张量的梯度与积分推导

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四. 张量的偏微分

(一)、梯度的计算

(1). 标量场.

$$\varphi(x_i) \qquad \vec{\nabla} = \frac{\partial}{\partial x_i}\vec{e_i}$$

右梯度:

$$\varphi\vec{\nabla} = \varphi(x_i)\frac{\partial}{\partial x_i}\vec{e_i} = \frac{\partial \varphi}{\partial x_i}\vec{e_i}$$

对标量偏:

左梯度:

$$\vec{\nabla}\varphi = \left(\frac{\partial}{\partial x_i}\vec{e_i}\right)\varphi(x_i) = \vec{e_i}\frac{\partial \varphi}{\partial x_i}$$

②. 对矢量场. $\vec{V} = V_i\vec{e_i}$

右梯度:

$$\vec{V}\otimes\vec{\nabla} = V_i\,\vec{e_i}\otimes\frac{\partial}{\partial x_j}\vec{e_j} = \frac{\partial}{\partial x_j}(V_i\vec{e_i})\otimes\vec{e_j}$$

$$= \frac{\partial V_i}{\partial x_j}\vec{e_i}\otimes\vec{e_j} + V_i\underbrace{\frac{\partial \vec{e_i}}{\partial x_j}}_{=0}\vec{e_i}\otimes\vec{e_j}$$

$$= \frac{\partial V_i}{\partial x_j}\vec{e_i}\otimes\vec{e_j} = V_{i,j}\,\vec{e_i}\otimes\vec{e_j}$$

左梯度:

$$\vec{\nabla}\otimes\vec{V} = \frac{\partial}{\partial x_i}\vec{e_i}\otimes V_j\vec{e_j} = \frac{\partial V_j}{\partial x_i}\vec{e_i}\otimes\vec{e_j}$$

$$= \frac{\partial V_i}{\partial x_j}\vec{e_j}\otimes\vec{e_i} = V_{i,j}\,\vec{e_j}\otimes\vec{e_i}$$

因而有: $\vec{V}\otimes\vec{\nabla} = (\nabla\otimes\vec{V})^T$

(3)、二阶张量 $T = T_{ij}\,\vec{e_i}\otimes e_j$

①: 对右梯度:

右梯度 $\vec{\vec{T}}\otimes\vec{\nabla} = T_{ij}\vec{e_i}\otimes\vec{e_j}\otimes\frac{\partial}{\partial x_k}\vec{e_k}$

$$= \frac{\partial T_{ij}}{\partial x_k}\vec{e_i}\otimes\vec{e_j}\otimes\vec{e_k}$$

左梯度: $\vec{\nabla}\otimes\vec{T} = \frac{\partial}{\partial x_i}\vec{e_i}$

梯度 $\nabla\varphi = \text{grad}\,\varphi$ 散度: div (divergent)

并矢 $\vec{\nabla}\otimes\vec{V}$ 点矢 $\vec{V}\cdot\vec{\nabla}$

(二)、散度的计算

(1). 对矢量场 $V = \vec{V_i}\vec{e_i}$

右散度: $\vec{V}\cdot\vec{\nabla} = V_i\vec{e_i}\cdot\frac{\partial}{\partial x_j}\vec{e_j}$

$$= \frac{\partial V_i}{\partial x_j}\vec{e_i}\cdot\vec{e_j} = \frac{\partial V_i}{\partial x_j}\delta_{ij} = \frac{\partial V_i}{\partial x_i} = V_{i,i}$$

左散度: $\vec{\nabla}\cdot\vec{V} = \left(\frac{\partial}{\partial x_i}\vec{e_i}\right)\cdot\left(V_j\vec{e_j}\right)$

$$= \frac{\partial V_j}{\partial x_i}\delta_{ij} = \frac{\partial V_i}{\partial x_i} = V_{i,i}$$

