Homework 1: Solutions

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Problem 1

Let $\vec{a}$,$\vec{b}$ be two vectors in $\mathbb{R}^3$.

1. Show that the vector $\vec{b} - \text{proj}_a b$ is orthogonal to $\vec{a}$.
   Solution:
   We need to prove that $\langle \vec{b} - \text{proj}_a b, \vec{a} \rangle = 0$.

   $$\langle \vec{b} - \text{proj}_a b, \vec{a} \rangle = \langle \vec{b}, \vec{a} \rangle - \langle \frac{\langle \vec{b}, \vec{a} \rangle}{||\vec{a}||^2} \vec{a}, \vec{a} \rangle = \langle \vec{b}, \vec{a} \rangle - \frac{\langle \vec{b}, \vec{a} \rangle}{||\vec{a}||^2} \langle \vec{a}, \vec{a} \rangle = 0$$

   the last part following from the fact that $\langle \vec{a}, \vec{a} \rangle = ||\vec{a}||^2$

2. Under what condition do we have $||\text{proj}_a b|| = ||\text{proj}_b a||$?
   Solution:
   If $||\text{proj}_a b|| = ||\text{proj}_b a||$, by definition we have

   $$||\frac{\langle \vec{b}, \vec{a} \rangle}{||\vec{a}||^2} \vec{a}|| = ||\frac{\langle \vec{b}, \vec{a} \rangle}{||\vec{b}||^2} \vec{b}||$$

   Now if the two vectors are orthogonal, both those values are equal to zero so the norm of their projection is the same. If they are not orthogonal, we can use the properties of the norm and simplify to get

   $$\frac{|\langle \vec{b}, \vec{a} \rangle|}{||\vec{a}||^2} ||\vec{a}|| = \frac{|\langle \vec{b}, \vec{a} \rangle|}{||\vec{b}||^2} ||\vec{b}|| \iff \frac{1}{||\vec{a}||} = \frac{1}{||\vec{b}||}$$

   which is realized if and only if they have the same norm. Therefore for the projections to have the same norm the vectors have to either be orthogonal or to have the same norm.

3. Under what condition do we have $\text{proj}_a b = \text{proj}_b a$?
   Solution:
   The condition is given by

   $$\frac{\langle \vec{b}, \vec{a} \rangle}{||\vec{a}||^2} \vec{a} = \frac{\langle \vec{b}, \vec{a} \rangle}{||\vec{b}||^2} \vec{b}$$

   which is realized if either the vectors are orthogonal and both quantities are equal to zero, or if

   $$\frac{\vec{a}}{||\vec{a}||^2} = \frac{\vec{b}}{||\vec{b}||^2}$$

   In other words the vectors have to be multiple of each other, therefore they have to lie on the same line. By definition of projection, the two projections will therefore be equal only if the two vectors are the same. Therefore the conditions for $\text{proj}_a b = \text{proj}_b a$ are either that the two vectors are orthogonal, or that they are the same vector.

Problem 2

Let $\vec{a} = (1, 1, -1)$, $\vec{b} = (1, -1, 1)$.

1. Find a ***unitary*** vector $\vec{n}$ which is orthogonal to both $\vec{a}$ and $\vec{b}$. How many vectors satisfy such properties? Draw the one you found.
   Solution:
   Let's use the cross product to find a vector which is perpendicular to both:
   $$\vec{p} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = (0, -2, -2)$$
   We normalize it to find a unitary vector $\vec{n} = (0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. There are two vectors satisfying those conditions, $\vec{n}$ and $-vecn$. To draw $\vec{n}$ just use the right hand rule.

2. Find the volume of the parallelepiped spanned by $\vec{a}, \vec{b}$ and $\vec{n}$.
   Solution:
   By a theorem seen in class
   $$Area_P = |\langle \vec{n}, \vec{a} \times \vec{b} \rangle| = |\langle \vec{p}, \frac{\vec{p}}{||\vec{p}||} \rangle = ||\vec{p}|| = 2\sqrt{2}$$

3. Find the area of the triangle spanned by $\vec{a}$ and $\vec{b}$.
   Solution:
   The third side of the triangle spanned by $\vec{a}$ and $\vec{b}$ is a diagonal of the parallelogram spanned by $\vec{a}$ and $\vec{b}$. We know that the area of that parallelogram is given by the norm of the cross product, therefore
   $$Area_T = \frac{||\vec{p}||}{2} = \sqrt{2}$$

Problem 3

1. Find the equation in parametric form and in symmetric form of the line $\boldsymbol{r_1}$ in $\mathbb{R}^3$ passing through the points $P_1 = (6, 1, -3)$ and $P_2 = (2, 4, 5)$.
   Solution:
   To find the equation of a line we need a point and a vector. Consider
   $$\vec{v_1} = \vec{P_1 P_2} = (2 - 6, 4 - 1, 5 - (-3)) = (-4, 3, 8)$$
   The parametric form is therefore given by
   $$\boldsymbol{r_1}(t) = (6 - 4t, 1 + 3t, -3 + 8t)$$

