EECS 461, Winter 2023, Problem Set 4: SOLUTIONS[1]

1. SOLUTION:

(a). In each case, the characteristic equation has three roots. For example, for $K = 10$, the roots lie at $-999, -1.89 \pm 2.53j$. The fast root associated with the electrical dynamics doesn't affect the response much, and the response is dominated by the other roots, which are associated with the integral controller and the motor shaft dynamics. For small values of K, these roots are real; for larger values of K, these roots are complex. When these roots are complex, these roots may be parameterized as

$$\begin{aligned} s_{\pm} &= x \pm jy \\ &= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}, \end{aligned}$$

and thus we can solve for natural frequency and damping from

$$\begin{aligned} \omega_n &= |s_+| = \sqrt{x^2+y^2} \\ \zeta &= -x/\omega_n \end{aligned}$$

For $K = 10, 50, 100$, two roots are complex, and we may use these formulas to compute the natural frequency and damping of these roots. The results are depicted in Table 1.

(b). Plots of the step and frequency responses for values of $K = 1, 10, 50, 100$ are shown in Figures 1-4, and labelled with the corresponding values of ω_n and ζ. Note that, as K increases,

- the natural frequency ω_n increases
- the damping coefficient ζ decreases
- the step response becomes both faster and more oscillatory.
- the frequency response has both a higher bandwidth and a larger peak

In particular, for $K = 10, 50, 100$ the roots are complex, and we may use the approximations above to estimate rise time and overshoot. These estimates, as well as the actual values obtained (approximately) from the step response plots, are given in Table 1.

K	ω_n	ζ	t_r (plot)	t_r (est.)	OS	OS (est.)
10	3.16	0.60	0.58	0.57	9%	9%
50	7.07	0.27	0.18	0.25	42%	40%
100	10	0.19	0.12	0.18	56%	54%

Table 1: Properties of Speed Step Response

The correspondence been estimates and actual values is reasonably close; the largest discrepancy is between the estimated and actual values of rise time for the lightly damped systems. In this case there is a more accurate approximation involving the damping coefficient that may be used.

[1] Revised November 5, 2022.

