Kinetic equation for finite systems of fermions with pairing

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Abstract

The solutions of the Wigner–transformed time–dependent Hartree–Fock–Bogoliubov equations are studied in the constant–Δ approximation, in spite of the fact that this approximation is known to violate particle–number conservation and gauge invariance. We find that there are two distinct classes of solutions: one that violates the particle–number symmetry and one that preserves the number of particles in the system. The two kinds of solutions can be easily separated. The number–preserving solutions are studied in detail in a one–dimensional model and are seen to violate the continuity equation. A simple prescription for restoring both broken symmetries is proposed. Explicit expressions for the eigenfrequencies of the correlated systems and for the density response function are derived and it is shown that the semiclassical analogous of the quantum single–particle spectrum, has an excitation gap of 2Δ, in agreement with the quantum result. The collective response is studied for a simplified form of the residual interaction.

1   Introduction

The problem of extending the Vlasov equation to systems in which pairing correlations play an important role has been tackled some time ago by Di Toro and Kolomietz [1] in a nuclear physics context and, more recently, by Urban and Schuck [2] for trapped fermion droplets. These last authors derived the TDHFB equations for the Wigner transform of the normal density matrix ϱ and of the pair correlation function κ (plus their time–reversal conjugates) and used them to study the dynamics of a spin-saturated trapped Fermi gas. In the time-dependent theory one obtains a system of four coupled differential equations for ϱ, κ, and their conjugates [2] and, if one wants an analytical solution, some approximation must be introduced. Here we try to find a solution

1

of the equations of motion derived by Urban and Schuck in the approximation in which the pairing field $\Delta(\mathbf{r}, \mathbf{p}, t)$ is treated as a constant. It is well known that such an approximation violates both particle–conservation and gauge invariance (see e.g. sect. 8–5 of [3] and [4]), nonetheless we study it because of its simplicity, with the aim of correcting the final results for its shortcomings. In Sct. 2, the basic equations are recalled and reformulated in terms of the even and odd components of the normal density ρ. In Sct. 3, the static limit is studied by following the approach of [5] and the constant–Δ approximation is introduced. In Sect. 4, the simplified dynamic equations resulting from the constant–Δ approxmation are derived and their solutions are studied. In Sect. 5, a simplified model of particles moving in a one–dimensional potential well is studied in detail. The effect of pairing correlations on the eigenfrequencies of the system and on the density response function are determined. It is shown that the problem of particle–number violation is due to one particular mode only, if this mode is omitted, the particle–number symmetry is restored. This restores only the *global* particle–number symmetry, not the *local* particle–number symmetry, which is related to the continuity equation. The local particle–number symmetry is automatically satisfied if the density fluctuations are expressed in terms of the current density through the continuity equation. This simple prescription (which, of course, solves also the global particle–number problem) is followed in order to evaluate the density response function of various one–dimensional systems of given density, but different sizes. It is shown that, in one dimension, the effects of pairing correlation depend strongly on the size of the system. We have in mind applications to nuclei, where the dependence of the effective radial potential on the nucleon angular momentum, gives effects that are similar to those of changing size in a one-dimensional system.

2   Basic equations

We assume that our system is saturated both in spin and isospin space and do not distinguish between neutrons and protons, so we can use directly the equations of motion of Urban and Schuck.

We start from the equations of motion derived in Ref. [2] for the Wigner–trasformed density matrices $\varrho = \varrho_(\mathbf{r}, \mathbf{p}, t)$ and $\kappa = \kappa(\mathbf{r}, \mathbf{p}, t)$, with the warning that the sign of κ that we are using agrees with that of Ref. [1], hence it is opposite to that of [2]. Moreover we find convenient to use the odd and even combinations of the normal density introduced in [2]:

$$\rho_{ev} = \frac{1}{2}[\rho(\mathbf{r}, \mathbf{p}, t) + \rho(\mathbf{r}, -\mathbf{p}, t)], \tag{1}$$

$$\rho_{od} = \frac{1}{2}[\rho(\mathbf{r}, \mathbf{p}, t) - \rho(\mathbf{r}, -\mathbf{p}, t)]. \tag{2}$$

Thus, the equations of motion given by Eqs.(15a...d) of Ref. [2] read

$$i\hbar\partial_t \rho_{ev} = i\hbar\{h, \rho_{od}\} - 2i\text{Im}[\Delta^*(\mathbf{r}, \mathbf{p}, t)\kappa] \tag{3}$$

$$
\begin{aligned}
i\hbar\partial_t\rho_{od} &= i\hbar\{h,\rho_{ev}\} + i\hbar\mathrm{Re}\{\Delta^*(\mathbf{r},\mathbf{p},t),\kappa\} && (4) \\
i\hbar\partial_t\kappa &= 2(h-\mu)\kappa - \Delta(\mathbf{r},\mathbf{p},t)(2\rho_{ev}-1) + i\hbar\{\Delta(\mathbf{r},\mathbf{p},t),\rho_{od}\}\,. && (5)
\end{aligned}
$$

Here h is the Wigner–transformed Hartree–Fock hamiltonian $h(\mathbf{r},\mathbf{p},t)$, while $\Delta(\mathbf{r},\mathbf{p},t)$ is the Wigner–transformed pairing field. Since the time–dependent part of κ is complex, $\kappa = \kappa_r + i\kappa_i$, the last equation gives two independent equations for the real and imaginary parts of κ.

Moreover, from the supplementary normalization condition ([6], p. 252)

$$
\mathcal{R}^2 = \mathcal{R} \tag{6}
$$

satisfied by the generalized density matrix $\mathcal{R}$, the two following independent equations are obtained:

$$
\begin{aligned}
\rho_{od}\kappa + i\frac{\hbar}{2}\{\rho_{ev},\kappa\} &= 0\,, && (7) \\
\rho_{ev}(\rho_{ev}-1) + \rho_{od}^2 + \kappa\kappa^* &= 0\,. && (8)
\end{aligned}
$$

We shall use the equations of motion (3–5), together with these equations, as our starting point.

First of all we notice that, in the limit of no pairing, both Δ and κ vanish, the third equation of motion reduces to a trivial identity, while the first two give the Vlasov equation for normal systems, expressed in terms of the odd and even components of ρ:

$$
\begin{aligned}
\partial_t\rho_{ev} &= \{h,\rho_{od}\}\,, && (9) \\
\partial_t\rho_{od} &= \{h,\rho_{ev}\}\,. && (10)
\end{aligned}
$$

A solution of the linearized Vlasov equation for normal systems (i. e. without pairing) has been obtained in Ref. [7] and our aim here is to study the changes introduced by the pairing interaction in the solution of [7].

Before studying the time–dependent problem, it is useful to look at the static limit.

