6.1 Mathematical definitions

Many of the theorems in the material about how to write proofs are facts about numbers that can be proven with standard algebra. Theorems and proofs throughout the rest of this material establish facts about a variety of mathematical objects such as graphs, functions, sequences, and sums. This section contains mathematical definitions related to many of the examples of theorems and proofs presented here. This section also reviews some basic mathematical concepts that are useful in the reasoning found in proofs.

Even and odd integers

Some of the theorems proven in this material are facts about even and odd integers. Although you may have a good understanding of whether an integer is odd or even, a formal mathematical definition provides an algebraic expression for odd or even numbers that can be used in proofs to establish that related integers are even or odd.

Definition 6.1.1: Even and odd integers.

An integer x is ***even*** if there is an integer k such that x = 2k.
An integer x is ***odd*** if there is an integer k such that x = 2k+1.

Definition 6.1.2: Parity.

The ***parity*** of a number is whether the number is odd or even. If two numbers are both even or both odd, then the two numbers have the ***same parity***. If one number is odd and the other is even, then the two numbers have ***opposite parity***.

PARTICIPATION ACTIVITY | 6.1.1: Even and odd integers.

1) Is the number 0 even or odd?

Check Show answer

2) Show that -73 is odd by giving an integer k such that -73 = 2k+1.

Check Show answer

3) Suppose that n is an integer. Is the number 2n + 3 even or odd?

Check Show answer

4) Do 7 and 22 have the same parity? (Write "yes" or "no".)

Check Show answer

Rational numbers

Some of the theorems proven in this material are facts about rational numbers. A number r is ***rational*** if there exist integers x and y such that $y \neq 0$ and $r = x/y$. Note that for a particular rational number r, the choice of x and y is not necessarily unique. For example, if r = .5, then r = 1/2 and r = 2/4.

PARTICIPATION ACTIVITY | 6.1.2: Rational numbers.

1) Is 0 a rational number? (Answer "yes" or "no")

Check Show answer

2) Suppose that r = .677 and y = 1000. Show that r is rational by

giving an integer x such that r = x/y.

Check Show answer

3) Suppose that r = -.05 and y = 100. Show that r is rational by giving an integer x such that r = x/y.

Check Show answer

Divides

Some of the theorems proven in this material are about where one integer divides another integer. The definition below gives a mathematical expression for an integer n when m divides n. The mathematical expression for n can then be used to prove related facts about n and m.

Definition 6.1.3: Divides.

An integer x ***divides*** an integer y if and only if x ≠ 0 and y = kx, for some integer k.

The fact that x divides y is denoted x|y. If x does not divide y, then that fact is denoted $x \nmid y$.

If x divides y, then y is said to be a ***multiple*** of x, and x is a ***factor*** or ***divisor*** of y.

PARTICIPATION ACTIVITY | 6.1.3: Does m divide n?

Indicate whether each expression is true or false.

1) 3|12

- True
- False

2) 5|-20

- True
- False

3) $11 \nmid 0$

- True
- False

4) $12 \nmid 3$

- True
- False

5) If x is an integer, then $1|x$.

- True
- False

6) There is an integer x such that $x \neq 0$ and $x \nmid x$.

- True
- False

7) If x and y are positive integers and $x > y$, then $x \nmid y$.

- True
- False

Prime and composite numbers

This material will also prove theorems about prime and composite numbers.

Definition 6.1.4: Prime and composite numbers.

An integer n is ***prime*** if and only if n > 1, and the only positive integers that divide n are 1 and n.

An integer n is ***composite*** if and only if n > 1, and there is an integer m such that 1 < m < n and m divides n.

PARTICIPATION ACTIVITY | 6.1.4: Prime and composite numbers.

Indicate whether each expression is true or false.

1) 1 is a prime number.
- True
- False

2) 2 is a prime number.
- True
- False

3) 9 is a prime number.
- True
- False

4) If n is an integer such that n > 1, then n is either prime or composite.
- True
- False

Inequalities

If x and c are real numbers, then exactly one of the following statements is true:

- $x < c$
- $x = c$
- $x > c$

The values of x and c can also be related using the symbols ≤ and ≥:

- $x \geq c$ if and only if $x = c$ or $x > c$. We say that x is ***at least*** c or x is ***greater than or equal to*** c.
- $x \leq c$ if and only if $x = c$ or $x < c$. We say that x is ***at most*** c or x is ***less than or equal to*** c.

In writing proofs, it is often necessary to consider the negation of an inequality, such as: "It is not true that $x > c$", where x and c are real numbers. It is convenient to translate statements with negations such as this into an equivalent statement that indicates what is true about the relative value of x and c.

- If it is not true that $x < c$, then $x = c$ or $x > c$, which is the same as saying that $x \geq c$.

- If it is not true that x > c, then x = c or x < c, which is the same as saying that x ≤ c.

If we know that x > c is true, then it must be the case that the statement "x > c or x = c" is also true, which is equivalent to x ≥ c. So if a proof established that x > c, then the proof can use the fact that x ≥ c. Similarly, if we know that x < c is true, then it must also be the case that x ≤ c is true.

A real number x is ***positive*** if and only if x > 0. A real number x is ***negative*** if and only if x < 0. A real number x is ***non-negative*** if and only if x ≥ 0. A real number x is ***non-positive*** if and only if x ≤ 0.

PARTICIPATION ACTIVITY | 6.1.5: Negating inequalities.

Animation content:

undefined

Animation captions:

1. If x is a real number, exactly one of the following statements is true: $x < 5$, $x = 5$, $x > 5$.
2. The statement $x < 5$ is false if and only if $x \geq 5$ is true.
3. The statement $x \leq 5$ is false if and only if $x > 5$ is true.

PARTICIPATION ACTIVITY | 6.1.6: Reasoning about inequalities.

Select the statement that is equivalent to each statement given below.

1) $x < -7$ or $x = -7$

- ○ $x \geq -7$
- ○ $x \leq -7$
- ○ $x < -7$

2) $x < 3$ is false.

- ○ $x \geq 3$
- ○ $x > 3$
- ○ $x = 3$

3) $x \geq 3$ is false.

- ○ $x \leq 3$
- ○ $x < 3$
- ○ $x > 3$

4) x is not positive.

- $x \leq 0$
- $x < 0$
- $x > 0$

5) x is non-negative.

- $x \leq 0$
- $x \geq 0$
- $x > 0$

PARTICIPATION ACTIVITY 6.1.7: Conditional statements with inequalities.

Indicate whether each statement below is true for every real number x.

1) If $x > 3$, then $x \geq 3$.

- True
- False

2) If $x \leq 4$, then $x < 4$.

- True
- False

3) If x is positive, then x is non-negative.

- True
- False

4) If $x = 3$, then $x \geq 3$.

- True
- False

5) If $x < 2$, then $x \leq 1$.

- True
- False

6) If $x \geq 4$, then $x > 3$.

○ True

Additional exercises

EXERCISE 6.1.1: Even and odd integers.

