# Laplace 方程边值问题的推导与例题

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定理 设 $\varphi(x,y)$ 是 Laplace 方程：$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0$ 的解：

则将 $\varphi(x,y)$ 共形映射为 $w(u,v)$ 则有

$$\frac{\partial^2\varphi}{\partial u^2}+\frac{\partial^2\varphi}{\partial v^2}=0$$

$\frac{\partial F}{\partial x}=F_u u'_x+F_v v'_x$

由 $\frac{\partial\varphi}{\partial u}$ 是 $u,v$ 函数，$\varphi=u+iv$

$\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial u}\right)=\left(\frac{\partial}{\partial u}\left(\frac{\partial\varphi}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial\varphi}{\partial u}\right)\frac{\partial v}{\partial x}\right)$

$=\frac{\partial^2\varphi}{\partial u^2}\frac{\partial u}{\partial x}+\frac{\partial^2\varphi}{\partial u\partial v}\cdot\frac{\partial v}{\partial x}$.

证明：

由 $w=f(z)=u(x,y)+iv(x,y)$ 由于 $\varphi(x,y)$ 满足 $\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0$.

则 $\frac{\partial\varphi}{\partial x}=\frac{\partial\varphi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\varphi}{\partial v}\frac{\partial v}{\partial x}$.

对 x 再次求导，再乘 $\frac{\partial u}{\partial x}$

$$\frac{\partial^2\varphi}{\partial x^2}=\frac{\partial}{\partial x}\left[\frac{\partial\varphi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\varphi}{\partial v}\frac{\partial v}{\partial x}\right]=\left[\frac{\partial^2\varphi}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial^2\varphi}{\partial u\partial v}\frac{\partial v}{\partial x}\right]\left(\frac{\partial u}{\partial x}\right)+\frac{\partial\varphi}{\partial u}\frac{\partial^2 u}{\partial x^2}+\left[\frac{\partial^2\varphi}{\partial v^2}\frac{\partial v}{\partial x}+\frac{\partial^2\varphi}{\partial u\partial v}\frac{\partial u}{\partial x}\right]\left(\frac{\partial v}{\partial x}\right)+\frac{\partial\varphi}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$=\varphi_{uu}u_x^2+2\varphi_{uv}u_xv_x+\varphi_{vv}v_x^2+\varphi_u u_{xx}+\varphi_v v_{xx}$$

$$\text{则 }\frac{\partial^2\varphi}{\partial y^2}=\varphi_{uu}u_y^2+2\varphi_{uv}u_yv_y+\varphi_{vv}v_y^2+\varphi_u u_{yy}+\varphi_v v_{yy}$$

由 $\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0$.

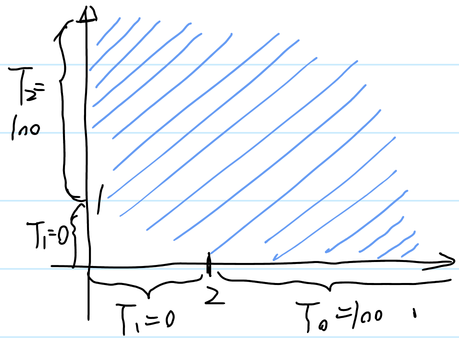
由 ①+②，则：$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=\varphi_{uu}(u_x^2+u_y^2)+2\varphi_{uv}(u_xv_x+u_yv_y)+\varphi_{vv}(u_x^2+v_x^2)+0=0$

由于 $u_{xx}+u_{yy}=v_{xx}+v_{yy}=0$

由于 $w=u+iv$ 为解析函数，则 $\longrightarrow$ $u_x=v_y$，$v_x=-u_y$，中间一项为 0

故：$(\varphi_{uu}+\varphi_{vv})\left[u_x^2+v_x^2\right]=0$，显有 $u_x^2+v_x^2\neq0$，则 $\varphi_{uu}+\varphi_{vv}=0$. 即：$\frac{\partial^2\varphi}{\partial u^2}+\frac{\partial^2\varphi}{\partial v^2}=0$.

例 1. 一块金属板位于 z 平面的第一象限，边界温度分布如图：




且定常温度分布必须满足 Laplace 方程：

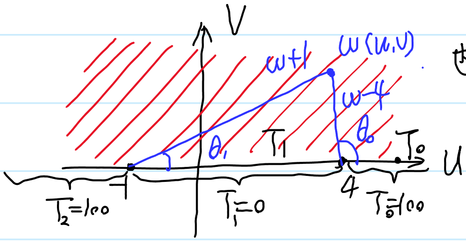
$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

求金属板上的定常温度分布：

解：我们使用 $w=z^2$ 将左图一象限内的平面映射为 $Im(z)>0$.

