壳体的内力和内矩平衡方程

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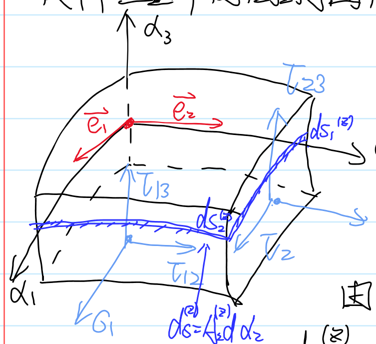
从壳体内沿 α_1, $\alpha_1+d\alpha_1$, α_2, $\alpha_2+d\alpha_2$ 主坐标线方向
取个垂直中曲面的面得到一个微元体



σ_1, σ_2 为微元体侧面上的正应力。

而在薄壳理论中，认为厚度方向忽略应力，即有：

$\sigma_3=0$.

由中曲面弧长 $ds_2=A_2\,d\alpha_2$,

$$ds_2^{(z)}=A_2^{(z)}d\alpha_2=\left(1+\frac{z}{R_2}\right)ds_2$$

此时：由 $A_2^{(z)}=\left(1+\frac{z}{R_2}\right)A_2$，则有：

$$N_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma^{(z)}dz\,ds_2^{(z)}$$ → 认为 $\sigma^{(z)}$ 在积分区间内均为 σ，则

当我们取单位弧长时，则 $ds_2=1$，

$$\begin{cases}N_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma\left(1+\frac{z}{R_2}\right)ds_2\,dz\\ N_{12}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{12}\left(1+\frac{z}{R_2}\right)ds_2\,dz\\ Q_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{13}\left(1+\frac{z}{R_2}\right)ds_2\,dz\end{cases}\quad\text{变为}\quad\begin{cases}N_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_1\left(1+\frac{z}{R_2}\right)dz\\ S_{12}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{12}\left(1+\frac{z}{R_2}\right)dz \quad ☆\\ Q_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{13}\left(1+\frac{z}{R_2}\right)dz\end{cases}$$

相应地 ⟹ 作用在壳体的力矩为：

$$\begin{cases}M_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_1 z\left(1+\frac{z}{R_2}\right)dz & \text{(弯矩)}\\ H_{12}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{12} z\left(1+\frac{z}{R_2}\right)dz & \text{(扭矩)}\end{cases}$$

因此，壳体总的内力与内矩可以用如下量描述：

$$\left.\begin{matrix}N_1, & S_{12} & Q_1 & M_1 & H_{12}\\ N_2 & S_{21} & Q_2 & M_2 & H_{21}\end{matrix}\right\}$$ 共有10个量，

※需要说明：即使有剪应力互等定律，但由于 R_1R_2 不相等，故 $S_{12}\neq S_{21}$, $H_{12}\neq H_{21}$

有关系

① $$N_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_1\left(1+\frac{z}{R_2}\right)dz=\sigma_1 h$$ ← 后面一项积分为 $\left(\frac{h^2}{2}-\frac{h^2}{2}\right)=0$

② $$M_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_1 z\left(1+\frac{z}{R_2}\right)dz=\sigma_1\frac{z^2}{2}\Big|_{-\frac{h}{2}}^{\frac{h}{2}}+\sigma_1\frac{z^3}{3R_2}\Big|_{-\frac{h}{2}}^{\frac{h}{2}}$$

$$=0+\frac{\sigma_1 h^3}{12R_2}\ \rightarrow R_2=\frac{\sigma_1 h^3}{12M_1}.$$

⟶ $$\sigma_1^{(z)}=\sigma_1\left(1+\frac{z}{R_2}\right)=\frac{N_1}{h}+\frac{12M_1}{h^3}z \quad ①$$

同理有 $$\sigma_2^{(z)}=\sigma_2\left(1+\frac{z}{R_1}\right)=\frac{N_2}{h}+\frac{12M_2}{h^3}z \quad ②$$

而：$$N_{12}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{12}\left(1+\frac{z}{R_2}\right)dz=\tau_{12}h \quad (1)$$

$$H_{12}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{12}z\left(1+\frac{z}{R_2}\right)dz=\tau_{12}\frac{h^3}{12R_1} \quad (2)$$

