Effective temperature of a stationary dissipative system: fully-developped turbulence.

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PACS. 05.70.Ln – Nonequilibrium and irreversible thermodynamics.
PACS. 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion.
PACS. 05.20.Jj – Statistical mechanics of classical fluids.

Abstract. – We measure an effective temperature $T_{\mathrm{eff.}}$ in a turbulent flow, where T_{eff} is defined according to Cugliandolo *et al.*, using a Fluctuation-Dissipation Relation (FDR). Although the hypothesis underlying the fluctuation-dissipation theorem does not strictly hold, we present experimental evidence that meaningful quantities can be extracted. T_{eff} is measured in a large Reynolds number turbulent air jet flow ($74000 \leq Re \leq 170000$), using the spectral fluctuations and response of a simple vibrating string. The wave-number spectra all collapse on a single master curve for all Reynolds numbers. This spectrum is in agreement with a very simple model derived from Kolmogorov's 1941 theory, predicting a $k^{-11/3}$ scaling.

Fluctuation-Dissipation relation. – The fluctuation-dissipation theorem holds strictly to equilibrium systems, that is systems having no internal heat fluxes ([1]). The recent theory of Cugliandolo *et. al.* [1] states that this theorem can be extended to probe the temperature of a system out of equilibrium, such as relaxing glasses after quench, or slowly "stirred" systems. They clearly state the relevance of such a definition of an *effective temperature* $T_{\mathrm{eff.}}$. Its definition comes from the Fluctuation-Dissipation Relation (FDR) [1, 2]:

$$\frac{\omega \ \langle \tilde{x}(\omega)^2 \rangle}{Im[H_{\mathrm{x,f}}(\omega)]} = 4 \ k_{\mathrm{B}} T_{\mathrm{eff.}}. \tag{1}$$

The dissipation is expressed by the imaginary part $Im[H_{x,f}]$ of the susceptibility of an observable $x(t)$ on the conjugate forcing $f(t)$. $\langle \tilde{x}^2 \rangle$ is the spectral energy density of the fluctuations x. The ratio of the two is the thermal agitation $k_B T_{\mathrm{eff}}$, where k_{B} is the Boltzman constant ([2]). $T_{\mathrm{eff.}}$, defined by this expression, is an "out of equilibrium temperature". Unlike the

([1])Other definitions are possible. It can be said that the system's state is fluctuating in its phase-space around a maximum of its entropy. Event though the entropy is not explicited, it exists and have a maximum (that might be time-dependant).
([2])Note that the Boltzman constant $k_{\mathrm{B}} \simeq 1.38\ 10^{-23}$ Joule/Kelvin links energy and temperature. It also links microscopic and macroscopic scales in statistical mechanics. It has a clear meaning in the case of kinetic theory of gases or Brownian motion, but may not be relevant in physical systems showing a continuous distribution of scales. It is given here purely by analogy, but we will not express $T_{\mathrm{eff.}}$ in Kelvins.

equilibrium case where the ratio of those ω-depending functions is independant of frequency, temperature here depends on frequency ω, (or equivalently time: see ageing processes for instance). This dependance reflects the presence of several relaxation time-scales in the process. As the fluctuation spectral density on one hand, and the response on the other hand can both be measured on an actual physical system, Eq.1 can be used to probe the temperature of the system even out of equilibrium. This has been done on several systems in the recent years, like glasses or granular materials [3–6]. In the present case, the relevant variables are not time-frequency, but position-wave number. Both are related by the dispersion law of the vibrating string: $k = \omega/c$, where c is the wave velocity in the string. As the modes of the string couples with turbulent structures, the wave number is related to scales of *eddies* of size r by: $k = 2\pi/r$.

In the present letter, we report on such measurements in a turbulent flow. A simple Melde string is coupled to the flow by viscous drag. A Melde string is a vibrating string with nearly fixed position at the boundaries, and constant tension. The drag excites the modes of the string while viscous coupling causes damping, like for Brownian motion, where excitation and damping are both caused by the same quantity. The relation between those two aspects is the Einstein formula, involving the temperature, and is nothing but a special case of FDT. In the case of turbulent flow, like glasses, there is no clear distinction between microscopic and macroscopic scales, excitations are continuously distributed over scales.

