The Complexity of Hamiltonian Cycle Problem in Digraps with Degree Bound Two is Polynomial Time

Guohun Zhu

Guilin University of Electronic Technology,
No.1 Jinji Road,Guilin, Guangxi, 541004,P.R.China
ccghzhu@guet.edu.cn

Abstract. The complexity Hamiltonian cycle problem (HCP) in digraph D with degree bound two is solved by two mappings. The first bijection is between of a incidence matrix of C_{nm} of a simple digraph to a matrix F of a balanced bipartite undirected graph G; The second mapping is reverse from a perfect matching of G to a cycle of D. It proves that the complexity of HCP in D is polynomial. and finding a second non-isomorphism Hamiltonian cycle from a given Hamiltonian digraph with degree bound two is also polynomial. Lastly it deduce $P = BPP = NP$ base on the results.

1 Introduction

It is well known that the Hamiltonian cycle problem(HCP) is one of the standard NP-complete problem [1]. As for digraph, even limited the digraph on these cases: planar digraphs with indegree 1 or 2 and outdegree 2 or 1 respectively, it is still on $NP-Complete$ which is proved by J.Plesník [2].

And also in [3], it is proved that the given a Hamiltonian cycle of a digraph, obtain another Hamiltonian cycle is $NP-Complete$.

Let us named a simple strong connected digraph with at most indegree 1 or 2 and outdegree 2 or 1 as Γ digraph. This paper solves the HCP of Γ graph with following main results.

Theorem 1. *Given a incidence matrix C_{nm} of Γ digraph, building a mapping:* $F = \begin{pmatrix} C^+ \\ -C^- \end{pmatrix}$, *then F is a incidence matrix of a undirected balanced bipartite graph $G(X, Y; E)$, which obeys the following properties:*

c1. $|X| = n$,$|Y| = n$,$|E| = m$
c2.

$$\forall x_i \in X \wedge 1 \leq d(x_i) \leq 2$$
$$\forall y_i \in Y \wedge 1 \leq d(y_i) \leq 2$$

c3. G has at most $\frac{n}{4}$ components which is length of 4.

Let us named the undirected balanced bipartite graph $G(X, Y : E)$ as projector graph of D.

Theorem 2. *Let G be the projector graph of a Γ graph $D(V, A)$, determining a Hamiltonian cycle in Γ digraph is equivalent to find a perfect match M in G and $r(C') = n - 1$, where C' is the incidence matrix of $D'(V, L) \subseteq D$ and $L = \{a_i | a_i \in D \wedge e_i \in M\}$.*

Let the each component of G corresponding to a boolean variable, a monotonic function $f(M)$ is build to represents the number of component in D. Based on this mapping, the complexity of Γ digraph is deduce as following.

Theorem 3. *Given the incidence matrix C_{nm} of a Γ digraph , the complexity of finding a Hamiltonian cycle existing or not is $O(n^4)$*

The concepts of cycle and rank of graph are given in section 2. Then the follows sections are the proof of above theorems. The last sections discuss the P versus NP in more detail.

2 Definition and properties

Throughout this paper we consider the finite simple (un)directed graph $D = (V, A)$ ($G(V, E)$, respectively), i.e. the graph has no multi-arcs and no self loops. Let n and m denote the number of vertices V and arcs A (edges E, respectively), respectively.

As conventional, let $|S|$ denote the number of a set S. The set of vertices V and set of arcs of A of a digraph $D(V, A)$ are denoted by $V = \{v_i | 1 \leq i \leq n\}$ and $A = \{a_j | (1 \leq j \leq m) \wedge a_j =< v_i, v_k >, (v_i \neq v_k \in V)\}$ respectively, where $< v_i, v_k >$ is a arc from v_i to v_k and a reverse arc is denoted by $\overleftarrow{a_k} =< v_k, v_i >$ if it exists . Let the out degree of vertex v_i denoted by $d^+(v_i)$, which has the in degree by denoted as $d^-(v_i)$ and has the degree $d(v_i)$ which equals $d^+(v_i) + d^-(v_i)$. Let the $N^+(v_i) = \{v_j | < v_i, v_j >\in A\}$, and $N^-(v_i) = \{v_j | < v_j, v_i >\in A\}$.

