CLUSTERING IN A STOCHASTIC MODEL OF ONE-DIMENSIONAL GAS

VLADISLAV V. VYSOTSKY

Abstract. We give a quantitative analysis of clustering in a stochastic model of one-dimensional gas. At time zero the gas consists of n identical particles, which are randomly distributed on the real line and have zero initial speeds. Particles begin to move under the forces of mutual attraction. At a collision particles *stick* together forming a new particle called *cluster* whose mass and speed are defined by the laws of conservation.

We are interested in the asymptotic behaviour of $K_n(t)$ as $n \to \infty$, where $K_n(t)$ denotes the number of clusters at time t in the system with n initial particles. The main result is a functional limit theorem for $K_n(t)$. Our proof is based on the discovered *localization property* of the aggregation process. This property states that the behavior of each particle is essentially defined only by the motion of neighbour particles.

Key words and phrases: sticky particles, particle systems, gravitating particles, number of clusters, aggregation process, adhesion.

2000 MSC: Primary 60K35, 82C22; secondary 60F17, 70F99.

1. Introduction

1.1. **Description of the model and the background.** We give a quantitative analysis of clustering in a stochastic model of one-dimensional gas. At time zero the gas consists of n point particles, each one of mass $\frac{1}{n}$. These particles are randomly distributed on the real line and have zero initial speeds. Particles begin to move under the forces of mutual attraction. When two or more particles collide, they *stick* together forming a new particle (called *cluster*) with mass and speed defined by the laws of mass and momentum conservation. Between collisions particles move according to the laws of Newtonian mechanics.

We suppose that the force of mutual attraction does not depend on distance and equals the product of masses; this is very natural for one-dimensional models. Thus at any moment, the acceleration of a particle is equal to difference of masses to the right and to the left of the particle.

Random initial positions of particles are usually described (see [8, 14, 23]) by the following natural models. In the *uniform* model, n particles are independently and uniformly spread on $[0, 1]$. In the *Poisson* model, particles are located at points $\frac{1}{n}S_1, \frac{1}{n}S_2, \ldots, \frac{1}{n}S_n$, where S_i is a standard exponential random walk. In other words, particles are located at points of first n jumps of a Poisson process with intensity n.

These two models are the most important and interesting. However, it is independence of distances between particles that plays the main role in the behaviour of the Poisson model, but not the type of distribution of these distances. Therefore we generalize the Poisson model introducing the *independent* model, where particles are also placed at $\frac{1}{n}S_1, \frac{1}{n}S_2, \ldots, \frac{1}{n}S_n$, but now S_i is an arbitrary random walk such that its i.i.d. increments X_i are non-negative and satisfy the normalization condition $\mathbb{E}X_i = 1$. Note that for all the mentioned models of initial positions, if we proceed to the limit as $n \to \infty$ we consider a system of total mass one consisting of, roughly speaking, infinitesimal particles homogeneously spread on $[0, 1]$.

Partly supported by the grant NSh-4222.2006.1.

As time goes, particles aggregate in clusters. Clusters become larger and larger while the number of clusters decreases. Finally, at some moment all clusters merge into a single cluster containing all initial particles. This aggregation process resembles formation of a star from dispersed space dust. Indeed, similar models of sticky particles have applications in astrophysics, where they could be used to describe formation of large-scale structures of the Universe, see Shandarin and Zeldovich [20] and the survey paper Vergassola et al. [22].

To be precise, these descriptions are based on the so called Burgers equation, which has interpretation in terms of sticky particles, see Brenier and Grenier [4] or E et al. [6]. The Burgers equation is a fundamental partial differential equation from fluid mechanics, and it is its direct relation with sticky particles systems that causes the constant interest to such systems. Sticky particles models are also used for numerical solving of other partial differential equations originating from pressureless gas dynamics, see Chertock et al. [5] for explanations and further references.

It should be mentioned that the aggregation process described by the sticky particles model is strongly connected with additive coalescence; see Bertoin [2] and the paper of Giraud [9] with the most recent results and references.

1.2. **Statement of problem and the results.** The most general question one could ask on the model is to describe the behaviour of the aggregation process. How fast it is? What are typical sizes of appearing clusters? Where do clusters appear most intensively, etc.? Numerous papers on the model, e.g., [8, 12, 14, 17, 23], are dedicated to probabilistic description of various properties of the aggregation process as the number of initial particles n tends to infinity. Thus the behaviour of a "typical" system consisting of a large number of particles is studied.

In the current paper, we are interested in the asymptotic behavior of $K_n(t)$, which denotes the number of clusters at time t in the system with n initial particles. This variable is a descreasing random step function satisfying $K_n(0) = n$ and $K_n(t) = 1$ for $t \geq T_n^{last}$, where T_n^{last} denotes the moment of the last collision. While calculating $K_n(t)$, we also count initial particles that have not experienced any collisions (in other words, $K_n(t)$ is the total number of particles existing at time t).

It is very important to know the behavior of $K_n(t)$. This gives a deep understanding of the aggregation process since the average size of a cluster at time t is $\frac{n}{K_n(t)}$.

At first we give a short deterministic example. Suppose that particles are located at points $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}$, i.e., $S_i = i$. Then by simple calculations we find that there would not be any collisions before $t = 1$. At the moment $t = 1$, all particles simultaneously merge into one cluster, that is $K_n(t) = n$ for $0 \leq t < 1$ and $K_n(t) = 1$ for $t \geq 1$.

But when the initial positions are random, the aggregation process behaves entirely differently. In [23] the author proved the following statement.

Fact 1. *There exists a deterministic function $a(t)$ such that both in the Poisson and the uniform models of initial positions, for any $t \geq 0$ we have*

$$\frac{K_n(t)}{n} \xrightarrow{\mathbb{P}} a(t), \qquad n \to \infty. \tag{1}$$

The function $a(t)$ is continuous, $a(0) = 1$, and $a(t) = 0$ for $t \geq 1$. Numerical simulations allow to conjecture that $a(t) = 1 - t^2$ for $0 \leq t \leq 1$.

The relation $a(t) = 0$ for $t > 1$ is not of surprise because from Giraud [8] we know that both in the Poisson and the uniform models, $T_n^{last} \xrightarrow{\mathbb{P}} 1$ in probability (the limit constant is so "fine" because of the proper scaling of the model). That is why we say that the instant $t = 1$ is *critical*;

note that this moment coincides with the moment of the unique collision in the deterministic model.

The aim of this paper is to improve the result of [23]. We generalize Fact 1 and prove it for the independent model. We will see that the limit function $a(t)$ depends on the distribution of X_i. For example, $a(t) = 1$ on $[0, \sqrt{\mu})$, where $\mu := \sup\{y : \mathbb{P}\{X_i < y\} = 0\}$; $a(t) \in (0, 1)$ on $(\sqrt{\mu}, 1)$; and $a(t) = 0$ on $(1, \infty)$.

Further, the recent results of the author [24] allow us to prove the conjecture from Fact 1 that $a^{Poiss}(t) = a^{Unif}(t) = 1 - t^2$ for $0 \le t \le 1$. There is an amazing contrast between the simplicity of this formula and the hard calculations one needs to obtain it. It is remarkable that now we know the limit function $a(t)$ for both main models of initial positions.

Our main goal is to sharpen (1) by finding the next term in the asymptotics of $K_n(t)$. The result is the following statement, where the standard symbol $\xrightarrow{\mathcal{D}}$ denotes weak convergence and D denotes the Skorohod space.

Theorem 1. *In independent models with continuous X_i satisfying $\mathbb{E}X_i^\gamma < \infty$ for some $\gamma > 4$, there exists a centered Gaussian process $K(\cdot)$ on $[0, 1)$ such that*

$$\frac{K_n(\cdot) - na(\cdot)}{\sqrt{n}} \xrightarrow{\mathcal{D}} K(\cdot) \quad \text{in } D[0, 1-\varepsilon] \text{ for all } \varepsilon \in (0, 1) \tag{2}$$

as $n \to \infty$. The process $K(\cdot)$ depends on the distribution of X_i. This process satisfies $K(0) = 0$ and has a.s. continuous trajectories. The covariance function $R(s, t)$ of $K(\cdot)$ is continuous on $[0, 1)^2$, $R(s, t) > 0$ on $(\sqrt{\mu}, 1)^2$, and $R(s, t) = 0$ on $[0, 1)^2 \setminus (\sqrt{\mu}, 1)^2$.

In the uniform model, (2) holds for some centered Gaussian process $K^{Unif}(\cdot)$ on $[0, 1)$. This process satisfies $K^{Unif}(0) = 0$ and has a.s. continuous trajectories. The covariance function $R^{Unif}(s, t)$ of $K^{Unif}(\cdot)$ is continuous on $[0, 1)^2$, and $R^{Unif}(s, t) = R^{Poiss}(s, t) - s^2t^2$.

Thus the Poisson and the uniform models lead to different limit processes $K^{Poiss}(\cdot)$ and $K^{Unif}(\cdot)$, although $a^{Poiss}(\cdot) = a^{Unif}(\cdot)$.

As an immediate corollary of Theorem 1 (see Billingsley [3, Sec. 15]), we get

$$\frac{K_n(t) - na(t)}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2(t)), \qquad n \to \infty \tag{3}$$

for any $t < 1$, where $\sigma^2(t) := R(t, t)$. It is possible to show that in independent models, (3) holds for all $t \neq 1$ under the less restrictive condition $\mathbb{E}X_i^2 < \infty$, with $\sigma^2(t) = 0$ for $t > 1$; the continuity of X_i is not required.

We also study the convergence of the left-hand side of (3) at the critical moment $t = 1$. Apparently, its limit is not Gaussian, but the answer to this complicated question is related to a curious but hardly provable conjecture on integrated random walks. In view of this non-Gaussianity, it seems impossible to prove the convergence of trajectories described by Theorem 1 on the whole $[0, 1]$ instead of $[0, 1 - \varepsilon]$; we refer to Section 7 for further discussions.

We finish this subsection with a note on scaling. In our case, masses of particles are $\frac{1}{n}$ and distances between them are of the order $\frac{1}{n}$. Let us rescale the independent model multiplying all masses and distances by n: We call the *expanding* model the system of particles of mass 1 each, initially located at points $S_1 - S_{[\frac{n}{2}]}, S_2 - S_{[\frac{n}{2}]}, \dots, S_n - S_{[\frac{n}{2}]}$. Positions of particles are shifted by $S_{[\frac{n}{2}]}$ because we want the system to expand as $n \to \infty$ "filling" the whole line but not only the positive half-line.

All results of our paper hold true for the expanding model. This is not of surprise because shifts change nothing, the rescaling of masses "contracts the time" by n times, and the rescaling of distances “expands the time” back by n times. For rigorous arguments, we refer the reader to Section 2 or to the note on scaling from Lifshits and Shi [14].

1.3. **The organization of the paper.** We describe the general method used to study systems of sticky particles in Section 2. This method is applied for studying the independent model in Section 3, where we investigate some properties of the aggregation process. It will be shown that *the aggregation process is highly local*, i.e., the behavior of a particle is essentially defined only by the motion of neighbour particles. The localization property suggests that we could somehow use limit theorems for weakly dependent variables to prove Fact 1 and Theorem 1 for the independent model. This will be done in Section 4. Then we prove Theorem 1 for the uniform model in Section 5. In Section 6 we study the number of clusters at the critical moment $t = 1$. Some open questions are discussed in Section 7, which finishes the paper.

2. The method of barycenters

In this section we briefly describe the method of barycenters, which is the main tool to study systems of sticky particles. This method was independently introduced by E et al. [6] and Martin and Piasecki [17] and then used in numerous papers on sticky particles model. The method of barycenters is also applicable to more general models, where particles could have nonzero initial speeds and different masses.

Let us start with several definitions. We always numerate particles from the left to the right, and identify particles with their numbers. A *block* of particles is a nonempty set $J \subset [1, n]$ consisting of consecutive numbers. For example, the block $(i, i + k]$ consists of particles $i + 1, \dots, i + k$. Note that there is no direct relation between blocks and clusters, e.g., different clusters could contain particles of a block; these clusters could even contain particles that do not belong to the block.

It is convenient to assume that initial particles do not vanish in collisions but continue to exist in created clusters. Then the coordinate $x_{i,n}(t)$ of the particle i could be defined as the coordinate of the cluster containing the particle at time t. The second subscript n always indicates the number of initial particles; we will omit this subscript as often as possible.

By $x_J(t) := |J|^{-1} \sum_{i\in J} x_i(t)$ denote the position of the *barycenter* of a block J at time t. Further, define

$$x^*_J(t) := x_J(0) + \frac{1}{2}\big(M^{(R)}_J - M^{(L)}_J\big)t^2,$$

where $M^{(R)}_J$ and $M^{(L)}_J$ are the total masses of particles initially located to the right and to the left of the block, respectively.

A block is *free from the right* up to time t if up to this time the block's particles did not collide with particles initially located to the right of the block. We similarly define blocks *free from the left*, and say that a block is *free* up to time t if it is both free from the right and from the left.

The next statement plays the key role in the analysis of systems of sticky particles: *The barycenter of a free block moves as an imaginary particle consisting of all particles of the block put together at the initial barycenter.* In a more precise and general way, we state that

Proposition 1. *If a block J is free from the right (resp. left) up to time t, then $x_J(s) \ge x^*_J(s)$ for $s \in [0, t]$ (resp. $x_J(s) \le x^*_J(s)$). If a block J is free up to time t, then $x_J(s) = x^*_J(s)$ for $s \in [0, t]$.*

This statement could be found, for example, in Lifshits and Shi [14], Proposition 4.1. The easy proof is based on the conservation property for momentum.