Indicate whether each integer n is even or odd. If n is even, show that n equals 2k, for some integer k. If n is odd, show that n equals 2k+1, for some integer k.

(a) $n = -1$

(b) $n = -101$

(c) $n = 258$

(d) $n = 1$

EXERCISE 6.1.2: Mathematical expressions that evaluate to even and odd integers.

In the expressions below, n is an integer. Indicate whether each expression has a value that is an odd integer or an even integer. Use the definitions of even and odd to justify your answer. You can assume that the sum, difference, or product of two integers is also an integer.

(a) $2n + 4$

(b) $4n+3$

(c) $10n^3 + 8n - 4$

(d) $-2n^2 - 5$

EXERCISE 6.1.3: Showing a number is rational.

Show that each number n is rational by showing that n is equal to the ratio of two integers, where the denominator is non-zero.

(a) n = .25

(b) n = -5

(c) n = .3274

(d) $n = \frac{\pi}{6\pi}$

EXERCISE 6.1.4: Listing the positive divisors of an integer.

For each integer n, list all the positive divisors of n

(a) n = -1

(b) n = 7

(c) n = 8

(d) n = 75

(e) n = -30

EXERCISE 6.1.5: Prime and composite numbers.

Indicate whether each number n is prime, composite or neither. If n is composite give a divisor of n that is less than n and greater than 1.

(a) n = 0

(b) n = 1

(c) n = 2

(d) n = 17

(e) n = 21

(f) n = 56328

(g) n = 59

EXERCISE 6.1.6: Reasoning about inequalities.

Give an equivalent statement for each statement. Your answer should be one of the following:

- $x < 7$
- $x \leq 7$
- $x > 7$
- $x \geq 7$

(a) It is not true that $x < 7$

(b) It is not true that $x \leq 7$

(c) It is not true that $x > 7$

(d) It is not true that $x \geq 7$

6.2 Introduction to proofs

A primary endeavor in mathematics is to prove theorems. A ***theorem*** is a statement that can be proven to be true. A ***proof*** consists of a series of steps, each of which follows logically from assumptions, or from previously proven statements, whose final step should result in the statement of the theorem being proven. The proof of a theorem may make use of ***axioms***, which are statements assumed to be true. A proof may also make use of previously proven theorems. Although mathematical proofs are typically expressed in English, the formalism of logic provides a good foundation for mathematical reasoning used in proving theorems. A proof should read like a verbal argument designed to convince a skeptical listener that an assertion is true.

In computer science, proofs are important for establishing that a program works as expected, showing that a cryptosystem is secure, or validating a set of inferences in artificial intelligence, to name a few applications.

The animation below gives the proof of the following theorem:

Theorem 6.2.1: A simple theorem.

Every positive integer is less than or equal to its square.

The statements of the proof itself are shown in black font. Comments which are not part of the proof are shown in red. Every proof should begin with a clear indication that the proof is starting and end with an indication that the proof is complete. In this material, every proof begins with the word **Proof:** and ends with the symbol ■.

PARTICIPATION ACTIVITY 6.2.1: A first proof.

Animation content:

undefined

Animation captions:

1. The theorem to be proven is stated before the proof. The word "Proof:" indicates that the proof is starting.
2. The first step names an arbitrary object in the domain and the given assumptions about the object.
3. The reasoning in a proof is stated in complete sentences. The proof is followed by an end of proof symbol.

Theorems that are universal or existential statements

Most theorems are an assertion about all the elements in a set and are therefore universal statements. A universal statement does not necessarily explicitly use words associated with a universal quantifier, such as "for all" or "for every". Consider the theorem below:

The sum of two positive real numbers is strictly greater than the average of the two numbers.

The statement of the theorem can be expressed in logic as:

$\forall x\ \forall y\ ((x > 0 \text{ and } y > 0) \rightarrow (x + y) > (x + y)/2)$

where the domain for x and y is the set of all real numbers.

Universal statements can also refer to more than one object which may come from different domains. For example, consider the statement:

If x and y are positive real numbers and n is a positive integer, then $(x + y)^n \geq x^n + y^n$.

The statement above concerns three numbers: x, y, and n. x and y are positive real numbers and n is a positive integer.

Some theorems are existential statements and assert that a number or object with certain properties exists. Consider for example the statement:

There is an integer that is equal to its square.

In logic, this statement would be expressed as:

$\exists x\ (x = x^2)$

where the domain for variable x is the set of all integers.

Although mathematical theorems are usually expressed in English, translating a statement into logic is often helpful in understanding exactly what the statement means. Understanding whether a theorem to be proven is a universal or existential statement is an important first step in proving that the theorem is true.

One of the questions below is about a perfect square. A number n is a ***perfect square*** if $n = k^2$ for some integer k.

PARTICIPATION ACTIVITY | 6.2.2: Identifying whether a statement is a universal or existential statement.

For each statement, indicate whether the statement is an existential or universal statement.

1) The product of two negative real numbers is a positive real number.

- Existential
- Universal

2) There is a perfect square that is the sum of two non-zero perfect squares.

- Existential
- Universal

3) If a real number x satisfies $0 < x < 1$, then $x^2 < x$.

- Existential
- Universal

Proofs of universal statements: proofs by exhaustion

If the domain of a universal statement is small, it may be easiest to prove the statement by checking each element individually. A proof of this kind is called a ***proof by exhaustion***. For example, consider the statement:

If n ∈ {-1, 0, 1}, then n^2 = |n|.

The notation n ∈ {-1, 0, 1} means that n is equal to -1, 0, or 1. It is straightforward to prove the above statement by verifying the equality for all three possible values of n. Here is the proof:

Proof 6.2.1: Proof by exhaustion.

Theorem: If n ∈ {-1, 0, 1}, then n^2 = |n|.

Proof.

Check the equality for each possible value of n:

- n = -1: $(-1)^2 = 1 = |-1|$.
- n = 0: $(0)^2 = 0 = |0|$.
- n = 1: $(1)^2 = 1 = |1|$. ■

PARTICIPATION ACTIVITY | 6.2.3: Proof by exhaustion.

Consider the following statement:

For every positive integer n less than 3, $(n + 1)^2 \geq 3^n$.

1) Which facts must be checked in a proof by exhaustion of the statement?

- ○
  - $1^2 \geq 3^0$
  - $2^2 \geq 3^1$
  - $3^2 \geq 3^2$
- ○
  - $2^2 \geq 3^1$
  - $3^2 \geq 3^2$
- ○
  - $2^2 \geq 3^1$
  - $3^2 \geq 3^2$
  - $4^2 \geq 3^3$

Proofs of universal statements: universal generalization

If the domain of a universal statement is a large or even infinite set, it becomes impractical or infeasible to prove the statement individually for each element in the domain. For this reason, the most common method for proving universal statements is to use universal generalization. A proof that uses ***universal generalization*** to prove a universal statement names an arbitrary object in the domain and proves the statement for that object. "Arbitrary" means that nothing is assumed about the object other than the assumptions that are given in the statement of the theorem.

Here is an example of the beginning of a proof that uses universal generalization.

