# Parametrized Post-Newtonian Expansion of Chern-Simons Gravity

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We investigate the weak-field, post-Newtonian expansion to the solution of the field equations in Chern-Simons gravity with a perfect fluid source. In particular, we study the mapping of this solution to the parameterized post-Newtonian formalism to 1 PN order in the metric. We find that the PPN parameters of Chern-Simons gravity are identical to those of general relativity, with the exception of the inclusion of a new term that is proportional to the Chern-Simons coupling parameter and the curl of the PPN vector potentials. We also find that the new term is naturally enhanced by the non-linearity of spacetime and we provide a physical interpretation for it. By mapping this correction to the gravito-electro-magnetic framework, we study the corrections that this new term introduces to the acceleration of point particles and the frame-dragging effect in gyroscopic precession. We find that the Chern-Simons correction to these classical predictions could be used by current and future experiments to place bounds on intrinsic parameters of Chern-Simons gravity and, thus, string theory.

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## I. INTRODUCTION

Tests of alternative theories of gravity that modify general relativity (GR) at a fundamental level are essential to the advancement of physics. One formalism that has had incredible success in this task is the parameterized post-Newtonian (PPN) framework [1, 2, 3, 4, 5, 6]. In this formalism, the metric of the alternative theory is solved for in the weak-field limit and its deviations from GR are expressed in terms of PPN parameters. Once a metric has been obtained, one can calculate predictions of the alternative theory, such as light deflection and the perihelion shift of Mercury, which shall depend on these PPN parameters. Therefore, experimental measurements of such physical effects directly lead to constraints on the parameters of the alternative theory. This framework, together with the relevant experiments, have already been successfully employed to constrain scalar-tensor theories (Brans-Dicke, Bekenstein) [7], vector-tensor theories (Will-Nordtvedt [8], Hellings-Nordtvedt [9]), bimetric theories (Rosen [10, 11]) and stratified theories (Ni [12]) (see [13] for definitions and an updated review.)

Only recently has this framework been used to study quantum gravitational and string-theoretical inspired ideas. On the string theoretical side, Kalyana [14] investigated the PPN parameters associated with the graviton-dilaton system in low-energy string theory. More recently, Ivashchuk, *et. al.* [15] studied PPN parameters in the context of general black holes and p-brane spherically symmetric solutions, while Bezerra, *et. al.* [16] considered domain wall spacetimes for low energy effective string theories and derived the corresponding PPN parameters for the metric of a wall. On the quantum gravitational side, Gleiser and Kozameh [17] and more recently Fan, *et. al.* [18] studied the possibility of testing gravitational birefringence induced by quantum gravity, which was proposed by Amelino-Camelia, *el. al.* [19] and Gambini and Pullin [20]. Other non-PPN proposals have been also put forth to test quantum gravity, for example through gravitational waves [21, 22, 23, 24, 25, 26, 27, 28], but we shall not discuss those tests here.

Chern-Simons (CS) gravity [29, 30] is one such extension of GR, where the gravitational action is modified by the addition of a parity-violating term. This extension is promising because it is required by all 4-dimensional compactifications of string theory [31] for mathematical consistency because it cancels the Green-Schwarz anomaly [32]. CS gravity, however, is not unique to string theory and in fact has its roots in the standard model, where it arises as a gravitational anomaly provided that there are more flavours of left handed leptons than right handed ones. Moreover the CS extension to GR can arise via the embedding of the three dimensional Chern-Simons topological current into a 4D space-time manifold, decsribed by Jackiw and Pi [30]

Chern-Simons gravity has been recently studied in the cosmological context. In particular, this framework was used to shed light on the anisotropies of the cosmic microwave background (CMB) [33, 34, 35] and the leptogenesis problem [34, 36, 37]. Parity violation has also been shown to produce birefringent gravitational waves [28, 29], where different polarizations modes acquire varying amplitudes. These modes obey different propagation equations because the imaginary sector of the classical dispersion relation is CS corrected. Different from [20], in CS birefringence the velocity of the gravitational wave remains that of light.

In this paper we study CS gravity in the PPN framework, extending the analysis of [38] and providing some missing details. In particular, we shall consider the effect of the CS correction to the gravitational field of, for instance, a pulsar, a binary system or a star in the weak-field limit. These corrections are obtained by solving the modified field equations in the weak-field limit for post-Newtonian (PN) sources, defined as those that are weakly-gravitating and slowly-moving [39]. Such an expansion requires the calculation of the Ricci and Cotton tensors to second order in the metric perturbation. We then find that CS gravity leads to the same gravitational field as that of classical GR and, thus, the same PPN parameters, except for the inclusion of a new term in the vectorial sector of the metric, namely

$$g_{0i}^{(CS)} = 2\dot{f}\,(\nabla \times V)_i\,, \tag{1}$$

where $\dot{f}$ acts as a coupling parameter of CS theory and $V_i$ is a PPN potential. We also show that this solution can be alternatively obtained by finding a formal solution to the modified field equations and performing a PN expansion, as is done in PN theory. The full solution is further shown to satisfy the additional CS constraint, which leads to equations of motion given only by the divergence of the stress-energy tensor.

The CS correction to the metric found here leads to an interesting interpretation of CS gravity and forces us to consider a new type of coupling. The interpretation consists of thinking of the field that sources the CS correction as a fluid that permeates all of spacetime. Then the CS correction in the metric is due to the "dragging" of such a fluid by the motion of the source. Until now, couplings of the CS correction to the angular momentum of the source had been neglected by the string theory community. Similarly, curl-type terms had also been considered unnecessary in the traditional PPN framework, since previous alternative gravity theories had not required it. As we shall show, in CS gravity and thus in string theory, such a coupling is naturally occurring. Therefore, a proper PPN mapping requires the introduction of a new curl-type term with a corresponding new PPN parameter of the type of Eq. (1).

A modification to the gravitational field leads naturally to corrections of the standard predictions of GR. In order to illustrate such a correction, we consider the CS term in the gravito-electro-magnetic analogy [40, 41], where we find that the CS correction accounts for a modification of gravitomagnetism. Furthermore, we calculate the modification to the acceleration of point particles and the frame dragging effect in the precession of gyroscopes. We find that these corrections are given by

$$\begin{aligned}
\delta a^i &= -\frac{3}{2}\frac{\dot{f}}{r}\frac{G\,m}{c^2\,r^2}\left(\frac{v}{c}\cdot n\right)\left(\frac{v}{c}\times n\right)^i, \\
\delta\Omega^i &= -\frac{\dot{f}}{r}\frac{G\,m}{c^3\,r^2}\left[3\left(\frac{v}{c}\cdot n\right)n^i - \frac{v^i}{c}\right],
\end{aligned} \tag{2}$$

where $m$ and $v$ are the mass and velocities of the source, while $r$ is the distance to the source and $n^i = x^i/r$ is a unit vector, with $\cdot$ and $\times$ the flat-space scalar and cross products. Both corrections are found to be naturally enhanced in regions of high spacetime curvature. We then conclude that experiments that measure the gravitomagnetic sector of the metric either in the weak-field (such as Gravity Probe B [42]) and particularly in the non-linear regime, will lead to a direct constraint on the CS coupling parameter $\dot{f}$. In this paper we develop the details of how to calculate these corrections, while the specifics of how to actually impose a constraint, which depend on the experimental setup, are beyond the scope of this paper.

The remainder of this paper deals with the details of the calculations discussed in the previous paragraphs. We have divided the paper as follows: Sec. II describes the basics of the PPN framework; Sec. III discusses CS modified gravity, the modified field equations and computes a formal solution; Sec. IV expands the field equations to second order in the metric perturbation; Sec. V iteratively solves the field equations in the PN approximation and finds the PPN parameters of CS gravity; Sec. VI discusses the correction to the acceleration of point particles and the frame dragging effect; Sec. VII concludes and points to future research.

The conventions that we use throughout this work are the following: Greek letters represent spacetime indices, while Latin letters stand for spatial indices only; semicolons stand for covariant derivatives, while colons stand for partial derivatives; overhead dots stand for derivatives with respects to time. We denote uncontrolled remainders with the symbol $\mathcal{O}(A)$, which stands for terms of order $A$. We also use the Einstein summation convention unless otherwise specified. Finally, we use geometrized units, where $G = c = 1$, and the metric signature $(-,+,+,+)$.

## II. THE ABC OF PPN

In this section we summarize the basics of the PPN framework, following [6]. This framework was first developed by Eddington, Robertson and Schiff [1, 6], but it came to maturity through the seminal papers of Nordtvedt and Will [2, 3, 4, 5]. In this section, we describe the latter formulation, since it is the most widely used in experimental tests of gravitational theories.

The goal of the PPN formalism is to allow for comparisons of different metric theories of gravity with each other and with experiment. Such comparisons become manageable through a slow-motion, weak-field expansion of the metric and the equations of motion, the so-called PN expansion. When such an expansion is carried out to sufficiently high but finite order, the resultant solution is an accurate approximation to the exact solution in most of the spacetime. This approximation, however, does break down for systems that are not slowly-moving, such as merging binary systems, or weakly gravitating, such as near the apparent horizons of black hole binaries. Nonetheless, as far as solar system tests are concerned, the PN expansion is not only valid but also highly accurate.