In the present situation, the vibrating string is a chain of independant harmonic oscillators. It is therefore a bit more than a spectrometer: it is a scale-dependant thermometer !

Turbulent flow and Melde string. – Since the pioneering work of Richardson, and later Kolmogorov [7, 8], it is well accepted that in a high enough Reynolds number (Re) flow, excitations should exhibit local isotropy, and therefore universal distribution in scale ([3]). We present in this work a new probe of turbulent flows, specifically on space-scale. Note that, by averaging on time, we loose all time information. As the excitation is applied all over the string, locality is lost. All we keep is a very high resolution on scale.

The statistical mechanics of turbulence is a rather old topic. (For a review, see the excellent Monin and Yaglom book [8], for instance.) Theoretical characterisation in terms of temperature has been proposed several times already: by J. Sommeria and R. Robert for 2D turbulence [9], and by B. Castaing for 3D turbulence [10], both on different basis.

Our system is a turbulent air jet flow. Diameter of the nozzle is 5 cm, and measurements are done 2 m downstream. (The jet-flow facility of the ENS-Lyon is detailed in length in [11].) The steal string is clamped on one end, as on the other end it goes over a ball-bearing and holds a 4 kg weight. It is therefore held at constant force through the jet, on a rigid aluminium stand. Close to the boundaries, piezo actuator/sensor are positioned in contact with the string by an X-Y micrometer lens positioner. The string used is a guitar E string of 0.2 mm diameter, the length of the vibrating part is $l = 60.3$ cm. (Force and length are chosen such that the string is tunned close to E, for better stability.)

Measurements. – Measurements are performed separately for *fluctuations* and *response*, respectively numerator and denominator of the FDR, Eq.1. When one of the piezos is the sensor and the other is the actuator, the modes of the string can be measured. This is the *response*. Only the resonance peaks are considered. Note that they are very narrow: the quality factor is aproximately $Q \simeq 4000$. This ensures a very good selection of wave-numbers. We have recently derived the theoretical response function of a Melde string with viscous

([3])At finite Re, corrections to expected Kolmogorov 1941 scalings are needed, but this is not the purpose of the present study.

damping, which is superimposed in Fig.1 with experimental spectrum. This derivation will be given in a future article. From the diameter of the string, the local Reynolds number is $Re \simeq 1$. We observed that the response with or without turbulence is the same, as the drag is viscous.

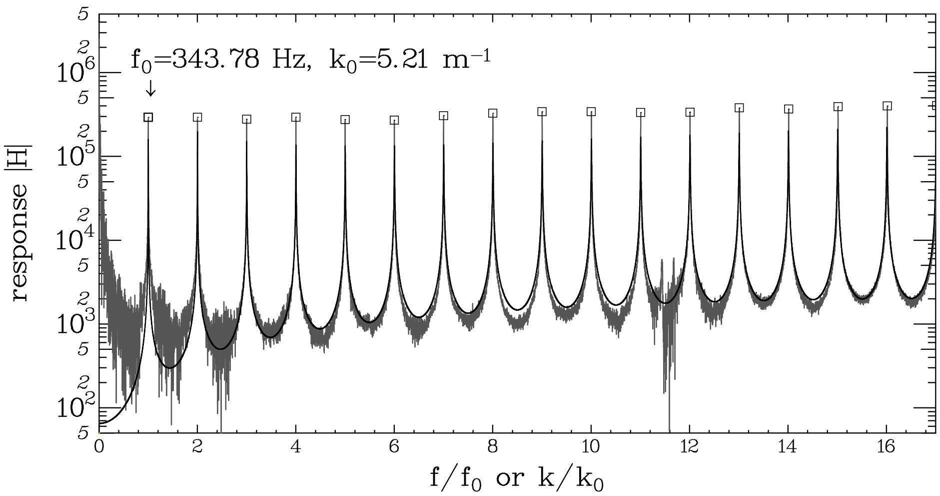

Fig. 1 – Modulus of the response function, with turbulence. The theoretical response function is superimposed on the experimental one.

