多尺度法

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在多尺度法中，我们引入表示不同尺度的时间变量，

$$T_n = \varepsilon^n t \quad (n = 0, 1, 2, L)$$

其中：对于不同的时间尺度，描述变化过程的不同节奏，

有：$$x(t, \varepsilon) = \sum_{n=0}^{m} \varepsilon^n x_n(T_0, T_1, L, T_m)$$

则对时间微分展开

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + L + \varepsilon^m \frac{\partial}{\partial T_m}$$

$$= D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + L + \varepsilon^m D_m$$

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + L + \varepsilon^m \frac{\partial}{\partial T_m}\right)$$

$$= (D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + L + \varepsilon^m D_m)^2$$

$$= D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_0 D_2 + 2 D_1 D_2) + \underbrace{L}_{\text{高次项}}$$

其中：$D_n = \frac{\partial}{\partial T_n} \quad (n = 0, 1, 2 \ldots m)$

将多尺度法展开的微分方程代入非线性振动微分方程

取自由振动的 Duffing 方程为例做如下推导：

$$\ddot{x} + \omega_0^2 (x + \varepsilon x^3) = 0,$$ 并取 $\omega_0 = 1$，

得：$\ddot{x} + x + \varepsilon x^3 = 0$，因此有：

$$\left[D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_0 D_2 + 2 D_1 D_2)\right] x$$

$+ x + \varepsilon x^3 = 0$，分别取零次，一次和二次项，

然后 ⟹ 将 x 展开并进行代入：

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

然后 ⟹ 将x展开成如下形式:

$$X = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \cdots$$

$$\left[D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_0 D_2 + 2 D_1 D_2)\right](X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \cdots)$$

$$+ (X_0 + \varepsilon X_1 + \varepsilon^2 X_2) + \varepsilon (X_0^3 + 3\varepsilon X_0^2 X_1 + 3\varepsilon^2 X_0 X_2 + 3\varepsilon^2 X_0 X_1^2 + \cdots) = 0,$$

取零次、一次和二次项,有:

① $D_0^2 X_0 + X_0 = 0$

②: $2 D_0 D_1 X_0 + D_0^2 X_1 + X_1 + X_0^3 = 0$

$$\Downarrow$$

$$D_0^2 X_1 + X_1 = -2 D_0 D_1 X_0 - X_0^3, \qquad (一次)$$

③: $(D_0 D_2 + 2 D_1 D_2) X_0 + 2 D_0 D_1 X_1 + D_0^2 X_2 + X_2 + 3 X_0^2 X_1 = 0$

$$\downarrow$$

$$D_0^2 X_2 + X_2 = -2 D_0 D_1 X_1 - 3 X_0^2 X_1 - (D_0 D_2 + 2 D_1 D_2) X_0$$

其中:零次方程: $D_0^2 X_0 + X_0 = 0$, 即: $\dfrac{\partial^2 X_0}{\partial T_0^2} + X_0 = 0$ $\quad \lambda_{1,2} = \pm i$

∴ 有: $\boxed{X_0 = A(T_1, T_2) e^{iT_0} + \bar{A}(T_1, T_2) e^{-iT_0}}$ ①

T_1, T_2 在前 $\quad$ T_0 分离出

→ 此时: $D_0 X_0 = \dfrac{\partial}{\partial T_0} X_0 = i A_1 e^{iT_0} - i \bar{A}_1 e^{-iT_0}$

代入②:有:

$$D_0^2 X_1 + X_1 = -2 D_0 D_1 X_0 - X_0^3$$

$$= -2i D_1 A_1 e^{iT_0} + i A_1^3 e^{3iT_0} - 3i A_1^2 \bar{A}_1 e^{iT_0} + cc$$

共轭项.

$$i^3 A_1^3 e^{3iT_0} - 3 i^3 A_1^2 \bar{A}_1 e^{iT_0}$$

$$\parallel$$

$$-i A_1^3 e^{3iT_0} + 3i A_1$$

$$= \boxed{-(2i D_1 A + 3i A^2 \bar{A})} e^{iT_0} + i A^3 e^{3iT_0} + cc$$

$$= \boxed{-(2iD_1A + 3iA\ A)}e + iAe + cc$$

cc为共轭项

为避免久期项，函数A必须满足

$$2iD_1A_1 + 3iA^2\bar{A} = 0 \longrightarrow \text{因此: } D_1A = -\frac{3}{2}A\bar{A}\text{，代入}$$

解x_1即可。