Due 2pm on 9/8/2022

Exercise 1. If X is a random variable with mean m and variance s^2 show $(X - m)/s$ has mean 0 and variance 1.

Solution

$E[(X - m)/s] = (E[X] - m)/s = 0$ since $E[X] = m$.

$\mathrm{Var}((X - m)/s) = \mathrm{Var}((X - m))/s^2 = \mathrm{Var}(X)/s^2 = 1$.

Recall the *cumulant* of a random variable X is $\kappa^X(s) = \log E[e^{sX}]$.

Exercise 2. *Let F be any positive random variable where $\log F$ has finite mean and variance. Show $F = fe^{-\kappa^X(s)+sX}$ where $f = E[F]$, $s^2 = \mathrm{Var}(\log F)$, and X has mean 0 and variance 1.*

Hint: $\log F = m + sX$ where X has mean 0 and variance 1.

Solution

$\mathrm{Var}(\log F) = s^2\,\mathrm{Var}(X) = s^2$.

$f = E[F] = E[e^{m+sX}] = e^m e^{\kappa(s)}$ so $e^m = fe^{-\kappa(s)}$.

Exercise 3. *If $F = fe^{-\kappa(s)+sX}$ show $F \le k$ if and only if $X \le (\log k/f + \kappa(s))/s$.*

Hint: Assume $s > 0$.

Solution

$$
\begin{aligned}
F &\le k \\
fe^{-\kappa(s)+sX} &\le k \\
-\kappa(s) + sX &\le \log k/f \\
sX &\le \log(k/f) + \kappa(s) \\
X &\le (\log(k/f) + \kappa(s))/s
\end{aligned}
$$

We call the functions $x(k; f, s) = (\log k/f + \kappa(s))/s$ the *moneyness* at k.

The Fischer Black model assumes X is normal and $s = \sigma\sqrt{t}$.

Exercise 4. *Find the formula for moneyness in the Black model.*

Recall if N is a normally distributed random variable then $E[e^N] = e^{E[N]+\mathrm{Var}(N)/2}$.

Solution

$\kappa(s) = \log E[e^{sX}] = \log e^{s^2/2} = s^2/2$.

$(\log k/f + \kappa(s))/s = (\log(k/f) + \sigma^2 t/2)/\sigma\sqrt{t}(\log(k/f)/\sigma\sqrt{t} + \sigma^2\sqrt{t}/2)$.