Derivation of the Total FEM Equation

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for the total FEM equation, the general form of it can be written as:

$$A_{mn} u_m = f_n,$$

where $A_{nm} = \int_0^h \frac{d\Phi_n}{dx} \cdot \frac{d\Phi_m}{dx} dx$

Note that we have $\{U_n^{(e)}\} = \Delta_{ni}^{(e)} \{u^{(e)}\}$

$\boxed{u^{(e)} = \Phi_i^{(e)} u_i}$

$\boxed{u_i^{(e)} = \Delta_{in}^{(e)} U_n}$ $(e=1,2,3,4)$

then $u_i^{(e)} = \Delta_{ni}^{(e)} \{u_i^{(e)}\}$

we use the element form of displacement:

the global vector of U

$u_i^{(e)} = \Delta_{in}^{(e)} U_n$

$\rightarrow \Phi_i^{(e)} = \Delta_{in}^{(e)} \Phi_n$

n is the n^{th} node in the global coordinate.

then we have that

$\boxed{\Phi_n = \Delta_{nj}^{(e)} \Phi_i^{(e)}}$ ☆

where $\Delta_{ni}^{(e)} = [\Delta_{in}^{(e)}]^T$.

thus we can substitute it into A_{nm}, then $A_{nm} = \int_0^h \frac{d\Phi_n}{dx} \frac{d\Phi_m}{dx} dx$

$$A_{nm} = \sum_{e=1}^{E} \int_{x_1^{(e)}}^{x_2^{(e)}} \frac{d\Phi_n}{dx} \frac{d\Phi_m}{dx} dx.$$

since we use Δ_{ni}, Δ_{mj} to transform the node i, j (in local coordinate) into m and n (global coordinate)

$$= \sum_{e=1}^{E} \int_{x_1^{(e)}}^{x_2^{(e)}} \frac{d(\Delta_{ni}\Phi_i^{(e)})}{dx} \cdot \frac{d(\Delta_{mj}\Phi_j^{(e)})}{dx} dx$$

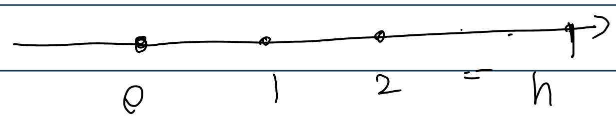
$$= \sum_{e=1}^{E} \boxed{\Delta_{ni} \int_{x_1^{(e)}}^{x_2^{(e)}} \frac{d\Phi_i^{(e)}}{dx} \frac{d\Phi_j^{(e)}}{dx} \Delta_{jm}} dx$$

$\Phi_j \Delta_{jm} = \Delta_{mj} \Phi_j$, where Δ_{jm} is the transform matrix.

then $A_{nm} = \sum_{e=1}^{E} \Delta_{ni} A_{ij}^{(e)} \Delta_{jm}$

also, we can derive that

$$f_n = -\int_0^h c\,\Phi_n\, dx = -\sum_{e=1}^{E} \int_{x_1^{(e)}}^{x_2^{(e)}} c\,\Phi_n\, dx$$



also we have $\Phi_n = \Delta_{ni} \Phi_i$

then $f_n = -\sum_{e=1}^{E} \int_{x_1^{(e)}}^{x_2^{(e)}} c\,\Delta_{ni} \Phi_i\, dx$

since we have that

$f_i^{(e)} = -\int_{x_1^{(e)}}^{x_2^{(e)}} c\,\Phi_i\, dx$, we have:

$$f_n = \sum_{e=1}^{E} \Delta_{ni}^{(e)} f_i^{(e)}$$

thus the result shows that $\Rightarrow$ the total FEM equation is:

where:

thus the result shows that $\Rightarrow$ the final FEM equation is:

$$\boxed{A_{nm} U_m = f_n}$$

where:

$$\begin{cases} A_{nm} = \sum_{e=1}^{E} \left[\Delta_{ni}^{(e)} A_{ij}^{(e)} \Delta_{jm}^{(e)} \right] \\ f_j = \sum_{e=1}^{E} \Delta_{ni}^{(e)} f_i^{(e)} \end{cases}$$

for examplex, we set $i=1$ and $j=1$

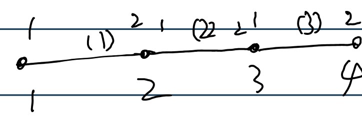
then:

$$A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ A_{21}^{(1)} & A_{22}^{(1)} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} & & \\ A_{21}^{(1)} & A_{22}^{(1)} & & \\ & & & \\ & & & \end{bmatrix}}_{6\times 6}$$

finally we can derive the total matrix as:

$$\begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} & & \\ A_{21}^{(1)} & A_{22}^{(1)} + A_{11}^{(2)} & A_{12}^{(2)} & \\ & A_{21}^{(2)} & A_{22}^{(2)} + A_{11}^{(3)} & A_{12}^{(3)} \\ & & A_{21}^{(3)} & A_{22}^{(3)} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} \end{bmatrix}$$

(using a 3-element structure as an example)



then we note that

$$A_{ij}^{(e)} \longrightarrow A_{nm}^{(e)}$$ is transformed into the global coordinates, and the $\{ij\}$ node in the local coordinate would then be added in the n^{th} line and m^{th} column.