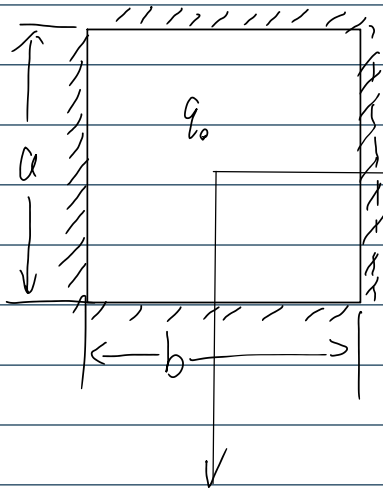


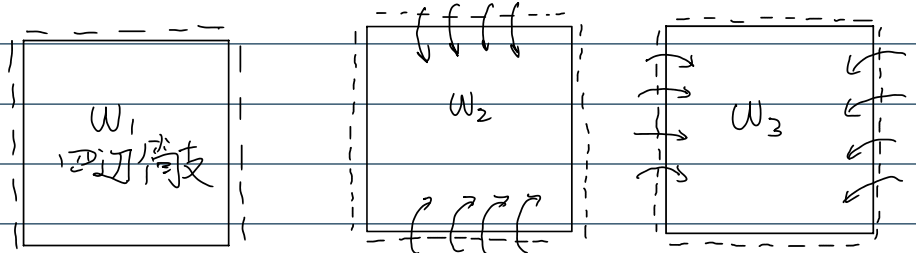
# 矩形薄板的叠加法原理示例

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如求解均布载荷下四边刚性固定的板，我们以说明叠加法的应用。

将此板分解为  $w_1, w_2, w_3$ ，则得四边简支和两个对边弯矩情况。



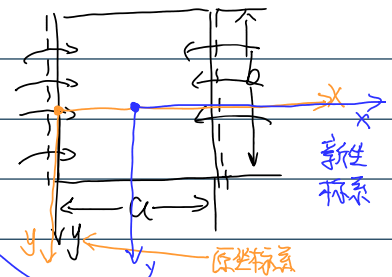
则四边简支显然已经求得，两弯矩对称，而边界条件变为：

$$\begin{cases} y = \pm \frac{b}{2} \text{ 处, } \frac{\partial w_1}{\partial y} + \frac{\partial w_2}{\partial y} + \frac{\partial w_3}{\partial y} = 0, \\ y = \pm \frac{a}{2} \text{ 处, } \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial x} + \frac{\partial w_3}{\partial x} = 0 \end{cases}$$

① 我们已经解出，四边简支板的挠度公式为：

$$w = \frac{4q_0a^4}{D\pi^5} \sum_{m=1,3,\dots}^{\infty} \frac{1}{m^5} \left[ 1 - \left(1 + \frac{\alpha_m}{2} \tanh \alpha_m\right) \frac{\cosh \frac{m\pi y}{a}}{\cosh \alpha_m} + \frac{m\pi y}{a} \frac{\sinh \frac{m\pi y}{a}}{\cosh \alpha_m} \right] \sin \frac{m\pi x}{a}$$

$\Rightarrow$  取  $x = x + \frac{a}{2}$ ，得右方 =  $\sin(\frac{m\pi x}{a} + \frac{1}{2}m\pi)$



由于当  $m=1,3,\dots$  时， $\sin(\frac{m\pi x}{a} + \frac{1}{2}m\pi) = (-1)^{\frac{m-1}{2}} \cos \frac{m\pi x}{a}$ ，代入，有：

$$w_1 = \frac{4q_0a^4}{D\pi^5} \sum_{m=1,3,\dots}^{\infty} \frac{(-1)^{\frac{m-1}{2}}}{m^5} \left[ 1 - \left(1 + \frac{\alpha_m}{2} \tanh \alpha_m\right) \frac{\cosh \frac{m\pi y}{a}}{\cosh \alpha_m} + \frac{m\pi y}{a} \frac{\sinh \frac{m\pi y}{a}}{2 \cosh \alpha_m} \right] \cos \frac{m\pi x}{a}$$

$$\left. \frac{\partial w_1}{\partial y} \right|_{y=\frac{b}{2}} = - \left. \frac{\partial w_1}{\partial y} \right|_{y=-\frac{b}{2}} = \frac{2q_0a^3}{D\pi^4} \sum_{m=1,3,\dots}^{\infty} \frac{(-1)^{\frac{m-1}{2}}}{m^4} \left( \frac{\alpha_m}{\cosh^2 \alpha_m} - \tanh \alpha_m \right) \cos \frac{m\pi x}{a}, \quad \alpha_m = \frac{m\pi b}{2a}$$

显然  $x = \frac{a}{2}$  处的转角只需将  $a, b$  互换， $\beta_m = \frac{n\pi a}{2b}$  代替上式即可。

即： $\left. \frac{\partial w_1}{\partial x} \right|_{x=\frac{a}{2}} = - \left. \frac{\partial w_1}{\partial x} \right|_{x=-\frac{a}{2}} = \frac{2q_0b^3}{D\pi^4} \sum_{n=1,3,\dots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^4} \left( \frac{\beta_n}{\cosh^2 \beta_n} - \tanh \beta_n \right) \cos \frac{n\pi y}{b}$  (先求解出均布载荷下的转角)