3 Static limit

In this section we follow the approach of Ref. [5]. At equilibrium we have

$$
\begin{aligned}
\rho_{ev} &= \rho_0(\mathbf{r},\mathbf{p})\,, && (11) \\
\rho_{od} &= 0\,, && (12) \\
h &= h_0(\mathbf{r},\mathbf{p})\,, && (13) \\
\kappa &= \kappa_0(\mathbf{r},\mathbf{p}) && (14) \\
\Delta_0 &= \Delta_0(\mathbf{r},\mathbf{p}) && (15)
\end{aligned}
$$

and equations (3–5) give

$$
\begin{aligned}
0 &= -2i\mathrm{Im}(\Delta_0^*\kappa_0)\,, && (16) \\
0 &= i\hbar\{h_0,\rho_0\} + i\hbar\mathrm{Re}\{\Delta_0^*,\kappa_0\}\,, && (17) \\
0 &= 2(h_0-\mu)\kappa_0 - \Delta_0(2\rho_0-1)\,, && (18)
\end{aligned}
$$

3

while Eqs. (7, 8) give

$$i\frac{\hbar}{2}\{\rho_0, \kappa_0\} = 0, \tag{19}$$

$$\rho_0(\rho_0 - 1) + |\kappa_0|^2 = 0. \tag{20}$$

Equation (16) is satisfied if we assume that Δ_0 and κ_0 are real quantities, while Eqs. (18) and (20), taken as a system, have the solution:[5]

$$\rho_0(\mathbf{r}, \mathbf{p}) = \frac{1}{2}\left(1 - \frac{h_0(\mathbf{r}, \mathbf{p}) - \mu}{E(\mathbf{r}, \mathbf{p})}\right) \tag{21}$$

$$\kappa_0(\mathbf{r}, \mathbf{p}) = -\frac{\Delta_0(\mathbf{r}, \mathbf{p})}{2E(\mathbf{r}, \mathbf{p})}, \tag{22}$$

with the quasiparticle energy

$$E(\mathbf{r}, \mathbf{p}) = \sqrt{\Delta_0^2(\mathbf{r}, \mathbf{p}) + (h_0(\mathbf{r}, \mathbf{p}) - \mu)^2}. \tag{23}$$

It can be easily checked that Eqs. (21, 22) satisfy also Eqs. (17) and (19), that is

$$\{h_0, \rho_0\} + \{\Delta_0, \kappa_0\} = 0 \tag{24}$$

and

$$\{\rho_0, \kappa_0\} = 0. \tag{25}$$

The (semi)classical equilibrium phase–space distribution is closely related to $\rho_0(\mathbf{r}, \mathbf{p})$:

$$f_0(\mathbf{r}, \mathbf{p}) = \frac{4}{(2\pi\hbar)^3}\rho_0(\mathbf{r}, \mathbf{p}) \tag{26}$$

and the statistical factor 4 takes into account the fact that there are two kinds of fermions.

The parametrer μ is determined by the condition

$$A = \int d\mathbf{r} d\mathbf{p} f_0(\mathbf{r}, \mathbf{p}). \tag{27}$$

3.1 Constant–Δ approximation

In a fully self–consistent approach, the pairing field $\Delta(\mathbf{r}, \mathbf{p}, t)$ is related to $\kappa(\mathbf{r}, \mathbf{p}, t)$, however here we introduce an approximation and replace the pairing field of the HFB theory with the phenomenological pairing gap of nuclei, hence in all our equations we put

$$\Delta(\mathbf{r}, \mathbf{p}, t) \approx \Delta_0(\mathbf{r}, \mathbf{p}) \approx \Delta = const., \tag{28}$$

with $\Delta \approx 1\text{MeV}$.

In the constant-Δ approximation the equilibrium distributions become

$$\rho_0(\epsilon) = \frac{1}{2}\left(1 - \frac{\epsilon - \mu}{E(\epsilon)}\right), \tag{29}$$

$$\kappa_0(\epsilon) = -\frac{\Delta}{2E(\epsilon)} \tag{30}$$

and the quasiparticle energy

$$E(\epsilon) = \sqrt{\Delta^2 + (\epsilon - \mu)^2}, \tag{31}$$

with

$$\epsilon = h_0(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V_0(\mathbf{r})$$

the particle energy in the equilibrium mean field.

In the following we shall use the relation, valid for finite Δ,

$$\kappa_0(\epsilon) = \frac{E^2(\epsilon)}{\Delta}\frac{d\rho_0(\epsilon)}{d\epsilon}. \tag{32}$$

In the limit of $\Delta \to 0$, we have $\kappa_0(\epsilon) \to 0$, $E(\epsilon) \to |\epsilon - \mu|$ and $\rho_0(\epsilon) \to \vartheta(\mu - \epsilon)$, as appropriate for normal Fermi systems at zero temperature.

4 Dynamics

Always in constant-Δ approximation, the time–dependent equations (3–5) become instead

$$i\hbar\partial_t\rho_{ev} = i\hbar\{h, \rho_{od}\} - 2i\Delta\mathrm{Im}(\kappa) \tag{33}$$

$$i\hbar\partial_t\rho_{od} = i\hbar\{h, \rho_{ev}\} \tag{34}$$

$$i\hbar\partial_t\kappa = 2(h - \mu)\kappa - \Delta(2\rho_{ev} - 1). \tag{35}$$

This is the simplified set of equations that we want to study here. The sum of the first two equations gives an equation that is similar to the Vlasov equation of normal systems, only with the extra term $-2i\Delta\mathrm{Im}(\kappa)$. This extra term couples the equation of motion of ρ with that of κ, thus, instead of a single differential equation (Vlasov equation), now we have a system of two coupled differential equations (for ρ and κ_i).

A good starting point for the solution of the normal Vlasov equation is to assume that the average mean field keeps its static value. This approximation, that was termed "zero–order" in [7], corresponds to the quantum single–particle approximation and is a preliminary step that allows us to include collective effects in a second stage. More explicitly, out of equilibrium the forces acting on a particle will differ from the forces acting in equilibrium, the extra forces can be divided into external and internal parts, accordingly

$$\delta h = \delta V^{ext} + \delta V^{int}. \tag{36}$$

The zero–order approximation is obtained by taking $\delta h = \delta V^{ext}$.

4.1 Even solution

As we shall show, there are two classes of solutions of Eqs. (33–35): a first class with $\rho_{od} = 0$ and a second class in which $\rho_{od} \neq 0$. Only the second class of solutions has a correspondence with the solutions of the normal Vlasov equation, while the solutions of the first class lead to a fluctuation of the number of particles in the system. In the following sections we shall assume that our system is slightly perturbed by a weak external force and will study its time evolution, in this section instead, we assume that there is no external driving field, but that the system is initially displaced from its equilibrium conditions (but with $\rho_{od} = 0$) and then evolves freely in time (according to the equilibrium hamiltonian h_0).

When $\rho_{od} = 0$ and $h = h_0$, Eqs. (33–35) read

$$\partial_t \rho = -2\frac{\Delta}{\hbar}\kappa_i \,, \tag{37}$$

$$0 = i\hbar\{h_0, \rho\} \,, \tag{38}$$

$$\partial_t \kappa = -\frac{i}{\hbar}[2(h_0 - \mu)\kappa - \Delta(2\rho - 1)] \,. \tag{39}$$

The second equation is satisfied if $\rho = \rho(h_0, t)$, while the last equation gives two equations for the real and imaginary parts of κ:

$$\partial_t \kappa_r = 2\frac{(h_0 - \mu)}{\hbar}\kappa_i \,, \tag{40}$$

$$\partial_t \kappa_i = -2\frac{(h_0 - \mu)}{\hbar}\kappa_r + \frac{\Delta}{\hbar}(2\rho - 1) \,. \tag{41}$$

Taking the time derivative of Eq. (37) gives

$$\partial_t^2 \rho = -2\frac{\Delta}{\hbar}\partial_t \kappa_i \tag{42}$$

and, by using the equations given above, it can be easily found that

$$\partial_t^3 \rho = -\left(\frac{2E(h_0)}{\hbar}\right)^2 \partial_t \rho = -\Omega^2(h_0)\partial_t \rho \,, \tag{43}$$

which implies that ρ and κ both oscillate with frequency

$$\Omega(h_0) = 2\frac{E(h_0)}{\hbar} \,. \tag{44}$$

It is interesting to note that this eigenfrequency does not depend explicitly on the size of the system.

