平均法推导Duffing方程的自由振动

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Duffing 方程的自由振动方程可以写为:

$$\ddot{x}+\omega_0^2(x+\varepsilon x^3)=0$$

则:平均法中:设x的表达式为:

(1) $x = A\cos(\omega_0 t-\phi)$,其中A,ϕ为时间的函数

若不考虑A,ϕ,则:

$$\begin{cases} \dot{x} = -A\omega_0\sin(\omega_0 t-\phi) \quad (2) \\ \ddot{x} = -A\omega_0^2\cos(\omega_0 t-\phi) \end{cases}$$

则:由(1)导数:

$\dot{x} = \dot{A}\cos(\omega_0 t-\phi) - A(\omega_0-\dot{\phi})\sin(\omega_0 t-\phi)$,代入有:

① $\dot{A}\cos(\omega_0 t-\phi)+A\dot{\phi}\sin(\omega_0 t-\phi)=0$

又(2)求导:

$\ddot{x} = -\dot{A}\omega_0\sin(\omega_0 t-\phi)-(\omega_0-\dot{\phi})A\omega_0\cos(\omega_0 t-\phi)$ → 代入:

由 $\ddot{x}+\omega_0^2(x+\varepsilon x^3)=0$,有:

$$-\dot{A}\omega_0\sin(\omega_0 t-\phi)-(\cancel{\omega_0}-\dot{\phi})A\omega_0\cos(\omega_0 t-\phi)+\cancel{\omega_0^2 A\cos(\omega_0 t-\phi)}$$
$$+\omega_0^2\varepsilon x^3=0$$

则有: $-\dot{A}\omega_0\sin(\omega_0 t-\phi)+\dot{\phi}A\omega_0\cos(\omega_0 t-\phi) = -\omega_0^2\varepsilon x^3$ ②.

联立方程①②,有:

$$\begin{cases} \dot{A}\cos(\omega_0 t-\phi)+A\dot{\phi}\sin(\omega_0 t-\phi)=0 \\ -\dot{A}\sin(\omega_0 t-\phi)+A\dot{\phi}\cos(\omega_0 t-\phi) = -\frac{\varepsilon}{\omega_0}(\omega_0^2 x^3) \end{cases}$$

得:

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$$\begin{cases}\dot{A} = \dfrac{\varepsilon}{\omega_0}(\omega_0^2 x^3)\sin(\omega_0 t-\phi) = \dfrac{\varepsilon}{\omega_0} Q \\ \dot{\phi} = -\dfrac{\varepsilon}{\omega_0 A}(\omega_0^2 x^3)\cos(\omega_0 t-\phi) = -\dfrac{\varepsilon}{\omega_0 A} P\end{cases}$$

故有：我们将 P, Q 进行积分并取平均值

$$P = \frac{1}{T}\int_0^T \omega_0^2 A^3 \cos^4(\omega_0 t-\phi) = \frac{2}{\pi}\omega_0^2 A^3 \times \frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{3\omega_0^2 A^3}{8}$$

$$Q = \frac{1}{T}\int_0^T \omega_0^2 A^3 \underbrace{\cos^3(\omega_0 t-\phi)\sin(\omega_0 t-\phi)\,dt}_{\text{凑} = d(\cos(\omega_0 t-\phi))} = 0$$

则：$\dot{A} = 0$, $\dot{\phi} = -\dfrac{\varepsilon}{\omega_0 A}\times\dfrac{3\omega_0^2 A^3}{8} = -\dfrac{3\varepsilon\omega_0 A^2}{8}$

即新的角速度为 $\omega_0 - \dot{\phi}$

$$= \omega_0 + \frac{3\varepsilon\omega_0 A^2}{8}$$