A Rigorous Time-Domain Analysis of Full–Wave Electromagnetic Cloaking (Invisibility) *†

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Abstract

There is currently a great deal of interest in the theoretical and practical possibility of cloaking objects from the observation by electromagnetic waves. The basic idea of these invisibility devices [8, 9, 13], [18] is to use anisotropic *transformation media* whose permittivity and permeability $\varepsilon^{\lambda\nu}, \mu^{\lambda\nu}$, are obtained from the ones, $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$, of isotropic media, by singular transformations of coordinates.
In this paper we study electromagnetic cloaking in the time-domain using the formalism of time-dependent scattering theory [23]. This formalism provides us with a rigorous method to analyze the propagation of electromagnetic wave packets with finite energy in *transformation media*. In particular, it allows us to settle in an unambiguous way the mathematical problems posed by the singularities of the inverse of the permittivity and the permeability of the *transformation media* on the boundary of the cloaked objects. Von Neumann's theory of self-adjoint extensions of symmetric operators plays an important role on this issue. We write Maxwell's equations in Schrödinger form with the electromagnetic propagator playing the role of the Hamiltonian. We prove that the electromagnetic propagator outside of the cloaked objects is essentially self-adjoint. This means that it has only one self-adjoint extension, A_Ω, and that this self-adjoint extension generates the only possible unitary time evolution, with constant energy, for finite energy electromagnetic waves, propagating outside of the cloaked objects.

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Moreover, A_Ω is unitarily equivalent to the electromagnetic propagator in the medium $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$. Using this fact, and since the coordinate transformation is the identity outside of a ball, we prove that the scattering operator is the identity. This implies that for any incoming finite-energy electromagnetic wave packet the outgoing wave packet is precisely the same. In other words, it is not possible to detect the cloaked objects in any scattering experiment where a finite-energy wave packet is sent towards the cloaked objects, since the outgoing wave packet that is measured after interaction is the same as the incoming one. Our results give a rigorous proof that the construction of [8, 9, 13], [18] perfectly cloaks passive and active devices from observation by electromagnetic waves. Actually, the cloaking outside is independent of what is inside the cloaked objects.
As is well known, self-adjoint extensions can be understood in terms of boundary conditions. Actually, for the electromagnetic fields in the domain of A_Ω the component tangential to the exterior of the boundary of the cloaked objects of both, the electric and the magnetic field have to be zero. This boundary condition is self-adjoint in our case because the permittivity and the permeability are degenerate on the boundary of the cloaked objects.
Furthermore, we prove cloaking for general anisotropic materials. In particular, our results prove that it is possible to cloak objects inside general crystals.

1 Introduction

There is currently a great deal of interest in the theoretical and practical possibility of cloaking objects from the observation by electromagnetic fields. The basic idea of these invisibility devices [8, 9, 13], [18] is to use anisotropic *transformation media* whose permittivity and permeability, $\varepsilon^{\lambda\nu}, \mu^{\lambda\nu}$, are obtained from the ones, $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$, of isotropic media, by singular transformations of coordinates. The singularities lie on the boundary of the objects to be cloaked. Here the *material interpretation* is taken. Namely, the $\varepsilon^{\lambda\nu}, \mu^{\lambda\nu}$ and the $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$, represent the components in flat Cartesian space of the permittivity and the permeability of physical media with *different material properties.* It appears that with existing technology it is possible to construct media as described above using artificially structured metamaterials. In [8, 9] a proof of cloaking was given for the conductivity equation -i.e., in the case of zero frequency- from detection by measurement of the Dirichlet to Neumann map that relates the value of the electric potential on the boundary to its normal derivative. The papers [13] and [18] consider electromagnetic waves in the geometrical optics approximation, i.e. for large frequencies. In [24] a experimental verification of cloaking is presented and [4] and

[5] give a numerical simulation. A rigorous prof of cloaking has already been given by [7] where fixed frequency waves were studied, i.e., in the frequency domain. They consider a class of finite energy solutions to Maxwell's equations in a bounded set, O, that contains the cloaked object on its interior, and they prove cloaking, at any frequency, with respect to the measurement of the Cauchy data of these solutions on the boundary of O. We give further comments on this paper below. For other results on this problem see [25] and [15]. In [16] cloaking of elastic waves is considered, and the history of invisibility is discussed.

In this paper we study electromagnetic cloaking in the time-domain using the formalism of time-dependent scattering theory [23]. This formalism provides us with a rigorous method to analyze the propagation of electromagnetic wave packets with finite energy in *transformation media*. In particular, it allows us to settle in an unambiguous way the mathematical problems posed by the singularities of the inverse of the permittivity and the permeability of the *transformation media* on the boundary of the cloaked objects. Von Neumann's theory of self-adjoint extensions of symmetric operators plays an important role on this issue. We write Maxwell's equations in Schrödinger form with the electromagnetic propagator playing the role of the Hamiltonian. We prove that the electromagnetic propagator outside of the cloaked objects is essentially self-adjoint. This means that it has only one self-adjoint extension, A_Ω, and that this self-adjoint extension generates the only possible unitary time evolution, with constant energy, for finite energy electromagnetic waves propagating outside of the cloaked objects. Moreover, A_Ω is unitarily equivalent to the electromagnetic propagator in the medium $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$. Using this fact, and since the coordinate transformation is the identity outside of a ball, we prove that the scattering operator is the identity. This implies that for any incoming finite-energy electromagnetic wave packet the outgoing wave packet is precisely the same. In other words, it is not possible to detect the cloaked objects in any scattering experiment where a finite-energy wave packet is sent towards the cloaked objects, since the outgoing wave packet that is measured after interaction is the same as the incoming one. Our results give a rigorous proof that the construction of [8, 9, 13], [18] perfectly cloaks passive and active devices from observation by electromagnetic waves. Actually, the cloaking outside is independent of what is inside the cloaked objects.

As is well known, self-adjoint extensions can be understood in terms of boundary conditions. Actually, for the electromagnetic fields in the domain of A_Ω the component tangential to the exterior of the boundary of the cloaked objects of both, the electric and the magnetic field have to be zero. This boundary condition is self-adjoint in our case because the permittivity and the permeability are degenerate on the boundary of the cloaked objects.

Furthermore, we prove cloaking for general anisotropic materials. In particular, our results prove that it is possible to cloak objects inside general crystals.

Even though, as mentioned above, the cloaking is independent of the cloaked objects, and in particular, the cloaking outside is not affected by the presence of passive and/or active devices inside the cloaked objects, we discuss the dynamics of electromagnetic waves inside the cloaked objects for completeness, since it helps to understand the above mentioned independence of cloaking from the properties of the cloaked objects.

We prove that every self-adjoint extension of the electromagnetic propagator in a *transformation medium* is the direct sum of the unique self-adjoint extension in the exterior of the cloaked objects, A_Ω, with some self-adjoint extension of the electromagnetic propagator in the interior of the cloaked objects. Each of these self-adjoint extensions corresponds to a possible unitary time evolution for finite energy electromagnetic waves. As is well known, the fact that time evolution is unitary assures us that energy is conserved. This results implies that the electromagnetic waves inside and outside of the cloaked objects completely decouple from each other. Actually, the electromagnetic waves inside the cloaked objects are not allowed to leave them, and viceversa, the electromagnetic waves outside can not go inside.

In terms of boundary conditions, this means that transmission conditions that link the electromagnetic fields inside and outside the cloaked objects are not allowed, since they do not correspond to self-adjoint extensions of the electromagnetic propagator, and then, they do not lead to a unitary dynamics that conserves energy. Furthermore, choosing a particular self-adjoint extension of the electromagnetic propagator of the cloaked objects amounts to choosing some boundary condition on the inside of the boundary of the cloaked objects. In other words, any possible unitary dynamics implies the existence of some boundary condition on the inside of the boundary of the cloaked objects. The particular boundary condition that nature will take depends on the specific properties of the metamaterial used to build the transformation media as well us on the properties of the media inside the cloaked objects. Note that this does not mean that we have to put any physical surface, a lining, on the surface of the cloaked object to enforce any particular boundary condition on the inside, since as we already mentioned this plays no role in the cloaking outside. It would be, however, of theoretical interest to see what the interior boundary condition turns out to be for specific cloaked objects and metamaterials.

Actually, we consider a slightly more general construction than the one of [8, 9, 13], [18] since we allow for a finite number of star-shaped cloaked objects.

