Lecture 9: Dynamical systems and Markov chains

Francesco Preta
08/04/2020

Discrete dynamical systems

Discrete difference equations

A discrete difference equation is an expression of the type

$$\mathbf{x}_{k+1} = f(k, \mathbf{x}_k, \mathbf{x}_{k-1}, \ldots, \mathbf{x}_{k-N})$$

that represents the growth of a given vector of variables depending on the value of that variable in previous discretized periods (up to $k - N$) and on the period k itself.

Example: predator-prey dynamic

Imagine that in an environment at each period k there are O_k owls and R_k rats. The population of owls is halved at each time if they can't eat rats, while rats replicate by a factor of 1.1 if they're not eaten by owls.

Example: predator-prey dynamic (continued)

We can write the system as

$$\begin{cases} O_{k+1} = 0.5O_k + 0.4R_k \\ R_{k+1} = 1.1R_k - 0.104O_k \end{cases}$$

If we consider $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$, we can write this system as $\mathbf{x}_{k+1} = A\mathbf{x}_k$, with

$$A = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix}$$

Time-independent systems

This type of difference equation is a linear time-independent system, since the function $f(k, \mathbf{x}_k, \mathbf{x}_{k-1}, \ldots, \mathbf{x}_{k-N})$ is linear and does not depend on k. These will be the systems we will be dealing with for the rest of our class.

Evolution of a system

A system of difference equations tells us the dynamics over time of a vector variable starting from a given state $\mathbf{x}_0$.

In the case of the previous example, we can choose $\mathbf{x}_0 = \begin{bmatrix} 20 \\ 50 \end{bmatrix}$ and observe the evolution.

Evolution at time k and long-term dynamics

in general, after the k-th period the vector $\mathbf{x}_k$ can be obtained by considering $\mathbf{x}_k = A\mathbf{x}_{k-1} = A^k\mathbf{x}_0$.

We are interested in the long-term effects of the dynamics of the system. In the previous example, for instance, we are interested in understanding whether the population can reach an equilibrium, if one of the two (or both) will go extinct or if both will thrive and replicate.

Eigendecomposition to understand the equilibrium

Suppose that A is diagonalizable. Then we can write $\mathbf{x}_0$ in the basis of eigenvectors $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$ and obtain

$$A^k\mathbf{x}_0 = c_1A^k\mathbf{v}_1 + c_2A^k\mathbf{v}_2 + \ldots + c_nA^k\mathbf{v}_n = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \ldots + c_n\lambda_n^k\mathbf{v}_n$$

Then taking the limit for $k \to \infty$ allows us to understand the long-term implications of our dynamics.

Example:

$$A = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix}$$

General analysis

In general, given a system of difference equations $\mathbf{x}_{k+1} = A\mathbf{x}_k$, a good strategy is to diagonalize the matrix of the system $A = P\Lambda P^{-1}$.

Then by considering the change of variables $\mathbf{y}_k = P^{-1}\mathbf{x}_k$ we can rewrite the system as $\mathbf{y}_{k+1} = \Lambda\mathbf{y}_k$, which corresponds to the equations

$$\begin{cases} y_{1,k+1} = \lambda_1 y_{1,k} \\ y_{2,k+1} = \lambda_2 y_{2,k} \\ \cdots \\ y_{n,k+1} = \lambda_n y_{n,k} \end{cases}$$

General solution of the diagonalized system

We know how to write a general solution to those equations, that is $y_{i,k} = \lambda_i^k y_{i,0}$ for $i = 1,\ldots,n$, where $y_{0,i}$ corresponds to the i-th coordinate of $\mathbf{x}_0$ in the basis given by the eigenvectors. That is,

$$\mathbf{y}_0 = P^{-1}\mathbf{x}_0$$

General solution of the system

Finally, we will find the general solution by rewriting $\mathbf{x}_k$ in the original basis, that is $\mathbf{x}_k = P\mathbf{y}_k$. This gives

$$\mathbf{x}_k = P\sum_{i=1}^{n} \lambda_i^k y_{k,i}(0)\mathbf{e}_i = \sum_{i=1}^{n} \lambda_i^k y_{k,i}(0)\mathbf{v}_i$$

The 2×2 case

In what follows, we will analyze different possibilities for the dynamics of diagonal 2×2 matrices, which will help understand what happens in the general, non-diagonal case for $n \geq 2$.

