Systems of linear equations: generalities and Gaussian elimination

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Lecture 3: Introduction to systems of linear equations and Gaussian elimination

Generalities

A system of m linear equation in n unknown is an object of the type

$$\begin{cases} A_{1,1}x_1 + A_{1,2}x_2 + ... + A_{1,n}x_n = b_1 \\ A_{2,1}x_1 + A_{2,2}x_2 + ... + A_{2,n}x_n = b_2 \\ ... \\ A_{m,1}x_1 + A_{m,2}x_2 + ... + A_{m,n}x_n = b_m \end{cases}$$

For given coefficients $\{A_{ij}\}_{i,j}$ and constants $\{b_j\}_j$, finding the solution to a system of linear equations is equivalent to finding values $(x_1, ..., x_n)$ that satisfy all the equations at once. We call all the possible values that satisfy a system of linear equations, the *set of solutions* of the system. Two systems are equivalent if they have the same set of solutions.

We can distinguish three cases:

- **One solution**: there is only one choice of $(x_1, ..., x_n)$ satisfying the system of linear equations.
- **Infinite solutions**: there are infinitely many choices of $(x_1, ..., x_n)$ satisfying the system of linear equations.
- **No solution**: there is no choice of $(x_1, ..., x_n)$ satisfying the system of linear equations.

In the first two cases, we say that the system is *consistent*, while if it has no solutions, the system is said to be *inconsistent*.

What is the intuition behind these three possibilities? One should think of it this way: each unknown represents a degree of freedom, while each equation represents an additional requirement to pin down a set of solutions, potentially reducing the degrees of freedom of the system. If there are too many requirements, then it's possible that no solutions exist. On the other hand, if the requirements are loose with respect to the degrees of freedom, then there could be an infinite number of solutions.

To complicate the matter even more, two equations can give the same requirement. Think about the case in which we have equations:

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 2x_2 &= 4 \end{aligned}$$

The second equation does not really give us more information to pin down the solution, as it's a linear combination of the first. Therefore more equations don't automatically mean less degrees of freedom. However, since we are dealing with linear combinations, we will develop tools to solve systems of linear equations in full generality using linear algebra.

If we have more unknowns than equations ($n > m$), then the system is said to be *under-determined*. If there are more equations than unknowns ($m > n$) then the system is *over-determined*. Finally, if $n = m$, the system is called *square*.

A system of linear equations can always be represented in matrix form as $A\mathbf{x} = \mathbf{b}$, where A is the $m \times n$ *coefficient matrix*, $\mathbf{x}$ is the $n \times 1$ vector of unknowns and $\mathbf{b}$ is the $m \times 1$ vector of *constants*. The matrix $[A|\mathbf{b}]$ is called *augmented matrix* of the system.

Notice that if $\mathbf{b} = \mathbf{0}$, then there is always a solution to the system given by $x_i = 0$ for $i = 1, ..., n$. In this case, the system is called *homogeneous*.

Example 1:

$$\begin{cases} 3x_1 - 2x_2 = -2 \\ 2x_1 - x_3 = 0 \\ -6x_1 + 4x_2 + 2x_3 = 4 \end{cases}$$

In this case the corresponding matrices are given by

$$A = \begin{bmatrix} 3 & -2 & 0 \\ 2 & 0 & -1 \\ -6 & 4 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$$

while the augmented matrix is the matrix

$$A' = \begin{bmatrix} 3 & -2 & 0 & -2 \\ 2 & 0 & -1 & 0 \\ -6 & 4 & 2 & 4 \end{bmatrix}$$

This is a square system, since the number of unknowns is the same as the number of equations.

Example 2:

$$\begin{cases} -4x_1 + 2x_3 + x_5 = 2 \\ 2x_1 + 3x_2 - x_4 = 8 \end{cases}$$

In this case the corresponding matrices are given by

$$A = \begin{bmatrix} -4 & 0 & 2 & 0 & 1 \\ 2 & 3 & 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

while the augmented matrix is the matrix

$$A' = \begin{bmatrix} -4 & 0 & 2 & 0 & 1 & 2 \\ 2 & 3 & 0 & -1 & 0 & 8 \end{bmatrix}$$

In this case the system is under-determined. Intuitively, this means that there is not enough information to have exactly one solution.

Example 3:

$$\begin{cases} 3x_1 + 2x_2 = 0 \\ 2x_1 - x_3 = 1 \\ -6x_1 + 4x_2 - 2x_3 = 0 \\ -2x_2 + x_3 = 1 \end{cases}$$

In this case the corresponding matrices are given by

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & -1 \\ -6 & 4 & -2 \\ 0 & -2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

while the augmented matrix is

$$A' = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 2 & 0 & -1 & 1 \\ -6 & 4 & -2 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

In this case the system is over-determined, meaning that there are more conditions than unknowns.

