoted to global investigation of some mutual relationships between the special Finsler spaces introduced in the preceding section. Some consequences are also drawn from these relationships.

We start with some immediate consequences from the definitions:
(a) A Locally Minkowskian manifold is a Berwald manifold.
(b) A Berwald manifold is a Landsberg manifold.
(c) A Landsberg manifold is a general Landsberg manifold.
(d) A Berwald manifold is C^h-recurrent (resp. P^*-Finsler).
(e) A P^*-manifold is a Landsberg manifold.
(f) A C-reducible (resp. C_2-like) manifold is semi-C-reducible.
(g) A semi-C-reducible manifold is quasi-C-reducible.
(h) A Finsler manifold of constant curvature is of scalar curvature.

The following two lemmas are useful for subsequent use.

Lemma 3.1. [25] *For every* $\overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M))$, *we have*:

(a) $P(\overline{\eta}, \overline{X})\overline{Y} = 0$, **(b)** $P(\overline{X}, \overline{\eta})\overline{Y} = 0$, **(c)** $P(\overline{X}, \overline{Y})\overline{\eta} = (\nabla_{\beta\overline{\eta}}T)(\overline{X}, \overline{Y})$.

Lemma 3.2. *If* ϕ *is the vector* π*-form defined by*

$$\phi(\overline{X}) := \overline{X} - L^{-1}\ell(\overline{X})\overline{\eta}, \ \ or \ \ \phi := I - L^{-1}\ell \otimes \overline{\eta}, \tag{3.1}$$

where ℓ *is the* π*-form given by* $\ell(\overline{X}) = L^{-1}g(\overline{X}, \overline{\eta})$, *then we have*:

(a) $\hbar(\overline{X}, \overline{Y}) = g(\phi(\overline{X}), \overline{Y})$, ***(b)*** $\phi(\overline{\eta}) = 0$, ***(c)*** $\phi \, o \, \phi = \phi$,

(d) $Tr(\phi) = n - 1$, ***(e)*** $\nabla_{\beta\overline{X}}\phi = 0$, ***(f)*** $\nabla_{\beta\overline{X}}\hbar = 0$.

As we have seen, a Landsberg manifold is general Landsberg. The converse is not true. Nevertheless, we have

Proposition 3.3. *A* C*-reducible general Landsberg manifold* (M, L) *is a Landsberg manifold.*

Proof. Since (M, L) is a C-reducible manifold, then, by Definition 2.4, Lemma 3.2, the symmetry of $\hbar$ and the non-degeneracy of g, we get

$$T(\overline{X}, \overline{Y}) = \frac{1}{n+1}\{\hbar(\overline{X}, \overline{Y})\overline{C} + C(\overline{X})\phi(\overline{Y}) + C(\overline{Y})\phi(\overline{X})\},$$

where $\overline{C}$ is the π-vector field defined by $g(\overline{C}, \overline{X}) := C(\overline{X})$. Taking the h-covariant derivative $\nabla_{\beta\overline{Z}}$ of both sides of the above equation, we obtain

$$\begin{aligned}(\nabla_{\beta\overline{Z}}T)(\overline{X}, \overline{Y}) = \frac{1}{n+1}\{&(\nabla_{\beta\overline{Z}}\hbar)(\overline{X}, \overline{Y})\overline{C} + \hbar(\overline{X}, \overline{Y})\nabla_{\beta\overline{Z}}\overline{C} + C(\overline{X})(\nabla_{\beta\overline{Z}}\phi)(\overline{Y}) + \\ &+(\nabla_{\beta\overline{Z}}C)(\overline{X})\phi(\overline{Y}) + C(\overline{Y})(\nabla_{\beta\overline{Z}}\phi)(\overline{X}) + (\nabla_{\beta\overline{Z}}C)(\overline{Y})\phi(\overline{X})\},\end{aligned}$$

from which, by setting $\overline{Z} = \overline{\eta}$ and taking into account the fact that $\nabla_{\beta\overline{Z}}\hbar = 0$ and that $\nabla_{\beta\overline{Z}}\phi = 0$ (Lemma 3.2), we get

$$(\nabla_{\beta\overline{\eta}}T)(\overline{X}, \overline{Y}) = \frac{1}{n+1}\{\hbar(\overline{X}, \overline{Y})\nabla_{\beta\overline{\eta}}\overline{C} + (\nabla_{\beta\overline{\eta}}C)(\overline{X})\phi(\overline{Y}) + (\nabla_{\beta\overline{\eta}}C)(\overline{Y})\phi(\overline{X})\}.$$

Now, under the given assumption that the (M, L) is a general Landsberg manifold, then $\nabla_{\beta\overline{\eta}} C = 0$ (Definition 2.8) and hence $\nabla_{\beta\overline{\eta}} \overline{C} = 0$. Hence $\nabla_{\beta\overline{\eta}} T = 0$ and the result follows. $\square$

Also, a Berwald manifold is Landsberg. The converse is by no means true, although we have no counter-examples. Finding a Landsberg manifold which is not Berwald is still an open problem. Nevertheless, we have

Proposition 3.4. [25] *A C-reducible Landsberg manifold (M, L) is a Berwald manifold.*

Combining the above two Propositions, we obtain the more powerful result :

Proposition 3.5. *A C-reducible general Landsberg manifold (M, L) is a Berwald manifold.*

Summing up, we get:

Theorem 3.6. *Let (M, L) be a C-reducible Finsler manifold. The following assertion are equivalent:*

(a) *(M, L) is a Berwald manifold.*

(b) *(M, L) is a Landsberg manifold.*

(c) *(M, L) is a general Landsberg manifold.*

We retrieve here a result of Matsumuoto [15], namely

Corollary 3.7. *If the h-curvature tensor R and hv-curvature tensor P of a C-reducible manifold vanish, then the manifold is Locally Minkowskian.*

Remark 3.8. [15] *It may be conjectured that a Finsler manifold will be Minkowskian if the h-curvature tensor R and hv-curvature tensor P vanish. As above seen the conjecture is verified already under somewhat strong condition "C-reducibility".*

Theorem 3.9. *Let (M, L) be a Finsler manifold. Then we have:*

(a) *A C-reducible manifold is P-reducible.*

(b) *A P-reducible general Landsberg manifold is Landsberg.*

Proof.

(a) Since (M, L) is C-reducible, then by Definition 2.4, we have

$$T(\overline{X}, \overline{Y}, \overline{Z}) = \frac{1}{n+1} \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} \{\hbar(\overline{X}, \overline{Y}) C(\overline{Z})\}.$$

Applying the h-covariant derivative $\nabla_{\beta\overline{W}}$ on both sides of the above equation, taking into account the fact that $(\nabla_{\beta\overline{W}} T)(\overline{X}, \overline{Y}, \overline{Z}) = g((\nabla_{\beta\overline{W}} T)(\overline{X}, \overline{Y}), \overline{Z})$ and that $\nabla_{\beta\overline{W}} \hbar = 0$, we obtain

$$g((\nabla_{\beta\overline{W}} T)(\overline{X}, \overline{Y}), \overline{Z}) = \frac{1}{n+1} \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} \{\hbar(\overline{X}, \overline{Y})(\nabla_{\beta\overline{W}} C)(\overline{Z})\}.$$

From which, by setting $\overline{W} = \overline{\eta}$ and noting that $P(\overline{X}, \overline{Y})\overline{\eta} = (\nabla_{\beta\overline{\eta}} T)(\overline{X}, \overline{Y})$, the result follows.

(b) Since (M,L) is a P-reducible manifold, then by Definition 2.11, taking into account the fact that g is nondegenerate, we obtain

$$P(\overline{X},\overline{Y})\overline{\eta}=\delta(\overline{X})\phi(\overline{Y})+\delta(\overline{Y})\phi(\overline{X})+\overline{\hbar}(\overline{X},\overline{Y})\,\overline{\zeta}, \tag{3.2}$$

where $\overline{\zeta}$ is the π-vector field defined by $g(\overline{\zeta},\overline{X}):=\delta(\overline{X})$.
Since $\delta(\overline{\eta})=0$, then $Tr\{\overline{Y}\longmapsto\delta(\overline{Y})\phi(\overline{X})+\hbar(\overline{X},\overline{Y})\,\overline{\zeta}\}=2\delta(\overline{X})$. Taking the trace of both sides of (3.2), using the fact that $P(\overline{X},\overline{Y})\overline{\eta}=(\nabla_{\beta\overline{\eta}}T)(\overline{X},\overline{Y})$ (Lemma 3.1) and that $Tr\{\overline{Y}\longmapsto(\nabla_{\beta\overline{\eta}}T)(\overline{X},\overline{Y})\}=(\nabla_{\beta\overline{\eta}}C)(\overline{X})$, we get

$$\delta(\overline{X})=\frac{1}{n+1}(\nabla_{\beta\overline{\eta}}C)(\overline{X}). \tag{3.3}$$

Now, from Equations (3.2) and (3.3), we have

$$g(P(\overline{X},\overline{Y})\overline{\eta},\overline{Z})=\frac{1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X},\overline{Y})(\nabla_{\beta\overline{\eta}}C)(\overline{Z})\}. \tag{3.4}$$

According to the given assumption that the manifold is general Landsberg, then $\nabla_{\beta\overline{\eta}}C=0$. Therefore, from (3.4), we get $P(\overline{X},\overline{Y})\overline{\eta}=0$ and hence the manifold is Landsberg. $\Box$

Proposition 3.10.

(a) *A C^h-recurrent manifold is a P^*-Finsler manifold.*

(b) *A general Landsberg P^*-Finsler manifold is a Landsberg manifold.*

Proof. The proof is straightforward and we omit it. $\Box$

Proposition 3.11. *A C_2-like Finsler manifold is a Berwald manifold if, and only if, the π-tensor field C is horizontally parallel.*

Proof. Let (M,L) be C_2-like. Then, $T(\overline{X},\overline{Y},\overline{Z})=\frac{1}{C(\overline{C})}C(\overline{X})C(\overline{Y})C(\overline{Z})$, from which $T(\overline{X},\overline{Y})=\frac{1}{C(\overline{C})}C(\overline{X})C(\overline{Y})\overline{C}$. Taking the h-covariant derivative of both sides, we get

$$\begin{aligned}(\nabla_{\beta\overline{Z}}T)(\overline{X},\overline{Y})=&\frac{-\nabla_{\beta\overline{Z}}C(\overline{C})}{C^4}C(\overline{X})C(\overline{Y})\overline{C}+\frac{1}{C(\overline{C})}(\nabla_{\beta\overline{Z}}C)(\overline{X})C(\overline{Y})\overline{C}+\\&+\frac{1}{C(\overline{C})}(\nabla_{\beta\overline{Z}}C)(\overline{Y})C(\overline{X})\overline{C}+\frac{1}{C(\overline{C})}C(\overline{X})C(\overline{Y})\nabla_{\beta\overline{Z}}\overline{C}.\end{aligned}$$

In view of this relation, $\nabla_{\beta\overline{Z}}T=0$ if, and only if, $\nabla_{\beta\overline{Z}}C=0$. Hence the result. $\Box$

Corollary 3.12. *A C_2-like general Landsberg manifold is a Landsberg manifold.*

In view of the above Theorems, we have:

Corollary 3.13. *The two notions of being Landsberg and general Landsberg coincide in the case of C-reducibility, P-reducibility, C_2-likeness or P^*-Finsler.*

As we know, a C-reducible Landsberg manifold is a Berwald manifold (Proposition 3.4). Moreover, A C_2-like Finsler manifold is a Berwald manifold if, and only if, the π-tensor field C is horizontally parallel (Proposition 3.11). We shall try to generalize these results to the case of semi-C-reduciblity.

Theorem 3.14. *A semi-C-reducible Finsler manifold is a Berwald manifold if, and only if, the characteristic scalar μ and the π-tensor field C are horizontally parallel.*

Proof. Firstly, if (M, L) is semi-C-reducible, then

$$T(\overline{X},\overline{Y},\overline{Z})=\frac{\mu}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X},\overline{Y})C(\overline{Z})\}+\frac{\tau}{C(\overline{C})}C(\overline{X})C(\overline{Y})C(\overline{Z}).$$

Taking the h-covariant derivative of both sides, noting that $\nabla_{\beta\overline{X}}\hbar=0$, we get

$$\begin{aligned}(\nabla_{\beta\overline{W}}T)(\overline{X},\overline{Y},\overline{Z}) = {} & \frac{1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X},\overline{Y})\{\mu(\nabla_{\beta\overline{W}}C)(\overline{Z})+(\nabla_{\beta\overline{W}}\mu)C(\overline{Z})\}\}+\\ & +\frac{\tau}{C^2}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{(\nabla_{\beta\overline{W}}C)(\overline{X})C(\overline{Y})C(\overline{Z})\}-\\ & -\{\frac{\nabla_{\beta\overline{W}}\,\mu}{C^2}+\frac{\tau\,\nabla_{\beta\overline{W}}C(\overline{C})}{C^4}\}C(\overline{X})C(\overline{Y})C(\overline{Z}).\end{aligned}$$

Now, if the characteristic scalar μ and the π-tensor field C are horizontally parallel, then $\nabla_{\beta\overline{W}}T=0$ and (M, L) is a Berwald manifold.