Case 1: both eigenvalues have absolute value greater than 1

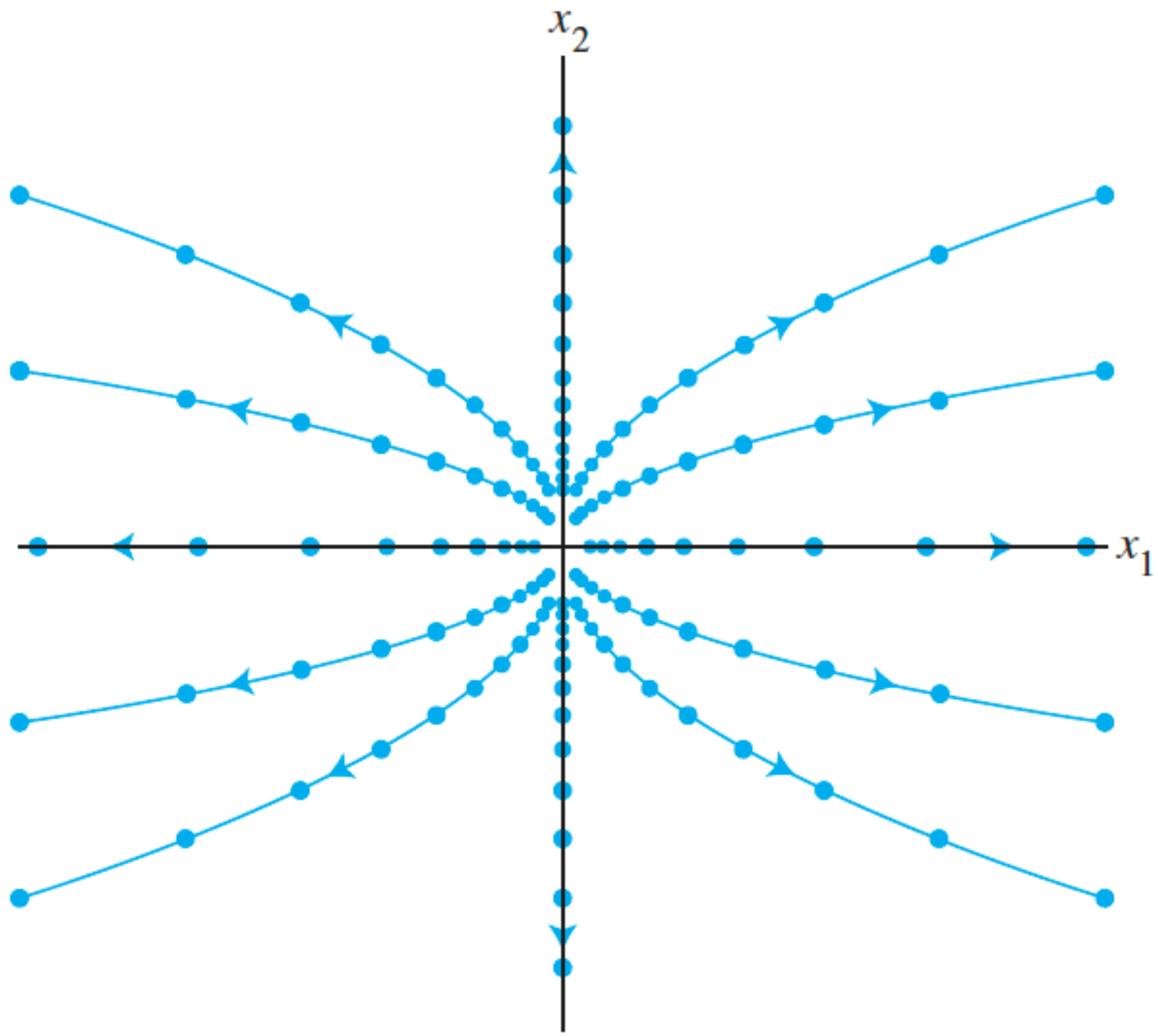
If both eigenvalues are greater than 1, then every starting point outside of the origin is such that the system will ultimately diverge to infinity.

