圆柱扭转的协调方程推导

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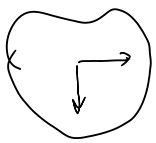
对端面：($z=\pm z_0$)

粗略去



$$\begin{cases} \sigma_x=0 \\ \sigma_y=0 \quad \sigma_z=0 \\ \tau_{xy}=0 \end{cases}$$

仅有 $\tau_{zx},\tau_{zy}\neq 0$，

则：代入平衡方程有：

$$③\quad \boxed{\frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{zy}}{\partial y}+0=0}$$

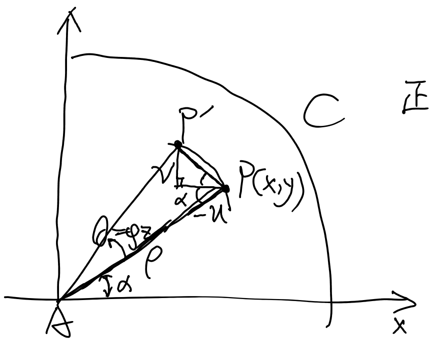
∴而由①/②方程有

$$\frac{\partial \tau_{zx}}{\partial z}=0,\quad \frac{\partial \tau_{zy}}{\partial z}=0$$

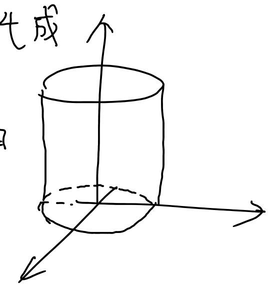
有：$\dfrac{\partial \tau_{zx}}{\partial x}=-\dfrac{\partial \tau_{zy}}{\partial y}$

此时：令 $\dfrac{\partial \phi}{\partial x}=\tau_{zy}$，$\dfrac{\partial \phi}{\partial y}=\tau_{zx}$ 则 ϕ 为普朗特应力函数。

有：$\tau_{zx}=\dfrac{\partial \phi}{\partial y}$，$\tau_{zy}=-\dfrac{\partial \phi}{\partial x}$，



有：截面的转角随柱体高度的变化成正比



有：转角 $\theta=\varphi z$，φ 为单位长度转角，是定值，

则有：$PP'=\rho\cdot\varphi z$，

则：近似认为转角 φ 为小量 φz

$$\therefore -u=PP'\cos(\tfrac{\pi}{2}-\alpha)=PP'\sin\alpha=\rho\varphi z\sin\alpha$$

$$v=PP'\sin(\tfrac{\pi}{2}-\alpha)=PP'\cos\alpha=\rho\varphi z\cos\alpha$$

此时有：

$$\begin{cases} u=-\rho\varphi z\sin\alpha=-y\varphi z \\ v=\rho\varphi z\cos\alpha=x\varphi z \end{cases}$$

有：

$$① \quad \boxed{\begin{cases} \gamma_{xz}=\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}=\dfrac{\partial w}{\partial x}-y\varphi \\ \gamma_{yz}=\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}=\dfrac{\partial w}{\partial y}+x\varphi \end{cases}}$$

体力分量为：

$$② \quad \boxed{\begin{cases} \sigma_x=\sigma_y=\sigma_z=\tau_{xy}=0 \\ \tau_{zx}=G\gamma_{xz}=G\left(\dfrac{\partial w}{\partial x}-y\varphi\right) \\ \tau_{zy}=G\gamma_{yz}=G\left(\dfrac{\partial w}{\partial y}+x\varphi\right) \end{cases}}$$

而由平衡方程有： 第3式，$\dfrac{\partial \sigma_z}{\partial z}=0$

$$\frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{zy}}{\partial y}+0=0$$

……对 y x 求导后再相减得：

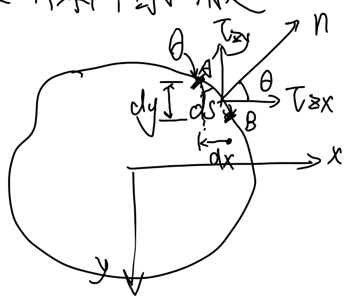
利用②中的方程；有：分别以2,3对y,x求导后再相减得：

$$\boxed{\frac{\partial \tau_{xz}}{\partial y} - \frac{\partial \tau_{zy}}{\partial x} = -2G\varphi}$$ 为φ满足的相容方程

将 $\tau_{xz} = \frac{\partial \phi}{\partial y}$, $\tau_{yz} = -\frac{\partial \phi}{\partial x}$ 代入得：

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\varphi}$$

边界条件的确定：



对于横截面,有:侧面边界条件

$\tau_{zx} \cdot l + \tau_{zy} \cdot m = 0$，设在边界上取线元ds,则：

其中：$l = \cos\langle n,x\rangle$， $m = \cos\langle n,y\rangle$.

$= \frac{dy}{ds}$ $\qquad\qquad = -\frac{dx}{ds}$

注意：从A→B点是以x→转到y的方向为正的方向.(因为顺时针,若 $y\uparrow \longrightarrow x$ 则逆)

如图中dx>0, dy>0. 代入：

有：$\tau_{zx}\frac{dy}{ds} + \tau_{zy} \cdot -\frac{dx}{ds} = 0$

$$\frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial x}\frac{dx}{ds} = 0 = \frac{d\psi}{ds} \Longrightarrow$$ 即有：$\psi = constant$,

由于应力应变分布仅与一阶导数有关，

则：取边界 $\boxed{\psi = 0}$ (沿周边C)

端部的弯矩条件为：

$$T = \iint_A (\tau_{zy}x - \tau_{zx}y)\,dxdy$$

$$= -\iint x\frac{\partial \psi}{\partial x}dxdy - \iint y\frac{\partial \psi}{\partial y}dxdy$$

$$= -\int dy \cdot \int x\frac{\partial \psi}{\partial x}dx - \int dx \int y\frac{\partial \psi}{\partial y}dy$$

进行分部积分 有：

$$= -\int dy \int x \frac{\partial \psi}{\partial x} dx - \int dx \int y \frac{\partial \psi}{\partial y} dy$$

进行分部积分，有：

$$-\left[\int dy \cdot \left[x\psi \Big|_{x_1}^{x_2} \right] - \iint \psi \, dx\,dy \right] - \left[\int dx \left[y\psi \Big|_{y_1}^{y_2} \right] - \iint \psi \, dx\,dy \right]$$

$$= 2\iint \psi \, dx\,dy$$ ← 由于在边界上有 $\psi = 0$，

则边界条件成为：$T = 2\iint \psi \, dx\,dy$