Placeholder Substructures II: Meta-Fractals, Made of Box-Kites, Fill Infinite-Dimensional Skies

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Abstract

Zero-divisors (ZDs) derived by Cayley-Dickson Process (CDP) from N-dimensional hypercomplex numbers (N a power of 2, and at least 4) can represent singularities and, as $N \longrightarrow \infty$, fractals – and thereby, scale-free networks. Any integer > 8 and not a power of 2 generates a meta-fractal or *Sky* when it is interpreted as the *strut constant* (S) of an ensemble of octahedral vertex figures called *Box-Kites* (the fundamental ZD building blocks). Remarkably simple bit-manipulation rules or *recipes* provide tools for transforming one fractal genus into others within the context of Wolfram's Class 4 complexity.

1 Introduction By Way of Reprise: From Box-Kites to ETs

The creation of 2^N-dimensional analogues of Complex Numbers (and it was not a trivial insight of 19th Century algebra that legitimate analogs *always* have dimension a power of 2) is handled by a now well-known algorithm called the Cayley-Dickson Process (CDP). Its name suggests a compressed account of its history: for Arthur Cayley – simultaneously with, but independently of, John Graves –

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jumped on Hamilton's initial generalization of the 2-D Imaginaries to the 4-D Quaternions within weeks of its announcement, producing – by the method later streamlined into Leonard Dickson's close-to-modern "cookie-cutter" procedure – the 8-D Octonions. The hope, voiced by no less than Gauss, had been that an infinity of new forms of Number were lurking out there, with wondrous properties just awaiting discovery, whose magical utility would more than compensate for the loss of things long taken for granted as their seekers ascended into higher dimensions. But such fantasies were quashed quite abruptly by Adolph Hurwitz's proof, just a few years before the 20th Century loomed, that it only took four dimension-doublings past the Real Number Line to find trouble: the 16-D Sedenions had zero-divisors, which meant division algebra itself broke down, which meant researchers were so at a loss to find anything good to say about such Numbers that nobody bothered to even give their 32-D immediate successors a name, much less investigate them seriously.

But it is with these 32-D "Pathions" (for short for "pathological," which we'll call them from now on) that our own account will pick up in this second part of our study of "placeholder substructures" (i.e., "zero divisors") For, due to a phenomenon we dubbed *carrybit overflow* in the first installment, strange yet predictable things are found to be afoot in the ZD equivalent of a "Cayley Table." As we'll see shortly, this is a listing, in a square array, of the ZD "emanations" (or lack of same) of all ZD "elements" with each other – all, that is, sharing membership in an ensemble defined not by a shared "identity element," but a common *strut constant*.

What we'll see is that the lacks are of the essence: for each doubling of N, the Emanation Table (ET) for the 2^{N+1}-ions of same strut-constant will contain that of its predecessor, leading to an infinite "boxes-within-boxes" deployment whose empty cells define, as N grows ever larger, an unmistakable *fractal limit*. The full algorithmic analysis of such Matrioshka-doll-like "meta-fractal" aspects – by the simple rules of what we'll call "recipe theory" (after the R, C, and P values related to the Row label, Column label, and their cell-specific Products in such Tables) – must await our third and last installment. But the colored-quilt-like graphics can be viewed by any interested readers at their leisure, in the Powerpoint slide-show online at Wolfram Science from our mid-June presentation at NKS 2006.[1] (The slide-show's title is almost identical to that of this monograph, as this latter is meant to be the "theorem/proof" exposition of that iconic, hence largely intuitive and empirically driven narration.)

What we'll need to undertake this voyage is a quick reprise of the results from Part I [2]. As the hardest part (as a hundred years of denial would imply) is finding the right way to think about the phenomenology of zero-division, not understanding its basic workings once they're hit upon, such a summary can be much more brief and easy to follow than the proofs required to produce and justify it. We need but grasp 3 rather simple things. First, we must internalize the path and vertex structure of an Octahedron – for, properly annotated and storyboarded, this will provide us with the Box-Kite representation that completely catalogs ZDs in the 16-D arena where they first emerged (and, as we'll see in our Roundabout Theorem herein, underwrites all higher-dimensional ZD emergences as well).

Second, instead of the cumbersome apparatus of CDP that one finds in algebra texts and the occasional software treatment, we offer two easy algebraic one-liners which (inspired by Dr. Seuss's "Thing 1" and "Thing 2"), we simply call "Rule 1" and "Rule 2" – which operate, in almost Pythagorean earnest, on triplets of integers (indices of associative triplets among our Hypercomplex Units, as we'll learn), and which, by so doing, accomplish everything the usual CDP tactics do, but without the all-too-frequent obfuscation. (There is also a very useful, albeit quite trivial, "Rule 0," which merely states that any integer-triple serving to index an associative triplet for one power of N will continue to do so for all higher powers. What makes this useful is its allowing us to recursively take triplet "givens" for lower-level 2^N-ions than those of current interest and toss them into the central circle of the third thing we must grasp.)

We'll need, that is, to be able to draw the simplest finite projective group's 7-line, 7-node representation, the so-called PSL(2,7) triangle. The Rules, plus the Triangle, applied to Box-Kite edge-tracings and nodal indices, are all we'll need. Indeed, the Box-Kite itself can be readily derived from the Triangle, by suppressing the central node, and then recognizing four correspondences. First, see the Triangle's 3 triple-noded sides – two vertices plus midpoint – as the sources of the Box-Kite's trio of "filled-in" triangles dubbed Trefoil Sails. Second, link the 1 triple-noded *circle* (which *is* a projective *line*, after all), wrapped around the suppressed center and threading the midpoints, as the 4th such triangle, the quite special "Zigzag Sail." Third, envision the 3 lines from midpoints to angles as underwriting the ZD-challenged part of the diagram (because ZDs housed at the midpoint node cannot mutually zero-divide any housed at the opposite, vertex, node), the *struts* (whence strut *constants*). Fourth and last, imagine the other four triangles of the Box-Kite (meeting, as with the first four, each to each, at corners only, like same-colored checkerboard squares) as the vents where the wind blows. They keep the kite afloat, letting the four prettily colored jib-shaped Sails show off, while the trio of wooden or plastic dowels that form the struts thanklessly provide the structural stability that makes the kite able to fly in the first place.

As Euclid knew well, 3 points determine a Triangle as well as a Circle – which is how we can glibly switch gears between representations based on these projective lines. But the easy convertibility of lines to circles is what *projective* means here – and is, as well, at the very heart of linking the above geometrical images to Imaginary Numbers. From Argand's diagram to Riemann's Sphere, this has been the essence of Complex geometry. On the latter image only, place a sphere on a flat tabletop, call the point of contact **S** (for "South"), and then direct rays from its polar opposite point **N**. Rays through the equator intersect the table in a circle whose radius we ascribe an absolute value of 1, with center $\mathbf{S} = 0$. This circle is just the trace of the usual $e^{i \cdot 2\pi \cdot \theta}$ exponential-orbit equation, with the i in the exponent, of course, being the standard Imaginary. Any diameter through this circle, extended indefinitely in either direction, is clearly a "projective pencil" of a circular motion in the plane containing both it and **N**, and centered on the latter.

What each "line," then, in the PSL(2,7) triangle represents is a coherent system interrelating 3 distinct imaginaries, one per nodal point: that is, a "Quaternion copy" *sans* the Reals (which latter, like our $\mathbf{N}, \mathbf{S}$ polar axis in the above, must stand "outside" the Number Space itself, since 3-D visualization is all used up by the nodes' dimensional requirements). Hence, the 7 lines are the 7 interconnected Quaternion copies which constitute the 8-D Octonions. And what makes this especially rich for our purposes is the built-in recursiveness of this Octonion-labeling scheme for higher-dimensional isomorphs, embedded in the sorts of ensembles we'll be needing ETs to investigate more thoroughly.

To see how this relates to actual integers, take the prototype of the 7 lines in the Triangle, and consider the Quaternions strictly from the vantage of CDP's Rule 1. The first task in studying any system of 2^N-ions is generating its units, so start with $N = 0$. Treat this singleton as the index of the Real axis: i_0, that is, is identically 1. Add a unit whose index = $2^0 = 1$ and we have the complex plane. Now, add in a unit whose index is the next available power of 2 – with $N = 1$, this is 2 itself. Call this unit and its index **G** for *Generator*, and declare this inductive rule: the index of the product of any two units is always the XOR of the indices of the units being multiplied; but, for any unit with index $u < \mathbf{G}$, the product of said unit, written on the left (right), with the Generator written on its right (left), has index equal to their indices' simple sum, and sign equal (opposite) to the product of the signs of their units': $i_1 \cdot i_2 = +i_3$, but $i_2 \cdot i_1 = -i_3$. But this is just a standard way of summarizing Quaternion multiplication.

