Homework 5

August 6, 2020

1 Discrete dynamical systems (20 points)

Consider the following discrete system:

$$\mathbf{x}_{k+1} = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} & 0 \\ \frac{3}{4} & \frac{5}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x}_k$$

1. Let $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. What is the behaviour of the system as $k \to \infty$?
2. Let $\mathbf{x}(0) = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$. What is the behaviour of the system as $k \to \infty$?
3. Let $\mathbf{x}(0) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. What is the general solution of the system? Depending on c_1, c_2 and c_3, what is the behaviour of the system as $k \to \infty$?

2 Continuous dynamical systems (20 points)

For each of the following systems, write the general solution for $\mathbf{x}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ and study the behaviour of the system as $t \to \infty$ depending on the choice of c_1 and c_2.

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \mathbf{x}(t) \tag{1}$$

$$\mathbf{x}'(t) = \begin{bmatrix} -2 & -5 \\ 1 & 4 \end{bmatrix} \mathbf{x}(t) \tag{2}$$

$$\mathbf{x}'(t) = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix} \mathbf{x}(t) \tag{3}$$

$$\mathbf{x}'(t) = \begin{bmatrix} -2 & 1 \\ -8 & 2 \end{bmatrix} \mathbf{x}(t) \tag{4}$$

3 Understanding Markov chains (20 points)

Suppose that there are 3 possible states of the weather in the fall: "sunny", "cloudy" and "rainy". Suppose that if the weather is sunny on one day, then on the following day it will be sunny with probability 0.4, cloudy with probability 0.3 and rainy with probability 0.3. Moreover, if the weather is cloudy on a given day, then it will never be sunny on the following day, but it might be cloudy again with a probability of 0.8 or rainy with a probability of 0.2. Finally, if the weather is rainy on any given day, it will be sunny on the following day with a probability of 0.2, cloudy with a probability of 0.3 and rainy with a probability of 0.5.

1. Write the transition matrix A corresponding to the Markov chain described above.

2. On any given day, what is the transition matrix B describing the probabilities of being in the states "sunny", "cloudy" and "rainy" two days from now?

3. Explain the interpretation of a **mixed state**, that is, a vector with all strictly positive coordinates that sum to one. In particular, if $\mathbf{v}$ is a mixed state, what is the meaning of $A\mathbf{v}$?

4. Without diagonalizing the matrix, is there any guarantee that the matrix A admits a steady state? In case there is, find the steady state, otherwise explain why there isn't.

5. Now, consider a matrix $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Interpret this matrix as a Markov chain for two states "sunny" and "rainy". What is the long-term dynamic if we start from a "sunny" state?

6. Does C admit a steady state?

4 SVD decomposition of scalar multiples (20 points)

1. Let λ be an eigenvalue of a square matrix A and $\alpha \in \mathbb{R} \setminus \{0\}$. Can you find an eigenvalue for αA?

2. Let $B = U\Sigma V^T$ be an SVD decomposition for a $m \times n$ matrix B. Can you find an SVD decomposition for αB?

3. Find an SVD decomposition for $B = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$.

4. Find an SVD decomposition for $C = \begin{bmatrix} 0 & \pi & -\pi \end{bmatrix}$

5 SVD decomposition of pseudo-inverse (20 points)

1. Let $A = U\Sigma V^T$ be an SVD decomposition for an $m \times n$ matrix A. Can you find an SVD decomposition for A^T?

2. Now suppose that the columns of A are linearly independent. Can you find an SVD for the pseudo-inverse $A^{\dagger} = (A^T A)^{-1} A^T$?

3. Find an SVD decomposition of

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

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4. Find an SVD decomposition for the pseudo-inverse $B^{\dagger}$.