Green 第二公式推导和基本解证明

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Green 第二公式:

$$\iint_{\Omega}(u\Delta v - v\Delta u)\,dx\,dy = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dl$$

证:

$$\begin{aligned} u\Delta v - v\Delta u &= u\nabla\cdot\nabla v - v\nabla\cdot\nabla u \\ &= \nabla\cdot(u\nabla v) - \cancel{\nabla u\nabla v} \\ &\quad - (\nabla\cdot(v\nabla u)) + \cancel{\nabla v\nabla u} \end{aligned}$$

$$= \nabla(u\nabla v - v\nabla u)$$

故有 $\iint_{\Omega}(u\Delta v - v\Delta u)\,dx\,dy = \iint_{\Omega}\nabla(u\nabla v - v\nabla u)\,dx\,dy$

$= \int_{\partial\Omega}(u\nabla v - v\nabla u)\,dl$ 其中 $\nabla v = \frac{\partial v}{\partial n}$

$$= \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dl$$

定理1.2 证: 设Γ是满足对应条件的u的解:有

由 $\iint_{R^2}\Gamma(x,y,\xi,\eta)(-\Delta\varphi(x,y))\,dx\,dy$, $\Delta\Gamma = 0$

$= \iint_{R^2}(\Delta\Gamma\,\varphi - \Gamma\Delta\varphi)\,dx\,dy$ 取2个球面 $\lim_{\varepsilon\to 0}\varepsilon$, $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}$

$= \int_{\partial R}\left(\Gamma\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial\Gamma}{\partial n}\right)dx\,dy$

$\longrightarrow \oint\left(\varphi\frac{\partial\Gamma}{\partial n} - \Gamma\frac{\partial\varphi}{\partial n}\right)dx\,dy$, 由 $\Gamma(\rho) = \frac{1}{2\pi}\ln\frac{1}{\rho}$

$\Rightarrow \oint_{\rho=\varepsilon}\left(\varphi\frac{\partial \Gamma}{\partial n}-\Gamma\frac{\partial \varphi}{\partial n}\right)dxdy$，由 $\Gamma(\rho)=\frac{1}{2\pi}\ln\frac{1}{\rho}$

$\therefore=\oint_{\rho=\varepsilon}\left(\varphi\cdot\frac{1}{2\pi}\rho\cdot-\frac{1}{\rho^2}-\frac{1}{2\pi}\ln\frac{1}{\rho}\frac{\partial\varphi}{\partial n}\right)dl$ 在 $\rho=\varepsilon\to 0$ 时

其中：后一项 又：$\frac{1}{\frac{1}{\rho}}\cdot-\frac{1}{\rho^2}=-\frac{1}{\rho}$

$\leq \cancel{2\pi}\varepsilon\cdot-\frac{1}{\cancel{2\pi}}\ln\frac{1}{\varepsilon}\frac{\partial\varphi}{\partial n}$

$-\varepsilon\ln\frac{1}{\varepsilon}\frac{\partial\varphi}{\partial n}$。增长缓慢，故极限为 0。

故结去：

在 $\xi\to 0$, $\varphi\to\varphi(\xi,\eta)$

$=\oint_{\rho=\varepsilon}\left(\varphi\frac{\partial\Gamma}{\partial n}\right)dl=\cancel{2\pi\varepsilon}\cdot\varphi(\xi,\eta)\cdot\frac{1}{\cancel{2\pi\varepsilon}}\to\varphi(\xi,\eta)$

得证；