Inverse matrices and determinants

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The goal of this lecture is to understand when it's possible to solve a system of linear equations through the use of matrix inverses. In particular, consider the matrix form of a system of linear equations $A\mathbf{x} = \mathbf{b}$. If there exists a matrix B such that $BA = I$, then a solution of the system could be obtained by multiplying both sides of the equation by B:

$$\mathbf{x} = B\mathbf{b}$$

This guarantees the existence of a solution, which, as we've seen, it's not always the case for systems of linear equations. That means that for some matrices A there is no B such that $BA = I$. Such a B is called a *left inverse* of A. We will first deal with the square matrix case, in which left inverse and right inverse coincide and then generalize to rectangular matrices.

The inverse of a 2×2 matrix

We start by trying to find a left inverse B for a 2×2 matrix A. This process can be done by solving a system of linear equations $BA = I$, which can be written as

$$\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which transforms into the system

$$\begin{cases} b_{1,1}a_{1,1} + b_{1,2}a_{2,1} = 1 \\ b_{1,1}a_{1,2} + b_{1,2}a_{2,2} = 0 \\ b_{2,1}a_{1,1} + b_{2,2}a_{2,1} = 0 \\ b_{2,1}a_{1,2} + b_{2,2}a_{2,2} = 1 \end{cases}$$

Remember that $a_{i,j}$ are the coefficients and $b_{i,j}$ the unknowns. We can apply the Gaussian elimination algorithm to obtain a solution for this system of 4 equations in 4 unknowns. The augmented matrix is, in general

$$A' = \begin{bmatrix} a_{1,1} & a_{2,1} & 0 & 0 & 1 \\ a_{1,2} & a_{2,2} & 0 & 0 & 0 \\ 0 & 0 & a_{1,1} & a_{2,1} & 0 \\ 0 & 0 & a_{1,2} & a_{2,2} & 1 \end{bmatrix}$$

Now we have to distinguish two cases:

Case 1: $a_{1,1}a_{2,2} - a_{2,1}a_{1,2} \neq 0$

In this case, notice that one between $a_{1,1}$ and $a_{1,2}$ has to be different from 0. If both are different from 0, A' can be reduced to echelon form in the following way:

$$\begin{bmatrix} a_{1,1} & a_{2,1} & 0 & 0 & 1 \\ 0 & a_{1,1}a_{2,2} - a_{2,1}a_{1,2} & 0 & 0 & -a_{1,2} \\ 0 & 0 & a_{1,1} & a_{2,1} & 0 \\ 0 & 0 & 0 & a_{1,1}a_{2,2} - a_{2,1}a_{1,2} & a_{1,1} \end{bmatrix}$$

In which case we can apply the last step of the algorithm to obtain a reduced echelon form for the matrix and get:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{a_{2,2}}{a_{1,1}a_{2,2}-a_{2,1}a_{1,2}} \\ 0 & 1 & 0 & 0 & -\frac{a_{1,2}}{a_{1,1}a_{2,2}-a_{2,1}a_{1,2}} \\ 0 & 0 & 1 & 0 & -\frac{a_{2,1}}{a_{1,1}a_{2,2}-a_{2,1}a_{1,2}} \\ 0 & 0 & 0 & 1 & \frac{a_{1,1}}{a_{1,1}a_{2,2}-a_{2,1}a_{1,2}} \end{bmatrix}$$

so that there exists a unique solution to the system of linear equations and we have

$$B = \frac{1}{a_{1,1}a_{2,2} - a_{2,1}a_{1,2}} \begin{bmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{bmatrix} \tag{1}$$