The PPN framework employs an order counting-scheme that is similar to that used in multiple-scale analysis [43, 44, 45, 46]. The symbol $\mathcal{O}(A)$ stands for terms of order $\epsilon^A$, where $\epsilon \ll 1$ is a PN expansion parameter. For convenience, it is customary to associate this parameter with the orbital velocity of the system $v/c = \mathcal{O}(1)$, which embodies the slow-motion approximation. By the Virial theorem, this velocity is related to the Newtonian potential $U$ via $U \sim v^2$, which then implies that $U = \mathcal{O}(2)$ and embodies the weak-gravity approximation. These expansions can be thought of as two independent series: one in inverse powers of the speed of light $c$ and the other in positive powers of Newton's gravitational constant.

Other quantities, such as matter densities and derivatives, can and should also be classified within this order-counting scheme. Matter density $\rho$, pressure $p$ and specific energy density $\Pi$, however, are slightly more complicated to classify because they are not dimensionless. Dimensionlessness can be obtained by comparing the pressure and the energy density to the matter density, which we assume is the largest component of the stress-energy tensor, namely $p/\rho \sim \Pi/\rho = \mathcal{O}(2)$. Derivatives can also be classified in this fashion, where we find that $\partial_t/\partial_x = \mathcal{O}(1)$. Such a relation can be derived by noting that $\partial_t \sim v^i \nabla_i$, which comes from the Euler equations of hydrodynamics to Newtonian order.

With such an order-counting scheme developed, it is instructive to study the action of a single neutral particle. The Lagrangian of this system is given by

$$\begin{aligned} L &= \left(g_{\mu\nu} u^\mu u^\nu\right)^{1/2}, \\ &= \left(-g_{00} - 2 g_{0i} v^i - g_{ij} v^i v^j\right)^{1/2} \end{aligned} \tag{3}$$

where $u^\mu = dx^\mu/dt = (1, v^i)$ is the 4-velocity of the particle and $v^i$ is its 3-velocity. From Eq. (3), note that knowledge of $L$ to $\mathcal{O}(A)$ implies knowledge of $g_{00}$ to $\mathcal{O}(A)$, $g_{0i}$ to $\mathcal{O}(A-1)$ and $g_{ij}$ to $\mathcal{O}(A-2)$. Therefore, since the Lagrangian is already known to $\mathcal{O}(2)$ (the Newtonian solution), the first PN correction to the equations of motion requires $g_{00}$ to $\mathcal{O}(4)$, $g_{0i}$ to $\mathcal{O}(3)$ and $g_{ij}$ to $\mathcal{O}(2)$. Such order counting is the reason for calculating different sectors of the metric perturbation to different PN orders.

A PPN analysis is usually performed in a particular background, which defines a particular coordinate system, and in an specific gauge, called the standard PPN gauge. The background is usually taken to be Minkowski because for solar system experiments deviations due to cosmological effects are negligible and can, in principle, be treated as adiabatic corrections. Moreover, one usually chooses a standard PPN frame, whose outer regions are at rest with respect to the rest frame of the universe. Such a frame, for example, forces the spatial sector of the metric to be diagonal and isotropic [6]. The gauge employed is very similar to the PN expansion of the Lorentz gauge of linearized gravitational wave theory. The differences between the standard PPN and Lorentz gauge are of $\mathcal{O}(3)$ and they allow for the presence of certain PPN potentials in the vectorial sector of the metric perturbation.

The last ingredient in the PPN recipe is the choice of a stress-energy tensor. The standard choice is that of a perfect fluid, given by

$$T^{\mu\nu} = (\rho + \rho\Pi + p)\, u^\mu u^\nu + p g^{\mu\nu}. \tag{4}$$

Such a stress-energy density suffices to obtain the PN expansion of the gravitational field outside a fluid body, like the Sun, or of compact binary system. One can show that the internal structure of the fluid bodies can be neglected to 1 PN order by the effacement principle [39] in GR. Such effacement principle might actually not hold in modified field theories, but we shall study this subject elsewhere [47].

With all these machinery, on can write down a super-metric [6], namely

$$\begin{aligned} g_{00} &= -1 + 2U - 2\beta U^2 - 2\xi \Phi_W + (2\gamma + 2 + \alpha_3 + \zeta_1 \\ &- 2\xi)\, \Phi_1 + 2\,(3\gamma - 2\beta + 1 + \zeta_2 + \xi)\, \Phi_2 \\ &+ 2\,(1 + \zeta_3)\, \Phi_3 + 2\,(3\gamma + 3\zeta_4 - 2\xi)\, \Phi_4 - (\zeta_1 - 2\xi)\, \mathcal{A}, \\ g_{0i} &= -\frac{1}{2}\,(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)\, V_i \\ &- \frac{1}{2}\,(1 + \alpha_2 - \zeta_1 + 2\xi)\, W_i, \\ g_{ij} &= (1 + 2\gamma U)\, \delta_{ij}, \end{aligned} \tag{5}$$

where $\delta_{ij}$ is the Kronecker delta and where the PPN potentials $(U, \Phi_W, \Phi_1, \Phi_2, \Phi_3, \Phi_4, \mathcal{A}, V_i, W_i)$ are defined in Appendix A. Equation (5) describes a super-metric theory of gravity, because it reduces to different metric theories, such as GR or other alternative theories [6], through the appropriate choice of PPN parameters $(\gamma, \beta, \xi, \alpha_1, \alpha_2, \alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4)$. One could obtain a more general form of the PPN metric by performing a post-Galilean transformation on Eq. (5), but such a procedure shall not be necessary in this paper.

The super-metric of Eq. (5) is parameterized in terms of a specific number of PPN potentials, where one usually employs certain criteria to narrow the space of possible potentials to consider. Some of these restriction include the following: the potentials tend to zero as an inverse power of the distance to the source; the origin of the coordinate system is chosen to coincide with the source, such that the metric does not contain constant terms; and the metric perturbations $h_{00}$, $h_{0i}$ and $h_{ij}$ transform as a scalar, vector and tensor. The above restrictions are reasonable, but, in general, an additional subjective condition is usually imposed that is based purely on simplicity: the metric perturbations are not generated by gradients or curls of velocity vectors or other generalized vector functions. As of yet, no reason had arisen for relaxing such a condition, but as we shall see in this paper, such terms are indeed needed for CS modified theories.

What is the physical meaning of all these parameters? One can understand what these parameters mean by calculating the generalized geodesic equations of motion and conservation laws [6]. For example, the parameter $\gamma$ measures how much space-curvature is produced by a unit rest mass, while the parameter $\beta$ determines how much "non-linearity" is there in the superposition law of gravity. Similarly, the parameter $\xi$ determines whether there are preferred-location effects, while $\alpha_i$ represent preferred-frame effects. Finally, the parameters $\zeta_i$ measure the amount of violation of conservation of total momentum. In terms of conservation laws, one can interpret these parameters as measuring whether a theory is fully conservative, with linear and angular momentum conserved ($\zeta_i$ and $\alpha_i$ vanish), semi-conservative, with linear momentum conserved ($\zeta_i$ and $\alpha_3$ vanish), or non-conservative, where only the energy is conserved through lowest Newtonian order. One can verify that in GR, $\gamma = \beta = 1$ and all other parameters vanish, which implies that there are no preferred-location or frame effects and that the theory is fully conservative.

A PPN analysis of an alternative theory of gravity then reduces to mapping its solutions to Eq. (5) and then determining the PPN parameters in terms of intrinsic parameters of the theory. The procedure is simply as follows: expand the modified field equations in the metric perturbation and in the PN approximation; iteratively solve for the metric perturbation to $\mathcal{O}(4)$ in $h_{00}$, to $\mathcal{O}(3)$ in $h_{0i}$ and to $\mathcal{O}(2)$ in $h_{ij}$; compare the solution to the PPN metric of Eq. (5) and read off the PPN parameters of the alternative theory. We shall employ this procedure in Sec. V to obtain the PPN parameters of CS gravity.

## III. CS GRAVITY IN A NUTSHELL

In this section, we describe the basics of CS gravity, following mainly [29, 30]. In the standard CS formalism, GR is modified by adding a new term to the gravitational action. This term is given by [30]

$$S_{CS} = \frac{m_{pl}^2}{64\pi} \int d^4x f \ ({}^\star R\, R)\,, \tag{6}$$

where $m_{pl}$ is the Planck mass, $f$ is a prescribed external quantity with units of squared mass (or squared length in geometrized units), $R$ is the Ricci scalar and the star stands the dual operation, such that

$$R^\star R = \frac{1}{2} R_{\alpha\beta\gamma\delta} \epsilon^{\alpha\beta\mu\nu} R^{\gamma\delta}{}_{\mu\nu}, \tag{7}$$

with $\epsilon_{\mu\nu\delta\gamma}$ the totally-antisymmetric Levi-Civita tensor and $R_{\mu\nu\delta\gamma}$ the Riemann tensor.

Such a correction to the gravitational action is interesting because of the unavoidable parity violation that is introduced. Such parity violation is inspired from CP violation in the standard model, where such corrections act as anomaly-canceling terms. A similar scenario occurs in string theory, where the Green-Schwarz anomaly is canceled by precisely such a CS correction [32], although CS gravity is not exclusively tied to string theory. Parity violation in CS gravity inexorably leads to birefringence in gravitational propagation, where here we mean that different polarization modes obey different propagation equations but travel at the same speed, that of light [29, 30, 36, 47]. If CS gravity were to lead to polarization modes that travel at different speeds, then one could use recently proposed experiments [17] to test this effect, but such is not the case in CS gravity. Birefringent gravitational waves, and thus CS gravity, have been proposed as possible explanations to the cosmic-microwave-background (CMB) anisotropies [36], as well as the baryogenesis problem during the inflationary epoch [33].

