欧拉定理证明

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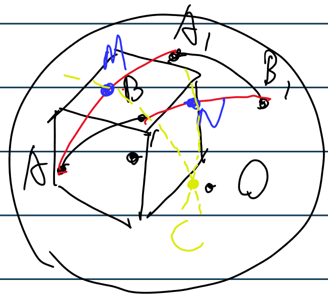
欧拉定理：作定点运动时，刚体任何位置的变化，可由此刚体绕过定点的某轴转动一次实现

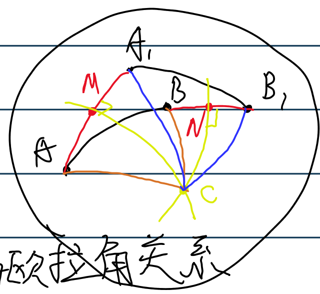


解：设刚体定点运动过程中，各点分别在以O为中心球上运动，只需使用一段圆弧$\overset{\frown}{AB}$代表运动轨迹，

首先做出弧$\overset{\frown}{AA_1}$，$\overset{\frown}{BB_1}$，取$\overset{\frown}{AA_1}$中点M，$\overset{\frown}{BB_1}$中点N，

然后过M做$CM \perp AA_1$且$CN \perp BB_1$

经连接得球面三角形ABC和A_1B_1C



显然C点是中垂线的交线，则

$AC = A_1C$，$BC = B_1C$，$AB = A_1B_1$

$\therefore \triangle ABC \cong \triangle A_1B_1C$

显然$\triangle A_1B_1C$可以由$\triangle ABC$经过绕C轴的一次旋转而得到。

方向余弦与欧拉角关系

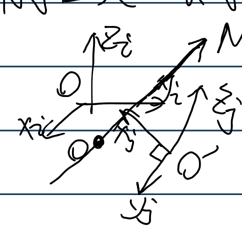
在进行欧拉角旋转，$Ox_jy_jz_j$相对$Ox_iy_iz_i$的欧拉轴ON可以使用单位矢$\vec{n}$表示

此时：给出ON，θ，

连体坐标系

为求解$[C^{ij}]$，取一z轴为ON的坐标系$Ox_ky_kz_k$，

并将其连同$x_iy_iz_i$旋转θ得到$O'x_jy_jz_j$，设$Ox_ky_kz_k$转至$Ox_my_mz_m$，则：



$$[C^{ik}] = [C^{jm}], = [C^{mj}]^T \qquad 显：[C^{mj}] = [C^{ik}]^T$$

则：

$$[C^{ij}] = [C^{ik}][C^{kj}] = [C^{ik}][C^{km}][C^{mj}]，$$ 利用$[C^{km}]$是绕轴转θ角，即：

$$[C^{km}] = \begin{bmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{bmatrix}，\quad 而：[C^{ij}] = [C^{ik}][C^{km}][C^{ik}]^T，$$

在k系下表示

我们设：ON的单位矢量$\vec{n} = (\vec{n_1}, \vec{n_2}, \vec{n_3})$，其中：$\vec{n_i} = \cos\langle x_i, z_k\rangle$，$\vec{n_j} = \cos\langle y_j, z_k\rangle$，$n_3 = \vec{n_k} = \cos\langle x_k, z_k\rangle$

则：

$$[C^{ik}] = \begin{bmatrix} a_{11} & a_{12} & n_1 \\ a_{21} & a_{22} & n_2 \\ a_{31} & a_{32} & n_3 \end{bmatrix}$$

