Logic and Computation – CS 2800
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Propositional logic

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Outline

- Logic: a brief history
- Propositional logic
- Boolean expressions
- Truth tables
- Satisfiability and validity
- Stronger, weaker, and equivalent formulas

Logic

Logic

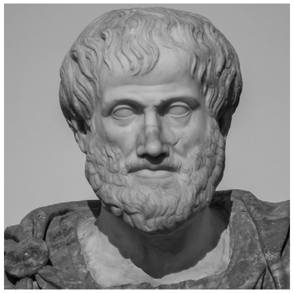
From Old French *logike*, from Latin *logica*, from Ancient Greek *λογική* (logikḗ, “logic”), from feminine of *λογικός* (logikós, “of or pertaining to speech or reason or reasoning, rational, reasonable”), from *λόγος* (lógos, “speech, reason”).

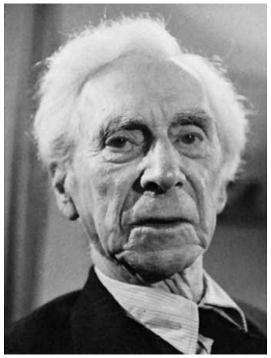
- What is logic?
  - A mathematical **language**: precise, unambiguous
  - A set of reasoning tools: deductions, proofs, …

- Why is logic important?
  - Foundation of mathematics
  - Foundation of computer science
  - Foundation of all science
  - Foundation of language? reason? intelligence?

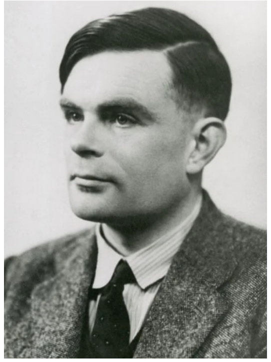
Logic – a brief history

- Very old:
  - Aristotle (384-322 BCE): syllogistic logic
  - His “Organon” books were the foundations of logic for 2000 years
  - Kant 1787: *“logic ..., since Aristotle, has been unable to advance a step and, thus, to all appearance has reached its completion.”*
  - Logic historian [Karl von Prantl]() (1820-1888) claimed that any logician who said anything new about logic was "confused, stupid or perverse.“



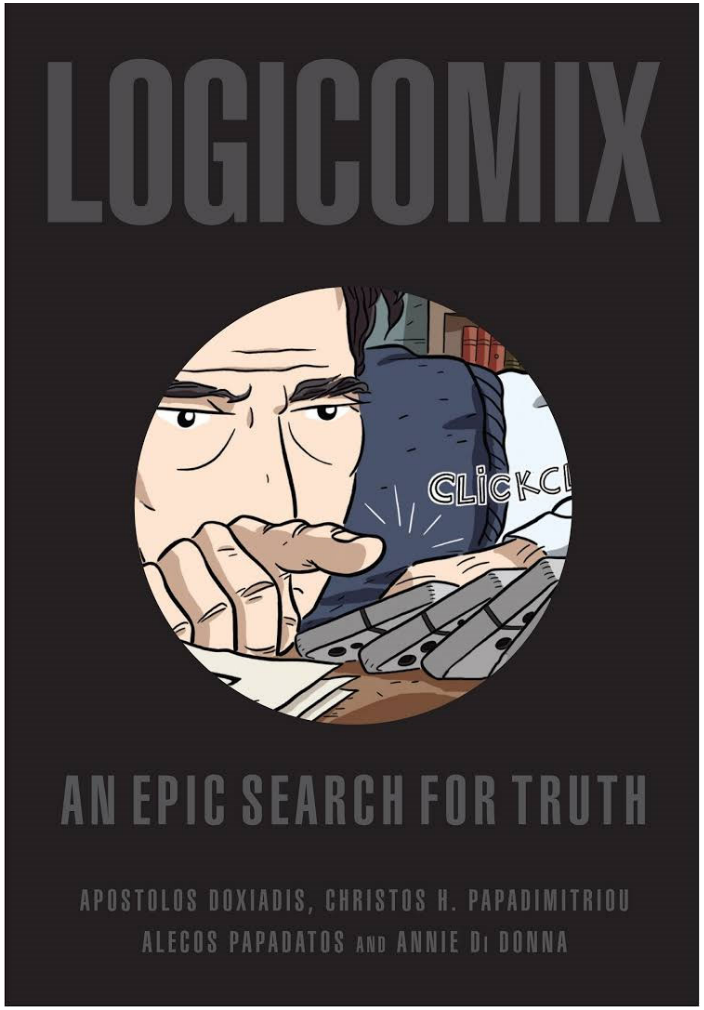


- Very new:
  - Russell’s paradox (1901)
  - Zermelo-Fraenkel set theory (1908-1920s)
  - Type theory (1908 – today)
  - Gödel’s incompleteness theorem (1930)
  - Turing machines (1936)
  - ACL2, Coq, LEAN, … (1970s-now, 1980s-now, 2013-now)

