薄板弯曲变分方程的推导

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我们已知：外力做的功：

$$\overline{W}=\iint_P q w\,dx\,dy-\int_{L_f}M_n\frac{\partial w}{\partial n}ds+\int_{L_f}\left(Q_n+\frac{\partial M_{nx}}{\partial s}\right)w\,ds$$

依据最小势能原理，在满足位移边界条件的所有可能位移中，真实的位移使总势能取最小值

$\longrightarrow$ 即：此时势能变分为0，

则 $\delta\Pi=\delta U-\iint_F q\,\delta w\,dx\,dy+\int_{L_f}M_n\frac{\partial(\delta w)}{\partial n}ds-\int_{L_f}\left(Q_n+\frac{\partial M_{nx}}{\partial s}\right)\delta w\,ds$，

我们知道：

$$U=\frac{D}{2}\iint_F\left\{(\nabla^2 w)^2-2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}dx\,dy$$

$$\delta U=D\iint_F\left\{(\nabla^2 w)(\nabla^2\delta w)-(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\delta w)}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\delta w)}{\partial x^2}-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2(\delta w)}{\partial x\partial y}\right]\right\}dx\,dy \quad ①$$

由 $(\nabla^2 w)(\nabla^2\delta w)=\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\left(\frac{\partial^2(\delta w)}{\partial x^2}+\frac{\partial^2(\delta w)}{\partial y^2}\right)=\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\delta w)}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2(\delta w)}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial^2(\delta w)}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\delta w)}{\partial y^2}$.

代入得 $=\delta U=\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\delta w)}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\delta w)}{\partial y^2}\right]+\nu\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\delta w)}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\delta w)}{\partial x^2}\right]+2(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2(\delta w)}{\partial x\partial y}$.

我们给出以下的变分变换：★

① $\iint_F\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2(\delta w)}{\partial x^2}dx\,dy=\iint_F\frac{\partial^4 w}{\partial x^4}\delta w\,dx\,dy+\oint_{L_f}\frac{\partial^2 w}{\partial x^2}\cos^2\alpha\frac{\partial(\delta w)}{\partial n}ds-\oint_{L_f}\frac{\partial}{\partial s}\left[\left(\frac{\partial^2 w}{\partial x^2}\sin\alpha\cos\alpha\right)-\frac{\partial^3 w}{\partial x^3}\cos\alpha\right]\delta w\,ds$,

② $\iint_F\frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial^2(\delta w)}{\partial y^2}dx\,dy=\iint\frac{\partial^4 w}{\partial y^4}\delta w\,dx\,dy+\oint_{L_f}\frac{\partial^2 w}{\partial y^2}\sin^2\alpha\frac{\partial(\delta w)}{\partial n}ds.$
$-\oint_{L_f}\left[\frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial y^2}\sin\alpha\cos\alpha\right)+\frac{\partial^3 w}{\partial y^3}\sin\alpha\right]\delta w\,ds.$

③ $\iint_F\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2(\delta w)}{\partial y^2}dx\,dy=\iint_F\frac{\partial^4 w}{\partial x^2\partial y}\delta w\,dx\,dy+\oint_{L_f}\frac{\partial^2 w}{\partial x^2}\sin^2\alpha\frac{\partial(\delta w)}{\partial n}ds$
$-\oint_{L_f}\left\{\frac{\partial}{\partial s}\left[\frac{\partial^2 w}{\partial x\partial y}(\sin^2\alpha-\cos^2\alpha)\right]+\frac{\partial^3 w}{\partial x^2\partial y}\sin\alpha\right\}\delta w\,ds$

④ $\iint_F\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\delta w)}{\partial x^2}dx\,dy=\iint_F\frac{\partial^4 w}{\partial x^2\partial y^2}\delta w\,dx\,dy+\oint_{L_f}\frac{\partial^2 w}{\partial y^2}\cos^2\alpha\frac{\partial(\delta w)}{\partial n}ds$
$+\oint_{L_f}\frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial x^2}\sin\alpha\cos\alpha-\frac{\partial^3 w}{\partial x\partial y^2}\cos\alpha\right)\delta w\,ds$

⑤ $2\iint_F\frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial^2(\delta w)}{\partial x\partial y}dx\,dy=2\iint_F\frac{\partial^4 w}{\partial x^2\partial y^2}\delta w\,dx\,dy+2\oint_{L_f}\frac{\partial^2 w}{\partial x\partial y}\sin\alpha\cos\alpha\frac{\partial(\delta w)}{\partial n}ds$
$+\oint_{L_f}\left\{\frac{\partial}{\partial s}\left[\frac{\partial^2 w}{\partial x\partial y}(\sin^2\alpha-\cos^2\alpha)\right]-\frac{\partial^3 w}{\partial x\partial y^2}\cos\alpha+\frac{\partial^3 w}{\partial x^2\partial y}\sin\alpha\right\}\delta w\,ds$

由①

最终有

$$⑤\ 2\iint_F \frac{\partial^2 w}{\partial x\partial y}\cdot\frac{\partial^2 \delta w}{\partial x\partial y}dxdy = 2\iint_F \frac{\partial^4 w}{\partial x^2\partial y^2}\delta w\,dxdy + 2\oint_{L_f}\frac{\partial^2 w}{\partial x\partial y}\sin\alpha\cos\alpha\frac{\partial \delta w}{\partial n}ds$$

$$+\oint_{L_f}\left\{\frac{\partial}{\partial s}\left[\frac{\partial^2 w}{\partial x\partial y}(\sin^2\alpha-\cos^2\alpha)\right]-\frac{\partial^3 w}{\partial x\partial y^2}\cos\alpha+\frac{\partial^3 w}{\partial x^2\partial y}\sin\alpha\right\}\delta w\,ds$$

由①

最终有:

$$\delta U = D\iint_F\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\left(\frac{\partial^2 \delta w}{\partial x^2}+\frac{\partial^2 \delta w}{\partial y^2}\right)-(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\delta w)}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\delta w)}{\partial x^2}-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2(\delta w)}{\partial x\partial y}\right]\right\}dxdy.$$

则有: $\delta\Pi = \delta U - \iint_A q\,\delta w\,dxdy + \underline{\oint_{L_f} M_n\frac{\partial(\delta w)}{\partial n}ds - \oint_{L_f}\left(Q_n+\frac{\partial M_{ns}}{\partial s}\right)\delta w\,ds}$

$$= \iint_A (D\nabla^2\nabla^2 w - q)\,\delta w\,dxdy + \oint_{L_f}\left\{D\left[(1-\nu)\left(\frac{\partial^2 w}{\partial x^2}\cos^2\alpha+2\frac{\partial^2 w}{\partial x\partial y}\cos\alpha\sin\alpha+\frac{\partial^2 w}{\partial y^2}\sin^2\alpha\right)+\nu\nabla^2 w\right]+M_n\right\}\frac{\partial(\delta w)}{\partial n}ds$$