Derivation of Lift Coefficient for symmetric airfoil

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We use the fundamental equation of airfoil theory as:

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{dz}{dx}\right)$$

for a symmetric airfoil, we have chord is equal to zero, or $\frac{dz}{dx}$ is zero, the

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{x-\xi} = V_\infty \alpha,$$

in this equation, we use θ to make transformation for ξ, we have:

$$\xi = \frac{c}{2}(1-\cos\theta), \quad \theta \in (0,\pi)$$

(note θ is from π to 0)

we also consider the x is a specified point on the chord, then we set the θ is equal to θ_0 at location x, or $x = \frac{c}{2}(1-\cos\theta_0)$, the equation becomes:

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)}{\frac{c}{2}(\cos\theta-\cos\theta_0)}\cdot\frac{c}{2}\sin\theta\,d\theta = V_\infty\alpha$$

$$\int_0^\pi \frac{\gamma(\theta)\sin\theta}{\cos\theta-\cos\theta_0}\,d\theta = V_\infty\alpha\cdot 2\pi$$

so we can solute $\gamma(\theta)$ as:

since $V_\infty \alpha \cdot 2\pi = \text{constant}$

$$V_\infty\alpha\cdot 2\pi = \int_0^\pi 2V_\infty\cdot\alpha\cdot d\theta,$$

then an available solution can be found as:

$$\int_0^\pi \left[\frac{\gamma(\theta)\sin\theta}{\cos\theta-\cos\theta_0} - 2V_\infty\alpha\right]d\theta = 0,$$

hence, we can use

$$\boxed{\gamma(\theta) = \frac{2V_\infty\alpha}{\sin\theta}(\cos\theta-\cos\theta_0)}$$

as solution,

in this equation, $\gamma(\theta)$ is a function related to θ but have no relationship with θ_0, however, the $\cos\theta_0$ satisfies for arbitary θ_0, then we need to find a value for θ_0 at the eq above to satisfy all the solution.

※ we give a important standard integral which is frequently used in airfoil theory as:

$$\bigstar \quad \underbrace{\int_0^\pi \frac{\cos n\theta}{\cos\theta-\cos\theta_0}\,d\theta = \pi\frac{\sin n\theta_0}{\sin\theta_0}}$$

then we set $\cos\theta_0 = C$

$$\int_0^\pi \frac{\cancel{\sin\theta}}{\cos\theta-\cos\theta_0}\cdot\frac{2V_\infty\alpha}{\cancel{\sin\theta}}(\cos\theta - C) = 2V_\infty\alpha.$$

$$\int_0 \frac{}{\cos\theta-\cos\theta_0}\cdot\frac{}{\sin\theta}(\cos\theta - C) = 2V_\infty\alpha$$

or $\cancel{2V_\infty\alpha}\int_0^\pi \frac{\cos\theta - C}{\cos\theta-\cos\theta_0} = \cancel{2V_\infty\alpha} \Rightarrow \int_0^\pi \frac{\cos\theta}{\cos\theta-\cos\theta_0} - C\int_0^\pi \frac{1}{\cos\theta-\cos\theta_0} = 0,$

we know it easily from ☆ that

$$\int_0^\pi \frac{\cos\theta}{\cos\theta-\cos\theta_0} = \pi\frac{\sin\theta_0}{\sin\theta_0} = \pi, \text{ and } \int_0^\pi \frac{1}{\cos\theta-\cos\theta_0} = 0.$$

← when $n=0$

hence the equation satisfies for all $C = \cos\theta_0 \in [-1, 1]$

it seems the choose of θ_0 is arbitary, However, we consider the Kutta Condition at trailing edge, which gives $\gamma(\theta=\pi)=0$, then $\lim_{x\to\pi}\frac{2V_\infty\alpha}{\sin x}(\cos x - \cos\theta_0) = 0$,

yields $\theta_0 = \pi$, $C = \cos\theta_0 = -1$.

then strength of circulation $\gamma(s)$ becomes:

$$\boxed{\gamma = \frac{2V_\infty\alpha}{\sin\theta}(\cos\theta+1)}$$

at location $\theta = \pi$, $\gamma = \lim_{x\to\pi}\frac{2V_\infty\alpha}{\sin\theta}(\cos\theta+1) = 0$ (Kutta Condition is satisfied).

then we calulate the lift coefficient for a thin and symmetric airfoil.

the circulation is derived by integral $\Rightarrow$

$$\Gamma = \int_0^c \gamma(\xi)\,d\xi = \int_0^\pi \gamma(\theta)\cdot\frac{c}{2}\sin\theta\,d\theta$$

$$= \int_0^\pi \frac{\cancel{2}}{\cancel{2}} cV_\infty\alpha(\cos\theta+1)\,d\theta = \boxed{\pi V_\infty \alpha c}$$

then the Lift per unit span is given:

$$L' = \rho_\infty V_\infty \Gamma = \rho_\infty V_\infty^2 \pi\alpha c,$$

lift coefficient: $C_l = \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 S} = \frac{2\pi\alpha c}{S} \xrightarrow{S = c\cdot(1)} \boxed{2\pi\alpha}$.

and the lift slope $= \boxed{\frac{dC_l}{d\alpha} = 2\pi}$