例题1.2

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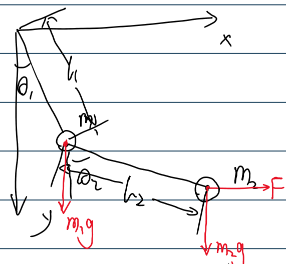
质量为 m_1, m_2 的两个小球，并使用绳系挂，在 m_2 上作用有水平方向已知的力 $F(t)$，建立运动微分方程：



①：系统使用 Lagrange 方程建立其运动微分方程，其一般形式为：

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_j}\right)-\frac{\partial T}{\partial q_j}=Q_j$$

为两个广义坐标；

显然为两自由度 $\rightarrow \theta_1, \theta_2$，设其对应的广义力为 Q_1, Q_2

$\Longrightarrow$ 有2个广义力对应的 Lagrange 方程

则：列写方程为：

$$\begin{cases}\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot\theta_1}\right)-\dfrac{\partial T}{\partial \theta_1}=Q_1\\[2ex]\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot\theta_2}\right)-\dfrac{\partial T}{\partial \theta_2}=Q_2\end{cases}\longrightarrow$$

(附：本题不是保守系统，若为保守系统可使用 $L=T-V$ 进行列写)

有：动能 T 表达式：

$$T=\frac{1}{2}m_1\cdot(l_1\dot\theta_1)^2+\frac{1}{2}m_2(\dot x_2^2+\dot y_2^2)$$

其中将 x_2, y_2 用 θ_1, θ_2 表达有：

$$x_2=l_1\sin\theta_1+l_2\sin\theta_2\rightarrow \dot x_2=l_1\cos\theta_1\dot\theta_1+l_2\cos\theta_2\dot\theta_2$$

$$y_2=l_1\cos\theta_1+l_2\cos\theta_2\rightarrow \dot y_2=-(l_1\sin\theta_1\dot\theta_1+l_2\sin\theta_2\dot\theta_2)$$

注意复合求导

$$=\frac{1}{2}m_1l_1^2\dot\theta_1^2+\frac{1}{2}m_2\left[(l_1\cos\theta_1\dot\theta_1+l_2\cos\theta_2\dot\theta_2)^2+(l_1\sin\theta_1\dot\theta_1+l_2\sin\theta_2\dot\theta_2)^2\right]$$

整理

$$T=\frac{1}{2}m_1l_1^2\dot\theta_1^2+\frac{1}{2}m_2\left[l_1^2\dot\theta_1^2+l_2^2\dot\theta_2^2+2l_1l_2(\sin\theta_1\sin\theta_2\dot\theta_1\dot\theta_2+\cos\theta_1\cos\theta_2\dot\theta_1\dot\theta_2)\right]$$

$$=\frac{1}{2}(m_1+m_2)l_1^2\dot\theta_1^2+\frac{1}{2}m_2l_2^2\dot\theta_2^2+m_2l_1l_2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)$$

则

$$\frac{\partial T}{\partial\dot\theta_1}=(m_1+m_2)l_1^2\dot\theta_1+m_2l_1l_2\dot\theta_2\cos(\theta_1-\theta_2)$$

$$\rightarrow\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta_1}\right)=(m_1+m_2)l_1^2\ddot\theta_1+m_2l_1l_2\ddot\theta_2\cos(\theta_1-\theta_2)-m_2l_1l_2\dot\theta_2\sin(\theta_1-\theta_2)\cdot(\dot\theta_1-\dot\theta_2)$$

$$=(m_1+m_2)l_1^2\ddot\theta_1+m_2l_1l_2\ddot\theta_2\cos(\theta_1-\theta_2)-m_2l_1l_2\sin(\theta_1-\theta_2)(\dot\theta_1\dot\theta_2-\dot\theta_2^2)$$

而

$$\frac{\partial T}{\partial\theta_1}=-m_2l_1l_2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2)$$

