DIRECT THEOREMS IN THE THEORY OF APPROXIMATION OF THE BANACH SPACE VECTORS BY ENTIRE VECTORS OF EXPONENTIAL TYPE

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ABSTRACT. For an arbitrary operator A on a Banach space $\mathfrak{X}$ which is a generator of C_0–group with certain growth condition at the infinity, the direct theorems on connection between the smoothness degree of a vector $x \in \mathfrak{X}$ with respect to the operator A, the order of convergence to zero of the best approximation of x by exponential type entire vectors for the operator A, and the k-module of continuity are given. Obtained results allows to acquire Jackson-type inequalities in many classic spaces of periodic functions and weighted L_p spaces.

1. INTRODUCTION

Direct and inverse theorems that establish a relationship between the degree of smoothness of a function with respect to the operator of differentiation and the rate of convergence to zero of its best approximation by trigonometric polynomials are well known in the theory of approximation of periodic functions. Jackson's inequality is one among such results.

N. P. Kuptsov proposed generalized notion of the module of continuity, expanded on (C_0) semigroups in Banach space [1]. Using this notion of module of continuity, N. P. Kuptsov [1] and A. P. Terekhin [2] proved generalized Jackson's inequalities for the cases of bounded group $\{U(t)\}_{t\in\mathbb{R}}$ and s-regular group.[1]

G. V. Radzievsky studied direct and inverse theorems [3, 4], using the notion of K-functional instead of module of continuity, but it should be noted that K-functional has two-sided estimates with regard to module of continuity at least for bounded (C_0) groups.

In the papers [5, 6] and [7] authors investigated the case of a group in a Hilbert space whose generator is a selfadjoint operator and established Jackson's type inequalities in Hilbert space and related spaces with positive and negative norms. Obtained inequalities are used to estimate the rate of convergence to zero of the best approximation of either finite smoothness vectors respect to the operator A or vectors of infinite smoothness by entire vectors of exponential type.

The question of the extension of the direct theorems to a largest possible class of operators naturally arises. In this paper as the largest possible class we consider the so-called *non-quasianalytic operators* [8], i.e. such operators A that generates group $\{U(t)\}_{t\in\mathbb{R}}$ of (C_0) class in the Banach space $\mathfrak{X}$, having constraint on the growth rate at

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[1]The group $U(t)$ is called s-regular if the resolvent of its generator A satisfies desired condition: $\exists s \in \mathbb{N}, \exists \theta \in \mathbb{R}: \quad \|R_\lambda(e^{i\theta}A^s)\| \leq \frac{C}{\mathrm{Im}\lambda}$.

the infinity:

$$\int_{-\infty}^{\infty} \frac{|\ln \|U(t)\||}{1+t^2} dt < \infty. \tag{1.1}$$

As was shown in [5], the set of entire vectors of exponential type of the non-quasianalytic operator A is dense in $\mathfrak{X}$, therefor the approximation problem by entire vectors of exponential type is substantial. On the other hand, in [9] it was shown that the condition (1.1) is close to the necessary condition, that is in case when (1.1) doesn't hold, the class of entire vectors doesn't necessary dense in $\mathfrak{X}$ and so corresponding approximation problem ceases to be substantial.

The purpose of this work is to obtain Jackson's-type inequalities in the case of the approximation of a vector of the Banach space by entire vectors of exponential type of the generator of non-quasianalytic C_0-group and, as a corollary, Jackson's-type inequalities in different classical functional spaces.

2. Preliminaries

Let A be a closed linear operator with dense domain of definition $\mathcal{D}(A)$ in the Banach space $(\mathfrak{X}, \|\cdot\|)$ over the field of complex numbers.

Let $C^\infty(A)$ denote the set of all infinitely differentiable vectors of the operator A, i.e.

$$C^\infty(A) = \bigcap_{n\in\mathbb{N}_0} \mathcal{D}(A^n), \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

For a number $\alpha > 0$ we set

$$\mathfrak{E}^\alpha(A) = \left\{x \in C^\infty(A) \,|\, \exists c = c(x) > 0 \ \forall k \in \mathbb{N}_0 \ \left\|A^k x\right\| \le c\alpha^k\right\}.$$

The set $\mathfrak{E}^\alpha(A)$ is a Banach space with respect to the norm

$$\|x\|_{\mathfrak{E}^\alpha(A)} = \sup_{n\in\mathbb{N}_0} \frac{\|A^n x\|}{\alpha^n}.$$

Then $\mathfrak{E}(A) = \bigcup_{\alpha>0} \mathfrak{E}^\alpha(A)$ is a linear locally convex space with respect to the topology of the inductiove limit of the Banach spaces $\mathfrak{E}^\alpha(A)$:

$$\mathfrak{E}(A) = \liminf_{\alpha\to\infty} \mathfrak{E}^\alpha(A).$$

Elements of the space $\mathfrak{E}(A)$ are called entire vectors of exponential type of the operator A. The type $\sigma(x, A)$ of a vector $x \in \mathfrak{E}(A)$ is defined as the number

$$\sigma(x, A) = \inf\{\alpha > 0 \,:\, x \in \mathfrak{E}^\alpha(A)\} = \limsup_{n\to\infty} \|A^n x\|^{\frac{1}{n}}.$$

Example 2.1. Let $\mathfrak{X}$ is one of the $L_p(2\pi)$ $(1 \le p < \infty)$ spaces of integrable in p-th degree over $[0, 2\pi]$, 2π-periodical functions or the space $C(2\pi)$ of continuous 2π-periodical functions (the norm in $\mathfrak{X}$ is defined in a standard way), and let A is the operator of differentiation in the space $\mathfrak{X}$ ($\mathcal{D}(A) = \{x \in \mathfrak{X} \cap AC(\mathbb{R}) \,:\, x' \in \mathfrak{X}\}$; $(Ax)(t) = \frac{dx}{dt}$, where $AC(\mathbb{R})$ denotes the space of absolutely continuous functions over $\mathbb{R}$). It can be proved that in such case the space $\mathfrak{E}(A)$ coincides with the space of all trigonometric polynomials, and for $y \in \mathfrak{E}(A)$ $\sigma(y, A) = \deg(y)$, where $\deg(y)$ is the degree of the trigonometric polynomial y.

In what follows, we always assume that the operator A is the generator of the group of linear continuous operators $\{U(t) \,:\, t \in \mathbb{R}\}$ of class (C_0) on $\mathfrak{X}$. We recall that belonging of the group to the (C_0) class means that for every $x \in \mathfrak{X}$ the vector-function $U(t)x$ is continuous on $\mathbb{R}$ with respect to the norm of the space $\mathfrak{X}$.

