Robotics

(Course viewgraphs)

João Silva Sequeira[1]

[1]joao.silva.sequeira@tecnico.ulisboa.pt

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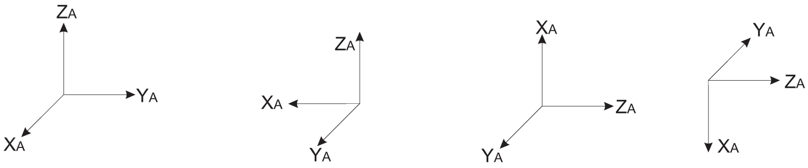
Rotation around moving axis: Non uniqueness – Example

The set of Euler angles $Z - Y - Z$ is not unique

Rotations: $R_Z(\frac{\pi}{2})$ $R_Y(\frac{\pi}{2})$ $R_Z(0)$

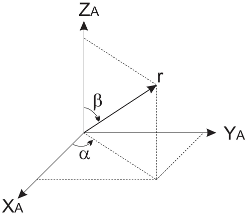
Rotation around moving axis: Non uniqueness – Example



Rotations: $R_Z(-\frac{\pi}{2})$ $R_Y(-\frac{\pi}{2})$ $R_Z(\pi)$

... 2 sequences of different rotations yield in the same orientation

Rotations around arbitrary axis



Step 1	Align with the axis Z
	$R_Z(-\alpha) \longrightarrow R_Y(-\beta)$
Step 2	Rotation around axis Z
	$R_Z(\theta)$
Step 3	Realign to initial direction
	$R_Y(\beta) \longrightarrow R_Z(\alpha)$

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Rotations around arbitrary axis

Let $r = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$ be a unit vector; from the previous figure

$$\sin(\alpha) = \frac{r_y}{\sqrt{r_x^2+r_y^2}} \qquad \cos(\alpha) = \frac{r_x}{\sqrt{r_x^2+r_y^2}} \qquad \sin(\beta) = \sqrt{r_x^2 + r_y^2} \qquad \cos(\beta) = r_z$$

substituting in $R_r(\theta)$

$$R_r(\theta) = \begin{bmatrix} r_x^2(1-c_\theta)+c_\theta & r_x r_y(1-c_\theta)-r_z s_\theta & r_x r_z(1-c_\theta)+r_y s_\theta \\ r_x r_y(1-c_\theta)+r_z s_\theta & r_y^2(1-c_\theta)+c_\theta & r_y r_z(1-c_\theta)-r_x s_\theta \\ r_x r_z(1-c_\theta)-r_y s_\theta & r_y r_z(1-c_\theta)+r_x s_\theta & r_z^2(1-c_\theta)+c_\theta \end{bmatrix}$$

Key property: $R_{-r}(-\theta) = R_r(\theta)$

Rotations around arbitrary axis

Given frames $\mathcal{F}_A$ and $\mathcal{F}_B$ is there a rotation axis, r, and an angle, θ, such that $\mathcal{F}_B = R_r(\theta)\ \mathcal{F}_A$?
The answer is yes (Euler theorem of rotation)

For elementary rotations, obviously:

$$\underbrace{R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix}} \quad \underbrace{R_Y(\theta) = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}} \quad \underbrace{R_Z(\theta) = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Orientation:

$\alpha = 0, \beta = 0, \gamma = \theta$

Rotation axis:

$r = (1, 0, 0)$

Orientation:

$\alpha = 0, \beta = \theta, \gamma = 0$

Rotation axis:

$r = (0, 1, 0)$

Orientation:

$\alpha = \theta, \beta = 0, \gamma = 0$

Rotations axis:

$r = (0, 0, 1)$

Rotations around arbitrary axis: Inverse problem

What is the rotation axis, and angle, that correspond to a given rotation matrix ?

