

Some aspects of the nonperturbative renormalization of the φ^4 model

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Abstract

A nonperturbative renormalization of the φ^4 model is considered. First we integrate out only a single pair of conjugated modes with wave vectors $\pm\mathbf{q}$. Then we are looking for the RG equation which would describe the transformation of the Hamiltonian under the integration over a shell $\Lambda - d\Lambda < k < \Lambda$, where $d\Lambda \rightarrow 0$. We show that the known Wegner–Houghton equation is consistent with the assumption of a simple superposition of the integration results for $\pm\mathbf{q}$. The renormalized action can be expanded in powers of the φ^4 coupling constant u in the high temperature phase at $u \rightarrow 0$. We compare the expansion coefficients with those exactly calculated by the diagrammatic perturbative method, and find some inconsistency. It causes a question in which sense the Wegner–Houghton equation is really exact.

1 Introduction

The renormalization group (RG) approach, perhaps, is the most extensively used one in numerous studies of critical phenomena [1, 2]. Particularly, the perturbative RG approach to the φ^4 or Ginzburg–Landau model is widely known [3, 4, 5, 6]. However, the perturbative approach suffers from some problems [7]. Therefore it is interesting to look for a nonperturbative approach. Historically, nonperturbative RG equations have been developed in parallel to the perturbative ones. These are so called exact RG equations (ERGE). The method of deriving such RG equations is close in spirit to the famous Wilson’s approach, where the basic idea is to integrate out the short-wave fluctuations corresponding to the wave vectors within $\Lambda/s < q < \Lambda$ with the upper (or ultraviolet) cutoff parameter Λ and the renormalization scale $s > 1$. The oldest nonperturbative equation of this kind, originally presented by Wegner and Houghton [8], uses the sharp momentum cutoff. Later, a similar equation with smooth momentum cutoff has been proposed by Polchinski [9]. The RG equations of this class are reviewed in [10].

According to the known classification [10, 11], there is another class of nonperturbative RG equations proposed by [12] and reviewed in [11]. Some relevant discussion can be found in [10], as well. Such equations describe the variation of an average effective action $\Gamma_k[\phi]$ depending on the running cutoff scale k . Here $\phi(\mathbf{x}) = \langle \varphi(\mathbf{x}) \rangle$ is the averaged order-parameter field (for simplicity, we refer to the case of scalar field). According to [12], the averaging is performed over volume $\sim k^{-d}$ such that the fluctuation degrees of freedom

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with momenta $q > k$ are effectively integrated out. In fact, the averaging over volume $\sim k^{-d}$ is the usual block-spin-averaging procedure of the real-space renormalization. At the same time, the fluctuations with $q \lesssim k$ are suppressed by a smooth infrared cutoff. As one can judge from [11], the existence of a deterministic relation between the configuration of external source $\{J(\mathbf{x})\}$ and that of the averaged order parameter $\{\phi(\mathbf{x})\}$ is (implicitly) assumed in the nonperturbative derivation of the RG flow equation. Namely, it is stated (see the text between (2.28) and (2.29) in [11]) that $\delta J(\mathbf{x})/\delta\phi(\mathbf{y})$ is the inverse of $\delta\phi(\mathbf{x})/\delta J(\mathbf{y})$, which has certain meaning as a matrix identity. To make this point clearer, let us consider a toy example $\vec{J} = A\vec{\phi}$, where $\vec{J} = (J(\mathbf{x}_1), J(\mathbf{x}_2), \dots, J(\mathbf{x}_N))$ and $\vec{\phi} = (\phi(\mathbf{x}_1), \phi(\mathbf{x}_2), \dots, \phi(\mathbf{x}_N))$ are N -component vectors and A is a matrix of size $N \times N$. In this case $\partial J(\mathbf{x}_i)/\partial\phi(\mathbf{x}_j)$ is the element A_{ij} of matrix A , whereas $\partial\phi(\mathbf{x}_i)/\partial J(\mathbf{x}_j)$ is the element $(A^{-1})_{ij}$ of the inverse matrix A^{-1} . In the continuum limit $N \rightarrow \infty$, this toy example corresponds to a linear dependence between $\{J(\mathbf{x})\}$ and $\{\phi(\mathbf{x})\}$. The calculation of derivative always implies the linearisation around some point, so that the matrix identity used in [11] (as a continuum limit in the above example) has a general meaning. However, it makes sense only if there exists a deterministic relation between the configurations of $\phi(\mathbf{x})$ and $J(\mathbf{x})$ or, in a mathematical notation, if there exist mappings $f : \{J(\mathbf{x})\} \rightarrow \{\phi(\mathbf{x})\}$ and $f^{-1} : \{\phi(\mathbf{x})\} \rightarrow \{J(\mathbf{x})\}$. On the other hand, according to the block-averaging, the values of $\phi(\mathbf{x})$ should be understood as the block-averages. These, of course, are not uniquely determined by the external sources, but are fluctuating quantities. So, we are quite sceptical about the exactness of such an approach of averaged effective action.