In [7] a very general construction for cloaking is introduced. In the case of Maxwell's equations all their constructions are made within the context of the permittivity and the permeability tensor densities being conformal to each other, i.e., multiples of each other by a positive scalar function. In particular, all isotropic media are included in this category. They mention that both for mathematical and practical reasons it would be very interesting to understand cloaking for general anisotropic materials in the absence of this assumption. In this paper we actually solve this problem, since we prove cloaking for all general anisotropic materials. In particular, our results prove that it is possible to cloak objects inside general crystals.

Note, moreover, that [7] also considers the cases of the Helmholtz equation. We do not discuss this problems here.

Furthermore, remark that the existing theorems in the uniqueness of inverse scattering do not apply under the present conditions.

Finally, let us discuss the relation between the frequency domain approach of [7] and the time-domain method presented here. In the case where the permittivity and the permeability are bounded above and below it is well known that the Cauchy data at a fixed frequency given on a surface that encloses the objects and the scattering matrix at the same frequency are equivalent data. See for example [17] and [26]. This equivalence is, however, not proven in the case where the permittivity and the permeability are degenerate on the boundary of the objects. In fact, it is perhaps even not true in general, since in this case it is possible that there are finite energy electromagnetic waves that are absorbed by the boundary of the objects as $t \to \pm\infty$. If this is true, the equivalence will not hold since the Cauchy data in a surface that encloses the objects will not contain information on the waves that are asymptotically absorbed by the boundary of the objects. It is a problem of independent interest to see if this actually happens or not for general degenerate permittivities and permeabilities. For an example of scattering by a bounded obstacle with a singular boundary and Neumann boundary condition, "the jelly roll", where this happens see [10]. For a similar situation in the scattering of electromagnetic waves by a Schwarzschild black-hole see [2]. Note that in our time-dependent approach we directly consider the finite-energy electromagnetic wave packets that are used in scattering experiments.

In the analysis of Maxwell's equations with permittivity and permeability that are independent of frequency the dispersion of the medium is not taken into account. This means that cloaking will hold for electromagnetic wave packets with a narrow enough range of frequencies, such that this assumption is valid.

The paper is organized as follows. In Section 2 we prove our results in electromagnetic cloaking. In Section 3 we consider the propagation of electromagnetic waves in the interior of the cloaked objects. In Section 4 we formulate cloaking as a boundary value problem outside of the cloaked objects for the Maxwell equations at a fixed frequency, following our analysis of the self-adjoint extensions of the electromagnetic propagator. In particular, we give the appropriate boundary condition on the outside of the boundary of the cloaked objects. Finally, in Section 5 we prove cloaking of infinite cylinders. This is of interest since this is the case considered in the experimental verification in [24] and in the numerical simulations of [4] and [5]. Of course, [24] only consider a slice of the cylinder. In Sections 3 and 4 we give further comments on the results of [7].

2 Electromagnetic Cloaking

Let us consider Maxwell's equations,

$$\nabla \times \mathbf{E} \;=\; -\frac{\partial}{\partial t}\mathbf{B},\;\nabla \times \mathbf{H} \;=\; \frac{\partial}{\partial t}\mathbf{D}, \tag{2.1}$$

$$\nabla \cdot \mathbf{B} \;=\; 0, \nabla \cdot \mathbf{D} \;=\; 0, \tag{2.2}$$

in a domain, $\Omega \subset \mathbb{R}^3$, as follows,

$$\Omega := \mathbb{R}^3 \setminus \cup_{j=1}^N K_j,\;\, K_j \cap K_l = \emptyset, j \neq l \tag{2.3}$$

where $K_j, j = 1, 2, \cdots, N$, are closed and bounded set, that are the objects to be cloaked. We assume that each K_j is star-shaped with center $\mathbf{c}_j$, i.e.,

$$K_j = \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \mathbf{c}_j + \rho\, g_j(\hat{\mathbf{z}})\; \hat{\mathbf{z}},\, 0 \leq \rho \leq 1,\, \hat{\mathbf{z}} \in \mathbb{S}^2 \right\}, \tag{2.4}$$

where $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{R}^3$. The $g_j, j = 1, 2, \cdots, N$, are twice continuously differentiable, bounded and positive functions, that are defined on $\mathbb{R}^3$. We suppose that

$$\partial K_j := \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \mathbf{c}_j + g_j(\hat{\mathbf{z}})\, \hat{\mathbf{z}},\;\, \hat{\mathbf{z}} \in \mathbb{S}^2 \right\}, \tag{2.5}$$

is a closed C^2 surface that divides $\mathbb{R}^3$ into two components with K_j the bounded one. The cloaked objects are denoted by

$$K := \cup_{j=1}^N K_j.$$

We designate the Cartesian coordinates of $\mathbf{x}$ by $x^\lambda, \lambda = 1, 2, 3$ and by $E_\lambda, H_\lambda, B^\lambda, D^\lambda, \lambda = 1, 2, 3$, respectively, the components of $\mathbf{E}, \mathbf{H}, \mathbf{B}$, and $\mathbf{D}$. As usual, we denote by $\varepsilon^{\lambda\nu}$ and $\mu^{\lambda\nu}$, respectively, the permittivity and the permeability. We have that,

$$D^\lambda = \varepsilon^{\lambda\nu} E_\nu,\;\; B^\lambda = \mu^{\lambda\nu} H_\nu, \tag{2.6}$$

where we use the standard convention of summing over repeated lower and upper indices.

We consider now a transformation from $\Omega_0 := \mathbb{R}^3 \setminus \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_N\}$ onto Ω that is a generalization of the transformation first used to obtain cloaking for the conductivity equation, i.e. at zero frequency, by [8, 9] and then by [18] for cloaking electromagnetic waves (for a related result in two dimensions using conformal mappings see [13]).

We define,

$$G_{j,\delta} := \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \mathbf{c}_j + \rho\, g_j(\hat{\mathbf{z}})\, \hat{\mathbf{z}},\, 1 \leq \rho \leq \delta,\, \hat{\mathbf{z}} \in \mathbb{S}^2 \right\}. \tag{2.7}$$

Clearly, $G_{j,1} = \partial K_j$.

For any $\mathbf{y} \in \mathbb{R}^3$ we denote, $\hat{\mathbf{y}} := \mathbf{y}/|\mathbf{y}|$. Let $y^\lambda, \lambda = 1, 2, 3$, designate the cartesian coordinates of $\mathbf{y} \in \Omega_0$. Then, for $0 < |\mathbf{y} - \mathbf{c}_j| \leq \delta - 1$, with $\delta > 1$, we define,

$$\mathbf{x} = \mathbf{x}(\mathbf{y}) = f(\mathbf{y}) := \mathbf{c}_j + (|\mathbf{y} - \mathbf{c}_j| + 1)\, g_j(\widehat{\mathbf{y} - \mathbf{c}_j})\, \widehat{\mathbf{y} - \mathbf{c}_j}. \tag{2.8}$$

Note that this transformation blows up the point $\mathbf{c}_j$ onto ∂K_j and that it sends the punctuated ball $\tilde{B}_{\mathbf{c}_j}(\delta - 1) := \{\mathbf{y} \in \mathbb{R}^3 : 0 < |\mathbf{y} - \mathbf{c}_j| \leq \delta - 1\}$ onto $G_{j,\delta}$. We take δ so close to one that,

$$\tilde{B}_{\mathbf{c}_j}(\delta - 1) \cap \tilde{B}_{\mathbf{c}_l}(\delta - 1) = \emptyset, \;\; G_{j,\delta} \cap G_{l,\delta} = \emptyset\,, j \neq l, 1 \leq j, l \leq N. \tag{2.9}$$

For $\mathbf{y} \in \mathbb{R}^3 \setminus \cup_{j=1}^N \tilde{B}_{\mathbf{c}_j}(\delta - 1)$ we define the transformation to be the identity, $\mathbf{x} = \mathbf{x}(\mathbf{y}) = f(\mathbf{y}) := \mathbf{y}$. Our transformation is a bijection from Ω_0 onto Ω. By $\mathbf{y} = \mathbf{y}(\mathbf{x}) := f^{-1}(\mathbf{x})$ we designate the inverse transformation. We denote the elements of the Jacobian matrix by $A^\lambda_{\lambda'}$,

$$A^\lambda_{\lambda'} := \frac{\partial x^\lambda}{\partial y^{\lambda'}}. \tag{2.10}$$

Note that the $A^\lambda_{\lambda'} \in C^1 \left(\Omega_0 \setminus \cup_{j=1}^N \partial \tilde{B}_{\mathbf{c}_j}(\delta - 1) \right)$ and that they have jump discontinuities at $\cup_{j=1}^N \partial \tilde{B}_{\mathbf{c}_j}(\delta - 1)$. This, however, will pose no problem for us. We designate by $A^{\lambda'}_{\lambda}$ the elements of the Jacobian of the inverse bijection, $\mathbf{y} = \mathbf{y}(\mathbf{x}) = f^{-1}(\mathbf{x})$,

$$A^{\lambda'}_{\lambda} := \frac{\partial y^{\lambda'}}{\partial x^\lambda} \in C^1 \left(\Omega \setminus \cup_{j=1}^N \partial G_{j,\delta} \right), \tag{2.11}$$

with jump discontinuities at $\cup_{j=1}^N \partial G_{j,\delta}$. [8, 9] and [18] considered the case where $N = 1, \mathbf{c}_1 = 0$ and $g_1 \equiv 1$.

