奇异摄动法推导Duffing方程自由振动

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解：Duffing 方程的自由振动为：

$$\boxed{\ddot{x}+\omega_0^2(x+\varepsilon x^3)=0} \quad ①$$

对于奇异摄动法：设 x 可展开为：（展到二阶）

$$x=x_0+\varepsilon x_1+\varepsilon^2 x_2+\cdots\cdots$$

$$\text{则}\ \varepsilon x^3=\varepsilon\cdot\left(x_0^3+\underbrace{3\varepsilon x_0^2 x_1}_{\text{一次项}}+\underbrace{3\varepsilon^2 x_1^2 x_0+3\varepsilon^2 x_0^2 x_2}_{\text{二次项}}+o(\varepsilon^3)\right)$$

由于ε在括号外

仅考虑零次与一次项，则：$\varepsilon x^3=\varepsilon x_0^3+3\varepsilon^2 x_0^2 x_1$

其中：为了避免久期项问题，可以将频率进行展开：

设 $\omega=\omega_0+\varepsilon\omega_1+\varepsilon^2\omega_2+\cdots\cdot$

则有：$\omega^2=\omega_0^2(1+\varepsilon\sigma_1+\varepsilon^2\sigma_2+\cdots+\cdots)$ 其中，σ_1为ε一次项系数，

即 $\sigma_1=\omega_0^2$，$\sigma_2=2\omega_0\omega_1+\omega_1^2$ 为二次项，…… 以此类推。

我们引入新的自变量 $\psi=\omega t$，此时：有：

$$\dot{x}=\frac{dx}{dt}=\frac{dx}{d\psi}\frac{d\psi}{dt}=\omega\frac{dx}{d\psi},$$

$$\therefore \ddot{x}=\frac{d}{dt}\left(\omega\frac{dx}{d\psi}\right)=\cancel{\frac{d\omega}{dt}}^{=0}\frac{dx}{d\psi}+\omega\frac{d}{dt}\left(\frac{dx}{d\psi}\right)=\omega\frac{d^2x}{d\psi^2}\frac{d\psi}{dt}=\omega^2\frac{d^2x}{d\psi^2}$$

而：Duffing 方程的自由振动为：

$\ddot{x}+\omega_0^2(x+\varepsilon x^3)=0$，代入上式：

$$\omega^2\frac{d^2x}{d\psi^2}+\omega_0^2(x+\varepsilon x^3)=0$$ ⟵ 在此式子中，即可代入对应的ω展开式

代入：$\omega_0^2(1+\varepsilon\sigma_1+\varepsilon^2\sigma_2+\varepsilon^3\sigma_3+\cdots)\frac{d^2x}{d\psi^2}+\omega_0^2(x+\varepsilon x^3)=0$

则有：

$$\omega_0^2(1+\varepsilon\sigma_1+\varepsilon^2\sigma_2)\cdot(\ddot{x}_0+\varepsilon\ddot{x}_1+\varepsilon^2\ddot{x}_2+\cdots)+\omega_0^2(x_0+\varepsilon x_1+\varepsilon^2 x_2+\varepsilon x_0^3+3\varepsilon^2 x_0^2 x_1)=0$$

①：零次项系数：$\omega_0^2\ddot{x}_0+\omega_0^2 x_0=0$ → 即：$\boxed{\ddot{x}_0+x_0=0}$. ①

②：一次项系数：$\omega_0^2\varepsilon\ddot{x}_1+\omega_0^2\varepsilon\sigma_1\ddot{x}_0+\omega_0^2\varepsilon x_1+\omega_0^2\varepsilon x_0^3=0$

即，$\ddot{x}_1+\sigma_1\ddot{x}_0+x_1+x_0^3=0$

即：$\boxed{\ddot{x}_1+x_1=-(\sigma_1\ddot{x}_0+x_0^3)}$ ②

③：二次项系数；$\omega_0^2\varepsilon^2\sigma_2\ddot{x}_0+\omega_0^2\varepsilon^2\sigma_1\ddot{x}_1+\omega_0^2\varepsilon^2\ddot{x}_2$

$+\omega_0^2\varepsilon^2 x_2+3\varepsilon^2\omega_0^2 x_0^2 x_1=0$

即：$\sigma_2\ddot{x}_0+\sigma_1\ddot{x}_1+\ddot{x}_2+x_2+3x_0^2 x_1=0$

则：$\boxed{\ddot{x}_2+x_2=-(\sigma_2\ddot{x}_0+\sigma_1\ddot{x}_1+3x_0^2 x_1)}$ ③

代入初始条件为 $x_0(0)=A$，$x_1(0)=0$，$x_2(0)=0$

$\dot{x}_0(0)=0$, $\dot{x}_1(0)=0$ $\dot{x}_2(0)=0$,

则：显然①方程解为：

$x_0 = A\cos\psi$ 而：$\ddot{x}_1 + x_1 = -(A\sigma_1\cos\psi + A^3\cos^3\psi)$

或：$\ddot{x}_1 + x_1 = -\left[(A\sigma_1 + \frac{3A^3}{4})\cos\psi + \frac{1}{4}A^3\cos 3\psi\right]$

