Due 2pm on 9/15/2022

Black-Scholes/Merton European option pricing.

You have probably already implemented this in an earlier course but this homework will provide you with a new angle that can be generalized to any model of stock price.

Exercise 1. Show $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$.

Hint: Let $I = \int_{-\infty}^{\infty} e^{-\pi x^2} dx$ so $I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-\pi y^2} dx dy$. Change to polar coordinates $x = r \cos \theta$, $y = r \sin \theta$ and show $I^2 = \int_{0}^{2\pi} \int_{0}^{\infty} e^{-\pi r^2} r dr d\theta$. Note $de^{-\pi r^2} = -2\pi r e^{-\pi r^2} dr$ so $\int r e^{-\pi r^2/2} = -e^{-\pi r^2}/2\pi$.

Exercise 2. Show $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\pi/\alpha}, \ \alpha > 0.$

The standard normal density function is $\phi(x)=e^{-x^2/2}/\sqrt{2\pi}>0, -\infty < x < \infty$. The above shows $\int_{-\infty}^{\infty}\phi(x)\,dx=1$ so ϕ is a probability density. The cumulative distribution function is $\Phi(x)=\int_{-\infty}^{x}f(u)\,du$. Let Z be the random variable defined by this so $P(Z\leq z)=\Phi(z)$ and

This function is not part of the standard C library, but erf is.

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Exercise 3. Show $\Phi(z) = (1 + \operatorname{erf}(z/\sqrt{2}))/2$.

For any function f, $E[f(Z)] = \int_{-\infty}^{\infty} f(z)e^{-z^2/2} dz/\sqrt{2\pi}$.

To compute all moments $\mu_n = E[Z^n]$ we can use the formula for the moment generating function $E[e^{sZ}] = \sum_{n=0}^{\infty} E[Z^n] s^n/n! = \sum_{n=0}^{\infty} \mu_n s^n/n!$.

Exercise 4. Show $E[e^{sZ}] = e^{s^2/2}$.

Hint: $sz - z^2/2 = s^2/2 - (z - s)^2/2$.

We have $(s^2/2)^n/n! = \mu_{2n} s^{2n}/(2n)!$ so $\mu_{2n} = (2n)!/2^n n!$. All odd moments are 0. The fourth moment is $\mu_4 = 4!/2^2 2! = 24/8 = 3$.

Exercise 5. Show $E[e^{sZ}f(Z)] = E[e^{sZ}]E[f(Z+s)].$

Hint: $sz - z^2/2 = s^2/2 - (z - s)^2/2$.

In the Fischer Black model the forward value of a stock at time t is

$$F_t = f e^{-\sigma^2 t/2 + \sigma B_t}$$

where $(B_t)_{t\geq 0}$ is standard Brownian motion. Recall $E[B_t]=0$ and $\mathrm{Var}(B_t)=t$, $t\geq 0$. The forward value of a put with strike k and expiration t is $E[\max\{k-F_t,0\}]=E[(k-F_t)^+]$. Since

$$F = fe^{-s^2/2 + sZ}$$

where $s = \sigma \sqrt{t}$ and Z is standard normal, has the same distribution as F_t we need to compute $E[(k-F)^+]$.

Exercise 6. Show $E[(k-F)^{+}] = kP(F \le k) - fP(Fe^{s^{2}} \le k)$.

Hint: Use $\max\{x,0\} = x^+ = x \mathbf{1}(x \ge 0)$ where $\mathbf{1}(x \ge 0) = 1$ if $x \ge 0$ and $\mathbf{1}(x \ge 0) = 0$ if x < 0. You will also need $E[e^{sZ}f(Z)] = E[e^{sZ}]E[f(Z+s)]$.

Exercise 7. Show $F \le k$ if and only if $Z \le (\log k/f + s^2/2)/s$ and $Fe^{s^2} \le k$ if and only if $Z + s \le (\log k/f + s^2/2)/s$.

Exercise 8. Follow the directions in this video.