Example:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix}$$

The origin as a repeller

The dynamic follows the figure:



In this case the origin is called a

Repeller for the system

Case 2: both eigenvalues have absolute values less than 1

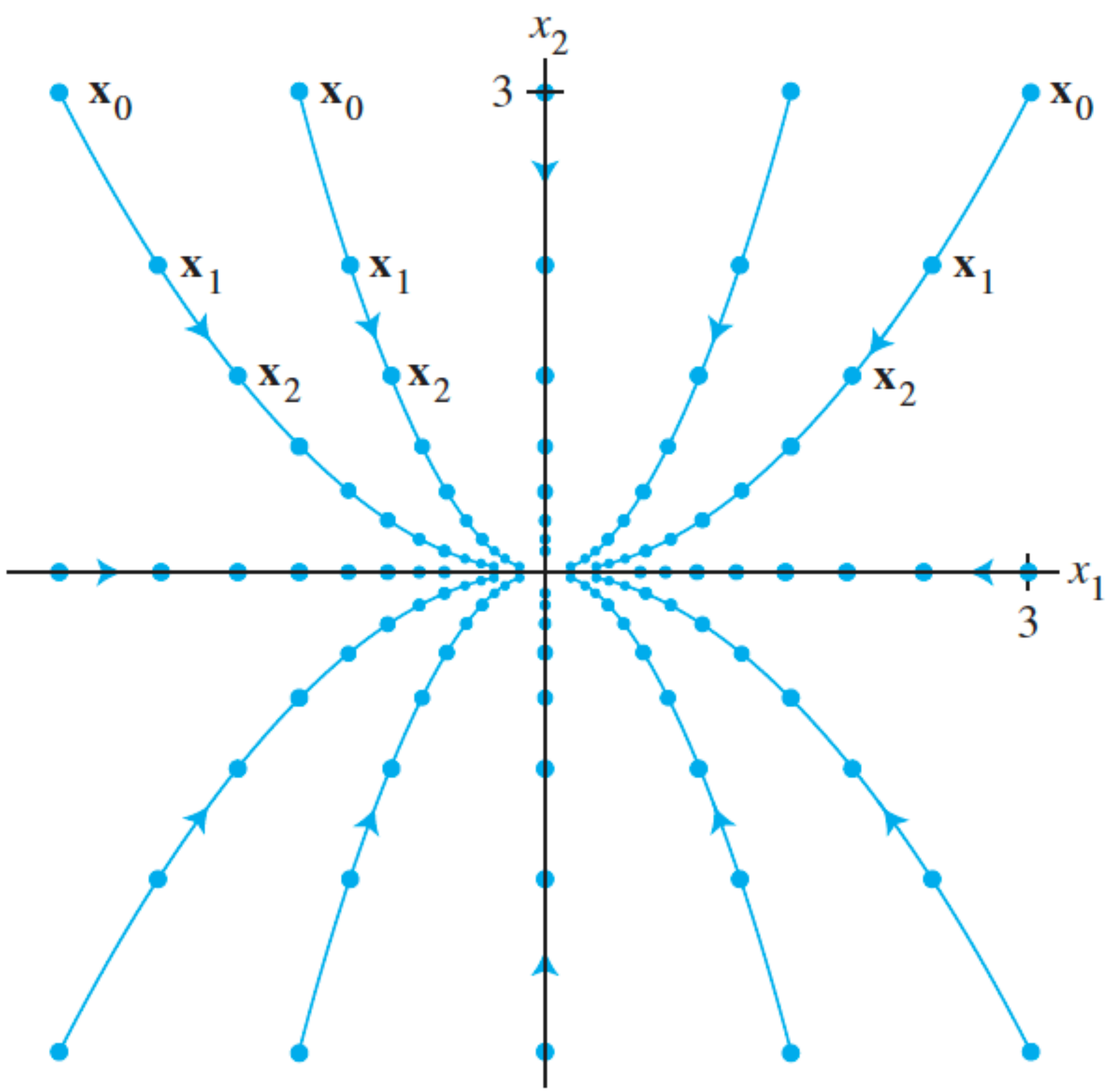
In this case, the origin is called an **attractor** for the system. No matter which starting point, the system will converge to the origin as $k \to \infty$.

