GROUP-THEORETICAL PROPERTIES OF NILPOTENT MODULAR CATEGORIES

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To Yuri Ivanovich Manin on his 70th birthday

Abstract. We characterize a natural class of modular categories of prime power Frobenius-Perron dimension as representation categories of twisted doubles of finite p-groups. We also show that a nilpotent braided fusion category $\mathcal{C}$ admits an analogue of the Sylow decomposition. If the simple objects of $\mathcal{C}$ have integral Frobenius-Perron dimensions then $\mathcal{C}$ is group-theoretical in the sense of [ENO]. As a consequence, we obtain that semisimple quasi-Hopf algebras of prime power dimension are group-theoretical. Our arguments are based on a reconstruction of twisted group doubles from Lagrangian subcategories of modular categories (this is reminiscent to the characterization of doubles of quasi-Lie bialgebras in terms of Manin pairs given in [Dr]).

1. introduction

In this paper we work over an algebraically closed field k of characteristic 0.

By a *fusion category* we mean a k-linear semisimple rigid tensor category $\mathcal{C}$ with finitely many isomorphism classes of simple objects, finite dimensional spaces of morphisms, and such that the unit object $\mathbf{1}$ of $\mathcal{C}$ is simple. We refer the reader to [ENO] for a general theory of such categories. A fusion category is *pointed* if all its simple objects are invertible. A pointed fusion category is equivalent to Vec_G^{ω}, i.e., the category of G-graded vector spaces with the associativity constraint given by some cocycle $\omega \in Z^3(G, k^\times)$ (here G is a finite group).

1.1. **Main results.**

Theorem 1.1. *Any braided nilpotent fusion category has a unique decomposition into a tensor product of braided fusion categories whose Frobenius-Perron dimensions are powers of distinct primes.*

The notion of *nilpotent* fusion category was introduced in [GN]; we recall it in Subsection 2.2. Let us mention that the representation category $\mathrm{Rep}(G)$ of a finite group G is nilpotent if and only if G is nilpotent. It is also known that fusion categories of prime power Frobenius-Perron dimension are nilpotent [ENO]. On the other hand, Vec_G^{ω} is nilpotent for *any* G and ω. Therefore it is *not* true that any nilpotent fusion category is a tensor product of fusion categories of prime power dimensions.

Theorem 1.2. *A modular category $\mathcal{C}$ with integral dimensions of simple objects is nilpotent if and only if there exists a pointed modular category $\mathcal{M}$ such that $\mathcal{C} \boxtimes \mathcal{M}$*

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is equivalent, as a braided tensor category, to the center of a fusion category of the form Vec_G^{ω} for a finite nilpotent group G.

We emphasize here that in general the equivalence in Theorem 1.2 does not respect the spherical structures (equivalently, twists) of the categories involved and thus *is not* an equivalence of modular categories. Fortunately, this is not a very serious complication since the spherical structures on $\mathcal{C}$ are easy to classify: it is well known that they are in bijection with the objects $X \in \mathcal{C}$ such that $X \otimes X = \mathbf{1}$, see [RT].

The category $\mathcal{M}$ in Theorem 1.2 is not uniquely determined by $\mathcal{C}$. However, there are canonical ways to choose $\mathcal{M}$. In particular, one can always make a canonical "minimal" choice for $\mathcal{M}$ such that $\dim(\mathcal{M}) = \prod_p p^{\alpha_p}$ with $\alpha_p \in \{0,1,2\}$ for odd p and $\alpha_2 \in \{0,1,2,3\}$, see Remark 6.11.

Theorem 1.3. *A modular category $\mathcal{C}$ is braided equivalent to the center of a fusion category of the form Vec_G^{ω} with G being a finite p-group if and only if it has the following properties:*

(i) *the Frobenius-Perron dimension of $\mathcal{C}$ is p^{2n} for some $n \in \mathbb{Z}^+$,*
(ii) *the dimension of every simple object of $\mathcal{C}$ is an integer,*
(iii) *the multiplicative central charge of $\mathcal{C}$ is 1.*

See Subsection 2.6 for the definition of multiplicative central charge. In order to avoid confusion we note that our definition of multiplicative central charge is different from the definition of central charge of a modular functor from [BK, 5.7.10]; in fact, the central charge from [BK] equals to the square of our central charge.

Remark 1.4. If $p \neq 2$ then it is easy to see that (i) implies (ii) (see, e.g., [GN]).

1.2. **Interpretation in terms of group-theoretical fusion categories and semisimple quasi-Hopf algebras.** The notion of group-theoretical fusion category was introduced in [ENO, O1]. Group-theoretical categories form a large class of well-understood fusion categories which can be explicitly constructed from finite group data (which justifies the name). For example, as far as we know, all currently known semisimple Hopf algebras have group-theoretical representation categories (however, there are semisimple quasi-Hopf algebras whose representation categories are not group-theoretical, see [ENO]).

Theorem 1.5. *Let $\mathcal{C}$ be a fusion category such that all objects of $\mathcal{C}$ have integer dimension and such that its center $\mathcal{Z}(\mathcal{C})$ is nilpotent. Then $\mathcal{C}$ is group-theoretical.*

Remark 1.6. A consequence of this theorem is the following statement: every semisimple (quasi-)Hopf algebra of prime power dimension is group-theoretical in the sense of [ENO, Definition 8.40]. This provides a partial answer to a question asked in [ENO].

1.3. **Idea of the proof.** We describe here the main steps in the proof of Theorem 1.3. First we characterize centers of pointed fusion categories in terms of Lagrangian subcategories and show that a modular category $\mathcal{C}$ is equivalent to the representation category of a twisted group double if and only if it has a Lagrangian (i.e., maximal isotropic) subcategory of dimension $\sqrt{\dim(\mathcal{C})}$. This result is reminiscent to the characterization of doubles of quasi-Lie bialgebras in terms of Manin pairs [Dr, Section 2].

Thus we need to show that a category satisfying the assumptions of Theorem 1.3 contains a Lagrangian subcategory. The proof is inspired by the following result for nilpotent metric Lie algebras (i.e, Lie algebras with an invariant non-degenerate scalar product) which can be derived from [KaO]: if $\mathfrak{g}$ is a nilpotent metric Lie algebra of even dimension then $\mathfrak{g}$ contains an abelian ideal $\mathfrak{k}$, which is Lagrangian (i.e., such that $\mathfrak{k}^{\perp} = \mathfrak{k}$). The relevance of metric Lie algebras to our considerations is explained by the fact that they appear in [Dr] as classical limits of quasi-Hopf algebras. In fact, our proof is a "categorification" of the proof of the above result. Thus we need some categorical versions of linear algebra constructions involved in this proof. Remarkably, the categorical counterparts exist for all notions required. For example the notion of orthogonal complement in a metric Lie algebra is replaced by the notion of centralizer in a modular tensor category introduced by M. Müger [Mu2].

1.4. **Organization of the paper.** Section 2 is devoted to preliminaries on fusion categories, which include nilpotent fusion categories, (pre)modular categories, centralizers, Gauss sums and central charge, and Deligne's classification of symmetric fusion categories.

In Section 3 we define the notions of isotropic and Lagrangian subcategories of a premodular category $\mathcal{C}$, generalizing the corresponding notions for a metric group (which is, by definition, a finite abelian group with a quadratic form). We then recall a construction, due to A. Bruguières [Br] and M. Müger [Mu1], which associates to a premodular category $\mathcal{C}$ the "quotient" by its centralizer, called a *modularization*. We prove in Theorem 3.4 an invariance property of the central charge with respect to the modularization. This result will be crucial in the proof of Theorem 6.5. We also study properties of subcategories of modular categories and explain in Proposition 3.9 how one can use maximal isotropic subcategories of a modular category $\mathcal{C}$ to canonically measure a failure of $\mathcal{C}$ to be hyperbolic (i.e., to contain a Lagrangian subcategory).

In Section 4 we characterize hyperbolic modular categories. More precisely, we show in Theorem 4.5 that for a modular category $\mathcal{C}$ there is a bijection between Lagrangian subcategories of $\mathcal{C}$ and braided tensor equivalences $\mathcal{C} \xrightarrow{\sim} \mathcal{Z}(\mathrm{Vec}_G^{\omega})$ (where G is a finite group, $\omega \in Z^3(G, K^{\times})$, and $\mathcal{Z}(\mathrm{Vec}_G^{\omega})$ is the center of Vec_G^{ω}). Note that the category $\mathcal{Z}(\mathrm{Vec}_G^{\omega})$ is equivalent to $\mathrm{Rep}(D^{\omega}(G))$ - the representation category of the twisted double of G [DPR].

We then prove in Theorem 4.8 that if $\mathcal{C}$ is a modular category such that $\dim(\mathcal{C}) = n^2$, $n \in \mathbb{Z}^+$, the central charge of $\mathcal{C}$ equals 1, and $\mathcal{C}$ contains a symmetric subcategory of dimension n, then either $\mathcal{C}$ is equivalent to the representation category of a twisted double of a finite group or $\mathcal{C}$ contains an object with non-integer dimension.

We also give a criterion for a modular category $\mathcal{C}$ to be group-theoretical. Namely, we show in Corollary 4.13 that $\mathcal{C}$ is group-theoretical if and only if there is an isotropic subcategory $\mathcal{E} \subset \mathcal{C}$ such that $(\mathcal{E}')_{ad} \subseteq \mathcal{E}$.

In Section 5 we study pointed modular p-categories. We give a complete list of such categories which do not contain non-trivial isotropic subcategories and analyze the values of their central charges. We then prove in Proposition 5.3 that a nondegenerate metric p-group (G, q) with central charge 1 such that $|G| = p^{2n}$, $n \in \mathbb{Z}^+$, contains a Lagrangian subgroup.

Section 6 is devoted to nilpotent modular categories. There we give proofs of our main results stated in 1.1 above. They are contained in Theorem 6.5, Theorem 6.6, Corollary 6.7, Theorem 6.10, and Theorem 6.12.

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2. Preliminaries

Throughout the paper we work over an algebraically closed field k of characteristic 0. All categories considered in this paper are finite, abelian, semisimple, and k-linear.

2.1. **Fusion categories.** For a fusion category $\mathcal{C}$ let $\mathcal{O}(\mathcal{C})$ denote the set of isomorphism classes of simple objects.

Let $\mathcal{C}$ be a fusion category. Its Grothendieck ring $K_0(\mathcal{C})$ is the free $\mathbb{Z}$-module generated by the isomorphism classes of simple objects of $\mathcal{C}$ with the multiplication coming from the tensor product in $\mathcal{C}$. The Frobenius-Perron dimensions of objects in $\mathcal{C}$ (respectively, $\mathrm{FPdim}(\mathcal{C})$) are defined as the Frobenius-Perron dimensions of their images in the based ring $K_0(\mathcal{C})$ (respectively, as $\mathrm{FPdim}(K_0(\mathcal{C}))$), see [ENO, 8.1]. For a semisimple quasi-Hopf algebra H one has $\mathrm{FPdim}(X) = \dim_k(X)$ for all X in $\mathrm{Rep}(H)$, and so $\mathrm{FPdim}(\mathrm{Rep}(H)) = \dim_k(H)$.

A fusion category is *pointed* if all its simple objects are invertible.

By a *fusion subcategory* of a fusion category $\mathcal{C}$ we understand a full tensor subcategory of $\mathcal{C}$. An example of a fusion subcategory is the maximal pointed subcategory $\mathcal{C}_{pt}$ generated by the invertible objects of $\mathcal{C}$.

A fusion category $\mathcal{C}$ is *pseudo-unitary* if its categorical dimension $\dim(\mathcal{C})$ coincides with its Frobenius-Perron dimension, see [ENO] for details. In this case $\mathcal{C}$ admits a canonical spherical structure (a tensor isomorphism between the identity functor of $\mathcal{C}$ and the second duality functor) with respect to which categorical dimensions of objects coincide with their Frobenius-Perron dimensions [ENO, Proposition 8.23]. The fact important for us in this paper is that a fusion category of an integer Frobenius-Perron dimension is automatically pseudo-unitary [ENO, Proposition 8.24].