While the symmetric form is given by

$$\frac{x-6}{-4} = \frac{y-1}{3} = \frac{z+3}{8}$$

2. Find the equation of a line which intersects $\boldsymbol{r_1}$ in one point and show that it is intersecting.
   Solution:
   We choose one point on the line, for example P_1 and a vector not parallel to $\vec{v_1}$, for instance $v_2 = (0,0,1)$. The line will therefore have equation $\boldsymbol{m}(s) = (6, 1, -3+s)$ To show that they intersect just show that $\boldsymbol{m}(s) = \boldsymbol{r_1}(t)$ if and only if $s = 0, t = 0$, that is the system of equations
   $$\begin{cases} -4t + 6 = & 6 \\ 3t + 1 = & 1 \\ 8t - 3 = & -3 + s \end{cases}$$
   has only one solution for $s = t = 0$.

3. Consider the line $\boldsymbol{r_2}(t) = (3 + 8t, 3 - 6t, -16)$. Write its symmetric form. Is it intersecting, skew or parallel to $\boldsymbol{r_1}$?
   Solution:
   $\boldsymbol{r_2}$ passes through the point $P_3 = (3, 3, -16)$ and has direction given by $\vec{v_3} = (8, -6, 0)$. Its symmetric form is given by
   $$\frac{x-3}{8} = \frac{y+6}{-6}, \quad z = -16$$
   Since $(8, -6, 0)$ and $(-4, 3, 8)$ are not multiple of each other, the two vectors are not parallel. To see if they are skew or intersecting, let us consider the system of equation $\boldsymbol{r_1}(t) = \boldsymbol{r_2}(s)$
   $$\begin{cases} -4t + 6 = 3 + 8s \\ 3t + 1 = 3 - 6s \\ 8t - 3 = \quad -16 \end{cases}$$
   By the third equation we get that if they intersect, $t = -\frac{13}{8}$. Substituting that value of t in the other two equations gives two different values of s, therefore the system has no solutions and the two lines are skew.

4. Determine whether the following lines are skew, parallel or intersecting:
   $$\boldsymbol{r_3} : \frac{x-1}{2} = \frac{y-3}{2} = \frac{z-2}{-1}$$
   $$\boldsymbol{r_4} : \frac{x-2}{1} = \frac{y-6}{-1} = \frac{z+2}{3}$$

Solution:
We first write them in parametric form

$$\begin{aligned}\boldsymbol{r_3}(t) &= (1+2t, 3+2t, 2-t)\\ \boldsymbol{r_4}(s) &= (2+s, 6-s, -2+3s)\end{aligned}$$

We have that $\boldsymbol{r_3}(t) = \boldsymbol{r_4}(s)$ is the following system of equations:

$$\begin{cases} 2t+1 = & 2+s\\ 2t+3 = & 6-s\\ -t+2 = & -2+3s\end{cases}$$

The system has only one solution for $t = 1$ and $s = 1$, therefore the two lines are intersecting.

Problem 4

1. Find the plane through the point $P_1 = (2, 5, -1)$ and containing the line

$$\boldsymbol{r}(t) = (4-t, 2, 3+2t)$$

. *Solution*:
Pick a point on the line $P_0 = r(0) = (4, 2, 3)$. We have two vectors on the plane, one given by the direction of the line $\vec{v} = (-1, 0, 2)$ and another one given by $\vec{w} = \vec{P_1P_0} = (-3, -5, 3)$, so that now we can find a normal vector given by

$$\vec{n} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ -1 & 0 & 2\\ -3 & -5 & 3\end{vmatrix} = (10, -3, 5)$$

so that the equation of the plane can be found as $\langle \vec{n}, (x, y, z) - P_1\rangle = 0$, that is

$$10(x-2) - 3(y-5) + 5(z+1) = 0$$

or, simplifying:

$$10x - 3y + 5z = 0$$

2. Find the plane that is parallel to the plane $x + y - z = 1$ and contains the point $P_2 = (1, 0, -1)$.
Solution:
We can either find 3 points on the plane $x + y - z = 1$, find two vectors on that plane and notice that if a plane is parallel to $x + y - z = 1$, it has the same normal vector

or use the fact that parallel planes have the same coefficients for the unknown terms. Therefore we need to find a plane of type $x+y-z=d$ such that P_2 is on that plane. That means that coordinates of P_2 have to satisfy the equation $x+y-z=d$, which gives us the condition $d=1+0-(-1)=2$. The desired plane is therefore $x+y-z=2$

3. Given the plane $x+y-z=1$, find whether the line with symmetric equations

$$\frac{x-2}{1}=\frac{y-6}{-1}=\frac{z+2}{3}$$

is parallel, orthogonal or neither.
Solution:
One strategy can be rewriting the line in parametric terms and see if it solves the equation of the plane for some t, as done in class. That means writing the line as

$$\boldsymbol{r}(t)=(2+t,6-t,-2+3t)$$

Now, substitute the coordinates in the plane to get

$$2+t+6-t-(-2+3t)=1$$

which is satisfied for $t=3$. Therefore the line is not parallel. Now, the vector of the line is given by $\vec{l}=(1,-1,3)$. We can consider two points on the plane, for instance $Q_1=(1,0,0)$ and $Q_2(0,1,0)$, which give us the vector $\vec{t}=(-1,1,0)$ and notice that $\langle\vec{t},\vec{l}\rangle=-2\neq 0$, which is a sufficient condition to show that the line is not orthogonal to the plane.