From $\quad r_{11} + r_{22} + r_{33} = \underbrace{(r_x^2 + r_y^2 + r_z^2)}_{=1} \; (1 - c_\theta) + 3c_\theta \quad$ one gets

$\theta = \mathrm{acos}(\frac{r_{11}+r_{22}+r_{33}-1}{2})$

From

$$r_{32} - r_{23} = r_y r_z(1 - c_\theta) + r_x s_\theta - r_y r_z(1 - c_\theta) + r_x s_\theta = 2r_x s_\theta$$
$$r_{13} - r_{31} = r_x r_z(1 - c_\theta) + r_y s_\theta - r_x r_z(1 - c_\theta) + r_y s_\theta = 2r_y s_\theta$$
$$r_{21} - r_{12} = r_x r_y(1 - c_\theta) + r_z s_\theta - r_x r_y(1 - c_\theta) + r_z s_\theta = 2r_z s_\theta$$

one gets

$$\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \frac{1}{2\sin(\theta)} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

For students to explore: Quaternions

- Consider the number, called a *quaternion*

$$q = (a, b, c, d) \equiv a\,\mathbf{1} + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}, \qquad \text{with } a, b, c, d \in \mathbb{R}$$

- It is possible to demonstrate that the quaternion

$$q = \cos(\theta/2)\mathbf{1} + \sin(\theta/2)\ (r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k})$$

  describes a rotation of θ around an axis (r_x, r_y, r_z)

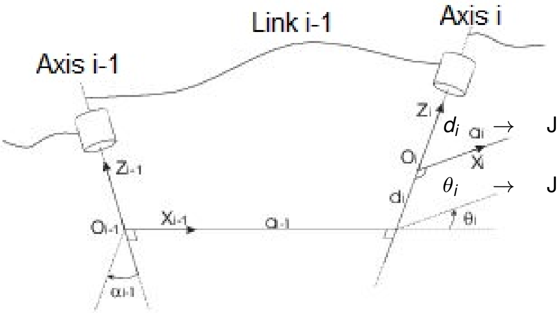
- There are advantages in using quaternions over Euler/Fixed angles (to be explored as homework)

Kinematics

Study of the motion due, exclusively, to the geometry of the robot

- How to represent the motion of the robot, i.e., what is the adequate space
- How to relate the different variables that express the different degrees of freedom
- How to relate the different parts of the structure of a robot
- What are the constraints imposed by the geometry of the robot

Serial manipulators – Modeling links and joints – The Denavit-Hartenberg (D-H) convention



$d_i \rightarrow$ Joint variable for linear joints

$\theta_i \rightarrow$ Joint variable for rotation joints

4 link parameters

Serial manipulators – Modelling links and joints – The D-H convention

$$a_i \equiv \text{Distance from } Z_i \text{ to } Z_{i+1} \text{ along } X_i$$

$$\alpha_i \equiv \text{Angle between } Z_i \text{ and } Z_{i+1} \text{ around } X_i$$

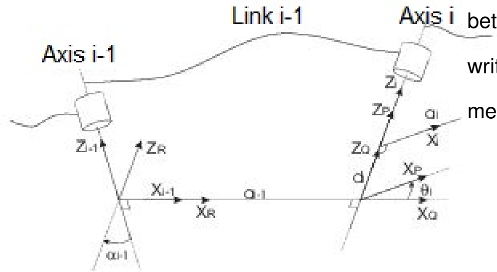
$$d_i \equiv \text{Distance from } X_{i-1} \text{ to } X_i \text{ along } Z_i$$

$$\theta_i \equiv \text{Angle between } X_{i-1} \text{ and } X_i \text{ around } Z_i$$

- The D-H convention does not yield a unique reference frame assignment (just change the base frame)
- The D-H convention can be defined in terms of axis other than X and Z
- Rule of thumb: try assigning the reference frames assuming zero value for the joints

The link transformation

Given the set of frames that describe the link variables what is the transformation that relates all of them ?



Using the auxiliary frames $\mathcal{F}_P$, $\mathcal{F}_Q$ and $\mathcal{F}_R$, the transformation between links $i-1$ and i can be written as the composition of elementary transformations

$$\mathcal{F}_{i-1} = R_X(\alpha_{i-1})\ \mathcal{F}_R$$

$$\mathcal{F}_R = \mathrm{Trans}_X(a_{i-1})\ \mathcal{F}_Q$$

$$\mathcal{F}_Q = R_Z(\theta_i)\ \mathcal{F}_P$$

$$\mathcal{F}_P = \mathrm{Trans}_Z(d_i)\ \mathcal{F}_i$$

The link transformation

$$R_X(\alpha_{i-1}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{\alpha_{i-1}} & -s_{\alpha_{i-1}} \\ 0 & s_{\alpha_{i-1}} & c_{\alpha_{i-1}} \end{bmatrix}$$

$$R_Z(\theta_i) = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_X(a_{i-1}) = \begin{bmatrix} a_{i-1} \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Trans}_Z(d_i) = \begin{bmatrix} 0 \\ 0 \\ d_i \end{bmatrix}$$

