Derivation of Nonlinear steady problem

Sunday, March 26, 2023 10:28 AM

the initial equation of the problem is:

$$\boxed{\alpha \frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x} = -P}$$

firstly we write it as:

$$\int \left[\alpha \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) + P \right] \delta u \, d\Omega = 0$$

where u is the speed of fluid

considering

$$\int_a^b \delta u \frac{\partial^2 u}{\partial x^2} dx = \left(\frac{\partial u}{\partial x}\delta u\right)\Big|_a^b - \int_a^b \frac{\partial(\delta u)}{\partial x}\frac{\partial u}{\partial x}dx .$$

then eq becomes:

$$\int_\Omega \left(\alpha \frac{\partial(\delta u)}{\partial x}\frac{\partial u}{\partial x} - P\delta u\right) d\Omega = \alpha\left(\frac{\partial u}{\partial x}\delta u\right)\Big|_a^b - \int_\Omega \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) d\Omega$$

$$F(x) = \int_a^b f(x)dx$$

also we set Γ_2 as the natural boundary condition,

$\longrightarrow$ then on Γ_1, $\frac{\partial u}{\partial x} = 0$,

partial integral

then we have $\boxed{\int_\Omega \delta u \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right)d\Omega = \frac{1}{2}u^2\delta u\Big|_a^b - \int_\Omega \frac{1}{2}u^2\frac{\partial(\delta u)}{\partial x}}$

we note here that: for 1-D problem

$$F(b) - F(a) = \int_\Gamma f(x)\, d\Gamma$$

hence:

$$\int_\Omega \left(\alpha\frac{\partial(\delta u)}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}u^2\frac{\partial(\delta u)}{\partial x} - P\delta u\right) d\Omega = \left(\alpha\frac{\partial u}{\partial x}\delta u - \frac{1}{2}u^2\delta u\right)\Big|_a^b$$

the right term is equal to $\int_\Gamma \left(\alpha\frac{\partial u}{\partial x} - \frac{1}{2}u^2\right)\delta u \, d\Gamma$

where $\Gamma = \Gamma_1 + \Gamma_2$, and $f(x)$ in Γ_1 is zero

then the weak form is:

$$\boxed{\int_\Omega \left(\alpha\frac{\partial(\delta u)}{\partial x}\frac{\partial u}{\partial x} - \frac{1}{2}u^2\frac{\partial(\delta u)}{\partial x} - P\delta u\right) d\Omega = \int_\Gamma \left(\alpha\frac{\partial u}{\partial x} - \frac{1}{2}u^2\right)\delta u \, d\Gamma}$$

we substitute

$$u^{(e)} = \Phi_i u_i^{(e)},$$

pay attention that $u^2 = \Phi_i\Phi_j u_i^{(e)}u_j^{(e)}$

then:

$$\int_\Omega \left(\alpha \frac{\partial \Phi_i}{\partial x}\frac{\partial \Phi_k}{\partial x}u_i^{(e)}\cancel{\delta u_k^{(e)}} - \frac{1}{2}\Phi_i\Phi_j\frac{\partial \Phi_k}{\partial x}u_i^{(e)}u_j^{(e)}\cancel{\delta u_k^{(e)}} - P\Phi_k\cancel{\delta u_k^{(e)}}\right)d\Omega = \int_{\Gamma_2}\left(\alpha\frac{\partial u}{\partial x} - \frac{1}{2}\Phi_i\Phi_j u_i u_j\right)\Phi_k\cancel{\delta u_k}d\Gamma$$

that is:

$$u_i^{(e)}\left[\int_{\Omega^{(e)}}\alpha\frac{\partial \Phi_i}{\partial x}\frac{\partial \Phi_k}{\partial x}d\Omega\right] + u_i^{(e)}u_j^{(e)}\left[\underbrace{-\int_{\Omega^{(e)}}\frac{1}{2}\Phi_i\Phi_j\frac{\partial \Phi_k}{\partial x}d\Omega}_{\text{negative}} + \int_{\Gamma_2}\frac{1}{2}\Phi_i\Phi_j\Phi_k d\Gamma\right] = \int_{\Omega^{(e)}}P\Phi_k\, d\Omega + \int_{\Gamma_2}\alpha\frac{\partial u}{\partial x}\Phi_k\, d\Gamma$$

we set $A_{ik}^{(e)} = \int_{\Omega^{(e)}}\alpha\frac{\partial \Phi_i}{\partial x}\frac{\partial \Phi_k}{\partial x}d\Omega$

$B_{ijk}^{(e)} = -\int_{\Omega^{(e)}}\frac{1}{2}\Phi_i\Phi_j\frac{\partial \Phi_k}{\partial x}d\Omega + \int_{\Gamma_2}\frac{1}{2}\Phi_i\Phi_j\Phi_k d\Gamma$ and $\boxed{f_k^{(e)} = \int_\Omega P\Phi_k\, d\Omega + \int_{\Gamma_2}\alpha\frac{\partial u}{\partial x}\Phi_k d\Gamma}$

then the equation becomes:

$$\boxed{A_{ik}^{(e)}u_i^{(e)} + B_{ijk}^{(e)}u_i^{(e)}u_j^{(e)} = f_k^{(e)}}$$

where

$$A_{ik}^{(e)} = \int_{\Omega^{(e)}}\alpha\frac{\partial \Phi_i^{(e)}}{\partial x}\frac{\partial \Phi_k^{(e)}}{\partial x}d\Omega$$

$$B_{ijk}^{(e)} = -\frac{1}{2}\int_{\Omega^{(e)}}\Phi_i^{(e)}\Phi_j^{(e)}\frac{\partial \Phi_k^{(e)}}{\partial x}d\Omega + \frac{1}{2}\int_{\Gamma_2^{(e)}}\Phi_i^{(e)}\Phi_j^{(e)}\Phi_k^{(e)}d\Gamma$$

and

$$f_k^{(e)} = \int_{\Omega^{(e)}}P\Phi_k^{(e)}d\Omega + \int_{\Gamma_2^{(e)}}\alpha\frac{\partial u}{\partial x}\Phi_k^{(e)}d\Gamma$$