The magnitude of the CS correction is controlled by the externally-prescribed quantity $f$, which depends on the specific theory under consideration. When we consider CS gravity as an effective quantum theory, then the correction is suppressed by some mass scale $M$, which could be the electro-weak scale or some other scale, since it is unconstrained. In the context of string theory, the quantity $f$ has been calculated only in conservative scenarios, where it was found to be suppressed by the Planck mass. In other scenarios, however, enhancements have been proposed, such as in cosmologies where the string coupling vanishes at late times [48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58], or where the field that generates $f$ couples to spacetime regions with large curvature [59, 60] or stress-energy density [28, 47]. For simplicity, we here assume that this quantity is spatially homogeneous and its magnitude is small but non-negligible, so that we work to first order in the string-theoretical correction. Therefore, we treat $\dot{f}$ as an independent perturbation parameter, [70] unrelated to $\epsilon$, the PN perturbation parameter.

The field equations of CS modified gravity can be obtained by varying the action with respect to the metric. Doing so, one obtains

$$G_{\mu\nu} + C_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{8}$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is a stress-energy tensor and $C_{\mu\nu}$ is the Cotton tensor. The latter tensor is defined via

$$C_{\mu\nu} = -\frac{1}{\sqrt{-g}} \left[ f_{,\sigma} \epsilon^{\sigma\alpha\beta}{}_{(\mu} D_\alpha R_{\nu)\beta} + (D_\sigma f_{,\tau}) \ {}^\star R^\tau{}_{(\mu}{}^\sigma{}_{\nu)} \right], \tag{9}$$

where parenthesis stand for symmetrization, $g$ is the determinant of the metric, $D_a$ stands for covariant differentiation and colon subscripts stand for partial differentiation.

Formally, the introduction of such a modification to the field equations leads to a new constraint, which is compensated by the introduction of the new scalar field degree of freedom $f$. This constraint originates by requiring that the divergence of the field equations vanish, namely

$$D^\mu C_{\mu\nu} = \frac{1}{8\sqrt{-g}} D_\nu f \ ({}^\star RR) = 0, \tag{10}$$

where the divergence of the Einstein tensor vanished by the Bianchi identities. If this constraint is satisfied, then the equations of motion for the stress-energy $D_\mu T^{\mu\nu}$ are unaffected by CS gravity. A common source of confusion is that Eq. (10) is sometimes interpreted as requiring that $R^\star R$ also vanish, which would then force the correction to the action to vanish. However, this is not the case because, in general, $f$ is an exact form ($d^2 f = 0$) and, thus, Eq. (10) only implies an additional constraint that forces all *solutions* to the field equations to have a vanishing $R^\star R$.

The previous success of CS gravity in proposing plausible explanations to important cosmological problems prompts us to consider this extension of GR in the weak-field regime. For this purpose, it is convenient to rewrite the field equations in trace-reversed form, since this form is most amenable to a PN expansion. Doing so, we find,

$$R_{\mu\nu} + C_{\mu\nu} = 8\pi \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \tag{11}$$

where the trace of the Cotton tensor vanishes identically and $T = g_{\mu\nu} T^{\mu\nu}$ is the four dimensional trace of the stress-energy tensor. To linear order, the Ricci and Cotton tensors are given by [30]

$$\begin{aligned} R_{\mu\nu} &= -\frac{1}{2} \Box h_{\mu\nu} + \mathcal{O}(h)^2, \\ C_{\mu\nu} &= -\frac{\dot{f}}{2} \tilde{\epsilon}^{0\alpha\beta}{}_{(\mu} \Box_\eta h_{\nu)\beta,\alpha} + \mathcal{O}(h)^2, \end{aligned} \tag{12}$$

where $\tilde{\epsilon}^{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol, with convention $\tilde{\epsilon}^{0123} = +1$, and $\Box_\eta = -\partial_t^2 + \eta^{ij}\partial_i\partial_j$ is the flat space D'Alambertian, with $\eta_{\mu\nu}$ the Minkowski metric. In Eq. (12), we have employed the Lorentz gauge condition $h_{\mu\alpha,}{}^{\alpha} = h_{,\mu}/2$, where $h = g^{\mu\nu} h_{\mu\nu}$ is the four dimensional trace of the metric perturbation.

The Cotton tensor changes the characteristic behavior of the Einstein equations by forcing them to become third order instead of second order. Third-order partial differential equations are common in boundary layer theory [43]. However, in CS gravity, the third-order contributions are multiplied by a factor of $f$ and we shall treat this function as a small independent expansion parameter. Therefore, the change in characteristics in the modified field equations can also be treated perturbatively, which is justified because eventhough $\dot{f}$ might be enhanced by standard model currents, extra dimensions or a vanishing string coupling, it must still carry some type of mass suppression.

The trace-reversed form of the field equations is useful because it allows us to immediately find a formal solution. Inverting the D'Alambertian operator we obtain

$$\mathcal{H}_{\mu\nu} = -16\pi\, \Box_\eta^{-1} \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) + \mathcal{O}(h)^2, \tag{13}$$

where we have defined an effective metric perturbation as

$$\mathcal{H}_{\mu\nu} \equiv h_{\mu\nu} + \dot{f} \tilde{\epsilon}^{0\alpha\beta}{}_{(\mu} h_{\nu)\beta,\alpha}. \tag{14}$$

Note that this formal solution is identical to the formal PN solution to the field equations in the limit $\dot{f} \to 0$. Also note that the second term in Eq. (14) is in essence a curl operator acting on the metric. This antisymmetric operator naturally forces the trace of the CS correction to vanish, as well as the 00 component and the symmetric spatial part.

From the formal solution to the modified field equations, we immediately identify the *only two possible non-zero CS contributions*: a coupling to the vector component of the metric $h_{0i}$; and coupling to the transverse-traceless part of the spatial metric $h_{ij}^{TT}$. The latter has already been studied in the gravitational wave context [29, 30, 47] and it vanishes identically if we require the spatial sector of the metric perturbation to be conformally flat. The former coupling is a new curl-type contribution to the metric perturbation that, to our knowledge, had so far been neglected both by the string theory and PPN communities. In fact, as we shall see in later sections, terms of this type will force us to introduce a new PPN parameter that is proportional to the curl of certain PPN potentials.

Let us conclude this section by pushing the formal solution to the modified field equations further to obtain a formal solution in terms of the actual metric perturbation $h_{\mu\nu}$. Combining Eqs. (13) and (14) we arrive at the differential equation

$$h_{\mu\nu} + \dot{f} \tilde{\epsilon}^{0\alpha\beta}{}_{(\mu} h_{\nu)\beta,\alpha} = -16\pi\, \Box_\eta^{-1} \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) + \mathcal{O}(h)^2. \tag{15}$$

Since we are searching for perturbations about the general relativistic solution, we shall make the ansatz

$$h_{\mu\nu} = h_{\mu\nu}^{(GR)} + \dot{f} \zeta_{\mu\nu} + \mathcal{O}(h)^2, \tag{16}$$

where $h_{\mu\nu}^{(GR)}$ is the solution predicted by general relativity

$$h_{\mu\nu}^{(GR)} \equiv -16\pi\, \Box_\eta^{-1} \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \tag{17}$$

and where $\zeta_{\mu\nu}$ is an unknown function we are solving for. Inserting this ansatz into Eq. (15) we obtain

$$\zeta_{\mu\nu} + \dot{f} \tilde{\epsilon}^{0\alpha\beta}{}_{(\mu} \zeta_{\nu)\beta,\alpha} = 16\pi \tilde{\epsilon}^{0\alpha\beta}{}_{(\mu} \partial_\alpha \Box_\eta^{-1} \left( T_{\nu)\beta} - \frac{1}{2} g_{\nu)\beta} T \right). \tag{18}$$

We shall neglect the second term on the left-hand side because it would produce a second order correction. Such conclusion was also reached when studying parity violation in GR to explain certain features of the CMB [35]. We thus obtain the formal solution

$$\zeta_{\mu\nu} = 16\pi \tilde{\epsilon}^{0\alpha\beta}{}_{(\mu} \partial_\alpha \Box_\eta^{-1} \left( T_{\nu)\beta} - \frac{1}{2} g_{\nu)\beta} T \right) \tag{19}$$

and the actual metric perturbation to linear order becomes

$$\begin{aligned} h_{\mu\nu} &= -16\pi\, \Box_\eta^{-1} \left( T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T \right) \\ &+ 16\pi \dot{f} \tilde{\epsilon}^{k\ell i} \Box_\eta^{-1} \left( \delta_{i(\mu} T_{\nu)\ell,k} - \frac{1}{2} \delta_{i(\mu} \eta_{\nu)\ell} T_{,k} \right) + \mathcal{O}(h)^2, \end{aligned} \tag{20}$$

where we have used some properties of the Levi-Civita symbol to simplify this expression. The procedure presented here is general enough that it can be directly applied to study CS gravity in the PPN framework, as well as possibly find PN solutions to CS gravity.

