Homework 2 Solutions

July 2020

1 Gaussian elimination (30 points)

Use the Gaussian algorithm to find the reduced echelon form of the following matrices and state if they have one solution, infinite solutions or no solution:

$$\begin{cases} x_1 - 2x_2 - x_3 + 3x_4 = 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 = 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 = 2 \end{cases} \tag{1}$$

$$\begin{cases} 3x_1 - 4x_2 + 2x_3 = 0 \\ -9x_1 + 12x_2 - 6x_3 = 0 \\ -6x_1 + 8x_2 - 4x_3 = 0 \end{cases} \tag{2}$$

$$\begin{cases} x_1 - 4x_2 + x_3 = 0 \\ x_1 + x_2 - x_3 = 2 \\ 3x_1 - 2x_2 = 5 \end{cases} \tag{3}$$

1.1 Solutions:

1

The system can be rewritten as

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ -2 & 4 & 5 & -5 \\ 3 & -6 & -6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

so that the augmented matrix on which to apply the Gaussian elimination algorithm is

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix}$$

we choose 1 as the first pivot and proceed to eliminate everything below it:

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix}$$

Now we choose 3 as second pivot and eliminate everything below it:

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Since there is a row of the type $[\mathbf{0}|b]$ this system has no solutions.

We can still find the reduced echelon form of the matrix to be

$$\begin{bmatrix} 1 & -2 & 0 & \frac{8}{3} & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

2

The system can be rewritten as

$$\begin{bmatrix} 3 & -4 & 2 \\ -9 & 12 & -6 \\ -6 & 8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is a homogeneous system so it has at least one solution. The augmented matrix is

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$$

which can be simplified by dividing the second row by 3 and the third by 2:

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ -3 & 4 & -2 & 0 \\ -3 & 4 & -2 & 0 \end{bmatrix}$$

By choosing the first pivot to be 3, we reduce the matrix to the form

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

therefore the system has infinitely many solutions that can be written as

$$\begin{cases} x_1 = \frac{4x_2 - 2x_3}{3} \\ x_2 \text{ free} \\ x_3 \text{ free} \end{cases}$$

3

The system can be rewritten as

$$\begin{bmatrix} 1 & -4 & 1 \\ 1 & 1 & -1 \\ 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -4 & 1 & 0 \\ 1 & 1 & -1 & 2 \\ 3 & -2 & 0 & 5 \end{bmatrix}$$

We choose the first element to be the first pivot and eliminate everything below it

$$\begin{bmatrix} 1 & -4 & 1 & 0 \\ 0 & 5 & -2 & 2 \\ 0 & 10 & -3 & 5 \end{bmatrix}$$

Then we choose 5 to be the second pivot and eliminate everything below it to obtain the matrix in echelon form:

$$\begin{bmatrix} 1 & -4 & 1 & 0 \\ 0 & 5 & -2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Now we eliminate everything above the pivot to get

$$\begin{bmatrix} 1 & 0 & 0 & \frac{11}{5} \\ 0 & 1 & 0 & \frac{4}{5} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

therefore the system has a unique solution of the form

$$\begin{cases} x_1 = \frac{11}{5} \\ x_2 = \frac{4}{5} \\ x_3 = 1 \end{cases}$$

2 Determinants (20 points)

Consider the following matrices:

$$A = \begin{bmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 11 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 3 & 5 & 4 & 1 \\ -6 & 5 & 5 & 0 \end{bmatrix}.$$

Compute $\det(A), \det(B)$ and $\det(C)$. Then, using the properties, compute $\det(AB)$, $\det(B^3)$,$\det(B^T C)$ and $\det(C^{-1})$.

Solution

$$\det(A) = \begin{vmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{vmatrix} = 12 \begin{vmatrix} 1 & 1 & -3 & -1 \\ 1 & 0 & 1 & -3 \\ -2 & 0 & -4 & 3 \\ 2 & 2 & -4 & -1 \end{vmatrix} = 12 \begin{vmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & -3 \\ -2 & 0 & -4 & 3 \\ 0 & 2 & 0 & 1 \end{vmatrix} =$$

$$12(- \begin{vmatrix} 1 & -1 & 0 \\ 0 & -4 & 3 \\ 2 & 0 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -3 \\ 2 & 0 & 1 \end{vmatrix}) = 12(10 - 10) = 0$$

$$\det(B) = \begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 11 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix} = 6 \begin{vmatrix} -1 & 1 & 3 & 0 \\ 3 & 2 & 3 & 0 \\ 11 & 2 & 6 & 2 \\ 4 & 1 & 4 & 1 \end{vmatrix} = 6 \begin{vmatrix} -1 & 1 & 3 & 0 \\ 3 & 2 & 3 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{vmatrix} =$$

$$6(- \begin{vmatrix} 3 & 3 & 0 \\ 3 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} + 2 \begin{vmatrix} -1 & 3 & 0 \\ 3 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix}) = 6(15 - 14) = 6$$

$$\det(C) = \begin{vmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 3 & 5 & 4 & 1 \\ -6 & 5 & 5 & 0 \end{vmatrix} = -2 \begin{vmatrix} 1 & 5 & 4 & 1 \\ 0 & 1 & 2 & 0 \\ 3 & 5 & 4 & 1 \\ -6 & 5 & 5 & 0 \end{vmatrix} = -2 \begin{vmatrix} 1 & 5 & 4 & 1 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ -6 & 5 & 5 & 0 \end{vmatrix} =$$

$$-4 \begin{vmatrix} 5 & 4 & 1 \\ 1 & 2 & 0 \\ 5 & 5 & 0 \end{vmatrix} = 20$$

Finally, by applying the properties we have

$$\begin{aligned}
&\det(AB) = \det(A)\det(B) = 0 \\
&\det(B^3) = \det(B)^3 = 6^3 \\
&\det(B^T C) = \det(B^T)\det(C) = \det(B)\det(C) = 120 \\
&\det(C^{-1}) = \det(C)^{-1} = \frac{1}{20}
\end{aligned}$$