Conversely, if (M, L) is a Berwald manifold, then $\nabla_{\beta\overline{X}}T=0$ and hence $\nabla_{\beta\overline{X}}C=0$, $\nabla_{\beta\overline{X}}\overline{C}=0$. These, together with the above equation, give

$$\nabla_{\beta\overline{W}}\mu\{\frac{1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X},\overline{Y})C(\overline{Z})\}-\frac{1}{C^2}C(\overline{X})C(\overline{Y})C(\overline{Z})\}=0,$$

which implies immediately that $\nabla_{\beta\overline{W}}\mu=0$. $\Box$

The following lemmas are useful for subsequent use

Lemma 3.15. *For all $\overline{X},\overline{Y}\in\mathfrak{X}(\pi(M))$, we have:*

(a) $[\gamma\overline{X},\gamma\overline{Y}]=\gamma(\nabla_{\gamma\overline{X}}\overline{Y}-\nabla_{\gamma\overline{Y}}\overline{X})$

(b) $[\gamma\overline{X},\beta\overline{Y}]=-\gamma(P(\overline{Y},\overline{X})\overline{\eta}+\nabla_{\beta\overline{Y}}\overline{X})+\beta(\nabla_{\gamma\overline{X}}\overline{Y}-T(\overline{X},\overline{Y}))$

(c) $[\beta\overline{X},\beta\overline{Y}]=\gamma(R(\overline{X},\overline{Y})\overline{\eta})+\beta(\nabla_{\beta\overline{X}}\overline{Y}-\nabla_{\beta\overline{Y}}\overline{X})$

Lemma 3.16. *For all $\overline{X},\overline{Y},\overline{Z},\overline{W}\in\mathfrak{X}(\pi(M))$ and $W\in\mathfrak{X}(\mathcal{T}M)$, we have:*

(a) $g((\nabla_W T)(\overline{X},\overline{Y}),\overline{Z})=g((\nabla_W T)(\overline{X},\overline{Z}),\overline{Y})$,

(b) $g(S(\overline{X},\overline{Y})\overline{Z},\overline{W})=-g(S(\overline{X},\overline{Y})\overline{W},\overline{Z})$.

Proof.

(a) From the definition of the covariant derivative, we get

$$\begin{aligned}g((\nabla_W T)(\overline{X},\overline{Y}),\overline{Z}) = {} & g(\nabla_W T(\overline{X},\overline{Y}),\overline{Z})-g(T(\nabla_W\overline{X},\overline{Y}),\overline{Z})-\\ & -g(T(\overline{X},\nabla_W\overline{Y}),\overline{Z}).\end{aligned}\tag{3.5}$$

Now, we have

$$\begin{aligned}g(\nabla_W T(\overline{X},\overline{Y}),\overline{Z}) & = W\cdot g(T(\overline{X},\overline{Y}),\overline{Z})-g(T(\overline{X},\overline{Y}),\nabla_W\overline{Z})\\ & = W\cdot g(T(\overline{X},\overline{Y}),\overline{Z})-g(T(\overline{X},\nabla_W\overline{Z}),\overline{Y}),\end{aligned}$$

Similarly,

$$g(T(\overline{X},\nabla_W\overline{Y}),\overline{Z})=W\cdot g(T(\overline{X},\overline{Z}),\overline{Y})-g(\nabla_W T(\overline{X},\overline{Z}),\overline{Y}).$$

Substituting these two equations into (3.5), noting the property that $g(T(\nabla_W\overline{X},\overline{Y}),\overline{Z})$ $= g(T(\nabla_W\overline{X},\overline{Z}),\overline{Y})$ (cf. §1), the result follows.

(b) follows directly from the general formula (which can be easily proved)

$$g(\mathbf{K}(X,Y)\overline{Z},\overline{W}) + g(\mathbf{K}(X,Y)\overline{W},\overline{Z}) = 0$$

by setting $X = \gamma\overline{X}$ and $Y = \gamma\overline{Y}$, where $\mathbf{K}$ is the classical curvature tensor of the Cartan connection as a linear connection in the pull-back bundle (cf. §1). $\square$

Proposition 3.17. *Let (M,L) be a C^h-recurrent Finsler manifold ($\nabla_{\beta\overline{X}}T = \lambda_0(\overline{X})T$). Then, we have*:

(a) *If $K_o := \lambda_o(\overline{\eta}) = 0$, then the hv-curvature tensor P is expressed in the form*:
$P(\overline{X},\overline{Y},\overline{Z},\overline{W}) = \lambda_o(\overline{Z})T(\overline{X},\overline{Y},\overline{W}) - \lambda_o(\overline{W})T(\overline{X},\overline{Y},\overline{Z})$
and the $(v)hv$-torsion $\widehat{P}$ vanishes.

(b) *If $K_o \neq 0$, then the $v(hv)$-torsion tensor $\widehat{P}$ is recurrent*:
$(\nabla_{\beta\overline{Z}}\widehat{P})(\overline{X},\overline{Y}) = (\lambda_o(\overline{Z}) + \frac{\nabla_{\beta\overline{Z}}K_o}{K_o})\widehat{P}(\overline{X},\overline{Y}).$

Proof.

(a) The hv-curvature tensor P can be written in the form [25]:

$$\begin{aligned} P(\overline{X},\overline{Y},\overline{Z},\overline{W}) = {} & g((\nabla_{\beta\overline{Z}}T)(\overline{X},\overline{Y}),\overline{W}) - g((\nabla_{\beta\overline{W}}T)(\overline{X},\overline{Y}),\overline{Z})+ \\ & +g(T(\overline{X},\overline{Z}),\widehat{P}(\overline{W},\overline{Y})) - g(T(\overline{X},\overline{W}),\widehat{P}(\overline{Z},\overline{Y})). \end{aligned}$$

Then, by using $\widehat{P}(\overline{X},\overline{Y}) = (\nabla_{\beta\overline{\eta}}T)(\overline{X},\overline{Y})$ (Lemma 3.1) and the C^h-recurrence condition, we get

$$\begin{aligned} P(\overline{X},\overline{Y},\overline{Z},\overline{W}) &= \lambda_o(\overline{Z})T(\overline{X},\overline{Y},\overline{W}) - \lambda_o(\overline{W})T(\overline{X},\overline{Y},\overline{Z}) - \\ &\quad -\lambda_o(\overline{\eta})\{g(T(\overline{X},\overline{W}),T(\overline{Y},\overline{Z})) - g(T(\overline{X},\overline{Z}),T(\overline{Y},\overline{W}))\} \\ &= \lambda_o(\overline{Z})T(\overline{X},\overline{Y},\overline{W}) - \lambda_o(\overline{W})T(\overline{X},\overline{Y},\overline{Z}) - \lambda_o(\overline{\eta})S(\overline{X},\overline{Y},\overline{Z},\overline{W}). \end{aligned}$$

Now, if $\lambda_o(\overline{\eta}) = 0$, then (a) follows from the above relation.

(b) If $K_o := \lambda_o(\overline{\eta}) \neq 0$, then by Lemma 3.1 and the recurrence condition, we have

$$\widehat{P}(\overline{X},\overline{Y}) = K_oT(\overline{X},\overline{Y}),$$

from which

$$(\nabla_{\beta\overline{Z}}\widehat{P})(\overline{X},\overline{Y}) = \{\nabla_{\beta\overline{Z}}K_o + K_o\lambda_o(\overline{Z})\}T(\overline{X},\overline{Y}).$$

Then, (b) follows from the above two equations. $\square$

Theorem 3.18. *Assume that (M,L) is C^h-recurrent. Then, the v-curvature tensor S is recurrent with respect to the h-covariant differentiation : $\nabla_{\beta\overline{X}}S = \theta(\overline{X})S$, where θ is a π-form of order one.*

Proof. One can easily show that : For all $X,Y,Z \in \mathfrak{X}(\mathcal{T}M)$,

$$\mathfrak{S}_{X,Y,Z}\{\mathbf{K}(X,Y)\rho Z + \nabla_X\mathbf{T}(Y,Z) + \mathbf{T}(X,[Y,Z])\} = 0.$$

Setting $X = \gamma\overline{X},\ Y = \gamma\overline{Y}$ and $Z = \beta\overline{Z}$ in the above equation, we get

$$\begin{aligned} S(\overline{X},\overline{Y})\overline{Z} = {} & \nabla_{\gamma\overline{Y}}T(\overline{X},\overline{Z}) - \nabla_{\gamma\overline{X}}T(\overline{Y},\overline{Z}) - \nabla_{\beta\overline{Z}}\mathbf{T}(\gamma\overline{X},\gamma\overline{Z}) - \\ & -\mathbf{T}(\gamma\overline{X},[\gamma\overline{Y},\beta\overline{Z}]) + \mathbf{T}(\gamma\overline{Y},[\gamma\overline{X},\beta\overline{Z}]) + \mathbf{T}([\gamma\overline{X},\gamma\overline{Y}],\beta\overline{Z}). \end{aligned}$$

Using Lemma 3.15 and the fact that $\mathbf{T}(\gamma\overline{X}, \gamma\overline{Z}) = 0$, the above equation reduces to

$$\begin{aligned} S(\overline{X},\overline{Y})\overline{Z} = {} & (\nabla_{\gamma\overline{Y}}T)(\overline{X},\overline{Z}) - (\nabla_{\gamma\overline{X}}T)(\overline{Y},\overline{Z}) + \\ & + T(\overline{X}, T(\overline{Y},\overline{Z})) - T(\overline{Y}, T(\overline{X},\overline{Z})). \end{aligned} \tag{3.6}$$

From which, since $g(T(\overline{X},\overline{Y}),\overline{Z}) = g(T(\overline{X},\overline{Z}),\overline{Y})$, we have

$$\begin{aligned} g(S(\overline{X},\overline{Y})\overline{Z},\overline{W}) = {} & g((\nabla_{\gamma\overline{Y}}T)(\overline{X},\overline{Z}),\overline{W}) - g((\nabla_{\gamma\overline{X}}T)(\overline{Y},\overline{Z}),\overline{W}) + \\ & + g(T(\overline{X},\overline{W}), T(\overline{Y},\overline{Z})) - g(T(\overline{Y},\overline{W}), T(\overline{X},\overline{Z})). \end{aligned}$$

Similarly,

$$\begin{aligned} g(S(\overline{X},\overline{Y})\overline{W},\overline{Z}) = {} & g((\nabla_{\gamma\overline{Y}}T)(\overline{X},\overline{W}),\overline{Z}) - g((\nabla_{\gamma\overline{X}}T)(\overline{Y},\overline{W}),\overline{Z}) + \\ & + g(T(\overline{X},\overline{Z}), T(\overline{Y},\overline{W})) - g(T(\overline{Y},\overline{Z}), T(\overline{X},\overline{W})). \end{aligned}$$

The above two equations, together with Lemma 3.16, yield

$$g((\nabla_{\gamma\overline{X}}T)(\overline{Y},\overline{Z}),\overline{W}) = g((\nabla_{\gamma\overline{Y}}T)(\overline{X},\overline{Z}),\overline{W}). \tag{3.7}$$

By (3.6) and (3.7), we obtain

$$S(\overline{X},\overline{Y},\overline{Z},\overline{W}) = g(T(\overline{X},\overline{W}), T(\overline{Y},\overline{Z})) - g(T(\overline{Y},\overline{W}), T(\overline{X},\overline{Z})). \tag{3.8}$$

Now, using the given assumption that the manifold is C^h-recurrent, Equation (3.8) implies that

$$\begin{aligned} (\nabla_{\beta\overline{X}}S)(\overline{Y},\overline{Z},\overline{V},\overline{W}) = {} & \nabla_{\beta\overline{X}}S(\overline{Y},\overline{Z},\overline{V},\overline{W}) - \\ & - S(\nabla_{\beta\overline{X}}\overline{Y},\overline{Z},\overline{V},\overline{W}) - S(\overline{Y},\nabla_{\beta\overline{X}}\overline{Z},\overline{V},\overline{W}) - \\ & - S(\overline{Y},\overline{Z},\nabla_{\beta\overline{X}}\overline{V},\overline{W}) - S(\overline{Y},\overline{Z},\overline{V},\nabla_{\beta\overline{X}}\overline{W}). \\ = {} & + \nabla_{\beta\overline{X}}g(T(\overline{Y},\overline{W}), T(\overline{Z},\overline{V})) - \nabla_{\beta\overline{X}}g(T(\overline{Z},\overline{W}), T(\overline{Y},\overline{V})) - \\ & - g(T(\nabla_{\beta\overline{X}}\overline{Y},\overline{W}), T(\overline{Z},\overline{V})) + g(T(\overline{Z},\overline{W}), T(\nabla_{\beta\overline{X}}\overline{Y},\overline{V})) - \\ & - g(T(\overline{Y},\overline{W}), T(\nabla_{\beta\overline{X}}\overline{Z},\overline{V})) + g(T(\nabla_{\beta\overline{X}}\overline{Z},\overline{W}), T(\overline{Y},\overline{V})) - \\ & - g(T(\overline{Y},\overline{W}), T(\overline{Z},\nabla_{\beta\overline{X}}\overline{V})) + g(T(\overline{Z},\overline{W}), T(\overline{Y},\nabla_{\beta\overline{X}}\overline{V})) - \\ & - g(T(\overline{Y},\nabla_{\beta\overline{X}}\overline{W}), T(\overline{Z},\overline{V})) + g(T(\overline{Z},\nabla_{\beta\overline{X}}\overline{W}), T(\overline{Y},\overline{V})). \\ = {} & g((\nabla_{\beta\overline{X}}T)(\overline{Y},\overline{W}), T(\overline{Z},\overline{V})) + g(T(\overline{Y},\overline{W}), (\nabla_{\beta\overline{X}}T)(\overline{Z},\overline{V})) - \\ & - g((\nabla_{\beta\overline{X}}T)(\overline{Z},\overline{W}), T(\overline{Y},\overline{V})) - g(T(\overline{Z},\overline{W}), (\nabla_{\beta\overline{X}}T)(\overline{Y},\overline{V})). \\ = {} & 2\lambda_o(\overline{X})S(\overline{Y},\overline{Z},\overline{V},\overline{W}) =: \theta(\overline{X})S(\overline{Y},\overline{Z},\overline{V},\overline{W}). \end{aligned}$$

