HECKE-CLIFFORD ALGEBRAS AND SPIN HECKE ALGEBRAS I: THE CLASSICAL AFFINE TYPE

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Abstract. Associated to the classical Weyl groups, we introduce the notion of degenerate spin affine Hecke algebras and affine Hecke-Clifford algebras, establish their PBW properties and describe the centers. We further develop connections of these algebras with the usual degenerate (i.e. graded) affine Hecke algebras of Lusztig by introducing a notion of degenerate covering affine Hecke algebras.

1. Introduction

1.1. The Hecke algebras associated to finite and affine Weyl groups are ubiquitous in diverse areas, including representation theories over finite fields, infinite fields of prime characteristic, p-adic fields, and Kazhdan-Lusztig theory for category $\mathcal{O}$. Lusztig [Lu1, Lu2] introduced the graded Hecke algebras, also known as the degenerate affine Hecke algebras, associated to a finite Weyl group W, and provided a geometric realization in terms of equivariant homology. The degenerate affine Hecke algebra of type A has also been defined earlier by Drinfeld [Dr] in connections with Yangians, and it has recently played an important role in modular representations of the symmetric group (cf. the books of Ariki and Kleshchev [Ar, Kle]).

In [W1], the second author introduced the degenerate spin affine Hecke algebra of type A, and related it to the degenerate affine Hecke-Clifford algebra introduced by Nazarov in his study of the representations of the spin symmetric group [Naz]. A quantum version of the spin affine Hecke algebra of type A has been subsequently constructed in [W2], and was shown to be related to the q-analogue of the affine Hecke-Clifford algebra (of type A) defined by Jones and Nazarov [JN].

1.2. The goal of this paper is to provide canonical constructions of the degenerate affine Hecke-Clifford algebras and degenerate spin affine Hecke algebras for all *classical* finite Weyl groups, which goes beyond the type A case, and then establish some basic properties of these algebras. The notion of spin Hecke algebras is arguably more fundamental while the notion of the Hecke-Clifford algebras is crucial for finding the right formulation of the spin Hecke algebras. We also construct the degenerate covering affine Hecke algebras which connect to both the degenerate spin affine Hecke algebras and the degenerate affine Hecke algebras of Lusztig.

1.3. Let us describe our constructions in some detail. The Schur multiplier for each finite Weyl group W has been computed by Ihara and Yokonuma [IY] (see [Kar]). We start with a distinguished double cover $\widetilde{W}$ for any finite Weyl group W:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \widetilde{W} \longrightarrow W \longrightarrow 1. \tag{1.1}$$

Denote $\mathbb{Z}_2 = \{1, z\}$. Assume that W is generated by $s_1, \ldots, s_n$ subject to the relations $(s_i s_j)^{m_{ij}} = 1$. The quotient $\mathbb{C}W^- := \mathbb{C}\widetilde{W}/\langle z+1\rangle$ is then generated by $t_1, \ldots, t_n$ subject to the relations $(t_i t_j)^{m_{ij}} = 1$ for m_{ij} odd, and $(t_i t_j)^{m_{ij}} = -1$ for m_{ij} even. In the symmetric group case, this double cover goes back to I. Schur [Sch]. Note that W acts as automorphisms on the Clifford algebra $\mathcal{C}_W$ associated to the reflection representation $\mathfrak{h}$ of W. We establish a (super)algebra isomorphism

$$\Phi^{fin} : \mathcal{C}_W \rtimes \mathbb{C}W \xrightarrow{\simeq} \mathcal{C}_W \otimes \mathbb{C}W^-,$$

extending an isomorphism in the symmetric group case (due to Sergeev [Ser] and Yamaguchi [Yam] independently) to all Weyl groups. The double cover $\widetilde{W}$ also appeared in Morris [Mo].

We formulate the notion of degenerate affine Hecke-Clifford algebras $\mathfrak{H}^{\mathfrak{c}}_W$ and spin affine Hecke algebras $\mathfrak{H}^-_W$, with unequal parameters in type B case, associated to Weyl groups W of type D and B. The algebra $\mathfrak{H}^{\mathfrak{c}}_W$ (and respectively $\mathfrak{H}^-_W$) contain $\mathcal{C}_W \rtimes \mathbb{C}W$ (and respectively $\mathbb{C}W^-$) as subalgebras. We establish the PBW basis properties for these algebras:

$$\mathfrak{H}^{\mathfrak{c}}_W \cong \mathbb{C}[\mathfrak{h}^*] \otimes \mathcal{C}_W \otimes \mathbb{C}W, \qquad \mathfrak{H}^-_W \cong \mathcal{C}[\mathfrak{h}^*] \otimes \mathbb{C}W^-$$

where $\mathbb{C}[\mathfrak{h}^*]$ denotes the polynomial algebra and $\mathcal{C}[\mathfrak{h}^*]$ denotes a noncommutative skew-polynomial algebra. We describe explicitly the centers for both $\mathfrak{H}^{\mathfrak{c}}_W$ and $\mathfrak{H}^-_W$. The two Hecke algebras are related by a (super)algebra isomorphism

$$\Phi : \mathfrak{H}^{\mathfrak{c}}_W \xrightarrow{\simeq} \mathcal{C}_W \otimes \mathfrak{H}^-_W$$

which extends the isomorphism Φ^{fin}. Such an isomorphism holds also for W of type A [W1].

We further introduce a notion of degenerate covering affine Hecke algebras $\mathfrak{H}^{\sim}_W$ associated to the double cover $\widetilde{W}$ of the Weyl group W of classical type. The algebra $\mathfrak{H}^{\sim}_W$ contains a central element z of order 2 such that the quotient of $\mathfrak{H}^{\sim}_W$ by the ideal $\langle z+1\rangle$ is identified with $\mathfrak{H}^-_W$ and its quotient by the ideal $\langle z-1\rangle$ is identified with Lusztig's degenerate affine Hecke algebras associated to W. In this sense, our covering affine Hecke algebra is a natural affine generalization of the central extension (1.1). A quantum version of the covering affine Hecke algebra of type A was constructed in [W2].

The results in this paper remain valid over any algebraically closed field of characteristic $p \neq 2$. In fact, most of the constructions can be made valid over the ring $\mathbb{Z}[\frac{1}{2}]$ (occasionally we need to adjoint $\sqrt{2}$).

1.4. This paper and [W1] raise many questions, including a geometric realization of the algebras $\mathfrak{H}^{\mathfrak{c}}_W$ or $\mathfrak{H}^-_W$ in the sense of Lusztig [Lu1, Lu2], the classification of the simple modules (cf. [Lu3]), the development of the representation theory, an extension to the exceptional Weyl groups, and so on. We remark that the modular representations of $\mathfrak{H}^{\mathfrak{c}}_W$ in the type A case including the modular representations of the spin symmetric group has been developed by Brundan and Kleshchev [BK] (also cf. [Kle]).

In a sequel [KW] to this paper, we will extend the constructions in this paper to the setup of rational double affine Hecke algebras (see Etingof-Ginzburg [EG]), generalizing and improving a main construction initiated in [W1] for the spin symmetric group. We also hope to quantize these degenerate spin Hecke algebras, reversing the history of developments from quantum to degeneration for the usual Hecke algebras.

1.5. The paper is organized as follows. In Section 2, we describe the distinguished covering groups of the Weyl groups, and establish the isomorphism theorem in the finite-dimensional case. We introduce in Section 3 the degenerate affine Hecke-Clifford algebras of type D and B, and in Section 4 the corresponding degenerate spin affine Hecke algebras. We then extend the isomorphism Φ^{fin} to an isomorphism relating these affine Hecke algebras, establish the PBW properties, and describe the centers of $\mathfrak{H}^{\mathfrak{c}}_W$ and $\mathfrak{H}^-_W$. In Section 5, we formulate the notion of degenerate covering affine Hecke algebras, and establish the connections to degenerate the spin and usual degenerate affine Hecke algebras.

Notations: $\mathbb{Z}_+$ denotes $\{0, 1, 2, \cdots\}$, and $\mathbb{Z}_2$ denotes $\mathbb{Z}/2\mathbb{Z}$.

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2. Spin Weyl groups and Clifford algebras

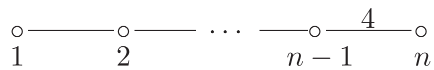
2.1. **The Weyl groups.** Let W be an (irreducible) finite Weyl group with the following presentation:

$$\langle s_1, \ldots, s_n | (s_i s_j)^{m_{ij}} = 1,\ m_{ii} = 1,\ m_{ij} = m_{ji} \in \mathbb{Z}_{\geq 2}, \text{for } i \neq j \rangle \tag{2.1}$$

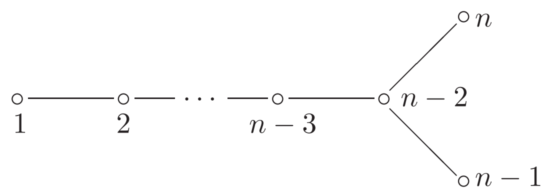
For a Weyl group W, the integers m_{ij} take values in $\{1, 2, 3, 4, 6\}$, and they are specified by the following Coxeter-Dynkin diagrams whose vertices correspond to the generators of W. By convention, we only mark the edge connecting i, j with $m_{ij} \geq 4$. We have $m_{ij} = 3$ for $i \neq j$ connected by an unmarked edge, and $m_{ij} = 2$ if i, j are not connected by an edge.

