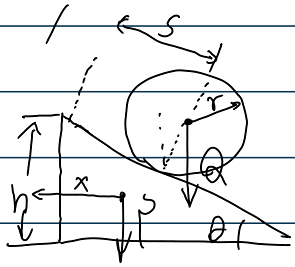
1章楔形体例题

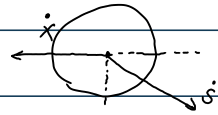
Friday, March 17, 2023 4:04 PM



有：系统上外力均保守力则

$$T = \frac{1}{2}\left(\frac{P}{g}\right)\cdot \dot{x}^2 + \frac{1}{2}\left(\frac{Q}{g}\right)\left[(\dot{x}-\dot{s}\cos\theta)^2 + (\dot{s}\sin\theta)^2\right]$$

转动动能 $\longrightarrow + \frac{1}{2} I \omega^2$,



有：$\omega = \dot{\alpha} = \frac{\dot{s}}{r}$,

则：$T = \frac{1}{2}\left(\frac{P}{g}\right)\dot{x}^2 + \frac{1}{2}\left(\frac{Q}{g}\right)\left[\dot{x}^2 - 2\dot{x}\dot{s}\cos\theta + \dot{s}^2\right] + \frac{1}{2}\cdot\overbrace{\boxed{\frac{1}{2}\left(\frac{Q}{g}\right) r^2}}^{I}\cdot\left(\frac{\dot{s}}{r}\right)^2$

此时势能 $V = 0 + Q\cdot\left[h - s\sin\theta + r\cos\theta\right]\left(+\frac{1}{3}ph\right)$ 可省略

则有：

$$L = T - V = \frac{1}{2}\left(\frac{P}{g}\right)\dot{x}^2 + \frac{1}{2}\left(\frac{Q}{g}\right)\left[\dot{x}^2 - 2\dot{x}\dot{s}\cos\theta + \frac{3}{2}\dot{s}^2\right]$$

$$- \left(Q\left[h - s\sin\theta + r\cos\theta\right] + \frac{1}{3}ph\right)$$

Lagrange 方程形式为：

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \left(\frac{\partial L}{\partial q_2}\right) = 0 \end{cases}$$

其中：$q_1 = x$

$q_2 = s$,

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ 得：$\frac{d}{dt}\left(\frac{P}{g}\cdot\dot{x} + \frac{Q}{g}\dot{x} - \frac{Q}{g}\dot{s}\cos\theta\right) = 0$

即：$\ddot{x}\left(\frac{P}{g} + \frac{Q}{g}\right) = \ddot{s}\cdot\cos\theta\cdot\frac{Q}{g}$

$$\therefore \boxed{\ddot{x} = \frac{Q}{P+Q}\ddot{s}\cos\theta} \quad ①$$

而：$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) + \frac{\partial L}{\partial s} = 0$, 由于 $L = T - V = \frac{1}{2}\left(\frac{P}{g}\right)\dot{x}^2 + \frac{1}{2}\left(\frac{Q}{g}\right)\left[\dot{x}^2 - 2\dot{x}\dot{s}\cos\theta + \frac{3}{2}\dot{s}^2\right]$

$- \left(Q\left[h - s\sin\theta + r\cos\theta\right] + \underline{\frac{1}{3}ph}\right)$

不影响

有：

$$\frac{d}{dt}\left(\frac{3}{2}\frac{Q}{g}\dot{s} - \frac{Q}{g}\dot{x}\cos\theta\right) - \left[+Q\sin\theta\right] = 0$$

$$\therefore \frac{3}{2}\frac{Q}{g}\ddot{s} - \frac{Q}{g}\ddot{x}\cos\theta - Q\sin\theta = 0$$

$$\therefore \boxed{\frac{3}{2}\ddot{s} - \ddot{x}\cos\theta = g\sin\theta} \quad ②$$

$$\therefore \boxed{\frac{3}{2}\ddot{s} - \ddot{x}\cos\theta = g\sin\theta} \quad ②$$

$$\therefore \frac{Q}{P+Q}\ddot{s}\cos^2\theta - \frac{3}{2}\ddot{s} = -g\sin\theta$$

此时:

$$\ddot{s}\left(\frac{3}{2} - \frac{Q\cos^2\theta}{P+Q}\right) = g\sin\theta$$

故得到: $\ddot{s}\left(\frac{3P+3Q-2Q\cos^2\theta}{2(P+Q)}\right) = g\sin\theta \to \ddot{s} = \frac{2(P+Q)}{3P+Q+2Q\sin^2\theta} g\sin\theta$

则 $\ddot{x} = \ddot{s}\cdot\frac{Q}{P+Q}\cos\theta = \frac{Q\sin 2\theta}{3P+Q+2Q\sin^2\theta} g$ 为楔形体加速度

(2)系统第一积分的求解: ①②

由于 Lagrange 方程中不显含时间t, 因此有能量积分

因 $\sum \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L = C_1$; 有:

$$T = \frac{P}{2g}\dot{x}^2 + \frac{Q}{2g}\dot{x}^2 - \frac{Q}{2g}\dot{x}\dot{s}\cos\theta + \frac{3}{4}\frac{Q}{g}\dot{s}^2$$

而:

$$V = \frac{1}{3}Ph + Q(h - s\sin\theta + r\cos\theta)$$

则广义积分方程为:

$\to T_0 = 0$

$T_2 - T_0 + V = \text{constant}$, 即:

$$\frac{P+Q}{2g}\dot{x}^2 - \frac{Q}{2g}\dot{x}\dot{s}\cos\theta + \frac{3}{4}\frac{Q}{g}\dot{s}^2 + \frac{1}{3}Ph + Q(h - s\sin\theta + r\cos\theta) = C_1$$

并将初始条件 $\dot{x}=0, \dot{s}=0, s=0$, 代入,

得:

$C_1 = \frac{1}{3}Ph + Q(h + r\cos\theta)$, 将其回代上式, 则:

$$\boxed{\frac{P+Q}{2g}\dot{x}^2 - \frac{Q}{2g}\dot{x}\dot{s}\cos\theta + \frac{3}{4}\frac{Q}{g}\dot{s}^2 - Qs\sin\theta = 0}$$

该式即为 Lagrange 方程的第一积分.

另外需要说明: Lagrange函数 L 中不含广义坐标 x, 因此有对x的循环积分式:

我们需要说明Lagrange函数L中不含广义坐标x，因此得对应的循环积分式：

即：$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$，

即：$\frac{d}{dt}\left(\frac{P+Q}{g}\cdot\dot{x} - \frac{Q}{2g}\dot{s}\cos\theta\right) = 0$

显然代入初值，有：

$(P+Q)\dot{x} - \frac{Q}{2}\dot{s}\cos\theta = 0$ 即为系统的循环积分