Lecture 10: Quadratic forms and Singular Value Decomposition

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Quadratic forms

Until now, we have mostly seen linear operations performed on $\mathbb{R}^n$. However, interesting theory can be developed around second degree operations, such as the sum of squares we already explored, or quadratic forms.

Definition 1. A *quadratic form* in $\mathbb{R}^n$ is an expression of the type

$$Q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle$$

for a symmetric matrix $A \in \mathbb{R}^{n \times n}$.

Example: let $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$, then

$$Q(\mathbf{x}) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 - 4x_1x_2 + 7x_2^2$$

A quadratic form has a particularly good expression if A is a diagonal matrix.

Example: let $A = \begin{bmatrix} 9 & 0 \\ 0 & -4 \end{bmatrix}$, then

$$Q(\mathbf{x}) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 9x_1^2 - 4x_2^2$$

We have the following classification of quadratic forms:

Definition 2. Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form, then Q is said to be

- *positive definite* if $Q(\mathbf{x}) > 0$ for every $\mathbf{x} \neq \mathbf{0}$.
- *semi-positive definite* if $Q(\mathbf{x}) \geq 0$ for every $\mathbf{x} \in \mathbb{R}^n$.
- *negative definite* if $Q(\mathbf{x}) < 0$ for every $\mathbf{x} \neq \mathbf{0}$.
- *semi-negative definite* if $Q(\mathbf{x}) \leq 0$ for every $\mathbf{x} \in \mathbb{R}^n$.
- *indefinite* otherwise.

Notice that it is much easier to classify a quadratic form if the corresponding matrix is diagonal.

Example: consider the quadratic forms associated to the matrices

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

. Then we have

$$Q_1(\mathbf{x}) = \langle \mathbf{x}, A_1\mathbf{x} \rangle = -x_1^2 - 4x_2^2$$

as you can see $Q_1(\mathbf{x}) < 0$ for every $\mathbf{x} \neq 0$. Moreover

$$Q_2(\mathbf{x}) = \langle \mathbf{x}, A_2\mathbf{x} \rangle = 9x_1^2 + 4x_2^2$$

in this case, $Q_2(\mathbf{x}) \geq 0$ for every $\mathbf{x}$, so it's at least semi-positive definite. However, notice that there exists $\mathbf{x} = \mathbf{e}_3 \neq 0$ such that $Q_2(\mathbf{x}) = 0$, so that the quadratic form is not positive definite. Finally,

$$Q_3(\mathbf{x}) = \langle \mathbf{x}, A_2\mathbf{x} \rangle = 5x_1^2 + 2x_2^2 - x_3^2$$

in this case there are vectors on which the form is positive (for example $\mathbf{e}_1$ and $\mathbf{e}_2$) and vectors on which the form is negative (for example $\mathbf{e}_3$). Therefore the form is indefinite.

Generally, if Q is a quadratic form associated to a non-diagonal matrix A, it's harder to say whether Q is positive (semi)-definite, negative (semi)-definite or indefinite just by looking at the explicit expression. However, since we are dealing with symmetric matrices, we can use its spectral property to change variables and diagonalize the quadratic form.

Example: consider the quadratic form Q associated to the matrix

$$A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$$

then we can diagonalize A with an orthonormal set of eigenvectors. In fact the eigenvalues of A solve the equation

$$\lambda^2 + 4\lambda - 21 = 0$$

so that $\lambda_1 = 3$ and $\lambda_2 = -7$. The matrix can be orthogonally diagonalizable by choosing

$$\mathbf{u}_1 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

so that

$$Q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

notice that this last expression can be rewritten as

$$Q(\mathbf{x}) = \begin{bmatrix} -\frac{2}{\sqrt{5}}x_1 + \frac{1}{\sqrt{5}}x_2 & \frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}}x_1 + \frac{1}{\sqrt{5}}x_2 \\ \frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_2 \end{bmatrix}$$

so that we can write it as

$$Q(\mathbf{x}) = 3(-\frac{2}{\sqrt{5}}x_1 + \frac{1}{\sqrt{5}}x_2)^2 - 7(\frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_2)^2$$

or, if we consider the change of variable

$$\begin{cases} y_1 = -\frac{2}{\sqrt{5}}x_1 + \frac{1}{\sqrt{5}}x_2 \\ y_2 = \frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_2 \end{cases}$$

then $Q(\mathbf{x}) = 3y_1^2 - 7y_2^2$, which is obviously indefinite.

