Midterm Solutions

COMS W3251, Summer Session 2

Due: Friday July 24th 2020, 4:59 PM EST

Instruction:

- You have 24 hours to complete this exam. You can use any time within the 24 hours, although the exam shouldn't take more than 3 hours.
- You may consult any non-living resources while you take the exam.
- Show full justification for every part. While you are allowed to use programs, we are grading each problem as if done by hand. Answers without justification will not be graded.
- You can either use the LaTeXtemplate given with the exam or write the answers on your own paper with each question/part clearly labeled.
- For questions and clarifications, send a private e-mail to the instructor (fp2428@columbia.edu) cc'ing the grader (ryan.chen@columbia.edu). We will try to answer as soon as possible.

Exercise 1 (20 points)

For a fixed vector $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3$, we define the cross product as

$$\mathbf{a} \times \mathbf{x} = \begin{bmatrix} a_2x_3 - a_3x_2 \\ a_3x_1 - a_1x_3 \\ a_1x_2 - a_2x_1 \end{bmatrix}$$

1. Prove that the cross product is linear and find a matrix A such that $A\mathbf{x} = \mathbf{a} \times \mathbf{x}$.
2. Consider the matrix A for $\mathbf{a} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and find a basis for its Kernel. What is its dimension?
3. What is the dimension of $\mathrm{Im}(A)$? Find a basis for it.

Solutions:

1. First we prove that it's linear. For every $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$

$$\begin{aligned}
\mathbf{a} \times (\alpha\mathbf{x} + \beta\mathbf{y}) &= \begin{bmatrix} a_2(\alpha x_3 + \beta y_3) - a_3(\alpha x_2 + \beta y_2) \\ a_3(\alpha x_1 + \beta y_1) - a_1(\alpha x_3 + \beta y_3) \\ a_1(\alpha x_2 + \beta y_2) - a_2(\alpha x_1 + \beta y_1) \end{bmatrix} \\
&= \alpha \begin{bmatrix} a_2x_3 - a_3x_2 \\ a_3x_1 - a_1x_3 \\ a_1x_2 - a_2x_1 \end{bmatrix} + \beta \begin{bmatrix} a_2y_3 - a_3y_2 \\ a_3y_1 - a_1y_3 \\ a_1y_2 - a_2y_1 \end{bmatrix} \\
&= \alpha(\mathbf{a} \times \mathbf{x}) + \beta(\mathbf{a} \times \mathbf{y})
\end{aligned}$$

so that the cross product is linear. The representation matrix is given by:

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

2. In this case, the representation becomes:

$$A = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

Then $\mathrm{Ker}(A)$ is given by the solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$. We can solve it by Gaussian elimination on the augmented matrix

$$A' = \begin{bmatrix} 0 & -2 & -1 & 0 \\ 2 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

which has echelon form

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and reduced echelon form

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the kernel is given by vectors in $\mathbb{R}^3$ of the type

$$\begin{cases} x_1 = \frac{x_3}{2} \\ x_2 = -\frac{x_3}{2} \\ x_3 \text{ free} \end{cases}$$

that can be rewritten as

$$\left\{ \begin{bmatrix} \alpha \\ -\alpha \\ 2\alpha \end{bmatrix} \mid \alpha \in \mathbb{R} \right\} = \mathrm{span}\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right) = \mathrm{span}(\mathbf{a})$$

3. By the rank-kernel theorem, $\dim(\mathrm{Im}(A)) = 3 - \dim(\mathrm{Ker}(A)) = 2$. It is generated by the pivot columns of A, so

$$\mathrm{Im}(A) = \mathrm{span}(\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix})$$

Exercise 2 (20 points)

1. Write the transformation T of $\mathbb{R}^2$ obtained first by a clock-wise rotation of $\frac{\pi}{3}$, then a dilatation of 2 along the x-axis.
2. Write the transformation S of $\mathbb{R}^2$ obtained first by a dilatation of 2 along the x-axis and then by a clock-wise rotation of $\frac{\pi}{3}$.
3. Are the two transformations the same? In case they are, find the inverse transformation. In case they are not, find the transformation Q such that $T = QS$. In case they are, find the inverse.

