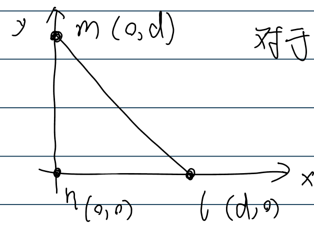
P34例题

Wednesday, March 22, 2023 5:46 PM



对于如图，厚为h的等腰直角三角形，直角边长d，单元的直角边为x、y轴，求其形状函数并建立单元刚度矩阵，

解：

设如图的等腰三角形各个节点的位移为$\{\delta\}^e$

即：$\{\delta\}^e = \begin{Bmatrix} u_l \\ v_l \\ u_m \\ v_m \\ u_n \\ v_n \end{Bmatrix}$，

设整个单元的位移函数为：

$$\begin{cases} u = a_1 + a_2 x + a_3 y \\ v = a_4 + a_5 x + a_6 y \end{cases}$$

则：$\{\delta\}^e = [A]\{a\}$

其中：

$$A = \begin{bmatrix} 1 & x_l & y_l & & & \\ & & & 1 & x_l & y_l \\ 1 & x_m & y_m & & & \\ & & & 1 & x_m & y_m \\ 1 & x_n & y_n & & & \\ & & & 1 & x_n & y_n \end{bmatrix}$$

有：对整个单元，有：$\{\delta\} = \begin{bmatrix} 1 & x & y & & & \\ & & & 1 & x & y \end{bmatrix}\{a\} = [M]\{a\}$，$= [M][A]^{-1}\{\delta\}^e$

其中令 $[M][A]^{-1} = [N]$ 为形函数，

有：

$$N_l = \frac{a_l + b_l x + c_l y}{2\Delta}$$ 并有 $$\Lambda = \begin{vmatrix} 1 & x_l & y_l \\ 1 & x_m & y_m \\ 1 & x_n & y_n \end{vmatrix} = \begin{vmatrix} 1 & d & 0 \\ 1 & 0 & d \\ 1 & 0 & 0 \end{vmatrix}$$

有：$a_l = 0$，$b_l = -\begin{vmatrix} 1 & d \\ 1 & 0 \end{vmatrix} = +d$，$c_l = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$

而 $\Delta = \frac{1}{2}|\Lambda| = \begin{vmatrix} 1 & d & 0 \\ 1 & 0 & d \\ 1 & 0 & 0 \end{vmatrix} = \frac{1}{2}d^2 = S_\Delta$

故有：$N_l = \frac{dx}{d^2} = \frac{x}{d}$，

$$N_m = \frac{a_m + b_m x + c_m y}{2\Delta}$$，$a_m = -\begin{vmatrix} d & 0 \\ 0 & 0 \end{vmatrix} = 0$，$b_m = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$，$c_m = -\begin{vmatrix} 1 & d \\ 1 & 0 \end{vmatrix} = d$

$$= \frac{dy}{d^2} = \frac{y}{d}$$

$$N_n = \frac{a_n + b_n x + c_n y}{2\Delta}$$，$a_n = d^2$，$b_n = -\begin{vmatrix} 1 & 0 \\ 1 & d \end{vmatrix} = -d$，$c_n = \begin{vmatrix} 1 & d \\ 1 & 0 \end{vmatrix} = -d$

$$= \frac{d^2 - dx - dy}{d^2} = \frac{d - x - y}{d} = 1 - \frac{1}{d}(x+y)$$

∴则代入：

$$\{\delta\} = [N]\{\delta^e\} = \begin{bmatrix} \frac{x}{d} & & \frac{y}{d} & & 1-\frac{x+y}{d} & \\ & \frac{x}{d} & & \frac{y}{d} & & 1-\frac{x+y}{d} \end{bmatrix} \begin{Bmatrix} u_l \\ v_l \\ u_m \\ v_m \\ u_n \\ \vdots \end{Bmatrix}$$

$$v_m,\ u_n,\ v_n$$

即:

$$N=\frac{1}{d}\begin{bmatrix} x & 0 & y & 0 & d-x-y & 0\\ 0 & x & 0 & y & 0 & d-x-y\end{bmatrix}$$

对于平面应力问题，可以得出应力矩阵为:

$$\text{而 } \varepsilon=\begin{bmatrix}\frac{\partial}{\partial x} & \\ & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}\quad \text{其中: } \begin{bmatrix}u\\v\end{bmatrix}=[N]\{\delta^e\}\quad \text{取}[\cdot][N]=B,$$