We take here the so called *material interpretation* and we consider our transformation as a bijection between two different spaces, Ω_0 and Ω. However, our transformation can be considered, as well, as a change of coordinates in Ω_0. Of course, these two point of view are mathematically equivalent. This means, in particular, that under our transformation the Maxwell equations in Ω_0 and in Ω will have the same invariance that they have under change of coordinates in three-space. See, for example, [21]. Let us denote by Δ the determinant of the Jacobian matrix (2.10). Then,

$$\Delta := \left(\frac{1+|\mathbf{y}-\mathbf{c}_j|}{|\mathbf{y}-\mathbf{c}_j|}\right)^2 \left(g(\widehat{\mathbf{y}-\mathbf{c}_j})\right)^3, \text{ for } 0 < |\mathbf{y}-\mathbf{c}_j| \leq \delta - 1. \tag{2.12}$$

This result is easily obtained rotating into a coordinate system such that, $\mathbf{y}-\mathbf{c}_j = (|\mathbf{y}-\mathbf{c}_j|, 0, 0)$. For $\mathbf{y} \in \Omega_0 \setminus \cup_{j=1}^N \tilde{B}_{\mathbf{c}_j}(\delta-1), \Delta \equiv 1$.

Let us denote by $\mathbf{E}_0, \mathbf{H}_0, \mathbf{B}_0, \mathbf{D}_0, \varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$, respectively, the electric and magnetic fields, the magnetic induction, the electric displacement, and the permittivity and permeability of Ω_0. The, $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$, are positive, hermitian matrices that are constant in Ω_0.

The electric field is a covariant vector that transforms as,

$$E_\lambda(\mathbf{x}) = A_\lambda^{\lambda'}(\mathbf{y}) E_{0,\lambda'}(\mathbf{y}). \tag{2.13}$$

The magnetic field $\mathbf{H}$ is a covariant pseudo-vector, but as we only consider space transformations with positive determinant, it also transforms as in (2.13). The magnetic induction $\mathbf{B}$ and the electric displacement $\mathbf{D}$ are contravariant vector densities of weight one that transform as

$$B^\lambda(\mathbf{x}) = (\Delta(\mathbf{y}))^{-1} A_{\lambda'}^{\lambda}(\mathbf{y}) B_0^{\lambda'}(\mathbf{y}), \tag{2.14}$$

with the same transformation for $\mathbf{D}$. The permittivity and permeability are contravariant tensor densities of weight one that transform as,

$$\varepsilon^{\lambda\nu}(\mathbf{x}) = (\Delta(\mathbf{y}))^{-1} A_{\lambda'}^{\lambda}(\mathbf{y})\, A_{\nu'}^{\nu}(\mathbf{y})\, \varepsilon_0^{\lambda'\nu'}(\mathbf{y}), \tag{2.15}$$

with the same transformation for $\mu^{\lambda\nu}$. The Maxwell equations (2.1, 2.2) are the same in both spaces Ω and Ω_0. Let us denote by $\varepsilon_{\lambda\nu}, \mu_{\lambda\nu}, \varepsilon_{0\lambda\nu}, \mu_{0\lambda\nu}$, respectively, the inverses of the corresponding permittivity and permeability. They are covariant tensor densities of weight minus one that transform as,

$$\varepsilon_{\lambda\nu}(\mathbf{x}) = \Delta(\mathbf{y}) A_{\lambda}^{\lambda'}(\mathbf{y})\, A_{\nu}^{\nu'}(\mathbf{y})\, \varepsilon_{0\lambda'\nu'}(\mathbf{y}),\ \mu_{\lambda\nu}(\mathbf{x}) = \Delta(\mathbf{y}) A_{\lambda}^{\lambda'}(\mathbf{y})\, A_{\nu}^{\nu'}(\mathbf{y})\, \mu_{0\lambda'\nu'}(\mathbf{y}). \tag{2.16}$$

Note that

$$\det \varepsilon^{\lambda\nu} = \Delta^{-1} \det \varepsilon_0^{\lambda\nu},\ \det \mu^{\lambda\nu} = \Delta^{-1} \det \mu_0^{\lambda\nu}, \tag{2.17}$$

$$\det \varepsilon_{\lambda\nu} = \Delta \det \varepsilon_{0\lambda\nu},\ \det \mu_{\lambda\nu} = \Delta \det \mu_{0\lambda\nu}. \tag{2.18}$$

We now introduce the Hilbert spaces of electric and magnetic fields with finite energy. The $\mathbf{E}_0, \mathbf{H}_0, \mathbf{B}_0, \mathbf{D}_0$, were defined in Ω_0, but since $\mathbb{R}^3 \setminus \Omega_0 = \{\mathbf{c}_j\}_{j=1}^N$ is of measure zero, we can consider them as defined in $\mathbb{R}^3$, what we do below.

We denote by $\mathcal{H}_{0E}$ the Hilbert space of all measurable, square integrable, $\mathbf{C}^3-$ valued functions defined on $\mathbb{R}^3$ with the scalar product,

$$\left(\mathbf{E}_0^{(1)}, \mathbf{E}_0^{(2)}\right)_{0E} := \int_{\mathbb{R}^3} E_{0\lambda}^{(1)}\, \varepsilon_0^{\lambda\nu}\, \overline{E_{0\nu}^{(2)}}\, d\mathbf{y}^3. \tag{2.19}$$

We similarly define the Hilbert space,$\mathcal{H}_{0H}$, of all measurable, square integrable, $\mathbf{C}^3-$ valued functions defined on $\mathbb{R}^3$ with the scalar product,

$$\left(\mathbf{H}_0^{(1)}, \mathbf{H}_0^{(2)}\right)_{0H} := \int_{\mathbb{R}^3} H_{0\lambda}^{(1)}\, \mu_0^{\lambda\nu}\, \overline{H_{0\nu}^{(2)}}\, d\mathbf{y}^3. \tag{2.20}$$

The Hilbert space of finite energy fields in $\mathbb{R}^3$ is the direct sum

$$\mathcal{H}_0 := \mathcal{H}_{0E} \oplus \mathcal{H}_{0H}. \tag{2.21}$$

Moreover, we designate by $\mathcal{H}_{\Omega E}$ the Hilbert space of all measurable, $\mathbf{C}^3-$ valued functions defined on Ω that are square integrable with the weight $\varepsilon^{\lambda\nu}$, with the scalar product,

$$\left(\mathbf{E}^{(1)}, \mathbf{E}^{(2)}\right)_{\Omega E} := \int_{\Omega} E_{\lambda}^{(1)}\, \varepsilon^{\lambda\nu}\, \overline{E_{\nu}^{(2)}}\, d\mathbf{x}^3. \tag{2.22}$$

Finally, we denote by $\mathcal{H}_{\Omega H}$ the Hilbert space of all measurable, $\mathbf{C}^3-$ valued functions defined on Ω that are square integrable with the weight $\mu^{\lambda\nu}$, with the scalar product,

$$\left(\mathbf{H}^{(1)}, \mathbf{H}^{(2)}\right)_{\Omega H} := \int_{\Omega} H^{(1)}_{\lambda}\, \mu^{\lambda\nu}\, \overline{H^{(2)}_{\nu}}\, d\mathbf{x}^3. \tag{2.23}$$

The Hilbert space of finite energy fields in Ω is the direct sum

$$\mathcal{H}_{\Omega} := \mathcal{H}_{\Omega E} \oplus \mathcal{H}_{\Omega H}. \tag{2.24}$$

