EECS 461, Winter 2023, Problem Set 1: SOLUTIONS[1]

1. Consider a thermocouple that gives an output voltage of 0.5 mV/°F. Suppose we wish to measure temperatures that range from −20°F to 120°F with a resolution of 0.5°F.

   (a) If we pass the output voltage through an n-bit A/D converter, what word length n is required in order to achieve this resolution? Note: since bits come in whole numbers, it may not be possible to achieve this resolution exactly. In that case round up to the smallest integer number of bits that achieves *at least* this resolution.

   SOLUTION: The range of temperatures that needs to be measured is $140^\circ \times 0.5$ mV$/^\circ F = 70$mV, with a resolution of $0.5^\circ \times 0.5$ mV/°F $= 0.25$mV. Hence n should be chosen so that

   $$\frac{70}{2^n} \leq 0.25 \quad \Rightarrow \quad n \geq 8.13.$$

   Since n must be an integer, a 9-bit wordlength is required.

   (b) Does you answer to (1a) change if the maximum binary value $1 \ldots 1$ is used to represent V_{max} uniquely? Explain.

   SOLUTION: The voltage range is now divided into one less interval than previously, so the resolution is

   $$\frac{V_{max}}{2^n - 1}.$$

   However, the answer is unchanged, as a 9-bit wordlength is still required:

   $$\frac{70}{2^n - 1} \leq 0.25 \quad \Rightarrow \quad n \geq 8.13.$$

   (c) Return to the scheme of (1a), and choose the minimum value of n that will achieve the desired resolution. The binary representation of −20°F is equal to zero. What is the lowest temperature that has a nonzero binary representation, assuming unipolar coding without centering?

   SOLUTION: With 9 bits, the least significant bit represents

   $$\frac{70\text{mV}}{2^9} = 0.137\text{mV}.$$

   The LSB corresponds to

   $$\frac{0.137\text{mV}}{0.5\text{mV}/^\circ\text{F}} = 0.274^\circ\text{F}.$$

   Hence the lowest temperature that has a nonzero binary representation is $-20^\circ + 0.274^\circ = -19.726^\circ$F.

   (d) How does the answer to (1c) change if centering is used?

   SOLUTION: With centering, the smallest voltage that yields a nonzero digital value corresponds to 1/2 LSB, or

   $$\frac{1}{2}\frac{70\text{mV}}{2^9} = 0.068\text{mV},$$

   which in turn represents

   $$\frac{0.068\text{mV}}{0.5\text{mV}/^\circ\text{F}} = 0.137^\circ\text{F}.$$

   Hence the lowest temperature that has a nonzero binary representation is $-20^\circ + 0.137^\circ = -19.863^\circ$F.

[1] Revised November 5, 2022.

2. Consider a rotating wheel with *one* mark painted on it, and suppose that the wheel rotates with a constant rate R revolutions/second in a counterclockwise (CCW) direction. Suppose also that we make a movie (as with a camcorder), which consists of still photographs taken at regularly spaced intervals of T seconds.

   We saw in class that if $T = 1/R$, then the wheel will appear to be stationary. Furthermore, in order that the movie depicts the correct direction of rotation of the wheel, it is necessary that $T < 1/2R$.

   Recall the movie from [2] in which the wheel had *four* arrows, 90° apart, painted on it, and suppose also that this wheel rotates at a constant rate CCW.

   (a). What is the smallest value of T that will now make the wheel appear stationary in the movie?

   (b). What is the largest value T can take in order that the direction of rotation appear correctly?

   (c). Suppose that the camcorder shoots pictures at a rate of 30 frames/second. What is the maximum value of rotation rate R for the wheel with one painted arrow that will allow the direction of rotation to appear correctly?

   (d). Suppose again that the camcorder shoots pictures at a rate of 30 frames/second. What is the maximum value of rotation rate R for the wheel with four painted arrows that will allow the direction of rotation to appear correctly?

   (e). How do the answers to these questions change if the wheel has N equally spaced arrows?

   SOLUTION:

   (a). Since the four arrows are indistinguishable, taking pictures at intervals equal to $T = 1/4R$ seconds will make the wheel appear stationary.

