Monte Carlo

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Abstract

Integrate using random variates

The *Monte Carlo* method of evaluating integrals using random variates was invented by Stan Ulam and Nick Metropolis while working on The Manhattan Project. It is based on the facts that if U is uniformly distributed on the interval $[0,1]$ then $E[f(U)] = \int_0^1 f(x)\,dx$ and if (X_j) are independent, identically distributed random variables then the average $(X_1 + \cdots + X_n)/n$ tends to $E[X]$.

If $F'(x) = f(x)$ then the fundamental theorem of calculus states $\int_0^1 f(x)\,dx = F(1) - F(0)$, however finding the anti-derivative, F, of f may be difficult. Monte Carlo estimates the integral by generating uniform $[0,1]$ variates u_1, …, u_n and computing the averages $(f(u_1) + \cdots + f(u_n))/n$. Replacing the numerical variates (u_j) by independent uniform random variables (U_j) lets us draw statistical conclusions. Clearly $E[\sum_1^n f(U_j)/n] = E[f(U)]$.

Exercise. *Show* $\mathrm{Var}(\sum_1^n f(U_j)/n) = \mathrm{Var}(f(U))/n$.

Hint. If random variables X and Y are independent then $f(X)$ and $g(Y)$ are independent for any functions f and g.

This is called the *weak law of large numbers* but it reveals an important general fact: **when trying to estimate a random variable using n samples the standard deviation is proportional to $1/\sqrt{n}$**.

Monte Carlo methods can be used for any random variable, not just uniform on $[0,1]$.

Exercise. *If X has cdf F then $E[g(X)] = E[g(F^{-1}(U))]$ where U is uniformly distributed on the interval* $[0,1]$.

Hint. Show X and $F^{-1}(U)$ have the same law.

Variance Reduction

Although variance is proportional to $1/n$ there are methods to reduce the constant of proprionality.

Antithetic Variates

If X and Y have the same law then $E[X] = E[Y]$ so $E[(X+Y)/2] = E[X] = E[Y]$ and $\operatorname{Var}((X+Y)/2) = \operatorname{Var}(X)/4 + \operatorname{Cov}(X,Y)/2 + \operatorname{Var}(Y)/4 = \operatorname{Var}(X)/2 + \operatorname{Cov}(X,Y)/2$. If $X = Y$ then $\operatorname{Var}((X+Y)/2) = \operatorname{Var}(X) = \operatorname{Var}(Y)$ and if $X = -Y$ then $\operatorname{Var}((X+Y)/2) = 0$.

Exercise. *If X and $-X$ have the same law and* $\operatorname{Cov}(f(X), f(-X)) < \operatorname{Var}(f(X))$ *then* $\operatorname{Var}((f(X) + f(-X))/2) < \operatorname{Var}(f(X))$.

The estimate of $E[f(X)]$ can be improved by averaging with the estimate of $E[f(-X)]$ if $\operatorname{Cov}(f(X), f(-X)) < \operatorname{Var}(f(X))$.

Black Model

The Fischer Black model for the forward price of a stock is $F_t = fe^{\sigma B_t - \sigma^2 t/2}$. The antithetic variate $F_t^* = fe^{-\sigma B_t - \sigma^2 t/2}$ can be used to reduce variance.

Control Variate

A *control variate* for a random variable X is a random variable Y that is close to X that has known mean and variance.

If X and Y are any random variables with non-zero variance then $E[X] = E[X - c(Y - E[Y])]$ for any $c \in \mathbf{R}$ and $\operatorname{Var}(X - c(Y - E[Y])) = \operatorname{Var}(X) - 2c\operatorname{Cov}(X, Y - E[Y]) + c^2 \operatorname{Var}(Y - E[Y])$.

Exercise. *Show this is minimized when* $c = \operatorname{Cov}(X,Y)/\operatorname{Var}(Y)$.

Hint. Take the derivative with respect to c and note $\operatorname{Var}(Y - E[Y]) = \operatorname{Var}(Y) > 0$.

Exercise. *Show the miniumum is* $\operatorname{Var}(X) - \operatorname{Cov}(X,Y)^2/\operatorname{Var}(Y)$.

Exercise. *If* $\operatorname{Var}(X) = \operatorname{Var}(Y) = \sigma^2$ *and ρ is the correlation of X and Y then* $\operatorname{Var}(X) - \operatorname{Cov}(X,Y)^2/\operatorname{Var}(Y) = \sigma^2(1 - \rho^2)$.

If Y is close to X then $\operatorname{Cov}(X,Y)$ is positive so $X - c(Y - E[Y])$ has smaller variance than X and sampling $X - c(Y - E[Y])$ would reduce the variance. Since $\operatorname{Cov}(X, Y - E[Y]) = \operatorname{Cov}(X,Y)$ and $\operatorname{Var}(Y - E[Y]) = \operatorname{Var}(Y)$ is known we only need to find $\operatorname{Cov}(X,Y)$. This can be estimated by Monte Carlo sampling of X and Y.