We now write the Maxwell's equations (2.1) in Schrödinger form. We first consider the case of $\mathbb{R}^3$. We denote by ε_0 and μ_0, respectively, the matrices with entries $\varepsilon_{0\lambda\nu}$ and $\mu_{0\lambda\nu}$. Recall that $(\nabla \times \mathbf{E})^{\lambda} = s^{\lambda\nu\rho}\left(\frac{\partial}{\partial x_\nu} E_\rho - \frac{\partial}{\partial x_\rho} E_\nu\right)$ where $s^{\lambda\nu\rho}$ is the permutation contravariant pseudo-density of weight -1 (see section 6 of chapter II of [21], where a different notation is used). By a_0 we denote the following formal differential operator,

$$a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} = i \begin{pmatrix} \varepsilon_0 \nabla \times \mathbf{H}_0 \\ -\mu_0 \nabla \times \mathbf{E}_0 \end{pmatrix}. \tag{2.25}$$

Here, as usual, we denote, $\varepsilon_0 \nabla \times \mathbf{H}_0 := \varepsilon_{0\lambda\nu}(\nabla \times \mathbf{H}_0)^{\nu}$, and $\mu_0 \nabla \times \mathbf{E}_0 = \mu_{0\lambda\nu}(\nabla \times \mathbf{E}_0)^{\nu}$. Then, equations (2.1) are equivalent to,

$$i\frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} = a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}. \tag{2.26}$$

Let us denote by $\mathbf{C}^1_0(\mathbb{R}^3)$ the set of all $\mathbf{C}^6-$valued continuously differentiable functions on $\mathbb{R}^3$ that have compact support. Then, a_0 with domain $\mathbf{C}^1_0(\mathbb{R}^3)$ is a symmetric operator in $\mathcal{H}_0$, i.e., $a_0 \subset a_0^*$. Moreover, it is essentially self-adjoint in $\mathcal{H}_0$, i.e., it has only one self-adjoint extension, that we denote by A_0. Its domain is given by,

$$D(A_0) = \left\{ \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} : a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in \mathcal{H}_0 \right\}, \tag{2.27}$$

and,

$$A_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} = a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}, \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in D(A_0), \tag{2.28}$$

where the derivatives are taken in distribution sense. These results follow easily from the fact that -via the Fourier transform- a_0 is unitarily equivalent to multiplication by a matrix valued function that is symmetric with respect to the scalar product of $\mathcal{H}_0$. Moreover, it follows from explicit computation that the only eigenvalue of A_0 is zero, that it has infinite multiplicity, and that,

$$\mathcal{H}_{0\perp} := (\text{kernel}\, A_0)^{\perp} = \left\{ \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in \mathcal{H}_0 : \frac{\partial}{\partial x_\lambda} \varepsilon_0^{\lambda\nu} E_{0\nu} = 0, \frac{\partial}{\partial x_\lambda} \mu_0^{\lambda\nu} H_{0\nu} = 0 \right\}. \qquad (2.29)$$

Furthermore, A_0 has no singular-continuous spectrum and its absolutely-continuous spectrum is $\mathbb{R}$. See, for example, [27, 28].

Taking any

$$\begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in \mathcal{H}_{0\perp} \cap D(A_0) \qquad (2.30)$$

we obtain a finite energy solution to the Maxwell equations (2.1, 2.2) as follows

$$\begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}(t) = e^{-itA_0} \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}. \qquad (2.31)$$

This is the unique finite energy solution with initial value at $t = 0$ given by (2.30). Note that as $e^{-itA_0}\mathcal{H}_{0\perp} \subset \mathcal{H}_{0\perp}$ equations (2.2) are satisfied for all times if they are satisfied at $t = 0$.

Let us now consider the case of Ω. We denote by ε and μ, respectively, the matrices with entries $\varepsilon_{\lambda\nu}$ and $\mu_{\lambda\nu}$.

We now define the following formal differential operator,

$$a_\Omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = i \begin{pmatrix} \varepsilon \nabla \times \mathbf{H} \\ -\mu \nabla \times \mathbf{E} \end{pmatrix}. \qquad (2.32)$$

Equations (2.1) are equivalent to,

$$i\frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = a_\Omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}.$$

Let us denote by $\mathbf{C}_0^1(\Omega)$ the set of all $\mathbf{C}^6-$valued continuously differentiable functions on Ω that have compact support. Then, a_Ω with domain $\mathbf{C}_0^1(\Omega)$ is a symmetric operator in $\mathcal{H}_\Omega$. To construct a unitary dynamics that preserves energy we have to analyse the self-adjoint extensions of a_Ω.

We denote by U_E the following unitary operator from $\mathcal{H}_{0E}$ onto $\mathcal{H}_{\Omega E}$,

$$\left(U_E \mathbf{E}_0\right)_\lambda(\mathbf{x}) := A_\lambda^{\lambda'} E_{0\lambda'}(\mathbf{y}), \tag{2.33}$$

and by U_H the unitary operator from $\mathcal{H}_{0H}$ onto $\mathcal{H}_{\Omega H}$,

$$\left(U_H \mathbf{H}_0\right)_\lambda(\mathbf{x}) := A_\lambda^{\lambda'} H_{0\lambda'}(\mathbf{y}). \tag{2.34}$$

Then,

$$U := U_E \oplus U_H \tag{2.35}$$

is a unitary operator from $\mathcal{H}_0$ onto $\mathcal{H}_\Omega$.

Moreover, U sends $\mathbf{C}_0^1(\Omega_0)$ onto $\mathbf{C}_0^1(\Omega)$, and, furthermore, by the invariance of Maxwell's equations,

$$a_\Omega = U\, a_{00}\, U^*, \tag{2.36}$$

where we denote by a_{00} the restriction of a_0 to $\mathbf{C}_0^1(\Omega_0)$. The operator a_{00} is essentially self-adjoint and its only self-adjoint extension is A_0. This follows from the essential self-adjointness of a_0 and from the fact that any function in $\mathbf{C}_0^1(\mathbb{R}^3)$ can be approximated in the graph norm of a_0 by functions in $\mathbf{C}_0^1(\Omega_0)$. To prove this take any continuously differentiable real-valued function, ϕ, defined on $\mathbb{R}$ such that, $\phi(y) = 0, |y| \leq 1$ and $\phi(y) = 1, |y| \geq 2$. Then, for any

$$\begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in \mathbf{C}_0^1(\mathbb{R}^3),$$

we have that,

$$\prod_{j=1}^N \phi(n|\mathbf{y} - \mathbf{c}_j|) \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in \mathbf{C}_0^1(\Omega_0)$$

and moreover,

$$\text{s-}\lim_{n\to\infty} \prod_{j=1}^N \phi(n|\mathbf{y} - \mathbf{c}_j|) \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} = \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix},$$

$$
\text{s-}\lim_{n\to\infty} a_0 \prod_{j=1}^{N} \phi(n|\mathbf{y}-\mathbf{c}_j|) \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} = a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix},
$$

where by s- lim we designate the strong limit in $\mathcal{H}_0$.

Then, as a_{00} is essentially self-adjoint, it follows from (2.36) that a_Ω is essentially self-adjoint, and that its unique self-adjoint extension, that we denote by A_Ω, satisfies

$$
A_\Omega = U\, A_0\, U^*. \tag{2.37}
$$

Hence, we have proven the following theorem.

THEOREM 2.1. *The operator a_Ω is essentially self-adjoint, and its unique self-adjoint extension, A_Ω, satisfies (2.37).*

The unitary equivalence given by (2.37) implies that A_Ω has the same spectral properties that A_0. Namely, it has no singular-continuous spectrum, the absolutely-continuous spectrum is $\mathbb{R}$ and the only eigenvalue is zero and it has infinite multiplicity. Moreover,

$$
\mathcal{H}_{\Omega\perp} := (\text{kernel}\, A_\Omega)^\perp = \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in \mathcal{H}_\Omega : \frac{\partial}{\partial x_\lambda} \varepsilon^{\lambda\nu} E_\nu = 0, \frac{\partial}{\partial x_\lambda} \mu^{\lambda\nu} H_\nu = 0 \right\}. \tag{2.38}
$$

Furthermore, taking any

$$
\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in \mathcal{H}_{\Omega\perp} \cap D(A_\Omega) \tag{2.39}
$$

we obtain a finite energy solution to the Maxwell equations (2.1, 2.2) as follows

$$
\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (t) = e^{-itA_\Omega} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}. \tag{2.40}
$$

This is the unique finite energy solution with initial value at $t = 0$ given by (2.39). Note that as $e^{-itA_\Omega}\mathcal{H}_{\Omega\perp} \subset \mathcal{H}_{\Omega\perp}$ equations (2.2) are satisfied for all times if they are satisfied at $t = 0$. We can consider more general solutions by considering the scale of spaces associated with A_Ω, but we do not go into this direction here.

