平面问题的按位移解法和按应力解法

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平面问题的两种基本求解：

1) 平衡方程：
$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x}+\dfrac{\partial \tau_{xy}}{\partial y}+f_x=0 \\ \dfrac{\partial \tau_{xy}}{\partial x}+\dfrac{\partial \sigma_y}{\partial y}+f_y=0 \end{cases}$$

2) 几何方程：$\varepsilon_x=\dfrac{\partial u}{\partial x}$ $\varepsilon_y=\dfrac{\partial v}{\partial y}$ $\gamma_{xy}=\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}$

→ 协调方程：$\left(\dfrac{\partial \varepsilon_x}{\partial y^2}+\dfrac{\partial \varepsilon_y}{\partial x^2}=\dfrac{\partial \gamma_{xy}}{\partial x\partial y}\right)$

3) 物理方程：(平面应力)
$$\begin{cases} \varepsilon_x=\dfrac{1}{E}(\sigma_x-\nu\sigma_y) \\ \varepsilon_y=\dfrac{1}{E}(\sigma_y-\nu\sigma_x) \\ \gamma_{xy}=\dfrac{\tau_{xy}}{G}=\dfrac{2(1+\nu)}{E}\tau_{xy} \end{cases}$$

边界
$$\begin{cases} u=\bar{u} \\ v=\bar{v} \end{cases}$$

$\gamma_{xy}=\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}$

由几何方程容易确定协调方程；

① 按位移求解，以 u, v 为基本未知量，

(1) 按位移表达平衡方程： 由于
$$\begin{cases} \sigma_x=\dfrac{E}{1-\nu^2}(\varepsilon_x+\nu\varepsilon_y) \\ \sigma_y=\dfrac{E}{1-\nu^2}(\varepsilon_y+\nu\varepsilon_x) \\ \tau_{xy}=\dfrac{E}{2(1+\nu)}\gamma_{xy} \end{cases}$$
，代入有：

$\dfrac{1-\nu}{2}+\nu$

$$\begin{cases} \dfrac{E}{1-\nu^2}\cdot\left(\dfrac{\partial \varepsilon_x}{\partial x}+\nu\dfrac{\partial \varepsilon_y}{\partial x}\right)+\dfrac{E}{2(1+\nu)}\dfrac{\partial \gamma_{xy}}{\partial y}+f_x=0 \\ \dfrac{E}{1-\nu^2}\left(\dfrac{\partial \varepsilon_y}{\partial y}+\nu\dfrac{\partial \varepsilon_x}{\partial y}\right)+\dfrac{E}{2(1+\nu)}\cdot\dfrac{\partial \gamma_{xy}}{\partial x}+f_y=0 \end{cases}$$

$\xrightarrow{\text{代换}}$

$$\boxed{\begin{cases} \dfrac{E}{1-\nu^2}\left[\dfrac{\partial^2 u}{\partial x^2}+\dfrac{(1-\nu)}{2}\dfrac{\partial^2 u}{\partial y^2}+\dfrac{1+\nu}{2}\dfrac{\partial^2 v}{\partial x\partial y}\right]+f_x=0 \\ \dfrac{E}{1-\nu^2}\left[\dfrac{\partial^2 v}{\partial y^2}+\dfrac{(1-\nu)}{2}\dfrac{\partial^2 v}{\partial x^2}+\dfrac{1+\nu}{2}\dfrac{\partial^2 u}{\partial x\partial y}\right]+f_y=0 \end{cases}}$$
(平衡方程)， ①

(2) 边界条件：
$$\begin{cases} l\sigma_x+m\tau_{xy}=\bar{f}_x \\ l\tau_{xy}+m\sigma_y=\bar{f}_y \end{cases}$$

