СЕМИНАР 11

③ [Листок 5]

Вычислить пределы:

a) $\lim\limits_{x\to 0}\left(\dfrac{\cos x}{\cos(2x)}\right)^{1/x^2}$ c) $\lim\limits_{x\to +\infty}\left(\dfrac{\ln(10+e^x)}{x}\right)^{\sqrt{e^{2x}+5}}$

b) $\lim\limits_{x\to 0}\dfrac{(1+x)^x-1}{1-\cos x}$ d) $\lim\limits_{x\to 1}\ln(e^x+x-1)^{\frac{1}{\sqrt[3]{x^7}-1}}$

Решение a) Заметим, что:

$$\lim_{x\to 0}\left(\frac{\cos x}{\cos(2x)}\right)^{1/x^2} = e^{\ln\left(\frac{\cos x}{\cos(2x)}\right)\cdot\frac{1}{x^2}}$$

$$= e^{\frac{\ln(\cos x)-\ln(\cos(2x))}{x^2}}$$

$$\left[\ \ln\cos x = \ln\left(1-2\sin^2\frac{x}{2}\right) = -2\sin^2\frac{x}{2} + \bar{o}\left(\sin^2\frac{x}{2}\right) = -2\sin^2\frac{x}{2} + \bar{o}(x^2)\ \ x\to 0 \right.$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{x^n}{n} + \bar{o}(x^n),\ x\to 0$$

$$\bar{o}(\sin^k x) = \bar{o}(x^k)\quad x\to 0,\ \forall k\in\mathbb{N}$$

$$f\in\bar{o}(\sin^k x) \Rightarrow \lim_{x\to 0}\frac{f}{\sin^k x} = 0 \Rightarrow$$

$$\left. \lim_{x\to 0}\frac{f}{x^k} = \lim_{x\to 0}\underbrace{\frac{f}{\sin^k x}}_{\to 0}\cdot\underbrace{\frac{\sin^k x}{x^k}}_{\to 1} = 0 \Rightarrow f\in\bar{o}(x^k).\ \right]$$

$$= e^{\frac{-2\sin^2\frac{x}{2}+\bar{o}(x^2)-(-2\sin^2 x+\bar{o}(x^2))}{x^2}}$$

$$= e^{2\cdot\underbrace{\frac{\sin^2 x}{x^2}}_{\to 1} - 2\cdot\underbrace{\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}}_{\to 1}\cdot\underbrace{\frac{\left(\frac{x}{2}\right)^2}{x^2}}_{=\frac{1}{4}} + \bar{o}(x^2)} = e^{2-\frac{2}{4}} = \boxed{e^{3/2}}$$

b) Заметим, что:

$$\lim_{x\to 0}\left\{\frac{(1+x)^x-1}{1-\cos x}\right. = \frac{e^{\ln(1+x)\cdot x}-1}{1-\cos x}$$

$$= \frac{1+\ln(1+x)\cdot x+\bar{\bar{o}}(\ln(1+x)\cdot x)-1}{1-\left(1-\frac{x^2}{2}+\bar{\bar{o}}(x^2)\right)}$$

$$\left[\begin{array}{l} e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}+\bar{\bar{o}}(x^n) \\ \cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots+(-1)^{n+1}\cdot\frac{x^{2n}}{(2n)!}+\bar{\bar{o}}(x^{2n+1}) \\ \bar{\bar{o}}(\ln(1+x)\cdot x) = \bar{\bar{o}}(x^2) \text{ т.к. } \lim_{x\to 0}\frac{\ln(1+x)}{x}=1 \end{array}\right]$$

$$= \frac{\ln(1+x)\cdot x+\bar{\bar{o}}(x^2)}{\frac{x^2}{2}+\bar{\bar{o}}(x^2)}$$

$$= \frac{\cancel{x^2}\left(\frac{\ln(1+x)}{x}+\bar{\bar{o}}(1)\right)}{\cancel{x^2}\left(\frac{1}{2}+\bar{\bar{o}}(1)\right)}$$

$$= \left.\frac{\overbrace{\frac{\ln(1+x)}{x}}^{\to 1}+\overbrace{\bar{\bar{o}}(1)}^{\to 0}}{\underbrace{\frac{1}{2}}_{\to 1/2}+\underbrace{\bar{\bar{o}}(1)}_{\to 0}}\right\} = \frac{1+0}{\frac{1}{2}+0} = \boxed{2}$$

c) Заметим, что:

$$\lim_{x\to+\infty}\left\{\left(\frac{\ln(10+e^x)}{x}\right)^{\sqrt{e^{2x}+5}} = \left(1+\underbrace{\frac{1}{x}\cdot\ln\left(1+\frac{10}{e^x}\right)}_{=f(x)}\right)^{e^x\cdot\sqrt{1+\frac{5}{e^{2x}}}} \ominus\right.$$

$$\ln(10+e^x) = \ln\left(e^x\left(1+\frac{10}{e^x}\right)\right) = \cancel{x} + \ln\left(1+\frac{10}{e^x}\right)$$

[если $g(x)\to 0$ при $x\to 0$, то $(1+g(x))^{\frac{1}{g(x)}} \to e$, $x\to 0$.