(2) 二阶张量 T

①: 右散度问题

$$\vec{\vec{T}}\cdot\nabla = T_{ij}\vec{e_i}\otimes\vec{e_j}\cdot\frac{\partial}{\partial x_k}\vec{e_k} = \frac{\partial T_{ij}}{\partial x_k}\delta_{jk}\vec{e_i} = \frac{\partial T_{ij}}{\partial x_j}\vec{e_i}$$

$$= T_{ij,j}\vec{e_i}$$

②: 左散度

$$\vec{\nabla}\cdot\vec{T} = \frac{\partial}{\partial x_i}\vec{e_i}\cdot T_{jk}\vec{e_j}\otimes\vec{e_k}$$

$$= \frac{\partial}{\partial x_i}\delta_{ij}T_{jk}\vec{e_k}$$

$$= \frac{\partial}{\partial x_j}T_{jk}\vec{e_k} = T_{jk,j}\vec{e_k} = \frac{\partial T_{ji}}{\partial x_j}\vec{e_i}$$

$$= T_{ji,j}\vec{e_i}$$

若 $T_{ij}=T_{ji}$,则相同,

③ 旋度:(Curl)

①: 对于矢量场 $\vec{V} = V_i\vec{e_i}$

$$\vec{V}\times\vec{\nabla} = V_i\vec{e_i}\times\frac{\partial}{\partial x_j}\vec{e_j} = V_i\frac{\partial}{\partial x_j}\varepsilon_{ijk}\vec{e_k}$$

$$= \frac{\partial V_i}{\partial x_j}\varepsilon_{ijk}\vec{e_k} = V_{i,j}\varepsilon_{ijk}\vec{e_k}$$

②: 左旋度:

$$\vec{\nabla}\times\vec{V} = \left(\frac{\partial}{\partial x_i}\vec{e_i}\right)\times V_j\vec{e_j} = \frac{\partial V_j}{\partial x_i}\varepsilon_{ijk}\vec{e_k}$$

$$= \frac{\partial V_i}{\partial x_j}\varepsilon_{jik}\vec{e_k} = -V_{i,j}\varepsilon_{ijk}\vec{e_k}.$$

有 $\vec{V}\times\vec{\nabla} = -\vec{\nabla}\times\vec{V}$

(2) 二阶张量: $T = T_{ij}\vec{e_j}\otimes\vec{e_k}$

右旋度

$$\vec{T}\times\nabla = T_{ij}\vec{e_i}\otimes\vec{e_j}\times\frac{\partial}{\partial x_k}\vec{e_k}$$

$$= \frac{\partial T_{ij}}{\partial x_k}\vec{e_i}\otimes\varepsilon_{jkm}\vec{e_m}$$

$$= \frac{\partial T_{ij}}{\partial x_k}\varepsilon_{jkm}\vec{e_i}\otimes\vec{e_m}$$

$$\vec{\nabla}\times\vec{T} = \frac{\partial}{\partial x_i}\vec{e_i}\times T_{jk}\vec{e_j}\otimes\vec{e_k}$$

$$\vec{\nabla} \times \vec{\vec{T}} = \frac{\partial}{\partial x_i} \vec{e_i} \times T_{jk} \vec{e_j} \otimes \vec{e_k}$$

$$= \frac{\partial}{\partial x_i} T_{jk} \, \varepsilon_{ijm} \, \vec{e_m} \otimes \vec{e_k}$$

令 $i=k$, $j=i$, $k=j$，则:

$$= \frac{\partial}{\partial x_k} T_{ij} \, \varepsilon_{kjm} \, \vec{e_m} \otimes \vec{e_j}$$

$$= \frac{\partial T_{ij}}{\partial x_k} \varepsilon_{kjm} \, \vec{e_m} \otimes \vec{e_j}$$

补充练习:

$$\nabla^2 = \Delta = \vec{\nabla} \cdot \vec{\nabla} = \left(\frac{\partial}{\partial x_i} \vec{e_i}\right) \cdot \left(\frac{\partial}{\partial x_j} \vec{e_j}\right)$$

$$= \frac{\partial^2}{\partial x_i \partial x_j} \delta_{ij}$$

$$= \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

标量

$$\underline{\vec{\nabla}(\vec{\nabla} \cdot \vec{u})} = (\vec{\nabla} \otimes \vec{\nabla}) \cdot \vec{u}$$

$$\vec{\nabla} \otimes \vec{\nabla} = \frac{\partial}{\partial x_i} \vec{e_i} \otimes \frac{\partial}{\partial x_j} \vec{e_j} = \frac{\partial^2}{\partial x_i \partial x_j} \vec{e_i} \otimes \vec{e_j}$$