The integration over fluctuation degrees with momenta $q > k$ does not alter the behavior of the infrared modes, directly related to the critical exponents. From this point of view, the approach based on the equations of Wegner–Houghton and Polchinski type seems to be more natural. These are widely believed to be the exact RG equations, although, in view of our currently presented results, it turns out to be questionable in which sense they are really exact. In any case, the nonperturbative RG equations cannot be solved exactly, therefore a suitable truncation is used. The convergence of several truncation schemes and of the derivative expansion has been widely studied in [13, 14, 15, 16, 17]. Here [17] refers to the specific approach of [12]. A review about all the methods of approximate solution can be found in [18].

Another problem is to test and verify the nonperturbative RG equations, comparing the results with the known exact and rigorous solutions, as well as with the results of the perturbation theory. In [15], the derivative expansion of the RG β -function has been considered, showing the agreement up to the second order between the perturbative results and those obtained from the Legendre flow equation, which also belongs to the same class of RG equations as the Wegner–Houghton and Polchinski equations. It has been stated in [13] that the critical exponent ν , extracted from the Wegner–Houghton equation in the local potential approximation, agrees with the ε -expansion up to the $O(\varepsilon)$ order, as well as with the $1/n$ ($1/N$ in the notations of [13]) expansion in the leading order. However, looking carefully on the results of [13], one should make clear that “the leading order of the $1/n$ expansion” in this case is no more than the zeroth order, whereas the expansion coefficient at $1/n$ is inconsistent with that proposed by the perturbative RG calculation at any fixed dimension d except only $d = 4$. The inconsistency could be understood from the point of view that the Wegner–Houghton equation has been solved approximately. Therefore it would be interesting to verify whether the problem is eliminated beyond the local potential approximation. One should also take into account that the perturbative RG theory is not rigorous and, therefore, we think that a possible inconsistency still would

not prove that something is really wrong with the nonperturbative RG equation. In any case, it is a remarkable fact that correct RG eigenvalue spectrum and critical exponents are obtained in the local potential approximation at $n \rightarrow \infty$ from the Wegner–Houghton equation [13], as well as from similar RG equations [19], in agreement with the known exact and rigorous results for the spherical model. It shows that some solutions, being not exact, nevertheless can lead to exact critical exponents. From this point of view, it seems also possible that some kind of approximations, made in the derivation of an RG equation, are not harmful for the critical exponents.

We propose a simple test of the Wegner–Houghton equation: to verify the expansion of the renormalized action S of the φ^4 model in powers of the coupling constant u at $u \rightarrow 0$ in the high–temperature phase. Such a test is rigorous, in the sense that the natural domain of validity of the perturbation theory is considered. We think that it would be quite natural to start with such a relatively simple and straightforward test before passing to more complicated ones, considered in [13, 15, 19]. We have made this simplest test in our paper and have found that the Wegner–Houghton equation fails to give all correct expansion coefficients. We have also proposed another derivation of the Wegner–Houghton equation (Secs. 2, 3). It is helpful to clarify the origin of the mentioned inconsistency. It is also less obscure from the point of view that the used assumptions and approximations are clearly stated. As regards the derivation in [8], at least one essential step is obscure and apparently contains an implicit approximation which, in very essence, is analogous to that pointed out in our derivation. We will discuss this point in Sec. 3.

2 An elementary step of renormalization

To derive a nonperturbative RG equation for the φ^4 model, we should start with some elementary steps, as explained in this section.

Consider the action $S[\varphi]$ which depends on the configuration of the order parameter field $\varphi(\mathbf{x})$ depending on coordinate \mathbf{x} . By definition, it is related to the Hamiltonian H of the model via $S = H/T$, where T is the temperature measured in energy units. In general, $\varphi(\mathbf{x})$ is an n –component vector with components $\varphi_j(\mathbf{x})$ given in the Fourier representation as $\varphi_j(\mathbf{x}) = V^{-1/2} \sum_{\mathbf{k} < \Lambda} \varphi_{j,\mathbf{k}} e^{i\mathbf{k}\mathbf{x}}$, where $V = L^d$ is the volume of the system, d is the spatial dimensionality, and Λ is the upper cutoff of the wave vectors. We consider the action of the Ginzburg–Landau form. For simplicity, we include only the φ^2 and φ^4 terms. The action of such φ^4 model is given by

$$S[\varphi] = \sum_{j,\mathbf{k}} \Theta(\mathbf{k}) \varphi_{j,\mathbf{k}} \varphi_{j,-\mathbf{k}} + u V^{-1} \sum_{j,l,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \varphi_{j,\mathbf{k}_1} \varphi_{j,\mathbf{k}_2} \varphi_{l,\mathbf{k}_3} \varphi_{l,-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3} , \quad (1)$$

where $\Theta(\mathbf{k})$ is some function of wave vector \mathbf{k} , e. g., $\Theta(\mathbf{k}) = r_0 + ck^2$ like in theories of critical phenomena [4, 5, 6, 7]. In the sums we set $\varphi_{l,\mathbf{k}} = 0$ for $k > \Lambda$.

