The birth of string theory

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Summary. In this contribution we go through the developments that in the years from 1968 to about 1974 led from the Veneziano model to the bosonic string theory. They include the construction of the N-point amplitude for scalar particles, its factorization through the introduction of an infinite number of oscillators and the proof that the physical subspace was a positive definite Hilbert space. We also discuss the zero slope limit and the calculation of loop diagrams. Lastly, we describe how it finally was recognized that a quantum relativistic string theory was the theory underlying the Veneziano model.

1 Introduction

The sixties was a period in which strong interacting processes were studied in detail using the newly constructed accelerators at Cern and other places. Many new hadronic states were found that appeared as resonant peaks in various cross sections and hadronic cross sections were measured with increasing accuracy. In general, the experimental data for strongly interacting processes were rather well understood in terms of resonance exchanges in the direct channel at low energy and by the exchange of Regge poles in the transverse channel at higher energy. Field theory that had been very successful in describing QED seemed useless for strong interactions given the big number of hadrons to accomodate in a Lagrangian and the strength of the pion-nucleon coupling constant that did not allow perturbative calculations. The only domain in which field theoretical techniques were successfully used was current algebra. Here, assuming that strong interactions were described by an almost chiral invariant Lagrangian, that chiral symmetry was spontaneously broken and that the pion was the corresponding Goldstone boson, field theoretical methods gave rather good predictions for scattering amplitudes involving pions at very low energy. Going to higher energy was, however, not possible with these methods.

Because of this, many people started to think that field theory was useless to describe strong interactions and tried to describe strong interacting processes with alternative and more phenomenological methods. The basic ingredients for describing the experimental data were at low energy the exchange of resonances in the direct channel and at higher energy the exchange of Regge poles in the transverse channel. Sum rules for strongly interacting processes were saturated in this way and one found good agreement with the experimental data that came from the newly constructed accelerators. Because of these successes and of the problems that field theory encountered to describe the data, it was proposed to construct directly the S matrix without passing through a Lagrangian. The S matrix was supposed to be constructed from the properties that it should satisfy, but there was no clear procedure on how to implement this construction[1]. The word "bootstrap" was often used as the way to construct the S matrix, but it did not help very much to get an S matrix for the strongly interacting processes.

One of the basic ideas that led to the construction of an S matrix was that it should include resonances at low energy and at the same time give Regge behaviour at high energy. But the two contributions of the resonances and of the Regge poles should not be added because this would imply double counting. This was called Dolen, Horn and Schmidt duality [2]. Another idea that helped in the construction of an S matrix was planar duality [3] that was visualized by associating to a certain process a duality diagram, shown in Fig. (1), where each meson was described by two lines representing the quark and the antiquark. Finally, also the requirement of crossing symmetry played a very important role.

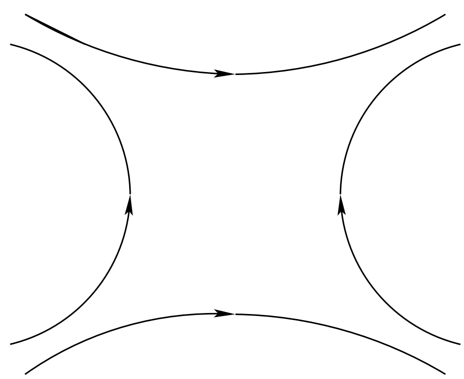

Fig. 1. Duality diagram for the scattering of four mesons

Starting from these ideas Veneziano [4] was able to construct an S matrix for the scattering of four mesons that, at the same time, had an infinite number of zero width resonances lying on linearly rising Regge trajectories and Regge behaviour at high energy. Veneziano originally constructed the model for the

[1] For a discussion of S matrix theory see Ref.s [1]

process $\pi\pi \rightarrow \pi\omega$, but it was immediately extended to the scattering of four scalar particles.

In the case of four identical scalar particles, the crossing symmetric scattering amplitude found by Veneziano consists of a sum of three terms:

$$A(s,t,u) = A(s,t) + A(s,u) + A(t,u) \tag{1}$$

where

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} = \int_0^1 dx x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1} \tag{2}$$

with linearly rising Regge trajectories

$$\alpha(s) = \alpha_0 + \alpha' s \tag{3}$$

This was a very important property to implement in a model because it was in agreement with the experimental data in a wide range of energies. s, t and u are the Mandelstam variables:

$$s = -(p_1+p_2)^2 \quad , \quad t = -(p_3+p_2)^2 \quad , \quad u = -(p_1+p_3)^2 \tag{4}$$

The three terms in Eq. (1) correspond to the three orderings of the four particles that are not related by a cyclic or anticyclic [2] permutation of the external legs. They correspond, respectively, to the three permutations: $(1234), (1243)$ and (1324) of the four external legs. They have only simple pole singularities. The first one has only poles in the s and t channels, the second only in the s and u channels and the third only in the t and u channels. This property follows directly from the duality diagram that is associated to each inequivalent permutation of the external legs. In fact, at that time one used to associate to each of the three inequivalent permutations a duality diagram where each particle was drawn as consisting of two lines that rappresented the quark and antiquark making up a meson. Furthermore, the diagram was supposed to have only poles singularities in the planar channels which are those involving adjacent external lines. This means that, for instance, the duality diagram corresponding to the permutation (1234) has only poles in the s and t channels as one can see by deforming the diagram in the plane in the two possible ways shown in figure (2).

This was a very important property of the duality diagram that makes it qualitatively different from a Feynman diagram in field theory where each diagram has only a pole in one of the three s, t and u channels and not simultaneously in two of them. If we accept the idea that each term of the sum in Eq. (1) is described by a duality diagram, then it is clear that we

[2] An anticyclic permutation corresponding, for instance, to the ordering (1234) is obtained by taking the reverse of the original ordering (4321) and then performing a cyclic permutation.

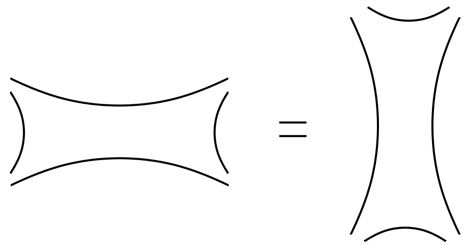

Fig. 2. The duality diagram contains both s and t channel poles

do not need to add terms corresponding to equivalent diagrams because the corresponding duality diagram is the same and has the same singularities. It is now clear that it was in some way implicit in this picture the fact that the Veneziano model corresponds to the scattering of relativistic strings. But at that time the connection was not obvious at all. The only S matrix property that the Veneziano model failed to satisfy was the unitarity of the S matrix. because it contained only zero width resonances and did not have the various cuts required by unitarity. We will see how this property will be implemented.

Immediately after the formulation of the Veneziano model, Virasoro [5] proposed another crossing symmetric four-point amplitude for scalar particles that consisted of a unique piece given by:

$$A(s,t,u) \sim \frac{\Gamma(-\frac{\alpha(u)}{2})\Gamma(-\frac{\alpha(s)}{2})\Gamma(-\frac{\alpha(t)}{2})}{\Gamma(1+\frac{\alpha(u)}{2})\Gamma(1+\frac{\alpha(s)}{2})\Gamma(1+\frac{\alpha(t)}{2})} \tag{5}$$

where

$$\alpha(s) = \alpha_0 + \alpha' s \tag{6}$$

The model had poles in all three s, t and u channels and could not be written as sum of three terms having poles only in planar diagrams. In conclusion, the Veneziano model satisfies the principle of planar duality being a crossing symmetric combination of three contributions each having poles only in the planar channels. On the other hand, the Virasoro model consists of a unique crossing symmetric term having poles in both planar and non-planar channels.

The attempts to construct consistent models that were in good agreement with the strong interaction phenomenology of the sixties boosted enormously the activity in this research field. The generalization of the Veneziano model to the scattering of N scalar particles was built, an operator formalism consisting of an infinite number of harmonic oscillators was constructed and the complete spectrum of mesons was determined. It turned out that the degeneracy of states grew up exponentially with the mass. It was also found that the N point amplitude had states with negative norm (ghosts) unless the intercept of the Regge trajectory was $\alpha_0 = 1$ [6]. In this case it turned out that the model was free of ghosts but the lowest state was a tachyon. The model was called in the literature the “dual resonance model”.

The model was not unitary because all the states were zero width resonances and the various cuts required by unitarity were absent. The unitarity was implemented in a perturbative way by adding loop diagrams obtained by sewing some of the external legs together after the insertion of a propagator. The multiloop amplitudes showed a structure of Riemann surfaces. This became obvious only later when the dual resonance model was recognized to correspond to scattering of strings.

But the main problem was that the model had a tachyon if $\alpha_0 = 1$ or had ghosts for other values of α_0 and was not in agreement with the experimental data: α_0 was not equal to about $\frac{1}{2}$ as required by experiments for the ρ Regge trajectory and the external scalar particles did not behave as pions satisfying the current algebra requirements. Many attempts were made to construct more realistic dual resonance models, but the main result of these attempts was the construction of the Neveu-Schwarz [7] and the Ramond [8] models, respectively, for mesons and fermions. They were constructed as two independent models and only later were recognized to be two sectors of the same model. The Neveu-Schwarz model still contained a tachyon that only in 1976 through the GSO projection was eliminated from the physical spectrum. Furthermore, it was not properly describing the properties of the physical pions.

Actually a model describing $\pi\pi$ scattering in a rather satisfactory way was proposed by Lovelace and Shapiro [9] [3]. According to this model the three isospin amplitudes for pion-pion scattering are given by:

$$A^0 = \frac{3}{2}\left[A(s,t) + A(s,u)\right] - \frac{1}{2}A(t,u)$$

$$A^1 = A(s,t) - A(s,u) \qquad\qquad A^2 = A(t,u) \tag{7}$$

where

$$A(s,t) = \beta \frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(t)-\alpha(s))} \qquad ; \qquad \alpha(s) = \alpha_0 + \alpha' s \tag{8}$$

The amplitudes in eq.(7) provide a model for $\pi\pi$ scattering with linearly rising Regge trajectories containing three parameters: the intercept of the ρ Regge trajectory α_0, the Regge slope α' and β. The first two can be determined by imposing the Adler's self-consistency condition, that requires the vanishing of the amplitude when $s = t = u = m_\pi^2$ and one of the pions is massless, and the fact that the Regge trajectory must give the spin of the ρ meson that is equal to 1 when $\sqrt{s}$ is equal to the mass of the ρ meson m_ρ. These two conditions determine the Regge trajectory to be:

$$\alpha(s) = \frac{1}{2}\left[1 + \frac{s - m_\pi^2}{m_\rho^2 - m_{\pi^2}}\right] = 0.48 + 0.885s \tag{9}$$

[3] See also Ref. [10].

Having fixed the parameters of the Regge trajectory the model predicts the masses and the couplings of the resonances that decay in $\pi\pi$ in terms of a unique parameter β. The values obtained are in reasonable agreement with the experiments. Moreover, one can compute the $\pi\pi$ scattering lenghts:

$$a_0 = 0.395\beta \qquad\qquad a_2 = -0.103\beta \tag{10}$$

and one finds that their ratio is within 10% of the current algebra ratio given by $a_0/a_2 = -7/2$. The amplitude in eq.(8) has exactly the same form as that for four tachyons of the Neveu-Schwarz model with the only apparently minor difference that $\alpha_0 = 1/2$ (for $m_\pi = 0$) instead of 1 as in the Neveu-Schwarz model. This difference, however, implies that the critical space-time dimension of this model is $d = 4$ [4] and not $d = 10$ as in the Neveu-Schwarz model. In conclusion this model seems to be a perfectly reasonable model for describing low-energy $\pi\pi$ scattering. The problem is, however, that nobody has been able to generalize it to the multipion scattering and therefore to get the complete meson spectrum.

As we have seen the S matrix of the dual resonance model was constructed using ideas and tools of hadron phenomenology of the end of the sixties. Although it did not seem possible to write a realistic dual resonance model describing the pions , it was nevertheless such a source of fascination for those who actively worked in this field at that time for its beautiful internal structure and consistency that a lot of energy was used to investigate its properties and for understanding its basic structure. It turned out with great surprise that the underlying structure was that of a quantum relativistic string.

The aim of this contribution is to explain the logic of the work that was done in the years from 1968 to 1974 [5] in order to uncover the deep properties of this model that appeared from the beginning to be so beautiful and consistent to deserve an intensive study.

This seems to me a very good way of celebrating the 65th anniversary of Gabriele who is the person who started and also contributed to develop the whole thing with his deep physical intuition.

2 Construction of the N-point amplitude

We have seen that the construction of the four-point amplitude is not sufficient to get information on the full hadronic spectrum because it contains only those hadrons that couple to two ground state mesons and does not see those intermediate states which only couple to three or to an higher number of ground state mesons [12]. Therefore, it was very important to construct the N-point amplitude involving identical scalar particles. The construction of

[4] This can be checked by computing the coupling of the spinless particle at the level $\alpha(s) = 2$ and seeing that it vanishes for $d = 4$.

[5] Reviews from this period can be found in Ref. [11]

the N-point amplitude was done in Ref. [13] (extending the work of Ref. [14]) by requiring the same principles that have led to the construction of the Veneziano model, namely the fact that the axioms of S-matrix theory be satisfied by an infinite number of zero width resonances lying on linearly rising Regge trajectories and planar duality.

The fully crossing symmetric scattering amplitude of N identical scalar particles is given by a sum of terms corresponding to the inequivalent permutations of the external legs:

$$A = \sum_{n=1}^{N_p} A_n \tag{11}$$

Also in this case two permutations of the external legs are inequivalent if they are not related by a cyclic or anticyclic permutation. N_p is the number of inequivalent permutations of the external legs and is equal to $N_p = \frac{(N-1)!}{2}$ and each term has only simple pole singularities in the planar channels. Each planar channel is described by two indices (i,j), to mean that it includes the legs $i, i+1, i+2 \ldots j-1, j$, by the Mandelstam variable

$$s_{ij} = -(p_i + p_{i+1} + \ldots + p_j)^2 \tag{12}$$

and by an additional variable u_{ij} whose role will become clear soon. It is clear that the channels (ij) and $(j+1, i-1)$ [6] are identical and they should be counted only once. In the case of N identical scalar particles the number of planar channels is equal to $\frac{N(N-3)}{2}$. This can be obtained as follows. The independent planar diagrams involving the particle 1 are of the type $(1,i)$ where $i = 2 \ldots N-2$. Their number is $N-3$. This is also the number of planar diagrams involving the particle 2 and not the 1. The number of planar diagrams involving the particle 3 and not the particles 1 and 2 is equal to $N-4$. In general the number of planar diagrams involving the particle i and not the previous ones from 1 to i-1 is equal to $N-1-i$. This means that the total number of planar diagram is equal to:

$$2(N-3) + \sum_{i=3}^{N-2} (N-1-i) = 2(N-3) + \sum_{i=1}^{N-4} i =$$

$$= 2(N-3) + \frac{(N-4)(N-3)}{2} = \frac{N(N-3)}{2} \tag{13}$$

If one writes down the duality diagram corresponding to a certain planar ordering of the external particles, it is easy to see that the diagram can have simultaneous pole singularities only in $N-3$ channels. The channels that allow simultaneous pole singularities are called compatible channels, the other

[6] This channel includes the particles $(j+1, \ldots, N, 1, \ldots i-1)$.

are called incompatible. Two channels (i,j) and (h,k) are incompatible if the following inequalities are satisfied:

$$i \leq h \leq j \quad ; \quad j+1 \leq k \leq i-1 \tag{14}$$

The aim is to construct the scattering amplitude for each inequivalent permutation of the external legs that has only pole singularities in the $\frac{N(N-3)}{2}$ planar channels. We have also to impose that the amplitude has simultaneous poles only in $N-3$ compatible channels. In order to gain intuition on how to proceed we rewrite the four-point amplitude in Eq. (2) as follows:

$$A(s,t) = \int_0^1 du_{12} \int_0^1 du_{23}\; u_{12}^{-\alpha(s_{12})-1} u_{23}^{-\alpha(s_{23})-1} \delta(u_{12}+u_{23}-1) \tag{15}$$

where u_{12} and u_{23} are the variables corresponding to the two planar channels (12) and (23) and the cancellation of simultaneous poles in incompatible channels is provided by the δ-function which forbids u_{12} and u_{23} to vanish simultaneously.