$\cos^3\psi = \frac{3}{4}\cos\psi + \frac{1}{4}\cos 3\psi$

取 $x_1 = B_1\cos\psi + B_2\cos 3\psi$

∴有：$\ddot{x}_1 + x_1 = \cancel{-B_1\cos\psi} - 9B_2\cos 3\psi + \cancel{B_1\cos\psi} + B_2\cos 3\psi$ =右：

则：将右移至左侧，有：

$$(A\sigma_1 + \frac{3A^3}{4})\cos\psi + (\frac{1}{4}A^3 - 8B_2)\cos 3\psi$$

为保证久期项为0，则 $A\sigma_1 + \frac{3A^3}{4} = 0$，$\sigma_1 = -\frac{3A^2}{4}$，

☆② $\frac{1}{4}A^3 - 8B_2 = 0$，有 $B_2 = \frac{A^3}{32}$，

此时：$\boxed{x_1 = B_1\cos\psi + \frac{A^3}{32}\cos 3\psi}$ ②的解，代入初始条件：$B_1 = -\frac{A^3}{32}$

代入③，即：$\rightarrow \ddot{x}_1 = \frac{A^3}{32}\cos\psi - \frac{9A^3}{32}\cos 3\psi$

$\ddot{x}_2 + x_2 = -[\sigma_2\ddot{x}_0 + \sigma_1\ddot{x}_1 + 3x_0^2x_1]$

此时有：

$$\ddot{x}_2 + x_2 = -\left[-\sigma_2 A\cos\psi + \frac{A^3}{32}\sigma_1\cos\psi - \sigma_1\frac{9A^3}{32}\cos 3\psi + 3A^2\cos^2\psi\left(-\frac{A^3}{32}\cos\psi + \frac{A^3}{32}\cos 3\psi\right)\right]$$

其中参考推导：$\cos^2\psi\cos 3\psi = \frac{1}{4}\cos 5\psi + \frac{1}{2}\cos 3\psi + \frac{1}{4}\cos\psi$

放入负号

有：

$$\ddot{x}_2 + x_2 = \sigma_2 A\cos\psi - \frac{A^3\sigma_1}{32}\cos\psi + \frac{9A^3\sigma_1}{32}\cos 3\psi + \frac{3A^5}{32}\left(\frac{1}{4}\cos\psi + \frac{3}{4}\cos 3\psi\right) - \frac{3A^5}{32}\left(\frac{1}{4}\cos 5\psi + \frac{1}{2}\cos 3\psi + \frac{1}{4}\cos\psi\right)$$

代入：$\sigma_1 = -\frac{3A^2}{4}$ 注意负号！

$$\rightarrow \left(\sigma_2 A + \frac{3A^5}{128} + \cancel{\frac{3A^5}{128}} - \cancel{\frac{3A^5}{128}}\right)\cos\psi + \left(-\frac{27A^5}{128} + \frac{9A^5}{128} - \frac{6A^5}{128}\right)\cos 3\psi - \frac{3A^5}{128}\cos 5\psi$$

首先为了避免久期项 有：$\sigma_2 A + \frac{3A^5}{128} = 0$，则 $\sigma_2 = -\frac{3A^4}{128}$，

则：取：$x_2 = C_1\cos\psi + C_2\cos 3\psi + C_3\cos 5\psi$，显然 C_1 由初始条件获得：

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$$\boxed{C_2\cos 3\psi - 9C_2\cos 3\psi} = \cos 3\psi\left(-\frac{27A^5}{128} + \frac{9A^5}{128} - \frac{6A^5}{128}\right) = -\frac{\overset{3}{\cancel{24}}A^5}{\underset{16}{\cancel{128}}}\cos 3\psi$$

$\overset{\shortparallel}{-8C_2}$

$$\boxed{C_2 = \frac{3A^5}{128}}$$

而：$C_3(1-25) = -\frac{3A^5}{128}$ $\Rightarrow$ $\boxed{C_3 = \frac{A^5}{1024}}$

$C_3 \times \overset{8}{\cancel{24}} = +\frac{\cancel{3}A^5}{128} \rightarrow C_3 = \frac{A^5}{1024}$

由 $x_2(0) = 0$ 得 $C_1 = -\frac{3A^5}{128} - \frac{A^5}{1024} = -\frac{25A^5}{1024}$

则：代入：

则：

$x_2 = -\frac{25A^5}{1024}\cos\psi + \frac{3A^5}{128}\cos 3\psi + \frac{A^5}{1024}\cos 5\psi$

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$$x_2 = -\frac{25A^5}{1024}\cos\psi + \frac{3A^5}{128}\cos3\psi + \frac{A^5}{1024}\cos5\psi$$

则：代入：

$$X = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \cdots$$

$$= A\cos\psi + \varepsilon\left(-\frac{A^3}{32}\cos\psi + \frac{A^3}{32}\cos3\psi\right) + \varepsilon^2\left(-\frac{25A^5}{1024}\cos\psi + \frac{3A^5}{128}\cos3\psi + \frac{A^5}{1025}\cos5\psi\right) + \cdots\cdots$$