Let us define a forward relation $\bowtie$ between two arcs as following, $a_i \bowtie a_j = v_k$ iff $a_i =< v_i, v_k > \wedge a_j =< v_k, v_j >$

It is obvious that $|a_i \bowtie a_i| = 0$. A pair of symmetric arcs $< a_i, a_j >$ are two arcs of a simple digraph if and only if $|a_i \bowtie a_j| = 1 \wedge |a_j \bowtie a_i| = 1$.

A *cycle* L is a set of arcs $(a_1, a_2, \ldots, a_q)$ in a digraph D, which obeys two conditions:

c1. $\forall a_i \in L, \exists a_j, a_k \in L \setminus \{a_i\},\ a_i \bowtie a_j \neq a_j \bowtie a_k \in V$
c2. $|\bigcup_{a_i \neq a_j \in L} a_i \bowtie a_j| = |L|$

If a cycle L obeys the following conditions, it is a *simple cycle.*

c3. $\forall L' \subset L$, L' does not satisfy both conditions $c1$ and $c2$.

A *Hamiltonian cycle* L is also a simple cycle of length $n = |V| \geq 2$ in digraph. As for simplify, this paper given a sufficient condition of Hamiltonian cycle in digraph.

Lemma 1. *If a digraph $D(V, A)$ include a sub graph $D'(V, L)$ with following two properties, the D is a Hamiltonian graph.*

c1. $\forall v_i \in D' \rightarrow d^+(v_i) = 1 \wedge d^-(v_i) = 1,$
c2. $|L| = |V| \geq 2$ *and* D' *is a strong connected digraph.*

A graph that has at least one Hamiltonian cycle is called a *Hamiltonian graph*. A graph G=($V; E$) is bipartite if the vertex set V can be partitioned into two sets X and Y (the bipartition) such that $\exists e_i \in E, x_j \in X, \forall x_k \in X \setminus \{x_j\}$, $(e_i \bowtie x_j \neq \emptyset \rightarrow e_i \bowtie x_k = \emptyset)$ $(e_i, Y$, respectively). if $|X| = |Y|$, We call that G is a balanced bipartite graph. A matching $M \subseteq E$ is a collection of edges such that every vertex of V is incident to at most one edge of M, a matching of balanced bipartite graph is perfect if $|M| = |X|$. Hopcroft and Karp shows that constructs a perfect matching of bipartite in $O((m+n)\sqrt(n))$ [4]. The matching of bipartite has a relation with neighborhood of X.

Theorem 4. *[5] A bipartite graph $G = (X, Y; E)$ has a matching from X into Y if and only if $|N(S)| \geq S$, for any $S \subseteq X$.*

Lemma 2. *A even length of simple cycle consist of two disjoin perfect matching.*

Lemma 3. *If a digraph $D(V, A)$ include a sub graph $D'(V, L)$ with following two properties, the D is a Hamiltonian graph.*

c1. $\forall v_i \in D' \rightarrow d^+(v_i) = 1 \wedge d^-(v_i) = 1,$
c2. $|L| = |V| \geq 2$ *and* D' *is a strong connected digraph.*

Two matrices representation for graphs are defined as follows.

Definition 1. *[6] The incidence matrix C of undirected graph G is a two dimensional $n \times m$ table, each row represents one vertex, each column represents one edge, the c_{ij} in C are given by*

$$c_{ij} = \begin{cases} 1, \text{ if } v_i \in e_j; \\ 0, \text{ otherwise.} \end{cases} \tag{1}$$

It is obvious that every column of an incidence matrix has exactly two 1 entries.

Definition 2. *[6] The incidence matrix C of directed graph D is a two dimensional $n \times m$ table, each row represents one vertex, each column represents one arc the c_{ij} in C are given by*

$$c_{ij} = \begin{cases} 1, & \text{if } < v_i, v_i > \bowtie a_j = v_i; \\ -1, & \text{if } a_j \bowtie < v_i, v_i >= v_i; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

It is obvious to obtain a corollary of the incidence matrix as following.

Corollary 1. *Each column of an incidence matrix of digraph has exactly one* 1 *and one* -1 *entries.*

Theorem 5. *[6] The* C *is the incidence matrix of a directed graph with* k *components the rank of* C *is given by*

$$r(C) = n - k \tag{3}$$

In order to convince to describe the graph D properties, in this paper, we denotes the $r(D) = r(C)$.