则:

$$\underline{\text{应变矩阵}}[B]=\begin{bmatrix}\frac{1}{d} & 0 & 0 & 0 & -\frac{1}{d} & 0\\ 0 & 0 & 0 & \frac{1}{d} & 0 & -\frac{1}{d}\\ 0 & \frac{1}{d} & \frac{1}{d} & 0 & -\frac{1}{d} & -\frac{1}{d}\end{bmatrix}=\frac{1}{d}\begin{bmatrix}1 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 & 0 & -1\\ 0 & 1 & 1 & 0 & -1 & -1\end{bmatrix}$$

则刚度矩阵可通过公式求得，有:

$$[k]^e=\begin{bmatrix}k_{ll} & k_{lm} & k_{ln}\\ k_{ml} & k_{mm} & k_{mn}\\ k_{nl} & k_{nm} & k_{nn}\end{bmatrix}\quad \text{其中}[k_{rs}]^e=\frac{Eh}{4(1-\mu^2)\Delta}\begin{bmatrix}b_rb_s+\frac{1-\mu}{2}c_rc_s, & \mu b_rc_s+\frac{1-\mu}{2}c_rb_s\\ \mu c_rb_s+\frac{1-\mu}{2}b_rc_s, & c_rc_s+\frac{1-\mu}{2}b_rc_s\end{bmatrix}$$

$$\text{有: } [b \; c]=\begin{array}{c} \\ 1\\ 2\\ 3\end{array}\begin{array}{c}\begin{array}{cc} b & c\end{array}\\ \begin{bmatrix} d & 0\\ 0 & d\\ -d & -d\end{bmatrix}\end{array}\quad \text{代入: 则:}$$

$$\begin{array}{c} \\ [k_{rs}]^e=\frac{Eh}{2(1-\nu^2)d^2}\end{array}\begin{array}{c}\begin{array}{cc} \quad\quad\quad\quad\quad\quad & r=1,s=2 \quad\quad\quad\quad r=1,s=3\end{array}\\ \begin{bmatrix} d^2 & 0 & 0, & \nu d^2 & -d^2 & -\nu d^2\\ 0 & \cancel{0}\ \frac{1-\mu}{2}d^2 & \frac{1-\nu}{2}d^2 & \frac{1-\nu}{2}d^2 & \frac{\nu-1}{2}d^2 & \frac{\nu-1}{2}d^2\\ \cancel{0} & \cancel{\nu d^2} & 0 & 0 & \frac{\nu-1}{2}d^2 & \frac{\nu-1}{2}d^2\\ \cancel{\frac{1-\nu}{2}d^2}, & \cancel{\frac{1-\nu}{2}} & 0 & d^2 & -\nu d^2 & -d^2\\ \cancel{-d^2} & \cancel{-\nu d^2} & \cancel{\frac{\nu-1}{2}d^2} & \frac{\nu-1}{2}d^2 & \frac{3-\nu}{2}d^2, & \frac{1+\nu}{2}d^2\\ \cancel{\frac{\nu-1}{2}d^2} & \cancel{\frac{\nu-1}{2}d^2} & \cancel{-\nu d^2} & -d^2 & \frac{1+\nu}{2}d^2 & \frac{3-\nu}{2}d^2\end{bmatrix}\end{array}$$

$r=2, S=1 \longrightarrow$
刚好反;

而

$$\frac{Eh}{2(1-\nu^2)d^2}\qquad d^2+\frac{1-\nu}{2}$$

则

$$k_{13}=\begin{bmatrix}-d^2, & -\nu d^2\\ \frac{1-\nu}{2}\cdot -d^2 & \frac{1-\nu}{2}-d^2\end{bmatrix}$$

$$k_{23}=\begin{bmatrix}-\frac{1-\nu}{2}d^2, & -\frac{1-\nu}{2}d^2\\ -\nu d^2, & -d^2\end{bmatrix}$$

即:

$$[k_{rs}]^e=\frac{Eh}{2(1-\nu^2)d^2}\begin{bmatrix}1 & 0 & 0 & \nu & -1 & -\nu\\ 0 & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & \frac{\nu-1}{2} & \frac{\nu-1}{2}\\ 0 & \frac{1-\nu}{2} & 0 & 0 & \frac{\nu-1}{2} & \frac{\nu-1}{2}\\ \nu & \frac{1-\nu}{2} & 0 & d^2 & -\nu & -1\\ -1 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -\nu & \frac{3-\nu}{2} & \frac{1+\nu}{2}\\ -\nu & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -1 & \frac{1+\nu}{2} & \frac{3-\nu}{2}\end{bmatrix}$$