②：边缘弯矩下简支板已经在单三角函数解法中给出，其解为：

$$w = \frac{a^2}{2\pi^2 D} \sum_{m=1}^{\infty} \frac{E_m}{m^2 \cosh \alpha_m} \left( \alpha_m \tanh \alpha_m \cosh \frac{m\pi y}{a} - \frac{m\pi y}{a} \sinh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a}$$

其中： $y = \pm \frac{b}{2}$  处  $M = \sum_{m=1,3,\dots}^{\infty} \frac{(-1)^{\frac{m-1}{2}}}{m^2} E_m \cos \frac{m\pi y}{a}$

四边简支边缘弯矩，则：

$$w = \frac{a^2}{2\pi^2 D} \sum_{m=1}^{\infty} \frac{E_m}{m^2 \cosh \alpha_m} \left( \alpha_m \tanh \alpha_m \cosh \frac{m\pi y}{a} - \frac{m\pi y}{a} \sinh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a}$$

则代入坐标， $x = x + \frac{a}{2}$ ，则同上理有：

$$w = \frac{a^2}{2\pi^2 D} \sum_{m=1}^{\infty} \frac{E_m}{m^2 \cosh \alpha_m} \left( \alpha_m \tanh \alpha_m \cosh \frac{m\pi y}{a} - \frac{m\pi y}{a} \sinh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a}$$

$$W = \frac{1}{2\pi^2 D} \sum_{m=1}^{\infty} \frac{m^2 \cosh \alpha_m}{\cosh^2 \alpha_m} \dots$$

则代入坐标,  $x = x + \frac{a}{2}$ , 则同上理有:

$$W_2 = \frac{a^2}{2\pi^2 D} \sum_{m=1,3,\dots} \frac{E_m (-1)^{\frac{m-1}{2}}}{m^2 \cosh \alpha_m} \left\{ \alpha_m \tanh \alpha_m \cosh \frac{m\pi y}{a} - \frac{m\pi y}{a} \sinh \frac{m\pi y}{a} \right\} \cos \frac{m\pi x}{a}$$

(求导烦, 直接给结果)

$$\theta \Big|_{x=\frac{b}{2}} = \frac{\partial W_2}{\partial y} \Big|_{y=\frac{b}{2}} = \frac{a}{2\pi^2 D} \sum_{m=1,3,\dots} \frac{E_m (-1)^{\frac{m-1}{2}}}{m} \cdot \left( \tanh \alpha_m + \frac{\alpha_m}{\cosh^2 \alpha_m} \right) \cos \frac{m\pi x}{a}$$

$$\theta \Big|_{x=\frac{a}{2}} = \frac{a}{2\pi^2 D} \sum_{m=1,3,\dots} \frac{E_m}{\cosh^2 \alpha_m} \left( b \sinh \alpha_m \cosh \frac{m\pi y}{a} - 2y \cosh \alpha_m \sinh \frac{m\pi y}{a} \right)$$

将  $\frac{\partial W_2}{\partial x}$  沿  $y$  方向展开成三角级数, 取括号内部分  $\downarrow$  由于是  $y$  偶函数, 可展开

$$b \sinh \alpha_m \cosh \frac{m\pi y}{a} - 2y \cosh \alpha_m \sinh \frac{m\pi y}{a} = \sum_{n=1,3,\dots} A_n \cos \frac{n\pi y}{b}$$

其中  $A_n$  表达式由 Fourier 展开获得 (偶展开)

$$A_n = \frac{2}{b} \int_{-\frac{b}{2}}^{\frac{b}{2}} \left( b \sinh \alpha_m \cosh \frac{m\pi y}{a} - 2y \cosh \alpha_m \sinh \frac{m\pi y}{a} \right) \cos \frac{n\pi y}{b} dy$$

$$= 16 \frac{n(-1)^{\frac{n-1}{2}}}{m^2 \pi^2} \cdot \frac{b^2}{a^2} \cdot \frac{1}{[(b/a)^2 + (n/m)^2]} \cosh^2 \alpha_m$$

我们将其代入上式, 则原式改写为:

$$\frac{\partial W_2}{\partial x} = -\frac{1}{4b} \sum_{m=1,3,\dots} \frac{E_m}{\cosh^2 \alpha_m} A_n \cos \frac{n\pi y}{b} = -\frac{4b^2}{\pi a^2 D} \sum_{m=1,3,\dots} \frac{E_m}{m^3} \cdot \frac{n(-1)^{\frac{n-1}{2}}}{[(b/a)^2 + (n/m)^2]} \cos \frac{n\pi y}{b}$$

$$\text{而: } M_x \Big|_{x=\pm \frac{a}{2}} = \sum_{n=1,3,5,\dots} (-1)^{\frac{n-1}{2}} F_n \cos \frac{n\pi y}{b}, \text{ 同理得 } W_2, \frac{\partial W}{\partial x_2} \text{ 和 } \frac{\partial W}{\partial x_3} \text{ (略去)}$$

最终代入边界条件即可。