Example:

$$A = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.2 \end{bmatrix}$$

The origin as an attractor

The dynamic is shown in the figure:



Case 3: one eigenvalue has absolute value greater than 1 and one in less than 1

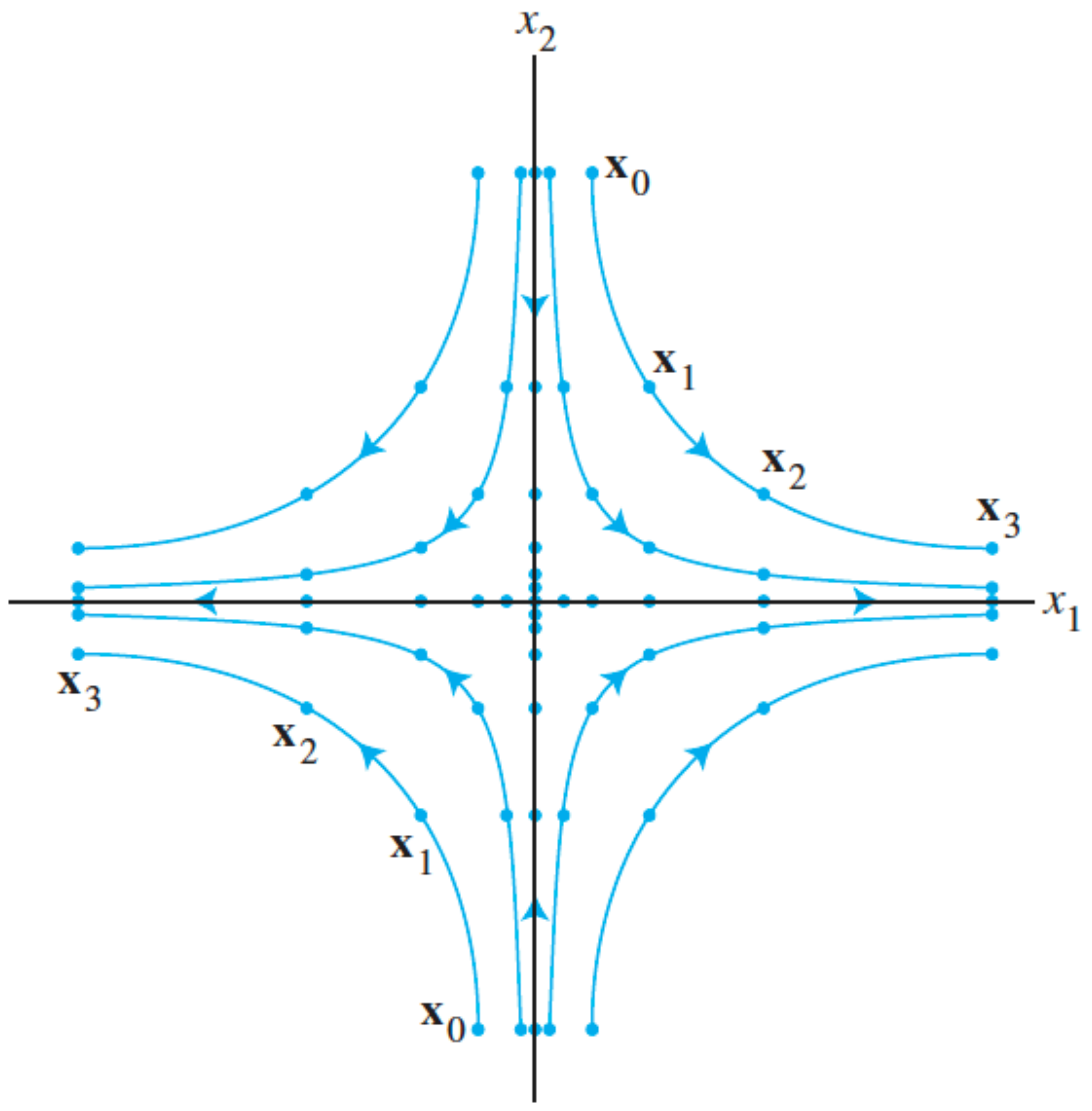
In this case, the origin is called a **saddle point** for the system. This means that as long the coordinates of the starting point give a nonzero value to the leading eigenvector, then the system will diverge. However, there is a direction corresponding to the eigenvector of the smaller eigenvalue such that if the initial condition lies on the subspace generated by this eigenvector, then the system will converge to $\mathbf{0}$.