Hence, the result follows. $\square$

Corollary 3.19. *In the course of the proof of Theorem* 3.18*, we have shown that* (Equations (3.7) and (3.8))*:*

(a) $(\nabla_{\gamma\overline{X}}T)(\overline{Y},\overline{Z}) = (\nabla_{\gamma\overline{Y}}T)(\overline{X},\overline{Z})$*,*

(b) $S(\overline{X},\overline{Y},\overline{Z},\overline{W}) = g(T(\overline{X},\overline{W}), T(\overline{Y},\overline{Z})) - g(T(\overline{Y},\overline{W}), T(\overline{X},\overline{Z}))$*.*

Corollary 3.20. *Let (M, L) be a C_2-like Finsler manifold. Then the the v-curvature tensor S vanishes.*

Proof. Substituting $T(\overline{X},\overline{Y}) = \frac{1}{C(\overline{C})}C(\overline{X})C(\overline{Y})\overline{C}$ in Corollary 3.19(b), we get the result. $\square$

Corollary 3.21. *Let (M, L) be a C-reducible manifold. Then,*

(a) *the v-curvature tensor S has the form*

$$S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \frac{1}{(n+1)^2}\{C^2\hbar(\overline{X}, \overline{W})\hbar(\overline{Y}, \overline{Z}) - C^2\hbar(\overline{Y}, \overline{W})\hbar(\overline{X}, \overline{Z}) + \\ +\hbar(\overline{X}, \overline{W})C(\overline{Y})C(\overline{Z}) + \hbar(\overline{Y}, \overline{Z})C(\overline{X})C(\overline{W}) - \\ -\hbar(\overline{Y}, \overline{W})C(\overline{X})C(\overline{Z}) - \hbar(\overline{X}, \overline{Z})C(\overline{Y})C(\overline{W})\}.$$

(b) *the vertical Ricc tensor Ric^v has the form*

$$Ric^v(\overline{X}, \overline{Y}) = \frac{(3-n)}{(n+1)^2}C(\overline{X})C(\overline{Y}) - \frac{(n-1)}{(n+1)^2}\,C^2\hbar(\overline{X}, \overline{Y}).$$

(c) *the vertical scalar curvature Sc^v has the form*

$$Sc^v = \frac{(2-n)}{(n+1)}\,C^2.$$

Theorem 3.22. *A Finsler manifold (M, L) is P-Symmetric if, and only if, the v-curvature tensor S satisfies the equation $\nabla_{\beta\overline{\eta}}S = 0$.*

Proof. One can show that: For all $X, Y, Z \in \mathfrak{X}(\mathcal{T}M)$,

$$\mathfrak{S}_{X,Y,Z}\{\nabla_Z\mathbf{K}(X, Y) - \mathbf{K}(X, Y)\nabla_Z - \mathbf{K}([X, Y], Z)\} = 0. \tag{3.9}$$

Setting $X = \gamma\overline{X}, Y = \gamma\overline{Y}$ and $Z = \beta\overline{Z}$ in the above equation, we get

$$\nabla_{\beta\overline{Z}}S(\overline{X}, \overline{Y})\overline{W} + \nabla_{\gamma\overline{Y}}P(\overline{Z}, \overline{X})\overline{W} - \nabla_{\gamma\overline{X}}P(\overline{Z}, \overline{Y})\overline{W} - \\ -S(\overline{X}, \overline{Y})\nabla_{\beta\overline{Z}}\overline{W} + P(\overline{Z}, \overline{Y})\nabla_{\gamma\overline{X}}\overline{W} - P(\overline{Z}, \overline{X})\nabla_{\gamma\overline{Y}}\overline{W} - \\ -\mathbf{K}([\gamma\overline{X}, \gamma\overline{Y}], \beta\overline{Z})\overline{W} - \mathbf{K}([\gamma\overline{Y}, \beta\overline{Z}], \gamma\overline{X})\overline{W} - \mathbf{K}([\beta\overline{Z}, \gamma\overline{X}], \gamma\overline{Y})\overline{W} = 0.$$

By using Lemma 3.15, the above relation reduces to

$$(\nabla_{\beta\overline{Z}}S)(\overline{X}, \overline{Y}, \overline{W}) + (\nabla_{\gamma\overline{Y}}P)(\overline{Z}, \overline{X}, \overline{W}) - (\nabla_{\gamma\overline{X}}P)(\overline{Z}, \overline{Y}, \overline{W}) + \\ +S(P(\overline{Z}, \overline{Y})\overline{\eta}, \overline{X})\overline{W} - S(P(\overline{Z}, \overline{X})\overline{\eta}, \overline{Y})\overline{W} + \\ +P(T(\overline{Y}, \overline{Z}), \overline{X})\overline{W} - P(T(\overline{X}, \overline{Z}), \overline{Y})\overline{W} = 0. \tag{3.10}$$

Setting $\overline{Z} = \overline{\eta}$ in the above equation, taking into account Lemma 3.1 and the fact that $T(\overline{X}, \overline{\eta}) = 0$ and that $(\nabla_{\gamma\overline{X}}P)(\overline{\eta}, \overline{Y}, \overline{Z}) = -P(\overline{X}, \overline{Y})\overline{Z}$, we get

$$P(\overline{X}, \overline{Y})\overline{Z} = P(\overline{Y}, \overline{X})\overline{Z} - (\nabla_{\beta\overline{\eta}}S)(\overline{X}, \overline{Y}, \overline{Z}). \tag{3.11}$$

The result follows immediately from (3.11). $\square$

According to (3.11) and Lemma 3.1, we have:

Corollary 3.23. *Let $\widehat{P}(\overline{X}, \overline{Y}) := P(\overline{X}, \overline{Y})\overline{\eta}$ and $\widehat{T}(\overline{X}, \overline{Y}) := (\nabla_{\beta\overline{\eta}}T)(\overline{X}, \overline{Y})$. Then the π-tensor fields $\widehat{P}$ and $\widehat{T}$ are symmetric.*

Theorem 3.18 and Theorem 3.22 give rise the following result.

Theorem 3.24. *Assume that a Finsler manifold (M, L) is C^h-recurrent and P-symmetric. If $\theta(\overline{\eta}) \neq 0$, then the v-curvature tensor S vanishes identically.*

Now, we shall prove the following lemma which provides some important and useful properties of the torsion tensor T and the v-curvature S:

Lemma 3.25. *For every $\overline{X}, \overline{Y}, \overline{Z}$ and $\overline{W} \in \mathfrak{X}(\pi(M))$, we have*

(a) $T(\overline{X}, \overline{Y}) = T(\overline{Y}, \overline{X})$,

(b) $T(\overline{\eta}, \overline{X}) = 0$,

(c) $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} S(\overline{X}, \overline{Y})\overline{Z} = 0$,

(d) $g(S(\overline{X}, \overline{Y})\overline{Z}, \overline{W}) = g(S(\overline{Z}, \overline{W})\overline{X}, \overline{Y})$,

(e) $S(\overline{\eta}, \overline{X})\overline{Y} = 0 = S(\overline{X}, \overline{\eta})\overline{Y}$,

(f) $(\nabla_{\gamma\overline{X}} S)(\overline{\eta}, \overline{Y})\overline{Z} = -S(\overline{X}, \overline{Y})\overline{Z}$, $(\nabla_{\gamma\overline{X}} S)(\overline{\eta}, \overline{X})\overline{\eta} = 0$.

(g) $S(\overline{X}, \overline{Y})\overline{Z} = -\frac{1}{2}\{(D_{\gamma\overline{X}} T)(\overline{Y}, \overline{Z}) - (D_{\gamma\overline{Y}} T)(\overline{X}, \overline{Z})\}$.
Consequently, S vanishes if and only if $(D_{\gamma\overline{X}} T)(\overline{Y}, \overline{Z}) = (D_{\gamma\overline{Y}} T)(\overline{X}, \overline{Z})$.

Proof.
(a) From Corollary 3.19(a), we have

$$(\nabla_{\gamma\overline{X}} T)(\overline{Y}, \overline{Z}) = (\nabla_{\gamma\overline{Y}} T)(\overline{X}, \overline{Z}).$$

Setting $\overline{Z} = \overline{\eta}$ and using the fact that $T(\overline{X}, \overline{\eta}) = 0$ and that $K\, o\, \gamma = id_{\mathfrak{X}(\pi(M))}$, the result follows.

(b) Follows from (a) together with the relation $T(\overline{X}, \overline{\eta}) = 0$.

(c) Setting $X = \gamma\overline{X}$, $Y = \gamma\overline{Y}$ and $Z = \gamma\overline{Z}$ in (3.9) and using Lemma 3.15, we get

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} (\nabla_{\gamma\overline{X}} S)(\overline{Y}, \overline{Z}, \overline{W}) = 0.$$

Again, setting $\overline{W} = \overline{\eta}$ in the above equation and using the fact that $S(\overline{X}, \overline{Y})\overline{\eta} = 0$ and that $K\, o\, \gamma = id_{\mathfrak{X}(\pi(M))}$, the result follows.

(d) Follows from Corollary 3.19(b), noting that T is symmetric.

(e) and **(f)** are clear.

(g) From the relation $D_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y} - T(\overline{X}, \overline{Y})$ [27], we get

$$(D_{\gamma\overline{X}} T)(\overline{Y}, \overline{Z}) = (\nabla_{\gamma\overline{X}} T)(\overline{Y}, \overline{Z}) - T(\overline{X}, T(\overline{Y}, \overline{Z})) + T(T(\overline{X}, \overline{Y}), \overline{Z}) + T(\overline{Y}, T(\overline{X}, \overline{Z})),$$

$$(D_{\gamma\overline{Y}} T)(\overline{X}, \overline{Z}) = (\nabla_{\gamma\overline{Y}} T)(\overline{X}, \overline{Z}) - T(\overline{Y}, T(\overline{X}, \overline{Z})) + T(T(\overline{Y}, \overline{X}), \overline{Z}) + T(\overline{X}, T(\overline{Y}, \overline{Z})).$$

The result follows from the above two equations, using Corollary 3.19 and the symmetry of T. $\square$

As a direct consequence of the above lemma, we have the

Corollary 3.26. *A P_2-like Finsler manifold is P-symmetric.*

Proposition 3.27. *Assume that (M, L) is C^v-recurrent. Then, the v-curvature tensor S is v-recurrent: $\nabla_{\gamma\overline{X}} S = \Psi(\overline{X})S$, Ψ being a $(1)\pi$-form. Consequently, S vanishes identically.*

Proof. Taking the v-covariant derivative of both sides of the relation in Corollary 3.19(b) and, then, using the assumption that $\nabla_{\gamma\overline{X}}T = \lambda_0(\overline{X})T$, we get

$$(\nabla_{\gamma\overline{X}}S)(\overline{Y},\overline{Z},\overline{V},\overline{W}) = 2\lambda_o(\overline{X})S(\overline{Y},\overline{Z},\overline{V},\overline{W}) =: \psi(\overline{X})S(\overline{Y},\overline{Z},\overline{V},\overline{W}),$$

which shows that S is v-recurrent.

Now, setting $\overline{V} = \overline{\eta}$ in the last equation, using the properties of S and noting that $K\,o\,\gamma = id_{\mathfrak{X}(\pi(M))}$, we conclude that $S = 0$. $\square$

The following result gives a characterization of Riemannian manifolds in terms of C^v-recurrence and C^0-recurrence.

Theorem 3.28.

(a) *A C^v-recurrent Finsler manifold is Riemannian,*

(b) *A C^0-recurrent Finsler manifold is Riemannian.*

Proof. (a) Since (M, L) is C^v-recurrent, then $(\nabla_{\gamma\overline{X}}T)(\overline{Y},\overline{Z}) = \lambda_o(\overline{X})T(\overline{Y},\overline{Z})$, from which, by setting $\overline{X} = \overline{\eta}$ and noting that $\nabla_{\gamma\overline{\eta}}T = -T$, we get

$$T(\overline{Y},\overline{Z}) = -\lambda_o(\overline{\eta})T(\overline{Y},\overline{Z}). \tag{3.12}$$

But since $(\nabla_{\gamma\overline{X}}T)(\overline{Y},\overline{Z}) = (\nabla_{\gamma\overline{Y}}T)(\overline{X},\overline{Z})$ (Corollary 3.19), then $\lambda_o(\overline{X})T(\overline{Y},\overline{Z}) = \lambda_o(\overline{Y})T(\overline{X},\overline{Z})$. Hence,

$$\lambda_o(\overline{\eta})T(\overline{Y},\overline{Z}) = 0. \tag{3.13}$$

Then, the result follows from (3.12) and (3.13).

(b) can be proved similarly. $\square$

Theorem 3.29. *For a Finsler manifold (M, L), the following assertions are equivalent*:

(a) *(M, L) is S^v-recurrent.*

(b) *The v-curvature tensor S vanishes identically.*

(c) *(M, L) is S^v-recurrent of the second order.*

Proof.

(a) $\Longrightarrow$ (b): If (M, L) is S^v-recurrent, then by Definition 2.7(a) we have

$$(\nabla_{\gamma\overline{W}}S)(\overline{X},\overline{Y},\overline{Z}) = \lambda(\overline{W})S(\overline{Y},\overline{X})\overline{Z},$$

from which, by setting $\overline{Z} = \overline{\eta}$, taking into account the fact that $S(\overline{X},\overline{Y})\overline{\eta} = 0$ and that $Ko\gamma = id_{\pi^{-1}(TM)}$, the result follows.