3 Divided incidence matrix and Projector incidence matrix

Firstly, let us divided the matrix of C into two groups.

$$C^+ = \{c_{ij} | c_{ij} \geq 0 \text{ otherwise } 0\ \} \tag{4}$$

$$C^- = \{c_{ij} | c_{ij} \leq 0 \text{ otherwise } 0\ \} \tag{5}$$

It is obvious that the matrix of C^+ represents the forward arc of a digraph and C^- matrix represents the backward arc respectively. A corollary is deduced as following.

Corollary 2. *A digraph* $D = (V, A)$ *is strong connected if and only if the rank of divided incidence matrix satisfies* $r(C^+) = r(C^-) = |V|$.

Secondly, let us combined the the C^+ and C^- as following matrix.

$$F = \begin{pmatrix} C^+ \\ -C^- \end{pmatrix} \tag{6}$$

In more additional, let F represents as a incidence of matrix of undirected graph $G(X, Y; E)$, and the F is named as *projector incidence matrix* of C and G is named as *projector graph* where X represents the vertices of C^+, Y represents the vertices of $-C^-$ respectively. In another words we build a mapping $F : D \to G$ and denotes it as $G = F(D)$. So the $F(D)$ has $2n$ vertices and m edges if D has n vertices and m arcs. We also build up a reverse mapping: $F^{-1} : G \to D$ When G is a projector graph. To simplify, we also denotes the vertex $x_i = F^{-1}(v_i)$ in graph D v_i in G.

4 Proof of Theorem 1

Firstly, let us prove the theorem 1.

Proof. c1. Since Γ digraph is strong connected, then each vertices of Γ digraph has at least one forward arcs, each row of C^+ has at least one 1 entries, and the U represents the C^+ , so

$$|U| = n$$

the same principle of C^-, each row of C^- has at least one -1 entries, and the V represents the C^- , so

$$|V| = n$$

Since the columns of F equal to the columns of C,

$$|E| = m$$

c2. Since the degree of each v_i of Γ digraph is $1 \leq d^+(v_i) \leq 2$,

$$\forall u_i \in U \wedge 1 \leq d(u_i) \leq 2$$

Since the degree of each v_i of Γ digraph is $1 \leq d^-(v_i) \leq 2$,

$$\forall v_i \in V \wedge 1 \leq d(v_i) \leq 2$$

c3. Since the G is $2n$ vertices and $n \leq m \leq \frac{3n}{2}$ edges, suppose there are k components in G with length of 4. Since $r(F) = \frac{3n}{2} - k \geq n$, since when $k = \frac{n}{2}$, the $4k = 2n > m$ edges, thus $k \leq \frac{n}{4}$.

Secondly, let us given the properties after mapping Hamiltonian cycle L of D into the sub graph M of projector graph G.

Lemma 4. *If a Hamiltonian cycle L of D mapping into a forest M of projector graph G, the forest M consist of $|L|$ number of trees which has only two node and one edge, and M has a unique perfect matching.*

Proof. Let the Γ digraph $D(V, A)$ has a sub digraph $D'(V, L)$ which exists one Hamiltonian cycle and $|L| = n$, the incidence matrix C of L could be permutation as follows.

$$C = \begin{pmatrix} 1 & 0 & 0 & \ldots 0 & -1 \\ -1 & 1 & 0 & \ldots 0 & 0 \\ 0 & -1 & 1 & \ldots 0 & 0 \\ 0 & 0 & -1 & \ldots 0 & 0 \\ 0 & 0 & 0 & \ldots 0 & 0 \\ 0 & 0 & 0 & \ldots -1 & 1 \end{pmatrix}. \tag{7}$$

Let

$$F = \begin{pmatrix} C^+ \\ -C^- \end{pmatrix}$$

It is obvious that each row of F has only one 1 entry and each column of F has two 1 entries.

According to theorem 1, F represents a balanced bipartite graph $G(X, Y; E)$ that each vertex has one edge connected, and each edge e_i connect on vertex $x_i \in X$, another in Y, in another words, $\exists e_i \in E\, x_j \in X$,$\forall x_k \in X \setminus \{x_j\}$, $e_i \bowtie$

$x_j \neq \emptyset \to e_i \bowtie x_k = \emptyset(e_i, Y$,respectively). According the matching definition, M is a matching, since $|E| = |L|$, E is a perfect matching. and pair of vertices between X and Y only has one edge, so M is a forest, and each tree has only two node with one edge.

