Homework 1

COMS W3251, Summer Session 2

Due: Monday July 13th 2020, 11:59 EST

Instructions: Compile typed (LaTeX highly recommended) or neatly handwritten solutions to all problems below. Show all of your steps and work, and submit them on Canvas by the deadline (late submissions are not accepted). Avoid turning it in last-minute to save time to check for errors and tag your problems. All submitted work should be your own—we will pursue suspected cases of academic dishonesty

1 Matrix and vector computations (20 points)

For each of the following expressions, compute the result. You can use properties of operations to simplify the expression:

$$\left(\begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}^T \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix} + 4I\right) e_2, \tag{1.1}$$

$$\left(\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}^T + \begin{bmatrix} 0 & -1 \\ 4 & 1 \end{bmatrix}^T\right)^T \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \tag{1.2}$$

$$\left\langle \left(6 \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} - 3I\right)(2e_1 + e_3), \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\rangle, \tag{1.3}$$

$$\langle 2e_1 + 3e_2 + 4e_3, 8e_2 - 6e_3 \rangle. \tag{1.4}$$

2 Vector spaces (20 points)

Let $P_3(\mathbb{R})$ be the set of all polynomials of **degree 3 or less** with real coefficients. Prove that $P_3(\mathbb{R})$ is a vector space when considering $+$ as point-wise sum and $\cdot$ as point-wise scalar multiplication (that is, if $p_1, p_2 \in P_3(\mathbb{R})$, then $p_1 + p_2$ is the polynomial such that $p_1 + p_2(x) = p_1(x) + p_2(x)$, while $\alpha \cdot p_1$ is the polynomial such that $\alpha \cdot p_1(x) = \alpha p_1(x)$). Can you find a finite subset of $P_3(\mathbb{R})$ that spans $P_3(\mathbb{R})$?

3 Linear functions (25 points)

Which of the following functions are linear? If they are, find the matrix A such that $T(x) = Ax$. If they are not, find a counterexample to the linearity condition.

$$T(x_1, x_2) = (x_1 + 4x_2, |x_2|) \tag{3.1}$$
$$T(x_1, x_2, x_3) = (x_1 - 8x_3, 0, 3x_2) \tag{3.2}$$
$$T(x_1, x_2, x_3) = (x_1 + 3, x_2, x_3 - x_1) \tag{3.3}$$
$$T(x_1, x_2) = x_1x_2 + x_2^2 \tag{3.4}$$
$$T(x_1) = (0, x_1, x_1^2, x_1^4) \tag{3.5}$$
$$T(x_1) = (x_1, x_1, 0, x_1) \tag{3.6}$$

4 Transformations of the plane (15 points)

Find the linear or affine transformations that satisfy the desired properties and write it in the form $T(x) = Ax + b$:

1. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ sending the origin to itself and a triangle of vertices $(0, 0), (1, 0), (0, 1)$ to a triangle of vertices $(0, 0), (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.
2. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ sending the origin to $(4, 1)$ and the unit square $[0, 1]\times[0, 1]$ to the parallelogram with vertices $(4, 1), (4, 2), (7, 2), (7, 3)$.

5 Flop count (20 points)

Let $A, B \in \mathbb{R}^{n\times n}$ and $x, y \in \mathbb{R}^n$. Following the order given by parenthesis, calculate the complexity of the following operations in term of flops.

$$\langle (A(Bx)), (\alpha y)\rangle \tag{5.1}$$
$$\alpha(\langle (AB)x, y\rangle \tag{5.2}$$

Which one has the lowest complexity? Can you reduce the complexity even more?