# EECS 461, Winter 2023, Problem Set 5: SOLUTIONS[1]

1. SOLUTION:

   Matlab code for the 2nd and 4th order Runge Kutta algorithms:

```
clear x

A = -2;  % state equation xdot = Ax + Bu
B = 2;
x0 = 0;  % initial condition
T = .1;  % integration step
u = 1;  % unit step at t = 0

x(1) = x0;
for n = 1:50
    k1 = A*x(n)+B*u;
    xh = x(n) + 0.5*T*k1;
    k2 = A*xh + B*u;
    x(n+1) = x(n) + T*k2;
end
xplot1 = x;

clear x

x(1) = x0;
N = 50;
for n = 1:N
    k1 = (A*x(n)+B*u)*T;
    xh = x(n)+ 0.5*k1;
    k2 = (A*xh + B*u)*T;
    xh = x(n)+ 0.5*k2;
    k3 = (A*xh + B*u)*T;
    xh = x(n)+ k3;
    k4 = (A*xh + B*u)*T;
    x(n+1) = x(n) + (1/6)*(k1+2*k2+2*k3+k4);
end
xplot2 = x;

tstart = 0;tfinal = 5;
options = simset('Solver','ode2');
sim('test_integration',[tstart tfinal],options)
tout1 = tout;simout1=simout;
options = simset('Solver','ode4');
sim('test_integration',[tstart tfinal],options)
tout2 = tout;simout2=simout;
figure(1)
clf
plot(T*[0:N],xplot1,'+',T*[0:N],xplot2,'*',tout1,simout1,'-',tout2,simout2,'--')
legend('2nd order Matlab','4th order Matlab','2nd order Simulink','4th order Simulink')
xlabel('time, seconds')
```

[1] Revised November 7, 2022.

```
figure(2)
clf
plot(T*[0:N],xplot1,'+',T*[0:N],xplot2,'*',tout1,simout1,'-',tout2,simout2,'--')
legend('2nd order Matlab','4th order Matlab','2nd order Simulink','4th order Simulink',2)
xlim([.75 1.2])
xlabel('time, seconds')
```

Plots of the simulations, including a detail showing agreement between the handcoded method and the Simulink model result are found in Figures 1-2.

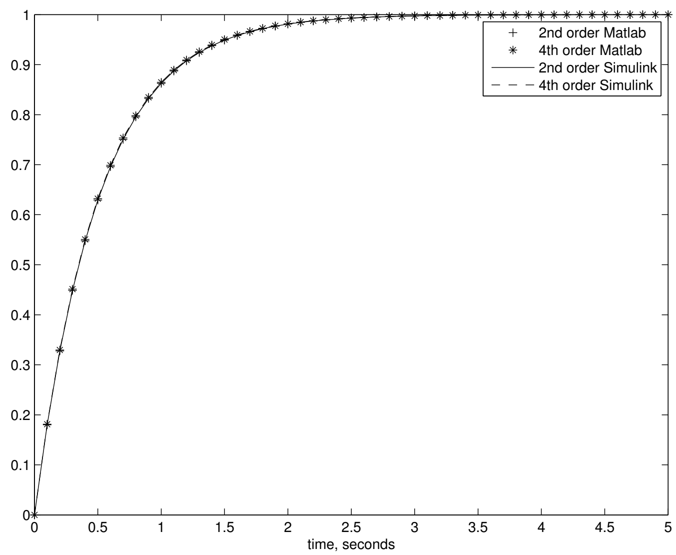


Figure 1: Simulation.