Thus, the general solution of Eqs. (33–35) with $\rho_{od} = 0$ and $h = h_0$ is

$$\rho_{ev}(\mathbf{r}, \mathbf{p}, t) = \rho(\mathbf{r}, \mathbf{p}, t) = \frac{1}{2}\left(1 - B(h_0)\frac{h_0 - \mu}{E(h_0)}\right)$$
$$+C(h_0)e^{i\Omega(h_0)t} + C^*(h_0)e^{-i\Omega(h_0)t} \tag{45}$$
$$\kappa(\mathbf{r}, \mathbf{p}, t) = a + be^{i\Omega(h_0)t} + ce^{-i\Omega(h_0)t} \,. \tag{46}$$

The parameters $B(h_0)$ and $C(h_0)$ are determined by the initial conditions, while a, b, c are related to $B(h_0)$ and $C(h_0)$ through the equations of motion, giving

$$a = B(h_0)\kappa_0(h_0) \,, \tag{47}$$

$$b = C(h_0)\frac{\Delta}{E(h_0) + (h_0 - \mu)} \,, \tag{48}$$

$$c = -C^*(h_0)\frac{\Delta}{E(h_0) - (h_0 - \mu)} \,. \tag{49}$$

The initial conditions that determine the parameters $B(h_0)$ and $C(h_0)$ are the initial values of ρ and κ and the initial value of the constraint (8) that, in this case reads

$$\rho(\rho - 1) + |\kappa|^2 = 0 \,. \tag{50}$$

It can be checked that, for the solution (45,46) this constraint holds at all times, provided that

$$B(h_0) = \sqrt{1 - 16|C(h_0)|^2\frac{E^2(h_0)}{\Delta^2}} \,, \tag{51}$$

(the other equation of the constraint , Eq. (7), is also obviously satisfied). The last equation places a restriction on the amplitude $B(h_0)$ of the time–dependent part of $\rho(\mathbf{r}, \mathbf{p}, t)$, restriction which is more severe for larger values of $E(h_0)/\Delta$. When $C(h_0) = 0$ we recover the standard static solutions (29,30).

Clearly the solution (45) induces a time–dependent part into the particle–number integral

$$\frac{4}{(2\pi\hbar)^3} \int d\mathbf{r} d\mathbf{p} \rho(\mathbf{r}, \mathbf{p}, t) = A + \delta A(t) \,, \tag{52}$$

thus it violates particle–number conservation. We shall comment on this point later.

4.2   Linearized equations

There is another class of solutions of Eqs. (33–35): the solutions with $\rho_{od} \neq 0$.

In order to study this second class of solutions, we assume that our system is initially at equilibrium, with densities given by Eqs. (29,30), and that at time $t = 0$ a weak external field of the kind

$$\delta V^{ext}(\mathbf{r}, t) = \beta\delta(t)Q(\mathbf{r}) \tag{53}$$

is applied to it. In a self–consistent approach, the fluctuation of the hamiltonian should be given by Eq. (36), but here we start with the zero–order approximation and assume

$$\delta h = \delta V^{ext}(\mathbf{r}, t) \,. \tag{54}$$

We want to solve Eqs. (33–35) in linear approximation, thus we study small fluctuations of the time–dependent quantities about their equilibrium values and neglect terms that are of second order in the fluctuations.

Hence, in Eqs. (33–35) we put:

$$h = h_0 + \delta h, \tag{55}$$

$$\rho_{ev} = \rho_0 + \delta\rho_{ev}, \tag{56}$$

$$\rho_{od} = \delta\rho_{od}, \tag{57}$$

$$\kappa = \kappa_0 + \delta\kappa = \kappa_0 + \delta\kappa_r + i\delta\kappa_i. \tag{58}$$

Then, the linearized form of Eqs. (33–35) is

$$i\hbar\partial_t\rho_{ev} = i\hbar\{h_0, \delta\rho_{od}\} - 2i\Delta\delta\kappa_i \tag{59}$$

$$i\hbar\partial_t\delta\rho_{od} = i\hbar\{h_0, \delta\rho_{ev}\} + i\hbar\{\delta h, \rho_0\} \tag{60}$$

$$-\hbar\partial_t\delta\kappa_i = 2(\epsilon - \mu)\delta\kappa_r + 2\kappa_0\delta h - 2\Delta\delta\rho_{ev}, \tag{61}$$

$$\hbar\partial_t\delta\kappa_r = 2(\epsilon - \mu)\delta\kappa_i. \tag{62}$$

Taking the sum of the first two equations gives

$$i\hbar\partial_t\delta\rho(\mathbf{r}, \mathbf{p}, t) = i\hbar\{h_0, \delta\rho(\mathbf{r}, \mathbf{p}, t)\} + i\hbar\{\delta h(\mathbf{r}, \mathbf{p}, t), \rho_0\} - 2i\Delta\delta\kappa_i(\mathbf{r}, \mathbf{p}, t), \tag{63}$$

which can be regarded as an extension of the linearized Vlasov equation studied in [7]. Compared to the same equation without pairing, in the constant–Δ approximation only the last term on the right–end side is added.

In order to make the comparison with [7] easier, from now on we change the normalization of the phase–space densities and define

$$f(\mathbf{r}, \mathbf{p}, t) = \frac{4}{(2\pi\hbar)^3}\rho(\mathbf{r}, \mathbf{p}, t) \tag{64}$$

$$\chi(\mathbf{r}, \mathbf{p}, t) = \frac{4}{(2\pi\hbar)^3}\kappa(\mathbf{r}, \mathbf{p}, t), \tag{65}$$

moreover, we put

$$F(\epsilon) = \frac{4}{(2\pi\hbar)^3}\rho_0(\epsilon). \tag{66}$$

In terms of the new functions Eqs. (63) and (61) read

$$i\hbar\partial_t\delta f(\mathbf{r}, \mathbf{p}, t) = i\hbar\{h_0, \delta f(\mathbf{r}, \mathbf{p}, t)\} + i\hbar\{\delta h(\mathbf{r}, \mathbf{p}, t), f_0\}$$
$$- 2i\Delta\delta\chi_i(\mathbf{r}, \mathbf{p}, t), \tag{67}$$

$$-\hbar\partial_t\delta\chi_i(\mathbf{r}, \mathbf{p}, t) = 2(\epsilon - \mu)\delta\chi_r(\mathbf{r}, \mathbf{p}, t) + 2\chi_0\delta h(\mathbf{r}, \mathbf{p}, t)$$
$$- 2\Delta\delta f_{ev}(\mathbf{r}, \mathbf{p}, t). \tag{68}$$

