Derivation of transformation between the local Coordinate and the Cartesian Coordinate of triangle Element

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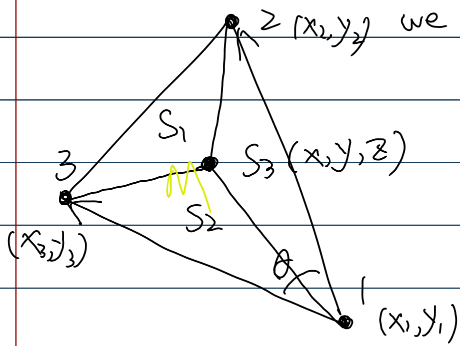

we have a three-node triangle element, and for each Node, we use

$$\Phi_1 = \frac{S_1}{A},\quad \Phi_2 = \frac{S_2}{A},\quad \Phi_3 = \frac{S_3}{A}$$

where

$$\xi_1 = \frac{S_1}{A}\quad \xi_2 = \frac{S_2}{A}\quad \xi_3 = \frac{S_3}{A}$$

we have $\Phi_1 = \xi_1,\ \Phi_2 = \xi_2,\ \Phi_3 = \xi_3$.

then the Area of the triangle can be calculated as:

$$\begin{cases} V_1 = (x_2 - x_1,\ y_2 - y_1) \\ V_2 = (x_3 - x_1,\ y_3 - y_1) \end{cases}$$

then $A = \frac{1}{2} V_1 V_2 \sin\theta$

$$= \frac{1}{2} |V_1 \times V_2|$$

$$= \frac{1}{2}\begin{vmatrix} i & j & k \\ x_2-x_1 & x_3-x_1 & 0 \\ y_2-y_1 & y_3-y_1 & 0 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ x_2-x_1 & x_3-x_1 & 0 \\ y_2-y_1 & y_3-y_1 & 0 \end{vmatrix}$$

$$= \frac{1}{2}\left[(x_2-x_1)(y_3-y_1) - (x_3-x_1)(y_2-y_1)\right]$$

$$= \frac{1}{2}\left[x_2y_3 - x_1y_3 - x_2y_1 + \cancel{x_1y_1} - x_3y_2 + x_1y_2 + x_3y_1 - \cancel{x_1y_1}\right]$$

$$= \frac{1}{2}\left[(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)\right]$$

$$= \frac{1}{2}|\Lambda| \quad \checkmark$$

$$|\Lambda| = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

thus we set the point M (x,y,z), and then

$$A_1 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix}$$

hence:

$$\Phi_1 = \frac{\frac{1}{2}|A_1|}{\frac{1}{2}|\Lambda|} = \frac{|A_1|}{\Lambda} = \frac{\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - x\begin{vmatrix} 1 & 1 \\ y_2 & y_3 \end{vmatrix} + y\begin{vmatrix} 1 & 1 \\ x_2 & x_3 \end{vmatrix}}{|\Lambda|}$$

finally

we have: $\Phi_1 = \frac{1}{|\Lambda|}(a_1 - b_1 x + c_1 y)$

$|\Lambda| = 2\Delta$

where a_1, b_1, c_1 are algebraic cofactor of $1, x$, and y, separately

and $|\Lambda| = 2S_{\Delta 123}$

we can derive other coefficients by the same process,

we can derive the expression of x, y by using local coordinates

we set $\begin{cases} x = \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3 \\ y = \beta_1\xi_1 + \beta_2\xi_2 + \beta_3\xi_3 \end{cases}$

and:

we set $\Big| \; y = \beta_1 \xi_1 + \beta_2 \xi_2 + \beta_3 \xi_3$

and:

$$x \Big|_{\xi_2=0,\ \xi_3=0} = x_1$$

, then we can easily have,

$$\boxed{\begin{cases} x = x_1 \Phi_1 + x_2 \Phi_2 + x_3 \Phi_3 \\ y = y_1 \Phi_1 + y_2 \Phi_2 + y_3 \Phi_3 \end{cases}}$$ ☆ (also ξ_1, ξ_2, ξ_3)