Logistic Distribution

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Every positive random variable F is parameterized by $F = fe^{-\kappa(s)+sX}$ where $f = E[F]$, $s^2 = \mathrm{Var}(\log F)$, X is a random variable with mean 0, variance 1, and $\kappa(s) = \log E[e^{sX}]$ is the cumulant of X.

The Black model of option pricing assumes X is standard normal. We investigate the case when X has a *logistic* distribution.

Logistic Distribution

The logistic cumulative distribution function with parameter σ is $P(X \leq x) = F(x) = 1/(1 + e^{-x/\sigma})$, $-\infty < x < \infty$ and has density function is $f(x) = e^{-x/\sigma}/\sigma(1 + e^{-x/\sigma})^2$. By symmetry, it has mean 0. It is not trivial to show its variance is $\pi^2\sigma^2/3$ so $\sigma = \sqrt{3}/\pi$ implies variance 1.

Solving $q = F(x) = 1/(1 + e^{-x/\sigma})$ for x gives the quantile function

$$\begin{aligned}
q &= F(x) = 1/(1 + e^{-x/\sigma}) \\
(1 + e^{-x/\sigma}) &= 1/q \\
e^{-x/\sigma} &= 1/q - 1 = (1 - q)/q \\
e^{x/\sigma} &= q/(1 - q) \\
x &= F^{-1}(q) = \sigma \log(q/(1 - q)) = \sigma(\log q - \log(1 - q))
\end{aligned}$$

Since $e^{x/\sigma} = q/(1 - q)$ and $dF(x) = dq$ we have the moment generating function is

$$\begin{aligned}
E[e^{sX}] &= \int_{-\infty}^{\infty} e^{sx}\, dF(x) \\
&= \int_0^1 (q/(1 - q))^{\sigma s}\, dq \\
&= B(1 + \sigma s, 1 - \sigma s)
\end{aligned}$$

where $B(\alpha, \beta) = \int_0^1 q^{\alpha-1}(1 - q)^{\beta-1}\, dq$ is the beta function and we use $\alpha - 1 = \sigma s$ and $\beta - 1 = -\sigma s$.

The Gamma function $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\, dx$, $\alpha > 0$, is related to the Beta function by $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$. Note $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, $\alpha > 0$. Euler's reflection formula for the Gamma function is $\Gamma(\alpha)\Gamma(1 - \alpha) = \pi/\sin(\pi\alpha)$.

Exercise. *Show* $E[e^{sX}] = \Gamma(1+\sigma s)\Gamma(1-\sigma s) = \pi\sigma s/\sin\pi\sigma s$, $-1/\sigma < s < 1/\sigma$.

The cumulant is $\kappa(s) = \log E[e^{sX}] = \log\pi\sigma s - \log\sin\pi\sigma s$. Using $\mathrm{Var}(X) = \kappa''(0)$ a non-trivial calculation gives $\mathrm{Var}(X) = \pi^2\sigma^2/3$.

Incomplete Beta Function

Share measure for the logistic is

$$\begin{aligned} P^s(X \le x) &= E[e^{-\kappa(s)+sX} 1(X \le x)] \\ &= \int_{-\infty}^{x} e^{-\kappa(s)+sz}\, dF(z) \\ &= e^{-\kappa(s)} \int_0^{F(x)} (q/(1-q))^{\sigma s}\, dq \\ &= B_{F(x)}(1+\sigma s, 1-\sigma s)/B(1+\sigma s, 1-\sigma s) \end{aligned}$$

where $B_u(\alpha,\beta) = \int_0^u q^{\alpha-1}(1-q)^{\beta-1}\, dq$ is the incomplete beta function.

This can be calculated using the hypergeometic function

$${}_2F_1(a,b;c;u) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{u^n}{n!}$$

where $(q)_n$ is the rising Pockhammer symbol defined by $(q)_0 = 1$ and $(q)_n = q(q+1)\cdots(q+n-1)$ if $n > 0$.

We have

$$B_u(\alpha,\beta) = \frac{x^\alpha}{\alpha}\, {}_2F_1(\alpha, 1-\beta; \alpha+1; u)$$

so

$$P^s(X \le x) = \frac{F(x)^{1+\sigma s}}{(1+\sigma s)B(1+\sigma s, 1-\sigma s)}\, {}_2F_1(1+\sigma s, \sigma s; 2+\sigma s; F(x))$$