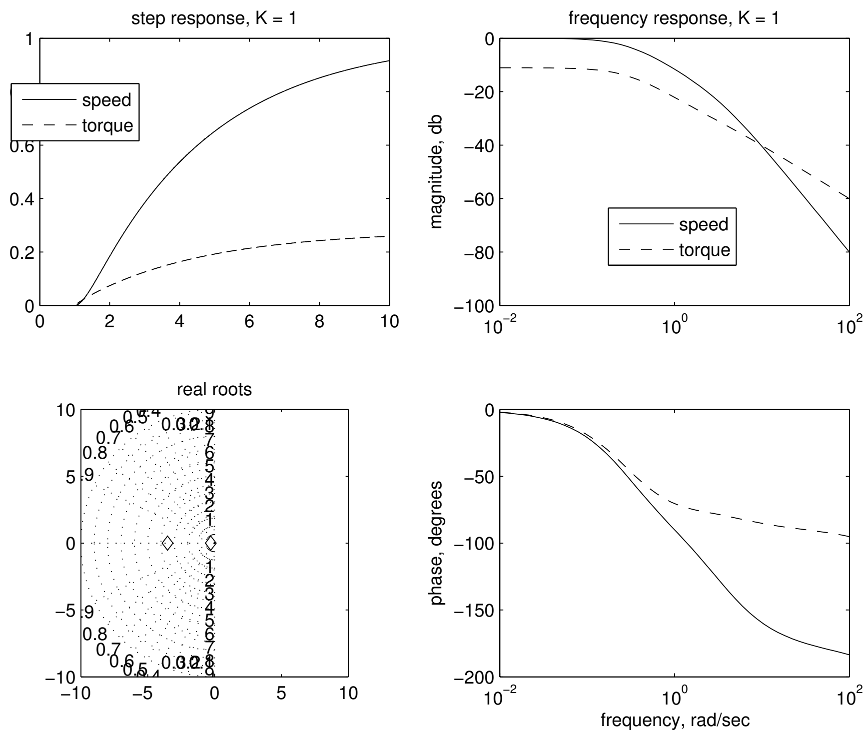

Figure 1: Response, $K = 1$

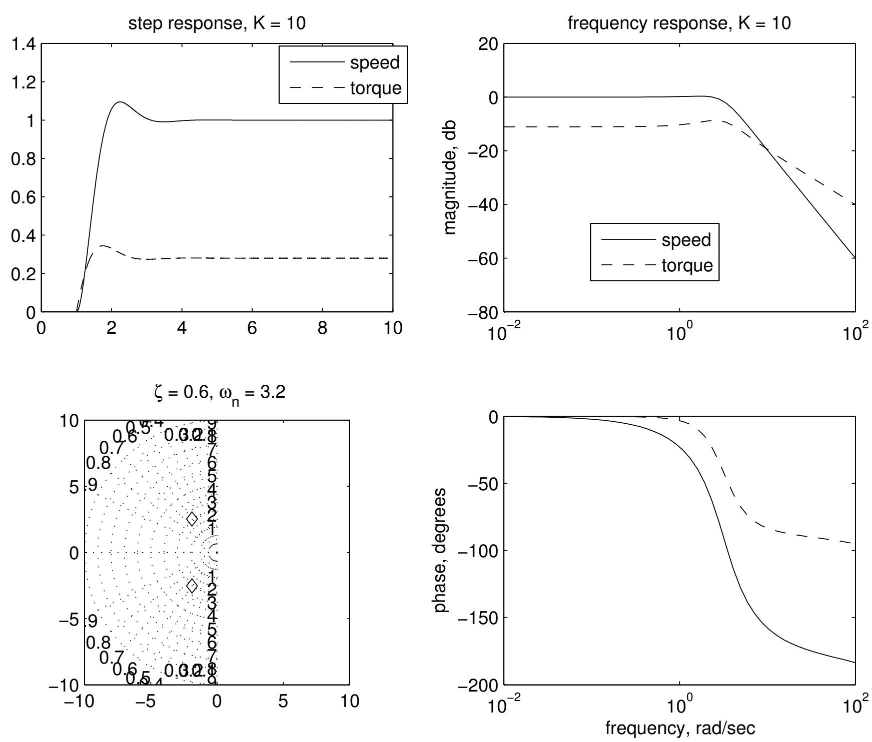

Figure 2: Response, $K = 10$

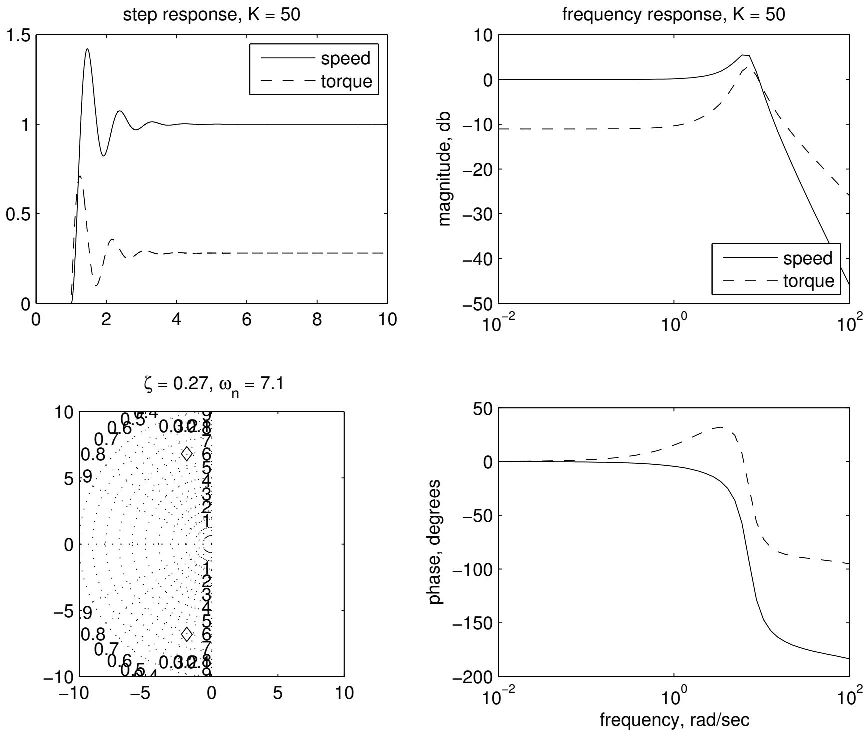

Figure 3: Response, $K = 50$

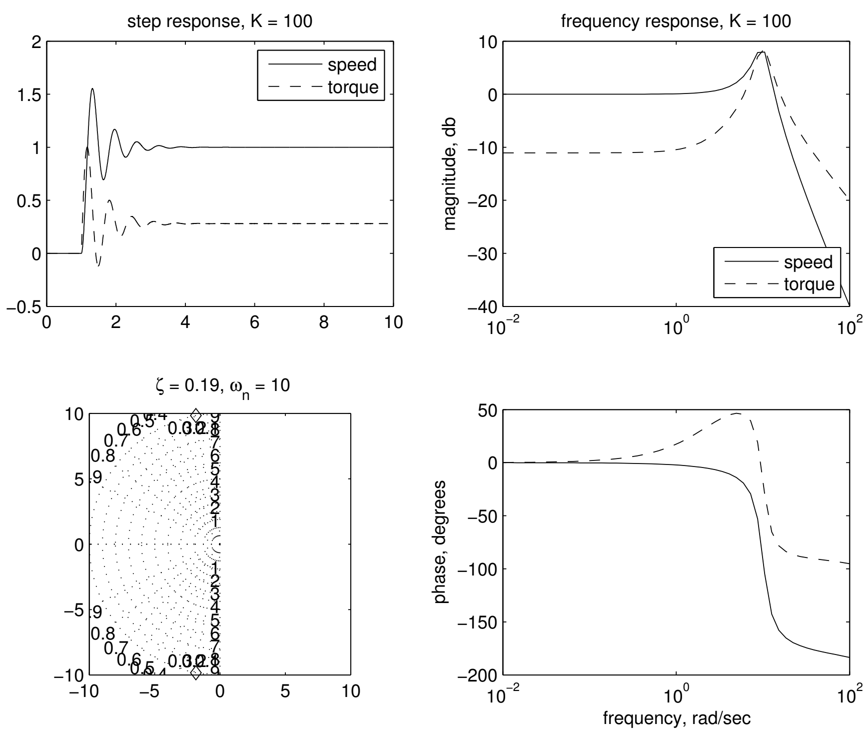

Figure 4: Response, $K = 100$

2. SOLUTION:

(a). The characteristic equation and characteristic roots are given by

$$s^2 + \frac{B}{M}s + \frac{K}{M} = 0$$

$$s_{\pm} = \frac{-B/M \pm \sqrt{(B/M)^2 - 4K/M}}{2}.$$

With $B = 0$ and $M = 1$, it follows that the roots are purely imaginary, $s_{\pm} = \pm j\sqrt{K}$, so that $\zeta = 0$ and $\omega_n = \sqrt{K}$.

