Homework 5 Solutions

August 18, 2020

1 Discrete dynamical systems (20 points)

Consider the following discrete system:

$$\mathbf{x}_{k+1} = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} & 0 \\ \frac{3}{4} & \frac{5}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x}_k$$

1. Let $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. What is the behaviour of the system as $k \to \infty$?

2. Let $\mathbf{x}(0) = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$. What is the behaviour of the system as $k \to \infty$?

3. Let $\mathbf{x}(0) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. What is the general solution of the system? Depending on c_1, c_2 and c_3, what is the behaviour of the system as $k \to \infty$?

Solutions:

we will solve part 3 first and then answer to part 1 and 2 as particular cases. We start by diagonalizing the system. The characteristic polynomial is

$$\begin{vmatrix} \frac{5}{4} - \lambda & \frac{3}{4} & 0 \\ \frac{3}{4} & \frac{5}{4} - \lambda & 0 \\ 0 & 0 & -1 - \lambda \end{vmatrix} = (-\lambda - 1)(\lambda - 2)(\lambda - \frac{1}{2})$$

So that we have $\lambda_1 = 2$, $\lambda_2 = \frac{1}{2}$ and $\lambda_3 = -1$. The eigenvector $\mathbf{v}_1$ solves the system having augmented matrix

$$\begin{bmatrix} \frac{-}{3}4 & \frac{3}{4} & 0 & 0 \\ \frac{3}{4} & -\frac{3}{4} & 0 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the corresponding unitary eigenvector is $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$. Moreover, the eigenvector $\mathbf{v}_2$ solves the system having augmented matrix

$$\begin{bmatrix} \frac{3}{4} & \frac{3}{4} & 0 & 0 \\ \frac{3}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & -\frac{3}{2} & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the unitary eigenvector is $\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$. Finally, the last eigenvector solves the system having augmented matrix

$$\begin{bmatrix} \frac{9}{4} & \frac{3}{4} & 0 & 0 \\ \frac{3}{4} & \frac{9}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the unitary eigenvector is $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then we have

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and since A is symmetric and we have chosen unitary eigenvectors, $P^{-1} = P^T$, so that

$$\mathbf{y}(0) = P^T\mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \frac{c_1+c_2}{\sqrt{2}} \\ \frac{c_1-c_2}{\sqrt{2}} \\ c_3 \end{bmatrix}$$

The general solution is therefore

$$\mathbf{x}_k = \frac{c_1 + c_2}{\sqrt{2}} 2^k \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + \frac{c_1 - c_2}{2^k \sqrt{2}} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + (-1)^k c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore we have the following: if $c_1 + c_2 \neq 0$, then the system diverges. If $c_1 + c_2 = 0$ and $c_3 = 0$, then the system converges towards the origin. Otherwise if $c_1 + c_2 = 0$ and $c_3 \neq 0$, then the system oscillates between two limit points: $\begin{bmatrix} 0 \\ 0 \\ -c_3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ c_3 \end{bmatrix}$.

Notice that Part 1 corresponds to the case in which the system diverges ($c_1 + c_2 = 4$). While Part 2 corresponds to the case in which the system oscillates between two limit points $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

2 Continuous dynamical systems (20 points)

For each of the following systems, write the general solution for $\mathbf{x}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ and study the behaviour of the system as $t \to \infty$ depending on the choice of c_1 and c_2.

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \mathbf{x}(t) \tag{1}$$

$$\mathbf{x}'(t) = \begin{bmatrix} -2 & -5 \\ 1 & 4 \end{bmatrix} \mathbf{x}(t) \tag{2}$$

$$\mathbf{x}'(t) = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix} \mathbf{x}(t) \tag{3}$$

$$\mathbf{x}'(t) = \begin{bmatrix} -2 & 1 \\ -8 & 2 \end{bmatrix} \mathbf{x}(t) \tag{4}$$

Solutions:

1. In this case we have

$$\begin{vmatrix} 2 - \lambda & 3 \\ -1 & -2 - \lambda \end{vmatrix} = \lambda^2 - 1$$

so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$. The first eigenvector solves a system with augmented matrix

$$\begin{bmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and an eigenvector $v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. The second eigenvector solves the system having augmented matrix

$$\begin{bmatrix} 3 & 3 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and second eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then the matrices for the diagonalization are

$$P = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix}, \qquad P^{-1} = \frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & -3 \end{bmatrix}$$

so the initial position in the new coordinates is

$$\mathbf{y}(0) = P^{-1}\mathbf{x}(0) = \frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -c_1 - c_2 \\ -c_1 - 3c_2 \end{bmatrix}$$