For $t \in \mathbb{R}_+$, we set

$$M_U(t) := \sup_{\tau\in\mathbb{R},\, |\tau|\le t} \|U(\tau)\|.$$

The estimation $\|U(t)\| \le Me^{\omega t}$ for some $M, \omega \in \mathbb{R}$ implies that $M_U(t) < \infty \, (\forall t \in \mathbb{R}_+)$. It is easy to see that the function $M_U(\cdot)$ has following properties:

1) $M_U(t) \ge 1$, $t \in \mathbb{R}_+$;
2) $M_U(\cdot)$ is monotonically non-decreasing on $\mathbb{R}_+$;
3) $M_U(t_1 + t_2) \le M_U(t_1) M_U(t_2)$, $t_1, t_2 \in \mathbb{R}_+$.

According to [1], for $x \in \mathfrak{X}$, $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$ we set

$$\omega_k(t, x, A) = \sup_{0 \le \tau \le t} \left\| \Delta_\tau^k x \right\|, \qquad \text{where} \tag{2.1}$$

$$\Delta_h^k = (U(h) - \mathbb{I})^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{j}{k} U(jh), \quad k \in \mathbb{N}_0,\ h \in \mathbb{R} \quad (\Delta_h^0 \equiv 1). \tag{2.2}$$

Moreover, let

$$\widetilde{\omega}_k(t, x, A) = \sup_{|\tau| \le t} \left\| \Delta_\tau^k x \right\|. \tag{2.3}$$

Remark 2.1. It is easy to see that in the case of the bounded group $\{U(t)\}$ ($\|U(t)\| \le M$, $t \in \mathbb{R}$) the quantities $\omega_k(t, x, A)$ and $\widetilde{\omega}_k(t, x, A)$ are equivalent within constant factor ($\omega_k(t, x, A) \le \widetilde{\omega}_k(t, x, A) \le M\, \omega_k(t, x, A)$), and in a case of isometric group ($\|U(t)\| \equiv 1$, $t \in \mathbb{R}$) these quantities coincides.

It is immediate from the definition of $\widetilde{\omega}_k(t, x, A)$ that for $k \in \mathbb{N}$:

1) $\widetilde{\omega}_k(0, x, A) = 0$;
2) for fixed x the function $\widetilde{\omega}_k(t, x, A)$ is non-decreasing and is continuous by the variable t on $\mathbb{R}_+$;
3) $\widetilde{\omega}_k(nt, x, A) \le \big(1 + (n-1) M_U((n-1)t)\big)^k \widetilde{\omega}_k(t, x, A) \quad (n \in \mathbb{N},\ t > 0)$;
4) $\widetilde{\omega}_k(\mu t, x, A) \le \big(1 + \mu M_U(\mu t)\big)^k \widetilde{\omega}_k(t, x, A) \quad (\mu, t > 0)$;
5) for fixed $t \in \mathbb{R}_+$ the function $\widetilde{\omega}_k(t, x, A)$ is continuous in x.

For arbitrary $x \in \mathfrak{X}$ we set, according to [7, 6],

$$\mathcal{E}_r(x, A) = \inf_{y \in \mathfrak{E}(A) \,:\, \sigma(y, A) \le r} \|x - y\|, \quad r > 0,$$

i.e. $\mathcal{E}_r(x, A)$ is the best approximation of the element x by entire vectors y of exponential type of the operator A for which $\sigma(y, A) \le r$. For fixed x $\mathcal{E}_r(x, A)$ does not increase and $\mathcal{E}_r(x, A) \to 0$, $r \to \infty$ for every $x \in \mathfrak{X}$ if and only if the set $\mathfrak{E}(A)$ of entire vectors of exponential type is dense in $\mathfrak{X}$. Particularly, as indicated above, the set $\mathfrak{E}(A)$ is dense in $\mathfrak{X}$ if the group $\{U(t) : t \in \mathbb{R}\}$ belongs to non-quasianalytic class.

3. Abstract Jackson's inequality in a Banach space

Theorem 3.1. *Suppose that $\{U(t) : t \in \mathbb{R}\}$ satisfies the condition (1.1). Then, $\forall k \in \mathbb{N}$ there exists a constant $\mathbf{m}_k = \mathbf{m}_k(A) > 0$, such that $\forall x \in \mathfrak{X}$ the following inequality holds:*

$$\mathcal{E}_r(x, A) \le \mathbf{m}_k \cdot \big(M_U(1/r)\big)^k \widetilde{\omega}_k\left(\frac{1}{r}, x, A\right), \quad r > 0. \tag{3.1}$$

Remark 3.1. If, additionally, the group $\{U(t)\}$ is bounded ($M_U(t) \le \widetilde{M} < \infty$, $t \in \mathbb{R}$), then the constant $M_U(1/r)$ in the inequality (3.1) may be changed to r-independent constant. When the group is unbounded, the constant can be changed to r-independent too, but only with the additional assumption $r \ge \varepsilon$ where $\varepsilon > 0$.

Integral kernels, constructed in [10], will be used in the proving of the theorem. Moreover, we need additional properties of these kernels, lacking in [10]. The following lemma shows how these kernels are constructed and continues the investigation of their properties.

In what follows we denote as $\mathfrak{Q}$ the class of functions $\alpha : \mathbb{R} \mapsto \mathbb{R}$, satisfying the following conditions:

I) $\alpha(\cdot)$ *is measurable and bounded on any segment* $[-T,T] \subset \mathbb{R}$.
II) $\alpha(t) > 0,\ t \in \mathbb{R}$.
III) $\alpha(t_1+t_2) \leq \alpha(t_1)\alpha(t_2), \quad t_1,t_2 \in \mathbb{R}$.
IV) $\int_{-\infty}^{\infty} \frac{|\ln(\alpha(t))|}{1+t^2} dt < \infty$.

Lemma 3.1. *Let $\alpha \in \mathfrak{Q}$. Then there exists such entire function $\mathcal{K}_\alpha : \mathbb{C} \mapsto \mathbb{C}$ that*

1) $\mathcal{K}_\alpha(t) \geq 0,\ t \in \mathbb{R}$;
2) $\int_{-\infty}^{\infty} \mathcal{K}_\alpha(t)\, dt = 1$;
3) $\forall r > 0\ \exists c_r = c_r(\alpha) > 0\ \forall z \in \mathbb{C} \quad |\mathcal{K}_\alpha(rz)| \leq c_r \frac{e^{r|\operatorname{Im} z|}}{\alpha(|z|)}$

Proof. Without lost of generality we may assume that the function $\alpha(t)$ satisfies additional conditions:

V) $\alpha(t) \geq 1,\ t \in \mathbb{R}$; [2]
VI) $\alpha(t)$ *is even on* $\mathbb{R}$ *and is monotonically increasing on* $\mathbb{R}_+$;
VII) $\left\|\alpha^{-1}\right\|_{L_1(\mathbb{R})} = \int_{-\infty}^{\infty} |\alpha^{-1}(t)| dt < \infty$.