Notice that in the previous example, we have used the orthogonal diagonalization of the symmetric variable A to induce a change of variable in the quadratic form that helps us understand its nature. Such procedure can always be applied, that is, if U is the orthogonal matrix having unit eigenvectors of A as columns, then we can consider the change of variable $\mathbf{y} = U^T\mathbf{x}$, to have

$$Q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle = \langle \mathbf{x}, U\Lambda U^T\mathbf{x} \rangle = \langle U^T\mathbf{x}, \Lambda U^T\mathbf{x} \rangle = \langle \mathbf{y}, \Lambda\mathbf{y} \rangle$$

this last inner product can be expressed as $\lambda_1 y_1^2 + ... + \lambda_n y_n^2$, for which the classification is possible on the basis of the coefficients $\lambda_1, ..., \lambda_n$. Therefore we have proved the following:

Theorem 1. *Let A be a symmetric matrix associated to the quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$. Then*

- *Q is positive (resp. negative) definite if all its eigenvalues are strictly greater (resp. less) than 0.*
- *Q is semi-positive (resp. semi-negative) definite if all its eigenvalues are greater (resp. less) than or equal to 0.*
- *Q is indefinite if it has eigenvalues with different signs.*

Example: consider the form $Q : \mathbb{R}^3 \to \mathbb{R}$ associated to the matrix

$$A = \begin{bmatrix} 9 & -4 & 4 \\ -4 & 7 & 0 \\ 4 & 0 & 11 \end{bmatrix}$$

in order to classify A, we need to find its eigenvalues. They solve the fundamental polynomial

$$\begin{vmatrix} 9-\lambda & -4 & 4 \\ -4 & 7-\lambda & 0 \\ 4 & 0 & 11-\lambda \end{vmatrix} = (9-\lambda)(7-\lambda)(11-\lambda) - 16(7-\lambda) - 16(11-\lambda) = 0$$

this expression can be reduced to

$$\lambda^3 - 27\lambda^2 + 207\lambda - 405 = 0$$

which can be factored as

$$(\lambda - 3)(\lambda^2 - 24\lambda + 135) = (\lambda - 3)(\lambda - 9)(\lambda - 15) = 0$$

so that the eigenvalues are $\lambda_1 = 15, \lambda_2 = 9$ and $\lambda_3 = 3$. Since they're all strictly positive, Q is positive definite.

Geometric interpretation of quadratic form

Consider a form in $Q : \mathbb{R}^2 \to \mathbb{R}$ having matrix $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Then the form will be

$$Q(\mathbf{x}) = ax_1^2 + 2bx_1x_2 + cx_2^2$$

this can be seen as the graph of a function in $\mathbb{R}^2$, thus corresponding to a quadric in $\mathbb{R}^3$, having a conic as a level set. A priori it's hard to say which quadric it is. However, through a change of variables we can diagonalize the quadratic form to obtain a form $Q(\mathbf{y}) = \lambda_1 y_1^2 + \lambda_2 y_2^2$. Then we have the following possibilities:

- If Q is positive or negative definite, its graph is a paraboloid in $\mathbb{R}^3$ and its level sets are ellipses, for the levels at which they're well-defined.
- If Q is semi-definite (but not positive or negative definite), then its graph is a parabolic cylinder and its level sets are parabolas.
- If Q is indefinite, then its graph is a hyperbolic paraboloid and its level sets are hyperbolas.

The diagonalization procedure can be used to classify a given conic

Example: classify the conic $5x_1 - 4x_1x_2 + 5x_2 = 48$. This is the level set $Q(\mathbf{x}) = 48$ for the quadratic form having matrix $A = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$. The eigenvalues of this matrix are given by $\lambda_1 = 7$ and $\lambda_2 = 3$, so that the form is positive definite and the conic is an ellipsis. Moreover, we can also find the principal axis of this ellipsis by considering the eigenvectors of the matrix. In particular, $\mathbf{v}_1$ is the unitary vector solving

$$\begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \end{bmatrix}$$

so that $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, while $\mathbf{v}_2$ solves

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \end{bmatrix}$$

so that $\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. The principal axis of the ellipsis are therefore given by the lines having directions $\mathbf{v}_1$ and $\mathbf{v}_2$ and the change of variables $\mathbf{y} = U^T\mathbf{x}$ corresponds to a transformation of $\mathbb{R}^2$ bringing the principal axis to the coordinate axis $x_2 = 0$ and $x_1 = 0$.