Step and frequency response plots are shown in Figures 5-6. Note that, as K increases, the period of oscillation becomes shorter, and the frequency of oscillation (as determined by the peak in the frequency response plot) becomes higher.

(b). With $M = 1$ and $K = 2$, the characteristic equation is

$$s^2 + Bs + 2 = 0,$$

with characteristic roots

$$s_{\pm} = \frac{-B \pm \sqrt{B^2 - 8}}{2}$$

Hence, for $0 < B < 2\sqrt{2}$, the roots are complex, with $\omega_n = \sqrt{2}$ and $\zeta = B/(2\sqrt{2})$. For $B \geq 2\sqrt{2}$ the roots are real. Step and frequency response plots are found in Figures 7-8. Note that, as B increases from 0, the time response becomes less oscillatory, and the peak in the frequency response decreases in magnitude.

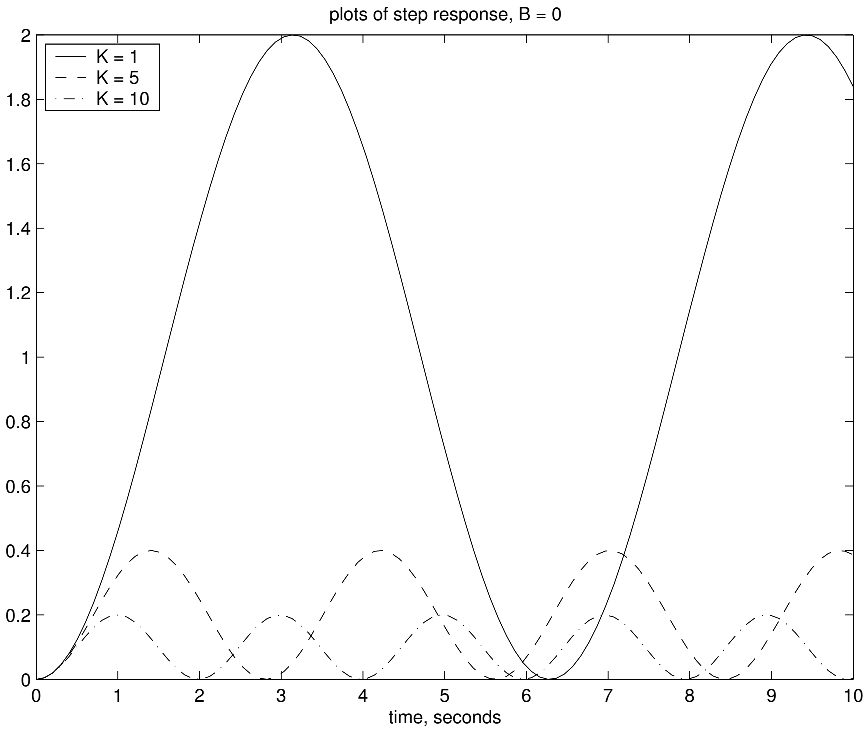

