矩形薄板的叠加法原理示例

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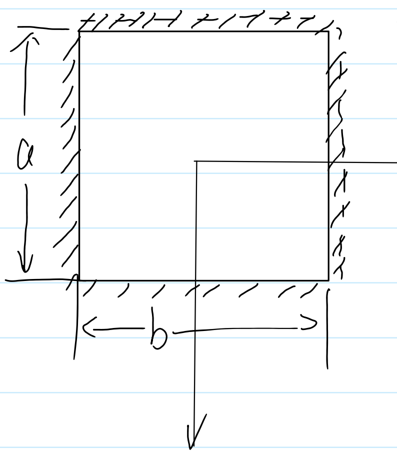
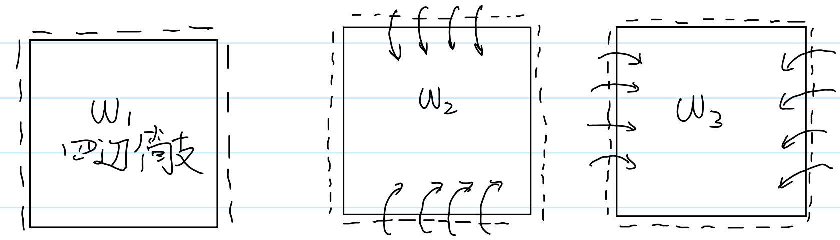


如求解均布载荷下四边刚性固定的板，我们以此说明叠加法的应用。

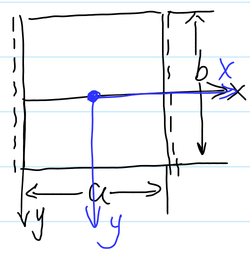
将此板分解为 w_1、w_2、w_3，则得四边简支和两个对边弯矩情况。



则四边简支显然已经求得，两弯矩对称，而边界条件变为：

$$\begin{cases} y=\pm\frac{b}{2}\text{处}，\ \dfrac{\partial w_1}{\partial y}+\dfrac{\partial w_2}{\partial y}+\dfrac{\partial w_3}{\partial y}=0， \\ y=\pm\frac{a}{2}\text{处}，\ \dfrac{\partial w_1}{\partial x}+\dfrac{\partial w_2}{\partial x}+\dfrac{\partial w_3}{\partial x}=0 \end{cases}$$

① 我们已经解出，四边简支板的挠度公式为：



$$w=\frac{4q_0a^4}{D\pi^5}\sum_{m=1,3,\dots}^{\infty}\frac{1}{m^5}\left[1-\left(1+\frac{\alpha_m}{2}\tanh\alpha_m\right)\frac{\cosh\frac{m\pi y}{a}}{\cosh\alpha_m}+\frac{m\pi y}{a}\frac{\sinh\frac{m\pi y}{a}}{\cosh\alpha_m}\right]\sin\frac{m\pi x}{a}$$

$\Rightarrow$ 取 $x=X+\frac{a}{2}$，得右方 $=\sin\left(\frac{m\pi X}{a}+\frac{1}{2}m\pi\right)$

当 $m=1,3,\cdots$ 时，$=(-1)^{\frac{m-1}{2}}\cos\frac{m\pi X}{a}$，代入，有：

$$w_1=\frac{4q_0a^4}{D\pi^5}\sum_{m=1,3,\dots}^{\infty}\frac{(-1)^{\frac{m-1}{2}}}{m^5}\left[1-\left(1+\frac{\alpha_m}{2}\tanh\alpha_m\right)\frac{\cosh\frac{m\pi y}{a}}{\cosh\alpha_m}+\frac{m\pi y}{a}\frac{\sinh\frac{m\pi y}{a}}{2\cosh\alpha_m}\right]\cos\frac{m\pi x}{a}.$$

$$\left.\frac{\partial w_1}{\partial y}\right|_{y=\frac{b}{2}}=-\left.\frac{\partial w_1}{\partial y}\right|_{y=-\frac{b}{2}}=\frac{2q_0a^3}{D\pi^4}\sum_{m=1,3,\dots}^{\infty}\frac{(-1)^{\frac{m-1}{2}}}{m^4}\left(\frac{\alpha_m}{\cosh^2\alpha_m}-\tanh\alpha_m\right)\cos\frac{m\pi x}{a}.$$

