蒙特卡洛法的样本方差表示推导

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考虑： $\hat{P}_f = \frac{1}{N}\sum_{i=1}^{n} I_F(X_i)$

$$\Rightarrow \quad V(\hat{P}_f) = \frac{1}{N^2} V\left(\sum_{i=1}^{n} I_F(X_i)\right)$$

显然有： $V\left(\sum_{i=1}^{n} I_F(X_i)\right) = \sum_{i=1}^{n} V(I_F(X_i))$

又：样本 X_i 与总体 X 独立同分布，即： $\sum_{i=1}^{n} V(I_F(X_i)) = nV(I_F(X_i))$

则有：

$$V(\hat{P}_f) = \frac{1}{N} V(I_F(X))$$

而：样本方差为：

$$S = \frac{1}{N-1}\sum_{i=1}^{n}(X_i - \bar{X})^2，\text{其中 } X_i = I_F(X_i)；\quad \bar{X} = \frac{1}{N}\sum_{i=1}^{n}(I_F(X_i))$$

∴有： $S = \frac{1}{N-1}\left[\sum_{i=1}^{n}\left(I_F(X_i) - \frac{1}{N}\sum_{i=1}^{n} I_F(X_i)\right)^2\right]$

显然：右 $= \sum_{i=1}^{n} I_F^2(X_i) - \sum_{i=1}^{n} 2I_F(X_i)\cdot\frac{1}{N}\sum_{i=1}^{n} I_F(X_i) + \sum_{i=1}^{n}\left(\frac{1}{N}\sum_{i=1}^{n} I_F(X_i)\right)^2$

$$= \sum_{i=1}^{n} I_F^2(X_i) - 2N\hat{P}_f^2 + N\hat{P}_f^2 = \underline{\sum_{i=1}^{n} I_F^2(X_i) - N\hat{P}_f^2}$$

故：

$$V(\hat{P}_f) = \frac{1}{N}\cdot\frac{1}{N-1}\left[\sum_{i=1}^{n} I_F^2(X_i) - N\hat{P}_f^2\right] = \left[\frac{1}{N(N-1)}\sum_{i=1}^{n} I_F^2(X_i) - \frac{1}{N-1}\hat{P}_f^2\right]$$

显：$I_F^2(X_i) = I_F(X_i)$

第一项变为 $\hat{P}_f\cdot\frac{1}{N-1}$

$$= \frac{1}{N-1}\left[\hat{P}_f - \hat{P}_f^2\right]$$

$$= \frac{\hat{P}_f - \hat{P}_f^2}{N-1} \Rightarrow \qquad V(\hat{P}_f) = \frac{\hat{P}_f - \hat{P}_f^2}{N-1}$$

对应变异系数 $Cov[\hat{P}_f]$ 与其估计值：

$$Cov[\hat{P}_f] = \frac{\sqrt{V(\hat{P}_f)}}{E(\hat{P}_f)} = \sqrt{\frac{1-\hat{P}_f}{\hat{P}_f(N-1)}}$$