

*Lecture notes from*

# **Matrix Algebra for Engineers**

The Hong Kong University of Science and Technology

Presented by



**Jeffrey R. Chasnov**

On the  web site

## Addition & multiplication of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

$$k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$


$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} \underline{ae} + bg & af + bh \\ ce + dg & cf + \underline{dh} \end{pmatrix}$$

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \underline{ae} + cf & be + df \\ ag + ch & bg + \underline{dh} \end{pmatrix}$$

$$(m \times n) \cdot (n \times p)$$

$$\rightarrow m \times p$$

$$C = AB$$

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$


Addition and Multiplication of Matrices



## Special matrices

Zero matrix:  $m \times n$

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$AX = O$$

$$m \times n \cdot n \times 1 = m \times 1$$

Identity matrix:  $n \times n$

$$AI = A = IA$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

Banded matrix  
eg. tridiagonal

$$\begin{pmatrix} d_1 & a_1 & 0 \\ b_1 & d_2 & a_2 \\ 0 & b_2 & d_3 \end{pmatrix}$$

Upper triangular matrix

$$U = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

$$L = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$

Special Matrices



## Transpose matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

$$a_{ij}^T = a_{ji}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}^T = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$$

$$(A^T)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T$$

## Symmetric matrix

$$A^T = A$$

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

## Skew-symmetric matrix

$$A^T = -A$$

$$A = \begin{pmatrix} 0 & b & c \\ -b & 0 & e \\ -c & -e & 0 \end{pmatrix}$$

Transpose Matrix



## Inner & outer products

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$u^T v = (u_1, u_2, u_3) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$u^T v = 0 \Rightarrow u, v \text{ orthogonal}$$

norm  $\|u\| = (u^T u)^{1/2} = \sqrt{u_1^2 + u_2^2 + u_3^2}$

$u$  is normalized if  $\|u\| = 1$

orthogonal + normalized

orthonormal

## outer product

$$uv^T = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} (v_1, v_2, v_3)$$

$$= \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix}$$

Inner and Outer Products





Inverse matrix

$\det A = ad - bc$

$A A^{-1} = I = A^{-1} A$

$(AB)^{-1} = B^{-1} A^{-1}$

$(A^T)^{-1} = (A^{-1})^T$

$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{matrix} \uparrow A & \uparrow A^{-1} & \uparrow I \end{matrix}$

$y_1 = -\frac{c}{a} x_1$

$y_2 = -\frac{a}{b} x_2$

$\frac{bc}{b} x_2 - \frac{ad}{b} x_2 = 1$

$x_2 = \frac{b}{bc-ad}$

$ax_1 + by_1 = 1 \checkmark$

$ax_2 + by_2 = 0 \leftarrow$

$cx_1 + dy_1 = 0 \leftarrow$

$cx_2 + dy_2 = 1 \leftarrow$

$\frac{ax_1}{a} + \frac{bc}{a} x_1 = 1$

$x_1 = \frac{d}{ad-bc}$

$y_1 = \frac{-c}{ad-bc}$

$x_2 = \frac{-b}{ad-bc}$

$y_2 = \frac{a}{ad-bc}$

Inverse Matrix



Orthogonal matrices

$Q^{-1} = Q^T$

$Q Q^T = I$

$Q^T Q = I$

$(Qx)^T (Qx)$

$= \|Qx\|^2$

$= x^T Q^T Q x$

$= x^T x$

$= \|x\|^2$

Orthogonal matrix preserves lengths

Pause

4:35 / 4:52

Orthogonal Matrices



## Orthogonal matrices example



$$x' = r \cos(\gamma + \theta)$$

$$= r(\cos\gamma \cos\theta - \sin\gamma \sin\theta)$$

$$= x \cos\theta - y \sin\theta$$

$$y' = r \sin(\gamma + \theta)$$

$$= r(\sin\gamma \cos\theta + \cos\gamma \sin\theta)$$

$$= y \cos\theta + x \sin\theta$$

$$R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$R_\theta$

Rotation Matrices



## permutation matrices

$$2 \times 2: \{1, 2\}, \{2, 1\}$$

$$I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

3x3:

$$\{1, 2, 3\}, \{1, 3, 2\}$$

$$\{2, 1, 3\}, \{2, 3, 1\}$$

$$\{3, 1, 2\}, \{3, 2, 1\}$$

$$6 = 3!$$

$$PA = (PI)A$$

Permutation Matrices

