Derivation of 2D characteristic functions

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for 2D characteristic functions, we firstly use the weak form:

$$\iint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial(\delta u)}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial(\delta u)}{\partial y}\right) d\Omega + \iint_\Omega p\,\delta u\, d\Omega = \int_{\Gamma_2} g\,\delta u\, d\Gamma$$

we substitute $u = \Phi_i^{(e)} u_i^{(e)}$ (displacement function) into it

then

$$\iint_\Omega \left(\frac{\partial \Phi_i}{\partial x}\frac{\partial \Phi_j}{\partial x} u_i^{(e)} \cancel{\delta u_j^{(e)}} + \frac{\partial \Phi_i}{\partial y} u_i^{(e)} \frac{\partial \Phi_j}{\partial y} \cancel{\delta u_j^{(e)}}\right) d\Omega + \iint_\Omega p\,\Phi_j \cancel{\delta u_j}\, d\Omega = \int_{\Gamma_2} g\,\Phi_j \cancel{\delta u_j}\, d\Gamma$$

then replace i by j and j by i, we have:

$$\iint_\Omega \left(\frac{\partial \Phi_i^{(e)}}{\partial x}\frac{\partial \Phi_j^{(e)}}{\partial x} + \frac{\partial \Phi_i^{(e)}}{\partial y}\frac{\partial \Phi_j^{(e)}}{\partial y}\right) u_j^{(e)}\, d\Omega + \iint_\Omega p\,\Phi_i^{(e)} d\Omega = \int_{\Gamma_2} g\,\Phi_i\, d\Gamma$$

then we set $A_{ij}^{(e)} = \iint_{\Omega^{(e)}} \left(\frac{\partial \Phi_i^{(e)}}{\partial x}\frac{\partial \Phi_j^{(e)}}{\partial x} + \frac{\partial \Phi_i^{(e)}}{\partial y}\frac{\partial \Phi_j^{(e)}}{\partial y}\right) d\Omega$

$$f_i^{(e)} = \underbrace{\int_{\Gamma_2^{(e)}} g\,\Phi_i^{(e)} d\Gamma}_{\text{the term of natural boundary condition.}} - \iint_{\Omega^{(e)}} p\,\Phi_i^{(e)} d\Omega$$

Note if Γ_2 not exist then this term would be zero.

We get

$$A_{ij}^{(e)} u_j^{(e)} = f_i^{(e)}$$

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then we can substitute the displacement function, which is:

$$\underbrace{u^{(e)} = \Phi_i^{(e)} u_i^{(e)}}_{} \Rightarrow \Phi_i^{(e)} = a_i + b_i x + c_i y.$$

then we have

$$\frac{\partial \Phi_i^{(e)}}{\partial x} = b_i,\quad \frac{\partial \Phi_j^{(e)}}{\partial x} = b_j \quad \frac{\partial \Phi_i^{(e)}}{\partial y} = c_i \quad \frac{\partial \Phi_i^{(e)}}{\partial y} = c_j$$

$$A_{ij}^{(e)} = \iint_\Omega \underbrace{b_i b_j + c_i c_j}_{\text{constant}}\, d\Omega$$

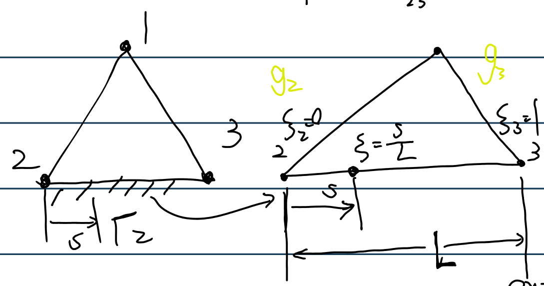
$$= (b_i b_j + c_i c_j)\, A^{(e)}$$

$$f_i^{(e)} = -\iint_{\Omega^{(e)}} p(a_i + b_i x + c_i y) + \underbrace{\int_{\Gamma_2^{(e)}} g(a_i + b_i x + c_i y)\, d\Gamma}_{\text{we can set this term as } I_i}$$

also if there's no Γ_2, $I_i = 0$. $I_i = \int_{\Gamma_2} g^{(e)} \Phi_i \, d\Gamma$

also if there exists the Γ_2 boundary, (note that in this condition, at least 2 nodes are on the boundary)

then $I_i = \int_{\overline{23}} g \Phi_i \, d\Gamma$



we can use the natural coordinate ξ which has relationship $\boxed{\xi = \frac{s}{L}}$

then Φ_i in the boundary $\overline{23}$ can be expressed linearly as:

$$\begin{cases} \Phi_1 = 0 \\ \Phi_2 = 1 - \xi \\ \Phi_3 = \xi \end{cases}$$

$\longrightarrow$ then we can use the boundary condition $g_2^{(e)}$ and $g_3^{(e)}$, the $g^{(e)}$ in boundary would be calculated:

$$g^{(e)} = g_2^{(e)} + (g_3^{(e)} - g_2^{(e)})\xi$$

so we have $\rightarrow$ using $I_i = \int_\Gamma g^{(e)} \Phi_i \, d\Gamma$

$$I_1 = 0 \quad , I_2 = L \int_0^1 [g_2^{(e)} + (g_3^{(e)} - g_2^{(e)})\xi](1-\xi) d\xi$$

then

$$I_3 = L \int_0^1 [g_2^{(e)} + (g_3^{(e)} - g_2^{(e)})\xi] \xi \, d\xi$$

$\xi - \xi^2$

$\frac{1}{2} - \frac{1}{3}$

then

$$I_2 = L \cdot \left[g_2^{(e)} \cdot \frac{1}{2} + (g_3^{(e)} - g_2^{(e)}) \left(\frac{\xi^2}{2} - \frac{\xi^3}{3} \right) \Big|_0^1 \right]$$

$$= L \cdot \left[\frac{g_2^{(e)}}{2} + \frac{1}{6}(g_3^{(e)} - g_2^{(e)}) \right] = L \cdot \left(\frac{1}{3} g_2^{(e)} + \frac{1}{6} g_3^{(e)} \right)$$

$$I_3 = L \cdot \left[\frac{1}{2} g_2^{(e)} + \frac{1}{3}(g_3^{(e)} - g_2^{(e)}) \right] = L \left(\frac{1}{6} g_2^{(e)} + \frac{1}{3} g_3^{(e)} \right)$$

we arrange the result above and get

$$\begin{cases} I_2 = \frac{L}{6}(2g_2^{(e)} + g_3^{(e)}) \\ I_3 = \frac{L}{6}(g_2^{(e)} + 2g_3^{(e)}) \end{cases}$$