Final Exam Solutions

COMS W3251, Summer Session 2

Due: Friday August 14th 2020, 4:59 PM EST

Instruction:

- You have 24 hours to complete this exam. You can use any time within the 24 hours, although the exam shouldn't take more than 3 hours.
- You may consult any non-living resources while you take the exam.
- Show full justification for every part. While you are allowed to use programs, we are grading each problem as if done by hand. Answers without justification will not be graded.
- You can either use the LaTeXtemplate given with the exam or write the answers on your own paper with each question/part clearly labeled.
- For questions and clarifications, send a private e-mail to the instructor (fp2428@columbia.edu) cc'ing the grader (ryan.chen@columbia.edu). We will try to answer as soon as possible.

Exercise 1 (20 points)

Consider the system of linear equations in matrix form

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -1 \\ 2 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ h \\ -3 \end{bmatrix}$$

1. Find the conditions on h under which the system admits a solution and find the solution $\mathbf{y}$.
2. For all h for which the system doesn't admit a solution, find the least square solution $\hat{\mathbf{x}}$ of the system (possibly depending on h).

Solutions:

1. Consider the augmented matrix

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ -1 & 1 & -1 & -4 \\ 2 & 0 & 3 & h \\ 0 & 2 & -1 & -3 \end{bmatrix}$$

which can be row-reduced to a matrix in echelon form of the type

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & h-3 \end{bmatrix}$$

so that the condition for a consistent system is $h = 3$. In this case, the solutions are obtained by finding the reduced echelon form of the matrix

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the solution is $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$.

2. In this case we want to solve the system $A^T A\mathbf{x} = A^T\mathbf{b}$. The coefficient matrix is

$$A^T A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \\ 2 & -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -1 \\ 2 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 9 \\ 0 & 6 & -1 \\ 9 & -1 & 15 \end{bmatrix}$$

while the constants are

$$A^T\mathbf{b} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \\ 2 & -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ -4 \\ h \\ -3 \end{bmatrix} = \begin{bmatrix} 3+2h \\ -11 \\ 5+3h \end{bmatrix}$$

Then we can row-reduce the system having augmented matrix

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 3+2h \\ 1 & 1 & 0 & 2 & -11 \\ 2 & -1 & 3 & -1 & 5+3h \end{bmatrix}$$

to its reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2+\frac{h}{3} \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & -1 & \end{bmatrix}$$

so that the general solution is $\hat{\mathbf{x}} = \begin{bmatrix} 2+\frac{h}{3} \\ -2 \\ -1 \end{bmatrix}$, which gives the same solution as before for $h = 3$.

Exercise 2 (20 points)

Consider a square matrix A having characteristic polynomial

$$p(\lambda) = (\lambda+1)^2(\lambda-2)(\lambda-3)^2$$

1. What is the dimension of A?
2. Find the trace and the determinant of A.
3. Do you have enough information to say whether A is invertible or not? In case you don't, what else do you need to know?
4. Do you have enough information to say whether A is diagonalizable or not? In case you don't, what else do you need to know?
5. Suppose that A is symmetric and classify the corresponding quadratic form $Q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle$.

Solutions:

1. The dimension of A is 5 since the polynomial has degree 5.

2. The eigenvalues are $\lambda_1 = \lambda_2 = 3, \lambda_3 = 2$ and $\lambda_4 = \lambda_5 = -1$. Then we have

$$\det(A) = \prod_{i=1}^{5} \lambda_i = (-1)^2 \cdot 2 \cdot 3^2 = 18$$

$$\mathrm{Tr}(A) = \sum_{i=1}^{5} \lambda_i = -1 - 1 + 2 + 3 + 3 = 6$$

I have accepted answers that would shift the sign of $\det(A)$ or $\mathrm{Tr}(A)$ if they considered $p(\lambda) = \det(A - \lambda I)$ for $\lambda = 0$ (I understand there was some ambiguity in this sense). However, I didn't accept a shift in sign of $\mathrm{Tr}(A)$ and $\det(A)$ if the eigenvalues had their signs shifted.

3. A is invertible, since $\det(A) \neq 0$.

4. No, we need to know the geometric multiplicities of $\lambda_1 = 3$ and $\lambda_4 = -1$.

5. Since there are eigenvalues with opposite signs, the quadratic form is indefinite.

Exercise 3 (20 points)

Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and let A be a matrix having $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as eigenvectors for the corresponding eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0.5$.

1. Find A.
2. Consider the discrete dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ and write the general solution for $\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$.
3. Does there exist $\mathbf{x}_0$ such that $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is a steady state of the system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ starting from $\mathbf{x}_0$? What about $\mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

Solutions:

1. Consider

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

one finds

$$P^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 & -1 \\ 3 & -1 & 1 \\ -2 & 2 & 2 \end{bmatrix}$$

and

$$A = P\Lambda P^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 3 & -1 & 1 \\ -2 & 2 & 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

2. Consider the starting point in the coordinates of the eigenvector basis

$$\mathbf{y}_0 = P^{-1}\mathbf{x}_0 = \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 \\ 3 & -1 & 1 \\ -2 & 2 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \frac{c_1+c_2-c_3}{4} \\ \frac{3c_1-c_2+c_3}{4} \frac{-c_1+c_2+c_3}{4} \end{bmatrix}$$

