1 Chapter 4.7 Variation of Parameters

1.1 Number 2

The problem statement is the following:

$$y'' + y = \tan(x) \tag{1}$$

Now one can refer to the auxiliary equation to get the associated homogeneous:

$$\begin{aligned} m^2 + 1 &= 0 \\ m^2 &= -1 \\ m &= \pm i \end{aligned}$$

From here we get the homogeneous part of our solution which is the following:

$$y_h = c_1 \cos(x) + c_2 \sin(x) \tag{2}$$

Now we can set up the Wronskian for the problem, based on the homogeneous part and nonhomogeneous part:

$$\begin{aligned} W &= \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} \\ &= \cos^2(x) + \sin^2(x) \\ &= 1 \end{aligned}$$

Unlike in problem 1, this one will be solved using the Wronskians:

$$\begin{aligned} u_1'(x) &= \frac{W_1}{W} = \frac{\begin{vmatrix} 0 & \sin(x) \\ \tan(x) & \cos(x) \end{vmatrix}}{1} \\ &= -\tan(x)\sin(x) \end{aligned}$$

Now integrating $u_1'(x)$:

$$\begin{aligned} \int \frac{-\sin^2(x)}{\cos(x)} dx &= \int -\frac{(1-\cos^2(x))}{\cos(x)} dx \\ &= -\ln\left(|\sec(x) + \tan(x)|\right) + \sin(x) \end{aligned}$$

The next part of the problem is to solve for u_2' by setting up the next Wronskian:

$$u_2'(x) = \frac{W_2}{W} = \frac{\begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & \tan(x) \end{vmatrix}}{1}$$
$$= \sin(x)$$

$$u_2(x) = \int u_2' dx = \int \sin(x) dx = -\cos(x) \tag{3}$$

$$y_p = y_1 u_1 + y_2 u_2 = \cos(x)(\sin(x) - \ln(|\sec(x) + \tan(x)|) + \cos(x)\sin(x)) \tag{4}$$

Therefore, the solution to the whole equation is the particular solution plus the homogeneous solution:

$$y = y_p + y_h = c_1 \cos(x) + c_2 \sin(x) + \cos(x)(\sin(x) - \ln(|\sec(x) + \tan(x)|) + \cos(x)\sin(x)) \tag{5}$$