Problem 1

Consider the following vector field in $\mathbb{R}^2$:

$$\boldsymbol{F}(x,y) = (\frac{\pi x}{(1+x^2)^2} - \frac{12}{5}y^5)\boldsymbol{i} + (4x^3y^2 + e^{cos(y)+y^3})\boldsymbol{j}$$

Let C be the simple closed curve starting in $(1,0)$ and obtained by the union of the following curves: the semicircle C_1 with equation $y = \sqrt{1-x^2}$, the segment C_2 from $(-1,0)$ to $(-\frac{\sqrt{2}}{2},0)$, the semicircle C_3 with equation $y = \sqrt{\frac{1}{2} - x^2}$ in the clockwise direction and the segment C_4 from $(\frac{\sqrt{2}}{2},0)$ to $(1,0)$.

1. Evaluate $\int_C \boldsymbol{F} \cdot d\gamma$.
   Solution:
   We can use Green's theorem to find $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 12x^2y^2 + 12y^2$ and integrate over the area D parametrized in polar coordinates as

   $$D = \{(\rho,\theta) | \frac{\sqrt{2}}{2} \leq \rho \leq 1 \,, 0 \leq \theta \leq \pi\}$$

   so that the integral becomes

   $$\int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\pi} 12\rho^5 \sin^2(\theta)\, d\theta\, d\rho = \frac{7\pi}{8}$$

2. Let C' be the curve obtained by the union of C_1, C_2 and C_3. Using your previous computation evaluate $\int_{C'} \boldsymbol{F} \cdot d\gamma$.
   Solution:
   By linearity of the integration on curves and Green's theorem:

   $$\int_{C'} \boldsymbol{F} \cdot d\gamma = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\, dA - \int_{C_4} \boldsymbol{F} \cdot d\gamma$$

   so we can parametrize C_4 and calculate the relative integral:

   $$C_4(t) = (t,0) \quad t \in [\frac{\sqrt{2}}{2}, 1]$$

   so that

   $$\int_{C_4} \boldsymbol{F} \cdot d\gamma = \int_{\frac{\sqrt{2}}{2}}^{1} \frac{\pi t}{(1+t^2)^2}\, dt = \frac{\pi}{12}$$

   and the final result is $\frac{7\pi}{8} - \frac{\pi}{12}$

Problem 2

Let $\boldsymbol{F}(x, y, z) = (x^2 - y, x + y^2, 0)$.

1. Evaluate the divergence and the curl.
   Solution:
   div$F = 2x + 2y$, while
   $$\text{curl}\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 - y & x + y^2 & 0 \end{vmatrix} = (0, 0, 2)$$

2. Compute the line integral $\int_C \boldsymbol{F} \cdot d\gamma$ for C the oriented segment from $(0, -1, 0)$ to $(0, 1, 0)$.
   Solution:
   We parametrize $C(t) = (0, t, 0)$ for $t \in [-1, 1]$ and evaluate the integral as
   $$\int_C \boldsymbol{F} \cdot d\gamma = \int_{-1}^{1} t^2 \, dt = \frac{2}{3}$$

3. What is the value of the integral if C is changed to the semicircle $x^2 + y^2 = 1$ with $x \geq 0$ on the plane $z = 0$, taken with the same orientation?
   Solution:
   The difference in the value between the two is given by the integral of the area enclosed by them, by Green's theorem. In this case $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2$, so that the area integral is twice the area of the semicircle, that is π. Therefore the integral on the semicircle is given by $\frac{2}{3} - \pi$. A direct calculation is also accepted.

Problem 3

Let $D = \{(x, y, z) | x^2 + y^2 \leq (z - 3)^2, 0 \leq z \leq 2\}$.

1. Calculate the volume of D.
   Solution:
   We can parametrize the domain in cylindrical coordinates in the following way
   $$D = \{(\rho, \theta, z) | \theta \in [0, 2\pi], z \in [0, 2], \rho \in [0, 3 - z]\}$$
   $$vol(D) = \int_0^{2\pi} \int_0^2 \int_0^{3-z} \rho \, d\rho \, dz \, d\theta = \frac{26\pi}{3}$$

2. Let $\boldsymbol{F}(x, y, z) = (z^4 - 2y, x - 6y^3, z)$ and consider S_1 to be the part of the boundary of D lying on the surface of the cone. Evaluate $\int_{S_1} \text{curl}\boldsymbol{F} \cdot dS$ (Hint: apply Stokes' theorem twice).
Solution:

$$\text{curl}\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ z^4 - 2y & x - 6y^3 & z \end{vmatrix} = (0, 4z^2, 3)$$