Back-substitution

A first method to solve systems of linear equations is given by back-substitution. This is the practice by which we use the equations of the systems to find a variable as a function of the others and substitute into the remaining equations to reduce the number of variables. Ideally, we want to minimize the number of variables present in each equation to understand if the system is consistent or not.

Example 1:

Consider the first example

$$\begin{cases} 3x_1 - 2x_2 = -2 \\ 2x_1 - x_3 = 0 \\ -6x_1 + 4x_2 + 2x_3 = 4 \end{cases}$$

we can use the second equation to obtain the condition $x_3 = 2x_1$ to substitute into the other equations, so that we get

$$\begin{cases} 3x_1 - 2x_2 = -2 \\ -x_1 + 2x_2 = 2 \\ x_3 = 2x_1 \end{cases} \tag{1}$$

Now from the second equation we get $x_1 = 2x_2 - 2$, that we can substitute in the first equation to get

$$6x_2 - 6 - 2x_2 + 2 = 0$$

which gives the solution $x_2 = 1$. We can now back-substitute into the previous conditions to get $x_1 = 0$ and $x_3 = 0$.

A fundamental intuition about substitution is that we can obtain an equivalent system by subtracting an equation from another in a way that erases one of the variables. In this example, starting from

$$\begin{cases} 3x_1 - 2x_2 = -2 \\ 2x_1 - x_3 = 0 \\ -6x_1 + 4x_2 + 2x_3 = 4 \end{cases}$$

we can add the second equation twice to the third to obtain

$$\begin{cases} 3x_1 - 2x_2 = -2 \\ 2x_1 - x_3 = 0 \\ -2x_1 + 4x_2 = 4 \end{cases}$$

which, upon dividing the third equation by 2 and changing order is equivalent to the system in (1).

The same sequence of operations can be taken in matrix notation to obtain an equivalent system. Ideally, we would like an augmented matrix of the type $[I|\mathbf{c}]$ where I is the identity matrix. In this example we would start from

$$A' = \begin{bmatrix} 3 & -2 & 0 & -2 \\ 2 & 0 & -1 & 0 \\ -6 & 4 & 2 & 4 \end{bmatrix}$$

First we want to transform A' to make it upper-triangular, so that each equation has a different leading coefficient. By mirroring the back-substitution method, we substitute the third row with itself plus twice the second row to get

$$\begin{bmatrix} 3 & -2 & 0 & -2 \\ 2 & 0 & -1 & 0 \\ -2 & 4 & 0 & 4 \end{bmatrix}$$

then we divide the third row by two:

$$\begin{bmatrix} 3 & -2 & 0 & -2 \\ 2 & 0 & -1 & 0 \\ -1 & 2 & 0 & 2 \end{bmatrix}$$

Now we substitute the first row with itself plus three times the third row to have the first coefficient disappear:

$$\begin{bmatrix} 0 & 4 & 0 & 4 \\ 2 & 0 & -1 & 0 \\ -1 & 2 & 0 & 2 \end{bmatrix}$$

then we divide the first row by 4:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & -1 & 0 \\ -1 & 2 & 0 & 2 \end{bmatrix}$$

and substitute the third row by itself minus twice the first row

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

Then, we multiply the third row by -1 and subtract it twice from the second

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

multiplying the second row by -1 and rearranging gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

which translates to the form

$$\begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 0 \end{cases}$$

Example 2:

Consider the second, underdetermined example. We have

$$\begin{cases} -4x_1 + 2x_3 + x_5 = 2 \\ 2x_1 + 3x_2 - x_4 = 8 \end{cases}$$

In order to find a set of solutions to this type of systems, we should find out how many of them are free variables and write the remaining ones as a function of those we have. This is a system of 2 equations in 5 unknowns, so it would have 3 degrees of freedom. In this sense, we can, for instance, write x_1 and x_2 as functions of x_3, x_4 and x_5. To do so, we need to eliminate x_1 from one of the equations, for instance by substituting $2x_1 = 8 - 3x_2 + x_4$ in the first equation to get

$$\begin{cases} 6x_2 - 2x_4 + 2x_3 + x_5 = 18 \\ 2x_1 = 8 - 3x_2 + x_4 \end{cases}$$

then by bringing everything to the other side and dividing by the corresponding coefficient we get

$$\begin{cases} x_2 = 3 - \frac{x_3}{3} + \frac{x_4}{3} - \frac{x_5}{6} \\ x_1 = 4 - \frac{3}{2}x_2 + \frac{x_4}{2} \end{cases}$$

by substituting the first into the second we get

$$\begin{cases} x_2 = 3 - \frac{x_3}{3} + \frac{x_4}{3} - \frac{x_5}{6} \\ x_1 = -\frac{1}{2} + \frac{x_3}{2} + \frac{x_5}{4} \end{cases}$$

which is the form we desire.

