轴对称问题相容方程的求解过程

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$$\nabla^4\varphi = \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2\varphi=0$$

由于 $\frac{\partial\varphi}{\partial\theta}=0$，代简：$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\varphi}{dr^2}+\frac{1}{r}\frac{d\varphi}{dr}\right)=0$

注意：其中计算时，不可直接相乘，如：$\frac{1}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\varphi}{dr}\right)$ 需使用复合函数求导法则。

得到：$\frac{d^4\varphi}{dr^4}+\frac{1}{r}\frac{d^3\varphi}{dr^3}+\underline{\frac{d}{dr^2}\left(\frac{1}{r}\frac{d\varphi}{dr}\right)}+\frac{1}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\varphi}{dr}\right)$

其中：$\frac{d}{dr^2}\left(\frac{1}{r}\frac{d\varphi}{dr}\right)=\frac{d}{dr}\left(-\frac{1}{r^2}\frac{d\varphi}{dr}+\frac{1}{r}\frac{d^2\varphi}{dr^2}\right)$

$$=\frac{2}{r^3}\frac{d\varphi}{dr}-\frac{1}{r^2}\frac{d^2\varphi}{dr^2}-\frac{1}{r^2}\frac{d^2\varphi}{dr^2}+\frac{1}{r}\frac{d^3\varphi}{dr^3}$$

而：$\frac{1}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\varphi}{dr}\right)=\frac{1}{r}\left(-\frac{1}{r^2}\frac{d\varphi}{dr}+\frac{1}{r}\frac{d^2\varphi}{dr^2}\right)$

注意：r,φ' 都必须为什，

此时：原式=

$$\frac{d^4\varphi}{dr^4}+\frac{1}{r}\frac{d^3\varphi}{dr^3}+\frac{2}{r^3}\frac{d\varphi}{dr}-\cancel{\frac{1}{r^2}\frac{d^2\varphi}{dr^2}}-\frac{1}{r^2}\frac{d^2\varphi}{dr^2}+\frac{1}{r}\frac{d^3\varphi}{dr^3}-\frac{1}{r^3}\frac{d\varphi}{dr}+\cancel{\frac{1}{r^2}\frac{d^2\varphi}{dr^2}}$$

得：$\frac{d^4\varphi}{dr^4}+\frac{2}{r}\frac{d^3\varphi}{dr^3}-\frac{1}{r^2}\frac{d^2\varphi}{dr^2}+\frac{1}{r^3}\frac{d\varphi}{dr}=0$，两边同乘 r^4，

即，Euler 齐次微分方程的形式：

$$r^4\frac{d^4\varphi}{dr^4}+2r^3\frac{d^3\varphi}{dr^3}-r^2\frac{d^2\varphi}{dr^2}+r\frac{d\varphi}{dr}=0$$

$\boxed{\text{解决方法：令 } r=e^t}$ → $t=\ln r$

☆ 则：$\frac{d\varphi}{dr}=\frac{d\varphi}{dt}\cdot\frac{dt}{dr}=\frac{1}{r}\frac{d\varphi}{dt}$

则：$\frac{d^2\varphi}{dr^2}=-\frac{1}{r^2}\frac{d\varphi}{dt}+\frac{1}{r^2}\frac{d^2\varphi}{dt^2}=\frac{1}{r^2}\left(\frac{d^2\varphi}{dt^2}-\frac{d\varphi}{dt}\right)$

$$\frac{d^3\varphi}{dr^3}=-\frac{2}{r^3}\left(\frac{d^2\varphi}{dt^2}-\frac{d\varphi}{dt}\right)+\frac{1}{r^3}\left(\frac{d^3\varphi}{dt^3}-\frac{d^2\varphi}{dt^2}\right)=\frac{1}{r^3}\left(\frac{d^3\varphi}{dt^3}-\frac{3d^2\varphi}{dt^2}+\frac{2d\varphi}{dt}\right)$$

$$\frac{d^4\varphi}{dr^4}=\frac{1}{r^4}\left[\left(\frac{d^4\varphi}{dt^4}-\frac{3d^3\varphi}{dt^3}+\frac{2d^2\varphi}{dt^2}\right)-3\left(\frac{d^3\varphi}{dt^3}-\frac{3d^2\varphi}{dt^2}+\frac{2d\varphi}{dt}\right)\right]$$

$$\frac{d^4\varphi}{dr^4} = \frac{1}{r^4}\left[\left(\frac{d^4\varphi}{dt^4} - \frac{3d^3\varphi}{dt^3} + \frac{2d^2\varphi}{dt^2}\right) - 3\left(\frac{d^3\varphi}{dt^3} - \frac{3d^2\varphi}{dt^2} + \frac{2d\varphi}{dt}\right)\right]$$

$$= \frac{1}{r^4}\left[\frac{d^4\varphi}{dt^4} - \frac{6d^3\varphi}{dt^3} + \frac{11d^2\varphi}{dt^2} - \frac{6d\varphi}{dt}\right]$$

此时全部代入:

$$\frac{d^4\varphi}{dt^4} - \frac{6d^3\varphi}{dt^3} + \frac{11d^2\varphi}{dt^2} - \cancel{\frac{6d\varphi}{dt}} + \frac{2d^3\varphi}{dt^3} - \frac{6d^2\varphi}{dt^2} + \cancel{\frac{4d\varphi}{dt}} - \frac{d^2\varphi}{dt^2} + \cancel{\frac{d\varphi}{dt} + \frac{d\varphi}{dt}} = 0$$

化简得 $\boxed{\frac{d^4\varphi}{dt^4} - 4\frac{d^3\varphi}{dt^3} + 4\frac{d^2\varphi}{dt^2} = 0}$ 　因 $\lambda^2 - 4\lambda + 4 = 0$

$\therefore \lambda_1 = \lambda_2 = 2$,

即: $\frac{d^2\varphi}{dt^2} = Ae^{2t} + Bte^{2t}$, 其中: A, B 为常数

则: $\varphi = Ae^{2t} + Bte^{2t} + Ct + D$ 　(打同)

$$\downarrow$$

$$At + Bte^{2t} + Ce^{2t} + D$$

$\therefore$ 因: $t = \ln r$

$\boxed{\text{则: } \varphi = A\ln r + Br^2\ln r + Cr^2 + D}$, 为应力函数初始形式。

此时:

$$\begin{cases} \sigma_r = \frac{1}{r}\frac{d\varphi}{dr} = \frac{1}{r}\left(\frac{A}{r} + 2Br\ln r + Br + 2Cr\right) \\ \quad = \frac{A}{r^2} + 2B\ln r + (B+2C) = \frac{A}{r^2} + B(1+2\ln r) + 2C \\ \sigma_\theta = \frac{d^2\varphi}{dr^2} = -\frac{A}{r^2} + 2B\ln r + 2B + B + 2C \\ \quad = -\frac{A}{r^2} + B(3+2\ln r) + 2C \\ \tau_{r\theta} = \tau_{\theta r} = 0 \end{cases}$$

为应力表达式,