裂纹的应力函数推导

Thursday, November 2, 2023 4:00 PM

首先：在弹性力学中，我们有下列基本方程

1). 平衡微分方程：

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x}+\dfrac{\partial \tau_{xy}}{\partial y}=0 \\ \dfrac{\partial \tau_{xy}}{\partial x}+\dfrac{\partial \sigma_y}{\partial y}=0 \end{cases}$$

2) 几何方程

$$\begin{cases} \varepsilon_x=\dfrac{\partial u}{\partial x} \\ \varepsilon_y=\dfrac{\partial v}{\partial y} \\ \gamma_{xy}=\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \end{cases}$$

3). 本构关系

$$\begin{cases} \varepsilon_x=\dfrac{1}{E}(\sigma_x-\nu\sigma_y) \\ \varepsilon_y=\dfrac{1}{E}(-\nu\sigma_x+\sigma_y) \\ \gamma_{xy}=\dfrac{\tau_{xy}}{G} \end{cases}$$ 其中 $G=\dfrac{E}{2(1+\nu)}$

4) 边界条件（应力边界条件）

有 $\begin{cases} \sigma_x m+\tau_{xy}n=P_x \\ \tau_{xy}m+\sigma_y n=P_y \end{cases} \longrightarrow \begin{aligned} m&=\cos\varphi \\ n&=\sin\varphi \end{aligned}$

①: Airy 应力函数：

由 $\begin{cases} \dfrac{\partial \sigma_x}{\partial x}=-\dfrac{\partial \tau_{xy}}{\partial y} \longrightarrow 我们可以取一个函数A，有：\dfrac{\partial A}{\partial y}=\sigma_x,\ \dfrac{\partial A}{\partial x}=-\tau_{xy} \\ \dfrac{\partial \sigma_y}{\partial y}=-\dfrac{\partial \tau_{xy}}{\partial x} \longrightarrow 取势函数B,\ \sigma_y=\dfrac{\partial B}{\partial x},\ \tau_{xy}=-\dfrac{\partial B}{\partial y} \end{cases}$

有：关系：$\dfrac{\partial A}{\partial x}=\dfrac{\partial B}{\partial y}$ 故可以取一个总的应力函数Φ，有 $\dfrac{\partial \Phi}{\partial y}=A,\ \dfrac{\partial \Phi}{\partial x}=B$，

$\therefore \sigma_x=\dfrac{\partial A}{\partial y}=\dfrac{\partial^2 \Phi}{\partial y^2}$，$\sigma_y=\dfrac{\partial B}{\partial x}=\dfrac{\partial^2\Phi}{\partial x^2}$，$\tau_{xy}=-\dfrac{\partial A}{\partial x}=-\dfrac{\partial B}{\partial y}=-\dfrac{\partial^2\Phi}{\partial x\partial y}$，

得到

$$\boxed{\sigma_x=\frac{\partial^2\Phi}{\partial y^2},\ \sigma_y=\frac{\partial^2\Phi}{\partial x^2},\ \tau_{xy}=-\frac{\partial^2\Phi}{\partial x\partial y}}$$

显然 由 $\varepsilon_x=\dfrac{\partial u}{\partial x}$，$\varepsilon_y=\dfrac{\partial v}{\partial y}$，$\gamma_{xy}=\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}$。则有相容方程 $\dfrac{\partial^2\gamma_{xy}}{\partial x\partial y}=\dfrac{\partial^2\varepsilon_x}{\partial y^2}+\dfrac{\partial^2\varepsilon_y}{\partial x^2}$

故：$\dfrac{\partial^2}{\partial y^2}\left(\dfrac{\sigma_x-\nu\sigma_y}{\cancel{E}}\right)+\dfrac{\partial^2}{\partial x^2}\left(\dfrac{\sigma_y-\nu\sigma_x}{\cancel{E}}\right)=\dfrac{\partial^2}{\partial x\partial y}\left(\dfrac{2(1+\nu)}{\cancel{E}}\tau_{xy}\right)$，代入：有：

$$\frac{\partial^4\Phi}{\partial x^4}+\frac{\partial^4\Phi}{\partial y^4}\cancel{-2\nu\frac{\partial^4\Phi}{\partial x^2\partial y^2}}=-2(1+\cancel{\nu})\frac{\partial^4\Phi}{\partial x^2\partial y^2}$$ 故：

$$\boxed{\frac{\partial^4\Phi}{\partial x^4}+2\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}=0}$$ 即 $\nabla^4\Phi=0$

