Homework 2

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Problem 1

Consider the following limits and prove if they exist or not. If they do, find the value of the limit.

1. $$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$$

   Solution:
   There are multiple ways to do this. One is noticing that $\frac{x^2}{x^2+y^2} \leq 1$, so that

   $$|\frac{x^2y}{x^2+y^2}| \leq |y| \to 0$$

   as (x, y) tends to $(0, 0)$.

2. $$\lim_{(x,y)\to\infty} \frac{x^2y}{x^2+y^2}$$

   Solution:
   Consider the line $y = 0$, which is a direction towards which we can go to infinity. In that case we have $f(x, 0) = 0$ so that the limit along that line is equal to 0. On the other hand, if you consider the line $y = x$ we have that $f(x, x) = \frac{x}{2}$ which tends to infinity as x goes to infinity. Therefore the limit does not exist.

3. $$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{3x^4+y^2}$$

   Solution:
   On the line $y = 0$ the function is constant and equal to 0, while on the parabola $y = x^2$ the function becomes $f(x, x^2) = \frac{1}{2}$, so that the limit does not exist.

Problem 2

Let $f(x, y) = \sqrt{16 - x^2 - y^2}$.

1. Find the domain of the function and sketch it on $\mathbb{R}^2$.
   Solution:

   $$D = \{(x, y)|16 - x^2 - y^2 \geq 0\}$$

   On $\mathbb{R}^2$ this is the interior (with boundary included) of the circle of radius 4.

2. Sketch the graph of the function and describe the relationship between the graph and the level curves. What is the range of the function?
   Solution:
   The level curves are given by the family
   $$\{(x, y) : x^2 + y^2 = k^2\}$$
   for k between 0 and 4. These are circles of radius k and the function has value $\sqrt{16-k^2}$ on such level curves. The graph of the function is given by $z = \sqrt{16-x^2-y^2}$ which can be identified as the upper emisphere of the sphere given by
   $$\text{Graph}(f) = \{(x, y, z)|x^2 + y^2 + z^2 = 16, \quad z > 0\}$$
   The level curve $x^2 + y^2 = 16 - k^2$ is the circle obtained by intersecting the upper emisphere with the plane $z = k$. The range of the function is given by $[0, 16]$.

Problem 3

Consider the function defined by

$$f(x, y) = \begin{cases} \frac{x^2y^4}{x^\alpha+y^{2\alpha}} & \text{if} \qquad (x, y) \neq (0, 0) \\ 0 & \qquad\qquad \text{if} \quad (x, y) = (0, 0) \end{cases}$$

where α is an even positive integer.

1. Consider $\alpha = 2$. Study continuity, derivability and differentiability of the function.
   Solution:
   $$f(x, y) = \begin{cases} \frac{x^2y^4}{x^2+y^4} & \text{if} \qquad (x, y) \neq (0, 0) \\ 0 & \qquad\qquad \text{if} \quad (x, y) = (0, 0) \end{cases}$$
   We can see that the function is differentiable everywhere except $(0, 0)$ because it's composition of continuous functions (polynomials and $\frac{1}{x}$). Therefore it is also derivable and continuous in $\mathbb{R}^2 - (0, 0)$. Let's prove it's differentiable in $(0, 0)$. We have that by an easy calculation using the definition $\nabla f(0, 0) = (0, 0)$, so that by using the definition of differentiability we need to prove
   $$\lim{(x, y) \to (0, 0)} \frac{x^2y^4}{(x^2 + y^4)\sqrt{x^2 + y^2}} = 0$$
   Notice that $\frac{y^4}{x^2+y^4} \leq 1$ and that $\frac{|x|}{x^2+y^2} \leq 1$, so that $\frac{x^2y^4}{(x^2+y^4)\sqrt{x^2+y^2}} \leq |x|$ which tends to 0 as (x, y) goes to $(0, 0)$ and the differentiability is proven. Since differentiable implies continuous and derivable, we have that the function is continuous, derivable and differentiable everywhere.

2. Consider $\alpha = 4$. Study continuity, derivability and differentiability of the function.
Solution:
$$f(x,y) = \begin{cases} \frac{x^2y^4}{x^4+y^8} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$
We can see that the function is differentiable everywhere except $(0,0)$ because it's composition of continuous functions (polynomials and $\frac{1}{x}$). Therefore it is also derivable and continuous in $\mathbb{R}^2 - (0,0)$. By definition of partial derivatives, it is definitely derivable in $(0,0)$, since we have
$$\lim_{h\to 0} \frac{f(h,0) - f(0,0}{h} = \lim_{h\to 0} 0 = 0$$
and
$$\lim_{h\to 0} \frac{f(0,h) - f(0,0}{h} = \lim_{h\to 0} 0 = 0$$
Therefore $\nabla f(0,0) = (0,0)$. Now, let us prove that the function is not continuous: on $x = 0$ it is constant and equal to 0, while on $x = y^2$ the function is constant and equal to $f(y^2, y) = 12$. For this reason, the function is not continuous on $(0,0)$ and therefore is not differentiable.