The facts that a_Ω is essentially self-adjoint and that its unique self-adjoint extension A_Ω is unitarily equivalent to the propagator A_0 of the homogeneous medium are strong statements.

They mean that the only possible unitary dynamics in Ω that preserves energy is given by (2.40) and that this dynamics is unitarily equivalent to the free dynamics in $\mathbb{R}^3$ given by (2.31). In fact, $\partial\Omega$ acts like a horizon for electromagnetic waves propagating in Ω in the sense that the dynamics is uniquely defined without any need to consider the cloaked objects $K = \cup_{j=1}^{N} K_j$. As we will prove below this implies electromagnetic cloaking for all frequencies in the strong sense that the scattering operator is the identity.

Since $D(A_\Omega) = UD(A_0)$, for any $(\mathbf{E}, \mathbf{H})^T \in D(A_\Omega)$ there is a $(\mathbf{E}_0, \mathbf{H}_0)^T \in D(A_0)$ such that

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = U \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}. \tag{2.41}$$

Then, it follows from (2.33, 2.34, 2.35) that

$$\mathbf{E} \times \mathbf{n} = 0, \mathbf{H} \times \mathbf{n} = 0, \text{ in } \partial K_+, \tag{2.42}$$

where ∂K_+ denotes the outside of the boundary of the cloaked objects, K, and $\mathbf{n}$ is the normal vector to ∂K_+, if $(\mathbf{E}_0, \mathbf{H}_0)$ are, for example, bounded near ∂K_+. That is to say, for electromagnetic fields in the domain of A_Ω the tangential components of both, the electric and the magnetic field vanish in the exterior of the boundary of the cloaked objects. This is a self-adjoint boundary condition because the permittivity and the permeability are degenerate on ∂K.

Let χ_Ω be the characteristic function of Ω, i.e., $\chi_\Omega(\mathbf{x}) = 1, \mathbf{x} \in \Omega, \chi_\Omega(\mathbf{x}) = 0, \mathbf{x} \in \mathbb{R}^3 \setminus \Omega$. We define,

$$\left(J \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \right)(\mathbf{x}) := \chi_\Omega(\mathbf{x}) \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}(\mathbf{x}). \tag{2.43}$$

By (2.8, 2.12, 2.15),

$$\left|\varepsilon^{\lambda\nu}(\mathbf{x})\right| \leq C, \quad \left|\mu^{\lambda\nu}(\mathbf{x})\right| \leq C, \quad \mathbf{x} \in \Omega.$$

Then, J is a bounded operator from $\mathcal{H}_0$ into $\mathcal{H}_\Omega$.

The wave operators are defined as follows,

$$W_{\pm} = \text{s-}\lim_{t \to \pm\infty} e^{itA_{\Omega}} \, J e^{-itA_0} P_{0\perp}, \tag{2.44}$$

where $P_{0\perp}$ denotes the projector onto $\mathcal{H}_{0\perp}$.

Let us designate by $\mathbf{W}^{1,2}(\mathbb{R}^3)$ the Sobolev space of $\mathbf{C}^6$ valued functions. We denote by I the identity operator on $\mathcal{H}_0$. Then,

LEMMA 2.2.

$$W_{\pm} = U P_{0\perp}. \tag{2.45}$$

Proof: Denote,

$$W(t) := e^{itA_{\Omega}} \, J \, e^{-itA_0} P_{0\perp}.$$

By (2.37), for any $\varphi \in \mathcal{H}_0$

$$W(t)\varphi = \psi(t) + U P_{0\perp}\varphi, \tag{2.46}$$

with

$$\psi(t) := U \, e^{itA_0} \, (U^* J - I) \, e^{-itA_0} P_{0\perp}\varphi. \tag{2.47}$$

Let B_R denote the ball of center zero and radius R in $\mathbb{R}^3$. Since for $|y| \geq R$, with R large enough, our transformation, $\mathbf{x} = f(\mathbf{y})$, is the identity, $\mathbf{x} = \mathbf{y}$, and in consequence, $A^{\lambda}_{\lambda'}(\mathbf{y}) = \delta^{\lambda}_{\lambda'}$ for $|\mathbf{y}| \geq R$, we have that,

$$(U^* J - I) = (U^* J - I) \, \chi_{B_R}. \tag{2.48}$$

It follows that,

$$\text{s-}\lim_{t \to \pm\infty} \psi(t) = U \, \text{s-}\lim_{t \to \pm\infty} e^{itA_0} \vartheta(t) \tag{2.49}$$

with,

$$\vartheta(t) := (U^* J - I) \, \chi_{B_R} e^{-itA_0} P_{0\perp}\varphi. \tag{2.50}$$

We have that,

$$\|\vartheta(t)\|_{\mathcal{H}_0} \leq \left\| J\chi_{B_R} e^{-itA_0} P_{0\perp}\varphi \right\|_{\mathcal{H}} + \left\| \chi_{B_R} e^{-itA_0} P_{0\perp}\varphi \right\|_{\mathcal{H}_0} \leq C \left\| \chi_{B_R} e^{-itA_0} P_{0\perp}\varphi \right\|_{\mathcal{H}_0}. \quad (2.51)$$

Then, as $(A_0+i)^{-1}P_{0\perp}$ is bounded from $\mathcal{H}_0$ into $\mathbf{W}^{1,2}(\mathbb{R}^3)$ [27] [28], it follows from the Rellich local compactness theorem that

$$\chi_{B_R}(A_0+i)^{-1}P_{0\perp}$$

is a compact operator in $\mathcal{H}_0$. Suppose that $\varphi \in D(A_0) \cap \mathcal{H}_{0\perp}$. Then,

$$\text{s-}\lim_{t\to\pm\infty} \chi_{B_R} e^{-itA_0} P_{0\perp}\varphi = \text{s-}\lim_{t\to\pm\infty} \chi_{B_R}(A_0+i)^{-1} P_{0\perp} e^{-itA_0}(A_0+i)\varphi = 0, \quad (2.52)$$

and whence, by (2.51),

$$\text{s-}\lim_{t\to\pm\infty} \vartheta(t) = 0, \quad (2.53)$$

and it follows that in this case,

$$\text{s-}\lim_{t\to\pm\infty} \psi(t) = 0. \quad (2.54)$$

By continuity this is also true for $\varphi \in \mathcal{H}_{0\perp}$.

Then, (2.45) follows from (2.46) and (2.54).

□

The scattering operator is defined as

$$S := W_+^* W_-. \quad (2.55)$$

COROLLARY 2.3.

$$S = P_{0\perp}. \quad (2.56)$$

Proof: This is immediate from (2.45) because $U^*U = I$.

□

Let us denote by $S_\perp$ the restriction of S to $\mathcal{H}_{0\perp}$. $S_\perp$ is the physically relevant scattering operator that acts in the Hilbert space $\mathcal{H}_{0\perp}$ of finite energy fields that satisfy equations (2.2). We designate by $I_\perp$ the identity operator on $\mathcal{H}_{0\perp}$. We have that,

COROLLARY 2.4.

$$S_\perp = I_\perp. \tag{2.57}$$

Proof: This follows from Corollary 2.4.

□

The fact that $S_\perp$ is the identity operator on $\mathcal{H}_{0\perp}$ means that there is perfect cloaking for all frequencies. Suppose that for very negative times we are given an incoming wave packet $e^{-itA_0}\varphi_-$, with $\varphi_- \in \mathcal{H}_{0\perp}$. Then, for large positive times the outgoing wave packet is given by $e^{-itA_0}\varphi_+$ with $\varphi_+ = S_\perp \varphi_-$. But, as $S = I$, we have that $\varphi_+ = \varphi_-$ and then,

$$e^{-itA_0}\varphi_- = e^{-itA_0}\varphi_+.$$

Since the incoming and the outgoing wave packets are the same there is no way to detect the cloaked objects K from scattering experiments performed in Ω.