We will now extend this procedure to the N-point amplitude. But for the sake of clarity let us start with the case of $N=5$ [14]. In this case we have 5 planar channels described by $u_{12}, u_{13}, u_{23}, u_{24}$ and u_{34}. Since we have only two compatible channels only two of the previous five variables are independent. We can choose them to be u_{12} and u_{13}. In order to determine the dependence of the other three variables on the two independent ones, we exclude simultaneous poles in incompatible channels. This can be done by imposing relations that prevent variables corresponding to incompatible channels to vanish simultaneously. A sufficient condition for excluding simultaneous poles in incompatible channels is to impose the conditions:

$$u_P = 1 - \prod_{\bar{P}} u_{\bar{P}} \tag{16}$$

where the product is over the variables $\bar{P}$ corresponding to channels that are incompatible with P. In the case of the five-point amplitude we get the following relations:

$$u_{23} = 1 - u_{34}u_{12} \quad ; \quad u_{24} = 1 - u_{13}u_{12}$$

$$u_{13} = 1 - u_{34}u_{24} \quad ; \quad u_{34} = 1 - u_{23}u_{13} \quad ; \quad u_{12} = 1 - u_{24}u_{23} \tag{17}$$

Solving them in terms of the two independent ones we get:

$$u_{23} = \frac{1-u_{12}}{1-u_{12}u_{13}} \quad ; \quad u_{34} = \frac{1-u_{13}}{1-u_{12}u_{13}} \quad ; \quad u_{24} = 1 - u_{12}u_{13} \tag{18}$$

In analogy with what we have done for the four-point amplitude in Eq. (15) we write the five-point amplitude as follows:

$$\int_0^1 du_{12} \int_0^1 du_{13} \int_0^1 du_{23} \int_0^1 du_{24} \int_0^1 du_{34} u_{12}^{-\alpha(s_{12})-1} u_{13}^{-\alpha(s_{13})-1} \times$$

$$\times u_{24}^{-\alpha(s_{24})-1} u_{23}^{-\alpha(s_{23})-1} u_{34}^{-\alpha(s_{34})-1} \times$$

$$\delta(u_{23} + u_{12}u_{34} - 1)\delta(u_{24} + u_{12}u_{13} - 1)\delta(u_{34} + u_{13}u_{23} - 1) \tag{19}$$

Performing the integral over the variables u_{23}, u_{24} and u_{34} we get:

$$\int_0^1 du_{12} \int_0^1 du_{13} u_{12}^{-\alpha(s_{12})-1} u_{13}^{-\alpha(s_{13})-1} \times$$

$$\times (1 - u_{12})^{-\alpha(s_{23})-1} (1 - u_{13})^{-\alpha(s_{13})-1} (1 - u_{12}u_{13})^{-\alpha(s_{24})+\alpha(s_{23})+\alpha(s_{34})} \tag{20}$$

We have implicitly assumed that the Regge trajectory is the same in all channels and that the external scalar particles have the same common mass m and are the lowest lying states on the Regge trajectory. This means that their mass is given by:

$$\alpha_0 - \alpha' p_i^2 = 0 \;\; ; \;\; p_i^2 \equiv -m^2 \tag{21}$$

Using then the relation:

$$\alpha(s_{23}) + \alpha(s_{34}) - \alpha(s_{24}) = 2\alpha' p_2 \cdot p_4 \tag{22}$$

we can rewrite Eq. (20) as follows:

$$B_5 = \int_0^1 du_2 \int_0^1 du_3 u_2^{-\alpha(s_2)-1} u_3^{-\alpha(s_3)-1} (1 - u_2)^{-\alpha(s_{23})-1} \times$$

$$\times (1 - u_3)^{-\alpha(s_{34})-1} \prod_{i=2}^{2} \prod_{j=4}^{4} (1 - x_{ij})^{2\alpha' p_i \cdot p_j} \tag{23}$$

where

$$s_i \equiv s_{1i} \;\; , \;\; u_i \equiv u_{1i} \; ; \; i = 2, 3 \;\; ; \;\; x_{ij} = u_i u_{i+1} \ldots u_{j-1}. \tag{24}$$

We are now ready to construct the N-point function [13]. In analogy with what has been done for the four and five-point amplitudes we can write the N-point amplitude as follows:

$$B_N = \int_0^1 \ldots \int_0^1 \prod_P [u_P^{-\alpha(s_P)-1}] \prod_Q \delta(u_Q - 1 + \prod_{\bar{Q}} u_{\bar{Q}}) \tag{25}$$

where the first product is over the $\frac{N(N-3)}{2}$ variables corresponding to all planar channels, while the second one is over the $\frac{(N-3)(N-2)}{2}$ independent δ-functions. The product in the δ-function is defined in Eq. (16).

The solution of all the non-independent linear relations imposed by the δ-functions is given by

$$u_{ij} = \frac{(1-x_{ij})(1-x_{i-1,j+1})}{(1-x_{i-1,j})(1-x_{i,j+1})} \tag{26}$$

where the variables x_{ij} are given in Eq. (24). Eliminating the δ-function from Eq. (25) one gets:

$$B_N = \prod_{i=2}^{N-2} \left[\int_0^1 du_i u_i^{-\alpha(s_i)-1} (1-u_i)^{-\alpha(s_{i,i+1})-1} \right] \prod_{i=2}^{N-3} \prod_{j=i+2}^{N-1} (1-x_{ij})^{-\gamma_{ij}} \tag{27}$$

where

$$\gamma_{ij} = \alpha(s_{ij}) + \alpha(s_{i+1;j-1}) - \alpha(s_{i;j-1}) - \alpha(s_{i+1;j}) \quad ; \quad j \geq i+2 \tag{28}$$

It is easy to see that

$$\alpha(s_{i,i+1}) = -\alpha_0 - 2\alpha' p_i \cdot p_{i+1} \quad ; \quad \gamma_{ij} = -2\alpha' p_i \cdot p_j \quad ; \quad j \geq i+2 \tag{29}$$

Inserting them in Eq. (27) we get:

$$B_N = \prod_{i=2}^{N-2} \left[\int_0^1 du_i u_i^{-\alpha(s_i)-1} (1-u_i)^{\alpha_0 - 1} \right] \prod_{i=2}^{N-2} \prod_{j=i+1}^{N-1} (1-x_{ij})^{2\alpha' p_i \cdot p_j} \tag{30}$$

This is the form of the N-point amplitude that was originally constructed. Then Koba and Nielsen [15] put it in the form that is more known nowadays. They constructed it using the following rules. They associated a real variable z_i to each leg i. Then they associated to each channel (i,j) an anharmonic ratio constructed from the variables $z_i, z_{i-1}, z_j, z_{j+1}$ in the following way

$$(z_i, z_{i+1}, z_j, z_{j+1})^{-\alpha(s_{ij})-1} = \left[\frac{(z_i - z_j)(z_{i-1} - z_{j+1})}{(z_{i-1} - z_j)(z_i - z_{j+1})} \right]^{-\alpha(s_{ij})-1} \tag{31}$$

and finally they gave the following expression for the N-point amplitude:

$$B_N = \int_{-\infty}^{\infty} dV(z) \prod_{(i,j)} (z_i, z_{i+1}, z_j, z_{j+1})^{-\alpha(s_{ij})-1} \tag{32}$$

where

$$dV(z) = \frac{\prod_1^N [\theta(z_i - z_{i+1}) dz_i]}{\prod_{i=1}^N (z_i - z_{i+2}) dV_{abc}} \; ; \; dV_{abc} = \frac{dz_a dz_b dz_c}{(z_b - z_a)(z_c - z_b)(z_a - z_c)} \tag{33}$$

and the variables z_i are integrated along the real axis in a cyclically ordered way: $z_1 \geq z_2 \ldots \geq z_N$ with a, b, c arbitrarily chosen.

The integrand of the N-point amplitude is invariant under projective transformations acting on the leg variables z_i:

$$z_i \rightarrow \frac{\alpha z_i + \beta}{\gamma z_i + \delta} \quad ; \quad i = 1 \ldots N \quad ; \quad \alpha\delta - \beta\gamma = 1 \tag{34}$$

This is because both the anharmonic ratio in Eq. (31) and the measure dV_{abc} are invariant under a projective transformation. Since a projective transformation depends on three real parameters, then the integrand of the N-point amplitude depends only on $N-3$ variables z_i. In order to avoid infinities, one has then to divide the integration volume with the factor dV_{abc} that is also invariant under the projective transformations. The fact that the integrand depends only on $N-3$ variables is in agreement with the fact that $N-3$ is also the maximal number of simultaneous poles allowed in the amplitude.

It is convenient to write the N-point amplitude in a form that involves the scalar product of the external momenta rather than the Regge trajectories. We distinguish three kinds of channels. The first one is when the particles i and j of the channel (i, j) are separated by at least two particles. In this case the channels that contribute to the exponent of the factor $(z_i - z_j)$ are the channels (i, j) with exponent equal to $-\alpha(s_{ij}) - 1$, $(i+1, j-1)$ with exponent $-\alpha(s_{i+1,j-1}) - 1$, $(i+1, j)$ with exponent $\alpha(s_{i+1,j}) + 1$ and $(i, j-1)$ with exponent $\alpha(s_{i,j-1}) + 1$. Adding these four contributions one gets for the channels where i and j are separated by at least two particles

$$-\alpha(s_{ij}) - \alpha(s_{i+1,j-1}) + \alpha(s_{i+1,j}) + \alpha(s_{i,j-1}) = 2\alpha' p_i \cdot p_j \tag{35}$$

The second one comes from the channels that are separated by only one particle. In this case only three of the previous four channels contribute. For instance if $j = i+2$ the channel $(i+1, j-1)$ consists of only one particle and therefore should not be included. This means that we would get:

$$-\alpha(s_{i;i+2}) - 1 + \alpha(s_{1+1;i+2}) + 1 + \alpha(s_{i;i+1} + 1) = 1 + 2\alpha' p_i \cdot p_{i+2} \tag{36}$$

Finally the third one that comes from the channels whose particles are adjacent, gets only contribution from:

$$-\alpha(s_{i;i+1}) - 1 = \alpha_0 - 1 + 2\alpha' p_i \cdot p_{i+1} \tag{37}$$

Putting all these three terms together in Eq. (32) and remembering the factor in the denominator in the first equation of (33) we get:

$$B_N = \int_{-\infty}^{\infty} \frac{\prod_1^N dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \prod_{i=1}^{N} \left[(z_i - z_{i+1})^{\alpha_0 - 1}\right] \prod_{j>i} (z_i - z_j)^{2\alpha' p_i \cdot p_j} \tag{38}$$

A convenient choice for the three variables to keep fixed is:

$$z_a = z_1 = \infty \;\; ; \;\; z_b = z_2 = 1 \;\; ; \;\; z_c = z_N = 0 \tag{39}$$

With this choice the previous equation becomes:

$$B_N = \prod_{i=3}^{N-1} \left[\int_0^1 dz_i \theta(z_i - z_{i+1}) \right] \prod_{i=2}^{N-1} (z_i - z_{i+1})^{\alpha_0 - 1} \times$$

$$\times \prod_{i=2}^{N-1} \prod_{j=i+1}^{N} (z_i - z_j)^{2\alpha' p_i \cdot p_j} \tag{40}$$

We now want to show that this amplitude is identical to the one given in Eq. (30). This can be done by performing the following change of variables:

$$u_i = \frac{z_{i+1}}{z_i} \;\; ; \;\; i = 2, 3 \ldots N-2 \tag{41}$$

that implies

$$z_i = u_2 u_3 \ldots u_{i-1} \;\; ; \;\; i = 3, 4 \ldots N-1 \tag{42}$$

Taking into account that the Jacobian is equal to:

$$\det \frac{\partial z}{\partial u} = \prod_{i=3}^{N-2} z_i = \prod_{i=2}^{N-3} u_i^{N-2-i} \tag{43}$$

using the following two relations:

$$\det \frac{\partial z}{\partial u} \prod_{i=2}^{N-1} (z_i - z_{i+1})^{\alpha_0 - 1} = \prod_{i=2}^{N-2} \left[u_i^{(N-1-i)\alpha_0 - 1} \right] \prod_{i=2}^{N-2} (1 - u_i)^{\alpha_0 - 1} \tag{44}$$

and

$$\prod_{i=2}^{N-1} \prod_{j=i+1}^{N} (z_j - z_i)^{2\alpha' p_i \cdot p_j} =$$

$$= \prod_{i=2}^{N-2} \prod_{j=i+1}^{N-1} (1 - x_{ij})^{2\alpha' p_i \cdot p_j} \prod_{i=2}^{N-2} u_i^{-\alpha(s_i) - (N-i-1)\alpha_0} \tag{45}$$

and the conservation of momentum

$$\sum_{i=1}^{N} p_i = 0 \tag{46}$$

together with Eq. (21), one can easily see that Eq.s (30) and (40) are equal.

The N-point amplitude that we have constructed in this section corresponds to the scattering of N spinless particles with no internal degrees of freedom. On the other hand it was known that the mesons were classified according to multiplets of an $SU(3)$ flavour symmetry. This was implemented by Chan and Paton [16] by multiplying the N-point amplitude with a factor, called Chan-Paton factor, given by

$$Tr(\lambda^{a_1}\lambda^{a_2}\ldots\lambda^{a_N}) \tag{47}$$

where the λ's are matrices of a unitary group in the fundamental representation. Including the Chan-Paton factors the total scattering amplitude is given by:

$$\sum_P Tr(\lambda^{a_1}\lambda^{a_2}\ldots\lambda^{a_N})B_N(p_1, p_2, \ldots p_N) \tag{48}$$

where the sum is extended to the $(N-1)!$ permutations of the external legs, that are not related by a cyclic permutations. Originally when the dual resonance model was supposed to describe strongly interacting mesons, this factor was introduced to represent their flavour degrees of freedom. Nowadays the interpretation is different and the Chan-Paton factor represents the colour degrees of freedom of the gauge bosons and the other massive particles of the spectrum.

The N-point amplitude B_N that we have constructed in this section contains only simple pole singularities in all possible planar channels. They correspond to zero width resonances located at non-negative integer values n of the Regge trajectory $\alpha(M^2) = n$. The lowest state located at $\alpha(m^2) = 0$ corresponds to the particles on the external legs of B_N. The spectrum of excited particles can be obtained by factorizing the N-point amplitude in the most general channel with any number of particles. This was done in Ref.s [17] and [18] finding a spectrum of states rising exponentially with the mass M. Being the model relativistic invariant it was found that many states obtained by factorizing the N-point amplitude were "ghosts", namely states with negative norm as one finds in QED when one quantizes the electromagnetic field in a covariant gauge. The consistency of the model requires the existence of relations satisfied by the scattering amplitudes that are similar to those obtained through gauge invariance in QED. If the model is consistent they must decouple the negative norm states leaving us with a physical spectrum of positive norm states. In order to study in a simple way these issues, we discuss in the next section the operator formalism introduced already in 1969 [19, 20, 21].

Before concluding this section let us go back to the non-planar four-point amplitude in Eq. (5) and discuss its generalization to an N-point amplitude. Using the technique of the electrostatic analogue on the sphere instead of on the disk Shapiro [22] was able to obtain a N-point amplitude that reduces to the four-point amplitude in Eq. (5) with intercept $\alpha_0 = 2$. The N-point amplitude found in Ref. [22] is:

$$\int \frac{\prod_{i=1}^{N} d^2 z_i}{dV_{abc}} \prod_{i<j} |z_i - z_j|^{\alpha' p_i \cdot p_j} \tag{49}$$

where

$$dV_{abc} = \frac{d^2 z_a d^2 z_b d^2 z_c}{|z_a - z_b|^2 |z_a - z_c|^2 |z_b - z_c|^2} \tag{50}$$

The integral in Eq. (49) is performed in the entire complex plane.

3 Operator formalism and factorization

The factorization properties of the dual resonance model were first studied by factorizing by brute force the N-point amplitude at the various poles [17, 18]. The number of terms that factorize the residue of the pole at $\alpha(s) = n$, increases rapidly with the value of n. In order to find their degeneracy it turned out to be convenient to first rewrite the N-point amplitude in an operator formalism. In this section we introduce the operator formalism and we rewrite the N-point amplitude derived in the previous section in this formalism.

The key idea [19, 20, 21] is to introduce an infinite set of harmonic oscillators and a position and momentum operators [7] which satisfy the following commutation relations:

$$[a_{n\mu}, a^{\dagger}_{m\nu}] = \eta_{\mu\nu} \delta_{nm} \quad ; \quad [\hat{q}_{\mu}, \hat{p}_{\nu}] = i\eta_{\mu\nu} \tag{51}$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric that we take to be $\eta_{\mu\nu} = (-1, 1, \dots 1)$. A state with momentum p is constructed in terms of a state with zero momentum as follows:

$$\hat{p}|p\rangle \equiv \hat{p} e^{ip\cdot\hat{q}}|0\rangle = p|p\rangle \;\; ; \;\; \hat{p}\,|0\rangle = 0 \tag{52}$$

normalized as [8]

$$\langle p|p'\rangle = (2\pi)^d \delta^{(d)}(p + p') \tag{53}$$

In order to avoid minus signs we use the convention that

$$\langle p| = \langle 0| e^{ip\cdot\hat{q}} \tag{54}$$

A complete and orthonormal basis of vectors in the harmonic oscillator space is given by

$$|\lambda_1, \lambda_2, \dots \lambda_i; p\rangle = \prod_n \frac{(a^{\dagger}_{\mu_n;n})^{\lambda_{n;\mu_n}}}{\sqrt{\lambda_{n,\mu_n}!}} e^{ip\hat{q}}|0, 0\rangle \tag{55}$$

[7] Actually the position and momentum operators were introduced in Ref. [23].

[8] Although we now use an arbitrary d we want to remind you that all original calculations were done for $d = 4$.

where the first $|0\rangle$ corresponds to the one annihilated by all annihilation operators and the second one to the state of zero momentum:

$$a_{\mu_n;n}|0,0\rangle = \hat{p}|0,0\rangle = 0 \tag{56}$$

Notice that Lorentz invariance forces to introduce also oscillators that create states with negative norm due to the minus sign in the flat Minkowski metric. This implies that the space spanned by the states in Eq. (55) is not positive definite. This is, however, not allowed in a quantum theory and therefore if the dual resonance model is a consistent quantum-relavistic theory we expect the presence of relations of the kind of those provided by gauge invariance in QED.

