Robotics

(Course viewgraphs)

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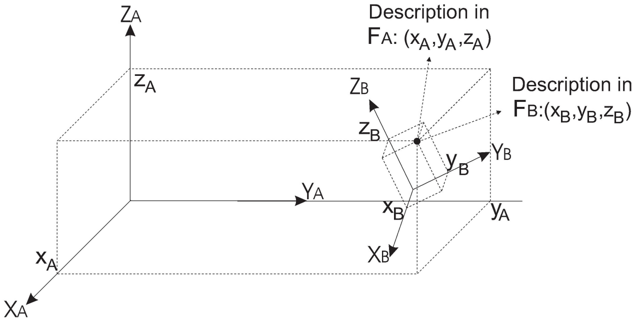
Basic math for robot manipulators (and not only)



Notation used in this course

$^AP = (x_A, y_A, z_A)$	Point in the 3D cartesian space $\mathcal{F}_A$
^A_BT	Transformation between coordinate systems $\mathcal{F}_B$ and $\mathcal{F}_A$
$^AP = \ ^A_BT \ ^BP$	Point ^AP is obtained from point ^BP through the transformation between reference frames $\mathcal{F}_B$ e $\mathcal{F}_A$

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Coordinate systems in $\mathbb{R}^3$

Coordinate system $\Leftrightarrow$ Set of 3 linearly independent vectors, that is, a base of $\mathbb{R}^3$ (only orthonormal basis are used in this course)

- A generic basis ${}^AX_B = \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}$ ${}^AY_B = \begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix}$ ${}^AZ_B = \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix}$ contains the description of the 3 coordinate axis of frame $\mathcal{F}_B$ in frame $\mathcal{F}_A$

- Given the vector ${}^BP = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ (in frame $\mathcal{F}_B$), the linear combination

  ${}^AP = \alpha{}^AX_B + \beta{}^AY_B + \gamma{}^AZ_B,$ yields vector ${}^BP$ in frame A

- Note that for ${}^BP = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, ${}^BP = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, ${}^BP = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ one obtains, ${}^AX_B$, ${}^AY_B$ e ${}^AZ_B$, respectively

- Note: the origin of the two frames is the same

Coordinate systems in $\mathbb{R}^3$

In matrix form $\quad {}^{A}P = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} {}^{B}P \quad$ and, in compact form,

$$ {}^{A}P = {}^{A}_{B}R \; {}^{B}P $$

${}^{A}_{B}R$ is named rotation matrix

Interpretation as rotation between frames $\mathcal{F}_A$ and $\mathcal{F}_B$

Matrix ${}^{A}_{B}R$ transforms points in $\mathcal{F}_B$ into points in $\mathcal{F}_A$

Interpretation as rotating a vector in $\mathcal{F}_A$

Matrix ${}^{A}_{B}R$ applies a rotation to a free vector in $\mathcal{F}_A$, yielding a new vector, also described in $\mathcal{F}_A$, but rotated relative to the first vector.

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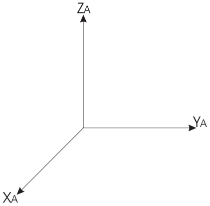
Rotations in $\mathbb{R}^3$ - Examples

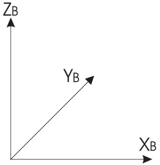


(a) Frame A

$\rightarrow$ rotation of $90°$ $\rightarrow$
(around Z_A)

$\leftarrow$ rotation of $-90°$ $\leftarrow$
(around Z_B)

(b)



(c) Frame B

$$ {}^{A}_{B}R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{B}_{A}R = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Rotations in $\mathbb{R}^3$ - Examples

Example: Consider vector ${}^{A}P = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and the product

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The resulting vector in $\mathcal{F}_A$, is rotated by $+90^\circ$ relative to the first vector

Conclusion: Product ${}^{A}_{B}R \ {}^{A}P$ yields a vector rotated relative to ${}^{A}P$