Let $\mathcal{C}$ and $\mathcal{D}$ be fusion categories. Recall that for a tensor functor $F : \mathcal{C} \to \mathcal{D}$ its image $F(\mathcal{C})$ is the fusion subcategory of $\mathcal{D}$ generated by all simple objects Y in $\mathcal{D}$ such that $Y \subseteq F(X)$ for some simple X in $\mathcal{C}$. The functor F is called *surjective* if $F(\mathcal{C}) = \mathcal{D}$.

2.2. **Nilpotent fusion categories.** For a fusion category $\mathcal{C}$ let $\mathcal{C}_{ad}$ be the trivial component in the universal grading of $\mathcal{C}$ (see [GN]). Equivalently, $\mathcal{C}_{ad}$ is the smallest fusion subcategory of $\mathcal{C}$ which contains all the objects $X \otimes X^*$, $X \in \mathcal{O}(\mathcal{C})$.

For a fusion category $\mathcal{C}$ we define $\mathcal{C}^{(0)} = \mathcal{C}$, $\mathcal{C}^{(1)} = \mathcal{C}_{ad}$, and $\mathcal{C}^{(n)} = (\mathcal{C}^{(n-1)})_{ad}$ for every integer $n \geq 1$. The non-increasing sequence of fusion subcategories of $\mathcal{C}$

$$\mathcal{C} = \mathcal{C}^{(0)} \supseteq \mathcal{C}^{(1)} \supseteq \cdots \supseteq \mathcal{C}^{(n)} \supseteq \cdots \tag{1}$$

is called the *upper central series* of $\mathcal{C}$. We say that a fusion category $\mathcal{C}$ is *nilpotent* if every non-trivial subcategory of $\mathcal{C}$ has a non-trivial group grading, see [GN]. Equivalently, $\mathcal{C}$ is nilpotent if its upper central series converges to Vec (the category of finite dimensional $k-$vector spaces), i.e., $\mathcal{C}^{(n)} = \text{Vec}$ for some n. The smallest such n is called the *nilpotency class* of $\mathcal{C}$. If $\mathcal{C}$ is nilpotent then every fusion subcategory $\mathcal{E} \subset \mathcal{C}$ is nilpotent, and if $F : \mathcal{C} \to \mathcal{D}$ is a surjective tensor functor, then $\mathcal{D}$ is nilpotent (see [GN]).

Example 2.1.
(1) Let G be a finite group and $\mathcal{C} = \text{Rep}(G)$. Then $\mathcal{C}$ is nilpotent if and only if G is nilpotent.
(2) Pointed categories are precisely the nilpotent fusion categories of nilpotency class 1. A typical example of a pointed category is Vec_G^{ω}, the category of finite dimensional vector spaces graded by a finite group G with the associativity constraint determined by $\omega \in Z^3(G, k^{\times})$.

In this paper we are especially interested in the following class of nilpotent fusion categories.

Example 2.2. Let p be a prime number. Any category of dimension $p^n, n \in \mathbb{Z}$, is nilpotent by [ENO, Theorem 8.28]. For representation categories of semisimple Hopf algebras of dimension p^n this follows from a result of A. Masuoka [Ma1].

By [GN], a nilpotent fusion category comes from a sequence of gradings, in particular it has an integer Frobenius-Perron dimension. It follows from results of [ENO] that a nilpotent fusion category $\mathcal{C}$ is pseudounitary.

2.3. **Premodular categories and modular categories.** Recall that a *braided* tensor category $\mathcal{C}$ is a tensor category equipped with a natural isomorphism $c : \otimes \cong \otimes^{\text{rev}}$ satisfying the hexagon diagrams [JS]. Let $c_{XY} : X \otimes Y \cong Y \otimes X$ with $X, Y \in \mathcal{C}$ denote the components of c.

A *balancing transformation*, or a *twist*, on a braided category $\mathcal{C}$ is a natural automorphism $\theta : \text{id}_{\mathcal{C}} \to \text{id}_{\mathcal{C}}$ satisfying $\theta_{\mathbf{1}} = \text{id}_{\mathbf{1}}$ and

$$\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) c_{YX} c_{XY}. \tag{2}$$

A braided fusion category $\mathcal{C}$ is called *premodular*, or *ribbon*, if it has a twist θ satisfying $\theta_X^* = \theta_{X^*}$ for all objects $X \in \mathcal{C}$.

The *S-matrix* of a premodular category $\mathcal{C}$ is $S = \{s_{XY}\}_{X,Y \in \mathcal{O}(\mathcal{C})}$, where s_{XY} is the quantum trace of $c_{YX}c_{XY}$, see [T]. Equivalently, the S-matrix can be defined as follows. For all $X, Y, Z \in \mathcal{O}(\mathcal{C})$ let N_{XY}^Z be the multiplicity of Z in $X \otimes Y$. For every object X let $d(X)$ denote its quantum dimension. Then

$$s_{XY} = \theta_X^{-1} \theta_Y^{-1} \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{XY}^Z \theta_Z d(Z). \tag{3}$$

The *categorical dimension* of $\mathcal{C}$ is defined by

$$\dim(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} d(X)^2. \tag{4}$$

One has $\dim(\mathcal{C}) \neq 0$ [ENO, Theorem 2.3].

Note 2.3. Below we consider only fusion categories with integer Frobenius-Perron dimensions of objects. Any such category $\mathcal{C}$ is pseudo-unitary (see 2.1). In particular, if $\mathcal{C}$ is braided then it has a canonical twist, which we will always assume chosen.

A premodular category $\mathcal{C}$ is called *modular* if the $S-$matrix is invertible.

Example 2.4. For any fusion category $\mathcal{C}$ its *center* $\mathcal{Z}(\mathcal{C})$ is defined as the category whose objects are pairs $(X, c_{X,-})$, where X is an object of $\mathcal{C}$ and $c_{X,-}$ is a natural family of isomorphisms $c_{X,V} : X \otimes V \cong V \otimes X$ for all objects V in $\mathcal{C}$ satisfying certain compatibility conditions (see e.g., [Kass]). It is known that the center of a pseudounitary category is modular.

2.4. **Pointed modular categories and metric groups.** Let G be a finite abelian group. Pointed premodular categories $\mathcal{C}$ with the group of simple objects isomorphic to G (up to a braided equivalence) are in the natural bijection with *quadratic forms* on G with values in the multiplicative group k^* of the base field. Here a quadratic form $q : G \to k^*$ is a map such that $q(g^{-1}) = q(g)$ and $b(g,h) := \frac{q(gh)}{q(g)q(h)}$ is a symmetric bilinear form, i.e., $b(g_1g_2,h) = b(g_1,h)b(g_2,h)$ for all $g_1,g_2,h \in G$. Namely, for $g \in G$ the value of $q(g)$ is the braiding automorphism of $g \otimes g$ (here by abuse of notation g denotes the object of $\mathcal{C}$ corresponding to $g \in G$). See [Q, Proposition 2.5.1] for a proof that if two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ produce the same quadratic form then they are braided equivalent (Quinn proves less canonical but equivalent statement). We will denote the category corresponding to a group G with quadratic form q by $\mathcal{C}(G,q)$ and call the pair (G,q) a *metric group*. The category $\mathcal{C}(G,q)$ is pseudounitary and hence has a spherical structure such that dimensions of all simple objects equal to 1; hence the categories $\mathcal{C}(G,q)$ always have a canonical ribbon structure. The category $\mathcal{C}(G,q)$ is modular if and only if the bilinear form $b(g,h)$ associated with q is non-degenerate (in this case we will say that the corresponding metric group is *non-degenerate*).

2.5. **Centralizers.** Let $\mathcal{K}$ be a fusion subcategory of a braided fusion category $\mathcal{C}$. In [Mu1, Mu2] M. Müger introduced the *centralizer* $\mathcal{K}'$ of $\mathcal{K}$, which is the fusion subcategory of $\mathcal{C}$ consisting of all the objects Y satisfying

$$c_{YX}c_{XY} = \mathrm{id}_{X\otimes Y} \qquad \text{for all objects } X \in \mathcal{K}. \tag{5}$$

If (5) holds we will say that objects X and Y *centralize* each other. In the case of a ribbon category $\mathcal{C}$, condition (5) is equivalent to $s_{XY} = d(X)d(Y)$, see [Mu2, Proposition 2.5]. Note that in the case of a pointed modular category the centralizer corresponds to the orthogonal complement. The subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called the *transparent* subcategory of $\mathcal{C}$ in [Br, Mu1].

For any fusion subcategory $\mathcal{K} \subseteq \mathcal{C}$ of a braided fusion category $\mathcal{C}$ let $\mathcal{K}^{co}$ be the *commutator* of $\mathcal{K}$ [GN], i.e., the fusion subcategory of $\mathcal{C}$ spanned by all simple objects $X \in \mathcal{C}$ such that $X \otimes X^* \in \mathcal{K}$. For example, if $\mathcal{C} = \mathrm{Rep}(G)$, G a finite group, then any fusion subcategory $\mathcal{K}$ of $\mathcal{C}$ is of the form $\mathcal{K} = \mathrm{Rep}(G/N)$ for some normal subgroup N of G, and $\mathcal{K}^{co} = \mathrm{Rep}(G/[G,N])$ (see [GN]). It follows from the definitions that $(\mathcal{K}^{co})_{ad} \subseteq \mathcal{K} \subseteq (\mathcal{K}_{ad})^{co}$.

Let $\mathcal{K}$ be a fusion subcategory of a pseudounitary modular category $\mathcal{C}$. It was shown in [GN] that

$$(\mathcal{K}_{ad})' = (\mathcal{K}')^{co}. \tag{6}$$

It was shown in [Mu2, Theorem 3.2] that for a fusion subcategory $\mathcal{K}$ of a modular category $\mathcal{C}$ one has $\mathcal{K}'' = \mathcal{K}$ and

$$\dim(\mathcal{K})\dim(\mathcal{K}') = \dim(\mathcal{C}). \tag{7}$$

The subcategory $\mathcal{K}$ is symmetric if and only if $\mathcal{K} \subseteq \mathcal{K}'$. It is modular if and only if $\mathcal{K} \cap \mathcal{K}' = \mathrm{Vec}$, in which case $\mathcal{K}'$ is also modular and there is a braided equivalence $\mathcal{C} \cong \mathcal{K} \boxtimes \mathcal{K}'$.

Let $\mathcal{C}$ be a modular category. Then by [GN, Corollary 6.9], $\mathcal{C}_{pt} = (\mathcal{C}_{ad})'$.

2.6. **Gauss sums and central charge in modular categories.** Let $\mathcal{C}$ be a modular category. For any subcategory $\mathcal{K}$ of $\mathcal{C}$ the *Gauss sums* of $\mathcal{K}$ are defined by

$$\tau^{\pm}(\mathcal{K}) = \sum_{X \in \mathcal{O}(\mathcal{K})} \theta_X^{\pm 1} d(X)^2. \tag{8}$$

Below we summarize some basic properties of twists and Gauss sums (see e.g., [BK, Section 3.1] for proofs).

Each θ_X, $X \in \mathcal{O}(\mathcal{C})$, is a root of unity (this statement is known as Vafa's theorem). The Gauss sums are multiplicative with respect to tensor product of modular categories, i.e., if $\mathcal{C}_1, \mathcal{C}_2$ are modular categories then

$$\tau^{\pm}(\mathcal{C}_1 \boxtimes \mathcal{C}_2) = \tau^{\pm}(\mathcal{C}_1)\tau^{\pm}(\mathcal{C}_2). \tag{9}$$

We also have that

$$\tau^{+}(\mathcal{C})\tau^{-}(\mathcal{C}) = \dim(\mathcal{C}). \tag{10}$$

When $k = \mathbb{C}$ the *multiplicative central charge* $\xi(\mathcal{C})$ is defined by

$$\xi(\mathcal{C}) = \frac{\tau^{+}(\mathcal{C})}{\sqrt{\dim(\mathcal{C})}}, \tag{11}$$

where $\sqrt{\dim(\mathcal{C})}$ is the positive root. If $\dim(\mathcal{C})$ is a square of an integer, then Formula (11) makes sense even if $k \neq \mathbb{C}$. By Vafa's theorem, $\xi(\mathcal{C})$ is a root of unity.