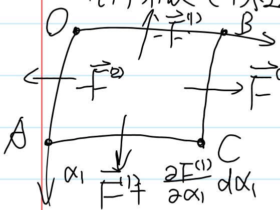
得 $$\tau_{12}^{(z)}=\tau_{12}\left(1+\frac{z}{R_2}\right)=\frac{N_{12}}{h}+\frac{12H_{12}}{h^3}z \quad ③$$

同理 $$\tau_{21}^{(z)}=\tau_{21}\left(1+\frac{z}{R_1}\right)=\frac{N_{21}}{h}+\frac{12H_{21}}{h^3}z \quad ④$$

薄膜　　弯曲应力、

当壳体微元体4条边上作用 F, M, Q 且表面力有 q_1, q_2, q_3 时，

当壳体微元各边上作用 F, M, Q 且表面力有 q_1, q_2, q_3 时,

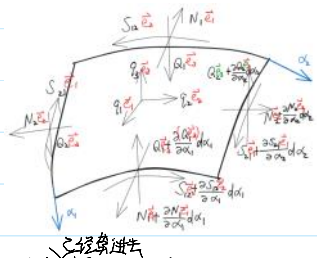


$\vec{F}^{(2)}+\frac{\partial \vec{F}^{(2)}}{\partial \alpha_2}d\alpha_2 \Rightarrow$ 其中, F 由正应力,剪力和弯矩共同构成,故有:

边界处力为:

$$\vec{F}^{(1)} = -(N_1\vec{e}_1+S_{12}\vec{e}_2+Q_1\vec{e}_3)A_2 d\alpha_2$$

$$\vec{F}^{(2)} = -(N_2\vec{e}_1+S_{21}\vec{e}_2+Q_2\vec{e}_3)A_1 d\alpha_1$$



注意!!!

整体偏导,而非展开偏导(时是向量,考虑 $\vec{e}_1, \vec{e}_2, \vec{e}_3$ 对 $\alpha_1, \alpha_2, \alpha_3$ 导数

由于 $A_1d\alpha_1$ (已经囊进去) 得到平衡方程为:

$$\left(\vec{F}^{(1)}+\frac{\partial \vec{F}^{(1)}}{\partial \alpha_1}d\alpha_1\right)-\vec{F}^{(1)}+\left(\vec{F}^{(2)}+\frac{\partial \vec{F}^{(2)}}{\partial \alpha_2}d\alpha_2\right)-\vec{F}^{(2)}+\vec{q}A_1A_2d\alpha_1d\alpha_2=\vec{0}$$

$$\Rightarrow \text{有:}\quad \boxed{\frac{\partial \vec{F}^{(1)}}{\partial \alpha_1}d\alpha_1+\frac{\partial \vec{F}^{(2)}}{\partial \alpha_2}d\alpha_2+\vec{q}A_1A_2d\alpha_1d\alpha_2=\vec{0}}$$

平衡方程

$$\frac{\partial}{\partial \alpha_1}\left[(N_1\vec{e}_1+S_{12}\vec{e}_2+Q_1\vec{e}_3)A_2\cancel{d\alpha_2}\right]\cancel{d\alpha_1}+\frac{\partial}{\partial \alpha_2}\left[(S_{21}\vec{e}_1+N_2\vec{e}_2+Q_2\vec{e}_3)A_1\cancel{d\alpha_1}\right]\cancel{d\alpha_2}+\vec{q}A_1A_2\cancel{d\alpha_1d\alpha_2}=0$$

$$\left[\frac{\partial(N_1A_2)}{\partial \alpha_1}\vec{e}_1+N_1A_2\frac{\partial \vec{e}_1}{\partial \alpha_1}+\frac{\partial(S_{12}A_2)}{\partial \alpha_1}\vec{e}_2+S_{12}A_2\frac{\partial \vec{e}_2}{\partial \alpha_1}+\frac{\partial(Q_1A_2)}{\partial \alpha_1}\vec{e}_3+Q_1A_2\frac{\partial \vec{e}_3}{\partial \alpha_1}\right]$$