We call the *merging time* $T_{j,n}$ of a particle j the moment when it sticks with the particle $j+1$. In other words, $T_{j,n}$ is the first moment when particles j and $j+1$ are contained in a common cluster; here $j \in [1, n-1]$. Proposition 4.3 from Lifshits and Shi [14], which is stated below, allows to calculate $T_{j,n}$.

Proposition 2. *For every $j \in [1, n-1]$, we have*

$$T_{j,n} = \min_{\substack{j<k\leq n \\ 0\leq l<j}} \left\{ s \geq 0 : \ x^*_{(j,k]}(s) = x^*_{(l,j]}(s) \right\}.$$

Thus $T_{j,n}$ is expressed by means of barycenters. Note that since

$$x^*_{(j,k]}(s) - x^*_{(l,j]}(s) = x_{(j,k]}(0) - x_{(l,j]}(0) - \frac{k-l}{2n} s^2, \tag{4}$$

each of the equations $x^*_{(j,k]}(s) = x^*_{(l,j]}(s)$ has a unique non-negative solution. We also mention that if the minimum is attained on some k and l, then a collision of two clusters consisting of the particles from the blocks $(l, j]$ and $(j, k]$, respectively, occurs at time $T_{j,n}$.

We will prove Proposition 2 since its proof is quite simple and perfectly illustrates the idea of the method of barycenters.

Proof. For any $u < T_{j,n}$, particles j and $j+1$ are contained in different clusters. Therefore for every $l < j$, the block $[l, j]$ is free from the right up to time u, and for every $k > j$, the block $[j+1, k]$ is free from the left. By Proposition 1,

$$x^*_{(l,j]}(u) \leq x_{(l,j]}(u) \leq x_j(u) < x_{j+1}(u) \leq x_{(j,k]}(u) \leq x^*_{(j,k]}(u),$$

and since, by (4), the function $x^*_{(j,k]}(s) - x^*_{(l,j]}(s)$ is decreasing for $s \geq 0$, we conclude that

$$u < \left\{ s \geq 0 : \ x^*_{(j,k]}(s) = x^*_{(l,j]}(s) \right\}.$$

By taking minimum over k and l, and taking supremum over u, we get $T_{j,n} \leq \min\{\dots\}$.

Let us prove the last inequality in the other direction. By definition of $T_{j,n}$, there exist an $l < j$ and a $k > j$ such that the blocks $(l, j]$ and $(j, k]$ are free up to time $T_{j,n}$ (clusters containing particles from these blocks collide exactly at time $T_{j,n}$). Then, in view of Proposition 1,

$$x^*_{(l,j]}(T_{j,n}) = x_{(l,j]}(T_{j,n}) = x_{(j,k]}(T_{j,n}) = x^*_{(j,k]}(T_{j,n}),$$

thus $T_{j,n} = \left\{ s \geq 0 : \ x^*_{(j,k]}(s) = x^*_{(l,j]}(s) \right\}$, and $T_{j,n} \geq \min\{\dots\}$. □

3. Study of the independent model. The localization property

At first, note that

$$K_n(t) = 1 + \sum_{i=1}^{n-1} \mathbb{1}_{\{t<T_{i,n}\}} \tag{5}$$

because at each moment $T_{i,n}$, the total number of clusters decreases by one. This representation plays the main role in the investigation of $K_n(t)$. Clearly, we need to study properties of the r.v.'s $T_{i,n}$ to prove limit theorems for $K_n(t)$. This study will be done in the current section.

3.1. **The initial study.** At first, let us simplify the representation for $T_{j,n}$ from Proposition 2. In this section we consider the independent model of initial positions, where $x_{j,n}(0) = \frac{1}{n}S_j$. Recall that S_j is a random walk with i.i.d. increments $\{X_j\}_{j\in\mathbb{Z}}$ (we will need the variables $\{X_j\}_{j\le 0}$ later).

Take a look at (4) and rewrite the initial distance between barycenters as

$$\begin{aligned} x_{(j,k]}(0) - x_{(l,j]}(0) &= \frac{1}{k-j}\sum_{i=j+1}^{k}\frac{1}{n}S_i - \frac{1}{j-l}\sum_{i=l+1}^{j}\frac{1}{n}S_i \\ &= \frac{1}{n}\left(\frac{1}{k-j}\sum_{i=j+1}^{k}(S_i - S_{j+1}) + \frac{1}{j-l}\sum_{i=l+1}^{j}(S_j - S_i) + (S_{j+1} - S_j)\right) \\ &= \frac{1}{n}\left(\frac{1}{k-j}\sum_{i=1}^{k-j-1}(S_{j+i+1} - S_{j+1}) + \frac{1}{j-l}\sum_{i=1}^{j-l-1}(S_j - S_{j-i}) + X_{j+1}\right); \end{aligned}$$

let us agree that $\sum_{\varnothing} := 0$. Further,

$$\begin{aligned} x_{(j,k]}(0) - x_{(l,j]}(0) &= \frac{1}{n}\left(\frac{1}{k-j}\sum_{i=1}^{k-j-1}\sum_{m=j+2}^{j+i+1}X_m + \frac{1}{j-l}\sum_{i=1}^{j-l-1}\sum_{m=j-i+1}^{j}X_m + X_{j+1}\right) \\ &= \frac{1}{n}\left(\frac{1}{k-j}\sum_{i=1}^{k-j-1}(k-j-i)X_{j+i+1} + \frac{1}{j-l}\sum_{i=1}^{j-l-1}(j-l-i)X_{j-i+1} + X_{j+1}\right), \end{aligned}$$

and from (4) we have

$$x^*_{(j,k]}(s) - x^*_{(l,j]}(s) = F_{k-j,j,j-l}(s),$$

where for $p, q \ge 1$ and $j \in \mathbb{Z}$ we denoted

$$F_{p,j,q}(s) := \frac{1}{p}\sum_{i=1}^{p-1}(p-i)X_{j+i+1} + \frac{1}{q}\sum_{i=1}^{q-1}(q-i)X_{j-i+1} + X_{j+1} - \frac{p+q}{2}s^2. \tag{6}$$

Now by Proposition 2, we get

$$T_{j,n} = \min_{\substack{j<k\le n\\ 0\le l<j}}\Big\{s \ge 0:\ F_{k-j,j,j-l}(s) = 0\Big\} = \min_{\substack{1\le k\le n-j\\ 1\le l\le j}}\Big\{s \ge 0:\ F_{k,j,l}(s) = 0\Big\}. \tag{7}$$

Notice that $F_{p,j,q}(0) \ge 0$ for all p, j, q, and $F_{p,j,q}(s)$ is decreasing for $s \ge 0$. This function could be also written in more convenient form

$$F_{p,j,q}(s) = \frac{1}{p}\sum_{i=1}^{p-1}(p-i)(X_{j+i+1} - s^2) + \frac{1}{q}\sum_{i=1}^{q-1}(q-i)(X_{j-i+1} - s^2) + (X_{j+1} - s^2). \tag{8}$$

3.2. **Localization property of the aggregation process.** We see that $T_{j,n}$ is a function of $X_2, \dots, X_n$, in other words, it is necessary to know the distances between all n particles to find $T_{j,n}$. Actually, the aggregation process is highly local, i.e., *the value of* $T_{j,n}$ *is essentially defined only by the initial distances between neighbour particles of* j (that is, by those X_i, for which $|j-i|$ is small enough).

To make this statement rigorous, we need to introduce the following notations. Let us put

$$T_j^{(M)} := \min_{1\le k,l\le M}\Big\{s \ge 0:\ F_{k,j,l}(s) = 0\Big\}, \qquad j \in \mathbb{Z},\ M \in \mathbb{N},$$

which is expressed in terms of the variables $\{X_i\}_{|j-i|\le M}$ only. Also, define

$$T_j := \inf_{k,l\ge 1}\Big\{s\ge 0:\ F_{k,j,l}(s)=0\Big\}, \qquad j\in\mathbb{Z},$$

which are, in some sense, merging times in the infinite system of particles[1]. It is clear that

$$T_j \le T_{j,n} \le T_j^{(j\wedge n-j)}, \qquad j,n\in\mathbb{N},\ j\le n \tag{9}$$

(by $\wedge$ and $\vee$ we denote minimum and maximum, respectively) and

$$T_j \le T_j^{(M)}, \qquad j\in\mathbb{Z},\ M\in\mathbb{N}. \tag{10}$$

Let us estimate the rate of convergence of $\mathbb{P}\big\{T_j\ne T_j^{(M)}\big\}$ to zero as the "radius of the neighbourhood" M tends to infinity. Thus we could "measure" the above-mentioned locality of the aggregation. In fact, for any $n\in\mathbb{N}$, $j\le n$, and $M\le j\wedge n-j$, by (9) we have $\mathbb{P}\{T_{j,n}\ne T_j^{(M)}\}\le \mathbb{P}\{T_j\ne T_j^{(M)}\}$.

Lemma 1. *Suppose $\mathbb{E}X_i^\gamma<\infty$ for some $\gamma\ge 1$. Then there exists a non-decreasing function $\rho(t)$ such that for any $t\in(0,1)$, $j\in\mathbb{Z}$, and $M\in\mathbb{N}$, we have*

$$\max\Big(\mathbb{P}\Big\{\mathbb{1}_{\{t\le T_j\}}\ne \mathbb{1}_{\{t\le T_j^{(M)}\}}\Big\},\ \mathbb{P}\Big\{T_j\ne T_j^{(M)}, T_j^{(M)}\le t\Big\}\Big)\le \rho(t)M^{1-\gamma}. \tag{11}$$

Moreover, for any $t<1$, the left-hand side of (11) *is $o(M^{1-\gamma})$.*

Proof. Let us estimate the first probability in the left-hand side of (11). By properties of $F_{k,j,l}(\cdot)$ and definitions of $T_j^{(M)}$ and T_j,

$$\begin{aligned}\mathbb{P}\Big\{\mathbb{1}_{\{t\le T_j\}}\ne \mathbb{1}_{\{t\le T_j^{(M)}\}}\Big\} &= \mathbb{P}\Big\{T_j<t\le T_j^{(M)}\Big\}\\ &= \mathbb{P}\Big\{\inf_{k,l\ge 1}F_{k,j,l}(t)<0,\ \min_{1\le k,l\le M}F_{k,j,l}(t)\ge 0\Big\}.\end{aligned}$$

By (8), this expression does not depend on j, and putting $j:=-1$,

$$\begin{aligned}&\mathbb{P}\Big\{\mathbb{1}_{\{t\le T_j\}}\ne \mathbb{1}_{\{t\le T_j^{(M)}\}}\Big\}\\ =\ &\mathbb{P}\Big\{\inf_{k\ge 1}\frac{1}{k}\sum_{i=1}^{k-1}(k-i)(X_i-t^2)+\inf_{l\ge 1}\frac{1}{l}\sum_{i=1}^{l-1}(l-i)(X_{-i}-t^2)+(X_0-t^2)<0,\\ &\min_{1\le k\le M}\frac{1}{k}\sum_{i=1}^{k-1}(k-i)(X_i-t^2)+\min_{1\le l\le M}\frac{1}{l}\sum_{i=1}^{l-1}(l-i)(X_{-i}-t^2)+(X_0-t^2)\ge 0\Big\}.\end{aligned}$$

[1]The T_j are merging times in the *infinite expanding model*. This model could be naturally defined as the limit of the expanding model as $n\to\infty$ (see Sec. 1). The problem is to define the accelerations of particles because each particle has infinite masses to the right and to the left of it.

Let us define the infinite expanding model. Initially, infinitely many immobile particles, each one of mass 1, are located at points $\{S_k\}_{k\in\mathbb{Z}}$ and have accelerations $\{2k\}_{k\in\mathbb{Z}}$; here $S_0:=0$ and $S_k:=-\sum_{i=1}^{-k}X_{-i}$ for negative integer k. In a collision, particles with masses m_i, speeds v_i, and accelerations a_i stick together forming a new particle with mass $M=\sum m_i$, speed $M^{-1}\sum v_i m_i$, and acceleration $M^{-1}\sum a_i m_i$. Between collisions particles move according to the second law of Newtonian mechanics.

Thus we do not care about the nature of accelerations. Note that from this definition it is readily seen that T_i form a stationary sequence.

