Lecture 9: Dynamical systems and Markov chains

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Discrete difference equations

A discrete difference equation is an expression of the type

$$\mathbf{x}_{k+1} = f(k, \mathbf{x}_k, \mathbf{x}_{k-1}, ..., \mathbf{x}_{k-N})$$

that represents the growth of a given vector of variables depending on the value of that variable in previous discretized periods (up to $k - N$) and on the period k itself.

Example: Predator-Prey dynamics

Imagine that in an environment at each period k there are O_k owls and R_k rats. The population of owls is halved at each time if they can't eat rats, while rats replicate by a factor of 1.1 if they're not eaten by owls. We can write the system as

$$\begin{cases} O_{k+1} = 0.5O_k + 0.4R_k \\ R_{k+1} = 1.1R_k - 0.104O_k \end{cases}$$

If we consider $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$, we can write this system as $\mathbf{x}_{k+1} = A\mathbf{x}_k$, with

$$A = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix}$$

This type of difference equation is a linear time-independent system, since the function $f(k, \mathbf{x}_k, \mathbf{x}_{k-1}, ..., \mathbf{x}_{k-N})$ is linear and does not depend on k. These will be the systems we will be dealing with for the rest of our class.

A system of difference equations tells us the dynamics over time of a vector variable starting from a given state $\mathbf{x}_0$. In the case of the previous example, choosing $\mathbf{x}_0 = \begin{bmatrix} 20 \\ 50 \end{bmatrix}$ tells us that after the first period we will have

$$\mathbf{x}_1 = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix} \begin{bmatrix} 20 \\ 50 \end{bmatrix} = \begin{bmatrix} 30 \\ 52.92 \end{bmatrix}$$

that is, 30 owls and about 52 rats. Then after the second period

$$\mathbf{x}_2 = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix} \begin{bmatrix} 30 \\ 52.92 \end{bmatrix} = \begin{bmatrix} 36.168 \\ 55.092 \end{bmatrix}$$

and, in general, after the k-th period the vector $\mathbf{x}_k$ can be obtained by considering $\mathbf{x}_k = A\mathbf{x}_{k-1} = A^k\mathbf{x}_0$.

We are interested in the long-term effects of the dynamics of the system. In the previous example, for instance, we are interested in understanding whether the population can reach an equilibrium, if one of the two (or both) will go extinct or if both will thrive and replicate. In order to do so, suppose that A is diagonalizable. Then we can write $\mathbf{x}_0$ in the basis of eigenvectors $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n$ and obtain

$$A^k\mathbf{x}_0 = c_1A^k\mathbf{v}_1 + c_2A^k\mathbf{v}_2 + ... + c_nA^k\mathbf{v}_n = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + ... + c_n\lambda_n^k\mathbf{v}_n$$

Then taking the limit for $k \to \infty$ allows us to understand the long-term implications of our dynamics.

Example: For $A = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix}$, the characteristic polynomial is

$$\lambda^2 - 1.6\lambda + 0.5916 = 0$$

having as solutions $\lambda_{1,2} = 0.8 \pm 0.22$, that is $\lambda_1 = 1.02$ and $\lambda_2 = 0.58$. The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

In our previous example $\mathbf{x}_0 = \begin{bmatrix} 20 \\ 50 \end{bmatrix}$ which is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ with

$$\mathbf{x}_0 = \frac{46}{11}\mathbf{v}_1 - \frac{48}{11}\mathbf{v}_2$$

That means that with this given initial state, the population dynamic in the long run goes as

$$\mathbf{x}_k = \frac{46}{11}(1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} - \frac{48}{11}(0.52)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

In this case, since $\lambda_1 > 1$ and $c_1 \neq 0$, $\mathbf{x}_k \to \infty$ and in particular both populations grow at a rate of 1.02. Since $0.58^k \to 0$ as $k \to \infty$, the only influence is given by the eigenvector corresponding to the eigenvalue greater than 1. In case $\mathbf{y}_0$ is a multiple of $\mathbf{v}_2$, for instance $\mathbf{y}_0 = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$, then the population grows like the second eigenvalue, that is $\mathbf{y}_k \sim 5(0.58)^k\mathbf{v}_2$, which goes to 0 as $k \to \infty$.

