A Unified Ensemble Approach to Classical Polarization Optics

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Abstract

Most practical optical systems are embedded in some media or the other and thus a statistical ensemble approach seems to offer a realistic model of such systems. It is observed that this approach is of the form of a classical version of the positive operator valued measures (POVM) of quantum density matrix for describing the effects of media in which the system is embedded. With suitable choice of the ensemble several optical outcomes are deduced, some of which do not belong to the well-known variety.

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The intensity and polarization of a beam of light passing through an isolated optical device undergoes a linear transformation, in general. The properties of the transformation form a much studied topic in the literature [1–15]. But this is an ideal situation because in general, the optical system is embedded in some media, such as atmosphere or other ambient material, through which the light passes and thus further modifies the properties of the beam. A statistical ensemble model of this type of situation was formulated two decades ago by Kim et al [16] but is not examined in any detail in the literature, to the best of our knowledge. The purpose of the present paper is to pursue this avenue and exhibit the power of this procedure in elucidating besides known processes, some hitherto unknown processes [15]. This had remained a mathematical possibility [15] but their physical implications were not understood before.

In order to motivate the utility of the proposed model, we first briefly describe the isolated system first. This also enables us to define the notations and terminology of the presentation. Following the standard procedure, let E_1 and E_2, defined here as a column matrix $\mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$, be the two components of the transverse electric field vector associated with a light beam propagating along the third direction. The coherency matrix is a positive semidefinite 2x2 hermitian matrix defined by

$$\mathbf{C} = \langle \mathbf{E} \otimes \mathbf{E}^{\dagger} \rangle \tag{1}$$

Expressing this in terms of the standard hermitian Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the unit matrix $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$\mathbf{C} = \frac{1}{2} \sum_{i=0}^{3} s_i \sigma_i = \frac{1}{2} \begin{pmatrix} s_0 + s_3 & s_1 - i s_2 \\ s_1 + i s_2 & s_0 - s_3 \end{pmatrix} \tag{2}$$

The physical significance of the quantities arising here are

$$\begin{aligned} s_0 &= \text{Tr } \mathbf{C}\sigma_{\mathbf{0}} = \text{Intensity of the beam} \\ s_i &= \text{Tr } \mathbf{C}\sigma_{\mathbf{i}} = \text{Components of Polarization vector } \tilde{\mathbf{s}} \text{ of the beam} \end{aligned} \tag{3}$$

Thus the coherency matrix completely specifies the physical properties of the light beam. The 4-vector $\mathbf{S} = \begin{pmatrix} s_0 \\ \vec{\mathbf{s}} \end{pmatrix}$ defined by Eq. (3) is the well known Stokes vector. Because $\mathbf{C}$ is hermitian, the Stokes vector is real. The positive semidefiniteness of C implies that the Stokes vector must satisfy the properties

$$s_0 > 0, \; s_0^2 - |\vec{\mathbf{s}}|^2 \geq 0 \tag{4}$$

A 2x2 complex matrix $\mathbf{J}$ called the Jones matrix represents the so-called deterministic optical device [17] or medium. When a light beam represented by $\mathbf{E}$ passes through such a medium, the transformed light beam is given by $\mathbf{E}' = \mathbf{JE}$. When a light beam is partially polarized, it is more conveniently represented by its coherency matrix $\mathbf{C}$. The transformation of the coherency matrix $\mathbf{C}$ by the Jones matrix $\mathbf{J}$ is given by

$$\mathbf{C}' = \mathbf{JCJ}^\dagger \tag{5}$$

Here $\mathbf{J}^\dagger$ is the hermitian conjugate of $\mathbf{J}$.

Alternatively, instead of the transformation given by Eq. (5), a transformation of the Stokes vector $\mathbf{S}$ expressed as a column matrix through a real 4x4 matrix $\mathbf{M}$ called the Mueller matrix is found useful [17].

$$\mathbf{S}' = \mathbf{MS} \tag{6}$$

Since Mueller matrix $\mathbf{M}$ here is determined through a Jones matrix, the condition (4) on the outgoing light beam (Stokes vector) is clearly satisfied. Explicitly using Eq. (3) and Eq. (5) we have,

$$s_i' = \mathrm{Tr}(\mathbf{C}'\sigma_i) = \mathrm{Tr}(\mathbf{JCJ}^\dagger\,\sigma_i) = \frac{1}{2}\sum_{j=0}^{3} \mathrm{Tr}(\mathbf{J}^\dagger\sigma_{\mathbf{i}}\mathbf{J}\sigma_j)s_j$$

Thus, we obtain the relationship

$$M_{ij} = \frac{1}{2}\mathrm{Tr}(\mathbf{J}^\dagger\sigma_{\mathbf{i}}\mathbf{J}\sigma_j)$$

between the elements of a Jones matrix and the elements of a Mueller matrix corresponding to it.

