1 Chapter 8

1.1 Section 2

Problem 1

To begin the problem, one of the first steps is to convert the following equation into matrix format:

$$\begin{cases} \frac{dx}{dt} = x + 2y \\ \frac{dy}{dt} = 4x + 3y \end{cases} \tag{1}$$

Converting this to the following format:

$$\frac{d}{dt}\tilde{\mathbf{X}} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2}$$

$$\tilde{\mathbf{X}}' = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \tilde{\mathbf{X}} \tag{3}$$

The next phase of solving the problem would be to do the following:

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$\det\left(\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix}\right) = 0$$
$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$
$$(\lambda - 5)(\lambda + 1) = 0$$
$$\lambda = -1, 5$$

These are your eigenvalues, and the next part of the problem would be to find the eigenvectors for the problem:

$$\left.\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix}\right|_{\lambda=5} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -4k_1 + 2k_2 &= 0 \\ k_1 &= 1 \\ k_2 &= 2 \end{aligned}$$

We have just arrived at our first eigenvector which is the following:
$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}$
Now to arrive at the other eigen vector we do the following:

$$\begin{aligned} \left.\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix}\right|_{\lambda=-1} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} 2k_1 + 2k_2 &= 0 \\ k_1 &= -1 \\ k_2 &= 1 \end{aligned}$$

Then the second eigenvector is the following:
$c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$
Thus the combined solution is the following:
$\tilde{\mathbf{x}} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$