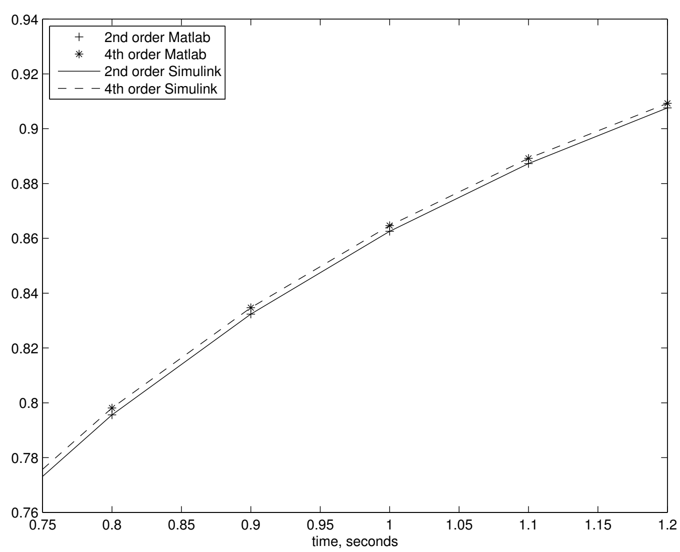


Figure 2: Detail of simulation.

2. (a) A Matlab simulation of the puck behavior, using Stateflow, is shown in Figure 3, and a comparison between this simulation and one using Simulink only is in Figure 4.

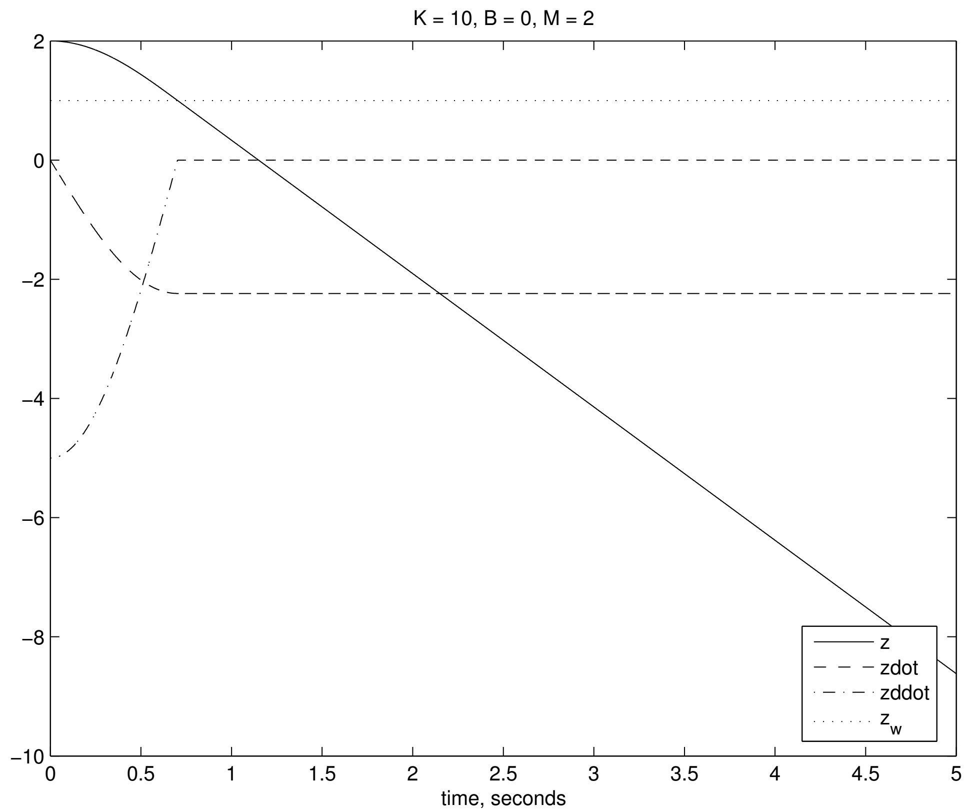


Figure 3: Simulation of the puck behavior obtained using the Stateflow model.

Even though the Stateflow and the Simulink implementations are functionally identical, they introduce slightly different amounts of computational delay. Zooming, as shown in Figure 5, reveals that the plot generated using Simulink lags that generated using Stateflow only.

(b) The Stateflow model is shown in Figure 6, and the response of the puck for $B = 0$ is in Figure 7.

