一维形式的熵流动控制方程

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气体内能：$U = C_v T$，声速 $C = \sqrt{\gamma RT}$，

而熵为 $S = \dfrac{P}{\rho^\gamma} = const$（等熵关系），

有：$c^2 = \gamma RT \xlongequal{P\frac{V}{m}=RT} \gamma \dfrac{P}{\rho} = \gamma \dfrac{S\rho^\gamma}{\rho} = S\gamma\rho^{\gamma-1}$

因此有：$C = \sqrt{S\gamma\rho^{\gamma-1}} = S^{\frac{1}{2}}\gamma^{\frac{1}{2}}\rho^{\frac{\gamma-1}{2}}$ 显有 $\dfrac{c}{\rho} = S^{\frac{1}{2}}\gamma^{\frac{1}{2}}\rho^{\frac{\gamma-3}{2}}$

故：$\boxed{\dfrac{dc}{d\rho} = \dfrac{\gamma-1}{2} S^{\frac{1}{2}}\gamma^{\frac{1}{2}}\rho^{\frac{\gamma-3}{2}} = \dfrac{\gamma-1}{2}\dfrac{c}{\rho}}$.

而：有均熵运动情况下的能量控制方程为：

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \dfrac{\partial(\rho u)}{\partial x} = 0, \longrightarrow \boxed{\dfrac{\partial\rho}{\partial t} + u\dfrac{\partial\rho}{\partial x} + \rho\dfrac{\partial u}{\partial x} = 0} \quad ① \\ \dfrac{\partial(\rho u)}{\partial t} + \dfrac{\partial(\rho u^2 + P)}{\partial x} = 0 \end{cases}$$

由于 $c^2 = \gamma RT = \gamma \dfrac{P}{\rho} \Rightarrow$ 由：$\dfrac{\partial P}{\partial \rho} = \dfrac{c^2}{\gamma}$

$\longrightarrow P = \dfrac{\rho c^2}{\gamma}$

故：$\dfrac{\partial P}{\partial \rho} = \dfrac{c^2}{\gamma}$，

则有：

$$u\frac{\partial\rho}{\partial t} + \rho\frac{\partial u}{\partial t} + \rho\frac{\partial u^2}{\partial x} + u^2\frac{\partial\rho}{\partial x} + \frac{\partial P}{\partial\rho}\cdot\frac{\partial\rho}{\partial x},$$

①②合并：$= u\left(\dfrac{\partial\rho}{\partial t} + u\dfrac{\partial\rho}{\partial x}\right) = -\left(\rho\dfrac{\partial u}{\partial x}\right)$ 故：

$$\rho\frac{\partial u}{\partial t} + \boxed{\rho\frac{\partial u^2}{\partial x} - \rho u\frac{\partial u}{\partial x}} + \frac{\partial P}{\partial\rho}\cdot\frac{\partial\rho}{\partial x} = 0$$

$$= \cancel{2}\rho u\frac{\partial u}{\partial x} - \cancel{\rho u\frac{\partial u}{\partial x}}$$

故：$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\frac{c^2}{\gamma}\frac{\partial \rho}{\partial x}=0$，

得：$\boxed{\frac{\partial u}{\partial t}+\frac{c^2}{\rho\gamma}\frac{\partial \rho}{\partial x}+u\frac{\partial u}{\partial x}=0}$ ②

该式可写为：

$$\begin{cases}\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0 & ① \\ \frac{\partial u}{\partial t}+\frac{c^2}{\rho\gamma}\frac{\partial \rho}{\partial x}+u\frac{\partial u}{\partial x}=0, & ②\end{cases}$$

$\frac{\partial U}{\partial t}+A\frac{\partial U}{\partial x}=0$，其中 $U=\begin{bmatrix}\rho \\ u\end{bmatrix}$ $A=\begin{bmatrix}u & \rho \\ \frac{c^2}{\rho\gamma} & u\end{bmatrix}$

其中：A有特征值：$|\lambda E-A|=\begin{bmatrix}\lambda-u & -\rho \\ -\frac{c^2}{\rho\gamma} & \lambda-u\end{bmatrix}$

$\therefore (\lambda-u)^2-\frac{c^2}{\gamma}=0$，

$\lambda_{1,2}=\underline{u\pm\sqrt{\frac{c^2}{\gamma}}}$