Example:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0.3 \end{bmatrix}$$

The origin as a saddle point

The dynamic of the system is shown in the figure:



Case 4: one eigenvalue has absolute value equal to 1

This case is of particular interest and will be an object of focus for Markov Chains.

Example:

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$$

Possible outcomes:

- If $|\lambda| > 1$, then $A^k\mathbf{x}_0 \to \infty$ as long as $c_1 \neq 0$, while if $c_1 = 0$, then $A^k\mathbf{x}_0 = c_2\mathbf{e}_2$for every $k \in \mathbb{N}$.
- If $|\lambda| < 1$, then $A^k\mathbf{x}_0 \sim c_2\mathbf{e}_2$, so that $A^k\mathbf{x}_0 \to c_2\mathbf{e}_2$ for $k \to \infty$, no matter the initial state.
- If $|\lambda| = 1$, then either A is the identity matrix (in which case there is no change no matter the initial point), or the state oscillates between $c_1\mathbf{e}_1 + c_2\mathbf{e}_2$ and$-c_1\mathbf{e}_1 + c_2\mathbf{e}_2$. In both cases, the system will not diverge.

Case 5: complex eigenvalues

In case of complex eigenvalues, there is a rotational component added to the dynamical system. In this case, both eigenvalues will have the same absolute value r.

Possible outcomes

- If $r > 1$, then the origin will be a **repeller** and the states of the system will spiral towards infinity.
- If $r = 1$, then the system will remain on an elliptical trajectory around the origin. There will not be a steady state, but the system will not diverge to infinity either.
- If $r < 1$, then the system will spiral inward towards the origin, which is an **attractor** of the system.

Conjugate form

We have seen in last lecture that rather than diagonalizing a matrix A with complex eigenvalues we can find a conjugate form

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

The three cases we just discussed depend on the value$r = \sqrt{a^2 + b^2}$, while the rotation will be given by θ such that $\cos(\theta) = \frac{a}{r}$ and $\sin(\theta) = \frac{b}{r}$.