Structure of a rotation matrix

Given a column vector $\begin{bmatrix} r_{i,j} \\ r_{i+1,j} \\ r_{i+2,j} \end{bmatrix}$ in a rotation matrix

$$\begin{aligned} r_{i,j} \quad & \rightarrow \quad \text{projection along } X_A \\ r_{i+1,j} \quad & \rightarrow \quad \text{projection along } Y_A \\ r_{i+2,j} \quad & \rightarrow \quad \text{projection along } Z_A \end{aligned}$$

The "projection along ..." operator corresponds to the <u>dot product</u>

$$\begin{array}{cccccc} r_{i,j} & = & \langle X_B, X_A \rangle & \equiv & X_B \cdot X_A \\ r_{i+1,j} & = & \langle X_B, Y_A \rangle & \equiv & X_B \cdot Y_A \\ r_{i+2,j} & = & \langle X_B, Z_A \rangle & \equiv & X_B \cdot Z_A \\ & & \downarrow & & \downarrow \end{array}$$

Both notations are common to represent the dot product

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Structure of a rotation matrix

The rotation matrix can be written

$$
{}^{A}_{B}R = \begin{bmatrix} {}^{A}X_B & {}^{A}Y_B & {}^{A}Z_B \end{bmatrix} = \begin{bmatrix} X_B \cdot X_A & Y_B \cdot X_A & Z_B \cdot X_A \\ X_B \cdot Y_A & Y_B \cdot Y_A & Z_B \cdot Y_A \\ X_B \cdot Z_A & Y_B \cdot Z_A & Z_B \cdot Z_A \end{bmatrix}
$$

Assuming $X_A, X_B, Y_A, Y_B, Z_A, Z_B$ unit vectors (normed basis) then the dot products in ${}^{A}_{B}R$ represent the <u>cosines of the angles formed by the vectors</u>

$$\downarrow$$

The columns in ${}^{A}_{B}R$ describe the axis of frame B in frame A

What about the lines ?

Structure of a rotation matrix

Using the cosine representation

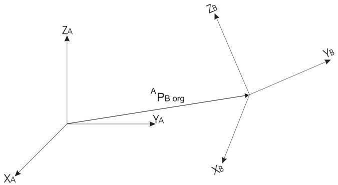
$$\left({}^{A}_{B}R\right)^{T} = \begin{bmatrix} X_B \cdot X_A & X_B \cdot Y_A & X_B \cdot Z_A \\ Y_B \cdot X_A & Y_B \cdot Y_A & Y_B \cdot Z_A \\ Z_B \cdot X_A & Z_B \cdot Y_A & Z_B \cdot Z_A \end{bmatrix} = \begin{bmatrix} X_A \cdot X_B & Y_A \cdot X_B & Z_A \cdot X_B \\ X_A \cdot Y_B & Y_A \cdot Y_B & Z_A \cdot Y_B \\ X_A \cdot Z_B & Y_A \cdot Z_B & Z_A \cdot Z_B \end{bmatrix} = {}^{B}_{A}R$$

The rotation ${}^{A}_{B}R^{-1}$ inverts the rotation.

Clearly

$${}^{B}_{A}R \equiv {}^{A}_{B}R^{-1} \equiv {}^{A}_{B}R^{T}$$

Transformation between frames I



- Rotation between frames given by

$${}^{A}_{B}R$$

- Translation between frames given by

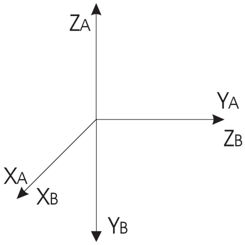
$${}^{A}P_{B\ org}$$

$${}^{A}P = {}^{A}_{B}R\ {}^{B}P + {}^{A}P_{B\ org}$$

Using a compact representation

$$\underset{\substack{\downarrow \\ \text{Point in} \\ \text{homogeneous} \\ \text{coordinates}}}{\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix}} = \underbrace{\begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{B\ org} \\ 0 & 1 \end{bmatrix}}_{\substack{\text{Homogeneous} \\ \text{transformation}}} \underset{\substack{\downarrow \\ \text{Point in} \\ \text{homogeneous} \\ \text{coordinates}}}{\begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}}$$