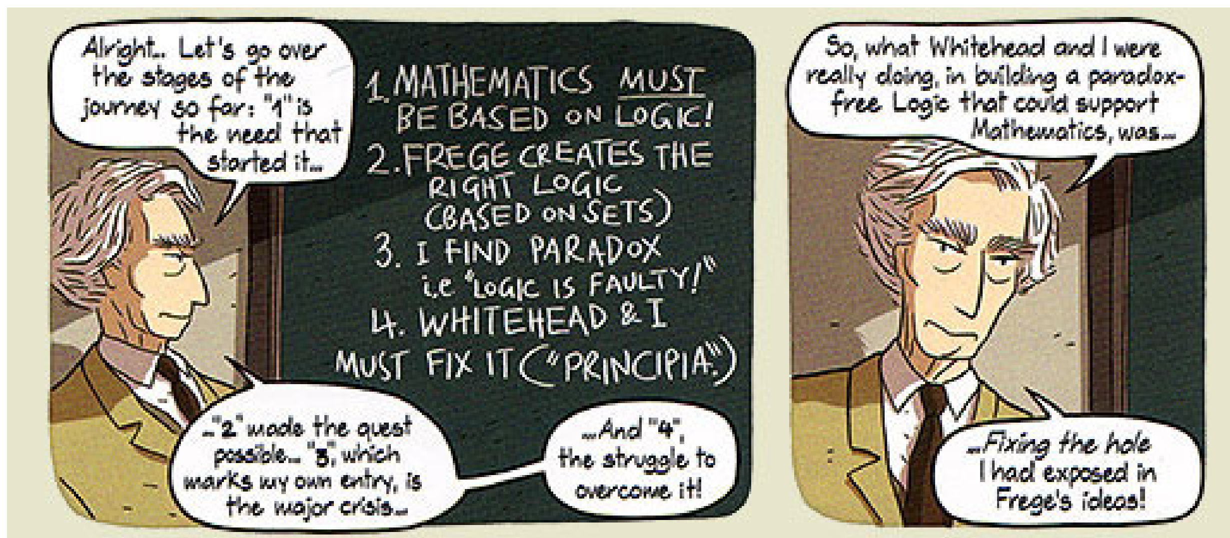


Russell's paradox

- Let S be the set that contains all sets that don't contain themselves.
- That is, let $S = \{X \mid X \notin X\}$
- Does $S \in S$?
  - Suppose not. Then $S \notin S$, therefore S matches X in the definition of S. Therefore, S must be a member of S. Contradiction!
  - Suppose yes. Then S should not be a member of S, since it is not the case that $S \notin S$. Contradiction again!
- Oops …

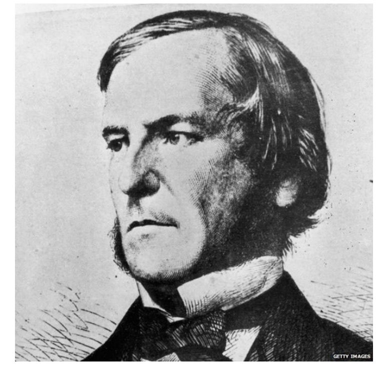

LOGICOMIX
CLICKCL
AN EPIC SEARCH FOR TRUTH
APOSTOLOS DOXIADIS, CHRISTOS H. PAPADIMITRIOU
ALECOS PAPADATOS AND ANNIE DI DONNA


ABOUT WHICH, WE CAN WELL ASK...
DOES IT CONT—
!?
WAIT A MINUTE!


Alright.. Let's go over the stages of the journey so far: "1" is the need that started it..
1. MATHEMATICS MUST BE BASED ON LOGIC!
2. FREGE CREATES THE RIGHT LOGIC (BASED ON SETS)
3. I FIND PARADOX i.e "LOGIC IS FAULTY!"
4. WHITEHEAD & I MUST FIX IT ("PRINCIPIA")
..."2" made the quest possible... "3", which marks my own entry, is the major crisis...
...And "4", the struggle to overcome it!
So, what Whitehead and I were really doing, in building a paradox-free Logic that could support Mathematics, was...
...Fixing the hole I had exposed in Frege's ideas!