It is easy to make sure that assumptions V),VII) and condition that the function $\alpha(t)$ is even in VI) don't confine the general case if one examined the function $\alpha_1(t) = \widetilde{\alpha}(t)\widetilde{\alpha}(-t)$, where $\widetilde{\alpha}(t) = (1+\alpha(t))(1+t^2)$. In [11, theorems 1 and 2] it had been proved that the monotony condition on $\alpha(t)$ in VI) didn't confine the general case.

It follows from VII) that

$$\alpha(t) \to \infty,\ t \to \infty. \tag{3.2}$$

Let $\beta(t) = \ln \alpha(t),\ t \in \mathbb{R}$. Conditions III)-VII) and (3.2) lead to conclusion that

$$\beta(t) > 0, \qquad \beta(-t) = \beta(t), \qquad \beta(t) \to \infty,\ t \to \infty;$$

$$\beta(t_1+t_2) \leq \beta(t_1) + \beta(t_2),\ t_1,t_2 \in \mathbb{R} \tag{3.3}$$

$$\int_1^\infty \frac{\beta(t)}{t^2} dt < \infty \tag{3.4}$$

Because of (3.3) there exists limit $\lim_{t\to\infty} \frac{\beta(t)}{t}$. And, by virtue of (3.4):

$$\lim_{t\to\infty} \frac{\beta(t)}{t} = 0. \tag{3.5}$$

Also, using (3.4) it is easy to check that

$$\sum_{k=1}^{\infty} \frac{\beta(k)}{k^2} < \infty, \tag{3.6}$$

moreover, all terms of the series (3.6) is positive. From the convergence of the series (3.6) follows the existence of such sequence $\{Q_n\}_{n=1}^{\infty} \subset \mathbb{R}$ that $Q_n > 1, \quad Q_n \to \infty,\ n \to \infty$ and

$$\sum_{k=1}^{\infty} \frac{\beta(k)}{k^2} Q_k = S < \infty. \tag{3.7}$$

We set

$$a_k := \frac{\beta(k) Q_k}{S\, k^2}, \quad k \in \mathbb{N}.$$

[2]As shown in [8], for non-quasianalytic groups the condition $\|U(t)\| \geq 1$ always holds, therefor in this paper the condition V) automatically takes place.

The definition of a_k and (3.7) result in equality

$$\sum_{k=1}^{\infty} a_k = 1. \tag{3.8}$$

We construct the sequence of functions, which, obviously, are entire for every $n \in \mathbb{N}$:

$$f_n(z) := \prod_{k=1}^{n} P_k(z), \quad \text{де } P_k(z) = \left(\frac{\sin \frac{a_k z}{2}}{\frac{a_k z}{2}} \right)^2, \ z \in \mathbb{C}, \, n \in \mathbb{N}.$$

Similarly to the proof of the Denjoy-Carleman theorem [12, p.378] it can be concluded that the sequence of (entire) functions $f_n(z)$ converges uniformly to the function

$$f(z) = \prod_{k=1}^{\infty} \left(\frac{\sin \frac{a_k z}{2}}{\frac{a_k z}{2}} \right)^2, \qquad z \in \mathbb{C}$$

in every disk $\{z \in \mathbb{C} \,|\, |z| \le R\}$. Thus, by Weierstrass theorem, the function $f(z)$ is entire.

Using the inequality $|\sin z| \le \min(1, |z|) e^{|\operatorname{Im} z|}$, $z \in \mathbb{C}$ and taking (3.8) into account, when $z \in \mathbb{C}$ and $r > 0$, we receive

$$|f(rz)| = \prod_{k=1}^{\infty} \left| \frac{\sin \frac{a_k r z}{2}}{\frac{a_k r z}{2}} \right|^2 \le \prod_{k=1}^{\infty} \left(\frac{2}{a_k r |z|} \min\left(1, \frac{a_k r |z|}{2}\right) e^{\frac{1}{2} a_k r |\operatorname{Im} z|} \right)^2 =$$

$$= e^{r|\operatorname{Im} z|} \prod_{k=1}^{\infty} \min^2\left(1, \frac{2}{a_k r |z|}\right) \le e^{r|\operatorname{Im}|} \prod_{k=1}^{N} \min^2\left(1, \frac{2}{a_k r |z|}\right)$$

for every $N \in \mathbb{N}$. Using the inequality $\min(1, a) \cdot \min(1, b) \le \min(1, ab)$, we get:

(3.9)

$$|f(rz)| \le e^{r|\operatorname{Im} z|} \min^2\left(1, \prod_{k=1}^{N} \frac{2}{a_k r |z|}\right) = e^{r|\operatorname{Im} z|} \min^2\left(1, \frac{2^N}{\left(\prod_{k=1}^{N} \frac{\beta(k) Q_k}{S\, k^2}\right) (r|z|)^N}\right) =$$

$$= e^{r|\operatorname{Im} z|} \min^2\left(1, \frac{2^N N!}{\frac{\beta(1)}{1} \cdots \frac{\beta(N)}{N} \left(\frac{r}{S}\right)^N |z|^N Q_1 \cdots Q_N}\right).$$

Because of the condition $Q_n \to \infty$, $n \to \infty$ there exists such number $n(r) \in \mathbb{N}$ that:

$$\forall n > n(r) \quad Q_n \ge \frac{4\sqrt{e} S}{r}. \tag{3.10}$$

It follows from (3.5) that there exists such number $T_0 \in (0, \infty)$ that:

$$\forall\, t > T_0 \quad \frac{\beta(t)}{t} \le 1. \tag{3.11}$$

In [10] the following statement was proved:

$$\forall\, t_1, t_2 \in \mathbb{R}_+ \quad t_1 \le t_2 \Rightarrow \frac{\beta(t_1)}{t_1} \ge \frac{1}{2} \frac{\beta(t_2)}{t_2}. \tag{3.12}$$