Figure 5: Step response, $B = 0$

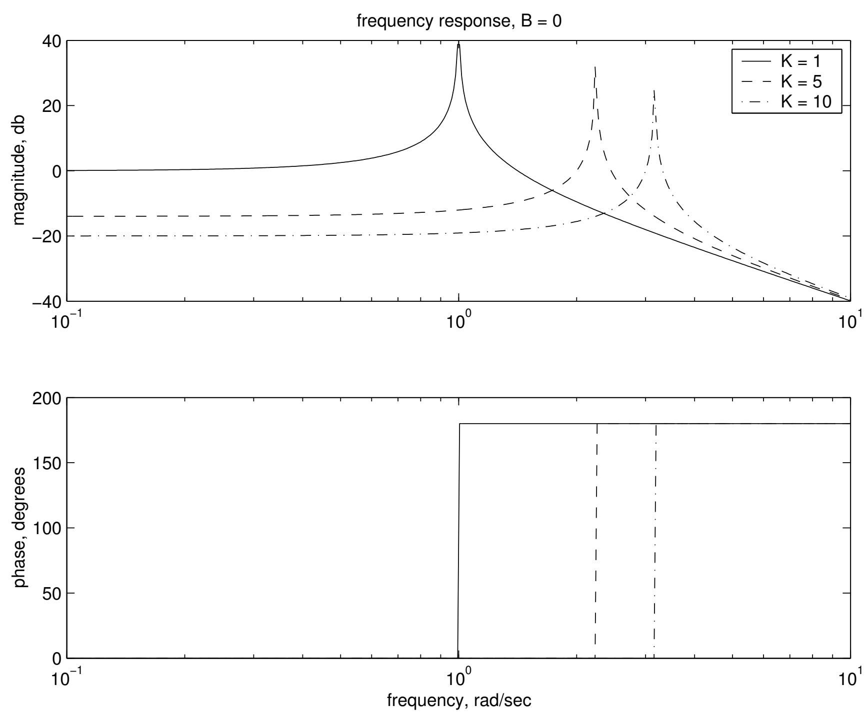

Figure 6: Frequency Response, $B = 0$

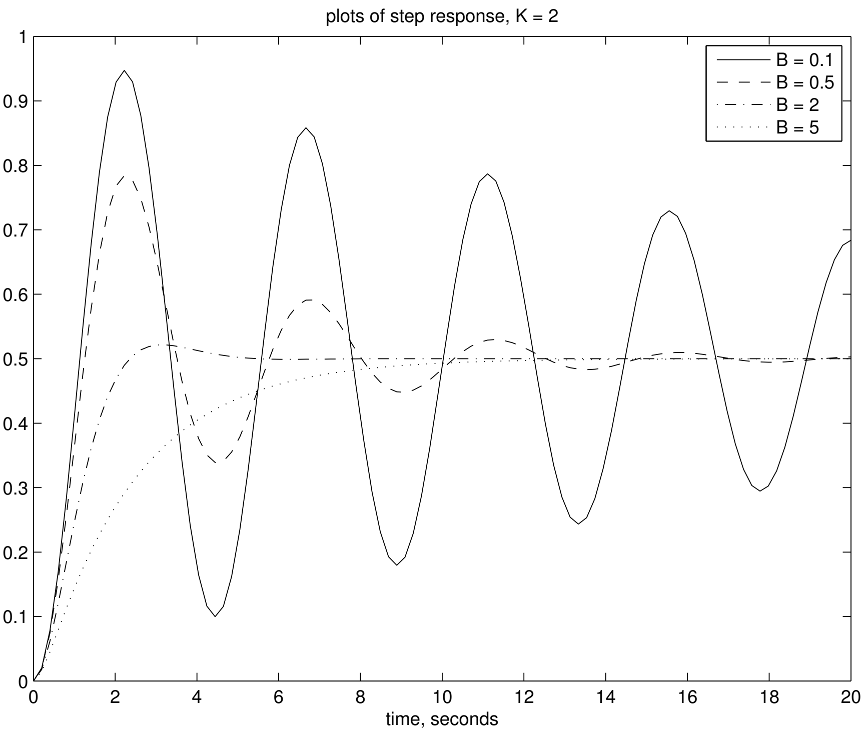

Figure 7: Step response, $K = 2$

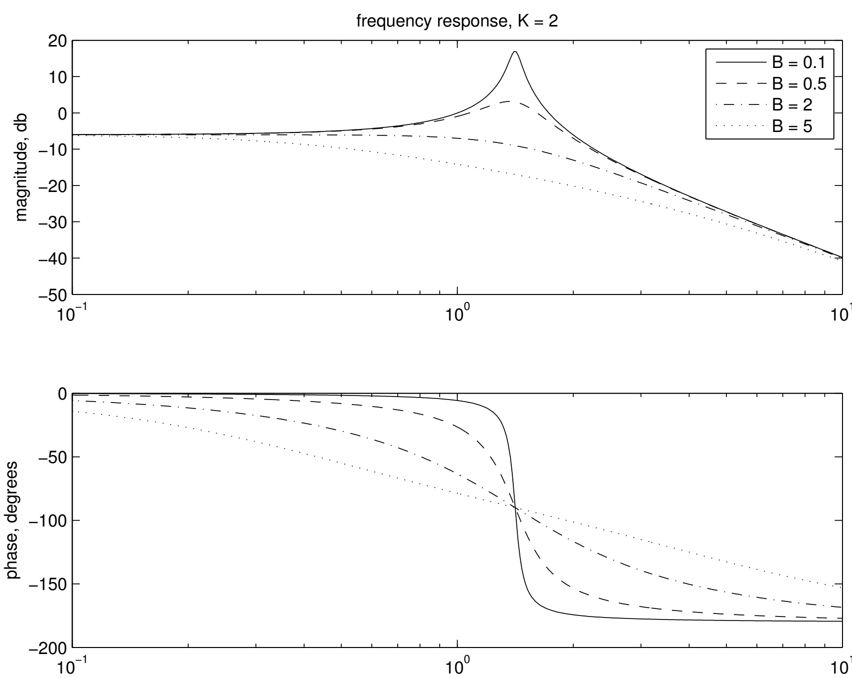

Figure 8: Frequency Response, $K = 2$

(c). With $B = 0$, the characteristic roots are those of the equation $s^2 + K/J = 0$, or $s = \pm j\sqrt{K/J}$. Hence the natural frequency of oscillations is $\omega_n = \sqrt{K/J}$ radians/second. The given values of

$J = 6.4 \times 10^{-4}$ N-m/(rad/sec^2) and $K = 10$ N-mm/degree are in incompatible units, hence we need to convert K to units of N-m/radian. Combining all this yields

$$\omega_n = \sqrt{\frac{10}{6.4 \times 10^{-4}} \frac{360 \times 0.001}{2\pi}} = 29.9 \text{ radians/second}, \tag{1}$$

or $f_n = 4.8$ Hz.

