1 Chapter 4.7 Variation of Parameters

1.1 Number 3

$$y'' + y = \sin(x) \tag{1}$$

$$\begin{aligned} m^2 + 1 &= 0 \\ m^2 &= -1 \\ m &= \pm i \end{aligned}$$

This yields the following as the homogeneous portion of the problem:

$$y_h = c_1 \cos(x) + c_2 \sin(x) \tag{2}$$

Then one can set up the Wronskian for the problem by doing the following:

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = 1 \tag{3}$$

Then one can begin to set up $u_1'(x)$ by doing the following:

$$u_1' = \frac{\begin{vmatrix} 0 & \sin(x) \\ \sin(x) & \cos(x) \end{vmatrix}}{1} = -\sin^2(x) \tag{4}$$

$$u_1 = \int -\sin^2(x)dx = -\int \frac{1}{2} - \frac{\cos(2x)}{2} dx = \frac{\sin(2x)}{4} - \frac{x}{2} \tag{5}$$

$$u_1 y_1 = \frac{\cos(x)\sin(2x)}{4} - \frac{x\cos(x)}{2} \tag{6}$$

Now I will set up the second $u_2'(x)$:

$$u_2' = \frac{\begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & \sin(x) \end{vmatrix}}{1} = \sin(x)\cos(x) \tag{7}$$

$$u_2 = \int u_2' dx = \int \sin(x)\cos(x)dx \tag{8}$$

Using U-Subtitution you get the following:

$$u_2 = \int u du = \frac{u^2}{2} = \frac{\sin^2(x)}{2} \tag{9}$$

$$y_2 u_2 = \frac{\sin^3(x)}{2} \tag{10}$$

Therefore the particular solution is the following:

$$y_p = \frac{\cos(x)\sin(2x)}{4} - \frac{x\cos(x)}{2} + \frac{\sin^3(x)}{2} \tag{11}$$

Adding our homogeneous and particular solutions, we get:

$$y = c_1\cos(x) + (c_2 + \frac{1}{2})\sin(x) - \frac{x\cos(x)}{2} \tag{12}$$

Which simplifies to:

$$y = c_1\cos(x) + c_2\sin(x) - \frac{x\cos(x)}{2} \tag{13}$$