薄板的变形能与余变形能公式导出

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首先：薄板变形能总表达式为：(考虑线性情况)

$$dV = \frac{1}{2}\sigma_x\varepsilon_x + \frac{1}{2}\sigma_y\varepsilon_y \cancel{+ \frac{1}{2}\sigma_z\varepsilon_z}$$

$$+\frac{1}{2}\tau_{xy}\gamma_{xy} \cancel{+\frac{1}{2}\tau_{xz}\gamma_{xz}+\frac{1}{2}\tau_{yz}\gamma_{yz}}$$

由薄板相关假设：可以排除其中的三项。

而 $U = \int dV$

$$= \int \frac{1}{2}\sigma_x d\varepsilon_x + \frac{1}{2}\sigma_y d\varepsilon_y + \frac{1}{2}\tau_{xy}\gamma_{xy}$$

由于：$\varepsilon_x = \frac{z}{\rho_x} = K_x \cdot z$

$\varepsilon_y = \frac{z}{\rho_y} = K_y \cdot z$

$\gamma_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} z = 2z\frac{\partial^2 w}{\partial x \partial y} = 2zK_x$

$$d\dot{U} = \int\left[\sigma_x z\, d(K_x) + \sigma_y z\, d(K_y) + 2\tau_{xy} z\, d(K_{xy})\right]$$

$$= M_x dK_x + M_y dK_y + 2M_{xy} dK_{xy},$$

显然有总能量：

$$U_{tot} = M_x K_x + M_y K_y + 2M_{xy}K_{xy}$$

$$\therefore dU_{tot} = K_x dM_x + M_x dK_x + K_y dM_y + M_y dK_y + 2K_{xy} dM_{xy} + 2M_{xy} dK_{xy}$$

因而：

$$\left\{\begin{aligned} d\bar{U} &= K_x dM_x + K_y dM_y + 2K_{xy} dM_{xy} \\ dU &= M_x dK_x + M_y dK_y + 2M_{xy} dK_{xy} \end{aligned}\right] ☆$$

我们利用挠度表达：有弯矩表达式：

$$\left.\begin{aligned} M_x &= D(K_x + \nu K_y) \\ M_y &= D(K_y + \nu K_x) \\ M_{xy} &= D(1-\nu)\cdot K_{xy} \end{aligned}\right]$$

代入有：

单位体积应变能微分

$$dU = D(K_x + \nu K_y)dK_x + D(K_y + \nu K_x)dK_y + 2D(1-\nu)K_{xy}\, dK_{xy}$$

$$U = D\iint\left[(K_x + \nu K_y)dK_x + (K_y + \nu K_x)dK_y + 2(1-\nu)K_{xy}\, dK_{xy}\right]$$

$\Rightarrow$ 由 $\nu(K_y dK_x + K_x dK_y) = \nu\, d(K_x K_y)$

相关，不能直积

乘积求导

$$+2(1-\nu)K_{xy}\,dK_{xy}\Big]$$

相关，不能直积 乘积求导

$$=D\left[\frac{1}{2}K_x^2+\frac{1}{2}K_y^2+K_xK_y+(1-\nu)K_{xy}^2\right]$$

$$=\frac{D}{2}\left[\underbrace{K_x^2+K_y^2+2K_xK_y}_{(\nabla^2 w)^2}+2(1-\nu)K_{xy}^2\right]$$

为单位体积应变能：

则有：在中面积分得总应变能，

$$U_{总}=\frac{D}{2}\iint_F\left[(\nabla^2 w)^2-2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right]dx\,dy$$

其中：F为板的面积，

我们说明：总的变形能密度：

$$\bar{V}+V=K_x\,dM_x+K_y\,dM_y+2K_{xy}\,dM_{xy}$$
$$+M_x\,dK_x+M_y\,dK_y+2M_{xy}\,dK_{xy}$$

显：

$$d\bar{V}=K_x\,dM_x+K_y\,dM_y+2K_{xy}\,dM_{xy}$$

由：$M_x+M_y=D(1+\nu)[K_x+K_y]$ ①

$$M_x-\nu M_y=D[(K_x+\cancel{\nu K_y}-\cancel{\nu K_y}-\nu^2K_x)]\ ②$$

∴ 由①②有：

$$D(1-\nu^2)K_x=M_x-\nu M_y\implies\boxed{K_x=\frac{M_x-\nu M_y}{D(1-\nu^2)}}$$

而：

$$K_y=\frac{M_x+M_y}{D(1+\nu)}-\frac{M_x-\nu M_y}{D(1-\nu^2)}=\frac{1}{D}\left[\frac{-\nu M_x+M_y}{1-\nu^2}\right]=\frac{M_y-\nu M_x}{D(1-\nu^2)}$$

