大挠度薄板的里兹法与中面变形能推导

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一、李兹法：

由薄板变形能：$U=\frac{1}{2}\iiint(\sigma_x^z\varepsilon_x^z+\sigma_y^z\varepsilon_y^z+\tau_{xy}^z\gamma_{xy}^z)\,dxdydz$

我们已经在大挠度问题基本方程中推出 $\sigma_x^z,\varepsilon_x^z,\cdots$ 表达式.

则 $$U=\frac{1}{2}\iiint\Big\{\Big[\sigma_x-\frac{Ez}{1-\nu^2}\Big(\frac{\partial^2w}{\partial x^2}+\nu\frac{\partial^2w}{\partial y^2}\Big)\Big]\cdot\Big(\varepsilon_x-z\frac{\partial^2w}{\partial x^2}\Big)$$
$$+\Big[\sigma_y-\frac{Ez}{1-\nu^2}\Big(\frac{\partial^2w}{\partial y^2}+\nu\frac{\partial^2w}{\partial x^2}\Big)\Big]\cdot\Big(\varepsilon_y-z\frac{\partial^2w}{\partial y^2}\Big)$$
$$+\Big[\tau_{xy}-\frac{Ez}{1+\nu}\frac{\partial^2w}{\partial x\partial y}\Big]\Big(\gamma_{xy}-2z\frac{\partial^2w}{\partial x\partial y}\Big)\Big\}\,dxdydz$$

将变形能U分解为薄膜力变形能与弯曲变形能之和，有：$\boxed{U=U_m+U_b}$

$$U_m=\frac{h}{2}\iint_A(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\tau_{xy}\gamma_{xy})\,dxdy \quad ①$$
← 对z积分得 展开
(其中：$\sigma_x,\sigma_y\cdots\varepsilon_x\cdots$ 为常数)

$$U_{bx}=\frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{Ez^2}{1-\nu^2}\iint\Big(\frac{\partial^2w}{\partial x^2}+\nu\frac{\partial^2w}{\partial y^2}\Big)\cdot\frac{\partial^2w}{\partial x^2} \qquad \int z^2=\frac{1}{3}z^3\Big|_{-\frac{h}{2}}^{\frac{h}{2}}=\frac{1}{3}\Big[\frac{h^3}{8}+\frac{h^3}{8}\Big]\Rightarrow\frac{Eh^3}{12(1-\nu^2)}=D$$
$$=\frac{D}{2}\iint\Big(\frac{\partial^2w}{\partial x^2}+\nu\frac{\partial^2w}{\partial y^2}\Big)\frac{\partial^2w}{\partial x^2}$$
三方向相加

则 $$U_b=\frac{D}{2}\iint\Big[\Big(\frac{\partial^2w}{\partial x^2}+\nu\frac{\partial^2w}{\partial y^2}\Big)\frac{\partial^2w}{\partial x^2}+\Big(\frac{\partial^2w}{\partial y^2}+\nu\frac{\partial^2w}{\partial x^2}\Big)\frac{\partial^2w}{\partial y^2}+2(1-\nu)\Big(\frac{\partial^2w}{\partial x\partial y}\Big)^2\Big]\,dxdydz$$
$$=\frac{D}{2}\iint\Big[\Big(\frac{\partial^2w}{\partial x^2}\Big)^2+\Big(\frac{\partial^2w}{\partial y^2}\Big)^2+2\nu\frac{\partial^2w}{\partial x^2}\frac{\partial^2w}{\partial y^2}+2(1-\nu)\Big(\frac{\partial^2w}{\partial x\partial y}\Big)^2\Big]\,dxdydz \quad ②$$

利用 $\varepsilon_x,\varepsilon_y,\gamma_{xy}$ 和 $\sigma_x,\sigma_y,\tau_{xy}$ 的关系. $\begin{cases}\varepsilon_x=\frac{1}{E}(\sigma_x-\nu\sigma_y)\\ \varepsilon_y=\frac{1}{E}(\sigma_y-\nu\sigma_x)\\ \gamma_{xy}=\frac{\tau_{xy}}{G}=\frac{2(1+\nu)}{E}\tau_{xy}\end{cases}$ 则中面变形能①式成为

$$U_m=\frac{h}{2E}\iint_A[(\sigma_x+\sigma_y)^2-2(1+\nu)(\sigma_x\sigma_y-\tau_{xy}^2)]\,dxdy$$
引入应力函数
$$\boxed{=\frac{h}{2E}\iint\Big\{[(\nabla^2F)^2-2(1+\nu)\Big[\frac{\partial^2F}{\partial x^2}\frac{\partial^2F}{\partial y^2}-\Big(\frac{\partial^2F}{\partial x\partial y}\Big)^2\Big]\Big\}\,dxdy}\text{，(应力表达)}$$

