Homework 4 solutions

July 2020

1 QR decompositions (20 points)

Consider the following matrix:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}$$

1. Justify why this matrix admits a QR decomposition and find it.
2. Does this matrix have a left pseudo-inverse or a right pseudo-inverse? Justify your answer.
3. Find the first column of the pseudo-inverse.

Solutions:

1. The matrix has maximum rank since

$$\begin{vmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \end{vmatrix} = -12 \neq 0$$

Therefore it admits a QR decomposition. Applying the Gram-Schmidt method on the columns gives

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\mathbf{u}_2' = \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix} - \left\langle \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\rangle \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = \begin{bmatrix} -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix}$$

$$\mathbf{u}_3' = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix} - \left\langle \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix} \right\rangle \begin{bmatrix} -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix} - \left\langle \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\rangle \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix}$$

$$\mathbf{u}_3 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

so that the decomposition is given by

$$Q = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2\sqrt{2}} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

$$R = \begin{bmatrix} 2 & 8 & 7 \\ 0 & 2\sqrt{2} & 3\sqrt{2} \\ 0 & 0 & 6 \end{bmatrix}$$

2. Since the columns of this matrix are all linearly independent, this matrix has a left pseudo-inverse. 3. The system we have to solve is

$$\begin{cases} 2a_{1,1} + 8a_{2,1} + 7a_{3,1} = \frac{1}{2} \\ 2\sqrt{2}a_{2,1} + 3\sqrt{2}a_{3,1} = -\frac{1}{2\sqrt{2}} \\ 6a_{3,1} = \frac{1}{2} \end{cases}$$

By back-substitution we get

$$\begin{cases} a_{1,1} = -\frac{25}{12} \\ a_{2,1} = \frac{1}{4} \\ a_{3,1} = \frac{1}{12} \end{cases}$$

2 Least square methods (20 points)

For each of the systems, answer the following questions.

$$\begin{cases} 4x_1 + x_3 = 9 \\ x_1 - 5x_2 + x_3 = 0 \\ 6x_1 + x_2 = 0 \\ x_1 - x_2 - 5x_3 = 0 \end{cases} \tag{1}$$

$$\begin{cases} x_1 + x_2 = 2 \\ x_1 - x_3 = 5 \\ x_2 + x_3 = 6 \\ -x_1 + x_2 + 2x_3 = 6 \end{cases} \tag{2}$$

1. Prove that the system does not admit a solution.
2. What type of system does the least square solution solve? Does it admit a solution?
3. Solve the system from part 2.

Solutions:

1.1. The first system has coefficient matrix

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -1 & 5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & 5 \end{bmatrix}$$

to prove that it does not admit solutions, it suffices to show that the augmented matrix has rank 4, since the coefficient matrix has rank at most 3. Then we have

$$\begin{vmatrix} 4 & 0 & 1 & 9 \\ -1 & 5 & 1 & 0 \\ 6 & 1 & 0 & 0 \\ 1 & -1 & 5 & 0 \end{vmatrix} = -9 \begin{vmatrix} -1 & 5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & 5 \end{vmatrix} = -9 * 148 \neq 0$$

1.2. The least square solution solves the system $A^T A\mathbf{x} = A^T\mathbf{b}$, for

$$A^T A = \begin{bmatrix} 4 & 1 & 6 & 1 \\ 0 & -5 & 1 & -1 \\ 1 & 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ -1 & 5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 54 & 0 & 10 \\ 0 & 27 & -10 \\ 10 & -10 & 27 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 4 & 1 & 6 & 1 \\ 0 & -5 & 1 & -1 \\ 1 & 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 36 \\ 0 \\ 9 \end{bmatrix}.$$

Let us find the rank of $A^T A$.

$$\begin{vmatrix} 54 & 0 & 10 \\ 0 & 27 & -10 \\ 10 & -10 & 27 \end{vmatrix} = 31266 \neq 0$$

therefore the system admits a unique solution.

1.3. We can use Cramer's method to find the solution, that is:

$$x_1 = \frac{\begin{vmatrix} 36 & 0 & 9 \\ 0 & 27 & -10 \\ 10 & -10 & 27 \end{vmatrix}}{31266} = \frac{1123}{1737}$$

$$x_2 = \frac{\begin{vmatrix} 54 & 0 & 10 \\ 36 & 0 & 9 \\ 10 & -10 & 27 \end{vmatrix}}{31266} = \frac{70}{1737}$$

$$x_3 = \frac{\begin{vmatrix} 54 & 0 & 10 \\ 0 & 27 & -10 \\ 36 & 0 & 9 \end{vmatrix}}{31266} = \frac{21}{193}$$

2.1. Let us find the rank of the coefficient matrix. Since $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0$, its rank is at least 2. However, both square submatrices obtained by adding rows and columns to this 2×2 submatrix have determinant equal to 0, that is

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{vmatrix} = 0$$

therefore A has rank 2. However, the augmented matrix has rank at least equal to 3, since

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & -1 & 5 \\ 1 & 1 & 6 \end{vmatrix} = -9 \neq 0$$

therefore the system doesn't admit solutions.

