壳体内力-内矩的本构公式推导

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利用壳体的位移和变形，建立壳体的本构方程

我们已经得到壳体的平衡方程如下：

内力的平衡方程为：

$$\begin{cases} \dfrac{\partial(N_1A_2)}{\partial\alpha_1}+\dfrac{\partial(A_1S_{21})}{\partial\alpha_2}+\dfrac{\partial A_1}{\partial\alpha_2}S_{12}-\dfrac{\partial A_2}{\partial\alpha_1}N_2+\dfrac{Q_1A_1A_2}{R_1}+A_1A_2q_1=0 \\ \dfrac{\partial(N_2A_1)}{\partial\alpha_2}+\dfrac{\partial(A_2S_{12})}{\partial\alpha_1}+\dfrac{\partial A_2}{\partial\alpha_1}S_{21}-\dfrac{\partial A_1}{\partial\alpha_2}N_1+\dfrac{Q_2A_1A_2}{R_2}+A_1A_2q_2=0 \\ \dfrac{\partial(Q_2A_1)}{\partial\alpha_2}+\dfrac{\partial(Q_2A_1)}{\partial\alpha_2}-A_1A_2\left(\dfrac{N_1}{R_1}+\dfrac{N_2}{R_2}\right)+A_1A_2q_3=0 \end{cases}$$

力矩的平衡方程：

$$\begin{cases} \dfrac{\partial(A_1M_2)}{\partial\alpha_2}+\dfrac{\partial(A_2H_{12})}{\partial\alpha_1}-M_1\dfrac{\partial A_1}{\partial\alpha_2}+H_{21}\dfrac{\partial A_2}{\partial\alpha_1}-Q_2A_1A_2=0 \\ \dfrac{\partial(A_2M_1)}{\partial\alpha_1}+\dfrac{\partial(A_1H_{21})}{\partial\alpha_2}-M_2\dfrac{\partial A_2}{\partial\alpha_2}+H_{12}\dfrac{\partial A_1}{\partial\alpha_2}-Q_1A_1A_2=0 \\ \dfrac{H_{12}}{R_1}-\dfrac{H_{21}}{R_2}=S_{21}-S_{12} \end{cases}$$

我们利用胡克定律以及应力的表示公式，(忽略 ε_3 方向的变形)，则有公式

1). 胡克定律

$$\begin{cases} \sigma_1=\dfrac{E}{1-\nu^2}\left(\varepsilon_1^{(z)}+\nu\varepsilon_2^{(z)}\right) \\ \sigma_2=\dfrac{E}{1-\nu^2}\left(\varepsilon_2^{(z)}+\nu\varepsilon_1^{(z)}\right) \\ \tau_{12}=\tau_{21}=G\,\omega^{(z)}=\dfrac{E}{2(1+\nu)}\omega^{(z)} \end{cases}$$

其中：

$$\begin{cases} \varepsilon_1^{(z)}=\dfrac{1}{1+\frac{z}{R_1}}\left[\varepsilon_1+K_1z\right] \\ \varepsilon_2^{(z)}=\dfrac{1}{1+\frac{z}{R_2}}\left[\varepsilon_2+K_2z\right] \\ \omega^{(z)}=\dfrac{1}{1+z/R_1}(\omega_1+\tau_1z)+\dfrac{1}{1+z/R_2}(\omega_2+\tau_2z) \end{cases}$$

2). 应力公式：(由内力内矩推导出)

$$N_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_1\left(1+\frac{z}{R_2}\right)dh=\frac{E}{1-\nu^2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\varepsilon_1^{(z)}+\nu\varepsilon_2^{(z)}\right)\left(1+\frac{z}{R_1}\right)dh,$$

※为了便于积分，取 $\frac{1}{1+\frac{z}{R_1}}$ ⟶ 展开到二阶：由：$(1+t)^\alpha=1+\alpha\cdot t+\frac{\alpha(\alpha-1)}{2!}t^2+\cdots$

取 $\alpha=-1$，有：$\left(1+\frac{z}{R_1}\right)^{-1}=1-\frac{z}{R_1}+\frac{-1\cdot-2}{2!}\left(\frac{z}{R_1}\right)^2=1-\frac{z}{R_1}+\left(\frac{z}{R_1}\right)^2$