The renormalization group (RG) transformation implies the integration over $\varphi_{j,\mathbf{k}}$ for some set of wave vectors with $\Lambda' < k < \Lambda$, i. e., the Kadanoff’s transformation, followed by certain rescaling procedure [4]. The action under the Kadanoff’s transformation is changed from $S[\varphi]$ to $S_{\text{tra}}[\varphi]$ according to the equation

$$e^{-S_{\text{tra}}[\varphi]} = \int e^{-S[\varphi]} \prod_{j,\Lambda' < k < \Lambda} d\varphi_{j,\mathbf{k}} . \quad (2)$$

Alternatively, one often writes $-S_{\text{tra}}[\varphi] + AL^d$ instead of $-S_{\text{tra}}[\varphi]$ to separate the constant part of the action AL^d . This, however, is merely a redefinition of S_{tra} , and for our purposes

it is suitable to use (2). Note that $\varphi_{j,\mathbf{k}} = \varphi'_{j,\mathbf{k}} + i\varphi''_{j,\mathbf{k}}$ is a complex number and $\varphi_{j,-\mathbf{k}} = \varphi_{j,\mathbf{k}}^*$ holds (since $\varphi_j(\mathbf{x})$ is always real), so that the integration over $\varphi_{j,\mathbf{k}}$ means in fact the integration over real and imaginary parts of $\varphi_{j,\mathbf{k}}$ for each pair of conjugated wave vectors \mathbf{k} and $-\mathbf{k}$.

The Kadanoff's transformation (2) can be split in a sequence of elementary steps $S[\varphi] \rightarrow S_{\text{tra}}[\varphi]$ of the repeated integration given by

$$e^{-S_{\text{tra}}[\varphi]} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-S[\varphi]} d\varphi'_{j,\mathbf{q}} d\varphi''_{j,\mathbf{q}} \quad (3)$$

for each j and $\mathbf{q} \in \Omega$, where Ω is the subset of independent wave vectors ($\pm\mathbf{q}$ represent one independent mode) within $\Lambda' < q < \Lambda$. Thus, in the first elementary step of renormalization we have to insert the original action (1) into (3) and perform the integration for one chosen j and $\mathbf{q} \in \Omega$. In an exact treatment we must take into account that the action is already changed in the following elementary steps.

For $\Lambda' > \Lambda/3$, we can use the following exact decomposition of (1)

$$S[\varphi] = A_0 + A_1\varphi_{j,\mathbf{q}} + A_1^*\varphi_{j,-\mathbf{q}} + A_2\varphi_{j,\mathbf{q}}\varphi_{j,-\mathbf{q}} + B_2\varphi_{j,\mathbf{q}}^2 + B_2^*\varphi_{j,-\mathbf{q}}^2 + A_4\varphi_{j,\mathbf{q}}^2\varphi_{j,-\mathbf{q}}^2, \quad (4)$$

where

$$A_0 = S|_{\varphi_{j,\pm\mathbf{q}}=0}, \quad (5)$$

$$A_1 = \left. \frac{\partial S}{\partial \varphi_{j,\mathbf{q}}} \right|_{\varphi_{j,\pm\mathbf{q}}=0} = 4uV^{-1} \sum'_{l,\mathbf{k}_1,\mathbf{k}_2} \varphi_{j,\mathbf{k}_1} \varphi_{l,\mathbf{k}_2} \varphi_{l,-\mathbf{q}-\mathbf{k}_1-\mathbf{k}_2}, \quad (6)$$

$$A_2 = \left. \frac{\partial^2 S}{\partial \varphi_{j,\mathbf{q}} \partial \varphi_{j,-\mathbf{q}}} \right|_{\varphi_{j,\pm\mathbf{q}}=0} = \Theta(\mathbf{q}) + \Theta(-\mathbf{q}) + 4uV^{-1} \sum'_{l,\mathbf{k}} (1 + 2\delta_{lj}) |\varphi_{l,\mathbf{k}}|^2, \quad (7)$$

$$B_2 = \left. \frac{1}{2} \frac{\partial^2 S}{\partial \varphi_{j,\mathbf{q}}^2} \right|_{\varphi_{j,\pm\mathbf{q}}=0} = 2uV^{-1} \sum'_{l,\mathbf{k}} (1 + 2\delta_{lj}) \varphi_{l,\mathbf{k}} \varphi_{l,-2\mathbf{q}-\mathbf{k}}, \quad (8)$$

$$A_4 = \left. \frac{1}{4} \frac{\partial^4 S}{\partial^2 \varphi_{j,\mathbf{q}} \partial^2 \varphi_{j,-\mathbf{q}}} \right|_{\varphi_{j,\pm\mathbf{q}}=0} = 6uV^{-1}. \quad (9)$$

Here the sums are marked by a prime to indicate that terms containing $\varphi_{j,\pm\mathbf{q}}$ are omitted. This is simply a splitting of (1) into parts with all possible powers of $\varphi_{j,\pm\mathbf{q}}$. The condition $\Lambda' > \Lambda/3$, as well as the existence of the upper cutoff for the wave vectors, ensures that terms of the third power are absent in (4). Besides, the derivation is performed formally considering all $\varphi_{l,\mathbf{k}}$ as independent variables.