Let $z \in \mathbb{C}$ and $|z| \ge \max\left(\beta^{[-1]}(n(r)), T_0\right)$, where $\beta^{[-1]}$ is the inverse function of the function β on $[0, \infty)$ (the inverse function $\beta^{[-1]}$ exists due to monotony of β on $[0, \infty)$). We substitute as N in (3.9) $N := [\beta(|z|)]$, where $[\cdot]$ denotes the integer part of a number. Then for $k \in \{1, \ldots, N\}$, in accordance with (3.11) and (3.12), we obtain $k \le N \le \beta(|z|) \le |z|$ and

$$\frac{\beta(k)}{k} \ge \frac{1}{2} \frac{\beta(|z|)}{|z|}. \tag{3.13}$$

Using (3.9),(3.10),(3.13) we find:

$$|f(rz)| \le e^{r|\operatorname{Im} z|}\left(\frac{2^N N!}{\left(\frac{1}{2}\frac{\beta(|z|)}{|z|}\right)^N \left(\frac{r}{S}\right)^N |z|^N Q_1\cdots Q_N}\right)^2 \le$$

$$\le e^{r|\operatorname{Im} z|}\left(\frac{2^N N!}{\left(\frac{1}{2}\frac{N}{|z|}\right)^N \left(\frac{r}{S}\right)^N |z|^N Q_1\cdots Q_N}\right)^2 = e^{r|\operatorname{Im} z|}\left(\frac{2^{2N} N!}{N^N \left(\frac{r}{S}\right)^N Q_1\cdots Q_N}\right)^2 \le$$

$$\le e^{r|\operatorname{Im} z|}\left(\frac{2^{2N}}{\left(\frac{r}{S}\right)^N Q_1\cdots Q_N}\right)^2 = e^{r|\operatorname{Im} z|}\left(\frac{\left(\frac{4S}{r}\right)^N}{Q_1\cdots Q_N}\right)^2.$$

Since $Q_n \ge 1$, the last inequality leads to

$$|f(rz)| \le e^{r|\operatorname{Im} z|}\left(\frac{\left(\frac{4S}{r}\right)^N}{\left(\frac{4\sqrt{e}S}{r}\right)^{N-n(r)}}\right)^2 = e^{r|\operatorname{Im} z|}\left(\frac{4\sqrt{e}S}{r}\right)^{2n(r)} e^{-[\beta(|z|)]} \le \tag{3.14}$$

$$\le e^{r|\operatorname{Im} z|}\left(\frac{4\sqrt{e}S}{r}\right)^{2n(r)} e^{-(\beta(|z|)-1)} = C_r^{(1)}\frac{e^{r|\operatorname{Im} z|}}{\alpha(|z|)},$$

where $C_r^{(1)} = e\left(\frac{4\sqrt{e}S}{r}\right)^{2n(r)}$. When $z \in \mathbb{C}$ and $|z| < \max\left(\beta^{[-1]}(n(r)), T_0\right)$, using (3.9), we get:

$$|f(rz)| \le e^{r|\operatorname{Im} z|} = e^{r|\operatorname{Im} z|}\frac{\alpha(|z|)}{\alpha(|z|)} \le e^{r|\operatorname{Im} z|}\frac{C_r^{(2)}}{\alpha(|z|)}, \tag{3.15}$$

where $C_r^{(2)} = \alpha(\max(\beta^{[-1]}(n(r)), T_0))$. It follows from (3.14), (3.15) that

$$|f(rz)| \le e^{r|\operatorname{Im} z|}\frac{C_r^{(0)}}{\alpha(|z|)}, \quad z \in \mathbb{C}, \qquad \text{where} \quad C_r^{(0)} = \max(C_r^{(1)}, C_r^{(2)}). \tag{3.16}$$

The inequality (3.16) and the condition VII) imply that $\|f\|_{L_1(\mathbb{R})} < \infty$. Thus it is enough to set $\mathcal{K}_\alpha(z) := \frac{1}{\|f\|_{L_1(\mathbb{R})}} f(z)$, $z \in \mathbb{C}$ and use (3.16) to finish the proof. □

Let $\alpha \in \mathfrak{Q}$, and $\mathcal{K}_\alpha : \mathbb{C} \mapsto \mathbb{C}$ is the function constructed by the function α in lemma 3.1. We set

$$\mathcal{K}_{\alpha,r}(z) := r\mathcal{K}_\alpha(rz), \quad z \in \mathbb{C},\ r \in (0,\infty).$$

The lemma 3.1 ensures us that the function $\mathcal{K}_{\alpha,r}$ has the following properties:

1) $\mathcal{K}_{\alpha,r}(t) \ge 0, \quad t \in \mathbb{R}$;
2) $\int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t)\,dt = 1$;
3) $\forall z \in \mathbb{C}\ |\mathcal{K}_{\alpha,r}(z)| \le rc_r\frac{e^{r|\operatorname{Im} z|}}{\alpha(|z|)}; \quad r > 0.$

Lemma 3.2. *$\forall r \in (0,\infty)$ there exists constant $\widetilde{c}_r = \widetilde{c}_r(\alpha) > 0$, such that $\forall n \in \mathbb{N}$ the following inequality holds:*

$$|\mathcal{K}_{\alpha,r}^{(n)}(t)| \le \widetilde{c}_r\frac{\sqrt{2\pi n}\,\alpha\left(\frac{n}{r}\right)}{\alpha(|t|)}r^n, \qquad t \in \mathbb{R}$$

Proof. In what follows in this proof we assume $t \in \mathbb{R}$, $r \in (0,\infty)$, $n \in \mathbb{N}$. Let

$$\gamma_{n,r}(t) := \left\{\zeta \in \mathbb{C} : |\zeta - t| = \frac{n}{r}\right\}.$$

Using Cauchy's integral theorem and Stirling's approximation for $n!$, we get

$$|\mathcal{K}^{(n)}_{\alpha,r}(t)| \leq \frac{n!}{2\pi}\oint_{\gamma_{n,r}(t)}\frac{|\mathcal{K}_{\alpha,r}(\xi)|}{|\xi-t|^{n+1}}|d\xi| = \frac{n!}{2\pi}\frac{r^{n+1}}{n^{n+1}}\oint_{\gamma_{n,r}(t)}|\mathcal{K}_{\alpha,r}(\xi)||d\xi| \leq$$
$$\leq \frac{c^{(!)}r^{n+1}}{\sqrt{2\pi n}}e^{-n}\oint_{\gamma_{n,r}(t)}|\mathcal{K}_{\alpha,r}(\xi)||d\xi|, \quad \text{where } c^{(!)} = \sup_{k\in\mathbb{N}}\frac{k!}{\sqrt{2\pi k}}\left(\frac{e}{k}\right)^k < e^{1/12}.$$