由 $z=x+iy \rightarrow w=z^2=x^2-y^2+2ixy \rightarrow u=x^2-y^2,\ v=2xy$

则在 w 平面上，2点变为 $(4,0)$，$(0,1)$ 变为 $(-1,0)$，边界条件也做左图映射。设 w 平面中一点为 $w(u,v)$，则 $\arg(w+1)=\theta_1$




$\arg(w-4)=\theta_0$

显然，当 w 为实数时，$w>4\rightarrow\theta_1=\theta_0=0$，

$w\in(-1,4)\longrightarrow\theta_1=0,\ \theta_0=\pi$，$w<-1\longrightarrow\theta_1=\theta_0=\pi$.（边界条件）

我们只需要找到一个满足 Laplace 方程和上述边界条件的解析函数 $T'$，即为合理的解.

取 $\boxed{T'=T_0+\frac{1}{\pi}(T_1-T_0)\,\theta_0+\frac{1}{\pi}(T_2-T_1)\,\theta_1}$ 故有：

取 $\boxed{T' = T_0 + \frac{1}{\pi}(T_1 - T_0)\,\theta_0 + \frac{1}{\pi}(T_2 - T_1)\,\theta_1}$ 故有：

$$T' = T_0 + \frac{1}{\pi}(T_1 - T_0)\arg(w-4) + \frac{1}{\pi}(T_2 - T_1)\arg(w+1)$$ ⟹ 先研究其解析性

利用 $\ln z = \ln|z| + i\cdot\arg z$，即 $T'$ 可以视为：

配凑：由于 $iT_0 + \frac{1}{\pi}(T_1 - T_0)\ln(w-4) + \frac{1}{\pi}(T_2 - T_1)\ln(w+1)$ 的虚部为 $T'$，

则取一个解析函数：将上面乘 $-i$ 有： ↳ 这个函数解析，故虚部是解析的。

$$T' = T_0 - \frac{i}{\pi}(T_1 - T_0)\ln(w-4) - \frac{i}{\pi}(T_2 - T_1)\ln(w+1)$$ ，代入：$T_0 = T_2 = 100$，$T_1 = 0$

$$T' = 100 + \frac{100}{\pi}\ln(w-4) - \frac{i}{\pi}\times 100\ln(w+1)$$

则 $T' = 100 - \frac{100}{\pi}\theta_0 + \frac{100}{\pi}\theta_1$ 在平面上为解析函数

$= \frac{100}{\pi}[\pi + \theta_1 - \theta_0]$ 其中 $\theta_0 = tg^{-1}\frac{v}{u-4} = tg^{-1}\frac{2xy}{x^2-y^2-4}$，$\theta_1 = tg^{-1}\frac{v}{u+1} = tg^{-1}\frac{2xy}{x^2-y^2+1}$

$$T' = \frac{100}{\pi}\left[\pi + tg\frac{v}{u+1} - tg\frac{v}{u-4}\right] \to tg\left(\frac{\pi T}{100}\right) = tg(\pi + \theta_1 - \theta_0) = tg(\theta_1 - \theta_0)$$

故 $$tg\frac{\pi T}{100} = \frac{tg\theta_1 - tg\theta_0}{1 + tg\theta_1 tg\theta_0} \xrightarrow[tg\theta_0 = \frac{2xy}{x^2-y^2-4}]{由\ tg\theta_1 = \frac{2xy}{x^2-y^2+1}} = \frac{\frac{2xy}{x^2-y^2+1} - \frac{2xy}{x^2-y^2-4}}{1 + \frac{2xy}{x^2-y^2+1}\cdot\frac{2xy}{x^2-y^2-4}}$$

$$= \frac{2xy(x^2-y^2-4) - 2xy(x^2-y^2+1)}{(x^2-y^2+1)(x^2-y^2-4) + 4x^2y^2} = \frac{-10xy}{(x^2-y^2+1)(x^2-y^2-4)+4x^2y^2} \Rightarrow T = -\frac{100}{\pi}tg^{-1}\left(\frac{10xy}{(x^2-y^2+1)(x^2-y^2-4)+4x^2y^2}\right)$$

解为：$$T = \begin{cases} \frac{100}{\pi}tg^{-1}B, & B>0 \\ \frac{100}{\pi}tg^{-1}B + \pi, & B<0 \end{cases}$$ 其中 $B = \frac{-10xy}{(x^2-y^2+1)(x^2-y^2-4)+4x^2y^2}$.