刚体定点运动速度加速度合成定理推导

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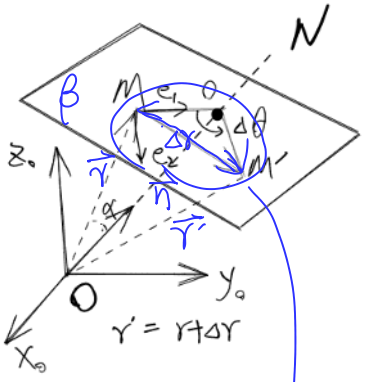
如图：做过 OM, OM' 的平面 β，并做 $\vec{e}_1 = \frac{\vec{M_0}}{|M_0|}$，$\vec{e}_2 = \frac{\vec{ON} \times \vec{OM}}{|\vec{ON}| \times |\vec{OM}|}$

（欧拉轴单位矢）



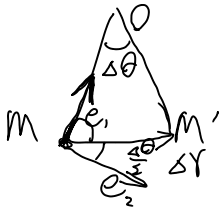
有：$|\vec{n} \times \vec{r}| = r\sin\alpha$

则 $|\vec{M_0}| = r\sin\alpha$

故有：$|\Delta r| = 2|M_0|\sin\frac{\Delta\theta}{2}$，

$= 2|r|\sin\alpha\sin\frac{\Delta\theta}{2}$

又：在上述矢量三角形内，有：



$\vec{e}_2$ 与 $\Delta\vec{r}$ 夹角 $\frac{\Delta\theta}{2}$

$\vec{e}_1$ 与 $\Delta\vec{r}$ 夹角 $\frac{\pi - \Delta\theta}{2}$

故由余弦定理：

$$\Delta r = |\Delta r| \cdot \cos\left(\frac{\pi - \Delta\theta}{2}\right)\vec{e}_1 + |\Delta r|\cos\left(\frac{\Delta\theta}{2}\right)\vec{e}_2,$$

$$\| \quad |\Delta r|\sin\frac{\Delta\theta}{2}\vec{e}_1 + |\Delta r|\cos\frac{\Delta\theta}{2}\vec{e}_2$$

有：$\vec{e}_1 = \frac{\vec{n} - \vec{r}}{|\vec{n} - \vec{r}|} = \frac{\vec{n} - \vec{r}}{r\sin\alpha}$

此时代入：$\vec{e}_2 = \frac{\vec{n} \times \vec{r}}{|\vec{n}||\vec{r}|\sin\alpha}$

$$\Delta r = 2|r|\sin\alpha\sin^2\frac{\Delta\theta}{2}\vec{e}_1 + 2|r|\sin\alpha\sin\frac{\Delta\theta}{2}\cos\frac{\Delta\theta}{2}\vec{e}_2$$

$$= 2|r|\sin\alpha\sin^2\frac{\Delta\theta}{2}(\vec{n} - \vec{r}) + |r|\sin\alpha\sin\Delta\theta \cdot \frac{\vec{n} \times \vec{r}}{|\vec{n}|}$$

$$= 2\sin^2\frac{\Delta\theta}{2}(\vec{n} - \vec{r}) + \frac{\vec{n} \times \vec{r}}{|\vec{n}|}\sin\Delta\theta$$

我们令 $|\vec{n}| = 1$，则：

$$\Delta r = 2\sin^2\frac{\Delta\theta}{2}(|r|\cos\alpha\vec{n} - \vec{r}) + \vec{n} \times \vec{r}\sin\Delta\theta$$

速度为 $\vec{v} = \lim_{\Delta t \to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$ → 绕 O 点做定点运动 故 $|r|$ 不变，α 不变

故：$\vec{v} = \lim_{\Delta t\to 0} \frac{(2\sin^2\frac{\Delta\theta}{2})}{\Delta t}\left(|\vec{r}|\cos\alpha\vec{n} - \vec{r}\right) + \lim_{\Delta t\to 0}\frac{(\vec{n}\sin\theta)}{\Delta t}\times\vec{r}$ 其中 由 $\lim AB = \lim A \lim B$

$= 2\lim_{\Delta t\to 0}\frac{\sin\frac{\Delta\theta}{2}}{\Delta t}\lim_{\Delta t\to 0}\left(\sin\frac{\Delta\theta}{2}\cdot|\vec{r}|\cos\alpha\vec{n} - \sin\frac{\Delta\theta}{2}\cdot\vec{r}\right) + \lim_{\Delta t\to 0}\left(\frac{\vec{n}\times\sin\Delta\theta}{\Delta t}\right)\times\vec{r}$

仅一阶小量

$= \cos\frac{\Delta\theta}{2}\cdot\frac{\Delta\theta}{\Delta t}\left[\lim_{\Delta t\to 0}\sin\frac{\Delta\theta}{2}\cdot|\vec{r}|\cos\alpha\vec{n} - \lim_{\Delta t\to 0}\sin\frac{\Delta\theta}{2}\cdot\vec{r}\right) + \lim_{\Delta t\to 0}\left(\frac{\vec{n}\sin\Delta\theta}{\Delta t}\right)\times\vec{r}$

$\to 0$; $\to 0$

$= \lim_{\Delta t\to 0}\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)\frac{\Delta\theta}{\Delta t}\ \vec{n}\times\vec{r}$ 这步处理

$= |\omega|\vec{n}\times\vec{r} = \vec{\omega}\times\vec{r}$.

故：$\vec{v} = \vec{\omega}\times\vec{r}$，

有：$\vec{a} = \frac{d\vec{v}}{dt} = \lim_{\Delta t\to 0}(\vec{\omega}\times\vec{r}) = \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt}$

此时：$= \vec{\varepsilon}\times\vec{r} + \vec{\omega}\times\vec{v} = \vec{\varepsilon}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$