General solution

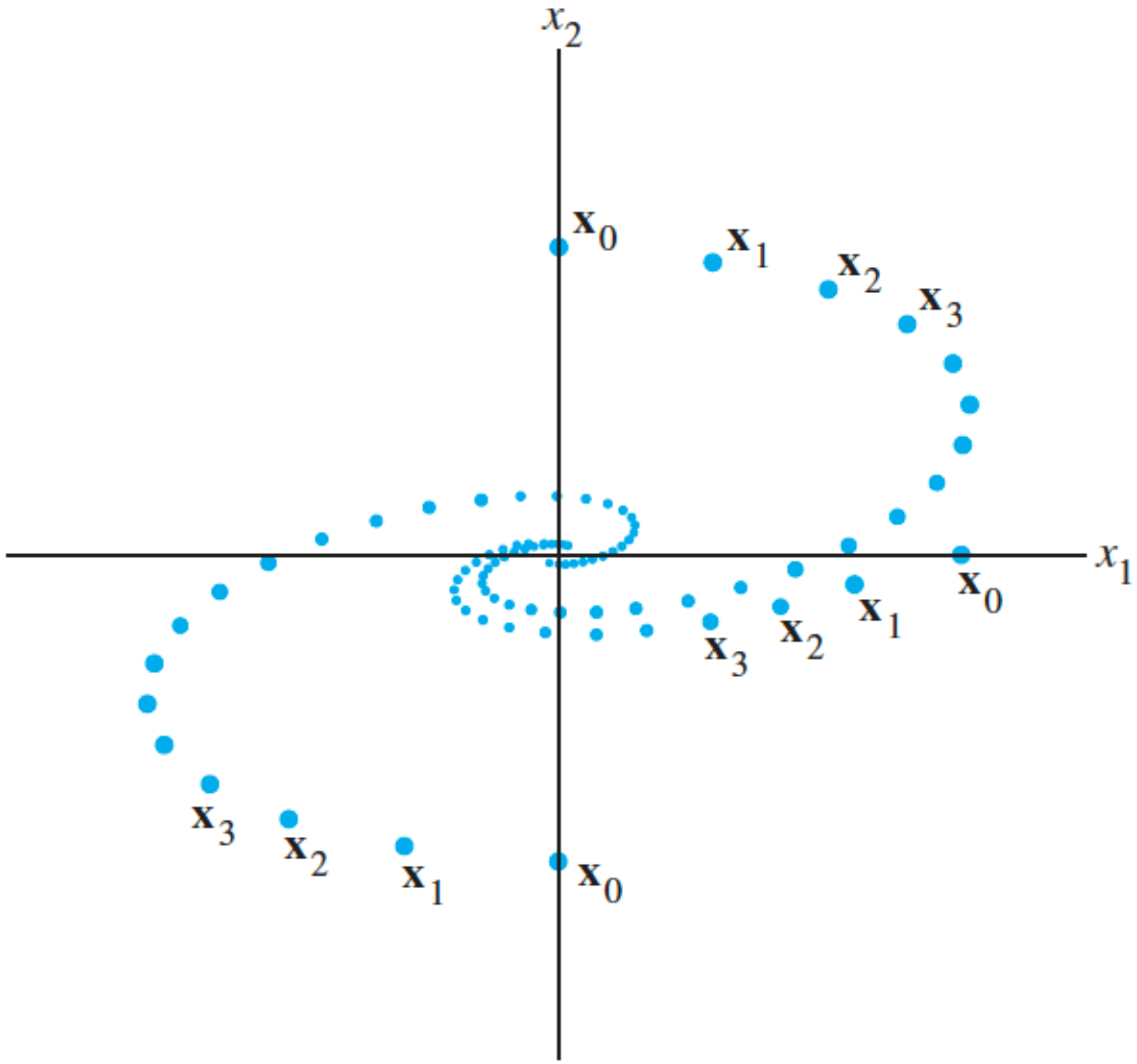
From a starting point $\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, we have

$$C^k \mathbf{x}_0 = r^k \begin{bmatrix} c_1 \cos(k\theta) - c_2 \sin(k\theta) \\ c_1 \sin(k\theta) + c_2 \cos(k\theta) \end{bmatrix}$$

Different choices of r give different orbits.

Spiraling solutions

The following picture shows a possibility for a dynamical system with complex eigenvalues and $r < 1$



Solving systems of difference equations: the general case

In order to understand the behaviour of a discrete dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ from the starting point $\mathbf{x}_0$ one has to take the following steps:

- Diagonalize the matrix A and find the basis of eigenvectors.
- Rewrite the initial state $\mathbf{x}_0$ in the coordinates given by the basis of eigenvectors, $\mathbf{y}_0 = P^{-1}\mathbf{x}_0$
- Analyze the general solution $\mathbf{y}_k = \sum_{i=1}^{n} y_i(0)\lambda_i^k \mathbf{v}_i$ and consider the following possibilities:

Solving systems of difference equations: the general case (continued)

1. If there exists an eigenvalue with absolute value greater than 1 whose corresponding coordinate of $\mathbf{y}_0$ is nonzero, then the system will diverge to infinity.
2. If there exists eigenvalues with absolute value equal to 1 for which the coordinates of $\mathbf{y}_0$ are nonzero and all the other eigenvalues whose coordinates are nonzero have absolute value less than 1, then the system will either converge to a single steady state (if all such eigenvalues are equal to 1) or stay on a steady orbit (if some of the eigenvalues are complex, or equal to -1).
3. Finally, if the only eigenvalues for which the coordinates are nonzero have absolute value less than one, then the system will converge to $\mathbf{0}$.

