Lecture 6: Change of basis, orthogonality and Gram-Schmidt algorithm

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Change of basis

We have defined a basis of a vector space as a minimal set of vectors that span the entire space. The canonical basis is a basis for $\mathbb{R}^n$, but it's not the only one.

Example: consider the set

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

We want to prove it's a basis of $\mathbb{R}^3$. In order to do so, we need to prove that we can write any $\mathbf{v} \in \mathbb{R}^3$ as a linear combination of these vectors, that is, for every $\mathbf{v} \in \mathbb{R}^3$ there exists a solution to the system of equations

$$y_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

This can be rewritten as

$$\begin{cases} y_1 + y_2 = v_1 \\ -y_1 + y_2 + y_3 = v_2 \\ y_1 + y_3 = v_3 \end{cases}$$

that is, if we consider the matrix $B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ having each element of the set as a column vector, we request that the system $B\mathbf{y} = \mathbf{v}$ has a solution for every $\mathbf{v} \in \mathbb{R}^3$. We have seen in previous classes that this is true if and only if $\det(B) \neq 0$. Since $\det(B) = 3$, we have proved that the set is a basis of $\mathbb{R}^3$. We have proved the case $n = 3$ of the following theorem:

Theorem 1. *Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\}$ be a set of vectors in $\mathbb{R}^n$. Then $\mathcal{B}$ is a basis of $\mathbb{R}^n$ if and only if they are linearly independent, which happens if and only if the determinant of the matrix having i-th column $\mathbf{b}_i$ is not zero.*

In order to prove that a given set of n vectors is a basis for $\mathbb{R}^n$, construct the matrix having each column corresponding to one of the vectors in the set and calculate the determinant. If it's nonzero, the set is a basis.

By the definition of a basis, we know that if $\mathcal{B} = \{\mathbf{b}_i\}_{i=1}^n$ is a basis, for each vector $\mathbf{v}$ there exists a unique set of coefficients y_i such that

$$\mathbf{v} = \sum_{i=1}^n y_i \mathbf{b}_i$$

Then the vector $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix}$ corresponds to the vector $\mathbf{v}$ in the coordinates given by the basis $\mathcal{B}$. Up until now, we have written every vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix}$ as a linear combination of the canonical basis, that is

$$\mathbf{v} = \sum_{i=1}^n v_i \mathbf{e}_i.$$

However, for a given basis $\{\mathbf{b}_i\}_{i=1}^n$, we want to be able to write every vector $\mathbf{v}$ as the corresponding linear combination

$$\mathbf{v} = \sum_{i=1}^n y_i \mathbf{b}_i.$$

This is what we call a *change of basis.*

In order to change coordinates from the canonical basis to $\mathcal{B}$, we need to solve the system $B\mathbf{y} = \mathbf{v}$, where B is the square matrix having $\mathbf{b}_i$ as i-th column. Since $\mathcal{B}$ is a basis, the solution exists for every $\mathbf{v}$ and is given by $\mathbf{y} = B^{-1}\mathbf{v}$. The matrix B^{-1} is called the *change of basis matrix.* For any given vector $\mathbf{v} = \sum_{i=1}^n v_i \mathbf{e}_i$, $B^{-1}\mathbf{v}$ is a vector $\mathbf{y}$ such that $\mathbf{v} = \sum_{i=1}^n y_i \mathbf{b}_i$.

Example: as in the previous example, consider

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

In order to find the change of basis matrix from the canonical basis to $\mathcal{B}$, we need to find the inverse of the matrix B having $\mathbf{b}_i$ as i-th column. Then we have

$$B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and through Gaussian elimination on the matrix

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

we get the echelon form

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & -1 & 1 & 2 \end{bmatrix}$$

and the reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

so that

$$B^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

In this way, if we want to write the vector $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ in the coordinates given by $\mathcal{B}$, we have

$$\mathbf{y} = B^{-1}\mathbf{v} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

and indeed

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Orthonormal basis

In general, the inverse matrix B^{-1} could be computationally expensive. The following definition will make the computation far more easy:

Definition 1. Let $\mathcal{U} = \{\mathbf{u}_i\}_{i=1}^n$ be a basis for $\mathbb{R}^n$. We say that $\mathcal{U}$ is an *orthogonal basis* if

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0 \qquad \text{for every } i \neq j.$$

Moreover, we say that $\mathcal{U}$ is *orthonormal* if it is orthogonal and

$$||\mathbf{u}_i|| = 1 \qquad \text{for every } i = 1, ..., n.$$

In short, one can say that a basis $\{\mathbf{u}_i\}_{i=1}^n$ is orthonormal if

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \delta_{ij}$$

Where δ_{ij} is called the *Kronecker delta*, taking value 1 when $i = j$ and 0 otherwise.

The canonical basis is an orthonormal basis, since if $i \neq j$, $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$, while $\langle \mathbf{e}_i, \mathbf{e}_i \rangle = 1$ for every $i = 1, ..., n$. However, there are other orthogonal basis which are not the canonical basis.

Example: consider the set

$$\mathcal{U} = \left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since

$$\begin{vmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$\mathcal{U}$ is a basis. Moreover,

$$\langle \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \rangle = 1 \qquad \langle \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \rangle = 1$$

$$\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle = 1 \qquad \langle \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \rangle = 0$$

$$\langle \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle = 0 \qquad \langle \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle = 0$$

Therefore this is an orthonormal basis.