Transformation between frames - Examples



$\mathcal{F}_B$ is rotated by $-\pi/2$

relative to $\mathcal{F}_A$ around

axis X_A

Using the cosine representation

$$
\begin{array}{lll}
X_B \cdot X_A = 1 & Y_B \cdot X_A = 0 & Z_B \cdot X_A = 0 \\
X_B \cdot Y_A = 0 & Y_B \cdot Y_A = 0 & Z_B \cdot Y_A = 1 \\
X_B \cdot Z_A = 0 & Y_B \cdot Z_A = -1 & Z_B \cdot Z_A = 0
\end{array}
$$

and hence $${}^A_B T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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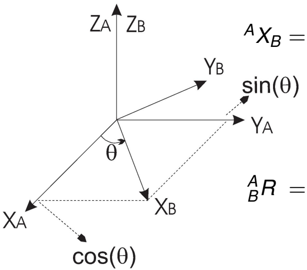
Transformation between frames – Examples



$$^A X_B = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix} \quad ^A Y_B = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix} \quad ^A Z_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$^A_B R = \left[^A X_B \; ^A Y_B \; ^A Z_B \right] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathcal{F}_B$ is rotated by θ relative to $\mathcal{F}_A$, around axis Z_A

and hence $$^A_B T = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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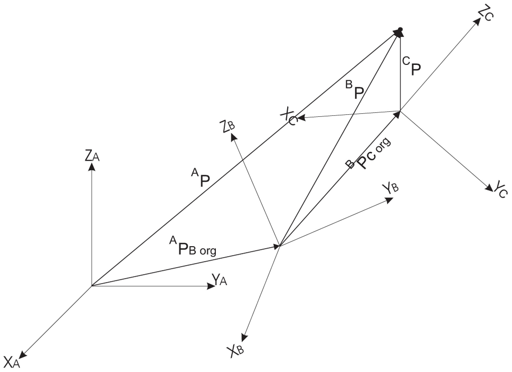
Composition of transformations



Given ^{C}P one wants to determine ^{A}P

Since the following relations

hold

$$^{B}P = {}^{B}_{C}T \; {}^{C}P$$

$$^{A}P = {}^{A}_{B}T \; {}^{B}P$$

then

$$^{A}P = {}^{A}_{B}T \; {}^{B}_{C}T \; {}^{C}P$$

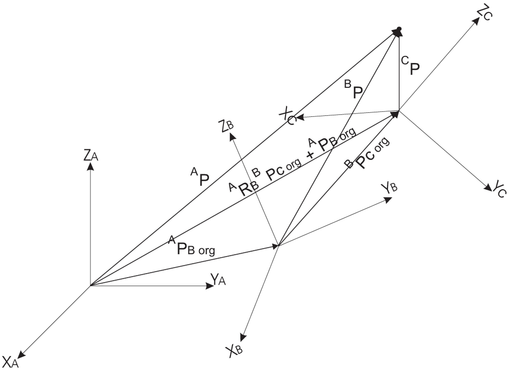
and hence

$$^{A}_{C}T = {}^{A}_{B}T \; {}^{B}_{C}T$$

Structure of the composed transformation

$$ {}^{A}_{C}T = {}^{A}_{B}T \; {}^{B}_{C}T $$

$$ {}^{A}_{C}T = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{B\ org} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}_{C}R & {}^{B}P_{C\ org} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R \; {}^{B}_{C}R & {}^{A}_{B}R \; {}^{B}P_{C\ org} + {}^{A}P_{B\ org} \\ 0 & 1 \end{bmatrix} $$