   (b). To obtain the correct direction of rotation, we must require that $T < 1/8R$ seconds.

   (c). The camcorder "samples" the position of the wheel every $T = 1/30$ seconds. For the wheel with one arrow, $T < 1/2R \Rightarrow R < 1/2T$, so that with $T = 1/30$ sec, we require $R < 15$ rev/sec.

   (d). For the wheel with four arrows, $T < 1/8R \Rightarrow R < 1/8T$, so that with $T = 1/30$ sec, we require $R < 15/4$ rev/sec.

   (e). The answer to (2a) becomes $T = 1/NR$ seconds, the answer to (2b) becomes $T < 1/2NR$ seconds, and the answer to (2d) becomes $T < 1/2NR \Rightarrow R < 1/2NT$, so that with $T = 1/30$ sec, we require $R < 15/N$ rev/sec.

3. Consider a sensor that generates an analog signal which must be sampled and passed through an A/D converter. We have seen that undersampling may cause the sampled signal to look very different from the original signal. In particular, noise at frequencies greater than the Nyquist frequency $f_N = 1/2T$ Hz will be "aliased" and appear as low frequency signals.

   In Lecture 2 we discussed the use of an analog lowpass filter to remove high frequency noise from the analog signal before sampling. Such a filter is referred to as an *antialiasing filter*.

   We also showed in Lecture 2 how a simple lowpass filter may be constructed using an RC circuit. (Antialiasing filters may also be built with active, op-amp circuits.)

   Suppose we wish to sample a sinusoidal signal with a sample period equal to T seconds. The sampling frequency is equal to $\omega_S = 2\pi/T$ radians/second, or $f_S = 1/T$ Hz.

   (a). Let $T = 0.09$ seconds, and suppose the sinusoid has frequency equal to 1 Hz: $x(t) = \sin(2\pi t)$. According to the Shannon sampling theorem, is it theoretically possible to reconstruct the sinusoid from its samples? That is, does the frequency of the sinusoid lie below the *Nyquist frequency* which is $f_N = 1/2T$ Hz, or $\omega_N = \pi/T$ radians/second?

   SOLUTION: The Nyquist frequency is $f_N = 1/(2 \times 0.09) = 5.56$ Hz. Hence the frequency of the sinusoid is below the Nyquist frequency, and it can theoretically be reconstructed from its samples.

(b). Now suppose that the signal we wish to sample is contaminated by 8 Hz noise: $x_{noisy}(t) = \sin(2\pi t) + 0.5\sin(16\pi t)$. How will the presence of the noise signal distort the sampled signal? What will the signal look like if the samples are passed through a zero order hold (ZOH)?

SOLUTION: Because the 8 Hz noise signal has a frequency above that of the Nyquist frequency (8 Hz $>$ 5.56 Hz), it will be "aliased" upon sampling, and appear as a lower frequency signal. It is interesting to determine the value of this lower frequency. To do so, note that two sinusoids, $\sin(\omega_1 t)$ and $\sin(\omega_2 t)$, will yield the same samples if $\sin(\omega_1 kT) = \sin(\omega_2 kT), k = 0, \pm 1, \pm 2, \ldots$. This fact implies that $\omega_1 T = \omega_2 T + 2\pi N$, for some (positive or negative) integer N. Hence after sampling the noise signal, whose frequency is $\omega_1 = 16\pi$, will be indistinguishable from any other signal whose frequency satisfies $\omega_2 = 16\pi - 2\pi N/.09$ for some integer N. This includes the value $N = 1$, and so the noise signal will be indistinguishable from a sinusoid of frequency $\omega_2 = 16\pi - 2\pi/.09 = -19.5$ rad/sec, or -3.11 Hz. Because the frequency spectrum has both positive and negative components, there will also be a component at $+3.11$ Hz.

(c). Use the Matlab m-file "aliasing.m" and the SIMULINK model "noisy_signal.slx" to compare the time responses of the clean signal, the signal contaminated with noise, and the reconstructed signal. Are the responses you see consistent with your analysis above? Explain.

