Problem 1

Let D be the domain in $\mathbb{R}^2$ bounded by the curves $y = 4 - x^2$, $y = x + 2$ and $y = 2 - x$.

1. Find the center of mass of D for a constant density.
   Solution:
   We can split the domain into two type I domains:

$$\{(x,y)| -1 \leq x \leq 0, 2 - x \leq y \leq 4 - x^2\} \cup \{(x,y)|0 \leq x \leq 1, x + 2 \leq y \leq 4 - x^2\}$$

   so that

$$m = \int_{-1}^{0}\int_{2-x}^{4-x^2} c\, dy\, dx + \int_{0}^{1}\int_{x+2}^{4-x^2} c\, dy\, dx = \frac{19c}{3}$$

   then

$$x_m = \frac{3}{19c}\int_{-1}^{0}\int_{2-x}^{4-x^2} cx\, dy\, dx + \frac{3}{19c}\int_{0}^{1}\int_{x+2}^{4-x^2} cx\, dy\, dx = 0$$

   which could also be seen by symmetry of the density and of the domain, while

$$y_m = \frac{3}{19c}\int_{-1}^{0}\int_{2-x}^{4-x^2} cy\, dy\, dx + \frac{3}{19c}\int_{0}^{1}\int_{x+2}^{4-x^2} cy\, dy\, dx = \frac{108}{35}$$

2. Evaluate

$$\iint_D f(x,y)dA$$

   for $f(x,y) = 2y^3x$.
   Solution:
   Calculating the integral for the same parametrization as before gives a high number of calculations. Therefore we can either notice that the domain is symmetric with respect to the y-axis and $f(-x,y) = -f(x,y)$, or we can use the parametrization into two type II domains

$$\{(x,y)|2 \leq y \leq 3, 2 - y \leq y \leq y - 2\} \cup \{(x,y)|3 \leq x \leq 4, -\sqrt{4-y} \leq y \leq \sqrt{4-y}\}$$

   to find that the integral is equal to 0.

Problem 2

Consider the domain D in $\mathbb{R}^2$ inside the circle of radius 1 centered at the origin and below the line $y = \frac{1}{2}$. Let $f(x,y) = \sin(x^2 + y^2)$.

1. Set up the integral as a union of type I domains, without computing it.
Solution:
The domain is given by the union of the following three type I domains:

$$D_1 = \{(x,y)| -1 \leq x \leq -\frac{\sqrt{3}}{2}, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}$$

$$D_2 = \{(x,y)| -\frac{\sqrt{3}}{2} \leq x \leq \frac{\sqrt{3}}{2}, -\sqrt{1-x^2} \leq y \leq \frac{1}{2}\}$$

$$D_3 = \{(x,y)| \frac{\sqrt{3}}{2} \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}$$

so that the integral can be written as

$$\int_{-1}^{-\frac{\sqrt{3}}{2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin(x^2+y^2)\, dy\, dx + \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int_{-\sqrt{1-x^2}}^{\frac{1}{2}} \sin(x^2+y^2)\, dy\, dx + \int_{\frac{\sqrt{3}}{2}}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin(x^2+y^2)\, dy\, dx$$

2. Set up the integral as a union of type II domains, without computing it.
Solution: The domain can be written as a unique type I domain:

$$\{(x,y)| -1 \leq y \leq \frac{1}{2}, -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}\}$$

so that the integral can be written as

$$\int_{-1}^{\frac{1}{2}} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sin(x^2+y^2)\, dx\, dy$$

3. Set up the integral in polar coordinates, without computing it.
Solution:
Notice that in this case we have ρ depending on θ for θ in $[\frac{\pi}{6}, \frac{5\pi}{6}]$, so that by applying some trigonometry we notice that we can rewrite the domain as the following union:

$$D_1 = \{(\rho,\theta)| 0 \leq \theta \frac{\pi}{6}, 0 \leq \rho \leq 1\}$$

$$D_2 = \{(\rho,\theta)| \frac{\pi}{6} \leq \theta \frac{5\pi}{6}, 0 \leq \rho \leq \frac{1}{2\sin\theta}\}$$

$$D_3 = \{(\rho,\theta)| \frac{5\pi}{6} \leq \theta \leq 2\pi, 0 \leq \rho \leq 1\}$$

so that the integral becomes

$$\int_0^{\frac{\pi}{6}} \int_0^1 \rho \sin(\rho^2)\, d\rho\, d\theta + \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_0^{\frac{1}{2\sin\theta}} \rho \sin(\rho^2)\, d\rho\, d\theta + \int_{\frac{5\pi}{6}}^{2\pi} \int_0^1 \rho \sin(\rho^2)\, d\rho\, d\theta$$

Problem 3

Let E be the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere of radius 1 centered in the origin. Let P be the center of mass of E.

