Euler方程组的基本微分格式推导

Thursday, June 1, 2023 7:31 PM

一维Euler方程组 $\frac{\partial U}{\partial t}+\frac{\partial f(U)}{\partial x}=0$

$$U=\begin{pmatrix}\rho\\ \rho u\\ E\end{pmatrix}\quad f(U)=\begin{pmatrix}\rho u\\ \rho u^2+p\\ u(E+p)\end{pmatrix}$$

利用：$E=\frac{p}{\gamma-1}+\frac{1}{2}\rho u^2$

则有：$p=(\gamma-1)\left(E-\frac{1}{2}\rho u^2\right)$

取：$U=\begin{pmatrix}\rho\\ \rho u\\ E\end{pmatrix}=\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix}$，则：$\rho=u_1$，$u=\frac{u_2}{u_1}$，$E=u_3$，代入上式：

$p=(\gamma-1)\left(u_3-\frac{1}{2}u_1\cdot\frac{u_2^2}{u_1^2}\right)$ 代入 $f(U)$ 中，有：

$$f(U)=\begin{bmatrix}u_2\\ \frac{u_2^2}{u_1}+(\gamma-1)\left(u_3-\frac{1}{2}\frac{u_2^2}{u_1}\right)\\ \frac{u_2}{u_1}\left(u_3+(\gamma-1)\left(u_3-\frac{1}{2}\frac{u_2^2}{u_1}\right)\right)\end{bmatrix}=\begin{bmatrix}u_2\\ \left(\frac{3-\gamma}{2}\right)\frac{u_2^2}{u_1}+(\gamma-1)u_3\\ \gamma\frac{u_2u_3}{u_1}+\frac{1}{2}(1-\gamma)\frac{u_2^3}{u_1^2}\end{bmatrix}$$

齐次函数，$f(\alpha U)=\alpha f(U)$

$$A=\underbrace{\frac{\partial f(U)}{\partial U}}_{\text{每一行分别对}u_1,u_2,u_3\text{求导，}}=\begin{bmatrix}0 & 1 & 0\\ -\frac{3-\gamma}{2}\left(\frac{u_2}{u_1}\right)^2 & (3-\gamma)\left(\frac{u_2}{u_1}\right) & \gamma-1\\ -\gamma\frac{u_2u_3}{u_1^2}-\frac{1}{2}(1-\gamma)\frac{u_2^3}{u_1^3} & \gamma\frac{u_3}{u_1}+(1-\gamma)\left(\frac{u_2}{u_1}\right) & \gamma\frac{u_2}{u_1}\end{bmatrix}$$

$a_{31}=\frac{\gamma-1}{2}u^3-\gamma\frac{u_2u_3}{u_1^2}$

$$=\begin{bmatrix}0 & 1 & 0\\ -\frac{3-\gamma}{2}u^2 & (3-\gamma)u & \gamma-1\\ \cdots & \cdots & \gamma u\end{bmatrix}$$

此时：若矩阵A可对角化：则：

$A=S^{-1}\Lambda S$，有：

$\frac{\partial U}{\partial t}+A\frac{\partial U}{\partial x}=0$　此时：$S\frac{\partial U}{\partial t}+\Lambda S\frac{\partial U}{\partial x}=0$，

$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0$ 此时: $S\frac{\partial U}{\partial t} + \Lambda S\frac{\partial V}{\partial x} = 0$,

故得到: $\frac{\partial v_j}{\partial t} + \lambda_j \frac{\partial v_j}{\partial x} = 0$,

此时:变为了独立的n个方程:

即差分公式: $\frac{\partial v}{\partial t} + \alpha\frac{\partial v}{\partial x} = 0$ 的由来