The function f_{ev} is given by the obvious extension of Eq. (1). In order to get a closed system of equations, we still need an extra equation for $\delta\chi_r(\mathbf{r}, \mathbf{p}, t)$. This can be obtained from the linearized form of the supplementary condition (8) that reads

$$\delta\rho_{ev}(2\rho_0 - 1) = -2\kappa_0\delta\kappa_r, \tag{69}$$

or

$$\delta\kappa_r(\mathbf{r}, \mathbf{p}, t) = \frac{1 - 2\rho_0(\epsilon)}{2\kappa_0}\delta\rho_{ev}(\mathbf{r}, \mathbf{p}, t) = -\frac{\epsilon - \mu}{\Delta}\delta\rho_{ev}(\mathbf{r}, \mathbf{p}, t). \tag{70}$$

8

The last expression has been obtained with the help of Eq. (18). In terms of the new functions f and χ, the last equation reads

$$\delta\chi_r(\mathbf{r},\mathbf{p},t) = -\frac{\epsilon - \mu}{\Delta}\delta f_{ev}(\mathbf{r},\mathbf{p},t)\,. \tag{71}$$

Equations (67, 68) and (71) are the set of coupled equations for the phase–space densities that we have to solve.

Replacing Eq. (71) into Eq. (68), and using Eq. (30), gives the following system of coupled differential equations:

$$\partial_t \delta f(\mathbf{r},\mathbf{p},t) = \{h_0, \delta f\} + \{\delta h, f_0\} - 2\frac{\Delta}{\hbar}\delta\chi_i(\mathbf{r},\mathbf{p},t)\,, \tag{72}$$

$$\partial_t \delta\chi_i(\mathbf{r},\mathbf{p},t) = \frac{E^2(\epsilon)}{\hbar\Delta}[\delta f(\mathbf{r},\mathbf{p},t) + \delta f(\mathbf{r},-\mathbf{p},t)] - 2\frac{\chi_0}{\hbar}\delta h(\mathbf{r},\mathbf{p},t)\,. \tag{73}$$

Taking the Fourier transform in time, gives

$$-i\omega\delta f(\mathbf{r},\mathbf{p},\omega) = \{h_0, \delta f\} + \{\delta h, f_0\} - 2\frac{\Delta}{\hbar}\delta\chi_i(\mathbf{r},\mathbf{p},\omega)\,, \tag{74}$$

$$-i\omega\delta\chi_i(\mathbf{r},\mathbf{p},\omega)) = \frac{E^2(\epsilon)}{\hbar\Delta}[\delta f(\mathbf{r},\mathbf{p},\omega) + \delta f(\mathbf{r},-\mathbf{p},\omega)]$$
$$- 2\frac{\chi_0}{\hbar}\delta h(\mathbf{r},\mathbf{p},\omega)\,, \tag{75}$$

or (for $\omega \neq 0$)

$$-i\omega\delta f(\mathbf{r},\mathbf{p},\omega) + \{\delta f, h_0\} = -i\omega d^2\frac{1}{2}[\delta f(\mathbf{r},\mathbf{p},\omega) + \delta f(\mathbf{r},-\mathbf{p},\omega)]$$
$$- F'(\epsilon)\{h_0, \delta h\} + i\omega d^2\frac{\Delta}{E^2(\epsilon)}\chi_0\delta h\,, \tag{76}$$

with

$$d^2 = \left(\frac{\Omega(\epsilon)}{\omega}\right)^2 \tag{77}$$

and $\Omega(\epsilon)$ from Eq. (44). In Eq. (76) we have used the relation $\{f_0, \delta h\} = F'(\epsilon)\{h_o, \delta h\}$.

By comparing Eq. (76) with Eq. (2.12) of [7], we note that the only effect of pairing in the constant–Δ approximation is that of adding to the normal Vlasov equation the terms proportional to d^2 in Eq. (76).

It is useful to study the solutions of Eq. (76) in one dimension. This is a preliminary step that helps to solve the more complicated three–dimensional problem.

5   One–dimensional systems

5.1   Zero–order solution

In one dimension, Eq.(76) reads

$$-i\omega\delta f(x,p_x,\omega) + \dot{x}\partial_x\delta f - \frac{dV_0(x)}{dx}\partial_{p_x}\delta f = \tag{78}$$

$$-i\omega d^2 \frac{1}{2}[\delta f(x, p_x, \omega) + \delta f(x, -p_x, \omega)] + F'(\epsilon)(\beta \frac{p_x}{m} \frac{dQ(x)}{dx} + i\omega d^2 \delta h),$$

with $\delta h(x, p_x, \omega) = \beta Q(x)$ (zero–order approximation). We have used the relation (32) in the last term. Obviously, in one dimension

$$F'(\epsilon) = \frac{4}{2\pi\hbar} \frac{d\rho_0}{d\epsilon}. \tag{79}$$

Changing variables from $(x, p_x) \to (x, \epsilon)$, with $\epsilon = \frac{p_x^2}{2m} + V_0(x)$, and putting

$$f^\pm = \delta f(x, \pm\sqrt{2m[\epsilon - V_0(x)]}, \omega)$$

gives

$$\frac{\partial f^+}{\partial x} - \frac{i\omega}{v(\epsilon, x)} f^+ = B^+(x) - \frac{i\omega}{v(\epsilon, x)} d^2 \frac{1}{2}[f^+ + f^-] \tag{80}$$

$$\frac{\partial f^-}{\partial x} + \frac{i\omega}{v(\epsilon, x)} f^- = B^-(x) + \frac{i\omega}{v(\epsilon, x)} d^2 \frac{1}{2}[f^+ + f^-], \tag{81}$$

with

$$v(\epsilon, x) = \sqrt{\frac{2}{m}[\epsilon - V_0(x)]}. \tag{82}$$

The analogous equations for normal systems are [7]

$$\frac{\partial f^+}{\partial x} - \frac{i\omega}{v(\epsilon, x)} f^+ = B(x) \tag{83}$$

$$\frac{\partial f^-}{\partial x} + \frac{i\omega}{v(\epsilon, x)} f^- = B(x), \tag{84}$$

with

$$B(x) = \beta F'(\epsilon)\left(\frac{dQ(x)}{dx}\right) \tag{85}$$

in zero–order approximation. The inhomogeneous terms in (80, 81) instead, are given by

$$B^\pm(x) = B(x) \pm d^2 C(x), \tag{86}$$

with (always in zero–order approximation)

$$C(x) = \frac{i\omega}{v(\epsilon, x)} \beta F'(\epsilon) Q(x). \tag{87}$$

Taking (half) the sum and difference of the two equations (80) and (81) gives

$$\partial_x f_{ev} - \frac{i\omega}{v(\epsilon, x)} f_{od} = B(x) \tag{88}$$

$$\partial_x f_{od} - \frac{i\omega}{v(\epsilon, x)}(1 - d^2) f_{ev} = d^2 C(x). \tag{89}$$

10

Putting $y = f_{od}$, this system gives the second–order equation

$$y" + \mathcal{P}(x)y' + \mathcal{Q}(x)y = \mathcal{R}(x),\tag{90}$$

with

$$\mathcal{P}(x) = \frac{\partial_x v(\epsilon, x)}{v(\epsilon, x)}\tag{91}$$

$$\mathcal{Q}(x) = \frac{\omega^2 - \Omega^2(\epsilon)}{v^2(\epsilon, x)}\tag{92}$$

$$\mathcal{R}(x) = \frac{i\omega}{v(\epsilon, x)}B(x).\tag{93}$$

Note that the non homogeneous term $\mathcal{R}(x)$ does not involve $C(x)$.