(b) $\Longrightarrow$ (a): Trivial.

(b) $\Longrightarrow$ (c): Trivial.

(c) $\Longrightarrow$ (b): If the given manifold (M, L) is S^v-recurrent of the second order, then by Definition 2.7(b) we get

$$\begin{aligned}
\Theta(\overline{X},\overline{Y})S(\overline{Z},\overline{V})\overline{W} &= (\overset{2}{\nabla}\overset{2}{\nabla}\, S)(\overline{Y},\overline{X},\overline{Z},\overline{V},\overline{W}) \\
&= \nabla_{\gamma\overline{Y}}(\nabla_{\gamma\overline{X}}S)(\overline{Z},\overline{V},\overline{W}) - (\nabla_{\gamma\nabla_{\gamma\overline{Y}}\overline{X}}S)(\overline{Z},\overline{V},\overline{W})- \\
&\quad -(\nabla_{\gamma\overline{X}}S)(\nabla_{\gamma\overline{Y}}\overline{Z},\overline{V},\overline{W}) - (\nabla_{\gamma\overline{X}}S)(\overline{Z},\nabla_{\gamma\overline{Y}}\overline{V},\overline{W})- \\
&\quad -(\nabla_{\gamma\overline{X}}S)(\overline{Z},\overline{V},\nabla_{\gamma\overline{Y}}\overline{W}).
\end{aligned}$$

By substituting $\overline{Z}=\overline{\eta}=\overline{W}$ in the above equation and using Lemma 3.25 and the fact that $S(\overline{X},\overline{Y})\overline{\eta}=0$, we get

$$S(\overline{X},\overline{Y})\overline{Z}=-S(\overline{Z},\overline{Y})\overline{X} \ \ and \ \ S(\overline{X},\overline{Y})\overline{Z}=-S(\overline{X},\overline{Z})\overline{Y}.$$

From this, together with the identity $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}S(\overline{X},\overline{Y})\overline{Z}=0$, the v-curvature tensor S vanishes identically. $\square$

In view of the above theorem we have:

Corollary 3.30.

(a) *An S^v-recurrent (resp. a second order S^v-recurrent) manifold (M,L) is S_3-like, provided that dim $M\geq 4$.*

(b) *An S^v-recurrent (resp. a second order S^v-recurrent) manifold (M,L) is S_4-like, provided that dim $M\geq 5$.*

Theorem 3.31. *If (M,L) is a P_2-like Finsler manifold, then the v-curvature tensor S vanishes or the hv-curvature tensor P vanishes. In the later case, the h-covariant derivative of S vanishes.*

Proof. As (M,L) is P_2-like, then $P(\overline{X},\overline{Y},\overline{\eta},\overline{W})=\alpha(\overline{\eta})T(\overline{X},\overline{Y},\overline{W})=:\alpha_oT(\overline{X},\overline{Y},\overline{W})$ and hence

$$\widehat{P}(\overline{X},\overline{Y})=\alpha_oT(\overline{X},\overline{Y}). \tag{3.14}$$

Now, setting $\overline{W}=\overline{\eta}$ into (3.10), we get

$$(\nabla_{\gamma\overline{Y}}\widehat{P})(\overline{Z},\overline{X})-(\nabla_{\gamma\overline{X}}\widehat{P})(\overline{Z},\overline{Y})-P(\overline{Z},\overline{X})\overline{Y}+P(\overline{Z},\overline{Y})\overline{X}-$$
$$-\widehat{P}(T(\overline{X},\overline{Z}),\overline{Y})+\widehat{P}(T(\overline{Y},\overline{Z}),\overline{X})=0.$$

Hence,

$$g((\nabla_{\gamma\overline{Y}}\widehat{P})(\overline{Z},\overline{X}),\overline{W})-g((\nabla_{\gamma\overline{X}}\widehat{P})(\overline{Z},\overline{Y}),\overline{W})-P(\overline{Z},\overline{X},\overline{Y},\overline{W})+$$
$$+P(\overline{Z},\overline{Y},\overline{X},\overline{W})-g(\widehat{P}(T(\overline{X},\overline{Z}),\overline{Y}),\overline{W})+g(\widehat{P}(T(\overline{Y},\overline{Z}),\overline{X}),\overline{W})=0.$$

From which, together with (3.14) and Definition 2.10, taking into account the relation $(\nabla_{\gamma\overline{Y}}\widehat{P})(\overline{Z},\overline{X})=(\nabla_{\gamma\overline{Y}}\alpha_o)T(\overline{Z},\overline{X})+\alpha_o(\nabla_{\gamma\overline{Y}}T)(\overline{Z},\overline{X})$, we obtain

$$g((\nabla_{\gamma\overline{Y}}\alpha_o)T(\overline{Z},\overline{X})+\alpha_o(\nabla_{\gamma\overline{Y}}T)(\overline{Z},\overline{X}),\overline{W})-g((\nabla_{\gamma\overline{X}}\alpha_o)T(\overline{Z},\overline{Y})+$$
$$+\alpha_o(\nabla_{\gamma\overline{X}}T)(\overline{Z},\overline{Y}),\overline{W})+\alpha(\overline{X})T(\overline{Z},\overline{Y},\overline{W})-\alpha(\overline{W})\,T(\overline{Z},\overline{Y},\overline{X})-\alpha(\overline{Y})T(\overline{Z},\overline{X},\overline{W})$$
$$+\alpha(\overline{W})\,T(\overline{X},\overline{Y},\overline{Z})-g(\alpha_oT(T(\overline{X},\overline{Z}),\overline{Y}),\overline{W})+g(\alpha_oT(T(\overline{Y},\overline{Z}),\overline{X}),\overline{W})=0.$$

Therefore, using Corollary 3.19,

$$(\nabla_{\gamma\overline{Y}}\alpha)(\overline{\eta})T(\overline{X},\overline{Z},\overline{W})-(\nabla_{\gamma\overline{X}}\alpha)(\overline{\eta})T(\overline{Y},\overline{Z},\overline{W})=\alpha_oS(\overline{X},\overline{Y},\overline{W},\overline{Z}).$$

It is to be observed that the left-hand side of the above equation is symmetric in the arguments $\overline{Z}$ and $\overline{W}$ while the right-hand side is skew-symmetric in the same arguments. Hence we have

$$\alpha_oS(\overline{X},\overline{Y},\overline{W},\overline{Z})=0, \tag{3.15}$$

$$\varepsilon(\overline{Y})T(\overline{X},\overline{Z},\overline{W})-\varepsilon(\overline{X})T(\overline{Y},\overline{Z},\overline{W})=0, \tag{3.16}$$

where ε is the π-form defined by $\varepsilon(\overline{Y}):=(\nabla_{\gamma\overline{Y}}\alpha)(\overline{\eta})$.

Now, If $\varepsilon \neq 0$, it follows from (3.16) that there exists a scalar function Υ such that $T(\overline{X},\overline{Y},\overline{Z}) = \Upsilon\,\varepsilon(\overline{X})\varepsilon(\overline{Y})\varepsilon(\overline{Z})$. Consequently, $T(\overline{X},\overline{Y}) = \Upsilon\,\varepsilon(\overline{X})\varepsilon(\overline{Y})\overline{\varepsilon}$, where $g(\overline{\varepsilon},\overline{X}) := \varepsilon(\overline{X})$. From which

$$\begin{aligned} S(\overline{X},\overline{Y},\overline{Z},\overline{W}) &= g(T(\overline{X},\overline{W}),T(\overline{Y},\overline{Z})) - g(T(\overline{Y},\overline{W}),T(\overline{X},\overline{Z})) \\ &= \Upsilon\,\varepsilon(\overline{X})\varepsilon(\overline{Y})\varepsilon(\overline{Z})\varepsilon(\overline{W})g(\overline{\varepsilon},\overline{\varepsilon}) - \Upsilon\,\varepsilon(\overline{X})\varepsilon(\overline{Y})\varepsilon(\overline{Z})\varepsilon(\overline{W})g(\overline{\varepsilon},\overline{\varepsilon}) = 0. \end{aligned}$$

On the other hand, if the v-curvature tensor $S \neq 0$, then it follows from (3.15) that $\varepsilon = 0$ and $\alpha(\overline{\eta}) = 0$. Hence, $\alpha = 0$ and the hv-curvature tensor P vanishes. In this case, it follows from the identity (3.10) that $\nabla_{\beta\overline{X}}S = 0$. $\square$

Proposition 3.32. *A P_2-like Finsler manifold (M,L) is a P^*-Finsler manifold.*

Proof. As (M,L) is P_2-like, then from (3.14), we have $\widehat{P}(\overline{X},\overline{Y}) = \alpha_o T(\overline{X},\overline{Y})$. Using Lemma 3.1, we get $(\nabla_{\beta\overline{\eta}}T)(\overline{X},\overline{Y}) = \alpha_0 T(\overline{X},\overline{Y})$, from which, by taking the trace, $\nabla_{\beta\overline{\eta}}C = \alpha_0 T$, where $\alpha_0 = \frac{\widehat{g}(\nabla_{\beta\overline{\eta}}C,C)}{C^2}$. Hence the result. $\square$

The next definition will be useful in the sequel.

Definition 3.33. *A π-tensor field Θ is positively homogenous of degree r in the directional argument y (symbolically, $h(r)$) if it satisfies the condition*

$$\nabla_{\gamma\overline{\eta}}\Theta = r\,\Theta, \quad or \quad D_{\gamma\overline{\eta}}\Theta = r\,\Theta.$$

Lemma 3.34. *Let (M,L) be a Finsler manifold, then we have*

(a) *The Finsler metric g (the angular metric tensor $\hbar$) is homogenous of degree* 0*,*

(b) *The v-curvature tensor S is homogenous of degree -2,*

(c) *The hv-curvature tensor P is homogenous of degree -1,*

(d) *The h-curvature tensor R is homogenous of degree* 0*,*

(e) *The $(h)hv$-torsion tensor T is homogenous of degree -1,*

(f) *The $(v)hv$-torsion tensor $\widehat{P}$ is homogenous of degree* 0*,*

(g) *The $(v)h$-torsion tensor $\widehat{R}$ is homogenous of degree* 1*.*

Lemma 3.35. *For every vector $(1)\pi$-form A, we have*

$$\begin{aligned} (\overset{1}{\nabla}\overset{1}{\nabla} A)(\overline{X},\overline{Y},\overline{Z}) - (\overset{1}{\nabla}\overset{1}{\nabla} A)(\overline{Y},\overline{X},\overline{Z}) = {} & A(R(\overline{X},\overline{Y})\overline{Z}) - R(\overline{X},\overline{Y})A(\overline{Z}) + \\ & + (\nabla_{\gamma\widehat{R}(\overline{X},\overline{Y})}A)(\overline{Z}). \end{aligned}$$

Deicke theorem [4] can be formulated globally as follows:

Lemma 3.36. *Let (M,L) be a Finsler manifold. The following assertions are equivalent:*

(a) *(M,L) is Riemannian,*

(b) *The $(h)hv$-torsion tensor T vanishes,*

(c) *The π-form C vanishes.*

Theorem 3.37. *Let (M,L) be Finsler manifold which is h-isotropic* (of scalar k_0) *and C^h-recurrent* (of recurrence vector λ_0). *Then, (M,L) is necessarily one of the following*:

(a) *A Riemannian manifold of constant curvature,*

(b) *A Finsler manifold of dimension 2,*

(c) *A Finsler manifold of dimensions $n \geq 3$ with vanishing scalar k_0 and $(\nabla_{\beta\overline{X}}\lambda_o)(\overline{Y}) = (\nabla_{\beta\overline{Y}}\lambda_o)(\overline{X})$.*

Proof. For a C^h-recurrent manifold, one can easily show that

$$\begin{aligned}&(\overset{1}{\nabla}\overset{1}{\nabla} T)(\overline{X},\overline{Y},\overline{Z},\overline{W}) - (\overset{1}{\nabla}\overset{1}{\nabla} T)(\overline{Y},\overline{X},\overline{Z},\overline{W}) = \\ &= \{(\nabla_{\beta\overline{X}}\lambda_o)(\overline{Y}) - (\nabla_{\beta\overline{Y}}\lambda_o)(\overline{X})\}T(\overline{Z},\overline{W}) =: \Psi(\overline{X},\overline{Y})T(\overline{Z},\overline{W}).\end{aligned}$$

From which, taking into account Lemma 3.35, we obtain

$$\begin{aligned}\Psi(\overline{X},\overline{Y})T(\overline{Z},\overline{W}) = {}& T(R(\overline{X},\overline{Y})\overline{Z},\overline{W}) + T(\overline{Z},R(\overline{X},\overline{Y})\overline{W}) - \\ &-R(\overline{X},\overline{Y})T(\overline{Z},\overline{W}) + (\nabla_{\gamma\hat{R}(\overline{X},\overline{Y})}T)(\overline{Z},\overline{W}).\end{aligned}$$

Now, as (M,L) is h-isotropic of scalar k_0, then the h-curvature tensor R has the form

$$R(\overline{X},\overline{Y})\overline{Z} = k_0\{g(\overline{X},\overline{Z})\overline{Y} - g(\overline{Y},\overline{Z})\overline{X}\}\,; \quad (n \geq 3).$$