действительно:

$$(1+g(x))^{\frac{1}{g(x)}} = e^{\overbrace{\ln(1+g(x))/g(x)}^{\to 1}} \to e, \quad x\to 0.$$]

$$\ominus \left.\underbrace{(1+f(x))^{\frac{1}{f(x)}}}_{\to e}{}^{\overbrace{e^x\cdot\sqrt{1+\frac{5}{e^{2x}}}\cdot f(x)}^{\to 0}}\right\} = e^0 = \boxed{1}$$

[$e^x\cdot\sqrt{1+\frac{5}{e^{2x}}}\cdot\frac{1}{x}\cdot\ln\left(1+\frac{10}{e^x}\right)$

$$= \underbrace{\frac{\ln\left(1+\frac{10}{e^x}\right)}{\frac{10}{e^x}}}_{\to 1}\cdot\underbrace{\frac{10}{e^x}\cdot e^x}_{=10}\cdot\underbrace{\sqrt{1+\frac{5}{e^{2x}}}}_{\to 1}\cdot\underbrace{\frac{1}{x}}_{\to 0}\to 0$$]

d) сделаем замену $x = t+1$, тогда имеем:

$$\lim_{t\to 0}\left\{\ln(e^{t+1}+t)^{\frac{1}{\sqrt[3]{t+1}-1}}\right.$$

$$\parallel$$

$$e^{\ln\left[\ln\left(e^{t+1}\left(1+\frac{t}{e^{t+1}}\right)\right)\right]\cdot\frac{1}{1/3\, t+\bar{o}(t)}}$$

$$\parallel$$

$$\left[\begin{array}{l}(1+x)^{\alpha}=1+\binom{\alpha}{1}x+\binom{\alpha}{2}x^2+\dots+\binom{\alpha}{n}x^n+\bar{o}(x^n), \text{ где}\\ \binom{\alpha}{k}=\frac{\alpha\cdot(\alpha-1)\cdots(\alpha-k+1)}{k!}\\ \Downarrow\\ \sqrt[3]{1+x}-1=1+\binom{1/3}{1}x+\bar{o}(x)-1=\frac{1}{3}x+\bar{o}(x)\end{array}\right]$$

$$\parallel$$

$$e^{\ln\left[1+\underbrace{t+\ln\left(1+\frac{t}{e^{t+1}}\right)}_{=f(t)}\right]\cdot\frac{3}{t+\bar{o}(t)}}$$

$$\parallel$$

$$\left[\text{т.к. } f(t)\to 0 \text{ при } t\to 0, \text{ то } \frac{\ln(1+f(t))}{f(t)}\to 1,\ t\to 0.\right]$$

$$\parallel$$

$$e^{\frac{\ln(1+f(t))}{f(t)}\cdot f(t)\cdot\frac{3}{t+\bar{o}(t)}}\Bigg\} = \boxed{e^{3+\frac{3}{e}}}$$

$$\left[\frac{f(t)}{t+\bar{o}(t)}=\frac{t+\ln\left(1+\frac{t}{e^{t+1}}\right)}{t+\bar{o}(t)}=\frac{1+\frac{\ln\left(1+\frac{t}{e^{t+1}}\right)}{\frac{t}{e^{t+1}}}\cdot\frac{t}{e^{t+1}}\cdot\frac{1}{t}}{1+\bar{o}(1)}\to\frac{1+\frac{1}{e}}{1+0},\ t\to 0.\right]$$

Вычислить пределы:

a) $\lim\limits_{n\to+\infty} n^{3/2}\cdot\sqrt{n+\operatorname{arctg}(1/n)}-\sqrt{n}$

b) $\lim\limits_{n\to+\infty} n^{p-1}\cdot\left[(n^p-1)^{1/p}-n\right]$, $p>0$

<u>Решение</u> a) Заметим, что:

$$\lim_{n\to+\infty}\left\{n^{3/2}\cdot\sqrt{n+\operatorname{arctg}1/n}-\sqrt{n}\right.$$

$$\parallel$$

$$n^{3/2}\cdot\frac{n+\operatorname{arctg}1/n-n}{\sqrt{n+\operatorname{arctg}1/n}+\sqrt{n}}$$