也可利用：$\sigma_x=\frac{E}{1-\nu^2}(\varepsilon_x+\nu\varepsilon_y)$，$\sigma_y=\frac{E}{1-\nu^2}(\varepsilon_y+\nu\varepsilon_x)$，$\tau_{xy}=\frac{E}{2(1+\nu)}\gamma_{xy}$

$$U_m=\frac{Eh}{2(1-\nu^2)}\iint\Big[\varepsilon_x^2+\varepsilon_y^2+2\nu\varepsilon_x\varepsilon_y+\frac{1-\nu}{2}\gamma_{xy}^2\Big]\,dxdy \quad \text{(应变表达)}$$

代入应变公式，即：$\begin{cases}\varepsilon_x=\frac{\partial u}{\partial x}+\frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2,\quad \varepsilon_y=\frac{\partial v}{\partial y}+\frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2\\ \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\end{cases}$

故：

$$U_m=\frac{Eh}{2(1-\nu^2)}\iint\Big[\Big(\frac{\partial u}{\partial x}\Big)^2+\frac{1}{4}\Big(\frac{\partial w}{\partial x}\Big)^4+\frac{\partial u}{\partial x}\Big(\frac{\partial w}{\partial x}\Big)^2\Big]+\Big[\Big(\frac{\partial v}{\partial y}\Big)^2+\frac{1}{4}\Big(\frac{\partial w}{\partial y}\Big)^4+\frac{\partial v}{\partial y}\Big(\frac{\partial w}{\partial y}\Big)^2\Big]+2\nu\Big(\frac{\partial u}{\partial x}+\frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2\Big)\Big(\frac{\partial v}{\partial y}+\frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2\Big)$$
$$+\frac{1-\nu}{2}\Big\{\Big(\frac{\partial u}{\partial y}\Big)^2+\Big(\frac{\partial v}{\partial x}\Big)^2+\Big(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Big)^2+2\Big[\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Big]\Big\}$$

$$+\frac{1-\nu}{2}\left\{\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)^2+2\left[\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\right\}$$

$$=\frac{Eh}{2(1-\nu^2)}\iint\left\{\left[\left(\frac{\partial u}{\partial x}\right)^2+\frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\frac{\partial v}{\partial y}\left(\frac{\partial w}{\partial y}\right)^2\right]+\frac{1}{4}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]^2\right.$$
$$+2\nu\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{1}{2}\left(\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial v}{\partial y}+\left(\frac{\partial w}{\partial y}\right)^2\frac{\partial u}{\partial x}\right)\right]$$
$$\left.+\frac{1-\nu}{2}\left\{\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\}\right\}dxdy$$

此时：作用于平板上的横向载荷 $q(x,y)$，外力功为：

$$W=\iint q(x,y)\,w\,dxdy$$

$$\frac{1-\nu}{2}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2+2\nu\cdot\frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2$$
$$=\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2$$，和上面合并

故薄板大挠度有：总势能

$$\boxed{\Pi=U_m+U_b-W}$$

在使用里兹法做近似解时，首先假设挠度函数 w 为

$$w=a_1\varphi_1(x,y)+a_2\varphi_2(x,y)+\cdots+a_n\varphi_n(x,y)=\sum_{i=1}^{n}a_i\varphi_i(x,y)$$

对满足的协调方程：

$$\frac{1}{E}\nabla^4F=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$，积分该式

将 w 代入 可得 $F(x,y)$ 表达式，并令：

$$\frac{\partial\Pi}{\partial a_1}=\frac{\partial\Pi}{\partial a_2}=\cdots=\frac{\partial\Pi}{\partial a_n}=0$$，即可解出参数并确定 w，