矩形薄板的初参数法求解过程

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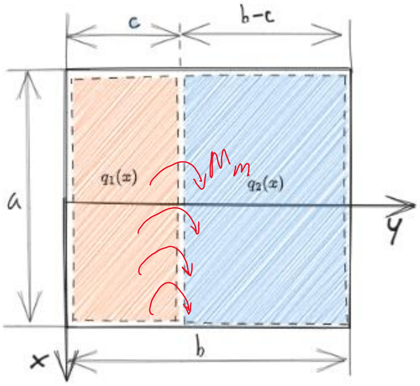
对于如下图的板，我们设



$w = w_1 + \bar{w}$，其中 $\bar{w}$ 在 $y \geq c$ 时 $\neq 0$.

则 $\nabla^2\nabla^2 \bar{w} = \dfrac{q_2(x,y) - q_1(x,y)}{D}$ $(y \geq c)$ ①

$\nabla^2\nabla^2 w_1 = \dfrac{q_1(x,y)}{D}$ ②，$(0 \leq y \leq c)$

则：②方程可利用两边简支的单三角级数解设出，

此时：设

$$\bar{w} = \sum_{m=1}^{\infty} \bar{f}_m(y) \sin\frac{m\pi x}{a}$$

$$\boxed{q_2(x,y) - q_1(x,y) = \sum_{m=1}^{\infty} \bar{q}_m(y) \sin\frac{m\pi x}{a}}，$$

由：$M(x) = -D\dfrac{\partial^2 \bar{w}(x,y)}{\partial y^2}$

需要说明的是：由于 $y=c$ 处，$\bar{w}=0$，则 $\dfrac{\partial \bar{w}}{\partial x} = 0$，才有剪力和弯矩边界条件，而展开为单三角级数可能并不合理

而：$M(x) = -D\dfrac{\partial^2 \bar{w}(x,y)}{\partial y^2}$　而：$V(x) = -D\dfrac{\partial^3 \bar{w}(x,y)}{\partial y^3}$

此时：取：$M(x) = \sum_{m=1}^{\infty} M_m \sin\dfrac{m\pi x}{a}$　$V(x) = \sum_{m=1}^{\infty} V_m \sin\dfrac{m\pi x}{a}$.

符合 $x=0$, $x=a$ 的 0弯矩条件　→ $x=0$, $x=a$ 无剪力

由于 $y=c$ 处有：

$$\boxed{\bar{f}_m(y) = 0，\text{而：}\bar{f}_m'(y) = 0 \quad -D\bar{f}_m''(y) = M_m，\quad -D\bar{f}_m^{(3)}(y) = V_m}$$ 边界 ☆

在 $y=c$ 时无边界条件 但 $x=0$, $x=a$ 处可利用 自然满足的边界

并有全板满足的微分方程为：$\dfrac{\bar{q}_m(y)}{D} = \bar{f}_m^{(4)}(y) + 2\left(\dfrac{m\pi}{a}\right)^2 \bar{f}_m^{(2)}(y) + \left(\dfrac{m\pi}{a}\right)^4 \bar{f}_m(y)$ ☆

上式是4阶线性微分方程组，则有4个线性无关的通解：

则：取 $\bar{f}_m(y) = \bar{f}_{1m}(y) + \bar{f}_{2m}(y)$，其中 $f_{1m}(y)$ 为特解，而：

$$\bar{f}_{2m}(y) = a_{mi} Y_1(y) + b_m Y_2(y) + c_m Y_3(y) + d_m Y_4(y)$$

则有：

$$Y_i(y) = A_{mi}\cosh\frac{m\pi y}{a} + B_m \sinh\frac{m\pi y}{a} + C_{mi}\frac{m\pi y}{a}\cosh\frac{m\pi y}{a} + D_{mi}\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}.$$

令 $y=0$ 时，令 Y_i 满足：

$Y_1(0) = 1$　$Y_1'(0) = 0$　$Y_1''(0) = 0$　$Y_1''' = 0$

0　$Y_2'(0) = \dfrac{m\pi}{a}$　$Y_3''(0) = \left(\dfrac{m\pi}{a}\right)^2$　$Y_4'''(0) = \left(\dfrac{m\pi}{a}\right)^3$，则得：

$$Y_1(y) = \cosh\frac{m\pi y}{a} - \frac{1}{2}\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}$$
$$Y_2(y) = \frac{3}{2}\sinh\frac{m\pi y}{a} - \frac{1}{2}\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}$$
$$Y_3(y) = \frac{1}{2}\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}$$
$$Y_4(y) = -\frac{1}{2}\sinh\frac{m\pi y}{a} + \frac{1}{2}\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}$$

