# Drive kinematics and simulation

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A useful reference: http://fbsbook.org.

## 1 Motor constants

A DC motor typically consists of a permanent magnet rotating relative to a conducting coil. As the magnet rotates, it induces an EMF in the coil. This 'back-EMF' is proportional to the angular velocity:

$$E = K_e \omega. \tag{1}$$

A DC motor can be modeled as a resistance $R$ in series with the back EMF. Therefore, given the applied voltage $V$ and current $I$:

$$V = IR + E = IR + K_e \omega. \tag{2}$$

In other words, when a voltage is applied to the motor, it increases in speed until the back-EMF (plus resistive losses) balance the applied voltage.

If the current with no load is $I_0$, then the maximum speed of the motor is

$$\omega_{\max} = \frac{V - I_0 R}{K_e}. \tag{3}$$

Also note that when the motor is stalled, $E = 0$, so the resistance can be found using the current drawn at the stall:

$$R = V/I_{\text{stall}}. \tag{4}$$

It is common to write $K_V = 1/K_e$. Then the motor speed for a given applied voltage is

$$\omega = K_V(V - IR). \tag{5}$$

In practice, the $IR$ term is usually small compared to $V$ and can be neglected, giving

$$\omega = K_V V. \tag{6}$$

$K_V$ is sometimes called the motor velocity constant, and has SI units of $1/V s$ (though it is sometimes given as RPM/$V$).

The torque of the motor is related to the current via the motor torque constant $K_T$:

$$\tau = K_T I. \tag{7}$$

This constant can be calculated from the torque required to stall the motor and the current draw at the stall:

$$K_T = \tau_{\text{stall}}/I_{\text{stall}}. \tag{8}$$

$K_T$ and $K_V$ are not independent; they are inversely related. The electrical power input to the motor is

$$P_{\text{in}} = VI = (IR + K_e\omega)I = I^2R + IK_e\omega \tag{9}$$

while the mechanical power out is

$$P_{\text{out}} = \tau\omega = IK_T\omega, \tag{10}$$

while losses are

$$P_{\text{loss}} = I^2R. \tag{11}$$

. But

$$P_{\text{in}} = P_{\text{out}} + P_{\text{loss}}, \tag{12}$$

which implies that

$$K_T = K_e = 1/K_V. \tag{13}$$

To take a specific example, the Falcon 500 motor operates at 12 V, with $I_0 = 1.5\,\text{A}$, $I_{\text{stall}} = 257\,\text{A}$, $\tau_{\text{stall}} = 4.69\,\text{N m}$, and $\omega_{\max} = 6380\,\text{RPM} = 668\,/\text{s}$. This then gives $R = V/I_{\text{stall}} = 0.047\,\Omega$, $K_V = \omega_{\max}/(V - I_0R) = 60.0\,/\text{Vs}$ and $K_T = \tau_{\text{stall}}/I_{\text{stall}} = 0.018\,\text{Nm/A}$. And $1/60 = 0.017$, so these are nearly inverse of each other. (Note that since power is voltage times current, VA = Nm/s or Vs = NM/A.)

## 2 A unicycle

Let's now apply this to a putative vehicle with a single wheel, constrained to move in one dimension. In addition to $R$, $K_V$, and $K_T$ of the motor, let $m$ be the mass, $r_w$ be the wheel radius, and $g$ be the gear ratio between the motor and the wheel.

Let $v$ be the (one-dimensional) velocity and $V$ be the voltage applied to the motor. Then we want to write:

$$m\dot{v} = c_1 v + c_2 V. \tag{14}$$

(Using the notation $\dot{v} = dv/dt$.) So the right-hand side represents the force acting on the vehicle.

Consider the $c_2$ term first. If the motor voltage is $V$, then the current is $I = V/R$, the motor torque is $\tau_m = IK_T = K_T V/R$, the wheel torque is $\tau_w = gK_T V/R$, and the force on the vehicle is $F = \tau_w/r_w$, which implies

$$c_2 = \frac{gK_T}{Rr_w}. \tag{15}$$

Now, consider the case where the voltage on the motor is zero. A rotating motor generates an EMF, so if the voltage is zero, then something must be absorbing the power produced by the motor; hence, the motor will act as a brake. We can thus write

$$m\dot{v} = -\frac{gK_T}{Rr_w}E, \tag{16}$$

where $E$ is the back-EMF. But $E = K_e\omega_m = g\omega_w/K_V = gv/r_w K_V$. Putting this together, we get

$$c_1 = -\frac{g^2 K_T}{K_V R r_w^2}. \tag{17}$$

To specify the state of the vehicle, we also need its position $s$. The equation for this is trivial:

$$\dot{s} = v. \tag{18}$$

## 3 State space representation

The dynamical variables describing the state of a system make up the *state space*. These variables contain sufficient information to find the future state of the system.

Formally, we write the state variables as an $n$-element vector $\mathbf{x}$. The system can also have control inputs, which we represent as a $p$-element vector $\mathbf{u}$. We can also make measurements on the system, which we represent as an $q$-element vector $\mathbf{y}$. The system can then be described by

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}, \mathbf{u}) && (19) \\ \mathbf{y} &= h(\mathbf{x}, \mathbf{u}) && (20) \end{aligned}$$

If $f$ and $h$ are time-invariant and linear, then we can write this as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} && (21) \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}. && (22) \end{aligned}$$

The matrix $\mathbf{A}$ is called the 'system' or 'dynamics' or 'state' matrix; $\mathbf{B}$ is called the 'input' or 'control' matrix; $\mathbf{C}$ is called the 'output' or 'sensor' matrix; and $\mathbf{D}$ is called the 'feedthrough' or 'feedforward' or 'direct' matrix.