In this paper we considered *transformation media* obtained from a singular transformation that blows up a finite number of points, by simplicity, and since this is the situation in the applications. Suppose that we have a transformation that is singular in a set of points that we call M and denote now $\Omega_0 := \mathbb{R}^3 \setminus M$. What we really used in the proofs is that $\mathbf{W}^{1,2}(\mathbb{R}^3) = \mathbf{W}_0^{1,2}(\Omega_0)$ where $\mathbf{W}_0^{1,2}(\Omega_0)$ denotes the completion of $\mathbf{C}_0^\infty(\Omega_0)$ in the norm of $\mathbf{W}^{1,2}(\mathbb{R}^3)$. We also assumed that $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$ are constant. What was actually needed is that a_0 is essentially self-adjoint. All our results hold under this more general conditions provided that in (2.44, 2.45) and (2.56) we replace $P_{0\perp}$ by the projector onto the absolutely-continuous subspace of A_0 and that we assume that $D(A_0) \cap \mathcal{H}_{0ac} \subset \mathbf{W}^{1,2}(\mathbb{R}^3)$, where we have denoted the absolutely-continuous subspace of A_0 by $\mathcal{H}_{0ac}$. Moreover, $S_\perp$ has to be defined as the restriction of S to $\mathcal{H}_{0ac}$ and in (2.57) $I_\perp$ has to be the identity operator on $\mathcal{H}_{0ac}$. Note that under these general assumptions A_0 could have non-zero eigenvalues and singular-continuous spectrum.

For example, $\mathbf{W}^{1,2}(\mathbb{R}^3) = \mathbf{W}_0^{1,2}(\Omega_0)$ if M has zero Sobolev one capacity [1, 11, 12]. Moreover, assume that the permittivity and the permeability tensor densities $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$ are bounded below and above. Under this condition a_0 is essentially self-adjoint. Furthermore, let us denote by $\hat{\mathcal{H}}_0$ the Hilbert space of finite energy solutions defined as in (2.21) but with $\varepsilon_0^{\lambda\nu} = \mu_0^{\lambda\mu} = \delta^{\lambda\mu}$. Let $\hat{A}_0, \hat{\mathcal{H}}_{0\perp}$ be, respectively, the electromagnetic propagator in $\hat{\mathcal{H}}_0$ and the orthogonal complement of its kernel. We have that $\mathcal{H}_0$ and $\hat{\mathcal{H}}_0$ are the same set of functions with equivalent norms. Furthermore, $D(A_0) = D(\hat{A}_0), \operatorname{kernel} \hat{A}_0 = \operatorname{kernel} A_0$. Moreover, $(\mathbf{E}_0, \mathbf{H}_0)^T \in \mathcal{H}_{0\perp}$ if and only if $\mathbf{E}_0 = \varepsilon_0 \hat{\mathbf{E}}_0, \mathbf{H}_0 = \mu_0 \hat{\mathbf{H}}_0$ for some $(\hat{\mathbf{E}}_0, \hat{\mathbf{H}}_0) \in \hat{\mathcal{H}}_{0\perp}$. As [27, 28] $D(\hat{A}_0) \cap \hat{\mathcal{H}}_{0\perp} \subset \mathbf{W}^{1,2}(\mathbb{R}^3)$ we have that $D(A_0) \cap \mathcal{H}_{0\perp} \subset \mathbf{W}^{1,2}(\mathbb{R}^3)$ if ε_0, μ_0 are bounded operators on $\mathbf{W}^{1,2}(\mathbb{R}^3)$ and this is true if the derivatives $\frac{\partial}{\partial y_\rho}\varepsilon_0, \frac{\partial}{\partial y_\rho}\mu_0$ are bounded operators on $\hat{\mathcal{H}}_0$ for $\rho = 1, 2, 3$. Note, furthermore, that $\mathcal{H}_{0ac} \subset \mathcal{H}_{0\perp}$.

3 Electromagnetic Waves Inside the Cloaked Objects

Let us now consider the propagation of electromagnetic waves in the cloaked objects. For this purpose we assume that in each K_j the permittivity and the permeability are given by $\varepsilon_j^{\lambda\nu}, \mu_j^{\lambda\nu}$, with inverses $\varepsilon_{j\lambda\nu}, \mu_{j\lambda\nu}$ and where ε_j, μ_j are the matrices with entries $\varepsilon_{j\lambda\nu}, \mu_{j\lambda\nu}$. Furthermore, we assume that $0 < \varepsilon^{\lambda\nu}, \mu^{\lambda\nu} \leq C, \mathbf{x} \in K_j$ and that for any compact set Q contained in the interior of K_j there is a positive constant C_Q such that $\det \varepsilon^{\lambda\nu} > C_Q, \det \mu^{\lambda\nu} > C_Q, \mathbf{x} \in Q$. In other words, we only allow for possible singularities of ε_j, μ_j on the boundary of K_j.

We designate by $\mathcal{H}_{jE}$ the Hilbert space of all measurable, $\mathbf{C}^3-$ valued functions defined on K_j that are square integrable with the weight $\varepsilon_j^{\lambda\nu}$, with the scalar product,

$$\left(\mathbf{E}_j^{(1)}, \mathbf{E}_j^{(2)}\right)_{jE} := \int_{K_j} E_{j\lambda}^{(1)}\, \varepsilon_j^{\lambda\nu}\, \overline{E_{j\nu}^{(2)}}\, d\mathbf{x}^3. \tag{3.1}$$

Similarly, we denote by $\mathcal{H}_{jH}$ the Hilbert space of all measurable, $\mathbf{C}^3-$ valued functions defined on K_j that are square integrable with the weight $\mu_j^{\lambda\nu}$, with the scalar product,

$$\left(\mathbf{H}_j^{(1)}, \mathbf{H}_j^{(2)}\right)_{jH} := \int_{K_j} H_{j\lambda}^{(1)}\, \mu_j^{\lambda\nu}\, \overline{H_{j\nu}^{(2)}}\, d\mathbf{x}^3. \tag{3.2}$$

The Hilbert space of finite energy fields in K_j is the direct sum

$$\mathcal{H}_j := \mathcal{H}_{jE} \oplus \mathcal{H}_{jH}, \tag{3.3}$$

and the Hilbert space in the cloaked objects K is the direct sum,

$$\mathcal{H}_K := \oplus_{j=1}^N \mathcal{H}_j.$$

The complete Hilbert space of finite energy fields including the cloaked objects is,

$$\mathcal{H} := \mathcal{H}_\Omega \oplus \mathcal{H}_K. \tag{3.4}$$

We now write (2.1) as a Schrödinger equation in each K_j as before. We define the following formal differential operator,

$$a_j \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} = i \begin{pmatrix} \varepsilon_j \nabla \times \mathbf{H}_j \\ -\mu_j \nabla \times \mathbf{E}_j \end{pmatrix}. \tag{3.5}$$

Equation (2.1) in K_j is equivalent to

$$i\frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} = a_j \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix}. \tag{3.6}$$

Let us denote by $\mathbf{C}_0^1(\hat{K}_j)$ the set of all $\mathbf{C}^6-$valued continuously differentiable functions on K_j that have compact support in the interior of K_j, that we denote by $\hat{K}_j := K_j \setminus \partial K_j$. Then, a_j with domain $C_0^1(\hat{K}_j)$ is a symmetric operator in $\mathcal{H}_j$. We denote,

$$a := a_\Omega \oplus_{j=1}^N a_j, \tag{3.7}$$

with domain,

$$D(a) := \left\{ \begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^N \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \in \mathbf{C}_0^1(\Omega) \oplus_{j=0}^N \mathbf{C}_0^1(\hat{K}_j) \right\}. \tag{3.8}$$

The operator a is symmetric in $\mathcal{H}$. The possible unitary dynamics that preserve energy for the whole system including the cloaked objects K are given by the self-adjoint extensions of a. Let us denote $\overline{a}$ the closure of a, with similar notation for $a_\Omega, a_j, j = 1, \cdots, N$. Then,

$$\overline{a} = A_\Omega \oplus_{j=1}^N \overline{a_j},$$

where we used the fact that as a_Ω is essentially self-adjoint, $\overline{a_\Omega} = A_\Omega$. The adjoint of a is given by,

$$D(a^*) = \left\{ \begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^N \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \in \mathcal{H} : \begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \in D(A_\Omega), a_j \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \in \mathcal{H}_j \right\}, \qquad (3.9)$$

and

$$a^* \left(\begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^N \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \right) = A_\Omega \begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^N a_j \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix}, \qquad (3.10)$$

for

$$\begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^N \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \in D(a^*). \qquad (3.11)$$