Example 2.5. The center $\mathcal{Z}(\mathcal{C})$ of any fusion category $\mathcal{C}$ (see Example 2.4) is a modular category with central charge 1 [Mu4, Theorem 1.2].

2.7. **Symmetric fusion categories.** The structure of symmetric fusion categories is known, thanks to Deligne's work [De]. Namely, let G be a finite group and let $z \in G$ be a central element such that $z^2 = 1$. Consider the category $Rep(G)$ with its standard symmetric braiding $\sigma_{X,Y}$. Then the map $\sigma'_{X,Y} = \frac{1}{2}(1 + z|_X + z|_Y - z|_X z|_Y)\sigma_{X,Y}$ is also a symmetric braiding on the category $Rep(G)$ (the meaning of the factor $\frac{1}{2}(1 + z|_X + z|_Y - z|_X z|_Y)$ is the following: if $z|_X$ or $z|_Y$ equals 1, then this factor is 1; if $z|_X = z|_Y = -1$ then this factor is (-1)). We will denote by $Rep(G, z)$ the category $Rep(G)$ with the commutativity constraint defined above.

Theorem 2.6. *(*[De]*) Any symmetric fusion category is equivalent (as a braided tensor category) to $Rep(G, z)$ for uniquely defined G and z. The categorical dimension of $X \in Rep(G, z)$ equals $Tr(z|_X)$ and $\dim(\mathcal{C}) = FPdim(\mathcal{C}) = |G|$.*

Now assume that the category $Rep(G, z)$ is endowed with a twist θ such that the dimension of any object is non-negative. It follows immediately from the theorem that $\theta_X = z|_X$. We have

Corollary 2.7. *Let $\mathcal{C}$ be a symmetric fusion category with the canonical spherical structure (see 2.3).*

(i) *If $\dim(\mathcal{C})$ is odd then $\theta_X = id_X$ for any $X \in \mathcal{C}$.*

(ii) *In general either $\theta_X = id_X$ for any $X \in \mathcal{C}$, or $\mathcal{C}$ contains a fusion subcategory $\mathcal{C}_1 \subset \mathcal{C}$ such that $FPdim(\mathcal{C}_1) = \frac{1}{2}FPdim(\mathcal{C})$ and $\theta_X = id_X$ for any $X \in \mathcal{C}_1$.*

Proof. As for (i), it is clear that $z = 1$. For (ii) one takes $\mathcal{C}_1 = Rep(G/\langle z \rangle) \subset \mathrm{Rep}(G)$. □

3. Isotropic subcategories and Bruguières-Müger modularization

3.1. Modularization.

Definition 3.1. Let $\mathcal{C}$ be a premodular category with braiding c and twist θ. A fusion subcategory $\mathcal{E}$ of $\mathcal{C}$ is called *isotropic* if θ restricts to the identity on $\mathcal{E}$, i.e., if $\theta_X = \mathrm{id}_X$ for all $X \in \mathcal{E}$. An isotropic subcategory $\mathcal{E}$ is called *Lagrangian* if $\mathcal{E} = \mathcal{E}'$. The category $\mathcal{C}$ is called *hyperbolic* if it has a Lagrangian subcategory and *anisotropic* if it has no non-trivial isotropic subcategories.

Remark 3.2. (a) When $\mathcal{C} = \mathcal{C}(G, q)$ is a pointed modular category defined in Example 2.4 then isotropic and Lagrangian subcategories of $\mathcal{C}$ correspond to isotropic and Lagrangian subgroups of (G, q), respectively. We discuss properties of pointed modular categories in Section 5.

(b) Let G be a finite group and let $\omega \in Z^3(G, k^\times)$. Consider the pointed fusion category Vec_G^ω. Its center $\mathcal{C} = \mathcal{Z}(\mathrm{Vec}_G^\omega)$ is a modular category. It contains a Lagrangian subcategory $\mathcal{E} \cong \mathrm{Rep}(G)$ formed by all objects in $\mathcal{C}$ which are sent to multiples of the unit object of Vec_G^ω by the forgetful functor $\mathcal{Z}(\mathrm{Vec}_G^\omega) \to \mathrm{Vec}_G^\omega$.

(c) It follows from the balancing axiom (2) that an isotropic subcategory $\mathcal{E} \subseteq \mathcal{C}$ is always symmetric. Conversely, if $\mathcal{E}$ is symmetric and $\dim(\mathcal{E})$ is odd then $\mathcal{E}$ is isotropic, see 2.7. In particular, if $\dim(\mathcal{C})$ is odd then any symmetric subcategory of $\mathcal{C}$ is isotropic.

(d) Recall that we assume that $\mathcal{C}$ is endowed with a canonical spherical structure, see 2.3. Any isotropic subcategory $\mathcal{E} \subset \mathcal{C}$ is equivalent, as a symmetric category, to $\mathrm{Rep}(G)$ for a canonically defined group G with its standard braiding and identical twist, see 2.7. In particular, if $\mathcal{E}$ is Lagrangian then $\dim(\mathcal{C}) = \dim(\mathcal{E})^2$ is a square of an integer.

Let $\mathcal{C}$ be a premodular category such that its centralizer $\mathcal{C}'$ is isotropic and dimensions of all objects $X \in \mathcal{C}'$ are non-negative. Let us recall a construction, due to A. Bruguières [Br] and M. Müger [Mu1], which associates to $\mathcal{C}$ a modular category $\bar{\mathcal{C}}$ and a surjective braided tensor functor $\mathcal{C} \to \bar{\mathcal{C}}$.

Let $G(\mathcal{C})$ be the unique (up to an isomorphism) group such that the category $\mathcal{C}'$ is equivalent, as a premodular category, to $\mathrm{Rep}(G(\mathcal{C}))$ with its standard symmetric braiding and identity twist.

Let A be the algebra of functions on $G(\mathcal{C})$. The group $G(\mathcal{C})$ acts on A via left translations and so A is a commutative algebra in $\mathcal{C}'$ and hence in $\mathcal{C}$.

Consider the category $\bar{\mathcal{C}} := \mathcal{C}_A$ of right A-modules in the category $\mathcal{C}$ (see, e.g., [KiO, 1.2]). It was shown in [Br, KiO, Mu1] that $\bar{\mathcal{C}}$ is a braided fusion category and that the "free module" functor

$$F : \mathcal{C} \to \bar{\mathcal{C}},\ X \mapsto X \otimes A \tag{12}$$

is surjective and has a canonical structure of a braided tensor functor. One can define a twist ϕ on $\bar{\mathcal{C}}$ in such a way that $\phi_Y = \theta_X$ for all $Y \in \mathcal{O}(\bar{\mathcal{C}})$ and $X \in$

$\mathcal{O}(\mathcal{C})$ for which $\mathrm{Hom}_{\mathcal{C}}(X, Y) \neq 0$. It follows that the category $\bar{\mathcal{C}}$ is modular, see [Br, Mu1, KiO] for details. We will call the category $\bar{\mathcal{C}}$ a *modularization* of $\mathcal{C}$.

Let d and $\bar{d}$ denote the dimension functions in $\mathcal{C}$ and $\bar{\mathcal{C}}$, respectively. For any object X in $\bar{\mathcal{C}}$ one has

$$\bar{d}(X) = \frac{d(X)}{d(A)}, \tag{13}$$

cf. [KiO, Theorem 3.5], [Br, Proposition 3.7].

Remark 3.3. Let $\mathcal{E}$ be an isotropic subcategory of a modular category $\mathcal{C}$. Then $\dim(\bar{\mathcal{E}'}) = \dim(\mathcal{C})/\dim(\mathcal{E})^2$ (see e.g. [KiO]).

3.2. **Invariance of the central charge.** In this subsection we prove an invariance property of the central charge with respect to modularization, which will be crucial in the sequel.

Theorem 3.4. *Let $\mathcal{C}$ be a modular category and let $\mathcal{E}$ be an isotropic subcategory of $\mathcal{C}$. Let $F : \mathcal{E}' \to \bar{\mathcal{E}'}$ be the canonical braided tensor functor from $\mathcal{E}'$ to its modularization. Then $\xi(\bar{\mathcal{E}'}) = \xi(\mathcal{C})$.*

Proof. Let A be the canonical commutative algebra in $\mathcal{E}$. We have $\dim(\mathcal{E}) = d(A)$. By definition, $\bar{\mathcal{E}'}$ is the category of left A-modules in $\mathcal{E}'$.

Let us compute the Gauss sums of $\bar{\mathcal{E}'}$:

$$\begin{aligned}
\dim(\mathcal{E})\tau^{\pm}(\bar{\mathcal{E}'}) &= \dim(\mathcal{E}) \sum_{Y \in \mathcal{O}(\bar{\mathcal{E}'})} \phi_Y^{\pm 1} \bar{d}(Y)^2 = \\
&= \sum_{Y \in \mathcal{O}(\bar{\mathcal{E}'})} \phi_Y^{\pm 1} d(Y) \bar{d}(Y) \\
&= \sum_{Y \in \mathcal{O}(\bar{\mathcal{E}'})} \phi_Y^{\pm 1} \left(\sum_{X \in \mathcal{O}(\mathcal{C})} \dim_k \mathrm{Hom}_{\mathcal{C}}(X, Y) d(X) \right) \bar{d}(Y) \\
&= \sum_{X \in \mathcal{O}(\mathcal{C})} \theta_X^{\pm 1} d(X) \left(\sum_{Y \in \mathcal{O}(\bar{\mathcal{E}'})} \dim_k \mathrm{Hom}_{\mathcal{C}}(X, Y) \bar{d}(Y) \right) \\
&= \sum_{X \in \mathcal{O}(\mathcal{C})} \theta_X^{\pm 1} d(X) \left(\sum_{Y \in \mathcal{O}(\bar{\mathcal{E}'})} \dim_k \mathrm{Hom}_{\bar{\mathcal{E}'}}(X \otimes A, Y) \bar{d}(Y) \right) \\
&= \sum_{X \in \mathcal{O}(\mathcal{C})} \theta_X^{\pm 1} d(X) \bar{d}(F(X)) = \tau^{\pm}(\mathcal{C}),
\end{aligned}$$

where we used the relation (13) and the fact that F is an adjoint of the forgetful functor from $\bar{\mathcal{E}'}$ to $\mathcal{E}'$.

Combining this with the equation $\dim(\bar{\mathcal{E}'}) = \dim(\mathcal{C})/\dim(\mathcal{E})^2$ (see Remark 3.3) we obtain the result. $\square$

3.3. **Maximal isotropic subcategories.** Let $\mathcal{C}$ be a modular category and let $\mathcal{L}$ be an isotropic subcategory of $\mathcal{C}$ which is maximal among isotropic subcategories of $\mathcal{C}$. Below we will show that the braided equivalence class of the modular category $\bar{\mathcal{L}'}$ (the modularization of $\mathcal{L}'$ by $\mathcal{L}$) is independent of the choice of $\mathcal{L}$.

Let $\mathcal{C}$ be a fusion category and let $\mathcal{A}$ and $\mathcal{B}$ be its fusion subcategories such that $X \otimes Y \cong Y \otimes X$ for all $X \in \mathcal{O}(\mathcal{A})$ and $Y \in \mathcal{O}(\mathcal{B})$. Let $\mathcal{A} \vee \mathcal{B}$ denote the fusion subcategory of $\mathcal{C}$ generated by $\mathcal{A}$ and $\mathcal{B}$, i.e., consisting of all subobjects of $X \otimes Y$, where $X \in \mathcal{O}(\mathcal{A})$ and $Y \in \mathcal{O}(\mathcal{B})$. Recall that the regular element of $K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ is $R_{\mathcal{C}} = \sum_{X \in \mathcal{O}(\mathcal{C})} d(X)X$. It is defined up to a scalar multiple by the property that $Y \otimes R_{\mathcal{C}} = d(Y)R_{\mathcal{C}}$ for all $Y \in \mathcal{O}(\mathcal{C})$ [ENO].