Now, set N = 2, making $\mathbf{G} = 4$. Applying the same logic, but slightly generalized, we get three more triplets of indices. Dispensing with the tedious overhead of explicitly writing the indices as subscripts to explicit copies of the letter i, these are written in cyclical positive order (CPO) as follows: $(1,4,5);(2,4,6);(3,4,7)$. (CPO is not mysterious: it just means read the triplet listing in left-right order, and so long as we multiply any unit with any such index by the unit whose index is to the right of it, the third term will result with signing as specified above: e.g., $i_4 \cdot i_5 = +i_1$; $i_4 \cdot i_3 = -i_7$.) We now have 4 of the Octonions' 7 triplets, forming labels on the nodes of 4 of PSL(2,7)'s lines. Call the central circle spanning the medians the Rule 0 line (the Quaternions' "starter kit" we just fed into our Rule 1 induction machine). Putting **G** = 4 in the center, the 3 lines through it are our Rule 1 triplets. If we further array the Quaternion index-set $(1,2,3)$ in clockwise order around the 4, starting from the left slope's midpoint at 10 o'clock, these lines are all oriented pointing into the angles. Now, with "Rule 2," let's construct the lines along the Triangle's sides.

Here's all that Rule 2 says: given an associative index-triplet (henceforth, *trip*) like the Quaternions' $(1,2,3)$, fix any one among them, then take its two CPO successors and add **G** to them. Swap the order of the resulting two new units, and you have a new trip. Hence, fixing 1, 2, and 3 in turn, in that order, Rule 2 gives us these 3 triplets: $(1,7,6);(2,5,7);(3,6,5)$. If you've drawn PSL(2,7) with the Octonion labels per the instructions in the last paragraph, you've already seen these 3 trips are the answers ... and now you know how and why they're oriented, too. (Clockwise, in parallel with the Rule 0 circle).

We've now laid out all the ingredients we need to do a basic run-through of Box-Kite properties. We'll merely state and describe them, rather than prove them (but we'll give the Roman numerals of the theorem numbers from last installment, for those who want to follow them). The first feature in need of elucidating, which should have those who've been reading attentively scratching their heads just about now, is this: the relations between the indices at the nodes of PSL(2,7) *qua* Octonion labeling scheme are clear enough; but how can these same labeled nodes serve to underwrite the 16-D Sedenion framework that Box-Kites reside in? The answer has two parts.

First part: since all Imaginaries have negative Reals as squares, Imaginaries whose products are zero must have different indices – meaning that the simple case (which we call "primitive" ZDs) will always involve products of *pairs* of differently-indexed units, whose respective planes share no points other than 0 [IV]. Second part: given any such ZD dyad, neither index can ever equal **G** [II]; and, one must have index $>$ **G**, while the other has index $<$ **G** [I, III]. The Octonion labeling scheme maps to the four Sails of a legitimate Sedenion Box-Kite [V], because it only provides the *low*-index labels at each of the 6 Octahedral vertices.

The 4 in the center of our example, meanwhile, is no longer the **G** for this setup, since that role in now played by 8 (the next power of 2 in the CDP induction). In the context of the Box-Kite scheme, it is now represented by a different letter: **S**, for *strut constant* – the only Octonion index *not* on a Box-Kite vertex.

Which is why, from one vantage, there are 7 distinct (but isomorphic) Box-Kites in Sedenion space: because we've 7 choices of which Octonion to suppress! 6 vertices times 7 gives us the *42 Assessors* of our first ZD paper [3], a term we'll use interchangeably with *dyad* throughout. We can, in fact, tug on the network of interconnected lines "wok-cooking" style, stirring things into and out of the hot oil in the center of the Box-Kite. (**S** as "Stir-fry constant"?) To find the "Octonion copy" labeling low indices on Box-Kite vertices where the 5, say, is suppressed, trace the line containing it and the 4, and "rotate": the 1 now goes from the left slope's midpoint to the bottom right angle, to be replaced by the 4 while the 5 heads for the middle, with CPO order (and hence, orientation of the line) remaining unchanged. Of the other 2 trips the 5 belongs to, only one will preserve midpoint-to-angle orientation along the 6 o'clock-to-midnight vertical: $(2,5,7)$, as one can check in an instance. (The two possibilities must orient oppositely when placed along the same line, since one is Rule 1, the other Rule 2.)

From this point, everything is forced. This is obviously a procedure that is trivial to automate, for any "Octonion copy," regardless of the ambient dimensionality the Box-Kite it underwrites might float in. This simple insight will be the basis, in fact, of our proof method, both in this paper and its sequel. Another simple insight will tell us how to find the *high*-index term for any vertex's dyad. Two indices per vertex leaves 4 that are suppressed: 0 (for the Reals), **G** and **S**, and the XOR (and also simple sum) of the latter two, which we'll shorthand **X**. These four clearly form a Quaternion copy – one, in fact, which has no involvement whatsoever in its containing Box-Kite's zero-divisions. Putting the index of the *one* among these which is *itself* an L-unit center stage gives us the full array of *L-index sets* (trips composed of those indices of a Sail's 3 vertices $<$ **G**) associated with the 4 Sails. Putting in **G** or **X**, then, must give us the full array of *U-index sets* ("U" as in "upper").

Since each node belongs to 3 lines in PSL(2,7), the strut constant belongs to 3 trips, each containing one term from the Rule 0 Zigzag Sail's L-index set, and one from the Vent which resides opposite it on the Box-Kite's octahedral frame. Three simple rules govern interactions of the Vent and Zigzag dyads sharing a strut. Writing the U- and L- index terms in upper and lower case respectively, we can symbolize their dyads as (V, v) and (Z, z) respectively. The "Three Viziers" (derived as side-effects of [VII], with one for each non-0 member of our ZD-free index set) read as follows:

$$\text{VZ1: } v \cdot z = V \cdot Z = \mathbf{S}$$
$$\text{VZ2: } Z \cdot v = V \cdot z = \mathbf{G}$$
$$\text{VZ3: } V \cdot v = z \cdot Z = \mathbf{X}.$$

The First Vizier motivates the term *strut constant*: for the same pattern obtains for it, regardless of the strut being investigated. The Second Vizier shows us that **G** connects *strut opposites*, always by Rule 1 logic. But clearly, the Third Vizier gives us the simplest way to answer any questions concerning the relations between indices within a dyad: the L- and U- indices of *any* dyad belong to the same trip as **X**, with CPO ordering determined by whether or not the dyad belongs to the Zigzag proper or the Vent opposite it.

With the Viziers, our toolkit is complete for all our later proofs. What's left to do still: get our hands messy with the plumbing, and then clean up with a last grand construct. Let's start with the plumbing, and add some notation. Label the Zigzag dyads with the letters A, B, C; label their strut-opposite terms in the Vent F, E, D respectively. Specify the diagonal lines containing all and only ZDs in any such dyad K as `(K, /)` and `(K, \)` – for $c \cdot (i_K + i_k)$ and $c \cdot (i_K - i_k)$ respectively, c an arbitrary real scalar. The twelve edges of the octahedral grid are so many pipes, through which course the two-way streets of *edge-currents*: for the 3 edges of the Zigzag (and the 3 defining the opposite Vent), currents joining arbitrary vertices M and N are called *negative*, since they have this form:

$$\texttt{(M,/)} \cdot \texttt{(N, \textbackslash)} = \texttt{(M, \textbackslash)} \cdot \texttt{(N,/)} = 0$$

Tracing the perimeter of the Zigzag with one's finger, performing ZD products in natural sequence – `(A, /)`·`(B, \)`, followed by the latter times `(C, /)`, then this times `(A, \)` and so forth – one should quickly see how the Zigzag's name was suggested. Suppressing all letters, one is left with just this cyclically repeating sequence: `/\/\/\`.