If $a_{1,2} = 0$, but $a_{1,1} \neq 0$, then A' is already in echelon form while its reduced echelon form becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{a_{1,1}} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{a_{2,1}}{a_{1,1}a_{2,2}} \\ 0 & 0 & 0 & 1 & \frac{1}{a_{2,2}} \end{bmatrix}$$

so that the solution still has the form of Equation (1). Finally, if $a_{1,1} = 0$, but $a_{1,2} \neq 0$, we can proceed to switch rows 1 and 2 and rows 3 and 4 to get an echelon form

$$A' = \begin{bmatrix} a_{1,2} & a_{2,2} & 0 & 0 & 0 \\ 0 & a_{2,1} & 0 & 0 & 1 \\ 0 & 0 & a_{1,2} & a_{2,2} & 1 \\ 0 & 0 & 0 & a_{2,1} & 0 \end{bmatrix}$$

whose reduced echelon form is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{a_{2,2}}{a_{1,2}a_{2,1}} \\ 0 & 1 & 0 & 0 & \frac{1}{a_{2,1}} \\ 0 & 0 & 1 & 0 & \frac{1}{a_{1,2}} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

whose general expression still follows Equation (1). The matrix B in Equation (1) is called the inverse of A or A^{-1} and has the property that $AA^{-1} = A^{-1}A = I$.

Case 2: $a_{1,1}a_{2,2} - a_{2,1}a_{1,2} = 0$

In this case, if $a_{1,1} \neq 0$, then we can reduce to echelon form in the same way as before to obtain

$$\begin{bmatrix} a_{1,1} & a_{2,1} & 0 & 0 & 1 \\ 0 & a_{1,1}a_{2,2} - a_{2,1}a_{1,2} & 0 & 0 & -a_{1,2} \\ 0 & 0 & a_{1,1} & a_{2,1} & 0 \\ 0 & 0 & 0 & a_{1,1}a_{2,2} - a_{2,1}a_{1,2} & a_{1,1} \end{bmatrix}$$

However, the fourth row will be of the type $[\mathbf{0}|c]$ for $c \neq 0$, which means that there is no solution to the equation.

If $a_{1,1} = 0$, then either $a_{1,2}$ or $a_{2,1} = 0$. In the first case the first and second lines of A' give a contradiction. In the second case, the first line has the form $[\mathbf{0}|b]$ for $b \neq 0$, so there are no solutions to the system.

In any case we have proved that if $a_{1,1}a_{2,2} - a_{2,1}a_{1,2} = 0$ there is no B such that $BA = I$. The converse is also true, that is

Theorem 1. *A 2×2 matrix A admits an inverse if and only if $a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \neq 0$*

The quantity $a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$ is called the *determinant* of A and its definition will be generalized to higher dimensional square matrices.

Before we proceed, notice that we have proved the following:

Theorem 2. *Let $A\mathbf{x} = \mathbf{b}$ be a 2×2 system. Then the system has a unique solution if and only if A is invertible, and the solution is $x = A^{-1}b$*

Consider the following system of linear equations:

$$\begin{cases} 3x_1 - x_2 = 4 \\ x_1 + x_2 = 0 \end{cases}$$

having matrix form $A\mathbf{x} = \mathbf{b}$

$$\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Solving it by substitution tells us that this system has a unique solution $x_1 = 1, x_2 = -1$. However, we will use this simplified example to show a more general approach to solving this system of equations. The determinant of the coefficient matrix is given by

$$a_{1,1}a_{2,2} - a_{2,1}a_{1,2} = 3 \cdot 1 - (-1) \cdot 1 = 4 \neq 0$$

Therefore we can find the inverse as

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

and the solutions to the system in the following way:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

which gives exactly the result that we wanted.

Now we want to generalize the results of Theorem 2 to a general $n \times n$ matrix. In order to do so, we need to introduce the definition of determinants for a general $n \times n$ matrix.