Taking into account (4), as well as the fact that $A_1 = A'_1 + iA''_1$ and $B_2 = B'_2 + iB''_2$ are complex numbers, the transformed action after the first elementary renormalization step reads

$$S_{\text{tra}}[\varphi] = A_0 - \ln \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[-2(A'_1 \varphi'_{j,\mathbf{q}} - A''_1 \varphi''_{j,\mathbf{q}}) \right] \exp \left[-A_2 (\varphi'_{j,\mathbf{q}}^2 + \varphi''_{j,\mathbf{q}}^2) \right] \right. \\ \times \exp \left[-2B'_2 (\varphi'_{j,\mathbf{q}}^2 - \varphi''_{j,\mathbf{q}}^2) + 4B''_2 \varphi'_{j,\mathbf{q}} \varphi''_{j,\mathbf{q}} \right] \\ \left. \times \exp \left[-A_4 (\varphi'_{j,\mathbf{q}}^2 + \varphi''_{j,\mathbf{q}}^2)^2 \right] d\varphi'_{j,\mathbf{q}} d\varphi''_{j,\mathbf{q}} \right\}. \quad (10)$$

Considering only the field configurations which are relevant in the thermodynamic limit $V \rightarrow \infty$, Eq. (10) can be simplified, omitting the terms with B_2 and A_4 . Really, using the coordinate representation $\varphi_{l,\mathbf{k}} = V^{-1/2} \int \varphi_l(\mathbf{x}) e^{-i\mathbf{k}\mathbf{x}} d\mathbf{x}$, we can write

$$B_2 = 2uV^{-1} \left\{ \left[\sum_{l,j} (1 + 2\delta_{lj}) \int \varphi_l^2(\mathbf{x}) e^{i\mathbf{q}\mathbf{x}} d\mathbf{x} \right] - 3\varphi_{j,-\mathbf{q}}^2 \right\}. \quad (11)$$

The quantity $V^{-1} \int \varphi_l^2(\mathbf{x}) e^{i\mathbf{q}\mathbf{x}} d\mathbf{x}$ is an average of $\varphi_l^2(\mathbf{x})$ over the volume with oscillating weight factor $e^{i\mathbf{q}\mathbf{x}}$. This quantity vanishes for relevant configurations in the thermodynamic limit: due to the oscillations, positive and negative contributions are similar in magnitude and cancel at $V \rightarrow \infty$. Since $\langle |\varphi_{j,-\mathbf{q}}|^2 \rangle = \langle |\varphi_{j,\mathbf{q}}|^2 \rangle$ is the Fourier transform of the two-point correlation function, it is bounded at $V \rightarrow \infty$ and, hence, $\varphi_{j,-\mathbf{q}}^2$ also is bounded for relevant configurations giving nonvanishing contribution to the statistical averages $\langle \cdot \rangle$ in the thermodynamic limit. Consequently, for these configurations, A_2 is a quantity of order $\mathcal{O}(1)$, whereas $V^{-1}\varphi_{j,-\mathbf{q}}^2$ and B_2 vanish at $V \rightarrow \infty$. Note, however, that the term with $A_4 = \mathcal{O}(V^{-1})$ in (10) cannot be neglected unless A_2 is positive. One can judge that the latter condition is satisfied for the relevant field configurations due to existence of the thermodynamic limit for the RG flow.

Omitting the terms with B_2 and A_4 , the integrals in (10) can be easily calculated. It yields

$$S_{\text{tra}}[\varphi] = S'[\varphi] + \Delta S_{\text{tra}}^{\text{el}}[\varphi], \quad (12)$$

where $S'[\varphi] = A_0$ is the original action, where only the $\pm\mathbf{q}$ modes of the j -th field component are omitted, whereas $\Delta S_{\text{tra}}^{\text{el}}[\varphi]$ represents the elementary variation of the action given by

$$\Delta S_{\text{tra}}^{\text{el}}[\varphi] = \ln \left(\frac{A_2}{\pi} \right) - \frac{|A_1|^2}{A_2}. \quad (13)$$

According to the arguments provided above, this equation is exact for the relevant field configurations with $A_2 > 0$ in the thermodynamic limit.

The contributions to (6) and (7) provided by modes with wave vectors \mathbf{k} , obeying two relations $\Lambda - d\Lambda < k < \Lambda$ and $\mathbf{k} \neq \pm\mathbf{q}$, are irrelevant in the thermodynamic limit at $d\Lambda \rightarrow 0$. It can be verified by the method of analysis introduced in Sec. 2. Hence, Eq. (13) can be written as

$$\Delta S_{\text{tra}}^{\text{el}}[\varphi] = \ln \left(\frac{\tilde{A}_2}{\pi} \right) - \frac{|\tilde{A}_1|^2}{\tilde{A}_2} + \delta S_{\text{tra}}^{\text{el}}[\varphi], \quad (14)$$

where

$$\tilde{A}_1 = P \frac{\partial S}{\partial \varphi_{j,\mathbf{q}}} = 4uV^{-1} \sum_{l,\mathbf{k}_1,\mathbf{k}_2}^{\Lambda-d\Lambda} \varphi_{j,\mathbf{k}_1} \varphi_{l,\mathbf{k}_2} \varphi_{l,-\mathbf{q}-\mathbf{k}_1-\mathbf{k}_2}, \quad (15)$$