In general, given a system of difference equations $\mathbf{x}_{k+1} = A\mathbf{x}_k$, a good strategy is to diagonalize the matrix of the system $A = P\Lambda P^{-1}$. Then by considering $\mathbf{y}_k = P^{-1}\mathbf{x}_k$ we can rewrite the system as $\mathbf{y}_{k+1} = \Lambda\mathbf{y}_k$. This corresponds to the equations

$$\begin{cases} y_{1,k+1} = \lambda_1 y_{1,k} \\ y_{2,k+1} = \lambda_2 y_{2,k} \\ ... \\ y_{n,k+1} = \lambda_n y_{n,k} \end{cases}$$

we know how to write a general solution to those equations, that is $y_{i,k} = \lambda_i^k y_{i,0}$ for $i = 1, ..., n$, where $y_{0,i}$ corresponds to the i-th coordinate of $\mathbf{x}_0$ in the basis given by the eigenvectors. Finally, we will find the general solution by rewriting $\mathbf{x}_k$ in the original basis, that is $\mathbf{x}_k = P\mathbf{y}_k$. This gives

$$\mathbf{x}_k = P\sum_{i=1}^{n} \lambda_i^k y_{k,i}(0)\mathbf{e}_i = \sum_{i=1}^{n} \lambda_i^k y_{k,i}(0)\mathbf{v}_i$$

The 2×2 case

In 2×2 matrices, a graphical interpretation of diagonal matrices can help us understand what happens to a dynamical system. Pictures are taken from the textbook Lay, Lay and McDonald - Linear Algebra and its applications, chapter 5.6.

Case 1: both eigenvalues have absolute value greater than 1

If both eigenvalues are greater than 1, then every starting point outside of the origin is such that the system will ultimately diverge to infinity. In particular, consider the following matrix:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix}.$$

For a given starting point $\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, we have that

$$A^k\mathbf{x}_0 = 2^k c_1\mathbf{e}_1 + 1.5^k c_2\mathbf{e}_2 \sim 2^k c_1\mathbf{e}_1$$

and this tends to infinity as $k \to \infty$. The dynamic is shown in Figure 1 In this

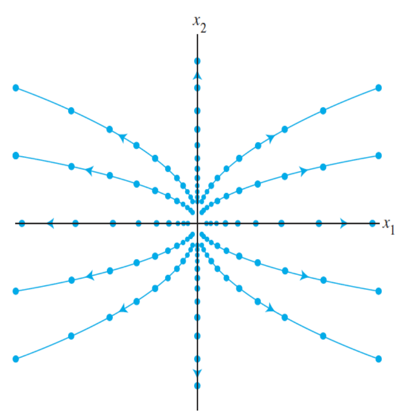

Figure 1: The origin as a repeller

case $\mathbf{0}$ is called a **repeller** for the system.

Case 2: both eigenvalues have absolute value less than 1

In this case, the origin is called an **attractor** for the system. No matter which starting point, the system will converge to the origin as $k \to \infty$. One example is given by the diagonal matrix

$$A = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.2 \end{bmatrix}$$

For a given starting point $\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, we have that

$$A^k \mathbf{x}_0 = 0.9^k c_1 \mathbf{e}_1 + 0.2^k c_2 \mathbf{e}_2 \sim 0.9^k c_1 \mathbf{e}_1$$

and this tends to 0 as $k \to \infty$. The dynamic is shown in Figure 2 In this case

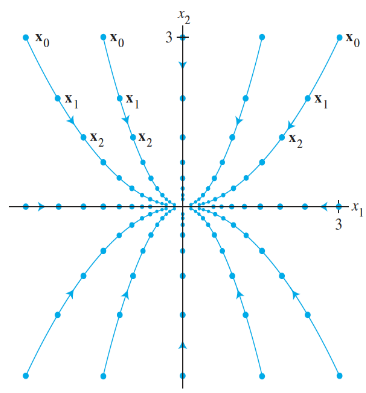

Figure 2: The origin as a attractor

0 is called an **attractor** for the system.

