Due 2pm on 9/15/2022

Black-Scholes/Merton European option pricing.

You have probably already implemented this in an earlier course but this homework will provide you with a new angle that can be generalized to any model of stock price.

Exercise 1. *Show* $\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1$.

Hint: Let $I = \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx$ so $I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-\pi y^2}\,dx\,dy$. Change to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and show $I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\pi r^2}\,r\,dr\,d\theta$. Note $de^{-\pi r^2} = -2\pi r e^{-\pi r^2}\,dr$ so $\int r e^{-\pi r^2/2} = -e^{-\pi r^2}/2\pi$.

Exercise 2. *Show* $\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\pi/\alpha}$, $\alpha > 0$.

The standard normal density function is $\phi(x) = e^{-x^2/2}/\sqrt{2\pi} > 0$, $-\infty < x < \infty$. The above shows $\int_{-\infty}^{\infty} \phi(x)\,dx = 1$ so ϕ is a probability density. The cumulative distribution function is $\Phi(x) = \int_{-\infty}^{x} f(u)\,du$. Let Z be the random variable defined by this so $P(Z \le z) = \Phi(z)$ and

This function is not part of the standard C library, but `erf` is.

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

Exercise 3. *Show* $\Phi(z) = (1 + \mathrm{erf}(z/\sqrt{2}))/2$.

For any function f, $E[f(Z)] = \int_{-\infty}^{\infty} f(z) e^{-z^2/2}\,dz/\sqrt{2\pi}$.

To compute all moments $\mu_n = E[Z^n]$ we can use the formula for the *moment generating function* $E[e^{sZ}] = \sum_{n=0}^{\infty} E[Z^n]s^n/n! = \sum_{n=0}^{\infty} \mu_n s^n/n!$.

Exercise 4. *Show* $E[e^{sZ}] = e^{s^2/2}$.

Hint: $sz - z^2/2 = s^2/2 - (z-s)^2/2$.

We have $(s^2/2)^n/n! = \mu_{2n}s^{2n}/(2n)!$ so $\mu_{2n} = (2n)!/2^n n!$. All odd moments are 0. The fourth moment is $\mu_4 = 4!/2^2 2! = 24/8 = 3$.

Exercise 5. *Show* $E[e^{sZ}f(Z)] = E[e^{sZ}]E[f(Z+s)]$.

Hint: $sz - z^2/2 = s^2/2 - (z-s)^2/2$.

In the Fischer Black model the forward value of a stock at time t is

$$F_t = f e^{-\sigma^2 t/2 + \sigma B_t}$$

where $(B_t)_{t\ge 0}$ is standard Brownian motion. Recall $E[B_t] = 0$ and $\mathrm{Var}(B_t) = t$, $t \ge 0$. The forward value of a put with strike k and expiration t is $E[\max\{k - F_t, 0\}] = E[(k - F_t)^+]$.
Since

$$F = f e^{-s^2/2 + sZ}$$

where $s = \sigma\sqrt{t}$ and Z is standard normal, has the same distribution as F_t we need to compute $E[(k - F)^+]$.

Exercise 6. *Show* $E[(k - F)^+] = kP(F \leq k) - fP(Fe^{s^2} \leq k)$.

Hint: Use $\max\{x, 0\} = x^+ = x1(x \geq 0)$ where $1(x \geq 0) = 1$ if $x \geq 0$ and $1(x \geq 0) = 0$ if $x < 0$. You will also need $E[e^{sZ} f(Z)] = E[e^{sZ}]E[f(Z + s)]$.

Exercise 7. *Show* $F \leq k$ *if and only if* $Z \leq (\log k/f + s^2/2)/s$ *and* $Fe^{s^2} \leq k$ *if and only if* $Z + s \leq (\log k/f + s^2/2)/s$.

Exercise 8. *Follow the directions in this video.*