Let us introduce the Fubini-Veneziano [23] operator:

$$Q_\mu(z) = Q_\mu^{(+)}(z) + Q_\mu^{(0)}(z) + Q_\mu^{(-)}(z) \tag{57}$$

where

$$Q^{(+)} = i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{a_n}{\sqrt{n}}z^{-n} \quad ; \quad Q^{(-)} = -i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{a_n^\dagger}{\sqrt{n}}z^{n}$$

$$Q^{(0)} = \hat{q} - 2i\alpha'\hat{p}\log z \tag{58}$$

In terms of Q we introduce the vertex operator corresponding to the external leg with momentum p:

$$V(z;p) =: e^{ip\cdot Q(z)} :\equiv e^{ip\cdot Q^{(-)}(z)} e^{ip\hat{q}} e^{+2\alpha'\hat{p}\cdot p\log z} e^{ip\cdot Q^{(+)}(z)} \tag{59}$$

and compute the following vacuum expectation value:

$$\langle 0,0|\prod_{i=1}^{N} V(z_i,p_i)|0,0\rangle \tag{60}$$

It can be easily computed using the Baker-Haussdorf relation

$$e^A e^B = e^B e^A e^{[A,B]} \tag{61}$$

that is valid if the commutator, as in our case, $[A,B]$ is a c-number. In our case the commutation relations to be used are:

$$[Q^{(+)}(z), Q^{(-)}(w)] = -2\alpha'\log\left(1-\frac{w}{z}\right) \tag{62}$$

and the second one in Eq. (51). Using them one gets:

$$V(z;p)V(w;k) =: V(z;p)V(w;k) : (z-w)^{2\alpha' p\cdot k} \tag{63}$$

and

$$\langle 0,0|\prod_{i=1}^{N} V(z_i,p_i)|0,0\rangle = \prod_{i>j}(z_i-z_j)^{2\alpha' p_i\cdot p_j}(2\pi)^d\delta^{(d)}(\sum_{i=1}^{N} p_i) \tag{64}$$

where the normal ordering requires that all creation operators be put on the left of the annihilation one and the momentum operator $\hat{p}$ be put on the right of the position operator $\hat{q}$. This means that

$$(2\pi)^d\delta^{(d)}(\sum_{i=1}^{N} p_i)B_N = \int_{-\infty}^{\infty}\frac{\prod_1^N dz_i\theta(z_i-z_{i+1})}{dV_{abc}}\prod_{i=1}^{N}\left[(z_i-z_{i+1})^{\alpha_0-1}\right]\times$$

$$\times\langle 0,0|\prod_{i=1}^{N} V(z_i,p_i)|0,0\rangle \tag{65}$$

By choosing the three variables z_a, z_b and z_c as in Eq. (39) we can rewrite the previous equation as follows:

$$(2\pi)^d\delta^{(d)}(\sum_{i=1}^{N} p_i)B_N = \int_0^1\prod_{i=3}^{N-1}dz_i\prod_{i=2}^{N-1}\theta(z_i-z_{i+1})\times$$

$$\times\prod_{i=2}^{N-1}\left[(z_i-z_{i+1})^{\alpha_0-1}\right]\langle 0,p_1|\prod_{i=2}^{N-1}V(z_i;p_i)|0,p_N\rangle \tag{66}$$

where we have taken $z_2=1$ and we have defined $(\alpha_0\equiv\alpha' p_i^2; i=1\ldots N)$:

$$\lim_{z_N\to 0}V(z_N;p_N)|0,0\rangle\equiv|0;p_N\rangle \quad ; \quad \langle 0;0|\lim_{z_1\to\infty}z_1^{2\alpha_0}V(z_1;p_1)=\langle 0,p_1| \tag{67}$$

Before proceeding to factorize the N-point amplitude let us study the properties under the projective group of the operators that we have introduced. We have already seen that the projective group leaves the integrand of the Koba-Nielsen representation of the N-point amplitude invariant. The projective group has three generators L_0, L_1 and L_{-1} corresponding respectively to dilatations, inversions and translations. Assuming that the Fubini-Veneziano fields $Q(z)$ transforms as a field with weight 0 (as a scalar) we can immediately write the commutation relations that $Q(z)$ must satisfy. This means in fact that, under a projective transformation, $Q(z)$ transforms as follows:

$$Q(z)\to Q^T(z)=Q\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) \quad ; \quad \alpha\delta-\beta\gamma=1 \tag{68}$$

Expanding for small values of the parameters we get:

$$Q^T(z)=Q(z)+(\epsilon_1+\epsilon_2 z+\epsilon_3 z^2)\frac{dQ(z)}{dz}+o(\epsilon^2) \tag{69}$$

This means that the three generators of the projective group must satisfy the following commutation relations with $Q(z)$:

$$[L_0, Q(z)] = z\frac{dQ}{dz} \;\; ; \;\; [L_{-1}, Q(z)] = \frac{dQ}{dz} \;\; ; \;\; [L_1, Q(z)] = z^2\frac{dQ}{dz} \qquad (70)$$

They are given by the following expressions in terms of the harmonic oscillators:

$$L_0 = \alpha'\hat{p}^2 + \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n \; ; \; L_1 = \sqrt{2\alpha'}\hat{p}\cdot a_1 + \sum_{n=1}^{\infty}\sqrt{n(n+1)}a_{n+1}\cdot a_n^\dagger \qquad (71)$$

and

$$L_{-1} = L_1^\dagger = \sqrt{2\alpha'}\hat{p}\cdot a_1^\dagger + \sum_{n=1}^{\infty}\sqrt{n(n+1)}a_{n+1}^\dagger \cdot a_n \qquad (72)$$

They annihilate the vacuum

$$L_0|0,0\rangle = L_1|0,0\rangle = L_{-1}|0,0\rangle = 0 \qquad (73)$$

that is therefore called the projective invariant vacuum, and satisfy the algebra that is called Gliozzi algebra [24][9]:

$$[L_0, L_1] = -L_1 \;\; ; \;\; [L_0, L_{-1}] = L_{-1} \;\; ; \;\; [L_1, L_{-1}] = 2L_0 \qquad (74)$$

The vertex operator with momentum p is a projective field with weight equal to $\alpha_0 = \alpha' p^2$. It transforms in fact as follows under the projective group:

$$[L_n, V(z,p)] = z^{n+1}\frac{dV(z,p)}{dz} + \alpha_0(n+1)z^n V(z,p) \;\; ; \;\; n = 0, \pm 1 \qquad (75)$$

or in finite form as follows:

$$UV(z,p)U^{-1} = \frac{1}{(\gamma z+\delta)^{2\alpha_0}} V\left(\frac{\alpha z+\beta}{\gamma z+\delta}, p\right) \qquad (76)$$

where U is the generator of an arbitrary finite projective transformation.

Since U leaves the vacuum invariant, by using Eq. (76) it is easy to show that:

$$\langle 0,0|\prod_{i=1}^{N} V(z_i', p)|0,0\rangle = \prod_{i=1}^{N}(\gamma z_i+\delta)^{2\alpha_0}\langle 0,0|\prod_{i=1}^{N} V(z_i,p)|0,0\rangle \qquad (77)$$

that together with the following equation:

$$\prod_{i=1}^{N} dz_i' \prod_{i=1}^{N}(z_i'-z_{i+1}')^{\alpha_0-1} = \prod_{i=1}^{N} dz_i \prod_{i=1}^{N-1}(z_i-z_{i+1})^{\alpha_0-1}\prod_{i=1}^{N}(\gamma z_i+\delta)^{-2\alpha_0} \qquad (78)$$

[9] See also Ref. [25].

implies that the integrand of the N-point amplitude in Eq. (65) is invariant under projective transformations.

We are now ready to factorize the N-point amplitude and find the spectrum of mesons.

From Eq.s (75) and (76) it is easy to derive the transformation of the vertex operator under a finite dilatation:

$$z^{L_0} V(1,p) z^{-L_0} = V(z,p) z^{\alpha_0} \tag{79}$$

Changing the integration variables as follows:

$$x_i = \frac{z_{i+1}}{z_i} \quad ; \quad i = 2,3 \ldots N-2 \quad ; \quad \det \frac{\partial z_i}{\partial x_j} = z_3 z_4 \ldots z_{N-2} \tag{80}$$

where the last term is the jacobian of the trasformation from z_i to x_i, we get from Eq.(66) the following expression:

$$A_N \equiv \langle 0, p_1 | V(1,p_2) D V(1,p_3) \ldots D V(1, p_{N-1}) | 0, p_N \rangle \tag{81}$$

where the propagator D is equal to:

$$D = \int_0^1 dx x^{L_0 - 1 - \alpha_0} (1-x)^{\alpha_0 - 1} = \frac{\Gamma(L_0 - \alpha_0) \Gamma(\alpha_0)}{\Gamma(L_0)} \tag{82}$$

and

$$A_N = (2\pi)^d \delta^{(d)} \left(\sum_{i=1}^{N} p_i \right) B_N \tag{83}$$

The factorization properties of the amplitude can be studied by inserting in the channel $(1, M)$ or equivalently in the channel $(M+1, N)$ described by the Mandelstam variable

$$s = -(p_1 + p_2 + \ldots p_M)^2 = -(p_{M+1} + p_{M+2} \ldots + p_N)^2 \equiv -P^2 \tag{84}$$

the complete set of states given in Eq. (55):

$$A_N = \sum_{\lambda,\mu} \langle p_{(1,M)} | \lambda, P \rangle \langle \lambda, P | D | \mu, P \rangle \langle \mu, P | p_{(M+1,N)} \rangle \tag{85}$$

where

$$\langle p_{(1,M)} | = \langle 0, p_1 | V(1,p_2) D V(1,p_3) \ldots V(1,p_M) \tag{86}$$

and

$$| p_{(M+1,N)} \rangle = V(1, p_{M+1}) D \ldots V(1, p_{N-1}) | p_N, 0 \rangle \tag{87}$$

Introducing the quantity:

$$R = \sum_{n=1}^{\infty} n a_n^{\dagger} \cdot a_n \tag{88}$$

it is possible to rewrite

$$\langle \lambda, P|D|\mu, P\rangle = \sum_{m=0}^{\infty} \langle \lambda, P| \frac{(-1)^m \begin{pmatrix} \alpha_0 - 1 \\ m \end{pmatrix}}{R + m - \alpha(s)} |\mu, P\rangle \tag{89}$$

where s is the variable defined in Eq. (84). Using this equation we can rewrite Eq. (85) as follows

$$A_N = \sum_{\lambda,\mu} \langle p_{(1,M)}|\lambda, P\rangle \sum_{m=0}^{\infty} \langle \lambda, P| \frac{(-1)^m \begin{pmatrix} \alpha_0 - 1 \\ m \end{pmatrix}}{R + m - \alpha(s)} |\mu, P\rangle \langle \mu, P|p_{(M+1,N)}\rangle \tag{90}$$

This expression shows that amplitude A_N has a pole in the channel $(1, M)$ when $\alpha(s)$ is equal to an integer $n \geq 0$ and the states $|\lambda\rangle$ that contribute to its residue are those satisfying the relation:

$$R|\lambda\rangle = (n - m)|\lambda\rangle \quad ; \quad m = 0, 1 \ldots n \tag{91}$$

The number of independent states $|\lambda\rangle$ contributing to the residue gives the degeneracy of states for each level n.

Because of manifest relativistic invariance the space spanned by the complete system of states in Eq. (55) contains states with negative norm corresponding to those states having an odd number of oscillators with timelike directions (see Eq. (51)). This is not consistent in a quantum theory where the states of a system must span a positive definite Hilbert space. This means that there must exist a number of relations satisfied by the external states that decouple a number of states leaving with a positive definite Hilbert space. In order to find these relations we rewrite the state in Eq. (87) going back to the Koba-Nielsen variables:

$$|p_{(1,M)}\rangle = \prod_{i=2}^{M-1} [\int dz_i \theta(z_i - z_{i+1})] \prod_{i=1}^{M-1} (z_i - z_{i+1})^{\alpha_0 - 1} \times$$

$$\times V(1, p_1) V(z_2, p_2) \ldots V(z_{M-1}, p_{M-1})|0, p_M\rangle \tag{92}$$

Let us consider the operator $U(\alpha)$ that generate the projective transformation that leaves the points $z = 0, 1$ invariant:

$$z' = \frac{z}{1 - \alpha(z - 1)} = z + \alpha(z^2 - z) + o(\alpha^2) \tag{93}$$

From the transformation properties of the vertex operators in Eq. (76) it is easy to see that the previous transformation leaves the state in Eq. (92) invariant:

$$U(\alpha)|p_{(1,M)}\rangle = |p_{(1,M)}\rangle \tag{94}$$

This means that the generator of the previous transformation annihilates the state in Eq. (92):

$$W_1|p_{(1,M)}\rangle = 0 \quad ; \quad W_1 = L_1 - L_0 \tag{95}$$

The explicit form of W_1 follows from the infinitesimal form of the transformation in Eq. (93). This condition that is of the same kind of the relations that on shell amplitudes with the emission of photons satisfy as a consequence of gauge invariance, implies that the residue at the pole in Eq. (90) can be factorized with a smaller number of states. It turns out, however, that a detailed analysis of the spectrum shows that negative norm states are still present. This can be qualitatively understood as follows. Due to the Lorentz metric we have a negative norm component for each oscillator. In order to be able to decouple all negative norm states we need to have a gauge condition of the type as in Eq. (95) for each oscillator. But the number of oscillators is infinite and, therefore, we need an infinite number of conditions of the type as in Eq. (95). It was found in Ref. [6] that, if we take $\alpha_0 = 1$, then one can easily construct an infinite number of operators that leave the state in Eq. (92) invariant. In the next section we will concentrate on this case.

4 The case $\alpha_0 = 1$

If we take $\alpha_0 = 1$ many of the formulae given in the previous section simplify. The N-point amplitude in Eq. (38) becomes:

$$B_N = \int_{-\infty}^{\infty} \frac{\prod_1^N dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \prod_{j>i} (z_i - z_j)^{2\alpha' p_i \cdot p_j} \tag{96}$$

that can be rewritten in the operator formalism as follows:

$$(2\pi)^4 \delta(\sum_{i=1}^N p_i) B_N = \int_{-\infty}^{\infty} \frac{\prod_1^N dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \langle 0,0| \prod_{i=1}^N V(z_i, p_i)|0,0\rangle \tag{97}$$

By choosing $z_1 = \infty, z_2 = 1$ and $z_N = 0$ it becomes

$$(2\pi)^4 \delta(\sum_{i=1}^N p_i) B_N =$$

$$= \int_0^1 \prod_{i=3}^{N-1} dz_i \prod_{i=2}^{N-1} \theta(z_i - z_{i+1}) \langle 0, p_1| \prod_{i=2}^{N-1} V(z_i; p_i)|0, p_N\rangle \tag{98}$$

where

$$\lim_{z_N \to 0} V(z_N; p_N)|0,0\rangle \equiv |0; p_N\rangle \quad ; \quad \langle 0;0| \lim_{z_1 \to \infty} z_1^2 V(z_1; p_1) = \langle 0, p_1| \tag{99}$$

Eq. (81) is as before, but now the propagator becomes:

$$D = \int dx x^{L_0 - 2} = \frac{1}{L_0 - 1} \tag{100}$$

This means that Eq. (89) becomes:

$$\langle \lambda, P|D|\mu, P\rangle = \langle \lambda, P|\frac{1}{L_0 - 1}|\mu, P\rangle \tag{101}$$

and Eq. (90) has the simpler form:

$$B_N = \sum_{\lambda} \langle p_{(1,M)}|\lambda, P\rangle\langle \lambda, P|\frac{1}{R - \alpha(s)}|\lambda, P\rangle\langle \lambda, P|p_{(M+1,N)}\rangle \tag{102}$$

B_N has a pole in the channel $(1, M)$ when $\alpha(s)$ is equal to an integer $n \geq 0$ and the states $|\lambda\rangle$ that contribute to its residue are those satisfying the relation:

$$R|\lambda\rangle = n|\lambda\rangle \tag{103}$$

Their number gives the degeneracy of the states contributing to the pole at $\alpha(s) = n$. The N-point amplitude can be written as:

$$B_N = \langle p_{(1,M)}|D|p_{(M+1,N)}\rangle \tag{104}$$

where

$$|p_{(1,M)}\rangle = \int \prod_{i=2}^{M-1} [dz_i \theta(z_i - z_{i+1})] \times$$

$$\times V(1, p_1) V(z_2, p_2) \dots V(z_{M-1}, p_{M-1}|0, p_M\rangle \tag{105}$$

Using Eq. (79) and changing variables from $z_i, i = 2 \dots M-1$ to $x_i = \frac{z_{i+1}}{z_i}, i = 1 \dots M-2$ with $z_1 = 1$ we can rewrite the previous equation as follows:

$$|p_{(1,M)}\rangle = V(1, p_1) D V(1, p_2) \dots D V(1, p_{M-1})|0, p_M\rangle \tag{106}$$

where the propagator D is defined in Eq. (100).

We want now to show that the state in Eq.s (105) and (106) is not only annihilated by the operator in Eq. (95), but, if $\alpha_0 = 1$ [6], by an infinite set of operators whose lowest one is the one in Eq. (95). We will derive this by using the formalism developed in Ref. [26] and we will follow closely their derivation.

