厚壁球壳应力公式推导

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已知平衡方程: $\frac{\partial\sigma_r}{\partial r}+2\frac{\sigma_r-\sigma_\theta}{r}=0$ 而协调方程为: $\frac{d\varepsilon_\theta}{dr}+\frac{\varepsilon_\theta-\varepsilon_r}{r}=0$

本构方程:
$$\begin{cases}\varepsilon_r=\frac{1}{E}(\sigma_r-2\nu\sigma_\theta)=\frac{du}{dr},\\ \varepsilon_\varphi=\varepsilon_\theta=\frac{1}{E}(\sigma_\theta-\nu(\sigma_r+\sigma_\theta))=\frac{u}{r},\end{cases}$$

代入本构方程, 则协调方程可表示为:

$$\frac{1}{E}\frac{d\sigma_\theta}{dr}-\frac{\nu}{E}\frac{d(\sigma_r+\sigma_\theta)}{dr}+\frac{1}{rE}(\sigma_\theta-\nu(\sigma_r+\sigma_\theta)-\sigma_r+2\nu\sigma_\theta)=0$$

$$\frac{1-\nu}{E}\frac{d\sigma_\theta}{dr}-\frac{\nu}{E}\frac{d\sigma_r}{dr}+\frac{1}{rE}(\sigma_\theta(1+\nu)-\sigma_r(1+\nu))=0$$

则: $\boxed{\frac{(1-\nu)d\sigma_\theta-\nu d\sigma_r}{dr}+\frac{(1+\nu)}{r}(\sigma_\theta-\sigma_r)=0}$ 为协调方程

此时: 代入平衡方程: $\frac{d\sigma_r}{dr}+2\frac{\sigma_r-\sigma_\theta}{r}=0$

则有:
$$\frac{\sigma_\theta-\sigma_r}{r}=\frac{1}{2}\frac{d\sigma_r}{dr}$$

$$\therefore\ \frac{(1-\nu)\,d\sigma_\theta}{dr}=\frac{\nu\, d\sigma_r}{dr}-(1+\nu)\frac{\sigma_\theta-\sigma_r}{r}$$

$$\frac{(1-\nu)d\sigma_\theta}{dr}=\frac{\nu\, d\sigma_r}{dr}-\frac{(1+\nu)}{2}\frac{d\sigma_r}{dr}$$

$$\frac{(1-\nu)d\sigma_\theta}{dr}=\frac{-1+\nu}{2}\frac{d\sigma_r}{dr}$$

即: $\cancel{(1-\nu)}\frac{(d\sigma_\theta+\frac{1}{2}d\sigma_r)}{dr}=0$

$$\frac{d\sigma_r}{dr}+2\frac{\sigma_r-\sigma_\theta}{r}=0$$

$\boxed{\frac{d\sigma_\theta}{dr}=-\frac{1}{2}\frac{d\sigma_r}{dr}}$ ☆, 则: 平衡方程改写为:

则凑成形式: $\left[\frac{d(\sigma_\theta-\sigma_r)}{dr}=-\frac{3}{2}\frac{d\sigma_r}{dr}\right]$

$$\frac{2}{3}\frac{d(\sigma_r-\sigma_\theta)}{dr}+\frac{2(\sigma_r-\sigma_\theta)}{r}=0$$

可代换 $\frac{d\sigma_r}{dr} = \frac{2}{3}\frac{d(\sigma_r-\sigma_\theta)}{dr}$

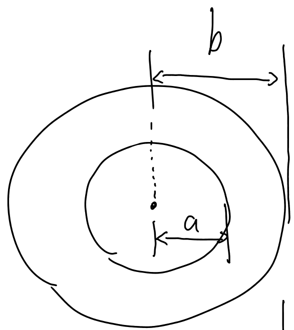
令 $\sigma_r-\sigma_\theta = t$,

$$\frac{2}{3}\frac{dt}{dr}+\frac{2t}{r}=0$$

$$\therefore \frac{dt}{dr} = -\frac{3t}{r} \Rightarrow \frac{1}{t}dt = -\frac{3}{r}dr$$

$$\ln t = -3\ln Cr \quad 设C=3B,$$

$$t = Cr^{-3}, = \frac{3B}{r^3}$$



则得: $\sigma_r-\sigma_\theta = \frac{3B}{r^3}$, 因此, σ_r, σ_θ 可以写成:

$$\boxed{\sigma_r = A - \frac{2B}{r^3} \qquad \sigma_\theta = A + \frac{B}{r^3}}$$

由: $\boxed{\sigma_r\big|_{r=a} = -P_{internal} = -P_i}$ $\sigma_r\big|_{r=b} = -P_{external} = -P_e$

有: $A - \frac{2B}{a^3} = -P_i$ $\qquad \therefore 2B\left(\frac{1}{b^3}-\frac{1}{a^3}\right) = P_e - P_i$

$A - \frac{2B}{b^3} = -P_e$,

$$B = \frac{1}{2}(P_e-P_i)\frac{a^3b^3}{a^3-b^3} = \frac{1}{2}(P_e-P_i)\frac{b^3}{1-\left(\frac{b}{a}\right)^3}$$

$\therefore A = -P_i + \frac{(P_e-P_i)b^3}{(a^6)}$ $\qquad$ 则: $A = -P_e + \frac{a^3}{a^3-b^3}(P_e-P_i)$

$$= \boxed{\frac{b^3P_e - a^3P_i}{a^3-b^3}}$$

Lamé解 参考课本.

得: $\sigma_r = \frac{b^3P_e - a^3P_i}{a^3-b^3} - \frac{(P_e-P_i)b^3}{r^3\left(1-\left(\frac{b}{a}\right)^3\right)}$

$$= \boxed{\frac{\left(\frac{b}{a}\right)^3P_e - P_i}{1-\left(\frac{b}{a}\right)^3} - \frac{(P_e-P_i)\left(\frac{b^3}{r^3}\right)}{1-\left(\frac{b^3}{a^3}\right)}}$$

若: $P_e=0 \Rightarrow \boxed{\frac{-P_i\left(1-\frac{b^3}{r^3}\right)}{1-\frac{b^3}{a^3}}} = \boxed{-P\frac{\left(\frac{b^3}{r^3}-1\right)}{\left(\frac{b^3}{a^3}-1\right)}}$

$$\sigma_\theta = A + \frac{B}{r^2} = \frac{\left(\frac{b}{a}\right)^3P_e - P_i}{1-\left(\frac{b}{a}\right)^3} + \frac{(P_e-P_i)\frac{b^3}{r^3}}{2\left[1-\left(\frac{b}{a}\right)^3\right]}$$

当 $P_e=0$时 $\Longrightarrow$ 则: $\sigma_\theta \Longrightarrow \frac{-2P_i - P_i\frac{b^3}{r^3}}{2\left[1-\left(\frac{b}{a}\right)^3\right]} = \boxed{-P_i\frac{\left(2+\frac{b^3}{r^3}\right)}{2\left[1-\left(\frac{b}{a}\right)^3\right]}} = \boxed{P_i\frac{\left[\frac{b^3}{2r^3}+1\right]}{\left[\frac{b^3}{a^3}-1\right]}}$