Derivation of the Element FEM characteristic equation

Thursday, March 23, 2023 5:09 PM

Substitute $u^{(e)} = u_i^{(e)} \Phi_i^{(e)}$ into the element FEM equation:

$$\underbrace{\int_{x_1^{(e)}}^{x_2^{(e)}} \frac{du}{dx}\frac{d(\delta u)}{dx} + c\,\delta u \, dx = 0}$$

then we have:

$$\int_{x_1^{(e)}}^{x_2^{(e)}} \left[\frac{d\Phi_i^{(e)}}{dx} u_i^{(e)} \frac{d\Phi_j}{dx}\delta u_j + c\,\delta u_j \Phi_j^{(e)}\right] dx = 0$$

since δu_j and δu_i not vary with dx: thus:

$$\delta u_j \int_{x_1^{(e)}}^{x_2^{(e)}} \left[u_i^{(e)} \frac{d\Phi_i^{(e)}}{dx} \cdot \frac{d\Phi_j}{dx} + c\,\Phi_j\right] dx = 0$$

Since the virtual displacement δu_i is arbitary, then

$$\int_{x_1^{(e)}}^{x_2^{(e)}} \left(u_i^{(e)} \frac{d\Phi_i^{(e)}}{dx}\frac{d\Phi_j}{dx} + c\,\Phi_j\right) dx = 0 \quad (\text{where } i = 1, 2)$$

then we have:

$$u_i^{(e)} \int_{x_1^{(e)}}^{x_2^{(e)}} \frac{d\Phi_i^{(e)}}{dx} \cdot \frac{d\Phi_j}{dx} = \int_{x_1^{(e)}}^{x_2^{(e)}} c\,\Phi_j \, ,$$ which can be written as:

firstly, we replace i with j and j with i

$$u_j^{(e)} \int_{x_1^{(e)}}^{x_2^{(e)}} \frac{d\Phi_j^{(e)}}{dx} \cdot \frac{d\Phi_i^{(e)}}{dx} = \int_{x_1^{(e)}}^{x_2^{(e)}} c\,\Phi_i \, dx$$

$$\boxed{A_{ij} u_j^{(e)} = f_i^{(e)}},$$

where

$$A_{ij} = \int_{x_1^{(e)}}^{x_2^{(e)}} \frac{d\Phi_i^{(e)}}{dx} \frac{d\Phi_j^{(e)}}{dx} dx$$

$$f_i = \int_{x_1^{(e)}}^{x_2^{(e)}} c\,\Phi_i \, dx$$

The equation is the general expression of the element FEM equation.

Also, if we use the natural coordinate, the expression becomes:

$$A_{ij}^{(e)} = \frac{1}{\Delta h}\int_0^1 \frac{d\Phi_i^{(e)}}{d\xi}\frac{d\Phi_j^{(e)}}{d\xi}\, d\xi$$

and $$f_i^{(e)} = -\Delta h \cdot c \int_0^1 \Phi_i \, d\xi$$

considering that
$$\begin{cases} \Phi_1 = 1-\xi \\ \Phi_2 = \xi \end{cases}$$
then the approximation function is

$$\boxed{A_{ij}^{(e)} = \frac{1}{\Delta h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad f_i^{(e)} = -\Delta h \cdot c \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}}$$