We want to find the solution of Eq. (90) satisfying the fixed–surface boundary conditions

$$f_{od}(x_1) = 0,\tag{94}$$
$$f_{od}(x_2) = 0.\tag{95}$$

By changing variable from x to $\tau(x)$, with

$$\tau(x) = \int_{x_1}^{x} dx' \frac{1}{v(\epsilon, x')}\tag{96}$$

and $x(\tau)$ its inverse, Eq. (90) becomes

$$\frac{\partial^2 y(\tau)}{\partial \tau^2} + \mathcal{Q}(x(\tau))y = \mathcal{R}(x(\tau)).\tag{97}$$

When formulated in this way, the problem of solving the system (80, 81) becomes equivalent to that of finding the eigenmodes of a stretched string satisfying appropriate boundary conditions, however here we have an extra constraint that has no equivalent in the stretched–string problem: particle–number conservation, that is, the integral (52) must be constant in time.

The eigenfrequencies of the system are determined in standard fashion by imposing the required boundary conditions on the solution of the associated homogeneous equation

$$\tilde{y} = a \cos \sqrt{\omega^2 - \Omega^2(\epsilon)}\tau(x) + b \sin \sqrt{\omega^2 - \Omega^2(\epsilon)}\tau(x),\tag{98}$$

this gives

$$\bar{\omega}_n(\epsilon) = n\omega_0(\epsilon)\sqrt{1 + \left(\frac{\Omega(\epsilon)}{n\omega_0(\epsilon)}\right)^2} \quad (n \neq 0)\tag{99}$$

$$\bar{\omega}_0(\epsilon) = \pm\Omega(\epsilon),\tag{100}$$

11

where n is an integer. The frequency $\omega_0(\epsilon) = 2\pi/T$ is the fundamental frequency of the uncorrelated system. The new eigenfrequencies should be compared to the eigenfrequencies of the normal system derived in [7]:

$$\omega_n(\epsilon) = n\omega_0(\epsilon)\,. \tag{101}$$

This comparison shows that the pairing correlations increase the absolute value of the system eigenfrequencies. If $\omega_0 >> \Omega$, like for nuclear giant resonances, for $n \neq 0$ the shift is rather small, in agreement with the findings of [1]. The mode $n = 0$ is the one which is most affected by the pairing correlations since the absolute value of the corresponding eigenfrequency changes from 0 to Ω.

It is interesting to note that, for an infinite uniform system, all the eigenfrequencies (99) tend to $\pm\Omega$, since in that case $\omega_0 \to 0$.

For any value of ω_0, the lowest eigenfrequency is Ω, which has a minimum value of $2\Delta/\hbar$. Thus, the excitation spectrum given by these equations has an enegy gap of 2Δ, in agreement with quantum calculations.

We turn now to the non homogeneous equation (90) and expand the external field $Q(x)$ as

$$Q(x) = \sum_{n=0}^{\infty} Q_n \cos n\omega_0\tau(x)\,, \tag{102}$$

with

$$Q_n = \frac{4}{T} \int_0^{\frac{T}{2}} d\tau Q(x(\tau)) \cos n\omega_0\tau = \frac{4}{T} \int_{x_1}^{x_2} \frac{dx}{v(\epsilon,x)} Q(x) \cos n\omega_0\tau(x) \tag{103}$$

for $n \neq 0$ and

$$Q_0 = \frac{2}{T} \int_{x_1}^{x_2} \frac{dx}{v(\epsilon,x)} Q(x)\,. \tag{104}$$

Since the external potential $Q(x)$ is defined only up to an arbitrary constant, we fix this constant by the requirement that

$$Q_0 = 0\,. \tag{105}$$

As we shall see, this is required by the condition that the number of particles remains constant in time.

The solution of the non homogeneous equation (90) can be expanded in a similar way:

$$f_{od}(x,\epsilon,\omega) = \sum_{n=0}^{\infty} B_n(\omega) \sin n\omega_0\tau(x)\,, \tag{106}$$

while

$$f_{ev}(x,\epsilon,\omega) = \sum_{n=0}^{\infty} A_n(\omega) \cos \omega_0\tau(x)\,. \tag{107}$$

12

The coefficients B_n and A_n are given by

$$B_n(\omega) = \frac{in\omega_0}{\omega}[A_n(\omega) - \beta F'(\epsilon)Q_n], \tag{108}$$

$$A_n(\omega) = -\beta F'(\epsilon)\frac{\bar{\omega}_n^2}{\omega^2 - \bar{\omega}_n^2}Q_n \qquad n \neq 0, \tag{109}$$

$$A_0(\omega) = -\beta F'(\epsilon)\frac{\Omega^2}{\omega^2 - \Omega^2}Q_0. \tag{110}$$

Consequently, if $Q_0 = 0$, we also have that $A_0(\omega) = 0$.
The fluctuation of the particle–number integral is given by

$$\delta A(\omega) = 2\int dx dp_x f_{ev}(x, \epsilon, \omega) \tag{111}$$

$$= 2\int_0^\infty d\epsilon A_0(\omega)\int_0^{T/2} d\tau + 2\sum_{n=1}^\infty \int_0^\infty d\epsilon A_n(\omega)\int_0^{T/2} d\tau \cos n\omega_0\tau.$$

Since the integrals $\int_0^{T/2} d\tau \cos n\omega_0\tau$ vanish when $n \neq 0$, the term containing A_0 is the only one giving a fluctuation of the number of particles. Clearly this mode corresponds to the even solution discussed in Sect. (4.1). However, since we can put $A_0(\omega) = 0$, particle–number conservation is ensured, at least in linear approximation.

Thus, the solution of the system (80, 81) satisfying the boundary conditions (94, 95) *and* particle–number conservation, is given by

$$f_{ev}(x, \epsilon, \omega) = \sum_{n=1}^\infty A_n(\omega)\cos n\omega_0\tau(x), \tag{112}$$

$$f_{od}(x, \epsilon, \omega) = \sum_{n=1}^\infty B_n(\omega)\sin n\omega_0\tau(x). \tag{113}$$

However, even if these solutions do not violate particle–number conservation, they are not acceptable as solutions of our problem because they do not satisfy the continuity equation

$$\partial_x j(x, \omega) - i\omega\delta\varrho(x, \omega) = 0. \tag{114}$$

The current density $j(x, \omega)$ involves only the odd part of $\delta f(x, p_x, \omega)$:

$$j(x, \omega) = \int dp_x \frac{p_x}{m}\delta f(x, p_x, \omega) = 2\int d\epsilon f_{od}(x, \epsilon, \omega), \tag{115}$$

while the density fluctuation involves only the even part:

$$\delta\varrho(x, \omega) = \int dp_x \delta f(x, p_x, \omega) = 2\int d\epsilon \frac{1}{v(\epsilon, x)}f_{ev}(x, \epsilon, \omega). \tag{116}$$

Thus, in order to allow for the constant–Δ approximation, we must change the equations of motion (33–35) and, consequently, Eq.s (88,89).