## IV. PN EXPANSION OF CS GRAVITY

In this section, we perform a PN expansion of the field equations and obtain a solution in the form of a PN series. This solution then allows us to read off the PPN parameters by comparing it to the standard PPN supermetric [Eq. (5)]. In this section we shall follow closely the methods of [6] and [61] and indices shall be manipulated with the Minkowski metric, unless otherwise specified.

Let us begin by expanding the field equations to second order in the metric perturbation. Doing so we find that the Ricci and Cotton tensors are given to second order by

$$\begin{aligned} R_{\mu\nu} &= -\frac{1}{2}\left[\Box_\eta h_{\mu\nu} - 2h_{\sigma(\mu,\nu)}{}^{\sigma} + h_{,\mu\nu}\right] - \frac{1}{2}\left[h^{\rho\lambda}\left(2h_{\rho(\mu,\nu)\lambda} - h_{\mu\nu,\rho\lambda} - h_{\rho\lambda,\mu\nu}\right) - \frac{1}{2}h^{\rho\lambda}{}_{,\mu}h_{\rho\lambda,\nu} + h^{\lambda}{}_{\mu,\rho}h^{\rho}{}_{\nu,\lambda} \right. \quad (21)\\ &- \left. h^{\rho}{}_{\mu,\lambda}h_{\rho\nu,}{}^{\lambda} + \frac{1}{2}\left(h^{,\lambda} - 2h^{\lambda\rho}{}_{,\rho}\right)\left(h_{\mu\nu,\lambda} - 2h_{\lambda(\mu,\nu)}\right)\right] + \mathcal{O}(h)^3, \\ C_{\mu\nu} &= -\frac{\dot{f}}{2}\tilde{\epsilon}^{0\alpha\beta}{}_{(\mu}\left(\Box_\eta h_{\nu)\beta,\alpha} - h_{\sigma\beta,\alpha\nu)}{}^{\sigma}\right) - \frac{\dot{f}}{2}\tilde{\epsilon}^{0\alpha\beta}{}_{(\mu}\left[h\left(\Box_\eta h_{\nu)\beta,\alpha} - h_{\sigma\beta,\alpha\nu)}{}^{\sigma}\right) + \frac{1}{2}\left(2h_{\nu)(\lambda,\alpha)} - h_{\lambda\alpha,\nu)}\right)\right. \\ &\times \left.\left(\Box_\eta h^{\lambda}{}_{\beta} - 2h_{\sigma}{}^{(\lambda}{}_{,\beta)}{}^{\sigma} + h_{,\beta}{}^{\lambda}\right) - 2\hat{Q}R_{\nu)\beta,\alpha}\right] - \frac{\dot{f}}{4}\tilde{\epsilon}^{\sigma\alpha\beta}{}_{(\mu}\left(2h^{0}{}_{(\sigma,\tau)} - h_{\sigma\tau,}{}^{0}\right)\left(h^{\tau}{}_{[\beta,\alpha]\nu)} - h_{\nu)[\beta,\alpha]}{}^{\tau}\right) \\ &- \frac{\dot{f}}{2}h_{\mu\lambda}\tilde{\epsilon}^{0\alpha\beta(\lambda}\left(\Box_\eta h_{\nu)\beta,\alpha} - h_{\sigma\beta,\alpha\nu)}{}^{\sigma}\right) - \frac{\dot{f}}{2}\tilde{\epsilon}^{0\alpha\beta(\mu}\left(\Box_\eta h^{\lambda)}{}_{\beta,\alpha} - h_{\sigma\beta,\alpha}{}^{\sigma\lambda)}\right)h_{\nu\lambda} + \mathcal{O}(h)^3. \quad (22) \end{aligned}$$

where index contraction is carried out with the Minkowski metric and where we have not assumed any gauge condition. The operator $\hat{Q}(\cdot)$ takes the quadratic part of its operand [of $\mathcal{O}(h)^2$] and it is explained in more detail in Appendix B, where the derivation of the expansion of the Cotton tensor is presented in more detail. In this derivation, we have used the definition of the Levi-Civita tensor

$$\begin{aligned} \epsilon_{\alpha\beta\gamma\delta} &= (-g)^{1/2}\tilde{\epsilon}_{\alpha\beta\gamma\delta} = \left(1 - \frac{1}{2}h\right)\tilde{\epsilon}_{\alpha\beta\gamma\delta} + \mathcal{O}(h)^2, \quad (23)\\ \epsilon^{\alpha\beta\gamma\delta} &= -(-g)^{-1/2}\tilde{\epsilon}^{\alpha\beta\gamma\delta} = -\left(1 + \frac{1}{2}h\right)\tilde{\epsilon}^{\alpha\beta\gamma\delta} + \mathcal{O}(h)^2. \end{aligned}$$

Note that the PN expanded version of the linearized Ricci tensor of Eq. (21) agrees with previous results [6]. Also note that if the Lorentz condition is enforced, several terms in both expressions vanish identically and the Cotton tensor to first order reduces to Eq. (12), which agrees with previous results [30].

Let us now specialize the analysis to the standard PPN gauge. For this purpose, we shall impose the following gauge conditions

$$\begin{aligned} h_{jk,}{}^{k} - \frac{1}{2}h_{,j} &= \mathcal{O}(4), \\ h_{0k,}{}^{k} - \frac{1}{2}h^{k}{}_{k,0} &= \mathcal{O}(5), \quad (24) \end{aligned}$$

where $h^k{}_k$ is the spatial trace of the metric perturbation. Note that the first equation is the PN expansion of one of the Lorentz gauge conditions, while the second equation is not. This is the reason why the previous equations where not expanded in the Lorentz gauge. Nonetheless, such a gauge condition does not uniquely fix the coordinate system, since we can still perform an infinitesimal gauge transformation that leaves the modified field equations invariant. One can show that the Lorentz and PPN gauge are related to each other by such a gauge transformation. In the PPN gauge, then, the Ricci tensor takes the usual form

$$\begin{aligned} R_{00} &= -\frac{1}{2}\nabla^2 h_{00} - \frac{1}{2}h_{00,i}h_{00,}{}^{i} + \frac{1}{2}h^{ij}h_{00,ij} + \mathcal{O}(6), \\ R_{0i} &= -\frac{1}{2}\nabla^2 h_{0i} - \frac{1}{4}h_{00,0i} + \mathcal{O}(5), \\ R_{ij} &= -\frac{1}{2}\nabla^2 h_{ij} + \mathcal{O}(4), \quad (25) \end{aligned}$$

which agrees with previous results [6], while the Cotton tensor reduces to

$$\begin{aligned} C_{00} &= \mathcal{O}(6), \\ C_{0i} &= -\frac{1}{4}\dot{f}\tilde{\epsilon}^{0kl}{}_{i}\nabla^2 h_{0l,k} + \mathcal{O}(5), \\ C_{ij} &= -\frac{1}{2}\dot{f}\tilde{\epsilon}^{0kl}{}_{(i}\nabla^2 h_{j)l,k} + \mathcal{O}(4), \quad (26) \end{aligned}$$

where $\nabla = \eta^{ij}\partial_i\partial_j$ is the Laplacian of flat space [see Appendix B for the derivation of Eq. (26).] Note again the explicit appearance of two coupling terms of the Cotton tensor to the metric perturbation: one to the transverse-traceless part of the spatial metric and the other to the vector metric perturbation. The PN expansions of the linearized Ricci and Cotton tensor then allow us to solve the modified field equations in the PPN framework.

## V. PPN SOLUTION OF CS GRAVITY

In this section we shall proceed to systematically solve the modified field equation following the standard PPN iterative procedure [6]. We shall begin with the 00 and $ij$ components of the metric to $\mathcal{O}(2)$, and then proceed with the $0i$ components to $\mathcal{O}(3)$ and the 00 component to $\mathcal{O}(4)$. Once all these components have been solved for in terms of PPN potentials, we shall be able to read off the PPN parameters adequate to CS gravity.

### A. $h_{00}$ and $h_{ij}$ to $\mathcal{O}(2)$

Let us begin with the modified field equations for the scalar sector of the metric perturbation. These equations are given to $\mathcal{O}(2)$ by

$$\nabla^2 h_{00} = -8\pi\rho, \tag{27}$$

because $T = -\rho$. Eq. (27) is the Poisson equation, whose solution in terms of PPN potentials is

$$h_{00} = 2U + \mathcal{O}(4). \tag{28}$$

Let us now proceed with the solution to the field equation for the spatial sector of the metric perturbation. This equation to $\mathcal{O}(2)$ is given by

$$\nabla^2 h_{ij} + \dot{f}\tilde{\epsilon}^{0kl}{}_{(i}\nabla^2 h_{j)l,k} = -8\pi\rho\delta_{ij}, \tag{29}$$

where note that this is the first appearance of a Cotton tensor contribution. Since the Levi-Civita symbol is a constant and $\dot{f}$ is only time-dependent, we can factor out the Laplacian and rewrite this equation in terms of the effective metric $\mathcal{H}_{ij}$ as

$$\nabla^2 \mathcal{H}_{ij} = -8\pi\rho\delta_{ij}, \tag{30}$$

where, as defined in Sec. III,

$$\mathcal{H}_{ij} = h_{ij} + \dot{f}\tilde{\epsilon}^{0kl}{}_{(i}h_{j)l,k}. \tag{31}$$

The solution of Eq. (30) can be immediately found in terms of PPN potentials as

$$\mathcal{H}_{ij} = 2U\delta_{ij} + \mathcal{O}(4), \tag{32}$$

which is nothing but Eq. (13). Recall, however, that in Sec. III we explicitly used the Lorentz gauge to simplify the field equations, whereas here we are using the PPN gauge. The reason why the solutions are the same is that the PPN and Lorentz gauge are indistinguishable to this order.