Inverting a transformation - ${}^{B}_{A}T = ({}^{A}_{B}T)^{-1}$

Option 1: Explicitly invert the matrix, i.e., ${}^{B}_{A}T = ({}^{A}_{B}T)^{-1}$

Option 2: Using the knowledge on the structure ${}^{B}_{A}T = \begin{bmatrix} {}^{B}_{A}R & {}^{B}P_{A\ org} \\ 0 & 1 \end{bmatrix}$

- As previously seen ${}^{B}_{A}R = {}^{A}_{B}R^{T}$

- Making $C \equiv A$ in ${}^{A}P_{C\ org} = {}^{A}_{B}R\ {}^{B}P_{C\ org} + {}^{A}P_{B\ org}$ yields

$$0 = {}^{A}_{B}R\ {}^{B}P_{A\ org} + {}^{A}P_{B\ org}$$

$$-\,{}^{A}P_{B\ org} = {}^{A}_{B}R\ {}^{B}P_{A\ org}$$

$${}^{B}P_{A\ org} = -\,{}^{A}_{B}R^{T}\ {}^{A}P_{B\ org}$$

and hence ${}^{B}_{A}T = \begin{bmatrix} {}^{A}_{B}R^{T} & -\,{}^{A}_{B}R^{T}\ {}^{A}P_{B\ org} \\ 0 & 1 \end{bmatrix}$

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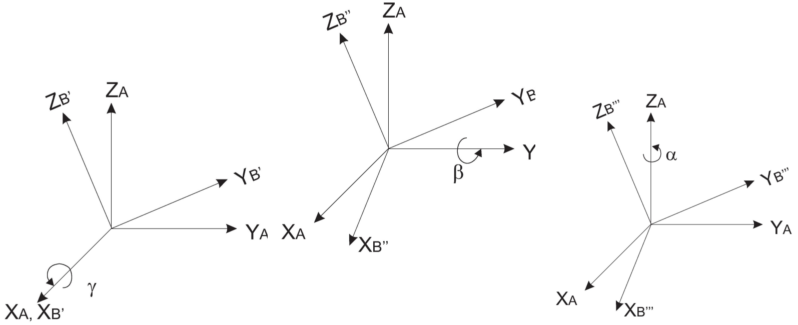
Rotations around fixed axis (angles $X - Y - Z$)



$$
{}^{A}P_{B'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix} {}^{A}P
$$

$$
{}^{A}P_{B'''} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^{A}P_{B''}
$$

$$
{}^{A}P_{B''} = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix} {}^{A}P_{B'}
$$

Rotations around fixed axis (angles $X - Y - Z$)

The sequence of transformations is described by the product (note that the input and output frames are the same)

$$
{}^{A}P_2 = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix} {}^{A}P_1
$$

that is $\quad {}^{A}R_{X-Y-Z}(\gamma, \beta, \alpha) = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$

where

$$
\begin{aligned}
c_\alpha &= \cos(\alpha) \\
s_\alpha &= \sin(\alpha) \\
c_\beta &= \cos(\beta) \\
s_\beta &= \sin(\beta) \\
c_\gamma &= \cos(\gamma) \\
s_\gamma &= \sin(\gamma)
\end{aligned}
$$

$$
\underbrace{{}^{A}P_2}_{\text{point after the rotation}} = {}^{A}R_{X-Y-Z}(\gamma, \beta, \alpha) \underbrace{{}^{A}P_1}_{\text{point before the rotation}}
$$

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Rotation around fixed axis: Order of rotations

... and if the sequence of rotations is not $X - Y - Z$?

Check what happens when the 1st rotation takes place around X and the 2nd goes around Y ...

$$\begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix} = \begin{bmatrix} c_\alpha c_\beta & -s_\alpha & c_\alpha s_\beta \\ s_\alpha c_\beta & c_\alpha & s_\alpha s_\beta \\ -s_\beta & 0 & c_\beta \end{bmatrix}$$