Let us denote by $\mathcal{K}_{\Omega\pm} := \mathrm{kernel}(i \mp a_\Omega^*), \mathcal{K}_{j\pm} := \mathrm{kernel}(i \mp a_j^*)$ the deficiency subspaces of a_Ω and $a_j, j = 1, \cdots, N$. Since a_Ω is essentially self-adjoint $\mathcal{K}_{\Omega\pm} = \{0\}$. Let $\mathcal{K}_\pm := \oplus_{j=1}^N \mathcal{K}_{j\pm}$ be the deficiency subspaces of $a_K := \oplus_{j=1}^N a_j$. Suppose that $\mathcal{K}_\pm$ have the same dimension. Then, it follows from Corollary 1 in page 141 of [22] that there is a one-to-one correspondence between self-adjoint extensions of a_K and unitary maps from $\mathcal{K}_+$ into $\mathcal{K}_-$. If V is such a unitary, then the corresponding self-adjoint extension A_{KV} is given by,

$$D(A_{KV}) = \left\{ \varphi + \varphi_+ + V\varphi_+ : \varphi \in D(\overline{a_K}), \varphi_+ \in \mathcal{K}_+ \right\},$$

and

$$A_K \varphi = \overline{a_K} \varphi + i\varphi_+ - iV\varphi_+.$$

Hence, since $\mathcal{K}_{\Omega\pm} = \{0\}$ and $\overline{a} = A_\Omega \oplus \overline{a_K}$ there is a one-to-one correspondence between self-adjoint extensions of a and unitary maps, V, from $\mathcal{K}_+$ into $\mathcal{K}_-$. The self-adjoint extension A_V corresponding to V is given by,

$$A_V = A_\Omega \oplus A_{KV}.$$

Thus, we have proven the following theorem.

THEOREM 3.1. *Every self-adjoint extension, A, of a is the direct sum of A_Ω and of some self-adjoint extension, A_K of a_K, i.e.,*

$$A = A_\Omega \oplus A_K. \tag{3.12}$$

This theorem tells us that the cloaked objects K and the exterior Ω are completely decoupled and that we are free to choose any boundary condition inside the cloaked objects K that makes a_K self-adjoint without disturbing the cloaking effect in Ω. Boundary conditions that make A_K self-adjoint are well known. See for example, [19], [20], [14] and [6].

It follows from explicit computation that zero is an eigenvalue of every A_K with infinite multiplicity and that,

$$\mathcal{H}_{K\perp} := (\text{kernel}\, A_K)^{\perp} = \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in \mathcal{H}_K : \frac{\partial}{\partial x_\lambda} \varepsilon_K^{\lambda\nu} E_\nu = 0, \frac{\partial}{\partial x_\lambda} \mu_K^{\lambda\nu} H_\nu = 0 \right\}, \tag{3.13}$$

where by $\varepsilon_K^{\lambda\nu}(\mathbf{x}) := \varepsilon_j^{\lambda\nu}(\mathbf{x})$ for $\mathbf{x} \in K_j$, and $\mu_K^{\lambda\nu}(\mathbf{x}) := \mu_j^{\lambda\nu}(\mathbf{x})$ for $\mathbf{x} \in K_j, j = 1, 2, \cdots, N$. It follows that zero is an eigenvalue of A with infinite multiplicity and that,

$$\mathcal{H}_\perp := (\text{kernel}\, A)^{\perp} = \mathcal{H}_{\Omega\perp} \oplus \mathcal{H}_{K\perp}. \tag{3.14}$$

For any $\varphi = \varphi_\Omega \oplus \varphi_K \in \mathcal{H}_\perp \cap D(A)$,

$$e^{-itA}\varphi = e^{-itA_\Omega}\, \varphi_\Omega \oplus e^{-itA_K}\, \varphi_K \tag{3.15}$$

is the unique solution of Maxwell's equations (2.1, 2.2) with finite energy that is equal to φ at $t = 0$. This shows once again that the dynamics in Ω and in K are completely decoupled. If at $t = 0$ the electromagnetic fields are zero in Ω, they remain equal to zero for all times, and viceversa. Actually, electromagnetic waves inside the cloaked objects are not allowed to leave them, and viceversa, electromagnetic waves outside can not go inside. This implies, in particular, that the presence of active devices inside the cloaked objects has no effect on the cloaking outside.

In terms of boundary conditions, this means that transmission conditions that link the electromagnetic fields inside and outside the cloaked objects are not allowed. Furthermore, choosing a particular self-adjoint extension of the electromagnetic propagator of the cloaked objects amounts to choosing some boundary condition on the inside of the boundary of the cloaked objects. In other words, any possible unitary dynamics implies the existence of some boundary condition on the inside of the boundary of the cloaked objects. The particular boundary condition that nature will take depends on the specific properties of the metamaterial used to build the transformation media as well us on the properties of the media inside the cloaked objects. Note that this does not mean that we have to put any physical surface, a lining, on the surface of the cloaked object to enforce any particular boundary condition on the inside, since as we already mentioned this plays no role in the cloaking outside. It would be, however, of theoretical interest to see what the interior boundary condition turns out to be for specific cloaked objects and metamaterials.

The fact that for the *single coating* there has to be boundary conditions on the inside of ∂K has already been observed by [7]. In Definition 4.1 of [7] a definition of finite energy solutions is given. Furthermore, is proven in Theorem 6.1 that in the case of the *single coating* -where the permittivity and the permeability are bounded above and below inside the cloaked object- the tangential components of the electric and the magnetic field of these solutions have to vanish in the inside of the boundary of the cloaked object. Note that in this case in order to have a self-adjoint extension of the electromagnetic propagator inside the cloaked object we are only allowed to require that either the tangential component of $\mathbf{E}$ or the tangential component of $\mathbf{H}$ vanishes, but not both.

These boundary conditions are called *hidden boundary conditions* in [7] where also the case of the Helmholtz equation is considered. In the case of Maxwell's equations they propose two solutions to this issue. One of them is a lining, i.e., a physical material on the boundary of the cloaked object that enforces a particular boundary condition, for example, they propose a lining by a perfect electric conductor. Note that this raises now the question of what is the boundary condition between the lining and the cloaking metamaterial. In fact, we face the same problem as before, since we can always consider that the lining is part of the cloaked objects, and then, the question of what is the appropriate boundary condition remains. The second proposal of [7] is a *double coating* that corresponds to surrounding both the inner and the outersurface of the cloaked objects with appropriately matched metamaterials. As our permittivities and permeabilities inside K are allowed to vanish as they approach ∂K the *double coating* fits in our formalism.

In Theorem 5.1 of [7] cloaking is proven for all frequencies and active devices, with the *double coating*, with respect to the Cauchy data of the finite energy solutions that they define in Definition 4.1.

Remark that there is no real contradiction between our results and the ones of [7]. Our results imply that there is always a *hidden boundary condition* on the inside of the boundary of the cloaked objects, that is imposed upon us by the fundamental principle of the conservation of the energy of the electromagnetic waves, that implies that time evolution has to be given by a unitary group generated by a self-adjoint extension of the electromagnetic propagator, and this amounts to a boundary condition at the inside of the boundary of the cloaked objects. Note that we do not exclude here the possibility that in some cases the electromagnetic propagator of the cloaked objects could be essentially self-adjoint, and in this situation the dynamics inside the cloaked objects will be uniquely defined. In this case the *hidden boundary condition* will be uniquely determined by the boundary conditions satisfied by the functions in the domain of the unique self-adjoint realization of the electromagnetic propagator in the cloaked objects. Note, however, that we have proven that the cloaking outside is actually independent of the cloaked objects.

4 Cloaking as a Boundary Value Problem

It is a question of independent interest to consider cloaking as a boundary value problem for the Maxwell' system at a fixed frequency

$$\nabla \times \mathbf{E} \;=\; ik\mathbf{B},\; \nabla \times \mathbf{H} = -ik\mathbf{D},\; k \neq 0, \tag{4.1}$$

$$\nabla \cdot \mathbf{B} \;=\; 0, \nabla \cdot \mathbf{D} = 0. \tag{4.2}$$

As we have already shown, cloaking is independent of the cloaked object, and this means that we only have to consider these equation in Ω. The main question now is to decide what is an appropriate class solutions with locally finite energy. Our analysis of the self-adjoint extensions of the electromagnetic propagator shows that we have to take solutions that are locally in the domain of A_Ω, that is to say that they are given by (2.41)

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = U \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}$$

with $(\mathbf{E}_0, \mathbf{H}_0)^T$ locally in the domain of A_0, that is to say, $(\mathbf{E}_0, \mathbf{H}_0)^T$ are in the domain of A_0 when multiplied by any function in $\mathbf{C}_0^\infty(\mathbb{R}^3)$. It follows from (2.42) that the solutions with locally finite energy have to satisfy the boundary condition,

$$\mathbf{E} \times \mathbf{n} = 0, \mathbf{H} \times \mathbf{n} = 0, \text{ in } \partial K_+,$$

where ∂K_+ is the outside of the boundary of the cloaked object. Note that we define in the same way solutions with (locally) finite energy in a bounded subset of Ω. In [7] a different definition of solutions with (locally) finite energy is given in Definition 4.1.

