Students:
Section 3.6 is a part of 1 assignment:
Reading Assignment 3

Requirements: PA CA
No due date

3.6 Logarithms and exponents

The ***exponential function*** $\exp_b$:**R** → $\mathbf{R}^+$ is defined as:

$$\exp_b(x) = b^x$$

where b is a positive real number and b ≠ 1. The parameter b is called the ***base of the exponent*** in the expression b^x. The input x to the function b^x is called the ***exponent***.

Figure 3.6.1: Properties of exponents.

For any positive real numbers b, c, and any real numbers x, and y, the following equalities are always true:

$$b^x b^y = b^{x+y}$$

$$(b^x)^y = b^{xy}$$

$$\frac{b^x}{b^y} = b^{x-y}$$

$$(bc)^x = b^x c^x$$

Feedback?

PARTICIPATION ACTIVITY | 3.6.1: Applying the properties of exponents.

Match mathematical expressions that have the same value.

Reset

Feedback?

CHALLENGE ACTIVITY | 3.6.1: Properties of exponents.

422102.2723990.qx3zqy7

Start

Ex: 5

1	2	3	4	5	6

Check Next

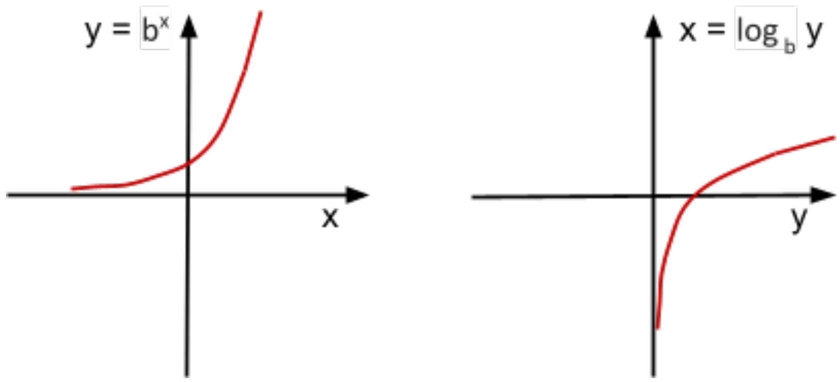
1 2 3 4 5 6

Feedback?

The exponential function is one-to-one and onto, and therefore has an inverse. The ***logarithm function*** is the inverse of the exponential function. For real number b > 0 and b ≠ 1, $\log_b$:$\mathbf{R}^+ \to \mathbf{R}$ is defined as:

The parameter b is called the ***base of the logarithm*** in the expression $\log_b y$.

Figure 3.6.2: Sketch of the exponential and logarithm function for b greater than 1.



Feedback?

Figure 3.6.3: Properties of logarithms.

For any positive numbers b, c, x, and y, such that b ≠ 1 and c ≠ 1, the following equalities are always true:

Feedback?

PARTICIPATION ACTIVITY | 3.6.2: Applying the properties of logarithms.

Match mathematical expressions that have the same value.

$\log_3 100$ **$\log_{10} 1000$** **$\log_3 1000$** **$\log_3 10$** **$\log_2 10$**

$\log_3 10 + \log_3 100$

$\log_3 1000 - \log_3 100$

$2 \cdot \log_3 10$

$\log_3 10 / \log_3 2$

3

Reset

Feedback?

CHALLENGE ACTIVITY | 3.6.2: Properties of logarithms.

422102.2723990.qx3zqy7

Start

Ex: 5

1	2	3	4	5	6

Check Next

1 2 3 4 5 6

Feedback?

A function f is said to be ***strictly increasing*** if whenever $x_1 < x_2$, then $f(x_1) < f(x_2)$. A function f is said to be ***strictly decreasing*** if whenever $x_1 < x_2$, then $f(x_1) > f(x_2)$. If $b > 1$, then the functions $f(x) = b^x$ and $f(x) = \log_b x$ are both strictly increasing. The fact that both functions are strictly increasing can help in approximating the value of the functions. For example, the value of $\log_3 200$ is not an integer and would be difficult to determine without a calculator. However, since $3^4 = 81$ and $3^5 = 243$, so by definition $\log_3 81 = 4$ and $\log_3 243 = 5$. Because the log function is strictly increasing and 81 < 200 < 243, the value of $\log_3 200$ is in between 4 and 5.

PARTICIPATION ACTIVITY | 3.6.3: Using the fact that exp and log are increasing to estimate values.

Use the equalities below to answer the questions without a calculator.

$2^2 = 4$	$2^5 = 32$	$2^8 = 256$
$2^3 = 8$	$2^6 = 64$	$2^9 = 512$
$2^4 = 16$	$2^7 = 128$	$2^{10} = 1024$

1) What is the value of ?

Check Show answer

2) What is the value of ?

Check Show answer

3) What is the value of ?

Check Show answer

4) What is the value of the smallest power of 2 that is greater than $2^{9.2}$? A power of 2 is a number that can be expressed as 2^k, where k is a non-negative integer.