Theorem: Every positive integer is less than or equal to its square.

Proof.
Let x be an integer such that $x > 0$.
[Steps showing that $x \leq x^2$]

Depending on the complexity of the proof, it can be useful to inform the reader about where the proof is going. The steps showing that $x \leq x^2$ could start out with an explicit statement in the proof that says: "We shall show that $x \leq x^2$". Whether or not the statement is included in the proof, it is useful to write down the fact you are trying to prove in precise mathematical language as part of your scratch work in order to know where your reasoning should lead.

One of the questions below is about consecutive integers. Two integers are ***consecutive*** if one of the numbers is equal to 1 plus the other number. For example, 4 and 5 are consecutive integers. However, 4 and 6 are not consecutive, nor are 4 and 4.

PARTICIPATION ACTIVITY 6.2.4: Universal generalization for proving universal statements.

1) Select a valid beginning to a proof of the following statement:

The average of two real numbers is less than or equal to at least one of the two numbers.

- Let x and y be two real numbers.
  [Steps showing that $(x + y)/2 \leq x + y$]
- Let x and y be two real numbers.
  [Steps showing that $(x + y)/2 \leq x$ or $(x + y)/2 \leq y$]
- Let x and y be two real numbers.
  [Steps showing that $(x + y)/2 \leq x$ and $(x + y)/2 \leq y$]

2) Select a valid beginning to a proof of the following statement:

The difference of two odd integers is even.

- Suppose that x and y are odd integers.
  [Steps showing that $(x - y)$ is even]

- Suppose that x and y are integers and that (x-y) is even.
  [Steps showing that x is odd and y is odd]
- Suppose that x and y are even integers.
  [Steps showing that (x - y) is odd]

3) Select a valid beginning to a proof of the following statement:

   Among any two consecutive integers, there is an odd number and an even number.

- Let x and y be two integers.
  [Step showing that x is odd and y is even or x is even and y is odd]
- Let x be an integer.
  [Steps showing that x is odd and x+1 is even]
- Let x be an integer.
  [Steps showing that x is odd and x+1 is even or x is even and x+1 is odd]

Counterexamples

It may be tempting to prove statements over a large or infinite domain by example as well. For example, consider the statement:

If n is an integer greater than 1, then $(1.1)^n < n^{10}$.

The statement certainly holds for n = 2 because

$$(1.1)^2 = 1.21 < 1024 = 2^{10}.$$

The inequality also holds for n = 100:

$$(1.1)^{100} \approx 13780.61 < 100000000000000000000 = 100^{10}.$$

In fact, the statement holds for every number all the way up through 685. However, for n = 686, the statement is false because $(1.1)^{686} > (686)^{10}$.

The example n = 686 is a counterexample for the statement that for every integer greater than 2, $(1.1)^n < n^{10}$. A ***counterexample*** is an assignment of values to variables that shows that a universal statement is false.

The example illustrates the danger in generalizing from examples because there can always be a counterexample that was not tried. The only way to be certain that a universal statement is true is a general proof that holds for all objects in the domain. A mathematician who does not know whether an unproven statement is true or false may divide his or her time between looking for a counterexample showing that the statement is false or a proof showing that the statement is true.

PARTICIPATION ACTIVITY | 6.2.5: Matching counterexamples to false statements.

Below is a list of false statements. Match each assignment to the statement for which it is a counterexample.

If unable to drag and drop, refresh the page.

x = 5 **x = 1** **x = -1**

For every real number x, $7 \cdot |x| = 7$

For all real numbers x, $x \neq x^2$

For all real numbers x, $-x \neq x^2$

Reset

Counterexamples for conditional statements

A counterexample for a conditional statement must satisfy all the hypotheses and contradict the conclusion.

Using the language of logic, consider the expression that says for every element x in a particular set (or domain), if the hypothesis H(x) is true for x, then the conclusion C(x) must also be true for x:

A counterexample for the expression above is a specific element d in the domain for variable x

$\forall x\,(H(x) \rightarrow C(x))$ such that $(H(d) \rightarrow C(d))$ is false. Using the laws of logic, it can be shown that $(H(d) \rightarrow C(d))$ is false if and only if $H(d)$ is true and $C(d)$ is false. Therefore a counterexample is a particular element of the domain that satisfies all the hypothesis of a conditional statement and does not satisfy the conclusion.

A counterexample for a statement with more than one hypothesis, such as

$$\forall x\,((H_1(x) \land H_2(x)) \rightarrow C(x)),$$

must be a particular element d that satisfies all the hypotheses and does not satisfy the conclusion: $H_1(d)$ and $H_2(d)$ are both true and $C(d)$ is false.

This reasoning derived from formal logic can be applied to statements expressed in English. For example, consider the statement:

For any real number x, if $x \geq 0$ and $x < 1$, then $x^2 < x$.

The assignment $x = 2$ would not be a counterexample to the statement above. Although the assignment $x = 2$ satisfies the hypothesis $x \geq 0$ and makes the conclusion ($x^2 < x$) false, the value $x = 2$ does not satisfy the hypothesis that $x < 1$. Similarly, the assignment $x = -1$ is also not a counterexample because $x = -1$ does not satisfy the hypothesis that $x \geq 0$. The assignment $x = 0$, is a counterexample because when $x = 0$, x is a real number, the hypothesis $x < 1$ and $x \geq 0$, are both true and the conclusion $x^2 < x$ is false.

PARTICIPATION ACTIVITY | 6.2.6: Identifying counterexamples for universal conditional statements.

For each statement, select the values for the variables that are a counterexample for the statement.

1) For any real numbers x and y, if $x < y$ and $0 < y$, then $x^2 < y^2$.
   - ○ $x = -3$ and $y = -2$
   - ○ $x = 1/2$ and $y = 1$
   - ○ $x = -1$ and $y = 1$
   - ○ $x = 3$ and $y = 1$

2) For any integer x, if x is positive, then $1/x < x$.
   - ○ $x = 1/2$
   - ○ $x = -1$
   - ○ $x = 1$

○ x = 2

Proving existential statements

A proof that shows that an existential statement is true is called an ***existence proof***. The most common type of existence proof is a constructive proof of existence. An existential statement asserts that there is at least one element in a domain that has some particular properties. A ***constructive proof of existence*** gives a specific example of an element in the domain or a set of directions to construct an element in the domain that has the required properties.

Theorem: There is an integer that can be written as the sum of the squares of two positive integers in two different ways.
Proof: Let n = 50. $50 = 1^2 + 7^2 = 5^2 + 5^2$. Therefore the integer 50 can be written as the sum of the squares of two positive integers in two different ways. ■

Here is another existential statement:

For every integer x, there is an integer y such that y + 3 = x.

This statement is expressed in logic using nested quantifiers: $\forall x\ \exists y\ (y + 3 = x)$, where the domain for both x and y is the set of all integers. The existence proof for this statement also uses universal generalization. The proof starts by defining the variable x to be an arbitrary integer. Then the proof gives a way to find y in terms of x such that y is an integer and y + 3 = x.

