Part I Q1 :

$x(n)$ can be written as the sum of $\delta(n)$

$$x(n) = \sum_{i=-\infty}^{\infty} x(i)\,\delta(n-i)$$

validation : Let j be an arbitrary input

$$LHS = x(j)$$

$$RHS = \sum_{i=-\infty}^{\infty} x(i)\,\delta(j-i)$$

Since $i=j \Rightarrow \delta(j-i)=0$

We are left with only 1 term where $i=j$

$$RHS = x(j)\,\delta(j-j) = x(j)\,\delta(0) = x(j) = LHS$$

So
$$T[x(n)] = T\left[\sum_{i=-\infty}^{\infty} x(i)\,\delta(n-i)\right]$$
$$= \sum_{i=-\infty}^{\infty} T[x(i)\,\delta(n-i)] \quad \text{(since T is linear)}$$
$$= \sum_{i=-\infty}^{\infty} x(i)\,T[\delta(n-i)]$$
$$= \sum_{i=-\infty}^{\infty} x(i)\,h(n-i)$$
$$= x * h \quad \text{(by definition of convolution)}$$
$$= h(n) * x(n) \quad \text{(law of commulative)}$$

□

Part I Q2 :

Let $P(x)$, $Q(x)$ be two polynomials with degree n, m

Let u be the vector representation of P with length $n+1$

Let v be the vector representation of Q with length $m+1$

Assume proper zero-padding and full-size convolution

By convention , $u[i] = \begin{cases} u[i] & 0 \le i \le n \\ 0 & i<0 \text{ or } i>n \end{cases}$ $v[i] = \begin{cases} v[i] & 0 \le i \le m \\ 0 & i<0 \text{ or } i>m \end{cases}$

The reason that it becomes 0 when index is going out of the bond is because we are doing zero-padding

Define $G(i) = [u * v](i)$

$$= \sum_{j=-\infty}^{\infty} u[j]\, v[i-j]$$

$$= \sum_{j=0}^{i} u[j]\, v[i-j] \quad \text{(by our convention)}$$

Since degrees of P and Q are n, m

$$P(x)\,Q(x) = \sum_{i=0}^{m+n} c_i x^i$$

where $$c_i = \sum_{j=0}^{i} p_j q_{i-j}$$

$$= \sum_{j=0}^{i} u[j]\, v[i-j]$$

$$= G(i)$$

Thus, polynomial multiplication is equivalent to convolve their vector representation

Part I Q3:

By definition of Laplacian. we know that

$L_i = I_i - Up(I_{i+1})$ (where $Up()$ is upsampling method)

$\Rightarrow I_i = L_i + Up(I_{i+1})$

Since we have all the L_i's, the minimum information we need to calculate I_0 is I_n

Listing first few terms.

$I_{n-1} = L_{n-1} + Up(I_n)$

$I_{n-2} = L_{n-2} + Up(L_{n-1} + Up(I_n))$

$= L_{n-2} + Up(L_{n-1}) + Up^2(I_n)$

$I_{n-3} = L_{n-3} + Up(L_{n-2} + Up(L_{n-1}) + Up^2(I_n))$

$= L_{n-3} + Up(L_{n-2}) + Up^2(L_{n-1}) + Up^3(I_n)$

Claim that $I_{n-k} = \sum_{i=0}^{k} Up^i(L_{n-k+i})$ for simplicity, assume $I_n = L_n$

Base case: when $k=0$

then $I_n = \sum_{i=0}^{0} Up^i(L_{n+i}) = L_n$ ✓

Inductive step: assume $I_{n-k+1} = \sum_{i=0}^{k-1} Up^i(L_{n-k+1+i})$ (I.H)

WTP: $I_{n-k} = \sum_{i=0}^{k} Up^i(L_{n-k+i})$

$I_{n-k} = L_{n-k} + Up(I_{n-k+1})$

$= L_{n-k} + Up\left(\sum_{i=0}^{k-1} Up^i(L_{n-k+1+i})\right)$ (by I.H)

$= L_{n-k} + \sum_{i=0}^{k-1} Up^{i+1}(L_{n-k+1+i})$

$= L_{n-k} + \sum_{i=1}^{k} Up^i(L_{n-k+i})$

$= \sum_{i=0}^{k} Up^i(L_{n-k+i})$ □

So, when $k=n \Rightarrow I_0 = \sum_{i=0}^{n} Up^i(L_i)$ when $L_n = I_n$

So the minimimum information we need from Gaussian pyramid is I_n

Q1 part 4: Need to show

$$I_{xx} + I_{yy} = I_{rr} + I_{r'r'}$$

$$\text{or } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 f}{\partial r'^2}$$