Rather than calculate the surface integral, we can apply Stokes' theorem to evaluate the function on the circles bounding it, with the correct orientation, which is counterclockwise (when seen from the top) for the one on $z = 2$ and clockwise for the other one. A direct calculation could be possible, but we can also apply Stokes' theorem once again to evaluate the integral on the two disks bounded by those circles. We know that the unit normal vector has form $(0, 0, -1)$ for the circle at the top and $(0, 0, 1)$ for the circle at the bottom, so that if we call them respectively S_2 and S_3, we get

$$\iint_{S_1} \text{curl}\boldsymbol{F} \cdot dS = \iint_{S_2} \text{curl}\boldsymbol{F} \cdot dS + \iint_{S_3} \text{curl}\boldsymbol{F} \cdot dS = 3Area(S_3) - 3Area(S_2) = 24\pi$$

Problem 4

Consider the vector field $\boldsymbol{F}(x, y, z) = (z \sin(x), -yz \cos(x), x^2 + y^2)$. Calculate $\iint_S \boldsymbol{F} \cdot dS$ where $S = \{(x, y, z) : z = -3(x^2 + y^2) + 3, x^2 + y^2 \leq 1\}$ oriented with the normal vector in the positive z-direction.
Solution:
The surface we are considering is the part of the paraboloid $z = -3(x^2 + y^2) + 3$ above the $z = 0$ plane. We can use the divergence theorem, remembering that we have to apply it to a closed solid region, so that if E is the solid region inside the paraboloid and above the $z = 0$ plane, the boundary is given by the paraboloid and by the unit disk on the $z = 0$ plane, that is,

$$\int_S \boldsymbol{F} \cdot dS = \iiint_E \text{div}F \, dV - \iint_P \boldsymbol{F} \cdot dS$$

where P is the unit disk on the plane with orientation pointing downward. We have

$$\text{div}F = z\cos(x) - z\cos(x) + 0 = 0$$

so that the first integral is 0. On the other hand, if we parametrize

$$p(u, v) = (u\cos(v), u\sin(v), 0) \quad u \in [0, 1] \quad v \in [0, 2\pi]$$

we get the correct orientation for $s_v \times s_u = (0, 0, -u)$ and the integral becomes

$$\iint_P \boldsymbol{F} \cdot dS = \int_0^1 \int_0^{2\pi} -u^3 \, dv \, du = -\frac{\pi}{2}$$

so that

$$\int_S \boldsymbol{F} \cdot dS = 0 - (-\frac{\pi}{2}) = \frac{\pi}{2}$$

Problem 5

Evaluate the integral $\iint_S (x^2 + y^2)\, dS$ where S is the unit sphere in $\mathbb{R}^3$, both with direct method and by using the divergence theorem.
Solution:
Direct method: we parametrize the sphere

$$s(\theta, \phi) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)) \quad \phi \in [0, \pi] \quad \theta \in [0, 2\pi]$$

and consider the outward orientation given by $s_\phi \times s_\theta$

$$s_\phi \times s_\theta = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos(\theta)\cos(\phi) & \sin(\theta)\cos(\phi) & -\sin(\phi) \\ -\sin(\theta)\sin(\phi) & \cos(\theta)\sin(\phi) & 0 \end{vmatrix} = (\sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi))$$

Then the integral becomes

$$\iint_S (x^2 + y^2)\, dS = \int_0^{2\pi} \int_0^{\pi} \sin^3(\phi)\, d\phi\, d\theta = \frac{8\pi}{3}$$

where the last calculation comes from substituting $\sin^2(\phi) = 1 - \cos^2(\phi)$.
Divergence theorem: We know that we can write the outward unit normal vector of S as $\vec{\boldsymbol{n}} = (\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}})$, so that for $\langle \boldsymbol{F}, \vec{\boldsymbol{n}} \rangle = x^2 + y^2$ we can use the vector field $\boldsymbol{F}(x, y, z) = (x\sqrt{x^2 + y^2 + z^2}, y\sqrt{x^2 + y^2 + z^2}, 0)$, so that $\iint_S (x^2 + y^2)\, dS = \iint_S \boldsymbol{F} \cdot dS$.
We can now apply the divergence theorem to get

$$\text{div}F = 2\sqrt{x^2 + y^2 + z^2} + \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + z^2}}$$

and the integral in the sphere becomes (in spherical coordinates)

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi} (2\rho^3 \sin(\phi) + \rho^3 \sin^3(\phi))\, d\phi\, d\theta\, d\rho = 2\pi \int_0^1 \rho^3\, d\rho \int_0^{\pi} (3\sin(\phi) - \sin(\phi)\cos^2(\phi))\, d\phi = \frac{8\pi}{3}$$