Derivations of the Stiffness Matrix in FEM

Saturday, March 11, 2023 2:40 PM

in the previous derivation, we have derived the linear form of the u^h as follows:

where d_A is the constant.

$$\boxed{u^h = \sum_{A=1}^{n} d_A N_A + q N_{n+1}} \rightarrow v^h = \sum_{B=1}^{n} d_B N_B \leftarrow \text{(Note that } v^h \in \mathcal{V}\text{, while } u^h \in \mathcal{S}\text{)}$$

$$\boxed{w^h = \sum_{A=1}^{n} C_A N_A} \rightarrow w^h \in \mathcal{V}^h$$

we substitute this equation into the Galerkin Equation, which is:

$$\boxed{a(w^h, v^h) = (w^h, l) + w^h(0)h - a(w^h, q^h)}$$

thus:

$$a\Big(\sum_{A=1}^{n} C_A \underbrace{N_A}_{\text{functions}}, \sum_{B=1}^{n} d_B N_B\Big) = \Big(\sum_{A=1}^{n} C_A N_A, l\Big) + \sum_{A=1}^{n} C_A N_A(0) h - a\Big(\sum_{A=1}^{n} C_A N_A, q N_{n+1}\Big)$$

We use the symmetry and bilinear quality of the equation, then:

$$\sum_{A=1}^{n} C_A \sum_{B=1}^{n} d_B\, a(N_A, N_B) = \sum_{A=1}^{n} C_A (N_A, l) + \sum_{A=1}^{n} C_A N_A(0) h - \sum_{A=1}^{n} C_A\, a(N_A, q N_{n+1})$$

that becomes:

$$\boxed{\sum_{A=1}^{n} C_A G_A = 0} \qquad (1.6.8)$$

where:

$$G_A = \sum_{B=1}^{n} d_B\, a(N_A, N_B) - (N_A, l) - N_A(0) h + a(N_A, q N_{n+1})$$

Now: Galerkin equations hold for all $w^h \in \mathcal{V}^h$.

since the constants C_A's are arbitary in equation (1.6.8), the term that contains C_A should all be identically zero. i.e. $G_A = 0$, then we can get the equation:

$$\boxed{\sum_{B=1}^{n} a(N_A, N_B) d_B = (N_A, l) + N_A(0) h - a(N_A, q N_{n+1})} \qquad (1.6.10)$$

$$\boxed{\sum_{B=1} a(N_A, N_B)\, d_B = (N_A, l) + N_A(0)\, h - a(N_A, q\, N_{n+1})} \quad (1.6.10)$$

In this equation, everything is known except d_B's. Thus the equation above constitutes a system of n equations in n unknowns. solve it by the numerical method, and this can be written in a more concise form as:

$$\begin{cases} K_{AB} = a(N_A, N_B) \implies \text{stiffness matrix} \\ F_A = (N_A, l) + N_A(0)\, h - a(N_A, N_{n+1})\, q \end{cases}$$

then the equation becomes:

$$\boxed{\sum_{B=1}^{n} K_{AB}\, d_B = F_A} \qquad (A = 1, 2, \cdots n)$$

where:

$$K = [K_{AB}] = \underbrace{\begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix}}_{\text{stiffness matrix}}, \quad F_A = \underbrace{\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}}_{\text{force vector}} \quad d_B = \underbrace{\begin{Bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{Bmatrix}}_{\text{displacement vector.}}$$

also we can write it as:

$$\boxed{Kd = F}$$