3 examples for the convergence analysis

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①: PPT P71, for a 1-D differential problem, as

$$\begin{cases} \frac{du}{dx} + A u = 0, & 0 < x \le a \\ u(0) = 1 \end{cases}$$

since $\frac{du}{dx} = -Au$, $\longrightarrow u = e^{-Ax}$ is the analytic solution.

for difference form, the solution becomes:

$$\frac{U_{i+1} - U_i}{\Delta x} = -A U_i \quad \therefore U_{i+1} = U_i(1 - A\Delta x)$$

that can be written as:

$$U_2 = U_1(1 - A\Delta x)$$

$$U_3 = U_2(1 - A\Delta x) = U(1 - A\Delta x)^2$$

$\vdots$

$$U_i = U_1(1 - A\Delta x)^{i-1} \xrightarrow{\text{we can set}} U_i = U_1\left(1 - A\frac{a}{n}\right)^{i-1}$$

$\Delta x = \frac{a}{n}$ here, thus:

then: $U_i = U_1\left(1 - A\frac{a}{n}\right)^{i-1} \Rightarrow$ boundary condition $U_0 = 1$

$$= U_0\left(1 - A\frac{a}{n}\right)^i$$

$$\longrightarrow U_i = \left(1 - A\frac{a}{n}\right)^i = \left(\left[1 - \frac{A}{\frac{n}{a}}\right]^{\frac{n}{a}}\right)^{\frac{a}{n}i} = \left[1 - \frac{A}{\frac{n}{a}}\right]^{\frac{n}{a}\cdot x_i}$$

since $\lim_{m\to\infty}\left[1 - \frac{A}{m}\right]^{m x_i} = e^{-A\cdot x_i}$ we have drawn the conclusion that the equation would converge unconditionally.

Example 2. (PPT P74) The equation of the convection,

for $\frac{\partial T}{\partial t} + \alpha\frac{\partial T}{\partial x} = 0$, we use the 1^{st} order backward difference method (FTBS), that is,

T^{n+1} T^n $T_i^n - T_{i-1}^n$ or T^{n+1} T^n [illegible] method (FTBS), that is,

$$\frac{T_i^{n+1}-T_i^n}{\Delta t}+\alpha\frac{T_i^n-T_{i-1}^n}{\Delta x} \text{ or } \Rightarrow T_i^{n+1}=T_i^n\left(\frac{\alpha\Delta t}{\Delta x}-1\right)-T_{i-1}^n\left(\frac{\alpha\Delta t}{\Delta x}\right)$$

the error of the equation is set as e_i^n

which is denoted as:

$$e_i^n = T_i^n - T(x_i, t_n)$$

also

$$\frac{T(x_i, t_n+\Delta t)-T(x_i,t_n)}{\Delta t}+\alpha\frac{T(x_i,t_n)-T(x_{i-1},t_n)}{\Delta x}=O(\Delta x,\Delta t)$$

hence:

$$\frac{e_i^{n+1}-e_i^n}{\Delta t}+\alpha\frac{e_i^n-e_{i-1}^n}{\Delta x}=O(\Delta x,\Delta t)$$ one-order precision.

or we can write the form of the error increase:

that is:

$$e_i^{n+1}=e_i^n\left(1-\frac{\alpha\Delta t}{\Delta x}\right)+e_{i-1}^n\left(\frac{\alpha\Delta t}{\Delta x}\right)+\Delta t\cdot O(\Delta x,\Delta t)$$

So we would judge the error of them in different situations:

only under the condition $0<\frac{\alpha\Delta t}{\Delta x}<1$ $\rightarrow$ or, $1-\frac{\alpha\Delta t}{\Delta x}>0$, $0<\frac{\alpha\Delta t}{\Delta x}<1$

$$|e_i^{n+1}|\leq\left(1-\frac{\alpha\Delta t}{\Delta x}\right)e_i^n+\left(\frac{\alpha\Delta t}{\Delta x}\right)e_{i-1}^n$$

$$\leq\left(1-\frac{\alpha\Delta t}{\Delta x}\right)\max|e_i^n|+\left(\frac{\alpha\Delta t}{\Delta x}\right)\max|e_{i-1}^n|+\Delta t\cdot O(\Delta x,\Delta t)$$

$$=\max|e_i^n|+O(\Delta x,\Delta t)$$

hence $\max|e_i^{n+1}|\leq\max|e_i^n|+\Delta t\, O(\Delta x,\Delta t)$

then:

$$\max|e_i^n|\leq\max|e_i|+(n-1)\Delta t\cdot O(\Delta x,\Delta t)$$

so the equation has the character of conditional convergence.

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example 3, (textbook P154)

for heat conduction equation: $\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial x^2}$

example 3, (textbook P.N 4)

for heat conduction equation: $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{\alpha \left(T_{i+1}^n + T_{i-1}^n - 2T_i^n\right)}{(\Delta x)^2} + O(\Delta t, \Delta x^2)$$

hence:

$$T_i^{n+1} = T_i^n + \frac{\alpha \Delta t}{(\Delta x)^2}\left(T_{i+1}^n + T_{i-1}^n - 2T_i^n\right)$$

we can just simply substitute e into T in solution process

that is:

$$e_i^{n+1} = e_i^n + \frac{\alpha \Delta t}{(\Delta x)^2}\left(e_{i+1}^n + e_{i-1}^n - 2e_i^n\right) = \left(1 - 2\frac{\alpha \Delta t}{(\Delta x)^2}\right)$$

we add e_2, e_3 -- together, then:

$$e_i^1 = e_i^0\left(1 - 2\frac{\alpha \Delta t}{(\Delta x)^2}\right) + \left(e_{i+1}^0 + e_{i-1}^0\right)\frac{\alpha \Delta t}{(\Delta x)^2}$$

$$e_i^2 \overset{\text{subs}}{=} e_i^1\left(1 - 2\frac{\alpha \Delta t}{(\Delta x)^2}\right) + \cdots - \quad = \cdots - \quad \cdots$$

at any given time, we define

$\left|\frac{e_i^{n+1}}{e_i^n}\right| \leq 1$, then the solution is stable

also, the error term can be written as:

$$e(x) = \sum_m A_m e^{ik_m x} \quad \text{and} \quad y = \sin\frac{2\pi x}{\lambda} \quad y = \sin k_m x$$

where we note that the error also represents a sine and cosine series since $e^{ik_m x} = \cos k_m x + \sin k_m x$

when $k_m = \frac{2\pi}{\lambda}$, and λ is the wave length,

also: $k_m = \frac{2\pi}{L} m$, L is the total length and m is the number.