将 ε_1 展开到 $\frac{z}{R_1}$ 二阶，则有：

(略去三次以上项)　取 $K_1-\frac{\varepsilon_1}{R_1}=K_1^*$

$$\varepsilon_1^{(z)}=\left(1-\frac{z}{R_1}+\left(\frac{z}{R_1}\right)^2+\cdots\right)(\varepsilon_1+K_1z)=\varepsilon_1+z\left(K_1-\frac{\varepsilon_1}{R_1}\right)-\frac{z^2}{R_1}\left(K_1-\frac{\varepsilon_1}{R_1}\right)$$

$$\varepsilon_2^{(z)}=\varepsilon_2+z\left(K_2-\frac{\varepsilon_2}{R_2}\right)-\frac{z^2}{R_2}\left(K_2-\frac{\varepsilon_2}{R_2}\right)$$

$K_2-\frac{\varepsilon_2}{R_2}=K_2^*$

$$\omega^{(z)}=\left(1-\frac{z}{R_1}+\left(\frac{z}{R_1}\right)^2+\cdots\right)(\omega_1+\tau_1z)+\left(1-\frac{z}{R_1}+\frac{z^2}{R_1^2}+\cdots\right)(\omega_2+\tau_2z)$$

$$=(\omega_1+\omega_2)+z\left(\tau_1-\frac{\omega_1}{R_1}+\tau_2-\frac{\omega_2}{R_2}\right)+\frac{z^2}{R_1}\left(\frac{\omega_1}{R_1}-\tau_1\right)+\frac{z^2}{R_2}\left(\frac{\omega_2}{R_2}-\tau_2\right)$$

$$\begin{cases} \varepsilon_1^{(z)}=\varepsilon_1+zK_1^*-\dfrac{z^2}{R_1}K_1^* \\ \varepsilon_2^{(z)}=\varepsilon_2+zK_2^*-\dfrac{z^2}{R_2}K_2^* \\ \omega^{(z)}=\omega+2z\tau^*-z^2\left(\dfrac{1}{R_1}+\dfrac{1}{R_2}\right)\left[\tau^*-\dfrac{\omega(R_1-R_2)^2}{2R_1R_2(R_1+R_2)}\right] \end{cases}$$

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$$N_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(1+\frac{z}{R_2}\right)\sigma_1^{(z)}dz,\ M_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_1^{(z)}z\left(1+\frac{z}{R_2}\right)dz,\ S_{12}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{12}\left(1+\frac{z}{R_2}\right)dz,\ H_{12}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{12}z\left(1+\frac{z}{R_2}\right)dz$$

$$N_2=\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(1+\frac{z}{R_1}\right)\sigma_1^{(z)}dz,\ M_2=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_1^{(z)}z\left(1+\frac{z}{R_1}\right)dz,\ S_{21}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{21}\left(1+\frac{z}{R_1}\right)dz,\ H_{12}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{21}z\left(1+\frac{z}{R_1}\right)dz.$$

代入:　仅以 N_1, M_1, S_{12} 和 H_{12} 的积分为例:

$$N_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\boxed{\left(1+\frac{z}{R_2}\right)}\left(\varepsilon_1^{(z)}+\nu\varepsilon_2^{(z)}\right)dh=\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\varepsilon_1^{(z)}+\nu\varepsilon_2^{(z)}\right)+\frac{z}{R_2}\left(\varepsilon_1^{(z)}+\nu\varepsilon_2^{(z)}\right)dh \quad \int z^2\left(\frac{K_1^*}{R_1}+\nu\frac{K_2^*}{R_2}\right)\leftarrow \text{用不到}.$$

$$=\frac{E}{1-\nu^2}\left\{\left[z(\varepsilon_1+\nu\varepsilon_2)\Big|_{-\frac{h}{2}}^{\frac{h}{2}}+\cancel{\frac{z^2}{2}(K_1^*+\nu K_2^*)}-\frac{z^3}{3}\left(\frac{K_1^*}{R_1}+\nu\frac{K_2^*}{R_2}\right)\right]+\left[\cancel{\frac{(\varepsilon_1+\nu\varepsilon_2)z^2}{2R_1}}+\frac{z^3}{R_2}(K_1^*+\nu K_2^*)\Big|_{-\frac{h}{2}}^{\frac{h}{2}}\right]\right.$$

