例题1.2

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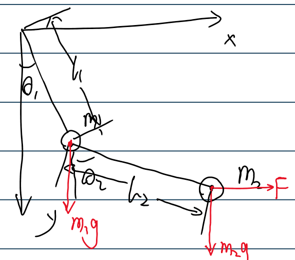
质量为 m_1，m_2 的两个小球，并使用绳系掌，在 m_2 上作用有水平方向已知的力 $F(t)$，建立运动微分方程：



①：系统使用 Lagrange 方程建立其运动微分方程，其一般形式为：

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=Q_j$$

为两个广义坐标；

显然为两自由度 → θ_1,θ_2，设其对应的广义力为 Q_1, Q_2

⟹ 有2个广义力对应的 Lagrange 方程

则：列写方程为：

$$\begin{cases}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)-\frac{\partial T}{\partial \theta_1}=Q_1\\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right)-\frac{\partial T}{\partial \theta_2}=Q_2\end{cases}$$

→ (附：本题不是保守系统，若为保守系统可使用 $L=T-V$ 进行列写)

有：动能 T 表达式：

$$T=\frac{1}{2}m_1(l_1\dot{\theta}_1)^2+\frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2)$$

其中将 x_2、y_2 用 θ_1,θ_2 表达有：

$$x_2=l_1\sin\theta_1+l_2\sin\theta_2 \to \dot{x}_2=l_1\cos\theta_1\dot{\theta}_1+l_2\cos\theta_2\dot{\theta}_2$$

$$y_2=l_1\cos\theta_1+l_2\cos\theta_2 \to \dot{y}_2=-(l_1\sin\theta_1\dot{\theta}_1+l_2\sin\theta_2\dot{\theta}_2)$$

注意复合求导

$$=\frac{1}{2}m_1l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2\left[(l_1\cos\theta_1\dot{\theta}_1+l_2\cos\theta_2\dot{\theta}_2)^2+(l_1\sin\theta_1\dot{\theta}_1+l_2\sin\theta_2\dot{\theta}_2)^2\right]$$

整理 $T=\frac{1}{2}m_1l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2+l_2^2\dot{\theta}_2^2+2l_1l_2(\sin\theta_1\sin\theta_2\dot{\theta}_1\dot{\theta}_2+\cos\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2)\right]$

$$=\frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2l_2^2\dot{\theta}_2^2+m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)$$

则 $\frac{\partial T}{\partial \dot{\theta}_1}=(m_1+m_2)l_1^2\dot{\theta}_1+m_2l_1l_2\dot{\theta}_2\cos(\theta_1-\theta_2)$

$$\to \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)=(m_1+m_2)l_1^2\ddot{\theta}_1+m_2l_1l_2\ddot{\theta}_2\cos(\theta_1-\theta_2)-m_2l_1l_2\dot{\theta}_2\sin(\theta_1-\theta_2)\cdot(\dot{\theta}_1-\dot{\theta}_2)$$

$$=(m_1+m_2)l_1^2\ddot{\theta}_1+m_2l_1l_2\ddot{\theta}_2\cos(\theta_1-\theta_2)-m_2l_1l_2\sin(\theta_1-\theta_2)(\dot{\theta}_1\dot{\theta}_2-\dot{\theta}_2^2)$$

而 $\frac{\partial T}{\partial \theta_1}=-m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)$

而：由

$$\sum\delta W=m_1g\cdot\delta(l_1\cos\theta_1)+m_2g\,\delta(l_1\cos\theta_1+l_2\cos\theta_2)+F\,\delta(l_1\sin\theta_1+l_2\sin\theta_2)$$

$$=-m_1gl_1\sin\theta_1\delta\theta_1-m_2gl_1\sin\theta_1\delta\theta_1-m_2gl_2\sin\theta_2\delta\theta_2+Fl_1\cos\theta_1\delta\theta_1+Fl_2\cos\theta_2\delta\theta_2$$

$$=(-m_1gl_1\sin\theta_1-m_2gl_1\sin\theta_1+F_1l_1\cos\theta_1)\delta\theta_1+(-m_2gl_2\sin\theta_2+Fl_2\cos\theta_2)\delta\theta_2$$

则有：$Q_1=-m_1gl_1\sin\theta_1-m_2gl_1\sin\theta_1+F_1l_1\cos\theta_1$

代入Lagrange方程有：

$$(m_1+m_2)l_1^2\ddot{\theta}_1+m_2l_1l_2\ddot{\theta}_2\cos(\theta_1-\theta_2)-m_2l_1l_2\sin(\theta_1-\theta_2)(\cancel{\dot{\theta}_1\dot{\theta}_2}-\dot{\theta}_2^2)\cancel{+m_2l_1l_2\sin(\theta_1-\theta_2)\cdot\dot{\theta}_1\dot{\theta}_2}=-m_1gl_1\sin\theta_1-m_2gl_1\sin\theta_1+F_1l_1\cos\theta_1$$

约去 l_1，则：

$$(m_1+m_2)l_1\ddot{\theta}_1+m_2l_2\ddot{\theta}_2\cos(\theta_1-\theta_2)-m_2l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)=-(m_1+m_2)g\sin\theta_1+F_1\cos\theta_1$$

则得到第①式：

$$(m_1+m_2)l_1\ddot{\theta}_1+m_2l_2\ddot{\theta}_2\cos(\theta_1-\theta_2)-m_2l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)+(m_1+m_2)g\sin\theta_1=F_1\cos\theta_1 \quad ①$$

则得到第①式:

$$(m_1+m_2)l_1\ddot{\theta}_1 + m_2 l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) - m_2 l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) + (m_1+m_2)g\sin\theta_1 = F_1\cos\theta_1 \quad ①$$

而:

$$Q_2 = -m_2 g l_2\sin\theta_2 + F l_2\cos\theta_2$$

而: $\frac{\partial T}{\partial\dot{\theta}_2} = m_2 l_2^2\dot{\theta}_2 + m_2 l_1 l_2\dot{\theta}_1\cos(\theta_1-\theta_2)$,

则 $$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}_2}\right) = m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2 l_1 l_2\dot{\theta}_1\sin(\theta_1-\theta_2)\cdot(\dot{\theta}_1-\dot{\theta}_2)$$

$$= m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2 l_1 l_2\sin(\theta_1-\theta_2)(\dot{\theta}_1^2-\dot{\theta}_1\dot{\theta}_2)$$

又: $\frac{\partial T}{\partial\theta_2} = \cancel{-}m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)\cdot\cancel{(-1)}$

则: $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}_2}\right) - \frac{\partial T}{\partial\theta_2} = Q_2$ 为 ②式;

代入得:

$$m_2 l_2^{\cancel{2}}\ddot{\theta}_2 + m_2 l_1\cancel{l_2}\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2 l_1\cancel{l_2}\sin(\theta_1-\theta_2)(\dot{\theta}_1^2\cancel{-\dot{\theta}_1\dot{\theta}_2}) \cancel{- m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)}$$

$$= -m_2 g\cancel{l_2}\sin\theta_2 + F\cancel{l_2}\cos\theta_2$$

故:

$$m_2 l_2\ddot{\theta}_2 + m_2 l_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2 l_1\sin(\theta_1-\theta_2)\dot{\theta}_1^2 + m_2 g\sin\theta_2 = F\cos\theta_2 \quad ②$$

则①,②式即为所求的运动微分方程