因而有：

$$d\bar{V}=\frac{12}{Eh^3}\int\left[(M_x-\nu M_y)\,dM_x+(M_y-\nu M_x)\,dM_y+2(1+\nu)M_{xy}\,dM_{xy}\right]$$

$$=\frac{12}{Eh^3}\left[\frac{1}{2}M_x^2+\frac{1}{2}M_y^2-\nu M_xM_y+(1+\nu)M_{xy}^2\right]$$

$$=\underbrace{\frac{6}{Eh^3}\left[M_x+M_y^2-2(1+\nu)(M_xM_y+M_{xy}^2)\right]}$$

为单位体积的余应变能，

而薄板余变形能为：

$$\bar{U}=\frac{6}{Eh^3}\iint_F\left[(M_x+M_y)^2-2(1+\nu)(M_xM_y-M_{xy}^2)\right]dx\,dy$$

$$\bar{U}=\frac{6}{Eh^3}\iint_F\left[(M_x+M_y)^2-2(1+\nu)(M_xM_y-M_{xy}^2)\right]dx\,dy$$

我们写出弯曲变形能和余变形能的表达式：

$$U=\frac{D}{2}\iint_F\left[(\nabla^2 w)^2-2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right]dx\,dy$$

$$\bar{U}=\frac{6}{Eh^3}\iint_F\left[(M_x+M_y)^2-2(1+\nu)(M_xM_y-M_{xy}^2)\right]dx\,dy.$$

我们考虑对于周边固定的任意多角形板（挠度=0时）

我们代换 $\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}\overset{以x变}{=}\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial w}{\partial x}\right)-\frac{\partial^3 w}{\partial y^2\partial x}\cdot\frac{\partial w}{\partial x}$ （拆开）

以y变

$$=\boxed{\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial y^2}\frac{\partial w}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial y\partial x}\frac{\partial w}{\partial x}\right)+\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2} \quad ①$$

$$=\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y}\right)-\frac{\partial^3 w}{\partial x^2\partial y}\frac{\partial w}{\partial y}$$

$$=\boxed{\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y}\right)-\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial w}{\partial y}\right)+\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2} \quad ②$$

我们取①②两式的平均值为 $\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}$ 的值.

则：U的第二项可以写成

$$\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2$$

$$=\frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial y^2}\right)-\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial y\partial x}\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y}\right)-\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial w}{\partial y}\right)\right]$$

我们将第二项积分，有：

$$\frac{1}{2}\iint_F\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial y^2}\frac{\partial w}{\partial x}-\frac{\partial^2 w}{\partial x\partial y}\frac{\partial w}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x\partial y}\frac{\partial w}{\partial x}-\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y}\right)\right]dx\,dy$$

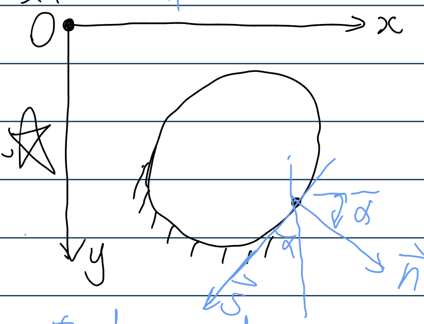
此时：我们由格林公式知：

此时：我们由格林公式知：

$$\iint_A \left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]dxdy = \oint_L Pdx+Qdy = \int\left[Q\cos\alpha - P(x,y)\sin\alpha\right]ds \ \star$$

$$=\frac{1}{2}\oint_L\left(\frac{\partial^2 w}{\partial x\partial y}\frac{\partial w}{\partial x}-\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y}\right)dx+\left(\frac{\partial^2 w}{\partial y^2}\frac{\partial w}{\partial x}-\frac{\partial^2 w}{\partial x\partial y}\frac{\partial w}{\partial y}\right)dy.$$