1. Set up the integrals for the (x, y, z) coordinates of P in cylindrical coordinates, without computing them.
   Solution:
   The domain is parametrized as:

$$D = \{(\theta, \rho, z) | 0 \leq \theta \leq 2\pi, 0 \leq \rho \leq \frac{\sqrt{2}}{2}, \rho \leq z \leq \sqrt{1 - \rho^2}\}$$

   so that the integral for the mass is given by

$$m = \int_0^{2\pi} \int_0^{\frac{\sqrt{2}}{2}} \int_\rho^{1-\rho^2} c\rho \, d\rho \, dz \, d\theta$$

   and the coordinates are given by

$$x_m = \frac{1}{m} \int_0^{2\pi} \int_0^{\frac{\sqrt{2}}{2}} \int_\rho^{1-\rho^2} c\rho^2 \cos\theta \, d\rho \, dz \, d\theta$$

$$y_m = \frac{1}{m} \int_0^{2\pi} \int_0^{\frac{\sqrt{2}}{2}} \int_\rho^{1-\rho^2} c\rho^2 \sin\theta \, d\rho \, dz \, d\theta$$

$$z_m = \frac{1}{m} \int_0^{2\pi} \int_0^{\frac{\sqrt{2}}{2}} \int_\rho^{1-\rho^2} c\rho z \, d\rho \, dz \, d\theta$$

2. Set up the integrals for the (x, y, z) coordinates of P in spherical coordinates, without computing them.
   Solution:
   The domain is parametrised in spherical coordinates as

$$D = \{(\theta, \phi, \rho) | 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{4}, 0 \leq \rho \leq 1\}$$

   so that the integrals become:

$$m = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 c\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

   and the coordinates become:

$$x_m = \frac{1}{m} \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 c\rho^3 \sin^2\phi \cos\theta \, d\rho \, d\phi \, d\theta$$

$$y_m = \frac{1}{m}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 c\rho^3 \sin^2\phi \sin\theta \, d\rho\, d\phi\, d\theta$$

and

$$z_m = \frac{1}{m}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 c\rho^3 \sin\phi \cos\phi \, d\rho\, d\phi\, d\theta$$

Problem 4

Consider the integral given by

$$\int_0^{2\pi}\int_0^1\int_0^{2-r} r\, dz\, dr\, d\theta$$

1. Sketch a domain in $\mathbb{R}^3$ on which we are calculating the integral.
   Solution:
   The domain corresponds to a cylinder with a cone cap in $\mathbb{R}^3$. In order to visualize it, fix a θ and draw the (r, z)-section.

2. Rewrite the domain in $dr\, dz\, d\theta$ (meaning you should write the boundaries of r as a function of z).
   Solution:
   We can rewrite the domain as

   $$D = \{(\theta, r, z) | 0 \le \theta \le 2\pi, 0 \le z \le 1, 0 \le \rho \le 1\} \cup \{(\theta, r, z) | 0 \le \theta \le 2\pi, 1 \le z \le 2, 0 \le \rho \le 2-z\}$$

   so that it becomes

   $$\int_0^{2\pi}\int_0^1\int_0^1 r\, dr\, dz\, d\theta + \int_0^{2\pi}\int_0^1\int_0^{2-z} r\, dr\, dz\, d\theta$$

3. Evaluate the integral. Is this the volume of a solid?
   Solution:

   $$\int_0^{2\pi}\int_0^1\int_0^{2-r} r\, dz\, dr\, d\theta = \frac{4}{3}\pi$$

   This is the volume in cylindrical coordinates of the solid described in the first part.