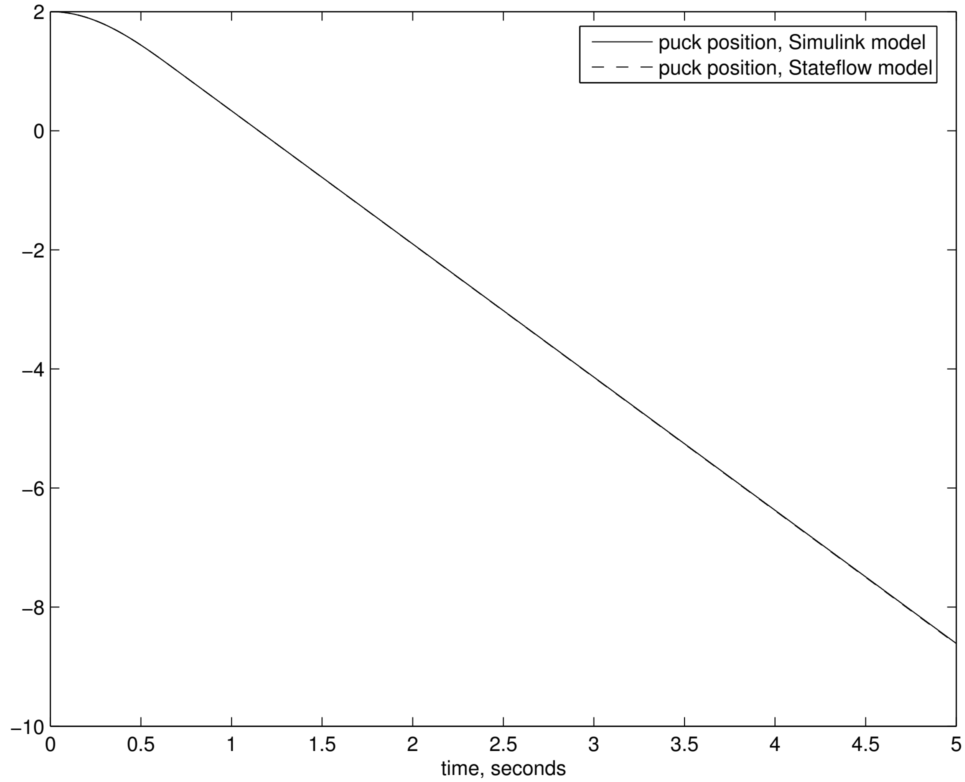


Figure 4: Comparison between the Stateflow and the Simulink only simulations.

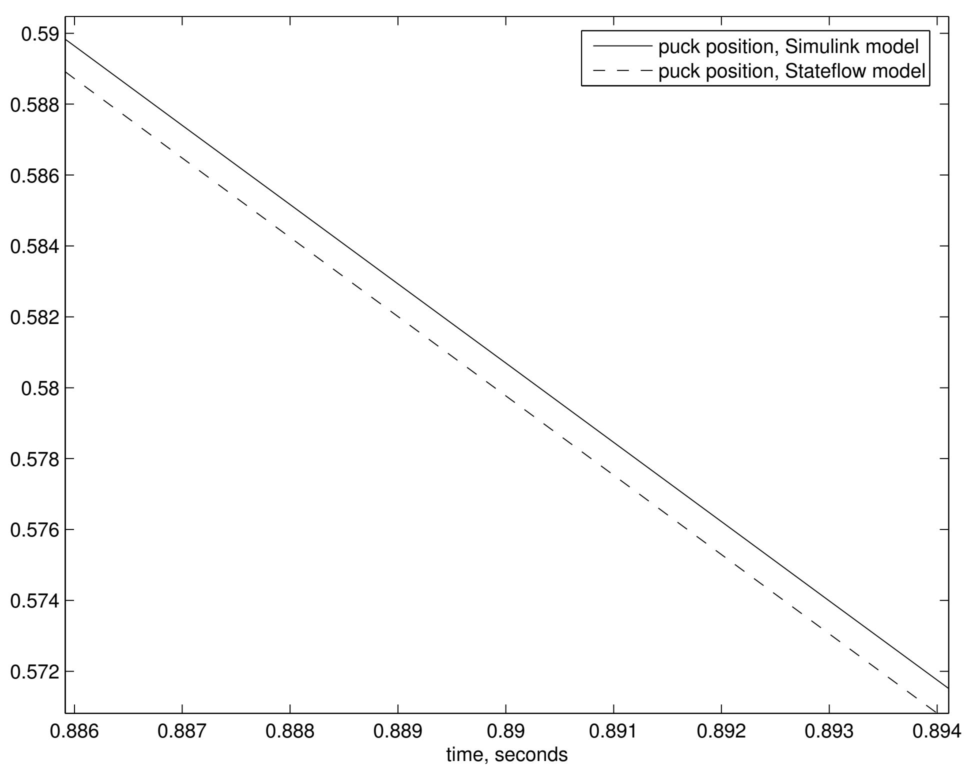


Figure 5: Comparison between the Stateflow and the Simulink only simulations (detail).

