Problem 1

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = x^2 + (y - 1)^2$.

1. *(10 points)* Identify the level curves of this function and sketch them in $\mathbb{R}^2$.
Solution:
The level curve $f(x, y) = k$ corresponds to $x^2 + (y - 1)^2 = k$ which is a circle of center $(0, 1)$ and radius $\sqrt{k}$.

2. *(10 points)* Identify the graph as a quadric in $\mathbb{R}^3$ and give a surface parametrization.
Solution:
The graph $z = x^2 + (y - 1)^2$ is a circular paraboloid. A possible parametrization is given by $s(u, v) = (u \cos(v), 1 + u \sin(v), u^2)$ for $u \in [0, +\infty), v \in [0, 2\pi]$.

3. *(10 points)* Find the equation of the plane tangent to the graph in $(3, 2)$.
Solution:
The value of the function in $(3, 2)$ if $f(3, 2) = 10$. The gradient of the function is given by $\nabla f = (2x, 2(y - 1))$ so that $\nabla f(3, 2) = (6, 2)$ and the equation of the plane is given by
$$z - 10 = 6(x - 3) + 2(y - 2)$$
that is, $6x + 2y - z = 12$.

4. *(10 points)* Find the absolute maximum and minimum of this function in the domain
$$D = \{(x, y)|\, (x - 1)^2 + y^2 \leq 1\}$$
Solution:
We first find the critical points $\nabla f(x, y) = (0, 0)$. In this case the only critical point is $(0, 1)$, which is not in the domain $((0 - 1)^2 + 1^2 \geq 1$. We then restrict to the boundary $(x - 1)^2 + y^2 = 1$ and use the method of Lagrange multipliers. Our Lagrangian will be
$$L(x, y; \lambda) = x^2 + (y - 1)^2 - \lambda[(x - 1)^2 + y^2 - 1]$$
from which we get the system of equations
$$\begin{cases} 2x - 2\lambda(x - 1) = 0 \\ 2(y - 1) + 2\lambda y = 0 \\ (x - 1)^2 + y^2 = 1 \end{cases}$$
Solving the system holds the two points $(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(1 + \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and evaluating the function on these points tells us that
$$f(1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = \frac{3 - 2\sqrt{2}}{2} < \frac{3 + 2\sqrt{2}}{2} = f(1 + \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$$
so that the first is the absolute minimum and the second is the absolute maximum.

Problem 2

Evaluate the following integrals:

1. *(10 points)* $\iint_S \boldsymbol{F} \cdot dS$ for S the closed surface bounding the region

$$\{(x, y, z)|\, z \geq x^2 + y^2,\, z \leq 4\}$$

and $\boldsymbol{F}(x, y, z) = (\cos(z) + xy^2, xe^{-z}, \sin(y) + x^2 z)$.
Solution:
We need to use the divergence theorem, which states that if E is the region above,

$$\iint_S \boldsymbol{F} \cdot dS = \iiint_E \text{div} F \, dV$$

We get that $divF = y^2 + x^2$ and if we parametrize E in cylindrical coordinates we get

$$E = \{(\rho, \theta, z)|\, \theta \in [0, 2\pi],\, \rho \in [0, 2],\, z \in [\rho^2, 4]\}$$

so that the integral becomes

$$\iiint_E \text{div} F \, dV = \int_0^{2\pi} \int_0^2 \int_{\rho^2}^4 \rho^3 \, dz \, d\rho \, d\theta = \frac{32}{3}\pi$$

2. *(10 points)* $\iint_S \text{curl}\boldsymbol{F} \cdot dS$ for S the part of the surface $z = x^2 + y^2$ that lies below the plane $z = 1$ and $\boldsymbol{F}(x, y, z) = (y^2, x, z^2)$.
Solution:
In this case we can either apply Stokes' theorem or calculate the integral directly. If we apply Stokes' theorem, we notice that the boundary curve is given by the circle on $z = 1$, oriented clockwise if we consider the outward orientation (the one with negative z-component). This is parametrized as $\gamma(t) = (\sin(t), \cos(t), 1)$ for $t \in [0, 2\pi]$, so that $\gamma'(t) = (\cos(t), -\sin(t), 0)$. Stokes' theorem tells us that

$$\iint_S \text{curl}\boldsymbol{F} \cdot dS = \int_\gamma \boldsymbol{F} \cdot d\gamma = \int_0^{2\pi} [\cos^2(t)\sin(t) - \sin^2(t)] \, dt = -\pi$$

No orientation was specified, so choosing the other orientation would have been fine too. The same result would have been obtained by evaluating the surface integral for $\text{curl}F = (0, 0, 1 - 2y)$.

