Euler-Lagrange 方程式推导

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对于泛函数 $I(y)=\int_{x_1}^{x_2} f(x,y,y')\,dx$，当有 $I(y)$ 有极值时，必要条件为 $\delta I=0$

故有：

$$\delta I=\int_{x_1}^{x_2}\delta f(x,y,y')\,dx$$

$$=\int_{x_1}^{x_2}\left[f(x,Y,Y')-f(x,y,y')\right]dx$$

其中：$f(x,Y,Y')$

$=f(x,y+\eta(x),y'+\eta'(x))$，展开得：

$$f(x,y+\epsilon\eta(x),y'+\epsilon\eta'(x))=f(x,y,y')+\epsilon\frac{\partial f}{\partial y}\eta(x)+\epsilon\frac{\partial f}{\partial y'}\eta'(x)+O(\epsilon^2)$$

有：

$$f(x,y+\varepsilon\eta(x),y'+\varepsilon\eta'(x))-f(x,y,y')$$

$$=\varepsilon\left(\frac{\partial f}{\partial y}\eta(x)+\frac{\partial f}{\partial y'}\eta'(x)\right)$$

$$\therefore\delta I=\int_{x_1}^{x_2}\delta f(x,y,y')\,dx=\int_{x_1}^{x_2}\varepsilon\left(\frac{\partial f}{\partial y}\eta(x)+\frac{\partial f}{\partial y'}\eta'(x)\right)dx$$

将 ε 提出，第二项分部有

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial y}\eta(x)\,dx+\cancel{\frac{\partial f}{\partial y}\eta(x)\Big|_{x_1}^{x_2}}-\int_{x_1}^{x_2}\frac{\partial f}{\partial y'\partial x}\eta(x)\,dx$$

我们认为端点处上 $\eta=0\longrightarrow\eta(x_1)=\eta(x_2)=0$.

$$\delta I=\epsilon\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\eta(x)-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\right]dx+O(\epsilon^2)$$

$$=\epsilon\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\,dx+O(\epsilon^2)=0$$

故有：

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$$

故有：

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$$ 为Euler-Lagrange方程.

上式为：$\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)-\frac{\partial f}{\partial y}=0}$，说明：

使用全微分式表示有

$$\frac{d}{dx}\left(\frac{\partial f(x,y,y')}{\partial y'}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right)+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right)\frac{dy}{dx}+\frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right)\frac{dy'}{dx}$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right)+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right)\frac{dy}{dx}+\frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right)\frac{d^2y}{dx^2}.$$

代入Lagrange方程，有：

$$\frac{\partial^2 f}{\partial x\partial y'}+\frac{\partial^2 f}{\partial y\partial y'}\frac{dy}{dx}+\frac{\partial^2 f}{\partial y'^2}\frac{d^2y}{dx^2}-\frac{\partial f}{\partial y}=0$$，整理为$\frac{dy}{dx}$方程

$$\Rightarrow \frac{\partial^2 f}{\partial y'^2}\frac{d^2y}{dx^2}+\frac{\partial^2 f}{\partial y\partial y'}\frac{dy}{dx}+\left(\frac{\partial^2 f}{\partial x\partial y'}-\frac{\partial f}{\partial y}\right)=0$$

其为一个非线性、非齐次的二阶常微分方程式。