Once the effective metric has been solved for, we can obtain the actual metric perturbation following the procedure described in Sec. III. Combining Eq. (31) with Eq. (32), we arrive at the following differential equation

$$h_{ij} + \dot{f}\tilde{\epsilon}^{0kl}{}_{(i}h_{j)l,k} = 2U\delta_{ij}. \tag{33}$$

We look for solutions whose zeroth-order result is that predicted by GR and the CS term is a perturbative correction, namely

$$h_{ij} = 2U\delta_{ij} + \dot{f}\zeta_{ij}, \tag{34}$$

where $\zeta$ is assumed to be of $\mathcal{O}(\dot{f})^0$. Inserting this ansatz into Eq. (33) we arrive at

$$\zeta_{ij} + \dot{f}\tilde{\epsilon}^{0kl}{}_{(i}\zeta_{j)l,k} = 0, \tag{35}$$

where the contraction of the Levi-Civita symbol and the Kronecker delta vanished. As in Sec. III, note that the second term on the left hand side is a second order correction and can thus be neglected to discover that $\zeta_{ij}$ vanishes to this order.

The spatial metric perturbation to $\mathcal{O}(2)$ is then simply given by the GR prediction without any CS correction, namely

$$h_{ij} = 2U\delta_{ij} + \mathcal{O}(4). \tag{36}$$

Physically, the reason why the spatial metric is unaffected by the CS correction is related to the use of a perfect fluid stress-energy tensor, which, together with the PPN gauge condition, forces the metric to be spatially conformally flat. In fact, if the spatial metric were not flat, then the spatial sector of the metric perturbation would be corrected by the CS term. Such would be the case if we had pursued a solution to 2 PN order, where the Landau-Lifshitz pseudo-tensor sources a non-conformal correction to the spatial metric [39], or if we had searched for gravitational wave solutions, whose stress-energy tensor vanishes [29, 36]. In fact, one can check that, in such a scenario, Eq. (30) reduces to that found by [29, 30, 36, 47] as $\rho \to 0$. We have then found that the weak-field expansion of the gravitational field outside a fluid body, like the Sun or a compact binary, is unaffected by the CS correction to $\mathcal{O}(2)$.

### B. $h_{0i}$ to $\mathcal{O}(3)$

Let us now look for solutions to the field equations for the vector sector of the metric perturbation. The field equations to $\mathcal{O}(3)$ become

$$\nabla^2 h_{0i} + \frac{1}{2}h_{00,0i} + \frac{1}{2}\dot{f}\tilde{\epsilon}^{0kl}{}_i\nabla^2 h_{0l,k} = 16\pi\rho v_i, \tag{37}$$

where we have used that $T^{0i} = -T_{0i}$. Using the lower order solutions and the effective metric, as in Sec. III, we obtain

$$\nabla^2 \mathcal{H}_{0i} + U_{,0i} = 16\pi\rho v_i, \tag{38}$$

where the vectorial sector of the effective metric is

$$\mathcal{H}_{0i} = h_{0i} + \frac{1}{2}\dot{f}\tilde{\epsilon}^{0kl}{}_{i}h_{0l,k}. \tag{39}$$

We recognize Eq. (38) as the standard GR field equation to $\mathcal{O}(3)$, except that the dependent function is the effective metric instead of the metric perturbation. We can thus solve this equation in terms of PPN potentials to obtain

$$\mathcal{H}_{0i} = -\frac{7}{2}V_i - \frac{1}{2}W_i, \tag{40}$$

where we have used that the superpotential $X$ satisfies $X_{,0j} = V_j - W_j$ (see Appendix A for the definitions.) Combining Eq. (39) with Eq. (40) we arrive at a differential equation for the metric perturbation, namely

$$h_{0i} + \frac{1}{2}\dot{f}\tilde{\epsilon}^{0kl}{}_{i}h_{0l,k} = -\frac{7}{2}V_i - \frac{1}{2}W_i. \tag{41}$$

Once more, let us look for solutions that are perturbation about the GR prediction, namely

$$h_{0i} = -\frac{7}{2}V_i - \frac{1}{2}W_i + \dot{f}\zeta_i, \tag{42}$$

where we again assume that $\zeta_i$ is of $\mathcal{O}(\dot{f})^0$. The field equation becomes

$$\zeta_i + \frac{1}{2}\dot{f}\left(\nabla \times \zeta\right)_i = \frac{1}{2}\left(\frac{7}{2}\left(\nabla \times V\right)_i + \frac{1}{2}\left(\nabla \times W\right)_i\right), \tag{43}$$

where $\left(\nabla \times A\right)^i = \epsilon^{ijk}\partial_j A_k$ is the standard curl operator of flat space. As in Sec. III, note once more that the second term on the left-hand side is again a second order correction and we shall thus neglect it. Also note that the curl of the $V_i$ potential happens to be equal to the curl of the $W_i$ potential. The solution for the vectorial sector of the actual gravitational field then simplifies to

$$h_{0i} = -\frac{7}{2}V_i - \frac{1}{2}W_i + 2\dot{f}\left(\nabla \times V\right)_i + \mathcal{O}(5). \tag{44}$$

We have arrived at the first contribution of CS modified gravity to the metric for a perfect fluid source. Chern-Simons gravity was previously seen to couple to the transverse-traceless sector of the metric perturbation for gravitational wave solutions [29, 30, 36, 47]. The CS correction is also believed to couple to Noether vector currents, such as neutron currents, which partially fueled the idea that this correction could be enhanced. However, to our knowledge, this correction was never thought to couple to vector metric perturbations. From the analysis presented here, we see that in fact CS gravity does couple to such terms, even if the matter source is neutrally charged. The only requirement for such couplings is that the source is not static, *ie.* that the object is either moving or spinning relative to the PPN rest frame so that the PPN vector potential does not vanish. The latter is suppressed by a relative $\mathcal{O}(1)$ because in the far field the velocity of a compact object produces a term of $\mathcal{O}(3)$ in $V_i$, while the spin produces a term of $\mathcal{O}(4)$. In a later section, we shall discuss some of the physical and observational implications of such a modification to the metric.

### C. $h_{00}$ to $\mathcal{O}(4)$

A full analysis of the PPN structure of a modified theory of gravity requires that we solve for the 00 component of the metric perturbation to $\mathcal{O}(4)$. The field equations to this order are

$$\begin{aligned} -\ &\frac{1}{2}\nabla^2 h_{00} - \frac{1}{2}h_{00,i}h_{00,i} + \frac{1}{2}h_{ij}h_{00,ij} = 4\pi\rho\,[1 \\ +\ &2\left(v^2 - U + \frac{1}{2}\Pi + \frac{3}{2}\frac{p}{\rho}\right)\Big], \end{aligned} \tag{45}$$

where the CS correction does not contribute at this order (see Appendix B.) Note that the $h_{0i}$ sector of the metric perturbation to $\mathcal{O}(3)$ does not feed back into the field equations at this order either. The terms that do come into play are the $h_{00}$ and $h_{ij}$ sectors of the metric, which are not modified to lowest order by the CS correction. The field equation, thus, reduce to the standard one of GR, whose solution in terms of PPN potentials is

$$h_{00} = 2U - 2U^2 + 4\Phi_1 + 4\Phi_2 + 2\Phi_3 + 6\Phi_4 + \mathcal{O}(6). \tag{46}$$

We have thus solved for all components of the metric perturbation to 1 PN order beyond the Newtonian answer, namely $g_{00}$ to $\mathcal{O}(4)$, $g_{0i}$ to $\mathcal{O}(3)$ and $g_{ij}$ to $\mathcal{O}(2)$.

### D. PPN Parameters for CS Gravity

We now have all the necessary ingredients to read off the PPN parameters of CS modified gravity. Let us begin by writing the full metric with the solutions found in the previous subsections:

$$\begin{aligned} g_{00} &= -1 + 2U - 2U^2 + 4\Phi_1 + 4\Phi_2 + 2\Phi_3 + 6\Phi_4 + \mathcal{O}(6), \\ g_{0i} &= -\frac{7}{2}V_i - \frac{1}{2}W_i + 2\dot{f}\left(\nabla \times V\right)_i + \mathcal{O}(5), \\ g_{ij} &= \left(1 + 2U\right)\delta_{ij} + \mathcal{O}(4). \end{aligned} \tag{47}$$

One can verify that this metric is indeed a solution of Eqs. (27), (29), (37) and (45) to the appropriate PN order and to first order in the CS coupling parameter. Also note that the solution of Eq. (47) automatically satisfies the constraint ${}^\star RR = 0$ to linear order because the contraction of the Levi-Civita symbol with two partial derivatives vanishes. Such a solution is then allowed in CS gravity, just as other classical solutions are [62], and the equations of motion for the fluid can be obtained directly from the covariant derivative of the stress-energy tensor.

