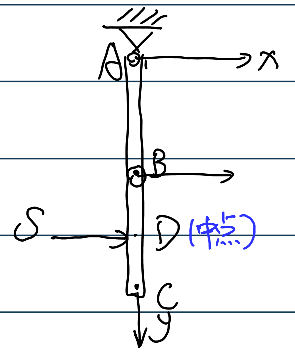
例题1.4

Friday, March 31, 2023 2:41 PM

如图，两均质细杆 AB, BC 长 l，质量均为 m，静止地悬挂在铅垂位置。此时，BC 杆中点 D 处受一碰撞，冲量为 S，求碰撞结束时，AB, BC 角速度：

解：

如图：系统自由度 2



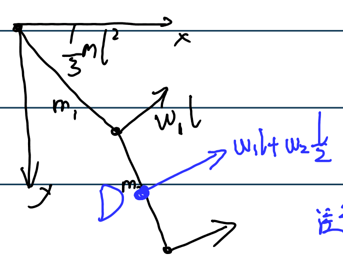
使用碰撞的 Lagrange 方程：

$$\Delta P_j = \hat{Q}_j$$

其中：$\Delta P_j = \left.\dfrac{\partial T}{\partial \dot{q}_{j}}\right|_{t_1}^{t_1+\Delta t}$ $\quad \hat{Q}_j = \sum_{i=1}^{n} S_i \dfrac{\partial r_i}{\partial q_j}$

有：对应于 θ_1 和 θ_2 的广义动量分别为：

$$\begin{cases} P_1 = \dfrac{\partial T}{\partial \dot{q}_1} \\ P_2 = \dfrac{\partial T}{\partial \dot{q}_2} \end{cases}$$



注意：计算公式是 $\frac{1}{2}I\omega^2$，而 $I = \frac{1}{3}ml^2$，$I = \frac{1}{12}ml^2$

注意：带有牵连速度的杆件，只能先计算出质心速度，再使用质心的转动惯量计算

即：$T = \frac{1}{2}\left(\frac{1}{3}m_1 l^2\right)\dot{\theta}_1^2 + \frac{1}{2}m_2\left(\dot{\theta}_1 l + \dot{\theta}_2 \frac{l}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{12}m_2 l^2 \dot{\theta}_2^2\right)$

$= \frac{1}{6}ml^2\dot{\theta}_1^2 + \frac{1}{2}m\left(\dot{\theta}_1^2 l^2 + \dot{\theta}_1\dot{\theta}_2 l^2 + \frac{\dot{\theta}_2^2}{4}l^2\right) + \frac{1}{24}ml^2\dot{\theta}_2^2$

$= \frac{2}{3}ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{6}m\dot{\theta}_2^2 l^2$

则有：$\begin{cases} P_1 = \frac{4}{3}ml^2\dot{\theta}_1 + \frac{1}{2}ml^2\dot{\theta}_2 \\ P_2 = \frac{1}{2}ml^2\dot{\theta}_1 + \frac{1}{3}ml^2\dot{\theta}_2 \end{cases}$ $\quad \begin{cases} P_1 = \frac{4}{3}ml^2\dot{\theta}_1 + \frac{1}{2}ml^2\dot{\theta}_2 \\ P_2 = \frac{1}{2}ml^2\dot{\theta}_1 + \frac{1}{3}ml^2\dot{\theta}_2 \end{cases}$

由于显然 $P_1|_{t=0} = P_2|_{t=0} = 0$，则：

$\Delta P_1 = \frac{4}{3}ml^2\omega_1 + \frac{1}{2}ml^2\omega_2$

$\Delta P_2 = \frac{1}{2}ml^2\omega_1 + \frac{1}{3}ml^2\omega_2$，

由 $\hat{Q}_j = \sum_{i=1}^{n} S_i \dfrac{\partial r_i}{\partial q_j}$

则 $\hat{Q}_j \delta q_j = \sum_{i=1}^{n} S_i \delta r_i = \Delta P \cdot \delta q_j$

有：$S \cdot \delta r_D = \hat{Q}_j \delta q_j = S\left(l\delta\theta_1 + \frac{1}{2}l\delta\theta_2\right)$ ☆ 难点：消 δr_D 并化为 $\delta\theta_1$, $\delta\theta_2$

有：$S \cdot \delta r_D = \hat{Q}_j \delta q_j = S(l\delta\theta_1 + \frac{1}{2}l\delta\theta_2)$ ☆ 难点：将 δr_D 表达为 $\delta\theta_1, \delta\theta_2$

则 $Sl\,\cancel{\delta\theta_1} = (\frac{4}{3}ml^2\omega_1 + \frac{1}{2}ml^2\omega_2)\cancel{\delta\theta_1}$

∴有：① : $\frac{4}{3}ml^2\omega_1 + \frac{1}{2}ml^2\omega_2 = Sl$

又：$S \cdot \frac{1}{2}l\,\cancel{\delta\theta_2} = (\frac{1}{2}ml^2\omega_1 + \frac{1}{3}ml^2\omega_2)\cancel{\delta\theta_2}$

② : $ml^2\omega_1 + \frac{2}{3}ml^2\omega_2 = Sl$

联①②：

有：

$$\frac{8}{3}ml^2\omega_1 - \frac{3}{2}ml^2\omega_1 = \frac{S}{2}l$$

即：$\frac{7}{6}ml^2\omega_1 = \frac{S}{2}l$

有：$\boxed{\omega_1 = \frac{3S}{7ml}}$

而：$\frac{2}{3}ml^2\omega_2 = \frac{4}{7}S$

$\boxed{\omega_2 = \frac{6S}{7ml}}$

故得到：

$$\omega_1 = \frac{3S}{7ml}, \quad \omega_2 = \frac{6S}{7ml}$$