When one of the piezos is the sensor and the other is not connected, the modes of the string excited by the turbulent wind have been measured, see Fig.2. From this spectrum of the *fluctuations*, resonance peaks are visible, together with many spurious vibrations that can be partly removed as explained below.

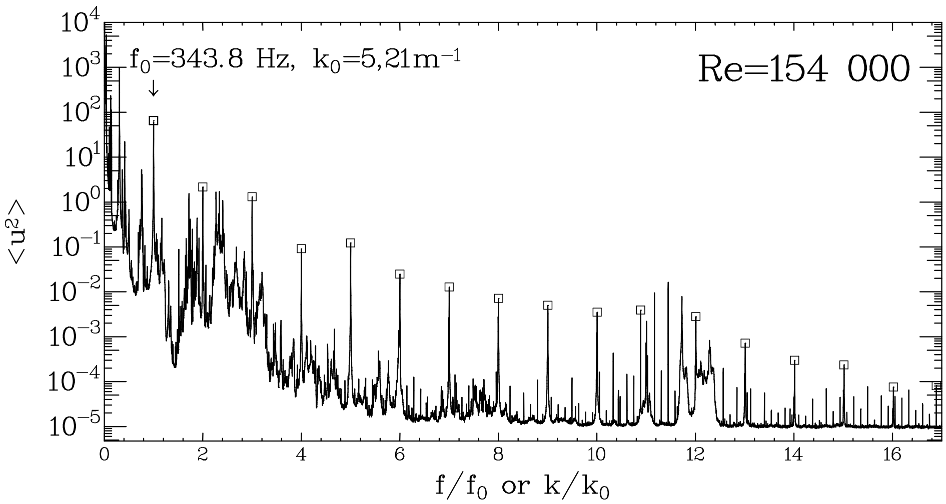

Fig. 2 – Spectrum of the modes excited by turbulence on the string, at $Re = 154000$.

Note that the highest resonance frequency visible in the fluctuations spectrum is a few kHz. That is the inverse of the excitation time of the string $\tau \simeq l/c$, not to be confused with the damping time. This is the high frequency cut-off of this probe taken as a *thermometer*.

An important experimental condition is to keep the system exactly the same to perform the two measurements of fluctuations and response, so that spurious effects cancel out. It is then possible to calculate the FDR ratio.

We observed that the spectrum of fluctuations is the same whether the axis of the piezos are in the direction of the flow or perpendiculars, whether they are parallels or perpendiculars to each other.

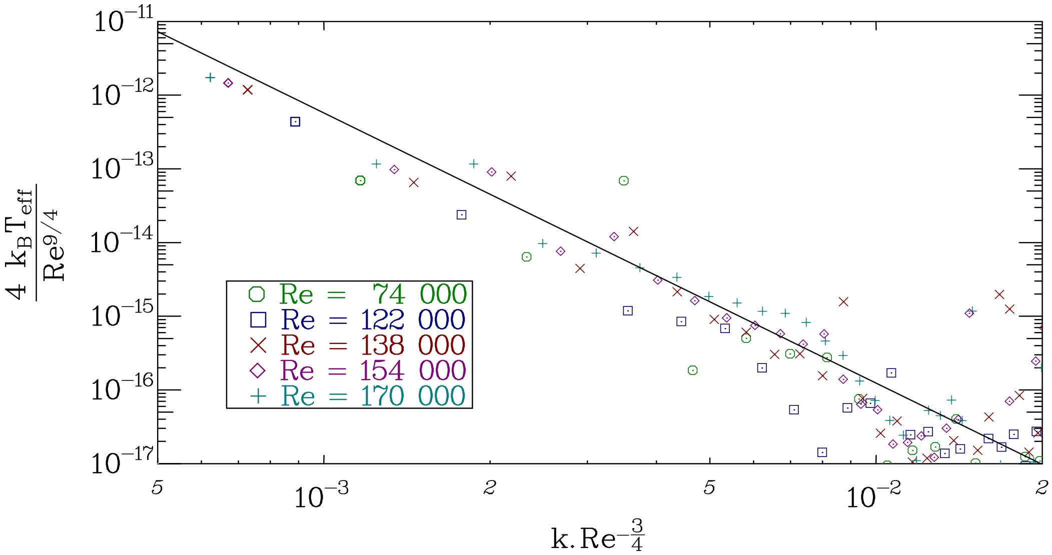

Fig. 3 – Spectrum of the *thermal agitation* of turbulence, rescaled with proper Reynolds number dependance: between 74000 and 170000. The solid line is a $k^{-11/3}$ power-law given as an eye guide.