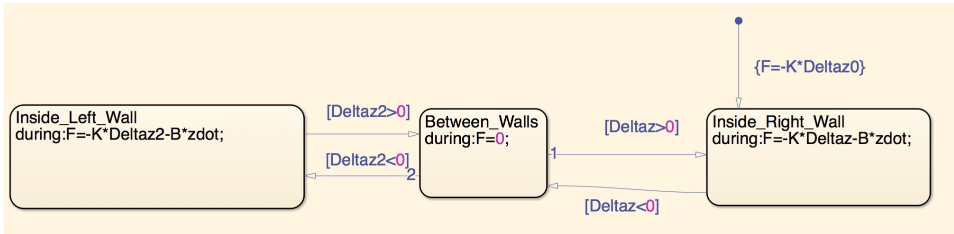


Figure 6: Stateflow Model for Two Virtual Walls

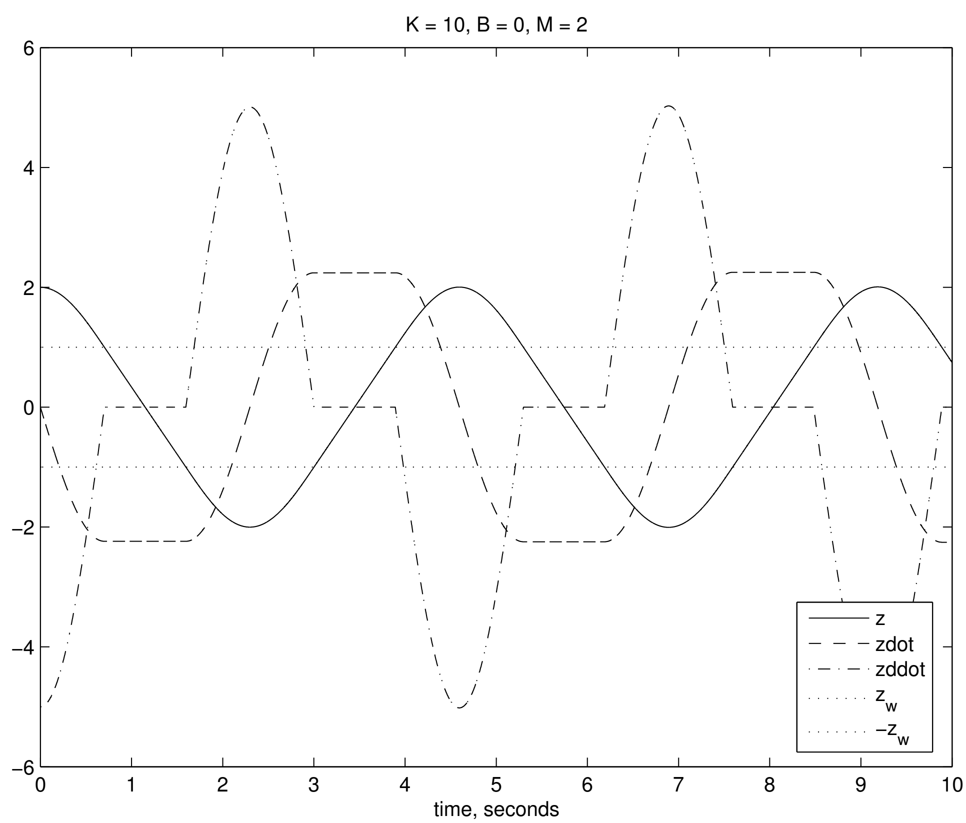


Figure 7: Simulation with two virtual walls.