Using property 3) of the function $K_{\alpha,r}$, the condition $t\in\mathbb{R}$ and the conditions III), VI) of the function α, one can find from the last inequality

$$|\mathcal{K}^{(n)}_{\alpha,r}(t)| \leq \frac{c^{(!)}r^{n+1}}{\sqrt{2\pi n}}e^{-n}rc_r\oint_{\gamma_{n,r}(t)}\frac{e^{r|\operatorname{Im}\xi|}}{\alpha(|\xi|)}|d\xi| =$$
$$= \frac{c^{(!)}r^{n+1}}{\sqrt{2\pi n}}e^{-n}\frac{rc_r}{\alpha(|t|)}\oint_{\gamma_{n,r}(t)}\frac{e^{r|\operatorname{Im}(\xi-t)|}\alpha(|(t-\xi)+\xi|)}{\alpha(|\xi|)}|d\xi| \leq$$
$$\leq \frac{c^{(!)}r^{n+1}}{\sqrt{2\pi n}}e^{-n}\frac{rc_r}{\alpha(|t|)}\oint_{\gamma_{n,r}(t)}e^{r|\operatorname{Im}(\xi-t)|}\alpha(|t-\xi|)|d\xi| \leq$$
$$\leq \frac{c^{(!)}r^{n+1}}{\sqrt{2\pi n}}e^{-n}\frac{rc_r}{\alpha(|t|)}\oint_{\gamma_{n,r}(t)}e^{n}\alpha\left(\frac{n}{r}\right)|d\xi| = \widetilde{c}_r\frac{\sqrt{2\pi n}\,\alpha\left(\frac{n}{r}\right)}{\alpha(|t|)}r^n,$$

where $\widetilde{c}_r = c^{(!)}rc_r$. □

Remark 3.2. If the function $\alpha(t)$ satisfies the conditions of the lemma 3.1, but, moreover, has the polynomial rate of growth at the infinity, i.e. $\exists m\in\mathbb{N}_0$, $\exists M>0$:

$$\alpha(t) \leq M(1+|t|)^{2m}, \quad t\in\mathbb{R}, \tag{3.17}$$

another integral kernel may be used:

$$\tilde{K}_\alpha(z) = \frac{1}{\mathrm{K}_m}\left(\frac{\sin\frac{z}{2m}}{\frac{z}{2m}}\right)^{2m}, \qquad \mathrm{K}_m = \int_{-\infty}^{\infty}\left(\frac{\sin\frac{x}{2m}}{\frac{x}{2m}}\right)^{2m}dx.$$

In much the same way to the proving of the lemmas 3.1 and 3.2 one can show that

$$\left|\tilde{K}_\alpha(rz)\right| \leq \tilde{C}_r\frac{e^{r|\operatorname{Im}z|}}{\alpha(|z|)}, \qquad \text{where } \tilde{C}_r = \frac{M}{\mathrm{K}_m}\left(1+\frac{2m}{r}\right)^{2m},$$

and

$$\left|\tilde{K}^{(n)}_{\alpha,r}(t)\right| \leq \tilde{c}_r\frac{\sqrt{2\pi n}\,\alpha(\frac{n}{r})}{\alpha(|t|)}r^n, \qquad \text{where } \tilde{c}_r = c^{(!)}r\tilde{C}_r,$$

that is to say, defined in such a way integral kernel satisfies lemmas 3.1 and 3.2.

Proof of the theorem 3.1. Let group $\{U(t) : t\in\mathbb{R}\}$ satisfies (1.1). Then it follows from the [11, theorems 1 and 2] that

$$\int_{-\infty}^{\infty}\frac{\ln(M_U(|t|))}{1+t^2}dt < \infty. \tag{3.18}$$

We fix arbitrary $k\in\mathbb{N}$ and set

$$\alpha(t) := \left(M_U(|t|)\right)^k(1+|t|)^{k+2}, \qquad t\in\mathbb{R}.$$

The function α is, obviously, even on $\mathbb{R}$. The condition (3.18) and the properties of the function $M_U(\cdot)$ imply $\alpha\in\mathfrak{Q}$, and, moreover,

$$\int_{-\infty}^{\infty}\frac{\left((1+|t|)M_U(|t|)\right)^k}{\alpha(t)}dt = \int_{-\infty}^{\infty}\frac{dt}{(1+|t|)^2} = 2. \tag{3.19}$$

Using lemma 3.1 (or the remark 3.2 if $\alpha(t)\leq M(1+|t|)^m$) for the function $\alpha(t)$, we construct a family of kernels $K_{\alpha,r}$.

In what follows, we assume that $x \in \mathfrak{X},\ r \in (0,\infty)$ and $n \in \{1,\dots,k\}$. We define

$$x_{r,n} := \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t)U(nt)x\,dt.$$

Let $\nu \in \mathbb{N}_0$. Let's prove that $x_{r,n} \in C^\infty(A) = \bigcap_{\nu\in\mathbb{N}_0} \mathcal{D}(A^\nu)$ and

$$A^\nu x_{r,n} = \frac{(-1)^\nu}{n^\nu}\int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}^{(\nu)}(t)U(nt)x\,dt. \tag{3.20}$$

It follows from the property 3) of the function $\mathcal{K}_{\alpha,r}$ and from lemma 3.2 that there exists such constant $\widetilde{C}(\nu,r) > 0$ that $\mathcal{K}_{\alpha,r}^{(\nu)}(t) \le \frac{\widetilde{C}(\nu,r)}{\alpha(t)}$, $t \in \mathbb{R}$. Thus, using (3.19), we get

$$\begin{aligned} \int_{-\infty}^{\infty} \left\|\mathcal{K}_{\alpha,r}^{(\nu)}(t)U(nt)x\right\| dt &\le \int_{-\infty}^{\infty} \frac{\widetilde{C}(\nu,r)}{\alpha(t)} \|U(t)\|^n \|x\|\,dt \le \\ &\le \widetilde{C}(\nu,r)\|x\| \int_{-\infty}^{\infty} \frac{M_U(|t|)^k}{\alpha(t)}dt \le 2\widetilde{C}(\nu,r)\|x\| < \infty. \end{aligned} \tag{3.21}$$

Therefor the integral $\int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}^{(\nu)}(t)U(nt)x\,dt$ converges. We define

$$x_{r,n}^{(\nu)} = \frac{(-1)^\nu}{n^\nu}\int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}^{(\nu)}(t)U(nt)x\,dt.$$