SOLUTION: When added to the "information" signal, the result will be to distort the signal as shown in Figure 1. Note that the effect of the aliased noise signal is to sometimes make the reconstructed

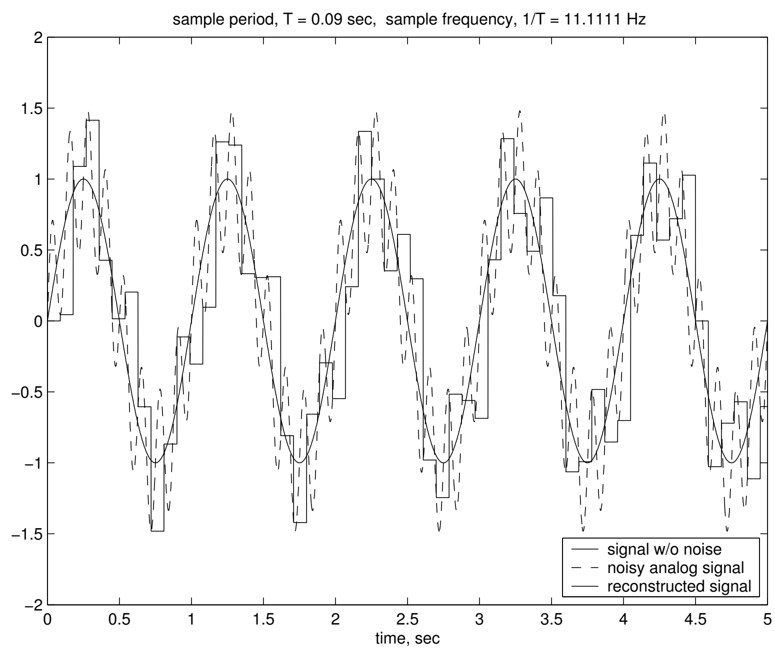

Figure 1: Noisy Signal Passed Through a Sample and Hold

signal have smaller amplitude than the "information" signal, and to sometimes have larger amplitude. If we were to plot the signal for a longer time period, we would see that this behavior persists.

(d). Modify the SIMULINK model "noisy_signal.slx" to create a new model, "filtered_signal.slx" that includes a lowpass filter. The frequency response of this filter, together with the resulting reconstructed signal, may be plotted using the m-file "aliasing_w_filter.m". What will the filtered signal look like after it is passed through a sample and hold? The lowpass filter in this figure can be adjusted by changing the parameter "RC". Try various values of RC, say $RC = .001, .01, .1, 1, 10$. What trends do you see?

SOLUTION: The reconstructed filtered signal will exhibit less distortion due to the aliased noise signal, but will also be altered in gain and phase due to the filter. Small values of RC will result in a high bandwidth filter that passes more of the noise. For example, setting $RC = 0.001$ results in a filter whose gain is nearly unity at both 1 Hz and 8 Hz, and the resulting reconstructed signal is almost identical to that in Figure 1. Setting $RC = 10$ results in a filter that greatly attenuates both signal and noise. An intermediate value, such as $RC = 0.1$, provides significant noise filtering, slightly diminishes the amplitude of the 1 Hz signal, and introduces roughly 30° phase lag into this signal.

(e). Design an antialiasing filter (i.e., pick a value of RC) that will reduce the noise component of the analog signal by (approximately) a factor of 10 (-20 db). (To do so, examine the frequency response plot of the filter.)

SOLUTION: The filter obtained by setting $RC = 1/5$ will reduce magnitude of the 8 Hz noise signal by -20 db. This may be determined by looking at the frequency response plots in Figure 2. The resulting time response, in Figure 3, shows that the reconstructed signal has a more regular appearance, and looks more like the filtered analog signal.