The general solution is therefore

$$\mathbf{x}(t) = -\frac{c_1 + c_2}{2} e^t \begin{bmatrix} -3 \\ 1 \end{bmatrix} - \frac{c_1 + 3c_2}{2} e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

If $c_1 + c_2 \neq 0$, then the system diverges to infinity. Otherwise it converges to the origin.

2. In this case we have

$$\begin{vmatrix} -2 - \lambda & -5 \\ 1 & 4 - \lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3$$

so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$. The first eigenvector solves a system with augmented matrix

$$\begin{bmatrix} -5 & -5 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and an eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The second eigenvector solves the system having augmented matrix

$$\begin{bmatrix} -1 & -5 & 0 \\ 1 & 5 & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and second eigenvector $\mathbf{v}_2 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$. Then the matrices for the diagonalization are

$$P = \begin{bmatrix} 1 & -5 \\ -1 & 1 \end{bmatrix}, \qquad P^{-1} = -\frac{1}{4} \begin{bmatrix} -1 & 1 \\ 5 & -1 \end{bmatrix}$$

so the initial position in the new coordinates is

$$\mathbf{y}(0) = P^{-1}\mathbf{x}(0) = -\frac{1}{4} \begin{bmatrix} -1 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} -c_1 + c_2 \\ 5c_1 - c_2 \end{bmatrix}$$

The general solution is therefore

$$\mathbf{x}(t) = \frac{c_1 - c_2}{4} e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{c_2 - 5c_1}{4} e^{-t} \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

If $c_1 - c_2 \neq 0$, then the system diverges to infinity. Otherwise it converges to the origin.

3. In this case we have

$$\begin{vmatrix} 4 - \lambda & -3 \\ 6 & -2 - \lambda \end{vmatrix} = \lambda^2 - 2\lambda + 10$$

so the eigenvalues are $\lambda_1 = 1 + 3i$ and $\lambda_2 = 1 - 3i$. The first eigenvector solves a system with augmented matrix

$$\begin{bmatrix} 3 - 3i & -3 & 0 \\ 6 & -3 - 3i & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & -\frac{1+i}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and an eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 + i \\ 2 \end{bmatrix}$, so that the matrix S for the desired conjugation is

$$S = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

and

$$S^{-1} = -\frac{1}{2}\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$$

The resulting rotation matrix is therefore

$$C = S^{-1}AS = -\frac{1}{2}\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$

so that $a = 1$ and $b = -3$. Then we can find $\mathbf{y}(0)$ as

$$\mathbf{y}(0) = -\frac{1}{2}\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} c_2 \\ 2c_1 - c_2 \end{bmatrix}$$

and the general result is given by

$$\mathbf{x}(t) = e^t(\frac{c_2}{2}\cos(3t) + \frac{2c_1 - c_2}{2}\sin(3t))\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^t(\frac{2c_1 - c_2}{2}\cos(3t) - \frac{c_2}{2}\sin(3t))\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

this can be written as

$$\mathbf{x}(t) = e^t\begin{bmatrix} c_1\cos(3t) + (c_1 - c_2)\sin(3t) \\ c_2\cos(3t) + (2c_1 - c_2)\sin(3t) \end{bmatrix}$$

which spirals out and diverges as $t \to \infty$ for every choice of c_1 and c_2.

