Homework 3

July 2020

1 Solutions of systems of linear equations (20 points)

For each of the following systems describe for which value of α and β the system admits one solution, infinite solutions or no solution. If the system has infinite solutions, state how many variables are free.

$$\begin{cases} x_1 - 2x_2 - x_3 = 0 \\ \alpha x_2 + 5x_3 = 3 \\ x_1 + 3x_2 + 6x_3 = 3\beta \end{cases} \tag{1}$$

$$\begin{cases} x_1 - 4x_2 + x_3 + \alpha x_4 = 0 \\ x_1 + x_2 - \beta x_3 - x_4 = 2 \\ x_1 - 2x_2 + x_4 = 5 \end{cases} \tag{2}$$

2 Kernel and Image of a matrix A (20 points)

Consider the following matrix:

$$A = \begin{bmatrix} 3 & 1 & 4 & -1 \\ 2 & -1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

1. Find a basis for $\text{Ker}(A)$. What is its dimension?
2. Find a basis for $\text{Im}(A)$. What is its dimension?
3. Find a basis for $\text{Ker}(A^T)$. What is its dimension?
4. Find a basis for $\text{Im}(A^T)$. What is its dimension?

3 Orthogonal sets (20 points)

Let H be a vector subspace of $\mathbb{R}^n$. We can define the orthogonal complement of H as

$$H^\perp := \{\mathbf{v} \in \mathbb{R}^n \mid \langle \mathbf{v}, \mathbf{h} \rangle = 0 \text{ for every } \mathbf{h} \in H\}$$

1. Prove that $H^{\perp}$ is a vector subspace of $\mathbb{R}^n$.
2. Consider A from the previous exercise and find a basis for $\mathrm{Ker}(A)^{\perp}$ in $\mathbb{R}^4$ and $\mathrm{Im}(A)^{\perp}$ in $\mathbb{R}^3$.
3. Prove that all the elements of the basis of $\mathrm{Ker}(A)^{\perp}$ are in $\mathrm{Im}(A^T)$.
4. *Bonus question (5 points)*: Prove that for every matrix A,
$$\mathrm{Ker}(A)^{\perp} = \mathrm{Im}(A^T)$$
$$\mathrm{Im}(A)^{\perp} = \mathrm{Ker}(A^T)$$
(Hint: you first need to show that for every $\mathbf{x}, \mathbf{y}$, $\langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^T\mathbf{y}\rangle$.)

4 Change of basis (20 points)

Consider the 4-dimensional vector space $P_3(\mathbb{R})$. We can treat the set of monomials $\mathcal{B} = \{1, x, x^2, x^3\}$ as a canonical basis for $P_3(\mathbb{R})$ in the sense that every polynomials
$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$
can be written as a vector
$$p = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$
Consider the polynomials
$$p_1(x) = x + 2 \qquad p_2(x) = 2x^2 - 1 \qquad p_3(x) = x^3 - 2x \qquad p_4(x) = 3$$

1. Write p_1, p_2, p_3, p_4 as vectors in the basis $\mathcal{B}$ and prove that they also form a basis $\mathcal{B}'$ for $P_3(\mathbb{R})$.
2. Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ and the change of basis matrix from $\mathcal{B}'$ to $\mathcal{B}$.
3. Write down the polynomial having coordinates $\begin{bmatrix} 1 \\ -3 \\ 0 \\ 4 \end{bmatrix}$ in $\mathcal{B}$ and the one with the same coordinates in $\mathcal{B}'$.

5 Gram-Schmidt algorithm (20 points)

Consider the following vectors
$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

1. Prove that the vectors in $\mathcal{C}$ are linearly independent.

2. Apply the Gram-Schmidt algorithm on $\mathcal{C}$ to find an orthonormal subset $\mathcal{U} \subset \mathbb{R}^4$.

3. What is the approximate complexity of applying the Gram-Schmidt algorithm on m vectors in $\mathbb{R}^n$?