② 利用复变函数表示应力函数：由于 $\Phi=\Phi(x,y)$，我们可以使用 $\begin{cases} z=x+iy \\ \bar z=x-iy \end{cases}$，来将复变函数合为一个。

由 $x=\dfrac{1}{2}(z+\bar z)$，$y=\dfrac{1}{2i}(z-\bar z)$，代入有：

$$\frac{\partial\Phi}{\partial x}=\frac{\partial\Phi}{\partial z}\frac{\partial z}{\partial x}+\frac{\partial\Phi}{\partial\bar z}\frac{\partial\bar z}{\partial x}=\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial\bar z}\right)\Phi$$

得：$\dfrac{\partial^2\Phi}{\partial x^2}=\left(\dfrac{\partial}{\partial z}+\dfrac{\partial}{\partial\bar z}\right)^2\Phi=\left(\dfrac{\partial^2}{\partial z^2}+2\dfrac{\partial^2}{\partial z\partial\bar z}+\dfrac{\partial^2}{\partial\bar z^2}\right)\Phi$

$$\frac{\partial\Phi}{\partial y}=\frac{\partial\Phi}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial\Phi}{\partial\bar z}\frac{\partial\bar z}{\partial y}=\left(i\frac{\partial}{\partial z}-i\frac{\partial}{\partial\bar z}\right)\Phi$$

$\dfrac{\partial^2\Phi}{\partial y^2}=i^2\left(\dfrac{\partial}{\partial z}-\dfrac{\partial}{\partial\bar z}\right)^2\Phi=-\left(\dfrac{\partial^2}{\partial z^2}-2\dfrac{\partial^2}{\partial z\partial\bar z}+\dfrac{\partial^2}{\partial\bar z^2}\right)\Phi$

$$\frac{\partial^2\Phi}{\partial x\partial y}=i\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial\bar z}\right)\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial\bar z}\right)\Phi=i\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial\bar z^2}\right)\Phi$$

得：$\nabla^4\Phi=\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)^2\Phi=\underset{\sim\sim\sim}{4^2\dfrac{\partial^4}{\partial z^2\partial\bar z^2}\Phi}=0$

故：∂^4 … Φ … 0 … $\partial^2\Phi$ … $zf(\bar z)+f(\bar z)\longrightarrow$ $\Phi=zf(\bar z)+f(\bar z)+\bar z f_3(z)+f_4(z)$

得：$\nabla^4\Phi = (\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})\Phi = 4\frac{\partial^4}{\partial z^2\partial\bar z^2}\Phi = 0$

故：$\frac{\partial^4}{\partial z^2\partial\bar z^2}\Phi = 0$，积分得到 $\frac{\partial^2\Phi}{\partial \bar z^2} = z f_1(\bar z) + f_2(\bar z) \longrightarrow \Phi = z f_1(\bar z) + f_2(\bar z) + \bar z f_3(z) + f_4(z)$

由于应力函数为实函数，故Φ的实、虚部必定是共轭的。利用含$z,\bar z$项共轭和含z项与$\bar z$项共轭，有：

② $\overline{z f_1(\bar z)} = \bar z f_3(z)$，$\overline{f_2(\bar z)} = f_4(z)$ ← 可取，该式成立

$\longrightarrow f_3(z) = \overline{f_1(\bar z)}$，$f_4(z) = \overline{f_2(\bar z)}$，代入得到 $\Phi = z f_1(\bar z) + \bar z\overline{f_1(\bar z)} + f_2(\bar z) + \overline{f_2(\bar z)}$

我们取 $f_1(\bar z) = \frac{1}{2}\varphi(\bar z)$，$f_2(\bar z) = \frac{1}{2}\theta(\bar z)$，则：$\Phi = \frac{1}{2}[z\varphi(\bar z) + \bar z\overline{\varphi(\bar z)} + \theta(\bar z) + \overline{\theta(\bar z)}]$

若取其它几个，可变为不同形式，

取其实部，有：$\Phi = \mathrm{Re}[z\varphi(\bar z) + \theta(\bar z)] \longrightarrow$ 另外可推出 $\mathrm{Re}[\bar z\varphi(z) + \theta(z)]$ 形式，略

