TABLE 11.1 Two z-Transforms

$f[n]$, $n \geqq 0$	$F(z)$
$u[n]$	$\dfrac{z}{z-1}$
a^n	$\dfrac{z}{z-a}$

Generally, we use a z-transform table to find inverse z-transforms, rather than using the inversion integral of (11.2). In any transform pair

$$f[n] \xleftrightarrow{\mathscr{Z}} F(z),$$

given $f[n]$, the transform is $F(z)$; given $F(z)$, the inverse transform is $f[n]$. For example, for the exponential function

$$a^n \xleftrightarrow{\mathscr{Z}} \frac{z}{z-a},$$

the z-transform of a^n is $z/(z-a)$; the inverse transform of $z/(z-a)$ is a^n for $n \geqq 0$.

Digital-Filter Example

We now use the z-transform to solve a first-order difference equation. However, first we must derive the real-shifting property of the z-transform. Consider the z-transform of a delayed function $f[n-n_0]u[n-n_0]$ for $n_0 \geqq 0$:

$$\begin{aligned}
\mathscr{Z}[f[n-n_0]u[n-n_0]] &= \sum_{n=0}^{\infty} f[n-n_0]u[n-n_0]z^{-n} \\
&= \sum_{n=n_0}^{\infty} f[n-n_0]z^{-n} \\
&= f[0]z^{-n_0} + f[1]z^{-n_0-1} + f[2]z^{-n_0-2} + \cdots \\
&= z^{-n_0}[f[0] + f[1]z^{-1} + f[2]z^{-2} + \cdots] = z^{-n_0}F(z).
\end{aligned}$$

For $n_0 \geqq 0$, we have the property

$$\mathscr{Z}[f[n-n_0]u[n-n_0]] = z^{-n_0}F(z). \tag{11.13}$$

Of course, n_0 must be an integer. We derive the real-shifting property for $n_0 < 0$ in Section 11.5.

The difference equation for the α-filter is

$$y[n] - (1-\alpha)y[n-1] = \alpha x[n], \tag{11.14}$$