Example 3.1

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We use an actual system of equations from fluid dynamics. Consider the irrotational, 2-dimensional and steady, inviscid flow of a compressible gas. The flow field is slightly perturbed from freestream conditions

For either subsonic freestream or supersonic freestream, the energy equations are given by:

$$\begin{cases} (1-M_\infty^2)\dfrac{\partial u'}{\partial x} + \dfrac{\partial v'}{\partial y} = 0 \\ \text{and} \quad \dfrac{\partial u'}{\partial y} - \dfrac{\partial v'}{\partial x} = 0 \end{cases}$$

where u' and v' are small perturbation velocities, measured relative to the freestream velocity,

That is the form of

$$\begin{cases} a_1 \dfrac{\partial u}{\partial x} + b_1 \dfrac{\partial u}{\partial y} + c_1 \dfrac{\partial v}{\partial x} + d_1 \dfrac{\partial v}{\partial y} = 0 \\ a_2 \dfrac{\partial u}{\partial x} + b_2 \dfrac{\partial u}{\partial y} + c_2 \dfrac{\partial v}{\partial x} + d_2 \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

then:

$$\begin{bmatrix} 1-M_\infty^2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ dx & dy & & \\ & & dx & dy \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

restating these values in terms of a, b and c from the equation above:

we have the formula:

$$a\,(dy)^2 + b\,dx\,dy + c\,dy^2$$

where:

$$a = a_1c_2 - a_2c_1 = M_\infty^2 - 1$$

$$b = -(a_1d_2 - a_2d_1 + b_1c_2 - b_2c_1) = 0$$

$$c = (b_1d_2 - b_2d_1) = -1$$

that is: $(1-M_\infty^2)\,dy^2 - dx^2 = 0$

from the quadratic formula, we have:

$$\frac{dy}{dx} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \pm \frac{1}{\sqrt{M_\infty^2 - 1}}$$

$$\frac{dy}{dx} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \pm \frac{1}{\sqrt{M_\infty^2-1}}$$

in the equation above, for the case of supersonic flow, $M_\infty > 1$, then there's two real characteristic directions through each point. and the slope of them are

$$\frac{1}{\sqrt{M_\infty^2-1}} \text{ and } -\frac{1}{\sqrt{M_\infty^2-1}}$$

also note that for the case of subsonic flow, the equations are elliptic.

then we apply the eigenvalue method, then we have the equation as:

$$\begin{bmatrix} 1-M_\infty^2 & \\ & -1 \end{bmatrix} \frac{\partial W}{\partial x} + \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \frac{\partial W}{\partial y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

, which is denoted by

$$[K]\frac{\partial W}{\partial x} + [M]\frac{\partial W}{\partial y} = 0$$

we use $[N] = [K]^{-1}[M] = \begin{bmatrix} \frac{1}{1-M_\infty^2} & \\ & -1 \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} = \begin{bmatrix} & \frac{1}{1-M_\infty^2} \\ -1 & \end{bmatrix}$

then

$$\frac{\partial W}{\partial x} + \underbrace{\begin{bmatrix} & \frac{1}{1-M_\infty^2} \\ -1 & \end{bmatrix}}_{=[N]} \frac{\partial W}{\partial y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

That is the matrix $[N]$ in the form of eigenvalue method.

We wish to examine the eigenvalues of $[N]$, or using

$$\left| [N] - \lambda [I] \right| = 0,$$

we substitute it into the matrix and then

$$\det \begin{bmatrix} -\lambda & \frac{1}{1-M_\infty^2} \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + \frac{1}{1-M_\infty^2} = 0, \quad \lambda_{1,2} = \pm \frac{1}{\sqrt{M_\infty^2-1}}$$

Note that for some systems of equations, the eigenvalues may be a mix of both real and complex values (that is system is neither hyperbolic nor elliptic.)