Determinants and Matrix inverse

Francesco Preta
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Inverse matrix to solve systems of linear equations

Consider the matrix form of a system of linear equations $A\mathbf{x} = \mathbf{b}$.

If there exists a matrix B such that $BA = I$, then a solution of the system could be obtained by multiplying both sides of the equation by B, that is:

$$\mathbf{x} = B\mathbf{b}$$

Such a B is called a **left inverse** of A.

Since we have seen that in some cases systems don't have solutions, not all A will have left inverses.

This lecture: square matrices

In this lecture we will deal with inverses of square matrices. This is a specific case, since for square matrices left and right inverses coincide.

We will give a necessary and sufficient condition for a square matrix to have an inverse along with methods to calculate it. In order to do so, we will have to introduce determinants and their properties.

The 2×2 case

We start by trying to find a left inverse B for a 2×2 matrix A. This process can be done by solving a system of linear equations $BA = I$, which can be written as

$$\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Augmented matrix

$$A' = \begin{bmatrix} a_{1,1} & a_{2,1} & 0 & 0 & 1 \\ a_{1,2} & a_{2,2} & 0 & 0 & 0 \\ 0 & 0 & a_{1,1} & a_{2,1} & 0 \\ 0 & 0 & a_{1,2} & a_{2,2} & 1 \end{bmatrix}$$

Case 1: $a_{1,1}a_{2,2} - a_{2,1}a_{1,2} \neq 0$

Inverse matrix

$$B = \frac{1}{a_{1,1}a_{2,2} - a_{2,1}a_{1,2}} \begin{bmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{bmatrix}$$

Case 2: $a_{1,1}a_{2,2} - a_{2,1}a_{1,2} = 0$

Existence of inverse

We have proved the following:

Theorem

A 2×2 matrix A admits an inverse if and only if $a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \neq 0$

The quantity $a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$ is called **determinant** of a 2×2 matrix.

Example:

$$\begin{cases} 3x_1 - x_2 = 4 \\ x_1 + x_2 = 0 \end{cases}$$

General case: $n \times n$ matrix

We want to prove a similar result for square matrix of any dimension. In order to do so, we need to introduce the general definition of **determinants.**

Determinants

Definition: Determinant

Let A be a $n \times n$ matrix with generic element $A_{i,j}$. If $n = 2$, then the **determinant** of A is defined as

$$\det(A) = A_{1,1}A_{2,2} - A_{1,2}A_{2,1}$$

if $n > 2$, let $\begin{bmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,n} \end{bmatrix}$ be the first row of A. Then

$$\det(A) = \sum_{j=1}^{n} A_{1,j}C_{1,j}$$

where $C_{i,j}$ is called the **co-factor** of the element $A_{i,j}$

Definition: co-factor

For a square matrix A, the **co-factor** $C_{i,j}$ of an element $A_{i,j}$ is obtained by the formula

$$C_{i,j} = (-1)^{i+j} \det(A^{i,j})$$

where $A^{i,j}$ is the square submatrix of A obtained by eliminating its i-th row and j-th column.

Equivalence of expansions

In fact, the choice of the row or the column is not important, since we have the following:

Theorem

Let A be a $n \times n$ matrix. Then for any row $\begin{bmatrix} A_{i,1} & A_{i,2} & \dots & A_{i,n} \end{bmatrix}$ we have

$$\det(A) = \sum_{j=1}^{n} A_{i,j} C_{i,j}$$

and for any column $\begin{bmatrix} A_{1,j} \\ A_{2,j} \\ \dots \\ A_{n,j} \end{bmatrix}$ we have $\det(A) = \sum_{i=1}^{n} A_{i,j} C_{i,j}$.

