Derivation for the Difference form of derivate

Friday, April 7, 2023 11:18 AM

for the equation of 1-order precision in the

Using Taylor's Series,

$$u_{i+1} = u_i + u_{ix}\Delta x + \frac{1}{2!}u_{ixx}\Delta x^2 + \frac{1}{3!}u_{ixxx}\Delta x^3 + \cdots \quad ①$$

$$u_{i-1} = u_i - u_{ix}\Delta x + \frac{1}{2!}u_{ixx}\Delta x^2 - \frac{1}{3!}u_{ixxx}\Delta x^3 + \cdots \quad ②$$

then we can easily derive:

1st-order front difference:

$$u_{ix} = \frac{u_{i+1}-u_i}{\Delta x} + \frac{1}{2!}u_{ixx}\Delta x + \cdots$$

$$= \frac{u_{i+1}-u_i}{\Delta x} + O(\Delta x)$$ has the precision of 1 order

also, for backward difference, we can easily derive:

$$u_{ix} = \frac{u_i - u_{i-1}}{\Delta x} + O(\Delta x)$$

Using ①−②, we have:

$$u_{ix} = \frac{u_{i+1}-u_{i-1}}{2\Delta x} + \frac{1}{3!}u_{ixxx}\Delta x^2 = \frac{u_{i+1}-u_{i-1}}{2\Delta x} + O(\Delta x^2)$$

is a second-order central difference equation.

if we sum eq ① and ② and eliminate the 1st order term

$$u_{i+1} + u_{i-1} = 2u + u_{ixx}\Delta x^2 + 2\times\frac{1}{4!}u_i^{(4)}\Delta x^4$$

then

$$u_{ixx} = \frac{u_{i+1}+u_{i-1}-2u}{\Delta x^2} + \underbrace{\frac{1}{12}u_i^{(4)}\Delta x^2} = \frac{u_{i+1}+u_{i-1}-2u}{\Delta x^2} + \underbrace{O(\Delta x^2)}_{\text{(2nd order truncacte error term)}}$$

for $U = U(x,y)$, we have the Taylor Series:

$$\rightarrow U_{i+1,j+1} = U_{ij} + \frac{\partial U_i}{\partial x}dx + \frac{\partial U_i}{\partial y}dy + \frac{1}{2!}\left[\frac{\partial^2 U_i}{\partial x^2}dx^2 + \frac{\partial^2 U_i}{\partial y^2}dy^2 + 2\frac{\partial^2 U_i}{\partial x \partial y}dxdy\right] + O(dx^2 y, dy x^2, dx^3, dy^3)$$

note its part of $U_{xy}dxdy + U_{yx}dxdy$

also consider that:

$$U_{i+1,j} = U_{ij} + \frac{\partial U}{\partial x}dx + \frac{1}{2!}\frac{\partial^2 U}{\partial x^2}dx^2$$

$$U_{i,j+1} = U_{ij} + \frac{\partial U_i}{\partial y}dy + \frac{1}{2!}\frac{\partial^2 U_i}{\partial y^2}dy^2$$

if we consider a 2nd order precision, we can strat from the 2nd order precision eq of 2nd order derivate.

[illegible]

if we consider a 2 order precision, we can strat from the 2 order precision eq of 2^{nd} order derivate:

$$U_{i+1,j+1} + U_{i-1,j-1} = 2U_{i,j} + \frac{\partial^2 U_i}{\partial x^2}dx^2 + \frac{\partial^2 U_i}{\partial y^2}dy^2 + 2\frac{\partial^2 U_i}{\partial x \partial y}dxdy \boxed{+O_1^3} + O_1^4 \quad ①$$

then

$$U_{i-1,j+1} + U_{i+1,j-1} = 2U_{ij} + \frac{\partial^2 U_i}{\partial x^2}dx^2 + \frac{\partial^2 U_i}{\partial y^2}dy^2 - 2\frac{\partial^2 U_i}{\partial x \partial y}dxdy \boxed{+O_2^3} + O_2^4 \quad ②$$

also note that O^3 in the equation above are:

$$O_1^3 = 2 \times \frac{1}{3!}[0+0+0+0] = 0$$

$$O_2^3 = 2 \times \frac{1}{3!}[0+0+0+0] = 0$$

subtracting ② from ① yields

$$U_{i+1,j+1} - U_{i-1,j+1} - U_{i+1,j-1} + U_{i-1,j-1} = 4\frac{\partial^2 U_i}{\partial x \partial y}dxdy + O^4$$

then

$$\frac{\partial^2 U_i}{\partial x \partial y} = \frac{U_{i+1,j+1} - U_{i-1,j+1} - U_{i+1,j-1} + U_{i-1,j-1}}{4\,dxdy} + \frac{O^4}{\Delta x \Delta y}$$

the error term is: $R = O(x^2, y^2)$

Also note that another method which is more comprehensive is using the second-order central derivate for the derivation of mixed derivate, that is: using

$$\left(\frac{\partial u}{\partial y}\right)\Big|_{i+1,j} = \frac{U_{i+1,j+1} - U_{i+1,j-1}}{2\Delta y}$$

where

$$U_{i+1,j+1} = u + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}dx^2 + \frac{\partial^2 u}{\partial y^2}dy^2 + 2\frac{\partial^2 u}{\partial x \partial y}dxdy\right) + \frac{1}{6}\left[\frac{\partial^3 u}{\partial x^3}dx^3 + 3\frac{\partial^3 u}{\partial x \partial y^2}dxdy^2 + 3\frac{\partial^3 u}{\partial x^2 \partial y}dx^2dy + \frac{\partial^3 u}{\partial y^3}dy^3\right] + \frac{1}{24}[\cdots]$$

also:

$$\left(\frac{\partial u}{\partial y}\right)\Big|_{i-1,j} = \frac{U_{i-1,j+1} - U_{i-1,j-1}}{2\Delta y}$$

we expend them more preciously that:

$$① \left(\frac{\partial u}{\partial y}\right)\Big|_{i+1,j} = \frac{1}{2dy}\left[2\frac{\partial u}{\partial y}dy + 2\frac{\partial^2 u}{\partial x \partial y}dxdy + \frac{\partial^3 u}{\partial x^2 \partial y}dx^2dy + \frac{1}{3}\frac{\partial^3 u}{\partial y^3}dy^3\right] + \frac{O^4}{2dy} = \frac{U_{i+1,j+1} - U_{i+1,j-1}}{2\Delta y}$$

$$② \left(\frac{\partial u}{\partial y}\right)\Big|_{i-1,j} = \frac{1}{2dy}\left[2\frac{\partial u}{\partial y}dy - 2\frac{\partial^2 u}{\partial x \partial y}dxdy + \frac{\partial^3 u}{\partial x^2 \partial y}dx^2dy + \frac{1}{3}\frac{\partial^3 u}{\partial y^3}dy^3\right] + \frac{O^4}{2dy} = \frac{U_{i-1,j+1} - U_{i-1,j-1}}{2\Delta y}$$

then we obtain the following equation by substituting dx, dy with Δx and Δy:

① − ②, hence:

$$\left(\frac{\partial u}{\partial y}\right)\Big|_{i+1,j} - \left(\frac{\partial u}{\partial y}\right)\Big|_{i-1,j} = \frac{1}{2\Delta y}\left[U_{i+1,j+1} - U_{i+1,j-1} - U_{i-1,j+1} + U_{i-1,j-1}\right] + \frac{O^4}{2\Delta y} = \frac{1}{\cancel{2\Delta y}}\left[\cancel{4}\,2\,\frac{\partial^2 u}{\partial x \partial y}\Delta x \cancel{\Delta y}\right]$$

$$\left(\frac{\partial u}{\partial y}\right)\Big|_{i+1,j} - \left(\frac{\partial u}{\partial y}\right)\Big|_{i-1,j} = \frac{1}{2\Delta y}\left[u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}\right] + \frac{O^4}{2\Delta y} = \frac{1}{\cancel{2\Delta y}}\left[\cancel{4}\frac{\partial^2 u}{\partial x \partial y}\Delta x \cancel{\Delta y}\right]$$

then: we have:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{4\Delta x \Delta y}\left[u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}\right] + \frac{O^4}{4\Delta x \Delta y}$$

the truncate error term is: $O\left(\frac{\Delta x^4, \Delta y^4}{\Delta x \Delta y}\right) = O^2$.

It has 2^{nd} order of precision.