拉格朗日变换式和拉格朗日方程的推导

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①：两大拉格朗日变换式：
$$\begin{cases}\dfrac{\partial \dot{r}_i}{\partial \dot{q}_j}=\dfrac{\partial r_i}{\partial q_j}\\[2ex] \dfrac{\partial \dot{r}_i}{\partial q_j}=\dfrac{d}{dt}\left(\dfrac{\partial r_i}{\partial q_j}\right)\end{cases}$$

推导：由于有公式：

$$r_i=r_i(q_1,q_2,\cdots\cdots q_k,t)\quad(i=1,2,\cdots n)$$

则有：$\dot{r}_i=\dfrac{\partial r_i}{\partial q_j}\boxed{\dfrac{\partial q_j}{\partial t}}\delta t+\dfrac{\partial r_i}{\partial t}\delta t$，$\dfrac{\partial q_j}{\partial t}\delta t \to \delta q_j$，显然有：$\boxed{\dfrac{\partial \dot{r}_i}{\partial \dot{q}_j}=\dfrac{\partial r_i}{\partial q_j}}$ ①式得证．

另外：
$$\dfrac{\partial \dot{r}_i}{\partial q_j}=\sum_{l=1}^{k}\dfrac{\partial^2 r_i}{\partial q_l\partial q_j}\overset{\dot{q}_l}{\dfrac{dq_l}{dt}}+\dfrac{\partial^2 r_i}{\partial t\partial q_j}\quad(1)$$
复合求导．

$\dfrac{\partial r_i}{\partial q_j}$ 由于 $\dfrac{\partial r_i}{\partial q_j}$ 是 (q_l,t) 的函数

并有：$\dfrac{\partial r_i}{\partial q_j}$，直接代入下式：

此时有：
$$\dfrac{d}{dt}\left(\dfrac{\partial r_i}{\partial q_j}\right)=\sum\dfrac{\partial}{\partial q_l}\left(\dfrac{\partial r_i}{\partial q_j}\right)\dfrac{dq_l}{dt}+\dfrac{\partial}{\partial t}\left(\dfrac{\partial r_i}{\partial q_j}\right)$$
$$=\sum_{l=1}^{k}\dfrac{\partial^2 r_i}{\partial q_j\partial q_l}\dot{q}_l+\dfrac{\partial^2 r_i}{\partial q_j\partial t}\quad(2)$$

显然有式(1)＝式(2)，故证得：$\dfrac{\partial \dot{r}_i}{\partial q_j}=\dfrac{\partial \dot{r}_i}{\partial q_j}$，②式得证．

接下来，我们使用两个拉格朗日变换式推导拉格朗日方程：

首先：设某一受理想约束的 n 质点系统，且 $q_1,q_2\cdots q_k$ 为广义坐标，有：对系统中任一质点 M，其相对于 O 点的矢径为

$$\vec{r}=\vec{r}(q_1,q_2\cdots q_k,t)，$$ 则有：对该式取变分：

有：$\delta\vec{r}_i=\displaystyle\sum_{j=1}^{k}\dfrac{\partial \vec{r}_i}{\partial q_j}\delta q_j$ （注意：对时间的变分为0）

代入动力学普遍方程：即：$\displaystyle\sum_{i=1}^{n}(F_i-m_ia_i)\cdot\delta r_i=0$

代入动力学普遍方程：即：$\sum_{i=1}^{n}(F_i - m_i a_i)\cdot \delta r_i = 0$

则：$\sum_{i=1}^{n} F_i \cdot \delta \vec{r}_i - \sum_{i=1}^{n}(m_i a_i \delta r_i) = 0$

有：$\left[\sum_{i=1}^{n} \vec{F}_i \sum_{j=1}^{k} \frac{\partial \vec{r}_i}{\partial q_j} - \sum_{i=1}^{n} m_i \ddot{r}_i \sum_{j=1}^{k} \frac{\partial \vec{r}_i}{\partial q_j}\right] \delta q_j = 0$

其中有

$$\sum_{i=1}^{n} F_i \delta \vec{r}_i = \sum_{i=1}^{n} F_i \sum_{j=1}^{k} \frac{\partial r_i}{\partial q_j} \delta q_j = \sum_{j=1}^{k}\left[\sum_{i=1}^{n} F_i \frac{\partial r_i}{\partial q_j}\right]\cdot \delta q_j$$

我们令广义力 $\boxed{Q_j = \sum_{i=1}^{n} F_i \frac{\partial r_i}{\partial q_j}}$

则上式变为：$\boxed{\sum_{i=1}^{n} F_i \delta \vec{r}_i = \sum_{j=1}^{k} Q_j \delta q_j}$

故所有主动力的虚功可由下式计算：

$$\boxed{\delta W = \sum_{j=1}^{k} Q_j \delta q_j}$$

代入得到：

$$\left[\sum_{j=1}^{k} Q_j - \sum_{i=1}^{n} m_i \ddot{r}_i \sum_{j=1}^{k} \frac{\partial r_i}{\partial q_j}\right]\cdot \delta q_j = 0$$

$\longrightarrow$ 第二项：$= -\sum_{j=1}^{k}\left[\sum_{i=1}^{n} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j}\right]$

并定义其中的：$\boxed{-\sum_{i=1}^{n} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} = Q_j'}$ 称为广义惯性力；

得到：$\sum_{j=1}^{k}\left[Q_j + Q_j'\right] \delta q_j = 0$

其中：对于广义惯性力，利用导数法则将其拆分：

由 $m_i \frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_j}\right) = m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} + m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)$

此时，有：

$$Q_j' = \sum_{i=1}^{n}\left[-m_i \frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_j}\right) + m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)\right]$$

※ 此时使用 Lagrange 求换公式，有：

(2) 此因为使用 Lagrange 我们引入 q，因：

$$Q_j' = \sum_{i=1}^{n}\left[-m_i\frac{d}{dt}\left(\dot{r}_i\frac{\partial \dot{r}_i}{\partial \dot{q}_j}\right) + m_i\dot{r}_i\frac{\partial \dot{r}_i}{\partial q_j}\right]$$

$$= \sum_{i=1}^{n}\left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left(-\frac{1}{2}m_i\dot{r}_i\cdot\dot{r}_i\right) + \frac{\partial}{\partial q_j}\left(\frac{1}{2}m_i\dot{r}_i\cdot\dot{r}_i\right)\right]$$

我们设系统的总动能为：T，并：$T=\sum_{i=1}^{n}\frac{1}{2}m_i\dot{r}_i\cdot\dot{r}_i$

将微分号提入：

则：$$Q_j' = -\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} + \frac{\partial T}{\partial q_j}$$

其中：$j=1,2,\cdots k$

又：利用 $Q_j + Q_j' = 0$，代入有：

$$\boxed{Q_j - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} + \frac{\partial T}{\partial q_j} = 0}$$

（广义坐标形式的动力学普遍方程）

或：

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

（第二类拉格朗日方程）