Then we compare the inequalities in the braces and obtain

$$\begin{aligned}
\mathbb{P}\Big\{\mathbb{1}_{\{t\le T_j\}}\neq \mathbb{1}_{\{t\le T_j^{(M)}\}}\Big\} &\le 2\mathbb{P}\Big\{\inf_{k>M}\frac{1}{k}\sum_{i=1}^{k-1}(k-i)(X_i-t^2)<\min_{1\le k\le M}\frac{1}{k}\sum_{i=1}^{k-1}(k-i)(X_i-t^2)\Big\}\\
&= 2\mathbb{P}\Big\{\inf_{k>M}\frac{1}{k}\sum_{i=1}^{k-1}(S_i-it^2)<\min_{1\le k\le M}\frac{1}{k}\sum_{i=1}^{k-1}(S_i-it^2)\Big\}\\
&\le 2\mathbb{P}\Big\{\inf_{k>M}\frac{1}{k}\sum_{i=1}^{k-1}(S_i-it^2)<\min_{k\in\{1,M\}}\frac{1}{k}\sum_{i=1}^{k-1}(S_i-it^2)\Big\}.
\end{aligned}$$

Now we rewrite the event in the last expression as

$$\begin{aligned}
&\Big\{\exists\, k>M:\ \frac{1}{k}\sum_{i=1}^{k-1}(S_i-it^2)<\min\Big(0,\frac{1}{M}\sum_{i=1}^{M-1}(S_i-it^2)\Big)\Big\}\\
=\ &\Big\{\exists\, k>M:\ \frac{1}{k}\sum_{i=1}^{M-1}(S_i-it^2)+\frac{1}{k}\sum_{i=M}^{k-1}(S_i-it^2)<\min\Big(0,\frac{1}{M}\sum_{i=1}^{M-1}(S_i-it^2)\Big)\Big\}.
\end{aligned}$$

Analyzing both cases $0\le \frac{1}{M}\sum_{i=1}^{M-1}(S_i-it^2)$ and $0>\frac{1}{M}\sum_{i=1}^{M-1}(S_i-it^2)$, we conclude that considered the event implies

$$\Big\{\exists\, k>M:\frac{1}{k}\sum_{i=M}^{k-1}(S_i-it^2)<0\Big\}=\Big\{\exists\, k>M:\sum_{i=M}^{k-1}(S_i-it^2)<0\Big\}.$$

Clearly, this event implies

$$\Big\{\exists\, i\ge M:S_i-it^2<0\Big\}=\Big\{\inf_{i\ge M}\frac{S_i}{i}<t^2\Big\},$$

thus combining all the estimates, we get

$$\mathbb{P}\Big\{\mathbb{1}_{\{t\le T_j\}}\neq \mathbb{1}_{\{t\le T_j^{(M)}\}}\Big\}\le 2\mathbb{P}\Big\{\inf_{i\ge M}\frac{S_i}{i}<t^2\Big\}. \tag{12}$$

Note that (12) was obtained without any assumptions on the moments of X_i.

Recall that we have $\mathbb{E}X_i=1$. Then the first part of (11) immediately follows from the classical result of Baum and Katz [1] (see Theorem 3 and Lemma):

Fact 2. *If $\mathbb{E}|X_i|^\gamma<\infty$ for some $\gamma\ge 1$ and $\mathbb{E}X_i=a$, then for any $\varepsilon>0$,*

$$\mathbb{P}\Big\{\sup_{i\ge k}\Big|\frac{S_i}{i}-a\Big|>\varepsilon\Big\}=o(k^{1-\gamma}),\qquad k\to\infty.$$

In addition, the series $\sum_{k=1}^\infty\mathbb{P}\big\{\sup_{i\ge k}\big|\frac{S_i}{i}-a\big|>\varepsilon\big\}$ converges for all $\varepsilon>0$ if $\gamma\ge 2$.

The proof of the second part of (11) is completely analogous, since

$$\begin{aligned}
\Big\{T_j\neq T_j^{(M)},T_j^{(M)}\le t\Big\} &= \Big\{T_j<T_j^{(M)}\le t\Big\}\\
&= \Big\{\inf_{1\le k,l}F_{k,j,l}\big(T_j^{(M)}\big)<0,\min_{1\le k,l\le M}F_{k,j,l}\big(T_j^{(M)}\big)=0,T_j^{(M)}\le t\Big\}.
\end{aligned}$$

We put $j:=-1$, repeat the estimates and get

$$\mathbb{P}\Big\{T_j\neq T_j^{(M)},T_j^{(M)}\le t\Big\}\le 2\mathbb{P}\Big\{\exists\, i\ge M:S_i-i\big[T_{-1}^{(M)}\big]^2<0,\ T_{-1}^{(M)}\le t\Big\}$$

instead of (12). Trivially, the right-hand side is majorized by $2\mathbb{P}\{\exists\, i \geq M : S_i - it^2 < 0\}$. $\square$

3.3. **The distribution function of T_0 in the Poisson model.** The amazing fact is that in the Poisson model, the distribution function of T_0 could be found explicitly. This is important because in the proof of Fact 1 for the independent model, we will see that the limit function $a(t)$ equals $\mathbb{P}\{T_0 > t\}$. We will also need twice continuous differentiability of $a^{Poiss}(t) = \mathbb{P}\{T_0^{Poiss} \geq t\}$ in the proof of Theorem 1 for the uniform model.

Lemma 2. *In the Poisson model, for $0 \leq t \leq 1$ we have*

$$\mathbb{P}\Big\{T_0 \geq t\Big\} = 1 - t^2. \tag{13}$$

For $t \geq 0$, $n \geq 2$, and $1 \leq j \leq n-1$,

$$\mathbb{P}\Big\{T_{j,n} \geq t\Big\} = e^{t^2}\mathbb{P}\Big\{\min_{1\leq k\leq j}\sum_{i=1}^{k}(S_i - it^2) \geq 0\Big\} \cdot \mathbb{P}\Big\{\min_{1\leq k\leq n-j}\sum_{i=1}^{k}(S_i - it^2) \geq 0\Big\}, \tag{14}$$

where S_i is a standard exponential random walk.

Proof. We start with (14). By (7), (8), and properties of $F_{k,j,l}(\cdot)$,

$$\begin{aligned}
\mathbb{P}\Big\{T_{j,n} \geq t\Big\} &= \mathbb{P}\Big\{\min_{\substack{1\leq k\leq n-j\\ 1\leq l\leq j}} F_{k,j,l}(t) \geq 0\Big\} \\
&= \mathbb{P}\Big\{\min_{1\leq k\leq n-j}\frac{1}{k}\sum_{i=1}^{k-1}(k-i)(X_{j+i+1} - t^2) \qquad (15)\\
&\qquad + \min_{1\leq l\leq j}\frac{1}{l}\sum_{i=1}^{l-1}(l-i)(X_{j-i+1} - t^2) + X_{j+1} - t^2 \geq 0\Big\}.
\end{aligned}$$

In the right-hand side of the last equality, by Y denote the first minimum and by $\tilde{Y}$ denote the second one.

Suppose X is a standard exponential r.v., Z is a non-negative r.v., and X and Z are independent; then

$$\mathbb{P}\{Z \leq X\} = \int_0^\infty \mathbb{P}\{Z \leq x\}e^{-x}dx = \int_0^\infty \mathbb{E}\,\mathbb{1}_{\{Z\leq x\}}e^{-x}dx = \mathbb{E}\int_0^\infty \mathbb{1}_{\{Z\leq x\}}e^{-x}dx = \mathbb{E}\,e^{-Z}.$$

Hence in view of independence of Y, $\tilde{Y}$, X_{j+1}, we get

$$\mathbb{P}\Big\{Y + \tilde{Y} + X_{j+1} - t^2 \geq 0\Big\} = \mathbb{E}\,e^{Y+\tilde{Y}-t^2} = e^{t^2}\mathbb{E}\,e^{Y-t^2}\,\mathbb{E}\,e^{\tilde{Y}-t^2};$$

therefore

$$\mathbb{P}\Big\{T_{j,n} \geq t\Big\} = e^{t^2}\mathbb{P}\Big\{Y + X_{j+1} - t^2 \geq 0\Big\}\,\mathbb{P}\Big\{\tilde{Y} + X_{j+1} - t^2 \geq 0\Big\}.$$

The proof of (14) is almost finished, because

$$\begin{aligned}
\mathbb{P}\Big\{\tilde{Y}+X_{j+1}-t^2\geq 0\Big\} &= \mathbb{P}\Big\{\min_{1\leq l\leq j}\frac{1}{l}\sum_{i=1}^{l-1}(l-i)(X_{j-i+1}-t^2)+X_{j+1}-t^2\geq 0\Big\} \\
&= \mathbb{P}\Big\{\min_{1\leq l\leq j}\Big(\sum_{i=1}^{l-1}(l-i)(X_{i+1}-t^2)+l(X_1-t^2)\Big)\geq 0\Big\} \qquad (16)\\
&= \mathbb{P}\Big\{\min_{1\leq l\leq j}\sum_{i=1}^{l}(l-i+1)(X_i-t^2)\geq 0\Big\}
\end{aligned}$$

and the last expression equals the first probability in the right-hand side of (14).

Let us now prove (13). From the definition of T_0 and $T_0^{(k)}$, we see that $\mathbb{1}_{\{t\leq T_0^{(k)}\}}\to\mathbb{1}_{\{t\leq T_0\}}$ a.s. as $k\to\infty$; then by (14),

$$\mathbb{P}\Big\{T_0\geq t\Big\}=e^{t^2}\mathbb{P}^2\Big\{\inf_{k\geq 1}\sum_{i=1}^{k}(S_i-it^2)\geq 0\Big\}.$$

Then we need to check that

$$\mathbb{P}\Big\{\inf_{k\geq 1}\sum_{i=1}^{k}(S_i-it)\geq 0\Big\}=\sqrt{1-t}e^{-t/2}$$

for $0\leq t\leq 1$. The complicated calculations of this probability take more then ten pages and therefore they were separated into the independent paper [24]. Although this question seems to be technical, the proof is based on original and really beautiful ideas. $\square$

3.4. **Some properties of the variables T_i.** In this subsection we prove several important properties of r.v.'s T_i.

1. The sequence T_i is stationary.

Proof. This immediately follows from the definition of T_i and stationarity of X_i (which are i.i.d.).

2. The common distribution function of T_i is defined by

$$\mathbb{P}\Big\{T_i\geq t\Big\}=\mathbb{P}\Big\{\inf_{k\geq 1}\frac{1}{k}\sum_{i=1}^{k-1}(k-i)(X_i-t^2)+\inf_{l\geq 1}\frac{1}{l}\sum_{i=1}^{l-1}(l-i)(X_{-i}-t^2)+(X_0-t^2)\geq 0\Big\}. \qquad (17)$$

Proof. This assertion is a consequence of (8).

3. We have $\mathbb{P}\big\{\sqrt{\mu}\leq T_i\leq 1\big\}=1$, but $\sup\big\{y:\mathbb{P}\{T_i<y\}=0\big\}=\sqrt{\mu}$ and $\inf\big\{y:\mathbb{P}\{T_i<y\}=1\big\}=1$; recall that $\mu=\sup\big\{y:\mathbb{P}\{X_i<y\}=0\big\}$. In addition, if $\mathbb{E}X_i^2<\infty$, then $\mathbb{P}\big\{T_i=1\big\}=0$.

Proof. First, the $\mathbb{P}\big\{\sqrt{\mu}\leq T_i\big\}=1$ is trivial, because both infima in (17) are non-positive.

Second, consider $\mathbb{P}\big\{T_i\geq t\big\}$ for $t\geq 1$. By taking into account that infima in (17) are non-positive, we obtain

$$\mathbb{P}\Big\{T_i\geq t\Big\}\leq\mathbb{P}\Big\{\inf_{k\geq 1}\frac{1}{k}\sum_{i=1}^{k-1}(k-i)(X_i-t^2)+(X_0-t^2)\geq 0\Big\}.$$

Then by the same arguments as in (16),

$$\mathbb{P}\Big\{T_i \ge t^2\Big\} \le \mathbb{P}\Big\{\inf_{k\ge 1}\sum_{i=1}^{k}(k-i+1)(X_i-t^2)\ge 0\Big\} = \mathbb{P}\Big\{\inf_{k\ge 1}\sum_{i=1}^{k}(S_i-it^2)\ge 0\Big\}.$$

By the strong law of large numbers, this probability is zero for all $t>1$.

If $t=1$ and $X_i^2<\infty$, then

$$\mathbb{P}\Big\{\inf_{k\ge 1}\sum_{i=1}^{k}(S_i-i)\ge 0\Big\} = \lim_{n\to\infty}\mathbb{P}\Big\{\min_{1\le k\le n}\sum_{i=1}^{k}(S_i-i)\ge 0\Big\} = \lim_{n\to\infty}\mathbb{P}\Big\{\min_{1\le k\le n}\frac{1}{n}\sum_{i=1}^{k}\frac{S_i-i}{\sqrt{n}}\ge 0\Big\},$$

and from the invariance principle we get

$$\mathbb{P}\Big\{T_i\ge 1\Big\}\le \mathbb{P}\Big\{\min_{0\le s\le 1}\int_0^s W(u)du\ge 0\Big\}.$$

It follows from the asymptotics of unilateral small deviations probabilities of an integrated Wiener process (see (40) and (41) below) that the last expression equals zero.

Third, to prove the assertions $\sup\big\{y:\mathbb{P}\{T_i<y\}=0\big\}=\sqrt{\mu}$ and $\inf\big\{y:\mathbb{P}\{T_i<y\}=1\big\}=1$, it is sufficient to prove that for any $t<\mathbb{E}X_i=1$, the common distribution of the i.i.d. infima in (17) has atom at zero. We have

$$\mathbb{P}\Big\{\inf_{k\ge 1}\frac{1}{k}\sum_{i=1}^{k-1}(k-i)(X_i-t^2)=0\Big\} = \mathbb{P}\Big\{\inf_{k\ge 1}\frac{1}{k}\sum_{i=1}^{k-1}(S_i-it^2)=0\Big\} \ge \mathbb{P}\Big\{\inf_{i\ge 1}\frac{S_i}{i}\ge t^2\Big\},$$

and it could be shown via the strong law of large numbers that the last probability is strictly positive for all $t<1$.

4. Suppose X_i is continuous. Then $T_j^{(k)}$ and $T_{j,n}$ are continuous (for any j,k and n), and the common distribution of T_j could have an atom only at 1. In addition, if $\mathbb{E}X_i^2<\infty$, then T_j are continuous.

Proof. By (6) and (7),

$$T_{j,n} = \min_{\substack{1\le k\le n-j\\ 1\le l\le j}}\sqrt{\frac{2}{k+l}\Big(\frac{1}{k}\sum_{i=1}^{k-1}(k-i)X_{j+i+1}+\frac{1}{l}\sum_{i=1}^{l-1}(l-i)X_{j-i+1}+X_{j+1}\Big)}, \tag{18}$$

hence $T_{j,n}$ is continuous as a minimum of a finite number of continuous r.v.'s. The $T_j^{(k)}$ are also continuous because $T_j^{(k)}\overset{\mathcal{D}}{=}T_{k,2k}$.

