有限元方法应力函数差分关系的推导

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①：应力函数表示的相容方程：$\nabla^4\phi=0$，即：（无体力）

$$\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}=0$$

或代入有重调和方程的差分形式：

$$20\phi_0-8(\phi_1+\phi_2+\phi_3+\phi_4)+2(\phi_5+\phi_6+\phi_7+\phi_8)+(\phi_9+\phi_{10}+\phi_{11}+\phi_{12})=0$$

②：平衡方程：$\left\{ l=\frac{dy}{ds},\quad m=-\frac{dx}{ds}\right.$，$\begin{cases} l\sigma_x+m\tau_{xy}=0 \\ l\tau_{xy}+m\sigma_y=0 \end{cases}$ 代入得：

$$\boxed{\begin{cases} \frac{dy}{ds}\left(\frac{\partial^2\phi}{\partial y^2}\right)+\frac{dx}{ds}\left(\frac{\partial^2\phi}{\partial x\partial y}\right)=f_x \\ -\frac{dy}{ds}\left(\frac{\partial^2\phi}{\partial x\partial y}\right)-\frac{dx}{ds}\left(\frac{\partial^2\phi}{\partial x^2}\right)=f_y \end{cases}}$$

对于左式，可以进行合并，成为：

有：

$$f_x=\frac{d}{ds}\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)dy+\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)dx\right]$$

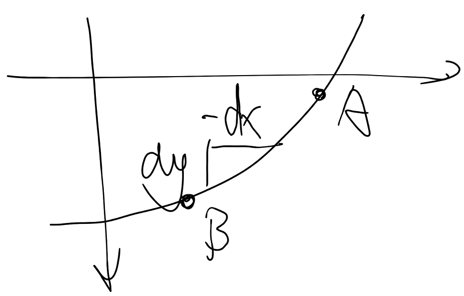
$$f_y=-\frac{d}{ds}\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)dy+\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)dx\right]$$

即有：

$$\begin{cases} f_x=\frac{d}{ds}\left[\frac{\partial\phi}{\partial y}\right]_s \\ f_y=-\frac{d}{ds}\left[\frac{\partial\phi}{\partial x}\right]_s \end{cases}$$

可积分有：

$$\boxed{\begin{aligned} \left(\frac{\partial\phi}{\partial y}\right)_s&=\int_A^B f_x\,ds \\ \left(\frac{\partial\phi}{\partial x}\right)_s&=-\int_A^B f_y\,ds \end{aligned}}\qquad ②$$



或有：

$$\begin{cases} \left.\frac{\partial\phi}{\partial y}\right|_B=\left.\frac{\partial\phi}{\partial y}\right|_A+\int_A^B f_x\,ds \\ \left.\frac{\partial\phi}{\partial x}\right|_B=\left.\frac{\partial\phi}{\partial x}\right|_A-\int_A^B f_y\,ds \end{cases}$$

有：利用在x,y方向上进行全微分和分部积分展开：

$$\Phi_B-\Phi_A=\underbrace{\int_A^B\frac{\partial\Phi}{\partial x}ds}_{(1)}-\underbrace{\int_A^B\frac{\partial\Phi}{\partial y}ds}_{(2)}$$

$$\int_A^B \frac{\partial \Phi}{\partial x} ds \quad \int_A^B \frac{\partial \Phi}{\partial y} ds$$

(1) (2)

利用：
$$\int_A^B x\frac{\partial \Phi}{\partial x} ds = \left(x\frac{\partial \Phi}{\partial x}\right)\Big|_A^B - \int_A^B x\cdot\frac{\partial}{\partial s}\left(\frac{\partial \Phi}{\partial x}\right) ds$$

将上式代換(1)(2)项则有：
$$\Phi_B - \Phi_A = \left(x\frac{\partial \Phi}{\partial x}\right)\Big|_A^B - \int_A^B x\cdot\frac{\partial}{\partial s}\left(\frac{\partial \Phi}{\partial x}\right) ds + \left(y\frac{\partial \Phi}{\partial y}\right)\Big|_A^B - \int_A^B y\frac{\partial}{\partial s}\left(\frac{\partial \Phi}{\partial y}\right) ds$$

利用②代換：
$$\boxed{\begin{cases}\left(\frac{\partial \Phi}{\partial x}\right)_s = -\int_A^B f_y ds \\ \left(\frac{\partial \Phi}{\partial y}\right)_s = \int_A^B f_x ds\end{cases}}$$

有
$$\Phi_B - \Phi_A = x_B\left(\frac{\partial \Phi}{\partial x}\right)\Big|_B - x_A\left(\frac{\partial \Phi}{\partial x}\right)\Big|_A + \int_A^B x f_y ds$$
$$+ y_B\left(\frac{\partial \Phi}{\partial y}\right)\Big|_B - y_A\left(\frac{\partial \Phi}{\partial y}\right)\Big|_A - \int_A^B y f_x ds$$

为 $\frac{\partial \Phi}{\partial x}$

$$= x_B\left[\left(\frac{\partial \Phi}{\partial x}\right)\Big|_A - \int_A^B f_y ds\right] - x_A\left(\frac{\partial \Phi}{\partial x}\right)\Big|_A + \int_A^B x f_y ds$$
$$+ y_B\left[\left(\frac{\partial \Phi}{\partial y}\right)\Big|_A + \int_A^B f_x ds\right] - y_A\left(\frac{\partial \Phi}{\partial y}\right)\Big|_A - \int_A^B y f_x ds$$

有应力函数计算关系导出式：
$$\boxed{\Phi_B = \Phi_A + (x_B - x_A)\left(\frac{\partial \Phi}{\partial x}\right)\Big|_A + (y_B - y_A)\left(\frac{\partial \Phi}{\partial y}\right)\Big|_A - \int_A^B (y - y_B) f_x ds + \int_A^B (x - x_B) f_y ds} \quad (e)$$

对于一般的应力函数，我们可以假想将函数Φ加上 $a+bx+cy$，并调整 a,b,c 三个系数，使得$\Phi_A=0$、$\left(\frac{\partial \Phi}{\partial x}\right)_A=0$、$\left(\frac{\partial \Phi}{\partial y}\right)_A=0$，此时上式化为

(一次项不影响应力分布)

$$\Phi_B - \Phi_A = \int_A^B (x - x_B) f_y ds - \int_A^B (y - y_B) ds \quad (5\text{-}13)$$

$$\Phi_B - \Phi_A = \int_A (x-x_B) f_y ds - \int_A (y-y_B) f_x ds \quad (5-13)$$

而有:

$$\left(\frac{\partial \Phi}{\partial y}\right)_B = \left(\frac{\partial \Phi}{\partial y}\right)_A + \int_A^B f_x ds$$

$$\Rightarrow \left(\frac{\partial \Phi}{\partial y}\right)_B = \int_A^B f_x ds \quad (5-11)$$

同理: $$\left(\frac{\partial \Phi}{\partial x}\right)_B = -\int_A^B f_y ds \quad (5-12)$$