Proof: Suppose that x is an integer. Let y = x - 3. Since x is an integer, x-3 is also an integer. Therefore y is an integer. Furthermore y + 3 = (x - 3) + 3 = x. ■

The two examples of existence proofs given above are both constructive. A ***nonconstructive proof of existence*** proves that an element with the required properties exists without giving a specific example. A common method for giving a nonconstructive existence proof is to show that the non-existence of an element with the required properties leads to a contradiction.

PARTICIPATION ACTIVITY | 6.2.7: Existential proofs.

Below is a list of true existential statements. Match each statement to the specific numbers that prove that the statement is true.

If unable to drag and drop, refresh the page.

1 and 2 **4 and 5** **0 and 1**

There are two consecutive integers whose sum is equal to the difference between the larger number and the smaller number.

There are two consecutive positive integers whose sum is equal to a perfect square.

There are two consecutive positive integers whose product is less than their sum.

Reset

Disproving existential statements

An existential statement asserts that there is at least one element in a domain that has some particular properties. In order to show that an existential statement is false, it is necessary to argue that every single element of the domain does not have the required properties. For example, consider the following statement:

There is a real number whose square is negative.

In order to disprove the statement above, it would be necessary to show that the square of every real number is not negative.

The reasoning above is an example of De Morgan's law. De Morgan's law says that the statement:

It is not true that there exists an element x in the domain with property P.

is equivalent to the statement:

Every element x in the domain does not have property P.

Therefore the approach to proving that an existential statement is false is the same as the approach to proving that a universal statement is true. In order to prove that the existential statement above is false, one would need to prove the following statement:

The square of every real number is greater than or equal to 0.

PARTICIPATION ACTIVITY 6.2.8: Proving an existential statement is false.

Select the statement that must be proven in order to show that the statement below is false.

1) There is an integer x that satisfies $x^2 - x = 1$.

- There is no integer x that satisfies $x^2 - x \neq 1$.
- There is an integer x that satisfies $x^2 - x \neq 1$.
- Every integer x satisfies $x^2 - x \neq 1$.
- Every integer x satisfies $x^2 - x = 1$.

2) There are two distinct integers whose sum is equal to their product.

Two integers x and y are distinct if $x \neq y$.

- For every pair of integers, x and y, if $x \neq y$, then $xy \neq x + y$.
- Every pair of distinct integers x and y satisfies $xy = x + y$.
- There are no two distinct integers x and y that satisfy $xy \neq x + y$.
- There are distinct integers x and y that satisfies $xy \neq x + y$.

CHALLENGE ACTIVITY 6.2.1: Introduction to proofs.

422102.2723990.qx3zqy7

Start

Suppose that the following fact is proven by exhaustion.

Theorem: Every odd integer in the range from 90 through 96 is composite.

Select the lines that would be included in the proof.

- [ ] 91 = (7)(13), so 91 is composite.
- [ ] 92 = (2)(46), so 92 is composite.
- [ ] 93 = (3)(31), so 93 is composite.
- [ ] 94 = (2)(47), so 94 is composite.
- [ ] 95 = (5)(19), so 95 is composite.

1 2 3

Check Next

Additional exercises

EXERCISE 6.2.1: Methods of proof.

Determine whether each statement is true or false. Provide a justification for each answer.

(a) Showing that a statement holds for a few cases is sufficient to prove a universal statement.

(b) Providing one example when the statement holds is sufficient to prove an existential statement.

(c) Providing one counterexample is sufficient to disprove a universal statement.

(d) Proof by exhaustion can be used to prove a universally quantified statement with a finite domain.

EXERCISE | 6.2.2: Proof by exhaustion.

Prove each statement using a proof by exhaustion.

(a) For every integer n such that $0 \leq n < 3$, $(n + 1)^2 > n^3$.

(b) For every integer n such that $0 \leq n < 4$, $2^{(n+2)} > 3^n$.

(c) For all positive integers $n \leq 4$, $(n+1)^3 \geq 3^n$.

EXERCISE | 6.2.3: Find a counterexample.

Find a counterexample to show that each of the statements is false.

(a) Every month of the year has 30 or 31 days.

(b) If n is an integer and n^2 is divisible by 4, then n is divisible by 4.

(c) For every positive integer x, $x^3 < 2^x$.

(d) Every positive integer can be expressed as the sum of the squares of two integers.

(e) The multiplicative inverse of a real number x, is a real number y such that xy = 1. Every real number has a multiplicative inverse.

EXERCISE 6.2.4: Restate the theorem.

Write each of the following statements as a precise mathematical statement. Use variable names to denote arbitrary numbers in the domain. Your statements should avoid mathematical terms in text (such as "square root" or "multiplicative inverse") and should be expressed using algebra.

(a) The square root of every positive number less than one is greater than the number itself.

(b) There is no largest integer.

(c) Every real number besides 0 has a multiplicative inverse.

(d) Among every three consecutive integers, there is a multiple of 3.
Avoid the use of the word "consecutive" in your statement.

EXERCISE 6.2.5: Proving existential statements.

Prove each existential statement given below.

(a) There are positive integers x and y such that $\frac{1}{x} + \frac{1}{y}$ is an integer.

(b) There is a positive integer x that is equal to the sum of all the positive integers less than x.

(c) There are integers m and n such that $\sqrt{m+n} = \sqrt{m} + \sqrt{n}$

(d) There are integers c and d such that 7c + 5d = 1.

(e) There are three positive integers, x, y, and z, that satisfy $x^2 + y^2 = z^2$

(f) There exists a negative integer that is equal to its cube.

(g) For every pair of real numbers, x and y, such that x ≠ 0, there exists a real number z such that xz + y = 0.

(h) For every pair of real numbers, x and y, there exists a real number z such that x - z = z - y.

(i) Show that there is an integer n such that n^2-1 is prime.

EXERCISE | 6.2.6: Disproving existential statements.

For each statement below state what needs to be proven in order to show that the existential statement is false. Your response should be a universal statement.

(a) There exists a negative integer that is equal to its square.

(b) There exists an integer that is smaller than every other integer.

(c) There are two distinct positive integers x and y such that $\frac{1}{x} + \frac{1}{y}$ is an integer.

(d) There are positive integers m and n such that $\sqrt{m+n} = \sqrt{m} + \sqrt{n}$

6.3 Best practices and common errors in proofs

Proof steps and assumptions

Most mathematical proofs make use of other facts that are assumed to be true. It is natural to ask what facts can be assumed to be true in writing proofs. A related question is how much detail should be provided in explaining how one step in a proof follows from previous steps. The answer to these questions depend on the intended audience. A proof written by a mathematician intended for publication in a journal that will be read by other mathematicians may assume facts that are inappropriate to assume in a proof to be read by students. If the reader is advanced, then small steps can be skipped under the assumption that the reader can fill in the details on his or her own. This material assumes a novice reader and tends to provide full explanations of each step. It is also good practice in first learning to write proofs to provide detailed arguments at each step.