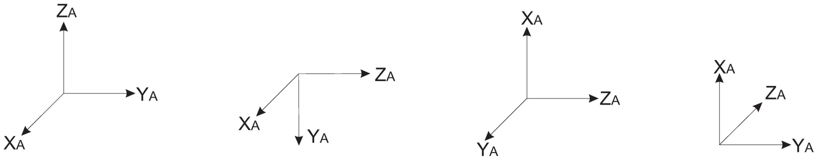
and compare when the rotation sequence is reversed

$$\begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix} \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\beta c_\alpha & -c_\beta s_\alpha & s_\beta \\ s_\alpha & c_\alpha & 0 \\ -s_\beta c_\alpha & -s_\beta s_\alpha & c_\beta \end{bmatrix}$$

... and conclude that the rotation sequence is important (recall that matrix product is non commutative) (note that in any case the result is a rotation matrix ...)

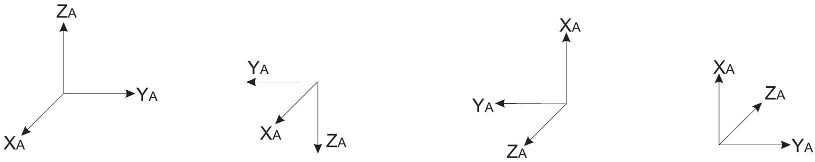
Rotation around fixed axis – Non uniqueness – Example

The set of fixed angles $X - Y - Z$ representing the orientation of a reference frame is not unique



Rotations: $R_X(-\frac{\pi}{2})$ $R_Y(-\frac{\pi}{2})$ $R_Z(\frac{\pi}{2})$

Rotation around fixed axis – Non uniqueness – Example



Rotations: $R_X(\pi)$ $R_Y(-\frac{\pi}{2})$ $R_Z(\pi)$

... 2 rotation sequences yield the same orientation

How many configurations for each orientations ? See ahead

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Rotation around fixed axis – Alternative conventions

What are the possible alternatives ?

Convention	Yes/No	Convention	Yes/No	Convention	Yes/No
$X - X - X$	$\times$	$X - X - Y$	$\times$	$X - X - Z$	$\times$
$X - Y - X$		$X - Y - Y$	$\times$	$X - Y - Z$	
$X - Z - X$		$X - Z - Y$		$X - Z - Z$	$\times$
$Y - X - X$	$\times$	$Y - X - Y$		$Y - X - Z$	
$Y - Y - X$	$\times$	$Y - Y - Y$	$\times$	$Y - Y - Z$	$\times$
$Y - Z - X$		$Y - Z - Y$		$Y - Z - Z$	$\times$
$Z - X - X$	$\times$	$Z - X - Y$		$Z - X - Z$	
$Z - Y - X$		$Z - Y - Y$	$\times$	$Z - Y - Z$	
$Z - Z - X$	$\times$	$Z - Z - Y$	$\times$	$Z - Z - Z$	$\times$

- There are 27 possible conventions of which only 12 are admissible
- A convention is admissible iff it is possible to determine 3 independent angles (which does not happen when there are consecutive rotations around the same axis)

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Fixed angles $X - Y - Z$: Inverse problem I

Given a rotation matrix $\,^{A}R_{\,X-Y-Z}(\gamma,\beta,\alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$ determine α, β e γ

By inspection

$$\left.\begin{aligned} \pm\sqrt{r_{11}^2 + r_{21}^2} &= c_\beta \\ r_{31} &= -s_\beta \end{aligned}\right\} \longrightarrow \quad \beta = \text{atan2}\left(-r_{31}, \pm\sqrt{r_{11}^2 + r_{21}^2}\right)$$

$$\left.\begin{aligned} r_{11} &= c_\alpha c_\beta \\ r_{21} &= s_\alpha c_\beta \end{aligned}\right\} \longrightarrow \quad \alpha = \text{atan2}\left(\frac{r_{21}}{c_\beta}, \frac{r_{11}}{c_\beta}\right), \qquad \text{if } c_\beta \neq 0$$

$$\left.\begin{aligned} r_{32} &= c_\beta s_\gamma \\ r_{33} &= c_\beta c_\gamma \end{aligned}\right\} \longrightarrow \quad \gamma = \text{atan2}\left(\frac{r_{32}}{c_\beta}, \frac{r_{33}}{c_\beta}\right), \qquad \text{if } c_\beta \neq 0$$

