偏微分方程的相似变换推导

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①: $T_{xx}+T_{yy}=0$

取 $\eta(x,y)=\frac{y}{x}$,

$$\begin{cases} T_x = T_\eta \eta_x = -\frac{y}{x^2}f'(\eta) \\ T_{xx} = \frac{2y}{x^3}f'(\eta)+\frac{y^2}{x^4}f''(\eta) \\ T_y = \frac{1}{x}f'(\eta) \\ T_{yy} = \frac{1}{x^2}f''(\eta) \end{cases}$$

则 $f(\eta) = T_{xx}+T_{yy} = \frac{y^2}{x^4}f''(\eta)+\frac{2y}{x^3}f'(\eta)+\frac{1}{x^2}f''(\eta)=0$

同乘 x^2 有:

$$\eta^2 f''(\eta)+2\eta f'(\eta)+f''(\eta)=0$$

$$\longrightarrow (\eta^2+1)f''(\eta)+2\eta f'(\eta)=0$$

② 二维椭圆型

$$T_{xx}+T_{xy}+T_{yy}=0$$

取 $S(x,y)=\frac{(2y-x)}{\sqrt{3}x}$ 取 $T=f(S)$

$$S_x = \frac{-\sqrt{3}x-(2y-x)\cdot\sqrt{3}}{3x^2} = \frac{2y}{\sqrt{3}x^2} \qquad S_{xy}=\frac{2}{\sqrt{3}x^2}$$

$$S_{xx} = \frac{-4\sqrt{3}xy}{3x^4} = -\frac{4y}{\sqrt{3}x^3}$$

$$S_y = \frac{2}{\sqrt{3}x} \qquad S_{yy}=0$$

$$T_{xx} = \left[f'(s)S(x)\right]' = f''(s)S_x^2+f'(s)S_{xx}$$

$$T_{xy} = \left[f'(s)\, S_x\right]_y = f''(s)\, S_x S_y + f'(s)\, S_{xy}$$

$$T_{yy} = \left[f'(s)\, S_y\right]_y = f''(s)\, S_y^2 + f'(s)\, S_{yy}$$

故:

$$T_{xx} + T_{xy} + T_{yy}$$

$$= f''(s)\left[S_x^2 + S_y^2 + S_x S_y\right] + f'(s)\left(S_{xx} + S_{xy} + S_{yy}\right)$$

$$= f''(s)\left(\frac{4y^2}{3x^4} + \frac{4}{3x^2} + \frac{4y}{3x^3}\right) + f'(s)\left(-\frac{4y}{\sqrt{3}x^3} + \frac{2}{\sqrt{3}x^2}\right)$$

$$= \frac{1}{x^2}\left(\frac{4x^2+4xy+4y^2}{3x^2}\right) f''(s) - 2 \cdot \frac{2y-x}{\sqrt{3}\,x} f'(s) \cdot \frac{1}{x^2} = 0$$

同乘 x^2

$$\left(\frac{x^2+4xy+4y^2}{3x^2} + 1\right) f''(s) - 2s f'(s) = 0$$

$$(s^2+1)\, f''(s) - 2s f'(s) = 0$$ 统一形式