Propositional logic

Propositional logic = Boolean logic



- George Boole (1815-1864)
  - Boolean algebra (1847)
- What's the simplest possible arithmetic?
  - Empty set? Set of one element? They are trivial.
  - Booleans = set of two elements = {true, false} = {T,F} = {0,1} = $\{\top, \bot\}$ = { t, nil} = …
  - Now our arithmetic becomes more interesting
  - There are many functions from Booleans to Booleans
- Rich domain, impressive list of applications
  - Logic, circuit design, SAT, verification, scheduling, AI, game theory, reliability, security, …

Boolean expressions (= Boolean formulas)

- The expressions of propositional logic
  - Recall: logic = language = set of expressions
  - To have a precise, unambiguous language, we must first define what are the syntactically valid expressions

- Atomic expressions:
  - The constants *true* and *false* (or 0 and 1, or tt and ff, …)
  - Propositional atoms or variables: p, q, r, …, or a, b, c, …, or x, y, z, …

- Composite expressions:
  - If ϕ and ψ are Boolean expressions, then we can combine them using the Boolean/propositional logic operators:
    - Negation: $\neg\phi$ ("not ϕ")
    - Conjunction: $\phi \land \psi$ ("ϕ and ψ")
    - Disjunction : $\phi \lor \psi$ ("ϕ or ψ")
    - Implication : $\phi \rightarrow \psi$ ("ϕ implies ψ"), also written $\phi \Rightarrow \psi$
    - Equivalence : $\phi \equiv \psi$ ("ϕ iff ψ"), also written $\phi \leftrightarrow \psi$
    - Exclusive or (xor) : $\phi \oplus \psi$ ("ϕ xor ψ")

Truth tables of basic Boolean operators

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \equiv q$
T	T	T
T	F	F
F	T	F
F	F	T

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Can you think of a new (different) truth table for a new (different) binary operator? unary operator?

- How many unary operators are there?
- How many binary operators are there?
- How many political parties do we need?

English usage

- In English “or” often means “exclusive or”
  - E.g., “you can have ice cream or a cookie” (implied: but you can’t have both)
  - In logic, or means at least one (see truth table!)
  - If you want to have “either-or” (exclusive or) use xor

English usage

- True or false?
  1. “if pigs can fly, then I am the president of the USA”
  2. “if I am the president of the USA, then pigs can fly”
  3. “if I am faculty at Northeastern, then pigs can fly”

- Answers:
  1. True: pigs cannot fly
  2. True: I am not president of the USA
  3. False: I am faculty at Northeastern, and pigs can’t fly

  - Truth table! (implication)
  - Some English speakers might say that the first two statements are false, since I am not the president / pigs can’t fly
  - Contrary to English, logic is precise => no debate

Implication

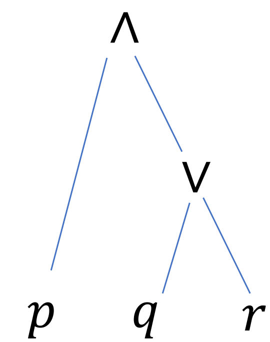
False => Anything!
False => False

- Cannot overemphasize its importance in logic
  - Make sure you understand its semantics
  - Truth table!

- True or false?
  - "if x is a natural number, then x>=0"
    - True
    - What about x=-1? Isn't it a counter-example since -1<0 ?
      - No, because -1 is NOT a natural number
    - The only way to make $p \to q$ false is to make p true and q false
    - Truth table!

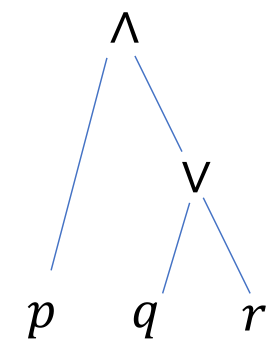
Boolean expressions

- Boolean expressions are really **trees**:



- To avoid ambiguities, use parentheses
  - e.g., $p \wedge (q \vee r)$

Parentheses



- Boolean expressions are really **trees**:

- To avoid ambiguities, use parentheses
  - e.g., $p \wedge (q \vee r)$

- When there is no ambiguity, don't use parentheses!
  - e.g., $p \wedge q \wedge r$ – why?

Precedence rules

- Sometimes we end up with too many parentheses
  - e.g., $\big((p \vee (\neg q)) \rightarrow r\big) \oplus ((\neg r) \rightarrow (q \wedge (\neg p)))$
- To avoid having to write too many parentheses, we establish precedence rules:
  - Negation $\neg$ binds strongest
  - Followed by conjunction and disjunction, $\wedge, \vee$
  - Followed by implication $\rightarrow$
  - Followed by iff and xor, $\equiv, \oplus$
- So we can rewrite the above equivalently as
  - $p \vee \neg q \rightarrow r \oplus \neg r \rightarrow q \wedge \neg p$

But: to me this is too confusing…
Often hard to remember the rules.

Use parentheses reasonably.
You will never be penalized for using "too many" parentheses
(unless we explicitly ask you to remove parentheses, as an exercise).

Syntax and semantics

- We have seen the syntax of propositional logic: Boolean expressions

- But what do these expressions **mean**?

- Propositional logic semantics:
  - **Boolean functions**
  - Can be represented as **truth tables**

Homework: build the truth table of this Boolean expression

$$\neg p \vee q \equiv p \Rightarrow q$$