Compare of 2D exact solution and numerical solution

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for 2D problems, $n=2$. In that case, we assume $w^h = c_1 N_1 + c_2 N_2$

for $N_1(x) = \begin{cases} 1-2x, & [0,\frac{1}{2}] \\ 0, & [\frac{1}{2},1] \end{cases}$ $N_2(x) = \begin{cases} 2x & [0,\frac{1}{2}] \\ 2(1-x) & [\frac{1}{2},1] \end{cases}$ $N_3(x) = \begin{cases} 0 & [0,\frac{1}{2}] \\ 2x-1 & [\frac{1}{2},1] \end{cases}$

$K = \begin{bmatrix} K_{11}, K_{12} \\ K_{21}, K_{22} \end{bmatrix}$ $F = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$ $d = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$

$K_{AB} = a(N_A, N_B) = \int_0^1 N_{A,x} N_{B,x}\, dx = \int_0^{\frac{1}{2}} N_{A,x} N_{B,x}\, dx + \int_{\frac{1}{2}}^1 N_{A,x} N_{B,x}\, dx$

显有: $N_{1,x} = -2$, $\therefore K_{11} = \int_0^{\frac{1}{2}} 4 \cdot dx = \frac{1}{2} \times 4 = 2$ $K_{12} = \int_0^{\frac{1}{2}} -2 \times 2\, dx + \int_{\frac{1}{2}}^1 0\, dx = -2$

$K_{21} = -2$, $K_{22} = \int_0^{\frac{1}{2}} 4\, dx + \int_{\frac{1}{2}}^1 4\, dx = 4$

$\therefore K = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} = 2\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \longrightarrow K^{-1} = \frac{1}{2}\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] = \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$-(N_A, l) + N_A(0)h - a(N_A, N_3)q$

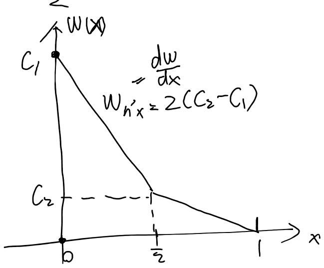
$\int_0^1 N_A l\, dx + \underbrace{N_A(0)h}_{0} - \int_{\frac{1}{2}}^1 N_{A,x} N_{3,x}\, q\, dx$

then $u^h = w^h + q N_3$

thus $\begin{cases} F_1 = \int_0^{\frac{1}{2}} (1-2x)\, l(x)\, dx + h \\ F_2 = \int_0^{\frac{1}{2}} 2x\, l(x)\, dx + \int_{\frac{1}{2}}^1 2(1-x)\, l(x)\, dx - \boxed{\int_{\frac{1}{2}}^1 -2\times 2\, dx \cdot q} \\ \quad = 2\int_0^{\frac{1}{2}} x\, l(x)\, dx + 2\int_{\frac{1}{2}}^1 (1-x)\, l(x)\, dx + 2q \end{cases}$

$+ 2q$

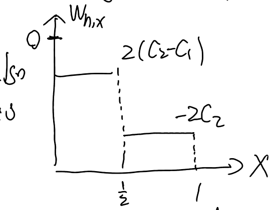
Note that due to the shape functions discontinuties in slope at $x = \frac{1}{2}$, It's convenient to express integrals over the subintervals $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$



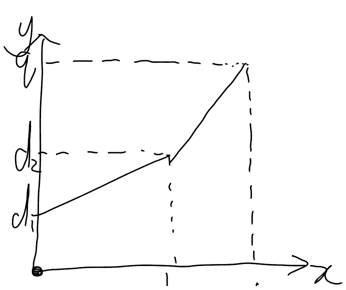
the w_h (weighting function) $w^h = c_1 N_1 + c_2 N_2$

$= \begin{cases} c_1(1-2x) + c_2(2x), & 0 \leq x \leq \frac{1}{2} \\ 2c_2(1-x) & \frac{1}{2} \leq x \leq 1 \end{cases}$

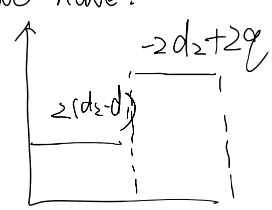
can also plot as



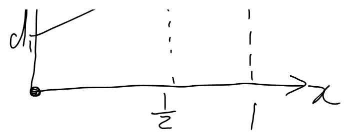
and also, the trial solution is $u^h = d_1 N_1 + d_2 N_2 + q N_3$



we have:



then we can easily obtain:



Also, we note that due to the shape functions' discontinuties, we always express the integrals / functions in two subintervals $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$, also note we needn't to worry about the derivate of N_A at $x=\frac{1}{2}$. This amount to employing the notion of a $\boxed{\text{generalized derivative}}$

In this example, the case can be shown as:

