均值与方差局部灵敏度估计

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$$\frac{\partial P_f}{\partial \theta_{X_i}^{(k)}} = \int_F I_F \frac{\partial f_X(x)}{\partial \theta_{X_i}^{(k)}} \cdot dx = E\left[\frac{I_F}{f_X(x)} \frac{\partial f_X(x)}{\partial \theta_{X_i}^{(k)}}\right]$$

总体期望 (母体)

①：失效概率 $P_f = \int_F f_X(x)\,dx$

样本均值代替总体均值：

显然：$E\left[\frac{\partial P_f}{\partial \theta_{X_i}^{(k)}}\right] = E\left[\int_F \frac{\partial f_X(x)}{\partial \theta_{X_i}^{(k)}} dX\right] \approx E\left[\frac{1}{N}\sum_{j=1}^{N} \frac{I_F(x_j)}{f_X(x_j)} \cdot \frac{\partial f_X(x_j)}{\partial \theta_{X_i}^{(k)}}\right]$ 转换为样本均值

N 为样本点个数

我们考虑到样本 x_j 均与母体 X 独立同分布，则有

$$E\left[\frac{\partial P_f}{\partial \theta_{X_i}^{(k)}}\right] = \underbrace{\frac{1}{n}\sum_{i=1}^{n}}_{\text{均相同，}\times n,\ \div n} \left[\frac{1}{N}\cdots\right] = \frac{1}{N}\cdots = \frac{\partial P_f}{\partial \theta_{X_i}^{(k)}} \text{(独立同分布)} = \boxed{\frac{\partial \hat{P}_f}{\partial \theta_{X_i}^{(k)}}}$$

① 灵敏度均值公式

方差估计值：使用样本方差代替总体方差：$V\left[\frac{\partial P_f}{\partial \theta_{X_i}^{(k)}}\right] \approx S^2(x) = \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right]$

关键：代出样本方差，利用样本方差公式计算。

由 $\underbrace{V\left[\frac{\partial \hat{P}_f}{\partial \theta_{X_i}^{(k)}}\right]}_{\text{样本方差}} \xrightarrow{\text{①代回}} V\left[\frac{1}{N}\sum_{j=1}^{N} \frac{I_F(x_j)}{f_X(x_j)} \frac{\partial f_X(x_j)}{\partial \theta_{X_i}^{(k)}}\right]$

直接变：含：$\left(\sum_{j=1}^{N} 4\cdots\right)^2$ 项，难求。

→ 由于样本与母体为独立同分布的，有：

☆ 注意!!! 虽然 $\frac{I_F(x_j)}{f_X(x_j)}$ 分布均相同，但改 $E\left[\frac{1}{N}\sum_{j=1}^{N}\cdots\right]$ 时，可去!!!

由于总体方差：

$$= \frac{1}{N^2}\sum_{j=1}^{n} V\left[\frac{I_F(x_j)}{f_X(x_j)} \cdot \frac{\partial f_X(x_j)}{\partial \theta_{X_i}^{(k)}}\right] \xrightarrow[\text{母体方差}]{\text{代换}} \frac{1}{N} V\left[\frac{I_F(x)}{f_X(x)} \frac{\partial f_X(x)}{\partial \theta_{X_i}^{(k)}}\right]$$

$$= \frac{1}{N} \cdot \frac{1}{N-1}\left[\sum_{j=1}^{N}\left(\frac{I_F(x_j)}{f_X(x_j)} \cdot \frac{\partial f_X(x_j)}{\partial \theta_{X_i}^{(k)}}\right)^2 - N\left(\frac{\partial \hat{P}_f}{\partial \theta_{X_i}^{(k)}}\right)^2\right] = \frac{1}{N-1}\left[\frac{1}{N}\sum_{j=1}^{N}\left(\frac{I_F(x_j)}{f_X(x_j)} \cdot \frac{\partial f_X(x_j)}{\partial \theta_{X_i}^{(k)}}\right)^2 - \left(\frac{\partial \hat{P}_f}{\partial \theta_{X_i}^{(k)}}\right)^2\right]$$

灵敏度方差公式.

另外，我们给出对均值和方差的相应估计公式：（相互独立正态随机变量；灵敏度）

由 $\frac{\partial \hat{P}_f}{\partial \mu_i} = E\left[\frac{I_F(x)}{f_X(x)} \frac{\partial f_X(x)}{\partial \mu_i}\right]$

由：$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_{x_i}} e^{-\frac{(x-\mu_{x_i})^2}{2\sigma_{x_i}^2}} \rightarrow \frac{\partial f_X(x)}{\partial \mu_{x_i}} = \frac{1}{\sqrt{2\pi}\,\sigma_{x_i}} e^{-\frac{(x-\mu_{x_i})^2}{2\sigma_{x_i}^2}} \cdot \frac{\cancel{2}(x-\mu_{x_i})}{\cancel{2}\sigma^2} \cdot -1$

同时：$\frac{1}{f_X(x)} \frac{\partial f_X(x)}{\partial \mu_{x_i}} = \frac{x-\mu_{x_i}}{\sigma_{x_i}^2}$ 均值灵敏度

对 σ 求导，则有：

$$\frac{\partial \hat{P}_f}{\partial \sigma_{x_i}} = \frac{1}{\sqrt{2\pi}} \cdot \left(-\frac{1}{\sigma_{x_i}^2}\right) e^{-\frac{(x-\mu_{x_i})^2}{2\sigma_{x_i}^2}} + \frac{1}{\sqrt{2\pi}\,\sigma_{x_i}} e^{-\frac{(x-\mu_{x_i})^2}{2\sigma_{x_i}^2}} \cdot -\frac{(x-\mu_{x_i})^2}{\cancel{2}} \cdot \cancel{-2} \cdot \frac{1}{\sigma_{x_i}^3}$$

$$\therefore \text{有：} \frac{1}{f_X(x)} \frac{\partial f_X(x)}{\partial \sigma_{x_i}} = -\frac{1}{\sigma_{x_i}} + \frac{(x-\mu_i)^2}{\sigma_{x_i}^3} = \frac{1}{\sigma_{x_i}}\left[\left(\frac{x_{ji}-\mu_i}{\sigma_{x_i}}\right)^2 - 1\right]$$

方差灵敏度