Derivation of the weak form of 2D Ellipse Boundary Problem

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since the problem is:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = p, & (x,y) \in \Omega \\ u|_{\Gamma_1} = \bar{u} \\ \frac{\partial u}{\partial n}\Big|_{\Gamma_2} = g. \end{cases}$$

and

in these boundary $\frac{\partial u}{\partial n}\Big|_{\Gamma_1} = 0$

then the weak form by writing the second-order derivation with the partial integral:

$$\iint_\Omega \left[\frac{\partial u}{\partial x}\frac{\partial(\delta u)}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial(\delta u)}{\partial y}\right] d\Omega + \iint_\Omega p\,\delta u\, d\Omega = \int_{\Gamma_2} g\,\delta u\, d\Gamma$$

this can be derivated by following process:

then we extract δu from the equation and

$$\iint_\Omega \delta u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) d\Omega - \iint_\Omega p\,\delta u\, d\Omega = 0$$

then we have $\nabla \cdot \underbrace{\left(\frac{\partial u}{\partial x}\delta u, \frac{\partial u}{\partial y}\delta u\right)}_{\text{vector product}}$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\delta u\right) = \delta u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + \frac{\partial u}{\partial x}\frac{\partial(\delta u)}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial(\delta u)}{\partial y}$$

then we have:

$$\iint_\Omega \delta u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) d\Omega = \iint_\Omega \nabla \cdot \left(\frac{\partial u}{\partial x}\delta u, \frac{\partial u}{\partial y}\delta u\right) d\Omega - \iint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial(\delta u)}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial(\delta u)}{\partial y}\right) d\Omega$$

we have $\iint_\Omega \nabla \cdot \left(\frac{\partial u}{\partial x}\delta u, \frac{\partial u}{\partial y}\delta u\right) d\Omega$

$$= \iint_\Omega \delta u \; \nabla \cdot \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) d\Omega$$

Gauss's law

$$\int (\nabla \cdot F)\, dV = \oint \vec{F} \cdot \vec{n}\, ds$$

(dx, dy) $\vec{n}$

Note that the Green equation gives:

$$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\Omega = \oint_L P\,dx + Q\,dy$$

we set $\vec{F} = (Q, P)$

and $\vec{n} = (dy, -dx)$

then $\nabla \cdot F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

then

$$\boxed{\iint_\Omega \nabla \cdot \vec{F}\, d\Omega = \oint \vec{F} \cdot \vec{n}_e\, ds}$$

so $\iint_\Omega \delta u\, \nabla\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) d\Omega = \oint_{\underset{\to 0}{\Gamma_1} + \Gamma_2} \delta u\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \vec{n}\, d\Gamma \Rightarrow \int_{\Gamma_2} \delta u \frac{\partial u}{\partial n} d\Gamma = \int_{\Gamma_2} \delta u \cdot g\, d\Gamma$

substitute the equation above and we will get:

So $\iint_{\Omega} \delta u \, \nabla\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) d\Omega = \oint_{\underbrace{\Gamma_1}_{\to 0}+\Gamma_2} \delta u \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\cdot n \, d\Gamma \Rightarrow \int_{\Gamma_2} \delta u \frac{\partial u}{\partial n} d\Gamma = \int_{\Gamma_2} \delta u \cdot g \, d\Gamma$

substitute the equation above and we will get:

$$\iint_{\Omega} p \, \delta u \, d\Omega = \iint_{\Omega} \delta u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \int_{\Gamma_2} g \, \delta u \, d\Gamma - \iint_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial (\delta u)}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial (\delta u)}{\partial y}\right) d\Omega$$

then we get the weak form:

$$\boxed{\iint_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial (\delta u)}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial (\delta u)}{\partial y}\right) d\Omega + \iint_{\Omega} p \, \delta u \, d\Omega = \int_{\Gamma_2} g \, \delta u \, d\Gamma}$$