①: for $l=0$,
$$F=\begin{Bmatrix} h \\ 2q \end{Bmatrix}$$

exact solution
$$\begin{cases} F_1=\int_0^{1/2}(1-2x)\,l(x)\,dx+h \\ F_2=2\int_0^{\frac{1}{2}}x\,l(x)\,dx+2\int_{\frac{1}{2}}^{1}(1-x)\,l(x)\,dx+2q \end{cases}$$

$$d=K^{-1}F=\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} h \\ 2q \end{Bmatrix}=\begin{bmatrix} h+q \\ \frac{h}{2}+q \end{bmatrix}$$

then it results in
$$\begin{cases} u^h=d_1N_1+d_2N_2+qN_3=(q+h)N_1+(\frac{h}{2}+q)N_2+qN_3 \\ \quad=q(N_1+N_2+N_3)+h(N_1+\frac{N_2}{2}) \end{cases}$$
and substitute N_1, N_2, N_3 into it $\to$ we get
$$u^h(x)=q+(1-x)h$$

②: for $l(x)=p=$ constant, $F_1=(\frac{1}{2}-\frac{1}{4})p+h=h+\frac{p}{4}$

$$F_2=2\int_0^{\frac{1}{2}}xp\,dx+2\int_{\frac{1}{2}}^{1}(1-x)p\,dx+2q$$
$$=\frac{p}{4}+2\underbrace{\left(x-\frac{x^2}{2}\right)\Big|_{\frac{1}{2}}^{1}}_{=2\times(\frac{1}{2}-(\frac{1}{2}-\frac{1}{8}))=\frac{1}{4}}p+2q$$
$$=\frac{p}{4}+\frac{p}{4}+2q=2q+\frac{p}{2},$$

then:
$$F=\begin{Bmatrix} h+\frac{p}{4} \\ 2q+\frac{p}{2} \end{Bmatrix},$$

So:
$$\{d\}=\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} \frac{p}{4}+h \\ \frac{p}{2}+2q \end{Bmatrix}=\begin{bmatrix} \frac{p}{2}+h+q \\ \frac{3p}{8}+\frac{h}{2}+q \end{bmatrix}$$

$h(1-2x)+hx=h(1-x)$

$$u^h=d_1N_1+d_2N_2+qN_3=\left(\frac{p}{2}+h+q\right)\cdot(1-2x)+\left(\frac{3p}{8}+\frac{h}{2}+q\right)\cdot 2x$$
for $x\in[0,\frac{1}{2}]$

for $x \in [0, \frac{1}{2}]$

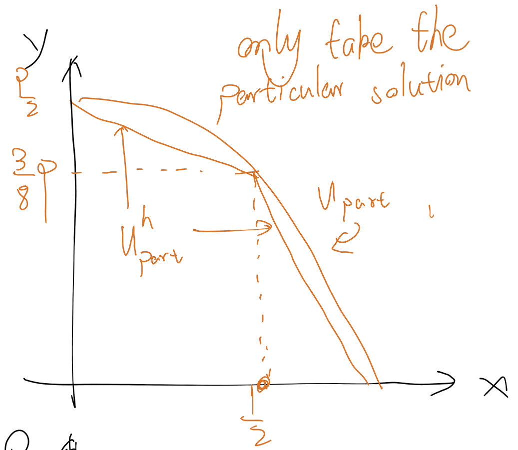
$$= q_f (1-x) h + u^h_{part}(x)$$

for $x \in [\frac{1}{2}, 1]$

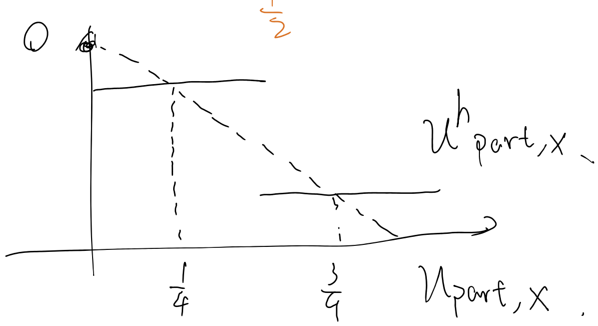
$$2(1-x)\left(\frac{3P}{8} + \frac{h}{2} + q\right) + (2x-1) q$$

$$= q_f (1-x) h + \frac{3P}{4}(1-x)$$

where: $\boxed{u^h_{part} = \frac{P}{2} N_1 + \frac{3P}{8} N_2}$ (particular solution)



so we can compare the exact solution and the particular solution in the left figure the agreement is achieved at $\boxed{x = 0, \frac{1}{2}}$, and the derivates coincide at $x = \frac{1}{4}$ and $x = \frac{3}{4}$.



also it can be derived ~~that~~ in the case $l(x) = qx$, the particular solution is same at point $0, \frac{1}{2}$ and 1, too;.