Binomial Model

Keith A. Lewis

Abstract

Discrete time random walk.

Binomial models are the simplest example of a stochastic process. They use

Let (Y_j) be independent random variables with $P(Y_j = 0) = P(Y_j = 1) = 1/2$. *Random walk* at time n is $V_n = Y_1 + \cdots + Y_n$ with $V_0 = 0$. The sample space is $\Omega = \{0,1\}^\infty$ where $Y_j(\omega) = \omega_j$ where $\omega = (\omega_j)$. The algebra generated by $Y_0, \ldots, Y_n$ can be identified with $\mathcal{A}_n = \{0,1\}^n$ indicating the outcome of each Y_j, $1 \le j \le n$.

Exercise. *Show* $P(V_n = k) = C(n,k)/2^n$, $k = 0, 1, \ldots, n$.

Hint. $C(n,k) = n!/(n-k)!k!$ is the number of choices of k items out of n items.

Fix $\nu\colon \mathbf{N} \to \mathbf{R}$ and let $v_j(i) = E[\nu(V_n) \mid V_j = i]$.

Exercise. *Show* $v_j(i) = v_{j+1}(i)/2 + v_{j+1}(i+1)/2$.

Hint. Use $V_{j+1} = V_j + Y_{j+1}$ to show $\{V_j = i\} = \{V_{j+1} = i\} \cup \{V_{j+1} = i+1\}$

Define *symmetric random walk* $W_n = n - 2V_n$. Let $X_j = 1 - 2Y_j$ so $W_n = X_1 + \cdots X_n$. Note $P(X_j = 1) = P(X_j = -1) = 1/2$.

Exercise. *Show* $E[W_n] = 0$ *and* $\operatorname{Var}(W_n) = n$.

Hint $E[X_j] = 0$ and $\operatorname{Var}(X_j) = 1$.

Exercise. *Show* $E[e^{sW_n}] = \cosh^n s$.

Hint. Recall $\cosh s = (e^s + e^{-s})/2$.

The *binomial model* for underlying F with n steps is $F = fe^{sW_n/\sqrt{n}}/\cosh^n(s/\sqrt{n})$.

Exercise. *Show* $E[F] = f$ *and* $\operatorname{Var}(\log F) = s^2$.

Let $Z_j = e^{sX_j/\sqrt{n}}/\cosh(s/\sqrt{n})$ and $F_j = f\Pi_{i=1}^j Z_i$.

Exercise. *Show* (F_j) *is a martingale and* $F_n = F$.

Hint. Show $E_j[F_{j+1}] = F_j E[Z_{j+1}]$. Recall $E[X \mid \mathcal{A}] = E[X]$ if X is independent of the algebra $\mathcal{A}$.

American Option

An *American option* on a futures with price F_t at time t specifies a payoff function ν and an expiration T. At any time $t \leq T$ the option holder can *exercise* the option and receive a cash flow of $\nu(F_t)$ at time t. We want to compute $v = \max_{\tau \leq T} E[\nu(F_\tau)]$ where τ is a stopping time. Recall τ is a stopping time if and only if $A_t = \{\tau = t\}$ is measurable at time t, $t \geq 0$.

Assume the option can only be exercised at discrete times $0 \leq t_0 < t_1 \cdots < t_n = T$. This makes the option *Bermudan*, somewhere between American and European. Let $v_j = \max_\tau E_{t_j}[\nu(F_\tau) \mid \tau \geq t_j]$ At time t_j the option holder can exercise and receive $\nu(F_{t_j})$ or continue to hold the option and receive v_{j+1} at t_{j+1}. We should exercise at t_j when $\nu(F_{t_j}) > E_{t_j}[v_{j+1}]$.

Exercise. *Show* $v_n = \nu(F_{t_n}) = \nu(F_T)$.

Using the procedure above determines v_j for $j < n$ and we can compute $v_0 = \max_\tau E[\nu(F_\tau)]$.