Example

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 1 \\ -1 & 0 & -2 \end{bmatrix}$$

Determinant of triangular matrices

If A is upper (lower) triangular, we can develop the determinant on the first column (row) recursively, to get the following:

Theorem

If A is a triangular $n \times n$ matrix, $\det(A)$ is the product of its diagonal elements

Example

$$A = \begin{bmatrix} 1 & 2 & 100 & -\pi \\ 0 & -1 & 299 & 9 \\ 0 & 0 & 2 & 0.1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Properties:

1. $\det(AB) = \det(A)\det(B)$
2. $\det(A) = \det(A^T)$
3. If B is obtained by interchanging two rows (or columns) of A, then $\det(B) = -\det(A)$
4. If B is obtained by multiplying a row (or column) of A by k, then $\det(B) = k\det(A)$. In particular,$\det(kA) = k^n\det(A)$.
5. If two rows (or column) of A are multiple of each other, then $\det(A) = 0$. The same is true if A has one row (or column) of all zeros.
6. If $\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b}$ are $n \times 1$ column vectors, then

$$\begin{vmatrix}\mathbf{a}_1 & \ldots & \mathbf{a}_i + \mathbf{b} & \ldots & \mathbf{a}_n\end{vmatrix} = \begin{vmatrix}\mathbf{a}_1 & \ldots & \mathbf{a}_i & \ldots & \mathbf{a}_n\end{vmatrix} + \begin{vmatrix}\mathbf{a}_1 & \ldots & \mathbf{b} & \ldots & \mathbf{a}_n\end{vmatrix}$$

and the same is true for rows.

Simplifying calculation of determinant

Theorem

Let A, B be two square $n \times n$ matrices. If B is obtained by substituting a row (or column) of A with the sum of itself and a multiple of another row, then $\det(B) = \det(A)$.

Example

Compute the determinant of

$$A = \begin{bmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{bmatrix}$$

Sarrus’ rule

Geometric interpretation

if A is a linear transformation of $\mathbb{R}^2$ we have for every parallelogram P in $\mathbb{R}^2$

Area$(A(P)) = |\det(A)|$Area(P).

For example, consider the action on the unit cube $[0,1] \times [0,1]$ of the dilatation

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Solving systems of linear equations through determinants

Definition

Let A be a $n \times n$ matrix and $\mathbf{b}$ a $n \times 1$ column vector. We call $A_j(\mathbf{b})$ the matrix obtained substituting the j-th column of A with $\mathbf{b}$.

Example:

$A = I$ and $\mathbf{x}$ be the column vector of x_i

Cramer's rule

Theorem

Consider a square system $A\mathbf{x} = \mathbf{b}$. If $\det(A) \neq 0$, then

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)}$$

Proof

Example:

$$\begin{cases} 3x_1 + 2x_2 + 4x_3 = 1 \\ -x_1 + x_2 = 3 \\ x_1 + x_2 - 2x_3 = 0 \end{cases}$$

Inverse square matrices

Definition

Let A be a $n \times n$ matrix. We say that A is **invertible** if there exists a matrix A^{-1} called **inverse matrix** such that

$$AA^{-1} = A^{-1}A = I$$

where I is the $n \times n$ identity matrix.

Properties

For square matrices we have the following properties:

1. Left and right inverse coincide.
2. The inverse is unique.
3. $(AB)^{-1} = B^{-1}A^{-1}$.
4. $(A^T)^{-1} = (A^{-1})^T$.
5. $\det(A^{-1}) = \det(A)^{-1}$.

Condition on invertibility

Theorem

Let A be a square $n \times n$ matrix. Then A is invertible if and only if $\det(A) \neq 0$ and its form is given by

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{1,1} & C_{2,1} & \cdots & C_{n,1} \\ C_{1,2} & C_{2,2} & \cdots & C_{n,2} \\ \cdots & \cdots & \cdots & \cdots \\ C_{1,n} & C_{2,n} & \cdots & C_{n,n} \end{bmatrix}$$

Proof (using Cramer's rule):

Example:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

Inverse Matrix Algorithm

Consider an augmented matrix $A' = [A \,|\, I]$ where I is the $n \times n$ identity matrix. Then find the reduced echelon form of A', A'_{ref}. If $\det(A) \neq 0$, A'_{ref} has the form $[I \,|\, A^{-1}]$ and so it will be sufficient to use the last n columns of A'_{ref} to obtain the inverse.

Example:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

To summarize:

Theorem

Consider the square system $A\mathbf{x} = \mathbf{b}$. The system has a unique solution if and only if $\det(A) \neq 0$ and the solution is given by $\mathbf{x} = A^{-1}\mathbf{b}$.

Future questions:

1. What about square matrices when $\det(A) = 0$?
2. Can we use matrix inverses to solve rectangular systems?