Homework 2

July 2020

1 Gaussian elimination (30 points)

Use the Gaussian algorithm to find the reduced echelon form of the following matrices and state if they have one solution, infinite solutions or no solution. In case they have solutions, write them explicitly:

$$\begin{cases} x_1 - 2x_2 - x_3 + 3x_4 = 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 = 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 = 2 \end{cases} \tag{1}$$

$$\begin{cases} 3x_1 - 4x_2 + 2x_3 = 0 \\ -9x_1 + 12x_2 - 6x_3 = 0 \\ -6x_1 + 8x_2 - 4x_3 = 0 \end{cases} \tag{2}$$

$$\begin{cases} x_1 - 4x_2 + x_3 = 0 \\ x_1 + x_2 - x_3 = 2 \\ 3x_1 - 2x_2 = 5 \end{cases} \tag{3}$$

2 Determinants (25 points)

Consider the following matrices:

$$A = \begin{bmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 11 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 3 & 5 & 4 & 1 \\ -6 & 5 & 5 & 0 \end{bmatrix}.$$

Compute $\det(A)$, $\det(B)$ and $\det(C)$. Then, using the properties, compute $\det(AB)$, $\det(B^3)$, $\det(B^T C)$ and $\det(C^{-1})$.

3 Cramer's rule (15 points)

Prove that the following square system of equations has a unique solution and find it using Cramer's rule:

$$\begin{cases} x_1 + 4x_2 - x_3 = 1 \\ x_1 - 2x_3 = 3 \\ -2x_1 + 3x_2 - 2x_3 = -1 \end{cases}$$

4 Matrix inverses (25 points)

Consider the square matrix

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 0 \\ 1 & -3 & -1 \end{bmatrix}$$

1. Find the inverse of A using the general formula.
2. Find the inverse of A using the Inverse Matrix algorithm.

5 Complexity of determinants (10 points)

1. Consider the system if linear equation $A\mathbf{x} = \mathbf{b}$, for A a $n \times n$ matrix and $\mathbf{b}$ a $n \times 1$ column vector. Let A' be the augmented matrix. Prove that the complexity of the computation to transform A' in echelon form is bounded above by $2n^3$ (we say that it's $O(n^3)$).
2. Now let B be a square $n \times n$ matrix. Prove that the computational complexity required to calculate the determinant through a co-factor expansion is of the order of $n!$.
3. Can you find a computation of the determinant having lower complexity than the co-factor expansion?