Writing this in matrix notation starts with

$$A' = \begin{bmatrix} -4 & 0 & 2 & 0 & 1 & 2 \\ 2 & 3 & 0 & -1 & 0 & 8 \end{bmatrix}$$

adding twice the second row to the first leads to

$$\begin{bmatrix} 0 & 6 & 2 & -2 & 1 & 18 \\ 2 & 3 & 0 & -1 & 0 & 8 \end{bmatrix}$$

Then we can divide everything by the leading coefficients to get

$$\begin{bmatrix} 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{6} & 3 \\ 1 & \frac{3}{2} & 0 & -\frac{1}{2} & 0 & 4 \end{bmatrix}$$

Finally, we subtract $\frac{3}{2}$ the first row from the second to get

$$\begin{bmatrix} 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{6} & 3 \\ 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{4} & -\frac{1}{2} \end{bmatrix}$$

whose lines can be shifted to obtain a matrix of the type

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{6} & 3 \end{bmatrix}$$

which translates into

$$\begin{cases} x_1 - \frac{x_3}{2} - \frac{x_5}{4} = -\frac{1}{2} \\ x_2 + \frac{x_3}{3} - \frac{x_4}{3} + \frac{x_5}{6} = 3 \end{cases}$$

Example 3:

Consider the third, overdetermined system:

$$\begin{cases} 3x_1 + 2x_2 = 0 \\ 2x_1 - x_3 = 1 \\ -6x_1 + 4x_2 - 2x_3 = 0 \\ -2x_2 + x_3 = 1 \end{cases}$$

From the second and fourth equation we obtain $x_1 = \frac{1+x_3}{2}$ and $x_2 = \frac{x_3-1}{2}$, that is

$$\begin{cases} 3x_1 + 2x_2 = 0 \\ x_1 = \frac{1+x_3}{2} \\ -6x_1 + 4x_2 - 2x_3 = 0 \\ x_2 = \frac{x_3-1}{2} \end{cases}$$

We can substitute these in the first and third equations to get

$$\begin{cases} x_3 = \frac{1}{5} \\ x_1 = \frac{1+x_3}{2} \\ x_3 = \frac{5}{3} \\ x_2 = \frac{x_3-1}{2} \end{cases}$$

which are in contradiction, so that the system admits no solution. In order to write this in matrix form, start from the augmented matrix

$$A' = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 2 & 0 & -1 & 1 \\ -6 & 4 & -2 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

First divide the second and fourth row by the leading coefficient:

$$\begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ -6 & 4 & -2 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

then subtract from the first line three times the second plus two times the fourth:

$$\begin{bmatrix} 0 & 0 & \frac{5}{2} & -\frac{1}{2} \\ 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ -6 & 4 & -2 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

and subtract from the third row -6 times the second plus 4 times the fourth

$$\begin{bmatrix} 0 & 0 & \frac{5}{2} & -\frac{1}{2} \\ 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -3 & 5 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Divide the first and third row by the leading coefficient to obtain

$$\begin{bmatrix} 0 & 0 & 1 & -\frac{1}{5} \\ 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

and subtract the first row from the third to get

$$\begin{bmatrix} 0 & 0 & 1 & -\frac{1}{5} \\ 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{22}{15} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

which can be written as the following system of equations:

$$\begin{cases} x_3 = -\frac{1}{5} \\ x_1 - \frac{x_3}{2} = \frac{1}{2} \\ 0 = -\frac{22}{15} \\ x_2 - \frac{x_3}{2} = -\frac{1}{2} \end{cases}$$

where we notice that the third equation is false, therefore no triple (x_1, x_2, x_3) satisfied these equations. Notice that the last passage was not considered in back-substitution. However, it will be useful to write a decision process for an algorithm regarding whether the system is consistent or not.

In matrix notation we have used the following row operations to switch between equivalent systems:

1. **Replacement**: replace a row by a nontrivial linear combination of itself with another row.

2. **Interchange**: interchange two rows.

3. **Scaling**: multiply a row by a nonzero real number.

Gaussian elimination

We want to write an algorithm that for any given system returns either the set of solution or the information that the system has no solution.