In general, for a given conic $ax_1^2 + 2bx_1x_2 + cx_2^2 = d$, it is always possible to classify the conic by analyzing the eigenvalues of $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and to find the principal axis corresponding to the directions of the eigenvectors. The transformation $\mathbf{y} = U^T\mathbf{x}$ corresponds to a transformation of $\mathbb{R}^2$ sending the principal axis to the coordinate axis.

Maximum and minimum of quadratic forms

Quadratic forms allow to understand another interpretation of eigenvalues. In particular, consider the diagonal form $Q(\mathbf{x}) = 9x_1^2+3x_2^2-2x_3^2$. We are interested in studying the maximum and the minimum of Q. Notice that if we look for optima in the whole space $\mathbb{R}^3$, then we can find arbitrarily large values (take for instance $\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$) or arbitrarily low values (take for instance $\begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}$). However, we can restrict our optimization on the unit sphere, that is, on vectors of unit norm. In other words, we want to find

$$M = \max_{||\mathbf{x}||=1} Q(\mathbf{x}) \text{ and } m = \min_{||\mathbf{x}||=1} Q(\mathbf{x})$$

Notice that since $3 \leq 9$ and $-2 \leq 9$, we have that

$$Q(\mathbf{x}) = 9x_1^2 + 3x_2^2 - 2x_3^2 \leq 9x_1^2 + 9x_2^2 + 9x_3^2 = 9||\mathbf{x}||$$

Since we are looking for the maximum on vectors of unit norm, we have $M \leq 9$. However, for $\mathbf{x} = \mathbf{e}_1$, we have $Q(\mathbf{e}_1) = 9$, so that $M = 9$. At the same time, since $9 \geq -2$ and $3 \geq -2$, we have

$$Q(\mathbf{x}) = 9x_1^2 + 3x_2^2 - 2x_3^2 \geq -2x_1^2 - 2x_2^2 - 2x_3^2 = -2||\mathbf{x}||$$

Since we are looking for the minimum on vectors of unit norm, we have $m \geq -2$. However, for $\mathbf{x} = \mathbf{e}_3$, we have $Q(\mathbf{e}_3) = -2$, so that $m = -2$.

The same property can be applied to any quadratic form, up to diagonalization of its corresponding symmetric matrix. In fact the following is true:

Theorem 2. *Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form with corresponding symmetric matrix A and let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be the eigenvalues of A in decreasing order. Then*

$$\lambda_1 = \max_{||\mathbf{x}||=1} \langle \mathbf{x}, A\mathbf{x} \rangle$$

$$\lambda_n = \min_{||\mathbf{x}||=1} \langle \mathbf{x}, A\mathbf{x} \rangle$$

Moreover, the point of maximum is the unitary eigenvector $\mathbf{v}_1$*, while the point of minimum is the unitary eigenvector* $\mathbf{v}_n$*.*

Proof. Consider the orthonormal diagonalization of $A = U\Lambda U^T$. By applying the change of variables $\mathbf{y} = U^T\mathbf{x}$, we have a diagonal form, for which maximum and minimum on vectors of unit length can be explicitly calculated as in the previous example (notice that $||U^T\mathbf{x}|| = ||\mathbf{x}||$ for every $\mathbf{x}$). Then the theorem is proved by noticing that the maximum value is reached for $\mathbf{e}_1 = U^T\mathbf{x}$, that is $\mathbf{x} = U\mathbf{e}_1 = \mathbf{v}_1$ and the minimum is $\mathbf{x} = U\mathbf{e}_n = \mathbf{v}_n$. □

A generalization of this theorem is the following:

Theorem 3 (Rayleigh quotient)**.** *Let* $Q : \mathbb{R}^n \to \mathbb{R}$ *be a quadratic form with corresponding symmetric matrix* A *and let* $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ *be the eigenvalues of* A *in decreasing order. Then*

$$\lambda_i = \max\{\langle \mathbf{x}, A\mathbf{x}\rangle \,|\, ||\mathbf{x}|| = 1, \langle \mathbf{x}, \mathbf{v}_j\rangle = 0 \text{ for all } j < i\}$$

Singular value decomposition

The singular value decomposition is one of the most important decompositions in applied linear algebra and it's a generalization of diagonalization for square matrices. It generalizes the idea of eigenvalues as optima, but for the maximization of the norm $||A\mathbf{x}||$, even if A is not a square matrix.