Solutions:

1. A clock-wise rotation by $\frac{\pi}{3}$ is given by

$$R = \begin{bmatrix} \cos(\frac{\pi}{3}) & \sin(\frac{\pi}{3}) \\ -\sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

while a dilatation along the x-axis is given by

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Applying the rotation first and then the dilatation yields:

$$T = DR = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{3} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

2. Applying the dilatation first and rotation afterwards yields:

$$S = RD = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ -\sqrt{3} & \frac{1}{2} \end{bmatrix}$$

3. The two transformations are not the same. To find the transformation Q such that $T = QS$, consider $Q = TS^{-1}$. Then

$$S^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Then we have

$$Q = TS^{-1} = \begin{bmatrix} 1 & \sqrt{3} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{7}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{8} & \frac{5}{8} \end{bmatrix}$$

Exercise 3 (20 points)

Consider the system

$$\begin{cases} 3x_1 + 2x_2 - x_3 = 1 \\ -2x_1 + \alpha x_2 + x_3 = 2 \\ x_1 + x_2 = 3 \end{cases}$$

1. Find the values of α for which this system has a unique solution, then use Cramer's rule to find the explicit solution as a function of α.
2. For all values of α for which the system doesn't have a unique solution, use Gaussian elimination to find out if the system has infinite solutions or no solution. If the system has infinite solutions, find an expression for them.

Solutions:

1. Consider the determinant:

$$\det(A) = \begin{vmatrix} 3 & 2 & -1 \\ -2 & \alpha & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1 + \alpha$$

For $\alpha \neq -1$ the system has only one solution. Then

$$x_1 = \frac{\begin{vmatrix} 1 & 2 & -1 \\ 2 & \alpha & 1 \\ 3 & 1 & 0 \end{vmatrix}}{1+\alpha} = \frac{3+3\alpha}{1+\alpha} = 3$$

$$x_2 = \frac{\begin{vmatrix} 3 & 1 & -1 \\ -2 & 2 & 1 \\ 1 & 3 & 0 \end{vmatrix}}{1+\alpha} = \frac{0}{1+\alpha} = 0$$

$$x_3 = \frac{\begin{vmatrix} 3 & 2 & 1 \\ -2 & \alpha & 2 \\ 1 & 1 & 3 \end{vmatrix}}{1+\alpha} = \frac{8\alpha+8}{1+\alpha} = 8$$

2. For $\alpha = -1$, consider the augmented matrix

$$A' = \begin{bmatrix} 3 & 2 & -1 & 1 \\ -2 & -1 & 1 & 2 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

we can apply the gaussian algorithm to obtain the echelon form

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the reduced echelon form

$$\begin{bmatrix} 1 & 0 & -1 & -5 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the system has solutions:

$$\begin{cases} x_1 = x_3 - 5 \\ x_2 = 8 - x_3 \\ x_3 \text{ free} \end{cases}$$

Exercise 4 (20 points)

Consider the system

$$\begin{cases} 4x_1 - 2x_2 + x_3 = 1 \\ x_1 + x_2 + x_3 = 0 \\ 3x_1 - 3x_2 = 2 \\ -x_1 - x_2 + x_3 = -1 \end{cases}$$

Use Rouché-Capelli's theorem to find whether the system has one solution, infinite solutions or no solution. In case it has infinite solutions, state how many variables are free.

Solutions:

We need to calculate the rank of the coefficient matrix

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ 3 & -3 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

We have that

$$\begin{vmatrix} 4 & -2 \\ 1 & 1 \end{vmatrix} = 6 \neq 0$$

so the rank is at least 2. Then we have

$$\begin{vmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ 3 & -3 & 0 \end{vmatrix} = 0$$

but

$$\begin{vmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix} = 12 \neq 0$$

therefore $\text{Rank}(A) = 3$. Now we need to find the rank of the augmented matrix A'. This is at least 3 because A is a submatrix of A'. We need to calculate the determinant

$$\det(A') = \begin{vmatrix} 4 & -2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 3 & -3 & 0 & 2 \\ -1 & -1 & 1 & -1 \end{vmatrix}$$

We can replace the third row with itself plus the second row minus the first row:

$$\begin{vmatrix} 4 & -2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & -1 \end{vmatrix}$$

and develop along the third row to get

$$\det(A') = -\begin{vmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix} = -12 \neq 0$$

Therefore $\text{Rank}(A') > \text{Rank}(A)$ and the system has no solutions.

Exercise 5 (20 points)

See the jupyter notebook attachment.