Use of momentum theory to calculate drag coefficient

1 Basic theory

Refer to figure 1, which shows a control volume (CV) around a non-lifting body in a fluid flow. The control volume extends to infinity in all directions. The streamtube above the body is of interest, and the distance of the streamtube exit above the centreline is y. This carries a mass flow rate $\dot{m}$, and at the upstream face of the streamtube the fluid velocity is the free stream velocity U, and the pressure is p_u, and at the downstream face the fluid velocity is some value u and the pressure is p_d. We can integrate all the momentum fluxes through the CV surfaces and all the pressures acting on them to find the net force on the control volume and hence the body. Since the upper and lower CV surfaces are at an infinite distance away from the body the pressure on the faces is ambient, so there is no net pressure force in the y direction, but there may be pressure force acting in the x direction due to the wake downstream or the upstream influence of the obstacle and wake. The net force in a given direction from Newton's 2nd Law as the rate of change of momentum in the given direction. Momentum fluxes through the upper and lower CV surfaces balance, so there is no lift force. We therefore only have to consider the x direction momentum flux through the upstream and downstream surfaces of the streamtube and force due to pressure in the x direction. The momentum flux per unit span of the body is the mass flow rate $\dot{m}$ times the velocity. At the downstream surface of the streamtube the mass flow rate is

$$\dot{m} = \rho u dy \tag{1}$$

The flow is steady, so this is also the mass flow rate entering the streamtube at the left hand side of the CV.

Newton's second law is

$$F = \Delta \dot{H} \tag{2}$$

where F is the force in the given direction and $\Delta \dot{H}$ is the rate of change of momentum in the same direction.

The force in the x direction is a body force on the control volume F_x plus pressure forces on the sides of the control volume, so

$$F = F_x + \int_{-\infty}^{\infty} p_u(y)dy - \int_{-\infty}^{\infty} p_d(y)dy \tag{3}$$

The momentum flux leaving the downstream face of the streamtube at the ordinate y is

$$\dot{H}_d = \dot{m}\, u \tag{4}$$

and the momentum flux entering the upstream face of the same streamtube is

$$\dot{H}_u = \dot{m}\, U \tag{5}$$

The rate of change of momentum through the streamtube is therefore

$$\Delta\dot{H} = \dot{m}u - \dot{m}U = \rho u(u - U)dy \tag{6}$$

and so the net force *on the control volume* in the x direction is the integral of this over the exit face of the control volume (since this is where we have defined the ordinate y), i.e.

$$F = \int_{-\infty}^{\infty} \rho u(u - U)dy \tag{7}$$

and

$$F_x + \int_{-\infty}^{\infty} p_u dy - \int_{-\infty}^{\infty} p_d dy = \int_{-\infty}^{\infty} \rho u(u - U)dy \tag{8}$$

The net force F_x on the *body* in the x-direction is the drag D exerted on the body. Then from Newton's 3rd Law and recognising the sign of D

$$D = \int_{-\infty}^{\infty} \rho u(U - u)dy + \int_{-\infty}^{\infty} p_u dy - \int_{-\infty}^{\infty} p_d dy \tag{9}$$

We will measure pressure relative to some reference value p_{ref} in the experiment, so

$$D = \int_{-\infty}^{\infty} \rho u(U - u)dy + \int_{-\infty}^{\infty} (p_u - p_{ref})dy - \int_{-\infty}^{\infty} (p_d - p_{ref})dy \tag{10}$$

Note that D is the drag per unit span on a 2-D body. The drag coefficient for a 2-D bluff body is defined as

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 d} \tag{11}$$

where d is the cylinder diameter for this case, so the drag coefficient is

$$C_D = 2\int_{-\infty}^{\infty} \frac{u}{U}\left(1 - \frac{u}{U}\right) d\left(\frac{y}{d}\right) + \int_{-\infty}^{\infty} (C_{p_u} - C_{p_d}) d\left(\frac{y}{d}\right) \tag{12}$$

where C_p is the pressure coefficient

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2} \tag{13}$$

Note that $\frac{u}{U}$ and C_p are functions of y-ordinate. The pressure only needs to be integrated where the local pressure does not equal free stream pressure p_∞. In general C_{p_u} (the upstream value) will be zero, and C_{p_d} may be non-zero dependent upon where the wake measurement probe is.

2 Implementation of theory in analysis of experimental data

2.1 Effect of error in wake static pressure

Careful measurement of the velocity and pressure in the wake will allow the above theory to be applied directly. However, the accurate and unambiguous measurement of the wake velocity and static pressure is not that straighforward. Dynamic pressure measurement is done easily using a Pitot-static probe. The Pitot measurement is the stagnation pressure, which is unambiguous. However, the static pressure measurement may be affected by the apparatus supporting the probe, because the supporting apparatus affects the oncoming flow. This is a remarkably strong effect, and is only negligible if the probes are very long and the other bodies are slender and streamlined. This means that it is good practice to measure the stagnation pressure, and only rely on a static pressure measurement for the wake velocity calculation if it is known that the static pressure has not been affected by the presence of any apparatus in the flow. Otherwise, a good method uses the wake stagnation pressure together with an assumption that the upstream static pressure can be extrapolated to far downstream (this requires a negligible streamwise static pressure gradient). The pressure contribution to the drag is nulled and the momentum drag is calculated using a modified wake velocity u' calculated based upon the wake stagnation pressure $p_{o_{wake}}$ and the upstream static pressure p_∞. In effect the pressure in the wake is allowed to relax to the ambient pressure. This gives the drag coefficient

$$C_D = 2\int_{-\infty}^{\infty} \frac{u'}{U}\left(1 - \frac{u'}{U}\right) d\left(\frac{y}{d}\right). \tag{14}$$

The modified wake velocity u' is

$$\frac{u'}{U} = \sqrt{\left(\frac{p_{o_{wake}} - p_{\infty}}{q_{\infty}}\right)}, \tag{15}$$

where q_{∞} and p_{∞} are the upstream dynamic and static pressure respectively.