(d). With virtual damping, the characteristic equation becomes $s^2+(B/J)s+K/J = 0$, or $s = (1/2)(-B/J \pm \sqrt{(B/J)^2 - 4K/J}$. For critical damping, we require that the roots be real and repeated, and thus that $(B/J)^2 - 4K/J = 0$, or $B = 2\sqrt{KJ}$. With our data, this becomes

$$B = 2\sqrt{(10 \times 0.001 \times 360/2\pi) \times 6.4 \times 10^{-4}} = 3.8 \times 10^{-2} \text{ N-m/(radian/second)}. \tag{2}$$

(e). With the convention we use for the PWM signal (100% means max torque in one direction, 0% means max torque in the opposite direction), it follows that changing the POL bit changes the sign of the commanded torque. Hence instead of negative feedback, we are effectively using positive feedback. The characteristic roots become $s = \pm\sqrt{K/J}$, and thus lie at $s = \pm 29.9$. Since one of these lies in the open right half plane, we see that the system is unstable.

3. SOLUTION:

(a).

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{J} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{k}{J} \end{bmatrix} \theta_z^*$$

(b). In general, $f_n = \sqrt{k/J}/2\pi$ Hz. The variables k and J need to be represented in compatible units; changing k into units compatible with J yields:

$$10 \frac{\text{N-mm}}{\text{degree}} \times \frac{1\text{N-m}}{1000\text{N-mm}} \times \frac{360\text{degrees}}{2\pi\text{radians}} = 0.573 \frac{\text{N-m}}{\text{radian}}.$$

Hence

$$f_n = \sqrt{k/J}/2\pi = 4.76 \text{ Hz}.$$

(c).

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{J} & -\frac{b_m}{J} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{k}{J} \end{bmatrix} \theta_z^*$$

(d).

$$\begin{aligned} \det(\lambda I - A) &= \lambda^2 + \frac{b_m}{J}\lambda + \frac{k}{J} \\ &= \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \\ \Rightarrow \omega_n &= \sqrt{k/J} \\ \zeta &= \frac{1}{2}\frac{b_m}{\sqrt{kJ}} \end{aligned}$$

Substituting numerical values yields

$$\zeta = \frac{1}{2}\frac{b_m}{\sqrt{kJ}} = \frac{1}{2} \times \frac{0.0012}{0.0191} = 0.0313$$

(e). For critical damping, we need that $\zeta = 1$, and thus we require

$$\begin{aligned} \frac{1}{2}\frac{b + b_m}{\sqrt{kJ}} &= 1 \\ \Rightarrow b &= 2\sqrt{kJ} - b_m. \end{aligned}$$

Hence we should set

$$b = 0.0371 \text{ N-m/(rad/sec)}.$$

4. SOLUTION:

(a). The characteristic equation is

$$Ms^2 + Bs + K = 0, \qquad \text{or} \qquad s^2 + (B/M)s + (K/M) = 0.$$

The natural frequency and damping may be found by equating coefficients, as in

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0.$$

It follows that

$$\omega_n = \sqrt{K/M}, \qquad \zeta = B/2\sqrt{MK}.$$

(b). With $K = 10$, the discriminant is $B^2 - 4MK = B^2 - 80$. Hence the roots will be real and distinct for $B > \sqrt{80}$, real and repeated for $B = \sqrt{80}$, and complex for $B < \sqrt{80}$.

(c). Plots of the response to initial conditions and characteristic roots for $B = 0$ (no damping), $B = 1$ (underdamped), and $B = 10$ (overdamped) are found in Figures 10-15. Note that you may monitor the location of the puck with respect to the wall by adding a scope to the SIMULINK diagram, as shown in Figure 9.

To understand why the puck sometimes leaves the wall, and sometimes not, suppose temporarily that the spring force is applied on both sides of the wall. In all cases the puck will eventually settle at the position of the wall ($z = z_w$), where the spring force is zero. If the roots are underdamped, then the puck will oscillate about its final position. If the roots are critically damped or over damped, then the puck will approach its final position monotonically, and thus never leave the wall. Now consider the virtual wall, for which the force drops to zero once the puck is outside the wall. In the case of underdamped roots, the puck will leave the wall with some velocity; once outside the wall, the applied force drops to zero, and the puck will continue to move away from the wall indefinitely. In the case of critically damped or overdamped roots, the puck will never leave the wall, and thus never enter the region of zero applied force. The difference between the two types of behavior thus depends on the value of the damping coefficient; for $K = 10$, the puck will not leave the wall if $B \geq \sqrt{80}$. Plots of the root locations and the resulting time response for $B = \sqrt{80}$ are found in Figures 16-17.