Problem 4

Consider the function $f(x,y) = xe^{xy} + y$

1. Find the plane tangent to the graph of the function in $P = (2,0)$ and calculate the linear approximation of the function in $(1.9, 0.1)$.
Solution:
The gradient of the function is $\nabla f(x,y) = ((1+y)e^{xy}, x^2e^{xy} + 1)$ so that its value in P is given by $\nabla f(2,0) = (1,5)$. The equation of the tangent plane at that point is therefore:
$$z - 2 = 1 \cdot (x-2) + 5 \cdot (y-0)$$
that is $z = x + 5y$. The linear approximation at the point $(1.9, 0.1)$ is given by $1.9 + 0.5 = 2.4$, which is pretty accurate because the function is differentiable at that point.

2. Find the directional derivative of the function at point P in the direction of $\vec{w} = (1,-2)$.
Solution:
Since the function is differentiable in P, we have
$$\partial_w f(2,0) = \langle (1,2), \frac{\vec{w}}{||\vec{w}||} \rangle = \langle (1,2), (\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle = -\frac{3}{\sqrt{5}}$$

3. Find the direction of maximal change for the function at point P.
Solution:

The direction of maximal change for a function at a given point is given by the gradient calculated in that point, that is $(1, 2)$.

4. Is there any direction $\vec{v}$ such that the directional derivative at point P in direction $\vec{v}$ is equal to 7? Explain your answer.
   Solution:
   No because there is no direction where the absolute value of the derivative is greater than the norm of the gradient and the norm of the gradient is $\sqrt{5} < 7$.

Problem 5

Consider the function $f(x, y) = (x^2 + y^2 - 1)(3x^2 + y^2 - 4)$

1. Find the critical points of f and classify them.
   Solution:
   First find the critical points through the equation $\nabla f(x, y) = 0$. This yields:

$$\nabla f(x, y) = (2x(6x^2 + 4y^2 - 7), 2y(4x^2 + 2y^2 - 5)) = (0, 0)$$

   We consider three possible cases:
   ***Case* 1**: $x = 0$
   If $x = 0$, the second equation gives the points $(0, 0), (0, \sqrt{\frac{5}{2}})$ and $(0, -\sqrt{\frac{5}{2}})$.
   ***Case* 2**: $y = 0$
   If $y = 0$, the second equation gives us the points $(0, 0), (\sqrt{\frac{7}{6}}, 0)$ and $(-\sqrt{\frac{7}{6}}, 0)$.
   ***Case* 1**: $x \neq 0, y \neq 0$:
   The two equations are now $6x^2 + 4y^2 - 7 = 0$ and $4x^2 + 2y^2 - 5 = 0$. If we consider the first equation minus two times the second we get $-2x^2 + 3 = 0$, so that $x^2 = \frac{3}{2}$. But then by substituting that condition in the second equation we get $2y^2 + 1 = 0$, which has no real solution.
   Now, let's classify the critical points using the hessian. We have that

$$H_f(x, y) = \begin{bmatrix} 36x^2 + 8y^2 - 14 & 16xy \\ 16xy & 8x^2 + 12y^2 - 10 \end{bmatrix}$$

   So that $H_f(0, 0) = \begin{bmatrix} -14 & 0 \\ 0 & -10 \end{bmatrix}$, which has positive determinant and negative first term: $(0, 0)$ is a point of local maximum.
   $H_f(\sqrt{\frac{7}{6}}, 0) = H_f(-\sqrt{\frac{7}{6}}, 0) = \begin{bmatrix} 28 & 0 \\ 0 & -\frac{2}{3} \end{bmatrix}$, which has negative determinant: $(\sqrt{\frac{7}{6}}, 0)$ and $(-\sqrt{\frac{7}{6}}, 0)$ are saddle points.

Finally, $H_f(0, \sqrt{\frac{5}{2}}) = H_f(0, -\sqrt{\frac{5}{2}}) = \begin{bmatrix} 6 & 0 \\ 0 & 20 \end{bmatrix}$, which has positive determinant and positive first term: $(0, \sqrt{\frac{5}{2}})$ and $(0, -\sqrt{\frac{5}{2}})$ are points of local minimum.

2. *Bonus question*: restrict your function to the domain $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 16\}$. Find the absolute maximum and absolute minimum of the function in this domain.
Solution:
Notice that all our critical points found in the previous point lie inside the domain, so we are left to analyze the behaviour on the boundary. We know that an absolute maximum and an absolute minimum have to exist because the domain is closed and bounded, represented by the interior of a circle of radius 4 with boundary included. Now, on the boundary we have that the function has the value $g(x) = 15(2x^2 + 12)$, obtained by substituting $x^2 + y^2 = 16$ in the expression for f for $x \in [-4, 4]$ (that is where x varies in the boundary). Now $g'(x) = 60x$ which has a local minimum in $x = 0$. The values on the boundaries are $g(-4) = g(4) = 660$, while $g(0) = 180$. On the other hand in the interior we have $f(0, 0) = 4, f(0, -\sqrt{\frac{5}{2}}) = f(0, \sqrt{\frac{5}{2}}) = -\frac{9}{4}$. Therefore we have that there are two global minima in the domain given by $(0, \sqrt{\frac{5}{2}})$ and $(0, -\sqrt{\frac{5}{2}})$ and two global maxima in the domain given by $(-4, 0)$ and $(4, 0)$.