Timoshenko对极坐标应力函数的证明

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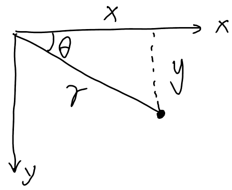
在直角坐标下，由于需要满足变形协调方程：

$$\frac{\partial \phi^4}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}=0$$



为将其转化至极坐标系下，

$2r\frac{\partial r}{\partial x}=2x$　有：$r^2=x^2+y^2$，$\theta=\arctan\frac{y}{x}$

则：$\frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta$；$\frac{\partial \theta}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot -\frac{y}{x^2}=-\frac{y}{x^2+y^2}=-\frac{r\sin\theta}{r^2}=-\frac{\sin\theta}{r}$

$\frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta$；$\frac{\partial \theta}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}=\frac{x}{r^2}=\frac{\cos\theta}{r}$

此时：
$$\frac{\partial\phi}{\partial x}=\underbrace{\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial\phi}{\partial\theta}\cdot\frac{\partial\theta}{\partial x}}=\cos\theta\cdot\frac{\partial\phi}{\partial r}-\frac{\sin\theta}{r}\frac{\partial\phi}{\partial\theta}$$

注意$\phi=\phi(r,\theta)$ 要用复合函数求导

$$\frac{\partial\phi}{\partial y}=\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial\phi}{\partial\theta}\cdot\frac{\partial\theta}{\partial y}=\sin\theta\frac{\partial\phi}{\partial r}+\frac{\cos\theta}{r}\frac{\partial\phi}{\partial\theta}$$

而同样地：$\frac{\partial\phi^2}{\partial x^2}=\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial x}\right)\frac{\partial r}{\partial x}+\frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{\partial x}\right)\cdot\frac{\partial\theta}{\partial x}$，其中注意：$\phi$为$r,\theta$函数

$$=\left[\cos\theta\cdot\frac{\partial^2\phi}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{\sin\theta}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right]\cdot\cos\theta$$
$$+\left[-\sin\theta\frac{\partial\phi}{\partial r}+\cos\theta\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{\cos\theta}{r}\frac{\partial\phi}{\partial\theta}-\frac{\sin\theta}{r}\frac{\partial^2\phi}{\partial\theta^2}\right]\cdot -\frac{\sin\theta}{r}$$

注意变号

$$=\cos^2\theta\frac{\partial^2\phi}{\partial r^2}+\frac{\sin\theta\cos\theta}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{\sin\theta\cos\theta}{r}\frac{\partial^2\phi}{\partial r\partial\theta}$$
$$+\frac{1}{r}\sin^2\theta\frac{\partial\phi}{\partial r}-\frac{\sin\theta\cos\theta}{r}\frac{\partial^2\phi}{\partial r\partial\theta}+\frac{\sin\theta\cos\theta}{r^2}\frac{\partial\phi}{\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$
$$=\cos^2\theta\frac{\partial^2\phi}{\partial r^2}+2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial\phi}{\partial\theta}-2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2\phi}{\partial r\partial\theta}+\frac{1}{r}\sin^2\theta\frac{\partial\phi}{\partial r}+\frac{\sin^2\theta}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\quad ☆①$$

而：$\frac{\partial^2\phi}{\partial y^2}=\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial y}\right)\frac{\partial r}{\partial y}+\frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{\partial y}\right)\cdot\frac{\partial\theta}{\partial y}$.

$$=\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial\phi}{\partial r}+\frac{\cos\theta}{r}\frac{\partial\phi}{\partial\theta}\right)\cdot\sin\theta+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial r}+\frac{\cos\theta}{r}\frac{\partial\phi}{\partial\theta}\right)\cdot\frac{\cos\theta}{r}$$

$$=\left[\sin\theta\frac{\partial^2\phi}{\partial r^2}-\frac{\cos\theta}{r^2}\cdot\frac{\partial\phi}{\partial\theta}+\frac{\cos\theta}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right]\sin\theta$$
$$+\left[\cos\theta\cdot\frac{\partial\phi}{\partial r}+\sin\theta\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{\sin\theta}{r}\frac{\partial\phi}{\partial\theta}+\frac{\cos\theta}{r}\frac{\partial^2\phi}{\partial\theta^2}\right]\frac{\cos\theta}{r}.$$

$$=\sin^2\theta\frac{\partial^2\phi}{\partial r^2}-\frac{\sin\theta\cos\theta}{r^2}\frac{\partial\phi}{\partial\theta}+\frac{\sin\theta\cos\theta}{r}\frac{\partial^2\phi}{\partial r\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial\phi}{\partial r}+\frac{\sin\theta\cos\theta}{r}\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{\sin\theta\cos\theta}{r^2}\frac{\partial\phi}{\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2\phi}{\partial\theta^2},$$
$$=\sin^2\theta\frac{\partial^2\phi}{\partial r^2}-2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial\phi}{\partial\theta}+2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2\phi}{\partial r\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial\phi}{\partial r}+\frac{\cos^2\theta}{r^2}\frac{\partial^2\phi}{\partial\theta^2},\quad ☆②$$

观察①②，有：

①+②：$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$　　也即：$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$

对比(1)(2)，得：

①+②: $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$.

