Homework 5

August 10, 2019

Problem 1

Consider the following vector field in $\mathbb{R}^2$:

$$\boldsymbol{F}(x,y) = (\frac{\pi x}{(1+x^2)^2} - \frac{12}{5}y^5)\boldsymbol{i} + (4x^3y^2 + e^{cos(y)+y^3})\boldsymbol{j}$$

Let C be the simple closed curve starting in $(1,0)$ and obtained by the union of the following curves: the semicircle C_1 with equation $y = \sqrt{1-x^2}$, the segment C_2 from $(-1,0)$ to $(-\frac{\sqrt{2}}{2}, 0)$, the semicircle C_3 with equation $y = \sqrt{\frac{1}{2} - x^2}$ in the clockwise direction and the segment C_4 from $(\frac{\sqrt{2}}{2}, 0)$ to $(1,0)$.

1. Evaluate $\int_C \boldsymbol{F} \cdot d\gamma$.
2. Let C' be the curve obtained by the union of C_1, C_2 and C_3. Using your previous computation evaluate $\int_{C'} \boldsymbol{F} \cdot d\gamma$.

Problem 2

Let $\boldsymbol{F}(x,y,z) = (x^2 - y, x + y^2, 0)$.

1. Evaluate the divergence and the curl.
2. Compute the line integral $\int_C \boldsymbol{F} \cdot d\gamma$ for C the oriented segment from $(0,-1,0)$ to $(0,1,0)$.
3. What is the value of the integral if C is changed to the semicircle $x^2 + y^2 = 1$ with $x \geq 0$ on the plane $z = 0$, taken with the same orientation?

Problem 3

Let $D = \{(x,y,z)|x^2 + y^2 \leq (z-3)^2, 0 \leq z \leq 2\}$.

1. Calculate the volume of D.
2. Let $\boldsymbol{F}(x,y,z) = (z^4 - 2y, x - 6y^3, z)$ and consider S_1 to be the part of the boundary of D lying on the surface of the cone. Evaluate $\iint_{S_1} \text{curl}\boldsymbol{F} \cdot dS$ (Hint: apply Stokes' theorem twice).

Problem 4

Consider the vector field $\boldsymbol{F}(x, y, z) = (z\sin(x), -yz\cos(x), x^2 + y^2)$. Calculate $\iint_S \boldsymbol{F} \cdot dS$ where $S = \{(x, y, z) : z = -3(x^2 + y^2) + 3, x^2 + y^2 \leq 1\}$ oriented with the normal vector in the positive z-direction.

Problem 5

1. Evaluate the integral $\int_S (x^2 + y^2)\, dS$ where S is the unit sphere in $\mathbb{R}^3$.
2. *Bonus Question:* Evaluate the previous integral with the divergence theorem.