Example:

$$\mathbf{x}_{k+1} = \frac{1}{9}\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}\mathbf{x}_k$$

Continuous dynamical systems

Differential equations

A similar procedure can be applied in the continuous case, in which we have differential equations rather than difference equations. In this sense, the derivative of each variable with respect to time is a function of all the other variables and possibly time itself, that is

$$\mathbf{x}'(t) = f(\mathbf{x}(t), t)$$

We will only treat the case in which the funcion is linear and independent on time, therefore the case in which $\mathbf{x}'(t) = A\mathbf{x}(t)$, for A a $n \times n$ coefficient matrix independent of time.

Example:

A particle is moving in a force field in $\mathbb{R}^2$ satisfying the following differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

If it starts from $\mathbf{x}(0) = \begin{bmatrix} 2.9 \\ 2.6 \end{bmatrix}$, where is it going to be at time $T = 4$?

Decoupling

In order to solve these types of differential equations, we need to apply a method which is similar to the one for systems of difference equations. In particular, we want to diagonalize the matrix to be able to decouple the single variables.

Differential equations in one variable

If we want to solve the system in one variable

$$\begin{cases} y'(t) = \lambda y(t) \\ y(0) = y_0 \end{cases}$$

then the general solution of this equation is given by $y(t) = y_0 e^{\lambda t}$.

Decoupling through diagonalization

In order to solve a continuous dynamical system $\mathbf{x}'(t) = A\mathbf{x}(t)$, we will diagonalize the matrix A and apply this solutions to the coordinates given by the basis of eigenvectors. In other words, let $A = P\Lambda P^{-1}$ a diagonalization of A. Then if $\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i$, P^{-1} is the change of basis matrix $P^{-1}\mathbf{x} = \mathbf{y} = \sum_{i=1}^{n} y_i \mathbf{e}_i$ such that $\sum_{i=1}^{n} y_i \mathbf{v}_i = \mathbf{x}$, where $\mathbf{v}_i$ is the eigenvector corresponding to λ_i in the i-th column of P.

Decoupling through diagonalization (continued)

Linearity of the derivative guarantees that

$$\mathbf{y}'(t) = P^{-1}\mathbf{x}'(t) = P^{-1}A\mathbf{x}(t) = P^{-1}P\Lambda P^{-1}\mathbf{x}(t) = \Lambda\mathbf{y}(t)$$

which can be rewritten as

$$\begin{cases} y_1'(t) = \lambda_1 y_1(t) \\ y_2'(t) = \lambda_2 y_2(t) \\ \cdots \\ y_n'(t) = \lambda_n y_n(t) \end{cases}$$

Solving a decoupled system

The system can be solved as

$$\mathbf{y}(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix} \mathbf{y}(0)$$

where $\mathbf{y}(0) = P^{-1}\mathbf{x}(0)$.

General solution of a continuous system

Finally we can write this in the original coordinates

$$\mathbf{x}(t) = P\mathbf{y}(t) = \sum_{i=1}^{n} y_i(0)e^{\lambda_i t}\mathbf{v}_i$$

where $y_i(0)$ is the i-th coordinate of the vector $\mathbf{y}(0) = P^{-1}\mathbf{x}(0)$.

Example:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \mathbf{x}(0) = \begin{bmatrix} 2.9 \\ 2.6 \end{bmatrix}$$

Limiting behavior for continuous dynamical systems

The behaviour of $\mathbf{x}(t)$ for $t \to \infty$ depends, as in the discrete case, on a combination of the eigenvalues and of the coordinates in the eigenvector basis. The main difference from the discrete case, however, is that the eigenvalues are on the exponent of the expression for the solution, rather than on the base.

Possibilities for real eigenvalues

- If there exists an $i = 1,...,n$ such that $\lambda_i > 0$ and $y_i(0) \neq 0$, then the system diverges to ∞.
- If all the eigenvalues λ_i for which the corresponding coordinate $y_i(0) \neq 0$ are such that $\lambda_i < 0$, then the system converges to the origin.
- If there exist coordinates $y_i(0) \neq 0$ with corresponding eigenvalues $\lambda_i = 0$ and all the other nonzero coordinates are such that $\lambda_j < 0$, then the solution converges to a steady state.