The Gaussian elimination algorithm does so by reducing the augmented matrix A' to a unique form, called *reduced echelon form* on which a decision rule can be applied. The reduction is done through the admissible row operations previously described. The treatment of this notes follows Lay, Lay and McDonald, Chapter 1.

In what follows we give some terminology used by the algorithm.

Definition 1. We say that a rectangular matrix is in *echelon form* if it has the following properties:

1. All nonzero rows are above all the rows of all zeros.
2. Each leading term in a row has higher column coefficient than the leading term of any previous row.
3. All entries in a column below the leading term of a row are zero.

Moreover, we say that a rectangular matrix is in *reduced echelon form* if the following is also true:

1. The leading term of each row is 1.
2. Each leading 1 is the only nonzero element of its column.

Examples of matrices in echelon form are:

$$\begin{bmatrix}1 & 0 & -1\\ 0 & -1 & 2\\ 0 & 0 & 3\end{bmatrix}, \qquad \begin{bmatrix}1 & 0 & -1\\ 0 & 0 & 4\\ 0 & 0 & 0\end{bmatrix}, \qquad \begin{bmatrix}1 & 0 & -1 & 10\\ 0 & 0 & 2 & 2\\ 0 & 0 & 0 & 4\end{bmatrix}.$$

Examples of matrices in reduced echelon form are:

$$\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 3\end{bmatrix}, \qquad \begin{bmatrix}1 & 3 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}, \qquad \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

The Gaussian elimination algorithm aims to transform the augmented matrix into its reduced echelon form and then implements a decision rule to understand if the system has no solution, one solution or infinitely many. In the previous examples, writing the corresponding systems of linear equation tells us that the first system has one solution, the second system has infinitely many solutions and the third has no solution.

Remember that we say that two systems are equivalent if they have the same set of solutions. In matrix notation, we say that two matrices are equivalent if they differ from one of the row operations we described before: replacement, interchange or scaling. The following theorem tells us that we can find a unique reduced echelon form for every augmented matrix A' on which we can apply our decision rule:

Theorem 1. *For every matrix A' there exists a unique equivalent matrix A'_{ref} in reduced echelon form.*

In fact, there are infinitely many matrices in echelon form that are equivalent to A, but only one in reduced echelon form. Finally we will use the following terms:

Definition 2. A *pivot position* in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A *pivot column* is a column containing a pivot position.

The Gaussian elimination algorithm works as follows:

1. Choose the leftmost non-zero column: this is a pivot column.
2. Choose a nonzero term in the pivot column to be the pivot and exchange rows till this element is in the pivot position.
3. Use row replacement operations to create all zeros below the pivot.
4. Consider the submatrix formed by the rows below the pivot and repeat the first 3 steps until there is no submatrix left.
5. Beginning with the rightmost pivot, create zeros above each pivot and scale so that the pivot is equal to one.

In order to see how it works we use the following example of an augmented matrix:

$$A' = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Step 1: Since the first column is not all zero, we consider it as the first pivot column and the position $A'_{1,1}$ as the first pivot position.

Step 2: We choose $A'_{4,1} = 1$ to be the pivot element and thus switch the fourth column with the first.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Step 3: We eliminate the first term from the second and third row. This can be done by adding the first row to the the second and by adding twice the first row to the third:

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Step 4: Reduce to the submatrix obtained by eliminating the first row and apply Step 1-3.

$$A'' = \begin{bmatrix} 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Step 1,2: Choose 2 to be the pivot.

Step 3: eliminate the second term from the second and third row:

$$\begin{bmatrix} 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Step 4: Reduce to the submatrix obtained by eliminating the first row and apply Step 1-3.

$$A''' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Step 1: Choose the fourth column as pivot column.

Step 2: Switch the rows so that -5 is in the pivot position

$$A''' = \begin{bmatrix} 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 3: No need for row reduction.

Step 4: The submatrix has only one row so it's already row-reduced. We get the echelon matrix to have the form:

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 5: We create 0 above each pivot and scale accordingly

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the matrix in the reduced echelon form, corresponding to the system

$$\begin{cases} x_1 - 3x_3 = 5 \\ x_2 + 2x_3 = -3 \\ x_4 = 0 \\ x_3 \text{ free} \end{cases}$$

which has infinitely many solutions depending on x_3 as the free parameter. In particular the reduced echelon form of a matrix tells us the following:

Theorem 2. *Let A be the reduced echelon form of a rectangular matrix. Then:*

- *If the first n rows of A are of the form $[I|\mathbf{c}]$ where I is the $n \times n$ identity matrix and $\mathbf{c}$ is a n-dimensional vector, while the following rows are rows of all zeros, then the corresponding system admits a unique solution, $x_i = c_i$ for $i = 1, ..., n$.*
- *If there exists a row of the form $[\mathbf{0}|b]$ for $b \neq 0$, then the corresponding system has no solutions.*
- *Otherwise, the system admits infinitely many solutions.*

Let us now apply the algorithm to our previous examples.