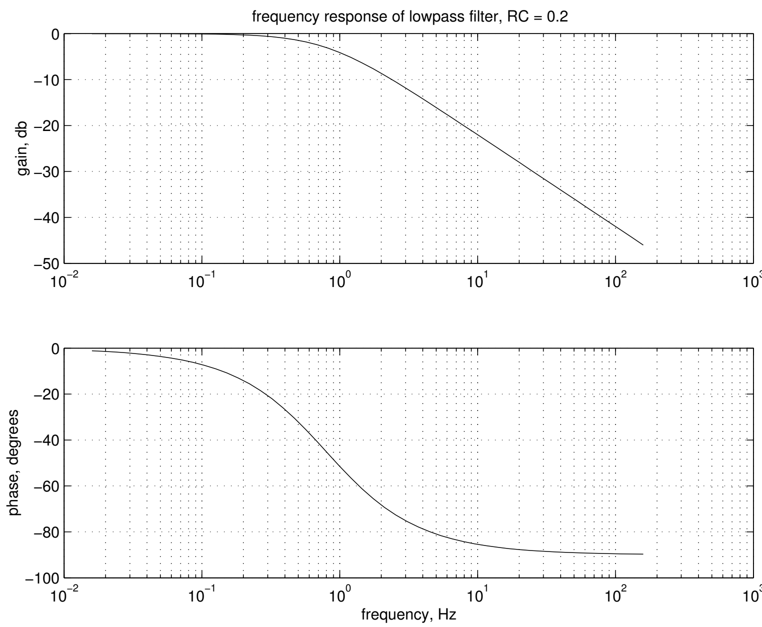

Figure 2: Frequency Response for $RC = 1/5$

Note that the anti-aliasing filter also changes the 1 Hz component of the signal. From the frequency response plots, we see that the filter will add 50° phase lag and 4 db gain reduction to a 1 Hz sinusoid. This explains why the filtered analog signal lags the original 1 Hz sinusoid and also has smaller amplitude.

The noise component of the signal can be further reduced by using a larger value of RC; however, a larger value of RC will also filter out more of the good signal.

(f). Examine the frequency spectra of the noisy sinusoid, the filtered sinusoid, and the output of the ZOH. In each case, you will see peaks in the response at various frequencies. Why are these peaks located where they are? To compute the frequency spectra, you may use the Matlab file "frequency_spectra.m", together with the auxiliary Matlab file "contfft.m", taken from [1]. Hand in one set of frequency spectra for the value of RC that achieves the desired attenuation of the noise signal.

SOLUTION: The frequency spectrum of the noisy sinusoid has components at ± 1 Hz, the frequency of the "good" signal, and at ± 8 Hz, the frequency of the "noise" signal. After filtering, the latter peaks are greatly reduced. After sampling and reconstruction, there are now small peaks at ± 3 Hz, due to the 8 Hz signal being aliased. The zero order hold reconstruction adds additional peaks at integer multiples of the sampling frequency away from the peaks at ± 1 and ± 3 Hz. This is responsible for the groups of four small peaks seen centered at ± 11 Hz, etc. (Recall these frequencies build the "corners" in the reconstructed signal.) To see the 3 Hz alias more clearly, in Figure 5 we plot the

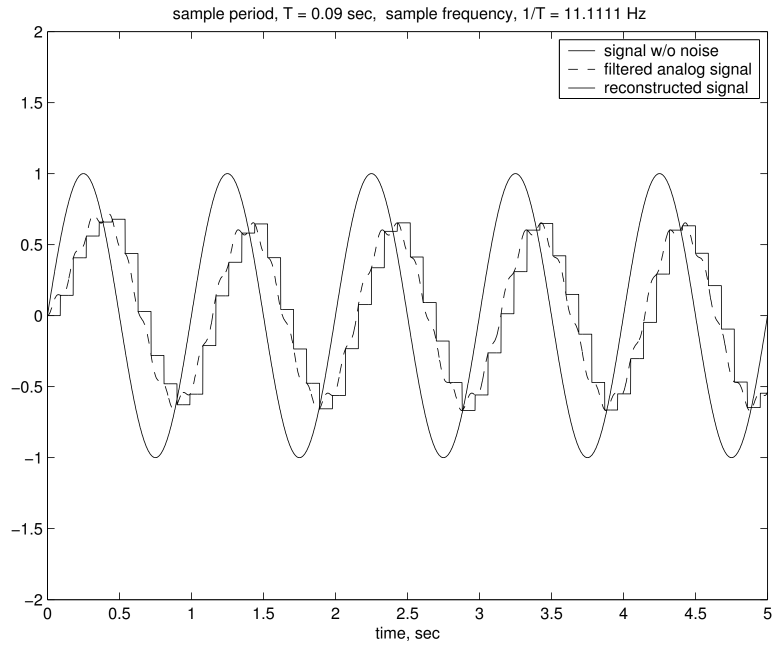

Figure 3: Reconstructed Signal for $RC = 1/5$

frequency spectrum of the reconstructed signal with no antialias filter present.