A_n

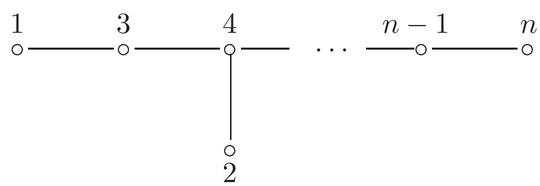
$B_n (n \geq 2)$



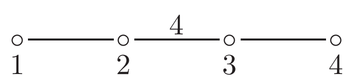
$D_n (n \geq 4)$



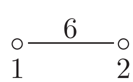
$E_{n=6,7,8}$



F_4



G_2



2.2. **A distinguished double covering of Weyl groups.** The Schur multipliers for finite Weyl groups W (and actually for all finite Coxeter groups) have been computed by Ihara and Yokonuma [IY] (also cf. [Kar]). The explicit generators and relations for the corresponding covering groups of W can be found in Karpilovsky [Kar, Table 7.1].

We shall be concerned about a distinguished double covering $\widetilde{W}$ of W:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \widetilde{W} \longrightarrow W \longrightarrow 1.$$

We denote by $\mathbb{Z}_2 = \{1, z\}$, and by $\tilde{t}_i$ a fixed preimage of the generators s_i of W for each i. The group $\widetilde{W}$ is generated by $z, \tilde{t}_1, \ldots, \tilde{t}_n$ with relations (besides the obvious relation that z is central of order 2) listed in the following table, which corresponds to setting the α_i for all i in Karpilovsky [Kar, Table 7.1] to be z.

W	Generators/Relations for $\widetilde{W}$
A_n	$\tilde{t}_i^2 = 1, 1 \leq i \leq n$, $\tilde{t}_i\tilde{t}_{i+1}\tilde{t}_i = \tilde{t}_{i+1}\tilde{t}_i\tilde{t}_{i+1}, 1 \leq i \leq n-1$ $\tilde{t}_i\tilde{t}_j = z\tilde{t}_j\tilde{t}_i$ if $m_{ij} = 2$
B_n $(n \geq 2)$	$\tilde{t}_i^2 = 1, 1 \leq i \leq n, \quad \tilde{t}_i\tilde{t}_{i+1}\tilde{t}_i = \tilde{t}_{i+1}\tilde{t}_i\tilde{t}_{i+1}, 1 \leq i \leq n-2$ $\tilde{t}_i\tilde{t}_j = z\tilde{t}_j\tilde{t}_i, 1 \leq i < j \leq n-1, m_{ij} = 2$ $\tilde{t}_i\tilde{t}_n = z\tilde{t}_n\tilde{t}_i, 1 \leq i \leq n-2$ $(\tilde{t}_{n-1}\tilde{t}_n)^2 = z(\tilde{t}_n\tilde{t}_{n-1})^2$
D_n $(n \geq 4)$	$\tilde{t}_i^2 = 1, 1 \leq i \leq n, \quad \tilde{t}_i\tilde{t}_j\tilde{t}_i = \tilde{t}_j\tilde{t}_i\tilde{t}_j$ if $m_{ij} = 3$ $\tilde{t}_i\tilde{t}_j = z\tilde{t}_j\tilde{t}_i, 1 \leq i < j \leq n, m_{ij} = 2, i \neq n-1$ $\tilde{t}_{n-1}\tilde{t}_n = z\tilde{t}_n\tilde{t}_{n-1}$
$E_{6,7,8}$	$\tilde{t}_i^2 = 1, 1 \leq i \leq n, \quad \tilde{t}_i\tilde{t}_j\tilde{t}_j\tilde{t}_i = \tilde{t}_j\tilde{t}_i\tilde{t}_j$ if $m_{ij} = 3$ $\tilde{t}_i\tilde{t}_j = z\tilde{t}_j\tilde{t}_i$ if $m_{ij} = 2$
F_4	$\tilde{t}_i^2 = 1, 1 \leq i \leq 4, \quad \tilde{t}_i\tilde{t}_{i+1}\tilde{t}_i = \tilde{t}_{i+1}\tilde{t}_i\tilde{t}_{i+1}$ $(i = 1, 3)$ $\tilde{t}_i\tilde{t}_j = z\tilde{t}_j\tilde{t}_i, 1 \leq i < j \leq 4, m_{ij} = 2$ $(\tilde{t}_2\tilde{t}_3)^2 = z(\tilde{t}_3\tilde{t}_2)^2$
G_2	$\tilde{t}_1^2 = \tilde{t}_2^2 = 1, \quad (\tilde{t}_1\tilde{t}_2)^3 = z(\tilde{t}_2\tilde{t}_1)^3$

The quotient algebra $\mathbb{C}W^- := \mathbb{C}\widetilde{W}/\langle z+1\rangle$ of $\mathbb{C}\widetilde{W}$ by the ideal generated by $z+1$ will be called the *spin Weyl group algebra* associated to W. Denote by $t_i \in \mathbb{C}W^-$ the image of $\tilde{t}_i$. The spin Weyl group algebra $\mathbb{C}W^-$ has the following uniform presentation: $\mathbb{C}W^-$ is the algebra generated by $t_i, 1 \leq i \leq n$, subject to the relations

$$(t_it_j)^{m_{ij}} = \begin{cases} 1, & \text{if } m_{ij} = 1, 3 \\ -1, & \text{if } m_{ij} = 2, 4, 6. \end{cases} \tag{2.2}$$

Note that $\dim \mathbb{C}W^- = |W|$. The algebra $\mathbb{C}W^-$ has a natural superalgebra (i.e. $\mathbb{Z}_2$-graded) structure by letting each t_i be odd.

By definition, the quotient by the ideal $\langle z-1\rangle$ of the group algebra $\mathbb{C}\widetilde{W}$ is isomorphic to $\mathbb{C}W$.

Example 2.1. Let W be the Weyl group of type A_n, B_n, or D_n, which will be assumed in later sections. Then the spin Weyl group algebra $\mathbb{C}W^-$ is generated by $t_1, \ldots, t_n$ with the labeling as in the Coxeter-Dynkin diagrams and the explicit relations summarized in the following table.

Type of W	Defining Relations for $\mathbb{C}W^-$
A_n	$t_i^2 = 1,\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$, $(t_i t_j)^2 = -1$ if $\|i-j\| > 1$
B_n	$t_1, \dots, t_{n-1}$ satisfy the relations for $\mathbb{C}W^-_{A_{n-1}}$, $t_n^2 = 1, (t_i t_n)^2 = -1$ if $i \neq n-1, n$, $(t_{n-1}t_n)^4 = -1$
D_n	$t_1, \dots, t_{n-1}$ satisfy the relations for $\mathbb{C}W^-_{A_{n-1}}$, $t_n^2 = 1, (t_i t_n)^2 = -1$ if $i \neq n-2, n$, $t_{n-2}t_n t_{n-2} = t_n t_{n-2} t_n$

2.3. **The Clifford algebra $\mathcal{C}_W$.** Denote by $\mathfrak{h}$ the reflection representation of the Weyl group W (i.e. a Cartan subalgebra of the corresponding complex Lie algebra $\mathfrak{g}$). In the case of type A_{n-1}, we will always choose to work with the Cartan subalgebra $\mathfrak{h}$ of gl_n instead of sl_n in this paper.

Note that $\mathfrak{h}$ carries a W-invariant nondegenerate bilinear form $(-,-)$, which gives rise to an identification $\mathfrak{h}^* \cong \mathfrak{h}$ and also a bilinear form on $\mathfrak{h}^*$ which will be again denoted by $(-,-)$. One standard way is to identify $\mathfrak{h}^*$ with a suitable subspace of the Euclidean space $\mathbb{C}^N$ and then describe the simple roots $\{\alpha_i\}$ for $\mathfrak{g}$ using a standard orthonormal basis $\{e_i\}$ of $\mathbb{C}^N$.

Denote by $\mathcal{C}_W$ the Clifford algebra associated to $(\mathfrak{h}, (-,-))$, which is regarded as a subalgebra of the Clifford algebra $\mathcal{C}_N$ associated to $(\mathbb{C}^N, (-,-))$. We shall denote by c_i the generator in $\mathcal{C}_N$ corresponding to $\sqrt{2}e_i$ and denote by β_i the generator of $\mathcal{C}_W$ corresponding to the simple root α_i normalized with $\beta_i^2 = 1$. In particular, $\mathcal{C}_N$ is generated by $c_1, \dots, c_N$ subject to the relations

$$c_i^2 = 1, \quad c_i c_j = -c_j c_i \text{ if } i \neq j. \tag{2.3}$$

The explicit generators for $\mathcal{C}_W$ are listed in the following table. Note that $\mathcal{C}_W$ is naturally a superalgebra with each β_i being odd.