The link transformation

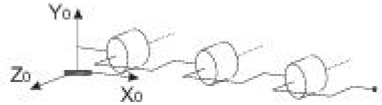
$$
{}^{i-1}_{i}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_{i-1}} & -s_{\alpha_{i-1}} & 0 \\ 0 & s_{\alpha_{i-1}} & c_{\alpha_{i-1}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

multiplying the matrices yields the link transformation

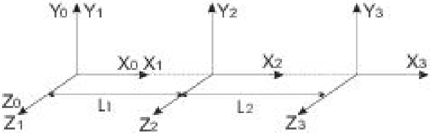
$$
{}^{i-1}_{i}T = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & a_{i-1} \\ s_{\theta_i} c_{\alpha_{i-1}} & c_{\theta_i} c_{\alpha_{i-1}} & -s_{\alpha_{i-1}} & -s_{\alpha_{i-1}} d_i \\ s_{\theta_i} s_{\alpha_{i-1}} & c_{\theta_i} s_{\alpha_{i-1}} & c_{\alpha_{i-1}} & c_{\alpha_{i-1}} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

In a serial structure, the composition of all the link transformations is the <u>direct kinematics</u> and describes completely the effect of the geometry of the device in the motion capabilities

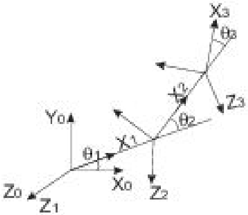
The D-H convention – Example



Planar manipulator with 3 revolute joints (RRR)



Manipulator's configuration when all joint variables are set to 0



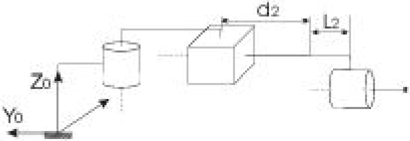
Frame assignment

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

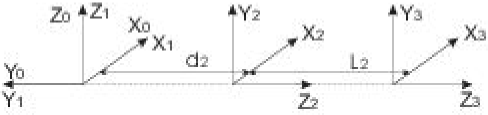
$$X_1 \equiv X_0 \Leftrightarrow \theta_1 = 0$$

Direct kinematics – Example

Direct kinematics – Example – Manipulator RPR



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$\pi/2$	0	d_2	0
3	0	0	L_2	θ_3



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Direct kinematics – Example – Manipulator RPR

$$
{}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\quad
{}^1_2T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\quad
{}^2_3T = \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0_3T = \begin{bmatrix} c_1c_3 & -c_1s_3 & s_1 & s_1(L_2+d_2) \\ s_1c_3 & -s_1s_3 & -c_1 & -c_1(L_2+d_2) \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

The frame assignment must be such that the D-H parameters can be determined without any ambiguity

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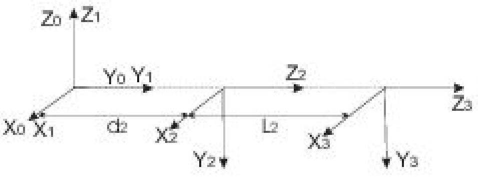
Direct kinematics – Example

Alternative frame assignment (D-H convention does not yield a unique assignment)



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$-\pi/2$	0	d_2	0
3	0	0	L_2	θ_3

- $\mathcal{F}_1$ aligns with $\mathcal{F}_0$ when $\theta_1 = 0$
- $\mathcal{F}_3$ aligns with $\mathcal{F}_2$ when $\theta_3 = 0$
- Axis Z are identified with the joint axis

Direct kinematics – Example

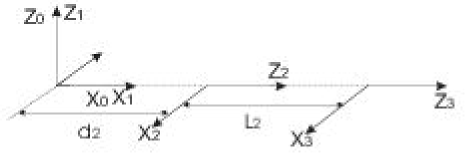
$$
{}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\quad
{}^1_2T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\quad
{}^2_3T = \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0_3T = \begin{bmatrix} c_1c_3 & -c_1s_3 & -s_1 & -s_1(L_2+d_2) \\ s_1c_3 & -s_1s_3 & c_1 & c_1(L_2+d_2) \\ -s_3 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Hint: Compare this transformation with the one obtained in the previous example.

Direct kinematics – Example

And if the frame assignment does not follow D-H ?