Currents along all 6 edges joining Zigzag and Vent dyads, on the contrary, connect similarly sloping diagonals, hence are called *positive*, yielding the shorthand sequence `///\\\` for Trefoil sail traversals:

$$\texttt{(Z,/)} \cdot \texttt{(V, /)} = \texttt{(Z, \textbackslash)} \cdot \texttt{(V,\textbackslash)} = 0$$

Consider the chain of ZD multiplications one can make along the Zigzag, between A and B, then B and C, then C and A, for $\mathbf{S} = 4$. The first term of this 6-cycle of zero products, once fully expanded, is writable thus:

$$(\texttt{A, /}) \cdot (\texttt{B, \textbackslash}) = (i_1 + i_{13}) \cdot (i_2 - i_{14}) = (i_3 - i_{15} + i_{15} - i_3) =$$
$$(\texttt{C, /}) - (\texttt{C, /}) = (\texttt{C, \textbackslash}) - (\texttt{C, \textbackslash}) = 0$$

We can readily see here where the notion of *emanation* arises: traversing the edge between any two vertices in a Sail yields a balance-pan pairing of oppositely signed instances of the terms at the Sail's third vertex ... the 0 being, then, an instance of "balanced bookkeeping" (whence the term "Assessor," our synonym for "dyad"). This suggests the spontaneous emanation of particle/anti-particle pairings from the quantum vacuum, rather than true "emptiness."

Finally, a side-effect of such "Sail dynamics" is this astonishing phenomenon: each Sail is an interlacing of 4 associative triplets. For the Zigzag, these are the L-index (a,b,c), plus the 3 U-index trips obtained by replacing *all but one* of these lowercase letters with their uppercase partners: ergo, (a,B,C); (A,b,C); (A,B,c). Ultimately this tells us that ZDs are extreme *preservers* of order, since they maintain associativity in rigorous lock-step patterns, for all 2^N-ions, no matter how close to ∞ their N might become. Put another way, the century-long aversion reaction experienced by virtually all mathematicians faced with zero-divisors was profoundly misguided.

2 Emanation Tables: Conventions for Construction

Theorem 7 guaranteed the simple structure of ETs: because any Assessor's uppercase index i_U is strictly determined by **G** and **S**, once we are given these two values, the table need only track interactions among the lowercase indices i_L. This will only lead to ambiguities in the very place these are meaningful: in the recursive articulation of a boxes-within-boxes tabulation of meta-fractal or Sky behaviors. In such cases, the overlaying will be as rich in significance as the multiplicity of sheets of a Riemann surface in complex analysis.

An ET does for ZD interactivity what a Cayley Table does for abstract groups: it makes things visible we otherwise could not see – and in a similar way. Each Assessor's L-index is entered (in a manner we'll soon specify) as a row (R) or column (C) value, with XOR products (P values) among them being placed in the "spreadsheet cell" (r,c) uniquely fixed by R and C. We've noted such values only get entered if P is the L-index of a legitimate emanation: that is, the Assessor it represents mutually zero-divides (forms *DMZ*s with, for "divisors making zero") *both* the Assessors represented by the R and C labels of its cell. (As already suggested, the natural use of the letters R, C, P here inspired calling the study of NKS-like "simple rules" for cooking fractals from their bit-strings *recipe theory*.)

Four conventions are used in building ETs: first, their labeling scheme obeys the same nested-parentheses ordering we've already used in designating Assessors A through F, with D, E, F the strut opposites of A, B, C in reverse of the order just written. The L-indices, then, are entered as labels running across the top and down the left. The label of the lowest L-index is placed flush left (abutting the ceiling), with the corresponding label of its strut opposite being entered flush right (atop the floor). As there will always be $\mathbf{G}-2$ (hence, an even number of) indices to enter, repeating this procedure after each pair has been copied to horizontal and vertical labels will completely exhaust them all.

Second convention: As the point of an ET is to display all legitimate DMZs, any cell whose R and C do *not* mutually zero-divide is left blank – even if, in fact, there *is* a well-defined XOR value. Hence, if R and C reference the same Assessor, the XOR of their L-indices will be 0; if they reference strut opposites, the XOR will be $\mathbf{S}$. But in both cases, the cell (hence, the P value) is left blank. All "normal" ETs, then, will have both long diagonals populated by blank cells, while all other cells are filled.

Third convention: the two ZD diagonals associated with any Assessor are not distinguished in the ET, although various protocols are possible that would make doing so easy. The reasons are parsimony and redundancy: rather than create longer, or twice as many, entries, we assume both entries for the same Box-Kite edge will contain the positive-sloping diagonal when the lower L-index appears as the row label, else the negative-sloping diagonal when the higher L-index appears first instead. Such niceties won't concern us much here: the key thing is that, in fact, all 24 filled cells of a Box-Kite's ET entries can be mapped one-to-one to its ZD diagonals. Recall, per Theorem 3, that both ZD diagonals of an Assessor form DMZs with the same Assessor, according to the same edge-sign logic. This leads us to the ...

Fourth convention: Although they are superfluous for many purposes, edge signs provide critical information for others, and so are indicated in all ETs provided here. Each of a Box-Kite's 12 edges conducts two currents – one per ZD diagonal – and does so according to one or the other orientational option. ZD diagonals are conventionally inscribed so that the horizontal axis of their Assessor plane is the L-indexed unit, while the vertical is the U-indexed unit. But even if this convention were reversed, the diagonal leading from lower left quadrant to upper right would still correspond to the state of synchrony implied by $\pm k(i_L + i_U)$: for some Assessor U, we write `(U,/)`. Conversely, the orthogonal diagonal indicative of anti-synchrony is written `(U,\)`. If DMZs formed by the Assessors bounding an edge are both of same kind, then we call the edge blue or notate it

[+]; if Assessors U and V only form DMZs from oppositely oriented ZD diagonals – $(\mathtt{U},/) \cdot (\mathtt{V},\backslash) = 0 \Leftrightarrow (\mathtt{U},\backslash) \cdot (\mathtt{V},/) = 0$ – then we call the edge red or notate it [-]. However, for ET purposes, since the red edges are the most informative (all-red-edged Zigzags providing the stable basis of Box-Kite structure, while all-red-edged DEF Vents play a key role in twist-product interpreting – a deep topic touched upon in Part I, which won't concern us further here), we leave them unmarked. The six blue edges bounding the hexagonal view of the Box-Kite, however, are preceded by an extra mark (best interpreted as a dash, rather than a minus sign). This has the pragmatic advantage that when zoomed, a large ET will have its entries with an extra mark become unreadable in many software systems (e.g., one sees only asterisks) – and so we want the unmarked entries to be those likely to be of most interest.

Since, given **X** (or, alternatively, **G** or N, and **S**), we can reconstruct a Box-Kite from just its Zigzag's L-index trip, gleaning this information from an ET is worth explaining. If a given row contains the indices of any such Zigzag L-trips, they will appear as the row label itself, plus two unmarked cell entries, with the column label of the one appearing as the content of the other. (If either cell in such a complementary set be marked with a dash, then we are dealing with a DEF Vent index.) Each Zigzag L-trip will also appear 3 times in an ET, once in each row whose label is one of its indices, its 2 non-label indices appearing in un-dashed cell entries each time.

Here is a readily interpreted emanation table. Having $6 = 2^3 - 2$ rows and columns, $\mathbf{G} = 8$, so $\mathbf{N} = 4$, making this a Sedenion ET (encoding, thereby, a single Box-Kite). And, since $2 \veebar 3 = 4 \veebar 5 = 6 \veebar 7 = 1$, the Strut Constant $\mathbf{S} = 1$ as well. A scan of the first row shows 6 and 5 unmarked, under headings 4 and 7 respectively; however, these two labels appear as cell values which *are* marked, making these edges that connect Assessors in the D, E, F Vent. In the fourth row of entries, though, column labels 5 and 3 contain cell values 3 and 5 respectively, both unmarked. With their row label 6, then, these form the Zigzag L-index set $(3,6,5)$, which hence must map to Assessors (A,B,C). Using the mirror-opposite logic of the labeling scheme to determine strut opposites, it is clear that the six row and column headings $(2,4,6,7,5,3)$ correspond, in that order, to the Assessors (F,D,B,E,C,A). (The unmarked contents 6 and 5 in the first row, having labels $(2,4)$ and $(2,7)$, thereby map to edges FD and FE, connecting DEF Vent Assessors as claimed.) Finally, the long diagonals are all empty: those cells in the diagonal beginning at the upper left all have identical row and column labels; those in the mirror-opposite slots, meanwhile, have labels which are strut-opposites. By our second convention, all these cells are left blank.