$$\parallel$$

$$\left.\frac{\underbrace{n\cdot\operatorname{arctg}1/n}_{\to 1}}{\underbrace{\sqrt{1+\frac{\operatorname{arctg}1/n}{n}}}_{\sqrt{1+0}}+1}\right\} = \frac{1}{\sqrt{1+0}+1} = \boxed{\frac{1}{2}}$$

b) Заметим, что:

$$\lim_{n\to+\infty}\left\{n^{p-1}\left[(n^p-1)^{1/p}-n\right] = n^p\left[\left(1-\frac{1}{n^p}\right)^{1/p}-1\right]\right. \ominus$$

$$\ominus\ n^p\cdot\left[\underbrace{1-\frac{1}{p}\cdot\frac{1}{n^p}+\bar{\bar{o}}\left(\frac{1}{n^p}\right)}-1\right] = -\frac{1}{p}+\bar{\bar{o}}(1)\Big\} = \boxed{-\frac{1}{p}}$$

$$(1+x)^\alpha = 1+\binom{\alpha}{1}x+\bar{\bar{o}}(x)_{x\to0}$$

⑤ [Листок 5]

Вычислить пределы:

a) $\lim\limits_{x\to 0} \dfrac{e-(1+x)^{1/x}}{\sin x}$ c) $\lim\limits_{x\to 0} \dfrac{x\sqrt[3]{\cos x}-\sin x}{x^5}$

b) $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin^2 x}-\dfrac{1}{x^2}\right)$ d) $\lim\limits_{x\to 0} \dfrac{(\cos x)^{\sin x}-\sqrt{1-x^3}}{x^6}$

Решение: a) Заметим, что:

$$\lim_{x\to 0}\left\{\frac{e-(1+x)^{1/x}}{\sin x} = \frac{e-e^{\ln(1+x)\cdot\frac{1}{x}}}{\sin x} \ominus\right.$$

$$\ominus \frac{e-e^{\left(\cancel{x}-\frac{x^2}{2}+\bar{o}(x^2)\right)\cdot\frac{1}{x}}}{\sin x} = \frac{e-e^{1-\frac{x}{2}+\bar{o}(x)}}{\sin x} \ominus$$

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\dots+(-1)^{n+1}\frac{x^n}{n}+\bar{o}(x^n),\ x\to 0$$

$$\ominus \frac{e\left(1-e^{-\frac{x}{2}+\bar{o}(x)}\right)}{\sin x} = \frac{e\left(1-\left[1\overbrace{-\frac{x}{2}+\bar{o}(x)}^{t}+\overbrace{\bar{o}\left(-\frac{x}{2}+\bar{o}(x)\right)}^{\bar{o}(t)}\right]\right)}{\sin x} \ominus$$

$$e^t = 1+t+\bar{o}(t),\quad t=-\frac{x}{2}+\bar{o}(x)$$

$$\ominus \left.\frac{e\left(+\frac{x}{2}+\bar{o}(x)\right)}{\sin x}\right\} = \boxed{\frac{e}{2}}$$

$$\bar{o}(x+\bar{o}(x)) = \bar{o}(\underline{O}(x)) = \bar{o}(x)$$

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b) Заметим, что:

$$\lim_{x\to 0}\left\{\frac{1}{\sin^2 x}-\frac{1}{x^2} = \frac{x^2-\sin^2 x}{x^2\cdot\sin^2 x} = \frac{x-\sin x}{x^2\cdot\sin x}\cdot\frac{x+\sin x}{\sin x} \ominus\right.$$

$$\ominus\ \frac{x-\left[x-\frac{x^3}{3!}+\bar{o}(x^3)\right]}{x^2\cdot\sin x}\cdot\frac{x+\sin x}{\sin x}=\underbrace{\frac{x^3}{6}\cdot\frac{1}{x^2\cdot\sin x}}_{\downarrow\ 1}\cdot\underbrace{\left(1+\frac{x}{\sin x}\right)}_{\downarrow\ 1+1}\Bigg\}=$$

$$=\frac{1}{6}\cdot 1\cdot 2=\boxed{\frac{1}{3}}$$

c) $$\lim_{x\to 0}\left\{\frac{x\cdot\sqrt[3]{\cos x}-\sin x}{x^5}\right.=\frac{x\sqrt[3]{\underbrace{1-\frac{x^2}{2!}+\frac{x^4}{4!}+\bar{o}(x^5)}_{\cos x}}-\sin x}{x^5}\ \ominus$$

$$\ominus\ \frac{x\left[1+\frac{1}{3}t+\frac{\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)}{2}\cdot t^2+\bar{o}(t^2)\right]-\sin x}{x^5}=$$