$$+\left[\frac{\partial(S_{21}A_1)}{\partial \alpha_2}\vec{e}_1+S_{21}A_1\frac{\partial \vec{e}_1}{\partial \alpha_2}+\frac{\partial(N_2A_1)}{\partial \alpha_2}\vec{e}_2+N_2A_1\frac{\partial \vec{e}_2}{\partial \alpha_2}+\frac{\partial(Q_2A_1)}{\partial \alpha_2}\vec{e}_3+Q_2A_1\frac{\partial \vec{e}_3}{\partial \alpha_2}\right]$$

$$\frac{\partial \vec{e}_1}{\partial \alpha_1}=-\frac{1}{A_2}\frac{\partial A_1}{\partial \alpha_2}\vec{e}_2-\frac{A_1}{R_1}\vec{e}_3 \qquad \frac{\partial \vec{e}_1}{\partial \alpha_2}=\frac{1}{A_1}\frac{\partial A_2}{\partial \alpha_1}\vec{e}_2$$

$$\frac{\partial \vec{e}_2}{\partial \alpha_1}=\frac{1}{A_2}\frac{\partial A_1}{\partial \alpha_2}\vec{e}_1 \qquad \frac{\partial \vec{e}_2}{\partial \alpha_2}=-\frac{1}{A_1}\frac{\partial A_2}{\partial \alpha_1}\vec{e}_1-\frac{A_2}{R_2}\vec{e}_3$$

$$\frac{\partial \vec{e}_3}{\partial \alpha_1}=\frac{A_1}{R_1}\vec{e}_1 \qquad \frac{\partial \vec{e}_3}{\partial \alpha_2}=\frac{A_2}{R_2}\vec{e}_2$$

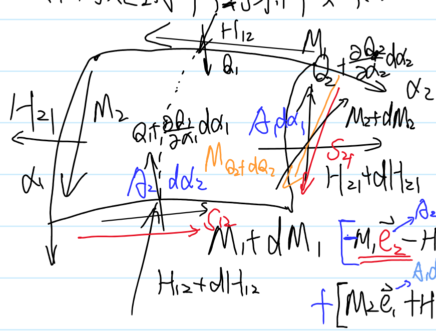
对 $\vec{e}_1$: $\frac{\partial(N_1A_2)}{\partial \alpha_1}+S_{12}\frac{\partial A_1}{\partial \alpha_2}+\frac{Q_1A_1A_2}{R_1}+\frac{\partial(A_1S_{21})}{\partial \alpha_2}-N_2\frac{\partial A_2}{\partial \alpha_1}+A_1A_2q_1=0$

$\vec{e}_2$: $-N_1\frac{\partial A_1}{\partial \alpha_2}+\frac{\partial(S_{12}A_2)}{\partial \alpha_1}+S_{21}\frac{\partial A_2}{\partial \alpha_1}+\frac{\partial(N_2A_2)}{\partial \alpha_2}+\frac{Q_2A_1A_2}{R_2}+A_1A_2q_2=0$

$\vec{e}_3$: $-N_1\frac{A_1A_2}{R_1}+\frac{\partial(Q_1A_2)}{\partial \alpha_1}-N_2\frac{A_1A_2}{R_2}+\frac{\partial(Q_2A_1)}{\partial \alpha_2}+A_1A_2q_3=0$

平衡方程的分量式.

而内矩的平衡条件如下:



首先: q 对于任一轴的力矩近似为

$q A_1A_2\, d\alpha_1 d\alpha_2 \cdot \frac{1}{2}A_1 d\alpha_1+\frac{1}{2}A_2d\alpha_2$

$\Rightarrow$ 为 $d\alpha$ 高阶微量 $\rightarrow$ 可略去, 内力矩考虑 Q, S, 则:

※ 弯矩和内力矩和为 0, 外矩考虑 M, H,

三阶, 略去

$$\left[M_1\vec{e}_2-H_{21}\vec{e}_2+\left[H_{21}\vec{e}_2+\frac{\partial(H_{21}\vec{e}_2)}{\partial \alpha_1}d\alpha_1\right]+\left[M_1\vec{e}_2+\frac{\partial M_1\vec{e}_2}{\partial \alpha_1}d\alpha_1\right]A_2d\alpha_2-(Q_1\vec{e}_2+\cdots)A_1d\alpha_1A_2d\alpha_2+(S_{12}\vec{e}_3+\cdots)A_1A_2d\alpha_1d\alpha_2\right]$$

$$+\left[M_2\vec{e}_1+H_{12}\vec{e}_1-\left[\left(M_2\vec{e}_1+\frac{\partial M_2\vec{e}_1}{\partial \alpha_2}d\alpha_2\right)+\left(H_{12}\vec{e}_1+\frac{\partial H_{12}\vec{e}_1}{\partial \alpha_2}d\alpha_2\right)\right]A_1d\alpha_1+(Q_2\vec{e}_1+\cdots)A_1d\alpha_1A_2d\alpha_2-(S_{21}\vec{e}_3+\cdots)A_1A_2d\alpha_1d\alpha_2\right]=0$$

$H_{12}+dH_{12}$

$$+\Big[M_2\vec{e}_1+H_{12}\vec{e}_1-\Big[\Big(M_2\vec{e}_1+\frac{\partial M_2\vec{e}_1}{\partial\alpha_2}d\alpha_2\Big)+\Big(H_{12}\vec{e}_1+\frac{\partial(H_{12}\vec{e}_1)}{\partial\alpha_1}d\alpha_1\Big)\Big]A_2 d\alpha_2+(Q_2\vec{e}_1+\cdots)A_1 d\alpha_1 A_2 d\alpha_2-(S_{21}\vec{e}_3+\cdots)A_1A_2\,d\alpha_1 d\alpha_2\Big]=0$$

$A_1d\alpha_1$ $A_2d\alpha_2$ $A_1d\alpha_1$

注意变的边长!

两排相加、

此时,略去 $d\alpha$ 的高阶小量,得到:

全部反方向

$$\left[\frac{\partial(A_1H_{21}\vec{e}_2)}{\partial\alpha_2}+\frac{\partial(A_2M_1\vec{e}_2)}{\partial\alpha_1}-Q_1A_1A_2\vec{e}_2+S_{12}A_1A_2\vec{e}_3\right]+\left[-\frac{\partial(A_1M_2\vec{e}_1)}{\partial\alpha_2}-\frac{\partial(A_2H_{12}\vec{e}_1)}{\partial\alpha_1}+Q_2A_1A_2\vec{e}_1-S_{21}A_1A_2\vec{e}_3\right]=0$$

代入微分公式(计算,略)有:

力矩平衡方程

$\vec{e}_2$: $\frac{\partial(A_1H_{21})}{\partial\alpha_2}+\frac{\partial(A_2M_1)}{\partial\alpha_1}+H_{12}\frac{\partial A_1}{\partial\alpha_2}-M_2\frac{\partial A_2}{\partial\alpha_1}-Q_1A_1A_2=0$ 化简有:

① $$\boxed{\frac{\partial(A_2M_1)}{\partial\alpha_1}+\frac{\partial(A_1H_{21})}{\partial\alpha_2}-M_2\frac{\partial A_2}{\partial\alpha_1}+H_{12}\frac{\partial A_1}{\partial\alpha_2}-Q_1A_1A_2=0}$$

$\vec{e}_1$: $-H_{21}\frac{\partial A_2}{\partial\alpha_1}+M_1\frac{\partial A_1}{\partial\alpha_2}-\frac{\partial(A_1M_2)}{\partial\alpha_2}-\frac{\partial(A_2H_{12})}{\partial\alpha_1}+Q_2A_1A_2=0$,得:

② $$\boxed{\frac{\partial(A_1M_2)}{\partial\alpha_2}+\frac{\partial(A_2H_{12})}{\partial\alpha_1}-M_1\frac{\partial A_1}{\partial\alpha_2}+H_{21}\frac{\partial A_2}{\partial\alpha_1}-Q_2A_1A_2=0}$$

$\vec{e}_3$: $-\frac{A_1A_2}{R_2}H_{21}+S_{12}A_1A_2+H_{12}\frac{A_2A_1}{R_1}-S_{21}A_1A_2=0$

③ $$\boxed{\frac{H_{12}}{R_1}-\frac{H_{21}}{R_2}=S_{21}-S_{12}}$$

由等式(切应力互等)

$\tau_{12}=\tau_{21}$

自然满足