Midterm solutions

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Problem 1

1. Parametric form:
$$\boldsymbol{r_1}(t) = (1+t, -t, 1)$$
Symmetric form:
$$\frac{x-1}{1} = \frac{y-0}{-1}, \quad z = 1$$

2. In order to find the parametric expression for this line we can consider the system
$$\begin{cases} x+y-z=2 \\ 2x+2y+z=3 \end{cases}$$
By multiplying the first equation by 2 and subtracting the second we get
$$\begin{cases} x+y-z=2 \\ \quad\quad -3z=1 \end{cases}$$
which gives us
$$\begin{cases} y=\frac{5}{3}-x \\ z=-\frac{1}{3} \end{cases}$$
If we use x as the parameter we can rewrite this as
$$_{r_2}(t) = (t, \frac{5}{3}-t, -\frac{1}{3})$$

3. We know that the plane contains the vector of $\boldsymbol{r_1}$, which is given by $\vec{v_1} = (1,-1,0)$. We need to find a second vector, for example by considering another point on $\boldsymbol{r_3}$, say $O = (0,0,0)$, which gives us $\vec{OP} = (1,0,1)$. Let's now find the orthonormal vector to the plane by considering the cross product of $\vec{v_1}$ and $\vec{OP}$
$$\vec{n} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (-1,-1,1)$$
Since the plane passes through the origin O, the equation of the plane is
$$\langle(-1,-1,1),(x,y,z)-(0,0,0)\rangle = 0$$
which is $-x-y+z=0$

4. In order to find the angle, we need to consider their scalar product, that is
$$\langle(1,1,1),(1,-1,0)\rangle = 0$$
therefore the two lines for a $\frac{\pi}{2}$ angle, that is, they are orthogonal. This line, however, is not orthogonal to the plane, as if we choose the vector $\vec{OP}$ lying on the plane we get
$$\langle(1,1,1),(1,0,1)\rangle = 2$$

Problem 2

Consider the function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = \sqrt{x^2 - y^2}$.

1. The domain D is given by
$$D = \{(x,y)|x^2 - y^2 \geq 0\}$$
that is, it corresponds to the quadrants on the right and on the left of the lines $x = \pm y$. Level curves are given by
$$L_k = \{(x,y)|x^2 - y^2 = k^2\}$$
which correspond to hyperbola for $k > 0$ and to lines for $k = 0$.

2. First evaluate the gradient in P, that is
$$\nabla f(1,0) = \left(\frac{x}{\sqrt{x^2-y^2}}, -\frac{y}{\sqrt{x^2-y^2}}\right)\Big|_{(x,y)=(1,0)} = (1,0)$$
Since f is differentiable in P, we know that the partial derivative in the direction of $\vec{v}$ is given by
$$\partial_v f(0,1) = \langle \nabla f(1,0), \vec{v} \rangle$$
which gives us the system of equations
$$\begin{cases} v_1 = \frac{1}{2} \\ v_1^2 + v_2^2 = 1 \end{cases}$$
solved by $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$. The highest possible value for the directional derivative is given by the norm of the gradient, which is 1, while the lowest possible value is given by the direction opposite to the gradient, which gives the value -1. In absolute value, the lowest possible value of the directional derivative is 0, in the direction orthogonal to the gradient.

3. f is not differentiable at the origin because the partial derivative in y does not exist, therefore it is not even derivable there.

4. By completing the square, we notice that we are asking to find the absolute maximum and minimum inside the circle centered in $(3,0)$ and of radius $\sqrt{19}$. A is not entirely contained in the domain, so we have to restrict to the intersection. Since $A \cap D$ is closed and bounded and f is continuous in such domain, there exist an absolute minimum and an absolute maximum. We notice that the absolute minimum can only be 0, since the function is always positive, therefore we have that the absolute minimum is on the boundary lines $A \cap \{(x,y)|x = \pm y\}$. Moreover, by the structure of the level curves, we notice that the absolute maximum has to be the furthest point for the origin, that is $(3 + \sqrt{19}, 0)$ where the function has value $\sqrt{28 + 6\sqrt{19}}$.

Problem 3

1. We need to set up a double integral in polar coordinates:

$$x = 1 + \rho\cos\theta, \quad y = 1 + \rho\sin\theta$$

for $\rho \in [0,1]$ and $\theta \in [0,2\pi]$. The integral becomes:

$$\int_0^{2\pi}\int_0^1 (30 - 3(1+\rho\cos\theta)^2 - (1+\rho\sin\theta)^2)\rho\,d\rho\,d\theta$$

whose result is 25π

2. We can parametrize this in spherical coordinates in the following way:

$$x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\phi$$

for $\rho \in [0,1]$,$\theta \in [0,2\pi]$ and $\phi \in [\frac{\pi}{4},\pi]$. The integral becomes

$$\int_0^{2\pi}\int_0^1\int_{\frac{\pi}{4}}^{\pi} \rho^2\sin\phi\,d\phi\,d\rho\,d\theta$$

which is equal to $\frac{2\pi}{3}(1-\frac{\sqrt{2}}{2})$.

3. In classical cylindrical coordinates this can be parametrized as

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad z = z$$

for $\theta \in [0,2\pi]$, $z \in [-1,\frac{\sqrt{2}}{2}]$ and $\rho \in [0,\sqrt{1-z^2}]$. The integral then becomes

$$\int_0^{2\pi}\int_{-1}^{\frac{\sqrt{2}}{2}}\int_0^{\sqrt{1-z^2}} \rho\,d\rho\,dz\,d\theta$$

whose result is $\frac{\pi}{12}(5\sqrt{2}+8)$.

4. In this case we notice that if we fix θ we have a section that can be parametrized in the following way:

$$\{0 \le \phi \le \frac{\pi}{4}, 0 \le \rho \le \frac{\sqrt{2}}{2\cos\phi}\} \cup \{\frac{\pi}{4} \le \phi\pi, 0 \le \rho \le 1\}$$

so that the integral can be set up in the following way:

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\sqrt{2}}{2\cos\phi}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\pi}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$