3.1.2 Pission 公式

对：$\begin{cases} \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t) & ① \\ u(x,0) = \varphi(x) & ② \end{cases}$，未提供 $\varphi(x)$ 边界

对①Fourier 变换，有　　由 $\left(\frac{d^m y}{dx^m}\right)^\wedge = (jw)^m$

$$\frac{\partial}{\partial t}\hat{u} - a^2 (jw)^2 \hat{u} = \hat{f}(w,t)$$

$$\begin{cases} ① & \frac{\partial}{\partial t}\hat{u} + a^2\lambda^2 \hat{u} = \hat{f}(\lambda,t) \\ ② & \hat{u}(\lambda, 0) = \hat{\varphi}(\lambda) \end{cases}$$

其中 $\hat{u}(\lambda,t)$ 为解 $u(x,t)$ 关于 x 的 Fourier 变式

对一阶偏微分方程，特征方程为 $\lambda + a^2\lambda^2 = 0$.

通解 取 $r = -a^2\lambda^2$，则 $\hat{u} = C \cdot e^{-a^2\lambda^2 t}$，有

$t=0$ 时，得：$C = \hat{\varphi}(\lambda t)$ 代入有：

$$-a^2\lambda^2 \hat{\varphi}_t(\lambda) e^{-a^2\lambda^2 t} + a^2\lambda^2 \hat{\varphi}(\lambda) e^{-a^2\lambda^2 t} = \hat{f}(\lambda,t)$$

$$a^2\lambda^2 e^{-a^2\lambda^2 t}(\varphi(\lambda) - \hat{\varphi}_t(\lambda)) = \int_{-\infty}^{+\infty} f(\lambda,t)\, e^{-j\lambda t}\, dt.$$

取一特解为 $u(x,t)$ 关于 x 的 Fourier 变式，为：

$$u = \int_0^t \hat{f}(\lambda,\tau)\, e^{-a^2\lambda^2(t-\tau)}\, d\tau \text{，则 } \frac{\partial u}{\partial t} + a^2\lambda^2 \hat{u}$$

由 ~~P(t)x+~~ $\frac{d\hat{u}}{dt} + P(t)\cdot \hat{u} = Q(t)$ $= \hat{f}(\lambda,t) + a^2\lambda^2 \int_0^t f(\lambda,\tau) e^{-a^2\lambda^2(t-\tau)} d\tau$

计算积分因子：$\mu(t) = e^{\int P(t)dt} = e^{\int a^2\lambda^2 dt}$ $\boxed{\text{两边同乘 } \mu(t) \text{ 有}}$ —一阶偏微分方程特解求法

同乘有：$\underbrace{e^{a^2\lambda^2 t}\frac{d\hat{u}}{dt} + e^{a^2\lambda^2 t} a^2\lambda^2 \hat{u}}_{} = e^{a^2\lambda^2 t}\hat{f}(\lambda,t)$

乘因子后，可合为一项

$$\frac{d}{dt}\left(\hat{u}\, e^{a^2\lambda^2 t}\right) = e^{a^2\lambda^2 t} \hat{f}(\lambda,t) \text{，两边积分 有}$$

$$\hat{u} e^{a^2\lambda^2 t} = \int_0^t e^{a^2\lambda^2 \tau} \hat{f}(\lambda,\tau)\, d\tau$$

故 $\hat{u} = \int_0^t \hat{f}(\lambda,\tau)\, e^{-a^2\lambda^2(t-\tau)} d\tau$ 为一特解 ✓.

故解为

$$\hat{u} = \hat{\varphi}(\lambda) e^{-a^2\lambda^2 t} + \int_0^t \hat{f}(\lambda,\tau) e^{-a^2\lambda^2(t-\tau)} d\tau.$$

由于 $\hat{u}(x,t)=\hat{\varphi}e^{-a^2\lambda^2 t}+\int_0^t (\hat{f}(\lambda,\tau)e^{-a^2\lambda^2(t-\tau)}d\tau$

则反变换有:

$$u(x,t)=\left(\hat{\varphi}e^{-a^2\lambda^2 t}\right)^{\vee}+\int_0^t\left(\hat{f}(\lambda,\tau)e^{-a^2\lambda^2(t-\tau)}\right)^{\vee}d\tau$$

考虑到 $(e^{-Ax^2})^{\wedge}=\frac{1}{\sqrt{2A}}e^{-\frac{\lambda^2}{4A}}$，取 $\frac{1}{4A}=a^2t \to A=\frac{1}{4a^2t}$

则 $\underbrace{\left(e^{-\frac{x^2}{4a^2t}}\right)^{\wedge}=\sqrt{2a^2t}\,e^{-a^2\lambda^2t}\to\left(e^{-a^2\lambda^2t}\right)^{\vee}=\frac{1}{a\sqrt{2t}}e^{-\frac{x^2}{4a^2t}}}$ 取为 $g(x,t)$

则有:

$$\boxed{\left(\hat{\varphi}e^{-a^2\lambda^2t}\right)^{\vee}=\left(\hat{\varphi}\hat{g}\right)^{\vee}}=\left((\varphi*g)^{\wedge}\right)^{\vee}=\varphi*g$$

卷积性质

$$=\underbrace{\frac{1}{a\sqrt{2t}}\int_{-\infty}^{+\infty}\varphi(\xi)\boxed{e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi}}\quad ①$$

$\frac{1}{a\sqrt{2t}}$ // $\Leftarrow K(x-\xi,t)$

同理有:

$\left(\hat{f}(\lambda,t)e^{-a^2\lambda^2(t-\tau)}\right)^{\vee}$，取 $A=\frac{1}{4a^2(t-\tau)}$，有:

$$=\underbrace{\frac{1}{a\sqrt{2(t-\tau)}}\int_{-\infty}^{+\infty}f(\xi,t)\boxed{e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}d\xi}}\quad ②$$

$\Rightarrow$ 令其 $=K(x-\xi,t-\tau)$

热核函数

代入 $u(x,t)$ 表达式中，则得:

$$u(x,t)=\int_{-\infty}^{+\infty}K(x-\xi,t)\varphi(\xi)d\xi$$
$$+\int_0^t\int_{-\infty}^{+\infty}K(x-\xi,t-\tau)f(\xi,t)\,d\xi\,d\tau$$