Type of W	N	Generators for $\mathcal{C}_W$
A_{n-1}	n	$\beta_i = \frac{1}{\sqrt{2}}(c_i - c_{i+1}), 1 \le i \le n-1$
B_n	n	$\beta_i = \frac{1}{\sqrt{2}}(c_i - c_{i+1}), 1 \le i \le n-1,\ \beta_n = c_n$
D_n	n	$\beta_i = \frac{1}{\sqrt{2}}(c_i - c_{i+1}), 1 \le i \le n-1,\ \beta_n = \frac{1}{\sqrt{2}}(c_{n-1} + c_n)$
E_8	8	$\beta_1 = \frac{1}{2\sqrt{2}}(c_1 + c_8 - c_2 - c_3 - c_4 - c_5 - c_6 - c_7)$ $\beta_2 = \frac{1}{\sqrt{2}}(c_1 + c_2), \beta_i = \frac{1}{\sqrt{2}}(c_{i-1} + c_{i-2}), 3 \le i \le 8$
E_7	8	the subset of β_i in E_8, $1 \le i \le 7$
E_6	8	the subset of β_i in E_8, $1 \le i \le 6$
F_4	4	$\beta_1 = \frac{1}{\sqrt{2}}(c_1 - c_2), \beta_2 = \frac{1}{\sqrt{2}}(c_2 - c_3)$ $\beta_3 = c_3, \beta_4 = \frac{1}{2}(c_4 - c_1 - c_2 - c_3)$
G_2	3	$\beta_1 = \frac{1}{\sqrt{2}}(c_1 - c_2), \beta_2 = \frac{1}{\sqrt{6}}(-2c_1 + c_2 + c_3)$

The action of W on $\mathfrak{h}$ and $\mathfrak{h}^*$ preserves the bilinear form $(-,-)$ and thus W acts as automorphisms of the algebra $\mathcal{C}_W$. This gives rise to a semi-direct product $\mathcal{C}_W \rtimes \mathbb{C}W$. Moreover, the algebra $\mathcal{C}_W \rtimes \mathbb{C}W$ naturally inherits the superalgebra structure by letting elements in W be even and each β_i be odd.

2.4. **The basic spin supermodule.**

Theorem 2.2. *Let W be a finite Weyl group. Then, there exists a surjective superalgebra homomorphism $\mathbb{C}W^- \xrightarrow{\Omega} \mathcal{C}_W$ which sends t_i to β_i for each i.*

Proof. It suffices to show that each β_i satisfies the same relations as for the t_i's, i.e. $(\beta_i\beta_j)^{m_{ij}} = 1$ for m_{ij} odd and $(\beta_i\beta_j)^{m_{ij}} = -1$ for m_{ij} even. This can be checked case by case using the explicit formulas of β_i in the Table of Section 2.3. □

Remark 2.3. In the type A, namely the symmetric group case, Theorem 2.2 goes back to I. Schur [Sch] (cf. [Joz]). Theorem 2.2 has appeared in a somewhat different form in Morris [Mo]. In [Mo], W is viewed as a subgroup of the orthogonal Lie group which preserves $(\mathfrak{h}, (-,-))$. The preimage of W in the spin group which covers the orthogonal group provides the double cover $\widetilde{W}$ of W, where the Atiyah-Bott-Shapiro construction of the spin group in terms of the Clifford algebra $\mathcal{C}_W$ was used to describe this double cover of W.

The superalgebra $\mathcal{C}_W$ has a unique (up to isomorphism) simple supermodule (i.e. $\mathbb{Z}_2$-graded module). By pulling it back via the homomorphism $\Omega : \mathbb{C}W^- \to \mathcal{C}_W$, we obtain a distinguished $\mathbb{C}W^-$-supermodule, called the basic spin supermodule. This is a natural generalization of the classical construction for $\mathbb{C}S_n^-$ due to Schur [Sch] (see [Joz]).

2.5. **A superalgebra isomorphism.** Given two superalgebras $\mathcal{A}$ and $\mathcal{B}$, we view the tensor product of superalgebras $\mathcal{A} \otimes \mathcal{B}$ as a superalgebra with multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|}(aa' \otimes bb') \qquad (a, a' \in \mathcal{A},\ b, b' \in \mathcal{B}) \tag{2.4}$$

where $|b|$ denotes the $\mathbb{Z}_2$-degree of b, etc. Also, we shall use short-hand notation ab for $(a \otimes b) \in \mathcal{A} \otimes \mathcal{B}$, $a = a \otimes 1$, and $b = b \otimes 1$.

Theorem 2.4. *We have an isomorphism of superalgebras:*

$$\Phi : \mathcal{C}_W \rtimes \mathbb{C}W \xrightarrow{\simeq} \mathcal{C}_W \otimes \mathbb{C}W^-$$

which extends the identity map on $\mathcal{C}_W$ and sends $s_i \mapsto -\sqrt{-1}\beta_i t_i$. The inverse map Ψ is the extension of the identity map on $\mathcal{C}_W$ which sends $t_i \mapsto \sqrt{-1}\beta_i s_i$.

We first prepare some lemmas.

Lemma 2.5. *We have $(\Phi(s_i)\Phi(s_j))^{m_{ij}} = 1$.*

Proof. Theorem 2.2 says that $(t_i t_j)^{m_{ij}} = (\beta_i\beta_j)^{m_{ij}} = \pm 1$. Thanks to the identities $\beta_j t_i = -t_i\beta_j$ and $\Phi(s_i) = -\sqrt{-1}\beta_i t_i$, we have

$$\begin{aligned}(\Phi(s_i)\Phi(s_j))^{m_{ij}} &= (-\beta_i t_i \beta_j t_j)^{m_{ij}} \\ &= (\beta_i\beta_j t_i t_j)^{m_{ij}} = (\beta_i\beta_j)^{m_{ij}}(t_i t_j)^{m_{ij}} = 1.\end{aligned}$$

□

Lemma 2.6. *We have $\beta_j\Phi(s_i) = \Phi(s_i)\, s_i(\beta_j)$ for all i, j.*

Proof. Note that $(\beta_i, \beta_i) = 2\beta_i^2 = 2$, and hence

$$\beta_j\beta_i = -\beta_i\beta_j + (\beta_j, \beta_i) = -\beta_i\beta_j + \frac{2(\beta_j, \beta_i)}{(\beta_i, \beta_i)}\beta_i^2 = -\beta_i s_i(\beta_j).$$

Thus, we have

$$\begin{aligned}\beta_j\Phi(s_i) &= -\sqrt{-1}\beta_j\beta_i t_i \\ &= -\sqrt{-1}t_i\beta_j\beta_i = \sqrt{-1}t_i\beta_i s_i(\beta_j) = \Phi(s_i)\, s_i(\beta_j).\end{aligned}$$

□

Proof of Theorem 2.4. The algebra $\mathcal{C}_W \rtimes \mathbb{C}W$ is generated by β_i and s_i for all i. Lemmas 2.5 and 2.6 imply that Φ is a (super) algebra homomorphism. Clearly Φ is surjective, and thus an isomorphism by a dimension counting argument.

Clearly, Ψ and Φ are inverses of each other. □

Remark 2.7. We were led to consider the distinguished double cover $\widetilde{W}$ in search of an isomorphism as in Theorem 2.4 and found Theorem 2.2 before learning about [Mo]. The type A case of Theorem 2.4 was due to Sergeev and Yamaguchi independently [Ser, Yam], and it played a fundamental role in clarifying the earlier observation in the literature (cf. [Joz, St]) that the representation theories of $\mathbb{C}S_n^-$ and $\mathcal{C}_n \rtimes \mathbb{C}S_n$ are essentially the same.

In the remainder of the paper, W is always assumed to be one of the classical Weyl groups of type A, B, or D.

3. Degenerate affine Hecke-Clifford algebras

In this section, we introduce the degenerate affine Hecke-Clifford algebras of type D and B, and establish some basic properties. The degenerate affine Hecke-Clifford algebra associated to the symmetric group S_n was introduced earlier by Nazarov under the terminology of the affine Sergeev algebra [Naz].