где $t=-\frac{x^2}{2}+\frac{x^4}{24}+\bar{o}(x^5)$

здесь мы взяли до t^2, т.к. $t^3\in\bar{o}(x^5)$

$$=\frac{x\left[1-\frac{1}{6}x^2+\frac{1}{3}\cdot\frac{1}{24}\cdot x^4+\bar{o}(x^5)-\frac{1}{9}\left(\frac{1}{4}x^4+\bar{o}(x^5)\right)+\bar{o}(x^4)\right]-\sin x}{x^5}$$

$$\left[\begin{array}{l} t^2=\frac{1}{4}x^4+\bar{o}(x^5)\\ \bar{o}(t^2)=\bar{o}\left(\frac{1}{4}x^4+\bar{o}(x^5)\right)=\bar{o}(x^4)\end{array}\right]$$

$$=\frac{\left[x-\frac{1}{6}x^3+\left(\frac{1}{24}\cdot\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{4}\right)\cdot x^5+\bar{o}(x^5)\right]-\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}+\bar{o}(x^5)\right]}{x^5}$$

$$\left.\frac{\overbrace{\left(\frac{1}{24}\cdot\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{4}-\frac{1}{5!}\right)}^{=\,-\frac{1}{45}}x^5+\bar{o}(x^5)}{x^5}\right\}=\boxed{-\frac{1}{45}}$$

d) Прежде всего заметим, что:

$$(\cos x)^{\sin x} = e^{\ln(\cos x)\cdot \sin x}$$

$$= e^{\ln\left(1-2\sin^2\frac{x}{2}\right)\cdot \sin x}$$

$$= e^{\left(-t - \frac{t^2}{2} + \underline{\underline{O}}(t^3)\right)\cdot \sin x}$$

где $t = 2\sin^2\frac{x}{2} = 2\cdot\left[\frac{x}{2} - \left(\frac{x}{2}\right)^3\cdot\frac{1}{6} + \bar{\bar{o}}(x^4)\right]^2 =$

$$= 2\left[\frac{x^2}{4} - \frac{x^4}{48} + \bar{\bar{o}}(x^5)\right] = \frac{x^2}{2} - \frac{x^4}{24} + \bar{\bar{o}}(x^5) = t$$

$$\Rightarrow t^2 = \left[\frac{x^2}{2} - \frac{x^4}{48} + \bar{\bar{o}}(x^5)\right]^2 = \frac{x^4}{4} + \bar{\bar{o}}(x^5)$$

$$\underline{\underline{O}}(t^3) = \underline{\underline{O}}(x^6) = \bar{\bar{o}}(x^5)$$

$$= e^{\left\{-\frac{x^2}{2} + \frac{x^4}{24} - \frac{1}{2}\left[\frac{x^4}{4}\right] + \bar{\bar{o}}(x^5)\right\}\cdot \sin x}$$

$$= e^{\left\{-\frac{x^2}{2} - \frac{1}{12}x^4 + \bar{\bar{o}}(x^5)\right\}\cdot\left(x - \frac{x^3}{3!} + \bar{\bar{o}}(x^4)\right)}$$

$$= e^{-\frac{x^3}{2} + \frac{1}{12}x^5 - \frac{1}{12}x^5 + \bar{\bar{o}}(x^6)}$$

$$= e^{-\frac{x^3}{2} + \bar{\bar{o}}(x^6)}$$

$$= 1 + \left(-\frac{x^3}{2} + \bar{\bar{o}}(x^6)\right) + \frac{1}{2}\left(-\frac{x^3}{2} + \bar{\bar{o}}(x^6)\right)^2 + \bar{\bar{o}}\left(-\frac{x^3}{2} + \bar{\bar{o}}(x^6)\right)^2$$

$$= 1 - \frac{x^3}{2} + \frac{1}{8}x^6 + \bar{\bar{o}}(x^6) = \cos x^{\sin x}$$

$$\sqrt{1 - x^3} = 1 - \frac{1}{2}x^3 - \frac{1}{8}x^6 + \bar{\bar{o}}(x^6)$$

Значит мы имеем:

$$\lim_{x\to 0}\left\{\frac{\cos x^{\sin x}-\sqrt{1-x^3}}{x^6} = \frac{\overbrace{\left(\cancel{1}-\cancel{\frac{x^3}{2}}+\frac{1}{8}x^6+\bar{o}(x^6)\right)}^{\cos x^{\sin x}}-\overbrace{\left(\cancel{1}-\cancel{\frac{1}{2}x^3}-\frac{1}{8}x^6+\bar{o}(x^6)\right)}^{\sqrt{1-x^3}}}{x^6}\right.$$

$$\left.=\frac{\left(\frac{1}{8}+\frac{1}{8}\right)x^6+\bar{o}(x^6)}{x^6}=\frac{\frac{1}{4}+\bar{o}(1)}{1}\right\}=\boxed{\frac{1}{4}}$$