Determinants

The quantity $a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$ in 2×2 matrices is called *determinant*. For a general $n \times n$ matrix, it's possible to define the determinant recursively. In order to do so, we need the following definition:

Definition 1. Let A be a $n \times n$ matrix with generic element $A_{i,j}$. If $n = 2$, then the *determinant* of A is defined as

$$\det(A) = A_{1,1}A_{2,2} - A_{1,2}A_{2,1}$$

if $n > 2$, let $\begin{bmatrix} A_{1,1} & A_{1,2} & ... & A_{1,n} \end{bmatrix}$ be the first row of A. Then

$$\det(A) = \sum_{j=1}^{n} A_{1,j} C_{1,j}$$

where $C_{i,j}$ is called the *cofactor* of the element $A_{i,j}$ obtained by

$$C_{i,j} = (-1)^{i+j} \det(A^{i,j})$$

where $A^{i,j}$ is the square submatrix of A obtained by eliminating its i-th row and j-th column.

In fact, the choice of the row or the column is not important, since we have the following:

Theorem 3. *Let A be a $n \times n$ matrix. Then for any row $\begin{bmatrix} A_{i,1} & A_{i,2} & ... & A_{i,n} \end{bmatrix}$ we have*

$$\det(A) = \sum_{j=1}^{n} A_{i,j} C_{i,j}$$

and for any column $\begin{bmatrix} A_{1,j} \\ A_{2,j} \\ ... \\ A_{n,j} \end{bmatrix}$ *we have*

$$\det(A) = \sum_{i=1}^{n} A_{i,j} C_{i,j}$$

This definition of determinant is called the *co-factor expansion* of the determinant and it requires to calculate inductively the determinants of n submatrix of size $n-1 \times n-1$.

Example: let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 1 \\ -1 & 0 & -2 \end{bmatrix}$. Then we can calculate the determinant using the first row:

$$\det(A) = 3 \cdot \begin{vmatrix} 0 & 1 \\ 0 & -2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} = 3$$

or equivalently, along the second column:

$$\det(A) = -1 \cdot \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} = 3$$

or the third row:

$$\det(A) = -1 \cdot \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} = -1 + 4 = 3$$

Notice that if A is upper (lower) triangular, we can develop the determinant on the first column (row) recursively, to get the following:

Theorem 4. *If A is a triangular $n \times n$ matrix,* $\det(A)$ *is the product of its diagonal elements*

For example, consider $A = \begin{bmatrix} 1 & 2 & 100 & -\pi \\ 0 & -1 & 299 & 9 \\ 0 & 0 & 2 & 0.1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ then by expanding repeatedly along the first column of each submatrix we get

$$\det(A) = 1 \cdot \begin{vmatrix} -1 & 299 & 9 \\ 0 & 2 & 0.1 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot (-1) \cdot \begin{vmatrix} 2 & 0.1 \\ 0 & 3 \end{vmatrix} = 1 \cdot (-1) \cdot 2 \cdot 3 = -6$$

This shows that it's important to choose a cofactor expansion that minimizes the number of calculations.

The following are properties of the determinant that we will use extensively. In what follows, A and B are $n \times n$ square matrices.

1. $\det(AB) = \det(A)\det(B)$.
2. $\det(A) = \det(A^T)$
3. If B is obtained by interchanging two rows (or columns) of A, then

$$\det(B) = -\det(A)$$

4. If B is obtained by multiplying a row (or column) of A by k, then

$$\det(B) = k\det(A)$$

In particular, $\det(kA) = k^n \det(A)$.

5. If $\mathbf{a}_1, ..., \mathbf{a}_n, \mathbf{b}$ are $n \times 1$ column vectors, then

$$\begin{vmatrix}\mathbf{a}_1 & ... & \mathbf{a}_i + \mathbf{b} & ... & \mathbf{a}_n\end{vmatrix} = \begin{vmatrix}\mathbf{a}_1 & ... & \mathbf{a}_i & ... & \mathbf{a}_n\end{vmatrix} + \begin{vmatrix}\mathbf{a}_1 & ... & \mathbf{b} & ... & \mathbf{a}_n\end{vmatrix}$$

6. If two rows (or column) of A are multiple of each other, then $\det(A) = 0$. The same is true if A has one row (or column) of all zeros.