则：

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$$[C_{ij}] = \begin{bmatrix} a_{11} & a_{12} & n_1 \\ a_{21} & a_{22} & n_2 \\ a_{31} & a_{32} & n_3 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ n_1 & n_2 & n_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\cos\theta + a_{12}\sin\theta & -a_{11}\sin\theta + a_{12}\cos\theta & n_1 \\ a_{21}\cos\theta + a_{22}\sin\theta & -a_{21}\sin\theta + a_{22}\cos\theta & n_2 \\ a_{31}\cos\theta + a_{32}\sin\theta & -a_{31}\sin\theta + a_{32}\cos\theta & n_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ n_1 & n_2 & n_3 \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}^2 + a_{12}^2)\cos\theta + n_1^2 & \cdots & \cdots \\ (a_{11}a_{21} + a_{12}a_{22})\cos\theta + \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{=n_3}\sin\theta + n_1 n_2 & (a_{21}^2 + a_{22}^2)\cos\theta + n_2^2 & \cdots \\ (a_{11}a_{31} + a_{12}a_{32})\cos\theta + \underbrace{(a_{11}a_{32} - a_{12}a_{31})}_{=-n_2}\sin\theta + n_1 n_3 & (a_{21}a_{31} + a_{22}a_{32})\cos\theta + (a_{21}a_{32} - a_{22}a_{31})\sin\theta + n_2 n_3 & (a_{31}^2 + a_{32}^2)\cos\theta + n_3^2 \end{bmatrix}$$

因于方向余弦阵为正交阵

有
$$\begin{cases} a_{11}^2 + a_{12}^2 + n_1^2 = 1 \\ a_{21}^2 + a_{22}^2 + n_2^2 = 1 \\ a_{31}^2 + a_{32}^2 + n_3^2 = 1 \\ a_{11}a_{21} + a_{12}a_{22} + n_1 n_2 = 0 \\ a_{11}a_{31} + a_{12}a_{32} + n_1 n_3 = 0 \\ a_{21}a_{31} + a_{22}a_{32} + n_2 n_3 = 0 \end{cases} \quad ①$$

$n_{j1} \cdot n_{j2}$

$n_{j1} \cdot n_{j3}$

由前一次 《刚体空间运动推导》：$e_m^i e_n^i = \delta_{mn}$

利用方向余弦阵性质（点乘）

$C_{1m}C_{1n} + C_{2m}C_{2n} + C_{3m}C_{3n} = \delta_{mn} = \begin{cases} 0 & (m \neq n) \\ 1 & (m = n) \end{cases}$

而：利用叉乘性质：

有：$[\tilde{e}_1^i]\{e_2^i\} = \{e_3^i\}$

$$\begin{bmatrix} & -a_{13} & a_{12} \\ +a_{13} & & -a_{11} \\ -a_{12} & a_{11} & \end{bmatrix} \begin{bmatrix} a_{21} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

此式可分别得：

$$\begin{cases} a_{12}a_{32} - a_{22}a_{13} = n_1 \\ a_{13}a_{21} - a_{32}a_{11} = n_2 \\ a_{11}a_{22} - a_{12}a_{21} = n_3 \end{cases} \quad ②$$

①② 代入有：

$$\begin{bmatrix} n_1^2(1-\cos\theta) + \cos\theta & n_1 n_2(1-\cos\theta) - n_3\sin\theta & n_1 n_3(1-\cos\theta) + n_2\sin\theta \\ n_1 n_2(1-\cos\theta) + n_3\sin\theta & n_2^2(1-\cos\theta) + \cos\theta & n_2 n_3(1-\cos\theta) - n_1\sin\theta \\ n_1 n_3(1-\cos\theta) - n_2\sin\theta & n_2 n_3(1-\cos\theta) + n_1\sin\theta & n_3^2(1-\cos\theta) + \cos\theta \end{bmatrix}$$

由此可推导出：

$$\begin{cases} \cos\theta = \frac{1}{2}(C_{11} + C_{22} + C_{33} - 1) \\ \sin\theta = \pm\frac{1}{2}\sqrt{(C_{11} + C_{22} + C_{33} + 1)(3 - C_{11} - C_{22} - C_{33})} \\ n_1 = \dfrac{C_{32} - C_{23}}{2\sin\theta} \\ n_2 = \dfrac{C_{13} - C_{31}}{2\sin\theta} \\ n_3 = \dfrac{C_{21} - C_{12}}{2\sin\theta} \end{cases}$$