One can see in Fig.3 the FDR ratio, $k_{\rm B}T_{\rm eff.}$, for several values of Re. The wave-number has been multiplied by an internal viscous scale $\eta \propto Re^{-3/4}$. The ordinate has been corrected by an estimated number of degrees of freedom: $(L/\eta)^3 \propto Re^{9/4}$. In other words, the *"thermal energy"* $k_{\rm B}T_{\rm eff.}$ is given by degree of freedom. Note that we suppose here as usual that the number of degrees of freedom is the number of particules of size η. A more realistic description should involve correlations between them, reducing this number. All the curves nicely collapse to a single power-law whose exponent we discuss in the following section.

Scaling law. – Among the main predictions of Kolmogorov 1941 theory, the power spectrum of velocity fluctuations scales as: $<\tilde{v}^2> \propto\ k^{-5/3}$ in the inertial range. $k = 2\pi/\lambda$ is the wave-number of the string, it is also the inverse scale at which the flow is probed. The couple of conjugate observables is displacement and forcing: $\{x(t); f(t)\}$.

At resonance, velocities of the string and the flow match. Displacement is linear in the drag forcing. The Melde string isn't dispersive ($\omega = 2\pi f = ck$). Therefore, displacement is $x = (2\pi v)/(ck)$, and its power spectrum is: $<\tilde{x}(\omega)^2> \propto <\tilde{v}(\omega)^2> k^{-2}\ \propto\ k^{-11/3}$. Because the response function of a Melde string with viscous damping is independant of frequency at resonance, the FDR ratio of Eq.1 is simply $k_{\rm B}T_{\rm eff.} \propto c\, k <\tilde{x}(\omega)^2> \propto\ k^{-11/3}$. This exponent is compatible with the spectrum we measured, as can be seen in Fig.3.

Discussion. – We have constructed the simplest "theoretician's thermometer" one can imagine; a Melde string can be seen as an ensemble of independant harmonic oscillators. At equilibrium with the flow, each mode of the string couples with the fluid at scale $r = 2\omega/c$, giving information much like a spectrometer, even though the flow itself is strongly out of equilibrium.

The definition of effective temperature according to Cugliandolo-Kurchan, based on the FDR, is used to measure a statistical *temperature of turbulence*. For this purpose and with our Melde string, fluctuations and dissipation are measured separately for all wave-numbers of the inertial range in the turbulent air jet flow. The *thermal* agitation $K_{\mathrm{B}}T_{\mathrm{eff}}$ is given by degree of freedom, versus the wave number. Those quantities are ploted in proper Reynolds number corrected axis, and exhibit a unique power law, when the Re is between 74000 and 170000. This range will soon be extended. The exponent is consistent with a value $-11/3$ given by a very simple model based on Kolmogorov 1941 theory.

If the equipartition theorem was applying, $k_{\mathrm{B}}T_{\mathrm{eff.}}/Re^{9/4}$ would be a constant. But there is no equipartition of energy because: 1- the system is out of equilibrium, 2- the degrees of freedom are not independant. Note that those two points are linked because of the non-linearity of Navier-Stokes equation.

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We acknowledge B. Castaing, E. Leveque, F. Delduc, S. Ciliberto for many discussions, and also F. Dumas for his help in the construction of positioning devices. Several students contributed to this study as part of their graduate lab-course, and are gratefully acknowledged: A. Louvet, G. Bordes, I. Dossmann, J. Perret, C. Cohen. We also thank the guitar maker D. Teyssot, from Lyon, who gently gave us his thinest E strings.

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