Weak form of unsteady problems

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$$\int_\Omega \left(\frac{\partial u}{\partial t}\delta u - \alpha \frac{\partial^2 u}{\partial x^2}\delta u\right) d\Omega = 0$$

then

$$\int_\Omega \left(\frac{\partial u}{\partial t}\delta u\right) - \int_\Omega \alpha \left(\frac{\partial^2 u}{\partial x^2}\delta u\right) d\Omega = 0$$

we have

$$\underbrace{\int_\Omega \alpha \left(\frac{\partial^2 u}{\partial x^2}\delta u\right) d\Omega = \int_{\Gamma_2} \alpha \left(\delta u \frac{\partial u}{\partial x}\right) d\Gamma - \int_\Omega \alpha \left(\frac{\partial(\delta u)}{\partial x}\frac{\partial u}{\partial x}\right) d\Omega}$$

then

$$\int_\Omega \left(\frac{\partial u}{\partial t}\delta u\right) + \int_\Omega \alpha \left(\frac{\partial(\delta u)}{\partial x}\frac{\partial u}{\partial x}\right) d\Omega = \int_{\Gamma_2} \alpha \left(\delta u \frac{\partial u}{\partial x}\right) d\Gamma$$

note that $u = u(x,t)$

we don't use the form $\underbrace{u^{(e)} = u_i^{(e)} \Phi_i(x,t)}$

we can use the form: $\boxed{u^{(e)} = \underbrace{u_i^{(e)}(t)\, \Phi_i(x)}}$

→ this is named "half-discrete method"

We have $\delta u^{(e)} = \Phi_i^{(e)}(x)\, \delta u_i^{(e)}(t)$, then

$$\int_\Omega \left(\frac{\partial u_i}{\partial t}\Phi_i^{(e)}(x)\, \Phi_j^{(e)}(x)\, \cancel{\delta u_j^{(e)}}\right) d\Omega + \int_\Omega \alpha \left(\frac{\partial u}{\partial x} \cdot \overbrace{\Phi_j\, \cancel{\delta u_j^{(e)}}}^{\delta u = \Phi_j \delta u_j}\right) d\Omega$$

$$= \int_{\Gamma_2} \alpha \left(\frac{\partial u}{\partial x}\, \Phi_j\right) d\Gamma$$

we have:

$$\int_{\Omega^{(e)}} \Phi_i^{(e)} \Phi_j^{(e)} d\Omega\, \dot{u}_i + \int_{\Omega^{(e)}} \alpha \left(\frac{\partial \Phi_i}{\partial x} \cdot \frac{\partial \Phi_j}{\partial x}\right) u_i^{(e)} d\Omega = \int_{\Gamma_2} \alpha \left(\frac{\partial u}{\partial x}\Phi_j\right) d\Gamma$$

$\dot{u}_i = \frac{\partial u_i^{(e)}}{\partial t}$

by letting:

$$A_{ij}^{(e)} = \int_{\Omega^{(e)}} \Phi_i \Phi_j\, d\Omega$$

$$B_{ij}^{(e)} = \int_{\Omega^{(e)}} \alpha \left(\frac{\partial \Phi_i}{\partial x}\frac{\partial \Phi_j}{\partial x}\right) d\Omega$$

$$f_j^{(e)} = \int_{\Gamma_2} \alpha \left(\frac{\partial u}{\partial x}\Phi_j\right) d\Gamma$$

we write the weak form:

$$\boxed{A_{ij}^{(e)} \dot{u}_i^{(e)} + B_{ij}^{(e)} u_i^{(e)} = f_j^{(e)}}$$