But in the case where a device cannot be represented by a Jones matrix, it is not possible to represent the transformation of the light beam through Eq. (5). In such a situation, the transformation of the light beam can only be represented through Eq. (6). Also, the

Mueller matrix $\mathbf{M}$ here is called non-deterministic and it can be any 4×4 matrix provided the outgoing light beam satisfies the condition (4). That is, a Mueller matrix is any 4×4 real matrix that transforms a Stokes vector into another Stokes vector.

There are many aspects of the relationships between these two formulations of the polarization optics and the complete characterization of Mueller matrices has been the subject matter of Ref. [1-16]. It was Gopala Rao et al [15] who presented a complete set of necessary and sufficient conditions for any 4x4 real matrix to be a Mueller matrix. In so doing, they found that there are two algebraic types of Mueller matrices called type I and type II. The type-I Mueller matrices contain all the known polarizing optical devices such as retarders, polarizers, analyzers, optical rotators etc., and thus are well understood. But type II Mueller matrices are yet to be physically realized and have remained as mere mathematical possibility. For the sake of completeness, we present here the characterization as well as categorization of these two types of Mueller matrices as is given in [15]. This will enable us to realize both these types in a unified manner in terms of the proposed ensemble approach promised in the Introductory paragraph.

I. A 4×4 real matrix M is called a type-I Mueller matrix iff

i) $M_{00} \geq 0$

ii) The G-eigenvalues $\rho_0,\ \rho_1,\ \rho_2,\ \rho_3$ of the real symmetric matrix $\mathbf{N}=\widetilde{\mathbf{M}}\mathbf{G}\mathbf{M}$ are all real. (Here, $\widetilde{\mathbf{M}}$ stands for the transpose of $\mathbf{M}$).

iii) The largest G-eigenvalue ρ_0 possesses a time-like G-eigenvector and the G-eigenspace of $\mathbf{N}$ contains one time-like and three space-like G-eigenvectors.

II. A 4×4 real matrix M is called a type-II Mueller matrix iff

i) $M_{00} > 0$.

ii) The G-eigenvalues $\rho_0,\ \rho_1,\ \rho_2,\ \rho_3$ of the real symmetric matrix $\mathbf{N}=\widetilde{\mathbf{M}}\mathbf{G}\mathbf{M}$ are all real.

iii) The largest G-eigenvalue ρ_0 possesses a null G-eigenvector and the G-eigenspace of $\mathbf{N}$ contains one null and two space-like G-eigenvectors.

iv) If $\mathbf{X}_0 = \mathbf{e}_0 + \mathbf{e}_1$ is the null G-eigenvector of $\mathbf{N}$ such that $\mathbf{e}_0$ is a time-like vector with positive zeroth component, $\mathbf{e}_1$ is a space-like vector G-orthogonal to $\mathbf{e}_0$, then $\widetilde{\mathbf{e}}_0\mathbf{N}\mathbf{e}_0 > 0$.

Despite the knowledge of a new category of Mueller matrices through the works of [13] and [15], not much attention is paid for realizing the corresponding devices. An experimental arrangement involving a parallel combination of deterministic optical devices (pure Mueller devices) and a two-beam splitter is proposed in [15] for realizing type-II Mueller devices. The extension of the above description to include the physical situations where the beam of light is subjected to the influence of a medium such as atmosphere was addressed by the work of Kim et al [16]. This work is applicable more generally as well and forms the basis of our discussion of the polarization optics in more general settings. Briefly, following [16], the ensemble construction involves associating a set of probabilities $\{p_e, \ \sum p_e = 1\}$ to describe the stochastic medium. Then a Jones device $\mathbf{J}_e$ associated with each element e of the ensemble gives a corresponding coherency matrix $\mathbf{C'}_e = \mathbf{J}_e\mathbf{C}\mathbf{J}_e^\dagger$. The ensemble averaged coherency matrix

$$\mathbf{C}_{av} = \sum_e p_e \mathbf{C'}_e = \sum_e p_e (\mathbf{J}_e\mathbf{C}\mathbf{J}_e^\dagger) \tag{7}$$

then describes the effects of the medium on the beam of light. In a similar fashion, the corresponding ensemble of Mueller matrices $\{\mathbf{M}_e\}$ associated with the ensemble of Jones matrices $\{\mathbf{J}_e\}$ is constructed and its ensemble averaged Mueller matrix is similarly formed as $\mathbf{M}_{av} = \sum_e p_e \mathbf{M}_e$. Since a linear combination of Mueller matrices with non-negative coefficients is also a Mueller matrix, the ensemble averaged Mueller matrix is a Mueller matrix [19].