	2	**4**	**6**	**7**	**5**	**3**
2		6	−4	5	−7	
4	6		−2	3		−7
6	−4	−2			3	5
7	5	3			−2	−4
5	−7		3	−2		6
3		−7	5	−4	6	

Before beginning an in-depth study of emanation tables by type, there is one general result that applies to them all – and whose proof will give us the chance to put the Three Viziers to good use. While seemingly quite concrete, we will use it in roundabout ways to simplify some otherwise quite complicated arguments, beginning with next section's Theorem 9. This Roundabout Theorem is our

Theorem 8. The number of filled cells in any emanation table is a multiple of 24.

Proof. Since 24 is the number of filled cells in a Sedenion Box-Kite, this is equivalent to claiming that CDP zero-divisors come in clusters no smaller than Box-Kites. We have already seen, in Theorem 5, that the existence of a DMZ implies the 3-Assessor system of a Sail, which further (as Theorem 7 spelled out) entails a system of 4 interlocking trips: the Sail's L-trip, plus 3 trips comprising each L-trip index plus the U-indices of its Assessor's 2 "sailing partners." Since we have an ET, we have a fixed **S** and fixed **G**. Hence, if we suppose our DMZ corresponds to a Zigzag edge-current, we immediately can derive its L-trip by Theorem 5, and all 3 Zigzag strut-opposites' L-indices by VZ 1, and all 6 U-indices by VZ 3. We then can test whether the Trefoil Sails' edge-currents are all DMZs as follows. As we wrote in Theorem 7, (u,v,w) maps to the Zigzag L-trip in CPO, but not necessarily in (a,b,c), order: hence, (u_{opp}, w_{opp}, v) is an L-trip, and can be mapped to any of the Trefoils. In other words, given the Zigzag's 3-fold rotational symmetry, proving the truth of the following arithmetical result proves the DMZ status of *all* Trefoil edges. Yet we can avail ourselves of all 3 Zigzag U-trips in proving it.

$$\begin{array}{c} (w_{opp} - W_{opp}) \\ \underline{(u_{opp} + U_{opp})} \\ -V \quad -v \\ \underline{+v \quad +V} \\ 0 \end{array}$$

The left bottom result is a given of the trip we started with. The result to its right is a three-step deduction from one of the Zigzag U-trips: use (u_{opp}, w, v_{opp});

Rule 2 gives $(u_{opp}, v_{opp}+G, w+G)$; the Second Vizier tells us this is (u_{opp}, V, W_{opp}); but the negative inner sign on the upper dyad reverses the sign this trip implies, yielding $+V$ for the answer.

The top results are derived similarly: find which of the 4 Zigzag trips underwrites the Vizier-derived "harmonic" which contains the pair of terms being multiplied, and flip signs as necessary. Hence, the top left uses (u, w_{opp}, v_{opp}), then applies Rule 2 and the Second Vizier to get $(-V)$, while the top right uses the Zigzag L-trip itself: $(u, v, w) \rightarrow (w+G, v, u+G) \rightarrow (W_{opp}, v, U_{opp})$ – which, multiplied by (-1), yields $(-v)$. ■

Remark. The implication that, regardless of how large N grows, ZDs only increase in their interconnectedness, rather than see their basic structures atrophy, flies in the face of a century's intuition based on the Hurwitz Proof. That there are no standalone edge-currents, nor even standalone Sails, bespeaks an astonishing (and hitherto quite unsuspected) stability in the realm of ZDs.

Corollary. An easy calculation makes it clear that the maximum number of filled cells in any ET for any 2^N-ions is just the square of a row or column's length in cells, minus twice the same number (to remove all the blanks in long diagonals): that is, $(2^{N-1}-2)(2^{N-1}-2) - 2 \cdot 2^{N-1} + 4 = (2^{2N-2} - 6 \cdot 2^{N-1} + 8) = (2^{N-1} - 4)(2^{N-1} - 2) = 4 \cdot (2^{N-2} - 1)(2^{N-2} - 2)$. By Roundabout, we now know this number is divisible by 24, hence indicates an integer number of Box-Kites. But two dozen into this number is just $(2^{N-2} - 1)(2^{N-2} - 2)/6$ – the *trip count* for the 2^{N-2}-ions! (See Section 2 of Part I.) We have, then, the very important Trip-Count Two-Step: *The maximum number of Box-Kites that can fill a* 2^N*-ion ET =* $Trip_{N-2}$. We will see just how important this corollary is next section.

3 ETs for $N > 4$ and S ≤ 7

One of the immediate corollaries of our CDP Rules for creating new triplets from old ones is something we might call the Zero-Padding Lemma: if two k-bit-long bitstring representations of two integers R and C being XORed are stuffed with the same number n of 0s between bits j and $j+1$, $0 \leq j \leq k$, their XOR will, but for the extra n bits of 0s in the same positions, be unchanged – and so will the sign of the product P of CDP-derived imaginary units with these three bit-strings representing their respective indices.

Examples. $(1,2,3) \to (2,4,6) \to (4,8,12)$ [Add 1, then 2, 0s to the right of each bitstring]

$(1,2,3) \to (1,4,5) \to (1,8,9)$ [Add 1, then 2, 0s just before the rightmost bit in each bitstring]

$(3,4,7) \to (3,8,11) \to (3,16,19)$ [Add 1, then 2, 0s just after the leftmost bit in each bitstring]

Proof. Rule 1 will create a new unit of index $\mathbf{G}+L$ from any unit of index $L < \mathbf{G}$, regardless of what power of 2 $\mathbf{G}$ might be. Rule 2, meanwhile, uses any power of 2 which exceeds all indices of the trip it would operate on, then adds this $\mathbf{G}$ to two of the members of the trip, creating a new trip with reversed orientation – one of an infinite series of such, differing only in the power of 2 (hence, position of the leftmost bit) used to construct them. The lemma, then, is an obvious restatement of the fundamental implications of the CDP Rules.

But creation of U-indices associated with L-indices in Assessor dyads is the direct result of creating new triplets with G+S as their middle term. Hence, if we call the current generator g and that of the next higher 2^N-ions $\mathbf{G}$ $(= 2 \cdot g)$, then if Assessors with L-indices u and v form DMZs in the Sedenions for a given strut constant $\mathbf{S}$, their U-indices will increment by g in the Pathions, and zero division will remain unaffected. By induction, the emanation table contents of the Sedenion (R,C,P) entries will remain unchanged for all N, for all fixed $\mathbf{S} \leq 7$. This leads us to

Theorem 9. All non-long-diagonal cell entries in all ETs for all N, for all fixed $\mathbf{S} \leq 7$, will be filled.

Proof. Keeping the same notation, the 2^N-ions will have g more Assessors than their predecessors, with indices ranging from g itself to $2g-1$ $(= G-1)$. Consider first some arbitrary Zigzag Assessor with L-index $z < g$, whose U-index is $\mathbf{G} + z \cdot \mathbf{S}$. (If it were a Vent Assessor, the second part of the expression would be reversed: $\mathbf{S} \cdot z$, per the First Vizier. This effects triplet orientation, but not absolute value of the index, however, and it is only the latter which matters at the moment.) Now consider the Assessor whose L-index is the lowest of those new to the 2^N-ions, g. We know it is a Vent Assessor, in all Box-Kites with $\mathbf{S} < \mathbf{g}$, of which there are 7 per each such $\mathbf{S}$ in the Pathions, 35 in the 64-D 2^6-ions, and so on: for it belongs to the trip $(\mathbf{S}, g, g+\mathbf{S})$ (Rule 1), so that its U-index appears on its immediate left in the triplet $(\mathbf{G}+g+\mathbf{S}, g, \mathbf{G}+\mathbf{S})$ (Rule 2 and last parentheses). Its U-index, then, is $\mathbf{G} + (g \veebar \mathbf{S})$, or (recall Rule 1) just $\mathbf{G}+g+\mathbf{S}$. We claim these Assessors form DMZs; or, writing out the arithmetic, that the following term-by-term multiplication is true:

$$\begin{array}{c} +g+(G+g+S) \\ \underline{+z+(G+z.S)} \\ -(G+g+z\cdot S)-(z+g) \\ \underline{+(z+g)+(G+g+z\cdot S)} \\ 0 \end{array}$$

Because one Assessor is assumed a Zigzag, while the other is proven a Vent, the inner signs will be the same. (Simple sign reversals, akin to those involving our frequently invoked binary variable *sg*, will let us generalize our proof to include the Vent-times-Vent case later.) Let's examine the terms one at a time, starting with the bottom line. Its left term is an obvious application of Rule 1, as $z < g$, the latter being the Generator of the prior CDP level which also contained z as an L-index. The term on bottom right we derive as follows: we know that z and its U-index partner in the 2^{N-1}-ions belong to the triplet mediated by $g+\mathbf{S}$: $(z, g+z\cdot\mathbf{S}, g+\mathbf{S})$. Supplementing this CPO expression by adding $\mathbf{G}$ to the right-hand terms (Rule 2), we get the triplet containing both multiplicands of the bottom-right quantity: $(z, \mathbf{G}+g+\mathbf{S}, \mathbf{G}+g+z\cdot\mathbf{S})$. The multiplicands appear in this trip in their order of application in forming the product; therefore, their resultant is a plus-signed copy of the trip's third term, as shown above.