Suppose (r, r') is rotated from (x, y) with a radius θ

Based on the rotation formula

$$\begin{cases} r = x\cos\theta - y\sin\theta \\ r' = x\sin\theta + y\cos\theta \end{cases}$$

starting with $\frac{\partial f}{\partial x}$

From MAT 237, we know

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial f}{\partial r'} \cdot \frac{\partial r'}{\partial x}$$

$$= \cos\theta \frac{\partial f}{\partial r} + \sin\theta \frac{\partial f}{\partial r'}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial f}{\partial r'} \cdot \frac{\partial r'}{\partial y}$$

$$= -\sin\theta \frac{\partial f}{\partial r} + \cos\theta \cdot \frac{\partial f}{\partial r'}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\cos\theta \frac{\partial f}{\partial r} + \sin\theta \frac{\partial f}{\partial r'}\right)$$

$$= \frac{\partial r}{\partial x} \cdot \frac{\partial}{\partial r}\left(\cos\theta \frac{\partial f}{\partial r} + \sin\theta \frac{\partial f}{\partial r'}\right) + \frac{\partial r'}{\partial x} \cdot \frac{\partial}{\partial r'}\left(\cos\theta \frac{\partial f}{\partial r} + \sin\theta \frac{\partial f}{\partial r'}\right)$$

$$= \cos\theta \cdot \cos\theta \cdot \frac{\partial^2 f}{\partial r^2} + \sin\theta\cos\theta \cdot \frac{\partial^2 f}{\partial r r'} + \sin\theta\cos\theta \frac{\partial^2 f}{\partial r' r} + \sin^2\theta \frac{\partial^2 f}{\partial r'^2}$$

$$= \cos^2\theta \frac{\partial^2 f}{\partial r^2} + 2\sin\theta\cos\theta \frac{\partial^2 f}{\partial r r'} + \sin^2\theta \frac{\partial^2 f}{\partial r'^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(-\sin\theta \frac{\partial f}{\partial r} + \cos\theta \cdot \frac{\partial f}{\partial r'}\right)$$

$$= \frac{\partial r}{\partial y}\cdot\frac{\partial}{\partial r}\left(-\sin\theta \frac{\partial f}{\partial r} + \cos\theta \cdot \frac{\partial f}{\partial r'}\right) + \frac{\partial r'}{\partial y}\cdot\frac{\partial}{\partial r'}\left(-\sin\theta \frac{\partial f}{\partial r} + \cos\theta \cdot \frac{\partial f}{\partial r'}\right)$$

$$= \sin^2\theta \frac{\partial^2 f}{\partial r^2} - \sin\theta\cos\theta \frac{\partial^2 f}{\partial r \partial r'} - \sin\theta\cos\theta \frac{\partial^2 f}{\partial r \partial r'} + \cos^2\theta \frac{\partial^2 f}{\partial r'^2}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (\sin^2\theta + \cos^2\theta)\frac{\partial^2 f}{\partial r^2} + (\sin^2\theta + \cos^2\theta)\frac{\partial^2 f}{\partial r'^2}$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 f}{\partial r'^2}$$

Report for part 2

Step 1: The first row of the figure is showing the result after calling gaussian filter function (see step 4). Parameters are size = 10, sigma = 3.

This is the function that I implemented for gaussian. It takes size and sigma as inputs and outputs a gaussian filter matrix. Details are commented in a1_code.py file.

```python
def gaussian_filter(size: int, sig: float):
    # Calculate the value for the left most and right most position and
    # using linspace to create the coresponding vector
    # for ex, 5 -> [-2, -1, 0, 1, 2]
    # 6 -> [-2.5, -1.5, -0.5, 0.5, 1.5, 2.5]
    right_end = (size - 1) / 2.0
    left_end = -right_end
    # The values in x are values that are going to be sub in gaussian formula
    x = np.linspace(left_end, right_end, size)

    # applying gaussian formula to all entries in x
    gaussian = np.exp(-0.5 * np.square(x) / np.square(sig))

    # taking product of 1d gaussian vector will give us 2D gaussian filter matrix
    gaussian_2d = np.outer(gaussian, gaussian)

    # normalize the entries
    gaussian_2d /= gaussian_2d.sum()
    return gaussian_2d
```

The result of images after taking a gaussian blur is listed in the second column in the figure (see step 4).