Now we study the continuity of T_j. By Property 3, it only remains to verify that $\mathbb{P}\big\{T_j\ge t\big\}$ is continuous on $[0,1)$. But $\mathbb{P}\big\{T_j^{(k)}\ge t\big\}-\mathbb{P}\big\{T_j\ge t\big\}=\mathbb{P}\big\{\mathbb{1}_{\{t\le T_j\}}\ne\mathbb{1}_{\{t\le T_j^{(k)}\}}\big\}$, and in view of (12),

$$\sup_{0\le t\le s}\Big|\mathbb{P}\Big\{T_j^{(k)}\ge t\Big\}-\mathbb{P}\Big\{T_j\ge t\Big\}\Big|\le\sup_{0\le t\le s}2\mathbb{P}\Big\{\inf_{m\ge k}\frac{S_m}{m}<t^2\Big\}=2\mathbb{P}\Big\{\inf_{m\ge k}\frac{S_m}{m}<s^2\Big\}$$

for every $s<1=\mathbb{E}X_i$. By the strong law of large numbers, the last expression tends to zero; then $\mathbb{P}\big\{T_j\ge t\big\}$ is continuous on $[0,s]$ as a uniform limit of continuous functions $\mathbb{P}\big\{T_j^{(k)}\ge t\big\}$. As far as $s<1$ is arbitrary, $\mathbb{P}\big\{T_j\ge t\big\}$ is continuous on $[0,1)$.

5. The $cov(\mathbb{1}_{\{s\le T_0\}}, \mathbb{1}_{\{t\le T_k\}})$ tends to zero as $k\to\infty$ for all $s,t\in[0,1)$. If, in addition, $\mathbb{E}X_i^\gamma<\infty$ for some $\gamma>1$, then for any $s,t\in[0,1)$ and $k\in\mathbb{N}$, we have

$$\big|cov(\mathbb{1}_{\{s\le T_0\}}, \mathbb{1}_{\{t\le T_k\}})\big| \le 2^\gamma\big(\rho(s)+\rho(t)\big)k^{1-\gamma}. \tag{19}$$

Proof. The idea is to approximate $\mathbb{1}_{\{s\le T_0\}}$ and $\mathbb{1}_{\{t\le T_k\}}$ by $\mathbb{1}_{\{s\le T_0^{(k/2)}\}}$ and $\mathbb{1}_{\{t\le T_k^{(k/2)}\}}$, respectively; here by $k/2$ we mean $\lceil k/2\rceil$, where $\lceil x\rceil=\min\{m\in\mathbb{Z}: m\ge x\}$. Note that $\mathbb{1}_{\{s\le T_0^{(k/2)}\}}$ and $\mathbb{1}_{\{t\le T_k^{(k/2)}\}}$ are independent because the first one is a function of $\{X_i\}_{i\le k/2}$ and the second is a function of $\{X_i\}_{i\ge k/2+1}$. Then we have

$$\begin{aligned}
\big|cov(\mathbb{1}_{\{s\le T_0\}}, \mathbb{1}_{\{t\le T_k\}})\big| &= \big|cov(\mathbb{1}_{\{s\le T_0\}}, \mathbb{1}_{\{t\le T_k\}}) - cov(\mathbb{1}_{\{s\le T_0^{(k/2)}\}}, \mathbb{1}_{\{t\le T_k^{(k/2)}\}})\big| \\
&\le \big|\mathbb{E}\big(\mathbb{1}_{\{s\le T_0\}}\mathbb{1}_{\{t\le T_k\}} - \mathbb{1}_{\{s\le T_0^{(k/2)}\}}\mathbb{1}_{\{t\le T_k^{(k/2)}\}}\big)\big| \\
&\quad + \big|\mathbb{E}\big(\mathbb{1}_{\{s\le T_0\}} - \mathbb{1}_{\{s\le T_0^{(k/2)}\}}\big)\big| + \big|\mathbb{E}\big(\mathbb{1}_{\{t\le T_k\}} - \mathbb{1}_{\{t\le T_k^{(k/2)}\}}\big)\big| \qquad (20)\\
&= \mathbb{P}\Big\{\mathbb{1}_{\{s\le T_0\}}\mathbb{1}_{\{t\le T_k\}} \ne \mathbb{1}_{\{s\le T_0^{(k/2)}\}}\mathbb{1}_{\{t\le T_k^{(k/2)}\}}\Big\} \\
&\quad + \mathbb{P}\Big\{\mathbb{1}_{\{s\le T_0\}} \ne \mathbb{1}_{\{s\le T_0^{(k/2)}\}}\Big\} + \mathbb{P}\Big\{\mathbb{1}_{\{t\le T_k\}} \ne \mathbb{1}_{\{t\le T_k^{(k/2)}\}}\Big\}.
\end{aligned}$$

But

$$\mathbb{P}\Big\{\mathbb{1}_{\{s\le T_0\}}\mathbb{1}_{\{t\le T_k\}} \ne \mathbb{1}_{\{s\le T_0^{(k/2)}\}}\mathbb{1}_{\{t\le T_k^{(k/2)}\}}\Big\} \le \mathbb{P}\Big\{\mathbb{1}_{\{s\le T_0\}} \ne \mathbb{1}_{\{s\le T_0^{(k/2)}\}} \bigcup \mathbb{1}_{\{t\le T_k\}} \ne \mathbb{1}_{\{t\le T_k^{(k/2)}\}}\Big\},$$

therefore by Lemma 1, the covariance tends to zero and assertion (19) holds true.

6. The r.v.'s $\{T_i\}_{i\in\mathbb{Z}}$, $\{T_i^{(k)}\}_{i\in\mathbb{Z}}$, and $\{T_{i,n}\}_{i=1}^{n-1}$ are *associated*; the author owes this observation to M. Lifshits.

Proof. Let us first recall the definition and some basic properties of associated variables. R.v.'s $\xi_1,\dots,\xi_m$ are *associated* if for any coordinate-wise nondecreasing functions $f,g:\mathbb{R}^m\to\mathbb{R}$ it holds that

$$cov\big(f(\xi_1,\dots,\xi_m),\, g(\xi_1,\dots,\xi_m)\big)\ge 0$$

(assuming that the left-hand side is well defined). An infinite set of r.v.'s is associated if any finite subset of its variables is associated. The following sufficient conditions of association are well known (see [7]):

(a) Independent variables are associated.
(b) Coordinate-wise nondecreasing functions (of finite number of arguments) of associated r.v.'s are associated.
(c) If the variables $\xi_{1,k},\dots,\xi_{m,k}$ are associated for every k and $(\xi_{1,k},\dots,\xi_{m,k}) \xrightarrow{\mathcal{D}} (\xi_1,\dots,\xi_m)$ as $k\to\infty$, then $\xi_1,\dots,\xi_m$ are associated.
(d) If two sets of associated variables are independent, then the union of these sets is also associated.

Thus $\{T_{i,n}\}_{i=1}^{n-1}$ are associated for every n by (a), (b), and (18). Analogously, $\{T_i^{(k)}\}_{i\in\mathbb{Z}}$ are associated for every k. Finally, since $T_i^{(k)}\to T_i$ a.s. as $k\to\infty$ for every i, (c) ensures the association of $\{T_i\}_{i\in\mathbb{Z}}$.

7. For any $s,t\in\mathbb{R}$ and $k\in\mathbb{Z}$,

$$cov(\mathbb{1}_{\{T_0\le s\}}, \mathbb{1}_{\{T_k\le t\}})\ge 0. \tag{21}$$

Proof. This inequality follows from $cov(\mathbb{1}_{\{T_0\le s\}}, \mathbb{1}_{\{T_k\le t\}}) = cov(\mathbb{1}_{\{s<T_0\}}, \mathbb{1}_{\{t<T_k\}})$, the association of T_0, T_k, and (b).

8. If $\mathbb{E}X_i^\gamma < \infty$ for some $\gamma \geq 2$, then the stationary sequence $\min\{T_i, t\}$ is strongly mixing for any $t < 1$, and the coefficients of strong mixing $\alpha(k)$ satisfy $\alpha(k) = o(k^{2-\gamma})$.
Proof. Recall that stationary r.v.'s ξ_i are *strongly mixing* if $\alpha(k) \to 0$ as $k \to \infty$, where $\alpha(k)$ are the coefficients of strong mixing defined as

$$\alpha(k) := \sup_{A \in \mathcal{F}^0_{-\infty},\, B \in \mathcal{F}^\infty_k} \big|\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)\big|;$$

here $\mathcal{F}^0_{-\infty} := \sigma(\xi_0, \xi_{-1}, \dots)$ and $\mathcal{F}^\infty_k := \sigma(\xi_k, \xi_{k+1}, \dots)$ are the σ-algebras of "past" and "future", respectively. It is readily seen that

$$\alpha(k) \leq \sup_{0 \leq f,g \leq 1} \big|cov\big(f(\xi_0, \xi_{-1}, \dots),\, g(\xi_k, \xi_{k+1}, \dots)\big)\big|, \tag{22}$$

where the supremum is taken over Borel functions $f, g : \mathbb{R}^\infty \to [0,1]$.

Let us estimate $\alpha(k)$ in the same way we estimated the left-hand side of (19). Fix some Borel functions $f, g : \mathbb{R}^\infty \to [0,1]$. We approximate the variables from the "past" $T_0 \wedge t$, $T_{-1} \wedge t$, $T_{-2} \wedge t$, $\dots$ by $T_0^{(k/2)} \wedge t$, $T_{-1}^{(k/2+1)} \wedge t$, $T_{-2}^{(k/2+2)} \wedge t$, $\dots$, respectively; for the variables from the "future", we use the analogous approximation. Then $f\big(T_0^{(k/2)} \wedge t, T_{-1}^{(k/2+1)} \wedge t, \dots\big)$ and $g\big(T_k^{(k/2)} \wedge t, T_{k+1}^{(k/2+1)} \wedge t, \dots\big)$ are independent because the first one is a function of $\{X_i\}_{i \leq k/2}$ and the second is a function of $\{X_i\}_{i \geq k/2+1}$. Now we argue in the same way as in (20) and get

$$\begin{aligned}
&\Big|cov\Big(f\big(T_0 \wedge t, T_{-1} \wedge t, \dots\big),\, g\big(T_k \wedge t, T_{k+1} \wedge t, \dots\big)\Big)\Big| \\
\leq\ & 2\mathbb{P}\Big\{\bigcup_{i=0}^\infty \big(T_{-i} \wedge t\big) \neq \big(T_{-i}^{(k/2+i)} \wedge t\big)\Big\} + 2\mathbb{P}\Big\{\bigcup_{i=0}^\infty \big(T_{k+i} \wedge t\big) \neq \big(T_{k+i}^{(k/2+i)} \wedge t\big)\Big\} \\
\leq\ & 4 \sum_{i=k/2}^\infty \mathbb{P}\Big\{\big(T_0 \wedge t\big) \neq \big(T_0^{(i)} \wedge t\big)\Big\}.
\end{aligned}$$

It remains to apply the formula of total probability to obtain

$$\begin{aligned}
\mathbb{P}\Big\{\big(T_0 \wedge t\big) \neq \big(T_0^{(i)} \wedge t\big)\Big\} &= \mathbb{P}\Big\{\big(T_0 \wedge t\big) \neq \big(T_0^{(i)} \wedge t\big), T_0^{(i)} \geq t\Big\} + \mathbb{P}\Big\{\big(T_0 \wedge t\big) \neq \big(T_0^{(i)} \wedge t\big), T_0^{(i)} < t\Big\} \\
&\leq \mathbb{P}\Big\{\mathbb{1}_{\{t \leq T_0\}} \neq \mathbb{1}_{\{t \leq T_0^{(i)}\}}\Big\} + \mathbb{P}\Big\{T_0 \neq T_0^{(i)}, T_0^{(i)} \leq t\Big\}.
\end{aligned}$$

Then we combine all the estimates, recall Lemma 1, and by (22) and arbitrariness of f, g, we get $\alpha(k) \leq 8 \sum_{i=k/2}^\infty o(i^{1-\gamma}) = o(k^{2-\gamma})$.

3.5. **The last collision.** We finish this section with a statement on convergence of the moments of the last collision.

Proposition 3. *If $\mathbb{E}X_i^2 < \infty$, then $T_n^{last} \xrightarrow{\mathbb{P}} 1$ as $n \to \infty$.*

Proof. Let us first prove that $\mathbb{P}\{T_n^{last} \geq t\} \to 0$ as $n \to \infty$ for all $t > 1$. Since $T_n^{last} = \max_{1 \leq j \leq n-1} T_{j,n}$, we have

$$\mathbb{P}\{T_n^{last} \geq t\} \leq \sum_{j=1}^{n-1} \mathbb{P}\{T_{j,n} \geq t\}. \tag{23}$$

Now estimate $\mathbb{P}\{T_{j,n} \geq t\}$. By taking into account that minima in (15) are non-positive and arguing as in (16), we get

$$\begin{aligned}\mathbb{P}\{T_{j,n} \geq t\} &\leq \mathbb{P}\Big\{\min_{1\leq k\leq j\,\vee\, n-j} \frac{1}{k}\sum_{i=1}^{k-1}(k-i)(X_{j+i+1}-t^2)+X_{j+1}-t^2 \geq 0\Big\}\\ &= \mathbb{P}\Big\{\min_{1\leq k\leq j\,\vee\, n-j} \sum_{i=1}^{k}(k-i+1)(X_i-t^2)\geq 0\Big\}\\ &\leq \mathbb{P}\Big\{\min_{1\leq k\leq n/2}\sum_{i=1}^{k}(S_i-it^2)\geq 0\Big\}.\end{aligned}$$

We claim that

$$\mathbb{P}\{T_{j,n}\geq t\} \leq \mathbb{P}\Big\{\sup_{i\geq \frac{t-1}{4t}n} \frac{S_i}{i} > \frac{1+t^2}{2}\Big\}; \tag{24}$$

recall $t>1$. Note that (24) was obtained without any assumptions on X_i.