5 Proof of Theorem 2

Proof. $\Rightarrow$ Let the Γ digraph $D(V, A)$ has a sub digraph $D'(V, L)$ which exists one Hamiltonian cycle and $|L| = n$, let matrix C' represents the incidence matrix of D', so $r(C') = n - 1$; According to lemma 4, the projector graph $F(D')$ has a perfect matching, thus $F(D)$ also has a perfect matching.

$\Leftarrow$ Let $G(X, Y; E)$ be a projector graph of the Γ graph $D(V, A)$,M is a perfect matching in G. Let $D'(V, L)$ be a sub graph of $D(V, A)$ and $L = \{a_i | a_i \in D \wedge e_i \in M\}$. Since $r(L) = n-1$, $D'(V, L)$ is a strong connected digraph. it deduces that $\forall v_i \in D'$,$d^+(v_i) \geq 1 \wedge d^-(v_i) \geq 1$. Suppose $\exists v_i \in D'$, $d^+(v_i) > 1$ ($d^-(v_i) > 1$ respectively), Since $|M| = n$, it deduces that $\sum_{i=1}^{n} d(v_i) > 2n + 1$, which imply that $|L| > n$. this is contradiction with $L = \{a_i | a_i \in D \wedge e_i \in M\}$ and $|M| = n$. So $\forall v_i \in D'$, $d^+(v_i) = d^-(v_i) = 1$, According the lemma 3, D' has a Hamiltonian cycle.

6 Number of perfect matching in projector graph

Let us considering the number of perfect matching in G . Firstly, let us considering a example as shown in figure 1.

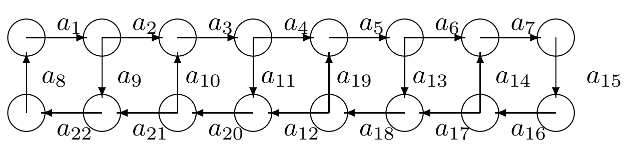


Figure 1. Original Digraph D

Then the projector graph is shown in figure 2.

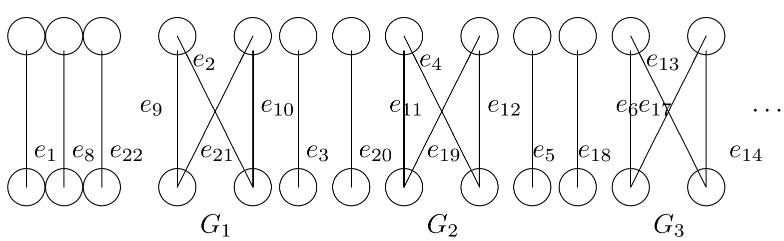


Figure 2. Projector graph G

Given a perfect matching M, each component(cycle) in G has two partition edges belong to M. Let us code component G_i which $|G_i| > 2$ and matching M to a binary variable.

$$G_i = \begin{cases} 1, \text{if } G_i \cap M = \{e_j, e_k, \ldots)\}; \\ 0, \text{if } G_i \cap M = \{e_l, e_q, \ldots)\}; \end{cases} \tag{8}$$

Now there are two cases for the number of perfect matching.

Label edge. In that cases, the $Code(M_1) = \{0, 0, 1\}$ is different with $Code(M_2) = \{0, 1, 0\}$. If there are k number of components(cycles), then there are 2^k perfect matching.

Unlabel edge. In that cases, the $Code(M_1) = \{0, 0, 1\}$ is isomorphic to $Code(M_2) = \{0, 1, 0\}$. The same principle that $Code(M_3) = \{0, 1, 1\}$ is isomorphic to $Code(M_4) = \{1, 1, 0\}$ but is not isomorphic to $Code(M_1)$.

Then let us summary the maximal number of perfect matching in these two cases.

Lemma 5. *The maximal number of labeled perfect matching in a projector graph G is $2^{\frac{n}{4}}$, but the maximal number of unlabeled perfect matching in a projector graph G is $\frac{n}{2}$.*

Proof. According to the theorem 1, there at most $\frac{n}{4}$ components with a components which is length of $k = 4$. When k=2, there are only one perfect matching in G; When $k = 4$, there are $\frac{n}{4}$ components which is C_4, and so on when $k = 6$, there are $\frac{n}{6}$ components which is C_6, etc, so on. According to the lemma 2, each simple cycle has divided the perfect matching into two class. So maximal number perfect matching in the non isomorphism cycle which is $2^{\frac{n}{4}}$. Since in unlabeled cases, every C_4 cycle is isomorphism, the maximal number of perfect matching is $2 * \frac{n}{4} = \frac{n}{2}$.