Check Show answer

Feedback?

Example 3.6.1: Exponential functions and population growth.

The growth of a population is often modeled mathematically by an exponential function. Consider an example in which the population of lizards on an island at the beginning of a study is denoted by a number p. After a time period of t days, the population is described by the function $liz(t) = p \cdot b^t$ for some number b. The number of lizards on the island must be an integer, but the function liz(t) will map positive real numbers to positive real numbers. It is understood that population models are approximate and the true population at time t will be an integer that is close to liz(t). The number b indicates how fast the population of lizards is increasing. The value of b will depend on the conditions on the island and the characteristics of the lizards. Developing a good mathematical model for population growth (finding an accurate function liz(t)) would likely require significant field work and study—but, once such a model was developed, it would be possible to answer questions such as: (approximately) how many days will it take until the population of lizards reaches n?

The value of t that satisfies the equation

is $\log_b(n/p)$. Therefore, the number of days for the population to reach n would be approximately $\log_b(n/p)$.

Feedback?

Divide-and-conquer is a common strategy in computer science in which a problem is solved for a large set of items by dividing the set of items into two evenly sized groups, solving the problem on each half and then combining the solutions for the two halves. For example, one approach to sorting a list of numbers is to divide the list in half, sort each half separately and then merge the two sorted lists. The logarithm function plays an important role in analyzing divide-and-conquer algorithms. Consider the following simpler scenario:

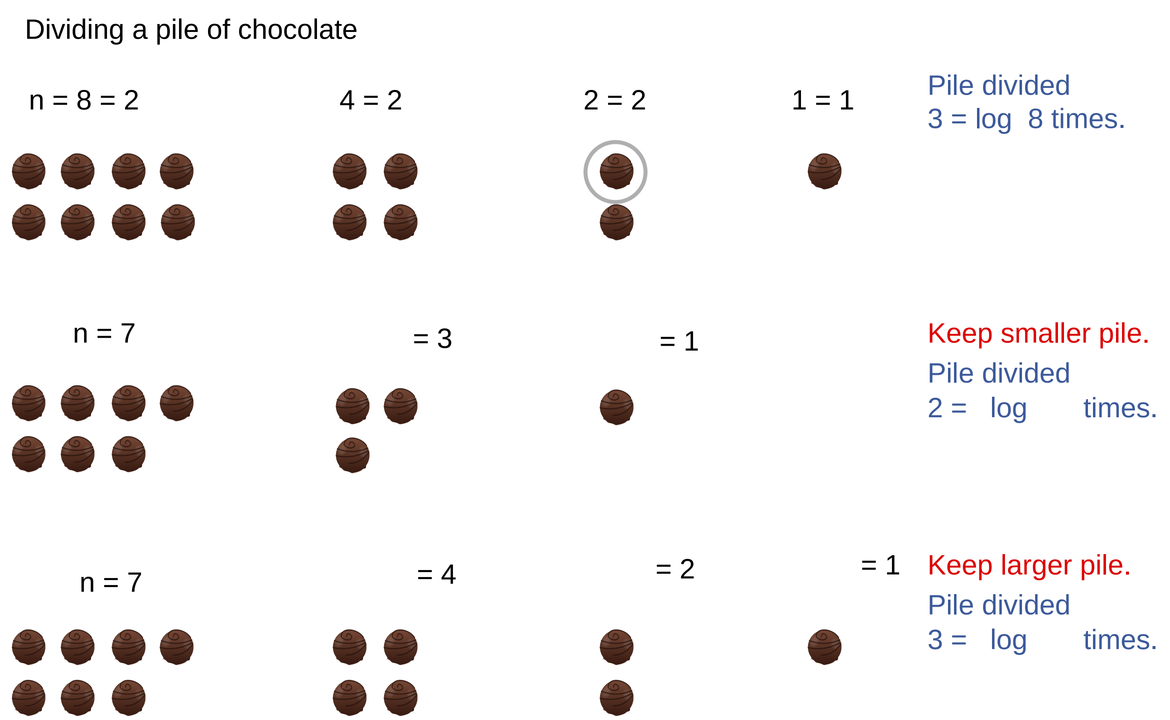
> Ingrid has a bag with n chocolates. n > 0, so the bag is not empty. Ingrid meets a friend and splits her chocolates into two evenly sized piles and gives her friend the chocolates in one of the piles and puts the rest in her bag. Then she meets another friend and splits her chocolates again. Every time she meets a friend, she shares half of her chocolates with the friend. How many friends can she meet and share half her chocolates before she is

down to one chocolate in her bag?

There is an ambiguity in the problem of Ingrid and her chocolates which requires clarification. If Ingrid has an odd number of chocolates, the chocolates can not be divided exactly evenly. For example, if Ingrid has 17 chocolates, she would divide them into one pile with 8 chocolates and one pile with 9 chocolates. In general, if there are n chocolates, one pile will have chocolates and the other pile will have . Does Ingrid keep the bigger or smaller pile for herself? Does the choice of which pile she chooses affect the number of friends with whom she can divide her pile?