From the above two equations, we get

$$\begin{aligned}\Psi(\overline{X},\overline{Y})T(\overline{Z},\overline{W}) = {}& k_0 g(\overline{X},\overline{Z})T(\overline{Y},\overline{W}) - k_0 g(\overline{Y},\overline{Z})T(\overline{X},\overline{W}) + k_0 g(\overline{X},\overline{W})T(\overline{Z},\overline{Y}) - \\ &-k_0 g(\overline{Y},\overline{W})T(\overline{Z},\overline{X}) - k_0 g(\overline{X},T(\overline{Z},\overline{W}))\overline{Y} + k_0 g(\overline{Y},T(\overline{Z},\overline{W}))\overline{X} \\ &+k_0 g(\overline{X},\overline{\eta})(\nabla_{\gamma\overline{Y}}T)(\overline{Z},\overline{W}) - k_0 g(\overline{Y},\overline{\eta})(\nabla_{\gamma\overline{X}}T)(\overline{Z},\overline{W}).\end{aligned} \tag{3.17}$$

Setting $\overline{Y} = \overline{\eta}$, noting that T is $h(-1)$ and $g(\overline{\eta},\overline{\eta}) = L^2$, we get

$$\begin{aligned}\Psi(\overline{X},\overline{\eta})T(\overline{Z},\overline{W}) = {}& -k_0 g(\overline{\eta},\overline{Z})T(\overline{X},\overline{W}) - k_0 g(\overline{\eta},\overline{W})T(\overline{Z},\overline{X}) - k_0 T(\overline{X},\overline{Z},\overline{W})\overline{\eta} - \\ &-k_0 g(\overline{X},\overline{\eta})T(\overline{Z},\overline{W}) - k_0 L^2(\nabla_{\gamma\overline{X}}T)(\overline{Z},\overline{W}).\end{aligned}$$

From which, we have

$$\begin{aligned}g(\overline{Y},\overline{\eta})\Psi(\overline{X},\overline{\eta})T(\overline{Z},\overline{W}) = {}& -k_0 g(\overline{Y},\overline{\eta})g(\overline{\eta},\overline{Z})T(\overline{X},\overline{W}) - k_0 g(\overline{Y},\overline{\eta})g(\overline{\eta},\overline{W})T(\overline{Z},\overline{X}) - \\ &-k_0 g(\overline{Y},\overline{\eta})T(\overline{X},\overline{Z},\overline{W})\overline{\eta} - k_0 g(\overline{Y},\overline{\eta})g(\overline{X},\overline{\eta})T(\overline{Z},\overline{W}) - \\ &-k_0 L^2 g(\overline{Y},\overline{\eta})(\nabla_{\gamma\overline{X}}T)(\overline{Z},\overline{W}),\end{aligned} \tag{3.18}$$

whereas

$$\begin{aligned}g(\overline{X},\overline{\eta})\Psi(\overline{Y},\overline{\eta})T(\overline{Z},\overline{W}) = {}& -k_0 g(\overline{X},\overline{\eta})g(\overline{\eta},\overline{Z})T(\overline{Y},\overline{W}) - k_0 g(\overline{X},\overline{\eta})g(\overline{\eta},\overline{W})T(\overline{Z},\overline{Y}) - \\ &-k_0 g(\overline{X},\overline{\eta})T(\overline{Y},\overline{Z},\overline{W})\overline{\eta} - k_0 g(\overline{X},\overline{\eta})g(\overline{Y},\overline{\eta})T(\overline{Z},\overline{W}) - \\ &-k_0 L^2 g(\overline{X},\overline{\eta})(\nabla_{\gamma\overline{Y}}T)(\overline{Z},\overline{W}).\end{aligned} \tag{3.19}$$

Now, from (3.17), (3.18) and (3.19), we obtain

$$\begin{aligned}&T(\overline{Z},\overline{W})\{L^2\Psi(\overline{X},\overline{Y}) - g(\overline{Y},\overline{\eta})\Psi(\overline{X},\overline{\eta}) + g(\overline{X},\overline{\eta})\Psi(\overline{Y},\overline{\eta})\} = \\ &= \mathfrak{U}_{\overline{X},\overline{Y}} k_0 L^2\{\hbar(\overline{X},\overline{Z})T(\overline{Y},\overline{W}) + \hbar(\overline{X},\overline{W})T(\overline{Y},\overline{Z}) - \phi(\overline{Y})\,T(\overline{X},\overline{Z},\overline{W})\}.\end{aligned}$$

Taking the trace of both sides of the above equation, we get

$$\begin{aligned}&C(\overline{Z})\{L^2\Psi(\overline{X},\overline{Y})-g(\overline{Y},\overline{\eta})\Psi(\overline{X},\overline{\eta})+g(\overline{X},\overline{\eta})\Psi(\overline{Y},\overline{\eta})\}=\\&=2k_0L^2\{\hbar(\overline{X},\overline{Z})C(\overline{Y})-\hbar(\overline{Y},\overline{Z})C(\overline{X})\}.\end{aligned}\tag{3.20}$$

Setting $\overline{Z}=\overline{C}$, taking into account the fact that $\hbar(\overline{X},\overline{C})=C(\overline{X})$, the above equation reduces to

$$C(\overline{C})\{L^2\Psi(\overline{X},\overline{Y})-g(\overline{Y},\overline{\eta})\Psi(\overline{X},\overline{\eta})+g(\overline{X},\overline{\eta})\Psi(\overline{Y},\overline{\eta})\}=0.$$

Now, if $C(\overline{C})=g(\overline{C},\overline{C})=0$, then $\overline{C}=0$ and so $C=0$. Consequently, by Lemma 3.36, (M,L) is a Riemannian manifold of constant curvature.

On the other hand, if (M,L) is not Riemannian, then we have

$$L^2\Psi(\overline{X},\overline{Y})-g(\overline{Y},\overline{\eta})\Psi(\overline{X},\overline{\eta})+g(\overline{X},\overline{\eta})\Psi(\overline{Y},\overline{\eta})=0.$$

From which, together with (3.20), we get

$$k_0\{\hbar(\overline{X},\overline{Z})C(\overline{Y})-\hbar(\overline{Y},\overline{Z})C(\overline{X})\}=0.\tag{3.21}$$

If $k_0\neq0$, then, by (3.21), $\hbar(\overline{X},\overline{Z})C(\overline{Y})=\hbar(\overline{Y},\overline{Z})C(\overline{X})$. Setting $\overline{Y}=\overline{C}$, we get $\hbar(\overline{X},\overline{Z})=\frac{1}{C^2}C(\overline{X})C(\overline{Z})$, which implies that $\dim M=2$.

If $k_0=0$, then $R=0$ and (3.17) yields $\Psi(\overline{X},\overline{Y})=0$, which means that $(\nabla_{\beta\overline{X}}\lambda_o)(\overline{Y})=(\nabla_{\beta\overline{Y}}\lambda_o)(\overline{X})$. $\square$

Now, we focus our attention to the interesting case (c) of the above theorem. In this case, the h-curvature tensor $R=0$ and hence the $(v)h$-torsion tensor $\widehat{R}=0$. Therefore, the equation (deduced from (3.9))

$$\begin{aligned}&(\nabla_{\gamma\overline{X}}R)(\overline{Y},\overline{Z},\overline{W})+(\nabla_{\beta\overline{Y}}P)(\overline{Z},\overline{X},\overline{W})-(\nabla_{\beta\overline{Z}}P)(\overline{Y},\overline{X},\overline{W})-\\&-P(\overline{Z},P(\overline{Y},\overline{X})\overline{\eta})\overline{W}+R(T(\overline{X},\overline{Y}),\overline{Z})\overline{W}-S(R(\overline{Y},\overline{Z})\overline{\eta},\overline{X})\overline{W}+\\&+P(\overline{Y},P(\overline{Z},\overline{X})\overline{\eta})\overline{W}-R(T(\overline{X},\overline{Z}),\overline{Y})\overline{W}=0.\end{aligned}$$

reduces to

$$\begin{aligned}&(\nabla_{\beta\overline{Y}}P)(\overline{Z},\overline{X},\overline{W})-(\nabla_{\beta\overline{Z}}P)(\overline{Y},\overline{X},\overline{W})-\\&-P(\overline{Z},\widehat{P}(\overline{Y},\overline{X}))\overline{W}+P(\overline{Y},\widehat{P}(\overline{Z},\overline{X}))\overline{W}=0.\end{aligned}$$

Setting $\overline{W}=\overline{\eta}$, we get

$$(\nabla_{\beta\overline{Y}}\widehat{P})(\overline{Z},\overline{X})-(\nabla_{\beta\overline{Z}}\widehat{P})(\overline{Y},\overline{X})-\widehat{P}(\overline{Z},\widehat{P}(\overline{Y},\overline{X}))+\widehat{P}(\overline{Y},\widehat{P}(\overline{Z},\overline{X}))=0.\tag{3.22}$$

Since (M,L) is C^h-recurrent, then, by Proposition 3.17, the $(v)hv$-torsion tensor $\widehat{P}$ satisfies the relations $(\nabla_{\beta\overline{Z}}\widehat{P})(\overline{X},\overline{Y})=(K_o\lambda_o(\overline{Z})+\nabla_{\beta\overline{Z}}K_o)T(\overline{X},\overline{Y})$ and $\widehat{P}(\overline{X},\overline{Y})=\lambda_o(\overline{\eta})T(\overline{X},\overline{Y})=K_oT(\overline{X},\overline{Y})$. From these, together with (3.22), we get

$$\begin{aligned}&(K_o\lambda_o(\overline{Y})+\nabla_{\beta\overline{Y}}K_o)T(\overline{Z},\overline{X})-(K_o\lambda_o(\overline{Z})+\nabla_{\beta\overline{Z}}K_o)T(\overline{X},\overline{Y})-\\&-K_o^2T(\overline{Z},T(\overline{X},\overline{Y}))+K_o^2T(\overline{Y},T(\overline{X},\overline{Z}))=0.\end{aligned}$$

Hence, by Corollary 3.19,

$$K_o^2S(\overline{Y},\overline{Z},\overline{X},\overline{W})=\mathfrak{U}_{\overline{Y},\overline{Z}}\{(K_o\lambda_o(\overline{Y})+\nabla_{\beta\overline{Y}}K_o)T(\overline{X},\overline{Z},\overline{W})\}.$$

As $S(\overline{Y},\overline{Z},\overline{X},\overline{W})$ is skew-symmetric in the arguments $\overline{X}$ and $\overline{W}$ while the right-hand side is symmetric in the same arguments, we obtain

$$K_o^2 S(\overline{Y},\overline{Z},\overline{X},\overline{W}) = 0, \tag{3.23}$$

$$\mathfrak{U}_{\overline{Y},\overline{Z}}\{(K_o\lambda_o(\overline{Y}) + \nabla_{\beta\overline{Y}}K_o)T(\overline{Z},\overline{X},\overline{W})\} = 0. \tag{3.24}$$

It follows from (3.23) and () that

$$P(\overline{X},\overline{Y},\overline{Z},\overline{W}) = \lambda_o(\overline{Z})T(\overline{X},\overline{Y},\overline{W}) - \lambda_o(\overline{W})T(\overline{X},\overline{Y},\overline{Z}).$$

On the other hand, if $K_o \neq 0$, then the v-curvature tensor S vanishes from (3.23). Next, it is seen from (3.24) that, if $\mathbf{V}(\overline{Y}) := K_o\lambda_o(\overline{Y}) + \nabla_{\beta\overline{Y}}K_o \neq 0$, then there exists a scalar function $\Upsilon = \frac{T(\overline{X},\overline{Z},\overline{W})T(\overline{X},\overline{Y},\overline{Z})T(\overline{Y},\overline{Z},\overline{W})}{(T(\overline{X},\overline{Y},\overline{W}))^2(\mathbf{V}(\overline{Z}))^3}$ such that

$$T(\overline{X},\overline{Y},\overline{W}) = \Upsilon\,\mathbf{V}(\overline{X})\mathbf{V}(\overline{Y})\mathbf{V}(\overline{W}).$$

Summing up, we have

Theorem 3.38. *Let (M,L) be a Finsler manifold of dimensions $n \geq 3$. If (M,L) is h-isotropic and C^h-recurrent, then*

(a) *the recurrence vector λ_o satisfies*: $(\nabla_{\beta\overline{X}}\lambda_o)(\overline{Y}) = (\nabla_{\beta\overline{Y}}\lambda_o)(\overline{X})$,

(b) the h-curvature tensor $R = 0$ and the $(v)h$-torsion tensor $\widehat{R} = 0$,

(c) the hv-curvature tensor P has the property that
$P(\overline{X},\overline{Y},\overline{Z},\overline{W}) = \lambda_o(\overline{Z})T(\overline{X},\overline{Y},\overline{W}) - \lambda_o(\overline{W})T(\overline{X},\overline{Y},\overline{Z})$,

(d) the $(v)hv$-torsion tensor $\widehat{P}(\overline{X},\overline{Y}) = K_oT(\overline{X},\overline{Y})$.

Moreover, if $K_o \neq 0$, then

(e) *the v-curvature tensor S vanishes,*

(f) *the $(h)hv$-torsion tensor T satisfies*: $T(\overline{X},\overline{Y},\overline{W}) = \Upsilon\,\mathbf{V}(\overline{X})\mathbf{V}(\overline{Y})\mathbf{V}(\overline{W})$.

By Definition 2.10 and Theorem 3.38, we immediately have:

Corollary 3.39. *A Finsler manifold (M,L) of dimension $n \geq 3$ which is h-isotropic and C^h-recurrent is necessarily P_2-like.*

Now, we define an operator $\mathbb{P}$ which aids us to investigate the R_3-like manifolds.