The following is a property implied by the previous ones that we will use extensively

Theorem 5. *Let A, B be two square $n \times n$ matrices. If B is obtained by substituting a row (or column) of A with the sum of itself and a multiple of another row, then* $\det(B) = \det(A)$

Proof. Let $\mathbf{a}_i$ to be the i-th column of A. Then for some $j \neq i$, $k \neq 0$ we have that

$$B = \begin{bmatrix}\mathbf{a}_1 & ... & \mathbf{a}_i + k\mathbf{a}_j & ... & \mathbf{a}_n\end{bmatrix}$$

then by property 5 we have

$$\det(B) = \det(A) + \begin{vmatrix}\mathbf{a}_1 & ... & k\mathbf{a}_j & ... & \mathbf{a}_n\end{vmatrix}$$

Since $k\mathbf{a}_j$ is a multiple of another column of the matrix $\begin{bmatrix}\mathbf{a}_1 & ... & k\mathbf{a}_j & ... & \mathbf{a}_n\end{bmatrix}$, by property 6 the determinant of such matrix is 0 and therefore $\det(B) = \det(A)$. By using property 2 we obtain the same result on rows. $\square$

Example: compute the determinant of $A = \begin{bmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{bmatrix}$.

The strategy to facilitate calculations is to turn to 0 as many elements as possible by using the properties. First notice that we can divide the first column by 2 to get

$$\det(A) = 2\begin{vmatrix} 1 & 5 & 4 & 1 \\ 2 & 7 & 6 & 2 \\ 3 & -2 & -4 & 0 \\ -3 & 7 & 7 & 0 \end{vmatrix}$$

then we can subtract the last column from the first to eliminate the first two elements

$$\det(A) = 2\begin{vmatrix} 0 & 5 & 4 & 1 \\ 0 & 7 & 6 & 2 \\ 3 & -2 & -4 & 0 \\ -3 & 7 & 7 & 0 \end{vmatrix} = 6\begin{vmatrix} 0 & 5 & 4 & 1 \\ 0 & 7 & 6 & 2 \\ 1 & -2 & -4 & 0 \\ -1 & 7 & 7 & 0 \end{vmatrix}$$

then we can subtract the third column from the second:

$$\det(A) = 6\begin{vmatrix} 0 & 1 & 4 & 1 \\ 0 & 1 & 6 & 2 \\ 1 & 2 & -4 & 0 \\ -1 & 0 & 7 & 0 \end{vmatrix}$$

Passing to rows, we can sum the fourth row to the third and subtract the second row from the first:

$$\det(A) = 6\begin{vmatrix} 0 & 0 & -2 & -1 \\ 0 & 1 & 6 & 2 \\ 0 & 2 & 3 & 0 \\ -1 & 0 & 7 & 0 \end{vmatrix}$$

Now we are ready for a co-factor expansion along the first column (the one with the highest number of zeros):

$$\det(A) = 6\begin{vmatrix} 0 & -2 & -1 \\ 1 & 6 & 2 \\ 2 & 3 & 0 \end{vmatrix} = 6(-8-3+12) = 6$$

In the final computation we have used **Sarrus' rule** which is a useful computational trick for the determinant of 3×3 matrices. It says that for

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

the determinant can be calculated as

$$\begin{aligned} \det(A) =& a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} \\ &- a_{3,1}a_{2,2}a_{1,3} - a_{2,1}a_{1,2}a_{3,3} - a_{1,1}a_{3,2}a_{2,3} \end{aligned}$$

which can be easily remembered through the use of diagonals.