其中 $Y_i(y)$ 称为**克雷洛夫函数**，其为齐次方程的解，同时 $Y_i(y-c)$ 也是齐次方程的解代换得到：

$$\bar{f}_m(y) = a_m Y_1(y-c) + b_m Y_2(y-c) + c_m Y_3(y-c) + d_m Y_4(y-c) + \bar{f}_{1m}(y)$$

此时 ⇒ 代入边界条件 ☆

即：$\bar{f}_m^{(2)}(y)\Big|_{y=c} = -\dfrac{M_m(y)}{D}$，$\bar{f}_m^{(3)}(y)\Big|_{y=c} = -\dfrac{V_m(y)}{D}$

→ 有：以弯矩为例：代入：$y=c$

$$C_m = -\left(\frac{a}{m\pi}\right)^2\left[\frac{V_m(y)}{D} + f_{1m}'''(c)\right]$$ 不要丢掉

$\bar{f}'''(c) = c_m Y_3'''(c) + \bar{f}'''(c) = \left(\dfrac{m\pi}{a}\right)^2 c_m + f'''(c) = -\dfrac{V_m(y)}{[illegible]}$，全部解 a_m, b_m, c_m, d_m 并代入，有 $\bar{f}_m$ 解为

→ 有: 以弯矩为例: 代入: $y=c$

$$\bar f_m'''(c) = C_m Y_3'''(0) + \bar f_{1m}'''(c) = \left(\frac{m\pi}{a}\right)^2 \cdot C_m + \bar f_{1m}'''(c) = -\frac{V_m(y)}{D}$$;

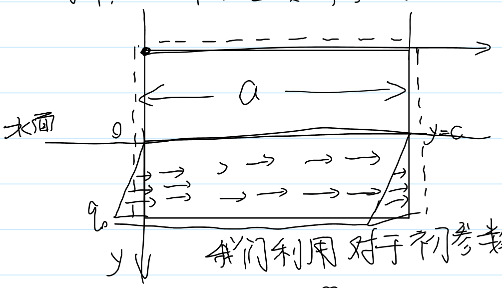
全部解 a_m, b_m, C_m, d_m 并代入, 有 $\bar f_m$ 解为

$$\bar f_m(y) = -\bar f_{1m}(c)\, Y_1(y-c) - \frac{a}{m\pi}\bar f_1'(c)\, Y_2(y-c) - \left(\frac{a}{m\pi}\right)^2\left(\bar f_{1m}''(c)+\frac{M_m}{D}\right) Y_3(y-c)$$
$$-\left(\frac{a}{m\pi}\right)^3\left(\bar f_{1m}'''(c)+\frac{V_m}{D}\right) Y_4(y-c) + \bar f_{1m}(y)$$ 为 $\bar f_m(y)$ 的解,

做法是取

$$w_2 = \sum_{m=1}^{\infty} f_m(y)\sin\frac{m\pi x}{a} + ||_{y>c}\sum_{m=1}^{\infty} \bar f_m(y)\sin\frac{m\pi x}{a} = \sum_{m=1}^{\infty}\Big[f_m(y)+\underbrace{\bar f_m(y)}_{(y\ge c)}\Big]\sin\frac{m\pi x}{a}$$

例1. 矩形板三边简支($x=0$, $x=a$, $y=0$)且在 $y\ge c$ 处, 承受静水压力作用, 求解整体的挠度:



解:

$$q_1(x,y) = 0 \quad (0\le y\le c)$$
$$\bar q(x,y) = q_0\frac{y-c}{b-c} \quad (c\le y\le b)$$

我们利用对于初参数法获得的 $\bar q(x,y)$, 有: 解为:

$$||_{y>c}\, w = ||_{y>c}\sum_{m=1}^{\infty}\bar f_m(y)\sin\frac{m\pi x}{a},$$

由载荷条件, 我们将其拆分为两部分:

$$q_2(x,y) - q_1(x,y) = q_0\frac{y-c}{b-c} = \sum_{m=1}^{\infty}\bar q_m(y)\sin\frac{m\pi x}{a} + 0$$, 此时有: 系数 q_m 计算为:

$$\bar q_m = \frac{2}{a}\int_0^a q_0\frac{y-c}{b-c}\sin\frac{m\pi x}{a}\,dx$$ (其中 $a=T$, 与傅氏展开相同). ? 推导;

$\to -\cos\theta = 2$.

$$挠度 = \frac{4q_0}{m\pi}\frac{y-c}{b-c}$$, 其中, $m=1,3,\cdots\to\infty$ 时, $\ne 0$, 其余为0, 则.