Lemma 3.5. *Let $\mathcal{C}$, $\mathcal{A}$, $\mathcal{B}$ be as above. Then* $\dim(\mathcal{A} \vee \mathcal{B}) = \frac{\dim(\mathcal{A})\dim(\mathcal{B})}{\dim(\mathcal{A}\cap\mathcal{B})}$.

Proof. It is easy to see that

$$R_{\mathcal{A}} \otimes R_{\mathcal{B}} = aR_{\mathcal{A}\vee\mathcal{B}}, \tag{14}$$

where the scalar a is equal to the multiplicity of the unit object $\mathbf{1}$ in $R_{\mathcal{A}} \otimes R_{\mathcal{B}}$, which is the same as the multiplicity of $\mathbf{1}$ in $\sum_{Z \in \mathcal{O}(\mathcal{A}\cap\mathcal{B})} d(Z)^2 Z \otimes Z^*$. Hence, $a = \dim(\mathcal{A} \cap \mathcal{B})$. Taking dimensions of both sides of (14) we get the result. □

Let $L(\mathcal{C})$ denote the lattice of fusion subcategories of a fusion category $\mathcal{C}$. For any two subcategories $\mathcal{A}$ and $\mathcal{B}$ their meet is their intersection and their joint is the category $\mathcal{A} \vee \mathcal{B}$.

Lemma 3.6. *Let $\mathcal{C}$ be a fusion category such that $X \otimes Y \cong Y \otimes X$ for all objects X, Y in $\mathcal{C}$. For all $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D} \in L(\mathcal{C})$ such that $\mathcal{D} \subseteq \mathcal{A}$ the following* modular law *holds true:*

$$\mathcal{A} \cap (\mathcal{B} \vee \mathcal{D}) = (\mathcal{A} \cap \mathcal{B}) \vee \mathcal{D}. \tag{15}$$

Proof. A classical theorem of Dedekind in lattice theory states that (15) is equivalent to the following statement: for all $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D} \in L(\mathcal{C})$ such that $\mathcal{D} \subseteq \mathcal{A}$, if $\mathcal{A} \cap \mathcal{B} = \mathcal{D} \cap \mathcal{B}$ and $\mathcal{A} \vee \mathcal{B} = \mathcal{D} \vee \mathcal{B}$ then $\mathcal{A} = \mathcal{D}$ (see e.g., [MMT]).

Let us prove the latter property. Take a simple object $X \in \mathcal{A}$. Then $X \in \mathcal{A} \vee \mathcal{B} = \mathcal{D} \vee \mathcal{B}$ so there are simple objects $D \in \mathcal{D}$ and $B \in \mathcal{B}$ such that X is contained in $D \otimes B$. Therefore, B is contained in $D^* \otimes X$ and so $B \in \mathcal{A}$. So $B \in \mathcal{A} \cap \mathcal{B} = \mathcal{D} \cap \mathcal{B} \subseteq \mathcal{D}$. Hence $X \in \mathcal{D}$, as required. □

Remark 3.7. When $\mathcal{C} = \mathrm{Rep}(G)$ is the representation category of a finite group G, Lemma 3.6 gives a well-known property of the lattice of normal subgroups of G.

The next lemma gives an analogue of a diamond isomorphism for the "quotients by isotropic subcategories."

Lemma 3.8. *Let $\mathcal{C}$ be a modular category, let $\mathcal{D}$ be an isotropic subcategory of $\mathcal{C}$ and let $\mathcal{B}$ be a subcategory of $\mathcal{D}'$. Let A, A_0 be the canonical commutative algebras in $\mathcal{D}$ and $\mathcal{D} \cap \mathcal{B}$, respectively.*

Then the category $\mathcal{B}_{A_0}$ of A_0-modules in $\mathcal{B}$ and the category $(\mathcal{D} \vee \mathcal{B})_A$ of A-modules in $\mathcal{D} \vee \mathcal{B}$ are equivalent as braided tensor categories.

Proof. Note that

$$\dim(\mathcal{B}_{A_0}) = \frac{\dim(\mathcal{B})}{\dim(\mathcal{D} \cap \mathcal{B})} = \frac{\dim(\mathcal{D} \vee \mathcal{B})}{\dim(\mathcal{D})} = \dim((\mathcal{D} \vee \mathcal{B})_A)$$

by Lemma 3.5.

Define a functor $H : \mathcal{B}_{A_0} \to (\mathcal{D}\vee\mathcal{B})_A$ by $H(X) = X \otimes_{A_0} A$, $X \in \mathcal{B}_{A_0}$. Then H has a natural structure of a braided tensor functor. Note that for $X = Y \otimes A_0$, $Y \in \mathcal{B}$ we have $H(X) = Y \otimes A$, i.e., the composition of H with the free A_0-module functor is the free A-module functor. The latter functor is surjective and, hence, so is H.

Since a surjective functor between categories of equal dimension is necessarily an equivalence (see [ENO, 5.7] or [EO, Proposition 2.20]) the result follows. □

Proposition 3.9. *Let C be a modular category and let $\mathcal{L}_1$, $\mathcal{L}_2$ be maximal among isotropic subcategories of $\mathcal{C}$. Then the modularization $\bar{\mathcal{L}_1'}$ and $\bar{\mathcal{L}_2'}$ are equivalent as braided fusion categories.*

Proof. Let $\mathcal{D} = \mathcal{L}_1$ and $\mathcal{B} = \mathcal{L}_1' \cap \mathcal{L}_2'$. By maximality of $\mathcal{L}_1, \mathcal{L}_2$ we have $\mathcal{L}_1' \cap \mathcal{L}_2 \subseteq \mathcal{L}_1$ and $\mathcal{L}_1 \cap \mathcal{L}_2' \subseteq \mathcal{L}_2$. Therefore, $\mathcal{D} \cap \mathcal{B} = \mathcal{L}_1 \cap \mathcal{L}_2$ and $\mathcal{D} \vee \mathcal{B} = \mathcal{L}_1' \cap (\mathcal{L}_1 \vee \mathcal{L}_2') = \mathcal{L}_1'$ by Lemma 3.6.

Let A_0 be the canonical commutative algebra in $\mathcal{L}_1 \cap \mathcal{L}_2$. Applying Lemma 3.8 we see that $\bar{\mathcal{L}_1'}$ is equivalent to the category $(\mathcal{L}_1' \cap \mathcal{L}_2')_{A_0}$ of A_0-modules in $\mathcal{L}_1' \cap \mathcal{L}_2'$. The proposition now follows by interchanging $\mathcal{L}_1$ and $\mathcal{L}_2$. ☐

Remark 3.10.
(i) We can call the modular category $\bar{\mathcal{L}_1'}$ constructed in the proof of Proposition 3.9 "the" *canonical modularization* corresponding to $\mathcal{C}$ (it measures the failure of $\mathcal{C}$ to be hyperbolic). The above proof gives a concrete equivalence $\bar{\mathcal{L}_1'} \cong \bar{\mathcal{L}_2'}$. But given another maximal isotropic subcategory $\mathcal{L}_3 \subset \mathcal{C}$ the composition of equivalences $\bar{\mathcal{L}_1'} \cong \bar{\mathcal{L}_2'}$ and $\bar{\mathcal{L}_2'} \cong \bar{\mathcal{L}_3'}$ is *not* in general equal to the equivalence $\bar{\mathcal{L}_1'} \cong \bar{\mathcal{L}_3'}$. This is why we put "the" above in quotation marks.
(ii) For a maximal isotropic subcategory $\mathcal{L} \subset \mathcal{C}$ the corresponding modularization does *not* have to be anisotropic, in contrast with the situation for metric groups. Examples illustrating this phenomenon are, e.g., the centers of non-group theoretical Tambara-Yamagami categories considered in [ENO, Remark 8.48].

4. Reconstruction of a twisted group double from a Lagrangian subcategory

4.1. **$\mathcal{C}$-algebras.** Let us recall the following definition from [KiO].

Definition 4.1. Let $\mathcal{C}$ be a braided fusion category. A $\mathcal{C}-$*algebra* is a commutative algebra A in $\mathcal{C}$ such that $\dim \mathrm{Hom}(\mathbf{1}, A) = 1$, the pairing $A \otimes A \to A \to \mathbf{1}$ given by the multiplication of A is non-degenerate, $\theta_A = \mathrm{id}_A$ and $\dim(A) \neq 0$.

Let $\mathcal{C}$ be a modular category, let A be a $\mathcal{C}-$algebra, and let $\mathcal{C}_A$ be the fusion category of right $A-$modules with the tensor product $\otimes_A$. The free module functor $F : \mathcal{C} \to \mathcal{C}_A$, $X \mapsto X \otimes A$ has an obvious structure of a *central functor*. By this we mean that there is a natural family of isomorphisms $F(X) \otimes_A Y \cong Y \otimes_A F(X)$, $X \in \mathcal{C}$, $Y \in \mathcal{C}_A$, satisfying an obvious multiplication compatibility, see e.g. [Be, 2.1]. Indeed, we have $F(X) = X \otimes A$, and hence $F(X) \otimes_A Y = X \otimes Y$. Similarly, $Y \otimes_A F(X) = Y \otimes X$. These two objects are isomorphic via the braiding of $\mathcal{C}_A$ (one can check that the braiding gives an isomorphism of A-modules using the commutativity of A).

Thus, the functor F extends to a functor $\tilde{F} : \mathcal{C} \to \mathcal{Z}(\mathcal{C}_A)$ in such a way that F is the composition of $\tilde{F}$ and the forgetful functor $\mathcal{Z}(\mathcal{C}_A) \to \mathcal{C}_A$.

Proposition 4.2. *The functor $\tilde{F} : \mathcal{C} \to \mathcal{Z}(\mathcal{C}_A)$ is injective (that is fully faithful).*

Proof. Consider $\mathcal{C}_A$ as a module category over $\mathcal{C}$ via F and over $\mathcal{Z}(\mathcal{C}_A)$ via $\tilde{F}$. We will prove the dual statement (see [ENO, Proposition 5.3]), namely that the functor $T : \mathcal{C}_A \boxtimes \mathcal{C}_A^{op} \to \mathcal{C}^*_{\mathcal{C}_A}$ dual to $\tilde{F}$ is surjective (here and below the superscript *op* refers to the tensor category with the opposite tensor product). Recall (see e.g. [O1]) that the category $\mathcal{C}^*_{\mathcal{C}_A}$ is identified with the category of $A-$bimodules. An explicit description of the functor T is the following: by definition, any $M \in \mathcal{C}_A$ is a right $A-$module. Using the braiding and its inverse one can define on M two structures of a left $A-$module: $A \otimes M \xrightarrow{c^{\pm 1}_{A,M}} M \otimes A \to M$. Both structures make M into an $A-$bimodule, and we will denote the two results by M_+ and M_-, respectively. Then we have $T(M \boxtimes N) = M_+ \otimes_A N_-$. In particular we see that the functor $\mathcal{C} \boxtimes \mathcal{C}^{op} \xrightarrow{F \boxtimes F} \mathcal{C}_A \boxtimes \mathcal{C}_A^{op} \xrightarrow{T} \mathcal{C}^*_{\mathcal{C}_A}$ coincides with the functor $\mathcal{C} \boxtimes \mathcal{C}^{op} \simeq \mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{C}^*_{\mathcal{C}_A}) \to \mathcal{C}^*_{\mathcal{C}_A}$ (see [O2]). Since the functor $\mathcal{Z}(\mathcal{C}^*_{\mathcal{C}_A}) \to \mathcal{C}^*_{\mathcal{C}_A}$ is surjective (see [EO, 3.39]) we see that the functor T is surjective. The proposition is proved. □

Remark 4.3. Note that since $\mathcal{C}$ and $\mathcal{Z}(\mathcal{C}_A)$ are modular we have a factorization $\mathcal{Z}(\mathcal{C}_A) = \mathcal{C} \boxtimes \mathcal{D}$, where $\mathcal{D}$ is the centralizer of $\mathcal{C}$ in $\mathcal{Z}(\mathcal{C}_A)$. One observes that $\mathcal{D}$ is identified with the category of "dyslectic" $A-$modules $\mathrm{Rep}^0(A)$, see [KiO, P].