3.1. The algebra $\mathfrak{H}^{\mathfrak{c}}_W$ of type A_{n-1}.

Definition 3.1. [Naz] Let $u \in \mathbb{C}$, and $W = W_{A_{n-1}} = S_n$ be the Weyl group of type A_{n-1}. The degenerate affine Hecke-Clifford algebra of type A_{n-1}, denoted by $\mathfrak{H}^{\mathfrak{c}}_W$ or $\mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}$, is the algebra generated by $x_1, \ldots, x_n$, $c_1, \ldots, c_n$, and S_n subject to the relation (2.3) and the following relations:

$$x_i x_j = x_j x_i \quad (\forall i, j) \tag{3.1}$$

$$x_i c_i = -c_i x_i,\ x_i c_j = c_j x_i \quad (i \neq j) \tag{3.2}$$

$$\sigma c_i = c_{\sigma i} \sigma \quad (1 \leq i \leq n, \sigma \in S_n) \tag{3.3}$$

$$x_{i+1} s_i - s_i x_i = u(1 - c_{i+1} c_i) \tag{3.4}$$

$$x_j s_i = s_i x_j \quad (j \neq i, i+1) \tag{3.5}$$

Remark 3.2. Alternatively, we may view u as a formal parameter and the algebra $\mathfrak{H}^{\mathfrak{c}}_W$ as a $\mathbb{C}(u)$-algebra. Similar remarks apply to various algebras introduced in this paper. Our convention $c_i^2 = 1$ differs from Nazarov's which sets $c_i^2 = -1$.

The symmetric group S_n acts as the automorphisms on the symmetric algebra $\mathbb{C}[\mathfrak{h}^*] \cong \mathbb{C}[x_1, \ldots, x_n]$ by permutation. We shall denote this action by $f \mapsto f^\sigma$ for $\sigma \in S_n, f \in \mathbb{C}[x_1, \ldots, x_n]$.

Proposition 3.3. *Let $W = W_{A_{n-1}}$. Given $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $1 \leq i \leq n-1$, the following identity holds in $\mathfrak{H}^{\mathfrak{c}}_W$:*

$$s_i f = f^{s_i} s_i + u \frac{f - f^{s_i}}{x_{i+1} - x_i} + u \frac{c_i c_{i+1} f - f^{s_i} c_i c_{i+1}}{x_{i+1} + x_i}.$$

It is understood here and in similar expressions below that $\frac{A}{g(x)} = \frac{1}{g(x)} \cdot A$. In this sense, both numerators on the right-hand side of the above formula are (left-)divisible by the corresponding denominators.

Proof. By the definition of $\mathfrak{H}^{\mathfrak{c}}_W$, we have that $s_i x_j^k = x_j^k s_i$ for any k if $j \neq i, i+1$. So it suffices to check the identity for $f = x_i^k x_{i+1}^l$. We will proceed by induction.

First, consider $f = x_i^k$, i.e. $l = 0$. For $k = 1$, this follows from (3.4). Now assume that the statement is true for k. Then

$$\begin{aligned}
s_i x_i^{k+1} &= \left(x_{i+1}^k s_i + u \frac{(x_i^k - x_{i+1}^k)}{x_{i+1} - x_i} + u \frac{(c_i c_{i+1} x_i^k - x_{i+1}^k c_i c_{i+1})}{x_{i+1} + x_i} \right) x_i \\
&= x_{i+1}^k \left(x_{i+1} s_i - u(1 - c_{i+1} c_i) \right) \\
&\quad + u \frac{(x_i^k - x_{i+1}^k)}{x_{i+1} - x_i} x_i + u \frac{(c_i c_{i+1} x_i^k - x_{i+1}^k c_i c_{i+1})}{x_{i+1} + x_i} x_i \\
&= x_{i+1}^{k+1} s_i + u \frac{(x_i^{k+1} - x_{i+1}^{k+1})}{x_{i+1} - x_i} + u \frac{(c_i c_{i+1} x_i^{k+1} - x_{i+1}^{k+1} c_i c_{i+1})}{x_{i+1} + x_i},
\end{aligned}$$

where the last equality is obtained by using (3.2) and (3.4) repeatedly.

An induction on l will complete the proof of the proposition for the monomial $f = x_i^k x_{i+1}^l$. The case $l = 0$ is established above. Assume the formula is true for $f = x_i^k x_{i+1}^l$. Then using $s_i x_{i+1} = (x_i s_i + u(1 + c_{i+1}c_i))$, we compute that

$$\begin{aligned} s_i x_i^k x_{i+1}^{l+1} &= \Bigg(x_i^l x_{i+1}^k s_i + u\frac{(x_i^k x_{i+1}^l - x_i^l x_{i+1}^k)}{x_{i+1} - x_i} \\ &\qquad + u\frac{(c_i c_{i+1} x_i^k x_{i+1}^l - x_i^l x_{i+1}^k c_i c_{i+1})}{x_{i+1} + x_i} \Bigg) \cdot x_{i+1} \\ &= x_i^l x_{i+1}^k (x_i s_i + u(1 + c_{i+1}c_i)) \\ &\quad + u\frac{(x_i^k x_{i+1}^{l+1} - x_i^l x_{i+1}^{k+1})}{x_{i+1} - x_i} + u\frac{(c_i c_{i+1} x_i^k x_{i+1}^{l+1} + x_i^l x_{i+1}^{k+1} c_i c_{i+1})}{x_{i+1} + x_i} \\ &= x_i^{l+1} x_{i+1}^k s_i + u\frac{(x_i^k x_{i+1}^{l+1} - x_i^{l+1} x_{i+1}^k)}{x_{i+1} - x_i} \\ &\qquad + u\frac{(c_i c_{i+1} x_i^k x_{i+1}^{l+1} - x_i^{l+1} x_{i+1}^k c_i c_{i+1})}{x_{i+1} + x_i}. \end{aligned}$$

This completes the proof of the proposition. □

The algebra $\mathfrak{H}_W^{\mathfrak{c}}$ contains $\mathbb{C}[\mathfrak{h}^*], \mathcal{C}_n$, and $\mathbb{C}W$ as subalgebras. We shall denote $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$ for $\alpha = (a_1, \ldots, a_n) \in \mathbb{Z}_+^n$, $c^\epsilon = c_1^{\epsilon_1} \cdots c_n^{\epsilon_n}$ for $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{Z}_2^n$.

Below we give a new proof of the PBW basis theorem for $\mathfrak{H}_W^{\mathfrak{c}}$ (which has been established by different methods in [Naz, Kle]), using in effect the induced $\mathfrak{H}_W^{\mathfrak{c}}$-module $\mathrm{Ind}_W^{\mathfrak{H}_W^{\mathfrak{c}}} \mathbf{1}$ from the trivial W-module $\mathbf{1}$. This induced module is of independent interest. This approach will then be used for type D and B.

Theorem 3.4. *Let $W = W_{A_{n-1}}$. The multiplication of subalgebras $\mathbb{C}[\mathfrak{h}^*], \mathcal{C}_n$, and $\mathbb{C}W$ induces a vector space isomorphism*

$$\mathbb{C}[\mathfrak{h}^*] \otimes \mathcal{C}_n \otimes \mathbb{C}W \xrightarrow{\simeq} \mathfrak{H}_W^{\mathfrak{c}}.$$

Equivalently, $\{x^\alpha c^\epsilon w | \alpha \in \mathbb{Z}_+^n, \epsilon \in \mathbb{Z}_2^n, w \in W\}$ forms a linear basis for $\mathfrak{H}_W^{\mathfrak{c}}$ (called a PBW basis).

Proof. Note that IND $:= \mathbb{C}[x_1, \ldots, x_n] \otimes \mathcal{C}_n$ admits an algebra structure by (2.3), (3.1) and (3.2). By the explicit defining relations of $\mathfrak{H}_W^{\mathfrak{c}}$, we can verify that the algebra $\mathfrak{H}_W^{\mathfrak{c}}$ acts on IND by letting x_i and c_i act by left multiplication, and $s_i \in S_n$ act by

$$s_i.(fc^\epsilon) = f^{s_i} c^{s_i \epsilon} + \left(u\frac{f - f^{s_i}}{x_{i+1} - x_i} + u\frac{c_i c_{i+1} f - f^{s_i} c_i c_{i+1}}{x_{i+1} + x_i} \right) c^\epsilon.$$

For $\alpha = (a_1, \ldots, a_n)$, we denote $|\alpha| = a_1 + \cdots + a_n$. Define a dictionary total ordering $<$ on the monomials $x^\alpha, \alpha \in \mathbb{Z}_+^n$, (or respectively on $\mathbb{Z}_+^n$), by declaring $x^\alpha < x^{\alpha'}$, (or respectively $\alpha < \alpha'$), if $|\alpha| < |\alpha'|$, or if $|\alpha| = |\alpha'|$ then there exists an $1 \le i \le n$ such that $a_i < a_i'$ and $a_j = a_j'$ for each $j < i$.

Note that the algebra $\mathfrak{H}^{\mathfrak{c}}_W$ is spanned by the elements of the form $x^\alpha c^\epsilon w$. It remains to show that these elements are linearly independent.

