Problem 1

Let γ be the curve obtained by intersecting the cylinder $x^2 + z^2 = 1$ with the plane $x = -y$.

1. Find a parametric representation of the curve.
   Solution:
   Since the curve lies on the cylinder we can consider the parametrisation $x(t) = \cos(t), z(t) = \sin(t)$ for $t \in [0, 2\pi]$. By the relation on the plane we get $y(t) = -x(t) = -\cos(t)$. Therefore the final representation is $\gamma(t) = (\cos(t), -\cos(t), \sin(t))$, for $t \in [0, 2\pi]$.

2. Show that the curve lies on the ellipsoid $x^2 + y^2 + 2z^2 = 2$.
   Solution: We only need to show that the coordinates satisfy this equation for each t. That is
   $$x(t)^2 + y(t)^2 + 2z(t)^2 = \cos^2(t) + (-\cos(t))^2 + 2\sin^2(t) = 2$$
   for each $t \in [0, 2\pi]$.

3. Find the *unit* tangent vector of γ at each point.
   Solution:
   $$\gamma'(t) = (-\sin(t), \sin(t), \cos(t)) \quad t \in [0, 2\pi]$$
   $||\gamma'(t)|| = \sqrt{2\sin^2(t) + \cos^2(t)} = \sqrt{1 + \sin^2(t)}$ so that the unit tangent vector becomes
   $$T(t) = \frac{1}{\sqrt{1 + \sin^2(t)}}(-\sin(t), \sin(t), \cos(t)) \quad t \in [0, 2\pi]$$

4. Write an expression for its arclength. The arclength is given by
   $$\int_0^{2\pi} \sqrt{1 + \sin^2(t)}\, dt$$

Problem 2

For each of the following surfaces, write a parametrization as a surface of revolution, express its grid curves and find its unit normal vector.

1. $x^2 - y^2 + z^2 = 0$.
   Solution:
   $$s(u, v) = (u\cos(v), u, u\sin(v)) \quad u \in \mathbb{R},\ v \in [0, 2\pi]$$
   u-curves are lines passing through the origin with angle given by v_0, v-curves are circles of height u_0 on the y-axis.
   $$s_u(u, v) = (\cos(v), 1, \sin(v))$$

$$s_v(u,v) = (-u\sin(v), 0, u\cos(v))$$

so that

$$s_u \times s_v = (u\cos(v), -u, u\sin(v))$$

and the unit normal vector is given by

$$\vec{n}(u,v) = \frac{1}{|u|\sqrt{2}}(u\cos(v), -u, u\sin(v))$$

2. $x^2 + y^2 - z^2 = 1$. *Solution:*

   We consider a branch of hyperbola $x = f(z) = \sqrt{z^2+1}$. Therefore we have

   $$s(u,v) = (\sqrt{u^2+1}\cos(v), \sqrt{u^2+1}\sin(v), u) \quad u \in \mathbb{R},\ v \in [0,2\pi]$$

   u-curves are one branch of hyperbola on a plane passing through the z-axis with angle given by v_0, v-curves are circles of height $\sqrt{u_0^2+1}$ on the y-axis.

   $$s_u(u,v) = (\frac{u}{\sqrt{u^2+1}}\cos(v), \frac{u}{\sqrt{u^2+1}}\sin(v), 1)$$

   $$s_v(u,v) = (-\sqrt{u^2+1}\sin(v), \sqrt{u^2+1}\cos(v), 0)$$

   so that

   $$s_u \times s_v = (-\sqrt{u^2+1}\cos(v), -\sqrt{u^2+1}\sin(v), u)$$

   and the unit normal vector is given by

   $$\vec{n}(u,v) = \frac{1}{\sqrt{2u^2+1}}(-\sqrt{u^2+1}\cos(v), -\sqrt{u^2+1}\sin(v), u)$$

3. $z^2 - y^2 - x^2 = 1$.
   Solution:
   We consider a branch of hyperbola $x = f(z) = \sqrt{z^2-1}$. Therefore we have

   $$s(u,v) = (\sqrt{u^2-1}\cos(v), \sqrt{u^2-1}\sin(v), u) \quad u \in (-\infty,-1] \cup [1,+\infty),\ v \in [0,2\pi]$$

   u-curves are two half-branches of hyperbola on a plane passing through the z-axis with angle given by v_0, v-curves are circles of height u on the z-axis.

   $$s_u(u,v) = (\frac{u}{\sqrt{u^2-1}}\cos(v), \frac{u}{\sqrt{u^2-1}}\sin(v), 1)$$

   $$s_v(u,v) = (-\sqrt{u^2-1}\sin(v), \sqrt{u^2-1}\cos(v), 0)$$

   so that

   $$s_u \times s_v = (-\sqrt{u^2-1}\cos(v), -\sqrt{u^2-1}\sin(v), u)$$

   and the unit normal vector is given by

   $$\vec{n}(u,v) = \frac{1}{\sqrt{2u^2-1}}(-\sqrt{u^2-1}\cos(v), -\sqrt{u^2-1}\sin(v), u)$$

4. $x = z^2 + y^2$.
Solution: This is a circular paraboloid around the $x-$ axis. One possible parametrization is given by

$$s(u, v) = (u, \sqrt{u}\cos(v), \sqrt{u}\sin(v)) \quad u \in [0, +\infty)\, v \in [0, 2\pi]$$

or equivalently, to make it more regular

$$s(u, v) = (u^2, u\cos(v), u\sin(v)) \quad u \in [0, +\infty)\, v \in [0, 2\pi]$$