此处积分保留到 h^3,
h^4 项略去。

$$=\frac{E}{1-\nu^2}\left[(\varepsilon_1+\nu\varepsilon_2)h-\frac{h^3}{12}\left(\frac{K_1^*}{R_1}+\nu\frac{K_2^*}{R_2}\right)+\frac{h^3}{12}\left(\frac{K_1^*}{R_2}+\nu\frac{K_2^*}{R_2}\right)\right]$$

$$=\frac{Eh}{1-\nu^2}\left[(\varepsilon_1+\nu\varepsilon_2)+\frac{h^2}{12}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)K_1^*\right]\quad ①$$

同理,应当有:

$$N_2=\frac{Eh}{1-\nu^2}\left[(\varepsilon_2+\nu\varepsilon_1)+\frac{h^2}{12}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)K_2^*\right].$$

$$M_1=\frac{E}{1-\nu^2}\left[\cancel{\frac{z^2}{2}(\varepsilon_1+\nu\varepsilon_2)}\Big|_{-\frac{h}{2}}^{\frac{h}{2}}+\frac{z^3}{3}(K_1^*+\nu K_2^*)+0\right]+\frac{1}{R_2}\left(\frac{z^3}{3}(\varepsilon_1+\nu\varepsilon_2)+0+\text{高}\right)$$

$$=\frac{Eh^3}{1-\nu^2}\left[\frac{1}{12}(K_1^*+\nu K_2^*)+\frac{1}{12R_2}(\varepsilon_1+\nu\varepsilon_2)\right]$$

$$M_1=\frac{Eh^3}{12(1-\nu^2)}\left[K_1^*+\nu K_2^*+\frac{1}{R_2}(\varepsilon_1+\nu\varepsilon_2)\right]\quad ③$$

同理 $M_2=\frac{Eh^3}{12(1-\nu^2)}\left[K_2^*+\nu K_1^*+\frac{1}{R_1}(\varepsilon_2+\nu\varepsilon_1)\right]$, ④　又: $\omega^{(z)}=\omega+2z\tau^*-z\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\left[\tau^*-\frac{\omega(R_1-R_2)^2}{2R_1R_2(R_1+R_2)}\right]$

$$S_{12}=\frac{E}{2(1+\nu)}\left[\omega h+\cdots\cancel{z^2}+\frac{2z^3}{3R_2}\Big|_{-\frac{h}{2}}^{\frac{h}{2}}\tau^*-\frac{z^3}{3}\Big|_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\left(\tau^*-\frac{\omega(R_1-R_2)^2}{2R_1R_2(R_1+R_2)}\right)\right.\quad \leftarrow\left(1+\frac{z}{R_2}\right)$$

$$=\frac{E}{2(1+\nu)}\left[\omega h+\frac{h^3}{12}\left[\frac{1}{R_2}-\frac{1}{R_1}\right]\tau^*+\frac{h^3}{12}\left(\frac{R_1+R_2}{R_1R_2}\right)\frac{(R_1-R_2)^2\omega}{2R_1R_2(R_1+R_2)}\right.\longrightarrow \text{由}\ \frac{R_1-R_2}{R_1R_2}=\left(\frac{1}{R_2}-\frac{1}{R_1}\right),\text{代入},$$

$$=\frac{Eh}{2(1+\nu)}\left[\omega+\frac{h^2}{12}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\left[\tau^*+\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\frac{\omega}{2}\right]\right]\quad ⑤$$

同理

$$S_{21}=\frac{Eh}{2(1+\nu)}\left[\omega+\frac{h^2}{12}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left[\tau^*+\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{\omega}{2}\right]\right]\quad ⑥$$

而: $H_{12}=\frac{E}{2(1+\nu)}\left[\frac{z^3}{3}2\tau^*+\omega\cdot\frac{z^3}{3R_2}\Big|\right]$

$$=\frac{Eh^3}{24(1+\nu)}\left[2\tau^*+\frac{\omega}{R_2}\right]\quad ⑦,\ \text{同理}\ H_{21}=\frac{Eh^3}{24(1+\nu)}\left[2\tau^*+\frac{\omega}{R_1}\right]\quad ⑧.$$

①~⑧为内力,内矩公式的表达,即内力,内矩的二阶解答
(即符拉索夫解答).

