薄板的应力公式与平衡方程推导

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由于 $\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$，代入

$$\sigma_x = \frac{-Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_y = \frac{-Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)$$

而：$\tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

$$= \frac{E}{2(1+\nu)} \cdot -2z\frac{\partial^2 w}{\partial x \partial y}$$

$$= -\frac{Ez}{1+\nu} \cdot \frac{\partial^2 w}{\partial x \partial y}$$

由广义胡克定律

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0,$$

∴有：$\frac{\partial \tau_{zx}}{\partial z} = -\frac{\partial \sigma_x}{\partial x} - \frac{\partial \tau_{yx}}{\partial y}$

$$= \frac{Ez}{1-\nu^2}\left(\frac{\partial^3 w}{\partial x^3} + \nu\frac{\partial^3 w}{\partial x \partial y^2}\right) + \frac{Ez}{1+\nu}\frac{\partial^3 w}{\partial x \partial y^2}$$

$$= \frac{Ez}{1-\nu^2} \cdot \frac{\partial^2 w}{\partial x^3} + \frac{Ez \cdot \cancel{\nu}}{1-\nu^2}\frac{\partial^3 w}{\partial x \partial y^2} + \frac{Ez(1-\cancel{\nu})}{1-\nu^2} \cdot \frac{\partial^3 w}{\partial x \partial y^2}$$

$$\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right) = \frac{Ez}{1-\nu^2}\frac{\partial}{\partial x}\nabla^2 w.$$

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}}$$

故：$\tau_{zx} = \int_{\frac{h}{2}}^{z} \frac{Ez}{1-\nu^2}\frac{\partial}{\partial x}\nabla^2 w \, dz$

$$= \frac{E}{2(1-\nu^2)} \cdot \left(z^2 - \frac{h^2}{4}\right)\frac{\partial}{\partial x}\nabla^2 w.$$

$$= \frac{6Q_x}{h^3}\left(z - \frac{h^2}{4}\right)$$

为 τ_{xz} 公式推导
τ_{yz} 同理

此时：容易给出各项应力方程；

⟶ 显然：$N_x = 0$，$N_{xy} = 0$，

而：$Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} \, dz$

$$= -\frac{E}{2(1-\nu^2)}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{h^2}{4} - z^2\right)\frac{\partial}{\partial x}\nabla^2 w$$

$$= -\frac{E}{2(1-\nu^2)}\left(\frac{h^2 z}{4} - \frac{z^3}{3}\right)\Bigg|_{-\frac{h}{2}}^{\frac{h}{2}} \cdot \frac{\partial}{\partial x}\nabla^2 w，代入：$$

$$\left(\frac{h^3}{8} - \frac{h^3}{24}\right) - \left(-\frac{h^3}{8} + \frac{h^3}{24}\right)$$

$$= \frac{h^3}{4} - \frac{h^3}{12} = \frac{h^3}{6},$$

$$= -\frac{E}{2(1-\nu^2)}\left(\frac{h^2}{4}z-\frac{z^3}{3}\right)\Big|_{-\frac{h}{2}}\cdot\frac{\partial}{\partial x}\nabla^2 w \text{，代入：}$$

$$= -\frac{Eh^3}{12(1-\nu^2)}\frac{\partial}{\partial x}\nabla^2 w$$，为横向剪力。

而：$$M_x = -\frac{E}{1-\nu^2}\left[\left(\frac{z^3}{3}\right)\Big|_{-\frac{h}{2}}^{\frac{h}{2}}\right]\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$= -\frac{E}{1-\nu^2}\left(\frac{h^3}{12}\right)\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)$$ 与 σ_x 比较

$$= -\frac{Eh^3}{12(1-\nu^2)}\left[\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right] \Longrightarrow = \frac{h^3}{12}\frac{\sigma_x}{z}$$

$$M_{xy} = -\frac{E}{1+\nu}\left(\frac{z^3}{3}\right)\Big|_{-\frac{h}{2}}^{\frac{h}{2}}\frac{\partial^2 w}{\partial x\partial y} = -\frac{Eh^3}{12(1+\nu)}\cdot\frac{\partial^2 w}{\partial x\partial y} \rightarrow = \frac{h^3}{12}\frac{\tau_{xy}}{z}$$

我们取弯曲刚度 $D=\frac{Eh^3}{12(1-\nu^2)}$ 表达上式，则有：

$$\boxed{\begin{aligned} &Q_x = -D\frac{\partial}{\partial x}(\nabla^2 w) && M_x = D(K_x+\nu K_y)\\ &Q_y = -D\frac{\partial}{\partial y}(\nabla^2 w) && M_y = D(K_y+\nu K_x)\\ & && M_{xy} = D(1-\nu)K_{xy}\end{aligned}}$$

☆弯曲刚度 $D=\frac{Eh^3}{12(1-\nu^2)}$

比较有：

$$\sigma_x = \frac{12M_x z}{h^3},\quad \tau_{xy} = \frac{12M_{xy}z}{h^3}\quad \tau_{xz} = \frac{6Q_x}{h^3}\left(\frac{h^2}{4}-z^2\right)$$

我们需要说明：

Q_x，Q_y 也可以通过平衡方程得到，有：

$$\begin{cases} Q_x = \frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}\\ Q_y = \frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}\end{cases}$$

此时，利用薄板的力平衡方程，有：

$$q\,dx\,dy - Q_x\,dy + \left(Q_x+\frac{\partial Q_x}{\partial x}dx\right)dy - Q_y\,dy + \left(Q_y+\frac{\partial Q_y}{\partial y}dy\right)dx = 0$$

故有：

$$q+\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}=0$$ 代入 $Q_x=-D\frac{\partial}{\partial x}(\nabla^2 w)$，$Q_y=-D\frac{\partial}{\partial y}(\nabla^2 w)$ 得到：

$$q = D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\nabla^2 w)$$，即 $\boxed{\nabla^4 w = \frac{q}{D}}$ 为平衡方程。

$q = D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\nabla^2 w)$，则 $\boxed{\nabla^4 w = \frac{q}{D}}$ [illegible]

我们利用应力方程公式☆，有：

$$M_x + M_y = (1+\nu) D (K_x + K_y)$$

$$= -(1+\nu) D \underbrace{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)}_{\nabla^2 w}$$

我们取 $M = -\frac{M_x + M_y}{1+\nu}$，☆ 则：$\nabla^2 w = -\frac{M_x + M_y}{(1+\nu) D}$

$$= \frac{M}{D},$$

又：$\nabla^4 w = \frac{q}{D}$，

$\therefore \nabla^2(\nabla^2 w) = \frac{\nabla^2 M}{D} = \frac{q}{D}$ 故：$\nabla^2 M = q$.

得：$\begin{cases} \nabla^2 w = \frac{M}{D} \\ \nabla^2 M = q. \end{cases}$

相同表达.