4. *Bonus question*: Consider the generic equation of a plane $ax+by+cz=d$, where at least one coefficient is nonzero. Prove that the vector normal to the plane is $\vec{n}=(a,b,c)$.
Solution:
Without loss of generality, suppose $a\neq 0$, so that we can rewrite the equation as $x-\frac{d}{a}+\frac{b}{a}y+\frac{c}{a}z=0$. One point satisfying this equation is given by $(\frac{d}{a},0,0)$, so that

$$\langle(1,\frac{b}{a},\frac{c}{a}),(x,y,z)-(\frac{d}{a},0,0)\rangle=0$$

satisfies the equation of a plane passing through $(\frac{d}{a},0,0)$ and with orthogonal vector $(1,\frac{b}{a},\frac{c}{a})$. Therefore our plane has $(1,\frac{b}{a},\frac{c}{a})$ as orthogonal vector. Since $\vec{n}=a\cdot(1,\frac{b}{a},\frac{c}{a})$, we have that $\vec{n}$ is also orthogonal.

Problem 5

Consider the function $f:D_1\subset\mathbb{R}^3\to\mathbb{R}$ given by

$$f(x,y,z)=\frac{\sqrt{x^2+y^2-4z^2}}{x^2+y^2}$$

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1. Find the domain D of f.
Solution:
We need to apply the rules to find the domain for square roots and denominators, that is, argument of a square root greater or equal to 0 and denominator different from 0. Then we have

$$D = \{(x, y, z)|x^2+y^2-4z^2 \geq 0, x^2+y^2 \neq= 0\} = \{(x, y, z)|x^2+y^2-4z^2 \geq 0, (x, y) \neq (0, 0)\}$$

2. Describe the domain in terms of quadrics and use the trace method to identify it.
Solution:
The inequality $x^2 + y^2 - 4z^2 \geq 0$ can be seen as a family of quadrics $x^2 + y^2 - 4z^2 = k$ for $k \geq 0$. We need to study the quadrics with the trace method.
Case 1: $k > 0$
xy plane: $z = z_0$, $x^2 + y^2 = k + 4z_0^2$. Since the right hand side of the equality is always positive, these are circles for all possible choices of z_0.
yz plane: $x = x_0$, $y^2 - 4z^2 = k - x_0^2$. These are hyperbolas for all values of x_0 except for $x_0 = \pm\sqrt{k}$ where these are lines. xz plane: $y = y_0$, $x^2 - 4z^2 = k - y_0^2$. These are hyperbolas for all values of y_0 except for $y_0 = \pm\sqrt{k}$ where these are lines.
Therefore if $k > 0$ we have a family of hyperboloids of one sheet.
Case 2: $k = 0$
xy plane: $z = z_0$, $x^2 + y^2 = 4z_0^2$. These are circles for all possible choices of z_0 except for $z_0 = 0$, where this is a point.
yz plane: $x = x_0$, $y^2 - 4z^2 = -x_0^2$. These are hyperbolas for all values of x_0 except for $x_0 = 0$ where these are lines. xz plane: $y = y_0$, $x^2 - 4z^2 = -y_0^2$. These are hyperbolas for all values of y_0 except for $y_0 = 0$ where these are lines.
Therefore if $k = 0$ we have a cone.
Finally, the condition $(x, y) \neq (0, 0)$ excludes a line corresponding to the z-axis, which intersects these quadrics only in the vertex of the cone. Therefore the domain can be represented as a family of hyperboloid of one sheet and a cone minus its vertex.

3. Consider $g : D_2 \subset \mathbb{R}^2 \to \mathbb{R}$ defined as $g(x, y) = f(x, y, 1)$ and write an expression for it. Find the domain D_2, draw it on the plane and calculate $\lim_{(x,y)\to\infty} g(x, y)$.
Solution:
$g(x, y) = \frac{\sqrt{x^2+y^2-4}}{x^2+y^2}$. Its domain is given by

$$D_2 = \{(x, y)|x^2 + y^2 - 4 \geq 0, (x, y) \neq (0, 0)\} = \{(x, y)|x^2 + y^2 \geq 4\}$$

which is the outside of a circle of radius 2 in $\mathbb{R}^2$. Since

$$|\frac{\sqrt{x^2 + y^2 - 4}}{x^2 + y^2}| \leq |\frac{\sqrt{x^2 + y^2}}{x^2 + y^2}| = \frac{1}{\sqrt{x^2 + y^2}}$$

and we know that the rightmost function tends to 0 as $(x, y) \to \infty$, we have that $\lim_{(x,y)\to\infty} g(x, y) = 0$