In both cases, u-curves are half-parabolas on planes through the x-axis with angle given by v_0, while v-curves are circle of height $\sqrt{x(u_0)}$. Let's use the second parametrisation which is more regular:

$$s_u(u, v) = (2u, \cos(v), \sin(v))$$
$$s_v(u, v) = (0, -u\sin(v), u\cos(v))$$

so that

$$s_u \times s_v = (u, -2u^2\cos(v), -2u^2\sin(v))$$

and the unit normal vector is given by

$$\vec{n}(u, v) = \frac{1}{\sqrt{1 + 4u^2}}(1, -2u\cos(v), -2u\sin(v))$$

Problem 3

Let S be the surface obtained by intersecting the cone S_1 given by the equation $z = 2 - \sqrt{x^2 + y^2}$ with the paraboloid S_2 of equation $z = x^2 + y^2$. Evaluate the surface area of S.
Solution:
We notice that the two surfaces intersect on the circle $\gamma(t) = (\cos(t), \sin(t), 1)$, so we can parametrize the two surfaces in the following way:

$$s_1(u, v) = (u\cos(v), u\sin(v), 2 - u) \quad u \in [1, 2]\, v \in [0, 2\pi]$$

$$s_2(u, v) = (u\cos(v), u\sin(v), u^2) \quad u \in [0, 1], v \in [0, 2\pi]$$

Now we calculate the normals:

$$\vec{n_1} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos(v) & \sin(v) & -1 \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix} = (u\cos(v), u\sin(v), u)$$

whose norm is $\sqrt{2u^2}$. Since we are integrating u in a positive domain, we have

$$A(S_1) = \int_0^1 \int_0^{2\pi} u\sqrt{2}\, dv\, du = \pi\sqrt{2}$$

On the other hand we have

$$\vec{n_1} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos(v) & \sin(v) & 2u \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix} = (-2u^2\cos(v), -2u^2\sin(v), u)$$

whose norm is $\sqrt{4u^4+u^2}$. Since we are integrating u in a positive domain, we have

$$A(S_2) = \int_1^2 \int_0^{2\pi} u\sqrt{4u^2+1}\, dv\, du = \frac{5\sqrt{5}-1}{6}\pi$$

where the last integral is solved by substitution $t = 4u^2$. The total area is therefore $A = \frac{5\sqrt{5}+6\sqrt{2}-1}{6}\pi$

Problem 4

Let $\boldsymbol{F}(x,y) = (x^2-y)\boldsymbol{i} + (x+y^2)\boldsymbol{j}$.

1. Compute the line integral
$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{r}$$
for γ the segment from $(0,-1)$ to $(0,1)$.
Solution:
We parametrize the segment as $\gamma(t) = (0, 2t-1)$, for $t \in [0,1]$. The line integral becomes:
$$\int_0^1 2(2t-1)^2\, dt = [\frac{(2t-1)^3}{3}]|_0^1 = 2$$

2. Compute the same integral for γ the unit semicircle for $x \geq 0$ from $(0,-1)$ to $(0,1)$.
Solution:
Now we parametrize the curve as $c(t) = (\sin(t), -\cos(t))$ for $t \in [0,\pi]$, so that $\gamma'(t) = (\cos(t), \sin(t))$. In this way the integral becomes
$$\int_0^\pi [(\sin^2(t)+\cos(t))\cos(t) + (\sin(t)+\cos^2(t))\sin(t)]\, dt = \pi + \frac{2}{3}$$

3. Is $\boldsymbol{F}$ conservative? Explain.
Solution:
F cannot be conservative or the integral would only depend on the endpoints. In this case two different curves with the same endpoints give two different values for the integral.

Problem 5

Consider the integral given by

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{r}$$

where $\boldsymbol{F}(x,y) = \frac{x}{x^2+y^2}\boldsymbol{i} + y\frac{1-x^2-y^2}{x^2+y^2}\boldsymbol{j}$ and γ is a curve in the domain of F.

1. What is the domain of $\boldsymbol{F}$? Is it simply connected?
   Solution:
   The domain is $\mathbb{R}^2$ minus the origin. It is not simply connected because for example the unit circle contains the origin, which is not in the domain.

2. Is $\boldsymbol{F}$ conservative? If so, what is its potential?
   Solution:
   We know that the partial derivative in x is given by $\frac{x}{x^2+y^2}$, so that by integrating we get $f(x,y) = \frac{1}{2}\ln(x^2+y^2) + C(y)$. By deriving this expression in y we get $C'(y) + \frac{y}{x^2+y^2} = y\frac{1-x^2-y^2}{x^2+y^2}$, which gives us $C'(y) = -\frac{y^3+x^2y}{x^2+y^2} = y$, which is solved for $C(y) = -\frac{y^2}{2}$. The potential is therefore $f(x,y) = \frac{1}{2}\ln(x^2+y^2) - \frac{y^2}{2}$.

3. Evaluate the integral for γ the circle of radius 1 in $\mathbb{R}^2$.
   Solution:
   By the fundamental theorem of calculus for line integrals, since F is conservative we have that the integral on a closed curve is equal to 0.

4. Evaluate the integral when γ is the parabola $y = 1 - x^2$ starting at $(-1,0)$ and ending at $(1,0)$.
   Solution:
   By the fundamental theorem of calculus for line integrals the value of the integral is equal to the difference between the values of the potential at the boundary, that is

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{r} = f(1,0) - f(-1,0) = 0$$