The proofs in this material are limited to the assumptions listed below. In addition, each step of a proof should apply at most one rule of algebra at a time so that the reader can follow the logic of the proof step by step. Sometimes a mathematical proof will make use of a fact or theorem that has been proven elsewhere. It is always important to provide a clear reference when using outside facts. It is up to your instructor as to whether you can make use of facts proven in this material or other exercises in writing your own proofs. The proofs in this material are for the most part self-contained in that the proofs only make use of the assumptions listed below.

Figure 6.3.1: Allowed assumptions in proofs.

The rules of algebra.
For example if x, y, and z are real numbers and x = y, then x+z = y+z.
The set of integers is closed under addition, multiplication, and subtraction.
In other words, sums, products, and differences of integers are also integers.

Every integer is either even or odd.
This fact is proven elsewhere in the material.
If x is an integer, there is no integer between x and x+1.
In particular, there is no integer between 0 and 1.
The relative order of any two real numbers.
For example 1/2 < 1 or 4.2 ≥ 3.7.
The square of any real number is greater than or equal to 0.
This fact is proven in a later exercise.

PARTICIPATION ACTIVITY | 6.3.1: Assumptions in proofs.

Indicate whether each assumption or proof step is valid.

1) If x is an integer and x > 0, then x ≥ 1.
- Valid
- Invalid

2) If x and y are even integers, then x+y is also even.
- Valid
- Invalid

3) If x and y are integers, then 3x-y+4 is also an integer.

- Valid
- Invalid

4) If x is a positive real number then x/2 < x.

Valid

The language of proofs

Every step in a proof requires justification. The reader needs to know if an assertion follows from an assumption of the proof, a definition, or a previously proven fact. The list below gives some common keywords and phrases that are used to explain the reasoning in a proof. In general, there are many different ways to write a valid proof of a theorem. Proofs can vary in notation or word choice, so the writer often has more than one option in selecting the specific words to be used in a proof.

Figure 6.3.2: Common keywords and phrases in proofs.

Thus and **therefore**

A statement that follows from the previous statement or previous few statements can be started with "Thus" or "Therefore".

- n and m are integers. Therefore, n+m is also an integer.
- n is a positive integer. Thus, n ≥ 1.

Other words that serve the same purpose are "it follows that", "then", "hence".

Let

New variable names are often introduced with the word "let". For example, "Let x be a positive integer".

Suppose

The word "suppose" can also be used to introduce a new variable. For example: "Suppose that x is a positive integer". Suppose is also used to introduce a new assumption, as in: "Suppose that x is odd", assuming that x has already been introduced as an integer earlier in the proof.

Since

If a statement depends on a fact that appeared earlier in the proof or in the assumptions of the theorem, it can be helpful to remind the reader of that fact before the statement. The phrase "because we know that" can serve the same purpose. For example, assuming that the facts x > 0 and y > z have been established earlier, a proof could say:

- "Since x > 0 and y > z, then xy > xz."
- "Because we know that x > 0 and y > z, then xy > xz."

By definition

A fact that is known because of a definition, can be started with the phrase "By definition". For example: "The integer m is even. By definition, m = 2k for some integer k."

By assumption

A fact that is known because of an assumption, can be started with the phrase "By assumption". For example: "By assumption, x is positive. Therefore x > 0."

In other words

Sometimes it is useful to rephrase a statement in a more specific way. The phrase "in other words" is useful in this context. For example: "We must show that the average of x and y is positive. In other words, we must show that (x+y)/2 > 0."

gives and **yields**

Sometimes a proof is clearer if even an algebraic step is justified. The words "gives" and "yields" are useful to say that one equation or inequality follows from another.

- Multiplying both sides of the inequality x > y by 2 gives 2x > 2y.
- Substituting m = 2k into m^2 yields $(2k)^2$
- Since z > 0, we can multiply both sides of the inequality x > y by z to get xz > yz.

PARTICIPATION ACTIVITY

6.3.2: Match the keyword to the correct location in a proof.

Below is a statement of a theorem and a proof with some phrases replaced by capital letters in red font.

Match each keyword to the place in the proof where the keyword best fits.

Theorem: If m and n are integers such that m divides n, then m divides n^2.

Proof.
Let m and n be two integers such that m divides n. __A__, m is nonzero and there exists an integer k such that n = km.

We will prove that m divides n^2. __B__, we will prove that there is an integer j such that n^2 = jm.

Plugging in the equation n = km into n^2 __C__ $n^2 = (km)^2 = (k^2m)m$

k and m are both integers. __D__, k^2m is also an integer.

__E__ n^2 = jm, where j = k^2m is an integer and m is nonzero, m divides n^2. ■

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Therefore, **Since** **By definition,** **gives** **In other words,**

A

B

C

D

E

Reset

Best practices in writing proofs

The list of best practices given below are important for writing valid proofs and for writing proofs that are easier for a reader to follow.

Figure 6.3.3: Best practices in writing proofs.

Indicate when the proof starts and ends.
In this material, every proof begins with the word **Proof:** and ends with the symbol ■.

Write proofs in complete sentences.
A proof should read like English text. In mathematical proofs, English sentences often conta mathematical expressions but those should read naturally as part of the sentence. For exam x is an integer that is greater than 0, then x ≥ 1."

Give the reader a roadmap of what has been shown, what is assumed, and where the proof is g
The beginning of a proof should always state what facts are assumed. It can also be helpful inform the reader what will be proven in the proof. If a proof is long, it is helpful to indicate at more points in the middle what has been proven and what has yet to be proven. For example have shown that n is a positive integer. Now we must establish that n is composite."

Introduce each variable when the variable is used for the first time.
Here are some examples of the introduction of a new variable:

- "Let x be a positive integer."
- "Since we know that m divides n, there is an integer k such that n = km." This sentence introduces the variable k. Variables m and n should already have been introduced.
- "Let s be the average of x and y: s = (x+y)/2." This sentence introduces the variable s. Variables x and y should already have been introduced.

A block of equations should be introduced with English text and each step that does not follow algebra should be justified.
In the example below the facts that n = k+1 and m ≥ 0 should be previously stated assumpt the proof or previously proven facts:

$$\text{Plugging in } n = k+1 \text{ into } n^2: \qquad n^2 = (k+1)^2$$

$$= k^2 + 2k + 1$$

$$\leq k^2 + 2k + 1 + m, \qquad \text{becau}$$

If the justification for a step does not fit easily on the line of the equation, the justification ca provided right after the block of equations.

PARTICIPATION ACTIVITY | 6.3.3: Proof example using best practices.

Animation content:

undefined

Animation captions:

1. The theorem to be proven is stated before the proof. The word "Proof:" indicates that the proof is starting.

2. The proof starts by stating what is assumed. Stating the fact to be proven is optional but often helpful in more complex proofs.
3. The definitions are applied to the assumptions that x divides y and x divides z, to give concrete expressions for y and z in terms of x.
4. The expressions for y and z are plugged in to y+z. The equation block is introduced with English text to form a complete sentence.
5. The proof concludes by arguing that j+k is an integer and we have established the fact that x divides y+z which what the proof promised to prove at the beginning.