3. Consider the equations of motion of a virtual world consisting of a virtual inertia, $J$, attached to the haptic wheel by a torsional spring with constant $k$

$$\ddot{\theta}_w + \frac{k}{J}\theta_w = \frac{k}{J}\theta_z, \tag{1}$$

where $\theta_w$ and $\theta_z$ denote the angles of the virtual and haptic wheels, respectively. Suppose that initially both wheels are at their reference locations, $\theta_w = \theta_z = 0$, and that at time $t = t_0$ the haptic wheel is turned to an angle $\theta_{z0}$ and held constant thereafter.

(a) Verify that the position of the virtual wheel satisfies the equation

$$\theta_w(t) = \theta_{z0}\left(1 - \cos(\omega_n(t - t_0))\right), \quad t \geq t_0, \tag{2}$$

where $\omega_n = \sqrt{k/J}$ radians/second.
SOLUTION: Taking the second derivative of (2) yields

$$\ddot{\theta}_w(t) = \theta_{z0}\omega_n^2 \cos(\omega_n(t - t_0)).$$

Using $\omega_n = \sqrt{k/J}$ yields

$$\theta_{z0}\omega_n^2 \cos(\omega_n(t - t_0)) + \frac{k}{J}\theta_{z0}\left(1 - \cos(\omega_n(t - t_0))\right) = \frac{k}{J}\theta_{z0},$$

which is the same as the differential equation (1) with $\theta_z(t) = \theta_{z0}$.

(b) Derive an expression for the torque acting on the virtual wheel as a function of time. What is the maximum magnitude of this torque?
SOLUTION: With $\theta_z(t) = \theta_{z0}$, the torque is given by

$$\begin{aligned} T(t) &= k(\theta_{z0} - \theta_w(t)) \\ &= k\theta_{z0}\cos(\omega_n(t - t_0)), \end{aligned}$$

and thus the maximum torque generated is equal to

$$T^{\max} = k\theta_{z0}.$$

4. State variable models can be applied in many situations. When modelling a mechanical system the state variables (or states) are generally chosen to be the position and velocity of each mass in the system. (If the system exhibits rotary motion, the states are chosen to be the angular position and angular velocity of each inertia.) If the system to be modelled is an electric circuit, such as the RLC circuit depicted in Figure 8, then the states may be chosen as the current through each inductor and the voltage across each capacitor

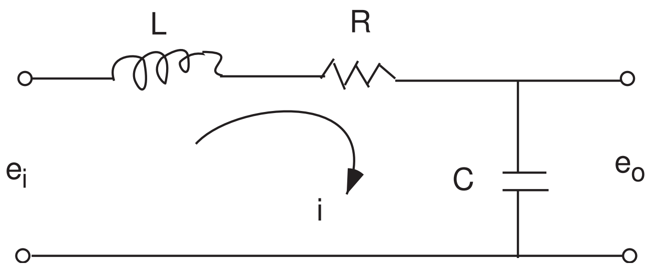


Figure 8: RLC circuit.

(a) Consider the RLC circuit shown in Figure 8. Kirchoff's laws state that the voltages $e_o$ and $e_i$ and the current $I$ are related by the equations:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i \tag{3}$$

$$\frac{1}{C}\int i\,dt = e_o. \tag{4}$$

Find the transfer function from input voltage $e_i$ to output voltage $e_o$. HINT: First find the transfer functions from $e_i$ to $i$ and from $i$ to $e_o$.
SOLUTION: Taking the Laplace transform of the differential equations (3)-(4) with initial conditions neglected yields the transfer functions

$$I(s) = \frac{sC}{LCs^2 + RCs + 1}E_i(s)$$
$$E_o(s) = \frac{1}{sC}I(s)$$

Combining these transfer functions in series yields

$$E_o(s) = \frac{1}{LCs^2 + RCs + 1}E_i(s)$$

(b) Find a second order differential equation relating input voltage $e_i$ to output voltage $e_o$.
SOLUTION: Working backwards from the transfer function derived in part (4a) yields

$$LC\ddot{e}_o + RC\dot{e}_o + e_o = e_i.$$

(c) Define state variables $x_1 = e_o$, $x_2 = \dot{e}_o$, input $u = e_i$, and output $y = e_o$. Find a state variable representation of the system in the form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

SOLUTION: It follows from the definition that $\dot{x}_1 = x_2$, and from the differential equation derived in part (4b) that $\dot{x}_2 = -(1/LC)x_1 - (R/L)x_2 + (1/LC)u$. In matrix notation these equations become

$$A = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1/LC \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

(d) For what values of the parameters $R$, $L$, and $C$ is the circuit a harmonic oscillator?