To prove (24), we check that

$$\Big\{\min_{1\leq k\leq n/2}\sum_{i=1}^{k}(S_i-it^2)\geq 0\Big\} \subset \Big\{\sup_{i\geq\frac{t-1}{4t}n}\frac{S_i}{i} > \frac{1+t^2}{2}\Big\}.$$

Assume the converse. Then, taking into account the non-negativity of S_i and denoting $c := \frac{t-1}{4t}$,

$$0 \leq \sum_{i=1}^{n/2}(S_i-it^2) = \sum_{i=1}^{cn}(S_i-it^2)+\sum_{i=cn+1}^{n/2}(S_i-it^2) \leq \sum_{i=1}^{cn}(S_{cn}-it^2)+\sum_{i=cn+1}^{n/2}\Big(i\frac{1+t^2}{2}-it^2\Big).$$

We estimate the last expression with

$$cnS_{cn} - \frac{(cn)^2}{2}t^2 - \frac{(n/2)^2-(cn)^2}{2}\cdot\frac{t^2-1}{2} \leq \frac{c^2}{2}n^2 - \frac{1/4-c^2}{2}\cdot\frac{t^2-1}{2}n^2.$$

It is simple to check that the last expression is negative, thus we got a contradiction.

Now from (23), (24), and Fact 2 follows that $\mathbb{P}\{T_n^{last}\geq t\} = \sum_{i=1}^{n-1} o\big((cn)^{-1}\big) = o(1)$ for all $t>1$.

Let us prove that $\mathbb{P}\{T_n^{last} < t\} \to 0$ as $n\to\infty$ for all $t<1$. Since $T_n^{last} = \max_{1\leq j\leq n-1} T_{j,n}$, we estimate

$$\begin{aligned}\mathbb{P}\{T_n^{last}<t\} &\leq \mathbb{P}\Big\{\max_{1\leq j\leq\sqrt{n}-1} T_{j\sqrt{n},n}<t\Big\}\\ &= \mathbb{P}\Big\{\max_{1\leq j\leq \sqrt{n}-1} T_{j\sqrt{n}}^{(\sqrt{n}/2)}<t\Big\} + \sum_{j=1}^{\sqrt{n}-1}\mathbb{P}\Big\{\mathbb{1}_{\{t\leq T_{j\sqrt{n},n}\}} \neq \mathbb{1}_{\{t\leq T_{j\sqrt{n}}^{(\sqrt{n}/2)}\}}\Big\}.\end{aligned}$$

In view of (9) and Lemma 1, the sum is $\sum_{j=1}^{\sqrt{n}-1} o(n^{-1/2}) = o(1)$, hence it is sufficient to prove that the first probability in the last expression tends to zero. For a fixed n, all $T_{j\sqrt{n}}^{(\sqrt{n}/2)}$ are independent because each one is a function of $\{X_i\}_{|j\sqrt{n}-i|\leq\sqrt{n}/2}$ (to be precise, of $X_{j\sqrt{n}-\sqrt{n}/2+2},\dots,X_{j\sqrt{n}+\sqrt{n}/2}$). Thus

$$\mathbb{P}\Big\{\max_{1\leq j\leq\sqrt{n}-1} T_{j\sqrt{n}}^{(\sqrt{n}/2)}<t\Big\} = \mathbb{P}^{\sqrt{n}-1}\Big\{T_{\sqrt{n}}^{(\sqrt{n}/2)}<t\Big\} \leq \mathbb{P}^{\sqrt{n}-1}\Big\{T_0<t\Big\},$$

which tends to zero. Indeed, $\mathbb{P}\{T_0<t\}<1$ by Property 3, Subsection 3.4. $\square$

4. The proofs of Fact 1 and Theorem 1 for the independent model

Recall that the number of clusters $K_n(t)$ is given by (5). Our idea is to study $\sum_{i=1}^{n-1} \mathbb{1}_{\{t<T_i\}}$ instead of $\sum_{i=1}^{n-1} \mathbb{1}_{\{t<T_{i,n}\}}$. Thus we deal with one sequence T_i and avoid considering the triangular array $T_{i,n}$.

Let us now prove Fact 1 for the independent model. We prove (1) for $t \neq 1$ without any additional assumptions on X_i; for $t = 1$, we require $\mathbb{E}X_i^2 < \infty$. The properties of the limit function $a(t)$ were studied in Subsection 3.4 (Properties 3 and 4).

Proof of Fact 1. We put $a(t) := \mathbb{P}\big\{T_0 > t\big\}$. Let us first prove assertion (1) for all $t < 1$. It is sufficient to prove that

$$\frac{K_n(t)}{n} - \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{t<T_i\}} \xrightarrow{\mathbb{P}} 0, \qquad n \to \infty. \tag{25}$$

Indeed, the stationary sequence $\mathbb{1}_{\{t<T_i\}}$ satisfies the law of large numbers because of Property 5 from Subsection 3.4 and the well-know result of S.N. Bernstein:

Fact 3. *The law of large numbers holds for r.v.'s ξ_i if there exists a sequence $r(k) \to 0$ such that $cov(\xi_i, \xi_j) \le r(|i-j|)$ for all $i, j \in \mathbb{N}$.*

By (5), we have

$$\left|\frac{K_n(t)}{n} - \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{t<T_i\}}\right| \le \frac{1}{n} + \frac{1}{n}\sum_{i=1}^{n-1}\big(\mathbb{1}_{\{t<T_{i,n}\}} - \mathbb{1}_{\{t<T_i\}}\big);$$

note that the sum in the right-hand side is non-negative because of (9). Then (25) immediately follows from the Chebyshev inequality provided that the expectation of the right-hand side tends to zero. Using (9) again, we obtain

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n-1}\mathbb{E}\big(\mathbb{1}_{\{t<T_{i,n}\}} - \mathbb{1}_{\{t<T_i\}}\big) &\le \frac{1}{n}\sum_{i=1}^{n-1}\big(\mathbb{E}\mathbb{1}_{\{t<T_i^{(i\wedge n-i)}\}} - \mathbb{E}\mathbb{1}_{\{t<T_i\}}\big) \\ &= \frac{1}{n}\sum_{i=1}^{n-1}\mathbb{P}\Big\{\mathbb{1}_{\{t<T_i\}} \neq \mathbb{1}_{\{t<T_i^{(i\wedge n-i)}\}}\Big\},\end{aligned}$$

which is $\frac{2}{n}\sum_{i=1}^{n/2} o(1) = o(1)$ by Lemma 1 (to be very precise, Lemma 1 deals with a little bit different indicators, but we can estimate the considered probability repeating the proof of Lemma 1 word-by-word).

Now we check that (1) holds for all $t > 1$. As far as by (24), it is true that $\mathbb{E}\frac{K_n(t)}{n} = \frac{1}{n}\sum_{i=1}^{n-1}\mathbb{P}\big\{T_{i,n} > t\big\} \to 0$ as $n \to \infty$, from the Chebyshev inequality follows $\frac{K_n(t)}{n} \xrightarrow{\mathbb{P}} a(t) = 0$.

To conclude the proof, it remains to check that (1) holds for $t = 1$ if $\mathbb{E}X_i^2 < \infty$. By Property 3 from Subsection 3.4, we have $a(1) = 0$ and $\mathbb{P}\big\{T_0 = 1\big\} = 0$; consequently, $a(t) = \mathbb{P}\big\{T_0 > t\big\}$ is continuous at $t = 1$. Then (1) is true for $t = 1$ because of $0 < \frac{K_n(1)}{n} \le \frac{K_n(t)}{n} \xrightarrow{\mathbb{P}} a(t)$ for any $t \in (0, 1)$ and $a(t) \to a(1) = 0$ as $t \nearrow 1$. $\square$

Now we prove Theorem 1 for the independent model. We think of $D[0, 1]$ as of a separable metric space equipped with the Skorohod metric d, which induces the Skorohod topology.

Proof of Theorem 1. At first, we prove (2). In view of representation (5) for $K_n(t)$, this assertion follows from the relation

$$\sup_{0\le t\le 1-\varepsilon}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1}\mathbb{1}_{\{t<T_{i,n}\}}-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{1}_{\{t<T_i\}}\right|\xrightarrow{\mathbb{P}}0\quad\text{for all } \varepsilon\in(0,1) \tag{26}$$

and the existence of a centered Gaussian process $K(\cdot)$ on $[0,1)$ such that

$$\frac{1}{\sqrt{n}}\left\{\sum_{i=1}^{n}\mathbb{1}_{\{t<T_i\}}-na(t)\right\}\xrightarrow{\mathcal{D}}K(\cdot)\quad\text{in } D[0,1-\varepsilon] \text{ for all } \varepsilon\in(0,1). \tag{27}$$

Indeed, if $Y_n\xrightarrow{\mathcal{D}}Y$ and $d(Y_n,Y_n')\xrightarrow{\mathbb{P}}0$ for some random elements Y_n,Y_n',Y of the separable metric space $D[0,1-\varepsilon]$, then $Y_n'\xrightarrow{\mathcal{D}}Y$; recall that $d(Y_n,Y_n')\le\sup_{t\in[0,1-\varepsilon]}|Y_n(t)-Y_n'(t)|$.

We start with (26). It is sufficient to prove that the expectation of the left-hand side tends to zero. Since the supremum of a sum does not exceed the sum of suprema, let us check that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1}\mathbb{E}\sup_{0\le t\le 1-\varepsilon}\left|\mathbb{1}_{\{t<T_{i,n}\}}-\mathbb{1}_{\{t<T_i\}}\right|\longrightarrow 0\quad\text{for all } \varepsilon\in(0,1). \tag{28}$$

By (9), we have

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le 1-\varepsilon}\left|\mathbb{1}_{\{t<T_{i,n}\}}-\mathbb{1}_{\{t<T_i\}}\right| &\le \mathbb{E}\sup_{0\le t\le 1-\varepsilon}\left(\mathbb{1}_{\{t<T_i^{(i\wedge n-i)}\}}-\mathbb{1}_{\{t<T_i\}}\right)\\ &= \mathbb{P}\left\{T_i\ne T_i^{(i\wedge n-i)},\, T_i\le 1-\varepsilon\right\}\\ &= \mathbb{P}\left\{T_i\ne T_i^{(i\wedge n-i)},\, T_i^{(i\wedge n-i)}<1-\varepsilon\right\}\\ &\quad+\mathbb{P}\left\{\mathbb{1}_{\{1-\varepsilon\le T_i\}}\ne\mathbb{1}_{\{1-\varepsilon\le T_i^{(i\wedge n-i)}\}}\right\},\end{aligned}$$

where we obtained the last equality applying the formula of total probability. Combining the estimates, by Lemma 1 we conclude that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1}\mathbb{E}\sup_{0\le t\le 1-\varepsilon}\left|\mathbb{1}_{\{t<T_{i,n}\}}-\mathbb{1}_{\{t<T_i\}}\right|\le\frac{2\rho(1-\varepsilon)}{\sqrt{n}}\sum_{i=1}^{n-1}(i\wedge n-i)^{1-\gamma}=\frac{4\rho(1-\varepsilon)}{\sqrt{n}}\sum_{i=1}^{n/2}i^{1-\gamma}.$$

The last expression is $O(n^{3/2-\gamma})$. Thus we proved (28), which implies (26).

Let us now prove (27). As long as

$$U_n(t):=-\frac{1}{\sqrt{n}}\left\{\sum_{i=1}^{n}\mathbb{1}_{\{t<T_i\}}-na(t)\right\}=\sqrt{n}\left\{\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{T_i\le t\}}-\big(1-a(t)\big)\right\},$$

$U_n(\cdot)$ is the empirical process of stationary r.v.'s T_i with the continuous common distribution function $1-a(t)$. Thus (27) is equivalent to the existence of a centered Gaussian process $K(\cdot)$ on $[0,1)$ such that

$$U_n(\cdot)\xrightarrow{\mathcal{D}}K(\cdot)\quad\text{in } D[0,1-\varepsilon] \text{ for all } \varepsilon\in(0,1) \tag{29}$$

(because $-K(\cdot)\stackrel{\mathcal{D}}{=}K(\cdot)$).

We will use the following result on the convergence of empirical processes from Lin and Lu [15, Sec. 12]. They attribute this statement to Q.-M. Shao, who published it in 1986, in Chinese.

Fact 4. *Let ξ_i be a sequence of stationary strongly mixing r.v.'s distributed on $[0,1]$, and let F be the common distribution function of ξ_i. Suppose $F(x)=x$ on $[0,1]$ (i.e., ξ_i are uniformly distributed) and the coefficients of strong mixing of the sequence $F(\xi_i)$ decay as $O(k^{-(2+\delta)})$ as $k\to\infty$ for some $\delta>0$. Then the empirical processes of ξ_i weakly converge in $D[0,1]$ to a centered Gaussian process with the covariance function $\sum_{i\in\mathbb{Z}} cov\big(\mathbb{1}_{\{\xi_0\le s\}},\mathbb{1}_{\{\xi_i\le t\}}\big)$.*

Remark. *The limit Gaussian process is a.s. continuous on $[0,1]$. Fact 4 also holds true if F is an arbitrary continuous distribution function.*

The a.s. continuity of the limit could be seen from the proof (compare the arguments of Lin and Lu [15] with the proof of Theorem 22.1 from Billingsley [3]). Then, since $F(\xi_i)$ is uniformly distributed on $[0,1]$ if F is continuous, Fact 4 holds true for any continuous F; see the proof of Theorem 22.1 by Billingsley [3] for explanations.