2.2. The least square solution solves the system $A^T A\mathbf{x} = A^T\mathbf{b}$ for

$$A^T A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & \\ 1 & 0 & -1 & \\ 0 & 1 & 1 & \\ -1 & 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 0 & 3 & 3 \\ -3 & 3 & 6 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix}$$

We have that $\text{Rank}(A^T A) = 2$ and so is the rank of the augmented matrix, therefore the system admits infinite solutions with one free variable.

2.3. We can solve the system by Gaussian elimination on the augmented matrix

$$\begin{bmatrix} 3 & 0 & -3 & 6 \\ 0 & 3 & 3 & 9 \\ -3 & 3 & 6 & 3 \end{bmatrix}$$

having echelon form

$$\begin{bmatrix} 3 & 0 & -3 & 6 \\ 0 & 3 & 3 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and reduced echelon form

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the solutions have the form

$$\begin{cases} x_1 = 2 + x_3 \\ x_2 = 3 - x_3 \\ x_3 \text{ free} \end{cases}$$

3 Diagonalization (20 points)

For each of the following matrices, prove if they are diagonalizable or not and in case they are, find the diagonalization.

$$\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}, \qquad \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}, \qquad \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Solutions:

1. Let $A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$. Then the two eigenvalues satisfy $\lambda_1 + \lambda_2 = (A) = 8$ and $\lambda_1\lambda_2 = \det(A) = 16$, which tells us that $\lambda_1 = \lambda_2 = 4$. Therefore there is a unique eigenvalue $\lambda = 4$ with algebraic multiplicity 2. In order to find the corresponding eigenvector, conider the system $A - 4I = \mathbf{0}$, with augmented matrix

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

which has solutions with one degree of freedom

$$E(4) = \{ \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} \}$$

so the geometric multiplicity is equal to one and A is not diagonalizable.

2. Let $B = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$. The characteristic polynomial is

$$\begin{vmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{vmatrix} = (1-\lambda)(\lambda+4)(\lambda-7)$$

so we have $\lambda_1 = 7, \lambda_2 = 1, \lambda_3 = -4$. Since all the eigenvalues are different, B is diagonalizable. Let us find the eigenvectors, for $\lambda_1 = 7$ we need to solve the following system

$$\begin{bmatrix} -2 & -2 & 3 & 0 \\ 0 & -6 & 0 & 0 \\ 6 & 7 & -9 & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the first eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$. Then the second eigenvector is any solution of the system

$$\begin{bmatrix} 4 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 7 & -3 & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 0 & \frac{3}{8} & 0 \\ 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which has a solution of the form $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 6 \\ 8 \end{bmatrix}$. Finally, we find the third eigenvector as a solution of the system

$$\begin{bmatrix} 9 & -2 & 3 & 0 \\ 0 & 5 & 0 & 0 \\ 6 & 7 & 2 & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the third eigenvector has the form $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$. Now we can find the matrix composing the diagonalization, starting from

$$P = \begin{bmatrix} 3 & -3 & -1 \\ 0 & 6 & 0 \\ 2 & 8 & 3 \end{bmatrix}$$

and we can compute P^{-1} by finding the reduced echelon form of the matrix

$$\begin{bmatrix} 3 & -3 & -1 & 1 & 0 & 0 \\ 0 & 6 & 0 & 0 & 1 & 0 \\ 2 & 8 & 3 & 0 & 0 & 1 \end{bmatrix}$$

which is

$$\begin{bmatrix} 1 & 0 & 0 & \frac{3}{11} & \frac{1}{66} & \frac{1}{11} \\ 0 & 1 & 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 & -\frac{2}{11} & -\frac{5}{11} & \frac{3}{11} \end{bmatrix}$$

so that the diagonalization of B is

$$B = \begin{bmatrix} 3 & -3 & -1 \\ 0 & 6 & 0 \\ 2 & 8 & 3 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} \frac{3}{11} & \frac{1}{66} & \frac{1}{11} \\ 0 & \frac{1}{6} & 0 \\ -\frac{2}{11} & -\frac{5}{11} & \frac{3}{11} \end{bmatrix}$$

3. Let $C = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$. Since C is triangular, we can see that it has 2 eigenvalues $\lambda_1 = 4, \lambda_2 = 2$, both with algebraic multiplicity equal to 2. In order to find the eigenvectors, for $\lambda_1 = 4$ consider the system

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \end{bmatrix}$$

having reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so that solutions have the form

$$E(4) = \Big\{ \begin{bmatrix} 2\alpha \\ \beta \\ 0 \\ \alpha \end{bmatrix}, \alpha, \beta \in \mathbb{R} \Big\}$$

so that a basis for $E(4)$ is given by $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{e}_2$. As for $\lambda_2 = 2$, the eigenvectors are the solutions of the system

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

having reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the solutions have the form

$$E(2) = \Big\{ \begin{bmatrix} 0 \\ 0 \\ \gamma \\ \delta \end{bmatrix}, \gamma, \delta \in \mathbb{R} \Big\}$$

and we can choose a basis as $\mathbf{v}_3 = \mathbf{e}_3$ and $\mathbf{v}_4 = \mathbf{e}_4$. Since for both eigenvalues the algebraic and geometric multiplicities coincide, the matrix is diagonalizable. In particular, we have

$$P = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

and the inverse is given by

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{bmatrix}$$

so that

$$C = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{bmatrix}$$

4 Multiplicities (20 points)

Consider the following matrix

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1. Find the eigenvalues of A and their corresponding algebraic multiplicity. Does this depend on h?