Fixed angles $X - Y - Z$: Inverse problem II

When $\cos(\beta) = 0$, $\sin(\beta) = \pm 1$

$$r_{12} = \pm\sin(\gamma \mp \alpha) \qquad r_{13} = \cos(\gamma \mp \alpha)$$
$$r_{22} = \cos(\gamma \mp \alpha) \qquad r_{23} = \pm\sin(\gamma \mp \alpha)$$

and hence it is possible to determine $\gamma \mp \alpha$

By convention,

$$\text{If } \beta = \pi/2 \quad \begin{cases} \alpha = 0 \\ \gamma = \operatorname{atan2}(r_{12}, r_{22}) \end{cases}$$

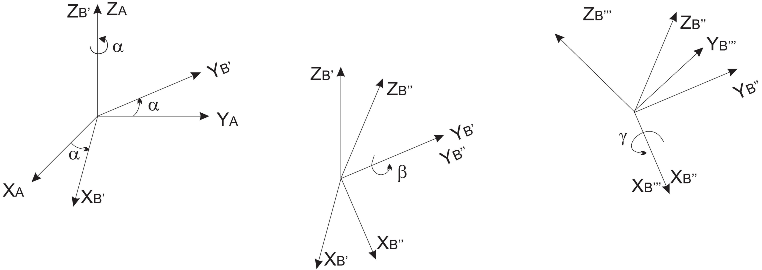
$$\text{Se } \beta = -\pi/2 \quad \begin{cases} \alpha = 0 \\ \gamma = -\operatorname{atan2}(r_{12}, r_{22}) \end{cases}$$

Fixed angles $X - Y - Z$: Inverse problem III

- The case seen before of non-uniqueness of the α, β, γ is an example of $\beta = \pm\pi/2$, $\alpha \pm \gamma = 0$

- Whenever $\beta \neq \pm\pi/2$ there are 2 solutions for α, β, γ

Rotations around moving axis (Euler angles $Z - Y - X$)



$$^{A}P = {}^{A}_{B'}R_{Z}\,{}^{B'}P$$

$$= \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^{B'}P$$

$$^{B'}P = {}^{B'}_{B''}R_{Y}\,{}^{B''}P$$

$$= \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix} {}^{B''}P$$

$$^{B''}P = {}^{B''}_{B'''}R_{X}\,{}^{B'''}P$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix} {}^{B'''}P$$

Rotations around moving axis (Euler angles $Z - Y - X$)

Concatenating the rotations (note that the input and output frames are different)

$^{A}P = {}^{A}_{B'}R_{Z}\,{}^{B'}_{B''}R_{Y}\,{}^{B''}_{B'''}R_{X}\,{}^{B'''}P = {}^{A}_{B'''}R_{ZYX}(\alpha,\beta,\gamma)\,{}^{B'''}P$

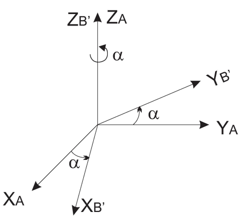
$$
{}^{A}_{B'''}R_{ZYX}(\alpha,\beta,\gamma) = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}
$$

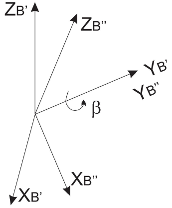
Conclusion 1: <u>Fixed angles $X - Y - Z$</u> yields a rotation matrix identical to <u>Euler angles $Z - Y - X$</u>

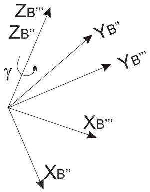
Note: The meaning of the angles is, of course, different between the two conventions

Conclusion 2: The inverse problem for Euler angles $Z - Y - X$ has the same solution of the fixed angles $X - Y - Z$ inverse problem