We now turn to the question of constructing the appropriate ensemble designed to describe a given physical situation. The simplest example of an ensemble is one where the elements of the ensemble are chosen entirely randomly, i.e., the system is described by a chaotic ensemble where the probabilities are all equal, $p_e = \frac{1}{n}$. The coherency matrix $\mathbf{C}_{av}$ of the light beam coming out from such a chaotic assembly is just an arithmetic average of the coherency matrices $\mathbf{C'}_e = \mathbf{J}_e\mathbf{C}\mathbf{J}_e^\dagger$ and hence

$$\mathbf{C}_{av} = \frac{1}{n}\sum_{e=1}^{n} \mathbf{J}_e\mathbf{C}\mathbf{J}_e^\dagger \tag{8}$$

More general models can be constructed depending on the environment that the beam of light is subjected to. For example, one may employ various types of filters or solid state systems through which the light passes through, and the assignment of the Jones matrices and the corresponding probabilities will then differ depending on the weights one wants to place on these elements.

We recall here that any complex 2×2 matrix qualifies to be a Jones matrix (except for the passivity requirement on the resultant electric vector) that transforms an electric vector. Therefore while choosing the constituent Jones matrices $\mathbf{J}_e$ of the ensemble $(p_e,\ \mathbf{J}_e)$, we have many choices. Restricting ourselves to an ensemble consisting of only two Jones devices which occur with equal probability $p_1 = 1/2$, $p_2 = 1/2$, we have found out that the resultant Mueller matrices can either be deterministic (type-I) or non-deterministic (type-I/type-II) matrices. We give in the foregoing (see Table I) a few Mueller matrices corresponding to several choices of Jones matrices in an ensemble $\mathbf{J}_e$ for some representative cases. This will also serve to show the generality of the ensemble procedure in capturing the physical situations discussed in [15].

In Table I, the Jones matrices chosen are so as to give pure Mueller (deterministic), type-I and type-II matrices in that order. We observe that an assembly of Jones matrices can result in a pure Mueller matrix if and only if all elements of the assembly correspond to the same optical device. This is because a transformation of the form $\mathbf{C}_{av} = \sum_e p_e(\mathbf{J}_e\mathbf{C}\mathbf{J}_e^\dagger)$ is equivalent to a transformation of the Stokes vector $\mathbf{S}$ through a Mueller matrix $\mathbf{M}_{av} = \sum p_e\mathbf{M}_e$. $\mathbf{M}_e$ being Mueller matrices corresponding to $\mathbf{J}_e$, $\mathbf{M}_{av}$=$\mathbf{M}_e$ when all $\mathbf{J}_e$'s are same. When the medium is represented by a pure Mueller matrix, the outgoing light beam will have the same degree of polarization as the incoming light beam. In fact, pure Mueller matrix is the simplest among type-I Mueller matrices. Not all type-I Mueller matrices preserve the degree of polarization of the incident light beam. For example, whereas the type-I matrix of example 2 converts any incident light beam into a linearly polarized light beam, the other 3 type-I matrices (examples 3 to 5) quoted in Table I in general transform completely polarized light beams into partially polarized light beams. Similarly type-II Mueller matrices change the polarization of the incident light beam in such a way that the degree of polarization is not preserved in general. For example, the type-II Mueller matrix 6 in Table I can be termed as a depolarizer matrix. It converts any incident light beam into an unpolarized light beam.