下面给出东南-铁木辛柯解答和诺符日洛夫解答

我们在内力-内矩本构关系中已经推导出壳体的内力,内矩

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$N_1, N_2, M_1, M_2, S_{12}, S_{21}, H_{12}, H_{21}$ 的表达式(精确解)

在积分时,由于每一个是叠加一个$\frac{z}{R}$后进行积分得到,若将$\frac{z}{R}$的小量略去,则有:

$$N_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_1 dz,\quad N_2=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_2 dz\quad S_{12}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{12} dz\quad H_{12}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{12} z dz$$

$$M_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_1 z dz\quad M_2=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_2 z dz.\quad S_{21}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{21} dz\quad H_{21}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{21} z dz$$

代入应力公式:

$$\sigma_1=\frac{E}{1-\nu^2}(\varepsilon_1^{(z)}+\nu\varepsilon_2^{(z)}),\ 由\varepsilon_1^{(z)}=\varepsilon_1+K_1 z$$

$$\sigma_2=\frac{E}{1-\nu^2}(\varepsilon_2^{(z)}+\nu\varepsilon_1^{(z)})\qquad \approx\varepsilon_1+K_1 z,$$

得: $\begin{cases}\sigma_1=\frac{Eh}{1-\nu^2}(\varepsilon_1+\nu\varepsilon_2) & ①\\ \sigma_2=\frac{Eh}{1-\nu^2}(\varepsilon_2+\nu\varepsilon_1) & ②\end{cases}$

同样有: $$M_1=\frac{E}{1-\nu^2}\frac{h^3}{12}(K_1-\nu K_2)=\frac{Eh^3}{12(1-\nu^2)}(K_1-\nu K_2)\quad ③$$

$$M_2=\frac{Eh^3}{12(1-\nu^2)}(K_2-\nu K_1)\quad ④$$

又: $\tau=G\gamma=G(\omega+2\tau z),=\frac{E}{2(1+\nu)}(\omega+2\tau z)=\tau_{12}=\tau_{21}$

从而: $$S_{12}=S_{21}=\frac{Eh}{2(1+\nu)}\omega,\quad ⑤$$

$$H_{12}=H_{21}=\frac{E}{2(1+\nu)}\cdot 2\tau\frac{z^3}{3}\Big|_{-\frac{h}{2}}^{\frac{h}{2}}=\frac{Eh^3}{12(1+\nu)}\tau\quad ⑥$$

①-⑥为东南-铁木辛柯解答,但是需要说明,此时在内力力矩与平衡方程中导出的公式

$\frac{H_{12}}{R_1}-\frac{H_{21}}{R_2}=S_{12}-S_{21}$ 并不能得到满足(不满足切应力互等定律)

我们考虑将解答进行优化,修改S_{12}和S_{21}的解答,对于$\frac{H_{12}}{R_1}-\frac{H_{21}}{R_2}=\frac{Eh^3}{12(1+\nu)}\tau\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$

因而我们可以在S_{12}和S_{21}上叠加修正项:

$$S_{12}=\frac{Eh}{2(1+\nu)}\omega-\frac{Eh^3}{12(1+\nu)}\cdot\frac{\tau}{R_2}=\frac{Eh}{2(1+\nu)}\left[\omega-\frac{h^2\tau}{6R_2}\right]$$

$$同\ S_{21}=\frac{Eh}{2(1+\nu)}\omega-\frac{Eh^3}{12(1+\nu)}\cdot\frac{\tau}{R_1}=\frac{Eh}{2(1+\nu)}\left[\omega-\frac{h^2\tau}{6R_1}\right]$$

为诺符且洛夫解。