$$\tilde{A}_2 = P \frac{\partial^2 S}{\partial \varphi_{j,\mathbf{q}} \partial \varphi_{j,-\mathbf{q}}} = \Theta(\mathbf{q}) + \Theta(-\mathbf{q}) + 4uV^{-1} \sum_{l,\mathbf{k}}^{\Lambda-d\Lambda} (1 + 2\delta_{lj}) |\varphi_{l,\mathbf{k}}|^2, \quad (16)$$

and $\delta S_{\text{tra}}^{\text{el}}[\varphi]$ is a vanishingly small correction in the considered limit. Here the operators P set to zero all $\varphi_{j,\mathbf{k}}$ within the shell $\Lambda - d\Lambda < k < \Lambda$ (i. e., the derivatives are evaluated at zero $\varphi_{j,\mathbf{k}}$ for \mathbf{k} within the shell), and the upper border $\Lambda - d\Lambda$ for sums implies that we set $\varphi_{l,\mathbf{k}} = 0$ for $k > \Lambda - d\Lambda$. The above replacements are meaningful, since they allow to obtain easily the Wegner-Houghton equation, as discussed in the following section.

3 Superposition hypothesis and the Wegner–Houghton equation

Intuitively, it could seem very reasonable that the result of integration over Fourier modes within the shell $\Lambda - d\Lambda < k < \Lambda$ at $d\Lambda \rightarrow 0$ can be represented as a superposition of elementary contributions given by (14), neglecting the irrelevant corrections $\delta S_{\text{tra}}^{\text{el}}[\varphi]$. We will call this idea the superposition hypothesis.

We remind, however, that strictly exact treatment requires a sequential integration of $\exp(-S[\varphi])$ over a set of $\varphi_{j,\mathbf{q}}$. The renormalized action changes after each such integration, and these changes influence the following steps. A problem is to estimate the discrepancy between the results of two methods: (1) the exact integration and (2) the superposition approximation. Since it is necessary to perform infinitely many integration steps in the thermodynamic limit, the problem is nontrivial and the superposition hypothesis cannot be rigorously justified.

Nevertheless, the summation of elementary contributions in accordance with the superposition hypothesis leads to the known Wegner–Houghton equation [8]. In this case the variation of the action due to the integration over shell reads

$$\Delta S_{\text{tra}}[\varphi] = \frac{1}{2} \sum_j \sum_{\Lambda - d\Lambda < q < \Lambda} \left[\ln \left(\frac{\tilde{A}_2(j, \mathbf{q})}{\pi} \right) - \frac{|\tilde{A}_1(j, \mathbf{q})|^2}{\tilde{A}_2(j, \mathbf{q})} \right]. \quad (17)$$

It is exactly consistent with Eq. (2.13) in [8]. The factor 1/2 appears, since only half of the wave vectors represent independent modes. Here we have indicated that the quantities \tilde{A}_1 and \tilde{A}_2 depend on the current j and \mathbf{q} . They depend also on the considered field configuration $[\varphi]$. If \tilde{A}_1 and \tilde{A}_2 are represented by the derivatives of $S[\varphi]$ (see (15) and (16)), then the equation is written exactly as in [8].

To avoid possible confusion, one has to make clear that the operators P in (15) and (16) influence the result, as discussed further on. It means that the equation where these operators are simply omitted, referred in the review paper [10] as the Wegner–Houghton equation, is not really the Wegner–Houghton equation.

The derivation in [8] is somewhat different. Instead of performing only one elementary step of integration first, the expansion of Hamiltonian in terms of all shell variables is made there. The basic method of [8] is to show that, in the thermodynamic limit at $d\Lambda \rightarrow 0$, the expansion consists of terms containing no more than two derivatives with respect to the field components. Moreover, it is assumed implicitly that only the diagonal terms with $\mathbf{k}' = -\mathbf{k}$ are important finally, when performing the summation over the wave vectors \mathbf{k}, \mathbf{k}' . It leads to Eq. (2.12) in [8]. The omitting of nondiagonal terms is equivalent to the superposition assumption we discussed already. Indeed, in this and only in this case the integration over the shell variables can be performed independently, as if the superposition principle were hold. Hence, essentially the same approximation is used in [8] as in our derivation, although it is not stated explicitly.

Our derivation refers to the φ^4 model, whereas in the form with derivatives the equation may have a more general validity, as supposed in [8]. Indeed, (14) remains correct for a generalized model provided that higher than second order derivatives of $S[\varphi]$ vanish for relevant field configurations in the thermodynamic limit. It, in fact, has been assumed and shown in [8]. Based on similar arguments we have used already, the latter assumption can be justified for certain class of models, for which the action is represented by a linear combination of φ^m -kind terms with wave-vector dependent weights and vanishing sum of

the wave vectors $\sum_{l=1}^m \mathbf{k}_l = \mathbf{0}$ related to the φ factors. In this case we have

$$\tilde{A}_1(j, \mathbf{q}) = \frac{\partial S}{\partial \varphi_{j, \mathbf{q}}} - \varphi_{j, -\mathbf{q}} \frac{\partial^2 S}{\partial \varphi_{j, \mathbf{q}} \partial \varphi_{j, -\mathbf{q}}}, \quad (18)$$

$$\tilde{A}_2(j, \mathbf{q}) = \frac{\partial^2 S}{\partial \varphi_{j, \mathbf{q}} \partial \varphi_{j, -\mathbf{q}}} \quad (19)$$

for the relevant configurations at $V \rightarrow \infty$ and $d\Lambda \rightarrow 0$. The second term in (18) appears because the derivative $\partial S / \partial \varphi_{j, \mathbf{q}}$ contains relevant terms with $\varphi_{j, -\mathbf{q}}$, which have to be removed. The influence of the operators P is seen from (15) and (18).