Step 2: Below is the function that I use for convolution. It uses valid option here so the result image will be smaller than the original one.

```
27
28  def convolution(image, filter):
29      assert filter.shape[0] == filter.shape[1], 'filter has to be k by k'
30      # flip the fiter first
31      filter = np.flip(filter)
32      k = filter.shape[0]
33      m, n = image.shape
34      # Since I'm doing valid option here, so the result img size will be smaller
35      m = m - k + 1
36      n = n - k + 1
37      res = np.zeros((m, n))
38      # loop through each entry to calculate the result
39      for i in range(m):
40          for j in range(n):
41              res[i][j] = np.sum(image[i:i + k, j:j + k] * filter)
42      return res
43
```

This function only works for the filters in k x k size, so it takes an 2D image and a filter as an input and returns the result image after applying the filter.

Below is the function of taking gradient magnitude. It takes image as input, convolve it with sober matrixes and calculates the magnitude for each entry.

```
44
45  def gradient_magnitude(image):
46      # input image is already in gray scale
47      # define sober filter
48      sober_x = np.array([[-1, 0, 1], [-2, 0, 2], [-1, 0, 1]])
49      sober_y = np.array([[-1, -2, -1], [0, 0, 0], [1, 2, 1]])
50      # using covolution with sober matrix to get derivatives.
51      g_x = convolution(image, sober_x)
52      g_y = convolution(image, sober_y)
53      # Calculate gradient matrix
54      res = np.sqrt(g_x ** 2 + g_y ** 2)
55      return res
56
```

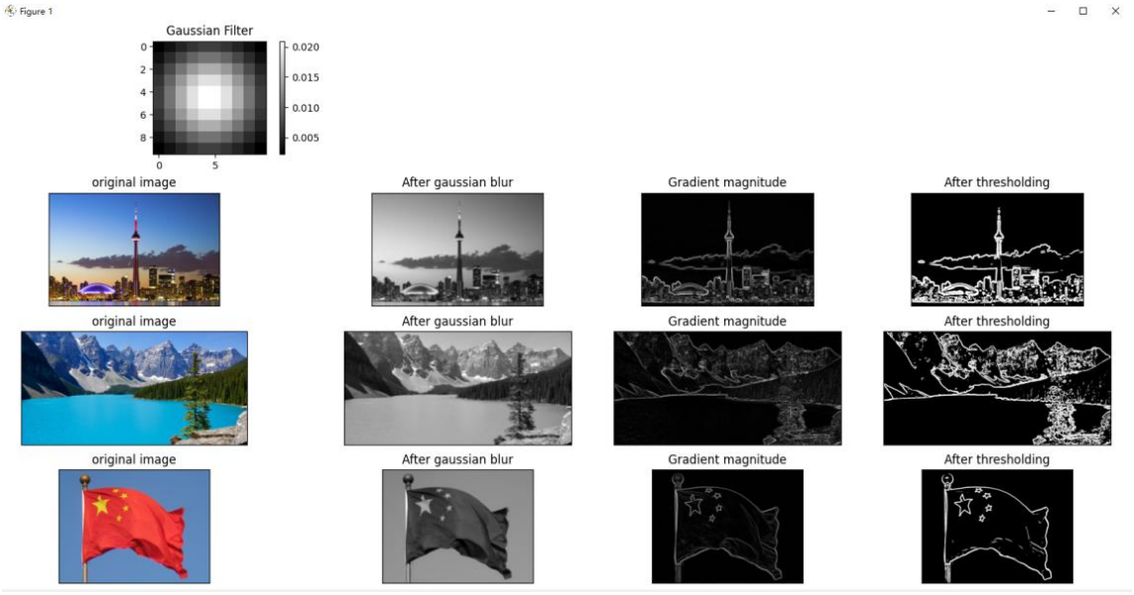
The result images are listed in the third column (see step 4).