Definition 3.40.

(a) *If ω is a π-tensor field of type* (1,p), *then $\mathbb{P}\cdot\omega$ is a π-tensor field of the same type defined by*:

$$(\mathbb{P}\cdot\omega)(\overline{X}_1,...,\overline{X}_p) := \phi(\omega(\phi(\overline{X}_1),...,\phi(\overline{X}_p))),$$

where ϕ is the vector π-form defined by (3.1).

(b) *If ω is a π-tensor field of type* (0,p), *then $\mathbb{P}\cdot\omega$ is a π-tensor field of the same type defined by*:

$$(\mathbb{P}\cdot\omega)(\overline{X}_1,...,\overline{X}_p) := \omega(\phi(\overline{X}_1),...,\phi(\overline{X}_p)).$$

Remark 3.41. *Since* $\phi(\phi(\overline{X}))=\phi(\overline{X})$ *for every* $\overline{X}\in\mathfrak{X}(\pi(M))$ (Lemma 3.2), *then the operator* $\mathbb{P}$ *is a projector* (i.e. $\mathbb{P}\cdot(\mathbb{P}\cdot\omega)=\mathbb{P}\cdot\omega$).

Definition 3.42. *A* π*-tensor field* ω *is said to be indicatory if it satisfies the condition*: $\mathbb{P}\cdot\omega=\omega$.

The following result gives a characterization of the indicatory property for certain types of π-tensor fields:

Lemma 3.43.

(a) *A vector* (2) π*-form* ω *is indicatory if, and only if,* $\omega(\overline{X},\overline{\eta})=0=\omega(\overline{\eta},\overline{X})$ *and* $g(\omega(\overline{X},\overline{Y}),\overline{\eta})=0$.

(b) *A scaler* (2) π*-form* ω *is indicatory if, and only if,* $\omega(\overline{X},\overline{\eta})=0=\omega(\overline{\eta},\overline{X})$.

Proof.
(a) Let ω be a vector (2) π-form. By Definition 3.40(a) and taking into account (3.1), we get

$$\begin{aligned}(\mathbb{P}\cdot\omega)(\overline{X},\overline{Y}) &= \phi(\omega(\phi(\overline{X}),\phi(\overline{Y})))\\ &= \phi\{\omega(\overline{X}-L^{-1}\ell(\overline{X})\overline{\eta},\overline{Y}-L^{-1}\ell(\overline{Y})\overline{\eta})\}\\ &= \phi\{\omega(\overline{X},\overline{Y})-L^{-1}\ell(\overline{Y})\omega(\overline{X},\overline{\eta})-\\ &\quad -L^{-1}\ell(\overline{X})\omega(\overline{\eta},\overline{Y})+L^{-2}\ell(\overline{X})\ell(\overline{Y})\omega(\overline{\eta},\overline{\eta})\}\\ &= \omega(\overline{X},\overline{Y})-L^{-2}g(\omega(\overline{X},\overline{Y}),\overline{\eta})\overline{\eta}-\phi\{L^{-1}\ell(\overline{Y})\omega(\overline{X},\overline{\eta})+\\ &\quad +L^{-1}\ell(\overline{X})\omega(\overline{\eta},\overline{Y})-L^{-2}\ell(\overline{X})\ell(\overline{Y})\omega(\overline{\eta},\overline{\eta})\}\end{aligned}\tag{3.25}$$

Now, if $\omega(\overline{X},\overline{\eta})=0=\omega(\overline{\eta},\overline{X})$ and $g(\omega(\overline{X},\overline{Y}),\overline{\eta})=0$, then (3.25) implies that $(\mathbb{P}\cdot\omega)(\overline{X},\overline{Y})=\omega(\overline{X},\overline{Y})$ and hence ω is indicatory.

On the other hand, if ω is indicatory, then $\omega(\overline{X},\overline{Y})=\phi(\omega(\phi(\overline{X}),\phi(\overline{Y})))$. From which, setting $\overline{X}=\overline{\eta}$ (resp. $\overline{Y}=\overline{\eta}$) and taking into account the fact that $\phi(\overline{\eta})=0$ (Lemma 3.2), we get $\omega(\overline{\eta},\overline{Y})=0$ (resp. $\omega(\overline{X},\overline{\eta})=0$). From this, together with $(\mathbb{P}\cdot\omega)(\overline{X},\overline{Y})=\omega(\overline{X},\overline{Y})$, Equation (3.25) implies that $L^{-2}g(\omega(\overline{X},\overline{Y}),\overline{\eta})\overline{\eta}=0$. Consequently, $g(\omega(\overline{X},\overline{Y}),\overline{\eta})=0$.
(b) The proof is similar to that of (a) and we omit it. $\Box$

Proposition 3.44. *For a Finsler manifold* (M,L)*, the following tensors are indicatory*:

(a) *The* π*-tensor field* ϕ,

(b) *The mixed torsion tensor* T,

(c) *The v-curvature tensor* S,

(d) *The angular metric tensor* $\hbar$,

(e) *The* π*-tensor field* $\mathbb{P}\cdot\omega$ *for every* π*-tensor field* ω.

Now, we define the following π-tensor fields:

$$\left.\begin{array}{rl} F: & F(\overline{X},\overline{Y}) := \frac{1}{n-2}\{Ric^h(\overline{X},\overline{Y}) - \frac{Sc^h g(\overline{X},\overline{Y})}{2(n-1)}\}, \\ F_o: & g(F_o(\overline{X}),\overline{Y}) := F(\overline{X},\overline{Y}), \\ F^a: & F^a(\overline{X}) := F(\overline{\eta},\overline{X}), \\ F^b: & F^b(\overline{X}) := F(\overline{X},\overline{\eta}), \\ m: & m(\overline{X},\overline{Y}) := (\mathbb{P}\cdot F)(\overline{X},\overline{Y}), \\ m_o: & g(m_o(\overline{X}),\overline{Y}) := m(\overline{X},\overline{Y}), \\ a: & a(\overline{X}) := L^{-1}(\mathbb{P}\cdot F^a)(\overline{X}), \\ \overline{a}: & g(\overline{a},\overline{Y}) := a(\overline{X}), \\ b: & b(\overline{X}) := L^{-1}(\mathbb{P}\cdot F^b)(\overline{X}), \\ \overline{b}: & g(\overline{b},\overline{X}) := b(\overline{X}), \\ c: & c := L^{-2}F(\overline{\eta},\overline{\eta}), \\ \widehat{R}: & \widehat{R}(\overline{X},\overline{Y}) := R(\overline{X},\overline{Y})\overline{\eta}, \\ H: & H(\overline{X}) := R(\overline{\eta},\overline{X})\overline{\eta} = \widehat{R}(\overline{\eta},\overline{X}). \end{array}\right\} \tag{3.26}$$

Remark 3.45. *One can show that* m, m_o, a *and* b *are indicatory and* $H(\overline{\eta}) = 0$.

Proposition 3.46. *If* (M,L) *is an* R_3*-like Finsler manifold, then the* π*-tensor field* F *can be written in the form*

$$F(\overline{X},\overline{Y}) = m(\overline{X},\overline{Y}) + \ell(\overline{X})a(\overline{Y}) + \ell(\overline{Y})b(\overline{X}) + c\,\ell(\overline{X})\ell(\overline{Y}). \tag{3.27}$$

Proof. The proof follows from Definitions 2.14 and 3.40(b), taking into account Equations (3.1) and (3.26). In more details:

$$\begin{aligned} (\mathbb{P}\cdot F)(\overline{X},\overline{Y}) &= F(\phi(\overline{X}),\phi(\overline{Y})) \\ &= F(\overline{X} - L^{-1}\ell(\overline{X})\overline{\eta}, \overline{Y} - L^{-1}\ell(\overline{Y})\overline{\eta}) \\ &= F(\overline{X},\overline{Y}) - L^{-1}\ell(\overline{Y})F(\overline{X},\overline{\eta}) - \\ &\quad -L^{-1}\ell(\overline{X})F(\overline{\eta},\overline{Y}) + L^{-2}\ell(\overline{X})\ell(\overline{Y})F(\overline{\eta},\overline{\eta}) \\ &= F(\overline{X},\overline{Y}) - L^{-1}\ell(\overline{Y})\{(\mathbb{P}\cdot F^b)(\overline{X}) + L^{-1}\ell(\overline{X})F(\overline{\eta},\overline{\eta})\} - \\ &\quad -L^{-1}\ell(\overline{X})\{(\mathbb{P}\cdot F^a)(\overline{Y}) + L^{-1}\ell(\overline{Y})F(\overline{\eta}\,\overline{\eta})\} + L^{-2}\ell(\overline{X})\ell(\overline{Y})F(\overline{\eta},\overline{\eta}) \\ &= F(\overline{X},\overline{Y}) - \ell(\overline{X})a(\overline{Y}) - \ell(\overline{Y})b(\overline{X}) - c\,\ell(\overline{X})\ell(\overline{Y}).\ \square \end{aligned}$$

Remark 3.47. *One can show that the* π*-tensor fields* a *and* b *satisfy the following relations*

$$\begin{aligned} F^a(\overline{X}) &= L\{a(\overline{X}) + c\,\ell(\overline{X})\}, \\ F^b(\overline{X}) &= L\{b(\overline{X}) + c\,\ell(\overline{X})\}. \end{aligned} \tag{3.28}$$

Proposition 3.48. *In an* R_3*-like Finsler manifold* (M,L)*, we have:*

(a) $R(\overline{X},\overline{Y})\overline{Z} = g(\overline{X},\overline{Z})F_o(\overline{Y}) + F(\overline{X},\overline{Z})\overline{Y} - g(\overline{Y},\overline{Z})F_o(\overline{X}) - F(\overline{Y},\overline{Z})\overline{X}$.

(b) $\widehat{R}(\overline{X},\overline{Y}) = g(\overline{X},\overline{\eta})F_o(\overline{Y}) + F(\overline{X},\overline{\eta})\overline{Y} - g(\overline{Y},\overline{\eta})F_o(\overline{X}) - F(\overline{Y},\overline{\eta})\overline{X}$.

(c) $H(\overline{Y}) = L^2F_o(\overline{Y}) + c\,L^2\overline{Y} - g(\overline{Y},\overline{\eta})F_o(\overline{\eta}) - F(\overline{Y},\overline{\eta})\overline{\eta}$.

(d) $F_o(\overline{X}) = m_o(\overline{X}) + \overline{a}\,\ell(\overline{X}) + L^{-1}b(\overline{X})\overline{\eta} + c\,L^{-1}\ell(\overline{X})\overline{\eta}$.

Consequently,

(e) $\widehat{R}(\overline{X},\overline{Y}) = L\{\ell(\overline{X})(m_o(\overline{Y}) + c\,\phi(\overline{Y})) + b(\overline{X})\phi(\overline{Y})\} -$
$\qquad - L\{\ell(\overline{Y})(m_o(\overline{X}) + c\,\phi(\overline{X})) + b(\overline{Y})\phi(\overline{X})\}.$

(f) $H(\overline{Y}) = L^2\{m_o(\overline{Y}) + c\,\phi(\overline{Y})\}.$

Proof.
(a) Since (M,L) is an R_3-like manifold, then by Definition 2.14, we have

$$\begin{aligned} R(\overline{X},\overline{Y},\overline{Z},\overline{W}) =& g(\overline{X},\overline{Z})F(\overline{Y},\overline{W}) - g(\overline{Y},\overline{Z})F(\overline{X},\overline{W}) + \\ &+ g(\overline{Y},\overline{W})F(\overline{X},\overline{Z}) - g(\overline{X},\overline{W})F(\overline{Y},\overline{Z}). \end{aligned}$$

From which, using the fact that $g(F_o(\overline{X}),\overline{Y}) = F(\overline{X},\overline{Y})$ and that the Finsler metric g is non-degenerate, the result follows.

(b) Follows from (a) by setting $\overline{Z} = \overline{\eta}$.

(c) Follows from (b) by setting $\overline{X} = \overline{\eta}$.

(d) By (3.27) and (3.26), we get
$g(F_o(\overline{X}),\overline{Y}) \;=\; g(m_o(\overline{X}),\overline{Y}) \,+\, g(\overline{a},\overline{Y})\,\ell(\overline{X}) \,+\, L^{-1}b(\overline{X})g(\overline{\eta},\overline{Y}) \,+\, c\,L^{-1}\ell(\overline{X})g(\overline{\eta},\overline{Y}).$
Hence, the result follows, from the non-degeneracy of g.

(e) Follows by substituting $F_o(\overline{X})$ (from (d)) and $F^b(\overline{X})$ (from (3.28)) into (b).

(f) Follows from (e) by setting $\overline{X} = \overline{\eta}$, taking into account Remark 3.45 and the fact that $\ell(\overline{\eta}) = L$. $\square$

Remark 3.49. *In view of* (3.26) *and Lemma* 3.2*, Definition* 2.13(a) *can be reformulated as follows:*
A Finsler manifold (M,L) is of scaler curvature if the π-tensor field H satisfies the relation $H(\overline{X}) = L^2\kappa\,\phi(\overline{X})$, where κ is a scalar function on $\mathcal{T}M$.

Definition 3.50. *A Finsler manifold (M,L) is said to be of perpendicular scalar* (*or of p-scalar*) *curvature if the h-curvature tensor R satisfies the condition*

$$(\mathbb{P}\cdot R)(\overline{X},\overline{Y},\overline{Z},\overline{W}) = R_o\{\hbar(\overline{X},\overline{Z})\hbar(\overline{Y},\overline{W}) - \hbar(\overline{X},\overline{W})\hbar(\overline{Y},\overline{Z})\}, \tag{3.29}$$

where R_o is a function called the perpendicular scalar curvature.