4. In this case we have

$$\begin{vmatrix} -2-\lambda & 1 \\ -8 & 2-\lambda \end{vmatrix} = \lambda^2 + 4$$

so the eigenvalues are $\lambda_1 = 2i$ and $\lambda_2 = -2i$. The first eigenvector solves a system with augmented matrix

$$\begin{bmatrix} -2-2i & 1 & 0 \\ -8 & 2-2i & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & \frac{-1+i}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and an eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1-i \\ 4 \end{bmatrix}$, so that the matrix S for the desired conjugation is

$$S = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$$

and

$$S^{-1} = \frac{1}{4}\begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix}$$

The resulting rotation matrix is therefore

$$C = S^{-1}AS = \frac{1}{4}\begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ -8 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

so that $a = 0$ and $b = -2$. Then we can find $\mathbf{y}(0)$ as

$$\mathbf{y}(0) = \frac{1}{4}\begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} c_2 \\ c_2 - 4c_1 \end{bmatrix}$$

and the general result is given by

$$\mathbf{x}(t) = (\frac{c_2}{4}\cos(2t) + \frac{c_2 - 4c_1}{4}\sin(2t))\begin{bmatrix} 1 \\ 4 \end{bmatrix} + (\frac{c_2 - 4c_1}{4}\cos(2t) - \frac{c_2}{4}\sin(2t))\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

this can be written as

$$\mathbf{x}(t) = \begin{bmatrix} c_1\cos(2t) + \frac{c_2 - 2c_1}{2}\sin(2t) \\ c_2\cos(2t) + (c_2 - 4c_1)\sin(2t) \end{bmatrix}$$

which stays on a fixed orbit as $t \to \infty$, depending on the initial point.

3 Understanding Markov chains (20 points)

Suppose that there are 3 possible states of the weather in the fall: "sunny", "cloudy" and "rainy". Suppose that if the weather is sunny on one day, then on the following day it will be sunny with probability 0.4, cloudy with probability 0.3 and rainy with probability 0.3. Moreover, if the weather is cloudy on a given day, then it will never be sunny on the following day, but it might be cloudy again with a probability of 0.8 or rainy with a probability of 0.2. Finally, if the weather is rainy on any given day, it will be sunny on the following day with a probability of 0.2, cloudy with a probability of 0.3 and rainy with a probability of 0.5.

1. Write the transition matrix A corresponding to the Markov chain described above.

2. On any given day, what is the transition matrix B describing the probabilities of being in the states "sunny", "cloudy" and "rainy" two days from now?

3. Explain the interpretation of a **mixed state**, that is, a vector with all strictly positive coordinates that sum to one. In particular, if $\mathbf{v}$ is a mixed state, what is the meaning of $A\mathbf{v}$?

4. Without diagonalizing the matrix, is there any guarantee that the matrix A admits a steady state? In case there is, find the steady state, otherwise explain why there isn't.

5. Now, consider a matrix $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Interpret this matrix as a Markov chain for two states "sunny" and "rainy". What is the long-term dynamic if we start from a "sunny" state?

6. Does C admit a steady state?

Solutions:

1.

$$A = \begin{bmatrix} 0.4 & 0 & 0.2 \\ 0.3 & 0.8 & 0.3 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

2.

$$B = A^2 = \begin{bmatrix} 0.22 & 0.04 & 0.18 \\ 0.45 & 0.7 & 0.45 \\ 0.33 & 0.26 & 0.37 \end{bmatrix}$$

3. For a mixed state $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, v_i represents the probability on a given day that the weather will be sunny (v_1), cloudy (v_2) or rainy (v_3). $A\mathbf{v}$ represents the probabilities of the weather being sunny, cloudy and rainy on the following day, given that on the first day the probabilities are respectively v_1, v_2 and v_3.

4. Yes, by the Perron-Frobenius theorem, since A is a stochastic matrix and A^2 has no component equal to 0, A admits a steady state. We can find it by solving the system having augmented matrix

$$\begin{bmatrix} -0.6 & 0 & 0.2 & 0 \\ 0.3 & -0.2 & 0.3 & 0 \\ 0.3 & 0.2 & -0.5 & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that $\mathbf{v}_1 = \begin{bmatrix} \frac{2}{11} \\ \frac{3}{11} \\ \frac{6}{11} \end{bmatrix}$ is the desired steady state.

5. If we start from sunny, then the system will continuously oscillate between sunny and rainy.

6. Yes, C admits a steady state, since $C - I$ has rank one. The desired steady state is

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

4 SVD decomposition of scalar multiples (20 points)

1. Let λ be an eigenvalue of a square matrix A and $\alpha \in \mathbb{R} \setminus \{0\}$. Can you find an eigenvalue for αA?
2. Let $B = U\Sigma V^T$ be an SVD decomposition for a $m \times n$ matrix B. Can you find an SVD decomposition for αB?
3. Find an SVD decomposition for $B = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$.
4. Find an SVD decomposition for $C = \begin{bmatrix} 0 & \pi & -\pi \end{bmatrix}$

Solutions:

1. $\alpha\lambda$ is an eigenvalue for αA, since, if $\mathbf{v}$ is the eigenvector of A for the eigenvalue λ, then

$$\alpha A\mathbf{v} = \alpha\lambda\mathbf{v}$$

so that $\mathbf{v}$ is an eigenvector of αA for the eigenvalue $\alpha\lambda$.