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$\times$	0	0	$\times$
3	0	0	L_2	θ_3

It is not possible to determine α_1 e θ_2, hence the link transformation obtained from D-H can not be used; still ...

Teaser:
Is it possible to put the manipulator in a configuration such that the D-H parameters can be determined ?

Direct kinematics – Example

$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^2_3T = \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \mathrm{Trans}_X(d_2)\ R_Z(-\pi/2)\ \ R_X(-\pi/2) $$

$$ = \begin{bmatrix} 1 & 0 & 0 & d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & d_2 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

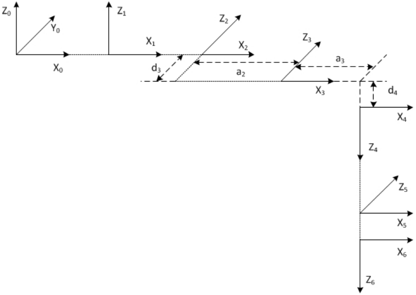
Direct kinematics – Example

- Note that ${}^{1}_{2}T$ can be easily obtained by inspection of the frames
- The full transformation is

$$
{}^{0}_{3}T = \begin{bmatrix} s_1 c_3 & -s_1 c_3 & c_1 & c_1(L_2 + d_2) \\ -c_1 c_3 & c_1 s_3 & s_1 & s_1(L_2 + d_2) \\ -s_3 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

This transformation contains all the information on the geometry of the manipulator

Direct kinematics – The Puma 560



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$-\pi/2$	0	0	θ_2
3	0	a_2	$-d_3$	θ_3
4	$-\pi/2$	a_3	d_4	θ_4
5	$\pi/2$	0	0	θ_5
6	$-\pi/2$	0	0	θ_6

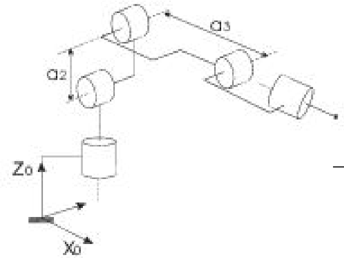
Direct kinematics – The Puma 560

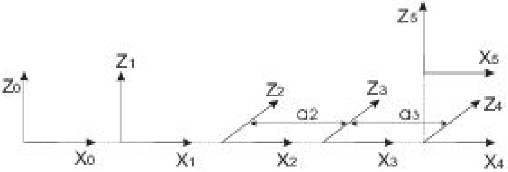
$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^2_3T = \begin{bmatrix} c_3 & -s_3 & 0 & a_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^3_4T = \begin{bmatrix} c_4 & -s_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s_4 & -c_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^4_5T = \begin{bmatrix} c_4 & -s_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^5_6T = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_6 & -c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_6T = {}^0_1T \; {}^1_2T \; {}^2_3T \; {}^3_4T \; {}^4_5T \; {}^5_6T $$

Direct kinematics – The Yasukawa L3





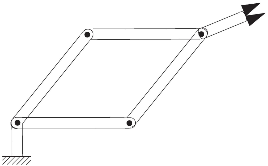
i	α_{i-1}	a_{i-1}	d_i	θ_i	θ_i value at rest
1	0	0	0	θ_1	0
2	$-\pi/2$	0	0	θ_2	$-\pi/2$
3	0	a_2	0	θ_3	$\pi/2$
4	0	a_3	0	θ_4	$\pi/2$
5	$\pi/2$	0	0	θ_5	0

Sometimes the rest configuration of the manipulator does not correspond to the joint variables being at ”zero”

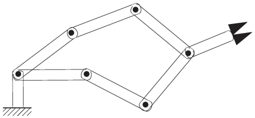
Inverse kinematics I

Goal:
Given a point in the workspace $(x, y, z, \alpha, \beta, \gamma)$, determine the corresponding point(s) in the joi
space

Positioning of the end-effector in an arbitrary point $(x, y, z, \alpha, \beta, \gamma)$ requires a minimum of 6 degrees of freedom



2 solutions



How many solutions ?