The advantage of an orthonormal basis $\mathcal{U} = \{\mathbf{u}_i\}_{i=1}^n$ is that the matrix U having i-th column equal to $\mathbf{u}_i$ is such that $U^T = U^{-1}$. We call such matrices *orthogonal*. In order to prove that this is the case, notice that

$$(U^T U)_{i,j} = \sum_{k=1}^{n} U^T_{i,k} U_{k,j} = \sum_{k=1}^{n} U_{k,i} U_{k,j} = \langle \mathbf{u}_i, \mathbf{u}_j \rangle = \delta_{i,j}$$

so that $U^T U = I$. Therefore $U^T = U^{-1}$ and the change of basis matrix becomes $U^{-1} = U^T$. We have just proved the following:

Theorem 2. *Let $\mathcal{U} = \{\mathbf{u}_i\}_{i=1}^n$ be an orthonormal basis. Then the change of basis matrix from the canonical basis to $\mathcal{U}$ is the matrix U^T having $\mathbf{u}_i$ as i-th row.*

Example: Write the vector $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ in the basis

$$\mathcal{U} = \left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

In order to do so, we need to consider the matrix

$$U = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then to find $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ such that $\mathbf{v} = y_1\mathbf{u}_1 + y_2\mathbf{u}_2 + y_3\mathbf{u}_3$, we need to consider the change of basis matrix

$$U^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$\mathbf{y} = U^T\mathbf{v} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{3\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

and in fact

$$\frac{\sqrt{2}}{2} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} - \frac{3\sqrt{2}}{2} \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

The Gram-Schmidt algorithm

Having an orthonormal basis decreases computational complexity with respect to basis that are not orthonormal. The following algorithm "normalizes" any given basis by returning an orthonormal basis associated to it.

Let $\{\mathbf{b}_i\}_{i=1}^n$ be any given basis of $\mathbb{R}^n$. We want to find an orthonormal basis $\{\mathbf{u}_i\}_{i=1}^n$. The initialization step is given by normalizing the first vector, that is

$$\mathbf{u}_1 = \frac{\mathbf{b}_1}{||\mathbf{b}_1||}$$

Then for each $i = 2, ..., n$, we first create a vector that is orthogonal to all the previous vectors

$$\mathbf{u}'_i = \mathbf{b}_i - \sum_{j<i} \langle \mathbf{b}_i, \mathbf{u}_j \rangle \mathbf{u}_j$$

and then we normalize it

$$\mathbf{u}_i = \frac{\mathbf{u}'_i}{||\mathbf{u}'_i||}$$

Notice that if $\{\mathbf{u}_j\}_{j=1}^{i-1}$ are orthogonal, then for $k < i$

$$\langle \mathbf{u}'_i, \mathbf{u}_k \rangle = \langle \mathbf{b}_i, \mathbf{u}_k \rangle - \sum_{j<i} \langle \mathbf{b}_i, \mathbf{u}_j \rangle \langle \mathbf{u}_j, \mathbf{u}_k \rangle = \langle \mathbf{b}_i, \mathbf{u}_k \rangle - \langle \mathbf{b}_i, \mathbf{u}_k \rangle = 0$$

so that $\{\mathbf{u}_j\}_{j=1}^{i}$ are orthogonal. We repeat this process for $i = 1, ..., n$ until we find the orthogonal basis.

Example: consider the basis $\mathcal{B}$ given by

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Applying the algorithm gives us

$$\mathbf{u}_1 = \frac{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}{|| \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} ||} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

then we pass to the second element:

$$\mathbf{u}'_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \rangle \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

and we normalize it so that the norm is equal to 1:

$$\mathbf{u}_2 = \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{|| \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} ||} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

Finally, for the third component we need to calculate

$$\mathbf{u}'_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \langle \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \rangle \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} - \langle \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \rangle \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

and the last element is:

$$\mathbf{u}_3 = \frac{\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}}{|| \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} ||} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

so that the orthogonal basis obtained through the Gram-Schmidt algorithm is

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}.$$

Notice that the Gram-Schmidt algorithm can be applied to any set of independent vectors, even if they are not a basis. The output will still be given by orthonormal vectors (which, of course, do not constitute a basis).

Example: consider the vectors

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \qquad \mathbf{b}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

The Gram-Schmidt algorithm gives us

$$\mathbf{u}_1 = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}}{|| \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} ||} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$$

then we pass to the second element:

$$\mathbf{u}'_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \langle \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix} \rangle \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

and we normalize it so that the norm is equal to 1:

$$\mathbf{u}_2 = \frac{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}}{|| \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} ||} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}$$

Finally, for the third component we need to calculate

$$\mathbf{u}_3' = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} - \langle \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix} \rangle \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix} - \langle \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} \rangle \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ -1 \end{bmatrix}$$

and the last element is:

$$\mathbf{u}_3 = \frac{\begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ -1 \end{bmatrix}}{|| \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ -1 \end{bmatrix} ||} = \begin{bmatrix} -\frac{2}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ -\frac{1}{\sqrt{15}} \\ -\frac{3}{\sqrt{15}} \end{bmatrix}$$

so that the output of the algorithm is

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} -\frac{2}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ -\frac{1}{\sqrt{15}} \\ -\frac{3}{\sqrt{15}} \end{bmatrix}$$

which is a set of 3 independent vectors in $\mathbb{R}^4$.