Then, using closedness of the operator A and integration by parts, one can find for $x \in \mathcal{D}(A)$ that $x_{r,n}^{(\nu)} \in \mathcal{D}(A)$ and

$$\begin{aligned} Ax_{r,n}^{(\nu)} &= \frac{(-1)^\nu}{n^\nu}\int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}^{(\nu)}(t)U(nt)Ax\,dt = \frac{(-1)^\nu}{n^\nu}\frac{1}{n}\int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}^{(\nu)}(t)(U(nt)x)'dt = \\ &= -\frac{(-1)^\nu}{n^\nu}\frac{1}{n}\int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}^{(\nu+1)}(t)U(nt)x\,dt = x_{r,n}^{(\nu+1)}. \end{aligned} \tag{3.22}$$

Let x is an arbitrary element of the space $\mathfrak{X}$. Then there exists the sequence $\{x_m\}_{m=1}^{\infty} \subset \mathcal{D}(A)$ such that $\|x_m - x\| \to 0, \quad m \to \infty$. Consequently, using the inequality (3.21) and the relation (3.22), one can get

$$\left\|(x_m)_{r,n}^{(\nu)} - x_{r,n}^{(\nu)}\right\| \le \frac{1}{n^\nu}\int_{-\infty}^{\infty} \left\|\mathcal{K}_{\alpha,r}^{(\nu)}(t)U(nt)(x_m - x)\right\| dt \le \frac{2\widetilde{C}(\nu,r)}{n^\nu}\|x_m - x\| \to 0;$$
$$\left\|A(x_m)_{r,n}^{(\nu)} - x_{r,n}^{(\nu+1)}\right\| = \left\|(x_m)_{r,n}^{(\nu+1)} - x_{r,n}^{(\nu+1)}\right\| \to 0, \quad m \to \infty.$$

Hence, taking into account the closedness of the operator A, we have:

$$x_{r,n}^{(\nu)} \in \mathcal{D}(A), \quad Ax_{r,n}^{(\nu)} = x_{r,n}^{(\nu+1)}. \tag{3.23}$$

One can get (3.20) from (3.23) by induction.

Using the relation (3.20) and lemma 3.2, one can find:

$$\begin{aligned} \|A^\nu x_{r,n}\| &\le \frac{\|x\|}{n^\nu}\int_{-\infty}^{\infty} \left|\mathcal{K}_{\alpha,r}^{(\nu)}(t)\right| \|U(nt)\|\,dt \le \\ &\le \frac{\|x\|}{n^\nu}\int_{-\infty}^{\infty} \widetilde{c}_r \frac{\sqrt{2\pi\nu}\,\alpha\left(\frac{\nu}{r}\right)}{\alpha(|t|)} r^\nu \|U(t)\|^n\,dt \le \\ &\le \widetilde{c}_r \|x\| \sqrt{2\pi\nu}\,\alpha\left(\frac{\nu}{r}\right)\left(\int_{-\infty}^{\infty} \frac{\|U(t)\|^k}{\alpha(t)}dt\right)\left(\frac{r}{n}\right)^\nu, \end{aligned} \tag{3.24}$$

where, accordingly to (3.19), $\int_{-\infty}^{\infty} \frac{\|U(t)\|^k}{\alpha(t)} dt \le 2 < \infty$. Since $\beta(t) = \ln(\alpha(t))$, $t \in \mathbb{R}$, as was mentioned in the proof of lemma 3.1, $\lim_{\tau\to\infty} \frac{\beta(\tau)}{\tau} = 0$ (cf. (3.5)). Thus

$$\lim_{\nu\to\infty} \left(\alpha\left(\frac{\nu}{r}\right)\right)^{1/\nu} = \lim_{\nu\to\infty} e^{\frac{1}{r}\left(\frac{r}{\nu}\beta\left(\frac{\nu}{r}\right)\right)} = e^{\frac{1}{r}\cdot 0} = 1.$$

Therefor from the relation (3.24) one can get:

$$\limsup_{\nu\to\infty} \left(\|A^\nu x_{r,n}\| \right)^{1/\nu} \le \frac{r}{n}.$$

The last inequality brings us to the conclusion that

$$x_{r,n} \in \mathfrak{E}(A) \quad \text{and} \quad \sigma(x_{r,n}, A) \le \frac{r}{n}. \tag{3.25}$$

For arbitrary $x \in \mathfrak{X}$ we define

$$\begin{aligned} \widetilde{x}_{r,k} := \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t)(x + (-1)^{k-1}(U(t) - \mathbb{I})^k x) dt = \\ = \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) \sum_{n=1}^{k} (-1)^{n+1} \binom{k}{n} U(nt) x\, dt \end{aligned} \tag{3.26}$$

(the absolute convergence by the norm of $\mathfrak{X}$ of the integral in the right part of (3.26) follows from the inequality (3.21), so the definition of the vector $\widetilde{x}_{r,k}$ is correct). Using the definition (3.26) one can get:

$$\widetilde{x}_{r,k} = \sum_{n=1}^{k} (-1)^{n+1} \binom{k}{n} \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) U(nt) x dt = \sum_{n=1}^{k} (-1)^{n+1} \binom{k}{n} x_{r,n}.$$

Therefor, accordingly to (3.25),

$$\widetilde{x}_{r,k} \in \mathfrak{E}(A) \quad \text{and} \quad \sigma(\widetilde{x}_{r,k}, A) \le r.$$

Hence for an arbitrary $x \in \mathfrak{X}$ we have:

$$\mathcal{E}_r(x, A) = \inf_{y \in \mathfrak{E}(A) \,:\, \sigma(y,A) \le r} \|x - y\| \le \|x - \widetilde{x}_{r,k}\|$$

Using (3.26), the property 2) of the kernel $\mathcal{K}_{\alpha,r}$ and (2.3), the last inequality implies:

$$\begin{aligned} \mathcal{E}_r(x, A) \le \left\| \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) x dt - \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) \big(x + (-1)^{k-1}(U(t) - \mathbb{I})^k x\big)\, dt \right\| \le \\ \le \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) \left\|(U(t) - \mathbb{I})^k x\right\| dt \le \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) \widetilde{\omega}_k(|t|, x, A)\, dt. \end{aligned}$$

So, in accordance with the property 4) of the function $\widetilde{\omega}_k(|t|, x, A)$,

$$\begin{aligned} \mathcal{E}_r(x, A) \le \int_{-\infty}^{\infty} \mathcal{K}_{\alpha,r}(t) \widetilde{\omega}_k \left(|rt| \frac{1}{r}, x, A \right) dt \le \\ \le \widetilde{\omega}_k \left(\frac{1}{r}, x, A \right) \int_{-\infty}^{\infty} \big(1 + |rt| M_U(|t|)\big)^k \mathcal{K}_{\alpha,r}(t) dt. \end{aligned} \tag{3.27}$$