Starting from Eq.s (70) Fubini and Veneziano realized that the generators of the projective group acting on a function of z are given by:

$$L_0 = -z\frac{d}{dz} \quad ; \quad L_{-1} = -\frac{d}{dz} \quad ; \quad L_1 = -z^2\frac{d}{dz} \tag{107}$$

They generalized the previous generators to an arbitrary conformal transformation by introducing the following operators, called Virasoro operators:

$$L_n = -z^{n+1}\frac{d}{dz} \tag{108}$$

that satisfy the algebra:

$$[L_n, L_m] = (n-m)L_{n+m} \tag{109}$$

that does not contain the term with the central charge! They also showed that the Virasoro operators satisfy the following commutation relations with the vertex operator:

$$[L_n, V(z,p)] = \frac{d}{dz}\left(z^{n+1}V(z,p)\right) \tag{110}$$

More in general actually they define an operator L_f corresponding to an arbitrary function $f(\xi)$ and $L_f = L_n$ if we choose $f(\xi) = \xi^n$. In this case the commutation relation in Eq. (110) becomes:

$$[L_f, V(z,p)] = \frac{d}{dz}\left(zf(z)V(z,p)\right) \tag{111}$$

By introducing the variable:

$$y = \int_A^z \frac{d\xi}{\xi f(\xi)} \tag{112}$$

where A is an arbitrary constant, one can rewrite Eq. (111) in the following form:

$$[L_f, zf(z)V(z,p)] = \frac{d}{dy}\left(zf(z)V(z,p)\right) \tag{113}$$

This implies that, under an arbitrary conformal transformation $z \to f(z)$, generated by $U = e^{\alpha L_f}$, the vertex operator transforms as:

$$e^{\alpha L_f} V(z,p)\, zf(z)\, e^{-\alpha L_f} = V(z',p)z'f(z') \tag{114}$$

where the parameter α is given by:

$$\alpha = \int_z^{z'} \frac{d\xi}{\xi f(\xi)} \tag{115}$$

On the other hand, this equation implies:

$$\frac{dz}{zf(z)} = \frac{dz'}{z'f(z')} \tag{116}$$

that, inserted in Eq. (114), implies that the quantity $V(z,p)\,dz$ is left invariant by the transformation $z \to f(z)$:

$$e^{\alpha L_f} V(z,p) dz e^{-\alpha L_f} = V(z',p)dz' \tag{117}$$

Let us now act with the previous conformal transformation on the state in Eq. (105). We get:

$$e^{\alpha L_f} |p_{(1,M)}\rangle = \int_0^1 \prod_{i=2}^{M-1} \left[dz_i \theta(z_i - z_{i+1}) \right] e^{\alpha L_f} V(1,p_1) e^{-\alpha L_f} \times$$

$$\times e^{\alpha L_f} V(z_2,p_2) e^{-\alpha L_f} \ldots\ldots e^{\alpha L_f} V(z_{M-1},p_{M-1}) e^{-\alpha L_f} e^{\alpha L_f} |0,p_M\rangle =$$

$$= \int_0^1 \prod_{i=2}^{M-1} \theta(z_i - z_{i+1}) \times e^{\alpha L_f} V(1,p_1) e^{-\alpha L_f} \times$$

$$\times V(z'_2,p_2) dz'_2 \ldots V(z'_{M-1},p_{M-1}) dz'_{M-1} e^{\alpha L_f} |0,p_M\rangle \tag{118}$$

where we have used Eq. (117). The previous transformation leaves the state invariant if both $z=0$ and $z=1$ are fixed points of the conformal transformation. This happens if the denominator in Eq. (115) vanishes when $\xi = 0,1$. This requires the following conditions:

$$f(1) = 0 \quad ; \quad \lim_{\xi \to 0} \xi f(\xi) = 0 \tag{119}$$

Expanding ξ near the poinr $\xi = 1$ we can determine the relation between z and z' near $z = z' = 1$. We get:

$$z' = \frac{z e^{-\alpha f'(1)}}{1 - z + z e^{-\alpha f'(1)}} \tag{120}$$

and from it we can determine the conformal factor:

$$\frac{dz'}{dz} = \frac{e^{-\alpha f'(1)}}{(1 - z + z e^{-\alpha f'(1)})^2} \to e^{\alpha f'(1)} \tag{121}$$

in the limit $z \to 1$. Proceeding in the same near the point $z = z' = 0$ we get:

$$z' = \frac{z f(0) e^{\alpha f(0)}}{f(0) + z f'(0)(1 - e^{\alpha f(0)}} \to z e^{\alpha f(0)} \tag{122}$$

in the limit $z \to 0$. This means that Eq. (118) becomes

$$e^{\alpha \left(L_f - f'(1) - f(0) \right)} |p_{(1,M)}\rangle = |p_{(1,M)}\rangle \tag{123}$$

A choice of f that satisfies Eq.s (119) is the following:

$$f(\xi) = \xi^n - 1 \tag{124}$$

that gives the following gauge operator:

$$W_n = L_n - L_0 - (n-1) \tag{125}$$

that annihilates the state in Eq. (105):

$$W_n |p_{1\dots M}\rangle = 0 \quad ; \quad n = 1 \dots \infty \tag{126}$$

These are the Virasoro conditions found in Ref. [6]. There is one condition for each negative norm oscillator and, therefore, in this case there is the possibility that the physical subspace is positive definite. An alternative more direct derivation of Eq. (126) can be obtained by acting with W_n on the state in Eq. (106) and using the following identities:

$$W_n V(1,p) = V(1,p)(W_n + n) \quad ; \quad (W_n + n) D = [L_0 + n - 1]^{-1} W_n \tag{127}$$

The second equation is a consequence of the following equation:

$$L_n x^{L_0} = x^{L_0 + n} L_n \tag{128}$$

Eq.s (127) imply

$$W_n V(1,p) D = V(1,p) [L_0 + n - 1]^{-1} W_n \tag{129}$$

This shows that the operator W_n goes unchanged through all the product of terms VD until it arrives in front of the term $V(1, p_{M-1})|0, p_M\rangle$. Going through the vertex operator it becomes $L_n - L_0 + 1$ that then annihilate the state

$$(L_n - L_0 + 1)|p_M, 0\rangle = 0 \tag{130}$$

This proves Eq. (126).

Using the representation of the Virasoro operators given in Eq. (108) Fubini and Veneziano showed that they satisfy the algebra given in eq. (109) without the central charge. The presence of the central charge was recognized by Joe Weis[10] in 1970 and never published. Unlike Fubini and Veneziano [26] he used the expression of the L_n operators in terms of the harmonic oscillators:

$$L_n = \sqrt{2\alpha' n}\hat{p} \cdot a_n + \sum_{m=1}^{\infty} \sqrt{m(n+m)} a_{n+m} \cdot a_m +$$

$$+ \frac{1}{2} \sum_{m=1}^{n-1} \sqrt{m(n-m)} a_{m-n} \cdot a_m \quad ; n \geq 0 \quad L_n = L_n^{\dagger} \tag{131}$$

[10] See noted added in proof in Ref. [26].

He got the following algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{d}{24}n(n^2-1)\delta_{n+m;0} \tag{132}$$

where d is the dimension of the Minkowski space-time. We write here d for the dimension of the Minkowski space, but we want to remind you that almost everybody working in a model for mesons at that time took for granted that the dimension of the space-time was $d = 4$. As far as I remember the first paper where a dimension $d \neq 4$ was introduced was Ref. [27] where it was shown that the unitarity violating cuts in the non-planar loop become poles that were consistent with unitarity if $d = 26$.

In the last part of this section we will generalize the factorization procedure to the Shapiro-Virasoro model whose N-point amplitude is given in Eq. (49). In this case we must introduce two sets of harmonic oscillators commuting with each other and only one set of zero modes satisfying the algebra [28] :

$$[a_{n\mu}, a^\dagger_{m\nu}] = [\tilde{a}_{n\mu}, \tilde{a}^\dagger_{m\nu}] = \eta_{\mu\nu}\delta_{nm} \quad ; \quad [\hat{q}_\mu, \hat{p}_\nu] = i\eta_{\mu\nu} \tag{133}$$

In terms of them we can introduce the Fubini-Veneziano operator

$$Q(z,\bar{z}) = \hat{q} - 2\alpha'\hat{p}\log(z\bar{z}) + i\frac{\sqrt{2\alpha'}}{2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left[a_n z^{-n} - a^\dagger_n z^n\right] +$$

$$+ i\frac{\sqrt{2\alpha'}}{2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left[\tilde{a}_n \bar{z}^{-n} - \tilde{a}^\dagger_n \bar{z}^n\right] \tag{134}$$

We can then introduce the vertex operator:

$$V(z,\bar{z};p) =: e^{ip\cdot Q(z,\bar{z})} : \tag{135}$$

and write the N-point amplitude in Eq. (95) in the following factorized form:

$$\int \frac{\prod_{i=1}^N d^2 z_i}{dV_{abc}} \langle 0| R\left[\prod_{i=1}^N V(z_i,\bar{z}_i,p_i))\right]|0\rangle =$$

$$= (2\pi)^4\delta^{(4)}(\sum_{i=1}^N p_i)\int \frac{\prod_{i=1}^N d^2 z_i}{dV_{abc}}\prod_{i<j}|z_i - z_j|^{\alpha' p_i\cdot p_j} \tag{136}$$

where the radial ordered product is given by

$$R\left[\prod_{i=1}^N V(z_i,\bar{z}_i,p_i))\right] = \prod_{i=1}^N V(z_i,\bar{z}_i,p_i))\prod_{i=1}^{N-1}\theta(|z_i| - |z_{i+1}|) + \ldots \tag{137}$$

and the dots indicate a sum over all permutations of the vertex operators.

By fixing $z_1 = \infty, z_2 = 1, z_N = 0$ we can rewrite the previous expression as follows:

$$\int \prod_{i=3}^{N-1} d^2 z_i \langle 0, p_1 | R \left[\prod_{i=2}^{N-1} V(z_i, \bar{z}_i, p_i)) \right] |0, p_N\rangle \tag{138}$$

For the sake of simplicity let us consider the term corresponding to the permutation $1, 2, \dots N$. In this case the Koba-Nielsen variables are ordered in such a way that $|z_i| \geq |z_{i+1}|$ for $i = 1, \dots N-1$. We can then use the formula:

$$V(z_i, \bar{z}_i, p_i)) = z_i^{L_0 - 1} \bar{z}_i^{\tilde{L}_0 - 1} V(1, 1, p_i) z_i^{-L_0} \bar{z}_i^{-\tilde{L}_0} \tag{139}$$

and change variables:

$$w_i = \frac{z_{i+1}}{z_i} \quad ; \quad |w_i| \leq 1 \tag{140}$$

to rewrite Eq. (138) as follows:

$$\langle 0, p_1 | V(1, 1, p_i 1) D V(1, 1, p_2) D \dots V(1, 1, p_{N-1}) |0, p_N\rangle \tag{141}$$

where

$$D = \int \frac{d^2 w}{|w|^2} \, w^{L_0 - 1} \bar{w}^{\tilde{L}_0 - 1} = \frac{2}{L_0 + \tilde{L}_0 - 2} \cdot \frac{\sin \pi (L_0 - \tilde{L}_0)}{L_0 - \tilde{L}_0} \tag{142}$$

We can now follow the same procedure for all permutations arriving at the following expression:

$$\langle 0, p_1 | P[V(1, 1, p_2) D V(1, 1, p_3) D \dots V(1, 1, p_{N-1})] |0, p_N\rangle \tag{143}$$

where P means a sum of all permutations of the particles.

If we want to consider the factorization of the amplitude on the pole at $s = -(p_1 + \dots p_M)^2$ we get only the following contribution:

$$\langle p_{(1 \dots M)} | D | p_{(M+1 \dots N)} \rangle \tag{144}$$

where

$$|p_{(M+1 \dots N)}\rangle = P[V(1, 1, p_{M+1}) D \dots V(1, 1, p_{N-1}] |0, p_N\rangle \tag{145}$$

and

$$\langle p_{(1 \dots M)} | = \langle 0, p_1 | P \left[V(1, 1, p_2) D \dots V(1, 1, p_M) \right] \tag{146}$$

The amplitude is factorized by introducing a complete set of states and rewriting Eq. (141) as follows:

$$\sum_{\lambda,\tilde{\lambda}}\langle p_{1\ldots M}|\lambda,\tilde{\lambda}\rangle\frac{2\pi\langle\lambda,\tilde{\lambda}|\delta_{L_0,\tilde{L}_0}|\lambda,\tilde{\lambda}\rangle}{L_0+\tilde{L}_0-2}\langle\lambda,\tilde{\lambda}|p_{(M+1,\ldots N)}\rangle \tag{147}$$

By writing

$$L_0=\frac{\alpha'}{4}\hat{p}^2+R \quad ; \quad \tilde{L}_0=\frac{\alpha'}{4}\hat{p}^2+\tilde{R} \tag{148}$$

with

$$R=\sum_{n=1}^{\infty} n a_n^{\dagger}\cdot a_n \quad ; \quad \tilde{R}=\sum_{n=1}^{\infty} n\tilde{a}_n^{\dagger}\cdot\tilde{a}_n \tag{149}$$

we can rewrite Eq. (147) as follows

$$\sum_{\lambda,\tilde{\lambda}}\langle p_{1\ldots M}|\lambda,\tilde{\lambda}\rangle\frac{2\pi\langle\lambda,\tilde{\lambda}|\delta_{R,\tilde{R}}|\lambda,\tilde{\lambda}\rangle}{R+\tilde{R}-\alpha(s)}\langle\lambda,\tilde{\lambda}|p_{(M+1,\ldots N)}\rangle \tag{150}$$

We see that the amplitude for the Shapiro-Virasoro model has simple poles only for even integer values of $\alpha_{SV}(s)=2+\frac{\alpha'}{2}s=2n\geq 0$ and the residue at the poles factorizes in a sum with a finite number of terms. Notice that the Regge trajectory of the Shapiro-Virasoro model has double intercept and half slope of that of the generalized Veneziano model.

5 Physical states and their vertex operators

In the previous section, we have seen that the residue at the poles of the N-point amplitudes factorizes in a sum of a finite number of terms. We have also seen that some of these terms, due to the Lorentz metric, correspond to states with negative norm. We have also derived a number of "Ward identities" given in Eq. (126) that imply that some of the terms of the residue decouple. The question to be answered now is: Is the space spanned by the physical states a positive norm Hilbert space? In order to answer this question we need first to find the conditions that characterize the on shell physical states $|\lambda,P\rangle$ and then to determine which are the states that contribute to the residue of the pole at $\alpha(s=-P^2)=n$. In other words, we have to find a way of characterizing the physical states and of eliminating the spurious states that decouple in Eq. (102) as a consequence of Eq.s (126). A state $|\lambda.P\rangle$ contributes at the residue of the pole in Eq.(102) for $\alpha(s=-P^2)=n$ if it is on shell, namely if it satisfies the following equations:

$$R|\lambda,P\rangle=n|\lambda,P\rangle \quad ; \quad \alpha(-P^2)=1-\alpha'P^2=n \tag{151}$$

that can be written in a unique equation:

$$(L_0 - 1)|\lambda, P\rangle = 0 \tag{152}$$

Because of Eq. (126) we also know that a state of the type:

$$|s, P\rangle = W_m^\dagger|\mu, P\rangle \tag{153}$$

is not going to contribute to the residue of the pole. We call it a spurious or unphysical state. We start constructing the subspace of spurious states that are on shell at the level n. Let us consider the set of orthogonal states $|\mu, P\rangle$ such that

$$R|\mu, P\rangle = n_\mu|\mu, P\rangle \quad ; \quad L_0|\mu, P\rangle = (1-m)|\mu, P\rangle \quad ; \quad 1 - \alpha' P^2 = n \tag{154}$$

where

$$m = n + n_\mu \tag{155}$$

In terms of these states we can construct the most general spurious state that is on shell at the level n. It is given by

$$|s, P\rangle = W_m^\dagger|\mu, P\rangle \quad ; \quad (L_0 - 1)|s, P\rangle = 0 \tag{156}$$

per any positive integer m. Using Eq. (154), eq. (156) becomes:

$$|s, P\rangle = L_m^\dagger|\mu, P\rangle \tag{157}$$

where $|\mu, P\rangle$ is an arbitrary state satisfying Eq.s (154).

A physical state $|\lambda, P\rangle$ is defined as the one that is orthogonal to all spurious states appearing at a certain level n. This means that it must satisfy the following equation:

$$\langle \lambda.P|L_\ell^\dagger|\mu, P\rangle = 0 \tag{158}$$

for any state $|\mu, P\rangle$ satisfying Eq.s (154). In conclusion, the on shell physical states at the level n are characterized by the fact that they satisfy the following conditions:

$$L_m|\lambda, P\rangle = (L_0 - 1)|\lambda, P\rangle = 0 \quad ; \quad 1 - \alpha' P^2 = n \tag{159}$$

These conditions characterizing the physical subspace were first found by Del Giudice and Di Vecchia [28] where the analysis described here was done.

In order to find the physical subspace one starts writing the most general on shell state contributing to the residue of the pole at level n in Eq. (154). Then one imposes Eq.s (159) and determines the states that span the physical subspace. Actually, among these states one finds also a set of zero norm states that are physical and spurious at the same time. Those states are of the form given in Eq. (157), but also satisfy Eq.s (159). It is easy to see that they are not really physical because they are not contributing to the residue of the pole at the level n. This follows from the form of the unit operator given in the space of the physical states by:

$$1 = \sum_{norm \neq 0} |\lambda, P\rangle\langle\lambda, P| + \sum_{zero} [|\lambda_0, P\rangle\langle\mu_0, P| + |\mu_0, P\rangle\langle\lambda_0, P|] \quad (160)$$

where $|\lambda_0, P\rangle$ is a zero norm physical and spurious state and $|\mu_0, P\rangle$ its conjugate state. A conjugate state of a zero norm state is obtained by changing the sign of the oscillators with timelike direction. Since $|\lambda_0, P\rangle$ is a spurious state when we insert the unit operator, given in Eq. (160), in Eq. (102) we see that the zero norm states never contribute to the residue because their contribution is annihilated either from the state $\langle p_{(1,M)}|$ or from the state $|p_{(M+1,N)}\rangle$. In conclusion, the physical subspace contains only the states in the first term in the r.h.s. of Eq. (160).

Let us analyze the first two excited levels. The first excited level corresponds to a massless gauge field. It is spanned by the states $\epsilon^\mu a^\dagger_{1\mu}|0, P\rangle$. In this case the only condition that we must impose is:

$$L_1 \epsilon^\mu a^\dagger_{1\mu}|0, P\rangle = 0 \Longrightarrow P \cdot \epsilon = 0 \quad (161)$$

Choosing a frame of reference where the momentum of the photon is given by $P^\mu \equiv (P, 0....0, P)$, Eq. (161) implies that the only physical states are:

$$\epsilon^i a^{+\dagger}_{1i}|0, P\rangle + \epsilon(a^\dagger_{1;0} - a^\dagger_{1;d-1})|0, P\rangle \quad ; \quad i = 1 \dots d-2 \quad (162)$$

where ϵ^i and ϵ are arbitrary parameters. The state in Eq. (162) is the most general state of the level $N = 1$ satisfying the conditions in Eq. (159). The first state in eq. (162) has positive norm, while the second one has zero norm that is orthogonal to all other physical states since it can be written as follows:

$$(a^\dagger_{1;0} - a^\dagger_{1;D-1})|0, P\rangle = L^\dagger_1|0, P\rangle \quad (163)$$

in the frame of reference where $P^\mu \equiv (P, ...0, P)$. Because of the previous property it is decoupled from the physical states together with its conjugate:

$$(a^\dagger_{1,0} + a^\dagger_{1,d-1})|0, P\rangle \quad (164)$$

In conclusion, we are left only with the transverse $d-2$ states corresponding to the physical degrees of freedom of a massless spin 1 state. At the next level $n = 2$ the most general state is given by:

$$[\alpha^{\mu\nu} a^\dagger_{1,\mu} a^\dagger_{1,\nu} + \beta^\mu a^\dagger_{2,\mu}]|0, P\rangle \quad (165)$$

If we work in the center of mass frame where $P^\mu = (M, \mathbf{0})$ we get the following most general physical state:

$$|Phys >= \alpha^{ij}[a^\dagger_{1,i} a^\dagger_{1,j} - \frac{1}{(d-1)}\delta_{ij} \sum_{k=1}^{d-1} a^\dagger_{1,k} a^\dagger_{1,k}]|0, P\rangle +$$

$$+\beta^i[a^\dagger_{2,i} + a^\dagger_{1,0}a^\dagger_{1,i}]|0, P >\rangle +$$

$$+\sum_{i=1}^{d-1}\alpha^{ii}\left[\sum_{i=1}^{d-1} a^\dagger_{1,i}a^\dagger_{1,i} + \frac{d-1}{5}(a^{\dagger 2}_{1,0} - 2a^\dagger_{2,0})\right]|0, P\rangle \tag{166}$$

where the indices i, j run over the $d-1$ space components. The first term in (166) corresponds to a spin 2 in $(d-1)$ dimensional space and has a positive norm being made with space indices. The second term has zero norm and is orthogonal to the other physical states since it can be written as $L_1^+ a^+_{1,i}|0, P\rangle$. Therefore it must be eliminated from the physical spectrum together with its conjugate, as explained above. Finally, the last state in (166) is spinless and has a norm given by:

$$2(d-1)(26-d) \tag{167}$$

If $d < 26$ it corresponds to a physical spin zero particle with positive norm. If $d > 26$ it is a ghost. Finally, if $d = 26$ it has a zero norm and is also orthogonal to the other physical states since it can be written in the form:

$$(2L^\dagger_2 + 3L^{\dagger 2}_1)|0 > \tag{168}$$

It does not belong, therefore, to the physical spectrum. The analysis of this level was done in Ref. [29] with $d = 4$. This did not allow the authors of Ref. [29] to see that there was a critical dimension.