Suppose $S := \sum a_{\alpha\epsilon w} x^\alpha c^\epsilon w = 0$ for a finite sum over α, ϵ, w. Now consider the action S on an element of the form $x_1^N x_2^{N^2} \cdots x_n^{N^n}$ for $N \gg 0$. Let $\tilde{w}$ be such that $(x_1^N x_2^{N^2} \cdots x_n^{N^n})^{\tilde{w}}$ is minimal among all possible w with $a_{\alpha\epsilon w} \neq 0$ for some α, ϵ. Let $\tilde{\alpha}$ be the smallest element among all α with $a_{\alpha\epsilon\tilde{w}} \neq 0$ for some ϵ. Then among all monomials in $S(x_1^N x_2^{N^2} \cdots x_n^{N^n})$, the monomial $x^{\tilde{\alpha}}(x_1^N x_2^{N^2} \cdots x_n^{N^n})^{\tilde{w}} c^\epsilon$ appears as a minimal term with coefficient $\pm a_{\tilde{\alpha}\epsilon\tilde{w}}$. It follows from $S = 0$ that $a_{\tilde{\alpha}\epsilon\tilde{w}} = 0$. This is only possible when all $a_{\alpha\epsilon w} = 0$, and hence the elements $x^\alpha c^\epsilon w$ are linearly independent. □

Remark 3.5. By the PBW Theorem 3.4, the $\mathfrak{H}^{\mathfrak{c}}_W$-module IND introduced in the above proof can be identified with the $\mathfrak{H}^{\mathfrak{c}}_W$-module induced from the trivial $\mathbb{C}W$-module. The same remark applies below to type D and B.

3.2. **The algebra $\mathfrak{H}^{\mathfrak{c}}_W$ of type D_n.** Let $W = W_{D_n}$ be the Weyl group of type D_n. It is generated by $s_1, \ldots, s_n$, subject to the following relations:

$$s_i^2 = 1 \quad (i \leq n-1) \tag{3.6}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (i \leq n-2) \tag{3.7}$$

$$s_i s_j = s_j s_i \quad (|i-j| > 1,\ i, j \neq n) \tag{3.8}$$

$$s_i s_n = s_n s_i \quad (i \neq n-2) \tag{3.9}$$

$$s_{n-2} s_n s_{n-2} = s_n s_{n-2} s_n, \quad s_n^2 = 1. \tag{3.10}$$

In particular, S_n is generated by $s_1, \ldots, s_{n-1}$ subject to the relations (3.6–3.8) above.

Definition 3.6. Let $u \in \mathbb{C}$, and let $W = W_{D_n}$. The degenerate affine Hecke-Clifford algebra of type D_n, denoted by $\mathfrak{H}^{\mathfrak{c}}_W$ or $\mathfrak{H}^{\mathfrak{c}}_{D_n}$, is the algebra generated by x_i, c_i, s_i, $1 \leq i \leq n$, subject to the relations (3.1–3.5), (3.6–3.10), and the following additional relations:

$$\begin{aligned} s_n c_n &= -c_{n-1} s_n \\ s_n c_i &= c_i s_n \quad (i \neq n-1, n) \\ s_n x_n + x_{n-1} s_n &= -u(1 + c_{n-1} c_n) \\ s_n x_i &= x_i s_n \quad (i \neq n-1, n). \end{aligned} \tag{3.11}$$

Proposition 3.7. *The algebra $\mathfrak{H}^{\mathfrak{c}}_{D_n}$ admits anti-involutions τ_1, τ_2 defined by*

$$\begin{aligned} \tau_1 &: s_i \mapsto s_i, \quad c_j \mapsto c_j, \quad x_j \mapsto x_j, \quad (1 \leq i \leq n); \\ \tau_2 &: s_i \mapsto s_i, \quad c_j \mapsto -c_j, \quad x_j \mapsto x_j, \quad (1 \leq i \leq n). \end{aligned}$$

Also, the algebra $\mathfrak{H}^{\mathfrak{c}}_{D_n}$ admits an involution σ which fixes all generators s_i, x_i, c_i except the following 4 generators:

$$\sigma : s_n \mapsto s_{n-1}, \quad s_{n-1} \mapsto s_n, \quad x_n \mapsto -x_n, \quad c_n \mapsto -c_n.$$

Proof. We leave the easy verifications on τ_1, τ_2 to the reader.

It remains to check that σ preserves the defining relations. Almost all the relations are obvious except (3.4) and (3.11). We see that σ preserves (3.4) as follows: for $i \leq n-2$,

$$\begin{aligned}
\sigma(x_{i+1}s_i - s_i x_i) &= x_{i+1}s_i - s_i x_i \\
&= u(1 - c_{i+1}c_i) = \sigma(u(1 - c_{i+1}c_i)); \\
\sigma(x_n s_{n-1} - s_{n-1}x_{n-1}) &= -x_n s_n - s_n x_{n-1} \\
&= u(1 + c_n c_{n-1}) = \sigma(u(1 - c_n c_{n-1})).
\end{aligned}$$

Also, σ preserves (3.11) since

$$\begin{aligned}
\sigma(s_n x_n + x_{n-1}s_n) &= -s_{n-1}x_n + x_{n-1}s_{n-1} \\
&= -u(1 - c_{n-1}c_n) = \sigma(-u(1 + c_{n-1}c_n)).
\end{aligned}$$

Hence, σ is an automorphism of $\mathfrak{H}^{\mathfrak{c}}_{D_n}$. Clearly $\sigma^2 = 1$. $\square$

The natural action of S_n on $\mathbb{C}[\mathfrak{h}^*] = \mathbb{C}[x_1, \ldots, x_n]$ is extended to an action of W_{D_n} by letting

$$x_n^{s_n} = -x_{n-1}, \quad x_{n-1}^{s_n} = -x_n, \quad x_i^{s_n} = x_i \quad (i \neq n-1, n).$$

Proposition 3.8. *Let $W = W_{D_n}$, $1 \leq i \leq n-1$, and $f \in \mathbb{C}[x_1, \ldots, x_n]$. Then the following identities hold in $\mathfrak{H}^{\mathfrak{c}}_W$:*

(1) $s_i f = f^{s_i} s_i + u\dfrac{f - f^{s_i}}{x_{i+1} - x_i} + u\dfrac{c_i c_{i+1} f - f^{s_i} c_i c_{i+1}}{x_{i+1} + x_i}$,

(2) $s_n f = f^{s_n} s_n - u\dfrac{f - f^{s_n}}{x_n + x_{n-1}} + u\dfrac{c_{n-1}c_n f - f^{s_n} c_{n-1}c_n}{x_n - x_{n-1}}$.

Proof. Formula (1) has been established by induction as in type A_{n-1}. Formula (2) can be verified by a similar induction. $\square$

3.3. **The algebra $\mathfrak{H}^{\mathfrak{c}}_W$ of type B_n.** Let $W = W_{B_n}$ be the Weyl group of type B_n, which is generated by $s_1, \ldots, s_n$, subject to the defining relation for S_n on $s_1, \ldots, s_{n-1}$ and the following additional relations:

$$s_i s_n = s_n s_i \quad (1 \leq i \leq n-2) \tag{3.12}$$

$$(s_{n-1}s_n)^4 = 1, \; s_n^2 = 1. \tag{3.13}$$

We note that the simple reflections $s_1, \ldots, s_n$ belongs to two different conjugacy classes in W_{B_n}, with $s_1, \ldots, s_{n-1}$ in one and s_n in the other.

Definition 3.9. Let $u, v \in \mathbb{C}$, and let $W = W_{B_n}$. The degenerate affine Hecke-Clifford algebra of type B_n, denoted by $\mathfrak{H}^{\mathfrak{c}}_W$ or $\mathfrak{H}^{\mathfrak{c}}_{B_n}$, is the algebra generated by x_i, c_i, s_i, $1 \leq i \leq n$, subject to the relations (3.1–3.5), (3.6–3.8), (3.12–3.13), and the following additional relations:

$$\begin{aligned} s_n c_n &= -c_n s_n \\ s_n c_i &= c_i s_n \quad (i \neq n) \\ s_n x_n + x_n s_n &= -\sqrt{2}\, v \\ s_n x_i &= x_i s_n \quad (i \neq n). \end{aligned}$$

The factor $\sqrt{2}$ above is inserted for the convenience later in relation to the spin affine Hecke algebras. When it is necessary to indicate u, v, we will write $\mathfrak{H}^{\mathfrak{c}}_W(u, v)$ for $\mathfrak{H}^{\mathfrak{c}}_W$. For any $a \in \mathbb{C}\backslash\{0\}$, we have an isomorphism of superalgebras $\psi : \mathfrak{H}^{\mathfrak{c}}_W(au, av) \to \mathfrak{H}^{\mathfrak{c}}_W(u, v)$ given by dilations $x_i \mapsto ax_i$ for $1 \le i \le n$, while fixing each s_i, c_i.

