because $x[k]$ is zero for $k < 0$. Then, from (11.4),

$$\begin{aligned}\mathcal{Z}[x[n]*y[n]] &= \sum_{n=0}^{\infty}[\sum_{k=0}^{\infty}x[k]y[n-k]]z^{-n} \\ &= \sum_{k=0}^{\infty}x[k][\sum_{n=0}^{\infty}y[n-k]z^{-n}],\end{aligned} \tag{11.37}$$

where the order of the summations is reversed in the last step. Next, we change variables on the inner summation, letting $m = (n - k)$. Then $n = m + k$ and

$$\begin{aligned}\mathcal{Z}[x[n]*y[n]] &= \sum_{k=0}^{\infty}x[k]\left[\sum_{m=-k}^{\infty}y[m]z^{-m-k}\right] \\ &= \sum_{k=0}^{\infty}x[k]z^{-k}\sum_{m=0}^{\infty}y[m]z^{-m} = X(z)Y(z).\end{aligned} \tag{11.38}$$

The lower limit $m = -k$ is changed to $m = 0$, because $y[m]$ is zero for $m < 0$. Hence, convolution transforms into multiplication in the z-domain. Examples of convolution are given later in this chapter when we consider linear systems.

Several properties of the z-transform have been developed. These properties are useful in generating tables of z-transforms and in applying the z-transform to the solutions of linear difference equations with constant coefficients. When possible, we model discrete-time physical systems with linear difference equations with constant coefficients; hence, these properties are useful in both the analysis and design of linear time-invariant physical systems. Table 11.4 gives properties for the z-transform and includes some properties in addition to those derived.

TABLE 11.4 Properties of the z-Transform

Name	Property
1. Linearity, (11.8)	$\mathcal{Z}[a_1f_1[n] + a_2f_2[n]] = a_1F_1(z) + a_2F_2(z)$
2. Real shifting, (11.13)	$\mathcal{Z}[f[n-n_0]u[n-n_0]] = z^{-n_0}F(z), \quad n_0 \geqq 0$
3. Real shifting, (11.25)	$\mathcal{Z}[f[n+n_0]u[n]] = z^{n_0}[F(z) - \sum_{n=0}^{n_0-1}f[n]z^{-n}]$
4. Complex shifting, (11.23)	$\mathcal{Z}[a^n f[n]] = F(z/a)$
5. Multiplication by n	$\mathcal{Z}[nf[n]] = -z\dfrac{dF(z)}{dz}$
6. Time scaling, (11.33)	$\mathcal{Z}[f[n/k]] = F(z^k)$, k a positive integer
7. Convolution, (11.38)	$\mathcal{Z}[x[n]*y[n]] = X(z)Y(z)$
8. Summation	$\mathcal{Z}[\sum_{k=0}^{n}f[k]] = \dfrac{z}{z-1}F(z)$
9. Initial value, (11.27)	$f[0] = \lim_{z\to\infty} F(z)$
10. Final value, (11.30)	$f[\infty] = \lim_{z\to 1}(z-1)F(z)$, if $f[\infty]$ exists