Here we do not include the second, i. e., the rescaling step of the RG transformation. It, however, can be easily calculated for any given action, as described, e. g., in [4]. It is not relevant for our further considerations.

4 The weak coupling limit

Here we consider the weak coupling limit $u \rightarrow 0$ of the model with $\Theta(\mathbf{k}) = r_0 + ck^2$ at a given positive r_0 , i. e., in the high temperature phase. In this case $\Delta S_{\text{tra}}[\varphi]$ can be expanded in powers of u . It is the natural domain of validity of the perturbation theory, and the expansion coefficients can be calculated exactly by the known methods applying the Feynman diagram technique and the Wick's theorem [4, 5, 11]. On the other hand, the expansion can be performed in (17). Our aim is to compare the results of both methods to check the correctness of (17), since the latter equation is based on assumptions.

Let us denote by $\Delta \tilde{S}_{\text{tra}}[\varphi]$ the variation of $S[\varphi]$ omitting the constant (independent of the field configuration) part. Then the expansion in powers of u reads

$$\Delta \tilde{S}_{\text{tra}}[\varphi] = \Delta S_1[\varphi] u + \left(\Delta S_2^{(a)}[\varphi] + \Delta S_2^{(b)}[\varphi] + \Delta S_2^{(c)}[\varphi] \right) u^2 + \mathcal{O}(u^3), \quad (20)$$




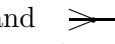
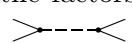
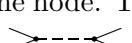
where the expansion coefficient at u^2 is split in three parts $\Delta S_2^{(a)}[\varphi]$, $\Delta S_2^{(b)}[\varphi]$, and $\Delta S_2^{(c)}[\varphi]$ corresponding to the φ^2 , φ^4 , and φ^6 contributions, respectively. The contribution of order u is related to the diagram , whereas the three second-order contributions — to the diagrams , , and . The diagram technique represents the expansion of $-S[\varphi]$ in terms of connected Feynman diagrams, where the coupled lines are associated with the Gaussian averages. In particular, the Fourier transformed two-point correlation function in the Gaussian approximation $G_0(\mathbf{k}) = \langle \varphi_{j, \mathbf{k}} \varphi_{j, -\mathbf{k}} \rangle_0 = 1/[2\Theta(\mathbf{k})]$ appears due to the integration over $\varphi'_{j, \mathbf{k}}$ and $\varphi''_{j, \mathbf{k}}$. It is represented as the coupling of lines, in such a way that each line related to the wave vector \mathbf{k} and vector-component j is coupled with another line having the wave vector $-\mathbf{k}$ and the same component j . Thus, if we integrate over $\varphi_{j, \mathbf{k}}$ within $\Lambda - d\Lambda < k < \Lambda$ in (2), then it corresponds to the coupling of lines in the same range of wave vectors in the diagram technique. According to the Wick's theorem, one has to sum over all possible couplings, which finally yields the summation (integration) over the wave vectors obeying the constraint $\Lambda - d\Lambda < k < \Lambda$ for each of the coupled lines associated with the factors $G_0(\mathbf{k})$. In the n -component case, it is suitable to represent the φ^4 vertex as , where the same index j is associated with two solid lines connected to one node. The above diagrams are given by the sum of all possible couplings of the vertices , yielding the corresponding topological pictures when the dashed lines shrink to points. In this case factor n corresponds to each closed loop of solid lines, which comes from the summation over j . For a complete definition of the

diagram technique, one has to mention that factors $-uV^{-1}$ are related to the dashed lines, $G_0(\mathbf{k})$ – to the coupled solid lines, and the fields $\varphi_{j,\mathbf{k}}$ – to the outer uncoupled solid lines. Besides, each diagram contains a combinatorial factor. For a diagram consisting of m vertices $\text{---}\text{---}\text{---}$, it is the number of all possible couplings of (numbered) lines, divided by $m!$.

At $d\Lambda \rightarrow 0$, the diagrammatic calculation for the n –component case yields


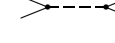
$$\Delta S_1[\varphi] = \frac{K_d \Lambda^{d-1}}{\Theta(\Lambda)} (n+2) d\Lambda \sum_{j,\mathbf{k}}^{\Lambda-d\Lambda} |\varphi_{j,\mathbf{k}}|^2 \quad (21)$$