# 4 Differential drive

Here, the system consists of two identical motors with wheels separated by distance $2r_b$, and the system can move in two dimensions. The mass is again $m$, and the moment of inertia is $I$.

The state variables here are the $x$ and $y$ positions, $s_x$ and $s_y$, the heading $\theta$, the positions of the two wheels $d_L$ and $d_R$, and their velocities $\dot{d}_L$, $\dot{d}_R$. We can write this as a vector: $\mathbf{x} = [s_x, s_y, \theta, \dot{d}_L, \dot{d}_R, d_L, d_R]^T$. The control inputs are the motor voltages: $\mathbf{u} = [V_L, V_R]^T$.

Using the results from the previous section, the forces on the vehicle produced by the left and right motors are

$$F_L = c_1\dot{d}_L + c_2V_L. \tag{23}$$

and similarly for the other motor. Given $F_L$ and $F_R$, the total force is $F_{\text{tot}} = F_L + F_R$ and the torque is $\tau = (F_L - F_R)r_b$. Therefore, the acceleration for one of the wheels is

$$\begin{aligned} \ddot{d}_L &= \frac{1}{m}F_{\text{tot}} + \frac{r_b}{I}\tau && (24) \\ &= \frac{1}{m}(F_L + F_R) + \frac{r_b^2}{I}(F_L - F_R) && (25) \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{m} + \frac{r_b^2}{I}\right) c_1 \dot{d}_L + \left(\frac{1}{m} - \frac{r_b^2}{I}\right) c_1 \dot{d}_R + & (26)\\
&\quad \left(\frac{1}{m} + \frac{r_b^2}{I}\right) c_2 V_L + \left(\frac{1}{m} - \frac{r_b^2}{I}\right) c_2 V_R \\
&= c_3 c_1 \dot{d}_L + c_4 c_1 \dot{d}_R + c_3 c_2 V_L + c_4 c_2 V_R, & (27)
\end{aligned}
$$

where

$$
\begin{aligned}
c_3 &= \left(\frac{1}{m} + \frac{r_b^2}{I}\right) & (28)\\
c_4 &= \left(\frac{1}{m} - \frac{r_b^2}{I}\right). & (29)
\end{aligned}
$$

Similarly,

$$
\ddot{d}_R = c_4 c_1 \dot{d}_L + c_3 c_1 \dot{d}_R + c_4 c_2 V_L + c_3 c_2 V_R. \tag{30}
$$

In addition, if we call the forward velocity $v = \frac{1}{2}(\dot{d}_L + \dot{d}_R)$, then

$$
\begin{aligned}
\dot{s}_x &= v \cos\theta & (31)\\
\dot{s}_y &= v \sin\theta & (32)\\
\dot{\theta} &= (\dot{d}_R - \dot{d}_L)/2r_b. & (33)
\end{aligned}
$$

For the subset of state variables $\mathbf{x} = [\dot{d}_L, \dot{d}_R, d_L, d_R]^T$, with $\mathbf{u} = [V_L, V_R]^T$, the dynamics can then be written in the notation of the previous section as

$$
\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{34}
$$

with

$$
\mathbf{A} = \begin{bmatrix} c_1 c_3 & c_1 c_4 & 0 & 0 \\ c_1 c_4 & c_1 c_3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{35}
$$

and

$$
\mathbf{B} = \begin{bmatrix} c_2 c_3 & c_2 c_4 \\ c_2 c_4 & c_2 c_3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{36}
$$

# 5 Runge-Kutta integration

Given

$$\frac{dy}{dt} = f'(y,t) \tag{37}$$

with an initial value of $y$ at some $t$, one can use Euler's method to integrate this in steps of $t$ of size $h$:

$$y_{n+1} = y_n + hf'(y_n, t_n). \tag{38}$$

In practice, however, this is not a good method to use. The error per step is $O(h^2)$, as can be seen from a series expansion:

$$y(t_n + h) = y(t_n) + \frac{dy}{dt}(t_n)h + O(h^2). \tag{39}$$

So over the $\sim 1/h$ steps needed to cover a fixed-size interval, the error is $O(h)$. This is not very accurate.

We can do better by first taking a trial step to the midpoint of the interval and then using the derivatives calculated there for entire step. That is,

$$\begin{aligned} k_1 &= hf'(t_n, y_n) && (40)\\ k_2 &= hf'(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}) && (41)\\ y_{n+1} &= y_n + k_2. && (42) \end{aligned}$$

This ends up canceling some the first order error terms, giving a step error of $O(h^3)$.

One can go further in canceling error terms. Fourth-order Runge-Kutta is defined by

$$\begin{aligned} k_1 &= hf'(t_n, y_n) && (43)\\ k_2 &= hf'(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}) && (44)\\ k_3 &= hf'(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}) && (45)\\ k_4 &= hf'(t_n + h, y_n + k_3) && (46)\\ y_{n+1} &= y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} && (47) \end{aligned}$$

and has a step error of $O(h^5)$. One can extend this further, but that is usually not found to be worthwhile.