We can now read off the PPN parameters of the CS modified theory by comparing Eq. (5) to Eq. (47). A visual inspection reveals that the CS solution is identical to the classical GR one, which implies that $\gamma = \beta = 1$, $\zeta = 0$ and $\alpha_1 = \alpha_2 = \alpha_3 = \xi_1 = \xi_2 = \xi_3 = \xi_4 = 0$ and there are no preferred frame effects. However, Eq. (5) contains an extra term that cannot be modeled by the standard PPN metric of Eq. (5), namely the curl contribution to $g_{0i}$. We then see that the PPN metric must be enhanced by the addition of a curl-type term to the $0i$ components of the metric, namely

$$\begin{aligned} g_{0i} &\equiv -\frac{1}{2}\left(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi\right) V_i \\ &- \frac{1}{2}\left(1 + \alpha_2 - \zeta_1 + 2\xi\right) W_i + \chi \left(r\nabla \times V\right)_i , \end{aligned} \tag{48}$$

where $\chi$ is a *new* PPN parameter and where we have multiplied the curl operator by the radial distance to the source, $r$, in order to make $\chi$ a proper dimensionless parameter. Note that there is no need to introduce any additional PPN parameters because the curl of $W_i$ equals the curl of $V_i$. In fact, we could have equally parameterized the new contribution to the PPN metric in terms of the curl of $W_i$, but we chose not to because $V_i$ appears more frequently in PN theory. For the case of CS modified gravity, the new $\chi$ parameter is simply

$$\chi = 2\frac{\dot{f}}{r}, \tag{49}$$

which is dimensionless since $\dot{f}$ has units of length. If an experiment could measure or place bounds on the value of $\chi$, then $\dot{f}$ could also be bounded, thus placing a constraint on the CS coupling parameter.

## VI. ASTROPHYSICAL IMPLICATIONS

In this section we shall propose a physical interpretation to the CS modification to the PPN metric and we shall calculate some GR predictions that are modified by this correction. This section, however, is by no means a complete study of all the possible consequences of the CS correction, which is beyond the scope of this paper.

Let us begin by considering a system of $A$ nearly spherical bodies, for which the gravitational vector potentials are simply [6]

$$\begin{aligned} V^i &= \sum_A \frac{m_A}{r_A} v_A^i + \frac{1}{2}\sum_A \left(\frac{J_A}{r_A^2} \times n_A\right)^i , \end{aligned} \tag{50}$$

$$W^i = \sum_A \frac{m_A}{r_A}\left(v_A \cdot n_A\right) n_A^i + \frac{1}{2}\sum_A \left(\frac{J_A}{r_A^2} \times n_A\right)^i ,$$

where $m_A$ is the mass of the $A$th body, $r_A$ is the field point distance to the $A$th body, $n_A^i = x_A^i/r_A$ is a unit vector pointing to the $A$th body, $v_A$ is the velocity of the $A$th body and $J_A^i$ is the spin-angular momentum of the $A$th body. For example, the spin angular momentum for a Kerr spacetime is given by $J^i = m^2 a^i$, where $a$ is the dimensionless Kerr spin parameter. Note that if $A = 2$ then the system being modeled could be a binary of spinning compact objects, while if $A = 1$ it could represent the field of the sun or that of a rapidly spinning neutron star or pulsar.

In obtaining Eq. (50), we have implicitly assumed a point-particle approximation, which in classical GR is justified by the effacement principle. This principle postulates that the internal structure of bodies contributes to the solution of the field equations to higher PN order. One can verify that this is indeed the case in classical GR, where internal structure contributions appear at 5 PN order. In CS gravity, however, it is a priori unclear whether an analogous effacement principle holds because the CS term is expected to couple with matter current via standard model-like interactions. If such is the case, it is possible that a "mountain" on the surface of a neutron star [63] or an r-mode instability [64, 65, 66] enhances the CS contribution. In this paper, however, we shall neglect these interactions, and relegate such possibilities to future work [47].

With such a vector potential, we can calculate the CS correction to the metric. For this purpose, we define the correction $\delta g_{0i} \equiv g_{0i} - g_{0i}^{(GR)}$, where $g_{0i}^{(GR)}$ is the GR prediction without CS gravity. We then find that the CS corrections is given by

$$\delta g_{0i} = 2\sum_A \frac{\dot{f}}{r_A}\left[\frac{m_A}{r_A}\left(v_A \times n_A\right)^i - \frac{J_A^i}{2r_A^2} + \frac{3}{2}\frac{\left(J_A \cdot n_A\right)}{r_A^2} n_A^i\right], \tag{51}$$

where the $\cdot$ operator is the flat-space inner product and where we have used the identities $\tilde{\epsilon}_{ijk}\tilde{\epsilon}_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ and $\tilde{\epsilon}_{ilk}\tilde{\epsilon}_{jlm} = 2\delta_{ij}$. Note that the first term of Eq. (51) is of $\mathcal{O}(3)$, while the second and third terms are of $\mathcal{O}(4)$ as previously anticipated. Also note that $\dot{f}$ couples both to the spin and orbital angular momentum. Therefore, whether the system under consideration is the Solar system ($v^i$ of the Sun is zero while $J^i$ is small), the Crab pulsar ($v^i$ is again zero but $J^i$ is large) or a binary system of compact objects (neither $v^i$ nor $J^i$ vanish), there will in general be a non-vanishing coupling between the CS correction and the vector potential of the system.

From the above analysis, it is also clear that the CS correction increases with the non-linearity of the spacetime. In other words, the CS term is larger not only for systems with large velocities and spins, but also in regions near the source. For a binary system, this fact implies that the CS correction is naturally enhanced in the last stages of inspiral and during merger. Note that this enhancement is *different* from all previous enhancements proposed, since it does not require the presence of charge [28, 47], a fifth dimension with warped compactifications [59, 60], or a vanishing string coupling [48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58]. Unfortunately, the end of the inspiral stage coincides with the edge of the PN region of validity and, thus, a complete analysis of such a natural enhancement will have to be carried out through numerical simulations.

In the presence of a source with the vector potentials of Eq. (50), we can write the vectorial sector of the metric perturbation in a suggestive way, namely

$$g_{0i} = \sum_A -\frac{7}{2}\frac{m_A}{r_A} v_A^i - \sum_A \frac{m_A}{6 r_A^2}\left(v_A - v_A^{(eff)}\right)^i - \frac{1}{2}\sum_A n_A^i \frac{m_A}{r_A} v_A^{(eff)} \cdot n_A - 2\sum_A \left[\frac{J_A^{(eff)}}{r_A^2} \times n_A\right]^i, \quad (52)$$

where we have defined an effective velocity and angular momentum vector via

$$v^i_{A(eff)} = v^i_A - 6\dot{f}\frac{J^i_A}{m_A r_A^2},$$
$$J^i_{A(eff)} = J^i_A - \dot{f} m_A v^i_A, \quad (53)$$

or in terms of the Newtonian orbital angular momentum $L^i_{A(N)} = r_A \times p_A$ and linear momentum $p^i_{A(N)} = m_A v^i_A$

$$L^i_{A(eff)} = L^i_{A(N)} - 6\dot{f}\left(n_A \times J_A\right)^i,$$
$$J^i_{A(eff)} = J^i_A - \dot{f} p^i_A. \quad (54)$$

From this analysis, it is clear that the CS corrections seems to couple to both a quantity that resembles the orbital and the spin angular momentum vector. Note that when the spin angular momentum vanishes the vectorial metric perturbation is identical to that of a spinning moving fluid, but where the spin is induced by the coupling of the orbital angular momentum to the CS term.

The presence of an effective CS spin angular momentum in non-spinning sources leads to an interesting physical interpretation. Let us model the field that sources $\dot{f}$ as a fluid that permeates all of spacetime. This field could be, for example, a model-independent axion, inspired by the quantity introduced in the standard model to resolve the strong CP problem [67]. In this scenario, then the fluid is naturally "dragged" by the motion of any source and the CS modification to the metric is nothing but such dragging. This analogy is inspired by the ergosphere of the Kerr solution, where inertial frames are dragged with the rotation of the black hole. In fact, one could push this analogy further and try to construct the shear and bulk viscosity of such a fluid, but we shall not attempt this here. Of course, this interpretation is to be understood only qualitatively, since its purpose is only to allow the reader to picture the CS modification to the metric in physical terms.