The analysis of the physical states can be easily extended [28] to the Shapiro-Virasoro model. In this case the physical conditions given in Eq. (159) for the open string, become [28]:

$$L_m|\lambda, \tilde{\lambda}\rangle = \tilde{L}_m|\lambda, \tilde{\lambda}\rangle = (L_0 - 1)|\lambda, \tilde{\lambda}\rangle = (\tilde{L}_0 - 1)|\lambda, \tilde{\lambda}\rangle = 0 \tag{169}$$

for any positive integer m. It can be easily seen from the previous equations that the lowest state of the Shapiro-Virasoro model is the vacuum $|0_a, 0_{\tilde{a}}, p\rangle$ corresponding to a tachyon with mass $\alpha' p^2 = 4$, while the next level described by the state $a^\dagger_{1\mu}\tilde{a}^\dagger_{1\nu}|0_a, 0_{\tilde{a}}, p\rangle$ contains massless states corresponding to the graviton, a dilaton and a two-index antisymmetric tensor $B_{\mu\nu}$.

Having characterized the physical subspace one can go on and construct a N-point scattering amplitude involving arbitrary physical states. This was done by Campagna, Fubini, Napolitano and Sciuto [30] where the vertex operator for an arbitrary physical state was constructed in analogy with what has been done for the ground tachyonic state. They associated to each physical state $|\alpha, P\rangle$ a vertex operator $V_\alpha(z, P)$ that is a conformal field with conformal dimension equal to 1:

$$[L_n, V_\alpha(z,p)] = \frac{d}{dz}\left(z^{n+1}V_\alpha(z,p)\right) \tag{170}$$

and reproduces the corresponding state acting on the vacuum as follows:

$$\lim_{z\to 0} V_\alpha(z;p)|0,0\rangle \equiv |\alpha; p\rangle \quad ; \quad \langle 0; 0| \lim_{z\to\infty} z^2 V_\alpha(z;p) = \langle \alpha, p| \tag{171}$$

It satisfies, in addition, the hermiticity relation:

$$V_{\alpha}^{\dagger}(z,P) = V_{\alpha}(\frac{1}{z}, -P)(-1)^{\alpha(-P^2)} \tag{172}$$

An excited vertex that will play an important role in the next section is the one associated to the massless gauge field. It is given by:

$$V_{\epsilon}(z,k) \equiv \epsilon \cdot \frac{dQ(z)}{dz} e^{ik\cdot Q(z)} \quad ; \quad k\cdot\epsilon = k^2 = 0 \tag{173}$$

Because of the last two conditions in Eq. (173) the normal order is not necessary. It is convenient to give the expression of $\frac{dQ(z)}{dz}$ in terms of the harmonic oscillators:

$$P(z) \equiv \frac{dQ(z)}{dz} = -i\sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \alpha_n z^{-n-1} \tag{174}$$

It is a conformal field with conformal dimension equal to 1. The rescaled oscillators α_n are given by:

$$\alpha_n = \sqrt{n} a_n \quad ; \quad \alpha_{-n} = \sqrt{n} a_n^{\dagger} \quad ; \quad n > 0 \quad ; \quad \alpha_0 = \sqrt{2\alpha'}\hat{p} \tag{175}$$

In terms of the vertex operators previously introduced the most general amplitude involving arbitrary physical states is given by [30]:

$$(2\pi)^4 \delta(\sum_{i=1}^{N} p_i) B_N^{ex} = \int_{-\infty}^{\infty} \frac{\prod_1^N dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \langle 0,0| \prod_{i=1}^{N} V_{\alpha_i}(z_i, p_i)|0,0\rangle \tag{176}$$

In the case of the Shapiro-Virasoro model the tachyon vertex operator is given in Eq. (135). By rewriting Eq. (134) as follows:

$$Q(z,\bar{z}) = Q(z) + \tilde{Q}(\bar{z}) \tag{177}$$

where

$$Q(z) = \frac{1}{2}\left[\hat{q} - 2\alpha'\hat{p}\log(z) + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left[a_n z^{-n} - a_n^{\dagger} z^n\right]\right] \tag{178}$$

and

$$\tilde{Q}(\bar{z}) = \frac{1}{2}\left[\hat{q} - 2\alpha'\hat{p}\log(\bar{z}) + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left[\tilde{a}_n \bar{z}^{-n} - \tilde{a}_n^{\dagger} \bar{z}^n\right]\right] \tag{179}$$

we can write the tachyon vertex operator in the following way:

$$V(z,\bar{z},p) =: e^{ip\cdot Q(z)} e^{ip\cdot\tilde{Q}(\bar{z})} : \tag{180}$$

This shows that the vertex operator corresponding to the tachyon of the Shapiro-Virasoro model can be written as the product of two vertex operators corresponding each to the tachyon of the generalized Veneziano model.

Analogously the vertex operator corresponding to an arbitrary physical state of the Shapiro-Virasoro model can always be written as a product of two vertex operators of the generalized Veneziano model:

$$V_{\alpha,\beta}(z,\bar{z},p) = V_{\alpha}(z,\frac{p}{2})V_{\beta}(\bar{z},\frac{p}{2}) \tag{181}$$

The first one contains only the oscillators α_n, while the second one only the oscillators $\tilde{\alpha}_n$. They both contain only half of the total momentum p and the same zero modes $\hat{p}$ and $\hat{q}$. The two vertex operators of the generalized Veneziano model are both conformal fields with conformal dimension equal to 1. If they correspond to physical states at the level $2n$, they satisfy the following relation ($n = \tilde{n}$):

$$\alpha' \frac{p^2}{4} + n = 1 \tag{182}$$

They lie on the following Regge trajectory:

$$2 - \frac{\alpha'}{2}p^2 \equiv \alpha_{SV}(-p^2) = 2n \tag{183}$$

as we have already seen by factorizing the amplitude in Eq. (150).

6 The DDF states and absence of ghosts

In the previous section we have derived the equations that characterize the physical states and their corresponding vertex operators. In this section we will explicitly construct an infinite number of orthonormal physical states with positive norm.

The starting point is the DDF operator introduced by Del Giudice, Di Vecchia and Fubini [31] and defined in terms of the vertex operator corresponding to the massless gauge field introduced in eq. (173):

$$A_{i,n} = \frac{i}{\sqrt{2\alpha'}} \oint_0 dz \epsilon_i^{\mu} P_{\mu}(z) e^{ik\cdot Q(z)} \tag{184}$$

where the index i runs over the $d-2$ transverse directions, that are orthogonal to the momentum k. We have also taken $\oint_0 \frac{dz}{z} = 1$. Because of the $\log z$ term appearing in the zero mode part of the exponential, the integral in Eq. (184), that is performed around the origin $z = 0$, is well defined only if we constrain the momentum of the state, on which $A_{i,n}$ acts, to satisfy the relation:

$$2\alpha' p \cdot k = n \tag{185}$$

where n is a non-vanishing integer.

The operator in Eq. (184) will generate physical states because it commutes with the gauge operators L_m:

$$[L_m, A_{n;i}] = 0 \tag{186}$$

since the vertex operator transforms as a primary field with conformal dimension equal to 1 as it follows from Eq. (170).

On the other hand it also satisfies the algebra of the harmonic oscillator as we are now going to show. From Eq. (184) we get:

$$[A_{n,i}, A_{m,j}] = -\frac{1}{2\alpha'} \oint_0 d\zeta \oint_\zeta dz \epsilon_i \cdot P(z) e^{ik\cdot Q(\zeta)} \epsilon_j \cdot P(\zeta) e^{ik'\cdot Q(\zeta)} \tag{187}$$

where

$$2\alpha' p \cdot k = n \quad ; \quad 2\alpha' p \cdot k' = m \tag{188}$$

and k and k' are supposed to be in the same direction, namely

$$k_\mu = n\hat{k}_\mu \quad ; \quad k'_\mu = m\hat{k}_\mu \tag{189}$$

with

$$2\alpha' p \cdot \hat{k} = 1 \tag{190}$$

Finally the polarizations are normalized as:

$$\epsilon_i \cdot \epsilon_j = \delta_{ij} \tag{191}$$

Since $\hat{k} \cdot \epsilon_i = \hat{k} \cdot \epsilon_j = \hat{k}^2 = 0$ a singularity for $z = \zeta$ can appear only from the contraction of the two terms $P(\zeta)$ and $P((z)$ that is given by:

$$\langle 0, 0| \epsilon_i \cdot P(z) \epsilon_j \cdot P(\zeta) |0, 0\rangle = -\frac{2\alpha' \delta_{ij}}{(z-\zeta)^2} \tag{192}$$

Inserting it in Eq. (187) we get:

$$[A_{n,i}, A_{m,j}] = \delta_{ij} in \oint_0 d\zeta \hat{k} \cdot P(\zeta) e^{-i(n+m))\hat{k}\cdot Q(\zeta)} =$$

$$= in\delta_{ij}\delta_{n+m;0} \oint_0 d\zeta \hat{k} \cdot P(\zeta) \tag{193}$$

where we have used the fact that the integrand is a total derivative and therefore one gets a vanishing contribution unless $n + m = 0$. If $n + m = 0$ from Eq.s (174) and (190) we get:

$$[A_{n,i}, A_{m,j}] = n\delta_{ij}\delta_{n+m;0} \quad ; \quad i, j = 1 \ldots d-2 \tag{194}$$

Eq. (194) shows that the DDF operators satisfy the harmonic oscillator algebra.

In terms of this infinite set of transverse oscillators we can construct an orthonormal set of states:

$$|i_1, N_1; i_2, N_2; \dots i_m, N_m\rangle = \prod_h \frac{1}{\sqrt{\lambda_h !}} \prod_{k=1}^{m} \frac{A_{i_k, -N_k}}{\sqrt{N_k}} |0, p\rangle \tag{195}$$

where λ_h is the multiplicity of the operator $A_{i_h, -N_h}$ in the product in Eq. (195) and the momentum of the state in Eq. (195) is given by

$$P = p + \sum_{i=1}^{m} \hat{k} N_i \tag{196}$$

They were constructed in four dimensions where they were not a complete system of states [11] and it took some time to realize that in fact they were a complete system of states if $d = 26$ [32, 33] [12]. Brower [32] and Goddard and Thorn [33] showed also that the dual resonance model was ghost free for any dimension $d \leq 26$. In $d = 26$ this follows from the fact that the DDF operators obviously span a positive definite Hilbert space (See Eq. (194)). For $d < 26$ there are extra states called Brower states [32]. The first of these states is the last state in Eq. (166) that becomes a zero norm state for $d = 26$. But also for $d < 26$ there is no negative norm state among the physical states. The proof of the no-ghost theorem in the case $\alpha_0 = 1$ is a very important step because it shows that the dual resonance model constructed generalizing the four-point Veneziano formula, is a fully consistent quantum-relativistic theory! This is not quite true because, when the intercept $\alpha_0 = 1$, the lowest state of the spectrum corresponding to the pole in the N-point amplitude for $\alpha(s) = 0$, is a tachyon with mass $m^2 = -\frac{1}{\alpha'}$. A lot of effort was then made to construct a model without tachyon and with a meson spectrum consistent with the experimental data. The only reasonably consistent models that came out from these attempts, were the Neveu-Schwarz [7] for mesons and the Ramond model [8] for fermions that only later were recognized to be part of a unique model that nowadays is called the Neveu-Schwarz-Ramond model. But this model was not really more consistent than the original dual resonance

[11] Because of this Fubini did not want to publish our result, but then he went to a meeting in Israel in spring 1971 giving a talk on our work where he found that the audience was very interested in our result and when he came back to MIT we decided to publish our result.

[12] I still remember Charles Thorn coming into my office at Cern and telling me: Paolo, do you know that your DDF states are complete if $d = 26$? I quickly redid the analysis done in Ref. [29] with an arbitrary value of the space-time dimension obtaining Eq.s (166) and (167) that show that the spinless state at the level $\alpha(s) = 2$ is decoupled if $d = 26$. I strongly regretted not to have used an arbitrary space-time dimension d in the analysis of Ref. [29] .

model because it still had a tachyon with mass $m^2 = -\frac{1}{2\alpha'}$. The tachyon was eliminated from the spectrum only in 1976 through the GSO projection proposed by Gliozzi, Scherk and Olive [34].

Having realized that, at least for the critical value of the space-time dimension $d = 26$, the physical states are described by the DDF states having only $d - 2 = 24$ independent components, open the way to Brink and Nielsen [35] to compute the value $\alpha_0 = 1$ of the Regge trajectory with a very physical argument. They related the intercept of the Regge trajectory to the zero point energy of a system with an infinite number of oscillators having only $d - 2$ independent components:

$$\alpha_0 = -\frac{d-2}{2}\sum_{n=1}^{\infty} n \tag{197}$$

This quantity is obviously infinite and, in order to make sense of it, they introduced a cutoff on the frequencies of the harmonic oscillators obtaining an infinite term that they eliminated by renormalizing the speed of light and a finite universal constant term that gave the intercept of the Regge trajectory. Instead of following their original approach we discuss here an alternative approach due to Gliozzi [36] that uses the ζ-function regularization. He rewrites Eq. (197) as follows:

$$\alpha_0 = -\frac{d-2}{2}\sum_{n=1}^{\infty} n = -\frac{d-2}{2}\lim_{s\to -1}\sum_{n=1}^{\infty} n^{-s} = -\frac{d-2}{2}\zeta_R(-1) = 1 \tag{198}$$

where in the last equation we have used the identity $\zeta_R(-1) = -\frac{1}{12}$ and we have put $d = 26$. Since the Shapiro-Virasoro model has two sets of transverse harmonic oscillators it is obvious that its intercept is twice that of the generalized Veneziano model.

Using the rules discussed in the previous section we can construct the vertex operator corresponding to the state in Eq. (195). It is given by:

$$V_{(i;N_i)}(z,P) = \prod_{i=1}^{m} \oint_z dz_i \epsilon_i \cdot P(z_i) e^{iN_i \hat{k}\cdot Q(z_i)} : e^{ip\cdot Q(z)} : \tag{199}$$

where the integral on the variable z_i is evaluated along a curve of the complex plane z_i containing the point z. The singularity of the integrand for $z_i = z$ is a pole provided that the following condition is satisfied.

$$2\alpha' p \cdot \hat{k} = 1 \tag{200}$$

The last vertex in Eq. (199) is the vertex operator corresponding to the ground tachyonic state given in Eq. (59) with $\alpha' p^2 = 1$.

Using the general form of the vertex one can compute the three-point amplitude involving three arbitrary DDF vertex operators. This calculation has been performed in Ref. [37] and since the vertex operators are conformal fields with dimension equal to 1 one gets:

$$\langle 0,0|V_{(i^{(1)}_{k_1};N^{(1)}_{k_1})}(z_1,P_1)V_{(i^{(2)}_{k_2};N^{(2)}_{k^{(2)}})}(z_2,P_2)V_{(i^{(3)}_{k_3};N^{(3)}_{k^{(3)}})}(z_3,P_3)|0,0\rangle =$$

$$= \frac{C_{123}}{(z_1-z_2)(z_1-z_3)(z_2-z_3)} \tag{201}$$

where the explicit form of the coefficient C_{123} is given by:

$$C_{123} = {}_1\langle 0,0|_2\langle 0,0|_3\langle 0,0|e^{\frac{1}{2}\sum_{r.s=1}^{3}\sum_{n,m=1}^{\infty} A^{(r)}_{-n;i}N^{rs}_{nm}A^{(s)}_{-m;i}+\sum_{i=1}^{3}P_i\cdot\sum_{n=1}^{\infty}A^{(r)}_{-n;i}} \times$$

$$\times\, e^{\tau_0\sum_{r=1}^{3}(\alpha' \Pi_r^2-1)}|N^{(1)}_{k_1},i^{(1)}_{k_1}\rangle_1|N^{(2)}_{k_2},i^{(2)}_{k_2}\rangle_2|N^{(3)}_{k_3},i^{(3)}_{k_3}\rangle_3 \tag{202}$$

where

$$N^{rs}_{nm} = -N^r_n N^s_m \frac{nm\alpha_1\alpha_2\alpha_3}{n\alpha_s+m\alpha_r} \quad ; \quad N^r_n = \frac{\Gamma(-n\frac{\alpha_{r+1}}{\alpha_r})}{\alpha_r n!\Gamma(1-n\frac{\alpha_{r+1}}{\alpha_r}-n)} \tag{203}$$

with

$$\Pi = P_{r+1}\alpha_r - P_r\alpha_{r+1} \quad ; \quad r=1,2,3 \tag{204}$$

Π is independent on the value of r chosen as a consequence of the equations:

$$\sum_{r=1}^{3}\alpha_r = \sum_{r=1}^{3}P_r = 0 \tag{205}$$

7 The zero slope limit

In the introduction we have seen that the dual resonance model has been constructed using rules that are different from those used in field theory. For instance, we have seen that planar duality implies that the amplitude corresponding to a certain duality diagram, contains poles in both s and t channels, while the amplitude corresponding to a Feynman diagram in field theory contains only a pole in one of the two channels. Furthermore, the scattering amplitude in the dual resonance model contains an infinite number of resonant states that, at high energy, average out to give Regge behaviour. Also this property is not observed in field theory. The question that was natural to ask, was then: is there any relation between the dual resonance model and field theory? It turned out, to the surprise of many, that the dual resonance model was not in contradiction with field theory, but was instead an extension of a certain number of field theories. We will see that the limit in which a field theory is obtained from the dual resonance model corresponds to taking the slope of the Regge trajectory α' to zero.