Though one cannot a priori state which choices of Jones matrices results in type-I or type-II, it is interesting to observe that all types of Mueller matrices result even in 2-element ensembles. It is not difficult to conclude that an ensemble with more Jones devices with different combination of weight factors can give rise to many more interesting Mueller matrices of all possible algebraic types. It would certainly be interesting to physically realize such systems. In fact, by noting that the 2x2 matrices can be expressed in terms

TABLE I: **Mueller matrices resulting from 2-element ensembles**

No.	$\mathbf{J}_1$	$\mathbf{J}_2$	$\mathbf{M} = p_1\mathbf{M}_1 + p_2\mathbf{M}_2;\ p_1 = p_2 = \frac{1}{2}$	Type of $\mathbf{M}$
1	$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}$	$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}$	$\frac{1}{6}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -1 & 2 \\ 0 & 2 & 2 & -1 \end{pmatrix}$	Pure Mueller
2	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	Type-I
3	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	Type-I
4	$\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$	$\frac{1}{6}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix}$	Type-I
5	$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}$	$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$	$\frac{1}{10}\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & -1 & 2 & 4 \\ 0 & 2 & -3 & 2 \\ 0 & 4 & 2 & -1 \end{pmatrix}$	Type-I
6	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	Type-II
7	$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$	$\frac{1}{4}\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	Type-II

of combinations of rotations and reflections, the corresponding physical devices would be optical elements (such as Nicol prisms, mirrors) which rotate the directions of polarization of the beam and/or reflect the beam.

We will now show that the phenomenology of the ensemble construction of Kim et al [16] described above has a fundamental theoretical underpinning if we make a formal identification of the coherency matrix with the density matrix description of a subsystem of a composite quantum system. The coherency matrix defined by Eqs. (1) and (2) resembles a quantum density matrix in that both describe a physical system by a hermitian, trace-class, and positive semi-definite matrix. The difference is that the quantum density matrix has unit trace while the coherency matrix has intensity of the beam as the value of the trace. The Jones matrix transformation is a general transformation of the coherency matrix which preserves its hermiticity and positive semi-definiteness but changes the values of the elements of the coherency matrix elements. The most general transformation of the density matrix ρ which preserves its hermiticity, positive semi-definiteness and also the unit trace is the positive operator valued measures (POVM) [18]:

$$\rho' = \sum_{i=1}^{n} \mathbf{V}_i \rho \mathbf{V}_i^\dagger; \qquad \sum_{i=1}^{n} \mathbf{V}_i^\dagger \mathbf{V}_i = \mathbf{E} \tag{9}$$

where V_i's are general matrices and $\mathbf{E}$ is the unit element in the Hilbert space.

More generally, one could relax the condition of preservation of the unit trace of the density matrix by examining the possibility of a contracting transformation, where the unit matrix condition on the POVM operators is replaced by an inequality. This mathematical theorem has a physical basis in the Kraus operator formalism [18] when we consider the Hamiltonian description of a composite interacting system $\mathbf{A}$, $\mathbf{B}$ described by a density matrix $\rho(A,\ B)$ and deduce the subsystem density matrix of $\mathbf{A}$ given by $\rho(A) = \mathrm{Tr}_B\ \rho(A,\ B)$. In this case, the Kraus operators are the explicit expressions of the POVM operators and contain the effects of interaction between the systems $\mathbf{A}$ and $\mathbf{B}$ in the description of the subsystem $\mathbf{A}$. It is thus clear that the phenomenology of Kim et al [16] has a correspondence with the Kraus formulation and the POVM theory. In order to make this association complete, we compare Eq. (9) with the expression given by Eq. (7). One then finds that apart from a phase factor, the POVM's for the coherency matrix may be chosen in the form

$$\mathbf{V}_i = \sqrt{p_i}\mathbf{J}_i, \qquad \sum_{i=1}^{n} \mathbf{V}_i^\dagger \mathbf{V}_i = \sum_{i=1}^{n} p_i \mathbf{J}_i^\dagger \mathbf{J}_i \tag{10}$$

In the construction of the Table I presented earlier, a simple model was proposed where all probabilities were chosen to be equal. Also, the condition on the sum over the Jones matrix combinations was set equal to unit matrix (except for example 7). In such cases, the intensity of the beam gets reduced by 1/n (n=2 for the examples presented in Table I) and the polarization properties of the beam gets changed by the Jones matrices, as was described earlier. With this identification, we have provided here an important interpretation and meaning to the phenomenology of the ensemble approach of Kim et al.[16].

In conclusion, we have established here a connection between the phenomenological ensemble approach of the coherency matrix and the POVM transformation [16] of quantum density matrix. This gives a fresh physical interpretation to the Types I and II of the Mueller matrix classification [15]. We plan on exploring further the POVM transformation in the description of quantum polarization optics.

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[19] This is because, each Mueller matrix $\mathbf{M}_e$ transforms a Stokes vector into another Stokes vector and a linear combination of Stokes vectors with non-negative coefficients is again a Stokes vector.