Example 1:

$$A' = \begin{bmatrix} 3 & -2 & 0 & -2 \\ 2 & 0 & -1 & 0 \\ -6 & 4 & 2 & 4 \end{bmatrix}$$

We want -6 to be the pivot, so we switch first and third row:

$$\begin{bmatrix} -6 & 4 & 2 & 4 \\ 3 & -2 & 0 & -2 \\ 2 & 0 & -1 & 0 \end{bmatrix}$$

then we make sure that all the element below it are equal to 0:

$$\begin{bmatrix} -6 & 4 & 2 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 4 & -1 & 4 \end{bmatrix}$$

then we switch second and third row to obtain a echelon form:

$$\begin{bmatrix} -6 & 4 & 2 & 4 \\ 0 & 4 & -1 & 4 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Now, by appropriate scaling and subtraction we can find the reduced echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

which gives us a matrix of the form $[I|\mathbf{c}]$ so that the system has the unique solution $x_i = c_i$, in this case

$$\begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 0 \end{cases}$$

Example 2:

$$A' = \begin{bmatrix} -4 & 0 & 2 & 0 & 1 & 2 \\ 2 & 3 & 0 & -1 & 0 & 8 \end{bmatrix}$$

We start by switching the first and second row

$$\begin{bmatrix} 2 & 3 & 0 & -1 & 0 & 8 \\ -4 & 0 & 2 & 0 & 1 & 2 \end{bmatrix}$$

We add twice the first row to the second row to obtain the echelon form of the matrix, with pivot in position $A'_{1,1}$ and $A'_{2,2}$.

$$\begin{bmatrix} 2 & 3 & 0 & -1 & 0 & 8 \\ 0 & 6 & 2 & -2 & 1 & 18 \end{bmatrix}$$

Now we can substitute the first row with two times itself minus the second to eliminate the second variable:

$$\begin{bmatrix} 4 & 0 & -2 & 0 & -1 & -2 \\ 0 & 6 & 2 & -2 & 1 & 18 \end{bmatrix}$$

Finally, we divide the first row by 4 and the second row by 6 to obtain the reduced echelon form:

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{6} & 3 \end{bmatrix}$$

which can be converted to a system in the form

$$\begin{cases} x_1 - \frac{x_3}{2} - \frac{x_5}{4} = -\frac{1}{2} \\ x_2 + \frac{x_3}{3} - \frac{x_4}{3} + \frac{x_5}{6} = 3 \end{cases}$$

with x_1 and x_2 function of the free variables x_3, x_4 and x_5. In this case the system has infinitely many solutions. In this case, the resulting matrix does not have the form $[I|\mathbf{c}]$ for I identity matrix and $\mathbf{c}$ column vector, but it doesn't admit any row of the form $[\mathbf{0}|b]$ for $b \neq 0$, therefore it is consistent with the decision process of Theorem 2.

Example 3:

$$A' = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 2 & 0 & -1 & 1 \\ -6 & 4 & 2 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

We start by switching the first and third row and then apply row-reductions to eliminate the first variable from all the equations except the first:

$$\begin{bmatrix} -6 & 4 & 2 & 0 \\ 0 & 4 & -1 & 3 \\ 0 & 8 & 2 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

Now we switch the second and third row and eliminate the second variable from the remaining equations:

$$\begin{bmatrix} -6 & 4 & 2 & 0 \\ 0 & 8 & 2 & 0 \\ 0 & 0 & -4 & 6 \\ 0 & 0 & 6 & 4 \end{bmatrix}$$

Finally, we eliminate the third variable from the last equation by first doubling it and then adding three times the third equation:

$$\begin{bmatrix} -6 & 4 & 2 & 0 \\ 0 & 8 & 2 & 0 \\ 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & 22 \end{bmatrix}$$

This is a echelon form of the augmented matrix. We can already see from the last row that there is a row of the form $[\mathbf{0}|b]$ with $b \neq 0$, so that the system has no solution. However, let's write this in reduced echelon form as an exercise:

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & \frac{3}{8} \\ 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 22 \end{bmatrix}$$

Notice that the last row didn't enter the process, so that we could have already decided that the system had no solutions.