Taking into account the properties of the function $M_U(\cdot)$, the definition of $\mathcal{K}_{\alpha,r}$, the lemma 3.1 and the equality (3.19), one can find:

$$\int_{-\infty}^{\infty}\big(1+|rt|M_U(|t|)\big)^k\mathcal{K}_{\alpha,r}(t)dt \le \int_{-\infty}^{\infty}\big(1+|rt|M_U(1/r)M_U(rt)\big)^k r\mathcal{K}_{\alpha}(rt)dt \le$$
$$\le \big(M_U(1/r)\big)^k\int_{-\infty}^{\infty}\big((1+\tau)M_U(\tau)\big)^k\mathcal{K}_{\alpha}(\tau)d\tau \le$$
$$\le \big(M_U(1/r)\big)^k c_1\int_{-\infty}^{\infty}\frac{\big((1+|\tau|)M_U(|\tau|)\big)^k}{\alpha(\tau)}d\tau = 2(M_U(1/r))^k c_1 < \infty.$$

In accordance with (3.27), the inequality (3.1) holds for all $r\in(0,\infty)$ with a constant $\mathbf{m}_k = 2c_1$. It should be noted that constant $\mathbf{m}_k$, indeed, depends on k, because due to 3.1, the constant $c_1 = c_1(\alpha)$ depends on the function $\alpha(t) = (M_U(|t|))^k(1+|t|)^{k+2}$. $\square$

Theorem 3.1 allows us to prove the analogue of the classic Jackson's inequality for m times differentiable functions:

Corollary 3.1. *Let $x\in\mathcal{D}(A^m)$, $m\in\mathbb{N}_0$. Then $\forall k\in\mathbb{N}_0$*

$$\mathcal{E}_r(x,A) \le \mathbf{m}_{k+m}\frac{M_U\left(\frac{m}{r}\right)}{r^m}\big(M_U(1/r)\big)^{k+m}\widetilde{\omega}_k\left(\frac{1}{r},A^m x,A\right),\quad r>0, \tag{3.28}$$

where the constants $\mathbf{m}_n$ $(n\in\mathbb{N})$ is the same as in the theorem 3.1.

Proof. Let $x\in\mathcal{D}(A^m)$ and $r\in\mathbb{R}_+$. By the theorem 3.1,

$$\mathcal{E}_r(x,A)\le \mathbf{m}_{k+m}\cdot\big(M_U(1/r)\big)^{k+m}\widetilde{\omega}_{k+m}\left(\frac{1}{r},x,A\right).$$

Let $t\in\mathbb{R}$, $0\le|t|\le\frac{1}{r}$. Then, using properties of the semigroups of the (C_0) class and properties of the function $M_U(t)$, one can get:

$$\left\|(U(t)-\mathbb{I})^{k+m}x\right\| = \left\|(U(t)-\mathbb{I})^m(U(t)-\mathbb{I})^k x\right\| \le$$
$$\le \int_0^t\cdots\int_0^t \|U(\xi_1+\cdots+\xi_m)\|\left\|(U(t)-\mathbb{I})^k A^m x\right\|d\xi_1\dots d\xi_m \le$$
$$\le M_U(m|t|)\left\|(U(t)-\mathbb{I})^k A^m x\right\|t^m \le \frac{M_U(\frac{m}{r})}{r^m}\widetilde{\omega}_k\left(\frac{1}{r},A^m x,A\right).$$

This implies $\widetilde{\omega}_{k+m}\left(\frac{1}{r},x,A\right) = \sup_{|t|\le\frac{1}{r}}\left\|(U(t)-\mathbb{I})^{k+m}x\right\| \le \frac{M_U(\frac{m}{r})}{r^m}\widetilde{\omega}_k\left(\frac{1}{r},A^m x,A\right)$, which proves the inequality (3.28). $\square$

By setting in the corollary 3.1 $k=0$ and taking into account that $\tilde{\omega}_0(\cdot,A^m x,A)\equiv \|A^m x\|$, one can conclude the following inequality:

Corollary 3.2. *Let $x\in\mathcal{D}(A^m)$, $m\in\mathbb{N}_0$. Then*

$$\mathcal{E}_r(x,A)\le\frac{\mathbf{m}_m}{r^m}\big(M_U(1/r)\big)^{2m}\|A^m x\|\qquad r>0, \tag{3.29}$$

where the constants $\mathbf{m}_n$ $(n\in\mathbb{N})$ is the same as in the theorem 3.1.

4. The examples of application of the abstract Jackson's inequality in particular spaces

Lets consider several examples of application of the theorem 3.1 in particular spaces.

4.1. **Jackson's inequalities in $L_p(2\pi)$ у $C(2\pi)$.**

Example 4.1. Let the space $\mathfrak{X}$ and the operator A are the same as in the example 2.1. Then for $x \in \mathfrak{X}$ the quantity $\mathcal{E}_r(x, A)$ is the value of the best approximation of function x by trigonometric polynomials whose degree does not exceed r with respect to the norm in $\mathfrak{X}$. It is generally known that differential operator A is a generator of (isometric) group of shifts in the space $\mathfrak{X}$:

$$(U(t)x)(\xi) = x(t+\xi), \qquad x \in \mathfrak{X};\ t, \xi \in \mathbb{R}$$
$$\|U(t)\| \equiv 1, \qquad t \in \mathbb{R}, \tag{4.1}$$

where $\|U(\cdot)\| = \|U(\cdot)\|_{\mathcal{L}(\mathfrak{X})}$ is the norm of the operator $U(t)$ in the space $\mathcal{L}(\mathfrak{X})$ of linear continuous operators over $\mathfrak{X}$. It follows from (4.1) that

$$\widetilde{\omega}_k(t, x, A) = \omega_k(t, x, A) = \sup_{0 \le h \le t} \left\| \sum_{j=0}^{k} (-1)^{k-j} \binom{j}{k} x(\cdot + jh) \right\|_{\mathfrak{X}}, \quad t \in \mathbb{R}_+,\ x \in \mathfrak{X}.$$

I.e., in that case, $\widetilde{\omega}_k(t, x, A)$ coincides with classic modulus of continuity of k-th degree in the space $\mathfrak{X}$.

Thus, from the theorem 3.1 and corollary 3.1 one can conclude all classic Jackson-type inequalities in the spaces $C(2\pi)$ та $L_p(2\pi)$ $1 \le p < \infty$.