An alternative interpretation can be given to the CS modification in terms of the gravito-electro-magnetic (GED) analogy [40, 41], which shall allow us to easily construct the predictions of the modified theory. In this analogy, one realizes that the PN solution to the linearized field equations can be written in terms of a potential and vector potential, namely

$$ds^2 = -\left(1 - 2\Phi\right)dt^2 - 4\left(A \cdot dx\right)dt + \left(1 + 2\Phi\right)\delta_{ij} dx^i dx^j, \quad (55)$$

where $\Phi$ reduces to the Newtonian potential $U$ in the Newtonian limit [41] and $A^i$ is a vector potential related to the metric via $A_i = -g_{0i}/4$. One can then construct GED fields in analogy to Maxwell's electromagnetic theory via

$$E^i = -\left(\nabla\Phi\right)^i - \partial_t\left(\frac{1}{2}A^i\right),$$
$$B^i = \left(\nabla \times A\right)^i, \quad (56)$$

which in terms of the vectorial sector of the metric perturbation becomes

$$E^i = -\left(\nabla\Phi\right)^i + \frac{1}{8}\dot{g}^i,$$
$$B^i = -\frac{1}{4}\left(\nabla \times g\right)^i, \quad (57)$$

where we have defined the vector $g^i = g_{0i}$. The geodesic equations for a test particle then reduce to the Lorentz force law, namely

$$F^i = -mE^i - 2m\left(v \times B\right)^i. \quad (58)$$

We can now work out the effect of the CS correction on the GED fields and equations of motion. First note that the CS correction only affects $g$. We can then write the CS modification to the Lorentz force law by defining $\delta a^i = a^i - a^i_{(GR)}$, where $a^i_{(GR)}$ is the acceleration vector predicted by GR, to obtain,

$$\delta a^i = \frac{1}{8}\delta\dot{g}^i + \frac{1}{2}\left(v \times \delta\Omega\right)^i, \quad (59)$$

where we have defined the angular velocity

$$\delta\Omega^i = \left(\nabla \times \delta g\right)^i. \quad (60)$$

The time derivative of the vector $g^i$ is of $\mathcal{O}(5)$ and can thus be neglected, but the angular velocity cannot and it is given by

$$\delta\Omega^i = -\sum_A \dot{f}\frac{m_A}{r_A^3}\left[3\left(v_A \cdot n_A\right) n^i_A - v^i_A\right], \quad (61)$$

which is clearly of $\mathcal{O}(3)$. Note that although the first term between square brackets cancels for circular orbits because $n^i_A$ is perpendicular to $v^i_A$ to Newtonian order, the second term does not. The angular velocity adds a correction to the acceleration of $\mathcal{O}(4)$, namely

$$\delta a^i = -\frac{3}{2}\sum_A \dot{f}\frac{m_A}{r_A^3}\left(v_A \cdot n_A\right)\left(v_A \times n_A\right)^i, \quad (62)$$

which for a system in circular orbit vanishes to Newtonian order. One could use this formalism to find the perturbations in the motion of moving objects by integrating Eq. (62) twice. However, for systems in a circular orbit, such as the Earth-Moon system or compact binaries, this correction vanishes to leading order. Therefore, lunar ranging experiments [68] might not be able to constraint $\dot{f}$.

Another correction to the predictions of GR is that of the precession of gyroscopes by the so-called Lense-Thirring or frame-dragging effect. In this process, the spin angular momentum of a source twists spacetime in such a way that gyroscopes are dragged with it. The precession angular velocity depends on the vector sector of the metric perturbation via Eq. (61). Thus, the full Lense-Thirring term in the precession angular velocity of precessing gyroscopes is

$$\Omega^i_{LT} = -\frac{1}{r_A^3} \sum_A J^i_{A(eff)} - 3 n^i_A \left( J_{A(eff)} \cdot n_A \right)^i . \quad (63)$$

Note that this angular velocity is identical to the GR prediction, except for the replacement $J^i_A \to J^i_{A(eff)}$. In CS modified gravity, then, the Lense-Thirring effect is not only produced by the spin angular momentum of the gyroscope but also by the orbital angular momentum that couples to the CS correction. Therefore, if an experiment were to measure the precession of gyroscopes by the curvature of spacetime (see, for example, Gravity Probe B [42]) one could constraint $\dot{f}$ and thus some intrinsic parameters of string theory. Note, however, that the CS correction depends on the velocity of the bodies with respect to the inertial PPN rest-frame. In order to relate these predictions to the quantities that are actually measured in the experiment, one would have to transform to the experiment's frame, or perhaps to a basis aligned with the direction of distant stars [6].

Are there other experiments that could be performed to measure such a deviation from GR? Any experiment that samples the vectorial sector of the metric would in effect be measuring such a deviation. In this paper, we have only discussed modifications to the frame-dragging effect and the acceleration of bodies through the GED analogy, but this need not be the only corrections to classical GR predictions. In fact, any predictions that depends on $g_{0i}$ indirectly, for example via Christoffel symbols, will probably also be modified unless the corrections is fortuitously canceled. In this paper, we have laid the theoretical foundations of the weak-field correction to the metric due to CS gravity and studied some possible corrections to classical predictions. A detailed study of other corrections is beyond the scope of this paper.

## VII. CONCLUSION

We have studied the weak-field expansion of the solution to the CS modified field equations in the presence of a perfect fluid PN source in the point particle limit. Such an expansion required that we linearize the Ricci and Cotton tensor to second order in the metric perturbation without any gauge assumption. An iterative PPN formalism was then employed to solve for the metric perturbation in this modified theory of gravity. We have found that CS gravity possesses the same PPN parameters as those of GR, but it also requires the introduction of a new term and PPN parameter that is proportional to the curl of the PPN vector potentials. Such a term is enhanced in non-linear scenarios without requiring the presence of standard model currents, large extra dimensions or a vanishing string coupling.

We have proposed an interpretation for the new term in the metric produced by CS gravity and studied some of the possible consequences it might have on GR predictions. The interpretation consists of picturing the field that sources the CS term as a fluid that permeates all of spacetime. In this scenario, the CS term is nothing but the "dragging" of the fluid by the motion of the source. Irrespective of the validity of such an interpretation, the inclusion of a new term to the weak-field expansion of the metric naturally leads to corrections to the standard GR predictions. We have studied the acceleration of point particles and the Lense-Thirring contribution to the precession of gyroscopes. We have found that both corrections are proportional to the CS coupling parameter and, therefore, experimental measurement of these effects might be used to constraint CS and, possibly, string theory.

Future work could concentrate on studying further the non-linear enhancement of the CS correction and the modifications to the predictions of GR. The PPN analysis performed here breaks down very close to the source due to the use of a point particle approximation in the stress energy tensor. One possible research route could consists of studying the CS correction in a perturbed Kerr background [69]. Another possible route could be to analyze other predictions of the theory, such as the perihelion shift of Mercury or the Nordtvedt effect. Furthermore, in light of the imminent highly-accurate measurement of the Lense-Thirring effect by Gravity Probe B, it might be useful to revisit this correction in a frame better-adapted to the experimental setup. Finally, the CS modification to the weak-field metric might lead to non-conservative effects and the breaking of the effacement principle [47], which could be studied through the evaluation of the gravitational pseudo stress-energy tensor. Ultimately, it will be experiments that will determine the viability of CS modified gravity and string theory.

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## APPENDIX A: PPN POTENTIALS

In this appendix, we present explicit expressions for the PPN potentials used to parameterize the metric in Eq. (5). These potentials are the following:

$$\begin{aligned}
U &\equiv \int \frac{\rho}{|\mathbf{x}-\mathbf{x}'|} d^3x', \\
V_i &\equiv \int \frac{\rho' v_i'}{|\mathbf{x}-\mathbf{x}'|} d^3x', \\
W_i &\equiv \int \frac{\rho' v_j' (x-x')^j (x-x')_i}{|\mathbf{x}-\mathbf{x}'|^3} d^3x', \\
\Phi_W &\equiv \int \rho' \rho'' \frac{(x-x')^i}{|\mathbf{x}-\mathbf{x}'|^3} \left( \frac{(x'-x'')_i}{|\mathbf{x}-\mathbf{x}''|} - \frac{(x-x'')_i}{|\mathbf{x}'-\mathbf{x}''|} \right) d^3x' d^3x'', \\
\Phi_1 &\equiv \int \frac{\rho' v'^2}{|\mathbf{x}-\mathbf{x}'|} d^3x', \qquad \Phi_2 \equiv \int \frac{\rho' U'}{|\mathbf{x}-\mathbf{x}'|} d^3x', \\
\Phi_3 &\equiv \int \frac{\rho' \Pi'}{|\mathbf{x}-\mathbf{x}'|} d^3x', \qquad \Phi_4 \equiv \int \frac{p'}{|\mathbf{x}-\mathbf{x}'|} d^3x', \\
\mathcal{A} &\equiv \int \frac{\rho' \left[ v_i' \, (x-x')^i \right]^2}{|\mathbf{x}-\mathbf{x}'|} d^3x', \\
X &\equiv \int \rho' |\mathbf{x}-\mathbf{x}'| d^3x'.
\end{aligned} \tag{A1}$$

These potentials satisfy the following relations

$$\begin{aligned}
\nabla^2 U &= -4\pi\rho, \qquad \nabla^2 V_i = -4\pi\rho v_i, \\
\nabla^2 \Phi_1 &= -4\pi\rho v^2, \qquad \nabla^2 \Phi_2 = -4\pi\rho U, \\
\nabla^2 \Phi_3 &= -4\pi\rho\Pi, \qquad \nabla^2 \Phi_4 = -4\pi p, \\
\nabla^2 X &= -2U
\end{aligned} \tag{A2}$$

The potential $X$ is sometimes referred to as the super-potential because it acts as a potential for the Newtonian potential.