References

[1] E. Kamen and B. Heck. *Fundamentals of Signals and Systems using MATLAB*. Prentice Hall, 1997.

[2] J. H. McClellan, R. W. Schafer, and M. A. Yoder. *DSP First: A Multimedia Approach*. Prentice-Hall, 1998.

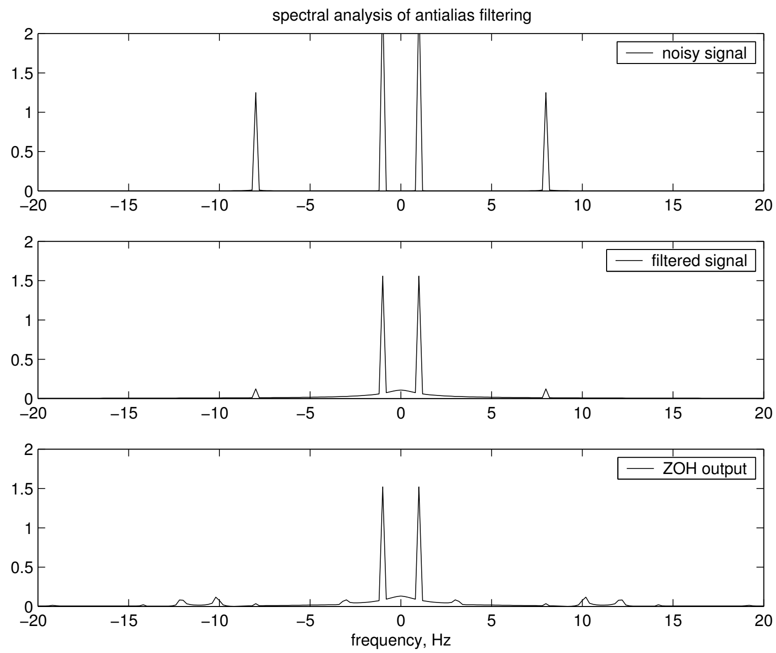

Figure 4: Fourier Transform of Reconstructed Signal

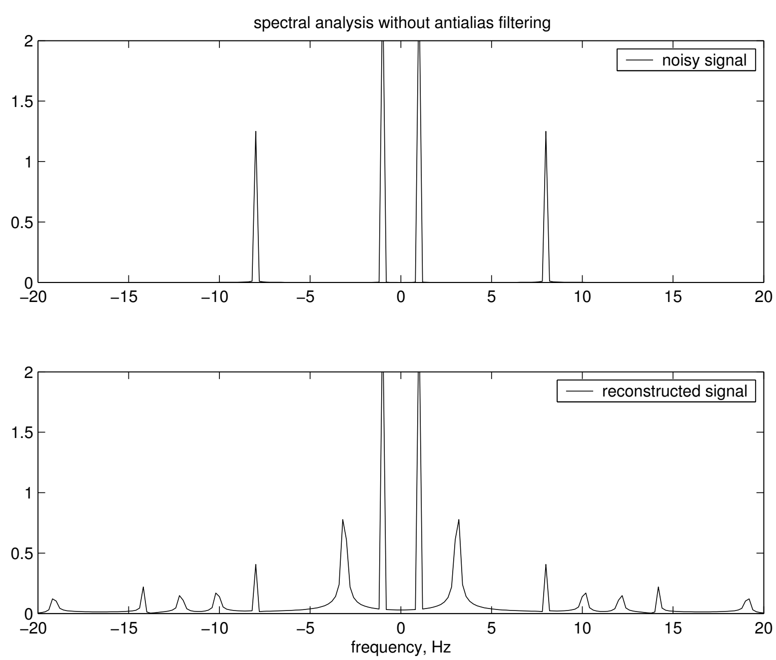

Figure 5: Frequency Spectrum of Reconstructed Signal with No Antialiasing Filter