Case 3: one eigenvalue has absolute value greater than 1 and one in less than 1

In this case, the origin is called a **saddle point** for the system. This means that as long the coordinates of the starting point give a nonzero value to the leading eigenvector, then the system will diverge. However, there is a direction corresponding to the eigenvector of the smaller eigenvalue such that if the initial condition lies on the subspace generated by this eigenvector, then the system will converge to **0**. This is the case of the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0.3 \end{bmatrix}$$

For a given starting point $\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, we have that

$$A^k \mathbf{x}_0 = 2^k c_1 \mathbf{e}_1 + 0.3^k c_2 \mathbf{e}_2$$

If $c_1 \neq 0$, then $A^k \mathbf{x}_0 \sim 2^k c_1 \mathbf{e}_1$, which tends to ∞ as $k \to \infty$. Otherwise, $A^k \mathbf{x}_0 \sim 0.3^k c_2 \mathbf{e}_2$, which tends to 0 as $k \to \infty$ The dynamic is shown in Figure 3

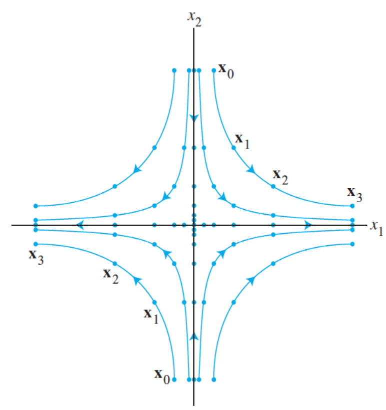

Figure 3: The origin as a saddle point

Case 4: one eigenvalue has absolute value equal to 1

This case is of particular interest and will be an object of focus for Markov Chains. In particular, let A be as follows:

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$$

For a given starting point $\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, we have that

$$A^k \mathbf{x}_0 = \lambda^k c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2$$

that gives rise to the following possibilities:

- if $|\lambda| > 1$, then $A^k \mathbf{x}_0 \to \infty$ as long as $c_1 \neq 0$, while if $c_1 = 0$, then $A^k \mathbf{x}_0 = c_2 \mathbf{e}_2$ for every $k \in \mathbb{N}$.
- If $|\lambda| < 1$, then $A^k \mathbf{x}_0 \sim c_2 \mathbf{e}_2$, so that $A^k \mathbf{x}_0 \to c_2 \mathbf{e}_2$ for $k \to \infty$, no matter the initial state.
- If $|\lambda| = 1$, then either A is the identity matrix (in which case there is no change no matter the initial point), or the state oscillates between $c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2$ and $-c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2$. In both cases, the system will not diverge.

Case 5: complex eigenvalues

In case of complex eigenvalues, there is a rotational component added to the dynamical system. In this case, both eigenvalues will have the same absolute value r. This allows for the following possibilities:

- If $r > 1$, then the origin will be a **repeller** and the states of the system will spiral towards infinity.
- If $r = 1$, then the system will remain on an elliptical trajectory around the origin. There will not be a steady state, but the system will not diverge to infinity either.
- If $r < 1$, then the system will spiral inward towards the origin, which is an **attractor** of the system.

We have seen in last lecture that rather than diagonalizing a matrix A with complex eigenvalues we can find a conjugate form

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

The three cases we just discussed depend on the value $r = \sqrt{a^2 + b^2}$, while the rotation will be given by θ such that $\cos(\theta) = \frac{a}{r}$ and $\sin(\theta) = \frac{b}{r}$. Then from a starting point $\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, we have

$$C^k \mathbf{x}_0 = r^k \begin{bmatrix} c_1 \cos(k\theta) - c_2 \sin(k\theta) \\ c_1 \sin(k\theta) + c_2 \cos(k\theta) \end{bmatrix}$$

Different choices of r give different orbits, coherently with the possibilities we enumerated above. The following Figure 4 shows the case $r < 1$.