Rotations around moving axis (Euler angles $Z-Y-Z$)







$$
{}^{A}P = {}^{A}_{B'}R_{Z}\;{}^{B'}P
= \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^{B'}P
$$

$$
{}^{B'}P = {}^{B'}_{B''}R_{Y}\;{}^{B''}P
= \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix} {}^{B''}P
$$

$$
{}^{B''}P = {}^{B''}_{B'''}R_{Z}\;{}^{B'''}P
= \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^{B'''}P
$$

Rotations around moving axis (Euler angles $Z-Y-Z$)

The composition of the 3 transformations yields

$$^{A}P = {}^{A}_{B'}R_{Z}\ {}^{B'}_{B''}R_{Y}\ {}^{B''}_{B'''}R_{Z}\ {}^{B'''}P = {}^{A}_{B'''}R_{ZYZ}(\alpha,\beta,\gamma)\ {}^{B'''}P$$

$$^{A}_{B'''}R_{ZYZ}(\alpha,\beta,\gamma) = \begin{bmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\ s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{bmatrix}$$

Euler angles $Z - Y - Z$: Inverse problem

The solution follows from using the same technique of previous cases

$$\begin{bmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\ s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Clearly

$$\gamma = \text{atan2}\,(r_{32}, -r_{31}) \qquad \text{if} \quad s_\beta > 0$$

$$\gamma = \text{atan2}\,(-r_{32}, r_{31}) \qquad \text{if} \quad s_\beta < 0$$

$$\beta = \text{atan2}\left(\pm\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right) = \text{atan2}\left(\pm\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right)$$

$$\alpha = \text{atan2}\,(r_{23}, r_{13}) \qquad \text{if} \quad s_\beta > 0$$

$$\alpha = \text{atan2}\,(-r_{23}, -r_{13}) \qquad \text{if} \quad s_\beta < 0$$

Euler angles $Z - Y - Z$: Inverse problem I

If $s_\beta = 0$ then $c_\beta = \pm 1$.

Assuming $c_\beta = 1$ comes for the rotation matrix

$$\begin{bmatrix} c_\alpha c_\gamma - s_\alpha s_\gamma & -c_\alpha s_\gamma - s_\alpha c_\gamma & 0 \\ s_\alpha c_\gamma + c_\alpha s_\gamma & -s_\alpha s_\gamma + c_\alpha c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{\alpha\gamma} & -s_{\alpha\gamma} & 0 \\ s_{\alpha\gamma} & c_{\alpha\gamma} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\alpha + \gamma = \operatorname{atan2}(r_{21}, r_{11}) \quad \text{or} \quad \alpha + \gamma = \operatorname{atan2}(-r_{12}, r_{22}) \quad \text{or}$$

$$\alpha + \gamma = \operatorname{atan2}(r_{21}, r_{22}) \quad \text{or} \quad \alpha + \gamma = \operatorname{atan2}(-r_{12}, r_{11})$$

Euler angles $Z - Y - Z$: Inverse problem II

Using $c_\beta = -1$ yields for the rotation matrix

$$\begin{bmatrix} -c_\alpha c_\gamma - s_\alpha s_\gamma & c_\alpha s_\gamma - s_\alpha c_\gamma & 0 \\ -s_\alpha c_\gamma + c_\alpha s_\gamma & s_\alpha s_\gamma + c_\alpha c_\gamma & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -\cos(\alpha-\gamma) & -\sin(\alpha-\gamma) & 0 \\ -\sin(\alpha-\gamma) & \cos(\alpha-\gamma) & 0 \\ 0 & 0 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\alpha - \gamma = \text{atan2}\left(-r_{21}, -r_{11}\right) \quad \text{or} \quad \alpha - \gamma = \text{atan2}\left(-r_{12}, r_{22}\right) \quad \text{or}$$

$$\alpha - \gamma = \text{atan2}\left(-r_{21}, r_{22}\right) \quad \text{or} \quad \alpha - \gamma = \text{atan2}\left(-r_{12}, -r_{11}\right)$$