

# Logic and Computation – CS 2800

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Propositional logic continued

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# Outline

- Truth tables
- Satisfiability and validity
- Stronger, weaker, and equivalent formulas

Homework: build the truth table of this Boolean expression

$$\neg p \vee q \equiv p \Rightarrow q$$

# Truth tables of Boolean formulas

- For every Boolean formula (= Boolean expression) we can construct its truth table
  - E.g., for the formula  $\neg p \vee q \equiv p \Rightarrow q$  we get the truth table:

p	q	$\neg p$	$\neg p \vee q$	$p \Rightarrow q$	$\neg p \vee q \equiv p \Rightarrow q$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

**Assignment:** assigns truth values to the propositional variables of the formula

**Subformula:** a subexpression (subtree)

# Size of truth tables

- How many rows does a truth table have?
  - Answer:  $2^n$ , where  $n$  is the number of inputs, i.e., the number of variables
- How many columns does a truth table have?
  - Answer: one column for every subformula
- If  $n$  is the number of inputs, how many different last columns can I have? (i.e., how many different  $n$ -ary Boolean functions can I have?)
  - Answer:  $2^{2^n}$  (make sure you understand why!)

# Syntax vs semantics

- Infinitely many Boolean **expressions**!
- Finitely many Boolean **functions**!
- Finitely many truth tables!
- Many Boolean expressions are **equivalent**: they represent the same Boolean function, i.e., they have the same last column in their truth table.
  - E.g.,  $p$  is equivalent to  $p \vee p$ , and also to  $p \wedge p$ , and also to  $p \vee p \vee p$ , etc.

# Satisfiability and validity

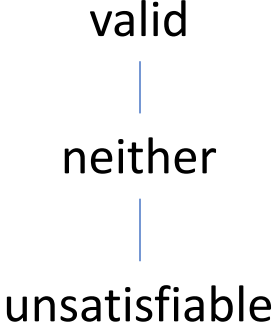
# Satisfiability and validity

- A Boolean formula (=Boolean expression) is called:
  - **Satisfiable**: when it is **sometimes true**
  - **Unsatisfiable**: when it is **never true**
  - **Valid** or a **tautology**: when it is **always true**
  - **Falsifiable**: when it is **sometimes false**
- “Sometimes”, “always”, “never” refer to the truth table of the formula:
  - Sometimes: exists assignment to make the formula true
  - Never: no assignment makes the formula true
  - Always: all assignments make the formula true



# Satisfiability and validity

- Every Boolean formula satisfies exactly two of the previous characterizations
- In particular, a Boolean formula is **exactly one** of the following:
  - **Satisfiable and valid**: always true
  - Satisfiable but not valid = **satisfiable and falsifiable**: sometimes true and sometimes false
  - **Unsatisfiable and falsifiable**: always false

- Think of it as a *lattice*:

```
graph TD; valid --- neither; neither --- unsatisfiable
```

# Examples – homework

- Think of examples of the three categories
- Valid:
- Satisfiable and falsifiable:
- Unsatisfiable:

# Stronger, weaker, and equivalent formulas

- We say that formula  $P$  is **stronger** than formula  $Q$  if the formula  $P \rightarrow Q$  is valid (i.e., a tautology). This is the same as saying that whenever  $P$  is true,  $Q$  is also true.
- We say that formula  $P$  is **weaker** than formula  $Q$  if the formula  $Q \rightarrow P$  is valid (i.e., a tautology). This is the same as saying that whenever  $Q$  is true,  $P$  is also true. This is also the same as saying that  $Q$  is stronger than  $P$ .
- We say that formula  $P$  is **equivalent** to formula  $Q$  if both formulas  $P \rightarrow Q$  and  $Q \rightarrow P$  are valid. This is the same as saying that  $P$  is both stronger and weaker than  $Q$ .
- The same terminology applies to non-Boolean formulas, e.g., we can say that  $x > 5$  is stronger than  $x > 0$ .
- Mini (ungraded) quizzes:
  - What is the strongest possible formula?
  - What is the weakest possible formula?

# Necessary and sufficient conditions

- In other settings you will hear the terms **necessary** and **sufficient**, usually followed by **conditions**
- $P$  is a **necessary** condition for  $Q$  if  $Q$  is stronger than  $P$
- $P$  is a **sufficient** condition for  $Q$  if  $P$  is stronger than  $Q$
- If  $P$  is both necessary and sufficient for  $Q$ , then the two are equivalent
- E.g.,  $x > 0$  is necessary for  $x > 1$ , and sufficient for  $x \geq 0$