Moving to the left-hand term of the top line, what trip do the multiplicands belong to? Within the prior generation, Rule 1 tells us that z's strut opposite, $z\cdot\mathbf{S}$, multiplies g on the left to yield $g+z\cdot\mathbf{S}$. Application of Rule 2 to the terms $\neq g$ reverses order and gives us this: $(\mathbf{G}+g+z\cdot\mathbf{S}, g, \mathbf{G}+z\cdot\mathbf{S})$. But what we've written above is the product of multiplying the third and second terms of the trip together, in CPO-reversed order; hence, the negative sign is correct. Finally, we get the negative of $(z+g)$ by similar tactics: the term is the U-index of z's strut-opposite Assessor in the prior CDP generation, hence belongs to the trip with this CPO expression: $(g+\mathbf{S}, z+g, z\cdot\mathbf{S})$. Rule 2 gives us $(\mathbf{G}+g+\mathbf{S}, \mathbf{G}+z\cdot\mathbf{S}, z+g)$. Hence, the product written above is properly signed.

Now, what effect does our initial assumption that z is the L-index of a Zigzag Assessor have on the argument? The lower-left term is obviously unaffected. But the upper-left term, perhaps less obviously, also is unchanged: while it seems to depend on $z\cdot\mathbf{S}$, in fact this is only used to define the L-index of z's strut opposite, which multiplies g on the left to precisely the same effect as z itself, both being less than it. The two terms on the right, just as clearly, *do* have their signs changed, for in both, the order relations of L- and U- indices *vis à vis* $\mathbf{G}+\mathbf{S}$ or $\mathbf{X}$ are necessarily invoked. But both signs on the right can be re-reversed to obtain the desired result if we change the inner sign of the topmost expression – which is to say, we have an effect analogous to that achieved in earlier arguments by use of the binary variable *sg*, as claimed.

Since one CDP level's **G** is the g of the next level up, the above demonstration clearly obtains, by the obvious induction, for all 2^N-ions including and beyond the Pathions. But what if one or both L-indices in a candidate DMZ pairing *exceed* g?

Rather than answer directly, we use the Roundabout Theorem of last section. Given a DMZ involving Assessors with L-indices $u < g$ and g, we are assured a full Box-Kite exists with a Trefoil L-trip $(u, g, g+u)$. The remaining Assessors, being their strut opposites, then have L-indices $u_{opp}, g+\mathbf{S}$, and $g+u\cdot\mathbf{S}$. As u varies from 1 to 7, skipping $\mathbf{S} < 8$, zero-padding assures us that all DMZs from prior CDP generations exist for higher N, for all L-indices $u, v < 8$. Only those Box-Kites created by zero-padding from prior-generation Box-Kites (of which there can be but 1 inherited per fixed **S** among the 7 found in the Pathions, for instance) will have all L-indices $< g$. For all others, the model shown with those having g as an L-index must obtain. Hence, only one strut will have L-indices $< g$, the rest being comprised of some w with L-index ≥ 8, the others deriving their L-indices from the XOR of w with the strut just mentioned, or with **S**.

But what will guarantee that any edge-currents will exist between arbitrary Assessors with L-indices $u < g$ and $g+k, 0 < k < g$, since there is not even *one* DMZ to be found among Assessors with L-indices $\leq g$ in the candidate Box-Kite they would share? We can now narrow the focus of our original question considerably, by making use of the curious computational fact we called the Trip-Count Two-Step.

In Part I's preliminary arguments concerning CDP, we showed that the number of associative triplets in a given generation of 2^N-ions, or $Trip_N$, can be derived from a simple combinatoric formula. Call the count of complete Box-Kites in an ET $BK_{N,S}$. For $\mathbf{S} < 8$, $BK_{N,S} = Trip_{N-2}$, provided all L-indices $g+k, 0 < k < g$, form DMZs in the candidate Box-Kites implied. To begin an induction, let us consider a new construction along familiar lines, which will provide us an easy way to comprehend the Pathion trip-systems of all $\mathbf{S} < 8$. Beginning with N = 5, we designate $Trip_{N-2}$ trips for each $\mathbf{S} < 8$ as type Rule 0, in the manner the singleton 2^2-ion trip $(1,2,3)$ was used in our introduction's "wok-cooking" discussion (which Part I, Section 5, used as the basis of its "slipcover proofs"). But now, instead of putting the Octonions' $\mathbf{G} = \mathbf{4}$ in the center of the PSL(2,7) triangle, we put the Sedenions' 8.

For consistency of examples, we continue to assume $\mathbf{S} = 1$, so we'll begin with $(3,6,5)$, the Zigzag L-trip for $\mathbf{S} = \mathbf{1}$ in the Sedenions, and also, by zero-padding, an L-trip Zigzag for 1 of the 7 Box-Kites with $\mathbf{S} = \mathbf{1}$ among the Pathions. Extending rays from the $(3,6,5)$ midpoints through the center creates Rule 1 trips which end in $11,14,13$: (a,b,c) get sent to (F,E,D) respectively. The Rule 2 trips along the sides, in order of Zigzag L-index inclusion, then correspond to Trefoil U-trips, all oriented clockwise. They read symbolically (literally) as follows: $EaD\ (14,3,13); DbF\ (13,6,11); FcE\ (11,5,14)$. We claim each of these 7 lines, when its nodes are attached to their strut opposites, map 1-to-1 to an $\mathbf{S} = \mathbf{1}$ Pathion Box-Kite. We have this as a given for the Rule 0 trip; we need to *explain* this for the Rule 1 trips (which Roundabout already tells us are Box-Kites); and, we need to *prove* it for the Rule 2 trips that make the sides. (And, once we *do* prove it, and frame the suitable induction for all higher N, the task which originally motivated us will be done: for these U-trips house the Assessors with L-indices $> g$, whose candidate Box-Kites don't include g.)

The Rule 1 trips, in all instances within this example, correspond to Assessor L-indices (a,d,e). With $g = 8$ at d, the Third Vizier tells us $c = 8 + \mathbf{S} =$ *Sedenion* $\mathbf{X}$. (a,b,c) thereby reads, within the Sedenions, as (a,A,X). But in the Pathions, all 3 terms are less than $\mathbf{G}$, hence can comprise an L-index trip for a Sail – and specifically, a Zigzag (else the order of A and $\mathbf{X}$ would be reversed). Similarly, the old Sedenion (f,F) are the new Pathion (f,e), with the new trip (f,c,e) being the Third Vizier's way of saying (f,X,F) from the Sedenions' vantage.

For the Rule 2 trips, we prove one relation in one of them a DMZ, which Roundabout tells us implies the whole Box-Kite, while symmetry allows us to assume the same of the other two. Consider, then, the *aDE* Trefoil U-trip, instantiated by $(3,13,14)$ in our example; specifically, compute the product of the Assessors containing a and $D = c+g$ as L-indices. Their U-indices within the Pathions must be $(\mathbf{G} + a \veebar \mathbf{S}) = (\mathbf{G} + f)$, and $(\mathbf{G} + g + c \veebar \mathbf{S}) = (\mathbf{G} + g + d)$ respectively. We write their dyads when multiplying with opposite inner signs, as we assume their DMZ is an edge in a Zigzag. We claim the truth of this arithmetic:

$$\begin{array}{c}
+(c+g) - (G+g+d) \\
\underline{+a \qquad + \quad (G+f)} \\
+(G+g+e) - (b+g) \\
\underline{+(b+g) - (G+g+e)} \\
0
\end{array}$$

Bottom left: $(a,b,c) \rightarrow (a, c+g, b+g)$ (Rule 2, with $N = 4$.)

Bottom right: $(a,d,e) \rightarrow (a, g+e, g+d) \rightarrow (a, \mathbf{G}+g+d, \mathbf{G}+g+e)$ (Rule 2 twice, $N = 4$, then $N = 5$.) Upper dyad's inner sign reverses that of product.