## APPENDIX B: LINEARIZATION OF THE COTTON TENSOR

In this appendix, we present some more details on the derivation of the linearized Cotton tensor to second order. We begin with the definition of the Cotton tensor [30] in terms of the symmetrization operator, namely

$$C^{\mu\nu} = -\frac{1}{\sqrt{-g}} \left[ (D_\sigma f)\, \epsilon^{\sigma\alpha\beta(\mu} D_\alpha R^{\nu)}{}_\beta + (D_{\sigma\tau} f)^\star \, R^{\tau(\mu|\sigma|\nu)} \right]. \tag{B1}$$

Using the symmetries of the Levi-Civita and Riemann tensor, as well as the fact that $f$ depends only on time, we can simplify the Cotton tensor to

$$\begin{aligned}
C^{\mu\nu} &= (-g)^{-1} \dot{f} \left[ \tilde{\epsilon}^{0\alpha\beta(\mu} R^{\nu)}{}_{\beta,\alpha} + \tilde{\epsilon}^{0\alpha\beta(\mu} \Gamma^{\nu)}_{\lambda\alpha} R^\lambda_\beta \right. \\
&+ \left. \frac{1}{2} \Gamma^0_{\sigma\tau} \tilde{\epsilon}^{\sigma\alpha\beta(\mu} R^{\nu)\tau}{}_{\alpha\beta} \right].
\end{aligned} \tag{B2}$$

Noting that the determinant of the metric is simply $g = -1 + h$, so that $(-g)^{-1} = 1 + h$, we can identify four terms in the Cotton tensor

$$\begin{aligned}
C^{\mu\nu}_A &= \dot{f} \tilde{\epsilon}^{0\alpha\beta(\mu} \left[ \hat{L} R^{\nu)}{}_{\beta,\alpha} \right], \\
C^{\mu\nu}_B &= \dot{f} \tilde{\epsilon}^{0\alpha\beta(\mu} h_{\rho\rho} \left[ \hat{L} R^{\nu)}{}_{\beta,\alpha} \right], \\
C^{\mu\nu}_C &= \dot{f} \tilde{\epsilon}^{0\alpha\beta(\mu} \left[ \hat{L} \Gamma^{\nu)}_{\lambda\alpha} \right] \left[ \hat{L} R^\lambda{}_\beta \right], \\
C^{\mu\nu}_D &= \frac{\dot{f}}{2} \tilde{\epsilon}^{\sigma\alpha\beta(\mu} \left[ \hat{L} \Gamma^0_{\sigma\tau} \right] \left[ \hat{L} R^{\nu)\tau}{}_{\alpha\beta} \right], \\
C^{\mu\nu}_E &= \dot{f} \tilde{\epsilon}^{0\alpha\beta(\mu} \left[ \hat{Q} R^{\nu)}{}_{\beta,\alpha} \right],
\end{aligned} \tag{B3}$$

where the $\hat{L}$ operator stands for the linear part of its operand, while the $\hat{Q}$ operator isolates the quadratic part of its operand. For example, if we act $\hat{L}$ and $\hat{Q}$ on $(1+h)^n$, where $n$ is some integer, we obtain

$$\left[ \hat{L} (1+h)^n \right] = nh, \; \left[ \hat{Q} (1+h)^n \right] = \frac{n(n-1)}{2} h^2 \tag{B4}$$

Let us now compute each of these terms separately. The first four terms are given by

$$\begin{aligned}
C^{\mu\nu}_A &= -\frac{\dot{f}}{2} \tilde{\epsilon}^{0\alpha\beta(\mu} \left( \Box_\eta h^{\nu)}{}_{\beta,\alpha} - h_{\sigma\beta,}{}^\nu{}_{\alpha\sigma} \right), \\
C^{\mu\nu}_B &= -\frac{\dot{f}}{2} h \tilde{\epsilon}^{0\alpha\beta(\mu} \left( \Box_\eta h^{\nu)}{}_{\beta,\alpha} - h_{\sigma\beta,}{}^\nu{}_{\alpha\sigma} \right), \\
C^{\mu\nu}_C &= -\frac{\dot{f}}{4} \tilde{\epsilon}^{0\alpha\beta(\mu} \left( h^{\nu)}{}_{\lambda,\alpha} + h^\nu{}_{\alpha,\lambda} - h_{\lambda\alpha,}{}^{\nu)} \right) \\
&\times \left( \Box_\eta h^\lambda{}_\beta - h_\sigma{}^\lambda{}_{,\beta}{}^\sigma - h_{\sigma\beta,}{}^{\lambda\sigma} + h_,{}^\lambda{}_\beta \right), \\
C^{\mu\nu}_D &= \frac{\dot{f}}{4} \tilde{\epsilon}^{\sigma\alpha\beta(\mu} \left( 2h^0{}_{(\sigma,\tau)} - h_{\sigma\tau,}{}^0 \right) \left( h^\tau{}_{[\beta,\alpha]}{}^\nu - h^\nu{}_{[\beta,\alpha]}{}^\tau \right).
\end{aligned} \tag{B5}$$

The last term of the Cotton tensor is simply the derivative of the Ricci tensor which we already calculated to second order in Eq. (21). In order to avoid notation clutter, we shall not present it again here, but instead we combine all the Cotton tensor pieces to obtain

$$\begin{aligned}
C^{\mu\nu} &= -\frac{\dot{f}}{2} \tilde{\epsilon}^{0\alpha\beta(\mu} \left( \Box_\eta h^{\nu)}{}_{\beta,\alpha} - h_{\sigma\beta,\alpha}{}^{\sigma\nu)} \right) - \frac{\dot{f}}{2} \tilde{\epsilon}^{0\alpha\beta(\mu} \left[ h \left( \Box_\eta h^{\nu)}{}_{\beta,\alpha} - h_{\sigma\beta,\alpha}{}^{\sigma\nu)} \right) + \frac{1}{2} \left( 2h^{\nu)}{}_{(\lambda,\alpha)} - h_{\lambda\alpha,}{}^{\nu)} \right) \right. \\
&\times \left. \left( \Box_\eta h^\lambda{}_\beta - 2h_\sigma{}^{(\lambda}{}_{,\beta)}{}^\sigma + h_{,\beta}{}^\lambda \right) - 2\hat{Q} R^{\nu)}{}_{\beta,\alpha} \right] + \frac{\dot{f}}{4} \tilde{\epsilon}^{\sigma\alpha\beta(\mu} \left( 2h^0{}_{(\sigma,\tau)} - h_{\sigma\tau,}{}^0 \right) \left( h^\tau{}_{[\beta,\alpha]}{}^{\nu)} - h^{\nu)}{}_{[\beta,\alpha]}{}^\tau \right) + \mathcal{O}(h)^3
\end{aligned} \tag{B6}$$

where its covariant form is

$$
\begin{aligned}
C_{\mu\nu} &= -\frac{\dot{f}}{2}\tilde{\epsilon}^{0\alpha\beta}{}_{(\mu}\left(\Box_\eta h_{\nu)\beta,\alpha} - h_{\sigma\beta,\alpha\nu)}{}^{\sigma}\right) - \frac{\dot{f}}{2}\tilde{\epsilon}^{0\alpha\beta}{}_{(\mu}\left[h\left(\Box_\eta h_{\nu)\beta,\alpha} - h_{\sigma\beta,\alpha\nu)}{}^{\sigma}\right) + \frac{1}{2}\left(2h_{\nu)(\lambda,\alpha)} - h_{\lambda\alpha,\nu)}\right)\right. \\
&\times \left.\left(\Box_\eta h^{\lambda}{}_{\beta} - 2h_{\sigma}{}^{(\lambda}{}_{,\beta)}{}^{\sigma} + h_{,\beta}{}^{\lambda}\right) - 2\hat{Q}R_{\nu)\beta,\alpha} + h_{\nu\lambda}\left(\Box_\eta h^{\lambda)}{}_{\beta,\alpha} - h_{\sigma\beta,\alpha}{}^{\sigma\lambda)}\right)\right] \\
&+ \frac{\dot{f}}{4}\tilde{\epsilon}^{\sigma\alpha\beta}{}_{(\mu}\left(2h^{0}{}_{(\sigma,\tau)} - h_{\sigma\tau,}{}^{0}\right)\left(h^{\tau}{}_{[\beta,\alpha]\nu)} - h_{\nu)[\beta,\alpha]}{}^{\tau}\right) - \frac{\dot{f}}{2}h_{\mu\lambda}\tilde{\epsilon}^{0\alpha\beta(\lambda}\left(\Box_\eta h_{\nu)\beta,\alpha} - h_{\sigma\beta,\alpha\nu)}{}^{\sigma}\right) + \mathcal{O}(h)^3. \quad \text{(B7)}
\end{aligned}
$$

For the PPN mapping of CS modified gravity, only the 00 component of the metric is needed to second order, which implies we only need $C_{00}$ to $\mathcal{O}(h)^2$. This component is given by

$$
\begin{aligned}
C_{00} &= \frac{\dot{f}}{4}\tilde{\epsilon}^{ijk}{}_{0}\left(2h^{0}{}_{(i,\ell)} - h_{i\ell,}{}^{0}\right)\left(h^{\ell}{}_{[k,j]0} - h_{0[k,j]}{}^{\ell}\right) \\
&- \frac{\dot{f}}{2}h_{0\ell}\tilde{\epsilon}^{0jk(\ell}\left(\Box_\eta h_{0k,j} - h_{ik,j0}{}^{i}\right) + \mathcal{O}(h)^3, \quad \text{(B8)}
\end{aligned}
$$

where in fact the last term vanishes due to the PPN gauge condition. Note that this term is automatically of $\mathcal{O}(6)$, which is well beyond the required order we need in $h_{00}$.


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