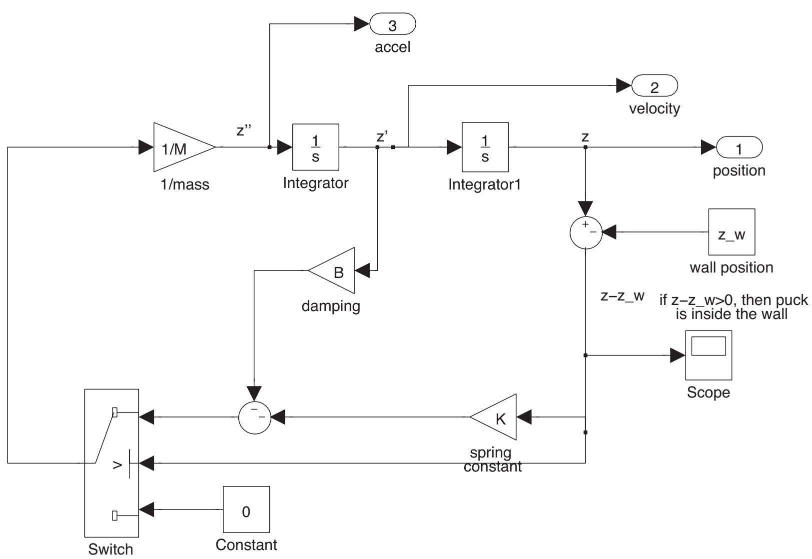

Figure 9: SIMULINK Model of Virtual Wall with Spring and Damping

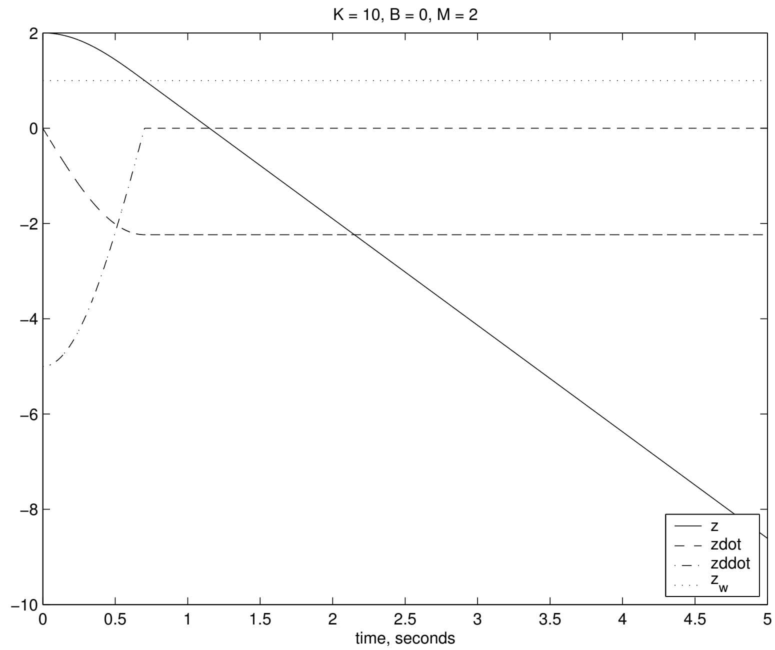

Figure 10: Response to Initial Condition with $B = 0$

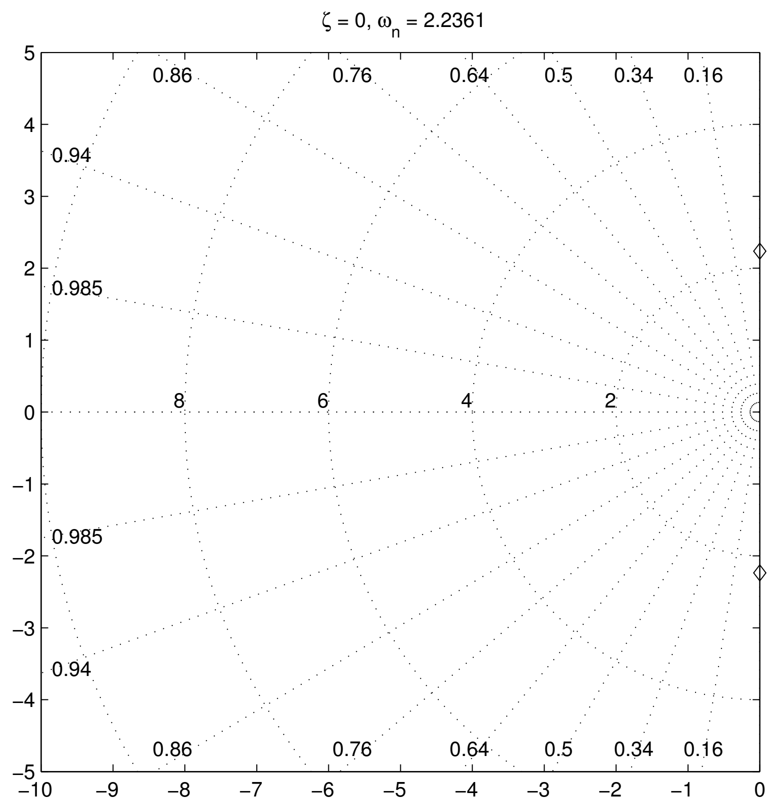

Figure 11: Characteristic Roots with $B = 0$