Top left: $(f,c,e) \to (e+g, c+g, f) \to (\mathbf{G}+f, c+g, \mathbf{G}+e+g)$ (Rule 2 twice, $N=4$, then $N=5$.)

Top right: $(f,d,b) \to (b+g, d+g, f) \to (b+g, \mathbf{G}+f, \mathbf{G}+g+d)$ (Rule 2 twice, $N=4$, then $N-5$.) Upper dyad's inner sign reverses that of product.

A similar brief exercise with either DMZ formed with the emanated Assessor will show it, too, has a negative inner sign with respect to a positive in its DMZ partner. Two negative edge-signs in one Sail means Zigzag (means *three* negative edge-signs, in fact). Our proof up through the Pathions is complete; we need only indicate the existence of a constructive mechanism for pursuing this same strategy as N grows arbitrarily large.

Consider now the same PSL(2,7) triangle, but in its center put a 16 ($= \mathbf{g} = \mathbf{G}/\mathbf{2}$ for the 64-D Chingons, after the 64 Hexagrams of the *I Ching*, to give them a name). Then, put all 7 of the Pathions' $\mathbf{S} = \mathbf{1}$ Zigzag L-trips into the Rule 0 circle. One gets $3 \cdot 7 = 21$ Rule 2 Zigzag L-trips, and the 10 integers $< g$ found in them and the 7 Rule 0 Zigzag L-trips implies there are 10 Rule 1 Trefoil L-trips, each associated with a distinct Box-Kite. But that would make for $7+21+10 = 38$ Zigzag L-trips, when we know there can only be 35. The extra 3 indicate there's some double-duty occurring: specifically, 3 of the Rule 1 Trefoil L-trips in fact designate not the standard (a,d,e), but (f,d,b), with $d = g = 16$ in each instance. When $(5,14,11)$ is fed into our "trip machine" as Rule 0 circle, both $(11,16,27)$ and $(14,16,30)$ map to (f,d,b) trips tied to Rule 0 Zigzag L-indices $(10,27,17)$ and $(15,30,17)$, whose (a,d,e) trips appear as rays on triangles for $(3,10,9)$ and $(3,13,14)$ respectively. $(11,16,27)$ also shows as an (f,d,b) with Rule 0 trip $(6,11,13)$. (Readers are encouraged to use the code in the appendix to [4], to generate ETs for low **S** and N. Trip-machining details for our $\mathbf{S} = \mathbf{1}$ example are in Appendix A.) For $N = 7$, use the 35 just-derived $\mathbf{S} = 1$ L-trips as Rule 0 circles with a central 32, and so on. ■

4 The Number Hub Theorem ($\mathbf{S} = 2^{N-2}$) for 2^N-ions

Given the lengths required to prove the fullness of ETs for $\mathbf{S} < \mathbf{8}$, it might be surprising to realize that the infinite number of cases for $\mathbf{S} = 2^{N-2}$ for all 2^N-ions are so simple to handle that they almost prove themselves. Yet the proof of this Number Hub Theorem, while technically trivial, has far-reaching implications.

Theorem 10. For all 2^N-ions with ZDs ($N > 3$), and $\mathbf{S} = g = \mathbf{G}/\mathbf{2}$, all non-long-diagonal entries in the emanation table are filled; more, each such filled cell in the ET's upper left quadrant is unmarked (indeed, indicates an edge-current in a

Zigzag); further, the row, column, and cell entries are isomorphic to those found in an unsigned, CDP-generated, multiplication table for the 2^{N-2}-ions; finally, the $Trip_{N-2}$ Zigzag L-index sets which underwrite its Box-Kites are precisely all and only those trips contained in said 2^{N-2}-ions, the ET effectively serving as their high-level atlas.

Proof. As the largest L-index of any Assessor is $2g-1$, and each **S** in the ETs in question is precisely g, then the row (column) labels will ascend from 1 to $g-1$ in simple increments from top to bottom (left to right) in the upper left quadrant, making its square of filled cells isomorphic to unsigned entries in the corresponding 2^{N-2}-ion multiplication table. Also, all these filled cells of the ET will only contain XORs of indices $< g$. Hence, all and only L-index trips will have the edges of their (necessarily Zigzag) Sails residing in said quadrant. All non-long-diagonal cells in the ET are meanwhile filled, since all candidate Assessors have form $M = (m, \mathbf{G} + g + m)$, and for any CPO triplet (a, b, c) whose row and column labels plus cell entry are contained in the upper left quadrant, it is easy to show that the following arithmetic is true:

$$\begin{array}{r} +b-(G+g+b) \\ +a\;+\;(G+g+a) \\ \hline +(G+g+c)-c \\ \underline{+c-(G+g+c)} \\ 0 \end{array}$$

Therefore, the $Trip_{N-2}$ Box-Kites, the Zigzag L-index set of each of which is one of the $Trip_{N-2}$ trips contained in the 2^{N-2}-ions, all have this simple form:

$$(a,b,c,d,e,f) = (a,b,c,g+c,g+b,g+a) \quad \blacksquare$$

Remarks. As will become ever more evident, powers of 2 – which is to say, singleton 1-bits in indefinitely long binary bitstrings – play a role in ZD number theory most readily analogized to that of primes in traditional studies. And while integer triples (from Pythagoras to Fermat) play a central role in prime-factor-based traditional studies, all XOR triplets at two CDP generations' remove from the power of 2 in question are collected by its ET in this new approach. All other integers sufficiently large (meaning > 8) are meanwhile associated with fractal signatures, to each of which is linked a unique infinite-dimensional space spanned by ZD diagonals. But can such a vantage truly be called Number Theory at all?

We say indeed it can: that it is, in fact, the "new kind of number theory" that must accompany Stephen Wolfram's New Kind of Science. In his massive 2002 book, he tells us that, common wisdom to the contrary, complex behavior can be derived from the simplest arithmetical behavior. The obstacle to seeing this resides in the common wisdom itself [5, p. 116]:

> $\cdots$ traditional mathematics makes a fundamental idealization: it assumes that numbers are elementary objects whose only relevant attribute is their size. But in a computer, numbers are not elementary objects. Instead, they must be represented explicitly, typically by giving a sequence of digits.

But that ultimately implies strings of 0's and 1's, where the matter of importance becomes which places in the string are held, and which are vacant: the original meaning of our decimal notation's sense of itself as placeholder arithmetic. The study of zero *divisors* – placeholder substructures – then becomes the natural way to investigate the composite characteristics of Numbers *qua* bitstrings. When we discover, in what follows, that *composite* integers (meaning those requiring multiple bits to be represented) are inherently linked, when seen as strut-constant bit-strings, with infinite-dimensional meta-fractals, the continuation of the quote on the following page should ring true:

> In traditional mathematics, the details of how operations performed on numbers affect sequences of digits are usually considered quite irrelevant. But $\cdots$ precisely by looking at such details, we will be able to see more clearly how complexity develops in systems based on numbers.

5 The Sand Mandala Flip-Book ($8 < \mathbf{S} < 16$, $N = 5$)

In the first concrete exploration of ZD phenomenology beyond the Sedenions [6, pp. 13-19], a startling set of patterns were discovered in the ETs for values of **S** beyond the "Bott limit": that is, for $8 < \mathbf{S} < 16$ (the upper bound being the **G** of the 32-D Pathions), the filled cells sufficed to define not 7, but only 3, Box-Kites for $N = 5$; more, the primary geometric figures in each such ET transformed into each other with each integer increment of **S**, in a manner exactly reminiscent of the flip-books which anticipated cartoon animation. While these seemed perplexing in mid-2002 when they were found, their logic is in fact profoundly simple.

