Homework 3 Solutions

July 2020

1 Solutions of systems of linear equations (20 points)

For each of the following systems describe for which value of α and β the system admits one solution, infinite solutions or no solution. If the system has infinite solutions, state how many variables are free.

$$\begin{cases} x_1 - 2x_2 - x_3 = 0 \\ \alpha x_2 + 5x_3 = 3 \\ x_1 + 3x_2 + 6x_3 = 3\beta \end{cases} \tag{1}$$

$$\begin{cases} x_1 - 4x_2 + x_3 + \alpha x_4 = 0 \\ x_1 + x_2 - \beta x_3 - x_4 = 2 \\ x_1 - 2x_2 + x_4 = 5 \end{cases} \tag{2}$$

Solutions:

1. Consider the coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 0 & \alpha & 5 \\ 1 & 3 & 6 \end{bmatrix}$$

It has rank at least 2, since

$$\begin{vmatrix} 1 & -1 \\ 0 & 5 \end{vmatrix} = 5 \neq 0$$

Its determinant is given by

$$\begin{vmatrix} 1 & -2 & -1 \\ 0 & \alpha & 5 \\ 1 & 3 & 6 \end{vmatrix} = 8\alpha - 25$$

so if $\alpha \neq \frac{25}{8}$, then the system has a unique solution. If $\alpha = \frac{25}{8}$, then the system could have either one solution or infinite solutions, depending on the rank of the adjoint matrix. Consider

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 5 & 3 \\ 1 & 6 & \beta \end{vmatrix} = 5\beta - 21$$

so that if $\beta = \frac{21}{5}$, then the system has infinitely many solutions with one free variable, while if $\beta \neq \frac{21}{5}$, then the system has no solutions.

2. In this case the coefficient matrix is given by

$$A = \begin{bmatrix} 1 & -4 & 1 & \alpha \\ 1 & 1 & -\beta & -1 \\ 1 & -2 & 0 & 1 \end{bmatrix}$$

Since $\begin{vmatrix} 1 & -4 \\ 1 & 1 \end{vmatrix} = 5 \neq 0$ the rank of this matrix is at least 2. Moreover, notice that

$$\begin{vmatrix} 1 & -4 & 0 \\ 1 & 1 & 2 \\ 1 & -2 & 5 \end{vmatrix} = 21 \neq 0$$

so the rank of the augmented matrix is 3. In order to have solutions, we need the coefficient matrix to also have rank 3. Then we get

$$\begin{vmatrix} 1 & -4 & 1 \\ 1 & 1 & -\beta \\ 1 & -2 & 0 \end{vmatrix} = 2\beta - 3 \qquad \begin{vmatrix} 1 & -4 & \alpha \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{vmatrix} = 7 - 3\alpha$$

Therefore if $\alpha \neq \frac{7}{3}$ or $\beta \neq \frac{3}{2}$, the system has infinite solutions with one free variable. Otherwise ($\alpha = \frac{7}{3}$ **and** $\beta = \frac{3}{2}$) the system has no solutions.

2 Kernel and Image of a matrix A (20 points)

Consider the following matrix:

$$A = \begin{bmatrix} 3 & 1 & 4 & -1 \\ 2 & -1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

1. Find a basis for $\mathrm{Ker}(A)$. What is its dimension?
2. Find a basis for $\mathrm{Im}(A)$. What is its dimension?
3. Find a basis for $\mathrm{Ker}(A^T)$. What is its dimension?
4. Find a basis for $\mathrm{Im}(A^T)$. What is its dimension?