The determinant has mostly a combinatorial definition. However, a geometric interpretation can be obtained by considering A as a linear transformation of $\mathbb{R}^n$ and its determinant as the change of area of a cube. For example, consider the dilatation

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

The unit cube $[0,1] \times [0,1]$ is mapped to the rectangle $[0,a] \times [0,b]$ which has area equal to ab. This is also the determinant of A and it is generally true that if A is a linear transformation we have for every parallelogram P in $\mathbb{R}^2$

$$\mathrm{Area}(A(P)) = |\det(A)|\mathrm{Area}(P).$$

Finally, we can use determinants to solve square systems of linear equations of the type $A\mathbf{x} = \mathbf{b}$. We start with a definition:

Definition 2. Let A be a $n \times n$ matrix and $\mathbf{b}$ a $n \times 1$ column vector. We call $A_j(\mathbf{b})$ the matrix obtained substituting the j-th column of A with $\mathbf{b}$

For example, let $A = I$ and $\mathbf{x}$ be the column vector of x_i. Then $I_j(\mathbf{x})$ is a matrix having the element $\mathbf{e}_i$ of the canonical basis in every column position, except for the j-th position, where it has $\mathbf{x}$.

Theorem 6 (Cramer's rule)**.** *Consider a square system $A\mathbf{x} = \mathbf{b}$. If $\det(A) \neq 0$, then*

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)}$$

Proof. Consider $I_i(\mathbf{x})$ We have that

$$AI_i(\mathbf{x}) = \begin{bmatrix} A\mathbf{e}_1 & ... & A\mathbf{x} & ... & A\mathbf{e}_n \end{bmatrix} = A_i(\mathbf{b})$$

Then by taking the determinants and noticing that $\det(I_i(\mathbf{x})) = x_i$ we obtain

$$\det(AI_i(\mathbf{x})) = \det(A_i(\mathbf{b}))$$

and since the determinant of the product is the product of the determinants, we have

$$\det(A)x_i = \det(A_i(\mathbf{b}))$$

which, upon the condition $\det(A) \neq 0$, gives the result. $\square$

Example: consider

$$\begin{cases} 3x_1 + 2x_2 + 4x_3 = 1 \\ -x_1 + x_2 = 3 \\ x_1 + x_2 - 2x_3 = 0 \end{cases}$$

Then $A = \begin{bmatrix} 3 & 2 & 4 \\ -1 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$ and the substitute matrices are

$$A_1(\mathbf{b}) = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix}, \quad A_2(\mathbf{b}) = \begin{bmatrix} 3 & 1 & 4 \\ -1 & 3 & 0 \\ 1 & 0 & -2 \end{bmatrix} \quad A_3(\mathbf{b}) = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

so that we have

$$\begin{aligned} x_1 &= \frac{\det(A_1(\mathbf{b}))}{\det(A)} = \frac{22}{-18} = -\frac{11}{9} \\ x_2 &= \frac{\det(A_2(\mathbf{b}))}{\det(A)} = \frac{-32}{-18} = \frac{16}{9} \\ x_3 &= \frac{\det(A_3(\mathbf{b}))}{\det(A)} = \frac{-5}{-18} = \frac{5}{18} \end{aligned}$$

Inverse square matrices

Now that we have defined the determinant for a general $n \times n$ matrix, it makes sense to define the inverse matrix in its full generality:

Definition 3. Let A be a $n \times n$ matrix. We say that A is *invertible* if there exists a matrix A^{-1} called *inverse matrix* such that

$$AA^{-1} = A^{-1}A = I$$

where I is the $n \times n$ identity matrix.

Notice that the inverse matrix is both a *left* and *right* inverse, meaning that it can be multiplied both on the left and on the right to obtain the identity matrix. The **properties** of the inverse matrix are the following:

1. A^{-1} is unique.
2. $(AB)^{-1} = B^{-1}A^{-1}$.
3. $(A^T)^{-1} = (A^{-1})^T$.
4. $\det(A^{-1}) = \det(A)^{-1}$.

The condition on invertibility is the same as for the 2×2 matrices.

Theorem 7. *Let A be a square $n \times n$ matrix. Then A is invertible if and only if* $\det(A) \neq 0$.

