张量的梯度与积分推导

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四. 张量的偏积分

(一)、梯度的计算

(1). 标量场.

$\varphi(x_i)$ $\boxed{\vec{\nabla} = \frac{\partial}{\partial x_i}\vec{e_i}}$

$$\begin{cases} \varphi\vec{\nabla} = \varphi\frac{\partial}{\partial x_i}\vec{e_i} = \frac{\partial\varphi}{\partial x_i}\vec{e_i} \\ \vec{\nabla}\varphi = \frac{\partial}{\partial x_i}\vec{e_i}\varphi = \frac{\partial\varphi}{\partial x_i}\vec{e_i} \end{cases}$$

右梯度:

$$\varphi\vec{\nabla} = \varphi(x_i)\frac{\partial}{\partial x_i}\vec{e_i} = \frac{\partial\varphi}{\partial x_i}\vec{e_i}$$

对标量场:

左梯度:

$$\vec{\nabla}\varphi = \left(\frac{\partial}{\partial x_i}\vec{e_i}\right)\varphi(x_i) = \vec{e_i}\frac{\partial\varphi}{\partial x_i}$$

②. 对矢量场. $V = V_i\vec{e_i} \Rightarrow$

右梯度:

$$\vec{V}\otimes\vec{\nabla} = V_i\vec{e_i}\otimes\frac{\partial}{\partial x_j}\vec{e_j} = \frac{\partial}{\partial x_j}(V_i\vec{e_i})\otimes\vec{e_j}$$

$V_i\vec{e_i}\otimes\frac{\partial}{\partial x_j}\vec{e_j}$

$$= \frac{\partial V_i}{\partial x_j}\vec{e_i}\otimes\vec{e_j} + V_i\underbrace{\frac{\partial\vec{e_i}}{\partial x_j}}_{=0}\vec{e_i}\otimes\vec{e_j}$$

$$= \frac{\partial V_i}{\partial x_j}\vec{e_i}\otimes\vec{e_j} = V_{ij}\,\vec{e_i}\otimes\vec{e_j}$$

左梯度:

$$\vec{\nabla}\otimes\vec{V} = \frac{\partial}{\partial x_i}\vec{e_i}\otimes V_j\vec{e_j} = \frac{\partial V_j}{\partial x_i}\vec{e_i}\otimes\vec{e_j}$$

$$= \frac{\partial V_i}{\partial x_j}\vec{e_j}\otimes\vec{e_i} = V_{i,j}\,\vec{e_j}\otimes\vec{e_i}$$

$$\vec{\nabla}\otimes\vec{V} = \frac{\partial}{\partial x_i}\vec{e_i}\otimes V_j\vec{e_j} = \frac{\partial V_j}{\partial x_i}\vec{e_i}\otimes\vec{e_j}$$

因而有:

$$\vec{V}\otimes\vec{\nabla} = (\nabla\otimes V)^T$$

(3)、二阶张量 $\overline{\overline{T}} = T_{ij}\vec{e_i}\otimes\vec{e_j}$

①. 对右梯度:

右梯度

$$\overline{\overline{T}}\otimes\vec{\nabla} = T_{ij}\,\vec{e_i}\otimes\vec{e_j}\otimes\frac{\partial}{\partial x_k}\vec{e_k}$$

$$= \frac{\partial T_{ij}}{\partial x_k}\vec{e_i}\otimes\vec{e_j}\otimes\vec{e_k}$$

左梯度:

$$\vec{\nabla}\otimes\vec{T} = \frac{\partial}{\partial x_i}\vec{e_i}$$

梯度 $\nabla\varphi = \mathrm{grad}\,\varphi$ 散度: div (divergent)

并矢 $\nabla\otimes\vec{V}$ 点矢 $\vec{V}\cdot\vec{\nabla}$

(二)、散度的计算

(1) 对矢量场 $V = \vec{V_i}\vec{e_i}$

右散度: $\vec{V}\cdot\vec{\nabla} = \vec{V_i}\vec{e_i}\cdot\frac{\partial}{\partial x_j}\vec{e_j}$

$$= \frac{\partial V_i}{\partial x_j}\vec{e_i}\cdot\vec{e_j} = \frac{\partial V_i}{\partial x_j}\delta_{ij} = \frac{\partial V_i}{\partial x_i} = V_{i,i}$$

左散度: $\vec{\nabla}\cdot\vec{V} = \left(\frac{\partial}{\partial x_i}\vec{e_i}\right)\cdot\left(V_j\vec{e_j}\right)$

$$= \frac{\partial V_j}{\partial x_i}\delta_{ij} = \frac{\partial V_i}{\partial x_i} = V_{i,i}$$