Then the general solution has the form

$$\mathbf{x}_k = \frac{c_1+c_2-c_3}{4}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{3c_1-c_2+c_3}{4}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{-c_1+c_2+c_3}{4}\left(\frac{1}{2}\right)^k\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

3. The long term equilibrium of this system is

$$\lim_{k\to\infty}\mathbf{x}_k = \frac{c_1+c_2-c_3}{4}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{3c_1-c_2+c_3}{4}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ \frac{c_1+c_2-c_3}{2} \\ \frac{c_1-c_2+c_3}{2} \end{bmatrix}$$

For $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ to be a steady state, we need to find if there are solutions to the system

$$\begin{cases} c_1 = 1 \\ \frac{c_1+c_2-c_3}{2} = 1 \\ \frac{c_1-c_2+c_3}{2} = 0 \end{cases}$$

In fact this systems admits infinite solutions of the type

$$\begin{cases} c_1 = 1 \\ c_2 = 1 + c_3 \end{cases}$$

so for any $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1+\alpha \\ \alpha \end{bmatrix}$ for $\alpha \in \mathbb{R}$, the long term equilibrium would be $\mathbf{y}$. On the other hand, for $\mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, there are no solutions to the system

$$\begin{cases} c_1 = 0 \\ \frac{c_1+c_2-c_3}{2} = 0 \\ \frac{c_1-c_2+c_3}{2} = 1 \end{cases}$$

so $\mathbf{z}$ is not a long-term equilibrium.

An easier way to solve this problem is to show that $A\mathbf{y} = \mathbf{y}$ and $A\mathbf{z} \neq \mathbf{z}$.

Exercise 4 (20 points)

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

1. Find a basis for $\mathrm{Ker}(A)$ and a basis for $\mathrm{Im}(A)$.
2. Find an SVD decomposition of $A = U\Sigma V^T$.

Solutions:

1. A basis for $\mathrm{Ker}(A)$ is given by the solutions of the system having augmented matrix

$$\begin{bmatrix} 2 & 1 & -1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix}$$

so that $\mathrm{Ker}(A) = \mathrm{span}(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix})$. On the other hand, $\mathrm{Im}(A)$ is generated by the column vectors corresponding to the pivot columns, so $\mathrm{Im}(A) = \mathrm{span}(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix})$.

2. In order to find an SVD decomposition, we need to diagonalize $A^T A$.

$$A^T A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -3 \\ 1 & 2 & 0 \\ -3 & 0 & 2 \end{bmatrix}$$

The corresponding eigenvalues solve the equation $p(\lambda) = 0$, for

$$p(\lambda) = \begin{bmatrix} 5-\lambda & 1 & -3 \\ 1 & 2-\lambda & 0 \\ -3 & 0 & 2-\lambda \end{bmatrix} = \lambda(2-\lambda)(\lambda-7)$$

so that $\lambda_1 = 7, \lambda_2 = 2$ and $\lambda_3 = 0$. We can find the corresponding unitary eigenvectors. For $\lambda_1 = 7$, $\mathbf{v}_1$ solves the system

$$\begin{bmatrix} -2 & 1 & -3 & 0 \\ 1 & -5 & 0 & 0 \\ -3 & 0 & -5 & 0 \end{bmatrix}$$

having reduced echelon form

$$\begin{bmatrix} 1 & 0 & \frac{5}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that $\mathbf{v}_1 = \begin{bmatrix} -\frac{5}{\sqrt{35}} \\ -\frac{1}{\sqrt{35}} \\ \frac{3}{\sqrt{35}} \end{bmatrix}$. For $\lambda_2 = 2$, the eigenvector solves the system with augmented matrix

$$\begin{bmatrix} 3 & 1 & -3 & 0 \\ 1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{bmatrix}$$

whose reduced echelon form is

$$\begin{bmatrix} 0 & 1 & -3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the unitary eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 0 \\ \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$. Finally, for $\lambda_3 = 0$ the eigenvector solves the system with augmented matrix

$$\begin{bmatrix} 5 & 1 & -3 & 0 \\ 1 & 2 & 0 & 0 \\ -3 & 0 & 2 & 0 \end{bmatrix}$$

whose reduced echelon form is

$$\begin{bmatrix} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the unitary eigenvector is $\mathbf{v}_3 = \begin{bmatrix} \frac{2}{\sqrt{14}} \\ -\frac{1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix}$. In this way we have

$$V = \begin{bmatrix} -\frac{5}{\sqrt{35}} & 0 & \frac{2}{\sqrt{14}} \\ -\frac{1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{14}} \\ \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{14}} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{7} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

Then we need to find U. In order to do so,

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{\sqrt{7}} \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{5}{\sqrt{35}} \\ -\frac{1}{\sqrt{35}} \\ \frac{3}{\sqrt{35}} \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix},$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

so that

$$U = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

and the SVD is

$$A = U\Sigma V^T = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{7} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} -\frac{5}{\sqrt{35}} & -\frac{1}{\sqrt{35}} & \frac{3}{\sqrt{35}} \\ 0 & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{14}} & -\frac{1}{\sqrt{14}} & \frac{3}{\sqrt{14}} \end{bmatrix}$$

Exercise 5 (20 points)

See the jupyter notebook attachment.