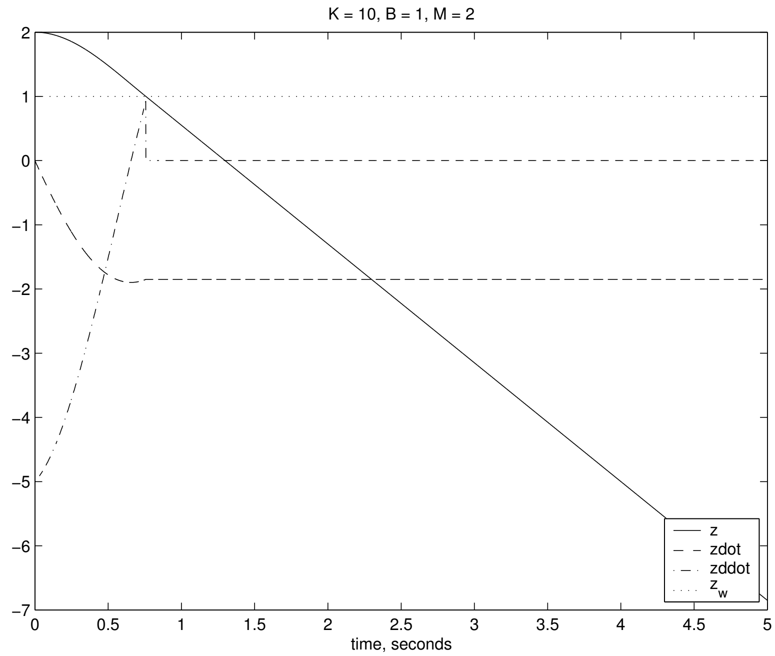

Figure 12: Response to Initial Condition with $B = 1$

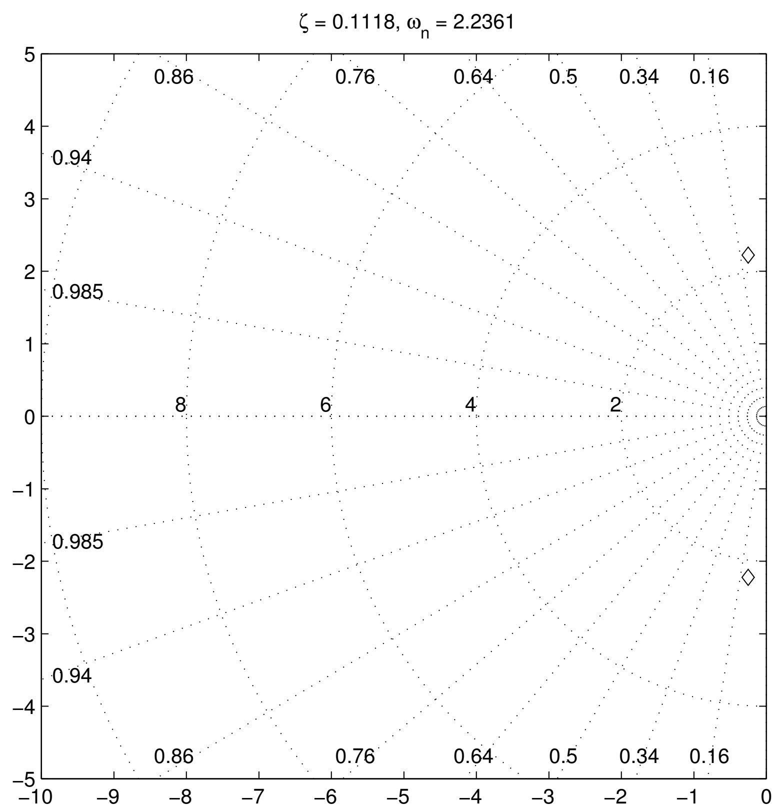

Figure 13: Characteristic Roots with $B = 1$

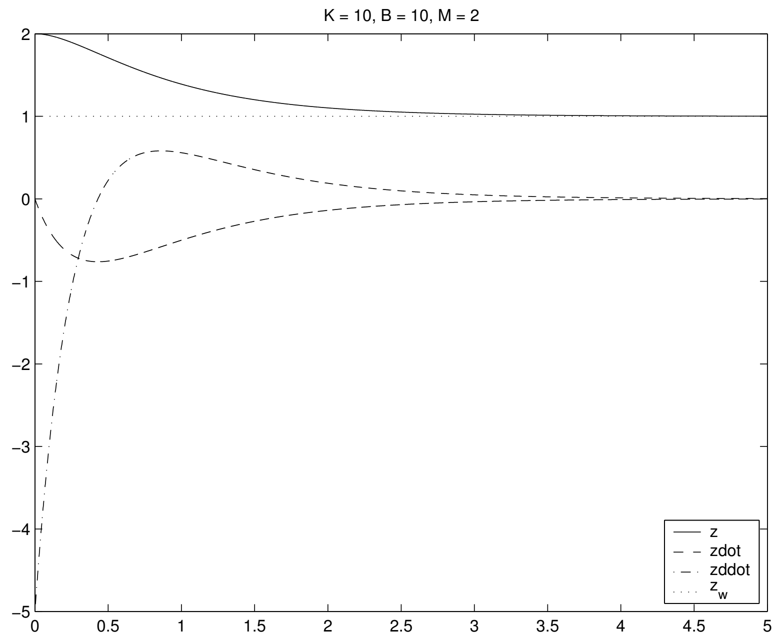

Figure 14: Response to Initial Condition with $B = 10$