We need to prove the convergence of the empirical process of T_i. It seems that the r.v.'s T_i are not strongly mixing, but by Property 8 from Subsection 3.4, we know that $\min\{T_i,1-\varepsilon\}$ are strongly mixing. The variables $\min\{T_i,1-\varepsilon\}$ are not continuous, and we have to "touch them up". Let us fix an $\varepsilon\in(0,1)$, and let α_i be i.i.d. r.v.'s, say, uniformly distributed on $[0,\varepsilon]$ and independent of all T_i; we define $\tilde T_i := \min\{T_i,1-\varepsilon\}+\mathbb{1}_{\{T_i\ge 1-\varepsilon\}}\,\alpha_i$.

The stationary variables $\tilde T_i$ are distributed on $[0,1]$, their common distribution function G is continuous, and the coefficients of strong mixing of $G(\tilde T_i)$ decrease as $o(k^{2-\gamma})$. The proof of the last assertion is the same as the proof of Property 8 from Subsection 3.4: We approximate the variables $G(\tilde T_0), G(\tilde T_{-1}),\dots$ from the "past" by $G(\tilde T_0^{(k/2)}), G(\tilde T_{-1}^{(k/2+1)}),\dots$; here $\tilde T_i^{(m)}$ are defined as $\tilde T_i^{(m)} := \min\{T_i^{(m)},1-\varepsilon\}+\mathbb{1}_{\{T_i^{(m)}\ge 1-\varepsilon\}}\,\alpha_i$. We use the analogous approximation for the variables from the "future", and then repeat word-by-word the arguments of the previous proof. Now, recalling that $\gamma>4$, we see that $\tilde T_i$ satisfy the assumptions of Fact 4, with the only difference that their distribution is not uniform.

By $\tilde U_n(\cdot)$ denote the empirical process of $\tilde T_i$; clearly, $\tilde U_n(\cdot)$ coincide with the empirical process $U_n(\cdot)$ of T_i on $[0,1-\varepsilon]$. By the Remark to Fact 4, we conclude that

$$\tilde U_n(\cdot)\xrightarrow{\mathcal{D}}\tilde K(\cdot)\ \text{in}\ D[0,1], \tag{30}$$

where $\tilde K(\cdot)$ is a centered Gaussian process with the covariance function

$$\tilde R(s,t) := \sum_{i\in\mathbb{Z}} cov\big(\mathbb{1}_{\{\tilde T_0\le s\}},\mathbb{1}_{\{\tilde T_i\le t\}}\big),$$

and that trajectories of $\tilde K(\cdot)$ are a.s. continuous on $[0,1]$.[2]

Define

$$R(s,t) := \sum_{i\in\mathbb{Z}} cov\big(\mathbb{1}_{\{T_0\le s\}},\mathbb{1}_{\{T_i\le t\}}\big), \tag{31}$$

[2]There exists a simpler and more elegant proof of (30). Note that $\{\tilde T_i\}_{i\in\mathbb{Z}}$ are associated as coordinate-wise nondecreasing functions of associated r.v.'s $\{T_i,\alpha_i\}_{i\in\mathbb{Z}}$, see (a), (b), and (d) from Property 6, Subsection 3.4. Then we can obtain (30) by the result of Louhichi [16] on the convergence of empirical processes of stationary associated r.v.'s ξ_i instead of using Fact 4. Her theorem requires only $cov\big(F(\xi_0),F(\xi_k)\big)=O(k^{-(4+\delta)})$, which could be proved analogously to Property 5, Subsection 3.4. Thus we avoid the complicated estimations of strong mixing coefficients, and the proof of (30) is much simpler. The only problem is that this proof requires $\gamma>5$.

We also note that the a.s. continuity of $\tilde K(\cdot)$ could be proved directly, without refering to the proof of Fact 4. The arguments are the same as in the proof of continuity of $K^{Unif}(\cdot)$ in Section 5.

which is evidently equal to $\tilde{R}(s,t)$ on $[0,1-\varepsilon]^2$. As far as $\tilde{R}(s,t)$ is positive definite and $\varepsilon > 0$ was arbitrary, the function $R(s,t)$ is positive definite on $[0,1)^2$. Hence (see Lifshits [13, Sec. 4]) there exists a centered Gaussian process $K(\cdot)$ on $[0,1)$ with the covariance function $R(s,t)$. The trajectories of $K(\cdot)$ are a.s. continuous on $[0,1)$ because of $K(\cdot) \stackrel{\mathcal{D}}{=} \tilde{K}(\cdot)$ on $[0,1-\varepsilon]$, arbitrariness of $\varepsilon > 0$, and a.s. continuity of $\tilde{K}(\cdot)$ on $[0,1]$.

Finally, by (30), $\tilde{U}_n(\cdot) = U_n(\cdot)$ on $[0,1-\varepsilon]$, $\tilde{K}(\cdot) \stackrel{\mathcal{D}}{=} K(\cdot)$ on $[0,1-\varepsilon]$, and a.s. continuity of $\tilde{K}(\cdot)$, we obtain (29). But (29) implies (27), thus the proof of (2) is finished.

It remains to study the properties of $R(s,t)$. We first check that $R(s,t)$ is continuous on $[0,1)^2$. The proof is analogous to the proof of continuity of $\mathbb{P}\{T_0 \ge t\}$ from Property 4, Subsection 3.4. The functions $cov\big(\mathbb{1}_{\{T_0 \le s\}}, \mathbb{1}_{\{T_i \le t\}}\big)$ are continuous on $[0,1)^2$ for every $i \ge 0$. Indeed, first, it is evident from the continuity of the joint distribution function of continuous variables $T_0^{(k)}$ and $T_i^{(k)}$ that $cov\big(\mathbb{1}_{\{T_0^{(k)} \le s\}}, \mathbb{1}_{\{T_i^{(k)} \le t\}}\big)$ is continuous for every k. Second, we have

$$\Big|cov\big(\mathbb{1}_{\{T_0 \le s\}}, \mathbb{1}_{\{T_i \le t\}}\big) - cov\big(\mathbb{1}_{\{T_0^{(k)} \le s\}}, \mathbb{1}_{\{T_i^{(k)} \le t\}}\big)\Big| \le 2\big(\rho(s) + \rho(t)\big)k^{1-\gamma}$$

for $s,t \in [0,1)$; the proof of this inequality is a verbatim copy of the proof of (19). Then the limit function $cov\big(\mathbb{1}_{\{T_0 \le s\}}, \mathbb{1}_{\{T_i \le t\}}\big)$ is continuous on $[0,1)^2$ as a uniform limit of continuous functions. In view of (19), $R(s,t)$ is continuous as a sum of uniformly converging series of continuous functions.

The strict positivity of $R(s,t)$ on $(\sqrt{\mu},1)^2$ trivially follows from (31), (21), and $cov\big(\mathbb{1}_{\{T_0 \le s\}}, \mathbb{1}_{\{T_0 \le t\}}\big) = a(s \vee t)\big(1 - a(s \wedge t)\big) > 0$; the last inequality holds by Property 3, Subsection 3.4. The $R(s,t) = 0$ on $[0,1)^2 \setminus (\sqrt{\mu},1)^2$ follows from $\mathbb{P}\{T_i \le \sqrt{\mu}\} = 0$, see Properties 3 and 4 from Subsection 3.4. □

We note that (3) holds for $t \ne 1$ under the less restrictive condition $\mathbb{E}X_i^2 < \infty$. The proof for $t < 1$ is almost the same: By (26), which is true for $\gamma > 3/2$, we conclude that (3) holds if the stationary associated sequence $\mathbb{1}_{\{t < T_i\}}$ satisfies the central limit theorem. Then we refer to the central limit theorem for stationary associated sequences by Newman [18]; his theorem requires only $R(t,t) < \infty$, i.e., convergence of the right-hand side of (31). This condition holds by (12) and Fact 2. For $t > 1$, assertion (3) with $\sigma^2(t) = 0$ immediately follows from Proposition 3.

Finally, note that the process $K(\cdot)$ is associated, i.e., the r.v.'s $\{K(t)\}_{t \in [0,1)}$ are associated. This holds by (2) and (c) from Property 6, Subsection 3.4 because the processes $\frac{K_n(\cdot) - na(\cdot)}{\sqrt{n}}$ are associated in view of (5), Property 6 from Subsection 3.4, and (b) from Property 6, Subsection 3.4.

5. The proof of Theorem 1 for the uniform model

There exists a simple method which allows to extend results obtained for the Poisson model to the uniform model, and vise versa. The method is based on the next statement (see Karlin [11, Sec. 9.1]).

Fact 5. *Let S_i be an exponential random walk. Then for any $k \ge 1$, we have*

$$\Big(\frac{S_1}{S_{k+1}}, \frac{S_2}{S_{k+1}}, \dots, \frac{S_k}{S_{k+1}}\Big) \stackrel{\mathcal{D}}{=} \big(U_{1,k}, U_{2,k}, \dots, U_{k,k}\big), \tag{32}$$

where $U_{i,k}$ are the order statistics of k i.i.d. random variables uniformly distributed on $[0,1]$. Moreover, the random vector in the left-hand side of (32) is independent of S_{k+1}.

Therefore if $x_{j,n}^{Poiss}(0) = \frac{1}{n}S_j$ are initial positions of particles in the Poisson model, then we have $x_{j,n}^{Unif}(0) = \frac{n}{S_{n+1}} \cdot x_{j,n}^{Poiss}(0)$ for initial positions of particles in the uniform model. By

Proposition 2 and (4), we conclude that

$$T_{j,n}^{Unif} = \beta_n^{-1} T_{j,n}^{Poiss}, \quad \beta_n := \sqrt{\frac{S_{n+1}}{n}}, \tag{33}$$

and hence, using (5), we get

$$K_n^{Unif}(t) = K_n^{Poiss}(\beta_n t). \tag{34}$$

Note that the process $K_n^{Unif}(\cdot)$ and the r.v. β_n are independent since values of the process are defined by $x_{1,n}^{Unif}(0), \dots, x_{n,n}^{Unif}(0)$, which are mutually independent of β_n by Fact 5.

Now we prove Theorem 1 for the uniform model.

Proof of Theorem 1. Denote

$$Y_n(t) := \frac{K_n^{Unif}(t) - na(t)}{\sqrt{n}}, \quad Z_n(t) := \sqrt{n}\big(a(t) - a(\beta_n t)\big);$$

it is very important that $Y_n(\cdot)$ and $Z_n(\cdot)$ are independent.

Fix an $\varepsilon \in (0,1)$. First, it follows from (2) for the Poisson model and (34) that in $D[0, 1-\varepsilon]$, we have

$$Y_n(\cdot) + Z_n(\cdot) \xrightarrow{\mathcal{D}} K^{Poiss}(\cdot). \tag{35}$$

Indeed, the process $Y_n(\cdot) + Z_n(\cdot)$ is obtained from $\frac{1}{\sqrt{n}}\big(K_n^{Poiss}(\cdot) - na(\cdot)\big)$ by the random time change $t \mapsto \beta_n t$; as far as $\|\beta_n t - t\|_{C[0,1-\varepsilon]} \xrightarrow{\mathbb{P}} 0$,

$$d\Big(Y_n(\cdot) + Z_n(\cdot), \frac{K_n^{Poiss}(\cdot) - na(\cdot)}{\sqrt{n}}\Big) \xrightarrow{\mathbb{P}} 0$$

by definition of the Skorohod metric d.

Second, using Fact 1, Lemma 2, and Property 4 from Subsection 3.4, we have $a^{Unif}(t) = a^{Poiss}(t) = \mathbb{P}\big\{T_0 \ge t\big\} = 1 - t^2$ for $0 \le t \le 1$, and by the central limit theorem,

$$Z_n(t) \xrightarrow{\mathcal{D}} t^2 \eta \tag{36}$$

in $D[0, 1-\varepsilon]$, where η is a standard Gaussian r.v.

We claim that (35), the independence of $Y_n(\cdot)$ and $Z_n(\cdot)$, and (36) yield the weak convergence of $Y_n(\cdot)$ in $D[0, 1-\varepsilon]$. Let us check the tightness of $Y_n(\cdot)$ and the convergence of their finite-dimensional distributions.

The tightness of $Y_n(\cdot)$ in $D[0, 1-\varepsilon]$ follows from $Y_n(\cdot) = \big(Y_n(\cdot) + Z_n(\cdot)\big) - Z_n(\cdot)$, (35), and (36). Indeed, by the Prokhorov theorem, (35) and (36) yield that both sequences $Y_n(\cdot) + Z_n(\cdot)$ and $-Z_n(\cdot)$ are tight. Recalling that by continuity of $a(\cdot)$, trajectories of $-Z_n(\cdot)$ are a.s. continuous, it remains to use the continuity of $+ : D \times C \to D$ and the fact that the image of a compact set under a continuous mapping is also a compact set.

Let us now study the convergence of finite dimensional distributions of $Y_n(\cdot)$. Recall that the characteristic function of a centered Gaussian vector in $\mathbb{R}^m$ is $e^{-\frac{1}{2}(R\mathbf{u},\mathbf{u})}$, where $\mathbf{u} \in \mathbb{R}^m$ and R is the covariance matrix of the vector. Then (35), the independence of $Y_n(\cdot)$ and $Z_n(\cdot)$, and (36) yield that for the characteristic function of a finite-dimensional distribution of $Y_n(\cdot)$, we have

$$\mathbb{E}e^{i(Y_n(\mathbf{t}),\mathbf{u})} \longrightarrow e^{-\frac{1}{2}\left(\left\{R^{Poiss}(t_j,t_k) - t_j^2 t_k^2\right\}_{j,k=1}^{m}\mathbf{u},\mathbf{u}\right)}, \tag{37}$$

where $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{t} = (t_1, \dots, t_m) \in [0, 1-\varepsilon]^m$, and $Y_n(\mathbf{t}) := \big(Y_n(t_1), \dots, Y_n(t_m)\big)$.