PARTICIPATION ACTIVITY 6.3.4: Applying best practices in writing proofs.

Theorem: The product of an even integer and an odd integer is even.

Identify how each proof of the theorem is in error or is missing text that could make the proof more readable.

1) **Proof.**

Since x is even, x = 2k for some integer k. Since y is odd, y = 2j+1 for some integer j.

Plugging in the expression 2k for x and 2j+1 for y into xy gives:
$xy = 2k(2j+1) = 2 \cdot k(2j+1) = 2(2jk+k)$.

Since j and k are integers, 2jk+k is also an integer. The expression xy is equal to two times an integer and is therefore even. ■

- There are no indications for the start and end of the proof.
- The proof is not written in complete sentences.
- A variable is used without being introduced.

2) **Proof.**

Let x be an even integer and y be an odd integer. We shall prove that xy is even.

Since x is even, x = 2k for some integer k. Since y is odd, y = 2j+1 for some integer j.

$xy = 2k(2j + 1) = 2 \cdot k(2j + 1) = 2(2jk + k)$.

Since j and k are integers, 2jk+k is also an integer. The expression xy is equal to two times an integer and is therefore even. ■

- There are no indications for the start and end of the proof.
- The assumptions about x and y are not stated.
- The proof is not written in complete sentences.
- A variable is used without being introduced.

3) Let x be an even integer and y be an odd integer. We shall prove that xy is even.

Since x is even, x = 2k for some integer k. Since y is odd, y = 2j+1, for some integer j.

Plugging in the expression 2k for x and 2j+1 for y into xy gives:
$xy = 2k(2j + 1) = 2 \cdot k(2j + 1) = 2(2jk + k)$.

Since j and k are integers, 2jk+k is also an integer. The expression xy is equal to two times an integer and is therefore even.

- There are no indications for the start and end of the proof.
- The assumptions about x and y are not stated.
- The proof is not written in complete sentences.
- A variable is used without being introduced.

4) **Proof.**
Let x and y be integers.

Since x is even, x = 2k for some integer k. Since y is odd, y = 2j+1 for some integer j.

Plugging in the expression 2k for x and 2j+1 for y into xy gives:

$$xy = 2k(2j+1) = 2 \cdot k(2j+1) = 2(2jk+k).$$

Since j and k are integers, 2jk+k is also an integer. The expression xy is equal to two times an integer and is therefore even. ■

- There are no indications for the start and end of the proof.
- The assumptions about x and y are not stated.
- The proof is not written in complete sentences.
- A variable is used without being introduced.

Existential instantiation

Existential instantiation is a law of logic that says if an object is known to exist, then that object can be given a name, as long as the name is not currently being used to denote something else. The definitions of odd and even numbers, rational numbers, and divides all use existential instantiation. If n is an odd integer, then n is equal to two times an integer plus 1. That is, n = 2k+1, for some integer k. Giving the integer k a name is an example of existential instantiation.

In the proof given in the animation above, the integers y and z may or may not be equal to each other. The proof uses two different variable names in applying the definition that x divides y and x divides z: y = kx for some integer k, and z = jx for some integer j. It would be a logical error to use the same variable name for both y and z: y = kx and z = kx. Using the same variable name k would imply that z = kx = y, which is not necessarily true. Therefore, it is important to use a different variable name every time existential instantiation is used.

PARTICIPATION ACTIVITY | 6.3.5: Valid and invalid uses of existential instantiation.

Indicate whether the following proof fragments contain valid or invalid uses of existential instantiation.

1) Since m^2 and n^2 are both even integers, m^2 = 2k for some integer k, and n^2= 2j for some integer j.

- Valid
- Invalid

2) Since m is an odd integer and n is an even integer, m = 2k+1 for some integer k, and n = 2k for some integer k.

Valid

Invalid

Common mistakes in proofs

Mistakes in arithmetic or basic algebra are some of the most common types of mistakes in proofs. It is important to check every step of a proof carefully for such errors. Here are some logical errors that are also common when first learning to write proofs.

Common mistake 6.3.1: Generalizing from examples.

Exploring specific examples is an important way to gain insight into a theorem to be proven. However, if a fact holds for some particular elements in a set, that does not necessarily imply that the fact holds for all the elements in a set. In proving universal statements, it is important to either check every element in the domain or prove that the fact holds true for a generic element in the domain. Here is an example of an invalid argument that generalizes from examples:

> $m = 8$ is an even integer since $8 = 2 \cdot 4$. $m^2 = 8^2 = 64$ is an even integer since $64 = 2 \cdot 32$. Therefore if n is an even integer, then n^2 is also an even integer.

Common mistake 6.3.2: Skipping steps.

It is important to justify every step of a proof using allowed assumptions. It is an error to assume a fact is true without proving a reason. Consider the example below:

> If n is an odd integer, then $n = 2k+1$ for some integer k. Therefore $n^2 = (2k+1)^2$ and n^2 is odd.

This argument omits the steps showing that $(2k+1)^2 = 2(2k^2 + 2k) + 1$ and arguing that $(2k^2 + 2k)$ is an integer. Establishing that $(2k+1)^2$ is equal to two times an integer plus 1 is required in order to establish that $(2k+1)^2$ is odd.

Common mistake 6.3.3: Circular reasoning.

A proof that uses circular reasoning uses the fact to be proven in the proof itself. Here is an example of a "proof" that supposedly shows that if an integer n is odd then n^2 is odd.

> If n is an odd integer, then n = 2k+1 for some integer k. Let $n^2 = 2j + 1$ for some integer j. Since n^2 is equal to two times an integer plus 1, then n^2 is odd.

This proof jumps to the conclusion that $n^2 = 2j + 1$ for some integer j. The fact that n^2 is odd is the fact that needs to be proven.

Common mistake 6.3.4: Assuming facts that have not yet been proven.

Every fact used in a proof must be previously proven and referenced or must be established within the proof. Here is an example of a "proof" that supposedly shows that if a number r is rational then r^2 is rational.

> Suppose r is a rational number. The product of any two rational numbers is rational. Therefore $r^2 = r \cdot r$ is also rational.

The fact that the product of two rational numbers is rational has not been established in the proof and therefore cannot be used in the reasoning of the proof.

PARTICIPATION ACTIVITY | 6.3.6: Find the mistake in the proof.

Theorem: The difference between two odd numbers is even.

Identify which line has a mistake in the proof of the theorem above.

1)

1. Let x and y be two odd integers. We shall show that x-y is even.
2. Since x is odd, then x = 2k+1 for some integer k. Since y is odd, then y = 2j+1 for some integer j.
3. Since x and y are both odd, x-y must be even.
4. Therefore the difference between two odd integers is

even.

- Line 1
- Line 2
- Line 3
- Line 4

2)

1. Since x is odd, then x = 2k+1 for some integer k. Since y is odd, then y = 2j+1 for some integer j.
2. Plug in the expressions 2k+1 and 2j+1 for x and y into x-y to get x-y = (2k+1)-(2j+1) = 2k-2j = 2(k-j).
3. Since j and k are integers, k-j is also an integer. Therefore x-y is two times an integer and x-y is even.