3 Cramer's rule (10 points)

Prove that the following square system of equations has a unique solution and find it using Cramer's rule:

$$\begin{cases} x_1 + 4x_2 - x_3 = 1 \\ x_1 - 2x_3 = 3 \\ -2x_1 + 3x_2 - 2x_3 = -1 \end{cases}$$

Solution

The coefficient matrix is given by

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 0 & -2 \\ -2 & 3 & -2 \end{bmatrix}$$

whose determinant is

$$\det(A) = \begin{vmatrix} 1 & 4 & -1 \\ 1 & 0 & -2 \\ -2 & 3 & -2 \end{vmatrix} = 27$$

since the determinant is different from 0, the system admits a unique solution. This can be computed using Cramer's rule as

$$x_1 = \frac{\begin{vmatrix} 1 & 4 & -1 \\ 3 & 0 & -2 \\ -1 & 3 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & -1 \\ 1 & 0 & -2 \\ -2 & 3 & -2 \end{vmatrix}} = \frac{29}{27}$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 3 & -2 \\ -2 & -1 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & -1 \\ 1 & 0 & -2 \\ -2 & 3 & -2 \end{vmatrix}} = -\frac{7}{27}$$

$$x_3 = \frac{\begin{vmatrix} 1 & 4 & 1 \\ 1 & 0 & 3 \\ -2 & 3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & -1 \\ 1 & 0 & -2 \\ -2 & 3 & -2 \end{vmatrix}} = -\frac{26}{27}$$

4 Complexity of determinants (20 points)

1. Consider the system if linear equation $A\mathbf{x} = \mathbf{b}$, for A a $n \times n$ matrix and $\mathbf{b}$ a $n \times 1$ column vector. Let A' be the augmented matrix. Prove that the complexity of the computation to transform A' in echelon form is bounded above by n^3 (we say that it's $O(2n^3)$).

2. Now let B be a square $n \times n$ matrix. Prove that the computational complexity required to calculate the determinant through a co-factor expansion is of the order of $n!$.

3. Can you find a computation of the determinant having lower complexity?

Solution

1. In order to pass from A' to the echelon form of A', for each line $k = 1, ..., n$, one has to do $n-k$ reductions. For each reduction there are $2(n-k)$ operations. Therefore the computation has flops given by

$$\sum_{k=1}^{n} 2(n-k)^2 = \sum_{k=1}^{n} 2n^2 \leq n \cdot 2n^2 = 2n^3$$

2. For a co-factor expansion of a determinant we have that after choosing a row or column

$$\det(B) = \sum_{k=1}^{n} B_{i,k} C_{i,k}$$

since each cofactor $C_{i,k}$ is a determinant of a $n-1$ matrix, we have to compute n determinants of a $n-1$ matrix and then consider a scalar product of them with the i-th row or column. That is

$$\text{flops(det(n)))} = n(\text{flops(det(n-1)))} + 2n - 1$$

By developing the formula, we get that flops(det(n))) are on the order of $n!$.

3. A square matrix in echelon form is always upper triangular, so we can first reduce the square matrix in echelon form and then calculate the determinant by multiplying the elements on the diagonal. Since this last operation has complexity n, the leading term is still n^3, which is less than $n!$.

5 Matrix inverses (20 points)

Consider the square matrix

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 0 \\ 1 & -3 & -1 \end{bmatrix}$$

1. Find the inverse of A using the general formula.

2. Find the inverse of A using the Inverse algorithm.

3. For a general $n \times n$ matrix, which of the two computations has the smallest computational complexity?

Solution

1

$$
\begin{aligned}
&C_{1,1} = -2 \qquad && C_{1,2} = -1 \qquad && C_{1,3} = 1 \\
&C_{2,1} = -12 && C_{2,2} = -5 && C_{2,3} = 3 \\
&C_{3,1} = -8 && C_{3,2} = -4 && C_{3,3} = 2
\end{aligned}
$$

Then we have $\det(A) = 2$, so that

$$
A^{-1} = \begin{bmatrix} -1 & -6 & -4 \\ -\frac{1}{2} & -\frac{5}{2} & -2 \\ \frac{1}{2} & \frac{3}{2} & 1 \end{bmatrix}
$$

2

Consider the matrix

$$
[A|I] = \begin{bmatrix} 1 & 0 & 4 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 1 & -3 & -1 & 0 & 0 & 1 \end{bmatrix}
$$

We apply the algorithm to row reduce the desired matrix, starting from the pivot in position $A_{1,1}$:

$$
\begin{bmatrix} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 & 1 & 0 \\ 0 & -3 & -5 & -1 & 0 & 1 \end{bmatrix}
$$

then we proceed by dividing the second row by 2 and row=reducing the third:

$$
\begin{bmatrix} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{2} & 1 \end{bmatrix}
$$

Finally, we make sure that the first three column are equal to the identity, to get

$$
\begin{bmatrix} 1 & 0 & 0 & -1 & -6 & -4 \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{5}{2} & -2 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{2} & 1 \end{bmatrix}
$$

which has the form $[I|A^{-1}]$.

3

From the previous exercise, row reduction takes about $O(n^3)$, while calculation of cofactor expansion takes $n(n-1)! = n!$, so computationally the second solution is more efficient.