The action of S_n on $\mathbb{C}[\mathfrak{h}^*] = \mathbb{C}[x_1, \ldots, x_n]$ can be extended to an action of W_{B_n} by letting

$$x_n^{s_n} = -x_n, \quad x_i^{s_n} = x_i, \quad (i \neq n).$$

Proposition 3.10. *Let $W = W_{B_n}$. Given $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $1 \le i \le n-1$, the following identities hold in $\mathfrak{H}^{\mathfrak{c}}_W$:*

(1) $s_i f = f^{s_i} s_i + u\dfrac{f - f^{s_i}}{x_{i+1} - x_i} + u\dfrac{c_i c_{i+1} f - f^{s_i} c_i c_{i+1}}{x_{i+1} + x_i}$,

(2) $s_n f = f^{s_n} s_n + v\dfrac{f - f^{s_n}}{2x_n}$.

Proof. The proof is similar to type A and D, and will be omitted. □

3.4. **PBW basis for $\mathfrak{H}^{\mathfrak{c}}_W$.** Note that $\mathfrak{H}^{\mathfrak{c}}_W$ contains $\mathbb{C}[\mathfrak{h}^*], \mathcal{C}_n, \mathbb{C}W$ as subalgebras. We have the following PBW basis theorem for $\mathfrak{H}^{\mathfrak{c}}_W$.

Theorem 3.11. *Let $W = W_{D_n}$ or $W = W_{B_n}$. The multiplication of subalgebras $\mathbb{C}[\mathfrak{h}^*], \mathcal{C}_n$, and $\mathbb{C}W$ induces a vector space isomorphism*

$$\mathbb{C}[\mathfrak{h}^*] \otimes \mathcal{C}_n \otimes \mathbb{C}W \longrightarrow \mathfrak{H}^{\mathfrak{c}}_W.$$

Equivalently, the elements $\{x^\alpha c^\epsilon w | \alpha \in \mathbb{Z}^n_+, \epsilon \in \mathbb{Z}^n_2, w \in W\}$ form a linear basis for $\mathfrak{H}^{\mathfrak{c}}_W$ (called a PBW basis).

Proof. Let us first assume $W = W_{D_n}$. We can verify by a direct lengthy computation that the $\mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}$-action on $\mathrm{IND} = \mathbb{C}[x_1, \ldots, x_n] \otimes \mathcal{C}_n$ (see the proof of Theorem 3.4) naturally extends to an action of $\mathfrak{H}^{\mathfrak{c}}_{D_n}$, where (compare Proposition 3.8) s_n acts by

$$s_n.(fc^\epsilon) = f^{s_n} c^{s_n \epsilon} - \left(u\frac{f - f^{s_n}}{x_n + x_{n-1}} - u\frac{c_{n-1}c_n f - f^{s_n} c_{n-1} c_n}{x_n - x_{n-1}} \right) c^\epsilon.$$

Clearly, the elements $x^\alpha c^\epsilon w$, where $\alpha \in \mathbb{Z}^n_+, \epsilon \in \mathbb{Z}^n_2, w \in W$, span $\mathfrak{H}^{\mathfrak{c}}_W$. It is known that W acts on $\mathbb{C}[x_1, \ldots, x_n]$ as linearly independent operators. Now a similar argument as in the proof of Theorem 3.4 applies here to establish the linear independence of $\{x^\alpha c^\epsilon w\}$.

The proof for $W = W_{B_n}$ is entirely analogous. We will omit the details except mentioning that the $\mathfrak{H}^{\mathfrak{c}}_{A_{n-1}}$-action on IND extends to an action of $\mathfrak{H}^{\mathfrak{c}}_{B_n}$, where (compare Proposition 3.10) s_n acts by

$$s_n.(fc^{\epsilon}) = f^{s_n}c^{s_n\epsilon} + v\frac{f - f^{s_n}}{2x_n}c^{\epsilon}.$$

□

3.5. **The even center for $\mathfrak{H}^{\mathfrak{c}}_W$.** The *even center* of a superalgebra A, denoted by $Z(A)$, is the subalgebra of even central elements of A.

Proposition 3.12. *Let $W = W_{D_n}$ or $W = W_{B_n}$. The even center $Z(\mathfrak{H}^{\mathfrak{c}}_W)$ of $\mathfrak{H}^{\mathfrak{c}}_W$ is isomorphic to $\mathbb{C}[x_1^2, \dots, x_n^2]^W$.*

Proof. We first show that every W-invariant polynomial f in $x_1^2, \dots, x_n^2$ is central in $\mathfrak{H}^{\mathfrak{c}}_W$. Indeed, f commutes with each c_i by (3.2) and clearly f commutes with each x_i. By Proposition 3.8 for type D_n or Proposition 3.10 for type B_n, $s_i f = f s_i$ for each i. Since $\mathfrak{H}^{\mathfrak{c}}_W$ is generated by c_i, x_i and s_i for all i, f is central in $\mathfrak{H}^{\mathfrak{c}}_W$ and $\mathbb{C}[x_1^2, \dots, x_n^2]^W \subseteq Z(\mathfrak{H}^{\mathfrak{c}}_W)$.

On the other hand, take an even central element $C = \sum a_{\alpha,\epsilon,w}x^{\alpha}c^{\epsilon}w$ in $\mathfrak{H}^{\mathfrak{c}}_W$. We claim that $w = 1$ whenever $a_{\alpha,\epsilon,w} \neq 0$. Otherwise, let $1 \neq w_0 \in W$ be maximal with respect to the Bruhat ordering in W such that $a_{\alpha,\epsilon,w_0} \neq 0$. Then $x_i^{w_0} \neq x_i$ for some i. By Proposition 3.8 for type D_n or Proposition 3.10 for type B_n, $x_i^2 C - C x_i^2$ is equal to $a_{\alpha,\epsilon,w_0}x^{\alpha}(x_i^2 - (x_i^{w_0})^2)c^{\epsilon}w_0$ plus a linear combination of monomials not involving w_0, hence nonzero. This contradicts to the fact that C is central. So we can write $C = \sum a_{\alpha,\epsilon}x^{\alpha}c^{\epsilon}$.

Since $x_i C = C x_i$ for each i, then (3.2) forces C to be in $\mathbb{C}[x_1, \dots, x_n]$. Now by (3.2) and $c_i C = C c_i$ for each i we have that $C \in \mathbb{C}[x_1^2, \dots, x_n^2]$. Since $s_i C = C s_i$ for each i, we then deduce from Proposition 3.8 for type D_n or Proposition 3.10 for type B_n that $C \in \mathbb{C}[x_1^2, \dots, x_n^2]^W$.

This completes the proof of the proposition. □

It will be very interesting to classify the simple modules of $\mathfrak{H}^{\mathfrak{c}}_W$ and to find a possible geometric realization. This was carried out by Lusztig [Lu1, Lu2, Lu3] for the usual degenerate affine Hecke algebra case.

4. Degenerate spin affine Hecke algebras

In this section we will introduce the degenerate spin affine Hecke algebra $\mathfrak{H}^{-}_W$ when W is the Weyl group of types D_n or B_n, and then establish the connections with the corresponding degenerate affine Hecke-Clifford algebras $\mathfrak{H}^{\mathfrak{c}}_W$. See [W1] for the type A case.

4.1. **The skew-polynomial algebra.** We shall denote by $\mathcal{C}[b_1, \dots, b_n]$ the $\mathbb{C}$-algebra generated by $b_1, \dots, b_n$ subject to the relations

$$b_i b_j + b_j b_i = 0 \quad (i \neq j).$$

This is naturally a superalgebra by letting each b_i be odd. We will refer to this as the *skew-polynomial algebra* in n variables. This algebra has a linear basis given by $b^\alpha := b_1^{k_1} \cdots b_n^{k_n}$ for $\alpha = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$, and it contains a polynomial subalgebra $\mathbb{C}[b_1^2, \dots, b_n^2]$.

4.2. **The algebra $\mathfrak{H}_W^-$ of type D_n.** Recall that the spin Weyl group $\mathbb{C}W^-$ associated to a Weyl group W is generated by $t_1, \dots, t_n$ subject to the relations as specified in Example 2.1.

Definition 4.1. Let $u \in \mathbb{C}$ and let $W = W_{D_n}$. The degenerate spin affine Hecke algebra of type D_n, denoted by $\mathfrak{H}_W^-$ or $\mathfrak{H}_{D_n}^-$, is the algebra generated by $\mathcal{C}[b_1, \dots, b_n]$ and $\mathbb{C}W^-$ subject to the following relations:

$$\begin{aligned} t_i b_i + b_{i+1} t_i &= u \quad (1 \le i \le n-1) \\ t_i b_j &= -b_j t_i \quad (j \ne i, i+1,\ 1 \le i \le n-1) \\ t_n b_n + b_{n-1} t_n &= u \\ t_n b_i &= -b_i t_n \quad (i \ne n-1, n). \end{aligned}$$

The algebra $\mathfrak{H}_W^-$ is naturally a superalgebra by letting each t_i and b_i be odd generators. It contains the type A degenerate spin affine Hecke algebra $\mathfrak{H}_{A_{n-1}}^-$ (generated by $b_1, \dots, b_n, t_1, \dots, t_{n-1}$) as a subalgebra.