Corollary 4.4. *Assume that* $\dim(A) = \sqrt{\dim(\mathcal{C})}$. *Then the functors* $\tilde{F} : \mathcal{C} \to \mathcal{Z}(\mathcal{C}_A)$ *and* $T : \mathcal{C}_A \boxtimes \mathcal{C}_A^{op} \to \mathcal{C}^*_{\mathcal{C}_A}$ *are tensor equivalences.*

Proof. We have already seen that $\dim(\mathcal{C}_A) = \frac{\dim(\mathcal{C})}{\dim(A)}$. Hence, $\dim(\mathcal{Z}(\mathcal{C}_A)) = \frac{\dim(\mathcal{C})^2}{\dim(A)^2} = \dim(\mathcal{C})$. Since $\tilde{F}$ is an injective functor between categories of equal dimension, it is necessarily an equivalence by [EO, Proposition 2.19]. Hence the dual functor T is also an equivalence. □

4.2. **Hyperbolic modular categories as twisted group doubles.** We are now ready to state and prove our first main result which relates hyperbolic modular categories and twisted doubles of finite groups.

Let $\mathcal{C}$ be a modular category. Consider the set of all triples (G, ω, F), where G is a finite group, $\omega \in Z^3(G, k^\times)$, and $F : \mathcal{C} \xrightarrow{\sim} \mathcal{Z}(\mathrm{Vec}_G^\omega)$ is a braided tensor equivalence. Let us say that two triples (G_1, ω_1, F_1) and (G_2, ω_2, F_2) are equivalent if there exists a tensor equivalence $\iota : \mathrm{Vec}_{G_1}^{\omega_1} \xrightarrow{\sim} \mathrm{Vec}_{G_2}^{\omega_2}$ such that $\mathcal{F}_2 \circ F_2 = \iota \circ \mathcal{F}_1 \circ F_1$, where $\mathcal{F}_i : \mathcal{Z}(\mathrm{Vec}_{G_i}^{\omega_i}) \to \mathrm{Vec}_{G_i}^{\omega_i}$, $i = 1, 2$, are the canonical forgetful functors.

Let $\mathrm{E}(\mathcal{C})$ be the set of all equivalences classes of triples (G, ω, F). Let $\mathrm{Lagr}(\mathcal{C})$ be the set of all Lagrangian subcategories of $\mathcal{C}$.

Theorem 4.5. *For any modular category* $\mathcal{C}$ *there is a natural bijection*

$$f : E(\mathcal{C}) \xrightarrow{\sim} Lagr(\mathcal{C}).$$

Proof. The map f is defined as follows. Note that each braided tensor equivalence $F : \mathcal{C} \xrightarrow{\sim} \mathcal{Z}(\mathrm{Vec}_G^\omega)$ gives rise to the Lagrangian subcategory $f(G, \omega, F)$ of $\mathcal{C}$ formed by all objects sent to multiples of the unit object $\mathbf{1}$ under the forgetful functor $\mathcal{Z}(\mathrm{Vec}_G^\omega) \to \mathrm{Vec}_G^\omega$. This subcategory is clearly the same for all equivalent choices of (G, ω, F).

Conversely, given a Lagrangian subcategory $\mathcal{E} \subseteq \mathcal{C}$ it follows from Deligne's theorem [De] that $\mathcal{E} = \mathrm{Rep}(G)$ for a unique (up to isomorphism) finite group G. Let $A = \mathrm{Fun}(G) \in \mathrm{Rep}(G) = \mathcal{E} \subset \mathcal{C}$. It is clear that A is a $\mathcal{C}-$algebra and $\dim(A) = \dim(\mathcal{E}) = \sqrt{\dim(\mathcal{C})}$. Then by Corollary 4.4, the functor $\tilde{F} : \mathcal{C} \to \mathcal{Z}(\mathcal{C}_A)$ is an equivalence.

Finally, let us show that $\mathcal{C}_A$ is pointed and $K_0(\mathcal{C}_A) = \mathbb{Z}G$. Note that there are $|G|$ non-isomorphic structures A_g, $g \in G$, of an invertible A-bimodule on A, since the category of A-bimodules in $\mathcal{E}$ is equivalent to Vec_G. For each A_g there is a pair X, Y of simple objects in $\mathcal{C}_A$ such that $T(X \boxtimes Y) = A_g$. Taking the forgetful functor to $\mathcal{C}_A$ we obtain $Y = X^*$ and X is invertible. Hence, for each $g \in G$ there is a unique invertible $X_g \in \mathcal{C}_A$ such that $T(X_g \boxtimes X_g^*) = A_g$, and therefore $g \mapsto X_g$ is an isomorphism of K_0 rings. Thus, $\mathcal{C}_A \cong \mathrm{Vec}_G^\omega$ for some $\omega \in \mathcal{Z}^3(G, k^\times)$. We set $h(\mathcal{E})$ to be the class of the equivalence $\tilde{F} : \mathcal{C} \xrightarrow{\sim} \mathcal{Z}(\mathcal{C}_A)$.

Let show that the above constructions f and h are inverses of each other. Let $\mathcal{E}$ be a Lagrangian subcategory of $\mathcal{C}$ and let A be the algebra defined in the previous paragraph. The forgetful functor from $\mathcal{C} \cong \mathcal{Z}(\mathcal{C}_A)$ to $\mathcal{C}_A$ is the free module functor, and so $f(h(\mathcal{E}))$ consists of all objects X in $\mathcal{C}$ such that $X \otimes A$ is a multiple of A. Since A is the regular object of $\mathcal{E}$, it follows that $f(h(\mathcal{E})) = \mathcal{E}$ and $f \circ h = \mathrm{id}$.

Proving that $h \circ f = \mathrm{id}$ amounts to a verification of the following fact. Let G be a finite group, let $\omega \in Z^3(G,\, k^\times)$, and let $A = \mathrm{Fun}(G)$ be the canonical algebra in $\mathrm{Rep}(G) \subset \mathcal{Z}(\mathrm{Vec}_G^\omega)$. Then the category of A-modules in $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ is equivalent to Vec_G^ω and the functor of taking the free A-module coincides with the forgetful functor from $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ to Vec_G^ω. This is straightforward and is left to the reader. □

Remark 4.6. Our reconstruction of the representation category of a twisted group double from a Lagrangian subcategory can be viewed as a categorical analogue of the following reconstruction of the double of a quasi-Lie bialgebra from a Manin pair (i.e., a pair consisting of a metric Lie algebra and its Lagrangian subalgebra) in the theory of quantum groups [Dr, Section 2].

Let $\mathfrak{g}$ be a finite-dimensional metric Lie algebra (i.e., a Lie algebra on which a nondegenerate invariant symmetric bilinear form is given). Let $\mathfrak{l}$ be a Lagrangian subalgebra of $\mathfrak{g}$. Then $\mathfrak{l}$ has a structure of a quasi-Lie bialgebra and there is an isomorphism between $\mathfrak{g}$ and the double $\mathfrak{D}(\mathfrak{l})$ of $\mathfrak{l}$. The correspondence between Lagrangian subalgebras of $\mathfrak{g}$ and doubles isomorphic to $\mathfrak{g}$ is bijective, see [Dr, Section 2] for details.

Remark 4.7. Given a hyperbolic modular category $\mathcal{C}$ there is no canonical way to assign to it a pair $(G,\, \omega)$ such that $\mathcal{C} \cong \mathcal{Z}(\mathrm{Vec}_G^\omega)$ as a braided fusion category. Indeed, it follows from [EG1] that there exist non-isomorphic finite groups G_1, G_2 such that $\mathcal{Z}(\mathrm{Vec}_{G_1}) \cong \mathcal{Z}(\mathrm{Vec}_{G_2})$ as braided fusion categories. (See also [N].)

Theorem 4.8. *Let $\mathcal{C}$ be a modular category such that $\dim(C) = n^2$, $n \in \mathbb{Z}^+$, and such that $\xi(\mathcal{C}) = 1$. Assume that $\mathcal{C}$ contains a symmetric subcategory $\mathcal{V}$ such that $\dim(\mathcal{V}) = n$. Then either $\mathcal{C}$ is the center of a pointed category or it contains an object with non-integer dimension.*

Proof. Assume that $\mathcal{V}$ is not isotropic. Then $\mathcal{V}$ contains an isotropic subcategory $\mathcal{K}$ such that $\dim(\mathcal{K}) = \frac{1}{2}\dim(\mathcal{V})$ (this follows from Deligne's description of symmetric categories, see 2.7). Hence the category $\bar{\mathcal{K}}'$ (modularization of $\mathcal{K}'$) has dimension 4 and central charge 1. It follows from the explicit classification given in Example 5.1 (b),(d) that the category $\bar{\mathcal{K}}'$ contains an isotropic subcategory of dimension 2; clearly this subcategory is equivalent to $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$. Let $A_1 = \mathrm{Fun}(\mathbb{Z}/2\mathbb{Z})$ be the commutative algebra of dimension 2 in this subcategory. Let $I : \bar{\mathcal{K}}' \to \mathcal{K}'$ be the right adjoint functor to the modularization functor $F : \mathcal{K}' \to \bar{\mathcal{K}}'$.

We claim that the object $A := I(A_1)$ has a canonical structure of a $\mathcal{C}-$algebra. Indeed, we have a canonical morphism in $\mathrm{Hom}(F(A), A_1) = \mathrm{Hom}(A, I(A_1)) = \mathrm{Hom}(A, A) \ni \mathrm{id}$. Using this one can construct a multiplication on A via $\mathrm{Hom}(A_1 \otimes A_1, A_1) \to \mathrm{Hom}(F(A) \otimes F(A), A_1) = \mathrm{Hom}(F(A \otimes A), A_1) = \mathrm{Hom}(A \otimes A, A)$. Since the functor F is braided it follows from the commutativity of A_1 that A is commutative. Other conditions from Definition 4.1 are also easy to check. In particular $\dim(A) = \dim(\mathcal{K})\dim(A_1) = \dim(\mathcal{V}) = \sqrt{\dim(\mathcal{C})}$. We also note that the category $Rep_{\mathcal{K}'}(A)$ contains precisely two simple objects (actually, the functor $M \mapsto I(M)$ is an equivalence of categories between $\mathrm{Rep}_{\bar{\mathcal{K}}'}(A_1)$ and $\mathrm{Rep}_{\mathcal{K}'}(A)$); we will call these two objects $\mathbf{1}$ (for A itself considered as an $A-$module) and δ. Clearly $\delta \otimes_A \delta = \mathbf{1}$.

By Corollary 4.4, we have an equivalence $\mathcal{C}^*_{\mathcal{C}_A} = \mathcal{C}_A \boxtimes \mathcal{C}_A^{op}$. Moreover, the forgetful functor $\mathcal{C}^*_{\mathcal{C}_A} \to \mathcal{C}_A$ corresponds to the tensor product functor $\mathcal{C}_A \boxtimes \mathcal{C}_A^{op} \to \mathcal{C}_A$. Now consider the subcategory $(\mathcal{K}')^*_{\mathcal{C}_A} \subset \mathcal{C}^*_{\mathcal{C}_A}$ (in other words $A-$bimodules in $\mathcal{K}'$); the forgetful functor above restricts to $S : (\mathcal{K}')^*_{\mathcal{C}_A} \to \mathrm{Rep}_{\mathcal{K}'}(A)$.

Let $M \in (\mathcal{K}')^*_{\mathcal{C}_A}$ be a simple object. We claim that there are three possibilities: 1) $S(M) = \mathbf{1}$, 2) $S(M) = \delta$ or 3) $S(M) = \mathbf{1} \oplus \delta$. Indeed, $M = X \boxtimes Y \in \mathcal{C}_A \boxtimes \mathcal{C}_A^{op}$ and $S(M) = X \otimes Y$ for some simple $X, Y \in \mathcal{C}_A$. Since $\mathbf{1}$ and δ are invertible the result is clear.