PARTICIPATION ACTIVITY 3.6.4: Dividing chocolates.

Start 2x speed



Captions

Feedback?

If it happens that the number of chocolates in Ingrid's bag is a power of two ($n = 2^k$ for non-negative integer k), then every time she meets a friend, the chocolates can be evenly divided into two piles. She starts with 2^k chocolates. After meeting the first friend, she is down to $2^k/2 = 2^{k-1}$ chocolates. After the next friend, she is down to $2^{k-1}/2 = 2^{k-2}$ chocolates. Each time she meets a friend, the exponent in the number of chocolates goes down by 1, until she is down to $1 = 2^0$ chocolates. She starts with 2^k and ends up with 2^0 chocolates. The number of friends she can meet is k = $\log_2$ n.

If n is not a power of 2, then $\log_2$ n is not an integer. Certainly, the number of encounters with friends has to be an integer. Also, if n is not a power of 2, Ingrid will be forced to divide her pile unevenly at some point in the process. If Ingrid always takes the larger half, then the number of times she can divide her pile of chocolates in half is . If she always takes the smaller half, then the number of times she can divide her pile of chocolates in half is .

Theorem 3.6.1: Dividing piles and the logarithm function.

Let n and b be positive integers with b > 1. Consider a process in which in each step, n is replaced with , until n < b. The process lasts for steps.

If instead in each step, n is replaced with , until n = 1. The process lasts for steps.

Feedback?

PARTICIPATION ACTIVITY 3.6.5: Dividing chocolates and the logarithm function.

1) Suppose that Ingrid starts with 54 chocolates. In each step, she divides the chocolates into two piles as evenly as possible. If the piles can not be divided perfectly evenly, then she always takes the larger pile. Then how many times can Ingrid divide her pile?

Check Show answer

2) Suppose that Ingrid has 27 chocolates. In every step, she meets two friends and divides her pile three ways between herself and her two

friends. How many steps until she is down to one chocolate?

Check Show answer

3) Suppose that Ingrid has 63 chocolates. In every step, she meets two friends and divides her pile three ways between herself and her two friends. She divides the chocolates as evenly as possible, but in the event that the chocolates can not be divided perfectly evenly into three piles, she takes the larger pile for herself. How many steps until she is down to one chocolate?

Check Show answer

Feedback?

Additional exercises

EXERCISE | 3.6.1: Find an equivalent expression - exponents.

For each expression, give an equivalent expression that is a power of 6. That is your answer should have the form 6^*, where * is an expression with numbers and possibly the variable k.

(a)

(b)

(c)

(d)

(e)

(f)

(g)

Feedback?

EXERCISE | 3.6.2: Find an equivalent expression - logarithms.

For each expression, give an equivalent expression that is of the form $\log_5(*)$, where * is an expression with numbers and possibly the variable k.

(a) $\log_5 k + \log_5 2$

(b) $2 \cdot \log_5 k$

(c) $\log_5 k - \log_5 7$

(d) $(\log_3 k)/(\log_3 5)$

(e) $(\log_3 (k^2))/(\log_3 25)$

Feedback?

EXERCISE | 3.6.3: Find an equivalent expression - exponents and logarithms.

For each expression, give an equivalent expression that does not use the log function.

(a) (Hint: Set the expression equal to y and take the log base 2 of both

sides of the equation. Remember that the log function is one-to-one, so if $\log_2 x$

sides of the equation. Remember that the log function is one to one, so if $\log_2 x$ $= \log_2 y$, then $x = y$.)

(b)

(c)

Feedback?

EXERCISE 3.6.4: Approximating the log function.

The table below shows integer powers of 3 up to 6:

$3^0 = 1$
$3^1 = 3$
$3^2 = 9$
$3^3 = 27$
$3^4 = 81$
$3^5 = 243$
$3^6 = 729$

Use the values in the table to calculate the values for the expressions below. You should not need a calculator!

(a)

(b)

(c)

(d)

Feedback?

EXERCISE 3.6.5: Repeated division.

(a) Start with the number $n = 54527$. Divide n by 5 and round the result down to an integer. Keep repeating the division and rounding step until the resulting integer. Keep repeating the division and rounding step until the resulting number is less than 5. How many divisions are performed? You can use a calculator for this problem, but you should not have to actually perform all of the divisions.

(b) Start with the number n = 54527. Divide n by 5 and round the result up to an integer. Keep repeating the division and rounding step until the resulting number is equal to 1. How many divisions are performed? You can use a calculator for this problem, but you should not have to actually perform all of the divisions.

(c) Suppose that you have a calculator that performs the log function but you do not know the base of the log. How can you use your calculator to solve the two problems above?

Feedback?

How was this section? Provide feedback

Activity summary for assignment: Reading Assignment 3 68 / 90 pts

No due date 68 / 90 pts submitted to canvas

Completion details