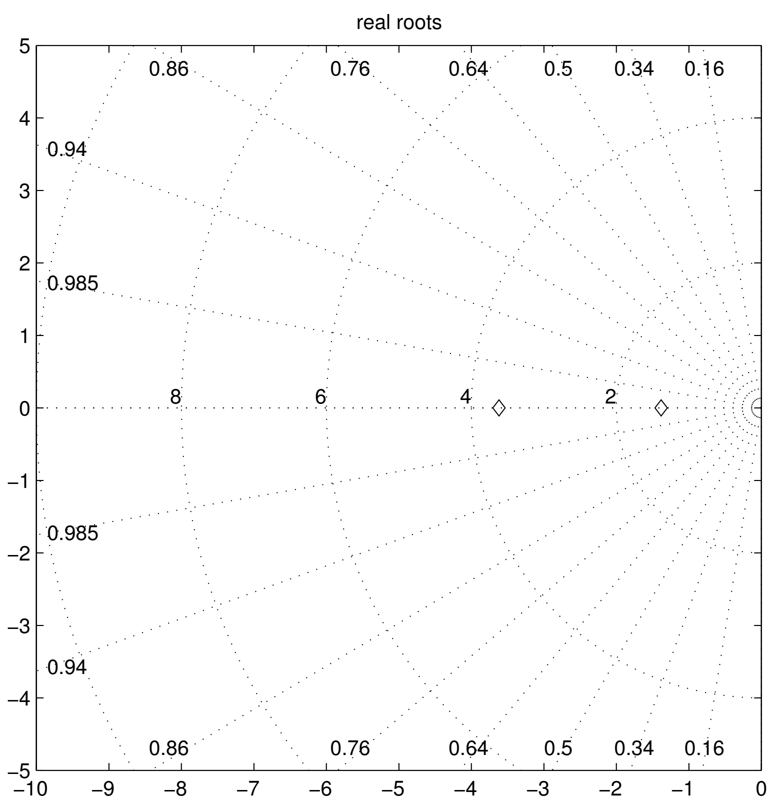

Figure 15: Characteristic Roots with $B = 10$

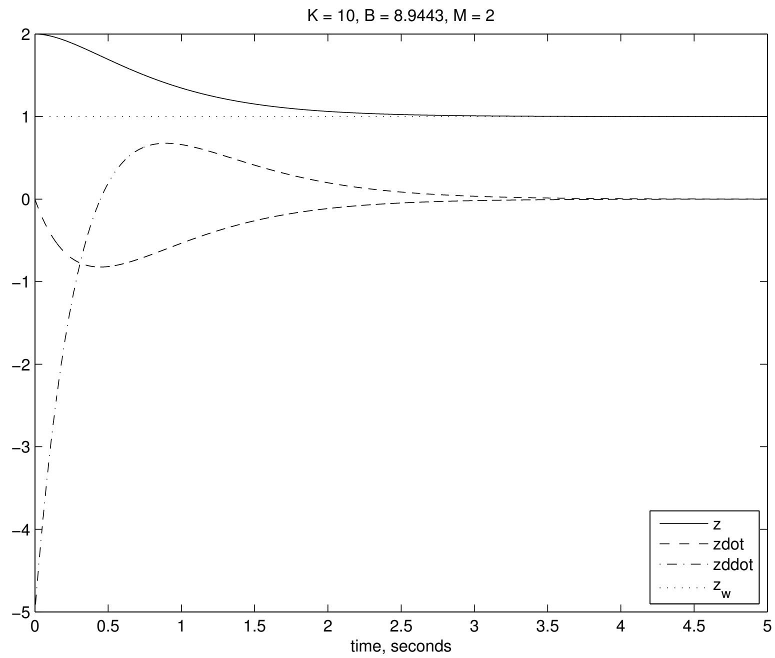

Figure 16: Response to Initial Condition with $B = \sqrt{80}$

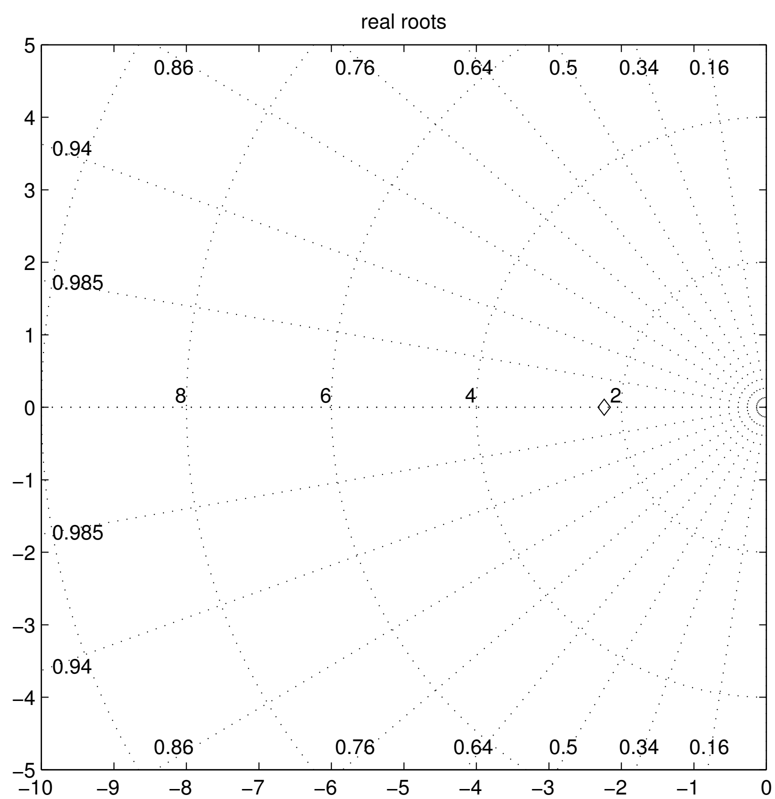

Figure 17: Characteristic Roots with $B = \sqrt{80}$