Definition 3.51. *A Finsler manifold (M,L) is said to be of s-ps curvature if (M,L) is both of scalar curvature and of p-scalar curvature.*

Proposition 3.52. *If $m_o(\overline{X}) = t\,\phi(\overline{X})$, then an R_3-like Finsler manifold is a Finsler manifold of s-ps curvature.*

Proof. Under the given assumption and taking into account Proposition 3.48(f), we have

$$H(\overline{X}) = L^2\kappa\phi(\overline{X}),\ \textit{with}\ \kappa = t + c.$$

Thus, the considered manifold is of scalar curvature.

Now, we prove that the given manifold is of p-scalar curvature. Applying the projection $\mathbb{P}$ on the h-curvature tensor R of an R_3-like manifold, we get

$$\begin{aligned} (\mathbb{P}\cdot R)(\overline{X},\overline{Y},\overline{Z},\overline{W}) &= R(\phi(\overline{X}),\phi(\overline{Y}),\phi(\overline{Z}),\phi(\overline{W})) \\ &= g(\phi(\overline{X}),\phi(\overline{Z}))(\mathbb{P}\cdot F)(\overline{Y},\overline{W}) + g(\phi(\overline{Y}),\phi(\overline{W}))(\mathbb{P}\cdot F)(\overline{X},\overline{Z}) - \\ &\quad - g(\phi(\overline{Y}),\phi(\overline{Z}))(\mathbb{P}\cdot F)(\overline{X},\overline{W}) - g(\phi(\overline{X}),\phi(\overline{W}))(\mathbb{P}\cdot F)(\overline{Y},\overline{Z}) \\ &= g(\phi(\overline{X}),\phi(\overline{Z}))m(\overline{Y},\overline{W}) + g(\phi(\overline{Y}),\phi(\overline{W}))m(\overline{X},\overline{Z}) - \\ &\quad - g(\phi(\overline{Y}),\phi(\overline{Z}))m(\overline{X},\overline{W}) - g(\phi(\overline{X}),\phi(\overline{W}))m(\overline{Y},\overline{Z}). \end{aligned} \tag{3.30}$$

Since

$$\begin{aligned} g(\phi(\overline{X}),\phi(\overline{Y})) &= g(\phi(\overline{X}),\overline{Y}-L^{-1}\ell(\overline{Y})\overline{\eta}) = g(\phi(\overline{X}),\overline{Y})-L^{-1}\ell(\overline{Y})g(\phi(\overline{X}),\overline{\eta}) \\ &= \hbar(\overline{X},\overline{Y})-L^{-1}\ell(\overline{Y})\hbar(\overline{X},\overline{\eta}) = \hbar(\overline{X},\overline{Y}), \end{aligned}$$

then, by using again the given assumption ($m_o = t\,\phi \Longrightarrow m = t\hbar$), Equation (3.30) reduces to

$$\begin{aligned} (\mathbb{P}\cdot R)(\overline{X},\overline{Y},\overline{Z},\overline{W}) &= \hbar(\overline{X},\overline{Z})m(\overline{Y},\overline{W})+\hbar(\overline{Y},\overline{W})m(\overline{X},\overline{Z})- \\ &\quad -\hbar(\overline{Y},\overline{Z})m(\overline{X},\overline{W})-\hbar(\overline{X},\overline{W})m(\overline{Y},\overline{Z}) \\ &= 2t\{\hbar(\overline{X},\overline{Z})\hbar(\overline{Y},\overline{W})-\hbar(\overline{Y},\overline{Z})\hbar(\overline{X},\overline{W})\}. \end{aligned}$$

Therefore, by taking $R_o = 2t$, we have

$$(\mathbb{P}\cdot R)(\overline{X},\overline{Y},\overline{Z},\overline{W}) = R_o\{\hbar(\overline{X},\overline{Z})\hbar(\overline{Y},\overline{W})-\hbar(\overline{Y},\overline{Z})\hbar(\overline{X},\overline{W})\}.$$

Consequently, the given manifold is of p-scalar curvature. $\Box$

Theorem 3.53. *If an R_3-like Finsler manifold (M, L) is of p-scalar curvature, then it is of s-ps curvature.*

Proof. Since the considered manifold is R_3-like, then, by the same procedure as in the proof of Proposition 3.52, we have

$$\begin{aligned} (\mathbb{P}\cdot R)(\overline{X},\overline{Y},\overline{Z},\overline{W}) &= \hbar(\overline{X},\overline{Z})m(\overline{Y},\overline{W})+\hbar(\overline{Y},\overline{W})m(\overline{X},\overline{Z})- \\ &\quad -\hbar(\overline{Y},\overline{Z})m(\overline{X},\overline{W})-\hbar(\overline{X},\overline{W})m(\overline{Y},\overline{Z}). \end{aligned} \tag{3.31}$$

On the other hand, since the considered manifold is of p-scalar curvature, then the h-curvature tensor satisfies

$$(\mathbb{P}\cdot R)(\overline{X},\overline{Y},\overline{Z},\overline{W}) = R_o\{\hbar(\overline{X},\overline{Z})\hbar(\overline{Y},\overline{W})-\hbar(\overline{Y},\overline{Z})\hbar(\overline{X},\overline{W})\}. \tag{3.32}$$

Now, from Equations (3.31) and (3.32), we obtain

$$\mathfrak{U}_{\overline{X},\overline{Y}}\{R_o\hbar(\overline{X},\overline{Z})\hbar(\overline{Y},\overline{W})-\hbar(\overline{X},\overline{Z})m(\overline{Y},\overline{W})-\hbar(\overline{Y},\overline{W})m(\overline{X},\overline{Z})\}=0.$$

Using (3.26) and the non-degeneracy of the metric tensor g, the above equation reduces to

$$\mathfrak{U}_{\overline{X},\overline{Y}}\{R_o\hbar(\overline{X},\overline{Z})\phi(\overline{Y})-\hbar(\overline{X},\overline{Z})m_o(\overline{Y})-m(\overline{X},\overline{Z})\phi(\overline{Y})\}=0. \tag{3.33}$$

Since the π-tensor fields ϕ, m and m_o are indicatory, then
$Tr\{\overline{Y}\longmapsto \hbar(\overline{X},\overline{Y})\phi(\overline{Z})\} = g(\overline{X},\phi(\overline{Z})) = \hbar(\overline{X},\overline{Z})$,
$Tr\{\overline{Y}\longmapsto \hbar(\overline{X},\overline{Y})m_o(\overline{Z})\} = m(\overline{X},\overline{Z})$,
$Tr\{\overline{Y}\longmapsto m(\overline{X},\overline{Y})\phi(\overline{Z})\} = m(\overline{X},\overline{Z})$.
Consequently, if we take the trace of both sides of Equation (3.33), making use of Lemma 3.43, we get

$$(n-2)R_o\hbar(\overline{X},\overline{Z})-(n-3)m(\overline{X},\overline{Z})-(n-1)t\,\hbar(\overline{X},\overline{Z})=0,$$

where $t := \frac{1}{n-1}Tr(m_o)$. From which, using (3.26) and Lemma 3.2, we get

$$(n-2)R_o\phi-(n-3)m_o-(n-1)t\,\phi=0. \tag{3.34}$$

Again, taking the trace of the above equation, we obtain

$$(n-1)(n-2)(R_o-2t)=0.$$

Substituting the above relation into (3.34), we get $m_o = t\,\phi$. Hence, by Proposition 3.52, the result follows. $\Box$

Theorem 3.54. *If an R_3-like Finsler manifold (M, L) is of scalar curvature, then it is of s-ps curvature.*

Proof. Since the given manifold is R_3-like, then the π-tensor H is given by (cf. Proposition 3.48):

$$H(\overline{X}) = L^2\{m_o(\overline{X}) + c\,\phi(\overline{X})\}. \tag{3.35}$$

And since the considered manifold is of scalar curvature, then

$$H(\overline{X}) = L^2\kappa\phi(\overline{X}). \tag{3.36}$$

From Equations (3.35) and (3.36), we deduce that $m_o(\overline{X}) = (\kappa - c)\phi(\overline{X}) =: t\phi(\overline{X})$. Hence, by Proposition 3.52, the result follows. $\square$

Now, let us define the π-tensor field

$$\begin{aligned}\Psi(\overline{X},\overline{Y},\overline{Z},\overline{W}) = R(\overline{X},\overline{Y},\overline{Z},\overline{W}) - \tfrac{1}{n-2}\mathfrak{U}_{\overline{X},\overline{Y}}\{g(\overline{X},\overline{Z})Ric^h(\overline{Y},\overline{W})+ \\ +g(\overline{Y},\overline{W})Ric^h(\overline{X},\overline{Z}) - rg(\overline{X},\overline{Z})g(\overline{Y},\overline{W})\},\end{aligned} \tag{3.37}$$

where $r = \frac{1}{n-1}\,Sc^h$. From Definition 2.14 and (3.37), we immediately obtain

Theorem 3.55. *An R_3-like Finsler manifold is characterized by*

$$\Psi(\overline{X},\overline{Y},\overline{Z},\overline{W}) = 0.$$

The tensor field Ψ in the above theorem being of the same form as the Weyl conformal tensor in Riemannian geometry, we draw the following

Theorem 3.56. *An R_3-like Riemannian manifold is conformally flat.*

Remark 3.57. *It should be noted that some important results of* [8], [9], [11], [13], [19], [20],*...,etc.* (*obtained in local coordinates*) *are retrieved from the above mentioned global results* (*when localized*).

Appendix. Local formulae

For the sake of completeness, we present in this appendix a brief and concise survey of the local expressions of some important geometric objects and the local definitions of the special Finsler manifolds treated in the paper.

Let $(U,(x^i))$ be a system of local coordinates on M and $(\pi^{-1}(U),(x^i,y^i))$ the associated system of local coordinates on TM. We use the following notations:
$(\partial_i) := (\frac{\partial}{\partial x^i})$: the natural basis of T_xM, $x \in M$,
$(\dot{\partial}_i) := (\frac{\partial}{\partial y^i})$: the natural basis of $V_u(\mathcal{T}M)$, $u \in \mathcal{T}M$,
$(\partial_i, \dot{\partial}_i)$: the natural basis of $T_u(\mathcal{T}M)$,
$(\overline{\partial}_i)$: the natural basis of the fiber over u in $\pi^{-1}(TM)$ ($\overline{\partial}_i$ is the lift of ∂_i at u).

To a Finsler manifold (M,L), we associate the geometric objects:
$g_{ij} := \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2 = \dot{\partial}_i\dot{\partial}_j E$: the Finsler metric tensor,
$C_{ijk} := \frac{1}{2}\dot{\partial}_k\, g_{ij}$: the Cartan tensor,
$\hbar_{ij} := g_{ij} - \ell_i\ell_j$ $(\ell_i := \partial L/\partial y^i)$: the angular metric tensor,
G^h: the components of the canonical spray,
$G^h_i := \dot{\partial}_i G^h$,
$G^h_{ij} := \dot{\partial}_j G^h_i = \dot{\partial}_j\dot{\partial}_i G^h$,
$(\delta_i) := (\partial_i - G^h_i\dot{\partial}_h)$: the basis of $H_u(\mathcal{T}M)$ adapted to G^h_i,
$(\delta_i, \dot{\partial}_i)$: the basis of $T_u(\mathcal{T}M) = H_u(\mathcal{T}M) \oplus V_u(\mathcal{T}M)$ adapted to G^h_i.

We have:
$\gamma(\overline{\partial}_i) = \dot{\partial}_i$,
$\rho(\partial_i) = \overline{\partial}_i$, $\rho(\dot{\partial}_i) = 0$, $\rho(\delta_i) = \overline{\partial}_i$,
$\beta(\overline{\partial}_i) = \delta_i$,
$J(\partial_i) = \dot{\partial}_i$, $J(\dot{\partial}_i) = 0$, $J(\delta_i) = \dot{\partial}_i$,
$h := \beta o \rho = dx^i \otimes \partial_i - G^i_j\, dx^j \otimes \dot{\partial}_i$ $\;v := \gamma o K = dy^i \otimes \dot{\partial}_i + G^i_j\, dx^j \otimes \dot{\partial}_i$.

We define:
$\gamma^h_{ij} := \frac{1}{2}\, g^{h\ell}(\partial_i\, g_{\ell j} + \partial_j\, g_{i\ell} - \partial_\ell\, g_{ij})$,
$C^h_{ij} := \frac{1}{2}\, g^{h\ell}(\dot{\partial}_i\, g_{\ell j} + \dot{\partial}_j\, g_{i\ell} - \dot{\partial}_\ell\, g_{ij}) = \frac{1}{2}\, g^{h\ell}\,\dot{\partial}_i\, g_{j\ell} = g^{h\ell}\, C_{ij\ell}$,
$\Gamma^h_{ij} := \frac{1}{2}\, g^{h\ell}(\delta_i\, g_{\ell j} + \delta_j\, g_{i\ell} - \delta_\ell\, g_{ij})$.

Then, we have:

- The canonical spray G: $G^h = \frac{1}{2}\,\gamma^h_{ij}\, y^i y^j$.
- The Barthel connection Γ: $G^h_i = \dot{\partial}_i G^h = \Gamma^h_{ij} y^j = G^h_{ij} y^j$.
- The Cartan connection $C\Gamma$: $(\,\Gamma^h_{ij}, G^h_i,\ C^h_{ij})$.