Proposition 4.2. *The algebra $\mathfrak{H}_{D_n}^-$ admits anti-involutions τ_1, τ_2 defined by*

$$\begin{aligned} \tau_1 &: t_i \mapsto -t_i, \quad b_i \mapsto -b_i \quad (1 \le i \le n); \\ \tau_2 &: t_i \mapsto t_i, \quad b_i \mapsto b_i \quad (1 \le i \le n). \end{aligned}$$

Also, the algebra $\mathfrak{H}_{D_n}^-$ admits an involution σ which swaps t_{n-1} and t_n while fixing all the remaining generators t_i, b_i.

Proof. Note that we use the same symbols τ_1, τ_2, σ to denote the (anti-) involutions for $\mathfrak{H}_{D_n}^-$ and $\mathfrak{H}_{D_n}^{\mathfrak{c}}$ in Proposition 3.7, as those on $\mathfrak{H}_{D_n}^-$ are the restrictions from those on $\mathfrak{H}_{D_n}^{\mathfrak{c}}$ via the isomorphism in Theorem 4.4 below. The proposition is thus established via the isomorphism in Theorem 4.4, or follows by a direct computation as in the proof of Proposition 3.7. □

4.3. **The algebra $\mathfrak{H}_W^-$ of type B_n.**

Definition 4.3. Let $u, v \in \mathbb{C}$, and $W = W_{B_n}$. The degenerate spin affine Hecke algebra of type B_n, denoted by $\mathfrak{H}_W^-$ or $\mathfrak{H}_{B_n}^-$, is the algebra generated by $\mathcal{C}[b_1, \dots, b_n]$ and $\mathbb{C}W^-$ subject to the following relations:

$$\begin{aligned} t_i b_i + b_{i+1} t_i &= u \quad (1 \le i \le n-1) \\ t_i b_j &= -b_j t_i \quad (j \ne i, i+1,\ 1 \le i \le n-1) \\ t_n b_n + b_n t_n &= v \\ t_n b_i &= -b_i t_n \quad (i \ne n). \end{aligned}$$

Sometimes, we will write $\mathfrak{H}_W^-(u, v)$ or $\mathfrak{H}_{B_n}^-(u, v)$ for $\mathfrak{H}_W^-$ or $\mathfrak{H}_{B_n}^-$ to indicate the dependence on the parameters u, v.

4.4. **A superalgebra isomorphism.**

Theorem 4.4. *Let $W = W_{D_n}$ or $W = W_{B_n}$. Then,*

(1) *there exists an isomorphism of superalgebras*

$$\Phi : \mathfrak{H}^{\mathfrak{c}}_W \longrightarrow \mathcal{C}_n \otimes \mathfrak{H}^-_W$$

which extends the isomorphism $\Phi : \mathcal{C}_n \rtimes \mathbb{C}W \longrightarrow \mathcal{C}_n \otimes \mathbb{C}W^-$ (in Theorem 2.4) and sends $x_i \longmapsto \sqrt{-2}c_i b_i$ for each i;

(2) *the inverse $\Psi : \mathcal{C}_n \otimes \mathfrak{H}^-_W \longrightarrow \mathfrak{H}^{\mathfrak{c}}_W$ extends $\Psi : \mathcal{C}_n \otimes \mathbb{C}W^- \longrightarrow \mathcal{C}_n \rtimes \mathbb{C}W$ (in Theorem 2.4) and sends $b_i \longmapsto \frac{1}{\sqrt{-2}} c_i x_i$ for each i.*

Theorem 4.4 also holds for $W_{A_{n-1}}$ (see [W1]).

Proof. We only need to show that Φ preserves the defining relations in $\mathfrak{H}^{\mathfrak{c}}_W$ which involve x_i's.

Let $W = W_{D_n}$. Here, we will verify two such relations below. The verification of the remaining relations is simpler and will be skipped. For $1 \le i \le n-1$, we have

$$\begin{aligned}
\Phi(x_{i+1}s_i - s_i x_i) &= c_{i+1}b_{i+1}(c_i - c_{i+1})t_i - (c_i - c_{i+1})t_i c_i b_i \\
&= (1 - c_{i+1}c_i)b_{i+1}t_i + (1 - c_{i+1}c_i)t_i b_i \\
&= u(1 - c_{i+1}c_i), \\
\Phi(s_n x_n + x_{n-1}s_n) &= (c_{n-1} + c_n)t_n c_n b_n + c_{n-1}b_{n-1}(c_{n-1} + c_n)t_n \\
&= -(1 + c_{n-1}c_n)t_n b_n - (1 + c_{n-1}c_n)b_{n-1}t_n \\
&= -u(1 + c_{n-1}c_n).
\end{aligned}$$

Now let $W = W_{B_n}$. For $1 \le i \le n-1$, as in the proof in type D_n, we have $\Phi(x_{i+1}s_i - s_i x_i) = u(1 - c_{i+1}c_i)$. Moreover, we have

$$\begin{aligned}
\Phi(s_n x_n + x_n s_n) &= \frac{\sqrt{-2}}{\sqrt{-1}} c_n t_n c_n b_n + \frac{\sqrt{-2}}{\sqrt{-1}} c_n b_n c_n t_n \\
&= \sqrt{2} c_n t_n c_n b_n + \sqrt{2} c_n b_n c_n t_n \\
&= -\sqrt{2}(t_n b_n + b_n c_n) = -\sqrt{2} v, \\
\Phi(s_n x_j) &= \frac{\sqrt{-2}}{\sqrt{-1}} c_n t_n c_j b_j = \sqrt{2} c_n t_n c_j b_j \\
&= \sqrt{2} c_j c_n t_n b_j = \sqrt{2} c_j b_j c_n t_n = \Phi(x_j s_n), \text{ for } j \neq n.
\end{aligned}$$

Thus Φ is a homomorphism of (super)algebras. Similarly, we check that Ψ is a superalgebra homomorphism. Observe that Φ and Ψ are inverses on generators and hence they are indeed (inverse) isomorphisms. □

4.5. **PBW basis for $\mathfrak{H}^-_W$.** Note that $\mathfrak{H}^-_W$ contains the skew-polynomial algebra $\mathcal{C}[b_1,\dots,b_n]$ and the spin Weyl group algebra $\mathbb{C}W^-$ as subalgebras. We have the following PBW basis theorem for $\mathfrak{H}^-_W$.

Theorem 4.5. *Let $W = W_{D_n}$ or $W = W_{B_n}$. The multiplication of the subalgebras $\mathbb{C}W^-$ and $\mathcal{C}[b_1,\dots,b_n]$ induces a vector space isomorphism*

$$\mathcal{C}[b_1,\dots,b_n]\otimes\mathbb{C}W^- \xrightarrow{\simeq} \mathfrak{H}^-_W.$$

Theorem 4.5 also holds for $W_{A_{n-1}}$ (see [W1]).

Proof. It follows from the definition that $\mathfrak{H}^-_W$ is spanned by the elements of the form $b^\alpha\sigma$ where σ runs over a basis for $\mathbb{C}W^-$ and $\alpha\in\mathbb{Z}^n_+$. By Theorem 4.4, we have an isomorphism $\psi:\mathcal{C}_n\otimes\mathfrak{H}^-_W\longrightarrow\mathfrak{H}^{\mathfrak{c}}_W$. Observe that the image $\psi(b^\alpha\sigma)$ are linearly independent in $\mathfrak{H}^{\mathfrak{c}}_W$ by the PBW basis Theorem 3.11 for $\mathfrak{H}^{\mathfrak{c}}_W$. Hence the elements $b^\alpha\sigma$ are linearly independent in $\mathfrak{H}^-_W$. □

4.6. **The even center for $\mathfrak{H}^-_W$.**

Proposition 4.6. *Let $W = W_{D_n}$ or $W = W_{B_n}$. The even center of $\mathfrak{H}^-_W$ is isomorphic to $\mathbb{C}[b_1^2,\dots,b_n^2]^W$.*

Proof. By the isomorphism $\Phi:\mathfrak{H}^{\mathfrak{c}}_W\to\mathcal{C}_n\otimes\mathfrak{H}^-_W$ (see Theorems 4.4) and the description of the center $Z(\mathfrak{H}^{\mathfrak{c}}_W)$ (see Proposition 3.12), we have

$$Z(\mathcal{C}_n\otimes\mathfrak{H}^-_W)=\Phi(Z(\mathfrak{H}^{\mathfrak{c}}_W))=\Phi(\mathbb{C}[x_1^2,\dots,x_n^2]^W)=\mathbb{C}[b_1^2,\dots,b_n^2]^W.$$

Thus, $\mathbb{C}[b_1^2,\dots,b_n^2]^W\subseteq Z(\mathfrak{H}^-_W)$.