Example: consider the matrix

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

the linear operator $A : \mathbb{R}^3 \to \mathbb{R}^2$ send spheres in $\mathbb{R}^3$ to ellipsis in $\mathbb{R}^2$. We are interested in understanding the length of the principal axis, corresponding to the maximum value of $||A\mathbf{x}||$ for $\mathbf{x} \in \mathbb{R}^3$ of unit norm and to the length of its orthogonal axis. By the previous part, we have that

$$||A\mathbf{x}||^2 = \langle A\mathbf{x}, A\mathbf{x}\rangle = \langle \mathbf{x}, A^T A\mathbf{x}\rangle$$

the maximum value of $||A\mathbf{x}||$, will therefore be given by the square root of the maximum eigenvalue of $A^T A$. In this case we have

$$A^T A = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 360, \lambda_2 = 90$ and $\lambda_3 = 0$. Then the length of the principal axis will be $\sqrt{\lambda_1} = 6\sqrt{10}$. The corresponding eigenvector solves the system

$$\begin{bmatrix} -280 & 100 & 40 & 0 \\ 100 & -190 & 140 & 0 \\ 40 & 140 & -160 & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the unit norm eigenvector for $\lambda_1 = 360$ is

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

and the principal axis in $\mathbb{R}^2$ has direction

$$A\mathbf{v}_1 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \end{bmatrix}$$

What about the other axis, can we find it as the image of the second eigenvector of A^TA? We will now develop a theory to show that this is indeed the case.

Definition 3. Let A be a $m \times n$ matrix. The *singular values* of A are the square roots of the eigenvalues of the $n \times n$ matrix A^TA, that is, if $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ are the eigenvalues of A^TA, then the singular values are $\sigma_i = \sqrt{\lambda_i}$ for $i = 1, ..., n$.

Notice that the square root is well-defined in $\mathbb{R}$, since A^TA is positive semi-definite, therefore all its eigenvalues are greater or equal to 0. This happens because

$$\langle \mathbf{x}, A^TA\mathbf{x} \rangle = \langle A\mathbf{x}, A\mathbf{x} \rangle = ||A\mathbf{x}||^2 \geq 0$$

Moreover, notice that if $\mathbf{v}_i$ is the unitary eigenvector of A^TA corresponding to the eigenvalue λ_i, then

$$\sigma_i = \sqrt{\lambda_i} = \sqrt{\langle \mathbf{v}_i, A^TA\mathbf{v}_i \rangle} = \sqrt{\langle A\mathbf{v}_i, A\mathbf{v}_i \rangle} = ||A\mathbf{v}_i||$$

Moreover, A preserves orthogonality of the eigenvectors, that is

Theorem 4. *Let $\mathbf{v}_1, ..., \mathbf{v}_n$ be an orthonormal basis of eigenvectors for A^TA in $\mathbb{R}^n$ and let $r \leq n$ be such that A has r nonzero singular values. Then $A\mathbf{v}_1, ..., A\mathbf{v}_r$ are an orthogonal basis for* $\mathrm{Im}(A)$.