2. Consider $\alpha\Sigma = \begin{bmatrix} \alpha\sigma_1 & 0 & ... & 0 \\ 0 & \alpha\sigma_2 & ... & 0 \\ ... & & & \\ 0 & 0 & ... & \alpha\sigma_n \end{bmatrix}$. This is matrix with value only on the principal diagonal. We have that

$$U\alpha\Sigma V^T = \alpha U\Sigma V^T = \alpha B$$

since U and V are still orthogonal and $\alpha\Sigma$ has elements only on the diagonal, $U\alpha\Sigma V^T$ is an SVD for αB.

3. Consider

$$B^T B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

the characteristic equation of this matrix is

$$\lambda^2(2 - \lambda) = 0$$

so that it admits $\lambda_1 = 2$ and $\lambda_2 = 0$ with multiplicity 2. The eigenvector $\mathbf{v}_1$ for λ_1 satisfies a system having augmented matrix

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

whose reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the unitary eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. As for $\lambda_2 = 0$, the corresponding eigenvectors solve the system with augmented matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

whose reduced echelon form is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and an orthonormal basis is given by $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. So now we have

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

so that

$$\mathbf{u}_1 = \frac{1}{\sigma} B\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = -1$$

and the SVD is

$$B = -1 \cdot \begin{bmatrix} \sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

4. By Part 2 we have

$$C = -1 \cdot \begin{bmatrix} \pi\sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

5 SVD decomposition of pseudo-inverse (20 points)

1. Let $A = U\Sigma V^T$ be an SVD decomposition for an $m \times n$ matrix A. Can you find an SVD decomposition for A^T?

2. Now suppose that the columns of A are linearly independent. Can you find an SVD for the pseudo-inverse $A^\dagger = (A^T A)^{-1} A^T$?

3. Find an SVD decomposition of

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

.

4. Find an SVD decomposition for the pseudo-inverse $B^\dagger$.

Solutions:

1.

$$A^T = V\Sigma^T U^T$$

so that the SVD decomposition for the transpose is the transpose of the SVD decomposition.

2. Let $A = U\Sigma V^T$. Then

$$(A^T A)^{-1}A^T = (V\Sigma^T U^T U\Sigma V^T)^{-1} V\Sigma^T U^T = V(\Sigma^T\Sigma)^{-1}V^T V\Sigma^T U^T = V(\Sigma^T\Sigma)^{-1}\Sigma^T U^T$$

which is a SVD for $A^\dagger$ provided that $(\Sigma^T\Sigma)^{-1}$ exists and is diagonal and $(\Sigma^T\Sigma)^{-1}\Sigma^T$ is diagonal. The existence of the inverse is given by A having maximum column rank (so all its singular values are nonzero) and an easy calculation shows that $(\Sigma^T\Sigma)^{-1}\Sigma^T$ is diagonal.

3. In order to find an SVD decomposition, consider

$$B^T B = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Then the matrix has an eigenvalue $\lambda_1 = 3$ with eigenvector $\mathbf{v}_1 = \mathbf{e}_2$ and an eigenvalue $\lambda_2 = 2$ with eigenvector $\mathbf{v}_2 = \mathbf{e}_1$. Then

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}, \qquad V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so that

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

we can complete the base by choosing $\mathbf{u}_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 - x_3 = 0 \end{cases}$$

which is solved by the unitary vector $\mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$. Then the SVD of B is

$$B = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

4. From Part 2, since B has maximum column rank, we need to consider $(\Sigma^T\Sigma)^{-1}\Sigma^T$. Now

$$\Sigma^T\Sigma = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

so that $(\Sigma^T\Sigma)^{-1} = \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$ and

$$(\Sigma^T\Sigma)^{-1}\Sigma^T = \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

and the SVD is

$$B^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$