We see that the matrix $\big\{R^{Poiss}(t_j,t_k)-t_j^2t_k^2\big\}_{j,k=1}^m$ is positive definite for any $\mathbf{t}=(t_1,\dots,t_m)\in[0,1-\varepsilon]^m$ and $m\geq 1$ because the absolute value of the left-hand side of (37) does not exceed one. Putting

$$R^{Unif}(s,t):=R^{Poiss}(s,t)-s^2t^2,$$

we have $\big\{R^{Poiss}(t_j,t_k)-t_j^2t_k^2\big\}_{j,k=1}^m=\big\{R^{Unif}(t_j,t_k)\big\}_{j,k=1}^m$, hence the function $R^{Unif}(s,t)$ is positive definite on $[0,1)^2$ since $\varepsilon>0$ was arbitrary. Then (see Lifshits [13, Sec. 4]) $R^{Unif}(s,t)$ is the covariance function of some centered Gaussian process $K^{Unif}(\cdot)$ on $[0,1)$.

Thus (37) yields the convergence of finite-dimensional distributions of [tight] $Y_n(\cdot)$ to finite-dimensional distributions of $K^{Unif}(\cdot)$. We check that $K^{Unif}(\cdot)\in C[0,1-\varepsilon]$ a.s. to conclude the proof of Theorem 1 for the uniform model.

For this purpose, let us prove that a.s., trajectories of $Y_n(\cdot)$ have jumps of size $\frac{1}{\sqrt{n}}$ only. In fact, the jumps of $Y_n(\cdot)$ coincide with the jumps of $\frac{1}{\sqrt{n}}K_n^{Unif}(\cdot)$, whose jumps are of size $\frac{1}{\sqrt{n}}$ if and only if $T_{j_1,n}^{Unif}\neq T_{j_2,n}^{Unif}$ for $1\leq j_1\neq j_2\leq n-1$. By (33), we need to verify that $T_{j_1,n}^{Poiss}\neq T_{j_2,n}^{Poiss}$ a.s. for $1\leq j_1\neq j_2\leq n-1$. This assertion follows from (18) if $H(k_1,j_1,l_1)\neq H(k_2,j_2,l_2)$ a.s. for $j_1\neq j_2$ and $k_1,k_2,l_1,l_2\geq 1$, where $H(k,j,l)$ denotes the expression under the square root in (18). But the $H(k_1,j_1,l_1)\neq H(k_2,j_2,l_2)$ a.s. is obvious: if the equality holds true, then a certain nontrivial linear combination of i.i.d. exponential X_i equals zero.

Then there exist a.s. continuous $\tilde{Y}_n(\cdot)$ such that $\sup_{t\in[0,1-\varepsilon]}\big|\tilde{Y}_n(t)-Y_n(t)\big|\leq\frac{1}{\sqrt{n}}$ a.s.; and $d\big(\tilde{Y}_n,Y_n\big)\leq\frac{1}{\sqrt{n}}$ a.s. By tightness, some subsequence $Y_{n_i}(\cdot)$ weakly converges to a random element $K'(\cdot)\in D[0,1-\varepsilon]$, and consequently, $\tilde{Y}_{n_i}(\cdot)\xrightarrow{\mathcal{D}}K'(\cdot)$. Then $K'(\cdot)$ is a.s. continuous because $1=\liminf\mathbb{P}\big\{\tilde{Y}_{n_i}(\cdot)\in C\big\}\leq\mathbb{P}\big\{K'(\cdot)\in C\big\}$ since $C\subset D$ is closed in the Skorohod topology. It remains to note that $K'(\cdot)\overset{\mathcal{D}}{=}K^{Unif}(\cdot)$ on $[0,1-\varepsilon]$, thus $K^{Unif}(\cdot)$ is a.s. continuous on $[0,1-\varepsilon]$. The proof of (2) is finished (recall that $\varepsilon\in(0,1)$ was taken arbitrary).

Clearly, $K^{Unif}(\cdot)$ is a.s. continuous on the whole interval $[0,1)$. The $R^{Unif}(s,t)=R^{Poiss}(s,t)-s^2t^2$ is continuous on $[0,1)^2$ because $R^{Poiss}(s,t)$ is. □

6. The number of clusters at the critical moment

Now we turn our attention to the number of clusters at the critical moment $t=1$. We are interested in the behavior of

$$\frac{K_n(1)-na(1)}{\sqrt{n}}=\frac{K_n(1)}{\sqrt{n}},$$

which is the left-hand side of (3) at $t=1$; here $a(1)=0$ under the assumption $EX_i^2<\infty$, see Property 3, Subsection 3.4.

We do not know if this sequence is weakly convergent. But we hope that it is, and have a naive guess that its limit is Gaussian because the limit in Theorem 1 was Gaussian. In view of $K_n(1)\geq 1$, this conjectural weak limit is nonnegative, and the only opportunity for it to be Gaussian is to be identically equal to zero. But the results of this section show that the limit is not zero, thus our guess on Gaussianity fails.

The study of convergence of $\frac{K_n(1)}{\sqrt{n}}$ is very complicated, that is why in this section, we consider the Poisson model only. First, let us prove the following statement.

Proposition 4. *In the Poisson model, we have* $\lim\limits_{n\to\infty}\mathbb{P}\big\{K_n(1)=1\big\}>0$.

Proof. On the one hand, $K_n(1) = 1$ is equivalent to $T^{last}_{n;Poiss} \le 1$, where $T^{last}_{n;Poiss}$ denotes the moment of the last collision in the Poisson model. On the other hand, a result by Giraud [8] states that in the uniform model,

$$\sqrt{n}\big(T^{last}_{n;Unif} - 1\big) \xrightarrow{\mathcal{D}} \sup_{0\le x\le 1}\left(\frac{1}{1-x}\int_x^1 \overset{\circ}{W}(y)dy - \frac{1}{x}\int_0^x \overset{\circ}{W}(y)dy\right) =: \tau,$$

where $\overset{\circ}{W}(\cdot)$ is a Brownian bridge. Now by (33) we have $T^{last}_{n;Unif} = \beta_n^{-1} T^{last}_{n;Poiss}$, hence

$$\sqrt{n}\big(\beta_n^{-1} T^{last}_{n;Poiss} - 1\big) \xrightarrow{\mathcal{D}} \tau. \tag{38}$$

But from the central limit theorem and the law of large numbers,

$$\sqrt{n}\big(\beta_n^{-1} - 1\big) = -\frac{S_{n+1} - n}{\sqrt{n}} \cdot \frac{n}{\sqrt{S_{n+1}}\big(\sqrt{S_{n+1}} + \sqrt{n}\big)} \xrightarrow{\mathcal{D}} \frac{\eta}{2}, \tag{39}$$

where η is a standard Gaussian r.v. and S_i is a standard exponential random walk that defines initial positions of particles. Since in view of Fact 5, $T^{last}_{n;Unif} = \beta_n^{-1} T^{last}_{n;Poiss}$ and β_n are independent, from (38), (39), and the law of large numbers it follows that

$$\sqrt{n}\big(T^{last}_{n;Poiss} - 1\big) \xrightarrow{\mathcal{D}} \tau - \frac{\eta}{2} \overset{\mathcal{D}}{=} \tau + \frac{\eta}{2};$$

here τ and η are independent. Thus

$$\lim_{n\to\infty} \mathbb{P}\Big\{K_n(1) = 1\Big\} = \lim_{n\to\infty} \mathbb{P}\Big\{T^{last}_{n;Poiss} \le 1\Big\} = \mathbb{P}\Big\{\tau + \frac{\eta}{2} \le 0\Big\} > 0.$$

□

The main advantage of the Poisson model is that by Lemma 2, we have $\mathbb{P}\{T_{j,n} \ge t\} = ep_j p_{n-j}$, where

$$p_k := \mathbb{P}\Big\{\min_{1\le m\le k} \sum_{i=1}^m (S_i - \mathbb{E}S_i) \ge 0\Big\}$$

and S_i is a standard exponential random walk. We say that the sequence of r.v.'s $\sum_{i=1}^m (S_i - \mathbb{E}S_i)$ is an *integrated random walk*; in fact, $\sum_{i=1}^m (S_i - \mathbb{E}S_i)$ signifies the area of first m steps of $S_i - \mathbb{E}S_i$. In the proof of Property 3, Subsection 3.4, we showed that $p_k \to 0$ as $k \to \infty$. Hence it is reasonable to call p_k the *unilateral small deviations probabilities of an integrated centered random walk.*

We need to obtain the asymptotics of $p_k \to 0$ to continue studying convergence of $\frac{K_n(1)}{\sqrt{n}}$. Unfortunately, the author can only make a conjecture, and the results of the rest of this section completely depend on the correctness of this conjecture.

Conjecture 1. *We have $p_k \sim c_1 k^{-1/4}$ as $k \to \infty$ for some $c_1 \in (0, \infty)$.*

Simulations shows that the conjecture is true, and $c_1 \approx 0.36$. The weaker form $p_k \asymp k^{-1/4}$ of Conjecture 1 was proved by Sinai [19], but only for an integrated symmetric Bernoulli random walk. It also interesting to note that by Isozaki and Watanabe [10], the unilateral small deviations probabilities of an integrated Wiener process have the same order as $T \to \infty$:

$$\mathbb{P}\Big\{\min_{0\le s\le T} \int_0^s W(u)du \ge -1\Big\} \sim c_2 T^{-1/4}, \tag{40}$$

for some $c_2 \in (0,\infty)$. We call the left-hand side of (40) the probability of a unilateral small deviation since

$$\mathbb{P}\Big\{\min_{0\le s\le T}\int_0^s W(u)du \ge -1\Big\} = \mathbb{P}\Big\{\min_{0\le s\le 1}\int_0^s W(u)du \ge -T^{-3/2}\Big\}. \tag{41}$$

By these results, we can also suppose that Conjecture 1 is true for another integrated centered random walks that satisfy some moment conditions.

We are now ready to prove a result on convergence of $\frac{K_n(1)}{\sqrt{n}}$.

Proposition 5. *Suppose Conjecture 1 holds true. Then in the Poisson model, we have*

$$\lim_{n\to\infty}\mathbb{E}\Big(\frac{K_n(1)}{\sqrt{n}}\Big) = c_3, \quad \sup_{n\ge 1}\mathbb{E}\Big(\frac{K_n(1)}{\sqrt{n}}\Big)^2 < \infty \tag{42}$$

for some $c_3 \in (0,\infty)$; the sequence $\frac{K_n(1)}{\sqrt{n}}$ is tight and uniformly integrable; and for any weakly converging subsequence of $\frac{K_n(1)}{\sqrt{n}}$, its limit takes value zero with positive probability, but is not identically equal to zero.

Numerical simulations allow to conjecture that $\frac{K_n(1)}{\sqrt{n}}$ is weakly convergent, and this convergence is quite fast. In Fig. 1 we present the (empirical) distribution function of $\frac{K_n(1)}{\sqrt{n}}$ for $n = 10\,000$. This function seems to be a good candidate for the distribution function of the supposed weak limit. The simulations performed for $n = 40\,000$ show a very hardly perceptible difference.

Note that if we weaken Conjecture 1 to $p_k \asymp k^{-1/4}$, then Proposition 5 also holds true, with the only difference that $\mathbb{E}\frac{K_n(1)}{\sqrt{n}} \asymp 1$.

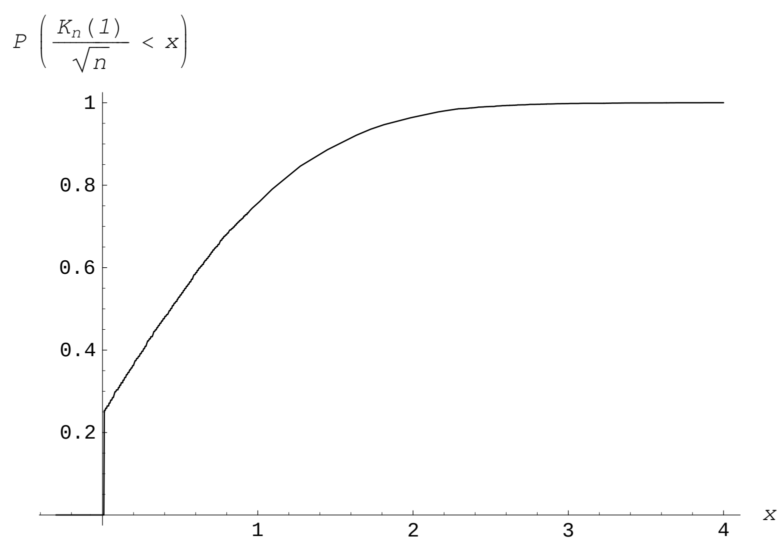

FIGURE 1. The distribution function of $\frac{K_n(1)}{\sqrt{n}}$ for $n = 10\,000$.