However, it would be nice to have an explicit formula, or an algorithm to calculate the inverse of a matrix. We provide both solutions, starting from a formula:

Theorem 8. *Let A be a $n \times n$ matrix with* $\det(A) \neq 0$. *Then*

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{1,1} & C_{2,1} & ... & C_{n,1} \\ C_{1,2} & C_{2,2} & ... & C_{n,2} \\ ... & ... & ... & ... \\ C_{1,n} & C_{2,n} & ... & C_{n,n} \end{bmatrix}$$

where $C_{i,j}$ is the cofactor of $A_{i,j}$ in A.

Notice that in the matrix, the cofactor $C_{i,j}$ is in the position (j,i). In order to prove it, we use Cramer's rule:

Proof. The matrix A^{-1} solves a system of linear equation of the type $AA^{-1} = I$. Now let $\mathbf{a^{-1}}_j$ be the j-th column of A^{-1}. We have $A\mathbf{a^{-1}}_j = \mathbf{e}_j$. But then by Cramer's rule,

$$\mathbf{a^{-1}}_{i,j} = \frac{\det(A_i(\mathbf{e}_j))}{\det(A)}$$

Now, we can calculate $\det(A_i(\mathbf{e}_j))$ along the i-th column and obtain

$$\det(A_i(\mathbf{e}_j)) = (-1)^{i+j} \det A^{j,i} = C_{j,i}$$

which proves our result. $\square$

For example we can try to calculate the inverse of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$. Then $\det(A) = -11$ Then an easy calculation gives $C_{1,1} = -5, C_{1,2} = -3, C_{1,3} = -6, C_{2,1} = -2, C_{2,2} = 1, C_{2,3} = 2, C_{3,1} = -4, C_{3,2} = 2, C_{3,3} = -7$. Therefore the inverse is

$$A^{-1} = -\frac{1}{11} \begin{bmatrix} -5 & -2 & -4 \\ -3 & 1 & 2 \\ -6 & 2 & -7 \end{bmatrix}$$

and we can verify that

$$-\frac{1}{11} \begin{bmatrix} -5 & -2 & -4 \\ -3 & 1 & 2 \\ -6 & 2 & -7 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Another way to find the inverse matrix is by using the **Inverse Matrix algorithm**. This algorithm works as follows: consider an augmented matrix $A' = [A|I]$ where I is the $n \times n$ identity matrix. Then find the reduced echelon form of A', A'_{ref}. If $\det(A) \neq 0$, A'_{ref} has the form $[I|A^{-1}]$ and so it will be sufficient to use the last n columns of A'_{ref} to obtain the inverse.

In the previous example we have

$$A' = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & -2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

we start by finding and echelon form for A'. For the first row:

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & -2 & -3 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

As for the second and third row:

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & -2 & -3 & 1 & 0 \\ 0 & 0 & 11 & 6 & -2 & 7 \end{bmatrix}$$

Now we can find the reduced echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{5}{11} & \frac{2}{11} & \frac{4}{11} \\ 0 & 1 & 0 & \frac{3}{11} & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 0 & 1 & \frac{6}{11} & -\frac{2}{11} & \frac{7}{11} \end{bmatrix}$$

so that by considering the last three columns we find the same expression

$$A^{-1} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} & \frac{4}{11} \\ \frac{3}{11} & -\frac{1}{11} & -\frac{2}{11} \\ \frac{6}{11} & -\frac{2}{11} & \frac{7}{11} \end{bmatrix}$$

In the case of square systems, we can summarize all these results in the following way:

Theorem 9. *Consider the square system $A\mathbf{x} = \mathbf{b}$. The system has a unique solution if and only if* $\det(A) \neq 0$ *and the solution is given by*

$$\mathbf{x} = A^{-1}\mathbf{b}$$

In the next lecture we will generalize this result by understanding what happens when $\det(A) = 0$ and what happens in the case of non-square systems.