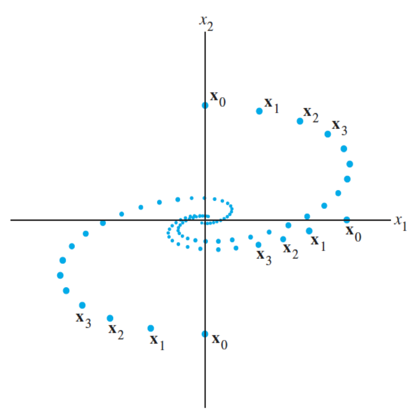

Figure 4: Orbits in the complex case, $r < 1$

Solving systems of difference equations: the general case

The previous analysis for the 2×2 diagonal case allows us to develop a general method to solve systems of difference equations. In order to understand the behaviour of a discrete dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ from the starting point $\mathbf{x}_0$ one has to take the following steps:

1. Diagonalize the matrix A and find the basis of eigenvectors.
2. Rewrite the initial state $\mathbf{x}_0$ in the coordinates given by the basis of eigenvectors.
3. If there exists an eigenvalue with absolute value greater than 1 whose corresponding coordinate of $\mathbf{y}_0$ is nonzero, then the system will diverge to infinity.
4. If there exists eigenvalues with absolute value equal to 1 for which the coordinates of $\mathbf{y}_0$ are nonzero and all the other eigenvalues whose coordinates are nonzero have absolute value less than 1, then the system will either converge to a single steady state (if all such eigenvalues are equal to 1) or stay on a steady orbit (if some of the eigenvalues are complex, or equal to -1).
5. Finally, if the only eigenvalues for which the coordinates are nonzero have absolute value less than one, then the system will converge to $\mathbf{0}$.

Example: Consider the system given by

$$\mathbf{x}_{k+1} = \frac{1}{9}\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}\mathbf{x}_k$$

What happens to the system when starting from the state $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 11 \\ -2 \end{bmatrix}$?

We start by diagonalizing the matrix. The characteristic polynomial is

$$\begin{vmatrix} \frac{7}{9} - \lambda & -\frac{2}{9} & 0 \\ -\frac{2}{9} & \frac{6}{9} - \lambda & \frac{2}{9} \\ 0 & \frac{2}{9} & \frac{5}{9} - \lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 - \frac{11}{9}\lambda + \frac{2}{9}$$

The solutions of $-\lambda^3 + 2\lambda^2 - \frac{11}{9}\lambda + \frac{2}{9} = 0$ are $\lambda_1 = 1$, $\lambda_2 = \frac{2}{3}$ and $\lambda_3 = \frac{1}{3}$. The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Then

$$P = \begin{bmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{bmatrix}$$

and

$$P^{-1} = \frac{1}{27}\begin{bmatrix} -6 & 6 & 3 \\ 6 & 3 & 6 \\ 3 & 6 & -6 \end{bmatrix}$$

so that the coordinates of $\mathbf{x}_0$ in the eigenvector basis are given by

$$\mathbf{y}_0 = \frac{1}{27}\begin{bmatrix} -6 & 6 & 3 \\ 6 & 3 & 6 \\ 3 & 6 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ 11 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Then as $k \to \infty$, since there exists an eigenvalue having value 1 and whose coordinate in the initial state is nonzero, while all the other eigenvalues have absolute value less than 1, the system will converge towards the steady state $y_1\mathbf{v}_1$.

$$A^k\mathbf{x}_0 \to \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}.$$

Continuous dynamical systems

A similar procedure can be applied in the continuous case, in which we have differential equations rather than difference equations. In this sense, the derivative of each variable with respect to time is a function of all the other variables and possibly time itself, that is

$$\mathbf{x}'(t) = f(\mathbf{x}(t), t)$$

We will only treat the case in which the funcion is linear and independent on time, therefore the case in which $\mathbf{x}'(t) = A\mathbf{x}(t)$, for A a $n \times n$ coefficient matrix independent of time.

Example: A particle is moving in a force field in $\mathbb{R}^2$ satisfying the following differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

If it starts from $\mathbf{x}(0) = \begin{bmatrix} 2.9 \\ 2.6 \end{bmatrix}$, where is it going to be at time $T = 4$?

In order to solve these types of differential equations, we need to apply a method which is similar to the one for systems of difference equations. In particular, we want to diagonalize the matrix to be able to decouple the single variables.

If we want to solve the system in one variable

$$\begin{cases} y'(t) = \lambda y(t) \\ y(0) = y_0 \end{cases}$$

then the general solution of this equation is given by $y(t) = y_0e^{\lambda t}$.

