

Cramer 法则证明

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对线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n = b_n \end{cases} \quad \text{其中 } D = |a_{ij}| \neq 0$$

$$\text{则: } x_j = \frac{D^{(j)}}{D} \quad j=1, 2, \dots, n$$

其中: $D^{(j)}$ 为将第 j 列换为 b_i 的行列式.

① 首先证明解的存在性:

我们有:

$$D^{(j)} = \begin{vmatrix} a_{11} & \dots & b_1 & \dots & a_{1n} \\ a_{21} & & b_2 & & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & b_n & \dots & a_{nn} \end{vmatrix} \xrightarrow{\text{按列展开}} D^{(j)} = b_1 A_{1j} + b_2 A_{2j} + \dots + b_n A_{nj}$$

此时: 只要说明 x_j 是对应解即可:

$$D^{(j)} = b_1 A_{1j} + b_2 A_{2j} + \dots + b_n A_{nj} \quad \text{则:}$$

$$a_{11}x_1 + \dots + a_{1n}x_n = a_{11} \frac{D^{(1)}}{D} + \dots + a_{1n} \frac{D^{(n)}}{D}$$

$$= \frac{a_{11}}{D} (b_1 A_{11} + b_2 A_{21} + \dots + b_n A_{n1}) + \frac{a_{12}}{D} (b_1 A_{12} + b_2 A_{22} + \dots + b_n A_{n2}) + \dots + \frac{a_{1n}}{D} (b_1 A_{1n} + b_2 A_{2n} + \dots + b_n A_{nn})$$

$$= \frac{1}{D} \begin{bmatrix} b_1 (a_{11}A_{11} + a_{12}A_{21} + \dots + a_{1n}A_{n1}) \\ b_2 (a_{11}A_{12} + a_{12}A_{22} + \dots + a_{1n}A_{n2}) \\ \vdots \\ b_n (a_{11}A_{1n} + a_{12}A_{2n} + \dots + a_{1n}A_{nn}) \end{bmatrix}$$

由定理:

有: 本列的元素与另一列元素代数余子式乘积之和为 0

则: 原式 而与本列元素的代数余子式 $\rightarrow D$

$$= \frac{b_i}{D} (a_{i1}A_{11} + \dots + a_{in}A_{in}) = b_i \quad \text{故 } x_j = \frac{D^{(j)}}{D} \text{ 是方程组的解}$$

②: 证明解的唯一性:

我们设 $x_1 = c_1, x_2 = c_2, \dots, x_n = c_n$ 是线性方程组的解,

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则有:

$$\begin{cases} a_{11}C_1 + a_{12}C_2 + \dots + a_{1n}C_n = b_1 \text{ ①} \\ a_{21}C_1 + a_{22}C_2 + \dots + a_{2n}C_n = b_2 \text{ ②} \\ \vdots \\ a_{n1}C_1 + a_{n2}C_2 + \dots + a_{nn}C_n = b_n \text{ ③} \end{cases}$$

我们取其中第 j 列元素的代数余子式 $A_{1j}, A_{2j}, \dots, A_{nj}$ 分别乘 ① ② ... ③,

$$\begin{cases} a_{11}A_{1j}C_1 + a_{12}A_{1j}C_2 + \dots + a_{1n}A_{1j}C_n = A_{1j}b_1 \\ a_{21}A_{2j}C_1 + \dots + a_{2n}A_{2j}C_n = A_{2j}b_2 \\ \vdots \\ a_{n1}A_{nj}C_1 + a_{n2}A_{nj}C_2 + \dots + a_{nn}A_{nj}C_n = A_{nj}b_n \end{cases}$$

相加有:

$$\sum_{k=1}^n \underbrace{a_{k1}A_{kj}}_{=0}C_1 + \sum_{k=1}^n \underbrace{a_{k2}A_{kj}}_{=0}C_2 + \dots + \sum_{k=1}^n a_{kn}A_{kj}C_n = \sum_{k=1}^n A_{kj}b_k$$

仅有第 j 项不为 0

$= D^{(j)}$

右端原是 $D^{(j)}$ 沿第 j 列展开式, 而左侧为

$$\sum_{k=1}^n a_{kj}A_{kj}C_j = D C_j = D^{(j)} \quad \text{故有: } C_j = \frac{D^{(j)}}{D}$$

解是唯一的.