Final Exam

COMS W3251, Summer Session 2

Due: Friday August 14th 2020, 4:59 PM EST

Instruction:

- You have 24 hours to complete this exam. You can use any time within the 24 hours, although the exam shouldn't take more than 3 hours.
- You may consult any non-living resources while you take the exam.
- Show full justification for every part. While you are allowed to use programs, we are grading each problem as if done by hand. Answers without justification will not be graded.
- You can either use the LaTeXtemplate given with the exam or write the answers on your own paper with each question/part clearly labeled.
- For questions and clarifications, send a private e-mail to the instructor (fp2428@columbia.edu) cc'ing the grader (ryan.chen@columbia.edu). We will try to answer as soon as possible.

Exercise 1 (20 points)

Consider the system of linear equations in matrix form

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -1 \\ 2 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ h \\ -3 \end{bmatrix}$$

1. Find the conditions on h under which the system admits a solution and find the solution $\mathbf{y}$.
2. For all h for which the system doesn't admit a solution, find the least square solution $\hat{\mathbf{x}}$ of the system (possibly depending on h).

Exercise 2 (20 points)

Consider a square matrix A having characteristic polynomial

$$p(\lambda) = (\lambda + 1)^2(\lambda - 2)(\lambda - 3)^2$$

1. What is the dimension of A?
2. Find the trace and the determinant of A.
3. Do you have enough information to say whether A is invertible or not? In case you don't, what else do you need to know?
4. Do you have enough information to say whether A is diagonalizable or not? In case you don't, what else do you need to know?
5. Suppose that A is symmetric and classify the corresponding quadratic form $Q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle$.

Exercise 3 (20 points)

Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and let A be a matrix having $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as eigenvectors for the corresponding eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0.5$.

1. Find A.
2. Consider the discrete dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ and write the general solution for $\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$.
3. Does there exist $\mathbf{x}_0$ such that $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is a steady state of the system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ starting from $\mathbf{x}_0$? What about $\mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

Exercise 4 (20 points)

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

1. Find a basis for $\mathrm{Ker}(A)$ and a basis for $\mathrm{Im}(A)$.
2. Find an SVD decomposition of $A = U\Sigma V^T$.

Exercise 5 (20 points)

See the jupyter notebook attachment.