代入整体的方程为:

$$f_m^{(4)}(y) - 2\left(\frac{m\pi}{a}\right)^2 \bar f_m^{(2)}(y) + \left(\frac{m\pi}{a}\right)^4 \bar f_m(y) = \frac{\bar q_m}{D} = \frac{4q_0}{m\pi D}\,\frac{y-c}{b-c}$$ ☆

我们使用初参数表达方程解; 初参数为:

① $f_m(y) = a_m Y_{1m}(y) + b_m Y_{2m}(y) + C_m Y_{3m}(y) + d_m Y_{4m}(y)$

边界条件表达: 显有

$M_{y=c} = V_{y=c} = 0$ (无突变), 则:

② $$f_m(y-c) = \bar f_{1m}(y)\, Y_{1m}(y-c) + \frac{a}{m\pi}\bar f_{1m}'(y)\, Y_2(y-c) + \left(\frac{a}{m\pi}\right)^2\left(\bar f_{1m}''(c)+0\right) Y_3(y-c)$$
$$+\left(\frac{a}{m\pi}\right)^3\left(\bar f_{1m}'''(c)+0\right) Y_4(y-c)。$$

则: 我们取特解:

? 如何取: $\bar f_{1m}(y) = \frac{4q_0a^4}{D(m\pi)^5}\cdot\frac{y-c}{b-c}$, 仅有 $\bar f_{1m}'(c) = \frac{1}{b-c}\cdot\frac{4q_0a^4}{D(m\pi)^5}$ 而 $f_{1m}(c) = \frac{4q_0a^4}{D(m\pi)^5}\cdot\frac{c-c}{b-c} = 0$

解出:

$$f_m(y) = \cancel{-f_{1m}(c)\,Y_1(y-c)} - \frac{a}{m\pi}f_1'(c)\,Y_2(y-c) - \left(\frac{a}{m\pi}\right)^2\cdot 0 - 0 + \bar f_{1m}(y)$$

$4q_0a^4$ $y-c$ a 1 $4q_0a^4$ $Y_2(y-c)$

$\bar{f}_m(y) = -J_{1m}(c)\, \cancel{Y_1(y-c)} - \frac{a}{m\pi} J_1(c) Y_2(y-c) - (m\pi) \cdots J_{1m}(y)$

$$= \frac{4q_0 a^4}{D(m\pi)^5} \cdot \frac{y-c}{b-c} - \frac{a}{m\pi} \cdot \frac{1}{b-c} \cdot \frac{4q_0 a^4}{D(m\pi)^5} Y_2(y-c)$$

$$\bar{f}_m(y) = \frac{4q_0 a^4}{D(m\pi)^5} \cdot \frac{y-c}{b-c} - \frac{4q_0 a^5}{D(m\pi)^6} \cdot \frac{1}{b-c} Y_2(y-c)$$

通解.

则: $w = w_1 + \bar{w}$

$$= \sum_{m=1,3,\cdots}^{\infty} \left[f_m(y) + 1\!\!1_{y>c}\, \bar{f}_m(y) \right] \sin\frac{m\pi x}{a},$$

$$= \sum_{m=1,3,\cdots}^{\infty} \left\{ \left[a_m Y_1(y) + b_m Y_2(y) + c_m Y_3(y) + d_m Y_4(y) \right] + 1\!\!1_{y>c} \left[\frac{4q_0 a^4}{D(m\pi)^5} \frac{y-c}{b-c} - \frac{4q_0 a^5}{D(m\pi)^6} \cdot \frac{1}{b-c} Y_2(y-c) \right] \right\} \sin\frac{m\pi x}{a}.$$

其中: a_m, b_m, c_m, d_m 由边界条件确定,有:

$y=0$处, $w=0$, $\frac{\partial^2 w}{\partial y^2} = 0$ 　其中 $Y_i(y) = A_{mi} \cosh\frac{m\pi y}{a} + B_{mi} \sinh\frac{m\pi y}{a} + C_{mi} \frac{m\pi y}{a} \cosh\frac{m\pi y}{a} + D_{mi} \sinh\frac{m\pi y}{a}$.

$y=b$处, $w=0$ $\frac{\partial w}{\partial y} = 0$, 　显有: $a_m = c_m = 0$??? 怎么来的,

由$y=b$的边界条件: $\Longrightarrow$ 解出b_m, d_m, 并代回上式, 而: $Y_2(y) = \frac{3}{2}\sinh\frac{m\pi y}{a} - \frac{1}{2}\frac{m\pi y}{a}\cosh\frac{m\pi x}{a}$.