③应力解和位移解：

由应力表达式代入，有：

$$\begin{cases} \sigma_x = -\left[\frac{\partial^2}{\partial z^2} - 2\frac{\partial^2}{\partial z\partial\bar z} + \frac{\partial^2}{\partial\bar z^2}\right]\Phi, \\ \sigma_y = \left[\frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z\partial\bar z} + \frac{\partial^2}{\partial\bar z^2}\right]\Phi \\ \tau_{xy} = -i\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial\bar z^2}\right)\Phi, \end{cases}$$

则有：

$$\begin{cases} \sigma_x + \sigma_y = 4\frac{\partial^2\Phi}{\partial z\partial\bar z} \quad ①, \\ \sigma_y - \sigma_x + 2i\tau_{xy} = 2\frac{\partial^2\Phi}{\partial\bar z^2} + \cancel{2\frac{\partial^2\Phi}{\partial\bar z^2}} + 2\left(\frac{\partial^2\Phi}{\partial\bar z^2}\right) - \cancel{2\frac{\partial^2\Phi}{\partial\bar z^2}} \\ \qquad = 4\frac{\partial^2\Phi}{\partial\bar z^2} \quad ② \end{cases}$$

由①②，代入 $\Phi = \mathrm{Re}[\bar z\varphi(z) + \theta(z)]$，

$$\begin{cases} \sigma_x + \sigma_y = 4\mathrm{Re}[\varphi'(z)] \quad (1) \\ \sigma_y - \sigma_x + 2i\tau_{xy} = 2(\bar z\varphi''(z) + \theta''(z)). \end{cases}$$

取 $\theta'(z) = \psi(z)$，则 $\psi'(z) = \theta''(z)$

$= 2(\bar z\varphi''(z) + \psi'(z)) \quad (3)$

↓取共轭

故 $\sigma_y - \sigma_x - 2i\tau_{xy} = 2(z\overline{\varphi''(z)} + \overline{\psi'(z)}) \quad (2)$　又：联(1),(2),有：

$$\boxed{\begin{aligned} &\tfrac{1}{2}[(1)+(3)] \Rightarrow \sigma_y + i\tau_{xy} = 2\mathrm{Re}[\varphi'(z)] + \bar z\varphi''(z) + \psi'(z) \\ &\tfrac{1}{2}[(1)+(2)] \Rightarrow \sigma_y - i\tau_{xy} = 2\mathrm{Re}[\varphi'(z)] + z\overline{\varphi''(z)} + \overline{\psi'(z)}, \end{aligned}}$$

此式可用于裂纹边界表达。

$(u-iv)' = u'-iv'$，$u'-iv'$

证：$\overline{f_2(z)}' = \overline{f_2'(z)}$

可交换

而位移分量有关系：

$$\varepsilon_x = \frac{\partial u_x}{\partial x} = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{E}[\sigma_x + \sigma_y - (1+\nu)\sigma_y]$$

$\longrightarrow E\varepsilon_x = \sigma_x + \sigma_y - (1+\nu)\frac{\partial^2\Phi}{\partial x^2}$，其中：$\sigma_x + \sigma_y = 4\mathrm{Re}[\varphi'(z)]$

由复变函数微积分关系：有：

$\frac{\partial \mathrm{Im}(\varphi(z))}{} \quad \frac{\partial \mathrm{Re}\,\varphi(z)}{} = \mathrm{Re}\, f'(z) \quad \frac{\partial \mathrm{Re} f(z)}{} = \frac{\partial \mathrm{Im} f(z)}{} = \mathrm{Im} f(z)$

因复变函数微积分关系：有：

$$\frac{\partial \mathrm{Im}(\varphi(z))}{\partial y}=\frac{\partial \mathrm{Re}\,\varphi(z)}{\partial x}=\mathrm{Re}\,f'(z),\quad -\frac{\partial \mathrm{Re} f(z)}{\partial x}=\frac{\partial \mathrm{Im} f(z)}{\partial y}=\mathrm{Im} f(z)$$

∴有：$\sigma_x+\sigma_y=4\frac{\partial}{\partial x}[\mathrm{Re}\,\varphi(z)]=2\frac{\partial}{\partial x}[\varphi(z)+\overline{\varphi}(z)]$

则 $$\boxed{E\varepsilon_x=2\frac{\partial}{\partial x}[\varphi(z)+\overline{\varphi}(z)]-(1-\nu)\frac{\partial^2\phi}{\partial x^2}}\quad ①,$$