13

5.2 Modified zero–order solution

By using the explicit expressions of f_{ev} and f_{od}, it can be easily checked that the continuity equation (114) is not satisfied by our solutions. It has been pointed out in [4] that this problem can be overcome by allowing for a coordinate and time dependence of Δ in the dynamical equations, however, in the spirit of a simplified approach, here we try to find another possible way out: we re–define the density fluctuations through Eq. (114)

$$\delta\bar{\varrho}(x,\omega) \equiv \frac{2}{i\omega}\int d\epsilon\,\partial_x f_{od}(x,\epsilon,\omega)\,. \tag{117}$$

In this way both the continuity equation and particle–number conservation are satisfied. By using equation (113) for f_{od}, we have

$$\delta\bar{\varrho}(x,\omega) = \int d\epsilon\,\frac{1}{v(\epsilon,x)}\sum_{n=1}^{\infty}\bar{A}_n(\omega)\cos n\omega_0\tau(x)\,, \tag{118}$$

with

$$\bar{A}_n(\omega) = \frac{n\omega_0}{i\omega}B_n(\omega) = -\beta F'(\epsilon)\frac{(n\omega_0)^2}{\omega^2 - \bar{\omega}_n^2}Q_n\,. \tag{119}$$

Defining

$$\bar{f}_{ev}(x,\epsilon,\omega) \equiv \sum_{n=0}^{\infty}\bar{A}_n(\omega)\cos n\omega_0\tau(x)\,, \tag{120}$$

gives

$$\delta\bar{\varrho}(x,\omega) = 2\int d\epsilon\,\frac{1}{v(\epsilon,x)}\bar{f}_{ev}(x,\epsilon,\omega)\,. \tag{121}$$

By using the explicit expressions of $A_n(\omega)$, $B_n(\omega)$ and $\bar{A}_n(\omega)$, the functions f_{ev}, f_{od} and $\bar{f}_{ev}$ can also be written as

$$f_{ev}(x,\epsilon,\omega) = -\frac{1}{2}\beta F'(\epsilon)\sum_{n=-\infty}^{n=+\infty}\frac{\bar{\omega}_n}{\omega - \bar{\omega}_n}Q_n\cos n\omega_0\tau(x)\,, \tag{122}$$

$$f_{od}(x,\epsilon,\omega) = -\frac{1}{2}\beta F'(\epsilon)\sum_{n=-\infty}^{n=+\infty}\frac{in\omega_0}{\omega - \bar{\omega}_n}Q_n\sin n\omega_0\tau(x)\,, \tag{123}$$

$$\bar{f}_{ev}(x,\epsilon,\omega) = -\frac{1}{2}\beta F'(\epsilon)\sum_{n=-\infty}^{n=+\infty}\left(\frac{n\omega_0}{\bar{\omega}_n}\right)^2\frac{\bar{\omega}_n}{\omega - \bar{\omega}_n}Q_n\cos n\omega_0\tau(x)\,. \tag{124}$$

Since $Q_0 = 0$, in the first formula it is irrelevant wether the term $n = 0$ is included into the sum or not.

It can be easily checked that

$$f_{ev}(x,\epsilon,-\omega) = f_{ev}(x,\epsilon,\omega)\,, \tag{125}$$

$$f_{od}(x,\epsilon,-\omega) = -f_{od}(x,\epsilon,\omega)\,, \tag{126}$$

$$\bar{f}_{ev}(x,\epsilon,-\omega) = \bar{f}_{ev}(x,\epsilon,\omega)\,. \tag{127}$$

14

From the expressions of f_{ev} and f_{od}, the solutions of the system (80, 81) can be easily obtained as

$$f^{\pm}(x, \epsilon, \omega) = f_{ev}(x, \epsilon, \omega) \pm f_{od}(x, \epsilon, \omega), \qquad (128)$$

while the combinations

$$\bar{f}^{\pm}(x, \epsilon, \omega) = \bar{f}_{ev}(x, \epsilon, \omega) \pm f_{od}(x, \epsilon, \omega), \qquad (129)$$

will satisfy a somewhat different system of equations. More explicitly, Eq.s (89, 88) are replaced by

$$\partial_x f_{od} - \frac{i\omega}{v(\epsilon, x)} \bar{f}_{ev} = 0, \qquad (130)$$

$$\partial_x \bar{f}_{ev} - \frac{i\omega}{v(\epsilon, x)} (1 - d^2) f_{od} = B(x) \qquad (131)$$

and their sum and difference give the equations for $\bar{f}^{\pm}$. Equation (130) ensures that the continuity equation is satisfied, while Eq. (131), together with (130), gives the same second order equation (90) as the original equations of motion.

In Eqs. (122–124), an infinitesimally small positive imaginary part $i\varepsilon$ should be added to ω in order to specify the integration path in the complex-ω plane when evaluating the corresponding time–dependent quantities.

It is convenient to express the solution of the problem discussed here in terms of a propagator analogous to that derived in [7]. The normal–density fluctuation $\delta \bar{\varrho}(x, \omega)$ is given by

$$\delta \bar{\varrho}(x, \omega) = 2 \int d\epsilon \frac{1}{v(\epsilon, x)} \bar{f}_{ev}(x, \epsilon, \omega) = \beta \int dx' \bar{D}^0(x, x', \omega) Q(x'). \qquad (132)$$

The second relation defines the propagator $\bar{D}^0(x, x', \omega)$. Introducing the explicit expression of the coefficients Q_n into Eq. (124) gives

$$\bar{D}^0(x, x', \omega) = 2 \int d\epsilon F'(\epsilon) \sum_n \frac{-2n\omega_0}{T} \frac{n\omega_0}{\bar{\omega}_n} \frac{\cos n\omega_0 \tau(x)}{v(\epsilon, x)} \frac{1}{\omega - \bar{\omega}_n + i\varepsilon} \frac{\cos n\omega_0 \tau(x')}{v(\epsilon, x')}. \qquad (133)$$

It can be easily checked that, in the limit of no pairing, the correlated propagotor (133) reproduces the uncorrelated zero–order propagator $D^0(x, x', \omega)$ of [7]:

$$\lim_{\Delta \to 0} \bar{D}^0(x, x', \omega) = D^0(x, x', \omega), \qquad (134)$$

with

$$D^0(x, x', \omega) = 2 \int d\epsilon F'(\epsilon) \sum_n \frac{-2n\omega_0}{T} \frac{\cos n\omega_0 \tau(x)}{v(\epsilon, x)} \frac{1}{\omega - n\omega_0 + i\varepsilon} \frac{\cos n\omega_0 \tau(x')}{v(\epsilon, x')}. \qquad (135)$$

From these propagators the corresponding strength functions are obtained as

$$S^0(\omega) = -\frac{1}{\pi} \text{Im} \int dx dx' Q(x) D^0(x, x', \omega) Q(x') \qquad (136)$$

15

and

$$\bar{S}^0(\omega) = -\frac{\pi}{2} \sum_n \int d\epsilon F'(\epsilon) n^2 \frac{\omega_0}{\bar{\omega}_n} Q_n^2 \delta(\omega - \bar{\omega}_n). \qquad (137)$$

As an illustrative example, here we compare the response functions of correlated and uncorrelated systems in three different cases. The three examples differ only in size, but have the same (linear) density. The static mean field is approximated by a square–well potential of length L ($L = L_1$, L_2, L_3 in the three cases considered), while the external field is assumed to be of the kind $Q(x) = x^2$.