Let us consider the scattering amplitude of four ground state particles in Eq. (1) that we rewrite here with the correct normalization factor:

$$A(s,t,u) = C_0 N_0^4 \left(A(s,t) + A(s,u) + A(t,u) \right) \tag{206}$$

where

$$N_0 = \sqrt{2} g (2\alpha')^{\frac{d-2}{4}} \tag{207}$$

is the correct normalization factor for each external leg, g is the dimensionless open string coupling constant that we have constantly ignored in the previous sections and C_0 is determined by the following relation:

$$C_0 N_0^2 \alpha' = 1 \tag{208}$$

that is obtained by requiring the factorization of the amplitude at the pole corresponding to the ground state particle whose mass is given in Eq. (21). Using Eq. (21) in order to rewrite the intercept of the Regge trajectory in terms of the mass of the ground state particle m^2 and the following relation satisfied by the Γ- function:

$$\Gamma(1+z) = z\Gamma(z) \tag{209}$$

we can easily perform the limit for $\alpha' \to 0$ of $A(s,t)$ obtaining:

$$\lim_{\alpha' \to 0} A(s,t) = \frac{1}{\alpha'} \left[\frac{1}{m^2 - s} + \frac{1}{m^2 - s} \right] \tag{210}$$

Performing the same limit on the other two planar amplitudes we get the following expression for the total amplitude in Eq. (206):

$$\lim_{\alpha' \to 0} A(s,t,u) = \left[\sqrt{2} g (2\alpha')^{\frac{d-2}{4}} \right]^2 \frac{2}{(\alpha')^2} \left[\frac{1}{m^2 - s} + \frac{1}{m^2 - s} + \frac{1}{m^2 - u} \right] \tag{211}$$

By introducing the coupling constant:

$$g_3 = 4g (2\alpha')^{\frac{d-6}{4}} \tag{212}$$

Eq. (211) becomes

$$\lim_{\alpha' \to 0} A(s,t,u) = g_3^2 \left[\frac{1}{m^2 - s} + \frac{1}{m^2 - s} + \frac{1}{m^2 - u} \right] \tag{213}$$

that is equal to the sum of the tree diagrams for the scattering of four particles with mass m of Φ^3 theory with coupling constant equal to g_3. We have shown that, by keeping g_3 fixed in the limit $\alpha' \to 0$, the scattering amplitude of four ground state particles of the dual resonance model is equal to the tree diagrams of Φ^3 theory. This proof can be extended to the scattering of N ground state particles recovering also in this case the tree diagrams of Φ^3 theory. It is also valid for loop diagrams that we will discuss in the next section. In conclusion, the dual resonance model reduces in the zero slope limit to Φ^3 theory. The proof that we have presented here is due to J. Scherk [38] [13]

A more interesting case to study is the one with intercept $\alpha_0 = 1$. We will see that, in this case, one will obtain the tree diagrams of Yang-Mills theory, as shown by Neveu and Scherk [40] [14].

Let us consider the three-point amplitude involving three massless gauge particles described by the vertex operator in Eq. (173). It is given by the sum of two planar diagrams. The first one corresponding to the ordering (123) is given by:

$$C_0 N_0^3 i^3 Tr\,(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}) \frac{\langle 0,0|V_{\epsilon_1}(z_1,p_1)V_{\epsilon_2}(z_2,p_2)V_{\epsilon_3}(z_3,p_3)|0,0\rangle}{\left[(z_1-z_2)(z_2-z_3)(z_1-z_3)\right]^{-1}} \tag{214}$$

Using momentum conservation $p_1+p_2+p_3=0$ and the mass shell conditions $p_i^2 = p_i \cdot \epsilon_i = 0$ one can rewrite the previous equation as follows:

$$C_0 N_0^3 Tr(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3})\sqrt{2\alpha'}\times$$

$$\times\left[(\epsilon_1\cdot\epsilon_2)(p_1\cdot\epsilon_3) + (\epsilon_1\cdot\epsilon_3)(p_3\cdot\epsilon_2) + (\epsilon_2\cdot\epsilon_3)(p_2\cdot\epsilon_1)\right] \tag{215}$$

The second contribution comes from the ordering 132 that can be obtained from the previous one by the substitution

$$Tr(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}) \to -Tr(\lambda^{a_1}\lambda^{a_3}\lambda^{a_2}) \tag{216}$$

Summing the two contributions one gets

$$C_0 N_o^3 Tr(\lambda^{a_1}[\lambda^{a_2},\lambda^{a_3}])\sqrt{2\alpha'}\times$$

$$\times\left[(\epsilon_1\cdot\epsilon_2)(p_1\cdot\epsilon_3) + (\epsilon_1\cdot\epsilon_3)(p_3\cdot\epsilon_2) + (\epsilon_2\cdot\epsilon_3)(p_2\cdot\epsilon_1)\right] \tag{217}$$

The factor

$$N_0 = 2g(2\alpha')^{(d-2)/4} \tag{218}$$

is the correct normalization factor for each vertex operator if we normalize the generators of the Chan-Paton group as follows:

$$Tr\left(\lambda^i\lambda^j\right) = \frac{1}{2}\delta^{ij} \tag{219}$$

[13] See also Ref. [39].
[14] See also Ref. [41].

It is related to C_0 through the relation [15]:

$$C_0 N_o^2 \alpha' = 2 \tag{220}$$

g is the dimensionless open string coupling constant. Notice that Eq.s (218) and (220) differ from Eq.s (207) and (208) because of the presence of the Chan-Paton factors that we did not include in the case of Φ^3 theory.

By using the commutation relations:

$$[\lambda^a, \lambda^b] = i f^{abc} \lambda^c \tag{221}$$

and the previous normalization factors we get for the three-gluon amplitude:

$$i g_{YM} f^{a_1 a_2 a_3} \left[(\epsilon_1 \cdot \epsilon_2)((p_1 - p_2) \cdot \epsilon_3 + \right.$$

$$\left. +(\epsilon_1 \cdot \epsilon_3)((p_3 - p_1) \cdot \epsilon_2) + (\epsilon_2 \cdot \epsilon_3)((p_2 - p_3) \cdot \epsilon_1) \right] \tag{222}$$

that is equal to the 3-gluon vertex that one obtains from the Yang-Mills action

$$L_{YM} = -\frac{1}{4} F^a_{\alpha\beta} F_a^{\alpha\beta} \quad , \quad F^a_{\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha + g_{YM} f^{abc} A^b_\alpha A^c_\beta \tag{223}$$

where

$$g_{YM} = 2g(2\alpha')^{\frac{d-4}{4}} \tag{224}$$

The previous procedure can be extended to the scattering of N gluons finding the same result that one gets from the tree diagrams of Yang-Mills theory. In the next section, we will discuss the loop diagrams. Also, in this case one finds that the h-loop diagrams involving N external gluons reproduces in the zero slope limit the sum of the h-loop diagrams with N external gluons of Yang-Mills theory.

We conclude this section mentioning that one can also take the zero slope limit of a scattering amplitude involving three and four gravitons obtaining agreement with what one gets from the Einstein Lagrangian of general relativity. This has been shown by Yoneya [43].

8 Loop diagrams

The N-point amplitude previously constructed satisfies all the axioms of S-matrix theory except unitarity because its only singularities are simple poles corresponding to zero width resonances lying on the real axis of the Mandelstam variables and does not contain the various cuts required by unitarity [1].

[15] The determination of the previous normalization factors can be found in the Appendix of Ref. [42].

In order to eliminate this problem it was proposed already in the early days of dual theories to assume, in analogy with what happens for instance in perturbative field theory, that the N-point amplitude was only the lowest order (the tree diagram) of a perturbative expansion and, in order to implement unitarity, it was necessary to include loop diagrams. Then, the one-loop diagrams were constructed from the propagator and vertices that we have introduced in the previous sections [44]. The planar one-loop amplitude with M external particles was computed by starting from a $(M+2)$-point tree amplitude and then by sewing two external legs together after the insertion of a propagator D given in Eq. (100). In this way one gets:

$$\int \frac{d^d P}{(2\alpha')^{d/2}(2\pi)^d} \sum_{\lambda} \langle P, \lambda | V(1, p_1) D V(1, p_2) \dots V(1, p_N) D | P, \lambda \rangle \quad (225)$$

where the sum over λ corresponds to the trace in the space of the harmonic oscillators and the integral in $d^d P$ corresponds to integrate over the momentum circulating in the loop. The previous expression for the one-loop amplitude cannot be quite correct because all states of the space generated by the oscillators in Eq. (51) are circulating in the loop, while we know that we should include only the physical ones. This was achieved first by cancelling by hand the time and one of the space components of the harmonic oscillators reducing the degrees of freedom of each oscillator from d to $d-2$ as suggested by the DDF operators at least for $d = 26$. This procedure was then shown to be correct by Brink and Olive [45]. They constructed the operator that projects over the physical states and, by inserting it in the loop, showed that the reduction of the degrees of freedom of the oscillators from d to $d-2$ was indeed correct. This was, at that time, the only procedure available to let only the physical states circulate in the loop because the BRST procedure was discovered a bit later also in the framework of the gauge field theories!

To be more explicit let us compute the trace in Eq. (225) adding also the Chan-Paton factor. We get:

$$(2\pi)^d \delta^{(d)} \left(\sum_{i=1}^{M} p_i \right) \frac{N Tr(\lambda^{a_1} \dots \lambda^{a_M})}{(8\pi^2 \alpha')^{d/2}} N_0^M \int_0^\infty \frac{d\tau}{\tau^{d/2+1}} [f_1(k)]^{2-d} k^{\frac{d-26}{12}} (2\pi)^M \times$$

$$\times \int_0^1 d\nu_M \int_0^{\nu_M} d\nu_{M-1} \dots \int_0^{\nu_3} d\nu_2 \; \tau^M \prod_{i<j} \left[e^{G(\nu_{ji})} \right]^{2\alpha' p_i \cdot p_j} \; ; k \equiv e^{-\pi\tau} \quad (226)$$

where $\nu_{ji} \equiv \nu_j - \nu_i$,

$$G(\nu) = \log \left(i e^{-\pi \nu^2 \tau} \frac{\Theta_1(i\nu\tau | i\tau)}{f_1^3(k)} \right) \quad ; \quad f_1(k) = k^{1/12} \prod_{n=1}^{\infty} (1 - k^{2n}) \quad (227)$$

and

$$\Theta_1(\nu|i\tau) = -2k^{1/4}\sin\pi\nu\prod_{n=1}^{\infty}\left(1-e^{2i\pi\nu}k^{2n}\right)\left(1-e^{-2i\pi\nu}k^{2n}\right)(1-k^{2n}) \quad (228)$$

Finally the normalization factor N_0 is given in Eq. (218). We have performed the calculation for an arbitrary value of the space-time dimension d. However, in this way one gets also the extra factor of $k^{\frac{d-26}{12}}$ appearing in the first line of Eq. (226) that implies that our calculation is actually only consistent if $d = 26$. In fact, the presence of this factor does not allow one to rewrite the amplitude, originally obtained in the Reggeon sector, in the Pomeron sector as explained below. In the following we neglect this extra factor, implicitly assuming that $d = 26$, but, on the other hand, still keeping an arbitrary d.

Using the relations:

$$f_1(k) = \sqrt{t}f_1(q) \quad ; \quad \Theta_1(i\nu\tau|i\tau) = i\Theta_1(\nu|it)t^{1/2}e^{\pi\nu^2/t} \quad (229)$$

where $t = \frac{1}{\tau}$ and $q \equiv e^{-\pi t}$, we can rewrite the one-loop planar diagram in the Pomeron channel. We get:

$$(2\pi)^d\delta^{(d)}\left(\sum_{i=1}^{M}p_i\right)\frac{NTr(\lambda^{a_1}\dots\lambda^{a_M})}{(8\pi^2\alpha')^{d/2}}N_0^M\int_0^\infty dt[f_1(q)]^{2-d}(2\pi)^M\times$$

$$\times\int_0^1 d\nu_M\int_0^{\nu_M}d\nu_{M-1}\dots\int_0^{\nu_3}d\nu_2\prod_{i<j}\left[-\frac{\Theta_1(\nu_{ji}|it)}{f_1^3(q)}\right]^{2\alpha' p_i\cdot p_j} \quad (230)$$

Notice that, by factorizing the planar loop in the Pomeron channel, one constructed for the first time what we now call the boundary state [46] [16]. This can be easily seen in the way that we are now going to describe. First of all, notice that the last quantity in Eq. (230) can be written as follows:

$$\prod_{i<j}\left[-\frac{\Theta_1(\nu_{ji}|it)}{f_1^3(q)}\right]^{2\alpha' p_i\cdot p_j} =$$

$$= \prod_{i<j}\left[-2\sin(\pi\nu_{ji})\prod_{n=1}^{\infty}\frac{\left(1-q^{2n}e^{2\pi i\nu_{ji}}\right)\left(1-q^{2n}e^{-2\pi i\nu_{ji}}\right)}{(1-q^{2n})^2}\right]^{2\alpha' p_i\cdot p_j} \quad (231)$$

This equation can be rewritten as follows:

$$\frac{Tr\left(\langle p=0|q^{2R}\prod_{i=1}^{M}:e^{ip_i\cdot Q(e^{2i\pi\nu_i})}:|p=0\rangle\right)i^M}{Tr\left(\langle p=0|q^{2N}|p=0\rangle\right)} \;;\; R = \sum_{n=1}^{\infty}na_n^{\dagger}\cdot a_n \quad (232)$$

[16] See also the first paper in Ref. [47].

where the trace is taken only over the non-zero modes and momentum conservation has been used. It must also be stressed that the normal ordering of the vertex operators in the previous equation is such that the zero modes are taken to be both in the same exponential instead of being ordered as in Eq. (59). By bringing all annihilation operators on the left of the creation ones, from the expression in Eq. (232) one gets ($z_i \equiv e^{2\pi i \nu_i}$):

$$(2\pi)^d \delta^{(d)} \left(\sum_{i=1}^{\infty} p_i \right) \prod_{i<j} (-2 \sin \pi \nu_{ji})^{2\alpha' p_i \cdot p_j} \times$$

$$\times \frac{\prod_{i.j} \prod_{n=1}^{\infty} Tr \left(q^{2n a_n^\dagger \cdot a_n} e^{\sqrt{2\alpha'} p_j \cdot \frac{a_n^\dagger}{\sqrt{n}} z_j^n} e^{-\sqrt{2\alpha'} p_i \cdot \frac{a_n}{\sqrt{n}} z_i^{-n}} \right)}{Tr \left(\langle p=0 | q^{2N} | p=0 \rangle \right)} \tag{233}$$

The trace can be computed by using the completeness relation involving coherent states $|f\rangle = e^{f a^\dagger} |0\rangle$:

$$\int \frac{d^2 f}{\pi} e^{-|f|^2} |f\rangle \langle f| = 1 \tag{234}$$

Inserting the previous identity operator in Eq. (233) one gets after some calculation:

$$(2\pi)^d \delta^{(d)} \left(\sum_{i=1}^{\infty} p_i \right) \prod_{i<j} (-2 \sin \pi \nu_{ji})^{2\alpha' p_i \cdot p_j} \times$$

$$\times \prod_{i.j=1}^{M} \prod_{n=1}^{\infty} e^{-2\alpha' p_i \cdot p_j e^{2\pi i n \nu_{ji}} \frac{q^{2n}}{n(1-q^{2n})}} \tag{235}$$

Expanding the denominator in the last exponent and performing the sum over n one gets:

$$(2\pi)^d \delta^{(d)} \left(\sum_{i=1}^{\infty} p_i \right) \prod_{i<j} (-2 \sin \pi \nu_{ji})^{2\alpha' p_i \cdot p_j} \times$$

$$\times \prod_{i.j} e^{2\alpha' p_i \cdot p_j \sum_{m=0}^{\infty} \log\left(1 - e^{2\pi i \nu_{ji}} q^{2(m+1)}\right)} \tag{236}$$

that is equal to the last line of Eq. (231) apart from the δ-function for momentum conservation. In conclusion, we have shown that Eq.s (231) and (232) are equal.