而：由

$$\sum\delta W=m_1g\cdot\delta(l_1\cos\theta_1)+m_2g\,\delta(l_1\cos\theta_1+l_2\cos\theta_2)+F\,\delta(l_1\sin\theta_1+l_2\sin\theta_2)$$

$$=-m_1gl_1\sin\theta_1\delta\theta_1-m_2gl_1\sin\theta_1\delta\theta_1-m_2gl_2\sin\theta_2\delta\theta_2+Fl_1\cos\theta_1\delta\theta_1+Fl_2\cos\theta_2\delta\theta_2$$

$$=(-m_1gl_1\sin\theta_1-m_2gl_1\sin\theta_1+F_1l_1\cos\theta_1)\delta\theta_1+(-m_2gl_2\sin\theta_2+Fl_2\cos\theta_2)\delta\theta_2$$

则有：

$$Q_1=-m_1gl_1\sin\theta_1-m_2gl_1\sin\theta_1+F_1l_1\cos\theta_1$$

代入Lagrange方程有：

$$(m_1+m_2)l_1^2\ddot\theta_1+m_2l_1l_2\ddot\theta_2\cos(\theta_1-\theta_2)-m_2l_1l_2\sin(\theta_1-\theta_2)(\cancel{\dot\theta_1\dot\theta_2}-\dot\theta_2^2)\cancel{+m_2l_1l_2\sin(\theta_1-\theta_2)\cdot\dot\theta_1\dot\theta_2}=-m_1gl_1\sin\theta_1-m_2gl_1\sin\theta_1+F_1l_1\cos\theta_1$$

约去 l_1，则：

$$(m_1+m_2)l_1\ddot\theta_1+m_2l_2\ddot\theta_2\cos(\theta_1-\theta_2)-m_2l_2\dot\theta_2^2\sin(\theta_1-\theta_2)=-(m_1+m_2)g\sin\theta_1+F_1\cos\theta_1$$

则得到第①式：

$$(m_1+m_2)l_1\ddot\theta_1+m_2l_2\ddot\theta_2\cos(\theta_1-\theta_2)-m_2l_2\dot\theta_2^2\sin(\theta_1-\theta_2)+(m_1+m_2)g\sin\theta_1=F_1\cos\theta_1 \quad ①$$

则得到第①式：

$$(m_1+m_2)l_1\ddot{\theta}_1+m_2l_2\ddot{\theta}_2\cos(\theta_1-\theta_2)-m_2l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)+(m_1+m_2)g\sin\theta_1=F_1\cos\theta_1 \quad ①$$

而：

$$Q_2=-m_2gl_2\sin\theta_2+Fl_2\cos\theta_2$$

而：$\frac{\partial T}{\partial\dot{\theta}_2}=m_2l_2^2\dot{\theta}_2+m_2l_1l_2\dot{\theta}_1\cos(\theta_1-\theta_2)$ ，

则 $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}_2}\right)=m_2l_2^2\ddot{\theta}_2+m_2l_1l_2\ddot{\theta}_1\cos(\theta_1-\theta_2)-m_2l_1l_2\dot{\theta}_1\sin(\theta_1-\theta_2)\cdot(\dot{\theta}_1-\dot{\theta}_2)$

$=m_2l_2^2\ddot{\theta}_2+m_2l_1l_2\ddot{\theta}_1\cos(\theta_1-\theta_2)-m_2l_1l_2\sin(\theta_1-\theta_2)(\dot{\theta}_1^2-\dot{\theta}_1\dot{\theta}_2)$

又：$\frac{\partial T}{\partial\theta_2}=\cancel{-}m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)\cdot\cancel{(-1)}$

则：$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}_2}\right)-\frac{\partial T}{\partial\theta_2}=Q_2$ 为 ②式；

代入有：

$m_2l_2^{\cancel{2}}\ddot{\theta}_2+m_2l_1\cancel{l_2}\ddot{\theta}_1\cos(\theta_1-\theta_2)-m_2l_1\cancel{l_2}\sin(\theta_1-\theta_2)(\dot{\theta}_1^2\cancel{-\dot{\theta}_1\dot{\theta}_2})\cancel{-m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)}$

$=-m_2g\cancel{l_2}\sin\theta_2+F\cancel{l_2}\cos\theta_2$

故：

$$m_2l_2\ddot{\theta}_2+m_2l_1\ddot{\theta}_1\cos(\theta_1-\theta_2)-m_2l_1\sin(\theta_1-\theta_2)\dot{\theta}_1^2+m_2g\sin\theta_2=F\cos\theta_2 \quad ②$$

则①，②式即为所求的运动微分方程。