二维正态分布的协方差和相关系数

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P19例2. 设(X,Y)服从二维正态分布,其概率密度

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2-2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right)$$

求 X,Y 的相关系数。

解：对二维正态分布,边缘概率密度

$$f_X(x)=\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right)\quad f_Y(y)=\frac{1}{\sqrt{2\pi}\sigma_2}\exp\left(-\frac{(x-\mu_2)^2}{2\sigma_2^2}\right)$$

从而: $E(X)=\mu_1,\ D(X)=\sigma_1^2,\ E(y)=\mu_2,\ D(y)=\sigma_2^2$

只需求 $E(XY)$ 即可; 有:

$$\int_{-\infty}^{+\infty}\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}xy\exp\ldots\quad X$$

和求 $Cov(X,Y)$: 由 $Cov(X,Y)=\underbrace{E[(X-E(X))(Y-E(Y))]}_{①}=E(XY)-E(X)E(Y)$

可积分:

$$\int_{-\infty}^{+\infty}\frac{(x-\mu_1)(y-\mu_2)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\ldots dydx$$ —— 仍拆分: $\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2-2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]$

$$=\left[\left(\frac{y-\mu_2}{\sigma_2}\right)-\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right]^2+(1-\rho^2)\left(\frac{x-\mu_1}{\sigma_1}\right)^2$$,代入有:

$$Cov(X,Y)=\int_{-\infty}^{+\infty}\frac{(x-\mu_1)(y-\mu_2)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{y-\mu_2}{\sigma_2}-\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right]^2\right\}dxdy$$

取: $\lambda=\frac{x-\mu_1}{\sigma_1}$, $t=\frac{1}{\sqrt{1-\rho^2}}\left(\frac{y-\mu_2}{\sigma_2}-\rho\frac{x-\mu_1}{\sigma_1}\right)$, 从而有: $dx=\sigma_1 d\lambda$, $dy=\sigma_2\sqrt{1-\rho^2}dt$

配凑: $\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2\sqrt{1-\rho^2}}=\lambda t+\frac{\rho}{\sqrt{1-\rho^2}}\left(\frac{x-\mu_1}{\sigma_1}\right)^2$,

$$代入=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\lambda t+\frac{\rho}{\sqrt{1-\rho^2}}\lambda^2\right)\exp\left(-\frac{\lambda^2}{2}-\frac{t^2}{2}\right),\ \sigma_1 d\lambda\cdot\sigma_2\sqrt{1-\rho^2}dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[(\sigma_1\sigma_2\sqrt{1-\rho^2})\lambda t+\sigma_1\sigma_2\rho\lambda^2\right]e^{-\frac{\lambda^2+t^2}{2}}d\lambda dt$$

$$=\frac{\sigma_1\sigma_2}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\sqrt{1-\rho^2}\lambda t+\rho\lambda^2\right]e^{-\frac{\lambda^2+t^2}{2}}d\lambda dt$$

分变量

$$=\frac{\sigma_1\sigma_2}{2\pi}\sqrt{1-\rho^2}\underbrace{\int_{-\infty}^{+\infty}\lambda e^{-\frac{\lambda^2}{2}}d\lambda\int_{-\infty}^{+\infty}te^{-\frac{t^2}{2}}dt}_{奇,积分为0}+\frac{\sigma_1\sigma_2\rho}{2\pi}\int_{-\infty}^{+\infty}\lambda^2e^{-\frac{\lambda^2}{2}}d\lambda\int_{-\infty}^{+\infty}e^{-\frac{t^2}{2}}dt$$

$$=\frac{\sigma_1\sigma_2\rho}{2\pi}\int_{-\infty}^{+\infty}\lambda^2e^{-\frac{\lambda^2}{2}}d\lambda\cdot\sqrt{2\pi}$$,其中 $\int_{-\infty}^{+\infty}\lambda^2e^{-\frac{\lambda^2}{2}}d\lambda=\int_{-\infty}^{+\infty}-\lambda\, d(e^{-\frac{\lambda^2}{2}})$

$$=-\lambda e^{-\frac{\lambda^2}{2}}\Big|_{-\infty}^{+\infty}+\int_{-\infty}^{+\infty}e^{-\frac{\lambda^2}{2}}d\lambda=\sqrt{2\pi}$$

$$= \frac{\sigma_1\sigma_2\rho}{2\pi}\int_{-\infty}^{+\infty} \lambda e^{-\frac{\lambda^2}{2}} d\lambda \cdot \sqrt{2\pi}$$，其中 $\int_{-\infty}^{+\infty} \lambda e^{-\frac{\lambda^2}{2}} d\lambda = \int_{-\infty}^{+\infty} -\lambda \, d(e^{-\frac{\lambda^2}{2}})$

$$= \lambda e^{-\frac{\lambda^2}{2}}\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{-\frac{\lambda^2}{2}} d\lambda = \sqrt{2\pi},$$

$$= \sigma_1\sigma_2\rho = Cov(X_1, X_2)$$

小结: $\rho_{XY} = \frac{Cov(X_1, X_2)}{\sigma_1 \sigma_2} = \rho$.