Midterm

COMS W3251, Summer Session 2

Due: Friday July 24th 2020, 4:59 PM EST

Instruction:

- You have 24 hours to complete this exam. You can use any time within the 24 hours, although the exam shouldn't take more than 3 hours.
- You may consult any non-living resources while you take the exam.
- Show full justification for every part. While you are allowed to use programs, we are grading each problem as if done by hand. Answers without justification will not be graded.
- You can either use the LaTeXtemplate given with the exam or write the answers on your own paper with each question/part clearly labeled.
- For questions and clarifications, send a private e-mail to the instructor (fp2428@columbia.edu) cc'ing the grader (ryan.chen@columbia.edu). We will try to answer as soon as possible.

Exercise 1 (20 points)

For a fixed vector $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3$, we define the cross product as

$$\mathbf{a} \times \mathbf{x} = \begin{bmatrix} a_2x_3 - a_3x_2 \\ a_3x_1 - a_1x_3 \\ a_1x_2 - a_2x_1 \end{bmatrix}$$

1. Prove that the cross product is linear and find a matrix A such that $A\mathbf{x} = \mathbf{a} \times \mathbf{x}$.
2. Consider the matrix A for $\mathbf{a} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and find a basis for its Kernel. What is its dimension?
3. What is the dimension of $\mathrm{Im}(A)$? Find a basis for it.

Exercise 2 (20 points)

1. Write the transformation T of $\mathbb{R}^2$ obtained first by a **clock-wise** rotation of $\frac{\pi}{3}$, then a dilatation of 2 along the x-axis.

2. Write the transformation S of $\mathbb{R}^2$ obtained first by a dilatation of 2 along the x-axis and then by a **clock-wise** rotation of $\frac{\pi}{3}$.

3. Are the two transformations the same? In case they are, find the inverse transformation. In case they are not, find the transformation Q such that $T = QS$.

Exercise 3 (20 points)

Consider the system

$$\begin{cases} 3x_1 + 2x_2 - x_3 = 1 \\ -2x_1 + \alpha x_2 + x_3 = 2 \\ x_1 + x_2 = 3 \end{cases}$$

1. Find the values of α for which this system has a unique solution, then use Cramer's rule to find the explicit solution, potentially as a function of α.

2. For all values of α for which the system doesn't have a unique solution, use Gaussian elimination to find out if the system has infinite solutions or no solution. If the system has infinite solutions, find an expression for them.

Exercise 4 (20 points)

Consider the system

$$\begin{cases} 4x_1 - 2x_2 + x_3 = 1 \\ x_1 + x_2 + x_3 = 0 \\ 3x_1 - 3x_2 = 2 \\ -x_1 - x_2 + x_3 = -1 \end{cases}$$

Use Rouché-Capelli's theorem to find whether the system has one solution, infinite solutions or no solution. In case it has infinite solutions, state how many variables are free.

Exercise 5 (20 points)

See the jupyter notebook attachment.