范德蒙德行列式的证明

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证

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & & x_n \\ x_1^2 & x_2^2 & & \vdots \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1\le j<i\le n} (x_i - x_j).$$

证明: 用归纳法进行证明.

二阶行列式有:

$$D_2 = \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1$$

我们假设 $D_{n-1} = \prod_{1\le i<j\le n-1} (x_i - x_j)$ 成立.

而有:

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & & x_n \\ x_1^2 & x_2^2 & & \vdots \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} \xrightarrow[\substack{\vdots \\ 2行 - x_1\cdot 1行}]{\substack{n行 - x_1\, n-1行 \\ n-1行 - x_1\, n-2行}} \begin{vmatrix} 1 & 1 & \cdot & 1 & \cdots & 1 \\ 0 & x_2-x_1 & & x_3-x_1 & \cdots & x_n-x_1 \\ 0 & x_2^{n-3}(x_2-x_1) & & \cdots & & x_n^{n-3}(x_n-x_1) \\ 0 & x_2^{n-2}(x_2-x_1) & & x_3^{n-2}(x_3-x_1) & \cdots & x_n^{n-2}(x_n-x_1) \end{vmatrix}$$

$$= (x_2-x_1)(x_3-x_1)\cdots(x_n-x_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & & \\ x_2^{n-2} & \cdots & & x_n^{n-2} \end{vmatrix} = (x_2-x_1)\cdots(x_n-x_1)\cdot \prod_{2\le j<i\le n} (x_i - x_j)$$

$$= \prod_{1\le j<i\le n} (x_i - x_j)$$