# 傅里叶变换性质证明

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0. 定义：$f^{\wedge}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$

$$f^{\vee}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{j\omega t} dt$$

6 对称性是显然的.

1. 有：$(af_1 + bf_2)^{\wedge} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (af_1(t) + bf_2(t)) e^{-j\omega t} dt$

$$= af_1^{\wedge}(t) + bf_2^{\wedge}(t) \checkmark$$

2. 微商：

有：$\left(\frac{df}{dx}\right)^{\wedge} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-j\omega t} df(t)$

$$\xrightarrow{\text{分部}} = \frac{1}{\sqrt{2\pi}} \left[ e^{-j\omega t} f(t) \Big|_{-\infty}^{+\infty} + j\omega \int_{-\infty}^{+\infty} e^{-j\omega t} f(t) dt \right]$$

因于 $f(t) \in L(-\infty, +\infty)$，故 $\lim_{x \to \infty} f(x) = 0$.

$$= j\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt = j\omega f^{\wedge}(x)$$

显然：$\frac{d^m f}{dx^m} = (j\omega)^m f^{\wedge}(x)$

3. 多项式相乘：

$$(xf(x))^{\wedge} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t f(t) e^{-j\omega t} dt.$$

又考虑到：$\frac{d}{d\omega} f^{\wedge}(\omega) = \frac{d}{d\omega} \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$

(系数略) $= \int_{-\infty}^{+\infty} f(t) \cdot -jt e^{-j\omega t} dt$

$$= -j \int_{-\infty}^{+\infty} t f(t) e^{-j\omega t} dt = -j(xf(x))^{\wedge}$$

$$= -j\int_{-\infty}^{+\infty} t f(t)\, e^{-j\omega t}\, dt = -j\,(x f(x))^{\wedge}$$

则：

$$\frac{d}{dt}(f(x))^{\wedge} = -j\,(x f(x))^{\wedge}$$

对应：

$$\frac{d^m}{d\omega^m}(f(x))^{\wedge} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\, \frac{d^m}{d\omega^m} e^{-j\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} (-jt)^m f(t)\, dt$$

$$= \frac{1}{j^m}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} t^m f(t)\, e^{-j\omega t}\, dt$$

$$\rightarrow (x^m f(x))^{\wedge} = j^m \frac{d}{d\omega^m}(f(x))^{\wedge}$$

4.平移，伸缩

$$(f(x-a))^{\wedge} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t-a)\, e^{j\omega t}\, dt.$$

取 $\lambda = t-a$, $\Rightarrow t = \lambda + a$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\lambda)\, e^{j\omega(\lambda+a)}\, d\lambda = e^{j\omega a} f^{\wedge}(x)$$

$$f\left(\frac{x}{\lambda}\right)^{\wedge} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f\left(\frac{t}{\lambda}\right) e^{j\omega t}\, dt \quad \text{取 } m = \frac{t}{\lambda}.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(m)\, e^{j\omega\lambda m} \cdot \lambda\, dm$$

$$= \lambda\, f^{\wedge}(x)(\lambda\omega)$$

即：$\hat{f}\left(\frac{x}{k}\right) = k\, f^{\wedge}(x)(k\omega)$

对应：$f^{\wedge}(kx) = \frac{1}{|k|} f^{\wedge}(x)\left(\frac{\omega}{|k|}\right)$

对应：$f^{\wedge}(kx)=\frac{1}{|k|}f^{\wedge}(x)\left(\frac{w}{|k|}\right)$

5.卷积性质：

$$(f*g)^{\wedge}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x-t)g(t)dt\ e^{-j\omega t}dt$$

由 Fubini 定理：

$$\|f*g\|_{L_1}=\int_{-\infty}^{\infty}dx\left|\int_{-\infty}^{+\infty}f(x-t)g(t)dt\right|$$

$$\leq\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}|f(x-t)g(t)|\,dt$$

$$\leq\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}|f(x-t)|\,|g(t)|\,dt$$

$$\leq\int_{-\infty}^{+\infty}|g(t)|dt\int_{-\infty}^{+\infty}|f(x-t)|dx$$

因 $L_1$ 范数定义 $=\sum_{i=1}^{n}x_i$，故 $|f(x)|_{L_1}=\int_{-\infty}^{+\infty}f(x)dx$

上式 $=\|g\|_{L_1}\|f\|_{L_1}$ 故有 $f*g\in L(-\infty,+\infty)$

此时：

$$(f*g)^{\wedge}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x-t)g(t)dt\ e^{-j\omega x}dx$$

配凑 $e^{-j\omega t}$

$$=\frac{1}{\sqrt{2\pi}}\underline{\int_{-\infty}^{+\infty}g(t)e^{-j\omega t}dt}\int_{-\infty}^{+\infty}f(x-t)e^{-j\omega(x-t)}dx$$

$$=\sqrt{2\pi}\ \hat{g}(\omega)\hat{f}(\omega)$$