Since the term $n = 0$ in the correlated propagator (133) vanishes, like for the uncorrelated propagator, the condition $Q_0 = 0$ is no longer necessary in order to preserve the particle–number.

For $n \neq 0$ the Fourier coefficients are given by

$$Q_n = \frac{4}{T} \int_0^L \frac{dx}{\sqrt{2\epsilon/m}} x^2 \cos n\omega_0 \tau(x) = (-)^n 4 \frac{L^2}{(n\pi)^2}. \qquad (138)$$

The density is chosen in such a way that in the uncorrelated limit the Fermi Energy ϵ_F has a value similar to that of nuclear matter: $\epsilon_F = 33 \, \text{MeV}$ and the three sizes L_1, L_2, L_3 are chosen so that

$$\begin{aligned}
\hbar\omega_0(\epsilon_F) &= 10 \, \text{MeV} & (L = L_1) & \qquad (139) \\
&= 1 \, \text{MeV} & (L = L_2) & \qquad (140) \\
&= .1 \, \text{MeV} & (L = L_3). & \qquad (141)
\end{aligned}$$

For the correlated systems, the value of Δ is 1 MeV, while $\mu = \epsilon_F$.

Figure 1 shows the main peak of the response function in the case $L = L_1$ ($\hbar\omega_0(\epsilon_F) = 10$ MeV, hence $\hbar\omega_0(\epsilon_F) >> \Delta$). Both peaks, concentrated near $\hbar\omega = 10\ MeV$, correspond to the mode $n = 1$ of the propagators (135) (dashed) and (133) (solid). In this case pairing correlations have a rather small effect: essentially they shift slightly the uncorrelated peak towards higher excitation energy, in agreement with Ref. [1].
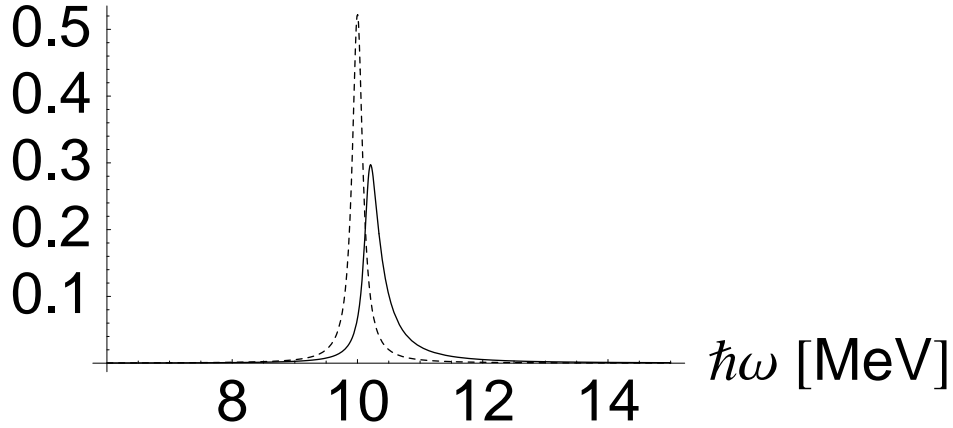

Figure 1: Uncorrelated and correlated strength functions (divided by L^4), as a function of excitation energy $\hbar\omega$ expressed in MeV. The peaks shown here correspond to the mode $n = 1$ of the two propagators (135) and (133).The peak on the left (dashed) should be a δ-function and it represents the response of a normal system given by the propagator (135); for numerical reasons we have used a finite value of $\varepsilon = 0.1$ MeV, thus inducing a small artificial smearing of the response. The peak on the right is obtained from the propagator (133) with the same value of ε. Its larger width is an effect of the pairing correlations.There are other similar peaks around $\hbar\omega = 20, 30 \dots$ MeV, but their strength decreases rapidly with increasing n.

Figure 2 instead shows the response in the case $L = L_2 = 10\,L_1$. In this case $\hbar\omega_0(\epsilon_F)$ is of the same order as the pairing parameter Δ. The pairing gap 2Δ at low excitation is clearly visible.
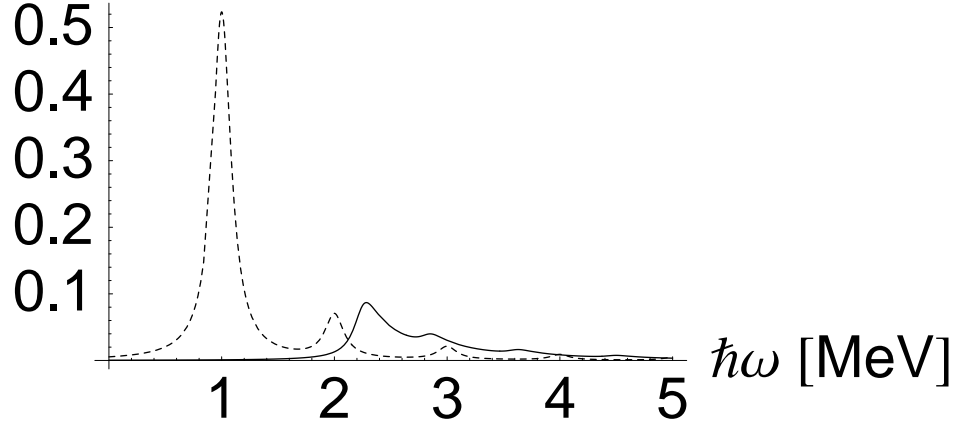

Figure 2: Same as Fig.1, but with $\hbar\omega_0(\epsilon_F) = 1\,\text{MeV}$ and including modes up to $n = 10$. The main peak of the uncorrelated strength function (dashed) at $\hbar\omega = 1\,\text{MeV}$ is pushed to higher energy by the pairing correlations and a gap of about 2Δ is created at low energy.

Finally Figure 3 shows the response in the case $L = L_3 = 10\,L_2$. In this case $\hbar\omega_0(\epsilon_F)$ is much smaller than Δ and we can clearly see that the correlated response is dominated by a peak around 2Δ, as expected for very large systems.
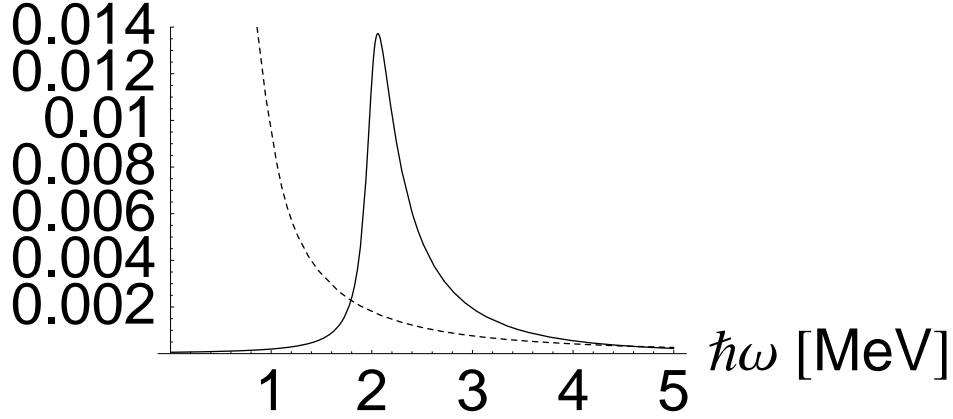

Figure 3: Same as Fig.2, but with $\hbar\omega_0(\mu) = .1\,\mathrm{MeV}$ and including modes up to $n = 10$. All the low–enegy peaks of the uncorrelated strength function (dashed, practically only the tail of the main peak at $\hbar\omega = .1\,\mathrm{MeV}$ is visible) are pushed to higher energy by the pairing correlations and a gap 2Δ is created at low energy.