The associated h-covariant (resp. v-covariant) derivative is denoted by $|$ (resp. $|$), where $K^i_{j|k} := \delta_k K^i_j + K^m_j \Gamma^i_{mk} - K^i_m \Gamma^m_{jk}$ and $K^i_j|_k := \dot{\partial}_k K^i_j + K^m_j C^i_{mk} - K^i_m C^m_{jk}$.

- The Berwald connection $B\Gamma$: $(\,G^h_{ij}, G^h_i,\ 0)$.

The associated h-covariant (resp. v-covariant) derivative is denoted by $\overset{*}{|}$ (resp. $\overset{*}{|}$), where $K^i_{j\overset{*}{|}k} := \delta_k K^i_j + K^m_j G^i_{mk} - K^i_m G^m_{jk}$ and $K^i_j\overset{*}{|}_k := \dot{\partial}_k K^i_j$.

We also have $G^h_{ij} = \Gamma^h_{ij} + C^h_{ij|k}\, y^k = \Gamma^h_{ij} + C^h_{ij|o}$, where $C^h_{ij|o} = C^h_{ij|k}\, y^k$.

For the Cartan connection, we have:
$(v)h$-torsion: $R^i_{jk} = \delta_k G^i_j - \delta_j G^i_k = \mathfrak{U}_{jk}\{\delta_k G^i_j\}$,
$(v)hv$-torsion: $P^i_{jk} = G^i_{jk} - \Gamma^i_{jk} = C^i_{jk|m} y^m = C^i_{jk|0}$,

$(h)hv$-torsion: $C^i_{jk} = 1/2\{g^{ri}\dot{\partial}_r g_{jk}\}$,
h-curvature: $R^i_{hjk} = \mathfrak{U}_{jk}\{\delta_k\Gamma^i_{hj} + \Gamma^m_{hj}\Gamma^i_{mk}\} - C^i_{hm}R^m_{jk}$,
hv-curvature: $P^i_{hjk} = \dot{\partial}_k\Gamma^i_{hj} - C^i_{hk|j} + C^i_{hm}P^m_{jk}$,
v-curvature: $S^i_{hjk} = C^m_{hk}C^i_{mj} - C^m_{hj}C^i_{mk} = \mathfrak{U}_{jk}\{C^m_{hk}C^i_{mj}\}$.

For the Berwald connection, we have:
$(v)h$-torsion: $R^{*i}_{\ \ jk} = \delta_k G^i_j - \delta_j G^i_k = \mathfrak{U}_{jk}\{\delta_k G^i_j\}$,
h-curvature: $R^{*i}_{\ \ hjk} = \mathfrak{U}_{jk}\{\delta_k G^i_{hj} + G^m_{hj}G^i_{mk}\}$,
hv-curvature: $P^{*i}_{\ \ hjk} = \dot{\partial}_k G^i_{hj} =: G^i_{hjk}$.

In the following, we give the ***local*** definitions of the special Finsler spaces treated in the paper. For each special Finsler space (M, L), we set its name, its defining property and a selected reference in which the local definition is located:

- Rimaniann manifold [22]: $g_{ij}(x,y) \equiv g_{ij}(x) \Longleftrightarrow C_{ijk} = 0 \ \Longleftrightarrow C_i := C^k_{ik} = 0$ (Deicke's theorem [4]).

- Minkowaskian manifold [22]: $g_{ij}(x,y) \equiv g_{ij}(y) \Longleftrightarrow C^i_{jk|h} = 0$ and $R^h_{ijk} = 0$.

- Berwald manifold [22]: $\Gamma^h_{ij}(x,y) \equiv \Gamma^h_{ij}(x)$ (i.e. $\dot{\partial}_k\Gamma^h_{ij} = 0$) $\Longleftrightarrow C^h_{ij|k} = 0$.

- C^h-recurrent manifold [13]: $C_{hij|k} = \mu_k C_{hij}$,
  where μ_j is a covariant vector field.

- P^*-Finsler manifold [7]: $C^h_{ij|0} = \lambda(x,y)C^h_{ij}$,
  where $\lambda(x,y) = \frac{P_iC^i}{C^2}$; $P_i := P^k_{ik} = C^k_{ik|0} = C_{i|0}$ and $C^2 = C_iC^i \neq 0$.

- C^v-recurrent manifold [13]: $C^i_{jk}|_l = \lambda_l\, C^i_{jk}$ or $C_{ijk}|_l = \lambda_l\, C_{ijk}$.

- C^0-recurrent manifold [13]: $C^i_{jk}\overset{*}{|}_l = \lambda_l\, C^i_{jk}$ or $C_{ijk}\overset{*}{|}_l = \lambda_l\, C_{ijk}$.

- Semi-C-reducible manifold ($\dim M \geq 3$) [18]:
  $$C_{ijk} = \frac{\mu}{(n+1)}(\hbar_{ij}C_k + \hbar_{jk}C_i + \hbar_{ki}C_j) + \frac{\tau}{C^2}C_iC_jC_k,\ C^2 \neq 0,$$
  where μ and τ are scalar functions satisfying $\mu + \tau = 1$.

- C-reducible manifold ($\dim M \geq 3$) [15]: $C_{ijk} = \frac{1}{n+1}(\hbar_{ij}C_k + \hbar_{jk}C_i + \hbar_{ki}C_j)$.

- C_2-like manifold ($\dim M \geq 2$) [17]: $C_{ijk} = \frac{1}{C^2}C_iC_jC_k,\ C^2 \neq 0$.

- quasi-C-reducible manifold ($\dim M \geq 3$) [23]: $C_{ijk} = A_{ij}C_k + A_{jk}C_i + A_{ki}C_j$,
  where $A_{ij}(x,y)$ is a symmetric tensor field satisfying $A_{ij}y^i = 0$.

- S_3-like manifold ($\dim M \geq 4$) [6]: $S_{lijk} = \frac{S}{(n-1)(n-2)}\{\hbar_{ik}\hbar_{lj} - \hbar_{ij}\hbar_{lk}\}$,
  where S is the vertical scalar curvature.

- S_4-like manifold ($\dim M \geq 5$) [6]: $S_{lijk} = \hbar_{lj}\mathbf{F}_{ik} - \hbar_{lk}\mathbf{F}_{ij} + \hbar_{ik}\mathbf{F}_{lj} - \hbar_{ij}\mathbf{F}_{lk}$,
  where $\mathbf{F}_{ij} := \frac{1}{n-3}\{S_{ij} - \frac{1}{2(n-2)}S\hbar_{ij}\}$; S_{ij} being the vertical Ricci tensor.

- S^v-recurrent manifold [20], [11]: $S_{hijk}|_m = \lambda_m S_{hijk}$,
  where $\lambda_j(x,y)$ is a covariant vector field.

- Second order S^v-recurrent manifold [20], [11]: $S_{hijk}|_m|_n = \Theta_{mn} S_{hijk}$,
  where $\Theta_{ij}(x,y)$ is a covariant tensor field.

- Landsberg manifold [7]: $P^h_{kji}\, y^k = 0 \Longleftrightarrow (\dot{\partial}_i \Gamma^h_{jk}) y^k = 0 \Longleftrightarrow C^h_{ij|k} y^k = 0$.

- General Landsberg manifold [10]: $P^r_{ijr} y^i = 0 \Longleftrightarrow C_{j|o} = 0$.

- P-symmetric manifold [19]: $P_{hijk} = P_{hikj}$.

- P_2-like manifold ($\dim M \geq 3$) [14]: $P_{hijk} = \alpha_h C_{ijk} - \alpha_i C_{hjk}$,
  where $\alpha_k(x,y)$ is a covariant vector field.

- P-reducible manifold ($\dim M \geq 3$) [19]: $P_{ijk} = \frac{1}{n+1}(\hbar_{ij}\, P_k + \hbar_{jk}\, P_i + \hbar_{ki}\, P_j)$,
  where $P_{ijk} = g_{hi} P^h_{jk}$.

- h-isotropic manifold ($\dim M \geq 3$) [13]: $R_{hijk} = k_o\{g_{hj} g_{ik} - g_{hk} g_{ij}\}$,
  for some scalar k_o, where $R_{hijk} = g_{il} R^l_{hjk}$.

- Manifold of scalar curvature [21]: $R_{ijkl}\, y^i y^k = k L^2 \hbar_{jl}$,
  for some function $k : \mathcal{T}M \longrightarrow \mathbb{R}$.

- Manifold of constant curvature [21]: the function k in the above definition is constant.

- Manifold of perpendicular scalar (*or of p-scalar*) curvature [8], [9]:
  $\mathbb{P} \cdot R_{hijk} := \hbar^l_h\, \hbar^m_i\, \hbar^n_j\, \hbar^r_k\, R_{lmnr} = R_o\{\hbar_{ik}\hbar_{hj} - \hbar_{ij}\hbar_{hk}\}$,
  where R_o is a function called a perpendicular scalar curvature.

- Manifold of *s-ps* curvature [8], [9]: (M,L) is both of scalar curvature and of p-scalar curvature.

- R_3-like manifold ($\dim M \geq 4$) [8]: $R_{hijk} = g_{hj} F_{ik} - g_{hk} F_{ij} + g_{ik} F_{hj} - g_{ij} F_{hk}$,
  where $F_{ij} := \frac{1}{n-2}\{R_{ij} - \frac{1}{2}\, r\, g_{ij}\}$; $R_{ij} := R^h_{ijh}$, $r := \frac{1}{n-1} R^i_i$.

References

[1] H. AKbar-Zadeh, *Les espaces de* F*insler et certaines de leurs généralisations*, Ann. Ec. Norm. Sup., Série 3, **80** (1963), 1–79.

[2] ———, *Sur les espaces de* F*insler isotropes*, C. R. Acad. Sc. Paris, série **A** (1979), 53–56.

[3] ———, *Initiation to global* F*insler geometry*, Elsevier, 2006.

[4] F. Brickell, *A new proof of* D*eicke's theorem on homogeneous functions*, Proc. Amer. Math. Soc., **16** (1965), 190-191.

[5] P. Dazord, *Propriétés globales des géodésiques des espaces de* F*insler*, Thèse d'Etat, (575) Publ. Dept. Math. Lyon, 1969.

[6] F. Ikedo, *On S_3- and S_4-like* F*insler spaces with the* T*-tensor of a special form*, Tensor, N. S., **35** (1981), 345–351.

[7] H. Izumi, *On* P^*-F*insler spaces,* I, Memoirs of the Defense Academy, Japan, No. **4, XVI** (1976), 133–138.

[8] H. Izumi and T. N. Srivastava, *On* R_3*-like* F*insler spaces*, Tensor, N. S., **32** (1978), 339–349.

[9] H. Izumi and M. Yoshida, *On* F*insler spaces of perpendicular scalar curvature*, Tensor, N. S., **32** (1978), 219–224.

[10] M. Kitayama, *Geometry of transformations of* F*insler metrics*, Ph. D. Thesis, Hokkaido University of Education, Kushiro, Japan, 2000.

[11] ———, *Indicatrices of* R*anders change*, 9th International Conference of Tensor Society, Sapporo, Japan, September 4-8, 2006.

[12] J. Klein and A. Voutier, *Formes extérieures génératrices de sprays*, Ann. Inst. Fourier, Grenoble, **18(1)** (1968), 241–260.

[13] M. Matsumoto, *On h-isotropic and* C^h*-recurrent* F*insler spaces*, J. Math. Kyoto Univ., **11** (1971), 1–9.

[14] ———, *On* F*insler spaces with curvature tensors of some special forms*, Tensor, N. S., **22** (1971), 201–204.

[15] ———, *On* C*-reducible* F*insler spaces*, Tensor, N. S., **24** (1972), 29–37.

[16] ———, *Foundations of* F*insler geometry and special* F*insler spaces*, Kaiseisha Press, Otsu, Japan, 1986.

[17] M. Matsumoto and S. Numata, *On semi-C-reducible* F*insler spaces with constant coefficients and C_2-like Finsler spaces*, Tensor, N. S., **34** (1980), 218–222.

[18] M. Matsumoto and C. Shibata, *On semi-*C*-reducibility,* T*-tensor* $= 0$ *and* S_4*-likeness of* F*insler spaces*, J. Math. Kyoto Univ., **19** (1979), 301–314.

[19] M. Matsumoto and Shimada, *On* F*insler spaces with the curvature tensors* P_{hijk} *and* S_{hijk} *satsfiying special conditions*, Rep. Math. Phys., **12** (1977), 77–87.

[20] A. Moór, *Über* F*insler* R*äume von* Z*weifach* R*ekurrenter* K*rümmung*, Acta Math. Acad. Sci. Hungaricae, **22** (1971), 453–465.

[21] S. Numata, *On* L*andesberg spaces of scalar curvature*, J. Korean Math. Soc., **12(2)** (1975), 97–100.

[22] H. Rund, *The differential geometry of* F*insler spaces*, Springer-Verlag, Berlin, 1959.

[23] C. Shibata, *On invariant tensors of β-changes of* F*insler metrics*, J. Math. Kyoto Univ., **24(1)** (1984), 163–188.

[24] A. A. Tamim, *General theory of* F*insler spaces with applications to* R*anders spaces*, Ph. D. Thesis, Cairo University, 1991.

[25] ______, *Special* F*insler manifolds*, J. Egypt. Math. Soc., **10(2)** (2002), 149–177.

[26] ______, *On* F*insler submanifolds*, J. Egypt. Math. Soc., **12(1)** (2004), 55–70.

[27] Nabil L. Youssef, S. H. Abed, and A. Soleiman, *A global theory of conformal* F*insler geometry*, Submitted. ArXiv No.: math. DG/0610052.