Using Eq. (231) we can rewrite Eq. (230) as follows:

$$\frac{N N_0^M Tr(\lambda^{a_1} \ldots \lambda^{a_M})}{(8\pi^2 \alpha')^{d/2}} \int_0^\infty dt [f_1(q)]^{2-d} (2\pi i)^M \int_0^1 d\nu_M \int_0^{\nu_M} d\nu_{M-1} \ldots$$

$$\ldots \int_0^{\nu_3} d\nu_2 \frac{\sum_\lambda \langle p=0,\lambda|q^{2R}\prod_{i=1}^{M} : e^{ip_i\cdot Q(e^{2i\pi\nu_i})} : |p=0,\lambda\rangle}{\sum_\lambda \langle p=0,\lambda|q^{2N}|p=0,\lambda\rangle} \tag{237}$$

where the sum over any state $|\lambda\rangle$ corresponds to taking the trace over the non-zero modes. If $d=26$ we can rewrite Eq. (237) in a simpler form:

$$\frac{NN_0^M Tr(\lambda^{a_1}\ldots\lambda^{a_M})}{(8\pi^2\alpha')^{d/2}} \int_0^\infty dt\,(2\pi i)^M \int_0^1 d\nu_M \int_0^{\nu_M} d\nu_{M-1}\ldots\int_0^{\nu_3} d\nu_2 \times$$

$$\times \sum_\lambda \langle p=0,\lambda|q^{2R-2}\prod_{i=1}^{M} : e^{ip_i\cdot Q(e^{2i\pi\nu_i})} : |p=0,\lambda\rangle \tag{238}$$

The previous equation contains the factor $\int dtq^{2R-2}$ that is like the propagator of the Shapiro-Virasoro model, but with only one set of oscillators as in the generalized Veneziano model. In the following we will rewrite it completely with the formalism of the Shapiro-Virasoro model. This can be done by introducing the Pomeron propagator:

$$\int_0^\infty dt\; q^{2N-2} = \frac{2}{\pi\alpha'}\hat{D} \;\;;\;\; \hat{D} \equiv \frac{\alpha'}{4\pi}\int \frac{d^2z}{|z|^2} z^{L_0-1}\bar{z}^{\tilde{L}_0-1}; |z| \equiv q = e^{-\pi t} \tag{239}$$

and rewriting the planar loop in the following compact form:

$$\langle B_0|\hat{D}|B_M\rangle \;\;;\;\; |B_0\rangle \equiv \frac{T_{d-1}}{2} N \prod_{n=1}^{\infty} e^{a_n^\dagger\cdot\tilde{a}_n^\dagger}|p=0,0_a,0_{\tilde{a}}\rangle \tag{240}$$

where $|B_0\rangle$ is the boundary state without any Reggeon on it,

$$T_{d-1} = \frac{\sqrt{\pi}}{2^{(d-10)/4}}(2\pi\sqrt{\alpha'})^{-d/2-1} \tag{241}$$

and $|B_M\rangle$ is instead the one with M Reggeons given by:

$$|B_M\rangle = N_0^M Tr(\lambda^{a_1}\ldots\lambda^{a_M})(2\pi i)^M \int_0^1 d\nu_M \int_0^{\nu_M} d\nu_{M-1}\ldots\int_0^{\nu_3} d\nu_2 \times$$

$$\times \prod_{i=1}^{M} : e^{ip_i\cdot Q(e^{2i\pi\nu_i})} : |B_0\rangle \tag{242}$$

We want to stress once more that the normal ordering in the previous equation is defined by taking the zero modes in the same exponential. Both the boundary states and the propagator are now states of the Shapiro-Virasoro model. This means that we have rewritten the one-loop planar diagram, where the states of the generalized Veneziano model circulate in the loop, as a tree diagram of the Shapiro-Virasoro model involving two boundary states and a propagator. This is what nowadays is called open/closed string duality.

Besides the one-loop planar diagram in Eq. (225), that is nowadays called the annulus diagram, also the non-planar and the non-orientable diagrams were constructed and studied. In particular the non-planar one, that is obtained as the planar one in Eq. (225) but with two propagators multiplied with the twist operator

$$\Omega = e^{L_{-1}}(-1)^{R} \ , \tag{243}$$

had unitarity violating cuts that disappeared [27] if the dimension of the space-time $d = 26$, leaving behind additional pole singularities. The explicit form of the non-planar loop can be obtained following the same steps done for the planar loop. One gets for the non-planar loop the following amplitude:

$$\langle B_R|\hat{D}|B_M\rangle \tag{244}$$

where now both boundary states contain, respectively, R and M Reggeon states. The additional poles found in the non-planar loop were called Pomerons because they occur in the Pomeron sector, that today is called the closed string channel, to distinguish them from the Reggeons that instead occur in the Reggeon sector, that today is called the open string sector of the planar and non-planar loop diagrams. At that time in fact, the states of the generalized Veneziano models were called Reggeons, while the additional ones appearing in the non-planar loop were called Pomerons. The Reggeons correspond nowadays to open string states, while the Pomerons to closed string states. These things are obvious now, but at that time it took a while to show that the additional states appearing in the Pomeron sector have to be identified with those of the Shapiro-Virasoro model. The proof that the spectrum was the same came rather early. This was obtained by factorizing the non-planar diagram in the Pomeron channel [46] as we have done in Eq. (244). It was found that the states of the Pomeron channel lie on a linear Regge trajectory that has double intercept and half slope of the one of the Reggeons. This follows immediately from the propagator $\hat{D}$ in Eq. (239) that has poles for values of the momentum of the Pomeron exchanged given by:

$$2 - \frac{\alpha'}{2}p^2 = 2n \tag{245}$$

that are exactly the values of the masses of the states of the Shapiro-Virasoro model [48], while the Reggeon propagator in Eq. (100) has poles for values of momentum equal to:

$$1 - \alpha' p^2 = n \tag{246}$$

However, it was still not clear that the Pomeron states interact among themselves as the states of the Shapiro-Virasoro model. To show this it was first necessary to construct tree amplitudes containing both states of the generalized Veneziano model and of the Shapiro-Virasoro model [49]. They reduced to the amplitudes of the generalized Veneziano (Shapiro-Virasoro) model if we have only external states of the generalized Veneziano (Shapiro-Virasoro) model. Those amplitudes are called today disk amplitudes containing both open and closed string states. They were constructed [49] by using for the Reggeon states the vertex operators that we have discussed in Sect. (5) involving one set of harmonic oscillators and for the Pomeron states the vertex operators given in Eq. (181) that we rewrite here:

$$V_{\alpha,\beta}(z,\bar{z},p) = V_{\alpha}(z,\frac{p}{2})V_{\beta}(\bar{z},\frac{p}{2}) \tag{247}$$

because now both component vertices contain the same set of harmonic oscillators as in the generalized Veneziano model. Furthermore, each of the two vertices is separately normal ordered, but their product is nor normal ordered. The amplitude involving both kinds of states is then constructed by taking the product of all vertices between the projective invariant vacuum and integrating the Reggeons on the real axis in an ordered way and the Pomerons in the upper half plane, as one does for a disk amplitude.

We have mentioned above that the two vertices are separately normal ordered, but their product is not normal ordered. When we normal order them we get, for instance for the tachyon of the Pomeron sector, a factor $(z-\bar{z})^{\alpha' p^2/2}$ that describes the Reggeon-Pomeron transition. This implies a direct coupling [51] between the $U(1)$ part of gauge field and the two-index antisymmetric field $B_{\mu\nu}$, called Kalb-Ramond field [50], of the Pomeron sector, that makes the gauge field massive [51].

It was then shown that, by factorizing the non-planal loop in the Pomeron channel, one reproduced the scattering amplitude containing one state of the Shapiro-Virasoro and a number of states of the generalized Veneziano model [52]. If we have also external states belonging to the generalized Shapiro-Virasoro model, then by factorizing the non-planar one loop amplitude in the pure Pomeron channel, one would obtain the tree amplitudes of the Shapiro-Virasoro model [52].

All this implies that the generalized Veneziano model and the Shapiro-Virasoro model are not two independent models, but they are part of the same and unique model. In fact, if one started with the generalized Veneziano model and added loop diagrams to implement unitarity, one found the appearence in the non-planar loop of additional states that had the same mass and interaction of those of the Shapiro-Virasoro model.

The planar diagram, written in Eq. (230) in the closed string channel, is divergent for large values of t. This divergence was recognized to be due to exchange, in the Pomeron channel, of the tachyon of the Shapiro-Virasoro model and of the dilaton [47]. They correspond, respectively, to the first two terms of the expansion:

$$[f_1(q)]^{-24} = e^{2\pi t} + 24 + O\left(e^{-2\pi t}\right) \tag{248}$$

The first one could be cancelled by an analytic continuation, while the second one could be eliminated through a renormalization of the slope of the Regge trajectory α' [47].

We conclude the discussion of the one-loop diagrams by mentioning that the one-loop diagram for the Shapiro-Virasoro model was computed by Shapiro [53] who also found that the integrand was modular invariant.

The computation of multiloop diagrams requires a more advanced technology that was also developed in the early days of the dual resonance model few years before the discovery of its connection to string theory. In order to compute multiloop diagrams one needs first to construct an object that was called the N-Reggeon vertex and that has the properties of containing N sets of harmonic oscillators, one for each external leg, and is such that, when we saturate it with N physical states, we get the corresponding N-point amplitude. In the following we will discuss how to determine the N-Reggeon vertex.

The first step toward the N-Reggeon vertex is the Sciuto-Della Selva-Saito [54] vertex that includes two sets of harmonic oscillators that we denote with the indices 1 and 2. It is equal to:

$$V_{SDS} = {}_2\langle x = 0, 0| : \exp\left(-\frac{1}{2\alpha'} \oint_0 dz X_2'(z) \cdot X_1(1-z) \right) : \tag{249}$$

where X is the quantity that we have called Q in Eq. (57) and the prime denotes a derivative with respect to z. It satisfies the important property of giving the vertex operator $V_\alpha(z=1)$ of an arbitrary state $|\alpha\rangle$ when we saturate it with the corresponding state:

$$V_{SDS}|\alpha\rangle_2 = V_\alpha(z=1) \tag{250}$$

A shortcoming of this vertex is that it is not invariant under a cyclic permutation of the three legs. A cyclic symmetric vertex has been constructed by Caneschi, Schwimmer and Veneziano [55] by inserting the twist operator in Eq. (243). But the 3-Reggeon vertex is not enough if we want to compute an arbitrary multiloop amplitude. We must generalize it to an arbitrary number of external legs. Such a vertex, that can be obtained from the one in Eq. (249) with a very direct procedure, or that can also be obtained by sewing together three-Reggeon vertices, has been written in its final form by Lovelace [56] [17]. Here we do not derive it, but we give directly its expression written in Ref. [56]:

$$V_{N,0} = \int \frac{\prod_{i=1}^N dz_i}{dV_{abc} \prod_{i=1}^N [V_i'(0)]} \prod_{i=1}^N [{}_i< x = 0, O_a|]\, \delta(\sum_{i=1}^N p_i)$$
$$\prod_{\substack{i,j=1 \\ i\neq j}}^N \exp\left[-\frac{1}{2} \sum_{n,m=0}^\infty a_n^{(i)} D_{nm}(\Gamma V_i^{-1} V_j)\, a_m^{(j)} \right] \tag{251}$$

[17] See also Ref. [57]. Earlier papers on the N-Reggeon can be found in Ref.s [58].

where $a_0^{(i)} \equiv \alpha_0^i = \sqrt{2\alpha'}\hat{p}_i$ is the momentum of particle i and the infinite matrix:

$$D_{nm}(\gamma) = \frac{1}{m!}\sqrt{\frac{m}{n}}\partial_z^m[\gamma(z)]^n|_{z=0} \; ; \; n,m = 1.. \; : \; D_{00}(\gamma) = -\log|\frac{D}{\sqrt{AD-BC}}|$$

$$D_{n0} = \frac{1}{\sqrt{n}}(\frac{B}{D})^n \quad ; \quad D_{0n} = \frac{1}{\sqrt{n}}(-\frac{C}{D})^n \quad ; \quad \gamma(z) = \frac{Az+B}{Cz+D} \tag{252}$$

is a "representation" of the projective group corresponding to the conformal weight $\Delta = 0$, that satisfies the eqs.:

$$D_{nm}(\gamma_1\gamma_2) = \sum_{l=1}^{\infty} D_{nl}(\gamma_1)D_{lm}(\gamma_2) + D_{n0}(\gamma_1)\delta_{0m} + D_{0m}(\gamma_2)\delta_{n0} \tag{253}$$

and

$$D_{nm}(\gamma) = D_{mn}(\Gamma\gamma^{-1}\Gamma) \qquad\qquad \Gamma(z) = \frac{1}{z} \tag{254}$$

Finally V_i is a projective transformation that maps $0, 1$ and ∞ into z_{i-1}, z_i and z_{i+1}.

The previous vertex can be written in a more elegant form as follows:

$$V_{N,0} = \int \frac{\prod_{i=1}^N dz_i}{dV_{abc}\prod_{i=1}^N [V_i'(0)]} \prod_{i=1}^N [{}_i\!< x = 0, O_a|] \, \delta(\sum_{i=1}^N p_i)$$

$$\exp\left\{\frac{i}{4\alpha'}\oint dz \partial X^{(i)}(z)\hat{p}_i \log V_i'(z)\right\}$$

$$\exp\left\{-\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^N \oint dz \oint dy \partial X^{(i)}(z)\log[V_i(z) - V_j(y)]\partial X^{(j)}(y)\right\} \tag{255}$$

where the quantities $X^{(i)}$ are what we called Q, namely the Fubini-Veneziano field, in the previous sections. The N-Reggeon vertex that satisfies the important property of giving the scattering amplitude of N physical particle when we saturate it with their corresponding states, is the fundamental object for computing the multiloop amplitudes. In fact, if we want to compute a M-loop amplitude with N external states, we need to start from the $(N+2M)$-Reggeon vertex and then we have to sew the M pairs together after having inserted a propagator D. In this way we obtain an amplitude that is not only integrated over the punctures z_i $(i = 1 \ldots N)$ of the N external states, but also over the additional $3h-3$ moduli corresponding to the punctures variables of the states that we sew together and the integration variable of the M propagators. h is the number of loops. The multiloop amplitudes have been obtained in this way already in 1970 [59, 60, 61] and, through the sewing procedure, one obtained functions, as the period matrix, the abelian differentials, the prime form, etc., that are well defined on Riemann surface! The only thing that was missing, was the correct measure of integrations over the $3h-3$ variables because it was technically not possible to let only the physical states to circulate in the loops. This problem was solved only much later [62, 63] when a BRST invariant formulation of string theory and the light-cone functional integral could be used for computing multiloops. They are two very different approaches that, however, gave the same result. For the sake of completeness we write here the planar h-loop amplitude involving M tachyons:

$$A_M^{(h)}(p_1,\ldots,p_M) = N^h \,\mathrm{Tr}(\lambda^{a_1}\cdots\lambda^{a_M})\; C_h\;\left[2g_s\,(2\alpha')^{(d-2)/4}\right]^M \times \int [dm]_h^M \prod_{i<j}\left[\frac{\exp\left(\mathcal{G}^{(h)}(z_i,z_j)\right)}{\sqrt{V_i'(0)\,V_j'(0)}}\right]^{2\alpha' p_i\cdot p_j} \quad , \tag{256}$$

where $N^h\,\mathrm{Tr}(\lambda^{a_1}\cdots\lambda^{a_M})$ is the appropriate $U(N)$ Chan-Paton factor, g is the dimensionless open string coupling constant, $\mathcal{C}_h$ is a normalization factor given by

$$C_h = \frac{1}{(2\pi)^{dh}}\; g_s^{2h-2}\frac{1}{(2\alpha')^{d/2}} \quad , \tag{257}$$

and $\mathcal{G}^{(h)}$ is the h-loop bosonic Green function

$$\mathcal{G}^{(h)}(z_i,z_j) = \log E^{(h)}(z_i,z_j) - \frac{1}{2}\int_{z_i}^{z_j}\omega^\mu\;(2\pi\mathrm{Im}\tau_{\mu\nu})^{-1}\int_{z_i}^{z_j}\omega^\nu \quad , \tag{258}$$

with $E^{(h)}(z_i,z_j)$ being the prime form, ω^μ ($\mu=1,\ldots,h$) the abelian differentials and $\tau_{\mu\nu}$ the period matrix. All these objects, as well as the measure on moduli space $[dm]_h^M$, can be explicitly written in the Schottky parametrization of the Riemann surface, and their expressions for arbitrary h can be found for example in Ref. [64]. It is given by

$$[dm]_h^M = \frac{1}{dV_{abc}}\prod_{i=1}^{M}\frac{dz_i}{V_i'(0)}\prod_{\mu=1}^{h}\left[\frac{dk_\mu\,d\xi_\mu\,d\eta_\mu}{k_\mu^2\,(\xi_\mu-\eta_\mu)^2}(1-k_\mu)^2\right] \times\;[\det(-i\tau_{\mu\nu})]^{-d/2}\;\prod_\alpha{}'\left[\prod_{n=1}^{\infty}(1-k_\alpha^n)^{-d}\prod_{n=2}^{\infty}(1-k_\alpha^n)^2\right] \quad . \tag{259}$$

where k_μ are the multipliers, ξ_μ and η_μ are the fixed points of the generators of the Schottky group,

9 From dual models to string theory

The approach presented in the previous sections is a real bottom-up approach. The experimental data were the driving force in the construction of the Veneziano model and of its generalization to N external legs. The rest of the work that we have described above consisted in deriving its properties. The result is, except for a tachyon, a fully consistent quantum-relativistic model that was a source of fascination for those who worked in the field. Although the model grew out of S-matrix theory where the scattering amplitude is the only observable object, while the action or the Lagrangian have not a central role, some people nevertheless started to investigate what was the underlying microscopic structure that gave rise to such a consistent and beautiful model. It turned out, as we know today, that this underlying structure is that of a quantum-relativistic string. However, the process of connecting the dual resonance model (actually two of them the generalized Veneziano and the Shapiro-Virasoro model) to string theory took several years from the original idea to a complete and convincing proof of the conjecture. The original conjecture was independently formulated by Nambu [20, 65], Nielsen [66] and Susskind [21] [18]. If we look at it in retrospective, it was at that time a fantastic idea that shows the enormous physical intuition of those who formulated it. On the other hand, it took several years to digest it before one was able to derive from it all the deep features of the dual resonance model. Because of this, the idea that the underlying structure was that of a relativistic string, did not really influence most of the research in the field up to 1973. Let me try to explain why.