It can be easily checked that the correlated zero–order response function obtained here gives the same energy–weighted sum rule as the uncorrelated one:

$$M_1 = \hbar^2 \int_0^\infty d\omega\,\omega \bar{S}^0(\omega) = \hbar^2 \int_0^\infty d\omega\,\omega S^0(\omega) = \frac{2}{3}\hbar^2 \frac{AL^2}{m}, \qquad (142)$$

while, if we had used the function f_{ev}, instead of $\bar{f}_{ev}$ in the calculation of the density fluctuations, we would have obtained an energy–weighted sum rule that is much larger than (142) when $\hbar\omega_0 << \Delta$.

5.3   Collective solution

When collective effects are taken into account, both factors $B(x)$ and $C(x)$ should change and become

19

$$B(x) = \beta F'(\epsilon)\Big(\frac{dQ(x)}{dx} + \frac{\partial \delta V^{int}(x,\omega)}{\partial x}\Big) \tag{143}$$

$$C(x) = \frac{i\omega}{v(\epsilon,x)}\beta F'(\epsilon)\Big(Q(x) + \delta V^{int}(x,\omega)\Big). \tag{144}$$

Up to now we have considered instead the zero–order approximation in which $B(x)$ and $C(x)$ are given by Eqs. (85) and (87).

In the zero–order solution discussed above, the fluctuation δh is given by Eq. (53), while one should take the form (36) in order to take into account collective effects. In this case the zero–order propagator (133) is replaced by the collective propagator $\bar{D}(x,x',\omega)$ satisfying the integral equation [7]

$$\bar{D}(x,x',\omega) = \bar{D}^0(x,x',\omega) + \int dy dz \bar{D}^0(x,y,\omega)v(y,z)\bar{D}(z,x',\omega). \tag{145}$$

The only difference with the uncorrelated case of [7] is that the zero–order propagator $\bar{D}^0$ is now given by Eq. (133) instead of (135).

Here we consider a simple separable interaction of the form

$$v(y,z) = \lambda Q(y)Q(z) \tag{146}$$

and define

$$\bar{D}_{QQ}(\omega) = \int dx dx' Q(x)\bar{D}(x,x',\omega)Q(x'), \tag{147}$$

then Eq.(145) gives

$$\bar{D}_{QQ}(\omega) = \frac{\bar{D}^0_{QQ}(\omega)}{1 - \lambda \bar{D}^0_{QQ}(\omega)}. \tag{148}$$

Compared to $D^0_{QQ}(\omega)$, the collective response function $D^0_{QQ}(\omega)$ has an extra pole, determined by the condition

$$1 - \lambda \bar{D}^0_{QQ}(\omega) = 0. \tag{149}$$

Depending on the value of λ, this collective pole can also be within the gap $\omega = [0, 2\Delta]$. Such a case is shown in Fig. 4, where the collective (solid) and zero–order (dashed) responses are shown for $\hbar\omega_0 = 1\text{MeV}$. The value of λ has been chosen so that the collective mode occurs a an energy $\hbar\omega = 0.5\text{MeV}$.
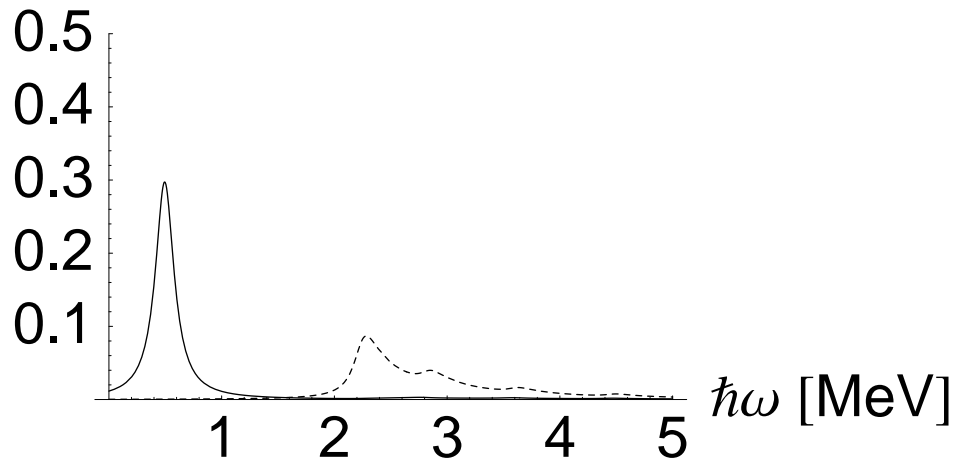
Figure 4: Strength of collective state within the gap (solid). The strength is normalized as in the other figures. The width of the collective state is due to the artificial smearing parameter $\varepsilon = 0.1\,\text{MeV}$. The dashed curve, which is the same as the full curve in Fig. 2, shows the zero–order response.

6   Conclusions

The solutions of the semiclassical time–dependent Hartree–Fock–Bogoliubov equations have been studied in a simplified model in which the pairing field $\Delta(\mathbf{r}, \mathbf{p}, t)$ is treated as a constant phenomenological parameter. Such an approximation is known to violate some important symmetries, like global and local particle–number conservation. In a linearized approach , we have shown that the global particle–number violation is related to one particular mode of the density fluctuations. Both global and local particle–number conservation can be restored by introducing a new density fluctuation that is related to the current density by the continuity equation. This prescription changes the strength associated with the various eigenmodes of the density fluctuations, but not the eigenfrequencies of the system. In one particular case that has been studied in detail, the energy–weighted sum rule calculated according to this prescription has exactly the same value as for normal, uncorrelated systems.

The qualitative features of the response depend essentially on the ratio between two characteristic frequencies of the system: $\Delta/\hbar$ and ω_0. For nuclei,

the parameter Δ is fixed by experiment and depends weakly on the size of the system, while the sigle–particle frequency ω_0 strongly depends on the system size. In one dimension, we must distinguish between small and large systems: in small systems, for which $\omega_0 >> \Delta/\hbar$, the effect of pairing correlations is rather small and gives essentially an increase of the value of the (zero–order) system eigenfrequencies, while in larger systems, for which $\omega_0 << \Delta/\hbar$, the zero–order system response is dominated by a single peak at a frequency $\approx 2\Delta/\hbar$. In all cases, the zero–order response (analogous to the quantum single–particle response) displays an excitation energy gap for $0 < \hbar\omega < 2\Delta$. The effects of this gap become important as the system becomes larger.

These considerations will change in three–dimensional sytems because in that case the frequency ω_0 does not depend solely on the system size (for example, in spherical systems there is also a dependence on the value of the particle angular momentum), thus in three dimensions we expect to find features typical both of small and large one–dimensional systems. Work for extending the present method to spherical systems is in progress.

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