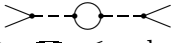
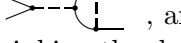
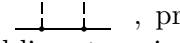
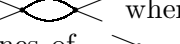
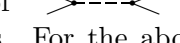
$$\Delta S_2^{(b)}[\varphi] = -4V^{-1} \sum_{j,l,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}^{\Lambda-d\Lambda} \varphi_{j,\mathbf{k}_1} \varphi_{j,\mathbf{k}_2} \varphi_{l,\mathbf{k}_3} \varphi_{l,-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3} \times [(n+4)Q(\mathbf{k}_1+\mathbf{k}_2, \Lambda, d\Lambda) + 4Q(\mathbf{k}_1+\mathbf{k}_3, \Lambda, d\Lambda)] \quad (22)$$

$$\Delta S_2^{(c)}[\varphi] = -8V^{-2} \sum_{i,j,l,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_5}^{\Lambda-d\Lambda} \varphi_{i,\mathbf{k}_1} \varphi_{i,\mathbf{k}_2} \varphi_{j,\mathbf{k}_3} \varphi_{j,\mathbf{k}_4} \varphi_{l,\mathbf{k}_5} \varphi_{l,-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4-\mathbf{k}_5} \times G_0(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) \mathcal{F}(|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|, \Lambda, d\Lambda), \quad (23)$$

where $K_d = S(d)/(2\pi)^d$, $S(d) = 2\pi^{d/2}/\Gamma(d/2)$ is the area of unit sphere in d dimensions, $\Theta(\Lambda)$ is the value of $\Theta(\mathbf{k})$ at $k = \Lambda$, whereas $\mathcal{F}(k, \Lambda, d\Lambda)$ is a cutoff function which has the value 1 within $\Lambda - d\Lambda < k < \Lambda$ and zero otherwise. The quantity Q is given by

$$Q(\mathbf{k}, \Lambda, d\Lambda) = V^{-1} \sum_{\Lambda-d\Lambda < q < \Lambda} G_0(\mathbf{q}) G_0(\mathbf{k}-\mathbf{q}) \mathcal{F}(|\mathbf{k}-\mathbf{q}|, \Lambda, d\Lambda). \quad (24)$$

Below we will give some details of calculation of (22), which is the most important term in our further discussion. To obtain this result, we have deciphered the  diagram as a sum of three diagrams of different topologies made of vertices , i. e.,

, , and , providing the same topological picture  when shrinking the dashed lines to points. Recall that any loop made of solid lines of  gives a factor n , and one needs also to compute the combinatorial factors. For the above three diagrams, the resulting factors are $4n$, 16 , and 16 , which enter the prefactors of Q in (22). To obtain the correct sign, we recall that the diagram expansion is for $-S[\varphi]$. The other diagrams are calculated in a similar way.

The expansion of (17) gives no contribution $\Delta S_2^{(a)}[\varphi]$, and we have skipped it in the diagrammatic calculation as an irrelevant term, which vanishes faster than $\propto d\Lambda$ at $d\Lambda \rightarrow 0$ in the thermodynamic limit $V \rightarrow \infty$. The expansion of the logarithm term in (17) yields $\Delta S_1[\varphi]$ exactly consistent with (21). Similarly, $\Delta S_2^{(c)}[\varphi]$ is exactly consistent with (23).

One has to remark that two propagators are involved in (24) and, therefore, the volume of summation region with nonvanishing cut function \mathcal{F} shrinks as $(d\Lambda)^2$ for a given nonzero wave vector \mathbf{k} at $d\Lambda \rightarrow 0$. However, there is a contribution linear in $d\Lambda$ for $\mathbf{k} = \mathbf{0}$. As a result, a contribution proportional to $d\Lambda$ appears in (22).

Note that the contributions (21) and (23) come from diagrams with only one coupled line. The term (22) is related to the diagram with two coupled lines. The expansion of (17) provides a different result for the corresponding part of $\Delta \tilde{S}_{\text{tra}}[\varphi]$:

$$\Delta S_2^{(b)}[\varphi] = -\frac{K_d \Lambda^{d-1} d\Lambda}{\Theta^2(\Lambda)} V^{-1} \sum_{j,l,\mathbf{k}_1,\mathbf{k}_2}^{\Lambda-d\Lambda} (n+4+4\delta_{jl}) |\varphi_{j,\mathbf{k}_1}|^2 |\varphi_{l,\mathbf{k}_2}|^2. \quad (25)$$

Note that (25) comes from the $\ln \tilde{A}_2$ term in (17), and the calculation is particularly simple in this case, since the related sum in (16) is independent of \mathbf{q} . Eq. (25) is obtained if we set $Q(\mathbf{k}, \Lambda, d\Lambda) \rightarrow \delta_{\mathbf{k}, \mathbf{0}} Q(\mathbf{0}, \Lambda, d\Lambda)$ in (22) (in this case only the diagonal terms $j = l$ are relevant when summing up the contributions with $Q(\mathbf{k}_1 + \mathbf{k}_3, \Lambda, d\Lambda)$, as it can be shown by an analysis of relevant real-space configurations, since $\langle \varphi_j(\mathbf{x}) \varphi_l(\mathbf{x}) \rangle = 0$ holds for $j \neq l$). It means that a subset of terms is missing in (25), as compared to (22). The following analysis will show that this discrepancy between (22) and (25) is important.

It is interesting to mention that (25) is obtained also by the diagrammatic perturbation method if we first integrate out only the mode with $\varphi_{j, \pm \mathbf{q}}$ and then formally apply the superposition hypothesis, as in the derivation of the Wegner–Houghton equation. It shows that the discrepancy between (25) and (22) arises because in one case the superposition hypothesis is applied, whereas in the other case it is not used.