Review the example 1 again, it is easy find that follow proposition.

Proposition 1. *Given two perfect matching $M1$ and $M2$ in projector graph G, if $code(M1) = code(M2)$, then the $r(F^{-1}(M1)) = r(F^{-1}(M2))$.*

7 Proof of Theorem 3

Now let us proof the theorem 3.

Proof. Let G be a project balanced bipartition of D. According theorem 1, the Γ graph is equivalent to find a perfect match M in a project G.

According to the lemma 5, the maximal number non isomorphism perfect matching in G is only n.

Thus it is only need exactly enumerate all of non isomorphism perfect matching M, then obtain the $value = r(F^{-1}(M))$,if $value = n - 1$, then the $e_i \in M$ is also $e_i \in C$, where $C \subset D$ is a Hamiltonian cycle.

Since the complexity of rank of matrix is $O(n^3)$, finding a simple cycle in a component with degree 2 is $O(n^2)$, and obtaining a perfect matching of a bipartite graph is $O(m+n)\sqrt(n) < O(n^2)$ [4]. Then all exactly algorithms need to calculate the n time $o(n^3)$. Thus the complexity is $O(n^4)$.

Since the non isomorphism perfect matching comes from the coding of edges in the component of G, it is not easy implementation.

Let us give two recursive equation to obtain a perfect matching M from G. Suppose there are k component $G_1, G_2, \dots G_k$ in G where G_i is a component with degree 2 and $|E_i| \geq 3$.

$$M' = \begin{cases} M(t) \otimes G_t, & G_t \text{ is a cycle} \\ M(t), & \text{otherwise.} \end{cases} \tag{9}$$

$$M(t+1) = \begin{cases} M', & \text{if } r(F^{-1}(M')) > r(F^{-1}(M(t))) \\ M(t), & \text{otherwise.} \end{cases} \tag{10}$$

where $t \leq k-1$, when $t=0$, $M(0)$ is the initial perfect matching from G.

When $rank(F^{-1}(M(t))) = n-1$, According the theorem 1, the $A = F^{-1}(M(t))$ is a Hamiltonian cycle solution. If all of $rank(F^{-1}(M(t))) < n-1$, then there has no Hamiltonian cycle in D.

Since the non isomorphism perfect matching M in G is poset, the function $rank(F^{-1}(M))$ in G is monotonic, so this approach is exactly approach.

Let us give a example to illustrate the approach in detail.

Example 1. Considering the digraph D in figure 1, then the projector graph G in figure 2.

Let $M(0) = \{e_1, e_8, e_{22}, e_9, e_{10}, e_3, e_{20}, e_{11}, e_{19}, e_5, e_{18}, e_6, e_{17}, e_7, e_{15}, e_{16}\}$.

Thus the $r(F^{-1}(M(0)) = n-3$. Let $M' = r(F^{-1}(M(0)\otimes G_3)$,then $r(F^{-1}(M') = n-4$, thus $M(1) = M(0)$ and then turn to G_2,G_1. At last it obtain the solution.

Considering the equation 10, let it substituted by following equations when $r(M') = n-1$ and $t < k-1$.

$$M(t+1) = M' \text{ if } r(F^{-1}(M')) \geq r(F^{-1}(M(t))) \tag{11}$$

It is obvious that all non-isomorphism Hamiltonian cycle could obtain by the repeat check equation 10 and equation $r(M') = n-1$.

In conversely, if a Hamiltonian cycle the Γ digraph is given, it represents a perfect matching M in its projector graph G. Thus the equation 10 and Theorem 3 follows a corollary.

Corollary 3. . *Given a Hamiltonian Γ digraph , the complexity of determining another non-isomorphism Hamiltonian cycle is polynomial time.*

8 The HCP in digraph with bound two

Let us extend the Theorem 3 to digraph with $d^+(v) = 2$ and $d^-(v)$ in this section.