Inverse kinematics II

- General 6R manipulators – 16 sols (real and complex), [Sinha et al, 2018], [Wang et al, 2006]

- 6R manipulators with the axis of the last 3 joints intersecting in a common point – 8 solutions in closed form, [Tsai, Morgan, 1985]

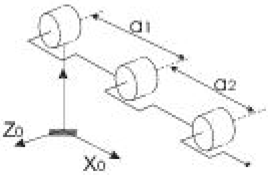
- 6R manipulators with any 3 adjacent axis parallel to each others – 8 solutions in closed form, [Tsai, Morgan, 1985]

L.-W. Tsai, A.P. Morgan
"Solving the Kinematics of the Most General Six-and Five-Degree-of-Freedom Manipulators by Continuation Methods"
Transactions of the ASME, Journal of Mechanisms, Transmissions, and Automation in Design, vol. 107, June 1985.

Sasanka Sinha, Rajeevlochana G. Chottawadigi, Subir Kumar Saha
"Inverse Kinematics for General 6R Manipulators in RoboAnalyzer"
5th Joint Int. Conf. on Multibody System Dynamics, June 24-28, Lisbon, Portugal

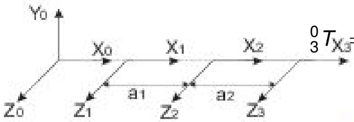
Wang Yan, Hang Lu-bin, Yang Ting-Li
"Inverse Kinematics Analysis of General 6R Serial Robot Mechanism Based on Groebner Base"
Journal Front. Mech. Eng. China, 1:115-124, 2006

Inverse kinematics – Algebraic method I



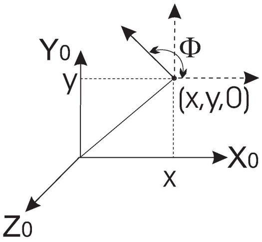
i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	a_1	0	θ_2
3	0	a_2	0	θ_3

Manipulator RRR planar



$$ {}^0_3T = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1c_1 + a_2c_{12} \\ s_{123} & c_{123} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Inverse kinematics – Algebraic method II



$$
{}^{\text{base}}_{\text{tool}}T = \begin{bmatrix} c_\Phi & -s_\Phi & 0 & x \\ s_\Phi & c_\Phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Which values for $\theta_1, \theta_2, \theta_3$ yield

$$
{}^{\text{base}}_{\text{tool}}T = {}^{0}_{3}T
$$

Inverse kinematics – Algebraic method

Equating the transformation "as seen by the end-effector" to the direct kinematics transformation results in

$$\begin{bmatrix} c_\Phi & -s_\Phi & 0 & x \\ s_\Phi & c_\Phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{123} & c_{123} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

from where one obtains

$$c_\Phi = c_{123} \qquad x = a_1 c_1 + a_2 c_{12}$$

$$s_\Phi = s_{123} \qquad y = a_1 s_1 + a_2 s_{12}$$

$$x^2 + y^2 = a_1^2 + a_2^2 + 2 a_1 a_2 c_2$$

$$c_2 = \frac{x^2+y^2-a_1^2-a_2^2}{2 a_1 a_2} \qquad s_2 = \pm\sqrt{1 - c_2^2}$$

$$\theta_2 = \operatorname{atan2}(s_2, c_2) \qquad \longrightarrow \text{ 2 solutions}$$

Inverse kinematics – Algebraic method

$$x = a_1 c_1 + a_2 c_1 c_2 - a_2 s_1 s_2 = \quad (a_1 + a_2 c_2)\, c_1 \quad - \quad a_2 s_2\, s_1 \quad = \quad K_1 c_1 - K_2 s_1$$

$$y = a_1 s_1 + a_2 c_1 s_2 + a_2 s_1 c_2 = \quad \underbrace{(a_1 + a_2 c_2)}_{K_1}\, s_1 \quad + \quad \underbrace{a_2 s_2}_{K_2}\, c_1 \quad = \quad K_1 s_1 + K_2 c_1$$

solving

$$c_1 = \frac{x + K_2 y}{K_1^2 + K_2^2} \qquad s_1 = \frac{y K_1 - K_2 x}{K_1^2 + K_2^2}$$

$\theta_1 = \text{atan2}\,(K_1 y - K_2 x, K_1 x + K_2 y)\,, \quad \forall\ \theta_2,$ note that it is always true that $K_1^2 + K_2^2 > 0$

to the remaining joint variable

$$\theta_1 + \theta_2 + \theta_3 = \text{atan2}\,(s_\Phi\,, c_\Phi) = \Phi \quad \longrightarrow \quad \theta_3 = \Phi - \theta_2 - \theta_1$$