显然 $x=\frac{a}{2}$ 处的转角只需将 a,b 互换（x,y），$\beta_n=\frac{n\pi a}{2b}$ 代替上式即可。

即：
$$\left.\frac{\partial w_1}{\partial x}\right|_{x=\frac{a}{2}}=-\left.\frac{\partial w_1}{\partial x}\right|_{x=-\frac{a}{2}}=\frac{2q_0b^3}{D\pi^4}\sum_{n=1,3,\dots}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n^4}\left(\frac{\beta_n}{\cosh^2\beta_n}-\tanh\beta_n\right)\cos\frac{n\pi y}{b},$$

②：边缘弯矩下简支板已经在单三角级数解法中给出，套用(2-22)，有：

四边简支边缘弯矩，则：

$$w=\frac{a^2}{2\pi^2D}\sum_{m=1}^{\infty}\frac{E_m}{m^2\cosh\alpha_m}\left(\alpha_m\tanh\alpha_m\cosh\frac{m\pi y}{a}-\frac{m\pi x}{a}\sinh\frac{m\pi x}{a}\right)\sin\frac{m\pi x}{a}$$ 换坐标

即有：

$$w_2=\frac{a^2}{2\pi^2D}\sum_{m=1,3,\dots}\frac{E_m(-1)^{\frac{m-1}{2}}}{m^2\cosh\alpha_m}\left\{\alpha_m\tanh\alpha_m\cosh\frac{m\pi y}{a}-\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right\}\cos\frac{m\pi x}{a}.$$

$$\text{则}\ \left.\frac{\partial w_2}{\partial y}\right|=\frac{a}{\ }\sum^{\infty}\frac{E_m(-1)^{\frac{m-1}{2}}}{\ }\cdot\left(\tanh\alpha_m+\frac{\alpha_m}{\cosh^2\alpha_m}\right)\cos\frac{m\pi x}{a}.$$

$$\theta\Big|_{y=\frac{b}{2}} = \frac{\partial w_2}{\partial y}\Big|_{y=\frac{b}{2}} = \frac{a}{2\pi D}\sum_{m=1,3,\dots}^{\infty}\frac{E_m(-1)^{\frac{m-1}{2}}}{m}\cdot\left(\tanh\alpha_m + \frac{\alpha_m}{\cosh^2\alpha_m}\right)\cos\frac{m\pi x}{a}.$$

$$\theta\Big|_{x=\frac{a}{2}} = \frac{\partial w}{\partial x} = -\frac{1}{4D}\sum_{m=1,3,\dots}^{\infty}\frac{E_m}{\cosh^2\alpha_m}\left(b\sinh\alpha_m\cosh\frac{m\pi y}{a} - 2y\cosh\alpha_m\sinh\frac{m\pi y}{a}\right)$$

将$\frac{\partial w_2}{\partial x}$沿y方向展开成三角级数,取:

$$b\sinh\alpha_m\cosh\frac{m\pi y}{a} - 2y\cosh\alpha_m\sinh\frac{m\pi y}{a} = \sum_{n=1,3,\dots}^{\infty}A_n\cos\frac{n\pi y}{b}$$

得:A_n表达式为:

$$A_n = \frac{2}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left(b\sinh\alpha_m\cosh\frac{m\pi y}{a} - 2y\cosh\alpha_m\sinh\frac{m\pi y}{a}\right)\cos\frac{m\pi y}{b}\,dy$$

$$= 16\,\frac{\cancel{a}\,n(-1)^{\frac{n-1}{2}}}{m^3\pi^2}\cdot\frac{b^2}{a^{\cancel{2}}}\cdot\frac{1}{[(b/a)^2+(n/m)^2]^2}\cosh^2\alpha_m$$

我们将其代入上式,则原式改写为:

$$\frac{\partial w_2}{\partial x} = -\frac{1}{4D}\sum_{m=1,3,\dots}^{\infty}\frac{E_m}{\cosh^2\alpha_m}A_n\cos\frac{n\pi y}{b} = -\frac{4b^2}{\pi a^2 D}\sum_{m=1,3,\dots}^{\infty}\frac{E_m}{m^3}\cdot\frac{n(-1)^{\frac{n-1}{2}}}{[(b/a)^2+(n/m)^2]^2}\cos\frac{n\pi y}{b}$$

消去

而:$M_x\Big|_{x=\pm\frac{a}{2}} = \sum_{n=1,3,5,\dots}^{\infty}(-1)^{\frac{n-1}{2}}F_n\cos\frac{n\pi y}{b}$,同理得 w_3,$\frac{\partial w}{\partial x_3}$和$\frac{\partial w}{\partial y_3}$(略去),

最终代入边界条件即可。