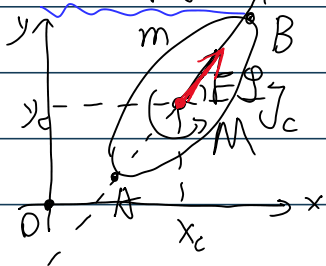


例题1.6

Friday, March 31, 2023 3:07 PM

如图, 质量为 m 的雪橇, 对其心质心轴的转动惯量为 J_c , 且雪橇速度始终沿 AB , 在 xOy 平面运动, 其上有 AB 方向力 F , 力偶 M , 设 F, M, t 均为已知函数, 确定其运动方程.



解: 由于受非完整约束, 不满足 Lagrange 方程,

可用 罗斯方程:

受非完整约束:

三自由度 x, y, φ , $k=3$

$$\text{则有: } \tan \varphi = \frac{\delta y}{\delta x} \quad \text{即: } \frac{\partial x}{\partial t} \tan \varphi - \frac{\partial y}{\partial t} = 0$$

$$\text{求: } \delta x \tan \varphi - \delta y = 0$$

$$\text{罗斯方程为: } \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j + \sum_{r=1}^m \lambda_r a_{rj} \quad (m=1)$$

$$\text{其中: 有: } a_{11} = \tan \varphi$$

$$a_{12} = -1$$

$$a_{13} = 0 \quad (\text{对 } \dot{\varphi} \text{ 导数}), \text{ 而 } T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} J_c \dot{\varphi}^2$$

难点: 广义力的获取方法:

$$\text{由虚功原理: } F \cos \varphi \delta x + F \sin \varphi \delta y + M \cdot \delta \varphi = \delta W = \sum_{j=1}^3 Q_j \delta q_j \quad \leftarrow k=3$$

$$\text{因此有: } Q_1 = F \cos \varphi \quad Q_2 = F \sin \varphi \quad Q_3 = M$$

列方程:

$$\begin{cases} m\ddot{x} = F \cos \varphi + \lambda_1 \tan \varphi & ① \\ m\ddot{y} = F \sin \varphi - \lambda_1 & ② \\ J_c \ddot{\varphi} = M & ③ \end{cases}$$

$$\text{联立约束方程: } \dot{x} \tan \varphi - \dot{y} = 0 \quad ④$$

这四个方程即为描述雪橇运动模型

下面讨论如何求解 λ : 联①④: 需要注意的是: 对 $\dot{x} \tan \varphi - \dot{y} = 0$ 求导时 $\Rightarrow \varphi$ 也要对时间求导;

对④求导: 复合求导:

$$\dot{x} \tan \varphi + \dot{x} \frac{1}{\cos^2 \varphi} \dot{\varphi} - \dot{y} = 0$$

$$\text{故: } m\ddot{x} \tan \varphi + \frac{m\dot{x}}{\cos^2 \varphi} \dot{\varphi} - m\ddot{y} = F \sin \varphi + \lambda_1 \tan \varphi + m\dot{x} \sec^2 \varphi \dot{\varphi} - F \sin \varphi + \lambda_1 = 0$$

$$\text{有: } \lambda_1 (1 + \tan^2 \varphi) = -m\dot{x} \sec^2 \varphi \dot{\varphi} \quad \uparrow \text{注意符号}$$

$$m\ddot{x} \sin \varphi + \frac{m\dot{x}\dot{\varphi}}{\cos \varphi} - m\dot{\varphi} = F \sin \varphi + m\dot{\varphi} \tan \varphi + \dots$$

$$\text{有: } \lambda_1 (1 + \tan^2 \varphi) = -m\dot{x} \sec^2 \varphi \quad \uparrow \text{注意符号}$$

$$\lambda_1 \frac{1}{\sec^2 \varphi} = -m\dot{x} \frac{1}{\cos^2 \varphi} \quad \text{即: } \lambda = m\dot{x} \quad \text{移项!}$$

因此我们求得: $\lambda_1 = m\dot{x}$

回代, 得:

$$\begin{cases} m\ddot{x} = F \cos \varphi - m\dot{x}\dot{\varphi} \tan \varphi \\ m\ddot{\varphi} = F \sin \varphi - m\dot{x}\dot{\varphi} \\ J_c \ddot{\varphi} = M \end{cases}$$

\Rightarrow

此时: 利用 J_c 已知, 首先: $\varphi = \varphi(t)$ 可求出:
即: $\varphi = \int \int \frac{M}{J_c} dt dt$

之后:

由 $m\ddot{x} = F \cos \varphi + m\dot{x}\dot{\varphi} \tan \varphi$, 解微分方程得 x ,

$$\text{即: } \ddot{x}_c = -\dot{x}\dot{\varphi} \tan \varphi + \frac{F}{m} \cos \varphi$$

在如右方程中, 为了解出 $x(t)$, 可令 $\dot{x}_c = p\varphi$

$$\dot{p}\varphi + p\dot{\varphi} + p\dot{\varphi} \tan \varphi = \frac{F}{m} \cos \varphi$$

$$\text{即: } (\dot{p} + p \tan \varphi) \varphi = \frac{F}{m} \cos \varphi - p\dot{\varphi}$$

$$\text{即: } \dot{p} = \cos \varphi, \text{ 则: } \dot{p} = -\dot{\varphi} \sin \varphi \quad \text{此时, } \varphi \text{ 前系数为 } 0$$

$$\text{则: } \frac{F}{m} \cos \varphi - \cos \varphi \cdot \dot{\varphi} = 0 \quad \therefore \dot{\varphi} = \frac{F}{m}, \quad \varphi = \frac{1}{m} \int_0^t F(t) dt + C$$

$$\therefore \text{有: } \dot{x}_c = \left[\frac{1}{m} \int_0^t F(t) dt + C \right] \cos \varphi \quad \text{同一次导数:} \quad \text{假设力 } \psi(t) + C$$

$$= [\psi(t) + C] \cos \varphi \quad \text{求得: } \dot{\varphi}_c = [\psi(t) + C] \sin \varphi$$

$$\text{故 } x_c = \int_0^t [\psi(t) + C] \cos \varphi dt \quad y_c = \int_0^t [\psi(t) + C] \sin \varphi dt$$

$$\varphi_c = \int_0^t \left[\frac{1}{m} \int_0^t F \sin \varphi - m\dot{\varphi} [\psi(t) + C] \cos \varphi \right] dt \quad \text{为运动方程}$$

$$\varphi = \varphi(t)$$