其中如图：



此时：也可写为：

$$=\frac{1}{2}\int\left[\left(\frac{\partial^2 w}{\partial y^2}\frac{\partial w}{\partial x}-\frac{\partial^2 w}{\partial x\partial y}\frac{\partial w}{\partial y}\right)\cdot\cos\alpha-\left(\frac{\partial^2 w}{\partial x\partial y}\frac{\partial w}{\partial x}-\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y}\right)\cdot\sin\alpha\right]ds \ \star$$

$$\begin{cases} dx = \cos\alpha\cdot dn - \sin\alpha\cdot ds \\ dy = \sin\alpha\cdot dn + \cos\alpha\, ds \end{cases}$$

我们知道：法线 n，切线 s 方向：有：

$$\begin{cases} \dfrac{\partial}{\partial x} = \cos\alpha\dfrac{\partial}{\partial n} - \sin\alpha\dfrac{\partial}{\partial s} \\ \dfrac{\partial}{\partial y} = \sin\alpha\dfrac{\partial}{\partial n} + \cos\alpha\dfrac{\partial}{\partial s} \\ \dfrac{\partial}{\partial n} = \dfrac{\partial}{\partial x}\cos\alpha + \dfrac{\partial}{\partial y}\sin\alpha \\ \dfrac{\partial}{\partial s} = -\dfrac{\partial}{\partial x}\sin\alpha + \dfrac{\partial}{\partial y}\cos\alpha \end{cases}$$

有：$dx = -ds\sin\alpha$

$dy = ds\cos\alpha$

$\Longrightarrow$ 我们利用这个给☆式变形：

$$\star = \frac{1}{2}\int\left[\frac{\partial w}{\partial x}\left(\frac{\partial^2 w}{\partial y^2}\cos\alpha-\frac{\partial^2 w}{\partial x\partial y}\sin\alpha\right)+\frac{\partial w}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}\sin\alpha-\frac{\partial^2 w}{\partial x\partial y}\cos\alpha\right)\right]dxdy$$

$$=\frac{1}{2}\iint_F\left[\frac{\partial w}{\partial x}\cdot\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial y}\right)+\frac{\partial w}{\partial y}\cdot-\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial x}\right)\right]dxdy$$

$$=\frac{1}{2}\iint_F\left[\frac{\partial w}{\partial x}\cdot\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial y}\right)-\frac{\partial w}{\partial y}\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial x}\right)\right]dxdy$$

代入 $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ 将 x,y 换 n,s

$$=\frac{1}{2}\iint_F\left[\left(\cos\alpha\frac{\partial w}{\partial n}-\sin\alpha\frac{\partial w}{\partial s}\right)\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial y}\right)-\left(\frac{\partial w}{\partial n}\sin\alpha+\frac{\partial w}{\partial s}\cos\alpha\right)\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial x}\right)\right]dxdy$$

中间部分：

$$\left(\cos\alpha\frac{\partial w}{\partial n}-\sin\alpha\frac{\partial w}{\partial s}\right)\frac{\partial}{\partial s}\left(\sin\alpha\frac{\partial w}{\partial n}+\cos\alpha\frac{\partial w}{\partial s}\right)-\left(\frac{\partial w}{\partial n}\sin\alpha+\frac{\partial w}{\partial s}\cos\alpha\right)\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial n}\cos\alpha-\frac{\partial w}{\partial s}\sin\alpha\right)$$

最终略去计算，有：

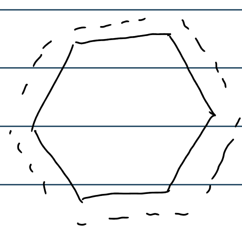
$$\longrightarrow\ =\frac{\partial w}{\partial n}\frac{\partial^2 w}{\partial s^2}-\frac{\partial w}{\partial s}\frac{\partial^2 w}{\partial n\partial s}+\frac{\partial\alpha}{\partial s}\left[\left(\frac{\partial w}{\partial n}\right)^2+\left(\frac{\partial w}{\partial s}\right)^2\right] \ \star$$

由于边界上(固支) $\frac{\partial w}{\partial n}=\frac{\partial w}{\partial s}=0$，则上式为0

简支时：

$\frac{\partial w}{\partial s}=\frac{\partial \alpha}{\partial s}=0$，上式仍为0

此时：在多边形板各边固支情况：有：



$$U=\frac{D}{2}\iint_F\left(\nabla^2 w\right)^2 dxdy$$