The difference between (22) and (25) can be better seen in the coordinate representation. In this case (22) reads

$$\begin{aligned} \Delta S_2^{(b)}[\varphi] = & - (4n + 16) \int \int \varphi^2(\mathbf{x}_1) R^2(\mathbf{x}_1 - \mathbf{x}_2) \varphi^2(\mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2 \\ & - 16 \sum_{j,l} \int \int \varphi_j(\mathbf{x}_1) \varphi_l(\mathbf{x}_1) R^2(\mathbf{x}_1 - \mathbf{x}_2) \varphi_j(\mathbf{x}_2) \varphi_l(\mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2, \end{aligned} \quad (26)$$

where

$$R(\mathbf{x}) = V^{-1} \sum_{\mathbf{q}} G_0(\mathbf{q}) \mathcal{F}(q, \Lambda, d\Lambda) e^{i\mathbf{q}\mathbf{x}} \quad (27)$$

is the Fourier transform of $G_0 \mathcal{F}$, and $\varphi^2(\mathbf{x}) = \sum_l \varphi_l^2(\mathbf{x})$. In three dimensions we have

$$R(\mathbf{x}) = \frac{\Lambda d\Lambda}{(2\pi)^2 \Theta(\Lambda)} \frac{1}{x} \sin(\Lambda x) \quad (28)$$

for any given \mathbf{x} at $d\Lambda \rightarrow 0$ and $L \rightarrow \infty$, where L is the linear system size. The continuum approximation (28), however, is not correct for $x \sim L$ and therefore, probably, should not be used for the evaluation of (26).

The coordinate representation of (25) is

$$\begin{aligned} \Delta S_2^{(b)}[\varphi] = & - \frac{K_d \Lambda^{d-1} d\Lambda}{\Theta^2(\Lambda)} \left[(n+4) \int \int \varphi^2(\mathbf{x}_1) V^{-1} \varphi^2(\mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2 \right. \\ & \left. + 4 \sum_j \int \int \varphi_j^2(\mathbf{x}_1) V^{-1} \varphi_j^2(\mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2 \right]. \end{aligned} \quad (29)$$

Eq. (29) represents a relevant contribution at $d\Lambda \rightarrow 0$, as it is proportional to $d\Lambda$. It is obviously not consistent with (26). In fact, the term (29) represents a mean-field interaction, which is proportional to $1/V$ and independent of the distance, whereas (26) corresponds to another non-local interaction given by $R^2(\mathbf{x}_1 - \mathbf{x}_2)$. Hence, the Wegner–Houghton equation (17) does not yield all correct expansion coefficients at $u \rightarrow 0$.

5 Discussion

The results of our test, stated at the end of Sec. 4, reveal some inconsistency between the Wegner–Houghton equation and the diagrammatic perturbation theory in the high

temperature phase at $u \rightarrow 0$. Since this is the natural domain of validity of the perturbation theory, there should be no doubts that it produces correct results here, which agree with (2). So, the results of our test point to some inconsistency between the Wegner–Houghton equation and (2), which causes a question in which sense the Wegner–Houghton equation is really exact. The same can be asked about the equations of Polchinski type, since these (as it is believed) are generalizations of the Wegner–Houghton equation to the case of smooth momentum cutoff. There is no contradiction with the tests of consistency made in [13, 15], since our test is independent and quite different. According to our derivation of the Wegner–Houghton equation and the related discussion, it turns out that the reason of the inconsistency, likely, is the superposition approximation (defined at the beginning of Sec. 3) used in our paper and analogous approximation implicitly used in [8]. Despite of this problem, the Wegner–Houghton equation is able to reproduce the exact RG eigenvalue spectrum and critical exponents of the spherical model at $n \rightarrow \infty$ [13]. This fact can be interpreted in such a way that the superposition approximation (or an analogous approximation) is valid to derive such nonperturbative RG equations, which can produce correct (exact) critical exponents in some limit cases, at least. From a general point of view, it concerns the fundamental question about the relation between the form of RG equation and the universal quantities. It has been verified in several known studies that the universal quantities are invariant with respect to some kind of variations in the RG equation, like changes in the shape of the momentum cutoff function. This property is known as the reparametrisation invariance [10]. Probably, the universal quantities are invariant also with respect to such a variation of the Wegner–Houghton equation, which makes it exactly consistent with (2). However, this is only a hypothesis.

6 Conclusions

1. The nonperturbative Wegner–Houghton RG equation has been rederived (Secs. 2 and 3), discussing explicitly some assumptions which are used here. In particular, our derivation assumes the superposition of small contributions provided by elementary integration steps over the short-wave fluctuation modes. We consider it as an approximation. As discussed in Sec. 3, the original derivation by Wegner and Houghton includes essentially the same approximation, although not stated explicitly.
2. According to our calculation in Sec. 4, the Wegner–Houghton equation is not completely consistent with the diagrammatic perturbation theory in the limit of small φ^4 coupling constant u in the high temperature phase. This fact, together with some other important results known from literature, is discussed in Sec. 5. Apart from critical remarks, a hypothesis has been proposed that the equations of Wegner–Houghton type, perhaps, can give exact universal quantities.

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