A common feature of the work of Ref.s [20, 66, 21] is the suggestion that the infinite number of oscillators, that one got through the factorization of the dual resonance model, naturally comes out from a two-dimensional free Lagrangian for the coordinate $X^\mu(\tau, \sigma)$ of a one-dimensional string, that is an obvious generalization of the Lagrangian that one writes for the coordinate $X^\mu(\tau)$ of a pointlike object in the proper-time gauge:

$$L \sim \frac{1}{2}\frac{dX}{d\tau} \cdot \frac{dX}{d\tau} \Longrightarrow L \sim \frac{1}{2}\left[\frac{dX}{d\tau} \cdot \frac{dX}{d\tau} - \frac{dX}{d\sigma} \cdot \frac{dX}{d\sigma}\right] \tag{260}$$

Being this theory conformal invariant the Virasoro operators were also constructed together with their algebra. In this very first formulation, however, the Virasoro generators L_n were just the generators associated to the conformal symmetry of the string world-sheet Lagrangian given in Eq. (260) as in any conformal field theory. It was not clear at all why they should imply the gauge conditions found by Virasoro or, in modern terms, why they should be zero classically. The basic ingredient to solve this problem was provided by Nambu [65] and Goto [68] who wrote the non-linear Lagrangian proportional

[18] See also Ref. [67].

to the area spanned by the string in the external target space. They proceeded in analogy with the point particle and wrote the following action:

$$S \sim \int \sqrt{-d\sigma_{\mu\nu} d\sigma^{\mu\nu}} \tag{261}$$

where

$$d\sigma_{\mu\nu} = \frac{\partial X_\mu}{\partial \zeta^\alpha}\frac{\partial X_\nu}{\partial \zeta^\beta} d\zeta^\alpha \wedge d\zeta^\beta = \frac{\partial X_\mu}{\partial \zeta^\alpha}\frac{\partial X_\nu}{\partial \zeta^\beta}\epsilon^{\alpha\beta} d\sigma d\tau \tag{262}$$

$X_\mu(\sigma, \tau)$ is the string coordinate and $\zeta^0 = \tau$ and $\zeta^1 = \sigma$ are the coordinates of the string worldsheet. $\epsilon^{\alpha\beta}$ is an antisymmetric tensor with $\epsilon^{01} = 1$. Inserting eq. (262) in (261) and fixing the proportionality constant one gets the Nambu-Goto action [65, 68]:

$$S = -cT \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2} \tag{263}$$

where

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau} \qquad\qquad X'^\mu \equiv \frac{\partial X^\mu}{\partial \sigma} \tag{264}$$

and $T \equiv \frac{1}{2\pi\alpha'}$ is the string tension, that replaces the mass appearing in the case of a point particle. In this formulation, the string Lagrangian is invariant under any reparametrization of the world-sheet coordinates σ and τ and not only under the conformal transformations. This, in fact, implies that the two-dimensional world-sheet energy-momentum tensor of the string is actually zero as we will show later on. But it took still a few years to connect the Nambu-Goto action to the properties of the dual resonance model. In the meantime an analogue model was formulated [69] that reproduced the tree and loop amplitudes of the generalized Veneziano model. This approach anticipated by several years the path integral derivation of dual amplitudes. It was very closely related to the functional integral formulation of Ref.s [70].

However, one needed to wait until 1973 with the paper of Goddard, Goldstone, Rebbi and Thorn [71], where the Nambu-Goto action was correctly treated, all its consequences were derived and it became completely clear that the structure underlying the dual resonance model was that of a quantum-relativistic string. The equation of motion for the string were derived from the action in Eq. (263) by imposing $\delta S = 0$ for variations such that $\delta X^\mu(\tau_i) = \delta X^\mu(\tau_f) = 0$. One gets:

$$\delta S = \int_{\tau_i}^{\tau_f} \left[\int_0^\pi d\sigma \left(-\frac{\partial}{\partial \tau}\frac{\partial L}{\partial \dot{X}^\mu} - \frac{\partial}{\partial \sigma}\frac{\partial L}{\partial X'^\mu} \right) \delta X^\mu + \frac{\partial L}{\partial X'^\mu} \delta X^\mu |_{\sigma=0}^{\sigma=\pi} \right] = 0 \tag{265}$$

where L is the Lagrangian in Eq. (263). Since δX^μ is arbitrary, from eq. (265) one gets the Euler-Lagrange equation of motion

$$\frac{\partial}{\partial\tau}\frac{\partial L}{\partial\dot{X}^{\mu}}+\frac{\partial}{\partial\sigma}\frac{\partial L}{\partial X'^{\mu}}\equiv\frac{\partial}{\partial\zeta^{\alpha}}\left(\frac{\partial L}{\partial(\frac{\partial X^{\mu}}{\partial\zeta^{\alpha}})}\right)=0 \tag{266}$$

and the boundary conditions

$$\frac{\partial L}{\partial X'^{\mu}}=0 \qquad or \qquad \delta X_{\mu}=0 \qquad at \qquad \sigma=0,\pi \tag{267}$$

for an open string and

$$X^{\mu}(\tau,0)=X^{\mu}(\tau,\pi) \tag{268}$$

for a closed string. In the case of an open string, the first kind of boundary condition in Eq.(267) corresponds to Neumann boundary conditions, while the second one to Dirichlet boundary conditions. Only the Neumann boundary conditions preserve the translation invariance of the theory and, therefore, they were mostly used in the early days of string theory. It must be stressed, however, that Dirichlet boundary conditions were already discussed and used in the early days of string theory for constructing models with off-shell states [72].

From Eq. (263) one can compute the momentum density along the string:

$$\frac{\partial L}{\partial\dot{X}^{\mu}}\equiv P_{\mu}=cT\frac{\dot{X}_{\mu}X'^{2}-X'_{\mu}(\dot{X}\cdot X')}{\sqrt{(\dot{X}\cdot X')^{2}-\dot{X}^{2}X'^{2}}} \tag{269}$$

and obtain the following constraints between the dynamical variables X^{μ} and P^{μ}:

$$c^{2}T^{2}x'^{2}+P^{2}=x'\cdot P=0 \tag{270}$$

They are a consequence of the reparametrization invariance of the string Lagrangian. Because of this one can choose the orthonormal gauge specified by the conditions:

$$\dot{X}^{2}+X'^{2}=\dot{X}\cdot X'=0 \tag{271}$$

that nowadays is called conformal gauge. In this gauge eq. (269) becomes:

$$P_{\mu}=cT\dot{X}_{\mu} \qquad\qquad \frac{\partial L}{\partial X'^{\mu}}=-cTX'_{\mu} \tag{272}$$

and therefore the eq. of motion in eq.(266) becomes:

$$\ddot{X}_{\mu}-X''_{\mu}=0 \tag{273}$$

while the boundary condition in eq.(267) becomes:

$$X'_{\mu}(\sigma=0,\pi)=0 \tag{274}$$

The most general solution of the eq. of motion and of the boundary conditions can be written as follows:

$$X^{\mu}(\tau,\sigma) = q^{\mu} + 2\alpha' p^{\mu}\tau + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}[a_n^{\mu}e^{-in\tau} - a_n^{+\mu}e^{in\tau}]\frac{cosn\sigma}{\sqrt{n}} \quad (275)$$

for an open string and

$$X^{\mu}(\tau,\sigma) = q^{\mu} + 2\alpha' p^{\mu}\tau + \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}[\tilde{a}_n^{\mu}e^{-2in(\tau+\sigma)} - \tilde{a}_n^{+\mu}e^{2in(\tau+\sigma)}]\frac{1}{\sqrt{n}}+$$

$$+\frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}[a_n^{\mu}e^{-2in(\tau-\sigma)} - a_n^{+\mu}e^{2in(\tau-\sigma)}]\frac{1}{\sqrt{n}} \quad (276)$$

for a closed string. This procedure really shows that, starting from the Nambu-Goto action, one can choose a gauge (the orthonormal or conformal gauge) where the equation of motion of the string becomes the two-dimensional D'Alembert equation in Eq. (273). Furthermore, the invariance under reparametrization of the Nambu-Goto action implies that the two-dimensional energy-momentum tensor is identically zero at the classical level (See Eq. (271)).

As the Lorentz gauge in QED the orthonormal gauge does not fix completely the gauge. We can still perform reparametrizations that leave in the conformal gauge: they are conformal transformatiuons. Introducing the variable $z = e^{i\tau}$ the generators of the conformal transformations for the open string can be written as follows:

$$L_n = \frac{1}{2\pi i}\oint dz z^{n+1}\left[-\frac{1}{4\alpha'}\left(\frac{\partial X^{\mu}}{\partial z}\right)^2\right] = \frac{1}{2}\sum_{m=-\infty}^{\infty}\alpha_{n-m}\cdot\alpha_m = 0 \quad (277)$$

where

$$\alpha_n^{\mu} = \begin{cases} \sqrt{n}a_n^{\mu} & if \;\; n>0 \\ \sqrt{2\alpha'}p^{\mu} & if \;\; n=0 \\ \sqrt{n}a_n^{\dagger\mu} & if \;\; n<0 \end{cases} \quad (278)$$

They are zero as a consequence of Eq.s (270) that in the conformal gauge become Eq.s (271). In the case of a closed string we get instead:

$$\tilde{L}_n = \frac{1}{2\pi i}\oint dz z^{n+1}\left[-\frac{1}{\alpha'}\left(\frac{\partial X^{\mu}}{\partial z}\right)^2\right] = 0 \quad (279)$$

$$L_n = \frac{1}{2\pi i}\oint d\bar{z}\bar{z}^{n+1}\left[-\frac{1}{\alpha'}\left(\frac{\partial X^{\mu}}{\partial \bar{z}}\right)^2\right] = 0 \quad (280)$$

In terms of the harmonic oscillators introduced in eq. (276) we get

$$L_n = \frac{1}{2}\sum_{m=-\infty}^{\infty} \alpha_m \cdot \alpha_{n-m} = 0 \quad ; \quad \tilde{L}_n = \frac{1}{2}\sum_{m=-\infty}^{\infty} \tilde{\alpha}_m \cdot \tilde{\alpha}_{n-m} = 0 \tag{281}$$

where for the non-zero modes we have used the convention in (278), while the zero mode is given by:

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{2\alpha'}\frac{p^\mu}{2} \tag{282}$$

In conclusion, the fact that we have reparametrization invariance implies that the Virasoro generators are classically identically zero. When we quantize the theory one cannot and also does not need to impose that they are vanishing at the operator level. They are imposed as conditions characterizing the physical states.

$$\langle Phys'|L_n|Phys\rangle = \langle Phys'|(L_0 - 1)|Phys\rangle = 0 \;\; ; \;\; n \neq 0 \tag{283}$$

These equations are satisfied if we require:

$$L_n|Phys >= (L_0 - 1)|Phys >= 0 \tag{284}$$

The extra factor -1 in the previous equations comes from the normal ordering as explained in Eq. (198).

The authors of Ref. [71] further specified the gauge by fixing it completely. They introduced the light-cone gauge specified by imposing the condition:

$$X^+ = 2\alpha' p^+ \tau \tag{285}$$

where

$$X^{\pm} = \frac{X^0 \pm X^{d-1}}{\sqrt{2}} \qquad\qquad X_{\pm} = \frac{X_0 \pm X_{d-1}}{\sqrt{2}} \tag{286}$$

In this gauge the only physical degrees of freedom are the transverse ones. In fact the components along the directions 0 and $d-1$ can be expressed in terms of the transverse ones by inserting Eq. (285) in the constraints in Eq. (271) and getting:

$$\dot{X}^- = \frac{1}{4\alpha' p^+}(\dot{X}_i^2 + X'^2_i) \qquad X'^- = \frac{1}{2\alpha' p^+}\dot{X}_i \cdot X'_i \tag{287}$$

that up to a constant of integration determine completely X^- as a function of X^i. In terms of oscillators we get

$$\alpha_n^+ = 0 \quad ; \quad \sqrt{2\alpha'}\alpha_n^- = \frac{1}{2p^+}\sum_{m=-\infty}^{\infty} \alpha_{n-m}^i \alpha_m^i \qquad n \neq 0 \tag{288}$$

for an open string and

$$\alpha_n^+ = \tilde{\alpha}_n^+ = 0 \qquad\qquad n \neq 0 \tag{289}$$

together with

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{2p^+} \sum_{m=-\infty}^{\infty} \alpha_{n-m}^i \alpha_m^i$$

$$\sqrt{2\alpha'}\tilde{\alpha}_n^- = \frac{1}{2p^+} \sum_{m=-\infty}^{\infty} \tilde{\alpha}_{n-m}^i \tilde{\alpha}_m^i \tag{290}$$

in the case of a closed string.

This shows that the physical states are described only by the transverse oscillators having only $d-2$ components. Those transverse oscillators correspond to the transverse DDF operators that we have discussed in Section 6. The authors of Ref. [71] also constructed the Lorentz generators only in terms of the transverse oscillators and they showed that they satisfy the correct Lorentz algebra only if the space-time dimension is $d = 26$. In this way the spectrum of the dual resonance model was completely reproduced starting from the Nambu-Goto action if $d = 26$! On the other hand, the choice of $d = 26$ is a necessity if we want to keep Lorentz invariance!

Immediately after this, the interaction was also included either by adding a term describing the interaction of the string with an external gauge field [73] or by using a functional formalism [74, 75].

In the following we will give some detail only of the first approach for the case of an open string. A way to describe the string interaction is by adding to the free string action an additional term that describes the interaction of the string with an external field.

$$S_{INT} = \int d^D y \Phi_L(y) J_L(y) \tag{291}$$

where $\Phi_L(y)$ is the external field and J_L is the current generated by the string. The index L stands for possible Lorentz indices that are saturated in order to have a Lorentz invariant action.

In the case of a point particle, such an interaction term will not give any information on the self-interaction of a particle.

In the case of a string, instead, we will see that S_{INT} will describe the interaction among strings because the external fields that can consistently interact with a string are only those that correspond to the various states of the string, as it will become clear in the discussion below.

This is a consequence of the fact that, for the sake of consistency, we must put the following restrictions on S_{INT}:

- It must be a well defined operator in the space spanned by the string oscillators.

- It must preserve the invariances of the free string theory. In particular, in the "conformal gauge" it must be conformal invariant.
- In the case of an open string, the interaction occurs at the end point of a string (say at $\sigma = 0$). This follows from the fact that two open strings interact attaching to each other at the end points.

The simplest scalar current generated by the motion of a string can be written as follows

$$J(y) = \int d\tau \int d\sigma \delta(\sigma) \delta^{(d)} [y^\mu - x^\mu(\tau, \sigma)] \tag{292}$$

where $\delta(\sigma)$ has been introduced because the interaction occurs at the end of the string. For the sake of simplicity we omit to write a coupling constant g in (292).

Inserting (292) in (291) and using for the scalar external field $\Phi(y) = e^{ik \cdot y}$ a plane wave, we get the following interaction:

$$S_{INT} = \int d\tau : e^{ik \cdot X(\tau,0)} : \tag{293}$$

where the normal ordering has been introduced in order to have a well defined operator. The invariance of (293) under a conformal transformation $\tau \to w(\tau)$ requires the following identity:

$$S_{INT} = \int d\tau : e^{ik \cdot X(\tau,0)} : = \int dw : e^{ik \cdot X(w,0)} : \tag{294}$$

or, in other words, that

$$: e^{ik \cdot X(\tau,0)} : \Longrightarrow w'(\tau) : e^{ik \cdot X(w,0)} : \tag{295}$$

This means that the integrand in Eq. (294) must be a conformal field with conformal dimension equal to one and this happens only if $\alpha' k^2 = 1$. The external field corresponds then to the tachyonic lowest state of the open string. Another simple current generated by the string is given by:

$$J_\mu(y) = \int d\tau \int d\sigma \delta(\sigma) \dot{X}_\mu(\tau, \sigma) \delta^{(d)}(y - X(\tau, \sigma)) \tag{296}$$

Inserting (296) in (291) we get

$$S_{INT} = \int d\tau \dot{X}_\mu(\tau, 0) \epsilon^\mu e^{ik \cdot X(\tau,0)} \tag{297}$$

if we use a plane wave for $\Phi_\mu(y) = \epsilon_\mu e^{ik \cdot y}$. The vertex operator in eq. (297) is conformal invariant only if

$$k^2 = \epsilon \cdot k = 0 \tag{298}$$

and, therefore, the external vector must be the massless photon state of the string. We can generalize this procedure to an arbitrary external field and the result is that we can only use external fields that correspond to on shell physical states of the string.

This procedure has been extended in Ref. [73] to the case of external gravitons by introducing in the Nambu-Goto action a target space metric and obtaining the vertex operator for the graviton that is a massless state in the closed string theory. Remember that, at that time, this could have been done only with the Nambu-Goto action because the σ-model action was introduced only in 1976 first for the point particle [76] and then for the string [77]. As in the case of the photon it turned out that the external field corresponding to the graviton was required to be on shell. This condition is the precursor of the equations of motion that one obtains from the σ-model action requiring the vanishing of the β-function [78].

One can then compute the probability amplitude for the emission of a number of string states corresponding to the various external fields, from an initial string state to a final one. This amplitude gives precisely the N-point amplitude that we discussed in the previous sections [73]. In particular, one learns that, in the case of the open string, the Fubini-Veneziano field is just the string coordinate computed at $\sigma = 0$:

$$Q^{\mu}(z) \equiv X^{\mu}(z, \sigma = 0) \;\; ; \;\; z = e^{i\tau} \tag{299}$$

In the case of a closed string we get instead:

$$Q^{\mu}(z, \bar{z}) \equiv X^{\mu}(z, \bar{z}) \;\; ; \;\; z = e^{2i(\tau-\sigma)} \; , \; \bar{z} = e^{2i(\tau+\sigma)} \tag{300}$$

Finally, let me mention that with the functional approach Mandelstam [74] and Cremmer and Gervais [79] computed the interaction between three arbitrary physical string states and reproduced in this way the coupling of three DDF states given in Eq. (202) and obtained in Ref. [37] by using the operator formalism. At this point it was completely clear that the structure underlying the generalized Veneziano model was that of an open relativistic string, while that underlying the Shapiro-Virasoro model was that of a closed relativistic string. Furthermore, these two theories are not independent because, if one starts from an open string theory, one gets automatically closed strings by loop corrections.

10 Conclusions

In this contribution, we have gone through the developments that led from the construction of the dual resonance model to the bosonic string theory trying as much as possible to include all the necessary technical details. This is because we believe that they are not only important from an historical point of view, but are also still part of the formalism that one uses today in many string calculations. We have tried to be as complete and objective as possible, but it could very well be that some of those who participated in the research of these years, will not agree with some or even many of the statements we made. We apologize to those we have forgotten to mention or we have not mentioned as they would have liked.

Finally, after having gone through the developments of these years, my thoughts go to Sergio Fubini who shared with me and Gabriele many of the ideas described here and who is deeply missed, and to my friends from Florence, Naples and Turin for a pleasant collaboration in many papers discussed here.

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