Proof. We start with convergence of expectation. On the one hand, by (5) and Lemma 2,

$$\mathbb{E}\Big(\frac{K_n(1)}{\sqrt{n}}\Big)=\frac{1}{\sqrt{n}}+\frac{e}{\sqrt{n}}\sum_{i=1}^{n-1}p_ip_{n-i}.$$

On the other hand,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n-1}i^{-1/4}(n-i)^{-1/4}=\frac{1}{n}\sum_{i=1}^{n-1}\Big(\frac{i}{n}\Big)^{-1/4}\Big(1-\frac{i}{n}\Big)^{-1/4}\longrightarrow \mathrm{B}(3/4,3/4),\qquad n\to\infty$$

as the integral sum of Beta function. Then it follows from Conjecture 1 and standard arguments that $\mathbb{E}\frac{K_n(1)}{\sqrt{n}}$ converges to $ec_1^2\mathrm{B}(3/4,3/4)>0$.

Now we check the uniform boundedness of $\mathbb{E}\big(\frac{K_n(1)}{\sqrt{n}}\big)^2$. By (5) and the convergence of $\mathbb{E}\frac{K_n(1)}{\sqrt{n}}$, it is sufficient to prove that

$$\sup_{n\ge 1}\frac{1}{n}\sum_{i,j=1,i\ne j}^{n-1}\mathbb{P}\Big\{T_{i,n}>1,T_{j,n}>1\Big\}<\infty. \tag{43}$$

Suppose $i<j$; then using (7) and properties of $F_{k,j,l}(\cdot)$, we get

$$\begin{aligned}\mathbb{P}\Big\{T_{i,n}>1,T_{j,n}>1\Big\} &= \mathbb{P}\Big\{\min_{\substack{1\le k\le n-i\\1\le l\le i}}F_{k,i,l}(1)>0,\ \min_{\substack{1\le k\le n-j\\1\le l\le j}}F_{k,j,l}(1)>0\Big\}\\ &\le \mathbb{P}\Big\{\min_{\substack{1\le k\le (j-i)/2\\1\le l\le i}}F_{k,i,l}(1)>0,\ \min_{\substack{1\le k\le n-j\\1\le l\le (j-i)/2}}F_{k,j,l}(1)>0\Big\};\end{aligned}$$

where by $(j-i)/2$ we mean $\lceil (j-i)/2\rceil$. The minima in the last expression are independent as functions of $\{X_m\}_{m\le (i+j)/2}$ and $\{X_m\}_{m\ge (i+j)/2+1}$, respectively; hence

$$\begin{aligned}\mathbb{P}\Big\{T_{i,n}>1,T_{j,n}>1\Big\} &\le \mathbb{P}\Big\{\min_{\substack{1\le k\le (j-i)/2\\1\le l\le i}}F_{k,i,l}(1)>0\Big\}\cdot\mathbb{P}\Big\{\min_{\substack{1\le k\le n-j\\1\le l\le (j-i)/2}}F_{k,j,l}(1)>0\Big\}\\ &= \mathbb{P}\Big\{T_{i,i+(j-i)/2}>1\Big\}\cdot\mathbb{P}\Big\{T_{(j-i)/2,n-j+(j-i)/2}>1\Big\}=e^2p_ip^2_{\lceil (j-i)/2\rceil}p_{n-j},\end{aligned}$$

where the first equality follows from (7) and the second follows from Lemma 2.

Recalling Conjecture 1, we get

$$\begin{aligned}\frac{1}{n}\sum_{i,j=1,i\ne j}^{n-1}\mathbb{P}\Big\{T_{i,n}>1,T_{j,n}>1\Big\} &\le \frac{1}{n}\sum_{i,j=1,i\ne j}^{n-1}e^2p_ip^2_{\lceil (j-i)/2\rceil}p_{n-j}\\ &\le \frac{c}{n}\sum_{i,j=1,i\ne j}^{n-1}i^{-1/4}\lceil (j-i)/2\rceil^{-1/2}(n-j)^{-1/4}\\ &\le \frac{c}{n^2}\sum_{i,j=1,i\ne j}^{n-1}\Big(\frac{i}{n}\Big)^{-1/4}\Big|\frac{j}{n}-\frac{i}{n}\Big|^{-1/2}\Big(1-\frac{j}{n}\Big)^{-1/4}\end{aligned}$$

for some $c>0$. The last expression is the integral sum converging to

$$c\int_0^1\int_0^1 x^{-1/4}|x-y|^{-1/2}(1-y)^{-1/4}dxdy,$$

and it is a simple exercise to check that the integral is finite. This concludes (43).

The tightness of $\frac{K_n(1)}{\sqrt{n}}$ follows from its positivity, (42), and the Chebyshev inequality. The uniform integrability follows from the second assertion of (42), see Billingsley [3, Sec. 5].

Finally, suppose $\frac{K_{n_i}(1)}{\sqrt{n_i}} \xrightarrow{\mathcal{D}} \xi$ for some sequence $n_i \to \infty$ and some r.v. ξ. Then by uniform integrability and (42), $\mathbb{E}\xi = c_3 > 0$, hence ξ is not identically equal to zero. But the distribution of ξ has an atom at zero since by Proposition 4 and properties of weak convergence,

$$\mathbb{P}\Big\{\xi = 0\Big\} = \lim_{\varepsilon \searrow 0} \mathbb{P}\Big\{\xi \le \varepsilon\Big\} \ge \lim_{\varepsilon \searrow 0} \limsup_{i \to \infty} \mathbb{P}\bigg\{\frac{K_{n_i}(1)}{\sqrt{n_i}} \le \varepsilon\bigg\} \ge \lim_{\varepsilon \searrow 0} \lim_{i \to \infty} \mathbb{P}\Big\{K_{n_i}(1) = 1\Big\} > 0.$$

□

7. Open questions

1. The number of clusters at the critical moment $t = 1$.

Here the main question is if Conjecture 1 holds true. This interesting problem is worth studying by itself.

But even if Conjecture 1 is true, we still do not have a proof of weak convergence of $\frac{K_n(1)}{\sqrt{n}}$, it is only known that this sequence is tight. The author strongly believes, relying on numerical simulations, that the limit exists, and it would be interesting to find it in an explicit form. Recall that the limit should be nontrivial by Proposition 5.

2. The weak convergence of trajectories of $\frac{K_n(\cdot)-na(\cdot)}{\sqrt{n}}$ on the whole interval $[0, 1]$.

The question of possibility to strengthen Theorem 1 by proving convergence of $\frac{K_n(\cdot)-na(\cdot)}{\sqrt{n}}$ in $D[0, 1]$ is very natural. Clearly, the answer is positive only if $\frac{K_n(1)}{\sqrt{n}}$ is weakly convergent (see Billingsley [3, Sec. 15]), thus we return to Question 1. But even if $\frac{K_n(1)}{\sqrt{n}}$ converges, its weak limit $K(1)$ is not Gaussian, hence the limit process $K(\cdot)$, which was Gaussian on $[0, 1)$, is not Gaussian on $[0, 1]$. Therefore it is doubtful that Theorem 1 is true in $D[0, 1]$. At least, one should provide a proof completely different from the presented one. It is also unclear what are the finite-dimensional distributions of the non-Gaussian $K(\cdot)$ on $[0, 1]$ since simulations show that $K(1)$ would not be independent with $K(t)$ for $t < 1$.

3. The number of clusters in the warm gas.

In the presented case, initial speeds of particles are zero. This model is often called the *cold* gas according to its zero initial temperature. We introduce a new model stating that initial speeds of particles are $a_n v_1, a_n v_2, \dots, a_n v_n$, where v_i are some i.i.d. r.v.'s and a_n is a sequence of normalization constants. This model, called the *warm* gas, was considered in many papers, e.g., [12, 14, 17, 23].

It is of great interest to study the $K_n(t)$ in the warm gas. In [23] the author proved that in the basic case, when $a_n = 1$ for all n and $\mathbb{E}v_i^2 < \infty$, we have $\frac{K_n(t)}{n} \xrightarrow{\mathbb{P}} 0$ for all $t > 0$. The question is to find a normalization of $K_n(t)$ leading to some nontrivial limit. Surely, this normalization depends on a_n, but it is very possible that there is an effect of phase transition similar to the one discovered by Lifshits and Shi [14]: If a_n are small enough, then the gas has a low temperature, and the normalization is the same as in the cold gas. If a_n are big enough, like in the basic case $a_n \equiv 1$, then the normalization and the behaviour of the gas differ entirely from the cold case.

The author believes that the localization property described in Section 3 would be helpful in study of these questions.

It is also interesting to compare the behaviour of $K_n(1)$ in the warm and in the cold gases; in the warm gas, the moment $t = 1$ plays the same “critical” role as in the cold gas, see Lifshits and Shi [14]. The variable $K_n(1)$ was studied by Suidan [21], who considered the warm gas with deterministic initial positions of particles (initial positions are $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$) and $a_n \equiv 1$. For this case, Suidan found the distribution of $K_n(1)$ and showed that $\mathbb{E}K_n(1) \sim \log n$. Recall that in the presented case, $\mathbb{E}K_n(1) \sim c_3\sqrt{n}$.

4. The number of clusters in ballistic systems of sticky particles.

A sticky particles model is called *ballistic* if it evolves according to the rules introduced in Section 1 but with no gravitation. Such models are in some sense more natural than gravitational ones because the gravitation's independence of distance is sometimes confusing. However, an unpublished paper of M. Lifshits and L. Kuoza shows that certain gravitational and ballistic models are tightly connected.

It seems interesting to study the number of clusters in the ballistic model. The author does not know any results in this field.

Acknowledgements

I am grateful to my adviser Mikhail A. Lifshits for drawing my attention to the subject and especially for his guidance. I also thank A.N. Tikhomirov, Ya.Yu. Nikitin, and Z. Kabluchko for discussions.

References

[1] L.E. Baum and M. Katz (1965) Convergence rates in the law of large numbers. Trans. Amer. Math. Soc., v. 120, pp. 108–123.
[2] J. Bertoin (2002) Self-attracting Poisson clouds in an expanding universe. Comm. Math. Phys., v. 232, pp. 59–81.
[3] P. Billingsley (1968) Convergence of Probability Measures. Wiley, New York.
[4] Y. Brenier and E. Grenier (1998) Sticky particles and scalar conservation laws, SIAM J. Numer. Anal., v. 35, pp. 2317–2328.
[5] A. Chertock, A. Kurganov and Yu. Rykov (2005) A new sticky particle method for pressureless gas dynamics. Preprint, available at www.math.tulane.edu/∼kurganov.
[6] W. E, Yu.G. Rykov and Ya.G. Sinai (1996) Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics. Comm. Math. Phys., v. 177, pp. 349–380.
[7] J.D. Esary, F. Proschan and D.W. Walkup (1967) Association of random variables, with applications. Ann. Math. Stat., v. 38, pp. 1466–1474.
[8] C. Giraud (2001) Clustering in a self-gravitating one-dimensional gas at zero temperature. J. Stat. Phys., v. 105, pp. 585–604.
[9] C. Giraud (2005) Gravitational clustering and additive coalescence. Stoch. Proc. Appl., v. 115, pp. 1302–1322.
[10] Y. Isozaki and S. Watanabe (1994) An asymptotic formula for the Kolmogorov Diffusion and a refinement of Sinai's estimates for the integral of Brownian motion. Proc. Japan Acad., v. 70, Ser. A, pp. 271–276.
[11] S. Karlin (1968) A First Course in Stochastic Processes. Academic Press, New York.
[12] L.V. Kuoza and M.A. Lifshits (2006) Aggregation in one-dimensional gas model with stable initial data. J. Math. Sci., v. 133, pp. 1298–1307.
[13] M.A. Lifshits (1995) Gaussian Random Functions. Kluwer, Dordrecht.
[14] M. Lifshits and Z. Shi (2005) Aggregation rates in one-dimensional stochastic systems with adhesion and gravitation. Ann. Probab., v. 33, pp. 53–81.
[15] Z. Lin and C. Lu (1996) Limit Theory for Mixing Dependent Random Variables. Kluwer, Boston.
[16] S. Louhichi (2000) Weak convergence for empirical processes of associated sequences.Ann. Inst. H. Poincare, Sec. B, v. 36, pp. 547–567.
[17] Ph.A. Martin and J. Piasecki (1996) Aggregation dynamics in a self-gravitating one-dimensional gas. J. Stat. Phys., v. 84, pp. 837–857.
[18] C.M. Newman (1980) Normal fluctuations and the FKG inequalities. Comm. Math. Phys., v. 74, pp. 119–128.

[19] Ya.G. Sinai (1992) Distribution of some functionals of the integral of a random walk. Theor. Math. Phys., v. 90, pp. 219–241.

[20] S.F. Shandarin and Ya.B. Zeldovich (1989) The large-scale structure of the universe: Turbulence, intermittency, structures in a self-gravitating medium. Rev. Mod. Phys., v. 61, pp. 185–220.

[21] T.M. Suidan (2001) A one-dimensional gravitationally interacting gas and the convex minorant of Brownian motion. Russ. Math. Surv., v. 56, pp. 687–708.

[22] M. Vergassola, B. Dubrulle, U. Frisch and A. Noullez (1994) Burgers' equation, Devil's staircases and the mass distribution function for large-scale structures. Astron. Astrophys., v. 289, pp. 325–356.

[23] V.V. Vysotsky (2006) On energy and clusters in stochastic systems of sticky gravitating particles. Theory Probab. Appl., v. 50, pp. 265–283.

[24] V.V. Vysotsky (2007) The area of exponential random walk and partial sums of uniform order statistics. Preprint, accepted to J. Math. Sci., available at arXiv:math/0612490v1 [math.PR].

Vladislav V. Vysotsky
Department of Probability Theory and Mathematical Statistics
Faculty of Mathematics and Mechanics
St. Petersburg State University
Bibliotechnaya pl., 2
Stary Peterhof, 198504
Russia
E-mail: vlad.vysotsky@gmail.com; vysotsky@vv9034.spb.edu