Solutions:

1. We need to solve the equation $A\mathbf{x} = \mathbf{0}$. The augmented matrix is

$$\begin{bmatrix} 3 & 1 & 4 & -1 & 0 \\ 2 & -1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

we can find its echelon form as

$$\begin{bmatrix} 3 & 1 & 4 & -1 & 0 \\ 0 & -5 & -5 & -4 & 0 \\ 0 & 0 & 0 & 24 & 0 \end{bmatrix}$$

and its reduced echelon form as

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

so that the general solution of $A\mathbf{x} = \mathbf{0}$ is given by:

$$\begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 \text{ free} \\ x_4 = 0 \end{cases}$$

so that we can write $\mathrm{Ker}(A) = \mathrm{span}(\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix})$. and $\dim(\mathrm{Ker}(A)) = 1$

2. We know that $\mathrm{Im}(A)$ is spanned by the pivot columns, so that we have

$$\mathrm{Im}(A) = \mathrm{span}(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix})$$

and $\dim(\mathrm{Im}(A)) = 3$.

3. We have that

$$A^T = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 0 \\ 4 & 1 & 1 \\ -1 & -2 & 1 \end{bmatrix}$$

Then we need to solve the system $A^T\mathbf{x} = \mathbf{0}$, having augmented matrix

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 4 & 1 & 1 & 0 \\ -1 & -2 & 1 & 0 \end{bmatrix}$$

whose echelon form is

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and whose reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that elements of $\mathrm{Ker}(A^T)$ have the unique solution

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

Therefore $\mathrm{Ker}(A^T) = \{\mathbf{0}\}$ and $\dim(\mathrm{Ker}(A^T)) = 0$.

4. We have that $\dim(\mathrm{Im}(A^T)) = 3$ and $\mathrm{Im}(A)$ is generated by the pivot columns of A^T:

$$\mathrm{Im}(A^T) = \mathrm{span}(\begin{bmatrix} 3 \\ 1 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix})$$

3 Orthogonal sets (20 points)

Let H be a vector subspace of $\mathbb{R}^n$. We can define the orthogonal complement of H as

$$H^\perp := \{\mathbf{v} \in \mathbb{R}^n \mid \langle \mathbf{v}, \mathbf{h} \rangle = 0 \text{ for every } \mathbf{h} \in H\}$$

1. Prove that $H^\perp$ is a vector subspace of $\mathbb{R}^n$.
2. Consider A from the previous exercise and find a basis for $\mathrm{Ker}(A)^\perp$ in $\mathbb{R}^4$ and $\mathrm{Im}(A)^\perp$ in $\mathbb{R}^3$.
3. Prove that all the elements of the basis of $\mathrm{Ker}(A)^\perp$ are in $\mathrm{Im}(A^T)$.
4. *Bonus question (5 points)*: Prove that for every matrix A,

$$\mathrm{Ker}(A)^\perp = \mathrm{Im}(A^T)$$
$$\mathrm{Im}(A)^\perp = \mathrm{Ker}(A^T)$$

(Hint: you first need to show that for every $\mathbf{x}, \mathbf{y}$, $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^T\mathbf{y} \rangle$.)

Solutions:

1. Clearly $\langle \mathbf{0}, \mathbf{h} \rangle = 0$ for every $\mathbf{h}$ in H, so $\mathbf{0} \in H^\perp$. Then if $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $H^\perp$, for every $\alpha, \beta \in \mathbb{R}$ we have

$$\langle \alpha\mathbf{v}_1 + \beta\mathbf{v}_2, \mathbf{h} \rangle = \alpha\langle \mathbf{v}_1, \mathbf{h} \rangle + \beta\langle \mathbf{v}_2, \mathbf{h} \rangle = 0$$

so any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is also in $H^\perp$.

2.

$$\mathrm{Ker}(A)^\perp = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mid -x_1 - x_2 + x_3 = 0 \}$$

Then the solution is given by

$$\begin{cases} x_1 = -x_2 + x_3 \\ x_2, x_3, x_4 \text{ free} \end{cases}$$

so that $\mathrm{Ker}(A)^\perp = \mathrm{span}(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix})$. As for $\mathrm{Im}(A)^\perp$, since $\mathrm{Im}(A) = \mathbb{R}^3$, we have that the only vector orthogonal to all of $\mathbb{R}^3$ is $\mathbf{0}$.