First, each such ET's **S** is just the **X** of one already seen in the Sedenions. We continue our convention of using g to indicate the **G** of the prior CDP generation, employ s for said generation's **S**, and reference all prior Assessor indices by suffixing their letters with asterisks. Then, since $\mathbf{S} = g + s$, the trip $(s, g, g + s)$ mandates, by the First Vizier, that g must belong to the Zigzag Sail if it's to be an Assessor L-index at all. Likewise, the Sedenion Vent L-indices, $f*, e*, d*$, must also be associated with Zigzag Assessors. By an argument exactly akin to that of last section, we then have 3 candidate Box-Kites to consider: since the 3 Vent L-indices are all less than g, they must be mapped to the 3 Assessors A, $g = 8$ must adhere to B (and $s = 1$ to E), while the L-indices of the C Assessors associated with $f*, e*, d*$ must be A*, B*, C* respectively. The proof is easy: taking the new A, C Assessors $= (f*, \mathbf{G} + g + a*)$ and $(g, \mathbf{G} + s)$ in that order as readily generalizable representatives, we do the arithmetic.

$$\begin{array}{c} +g - (\mathbf{G}+s) \\ \underline{+f* + (\mathbf{G}+g+a*)} \\ +(\mathbf{G}+a*) \quad -(f*+g) \\ \underline{+(f*+g) - (\mathbf{G}+a*)} \\ 0 \end{array}$$

The bottom left is just Rule 1. For the bottom right, start with the First Vizier: $(f*, a*, s) \to (f*, \mathbf{G}+s, \mathbf{G}+a*) \to (f*) \cdot (-(\mathbf{G}+s)) = -(\mathbf{G}+a*)$. The top left is derived thus: $(a*,\ g,\ g+a*) \to (g,\ \mathbf{G}+a*,\ \mathbf{G}+g+a*) \to (\mathbf{G}+g+a*) \cdot g = +(\mathbf{G}+a*)$. Finally, $(a*, s, f*) \to (g+a*, g+f*, s) \to (\mathbf{G}+g+a*, \mathbf{G}+s, g+f*)$, but the negative inner sign of the top dyad reverses sign as shown.

The 3 Box-Kites thus derived are the only among the 7 candidates to be viable: for the Zigzag L-index of the $\mathbf{S} = \mathbf{1}$ Sedenion Box-Kite does *not* underwrite a Sail; hence, by what lawyers would call a "fruit of the poisoned tree" argument, neither do the 3 U-trips associated with the same failed Zigzag. Using A* and B*, then invoking the Roundabout Theorem, we see this readily:

$$\begin{array}{c} +b* + (\mathbf{G}+g+e*) \\ \underline{+a* + (\mathbf{G}+g+f*)} \\ -(\mathbf{G}+g+d*) - c* \\ \underline{+c* - (\mathbf{G}+g+d*)} \\ \text{NOT ZERO (only c*'s cancel)} \end{array}$$

With the appending of two successive bits to the left, the bottom-left and top-right products are identical to those obtaining without the $(\mathbf{G}+g)$ being included. Similarly, the top-left product uses Rule 2 twice, to similar effect, but with $(\mathbf{G}+g)$ included in the outcome: since $(f*, d*, b*)$ is CPO, we then get $-(\mathbf{G}+g+d*)$. For the top-right result, meanwhile, the two high bits induce a double reversal, then are killed by XOR, leaving the product the same as if they hadn't been there: $(f*, c*, e*) \rightarrow (g+e*, c*, g+f*) \rightarrow (\mathbf{G}+g+f*, c*, \mathbf{G}+g+e*)$, hence $-c*$. We have an argument reminiscent of Theorem 2: depending on the inner sign of the upper dyad, one pair of products cancels or the other, but not both.

We see, then, that the construction given without explanation at the end of Part I is correct. The arguments given there concerning the vital relationship of a Box-Kite's non-ZD structures to semiotic modeling suggest that this "offing" (to use the appropriately binary slang linked to Mafia hitmen) of a Zigzag's 4 triplets should have a similarly significant role to play in such modeling. This has bearing not just on semiotic, but physical models, since the key dynamic fact implicit in the Zigzag L- and U- trips (or just Z-trips henceforth) is their similarity of orientation: since $(a,b,c);(a,B,C);(A,b,C);(A,B,c)$ are all CPO as written, we are effectively allowed to do pairwise swaps of upper- and lower- case lettering among them without inducing anything a physicist might deem observable (e.g., a 180° reversal or "spin quantum"). This condition of *trip sync* breaks down as soon as we attempt to allow similar swapping between Z-trips and their Trefoil compatriots: in particular, those 2 which don't share an Assessor with the Zigzag. The toy model of [7] would use these features to designate the basis of a "Creation Pressure" that leads to the output of the string theorist's $E_8 \times E_8$ symmetry. This symmetry, as discussed there, breaks in the standard models when one of the primordial E_8's decays into an E_6 – which has 72 roots to parallel the 72 filled cells of our Sand Mandalas. For present purposes, the key aspect of this correspondence is that, in ZD theory at least, the explosion of a singleton Box-Kite into a Sand Mandalic trinity throws the off-switch on the source of the dynamics: the Z-trips which underwrite trip sync no longer even underwrite Box-Kites. The whole scenario suggests nothing so much as those boxes which, when opened by pushing an external lever, emit an arm which pulls up on the same lever, forcing the box to close and the arm to return to its hiding place inside it.

Let's turn now to the ET graphics of the flip-book sequence, so suggestive of cellular automata. For each of the 7 ET's in question, all labels $< g$ are monotonically increasing, since $\mathbf{S}$, and hence their strut opposites, exceed them all. But the only filled (but for long-diagonal crossings) rows and columns will be those with labels equal to $\mathbf{S}-\mathbf{g}=s$ and its strut-opposite g, for these L-indices reside at E and B respectively in all 3 Box-Kites in the ensemble, hence either dyad containing one of them makes DMZs within each of the trio's (a,d,e) and (f,d,b) Sails, filling all 12 ($= 2^4 - 2$, minus 2 for diagonals) fillable cells in each row or column tagged with these Assessors' label. Thus, as s is incremented, two parallel sets of perpendicular lines of ET cells start off defining a square missing its corners, then these parallels move in unit increments toward each other, until they form a 2-ply crossbar once $s = 7$ ($\mathbf{S} = \mathbf{15}$). 24 cells each have row label R or column label C $= s$; 24 reside in lines with label $= g$; and 24 more have their contents P $= s$ or g: these last have an orderliness that is less obvious, but by the last ET in the flip-book, they have arrayed themselves to form the edges of a diamond, orthogonal to the long diagonals and meeting up with the crossbar at its four corners, with $s = 7$ values filling the upward-pointing edges, and $g = 8$'s those sloping down.

The graphics for the flip-book first appeared in [6, p. 15]; they were recycled on p. 13 of [8]; larger, easily-read versions of these ETs were then included (along with numerous other Chingon-based flip-books and other graphics we'll discuss later) as Slides 25-31 of the Powerpoint presentation comprising [1], delivered at Wolfram Science's June 15-18, 2006, NKS conference in Washington, D.C. All three of these resources are available online, and the reader is especially encouraged to explore the last, whose 78 slides can be thought of as the visual accompaniment to this monograph. (Henceforth, references to numbered Slides will be to those contained and indexed in it.)

6 64-D Spectrography: 3 Ingredients for "Recipe Theory"

In a manner clearly related to Bott periodicity, strut constants fall into types demarcated by multiples of 8. But unlike the familiar modulo 8 categorization of types demonstrated, perhaps most familiarly, in the Clifford algebras of various dimensions, the situation with zero-divisors concerns not typology (which keeps producing new patterns at all dimensions), but granularity. As we shall see, emanation tables for $\mathbf{S} > 8$ (and not a power of 2), aside from diagonally aligned cells in otherwise empty stretches, display checkerboard layouts of parallel and perpendicular *near-solid lines* (NSLs), whose cells all have emanations save for a pair of long-diagonal crossings, and whose visual rhythms are strictly governed by $\mathbf{S}$ and 8 or the latter's higher multiples.

The rule we found in the 32-D Pathions for the Sand Mandalas indicates that the basic pattern (and $BK_{5,\,\mathbf{S}}$ for $8 < \mathbf{S} < 16$) is "essentially the same" for all of them. We put the qualifying phrase in quotes, as it is an open question at this point what features, residing at what depth, *are* indeed "the same," and which are different. For the moment, we will invoke the term *spectrographic equivalence* as a sort of promissory note, hoping to stuff ever more elements into its grab-bag of properties, beginning with two. First is something at once intuitively obvious but not readily proven. (We will include a corollary to a later theorem when we have done so). Since the first 8 possible strut-constant values all display maximally-filled ETs, and since anomalies displayed by higher values are strictly side-effects of bits to the left of the 8-bit (which are, of course, its multiples), it is natural to assume that any recursive induction upon simpler forms will echo this "octave" structure: that each time **S** passes a new multiple of 8, it participates in a new type. (As with the Sand Mandalas, we will see this means that $BK_{N,\,\mathbf{S}}$ for the new 7- or 8-element spectral band of new forms will differ from that found in its predecessor band.) This will lead, in the most clear-cut cases – $\mathbf{S} = 15$, or a multiple of 8 not a power of 2, say – to grids composed of 8 x 8 boxes some or all of whose borders are NSLs.

