N维正态随机变量的概率密度推导

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对于n维随机变量 为考虑其概率密度的一般形式，

先考虑二维正态随机变量的一般形式。

$$f(x_1,x_2)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-\mu}{\sigma_1}\right)\left(\frac{x_2-\mu}{\sigma_2}\right)+\left(\frac{x_2-\mu}{\sigma_2}\right)^2\right]\right\}$$

考虑写成矩阵形式.

$$\boxed{X=\begin{pmatrix}x_1\\x_2\end{pmatrix}\qquad \mu=\begin{pmatrix}\mu_1\\\mu_2\end{pmatrix}},\quad 则\ D_1=\sigma_1\quad D_2=\sigma_2.$$

∴ 又：$Cov(X,Y)$ 已推出为，$\sigma_1\sigma_2\rho$，$\rho_{XY}=\rho$.

$$则协方差阵\quad C=\begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}$$

显然：$\sigma_1\sigma_2$ 相关系数 $\rho=1$.

显有：

$$\det C=\sigma_1^2\sigma_2^2-\rho^2\sigma_1^2\sigma_2^2=(1-\rho^2)\sigma_1^2\sigma_2^2$$

故：

求逆时变负

$$C^{-1}=\frac{1}{\det C}\begin{bmatrix}\sigma_2^2 & -\rho\sigma_1\sigma_2\\ -\rho\sigma_1\sigma_2 & \sigma_1^2\end{bmatrix}=\frac{1}{(1-\rho^2)}\begin{bmatrix}\frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2}\\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2}\end{bmatrix}$$

故：

$$x-\mu=\begin{bmatrix}x_1-\mu_1\\x_2-\mu_2\end{bmatrix},$$

则二维正态变量可以表达为：

考虑：$$(X-\mu)^T\frac{1}{1-\rho^2}\begin{bmatrix}\frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2}\\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2}\end{bmatrix}(X-\mu)$$

$$=\frac{1}{1-\rho^2}\underbrace{[x_1-\mu_1,x_2-\mu_2]\begin{bmatrix}\frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2}\\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2}\end{bmatrix}}_{\frac{1}{1-\rho^2}\left[\frac{x_1-\mu_1}{\sigma_1^2}-\rho\frac{x_2-\mu_2}{\sigma_2}\cdot\frac{1}{\sigma_1},\ \frac{x_2-\mu_2}{\sigma_2^2}-\rho\frac{x_1-\mu_1}{\sigma_2}\right]}\begin{bmatrix}x_1-\mu_1\\x_2-\mu_2\end{bmatrix}=\frac{1}{1-\rho^2}\left[\left(\frac{x_1-\mu}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-\mu}{\sigma_1}\right)\left(\frac{x_2-\mu}{\sigma_2}\right)+\left(\frac{x_2-\mu}{\sigma_2}\right)^2\right]$$

故：$$f(x_1,x_2)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2}(x-\mu)^TC^{-1}(x-\mu)\right]=\frac{1}{(\sqrt{2\pi})^2(\det C)^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}(x-\mu)^TC^{-1}(x-\mu)\right\}$$

负

故N维随机变量概率密度

$$f(x_1,x_2,\cdots x_n)=\frac{1}{(\sqrt{2\pi})^n(\det)^{\frac{1}{2}}}\exp\left[-\frac{1}{2}(x-\mu)^TC^{-1}(x-\mu)\right]$$