In order to solve a continuous dynamical system $\mathbf{x}'(t) = A\mathbf{x}(t)$, we will diagonalize the matrix A and apply this solutions to the coordinates given by the basis of eigenvectors. In other words, let $A = P\Lambda P^{-1}$ a diagonalization of A. Then if $\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i$, P^{-1} is the change of basis matrix $P^{-1}\mathbf{x} = \mathbf{y} = \sum_{i=1}^n y_i \mathbf{e}_i$ such that $\sum_{i=1}^n y_i \mathbf{v}_i = \mathbf{x}$, where $\mathbf{v}_i$ is the eigenvector corresponding to λ_i in the i-th column of P. Linearity of the derivative guarantees that

$$\mathbf{y}'(t) = P^{-1}\mathbf{x}'(t) = P^{-1}A\mathbf{x}(t) = P^{-1}P\Lambda P^{-1}\mathbf{x}(t) = \Lambda \mathbf{y}(t)$$

which can be rewritten as

$$\begin{cases} y_1'(t) = \lambda_1 y_1(t) \\ y_2'(t) = \lambda_2 y_2(t) \\ \dots \\ y_n'(t) = \lambda_n y_n(t) \end{cases}$$

this practice is called **decoupling** because now we have n equations relating the derivative of each variable to the variable itself, without any dependence on mixed components. Then, as in the one-dimensional case this can be solved as

$$\mathbf{y}(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix} \mathbf{y}(0)$$

where $\mathbf{y}(0) = P^{-1}\mathbf{x}(0)$. Finally we can write this in the original coordinates

$$\mathbf{x}(t) = P\mathbf{y}(t) = \sum_{i=1}^n y_i(0) e^{\lambda_i t} \mathbf{v}_i$$

where $y_i(0)$ is the i-th coordinate of the vector $\mathbf{y}(0) = P^{-1}\mathbf{x}(0)$.

Example: consider A from the previous example. Then the characteristic equation is:

$$\lambda^2 - 5\lambda - 6 = 0$$

which has solutions $\lambda_1 = 6$ and $\lambda_2 = -1$. The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore we have

$$P = \begin{bmatrix} -5 & 1 \\ 2 & 1 \end{bmatrix}, \qquad P^{-1} = -\frac{1}{7}\begin{bmatrix} 1 & -1 \\ -1 & -5 \end{bmatrix}$$

In this way we have

$$\mathbf{y}(0) = -\frac{1}{7}\begin{bmatrix} 1 & -1 \\ -1 & -5 \end{bmatrix}\begin{bmatrix} 2.9 \\ 2.6 \end{bmatrix} = \begin{bmatrix} -\frac{3}{70} \\ \frac{188}{70} \end{bmatrix}$$

so that the general solution of the differential equation is

$$\mathbf{x}(t) = -\frac{3}{70}e^{6t}\begin{bmatrix}-5\\2\end{bmatrix} + \frac{188}{70}e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$$

and in order to get the value at time $T = 4$ it's sufficient to just input $t = 4$ in the general expression.

Limiting behaviour for continuous dynamical systems

As in the discrete case, we are interested in the limiting behaviour of the dynamical system for $t \to \infty$. Recall that the general form of the solution is

$$\mathbf{x}(t) = \sum_{i=1}^{n} y_i(0)e^{\lambda_i t}\mathbf{v}_i$$

Then the behaviour of $\mathbf{x}(t)$ for $t \to \infty$ depends, as in the discrete case, on a combination of the eigenvalues and of the coordinates in the eigenvector basis. The main difference from the discrete case, however, is that the eigenvalues are on the exponent of the expression for the solution, rather than on the base. The following cases cover the theory for real eigenvalues:

- If there exists an $i = 1, ..., n$ such that $\lambda_i > 0$ and $y_i(0) \neq 0$, then the system diverges to ∞.
- If all the eigenvalues λ_i for which the corresponding coordinate $y_i(0) \neq 0$ are such that $\lambda_i < 0$, then the system converges to the origin.
- If there exist coordinates $y_i(0) \neq 0$ with corresponding eigenvalues $\lambda_i = 0$ and all the other nonzero coordinates are such that $\lambda_j < 0$, then the solution converges to a steady state.