Now let $C\in Z(\mathfrak{H}^-_W)$. Since C is even, C commutes with $\mathcal{C}_n$ and thus commutes with the algebra $\mathcal{C}_n\otimes\mathfrak{H}^-_W$. Then $\Psi(C)\in Z(\mathfrak{H}^{\mathfrak{c}}_W)=\mathbb{C}[x_1^2,\dots,x_n^2]^W$, and thus, $C=\Phi\Psi(C)\in\Phi(\mathbb{C}[x_1^2,\dots,x_n^2]^W)=\mathbb{C}[b_1^2,\dots,b_n^2]^W$. □

In light of the isomorphism Theorem 4.4, the problem of classifying the simple modules of the spin affine Hecke algebra $\mathfrak{H}^-_W$ is equivalent to the classification problem for the affine Hecke-Clifford algebra $\mathfrak{H}^{\mathfrak{c}}_W$. It remains to be seen whether it is more convenient to find the geometric realization of $\mathfrak{H}^-_W$ instead of $\mathfrak{H}^{\mathfrak{c}}_W$.

5. Degenerate covering affine Hecke algebras

In this section, the degenerate covering affine Hecke algebras associated to the double covers $\widetilde{W}$ of classical Weyl groups W are introduced. It has as its natural quotients the usual degenerate affine Hecke algebras $\mathfrak{H}_W$ [Dr, Lu1, Lu2] and the spin degenerate affine Hecke algebras $\mathfrak{H}^-_W$ introduced by the authors.

Recall the distinguished double cover $\widetilde{W}$ of a Weyl group W from Section 2.2.

5.1. **The algebra $\mathfrak{H}_W^\sim$ of type A_{n-1}.**

Definition 5.1. Let $W = W_{A_{n-1}}$, and let $u \in \mathbb{C}$. The degenerate covering affine Hecke algebra of type A_{n-1}, denoted by $\mathfrak{H}_W^\sim$ or $\mathfrak{H}_{A_{n-1}}^\sim$, is the algebra generated by $\tilde{x}_1, \dots, \tilde{x}_n$ and $z, \tilde{t}_1, \dots, \tilde{t}_{n-1}$, subject to the relations for $\widetilde{W}$ and the additional relations:

$$z\tilde{x}_i = \tilde{x}_i z, \quad z \text{ is central of order } 2 \tag{5.1}$$

$$\tilde{x}_i\tilde{x}_j = z\tilde{x}_j\tilde{x}_i \quad (i \neq j) \tag{5.2}$$

$$\tilde{t}_i\tilde{x}_j = z\tilde{x}_j\tilde{t}_i \quad (j \neq i, i+1) \tag{5.3}$$

$$\tilde{t}_i\tilde{x}_{i+1} = z\tilde{x}_i\tilde{t}_i + u. \tag{5.4}$$

Clearly $\mathfrak{H}_W^\sim$ contains $\mathbb{C}\widetilde{W}$ as a subalgebra.

5.2. **The algebra $\mathfrak{H}_W^\sim$ of type D_n.**

Definition 5.2. Let $W = W_{D_n}$, and let $u \in \mathbb{C}$. The degenerate covering affine Hecke algebra of type D_n, denoted by $\mathfrak{H}_W^\sim$ or $\mathfrak{H}_{D_n}^\sim$, is the algebra generated by $\tilde{x}_1, \dots, \tilde{x}_n$ and $z, \tilde{t}_1, \dots, \tilde{t}_n$, subject to the relations (5.1–5.4) and the following additional relations:

$$\tilde{t}_n\tilde{x}_i = z\tilde{x}_i\tilde{t}_n \quad (i \neq n-1, n)$$
$$\tilde{t}_n\tilde{x}_n = -\tilde{x}_{n-1}\tilde{t}_n + u.$$

5.3. **The algebra $\mathfrak{H}_W^\sim$ of type B_n.**

Definition 5.3. Let $W = W_{B_n}$, and let $u, v \in \mathbb{C}$. The degenerate covering affine Hecke algebra of type B_n, denoted by $\mathfrak{H}_W^\sim$ or $\mathfrak{H}_{B_n}^\sim$, is the algebra generated by $\tilde{x}_1, \dots, \tilde{x}_n$ and $z, \tilde{t}_1, \dots, \tilde{t}_n$, subject to the relations (5.1–5.4) and the following additional relations:

$$\tilde{t}_n\tilde{x}_i = z\tilde{x}_i\tilde{t}_n \quad (i \neq n)$$
$$\tilde{t}_n\tilde{x}_n = -\tilde{x}_n\tilde{t}_n + v.$$

5.4. **PBW basis for $\mathfrak{H}_W^\sim$.**

Proposition 5.4. *Let $W = W_{A_{n-1}}, W_{D_n}$, or W_{B_n}. Then the quotient of the covering affine Hecke algebra $\mathfrak{H}_W^\sim$ by the ideal $\langle z-1\rangle$ (respectively, by the ideal $\langle z+1\rangle$) is isomorphic to the usual degenerate affine Hecke algebras $\mathfrak{H}_W$ (respectively, the spin degenerate affine Hecke algebras $\mathfrak{H}_W^-$).*

Proof. Follows by the definitions in terms of generators and relations of all the algebras involved. □

Theorem 5.5. *Let $W = W_{A_{n-1}}, W_{D_n}$, or W_{B_n}. Then the elements $\tilde{x}^\alpha\tilde{w}$, where $\alpha \in \mathbb{Z}_+^n$ and $\tilde{w} \in \widetilde{W}$, form a basis for $\mathfrak{H}_W^\sim$ (called a PBW basis).*

Proof. By the defining relations, it is easy to see that the elements $\tilde{x}^\alpha\tilde{w}$ form a spanning set for $\mathfrak{H}_W^\sim$. So it remains to show that they are linearly independent.

For each element $t \in W$, denote the two preimages in $\widetilde{W}$ of t by $\{\tilde{t}, z\tilde{t}\}$. Now suppose that

$$0 = \sum a_{\alpha,\tilde{t}} \tilde{x}^{\alpha} \tilde{t} + b_{\alpha,\tilde{t}} z \tilde{x}^{\alpha} \tilde{t}.$$

Let I^+ and I^- be the ideals of $\mathfrak{H}_{\widetilde{W}}$ generated by $z-1$ and $z+1$ respectively. Then by Proposition 5.4, $\mathfrak{H}_{\widetilde{W}}/I^+ \cong \mathfrak{H}_W$ and $\mathfrak{H}_{\widetilde{W}}/I^- \cong \mathfrak{H}^-_W$. Consider the projections:

$$\Upsilon_+ : \mathfrak{H}_{\widetilde{W}} \longrightarrow \mathfrak{H}_{\widetilde{W}}/I^+, \qquad \Upsilon_- : \mathfrak{H}_{\widetilde{W}} \longrightarrow \mathfrak{H}_{\widetilde{W}}/I^-.$$

By abuse of notation, denote the image of $\tilde{x}^{\alpha}$ in $\mathfrak{H}_W$ by x^{α}. Observe that

$$0 = \Upsilon_+ \left(\sum (a_{\alpha,\tilde{t}} \tilde{x}^{\alpha} \tilde{t} + b_{\alpha,\tilde{t}} \tilde{x}^{\alpha} z \tilde{t}) \right) = \sum (a_{\alpha,\tilde{t}} + b_{\alpha,\tilde{t}}) x^{\alpha} t \in \mathfrak{H}_W.$$

Since it is known [Lu1] that $\{x^{\alpha} t | \alpha \in \mathbb{Z}^n_+ \text{ and } t \in W\}$ form a basis for the usual degenerate affine Hecke algebra $\mathfrak{H}_W$, $a_{\alpha,\tilde{t}} = -b_{\alpha,\tilde{t}}$ for all α and t. Similarly, denoting the image in $\mathbb{C}W^-$ of $\tilde{t}$ by $\bar{t}$, we have

$$0 = \Upsilon_- \left(\sum (a_{\alpha,\tilde{t}} \tilde{x}^{\alpha} \tilde{t} + b_{\alpha,\tilde{t}} \tilde{x}^{\alpha} z \tilde{t}) \right) = \sum (a_{\alpha,\tilde{t}} - b_{\alpha,\tilde{t}}) x^{\alpha} \bar{t} \in \mathfrak{H}^-_W.$$

Since $\{x^{\alpha} \bar{t}\}$ is a basis for the spin degenerate affine Hecke algebra $\mathfrak{H}^-_W$, we have $a_{\alpha,\tilde{t}} = b_{\alpha,\tilde{t}}$ for all α and t. Hence, $a_{\alpha,\tilde{t}} = b_{\alpha,\tilde{t}} = 0$, and the linear independence is proved. □

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