Derivation for transformation of 2-D computational plane

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since we have

$$\begin{cases} x = x(\xi, \eta, \tau) \\ y = y(\xi, \eta, \tau) \\ t = t(\tau) \end{cases} \quad \text{and} \quad \begin{cases} \xi = \xi(x, y, t) \\ \eta = \eta(x, y, t) \\ \tau = \tau(t) \end{cases}$$

then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} + \underbrace{\frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial x}}_{\text{this term is } 0}$$

hence

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x}, \quad \frac{\partial}{\partial y} = \cdots$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t},$$

we have:

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x}\right]$$

attention that it's a function of ξ and η, so we can derive the derivation of ξ and η

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \xi}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \xi}\frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \eta}\right)\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial x}\right)$$

$$= \frac{\partial \xi}{\partial x}\frac{\partial^2}{\partial x \partial \xi} + \frac{\partial^2 \xi}{\partial x^2}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta \partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 \eta}{\partial x^2}\cdot\frac{\partial}{\partial \eta}$$

we express: $B = \frac{\partial^2}{\partial x \partial \xi} = \frac{\partial^2}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2}{\partial \xi \partial \eta}\cdot\frac{\partial \eta}{\partial x}$ (5.7)

if we want to derive for x, not only ξ but also η should be taken into consideration

and

$$C = \frac{\partial^2}{\partial x \partial \eta} = \frac{\partial^2}{\partial \eta^2}\frac{\partial \eta}{\partial x} + \frac{\partial^2}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x} \quad (5.8)$$

we can write the result as:

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\partial^2}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \xi}\left(\frac{\partial^2 \xi}{\partial x^2}\right) + \frac{\partial^2}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial^2}{\partial \xi \partial \eta}\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial x}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial^2 \eta}{\partial x^2}\right)$$

$$\boxed{\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x^2}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)^2 + 2\left(\frac{\partial^2}{\partial \eta \partial \xi}\right)\left(\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x}\right)}$$

similarly:

$$\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + \frac{\partial^2}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial^2}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + \frac{\partial^2}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + 2\left(\frac{\partial^2}{\partial \eta \partial \xi}\right)\left(\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial y}\right)$$

similarly:

$$\boxed{\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + \frac{\partial^2}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial^2}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + \frac{\partial^2}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + 2\left(\frac{\partial^2}{\partial \eta \partial \xi}\right)\left(\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial y}\right)}$$

also we give

$$\boxed{\begin{aligned}\frac{\partial^2}{\partial x \partial y} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x \partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x \partial y}\right) + \frac{\partial^2}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right) + \frac{\partial^2}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right) \\ + \frac{\partial^2}{\partial \xi \partial \eta}\left[\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right]\end{aligned}}$$

These are the transformation of the derivate from x, y to ξ, η.