Example: in the previous example, for $\mathbf{x}_0 = \begin{bmatrix}2.9\\2.6\end{bmatrix}$, we have

$$\mathbf{x}(t) = -\frac{3}{70}e^{6t}\begin{bmatrix}-5\\2\end{bmatrix} + \frac{188}{70}e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$$

Since there is a positive eigenvalue whose corresponding coordinate in the eigenvector basis is nonzero, the system will diverge to infinity as $t \to \infty$. In general, however, the second eigenvalue is negative and therefore there exist initial conditions (for example $\mathbf{x}(0) = \begin{bmatrix}3\\3\end{bmatrix}$) for which the system will converge to the origin. In this sense, the origin is called a **saddle point**, as in the discrete case. The same generalizations can be made for **attractor** and **repeller**

Complex 2×2 case

Suppose that A is a square 2×2 matrix with a pair of complex eigenvalues. Then we know there exists a matrix of the form

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

obtained by conjugating A through a matrix $S = \begin{bmatrix} \mathrm{Re}(\mathbf{v}_1) & \mathrm{I}(\mathbf{v}_1) \end{bmatrix}$. Since A has complex eigenvalues, S has maximum rank, so S^{-1} induces a change of basis from the canonical basis to $\{\mathrm{Re}(\mathbf{v}_1), \mathrm{I}(\mathbf{v}_1)\}$. In such basis, we have a linear differential equation of the type

$$\begin{cases} y_1'(t) = ay_1(t) - by_2(t) \\ y_2'(t) = by_1(t) + ay_2(t) \\ \mathbf{y}(0) = S^{-1}\mathbf{x}(0) \end{cases}$$

whose solution is determined by

$$\mathbf{y}(t) = e^{at} \begin{bmatrix} y_1(0)\cos(bt) - y_2(0)\sin(bt) \\ y_1(0)\sin(bt) + y_2(0)\cos(bt) \end{bmatrix}$$

Then the dynamics of this system depends on the values of a, with the following possibilities:

- If $a > 0$, then the system diverges as $t \to \infty$.
- If $a < 0$, then the system converges to the origin as $t \to \infty$.
- Finally, if $a = 0$, the system stays on a fixed orbit.

Then the general solution in the original coordinates is given by $\mathbf{x}(t) = S\mathbf{y}(t)$, that is

$$\mathbf{x}(t) = e^{at}(y_1(0)\cos(bt) - y_2(0)\sin(bt))\mathrm{Re}(\mathbf{v}_1) + e^{at}(y_1(0)\sin(bt) + y_2(0)\cos(bt))\mathrm{I}(\mathbf{v}_1)$$

Example: consider the dynamical system

$$\mathbf{x}'(t) = \begin{bmatrix} -2 & -2.5 \\ 10 & -2 \end{bmatrix} \mathbf{x}(t), \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

We can find its eigenvalues to solve the equation

$$\lambda^2 + 4\lambda + 29 = 0$$

which gives $\lambda_{1,2} = -2 \pm 5i$. Then we can consider the eigenvector for $\lambda_1 = -2 + 5i$ which solves an homogeneous system of linear equations having augmented matrix

$$\begin{bmatrix} -5i & -2.5 & 0 \\ 10 & -5i & 0 \end{bmatrix}$$

since the second row is a multiple of the first by a factor of $2i$, a solution is given by

$$\mathbf{v}_1 = \begin{bmatrix} i \\ 2 \end{bmatrix}$$

so that the matrix S is

$$S = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

and therefore

$$S^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

so that

$$C = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & -2.5 \\ 10 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ -5 & -2 \end{bmatrix}$$

Now we have

$$\mathbf{y}(0) = S^{-1}\mathbf{x}(0) = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 3 \end{bmatrix}$$

so that the general solution has the form

$$\mathbf{x}(t) = e^{-2t}(1.5\cos(-5t) - 3\sin(-5t)) \begin{bmatrix} 0 \\ 2 \end{bmatrix} + e^{-2t}(3\cos(-5t) + 1.5\sin(-5t)) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

or rather

$$\mathbf{x}(t) = e^{-2t} \begin{bmatrix} 3\cos(5t) - 1.5\sin(5t) \\ 3\cos(5t) + 6\sin(5t) \end{bmatrix}.$$

Since $a = -2 < 0$, this system will converge to the origin, which works as an attractor.