- Line 1
- Line 2
- Line 3

3)

1. Let x and y be two odd integers. We shall show that x-y is even.
2. Since x is odd, then x = 2k+1 for some integer k. Since y is odd, then y = 2k+1 for some integer k.
3. Plug in the expressions 2k+1 and 2k+1 for x and y into x-y to get x-y = (2k+1)-(2k+1) = 2k-2k = 2(k-k).
4. Since k is an integer, k-k is also an integer. Therefore x-y is two times an integer and x-y is even.

- Line 1
- Line 2
- Line 3
- Line 4

4)

1. Let x and y be two odd integers. We shall show that x-y is even.
2. Since x is odd, then x = 2k+1 for some integer k. Since y is odd, then y = 2j+1 for some integer j.
3. Plug in the expressions 2k+1 and 2j+1 for x and y into x-y to get x-y = (2k+1)-(2j+1) = 2k-2j = 2(k-j).
4. Since x-y is two times an integer, then x-y is even.

- Line 1
- Line 2
- Line 3
- Line 4

Additional exercises

EXERCISE | 6.3.1: Fill in the words to form a complete proof.

Use the given equations in a complete proof of each theorem. Your proof should be expressed in complete English sentences.

(a) **Theorem:** If a, b, and c are integers such that $a^3|b$ and $b^2|c$, then $a^6|c$.
$b = ka^3$
$c = jb^2$
$c = jb^2 = j(ka^3)^2 = (jk^2)a^6$

(b) **Theorem:** If m and n are integers such that m|n, then $m|(5n^3 - 2n^2 + 3n)$.
$n = km$
$5n^3 - 2n^2 + 3n = 5(km)^3 - 2(km)^2 + 3(km) = 5k^3m^3 - 2k^2m^2 + 3km = (5$

(c) **Theorem:** If n is an odd integer, then 4 divides n^2-1.
$n = 2k + 1$
$n^2 - 1 = (2k + 1)^2 - 1 = (4k^2 + 4k + 1) - 1 = 4(k^2 + k)$

(d) **Theorem:** The sum of the squares of any two consecutive integers is odd.
$x^2 + (x + 1)^2 = x^2 + (x^2 + 2x + 1) = 2x^2 + 2x + 1 = 2(x^2 + x) + 1$

EXERCISE 6.3.2: Find the mistake in the proof - integer division.

Theorem: If w, x, y, z are integers where w divides x and y divides z, then wy divides xz.

For each "proof" of the theorem, explain where the proof uses invalid reasoning or skips essential steps.

(a) **Proof.**
Let w, x, y, z be integers such that w divides x and y divides z. Since, by assumption, w divides x, then x = kw for some integer k and w ≠ 0. Since, by assumption, y divides z, then z = ky for some integer k and y ≠ 0. Plug in the expression kw for x and ky for z in the expression xz to get

$$xz = (kw)(ky) = (k^2)(wy)$$

Since k is an integer, then k^2 is also an integer. Since w ≠ 0 and y ≠ 0, then wy ≠ 0. Since xz equals wy times an integer and wy ≠ 0, then wy divides xz. ■

(b) **Proof.**
Let w, x, y, z be integers such that w divides x and y divides z. Since, by assumption, w divides x, then x = kw for some integer k and w ≠ 0. Since, by assumption, y divides z, then z = jy for some integer j and y ≠ 0. Since w ≠ 0 and y ≠ 0, then wy ≠ 0. Let m be an integer such that xz = m·wy. Since xz equals wy times an integer and wy ≠ 0, then wy divides xz. ■

(c) **Proof.**
Let w, x, y, z be integers such that w divides x and y divides z. Since, by assumption, w divides x, then x = kw for some integer k and w ≠ 0. Since, by assumption, y divides z, then z = jy for some integer j and y ≠ 0. Plug in the expression kw for x and jy for z in the expression xz to get

$$xz = (kw)(jy)$$

Since w ≠ 0 and y ≠ 0, then wy ≠ 0. Since xz equals wy times an integer and wy ≠ 0, then wy divides xz. ■

(d) **Proof.**
Let w, x, y, z be integers such that w divides x and y divides z. Since w divides x, then x = kw and w ≠ 0. Since y divides z, then z = jy and y ≠ 0. Plug in the expression kw for x and jy for z in the expression xz to get

$$xz = (kw)(jy) = (kj)(wy)$$

Since k and j are integers, then kj is also an integer. Since w ≠ 0 and y ≠ 0, then wy ≠ 0. Since xz equals wy times an integer and wy ≠ 0, then wy divides xz. ■

EXERCISE 6.3.3: Find the mistake in the proof - odd and even numbers.

Theorem: If n and m are odd integers, then $n^2 + m^2$ is even

For each "proof" of the theorem, explain where the proof uses invalid reasoning or skips essential steps.

(a) **Proof.**
m = 7 is odd because 7 = 2·3+1. n = 9 is odd because 9 = 2·4 + 1.

$$7^2 + 9^2 = 49 + 81 = 130 = 2 \cdot 65$$

Since $7^2 + 9^2$ is equal to 2 times an integer, $7^2 + 9^2$ is even. Therefore the theorem is true. ■

(b) **Proof.**
Let n and m be odd integers. Since n is an odd integer, then n = 2k+1. Since m is an odd inte
Plugging in 2k+1 for n and 2j+1 for m into the expression $n^2 + m^2$ gives

$$n^2 + m^2 = (2k+1)^2 + (2j+1)^2 = 4k^2 + 4k + 1 + 4j^2 + 4j + 1 = 2(2k^2 + 2$$

Since k and j are integers, $2k^2 + 2k + 2j^2 + 2j + 1$ is also an integer. Since $n^2 + m^2$ is eq
integer, then $n^2 + m^2$ is an even integer. ■

(c) **Proof.**
Let n and m be odd integers. Since n is an odd integer, then n = 2k+1 for some integer k. Since m is an odd integer, then m = 2j+1 for some integer j. Since n^2 = n·n and n is odd, then n^2 is also odd because the product of two odd integers is odd. Since m^2 = m·m and m is odd, then m^2 is also odd because the product of two odd integers is odd. Since the sum of two odd integers is even, $n^2 + m^2$ is even. ■

(d) **Proof.**
Let n and m be odd integers. Since n is an odd integer, then n = 2k+1 for some integer k. Since m is an odd integer, then m = 2j+1 for some integer j. Plugging in 2k+1 for n and 2j+1 for m into the expression $n^2 + m^2$ gives

$$n^2 + m^2 = (2k+1)^2 + (2j+1)^2$$

Since $n^2 + m^2$ is equal to two times an integer, then $n^2 + m^2$ is an even integer. ■

(e) **Proof.**
Let n and m be odd integers. Since n is an odd integer, then n = 2j+1 for some integer j. Since
an odd integer, then m = 2j+1 for some integer j. Plugging in 2j+1 for n and 2j+1 for m into th
expression $n^2 + m^2$ gives

$$n^2 + m^2 = (2j+1)^2 + (2j+1)^2 = 4j^2 + 4j + 1 + 4j^2 + 4j + 1 = 2(4j^2 + 4j$$

Since j is an integer, $4j^2 + 4j + 1$ is also an integer. Since $n^2 + m^2$ is equal to two times an integer, then $n^2 + m^2$ is an even integer. ■

6.4 Writing direct proofs

The form of direct proofs

Many mathematical theorems take the form of a conditional statement in which a conclusion follows from a set of hypotheses. Theorems of this kind can be expressed as $p \rightarrow c$, where p is a proposition which asserts that a set of hypotheses are true and c is the conclusion. Sometimes p is referred to as "the hypothesis" for simplicity. In a ***direct proof*** of a conditional statement, the hypothesis p is assumed to be true and the conclusion c is proven as a direct result of the assumption.