3. In order to prove this, we need to prove that every element of the basis of $\mathrm{Ker}(A)^\perp$ is linearly dependent on the element of the basis of $\mathrm{Im}(A^T)$. In particular, we can consider the corresponding 4×4 matrices and prove that their determinant is equal to 0.

$$\begin{vmatrix} 3 & 2 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 4 & 1 & 1 & 0 \\ -1 & -2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 4 & 1 & 1 \\ -1 & -2 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 1 \\ -1 & -2 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 3 & 2 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 4 & 1 & 1 & 1 \\ -1 & -2 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & 0 \\ 4 & 1 & 1 \\ -1 & -2 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & -2 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 3 & 2 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 4 & 1 & 1 & 0 \\ -1 & -2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -1 & 0 \\ 4 & 1 & 1 \end{vmatrix} = 0$$

therefore $\mathrm{Ker}(A)^\perp = \mathrm{Im}(A^T)$.

4. As suggested by the hint, consider

$$\langle A\mathbf{x}, \mathbf{y}\rangle = \sum_{j=1}^{m}\sum_{i=1}^{n} A_{j,i}x_i y_j = \sum_{i=1}^{n}\sum_{j=1}^{m} A^T_{i,j} y_j x_i = \langle \mathbf{x}, A^T\mathbf{y}\rangle$$

Then if $\mathbf{z} \in \mathrm{Im}(A^T)$, there exists $\mathbf{y}$ such that $\mathbf{z} = A^T\mathbf{y}$, so that for every $\mathbf{x} \in \mathrm{Ker}(A)$, we have

$$\langle \mathbf{x}, \mathbf{z}\rangle = \langle \mathbf{x}, A^T\mathbf{y}\rangle = \langle A\mathbf{x}, \mathbf{y}\rangle = 0$$

so that $\mathbf{z} \in \mathrm{Ker}(A)^\perp$ and $\mathrm{Im}(A^T) \subset \mathrm{Ker}(A)^\perp$. Moreover, if $\mathbf{z} \in \mathrm{Im}(A^T)^\perp$, then for every $\mathbf{y}$

$$0 = \langle \mathbf{z}, A^T\mathbf{y}\rangle = \langle A\mathbf{z}, \mathbf{y}\rangle$$

so that $\mathbf{z} \in \mathrm{Ker}(A)$ and $\mathrm{Im}(A^T)^\perp \subset \mathrm{Ker}(A)$.

Now, we need to prove the following properties:

$$U \subset V \implies V^\perp \subset U^\perp \tag{3}$$

$$U^{\perp\perp} = U \tag{4}$$

In order to prove the first, notice that if $\mathbf{x} \in V^\perp$, then $\langle \mathbf{x}, \mathbf{v}\rangle = 0$ for every $\mathbf{v}$ in V. But since $U \subset V$, then $\langle \mathbf{x}, \mathbf{u}\rangle = 0$ for every $\mathbf{u}$ in U and therefore $\mathbf{x} = U^\perp$.

As for the second, notice that $U \subset U^{\perp\perp}$, since all elements of U are perpendicular to $U^\perp$. But then since $U^{\perp\perp} \cap U^\perp = \{\mathbf{0}\}$, $U = U^{\perp\perp}$.

Finally, we have

$$\mathrm{Im}(A^T) \subset \mathrm{Ker}(A)^\perp \subset \mathrm{Im}(A^T)^{\perp\perp} = \mathrm{Im}(A^T)$$

which proves $\mathrm{Im}(A^T) = \mathrm{Ker}(A)^\perp$. By considering the orthogonal of the transpose, we get the other identity:

$$\mathrm{Im}(A) = \mathrm{Ker}(A^T)^\perp \implies \mathrm{Im}(A)^\perp = \mathrm{Ker}(A^T)$$

4 Change of basis (20 points)

Consider the 4-dimensional vector space $P_3(\mathbb{R})$. We can treat the set of monomials $\mathcal{B} = \{1, x, x^2, x^3\}$ as a canonical basis for $P_3(\mathbb{R})$ in the sense that every polynomials

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

can be written as a vector

$$p = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Consider the polynomials

$$p_1(x) = x + 2 \qquad p_2(x) = 2x^2 - 1 \qquad p_3(x) = x^3 - 2x \qquad p_4(x) = 3$$

1. Write p_1, p_2, p_3, p_4 as vectors in the basis $\mathcal{B}$ and prove that they also form a basis $\mathcal{B}'$ for $P_3(\mathbb{R})$.

2. Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ and the change of basis matrix from $\mathcal{B}'$ to $\mathcal{B}$.

3. Write down the polynomial having coordinates $\begin{bmatrix} 1 \\ -3 \\ 0 \\ 4 \end{bmatrix}$ in $\mathcal{B}$ and the one with the same coordinates in $\mathcal{B}'$.

Solutions:

1. The desired basis is

$$p_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad p_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \qquad p_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \qquad p_4 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In order to prove that it's a basis, consider the matrix B having each one of them as columns. Then

$$\begin{vmatrix} 2 & -1 & 0 & 3 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -3 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -6 \neq 0$$

therefore $\mathcal{B}'$ is a basis for $P_4(\mathbb{R})$.

2. The change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ is B^{-1}, that is,

$$B^{-1} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{6} & -\frac{2}{3} \end{bmatrix}$$

as for the change of basis matrix from $\mathcal{B}'$ to $\mathcal{B}$, that is $(B^{-1})^{-1} = B$.

3. In $\mathcal{B}$ the polynomial is $p(x) = 1 - 3x + 4x^3$, while in $\mathcal{B}'$, it's

$$\tilde{p}(x) = 2 + x - 3(-1 + 2x^2) + 4(3) = -6x^2 + x + 17$$

5 Gram-Schmidt algorithm (20 points)

Consider the following vectors

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

1. Prove that the vectors in $\mathcal{C}$ are linearly independent.

2. Apply the Gram-Schmidt algorithm on $\mathcal{C}$ to find an orthonormal subset $\mathcal{U} \subset \mathbb{R}^4$.

3. What is the approximate complexity of applying the Gram-Schmidt algorithm on m vectors in $\mathbb{R}^n$?

Solutions:

1. Consider the matrix C having each vector as column. We need to prove that C has maximum rank. By considering the first three rows we have

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 4 & 1 \end{vmatrix} = -6 \neq 0$$

therefore C has maximum rank and the three vectors are linearly independent.

2. We have

$$\mathbf{u}_1 = \frac{\mathbf{c}_1}{||\mathbf{c}_1||} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

then

$$\mathbf{u}_2' = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 0 \end{bmatrix} - \langle \begin{bmatrix} 1 \\ -1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix} \rangle \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 0 \end{bmatrix}$$

and

$$\mathbf{u}_2 = \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \\ 0 \end{bmatrix}$$

Finally

$$\mathbf{u}_3' = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \langle \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix} \rangle \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix} - \langle \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \\ 0 \end{bmatrix} \rangle \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} \\ \frac{7}{6} \\ \frac{1}{3} \\ -1 \end{bmatrix}$$

so that

$$\mathbf{u}_3 = \begin{bmatrix} -\frac{1}{3\sqrt{10}} \\ \frac{7}{3\sqrt{10}} \\ \frac{2}{3\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \end{bmatrix}$$

3.For every vector $\mathbf{c}_i$, for $i = 1, ..., m$ we need to compute i scalar products, i scalar multiplications and $i-1$ sums, then the normalization takes approximately $2n$ flops, so that

$$tot = \sum_{i=1}^{m} [(2n-1)i + ni + n(i-1) + 2n] = nm + (4n-1)\sum_{i=1}^{m} i =$$
$$nm - m + (4n-1)\frac{m(m+1)}{2}$$

which is of the order of $2nm^2$, or $2n^3$ when $m = n$.