Theorem 6. *The complexity of finding a Hamiltonian cycle existing or not in a digraph with degree $d^+(v) \leq 2$ and $d^-(v) \leq 2$ is polynomial time.*

Proof. Suppose a digraph $D(V, A)$ having a vertex v_i is shown as figure 3, which is $d^(v_i) = 2 \wedge d^-(v_i) = 2$

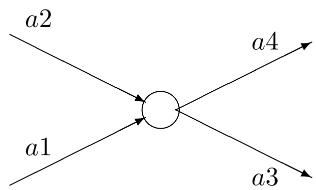
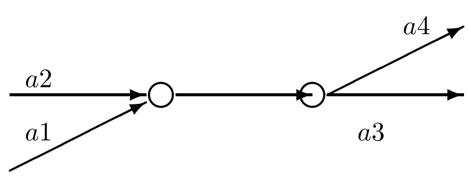


Figure 3. A vertex with degree than 2

Let us spilt this vertex to two vertices that one of vertex has degree with in degree 2 or out degree 1 , another vertex has degree with in degree 1 or out degree 2 as shown in figure 4. Then the D is derived to a new Γ graph S.



Figrue 4 A vertex in D is mapping to a vertex in Γ digraph

It is obvious that each vertex in the Γ graph S has increase 1 vertices and 1 arcs of D. Suppose the worst cases is each vertex in D has in degree 2 and out degree 2, the total vertices in S has $2n$ vertices.

According to the theorem 3, obtain a Hailtonian cycle L' in S is no more then $O(n^4)$, then the D will has a Hamiltonian cycle $L' = L \cap A$.

9 Discussion P versus NP

The P versus NP is a famous open problem in computer science and mathematics, which means to determine whether very language accepted by some nondeterministic algorithm in polynomial time is also accepted by some deterministic algorithm in polynomial time [7]. Cook give three proposition for the P versus NP.

Proposition 2. *If L is NP-complete and $L \in P$, then $P = NP$.*

Let $< B_n >$ be a family of Boolean circuits such that B_n has n inputs and consist of by circuit gates only $\{AND, OR, NOT\}$.

Proposition 3. *If L is a language over$\{0, 1\}$ which can be computed by a family $< B_n >$ of Boolean circuits of size $O(n)$, then $P \neq NP$.*

Let E be the class of languages recognizable in exponential time;

Proposition 4. *If there is $L \in E$ and $e > 0$ such that for every circuit family $< B_n >$ computing L and for all sufficiently large n, B_n has at least 2^{en} gages, then $BPP = P$, otherwise $P \neq NP$.*

According above three proposition and the results above section, P versus NP has a answer.

Theorem 7. $P = BPP = NP$

Proof. Let us proof in two steps.

proof of $P = NP$. As the result of [2], the complexity of HCP in digraph with bound two is $NP-complete$. According the theorem 6, the complexity of HCP in digraph with bound two is P, thus according to proposition 2, $P = NP$.

proof $P = BPP$. According to the lemma 5, the number of labeled perfect matching M is exponential and the worst cases if $2^{\frac{n}{4}}$, so if n is large enough, every circuit family $< B_n >$ computing $r(F^{-1}(M))$ in $O(\frac{n^3}{w})$, then B_n has at least $2^{\frac{wn}{4}}$ gages, where $w > 0$. Thus $BPP = P$

According to the equation 10, the size of matrix is $O(nm) \geq O(n^2)$, thus $P = NP$ deduce that general HCP can not computed in linear size.

In fact, the [2] proves that $3SAT \preceq_p HCP\ of\ \Gamma\ digraph$, since $3SAT$ is a NPC problem, which also implies that $P = NP$.

10 Conclusion

According to the theorem 6, the complexity of determining a Hamiltonian cycle existence or not in digraph with bound degree two is in polynomial time. And according to the theorem 7, $P = BPP = NP$.

11 Acknowledgements

My thanks to Prof. Jørgen Bang-Jensen who called mine attention to the paper [2].

References

1. Lawler E. L., Lenstra J. K., i.K.A.H.G.S.D.: Computational Complexity. John-Wiley and Sons., London (1985)
2. J.Plesník: The np-completeness of the hamiltonian cycle problem in planar digraphs with degree bound two. Journal Information Processing Letters **8** (1978) 199–201
3. Cristina Bazgan, M.S., Tuza, Z.: On the approximation of finding a(nother) hamiltonian cycle in cubic hamiltonian graphs. Journal of Algorithms **31** (1999) 249–268
4. Hopcroft, J., Karp, R.: An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. SIAM J. Comput. **2** (1973)
5. Hall, P.: On representative of subsets. Journal Landon Math. **10** (1935) 26–30
6. Pearl, M.: Matrix Theory and Finite Mathematics. McGraw-Hill, New York (1973)
7. Cook, S.A.: The P versus NP problem. (2000)