为单元的刚度矩阵

附注: 实际上，对于平面问题，若材料物理性质相同，则几何形状相似的单元都有相同的位移刚度矩阵。

我们在上述题目中，可取: $\{\delta\}^e=[1,0,0,0,0,0]^T$，则由

$$[N]=\frac{1}{d}\begin{bmatrix} x & 0 & y & 0 & d-x-y & 0\\ 0 & x & 0 & y & 0 & d-x-y\end{bmatrix},\quad \text{则由 } \{\delta\}=[N]\{\delta^e\}$$

代入得:

$$u=\frac{x}{d},\quad v=0,$$

$$Eh\ \begin{Bmatrix}1\\0\end{Bmatrix}$$

代入得：

$u = \frac{x}{d}, \quad v = 0,$

显然：

$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{1}{d}, \quad \varepsilon_y = 0,$

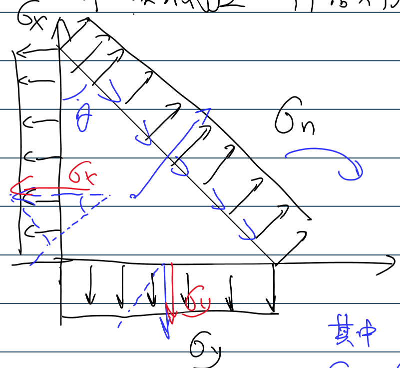
即：该式对应一种沿x方向的均匀拉伸状态；

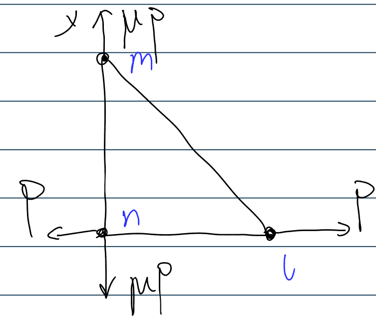
将其代入 $\{P\}^e = [k]^e\{\delta\}^e \Rightarrow \boxed{P_e = \frac{Eh}{2(1-\nu^2)d^2} \cdot \begin{Bmatrix} 1 \\ 0 \\ 0 \\ \nu \\ -1 \\ -\nu \end{Bmatrix}}$

我们可以令 $P = \frac{Eh}{2(1-\nu^2)d^2}$，

则：$P_i = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$，$P_m = \begin{Bmatrix} 0 \\ \nu P \end{Bmatrix}$，$P_n = \begin{Bmatrix} -P \\ -\nu P \end{Bmatrix}$

如右图所示





其中

$\sigma_n = \sigma_x \cos45° \cos45° + \sigma_y \sin45° \cos45°$

$= \frac{\sigma_x + \sigma_y}{2}$

利用应力矩阵

$[S] = [D][\varepsilon] = \frac{E}{2\Delta(1-\mu^2)} \begin{bmatrix} b_i & \mu c_i \\ \mu b_i & c_i \\ \frac{1-\mu}{2} c_i & \frac{1-\mu}{2} b_i \end{bmatrix}$ 其中 $[b \,\vdots\, c] = \begin{array}{c|cc} & b & c \\ \hline 1 & d & 0 \\ 2 & 0 & d \\ 3 & -d & -d \end{array}$

故有：

$[S] = \frac{E}{(1-\mu^2)d^2} \begin{bmatrix} d & 0 & 0 & \mu d & -d & -\mu d \\ \mu d & 0 & 0 & d & -\mu d & -d \\ 0 & \frac{1-\mu}{2}d & \frac{1-\mu}{2}d & 0 & -\frac{1-\mu}{2}d & -\frac{1-\mu}{2}d \end{bmatrix} = \frac{E}{\cancel{2}(1-\mu^2)d} \begin{bmatrix} 1 & 0 & 0 & \mu & -1 & -\mu \\ \mu & 0 & 0 & 1 & -\mu & -1 \\ 0 & \frac{1-\mu}{2} & \frac{1-\mu}{2} & 0 & \frac{\mu-1}{2} & \frac{\mu-1}{2} \end{bmatrix}$

2消去

因：$[\sigma] = [S]\{\delta\}$，代入有：$\sigma_x = \sigma$，

$\sigma_y = \mu\sigma$，其中 $\sigma = \frac{E}{\cancel{2}(1-\mu^2)d}$

$\tau_{xy} = 0$

故有：$\sigma_n = \frac{\sigma_x + \sigma_y}{2} = \frac{E(1+\mu)}{2(1-\mu^2)d} = \boxed{\frac{E}{2(1-\mu)d}}$

$\tau_n = \sigma_x \cos45° \sin45° - \sigma_y \sin45° \sin45°$

$= \frac{1}{2}(\sigma_x - \sigma_y) = \frac{E(1-\nu)}{2(1-\mu^2)d} \xrightarrow{\text{则有}} \tau_n = \frac{E}{2(1+\mu)d}$

为应力分布