Proof. We have that for $i \neq j$ and $i, j \leq r$

$$\langle A\mathbf{v}_j, A\mathbf{v}_i \rangle = \langle \mathbf{v}_j, A^TA\mathbf{v}_i \rangle = \lambda_i \langle \mathbf{v}_j, \mathbf{v}_i \rangle = 0$$

□

By this theorem, we see that the minor principal axis of the ellipsis is given by the image of the second eigenvector of A^TA. In this case $\lambda_2 = 90$, which means that $\sigma_2 = ||A\mathbf{v}_2|| = 3\sqrt{10}$. Moreover, $\mathbf{v}_2$ is the unitary vector satisfying the system having augmented matrix

$$\begin{bmatrix} -10 & 100 & 40 & 0 \\ 100 & 80 & 140 & 0 \\ 40 & 140 & 110 & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that $\mathbf{v}_2 = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$ and the second axis corresponds to the direction

$$A\mathbf{v}_2 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$$

Given a matrix A we can always find a (left) singular value decomposition:

Theorem 5 (Singular Value Decomposition)**.** *Let A be a $m \times n$ matrix having $r \leq \min\{m, n\}$ nonzero singular values $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$. Then there exists an orthogonal $n \times n$ matrix V and an orthogonal $m \times m$ matrix U such that*

$$A = U\Sigma V^T$$

where Σ is a $m \times n$ matrix having σ_i on the i-th element of its diagonal and 0 *everywhere else.*

The process to construct a singular value decomposition is the following:
Step 1: find the eigenvalues and the unitary eigenvectors of the symmetric matrix A^TA. These will be $\lambda_1 \geq \lambda_2 \geq \lambda_r > 0$ and $\lambda_{r+1} = ... = \lambda_n = 0$ and the corresponding $\mathbf{v}_1, ..., \mathbf{v}_n$.
Step 2: construct Σ and V. Σ is the matrix such that $\Sigma_{i,i} = \sigma_i = \sqrt{\lambda_i}$ for $i = 1, ..., r$ and 0 otherwise, while V is the matrix having $\mathbf{v}_i$ as i-th column.
Step 3: for $i = 1, ..., r$, find $\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i$.
Step 4: complete $\mathbf{u}_1, ..., \mathbf{u}_r$ to a basis or $\mathbb{R}^m$ and write U the matrix having $\mathbf{u}_i$ as i-th column.

Example: for our previous A, the eigenvalues of A^TA are $\lambda_1 = 360, \lambda_2 = 90$ and $\lambda_3 = 0$. We have already found $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$. We need to find $\mathbf{v}_3$ either by solving the system $A\mathbf{x} = \mathbf{0}$, or by finding the unit vector which is orthogonal to the first two eigenvectors, that is

$$\begin{cases} v_1 + 2v_2 + 2v_3 = 0 \\ -2v_1 - v_2 + 2v_3 = 0 \end{cases}$$

which gives us the solution $\mathbf{v}_3 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$. So we have

$$V = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

as for U, we only need to normalize the two independent vectors we found before:

$$U = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix}$$

so that

$$A = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

The reason why this happens is because we have

$$\begin{aligned} U\Sigma &= \begin{bmatrix} A\mathbf{v}_1 & \ldots & A\mathbf{v}_r & \mathbf{0} \end{bmatrix} = AV \\ U\Sigma V^T &= AVV^T = A \end{aligned}$$

Example: find a Singular Value Decomposition of

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

We have

$$A^T A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

Then $\text{Rank}(A^T A) = 1$ and its nonzero eigenvalue is $\lambda_1 = 18$ (just by looking at the trace). The unitary eigenvector is obtained by solving a system having augmented matrix equal to

$$\begin{bmatrix} -9 & -9 & 0 \\ -9 & -9 & 0 \end{bmatrix}$$

so that $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$. On the other hand the eigenvector for $\lambda_2 = 0$ corresponds to $\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. So we have

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now consider

$$\mathbf{u}_1 = \frac{1}{3\sqrt{2}} A\mathbf{v}_1 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

Finally, we need to consider two elements of $\mathbb{R}^3$ orthogonal to $\mathbf{u}_1$. These are the solutions of

$$u_1 - 2u_2 + 2u_3 = 0$$

a (non-orthogonal) basis for this solutions is $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{b}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Applying Gram-Schmidt gives

$$\mathbf{u}_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}$$

$$\mathbf{u}_3' = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \langle \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} \rangle \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ \frac{4}{5} \\ 1 \end{bmatrix}$$

$$\mathbf{u}_3 = \begin{bmatrix} -\frac{2}{3\sqrt{5}} \\ \frac{4}{3\sqrt{5}} \\ \frac{\sqrt{5}}{3} \end{bmatrix}$$

so that the desired SVD decomposition is

$$A = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{\sqrt{5}}{3} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$