The Perron-Frobenius Theorem and applications to Markov chains

The theory behind Markov chains could occupy a course on its own. In this small section, we analyze Markov chains as a particular type of difference equations and use the Perron-Frobenius theorem to show when they are guaranteed to admit an equilibrium.

A Markov chain is a discrete difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$, in which the transition matrix A is a **stochastic matrix**, that is, all columns have non-negative entries which sum to one. In general, a process is called **Markov** if the current state only depends from the previous one and not from all the other before it and is called N-Markov if the current states depend from up to N states before itself, but no other state.

One example of applications of Markov chains is given by migration dynamics.

Example: let x_1 represent population in an urban area, x_2 population in a suburban area and x_3 population in a rural area. Then suppose that these populations follow the dynamics given by

$$\mathbf{x}_{k+1} = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.7 \end{bmatrix} \mathbf{x}_k$$

This means that at the end of any given period, 70% of the urban population will be composed by people who lived in the city already, 10% by people moving from the suburbs and 20% by people from the countryside (first row). By reading the first column, we see that 70% of the urban population will stay in the city, 20% will move to the suburbs and 10% to the countryside. Notice that with this interpretation the requirement is that the values of each column sum to 1 (or 100%. The second column and third column will show how the population of each area is moving by the end of period k, while the second and third rows tell us the composition of the population of each area. By considering an initial state $\mathbf{x}_0$, we can study the dynamic of this population.

Intuitively, it seems that there is no way for the system to diverge to infinity or to converge to zero, as the total population will be preserved, even if split differently between the possible states.

Definition 1. A state $\bar{\mathbf{x}}$ is called a *steady state* for a Markov chain if $A\bar{\mathbf{x}} = \bar{\mathbf{x}}$, that is $\bar{\mathbf{x}}$ is an eigenvector for $\lambda = 1$.

In particular, under the following conditions there will always be a **steady state** for the Markov chain:

Theorem 1 (Perron-Frobenius Theorem)**.** *Let A be a stochastic matrix such that a positive power of A has all strictly positive entries. Then $\lambda_1 = 1$ is the highest eigenvalue of A having a unique eigenvector with all positive entries.*

The steady state of the Markov chain is therefore the unique eigenvector associated to the eigenvalue $\lambda_1 = 1$. Under the regularity condition stated above, this means that a Markov chain always admits a unique steady state and that no matter where we start from, up to a multiplicative constant the system will get to the steady state.

Example: In the previous example we have

$$A = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.7 \end{bmatrix}$$

which respects the conditions of the Perron-Frobenius theorem. Therefore it admits a unique eigenvector for the eigenvalue $\lambda = 1$. In order to find it, we need to solve the system having augmented matrix

$$\begin{bmatrix} -0.3 & 0.1 & 0.2 & 0 \\ 0.2 & -0.2 & 0.1 & 0 \\ 0.1 & 0.1 & -0.3 & 0 \end{bmatrix}$$

whose reduced echelon form is

$$\begin{bmatrix} 1 & 0 & -\frac{5}{4} & 0 \\ 0 & 1 & -\frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the eigenspace is spanned by

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}$$

If the starting state is $\mathbf{x}(0) = \begin{bmatrix} 1000 \\ 500 \\ 500 \end{bmatrix}$, the final state will be a multiple α of $\mathbf{v}_1$ such that

$$(5 + 7 + 4)\alpha = (1000 + 500 + 500)$$

therefore the steady state is given by $\bar{\mathbf{x}} = 125\mathbf{v}_1 = \begin{bmatrix} 625 \\ 875 \\ 500 \end{bmatrix}$. In fact we can check

$$\begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} 625 \\ 875 \\ 500 \end{bmatrix} = \begin{bmatrix} 625 \\ 875 \\ 500 \end{bmatrix}$$

which means that if the city population is formed by 625,000 people, the suburban population by 875,000 people and the rural population by 500,000 people, then the system will be in equilibrium.