而：$\varepsilon_y=\frac{\partial u_y}{\partial y}=\frac{1}{E}(\sigma_y-\nu\sigma_x)\Rightarrow E\varepsilon_y=\sigma_y-\nu\sigma_x$

$$=\sigma_x+\sigma_y-(1+\nu)\sigma_x$$

由 $\sigma_x+\sigma_y=4\mathrm{Re}[\varphi'(z)]=4\frac{\partial \mathrm{Im}(\varphi(z))}{\partial y}=2\frac{\partial}{\partial y}\left[\frac{\varphi(z)-\overline{\varphi}(z)}{i}\right]$

$$=-2i\frac{\partial}{\partial y}[\varphi(z)-\overline{\varphi}(z)]$$

则：$$\boxed{E\varepsilon_y=-2i\frac{\partial}{\partial y}[\varphi(z)-\overline{\varphi}(z)]-(1+\nu)\frac{\partial^2\phi}{\partial y^2}}\quad ②$$

两边同时积分，有：

$$\begin{cases}E\,u_x=2[\varphi(z)+\overline{\varphi}(z)]-(1+\nu)\frac{\partial\phi}{\partial x}\\ E\,u_y=-2i[\varphi(z)-\overline{\varphi}(z)]-(1+\nu)\frac{\partial\phi}{\partial y}\end{cases}$$

则有：

$$\boxed{E(u_x+iu_y)=2[\varphi(z)+\overline{\varphi}(z)]+2[\varphi(z)-\overline{\varphi}(z)]-(1+\nu)\left(\frac{\partial\phi}{\partial x}+i\frac{\partial\phi}{\partial y}\right)}\quad \triangle$$

我们知道：$\frac{\partial\phi}{\partial x}=\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial\bar z}\right)\phi$，$\frac{\partial\phi}{\partial y}=\left(i\frac{\partial}{\partial z}-i\frac{\partial}{\partial\bar z}\right)\phi$，有：

$$\phi=\frac{1}{2}[\bar z\varphi_1(z)+z\overline{\varphi_1(z)}+\theta_1(z)+\overline{\theta(z)}]$$，上式等于：☆

代入：

$$\frac{\partial\phi}{\partial x}+i\frac{\partial\phi}{\partial y}=\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial\bar z}-\frac{\partial}{\partial z}+\frac{\partial}{\partial\bar z}\right)\phi=2\frac{\partial\phi}{\partial\bar z}\quad ☆$$

要注意的是：$\overline{\varphi_1(z)}$，$\overline{\theta(z)}$ 实质上是 $\bar z$ 的函数，

故：对 $\overline{\varphi_1(z)}$，$\overline{\theta(z)}$ 均对 $\bar z$ 求导：

因：$\overline{f_1(z)}=f_3(\bar z)$ 之

$\overline{f_1'(z)}=f_3'(\bar z)$

$$\frac{\partial z\overline{\varphi_1(z)}}{\partial\bar z}=\frac{\partial z f_3(\bar z)}{\partial\bar z}=zf_3'(\bar z)$$

$$=z\overline{f_1'(z)}=z\overline{\varphi_1'(z)}$$

故：$\frac{\partial\overline{\varphi_1(z)}}{\partial\bar z}=\overline{\varphi_1'(z)}$

$= \bar{z}\, f_1(z) = \bar{z}\, \varphi_1'(z)$

故：$\dfrac{\partial \overline{\varphi_1(z)}}{\partial \bar{z}} = \overline{\varphi_1'(z)}$，

而 $\dfrac{\partial \overline{\theta(z)}}{\partial \bar{z}} = \overline{\theta'(z)}$ （同理），代入有：

$$\frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y} = 2 \times \frac{1}{2}\left[\varphi(z) + z\,\overline{\varphi'(z)} + \overline{\theta'(z)}\right]$$

，代入△式，有：

$$E(u_x + i u_y) = 4\varphi(z) - (1+\nu)\left[\varphi(z) + z\,\overline{\varphi'(z)} + \overline{\theta'(z)}\right]$$

$$= (3-\nu)\varphi(z) - (1+\nu)\left[z\,\overline{\varphi'(z)} + \overline{\theta'(z)}\right]$$

，

得到 Kolosov / Muskhelishvi 位移函数为：

$$\boxed{E(u_x + i u_y) = (3-\nu)\varphi(z) - (1+\nu)\left[z\,\overline{\varphi'(z)} + \overline{\psi(z)}\right]}$$