2.2 Analysis of wake data for this experiment

It is known that the wake static pressure in this experiment is subject to error due to the mounting apparatus, so the static pressure correction method suggested above should be used. Dynamic and static pressure q_{wake} and p_{wake} are measured in the experiment, so the wake stagnation pressure is

$$p_{o_{wake}} = q_{wake} + p_{wake}. \tag{16}$$

The stagnation pressure is unchanged by the effect of the mounting apparatus. The modified wake velocity u' is then

$$\frac{u'}{U} = \sqrt{\left(\frac{q_{wake} + p_{wake} - p_{\infty}}{q_{\infty}}\right)}. \tag{17}$$

Note that the unmodified wake velocity $\frac{u}{U}$ is calculated without the pressure correction, so

$$\frac{u}{U} = \sqrt{\frac{q_{wake}}{q_{\infty}}}. \tag{18}$$

Use the following procedure:

1. Convert data for q_{wake}, p_{wake}, p_{∞} and q_{∞} into consistent units from the raw experimental voltage data.

2. Compute u'/U and the wake momentum deficit parameter $\frac{u'}{U}\left(1 - \frac{u'}{U}\right)$ across the wake.

3. To check the data, plot u'/U and the wake momentum deficit parameter $\frac{u'}{U}\left(1 - \frac{u'}{U}\right)$ across the wake. Ideally $u'/U = 1$ outside the wake , and $\frac{u'}{U}\left(1 - \frac{u'}{U}\right) = 0$ outside the wake. The edge of the wake may not be well defined, and the parameters may not be 0 or 1 outside the wake. What is the wake width?

4. *To perform the wake momentum deficit integration for the drag, the best thing to do is to integrate the wake momentum deficit only in the regions where it is greater than zero.*

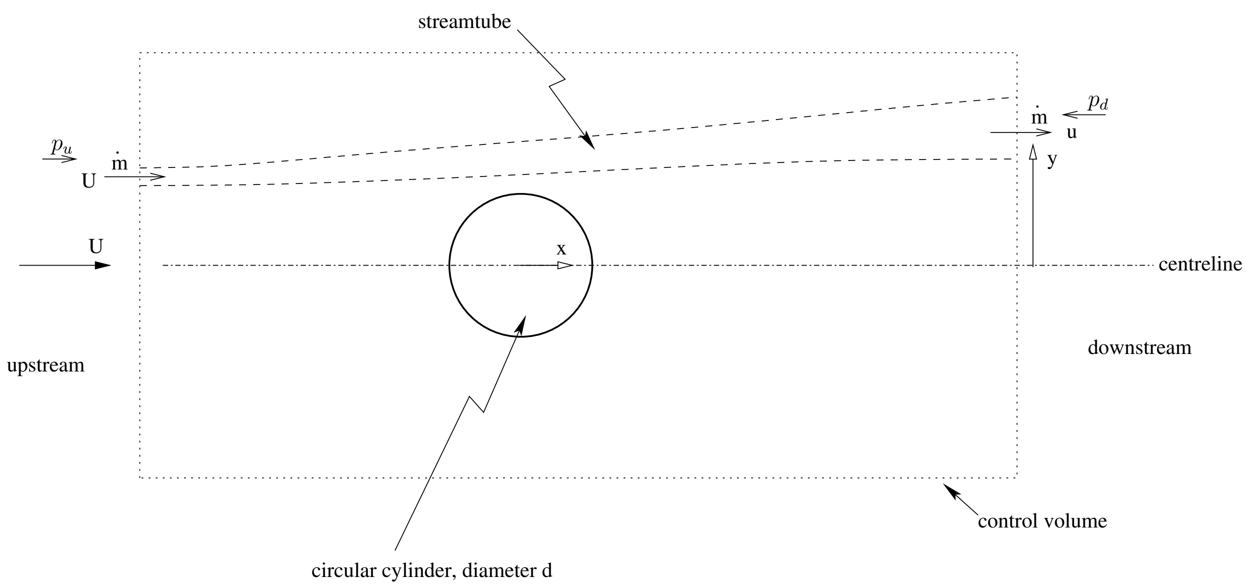

Figure 1: Control volume around an object in a fluid flow (circular cylinder) with notation. The streamtube carries mass flow $\dot{m}$. The downstream and upstream faces of the control volume have local static pressure p_d and p_u respectively, the fluid velocity entering the streamtube is the wind tunnel speed U, and fluid velocity leaving the streamtube is the measure wake velocity u.