(2) 二阶张量 T

①: 右散度问题

$$\overline{\overline{T}}\cdot\nabla = T_{ij}\vec{e_i}\otimes\vec{e_j}\cdot\frac{\partial}{\partial x_k}\vec{e_k} = \frac{\partial T_{ij}}{\partial x_k}\delta_{jk}\vec{e_i} = \frac{\partial T_{ij}}{\partial x_j}\vec{e_i}$$

$$= T_{ij,j}\vec{e_i}$$

②: 左散度

$$\vec{\nabla}\cdot\vec{T} = \frac{\partial}{\partial x_i}\vec{e_i}\cdot T_{jk}\vec{e_j}\otimes\vec{e_k}$$

$$= \frac{\partial}{\partial x_i}\delta_{ij}T_{jk}\vec{e_k}$$

$$= \frac{\partial}{\partial x_j}T_{jk}\vec{e_k} = T_{jk,j}\vec{e_k} = \frac{\partial T_{ji}}{\partial x_j}\vec{e_i}$$

$$= T_{ji,j}\vec{e_i}$$ 若 $T_{ij} = T_{ji}$, 则相同,

③ 旋度: (Curl)

①: 对于矢量场 $\vec{V} = V_i\vec{e_i}$

$$\vec{V}\times\vec{\nabla} = V_i\vec{e_i}\times\frac{\partial}{\partial x_j}\vec{e_j} = V_i\frac{\partial}{\partial x_j}\varepsilon_{ijk}\vec{e_k}$$

$$= \frac{\partial V_i}{\partial x_j}\varepsilon_{ijk}\vec{e_k} = V_{i,j}\varepsilon_{ijk}\vec{e_k}$$

$$= \frac{\partial V_i}{\partial x_j} \varepsilon_{ijk} \vec{e_k} = V_{i,j} \varepsilon_{ijk} \vec{e_k}$$

②：左旋度：

$$\vec{\nabla} \times \vec{V} = \left(\frac{\partial}{\partial x_i} \vec{e_i}\right) \times V_j \vec{e_j} = \frac{\partial V_j}{\partial x_i} \varepsilon_{ijk} \vec{e_k}$$

$$= \frac{\partial V_i}{\partial x_j} \varepsilon_{jik} \vec{e_k} = -V_{i,j} \varepsilon_{ijk} \vec{e_k}.$$

有 $\vec{V} \times \vec{\nabla} = -\vec{\nabla} \times \vec{V}$

(2)、二阶张量：$T = T_{ij} \vec{e_j} \otimes \vec{e_k}$

右旋度

$$\vec{T} \times \nabla = T_{ij} \vec{e_i} \otimes \vec{e_j} \times \frac{\partial}{\partial x_k} \vec{e_k}$$

$$= \frac{\partial T_{ij}}{\partial x_k} \vec{e_i} \otimes \varepsilon_{jkm} \vec{e_m}$$

$$= \frac{\partial T_{ij}}{\partial x_k} \varepsilon_{jkm} \vec{e_i} \otimes \vec{e_m}$$

$$\vec{\nabla} \times \vec{T} = \frac{\partial}{\partial x_i} \vec{e_i} \times T_{jk} \vec{e_j} \otimes \vec{e_k}$$

$$= \frac{\partial}{\partial x_i} T_{jk} \varepsilon_{ijm} \vec{e_m} \otimes \vec{e_k}$$

令 $i=k, j=i, k=j$，则：

$$= \frac{\partial}{\partial x_k} T_{ij} \varepsilon_{kjm} \vec{e_m} \otimes \vec{e_j}$$

$$= \frac{\partial T_{ij}}{\partial x_k} \varepsilon_{kjm} \vec{e_m} \otimes \vec{e_j}$$

补充练习：

$$\nabla^2 = \triangle = \vec{\nabla} \cdot \vec{\nabla} = \left(\frac{\partial}{\partial x_i} \vec{e_i}\right) \cdot \left(\frac{\partial}{\partial x_j} \vec{e_j}\right)$$

$$= \frac{\partial^2}{\partial x_i \partial x_j} \delta_{ij}$$

$$= \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

矢量.

$$\underbrace{\vec{\nabla}(\vec{\nabla} \cdot \vec{u})} = (\vec{\nabla} \otimes \vec{\nabla}) \cdot \vec{u}$$

$$\vec{\nabla} \otimes \vec{\nabla} = \frac{\partial}{\partial x_i} \vec{e_i} \otimes \frac{\partial}{\partial x_j} \vec{e_j} = \frac{\partial^2}{\partial x_i \partial x_j} \vec{e_i} \otimes \vec{e_j}$$

$$\vec{\nabla} \otimes \vec{\nabla} = \frac{\partial}{\partial x_i} \vec{e}_i \otimes \frac{\partial}{\partial x_j} \vec{e}_j = \frac{\partial^2}{\partial x_i \partial x_j} \vec{e}_i \otimes \vec{e}_j$$