Step 3: Below is the function that I use to apply the threshold.

```
58  def threshold(image, epsilon):
59      # calcutate the matrix mean
60      t = image.mean()
61      # initial value of t_{i - 1} should be inf so the first iteration will start
62      t_last = float('inf')
63      while abs(t - t_last) > epsilon:
64          t_last = t
65          # calcutate low average and high average with last t value
66          low_avg = image.mean(where=image < t_last)
67          high_avg = image.mean(where=image >= t_last)
68          t = (low_avg + high_avg) / 2
69      # map every entry to 0 or 255 based on the condition
70      mapping_func = np.vectorize(lambda x: 255 if x >= t else 0)
71      res = mapping_func(image)
72      return res
73
74
```

It basically follows the logic described in assignment. The epsilon that I used in the figure above is 0.0001. These parameters can be altered in line 76 – 78 in a1_code.py

Step 4: testing



This is the result after running the program. The second and the third rows are images provided on quercus and the fourth row is the image that I chose. There are 4 columns:

i) The original image.

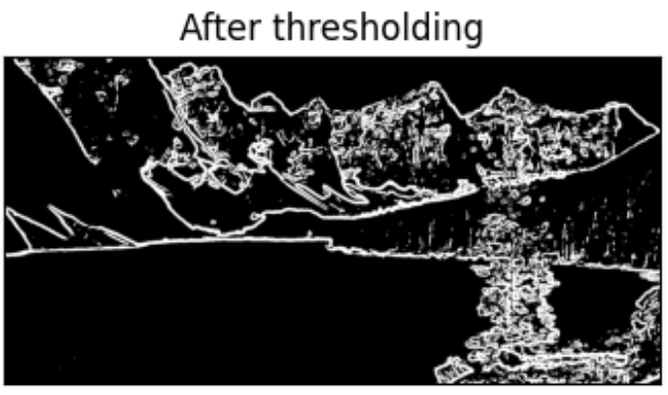
ii) The image after taking gaussian blur.

iii) The image after taking gradient magnitude.

ix) The image after applying threshold.

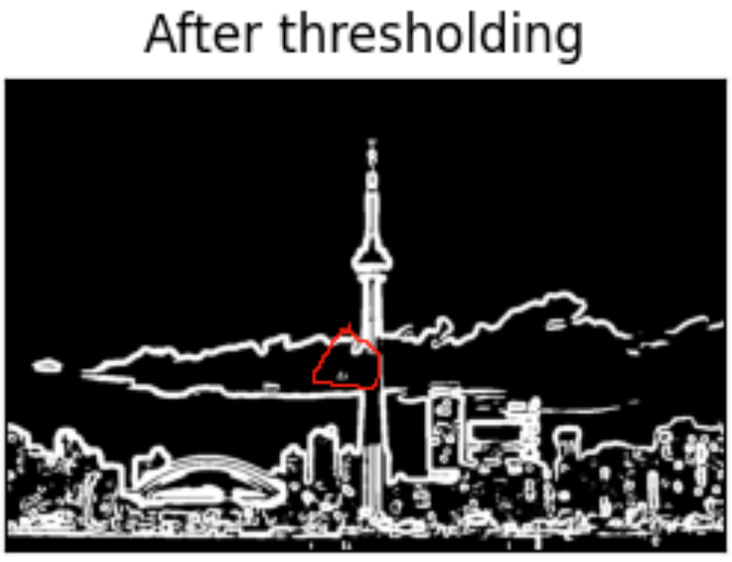
Also shape of images in each step are printed out.

```
The shape of original image 1 is  (650, 980, 3)
The shape of image 1 after applying gaussian blur is  (626, 956)
The shape of image 1 after taking gradient magnitude is  (624, 954)
The shape of image 1 after applying threshold is  (624, 954)
The shape of original image 2 is  (900, 1800, 3)
The shape of image 2 after applying gaussian blur is  (876, 1776)
The shape of image 2 after taking gradient magnitude is  (874, 1774)
The shape of image 2 after applying threshold is  (874, 1774)
The shape of original image 3 is  (960, 1280, 3)
The shape of image 3 after applying gaussian blur is  (936, 1256)
The shape of image 3 after taking gradient magnitude is  (934, 1254)
The shape of image 3 after applying threshold is  (934, 1254)
```



Strength: Most of the information is preserved. By just looking at the result image, we could image what it is in original image. For example, I know that the first image is a picture of city with a lot of buildings and a tower that is much taller than other building. Also, there are clouds in the background. For the second image, I know that it should be picture of a tree and mountains.

Weakness: Some information is lost as well. For example,



The part circled in red, this edge doesn't survive the threshold because it is too weak. However, it should be preserved because we know that tower is not part of the cloud. Also,

This part is a bit noisy. If we increase the sigma value, that will risk of losing important information. The values of parameters are hard to control.