Derivation of the t-Distribution

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Student's t-distribution was introduced in 1908 by William Sealy Goset. The statistc variable t is defined by

$$t = \frac{u}{\sqrt{v/n}},$$

where u is a variable of the standard normal distribution $g(u)$, and v be a variable of the χ^2 distribution $T_n(v)$ of of the n degrees of freedom. Thus, we can express the distribution function of t in terms of $g(u)$ and $T_n(v)$.

The distribution function $g(u)$ and $T_n(v)$ are represented by

联合密度 $f(u,v) = g(u)T_n(v)$

$$g(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \qquad (1)$$

$u \sim N(0,1)$

and

$$T_n(v) = \frac{1}{2^{n/2}\Gamma(n/2)} v^{(n-2)/2} e^{-v/2} \qquad (2)$$

$v \sim \chi^2(n)$

由δ函数性质: $\int_{-\infty}^{+\infty} \delta(t-t_0) f(t) = f(t_0)$

$t = \frac{u}{\sqrt{v/n}}$

respectivley.

设: $P\left(t \le \frac{u}{\sqrt{v/n}}\right)$

$F(t) = \int_{-\infty}^{+\infty} \int_{\infty}^{n(\frac{u}{t})^2} (u,v)\, dv$

The probabilistic function $f_n(t)$ of $t = u/\sqrt{v/n}$ can then be written as

$$f_n(t) = \int \delta\left(t - \frac{u}{\sqrt{v/n}}\right) g(u)T_n(v)\, dudv,$$

$t = \frac{u}{\sqrt{v/n}}$

联合密度

where $\delta\left(t - u/\sqrt{v/n}\right)$ is a Dirac's delta function.

Before integrate over u, we introduce a variable $y = u/\sqrt{v/n}$, and integrate over y instead of u.

$$\begin{aligned} f_n(t) &= \int \sqrt{\frac{v}{n}} \delta\,(t - y)\, g(\sqrt{v/n}\, y) T_n(v)\, dydv \\ &= \int \sqrt{\frac{v}{n}} g(\sqrt{v/n}\, t) T_n(v)\, dv \end{aligned}$$

Now, substituting eq.(1) and eq.(2) into this equation, we get

$$f_n(t) = \frac{1}{\sqrt{2\pi n}\, 2^{n/2}\Gamma(n/2)} \int_0^\infty v^{(n-1)/2} e^{-(1+t^2/n)v/2}\, dv$$

Furthermore, we rewrite the equation using a variable $x = \left(1 + \frac{t^2}{n}\right)\frac{v}{2}$ instead of v, to find that

$$f_n(t) = \frac{(1+t^2/n)^{-(n+1)/2}}{\sqrt{\pi n}\,\Gamma(n/2)} \int_0^\infty x^{(n+1)/2-1} e^{-x}\, dx.$$

The integrand of the right hand side can be represented by using the gamma function,

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx.$$

Thus we obtain

$$f_n(t) = \frac{\Gamma(n+1)/2)}{\sqrt{\pi n}\,\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$$

Furthermore, gamma functions are repsented by beta funtion as

$$B(1/2,\, n/2) = \frac{\Gamma(1/2)\,\Gamma(n/2)}{\Gamma((n+1)/2)} = \frac{\sqrt{\pi}\,\Gamma(n/2)}{\Gamma((n+1)/2)}.$$

So we finaly otain the studen's t-distribution function as follows,

$$f_n(t) = \frac{1}{\sqrt{n}\, B(1/2,\, n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$$

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We show the t distribution function as Figure 1. For comparison, we also show the standard normal distribution function ($N(0,1)$ - solid line).

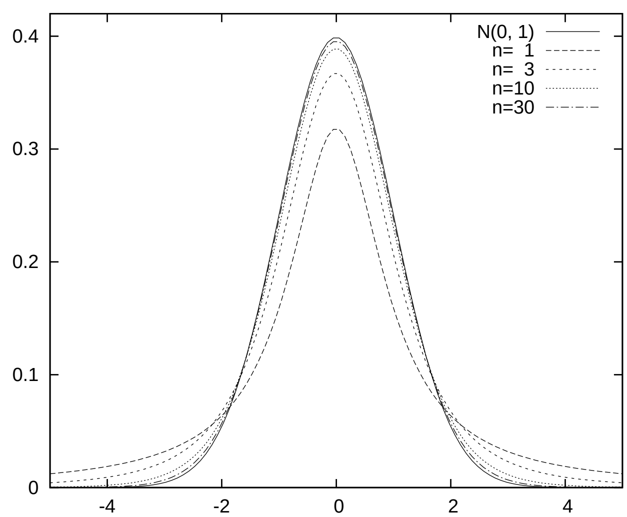

Figure 1: $t$distribution function $y = f_n(t)$ of $n = 1,\ 3,\ 10,\ 30$ degrees of freedom, and the standard normal distribution function$N(0,\ 1)$as solid line.