How we determine which cases are clear-cut, meanwhile, and why and how we might want or need to privilege them, leads to our second property to include up-front in our grab-bag. In a manner reminiscent of the various tricks – like minors and cofactors – used in classical matrix theory to prove two matrices are equivalent, we can transform members of a spectral band into each other by certain formal methods of hand-waving. With the Sand Mandalas, for instance, we could replace concrete indices in the row and column labels with abstract designations referencing the (a,b,c) values of each of their 3 Box-Kites, listed in one of a number of predetermined orders: by least-first CPO ordering of such (a,b,c) triplets, in a sequence determined by the Zigzag L-trip of the Sedenion Box-Kite we can derive them from, for instance (which is equivalent to the 3 sand-mandalic Box-Kites' d values, as we've seen).

Since which cells are filled is strictly determined by **S** and **G**, such designations eliminate all individuality among the ETs in question. Hence, if certain display features of one of them seem convenient, we can convert its "tone row" of indices populating its row and column labels into an abstract layout, governed by which index is associated with which Assessor, in the manner sketched last paragraph. We could then use *this* layout as the template for re-writes of all other ETs in the same spectral band, knowing that results obtained using the specific instantiation of the band could thereby be converted into exactly analogous ones for the other band-members.

We will, in fact, implicitly adopt this tactic by using $\mathbf{S} = \mathbf{1}$ as an exemplary "for instance" in numerous arguments, while employing the highest-valued **S** found among the Sand Mandalas, 15, to simplify the visualizing (and calculating) of recursive pattern creation for fixed-**S**, growing N sequences. ($\mathbf{S} = \mathbf{15}$ is chosen because it has all its low bits filled, hence all XORs are derived by simple subtraction, leaving carrybit overflow to show itself only in what matters most to us: the turning off of 4 candidate Box-Kites in the Pathions, and – as we will show two sections hence – 16 in the Chingons, and 4^{N-4} in all higher 2^N-ions.) Where we termed, for reasons already explained, the fixed-N, growing **S** sequences *flip-books*, we designate these new displays (for reasons we'll justify shortly) *balloon-rides*.

While there is but one abstract type for the Sedenions, with one Box-Kite for each of the 7 possible **S** values, a second spectral band emerges in the Pathions to include the Sand Mandalas, and two more are added for the 64-D Chingons. By induction from the universally shared first band for all $N > 3$, where there are $Trip_{N-2}$ Box-Kites in each ET, for each $\mathbf{S} \leq \mathbf{8}$, the first new spectrographic addition includes the upper multiple of 8 that bounds it, since it is not a power of 2: $16 < \mathbf{S} \leq 24$. The second new range, though, is bounded by **G**, hence does not include it, as it is tautologically a power of 2 (which powers, as we saw two sections ago, comply with a type all their own, with the same Box-Kite-count formula as for the lowest spectral band): $24 < \mathbf{S} < 32$.

Each of these two new bands displays a distinctive feature which underwrites one of the three key ingredients for the *recipe theory* we are ultimately aiming for. We call these, for **S** ascending, *(s,g)-modularity* and *hide/fill involution* respectively. The third key ingredient, meanwhile, resides in the band that first emerges in the Pathions – and whose echo in the Chingons has recapitulative features sufficiently rich as to merit the name of *recursivity*. We will be devoting Part III's first post-introductory section to a thorough treatment of the simplest instance of this third ingredient, showing how to ascend into the meta-fractal we call the Whorfian Sky (named for the great theorist of linguistics, Benjamin Lee Whorf, whose last-ever lecture on "Language, mind and reality" described the layering of meaning in language in a manner strongly suggesting something akin to it). Among many visionary passages in his descriptions of a future cross-disciplinary science, the following seems most apt to serve as the lead-in quote for the third and final sweep of our argument [9]:

> Patterns form wholes, akin to the Gestalten of psychology, which are embraced in larger wholes in continual progression. Thus the cosmic picture has a serial or hierarchical character, that of a progression of planes or levels. Lacking recognition of such serial order, different sciences chop segments, as it were, out of the world, segments which perhaps cut across the direction of the natural levels, or stop short when, upon reaching a major change of level, the phenomena become of quite different type, or pass out of the ken of the older observational methods. But $\cdots$ the facts of the linguistic domain compel recognition of serial planes, each explicitly given by an order of patterning observed. It is as if, looking at a wall covered with fine tracery of lacelike design, we found that this tracery served as the ground for a bolder pattern, yet still delicate, of tiny flowers, and that upon becoming aware of this floral expanse we saw that multitudes of gaps in it made another pattern like scrollwork, and that groups of scrolls made letters, the letters if followed in a proper sequence made words, the words were aligned in columns which listed and classified entities, and so on in continual cross-patterning until we found this wall to be – a great book of wisdom! [10, p. 248]

Appendix A: Genealogy of $\mathbf{S} = 1$ Box-Kites

$N = 4$: Unique Quaternion L-index set $(1,2,3)$ fed as Rule 0 circle into PSL(2,7) with central $g = 4$, yielding 7 Octonions trips, each with a different $\mathbf{S}$. For $\mathbf{S} = \mathbf{1}$, have $(3,6,5)$, which becomes singleton Rule 0 for next level.

$N = 5$: $(3,6,5)$ fed as Rule 0 circle into PSL(2,7) with central $g = 8$ yields 3 Rule 2 L-trips as triangle's sides, which (upon affixing their strut opposites as L-indices) generate (along with zero-padded $(3,6,5)$) 4 Box-Kites with $\mathbf{X} = \mathbf{G} + \mathbf{1} = \mathbf{17}$. Triangle's medians become (a,d,e) Trefoil L-index sets of 3 Rule 1 $\mathbf{S} = \mathbf{1}$ Box-Kites, making 7 in all. These Zigzag L-index sets become Rule 0 trips for the next level, and are:

$$\begin{gathered}\text{Rule 0: } (3,6,5) \\ \text{Rule 1: } (3,10,9);\ (6,15,9);\ (5,12,9) \\ \text{Rule 2: } (3,13,14);\ (6,11,13);\ (5,14,11)\end{gathered}$$

$N = 6$: The 7 $N = 5$ Zigzag L-index sets just listed are fed as Rule 0 circles into

PSL(2,7) triangles with central $g = 16$, and are Zigzag L-index sets in their own right for Box-Kites with $\mathbf{X} = \mathbf{G} + \mathbf{1} = \mathbf{33}$.

10 Rule 1 medians, 3 redundant (as they generate (f, d, b)'s where (a, d, e)'s are also given: $(14, 16, 30)$* and $(11, 16, 27)$** in $(5, 14, 11)$'s triangle, the latter also in $(6, 11, 13)$'s). They are associated with these 7 Zigzag L-index sets:

$$(3,18,17);\ (5,20,17);\ (6,23,17);\ (9,24,17);$$
$$(10,27,17)*;\ (12,29,17);\ (15,30,17)**$$

Rule 2 sides: 3 per each Rule 0 trip, as follows:

$$(3,6,5) \to (3,21,22);\ (6,19,21);\ (5,22,19)$$
$$(3,10,9) \to (3,25,26);\ (10,19,25);\ (9,26,19)$$
$$(6,15,9) \to (6,25,31);\ (15,22,25);\ (9,31,22)$$
$$(5,12,9) \to (5,25,28);\ (12,21,25);\ (9,28,21)$$
$$(3,13,14) \to (3,30,29);\ (13,19,30);\ (14,29,19)$$
$$(6,11,13) \to (6,29,27);\ (11,22,29);\ (13,27,22)$$
$$(5,14,11) \to (5,27,30);\ (14,21,27);\ (11,30,21)$$

$N = 7$: Feed the just-listed 35 Zigzag L-index sets to PSL(2,7)'s with $g = 32$, as Rule 0 circles, thereby generating the 155 $\mathbf{S} = \mathbf{1}$ Zigzags found in the 2^7-ions, or Routions – named for the site of the Internet Bubble's once-famed "Massachusetts Miracle," Route 128 – and so on.

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[3] Robert P. C. de Marrais, "The 42 Assessors and the Box-Kites They Fly," arXiv:math.GM/0011260.

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[10] Benjamin Lee Whorf, *Language, Thought, and Reality*, edited by John B. Carroll (M.I.T. Press, Cambridge MA, 1956).