线性化声学波动方程的推导

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①：对流动有：

质量守恒：$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u_i)}{\partial x_i}=0$

$$\frac{\partial \rho}{\partial t}+\rho\frac{\partial u_i}{\partial x_i}+u_i\frac{\partial \rho}{\partial x_i}=0,\quad \rho=\rho_0+\rho^*$$

$$u_i=u_i^*$$

动量守恒：

$$\frac{\partial(\rho u_i)}{\partial t}+\frac{\partial(\rho u_i u_j)}{\partial x_j}+\frac{\partial p_i}{\partial x_i}=0$$

令 $\rho(x,t)=\rho_0(x)+\rho^*(x,t)$，$u_i=u_i^*(x,t)$

$$\frac{\partial \rho^*}{\partial t}+u_i^*\frac{\partial(\rho+\rho^*(x,t))}{\partial x_i}+(\rho_0+\rho^*)\frac{\partial u_i^*}{\partial x_i}=0$$

忽略高阶小扰动

$$\longrightarrow \frac{\partial \rho^*}{\partial t}+u_i^*\frac{\partial \rho_0}{\partial x_i}+\rho_0\frac{\partial u_i^*}{\partial x_i}=0$$

★我们考虑背景密度 ρ_0 为常数情况：

$\frac{\partial \rho_0}{\partial x_i}=0$，故得：$\frac{\partial \rho^*}{\partial t}+\rho_0\frac{\partial u_i^*}{\partial x_i}=0$ ①

② 动量方程

$$\rho\frac{\partial u_i}{\partial t}+\rho u_j\frac{\partial u_i}{\partial x_j}+\frac{\partial p}{\partial x_i}=0 \quad \text{展开}$$

$$(\rho_0+\rho^*)\frac{\partial(u_i^*)}{\partial t}+(\rho_0+\rho^*)\left[u_j^*\frac{\partial u_i^*}{\partial t}\right]+\frac{\partial(p_0+p^*)}{\partial x_i}=0$$

同理，认为 P_0 为常数 $\Longrightarrow \frac{\partial P_0}{\partial x_i}=0$

$$\rho_0\frac{\partial u_i^*}{\partial t}+\underbrace{\rho_0 u_j^*\frac{\partial u_i^*}{\partial t}}_{\text{仍然认为小，忽略}}+\frac{\partial p^*}{\partial x_i}=0$$

仍然认为小，忽略

$$\Longrightarrow \rho_0 \frac{\partial u_i^*}{\partial t} + \frac{\partial P^*}{\partial x_i} = 0 \quad ②$$

③：本构方程：

$$\frac{dP}{d\rho} = \frac{P}{\rho}\gamma$$ $\Longrightarrow$ 做微进的简化，由于 P_0 ρ_0 均接

$$dp^* = \frac{P_0}{\rho_0}\gamma\, d\rho^*$$

积分 $\Longrightarrow$ $$p^* = \frac{P_0}{\rho_0}\gamma \rho^* \quad ③ \quad \longrightarrow \rho^* = \frac{\rho_0}{P_0\gamma} p^*$$

联①-③有：

$$\begin{cases} \rho_0 \dfrac{\partial u_i}{\partial t} + \dfrac{\partial p^*}{\partial x_i} = 0 & (1) \\ \dfrac{\rho_0}{P_0\gamma} \dfrac{\partial p^*}{\partial t} + \rho_0 \dfrac{\partial u_i^*}{\partial x_i} = 0 & (2) \end{cases}$$

③→①

对消 p^*，(1) 两边对 t 导，(2) 两边对 x 导，有：

$$\rho_0 \frac{\partial u_i}{\partial t^2} + \cancel{\frac{\partial^2 p^*}{\partial x_i \partial t}} - \cancel{\frac{\partial^2 p^*}{\partial t \partial x_i}} - P_0\gamma \frac{\partial^2 u_i^*}{\partial x_i^2} = 0$$

$$\Longrightarrow \frac{\rho_0}{P_0\gamma}\frac{\partial^2 u_i^*}{\partial t^2} - \frac{\partial^2 u^*}{\partial x_i^2} = 0, \quad 取\ c = \sqrt{\frac{\gamma P_0}{\rho_0}}$$

$$\longrightarrow \frac{1}{c^2}\frac{\partial^2 u^*}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

Helmholtz 方程：

$$(\nabla^2 + k^2)\tilde{p}' = 0$$

$$\left(R''\phi + \frac{1}{\gamma}R'\phi + \frac{1}{\gamma^2}R\phi'' + k^2\right)\tilde{p}' = 0$$

$$\left(\gamma^2 R''\phi + \gamma R'\phi + R\phi'' + \gamma^2 k^2 R\phi\right) = 0$$

$$\gamma^2 R''\phi + \gamma R'\phi + R\phi'' + \gamma^2 k^2 R\phi = 0$$

$$r^2 R''\phi + rR'\phi + R\phi'' + r^2k^2R\phi = 0$$

☆ 分开 r, ϕ 到两边，并同除

$$r^2R''\phi + rR'\phi + r^2k^2R\phi = -R\phi''$$

$$r^2\frac{R''}{R} + \frac{rR'}{R} + r^2k^2 = -\frac{\phi''}{\phi} = n^2$$

取上式 $= n^2$，则 $\phi'' + n^2\phi = 0$，

$$\phi = A\sin n\phi + B\cos n\phi = \Phi_0 e^{in\phi}$$

而：$r^2\frac{R''}{R} + r\frac{R'}{R} + (r^2k^2 - n^2) = 0$

令 $x = kr$，有：

$R(r) = y(x)$ $\longrightarrow$ 此时 $y_r = ky_x$

$$\frac{x^2}{k^2}\frac{y''}{y}k^2 + \frac{x}{k}\frac{y'}{y}k + x^2 - n^2 = 0$$

$$\boxed{x^2y'' + xy' + (x^2-n^2)y = 0}, \quad \forall n \in N,$$

为 Bessel 方程.