2. Find the geometric multiplicity of each eigenvalue. Does this depend on h?

3. For the values of h for which all algebraic multiplicites are equal to the corresponding geometric multiplicity, find a diagonalization of A.

Solutions:

1. The eigenvalues are on the diagonal, since A is an upper triangular matrix. These are $\lambda_1 = 5$ with algebraic multiplicity equal to 2, $\lambda_2 = 3$ with multiplicity 1 and $\lambda_3 = 1$ with multiplicity 1. These do not depend on h

2. First notice that the geometric multiplicities for $\lambda_2 = 3$ and $\lambda_3 = 1$ do not depend on h. For the first one, the eigenvectors solve the system

$$\begin{bmatrix} 2 & -2 & 6 & -1 & 0 \\ 0 & 0 & h & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

which, independently on h, has reduced echelon form

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the eigenvector has the form $\mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. As for $\lambda_3 = 1$, in order to find the eigenvector we need to solve the system

$$\begin{bmatrix} 4 & -2 & 6 & -1 & 0 \\ 0 & 2 & h & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{7+h}{4} & 0 \\ 0 & 1 & 0 & -\frac{h}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so that one solution has the form $\mathbf{v}_3 = \begin{bmatrix} 7+h \\ 2h \\ -4 \\ 4 \end{bmatrix}$ which is always different from $\mathbf{0}$. As for the geometric multiplicity of $\lambda_1 = 5$, we need to solve the system

$$\begin{bmatrix} 0 & -2 & 6 & -1 & 0 \\ 0 & -2 & h & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix}$$

which can be row reduced until

$$\begin{bmatrix} 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & h-6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we have two possibilities: if $h = 6$, then we have a solution of the type

$$\begin{bmatrix} 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives a two-dimensional eigenspace spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}$. The geometric multiplicity in this case is equal to 2 and therefore A is diagonalizable. Otherwise, if $h \neq 6$ there is an additional pivot column and the reduced echelon form is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so that $E(5)$ is spanned by only $\mathbf{v}_1$ and the geometric multiplicity is equal to 1. Therefore the dimension of $E(5)$ depends on h.

3. For $h = 6$ we consider

$$P = \begin{bmatrix} 1 & 0 & 13 & 1 \\ 0 & 3 & 12 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

then we can find the inverse by finding the reduced echelon form of

$$\begin{bmatrix} 1 & 0 & 13 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 12 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

which is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & \frac{13}{8} & \frac{11}{4} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{8} & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -6 \end{bmatrix}$$

so that

$$A = \begin{bmatrix} 1 & 0 & 13 & 1 \\ 0 & 3 & 12 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{13}{8} & \frac{11}{4} \\ 0 & \frac{1}{3} & \frac{1}{2} & 1 \\ 0 & 0 & -\frac{1}{8} & \frac{1}{4} \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

5 Eigenvalues of transformations in $\mathbb{R}^2$ (20 points)

For each of the following transformations in $\mathbb{R}^2$, find the eigenvalues, eigenvectors and, in case it's possible and the eigenvalues are real, a diagonalization:

1. A horizontal shear transformation by $\frac{3}{2}$.
2. A counter-clockwise rotation by $\frac{\pi}{3}$.
3. A reflection across the line $y = -x$.

Solutions:

1. We have

$$A = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$$

so that A has a unique eigenvalue $\lambda = 1$ with algebraic multiplicity equal to 1. The dimension of its eigenspace is 1, since the solutions of the system

$$\begin{bmatrix} 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are multiples of $\mathbf{e}_1$.

2. We have

$$B = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

The eigenvalues solve the equation

$$\lambda^2 - \lambda + 1 = 0$$

which has complex solutions

$$\lambda_{1,2} = \frac{1 \pm \sqrt{3}i}{2}$$

Since these are complex solutions, we don't have to diagonalize the matrix. However we can find the eigenvectors as solutions of

$$\begin{bmatrix} -\frac{\sqrt{3}}{2}i & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}i \end{bmatrix}$$

which is given by $\mathbf{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$, and the solution of

$$\begin{bmatrix} \frac{\sqrt{3}}{2}i & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2}i \end{bmatrix}$$

given by $\mathbf{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

3. We have

$$C = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

In this case, the characteristic polynomial is given by $p(\lambda) = \lambda^2 - 1$, which has solution $\lambda = 1$ with algebraic multiplicity 2. The system

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

has only a one-dimensional solution of multiples of $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so the matrix is not diagonalizable.