5 Cloaking an Infinite Cylinder

We discuss now the case of an infinite cylinder. For simplicity we consider one cylinder centered at zero and with its axis the vertical line $L := (0, 0, x^3), x^3 \in \mathbb{R}$. Then,

$$K := \left\{ \mathbf{x} = (x^1, x^2, x^3) \in \mathbb{R}^3 : |x^1| + |x^2| \leq 1, x^3 \in \mathbb{R} \right\}, \ \Omega := \mathbb{R}^3 \setminus K. \tag{5.3}$$

The set Ω_0 is now given by,

$$\Omega_0 = \mathbb{R}^3 \setminus L. \tag{5.4}$$

Let us denote by $\underline{\mathbf{x}} := (x,^1, x^2)$ the vectors in the $x^1 - x^2$ plane and $\hat{\underline{\mathbf{x}}} := \underline{\mathbf{x}}/|\underline{\mathbf{x}}|$. The transformation (2.8) is replaced by

$$\mathbf{x} = \mathbf{x}(\mathbf{y}) = f(\mathbf{y}) := \begin{cases} \underline{\mathbf{x}} = (|\underline{\mathbf{y}}| + 1)\hat{\underline{\mathbf{y}}}, \\ x_3 = y_3, \end{cases} \tag{5.5}$$

for $0 < |\underline{\mathbf{y}}| \leq 1$. This transformation blows up the line L onto ∂K and that it sends $K \setminus L$ onto $K_2 \setminus K$ where

$$K_2 := \left\{ \mathbf{x} = (x^1, x^2, x^3) \in \mathbb{R}^3 : |x^1| + |x^2| \leq 2, x^3 \in \mathbb{R} \right\}.$$

For $\mathbf{y} \in \mathbb{R}^3 \setminus K$ we define the transformation to be the identity, $\mathbf{x} = \mathbf{y}$.

The Hilbert spaces of finite energy electromagnetic fields, the unitary operator U and a_0, A_0, a_Ω, a_K, a, are defined as in Section 2. Note that (2.36) holds in this case.

THEOREM 5.1. *The operator a_Ω is essentially self-adjoint, and its unique self-adjoint extension, A_Ω, satisfies*

$$A_\Omega = U\, A_0\, U^*. \tag{5.6}$$

Proof: The theorem is proven as Theorem 2.1 observing that $W^{1,2}(\mathbb{R}^2) = W_0^{1,2}(\mathbb{R}^2 \setminus 0)$ since $\{0\}$ has zero Sobolev one capacity in $\mathbb{R}^2$ [1, 11, 12].

□

Equations (2.41, 2.42) hold and the formulation of cloaking as a boundary value problem is the same as the one given in Section 4. Furthermore Theorem 3.1 is also true, the proof is the same.

We now consider the wave and the scattering operators. For simplicity we assume below that $\varepsilon_0^{\lambda\nu} = \tilde{\varepsilon}\, \delta^{\lambda\nu}, \mu_0^{\lambda\nu} = \tilde{\mu}\, \delta^{\lambda\nu}$.

LEMMA 5.2.

$$W_\pm = U P_{0\perp}. \tag{5.7}$$

Proof: The lemma is proven as in the proof of Lemma 2.2, but in (2.48, 2.49, 2.50, 2.51) we have to replace χ_{B_R}, by χ_{C_R} where, $C_R := \{\mathbf{y} \in \mathbb{R}^3 : |\underline{\mathbf{y}}| \leq R\}$ for R large enough. Now we can not prove (2.52, 2.53, 2.54) by compactness arguments because K is unbounded. Instead we use propagation estimates for A_0. The following results are well known. See for example [3, 27, 28, 29] where the general anisotropic case is considered. For any $\varphi \in \mathcal{H}_{0\perp}$,

$$e^{-itA_0}\varphi = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ik\cdot\mathbf{y}} \left(e^{-i\omega_+(k)t} P_+(k)\hat{\varphi}(k) + e^{-i\omega_-(k)t} P_-(k)\hat{\varphi}(k) \right) d^3k \tag{5.8}$$

where $\hat{\varphi}$ is the Fourier transform of φ, $\omega_\pm(k) = \pm|k|c$ with $c := (\tilde{\varepsilon}\tilde{\mu})^{-1/2}$, and $P_\pm(k)$ are projectors on $\mathbb{R}^3$ that are infinitely differentiable for $k \in \mathbb{R}^3 \setminus 0$. Suppose that $\hat{\varphi} \in \mathbf{C}_0^\infty(\mathbb{R}^3 \setminus L)$ and let O be a bounded open set such that $\overline{O} \subset \mathbb{R}^3 \setminus L$ and support $\varphi \subset O$. Denote

$$\hat{O} := \left\{ \frac{k}{|k|} : k \in O \right\}.$$

Then by the (non) stationary phase Theorem (see the Corollary to Theorem XI.14 of [23]), for any $n = 1, 2, \cdots$ there is a constant C_n such that

$$\left| \left(e^{-itA_0} \varphi \right) (\mathbf{y}) \right| \leq C_n \left(1 + |\mathbf{y}| + |t| \right)^{-n}, \ \pm \frac{\mathbf{y}}{ct} \notin \hat{O}. \tag{5.9}$$

We write,

$$\chi_{C_R} e^{-itA_0} \varphi = \phi_1 + \phi_2 \tag{5.10}$$

with

$$\phi_1 := \chi_{(\pm \mathbf{y}/(ct) \notin \hat{O})} \chi_{C_R} e^{-itA_0} \varphi \tag{5.11}$$

and

$$\phi_2 := \chi_{(\pm \mathbf{y}/(ct) \in \hat{O})} \chi_{C_R} e^{-itA_0} \varphi. \tag{5.12}$$

by (5.9)

$$\text{s-}\lim_{t \to \pm\infty} \phi_1 = 0. \tag{5.13}$$

Note, furthermore, that there is an $\epsilon > 0$ such that $|\underline{k}| \geq \epsilon$ for any $k \in \hat{O}$. Then, for any $\pm \frac{\mathbf{y}}{ct} \in \hat{O}$, $|\underline{\mathbf{y}}| \geq c|t|\epsilon$. It follows that there is a T such that

$$\phi_2 = 0, \ \text{for} |t| \geq T. \tag{5.14}$$

By (5.10, 5.13, 5.14)

$$\text{s-}\lim_{t \to \infty} \chi_{C_R} e^{-itA_0} \varphi = 0, \tag{5.15}$$

and (2.53, 2.54) follow. Note that $P_{0\perp}$ is not needed because $\varphi \in \mathcal{H}_{0\perp}$. By continuity this is true for all $\varphi \in \mathcal{H}_{0\perp}$. Then, (5.7) follows from (2.46, 2.54).

□

COROLLARY 5.3.

$$S = P_{0\perp}. \tag{5.16}$$

Proof: This is immediate from (5.7) because $U^* U = I$.

□

Let us denote by $S_\perp$ the restriction of S to $\mathcal{H}_{0\perp}$. $S_\perp$ is the physically relevant scattering operator that acts in the Hilbert space $\mathcal{H}_{0\perp}$ of finite energy fields that satisfy equations (2.2). We designate by $I_\perp$ the identity operator on $\mathcal{H}_{0\perp}$. We have that,

COROLLARY 5.4.

$$S_\perp = I_\perp. \tag{5.17}$$

Proof: This follows from Corollary 5.3.

□

Again, the fact that $S_\perp$ is the identity operator on $\mathcal{H}_{0\perp}$ means that there is perfect cloaking for all frequencies.

Observe that all the remarks about finite energy solutions, cloaking and *hidden boundary conditions* that we made in Sections 2, 3, and 4 remain true in the case of a cylinder. We do not repeat them here. In Theorem 7.1 of [7] cloaking is proven for all frequencies with respect to the Cauchy data of the finite energy solutions that they define in Definition 4.1 and furthermore, in Theorem 8.2, they prove cloaking for all frequencies with the SHS boundary condition with respect to the Cauchy data of the finite energy solutions that they define in Definition 8.1.

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