Now, notice that if there exists M as in case 3) then we have $X = Y^*$ and $\dim(X) = \dim(Y) = \sqrt{2}$. Thus the category $\mathcal{C}_A$ contains an object with non-integer dimension, which implies that the category $\mathcal{C}$ contains an object with non-integer dimension (see e.g. [ENO, Corollary 8.36]), and the theorem is proved in this case. Hence we will assume that for any $M \in (\mathcal{K}')^*_{\mathcal{C}_A}$ only 1) or 2) holds. This implies that all objects of $(\mathcal{K}')^*_{\mathcal{C}_A}$ are invertible. Note that $\dim((\mathcal{K}')^*_{\mathcal{C}_A}) = \dim(K') = 2\sqrt{\dim(\mathcal{C})}$ and hence we have precisely $2\sqrt{\dim(\mathcal{C})}$ simple objects. Consider all objects $M \in (\mathcal{K}')^*_{\mathcal{C}_A}$ such that $S(M) = \mathbf{1}$; it is easy to see that there are precisely $\sqrt{\dim(C)}$ of those (indeed, $X \boxtimes Y \mapsto X \boxtimes (Y \otimes_A \delta)$ gives a bijection between simple bimodules M with $S(M) = \mathbf{1}$ and simple bimodules M with $S(M) = \delta$). Let G be the group of isomorphism classes of all objects $M \in (\mathcal{K}')^*_{\mathcal{C}_A}$ with $S(M) = \mathbf{1}$ (thus $|G| = \sqrt{\dim(\mathcal{C})}$). Any object of this type is of the form $X_g \boxtimes (X_g)^*$ for some invertible $X_g \in \mathcal{C}_A$. Thus we already constructed $\sqrt{\dim(\mathcal{C})}$ invertible simple objects in $\mathcal{C}_A$. Since $\dim(\mathcal{C}_A) = \sqrt{\dim(\mathcal{C})}$ the objects X_g exhaust all simple objects in $\mathcal{C}_A$. By Corollary 4.4, we are done. □

4.3. **A criterion for a modular category to be group-theoretical.** Let $\mathcal{C}$ be a modular category. It is known that the entries of the S-matrix of $\mathcal{C}$ are cyclotomic integers [CG, dBG]. Hence, we may identify them with complex numbers. In particular, the notions of complex conjugation and absolute value of the elements of the S-matrix make sense.

Remark 4.9. Let $\mathcal{K} \subseteq \mathcal{C}$ be a fusion subcategory. Recall from [GN] that $(\mathcal{K}_{ad})'$ is spanned by simple objects Y such that $|s_{XY}| = d_X d_Y$ for all simple X in $\mathcal{K}$. In this case the ratio $b(X, Y) := s_{XY}/(d_X d_Y)$ is a root of unity. Furthermore, for all simple $X \in \mathcal{K}$, $Y_1, Y_2 \in \mathcal{K}'_{ad}$ and any simple subobject Z of $Y_1 \otimes Y_2$ we have

$$b(X, Y_1)b(X, Y_2) = b(X, Z), \tag{16}$$

as explained in [Mu2].

Lemma 4.10. *Let $\mathcal{C}$ be a modular category and let $\mathcal{K} \subseteq \mathcal{C}$ be a fusion subcategory such that $\mathcal{K} \subseteq (\mathcal{K}_{ad})'$.*

(1) *There is a grading $\mathcal{K} = \oplus_{g \in G} \mathcal{K}_g$ such that $\mathcal{K}_1 = \mathcal{K}' \cap \mathcal{K}$.*
(2) *There is a non-degenerate symmetric bilinear form b on G such that $b(g, h) = s_{XY}/(d_X d_Y)$ for all $X \in \mathcal{K}_g$ and $Y \in \mathcal{K}_h$.*

(3) *If $\mathcal{K}' \cap \mathcal{K}$ is isotropic then there is a non-degenerate quadratic form q on G such that $q(g) = \theta_X$ for all $X \in \mathcal{K}_g$. In this case b is the bilinear form corresponding to q.*

Proof. Since $\mathcal{K}_{ad} \subseteq \mathcal{K}' \cap \mathcal{K} \subseteq \mathcal{K}$ the assertion (1) follows from [GN].

Let $b(X, Y) = s_{XY}/(d_X d_Y)$ for all simple $X, Y \in \mathcal{K}$. Clearly, b is symmetric and $b(X, Y) = 1$ for all simple X in $\mathcal{K}$ if and only if $Y \in \mathcal{K}' \cap \mathcal{K} = \mathcal{K}_1$. To prove (2) it suffices to check that b depends only on $h \in G$ such that $Y \in \mathcal{K}_h$ (then the G-linear property follows from (16)). Let Y_1, Y_2 be simple objects in $\mathcal{K}_h$. Then $Y_1 \otimes Y_2^* \in \mathcal{K}' \cap \mathcal{K}$ and so $b(X, Y_1)b(X, Y_2^*) = 1$, whence $b(X, Y_1) = b(X, Y_2)$, as desired.

Finally, (3) is a direct consequence of our discussion in Section 3.1. □

For a subcategory $\mathcal{K} \subseteq \mathcal{C}$ satisfying the hypothesis of Lemma 4.10 let $(G_{\mathcal{K}}, b_{\mathcal{K}})$ be the corresponding abelian grading group and bilinear form. Note that if such $\mathcal{K}$ is considered as a subcategory of $\mathcal{C}^{rev}$ then the corresponding bilinear form is $(G_{\mathcal{K}}, b_{\mathcal{K}}^{-1})$.

Theorem 4.11. *Let $\mathcal{C}$ be a modular category. Then symmetric subcategories of $\mathcal{Z}(\mathcal{C}) \cong \mathcal{C} \boxtimes \mathcal{C}^{rev}$ of dimension $\dim(\mathcal{C})$ are in bijection with triples $(\mathcal{L}, \mathcal{R}, \iota)$, where $\mathcal{L} \subseteq \mathcal{C}$, $\mathcal{R} \subseteq \mathcal{C}^{rev}$ are symmetric subcategories such that $(\mathcal{L}')_{ad} \subseteq \mathcal{L}$, $(\mathcal{R}')_{ad} \subseteq \mathcal{R}$, and $\iota : (G_{\mathcal{L}'}, b_{\mathcal{L}'}) \cong (G_{\mathcal{R}'}, b_{\mathcal{R}'})$ is an isomorphism of bilinear forms.*

Namely, any such subcategory is of the form

$$\mathcal{D}_{\mathcal{L},\mathcal{R},\iota} = \oplus_{g \in G_{\mathcal{L}'}} \mathcal{L}_g \boxtimes \mathcal{R}_{\iota(g)}. \tag{17}$$

Proof. Let $X_1 \boxtimes Y_1$ and $X_2 \boxtimes Y_2$ be two simple objects of $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$. They centralize each other if and only if

$$|s_{X_1 X_2}| = d_{X_1} d_{X_2}, \tag{18}$$

$$|s_{Y_1 Y_2}| = d_{Y_1} d_{Y_2}, \quad \text{and} \tag{19}$$

$$\frac{s_{X_1 X_2}}{d_{X_1} d_{X_2}} \frac{s_{Y_1 Y_2}}{d_{Y_1} d_{Y_2}} = 1. \tag{20}$$

Let $\mathcal{D}$ be a symmetric subcategory of $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ and let $\mathcal{L}$ (respectively, $\mathcal{R}$) be the centralizers of fusion subcategories of $\mathcal{C}$ (respectively, $\mathcal{C}^{\mathrm{rev}}$) formed by left (respectively, right) tensor factors of simple objects in $\mathcal{D}$. By conditions (18), (19), and Remark 4.9 we must have ${\mathcal{L}'}_{ad} \subseteq \mathcal{L}$ and ${\mathcal{R}'}_{ad} \subseteq \mathcal{R}$. Hence, Lemma 4.10 gives gradings $\mathcal{L}' = \oplus_{g \in G_{\mathcal{L}}} (\mathcal{L}')_g$ with $(\mathcal{L}')_1 = \mathcal{L}' \cap \mathcal{L}$ and $\mathcal{R}' = \oplus_{g \in G_{\mathcal{R}}} (\mathcal{R}')_g$ with $(\mathcal{R}')_1 = \mathcal{R}' \cap \mathcal{R}$. The condition (20) gives an isomorphism of bilinear forms $\iota : (G_{\mathcal{L}'}, b_{\mathcal{L}'}) \cong (G_{\mathcal{R}'}, b_{\mathcal{R}'})$ which is well-defined be the property that whenever $X \in (\mathcal{L}')_g$ and $Y \in \mathcal{R}'$ are simple objects such that $X \boxtimes Y \in \mathcal{D}$ then $Y \in (\mathcal{R}')_{\iota(g)}$.

Note that

$$\mathcal{D} \subseteq \oplus_{g \in G_{\mathcal{L}'}} \mathcal{L}_g \boxtimes \mathcal{R}_{\iota(g)}, \tag{21}$$

and hence

$$\dim(\mathcal{D}) \leq \dim(L \cap \mathcal{L}') \dim(\mathcal{R} \cap \mathcal{R}') |G_{\mathcal{L}'}| = \dim(\mathcal{L}') \dim(\mathcal{R} \cap \mathcal{R}').$$

The same inequality holds with $\mathcal{L}$ and $\mathcal{R}$ interchanged. Therefore,

$$\dim(\mathcal{C})^2 = \dim(\mathcal{D})^2 \leq \dim(\mathcal{L}') \dim(\mathcal{L}' \cap \mathcal{L}) \dim(\mathcal{R}') \dim(\mathcal{R} \cap \mathcal{R}') \leq \dim(\mathcal{C})^2.$$

Here the first inequality becomes equality if and only if the inclusion in (21) is an equality and the second inequality becomes equality if and only if $\mathcal{L}' \cap \mathcal{L} = \mathcal{L}$ and $\mathcal{R}' \cap \mathcal{R} = \mathcal{R}$, i.e., when $\mathcal{L}$ and $\mathcal{R}$ are symmetric. □

Remark 4.12. The subcategory $\mathcal{D}_{\mathcal{L},\mathcal{R},\iota}$ constructed in Theorem 4.11 is Lagrangian if and only if $\mathcal{L}$ and $\mathcal{R}$ are isotropic subcategories of $\mathcal{C}$ and ι is an isomorphism of metric groups.

Corollary 4.13. *Let $\mathcal{C}$ be a modular category. The following conditions are equivalent:*

(i) *$\mathcal{C}$ is group-theoretical.*
(ii) *There is a finite group G and a 3-cocycle $\omega \in Z^3(G,\, k^\times)$ such that $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(Vec_G^\omega)$ as a braided fusion category.*
(iii) *$\mathcal{C} \boxtimes \mathcal{C}^{rev}$ contains a Lagrangian subcategory.*
(iv) *There is an isotropic subcategory $\mathcal{E} \subset \mathcal{C}$ such that $(\mathcal{E}')_{ad} \subseteq \mathcal{E}$.*

Proof. The equivalence (i)⇔(ii) is a consequence of [ENO], (ii)⇔(iii) follows from Theorem 4.5, and (iii)⇔(iv) follows from taking $\mathcal{E} = \mathcal{R} = \mathcal{L}$ and $\iota = \mathrm{id}_{G_{\mathcal{E}'}}$ in Theorem 4.11, cf. Remark 4.12. □

Combining the above criterion with Theorem 4.8 we obtain the following useful characterization of group-theoretical modular categories.

Corollary 4.14. *A modular category $\mathcal{C}$ is group-theoretical if and only if simple objects of $\mathcal{C}$ have integral dimension and there is a symmetric subcategory $\mathcal{L} \subset \mathcal{C}$ such that $(\mathcal{L}')_{ad} \subseteq \mathcal{L}$.*

5. Pointed modular categories

In this section we analyze the structure of pointed modular categories, their central charges, and Lagrangian subgroups. Recall that such categories canonically correspond to metric groups [Q].