Example:

The solution to the previous example is

$$\mathbf{x}(t) = -\frac{3}{70}e^{6t}\begin{bmatrix} -5 \\ 2 \end{bmatrix} + \frac{188}{70}e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Complex eigenvalues

Suppose that A is a square 2×2 matrix with a pair of complex eigenvalues. Then we know there exists a matrix of the form

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

obtained by conjugating A through a matrix $S = \begin{bmatrix} \text{Re}(\mathbf{v}_1) & \text{I}(\mathbf{v}_1) \end{bmatrix}$.

Change of basis through S

Since A has complex eigenvalues, S has maximum rank, so S^{-1} induces a change of basis from the canonical basis to $\{\text{Re}(\mathbf{v}_1), \text{I}(\mathbf{v}_1)\}$.

In such basis, we have a linear differential equation of the type

$$\begin{cases} y_1'(t) = ay_1(t) - by_2(t) \\ y_2'(t) = by_1(t) + ay_2(t) \\ \mathbf{y}(0) = S^{-1}\mathbf{x}(0) \end{cases}$$

Solution to the differential equation

The solution is determined by

$$\mathbf{y}(t) = e^{at} \begin{bmatrix} y_1(0)\cos(bt) - y_2(0)\sin(bt) \\ y_1(0)\sin(bt) + y_2(0)\cos(bt) \end{bmatrix}$$

Long-term dynamics of the complex system

The dynamics of this system depends on the values of a, with the following possibilities:

- If $a > 0$, then the system diverges as $t \to \infty$.
- If $a < 0$, then the system converges to the origin as $t \to \infty$.
- Finally, if $a = 0$, the system stays on a fixed orbit.

General solution for the complex system

Then the general solution in the original coordinates is given by $\mathbf{x}(t) = S\mathbf{y}(t)$, that is

$$\mathbf{x}(t) = e^{at}(y_1(0)\cos(bt) - y_2(0)\sin(bt))\text{Re}(\mathbf{v}_1) + e^{at}(y_1(0)\sin(bt) + y_2(0)\cos(bt))\text{I}(\mathbf{v}_1)$$

Example:

$$\mathbf{x}'(t) = \begin{bmatrix} -2 & -2.5 \\ 10 & -2 \end{bmatrix} \mathbf{x}(t), \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Basics of Markov chains

What is a Markov chain?

A Markov chain is a discrete difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$, in which the transition matrix A is a **stochastic matrix**, that is, all columns have non-negative entries which sum to one. In general, a process is called **Markov** if the current state only depends from the previous one and not from all the other before it and is called N-Markov if the current states depend from up to N states before itself, but no other state.

Example: migration dynamics

Let x_1 represent population in an urban area, x_2 population in a suburban area and x_3 population in a rural area. Then suppose that these populations follow the dynamics given by

$$\mathbf{x}_{k+1} = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.7 \end{bmatrix} \mathbf{x}_k$$

Long-term behavior of a Markov chain

Intuitively, it seems that there is no way for the system to diverge to infinity or to converge to zero, as the total population will be preserved, even if split differently between the possible states.

Steady state

Definition

A state $\bar{\mathbf{x}}$ is called a **steady state** for a Markov chain if $A\bar{\mathbf{x}} = \bar{\mathbf{x}}$, that is $\bar{\mathbf{x}}$ is an eigenvector for $\lambda = 1$.

Existence of steady state

In particular, under the following conditions there will always be a steady state for the Markov chain:

Perron-Frobenius Theorem

Let A be a stochastic matrix such that a positive power of A has all strictly positive entries. Then $\lambda_1 = 1$ is the highest eigenvalue of A having a unique eigenvector with all positive entries.

Getting to the steady state

The steady state of the Markov chain is therefore the unique eigenvector associated to the eigenvalue $\lambda_1 = 1$. Under the regularity condition stated in the theorem, this means that a Markov chain always admits a unique steady state and that no matter where we start from, up to a multiplicative constant the system will get to the steady state.

Example:

$$A = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.7 \end{bmatrix}$$