3. *(10 points)* The volume of the region

$$E = \{(x, y, z)|\, x^2 + y^2 \leq 4,\, z \leq \sqrt{x^2 + y^2},\, z \geq 0\}$$

Solution:
We can parametrize the region in cylindrical coordinates as

$$E = \{(\rho, \theta, z) |\, \theta \in [0, 2\pi],\, \rho \in [0, 2],\, z \in [0, \rho]\}$$

which gives the integral

$$V(E) = \int_0^{2\pi} \int_0^2 \int_0^\rho \rho \, dz \, d\rho \, d\theta = \frac{16}{3}\pi$$

Problem 3

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field

$$\boldsymbol{F}(x, y) = (e^{x-y} + \frac{y^3}{3})\boldsymbol{i} + (xy^2 - e^{x-y})\boldsymbol{j}$$

and γ be the curve starting at $(0, 1)$, following the parabola $x = y^2 - 1$ until $(0, -1)$ and then following the unit circle counterclockwise to the point $(1, 0)$.

1. *(10 points)* Parametrize γ piecewise and set up the integral for its arclength.
   Solution:
   We can parametrize γ piecewise as

   $$\begin{aligned} \gamma_1(t) &= (t^2 - 1, -t) && t \in [-1, 1] \\ \gamma_2(t) &= (\cos(t), \sin(t)) && t \in [\tfrac{3}{2}\pi, 2\pi] \end{aligned}$$

   The corresponding tangent vectors are $\gamma_1'(t) = (2t, -1)$ and $\gamma_2'(t) = (-\sin(t), \cos(t))$, so that the arclength is given by

   $$l(\gamma) = l(\gamma_1) + l(\gamma_2) = \int_{-1}^{1} \sqrt{1 + 4t^2} \, dt + \int_{\frac{3}{2}\pi}^{2\pi} \sqrt{\sin^2(t) + \cos^2(t)} \, dt$$

2. *(10 points)* Prove whether $\boldsymbol{F}$ is conservative or not.
   Solution:
   Since the domain of $\boldsymbol{F}$ is $\mathbb{R}^2$ and

   $$\frac{\partial P}{\partial y} = -e^{x-y} + y^2 = \frac{\partial Q}{\partial x}$$

   we have that $\boldsymbol{F}$ is conservative.

3. *(10 points)* Evaluate $\int_\gamma \boldsymbol{F} \cdot d\gamma$.
Solution:
We can just find a potential and apply the fundamental theorem of calculus for line integrals. In this case $f(x,y) = e^{x-y} + \frac{xy^3}{3}$ is our potential, since $\nabla f(x,y) = \boldsymbol{F}$. Therefore

$$\int_\gamma \boldsymbol{F} \cdot d\gamma = f(1,0) - f(0,1) = e - e^{-1}$$

4. *(5 points)* Consider $\boldsymbol{G}(x,y) = \boldsymbol{F}(x,y) + (2y, 3x)$. Prove if $\boldsymbol{G}$ is conservative or not.
Solution:
In this case we have that

$$\frac{\partial P}{\partial y} = -e^{x-y} + y^2 + 2$$
$$\frac{\partial Q}{\partial x} = -e^{x-y} + y^2 + 3$$

so that they are not equal and therefore the vector field $\boldsymbol{G}$ is not conservative.

5. *(5 points)* Evaluate $\int_\gamma \boldsymbol{G} \cdot d\gamma$.
Solution:
Notice that $\int_\gamma \boldsymbol{G} \cdot d\gamma = \int_\gamma \boldsymbol{F} \cdot d\gamma + \int_\gamma (2y, 3x) \cdot d\gamma$. Since we know the first value, we can calculate the second using Green's theorem and then subtracting the integral over the quarter of circle from $(1,0)$ to $(0,1)$ that we need to close the curve. Let γ' be the closed curve (given by γ and the quarter of circle we just mentioned). We have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ so that by Green's theorem

$$\int_{\gamma'} (2y, 3x) \cdot d\gamma = \int_D dA = Area(D)$$

where D is the domain enclosed by γ'. This is given by

$$Area(D) = \int_{-1}^{1} \int_0^{1-y^2} dx\, dy + Area(Semicircle) = \frac{\pi}{2} + \frac{4}{3}$$

On the other hand, the line integral in the additional quarter of circle is given by

$$\int_0^{\frac{\pi}{2}} [] - 2\sin^2(t) + 3\cos^2(t)]\, dt = \frac{\pi}{4}$$

Piecing everything together we get

$$\int_\gamma \boldsymbol{G} \cdot d\gamma = \int_\gamma \boldsymbol{F} \cdot d\gamma + \int_\gamma (2y, 3x) \cdot d\gamma = e - e^{-1} + \frac{\pi}{2} + \frac{4}{3} - \frac{\pi}{4} = e - e^{-1} + \frac{\pi}{4} + \frac{4}{3}$$