Let $G = \mathbb{Z}/n\mathbb{Z}$. The corresponding braided categories of the form $\mathcal{C}(\mathbb{Z}/n\mathbb{Z}, q)$ are completely classified by numbers $\sigma = q(1)$ such that $\sigma^n = 1$ (n is odd) or $\sigma^{2n} = 1$ (n is even). Then the braiding of objects corresponding to $0 \le a, b < n$ is the multiplication by σ^{ab} and the twist of the object a is the multiplication by σ^{a^2} (see [Q]). We will denote the category corresponding to σ by $\mathcal{C}(\mathbb{Z}/n\mathbb{Z}, \sigma)$.

Example 5.1. (a) Let $G = \mathbb{Z}/2\mathbb{Z}$. There are 4 possible values of σ: $\pm 1, \pm i$. The categories $\mathcal{C}(\mathbb{Z}/2\mathbb{Z}, \pm i)$ are modular with central charge $\frac{1\pm i}{\sqrt{2}}$ and the categories $\mathcal{C}(\mathbb{Z}/2\mathbb{Z}, \pm 1)$ are symmetric. The category $\mathcal{C}(\mathbb{Z}/2\mathbb{Z}, 1)$ is isotropic and the category $\mathcal{C}(\mathbb{Z}/2\mathbb{Z}, -1)$ is not.

(b) Let $G = \mathbb{Z}/4\mathbb{Z}$. The twist of the object $2 \in \mathbb{Z}/4\mathbb{Z}$ is $\sigma^4 = \pm 1$. If this twist is -1 then σ is a primitive 8th root of 1 and the corresponding category is modular; its Gauss sum is $1 + \sigma + \sigma^4 + \sigma^9 = 2\sigma$ and the central charge is σ. Note that if $\sigma^4 = 1$ then the category $\mathcal{C}(\mathbb{Z}/4\mathbb{Z}, \sigma)$ contains a nontrivial isotropic subcategory.

(c) Let $G = \mathbb{Z}/2^k\mathbb{Z}$ with $k \ge 3$. Since the twist of the object 2^{k-1} is $\sigma^{2^{2k-2}} = 1$, the category $\mathcal{C}(\mathbb{Z}/2^k\mathbb{Z}, \sigma)$ always contains a nontrivial isotropic subcategory.

(d) Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. There are five modular categories with this group. We give for each of them the list of values of q on nontrivial elements of G:

(1) $\mathcal{C}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, i)$: the values of q are $i, i, -1$, and the central charge is i.

(2) $\mathcal{C}(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}, -i)$: the values of q are $-i, -i, -1$, and the central charge is $-i$.
(3) $\mathcal{C}(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}, -1)$: the values of q are $-1, -1, -1$, and the central charge is -1.
(4) $\mathcal{C}(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}, 1)$: the values of q are $i, -i, 1$, and the central charge 1.
(5) The double of $\mathbb{Z}/2\mathbb{Z}$: the values of q are $1, 1, -1$, and the central charge 1.

In this list, each category of central charge 1 contains a nontrivial isotropic subcategory while the others contain a nontrivial symmetric (but not isotropic) subcategory.

(e) Let $G=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$. Assume that the category $\mathcal{C}(G,q)$ does not contain a nontrivial isotropic subcategory. Then $\mathcal{C}(G,q)$ is equivalent to $\mathcal{C}(\mathbb{Z}/4\mathbb{Z},\sigma)\boxtimes\mathcal{C}(\mathbb{Z}/2\mathbb{Z},\pm i)$ where σ is a primitive 8th root of 1. The possible central charges are ± 1 and $\pm i$.

(f) Let $G=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Assume that the category $\mathcal{C}(G,q)$ does not contain a nontrivial isotropic subcategory. Then $\mathcal{C}(G,q)$ is equivalent to $\mathcal{C}(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},\sigma)\boxtimes\mathcal{C}(\mathbb{Z}/2\mathbb{Z},\sigma')$, where $\sigma'=\pm i$ and $\sigma\neq 1,-\sigma'$.

Example 5.2. Let p be an odd prime.

(a) Let $G=\mathbb{Z}/p\mathbb{Z}$. The category $\mathcal{C}(\mathbb{Z}/p\mathbb{Z},\sigma)$ is modular for $\sigma\neq 1$ and is isotropic for $\sigma=1$. The central charge of the modular category $\mathcal{C}(\mathbb{Z}/p\mathbb{Z},\sigma)$ is ± 1 for $p=1 \mod 4$ and $\pm i$ for $p=3 \mod 4$.

(b) Let $G=\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$. There are two modular pointed categories with underlying group G. One has central charge 1 (and is equivalent to the center of $\mathbb{Z}/p\mathbb{Z}$), and the other one has central charge -1.

Recall that for a metric group (G,q) its Gauss sum is $\tau^{\pm}(G,\, q)=\sum_{a\in G} q(a)^{\pm 1}$. A subgroup H of G is called *isotropic* if $q|_H=1$. An isotropic subgroup is called *Lagrangian* if $H^{\perp}=H$.

The following proposition is well known.

Proposition 5.3. *Let $(G,\, q)$ be a non-degenerate metric group such that $|G|=p^{2n}$ where p is a prime number and $n\in\mathbb{Z}^{+}$. Suppose that $\tau^{\pm}(G,q)=\sqrt{|G|}$ (i.e., the central charge of G is 1). Then G contains a Lagrangian subgroup.*

Proof. It suffices to prove that G contains a non-trivial isotropic subgroup H, then one can pass to $H^{\perp}/H$ and use induction.

Assume that p is odd. Assume that G contains a direct summand $\mathbb{Z}/p^k\mathbb{Z}$ with $k>1$. Then the subgroup $\mathbb{Z}/p\mathbb{Z}\subset\mathbb{Z}/p^k\mathbb{Z}$ is isotropic, since otherwise it is a non-degenerate metric subgroup of G and hence can be factored. Thus we are reduced to the case when G is a direct sum of k copies of $\mathbb{Z}/p\mathbb{Z}$. When $k>2$, the quadratic form on G is isotropic (by the Chevalley - Waring theorem). Thus we are reduced to the case $k=2$, which is easy (see Example 5.2 (b)).

Assume now that $p=2$. Again assume that G contains a direct summand $\mathbb{Z}/2^k\mathbb{Z}$ with $k>1$. Again the subgroup $\mathbb{Z}/2\mathbb{Z}\subset\mathbb{Z}/2^k\mathbb{Z}$ is inside its orthogonal complement; moreover it is isotropic if $k\geq 3$. If $k=2$ and the subgroup $\mathbb{Z}/2\mathbb{Z}\subset\mathbb{Z}/4\mathbb{Z}$ is not isotropic then the subgroup $\mathbb{Z}/4\mathbb{Z}$ is a non-degenerate metric subgroup and hence factors out; let $G=G_1\oplus\mathbb{Z}/4\mathbb{Z}$ be the corresponding decomposition of G. If G_1 contains $\mathbb{Z}/2\mathbb{Z}$ such that $\mathbb{Z}/2\mathbb{Z}\subseteq\mathbb{Z}/2\mathbb{Z}^{\perp}$ then we are done: if this subgroup is not isotropic then the diagonal subgroup $\mathbb{Z}/2\mathbb{Z}\subset\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\subset G_1\oplus\mathbb{Z}/4\mathbb{Z}$ is isotropic. Thus G_1 is a sum of $\mathbb{Z}/2\mathbb{Z}$'s and each summand is non-degenerate. But note that the central charge of a non-degenerate metric group $\mathbb{Z}/4\mathbb{Z}$ is a primitive eighth root of 1 (see Example 5.1 (b)) which is also the central charge of a non-degenerate metric $\mathbb{Z}/2\mathbb{Z}$ (see Example 5.1 (a)). This implies that the number of $\mathbb{Z}/2\mathbb{Z}$ summands in G_1 is odd which is impossible since the order of G is a square. Thus we are reduced to the case when G is a sum of k copies of $\mathbb{Z}/2\mathbb{Z}$. In this case all possible values of the quadratic form q are $\pm 1, \pm i$ and since $\tau^+(G,q) = 2^{k/2}$, there is at least one non-identity $a \in G$ with $q(a) = 1$. So the subgroup generated by a is isotropic. The proposition is proved. □

6. Nilpotent modular categories

In this section we prove our main results, stated in 1.1, and derive a few corollaries.

Recall the definitions of $\mathcal{K}_{ad}$ and $\mathcal{K}^{co}$ from 2.5.

Proposition 6.1. *Let $\mathcal{C}$ be a nilpotent modular category. Then for any maximal symmetric subcategory $\mathcal{K}$ of $\mathcal{C}$ one has $(\mathcal{K}')_{ad} \subseteq \mathcal{K}$. Equivalently, there is a grading of $\mathcal{K}'$ such that $\mathcal{K}$ is the trivial component:*

$$\mathcal{K}' = \oplus_{g\in G}\, \mathcal{K}'_g, \qquad \mathcal{K}'_1 = \mathcal{K}. \tag{22}$$

Proof. The two conditions are equivalent since by [GN] the adjoint subcategory is the trivial component of the universal grading.

Let $\mathcal{K}$ be a symmetric subcategory of $\mathcal{C}$, i.e., such that $\mathcal{K} \subseteq \mathcal{K}'$. Assume that $(\mathcal{K}')_{ad}$ is not contained in $\mathcal{K}$. It suffices to show that $\mathcal{K}$ is not maximal.

Let $\mathcal{E} = (\mathcal{K}^{co} \cap (\mathcal{K}')_{ad}) \vee \mathcal{K}$. Clearly, $\mathcal{K} \subseteq \mathcal{E} \subseteq \mathcal{K}'$. We have

$$\begin{aligned} \mathcal{E}' &= ((\mathcal{K}^{co} \cap (\mathcal{K}')_{ad}) \vee \mathcal{K})' \\ &= \mathcal{K}' \cap ((\mathcal{K}^{co})' \vee ((\mathcal{K}')_{ad})') \\ &= \mathcal{K}' \cap ((\mathcal{K}')_{ad} \vee \mathcal{K}^{co}) \\ &= (\mathcal{K}' \cap \mathcal{K}^{co}) \vee (\mathcal{K}')_{ad}, \end{aligned}$$

where we used the modular law of the lattice $L(\mathcal{C})$ from Lemma 3.6. Since $\mathcal{K} \subseteq \mathcal{K}' \cap \mathcal{K}^{co}$ and $\mathcal{K}^{co} \cap (\mathcal{K}')_{ad} \subseteq (\mathcal{K}')_{ad}$ we see that $\mathcal{E} \subseteq \mathcal{E}'$, i.e., $\mathcal{E}$ is symmetric.

Let n be the largest positive integer such that $(\mathcal{K}')^{(n)} \not\subseteq \mathcal{K}$. Such n exists by our assumption and the nilpotency of $\mathcal{K}'$. We claim that $(\mathcal{K}')^{(n)} \subseteq \mathcal{K}^{co}$. Indeed,

$$(\mathcal{K}')^{(n)} \subseteq ((\mathcal{K}')^{(n+1)})^{co} \subseteq \mathcal{K}^{co}$$

since $\mathcal{D} \subseteq (\mathcal{D}_{ad})^{co}$ for every subcategory $\mathcal{D} \subseteq \mathcal{C}$. Therefore, $\mathcal{K}^{co} \cap (\mathcal{K}')^{(n)} = (\mathcal{K}')^{(n)}$ is not contained in $\mathcal{K}$ and

$$\mathcal{K} \subsetneq (\mathcal{K}^{co} \cap (\mathcal{K}')^{(n)}) \vee \mathcal{K} \subseteq (\mathcal{K}^{co} \cap (\mathcal{K}')_{ad}) \vee \mathcal{K} = \mathcal{E},$$

which completes the proof. □

Recall that in a fusion category whose dimension is an *odd* integer the dimensions of all objects are automatically integers [GN, Corollary 3.11].