也即：$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta}$

故有：

$$\boxed{\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}}$$

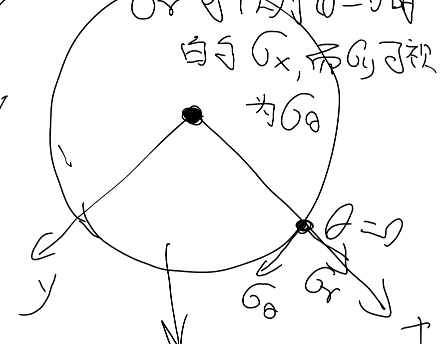
★重要公式！

则有：$\frac{\partial^4\phi}{\partial x^2}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial\theta^2}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)$

则极坐标下的协调方程可表达为：

$$\nabla^4\phi=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2\phi=0$$

说明：对圆盘上任意一点，σ_r 可视为 $\theta=0$ 时的 σ_x，而 σ_y 可视为 σ_θ



而：我们令

$$\sigma_r=\sigma_x\Big|_{\theta=0}=\left(\frac{\partial^2\phi}{\partial y^2}\right)\Big|_{\theta=0}=\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\sigma_\theta=\sigma_y\Big|_{\theta=0}=\left(\frac{\partial^2\phi}{\partial x^2}\right)\Big|_{\theta=0}=\frac{\partial^2\phi}{\partial r^2}$$

应力函数具有坐标变换不变性！

因此可以取不同的坐标轴

显然 σ_r,σ_θ 即为　　同理 $\tau_{xy}=-\left(\frac{\partial^2\phi}{\partial x\partial y}\right)\Big|_{\theta=0}$

$$-\frac{\partial^2\phi}{\partial x\partial y}=\left[\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial\phi}{\partial r}-\frac{\sin\theta}{r}\cdot\frac{\partial\phi}{\partial\theta}\right)\cdot\sin\theta+\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial\phi}{\partial r}-\frac{\sin\theta}{r}\frac{\partial\phi}{\partial\theta}\right)\cdot\frac{\cos\theta}{r}\right]\times(-1)$$

$$=-1\times\left[\left(\frac{\partial^2\phi}{\partial r^2}\cos\theta+\frac{\sin\theta}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{\sin\theta}{r}\frac{\partial^2\phi}{\partial\theta\partial r}\right)\sin\theta+\left(-\sin\theta\frac{\partial\phi}{\partial r}+\cos\theta\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{\cos\theta}{r}\frac{\partial\phi}{\partial\theta}-\frac{\sin\theta}{r}\frac{\partial^2\phi}{\partial\theta^2}\right)\frac{\cos\theta}{r}\right]$$

$$=-\left[\frac{\partial^2\phi}{\partial r^2}\sin\theta\cos\theta+\frac{\sin^2\theta}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{\sin^2\theta}{r}\frac{\partial^2\phi}{\partial\theta\partial r}-\frac{\sin\theta\cos\theta}{r}\frac{\partial\phi}{\partial r}+\frac{\cos^2\theta}{r}\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{\cos^2\theta}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right]$$

$$=-\left(\frac{\partial^2\phi}{\partial r^2}-\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)\sin\theta\cos\theta-\frac{(\cos^2\theta-\sin^2\theta)}{r}\frac{\partial^2\phi}{\partial r\partial\theta}+\frac{(\cos^2\theta-\sin^2\theta)}{r^2}\frac{\partial\phi}{\partial\theta}+\frac{\sin\theta\cos\theta}{r}\frac{\partial\phi}{\partial r}$$

$$=-\left(\frac{\partial^2\phi}{\partial r^2}-\frac{1}{r}\frac{\partial\phi}{\partial r}-\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)\sin\theta\cos\theta+(\cos^2\theta-\sin^2\theta)\left[\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right]$$

令 $\theta=0$，得切应力：

$$\tau \quad \frac{1}{\ }\frac{\partial\phi}{\ }\quad\frac{1}{\ }\frac{\partial^2\phi}{\ }$$

$$\nu = \; r^2 \partial\theta - r\, \partial r \partial\theta \, ,$$