代入有边界条件：

$$\boxed{\begin{cases} \dfrac{E}{1-\nu^2}\left[l\left(\dfrac{\partial u}{\partial x}+\nu\dfrac{\partial v}{\partial y}\right)+m\dfrac{1-\nu}{2}\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)\right]=\bar{f}_x\,(s) \\ \dfrac{E}{1-\nu^2}\left[l\dfrac{1-\nu}{2}\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)+m\left(\dfrac{\partial v}{\partial y}+\nu\dfrac{\partial u}{\partial x}\right)\right]=\bar{f}_y\,(s) \end{cases}}$$
②

③：$u=\bar{u}$，$v=\bar{v}$

故①②③为按位移求解平面问题基本方程

② ：按应力求解：以 σ_x, σ_y, τ_{xy} 为基本未知量。

平衡方程：
$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x}+\dfrac{\partial \tau_{xy}}{\partial y}+f_x=0 & ① \\ \dfrac{\partial \tau_{xy}}{\partial x}+\dfrac{\partial \sigma_y}{\partial y}+f_y=0, \end{cases}$$

协调方程：
$$\dfrac{\partial^2 \varepsilon_x}{\partial y^2}+\dfrac{\partial^2 \varepsilon_y}{\partial x^2}=\dfrac{\partial^2 \gamma_{xy}}{\partial x\partial y}$$

其中：$\gamma_{xy}=\dfrac{2(1+\nu)}{E}\tau_{xy}$ 由平衡方程：

$\rightarrow \tau_{xy}=\dfrac{E}{2(1+\nu)}\gamma_{xy}$

代入：
$$\begin{cases} \varepsilon_x=\dfrac{1}{E}(\sigma_x-\nu\sigma_y) \\ \ldots \end{cases}$$

代入：
$$\begin{cases} \varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \\ \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \\ \gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E}\tau_{xy} \end{cases}$$

$$\frac{\partial \tau_{xy}}{\partial y} = -\frac{\partial \sigma_x}{\partial x} - f_x$$

$$\frac{\partial \tau_{xy}}{\partial x} = -\frac{\partial \sigma_y}{\partial y} - f_y$$

$$\rightarrow \tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

则：$\frac{\partial \sigma_x}{\partial x} + \frac{E}{2(1+\nu)}\frac{\partial \gamma_{xy}}{\partial y} + f_x = 0$

即：$\frac{\partial \gamma_{xy}}{\partial y} = -\frac{2(1+\nu)}{E}\frac{\partial \sigma_x}{\partial x} - \frac{2(1+\nu)}{E}f_x$

则：$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = -\frac{2(1+\nu)}{E}\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{2(1+\nu)}{E}\frac{\partial f_x}{\partial x}$. ①

同理：$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = -\frac{2(1+\nu)}{E}\frac{\partial^2 \sigma_y}{\partial y^2} - \frac{2(1+\nu)}{E}\frac{\partial f_y}{\partial y}$ ②

①+②：$\boxed{\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = -\frac{1+\nu}{E}\left[\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right) + \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)\right]}$

代入协调方程：有：

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\cancel{\frac{1}{E}}\left(\frac{\partial^2 \sigma_x}{\partial y^2} - \nu\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - \nu\frac{\partial^2 \sigma_x}{\partial x^2}\right) = -\frac{1+\nu}{\cancel{E}}\left[\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right) + \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)\right]$$

$$\frac{\partial^2 \sigma_x}{\partial y^2} - \nu\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - \nu\frac{\partial^2 \sigma_x}{\partial x^2} + (1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right) = -(1+\nu)\left[\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right]$$

化简，有：

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1+\nu)\left[\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right]$$ 为平面应力相容方程

②

代 $\nu = \frac{\nu}{1-\nu}$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left[\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right]$$ 为平面应变相容方程，

③：$\begin{cases} l\sigma_x + m\tau_{xy} = \bar{p}_x \\ l\tau_{xy} + m\sigma_y = \bar{p}_y \end{cases}$ 为边界条件

①②③ 按应力求解平面问题的基本方程。