Corollary 6.2. *A nilpotent modular category $\mathcal{C}$ with integral dimensions of simple objects is group-theoretical.*

Proof. This follows immediately from Corollary 4.14 and Proposition 6.1. □

Remark 6.3. It follows from Corollary 4.13 that a nilpotent modular category $\mathcal{C}$ with integral dimensions of simple objects contains an isotropic subcategory $\mathcal{E}$ such that $(\mathcal{E}')_{ad} \subseteq \mathcal{E}$. The corresponding grading

$$\mathcal{E}' = \oplus_{h\in H}\, \mathcal{E}'_h, \qquad \mathcal{E}'_1 = \mathcal{E}, \tag{23}$$

gives rise to a non-degenerate quadratic form q on H defined by $q(h) = \theta_V$ for any non-zero $V \in \mathcal{C}_h$. We have a braided equivalence $\bar{\mathcal{E}}' \cong \mathcal{C}(H, q)$.

We may assume that $\mathcal{E}$ is maximal among isotropic subcategories of $\mathcal{C}$. In this case, Proposition 3.9 implies that the isomorphism class of the above metric group (H, q) does not depend on the choice of the maximal isotropic subcategory $\mathcal{E}$.

Corollary 6.4. *The central charge of a modular nilpotent category with integer dimensions of objects is always an* 8*th root of* 1*. Moreover, the central charge of a modular $p-$category is ± 1 if $p = 1 \bmod 4$ and ± 1, $\pm i$ if $p = 3 \bmod 4$. The central charge of a modular $p-$category of dimension p^{2k}, $k \in \mathbb{Z}^+$ with odd p is ± 1.*

Proof. By Remark 6.3 and Theorem 3.4 the central charge always equals the central charge of some pointed category, so the first claim follows from Examples 5.1-5.2. The second and third claims follow from Example 5.2. □

Theorem 6.5. *Let $\mathcal{C}$ be a modular category with integral dimensions of simple objects. Then $\mathcal{C}$ is nilpotent if and only if there exists a pointed modular category $\mathcal{M}$ such that $\mathcal{C} \boxtimes \mathcal{M}$ is equivalent (as a braided fusion category) to $\mathcal{Z}(Vec_G^\omega)$, where G is a nilpotent group.*

Proof. Note that for a nilpotent group G the category $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ is a tensor product of modular p-categories and, hence, is nilpotent. So if $\mathcal{C} \boxtimes \mathcal{M} \cong \mathcal{Z}(\mathrm{Vec}_G^\omega)$ then $\mathcal{C}$ is nilpotent (as a subcategory of a nilpotent category).

Let us prove the converse implication. Pick an isotropic subcategory $\mathcal{E} \subset \mathcal{C}$ such that $(\mathcal{E}')_{ad} \subseteq \mathcal{E}$ (such a subcategory exists by Remark 6.3). There is a metric group (H, q) such that $\bar{\mathcal{E}}' \cong \mathcal{C}(H, q)$. Let $\mathcal{E}' = \oplus_{h\in H}\, \mathcal{E}'_h$, where $\mathcal{E}_1 = \mathcal{E}$ be the corresponding grading from (23).

Let $\mathcal{M}$ be the reversed category of $\bar{\mathcal{E}}'$ (i.e., with the opposite braiding and twist). Then $\mathcal{M} \cong \mathcal{C}(H, q^{-1})$ and $\xi(\mathcal{M}) = \xi(\mathcal{C}(H, q))^{-1} = \xi(\mathcal{C})^{-1}$ by Theorem 3.4.

The modular category $\mathcal{C}_{new} = \mathcal{C} \boxtimes \mathcal{M}$ is nilpotent and $\xi(\mathcal{C}_{new}) = 1$. The category $\mathcal{E}_{new} := \oplus_{h\in H}\, \mathcal{E}_h \boxtimes h$ is a Lagrangian subcategory of $\mathcal{C}_{new}$ and the required statement follows from Theorem 4.5. □

Let p be a prime number.

Theorem 6.6. *A modular category $\mathcal{C}$ is equivalent to the center of a fusion category of the form Vec_G^ω with G being a p-group if and only if it has the following properties:*

(i) *the Frobenius-Perron dimension of $\mathcal{C}$ is p^{2n} for some $n \in \mathbb{Z}^+$,*
(ii) *the dimension of every simple object of $\mathcal{C}$ is an integer,*
(iii) *the multiplicative central charge of $\mathcal{C}$ is* 1.

Proof. It is clear that for any finite p-group G and $\omega \in Z^3(G, k^\times)$ the modular category $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ satisfies properties (i) and (ii). The central charge of $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ equals 1 by [Mu4, Theorem 1.2].

Let us prove the converse. Suppose that $\mathcal{C}$ satisfies conditions (i), (ii), and (iii). Let $\mathcal{E}$ be an isotropic subcategory of $\mathcal{C}$ such that $(\mathcal{E}')_{ad} \subseteq \mathcal{E}$ (such an $\mathcal{E}$ exists by Remark 6.3). There is a grading $\mathcal{E}' = \oplus_{h\in H}\, \mathcal{E}'_h$ with $\mathcal{E}'_1 = \mathcal{E}$ and θ being constant on each $\mathcal{E}'_h$, $h \in H$. Note that H is a metric p-group whose order is a square. By Proposition 5.3 it contains a Lagrangian subgroup H_0, whence $\oplus_{h\in H_0} \mathcal{E}'_h$ is a Lagrangian subcategory of $\mathcal{C}$.

Thus, $\mathcal{C} \cong \mathrm{Vec}_G^\omega$ for some G and ω by Theorem 4.5. Since $|G|^2 = \dim(\mathrm{Vec}_G^\omega) = \dim(\mathcal{C})$ it follows that G is a p-group. $\square$

Finally, we apply our results to show that certain fusion categories (more precisely, representation categories of certain semisimple quasi-Hopf algebras) are group-theoretical and to obtain a categorical analogue of the Sylow decomposition of nilpotent groups.

Corollary 6.7. *Let $\mathcal{C}$ be a fusion category with integral dimensions of simple objects and such that $\mathcal{Z}(\mathcal{C})$ is nilpotent. Then $\mathcal{C}$ is group-theoretical.*

Proof. By Corollary 6.2 the category $\mathcal{Z}(\mathcal{C})$ is group-theoretical. Hence, $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ is group theoretical (as a dual category of $\mathcal{Z}(\mathcal{C})$, see [ENO]). Therefore, $\mathcal{C}$ is group-theoretical (as a fusion subcategory of $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$). $\square$

Corollary 6.8. *Let $\mathcal{C}$ be a fusion category of dimension p^n, $n \in \mathbb{Z}^+$, such that all objects of $\mathcal{C}$ have integer dimension (this is automatic if $p > 2$). Then $\mathcal{C}$ is group-theoretical.*

In other words, semisimple quasi-Hopf algebras of dimension p^n are group-theoretical.

Remark 6.9. Semisimple Hopf algebras of dimension p^n were studied by several authors, see e.g., [EG2], [Kash], [Ma1], [Ma2], [MW], [Z].

From Corollary 6.2 we obtain the following Sylow decomposition.

Theorem 6.10. *Let $\mathcal{C}$ be a braided nilpotent fusion category such that all objects of $\mathcal{C}$ have integer dimension. Then $\mathcal{C}$ is group-theoretical and has a decomposition into a tensor product of braided fusion categories of prime power dimension. If the factors are chosen in such a way that their dimensions are relatively prime, then such a decomposition is unique up to a permutation of factors.*

Proof. It was shown in [GN, Theorem 6.11] that the center of a braided nilpotent fusion category is nilpotent. Hence, $\mathcal{Z}(\mathcal{C})$ is group-theoretical by Corollary 6.2. Since $\mathcal{C}$ is equivalent to a subcategory of $\mathcal{Z}(\mathcal{C})$, it is group-theoretical by [ENO, Proposition 8.44]. This means that there is a group G and $\omega \in Z^3(G, k^*)$ such that $\mathcal{C}$ is dual to Vec_G^ω with respect to some indecomposable module category. The group G is necessarily nilpotent since $\mathrm{Rep}(G) \subseteq \mathcal{Z}(\mathrm{Vec}_G^\omega) \cong \mathcal{Z}(\mathcal{C})$. Hence, G is isomorphic to a direct product of its Sylow p-subgroups, $G = G_1 \times \cdots \times G_n$, and so Vec_G^ω is equivalent to a tensor product of p-categories. It follows from [ENO, Proposition 8.55] that the dual category $\mathcal{C}$ is also a product of fusion p-categories, as desired.

Now suppose that $\mathcal{C}$ is decomposed into factors of prime power Frobenius-Perron dimension, $\mathcal{C} \simeq \boxtimes_p \mathcal{C}_p$. It is easy to see that the objects from $\mathcal{C}_p \subset \mathcal{C}$ are characterized by the following property:

$$X \in \mathcal{C}_p \text{ if and only if there exists } k \in \mathbb{Z}^+ \text{ such that } \mathrm{Hom}(\mathbf{1}, X^{\otimes^{p^k}}) \neq 0. \tag{24}$$

This shows that the decomposition in question is unique. $\square$

Remark 6.11. Let $\mathcal{C}$ be a nilpotent modular category with integral dimensions of simple objects. We already mentioned in the introduction that the choice of a tensor complement $\mathcal{M}$ satisfying $\mathcal{C} \boxtimes \mathcal{M} \cong \mathrm{Vec}_G^\omega$ is not unique. In the proof of Theorem 6.5 such $\mathcal{M}$ can be chosen canonically as the category opposite to the canonical modularization corresponding to a maximal isotropic subcategory of $\mathcal{C}$, see Proposition 3.9.

Another canonical way is to choose an $\mathcal{M}$ of minimal possible dimension. This is done as follows. By Theorem 6.10, we have $\mathcal{C} = \boxtimes_p \mathcal{C}_p$ and $\mathcal{M} = \boxtimes_p \mathcal{M}_p$, where $\mathcal{C}_p, \mathcal{M}_p$ are modular p-categories. By Theorem 6.6, $\mathcal{M}_p$ has to be chosen in such a way that $\dim(\mathcal{C}_p)\dim(\mathcal{M}_p)$ is a square and $\xi(\mathcal{M}_p) = \xi(\mathcal{M}_p)^{-1}$. It follows from Examples 5.1, 5.2 and Corollary 6.4 that there is a unique such choice of $\mathcal{M}_p$ with minimal $\dim(\mathcal{M}_p)$, in which case $\dim(\mathcal{M}_p) \in \{1, p, p^2\}$ for odd p and $\dim(\mathcal{M}_2) \in \{1, 2, 4, 8\}$.

Theorem 6.12. *Let $\mathcal{C}$ be a braided nilpotent fusion category. Then $\mathcal{C}$ has a unique decomposition into a tensor product of braided fusion categories of prime power dimension.*

Proof. According to Theorem 6.10 the result is true if the dimensions of simple objects of $\mathcal{C}$ are integers. In general, define subcategories $\mathcal{C}_p \subset \mathcal{C}$ by condition (24) above. For a simple object $X \in \mathcal{C}$ it is known (see [GN]) that $\mathrm{FPdim}(X) = \sqrt{N}$, $N \in \mathbb{N}$. Thus $X \boxtimes X \in \mathcal{C} \boxtimes \mathcal{C}$ has an integer dimension. The category $\mathcal{C} \boxtimes \mathcal{C}$ contains a fusion subcategory $(\mathcal{C} \boxtimes \mathcal{C})^{int}$ consisting of all objects with integer dimension, see [GN]. We can apply Theorem 6.10 to the category $(\mathcal{C} \boxtimes \mathcal{C})^{int}$ and obtain a unique decomposition $X = \otimes_p X_p$ with $X_p \in \mathcal{C}_p$. The theorem is proved. □

Corollary 6.13. *Let $\mathcal{C}$ be a braided nilpotent fusion category. Assume that $X \in \mathcal{C}$ is simple and its dimension is not integer. Then FPdim$(X) \in \sqrt{2}\mathbb{Z}$.*

Proof. This follows immediately from Theorem 6.12 since if a category of prime power Frobenius-Perron dimension p^k contains an object of a non-integer dimension then $p = 2$, see [ENO]. □

Example 6.14. It is easy to see that the Tambara-Yamagami categories from [TY] are nilpotent and indecomposable into a tensor product. Thus Theorem 6.12 implies that if such a category admits a braiding, then its dimension should be a power of 2 (since the dimension of a Tambara-Yamagami category is always divisible by 2). A stronger result is contained in [S].

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