4.2. **Jackson's inequalities of the approximation by entire functions of exponential type in the space $L_p(\mathbb{R}, \mu^p)$.** We consider real-values function $\mu(t)$ satisfying following conditions:

1) $\mu(t) \ge 1, \quad t \in \mathbb{R}$;
2) $\mu(t)$ is even, monotonically non-decreasing when $t > 0$;
3) $\mu(t)$ satisfies naturally occurring in many applications condition $\mu(t+s) \le \mu(t) \cdot \mu(s),\ s, t \in \mathbb{R}$.
4) $\int_{-\infty}^{\infty} \frac{\ln \mu(t)}{1+t^2}\, dt < \infty$

or alternatively, instead of 4), the equivalent condition holds

4') $\sum_{k=1}^{\infty} \frac{\ln \mu(k)}{k^2} < \infty$.

Lets consider several important classes of functions satisfying conditions 1)–4).

1. Constant function $\mu(t) \equiv 1, \quad t \in \mathbb{R}$.

2. Functions with polynomial growth rate at the infinity. It is easy to check that for such functions following estimate holds: $\exists k \in \mathbb{N},\ \exists M \ge 1$:

$$\mu(t) \le M(1+|t|)^k, \quad t \in \mathbb{R}.$$

3. Functions of the form

$$\mu(t) = e^{|t|^\beta}, \quad 0 < \beta < 1,\ t \in \mathbb{R}.$$

4. $\mu(t)$ represented as a power series for $t > 0$. I.e.,

$$\mu(t) = \sum_{n=0}^{\infty} \frac{|t|^n}{m_n},$$

where $\{m_n\}_{n \in \mathbb{N}}$ is a sequence of positive real numbers satisfying two conditions:

- $m_0 = 1,\ m_n^2 \le m_{n-1} \cdot m_{n+1},\ n \in \mathbb{N}$;
- $\forall k, l \in \mathbb{N}\ \frac{(k+l)!}{m_{k+l}} \le \frac{k!}{m_k} \frac{l!}{m_l}$.

The function $\mu(t)$, defined above, obviously satisfies conditions 1) and 2). The condition $\forall k, l \in \mathbb{N}\ \frac{(k+l)!}{m_{k+l}} \le \frac{k!}{m_k} \frac{l!}{m_l}$ implies

$$\sum_{k=0}^{n} \frac{t^k s^{n-k} n!}{k!(n-k)!\, m_n} \le \sum_{k=0}^{n} \frac{t^k s^{n-k}}{m_k m_{n-k}}, \tag{4.2}$$

and it is easy to see that condition 3) follows from the inequality (4.2). The Denjoy - Carleman theorem [12, p.376] asserts that the following conditions are equivalent:

a) $\mu(t)$ satisfies condition 4);
b) $\sum_{n=1}^{\infty}\left(\frac{1}{m_n}\right)^{1/n} < \infty$;
c) $\sum_{n=1}^{\infty}\frac{m_{n-1}}{m_n} < \infty$.

5. $\mu(t)$ as a module of an entire function with zeroes on the imaginary axis. We consider

$$\omega(t) = C\prod_{k=1}^{\infty}\left(1-\frac{t}{it_k}\right), \quad t\in\mathbb{R},$$

where $C \geq 1,\ 0 < t_1 \leq t_2 \leq \dots,\ \sum_{k=1}^{\infty}\frac{1}{t_k} < \infty$. We set $\mu(t) := |\omega(t)|$. Then $\mu(t)$ satisfies conditions 1) – 3), and, as shown in [8], $\mu(t)$ satisfies the condition 4) also.

Lets proceed to the description of the spaces $L_p(\mathbb{R},\mu^p)$. Let the function $\mu(t)$ satisfies conditions 1) – 4). One can consider the space $L_p(\mathbb{R},\mu^p)$ of the functions $x(s),\ s\in\mathbb{R}$, integrable in p-th degree with the weight μ^p:

$$\|x\|^p_{L_p(\mathbb{R},\mu^p)} = \int_{-\infty}^{\infty}|x(s)|^p\mu^p(s)\,ds.$$

$L_p(\mathbb{R},\mu^p)$ is the Banach space. We consider the differential operator A ($\mathcal{D}(A) = \{x \in L_p(\mathbb{R},\mu^p)\cap AC(\mathbb{R}):\ x' \in L_p(\mathbb{R},\mu^p)\},\ (Ax)(t) = \frac{dx}{dt}$). As in example 4.1, the operator A generates the group of shifts $\{U(t)\}_{t\in\mathbb{R}}$ in the space $L_p(\mathbb{R},\mu^p)$. But in contrast to example 4.1, this group doesn't bounded. Indeed, lets consider

$$x(s) = \begin{cases}1, & s\in[0,1],\\ 0, & s\notin[0,1].\end{cases}$$

Obviously, $x(s)\in L_p(\mathbb{R},\mu^p)$, but for $t > 1$

$$\|U(t)x\|^p = \int_{-\infty}^{\infty}|x(t+s)|^p\mu^p(s)\,ds = \int_{t-1}^{t}\mu^p(s) \geq \mu^p(t-1)\to\infty,\ t\to\infty.$$

On the other hand, because of the property 3),

$$\|U(t)x\|^p = \int_{-\infty}^{\infty}|x(t+s)|^p\mu^p(s)\,ds \leq \mu^p(-t)\int_{-\infty}^{\infty}|x(t+s)|^p\mu^p(t+s)\,ds = \big(\mu(-t)\big)^p\|x\|^p,$$

so $\|U(t)\|_{L_p(\mathbb{R},\mu^p)} \leq \mu(-t) = \mu(|t|), \quad t\in\mathbb{R}.$ [3]

By the same way as in the example 4.1, modules of continuity ω_k and $\tilde{\omega}_k$ coincides with classic ones, but in contrast to the example 4.1, they don't equal mutually.The space $\mathfrak{E}(A)$ consists of fast decrescent at the infinity entire functions. The examples of such functions have been given in [8]. By applying the theorem 3.1 one can get

Corollary 4.1. *$\forall k\in\mathbb{N}$ there exists constant $\mathbf{m}_k(p,\mu) > 0$ such that $\forall f\in L_p(\mathbb{R},\mu^p)$*

$$\mathcal{E}_r(f) \leq \mathbf{m}_k\cdot\big(\mu\,(1/r)\,\big)^k\tilde{\omega}_k\left(\frac{1}{r},x,A\right), \quad r>0.$$

[3] If $\mu(t)$ is continuous and $\mu(0)=1$, it is possible to show in a similar manner that $\|U(t)\|_{L_p(\mathbb{R},\mu^p)} = \mu(|t|)$.

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