SOLUTION: The circuit will be an oscillator when $R = 0$, in which case the characteristic roots are $\pm j/\sqrt{LC}$.

(e) Let parameter values be chosen so that the circuit is an oscillator with natural frequency $\omega_n = 10$ radians/second. Suppose that the circuit is simulated on a computer using Forward Euler integration with simulation step size $T$ seconds. Where do the eigenvalues of the discrete system lie, as a function of $T$, for the specified natural frequency?

SOLUTION: With $\omega_n = 10$, we have that $LC = 0.01$. Hence, with $R = 0$ for a harmonic oscillator, it follows that

$$A = \begin{bmatrix} 0 & 1 \\ -100 & 0 \end{bmatrix}.$$

The discrete system with Forward Euler has eigenvalues at those of the matrix $A_d = I + AT$, and which lie at $1 \pm j\omega_n T = 1 \pm j10T$.

(f) How should we change the values of the circuit parameters if we wish its discrete approximation to be a harmonic oscillator?

SOLUTION: We should change the resistor so that the circuit has a damping coefficient that satisfies $\zeta = T\omega_n/2$. To relate the damping coefficient to the value of $R$, we note that the characteristic equation of the matrix $A$ derived in part (4c) is given by $\lambda^2 + (R/L)\lambda + (1/LC) = 0$. Since $2\zeta\omega_n = R/L$, it follows that we must choose $R$ so that $R = LT\omega_n^2$. Since $\omega_n^2 = 1/LC$, this is equivalent to choosing $R = T/C$.

5. (a) The system output satisfies $Y(s) = G(s)R(s)$, where

$$G(s) = \frac{\alpha}{s+\alpha},$$

and the differential equation

$$\dot{y} = -\alpha y + \alpha r.$$

The system has one characteristic root, at $s = -\alpha$. Since $\alpha > 0$, the system is stable. The steady state response to an input $r(t) = 1$ is given by the DC gain: $y_{ss} = G(0) = 1$.

(b) The time constant is given by $\tau = 1/\alpha$. Hence larger values of $\alpha$ will imply shorter time constants. Since the time constant describes the rate at which $y(t) \to y_{ss}$, a larger value of $\alpha$ will imply faster convergence.

(c) For $\tau = 0.1$, set $\alpha = 10$. The response will converge like the decaying exponential $e^{-10t}$.

(d) The discrete system output satisfies $Y(z) = G(z)R(z)$, where

$$G(z) = \frac{\alpha T}{z - 1 + \alpha T}.$$

The difference equation is

$$y((k+1)T) = (1 - \alpha T)y(kT) + \alpha T r(kT).$$

(e) The discrete system has one characteristic root, at $z = 1 - \alpha T$. For stability this root must lie within the unit circle. Since $\alpha > 0$ and $T > 0$, we need only check to see whether $-1 < 1 - \alpha T$. This condition is satisfied for

$$T < 2/\alpha.$$

Hence large values of $\alpha$, which correspond to short time constants, require faster sampling.

(f) We need to choose $\alpha$ so that the characteristic root satisfies

$$\begin{aligned} 1 - \alpha T &= e^{-10T} \\ \Rightarrow \alpha &= \frac{1 - e^{-10T}}{T}. \end{aligned}$$

For small values of $T$, a Taylor series expansion of the exponential implies that

$$\begin{aligned} \frac{1 - e^{-10T}}{T} &= \frac{1 - (1 + (-10T) + (-10T)^2/2! + (-10T)^3/3! + \dots}{T} \\ &\approx 10, \end{aligned}$$

so that the two values of $\alpha$ are approximately equal for sufficiently fast sampling.