Many theorems are conditional statements that also have a universal quantifier such as:

For every integer n, if n is odd then n^2 is odd.

The domain of variable n is the set of all integers. If D(n) is the predicate that says that n is odd, then the statement is equivalent to the logical expression: $\forall n\ (D(n) \rightarrow D(n^2))$. A direct proof of the theorem starts with n, an arbitrary integer, assumes that n is odd, and then proves that n^2 is odd. Frequently, the universal quantifier and domain are expressed as part of the hypothesis, as in

If n is an odd integer, then n^2 is an odd integer.

The universal quantifier does not always occur at the beginning of the theorem statement. The sentence below would be an equivalent way of expressing the theorem:

The square of every odd integer is also odd.

Typically the first few sentences of a direct proof names one or more generic objects, states all the assumptions about that object (including the assumption that the object is in the given domain), and then states what will be proven. For example, a proof of the statement above could begin with the following sentences:

Let n be an integer that is odd. We will show that n^2 is also odd.

PARTICIPATION ACTIVITY 6.4.1: Assumptions in a direct proof.

Match each theorem statement with the correct set of assumptions in a direct proof of the theorem.

If unable to drag and drop, refresh the page.

The average of two real numbers is greater than or equal to at least one of the numbers.

The difference between two distinct real numbers is not equal to zero.

The product of two positive real numbers is a positive number.

Let x and y be real numbers such that x > 0 and y > 0.

Let x and y be real numbers.

Let x and y be real numbers such that x ≠ y.

Reset

Writing direct proofs

After the assumptions are stated, a direct proof proceeds by proving the conclusion is true. Consider again the theorem:

The square of every odd integer is also odd.

The proof of the theorem begins by naming a generic object n, stating all the assumptions about n, and then what will be proven.

Let n be an integer that is odd. We will show that n^2 is also odd.

The next step is to express the fact that n is odd using the mathematical definition that n = 2k+1 for some integer k. This step provides an algebraic expression for n = (2k+1) which makes it possible to show that n^2 is odd by a series of algebraic steps. Since the goal of the proof is to show that n^2 is odd, the algebraic steps leads toward an equation of the form n^2 = 2m+1, where m is an integer.

The proof ends with a statement saying that we have shown that n^2 is odd, which is the fact that the proof claimed would be proven at the beginning.

PARTICIPATION ACTIVITY | 6.4.2: A direct proof.

Animation content:

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Animation captions:

1. An equivalent statement of the theorem to be proven is that if n is an odd integer, then n^2 is an odd integer.
2. In a direct proof, the hypothesis that n is an odd integer is assumed, and the conclusion that n^2 is an odd integer is proven.
3. The first step names an arbitrary object in the domain, states the assumptions about the object, and then states what will be shown in the proof.
4. Apply the definition of an odd integer to obtain an algebraic expression for n which is that n = 2k+1, where k is an integer.
5. Plug in n = 2k+1 into n^2, to obtain an expression of n^2 in terms of k.
6. Use algebra to express n^2 as two times an integer plus 1.
7. Reason that n^2 has been expressed as two times an integer plus 1, and therefore, n^2 is odd.

PARTICIPATION ACTIVITY | 6.4.3: Order the steps of a direct proof.

Below are the steps of a direct proof of the following theorem:

Theorem: The difference between two even integers is even.

Put the steps of the proof in the correct order so that each step follows from previous steps in the proof.

If unable to drag and drop, refresh the page.

Let x and y be even integers. We shall prove that x-y is even. **(2k) - (2j) = 2(k-j)**

Since x-y is equal to 2m, where m = k-j is an integer, x-y is even.

Since x is even, there is an integer k such that x = 2k. Since y is even, there is an integer j suc

x-y = (2k) - (2j) **Since k and j are integers, then k-j is also an integer.**

Step 1

Step 2

Step 3

Step 4

Step 5

Step 6

Reset

The process of writing proofs

One of the hardest parts of writing proofs is knowing where to start. If a proof can be any sequence of logical steps, how is the prover to know which sequence of steps will lead to a proof of the theorem? Fortunately, many proofs follow one of a relatively small number of patterns. Following one of the common patterns helps give the proof structure and the prover some direction. This material will give a brief overview of some of the common patterns of mathematical proofs. The types of proofs presented are not an exhaustive list, but rather give a first glimpse of the kinds of mathematical arguments used in proving theorems.

Proofs in their final form are expressed in their simplest and most direct way; however, proofs are rarely conceived in their simplest form. Coming up with proofs requires trial and error, even for experienced mathematicians. Often the process includes experimenting with small examples in order to develop intuition about a more general rule. The process almost always entails some dead ends along the way.

The next example gives a direct proof for the theorem given below. The video and final proof illustrate that the process of coming up with the steps for a proof can look very different than the final proof.

Theorem 6.4.1: An algebraic theorem to be proven by a direct proof.

If x and y are positive real numbers, then

$$\frac{x}{y} + \frac{y}{x} \geq 2.$$

Video 6.4.1: Scratch work for the proof.

Proof 6.4.1: A direct proof of the algebraic theorem.

Theorem: If x and y are positive real numbers, then

$$\frac{x}{y} + \frac{y}{x} \geq 2.$$

Proof.

Let x and y be positive real numbers. Since x and y are real, x - y is also a real number. Therefore $(x - y)^2 \geq 0$, because the square of any real number is greater than or equal to 0. Multiplying out the left hand side of the inequality gives

$$x^2 - 2xy + y^2 \geq 0.$$

Since x and y are both greater than 0, we can divide both sides of the inequality by xy to get

$$\frac{x}{y} - 2 + \frac{y}{x} \geq 0.$$

Adding 2 to both sides, gives the conclusion of the theorem:

$$\frac{x}{y} + \frac{y}{x} \geq 2.$$

■

PARTICIPATION ACTIVITY | 6.4.4: Order the steps of a direct proof.

Below are the steps of a direct proof of the following theorem:

Theorem: If x is a real number and x - 3 = 0, then $x^2 - 2x - 3 = 0$.

Put the steps of the proof in the correct order so that each step follows from previous steps in the proof.

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Plug in x = 3 into the expression $x^2 - 2x - 3$. **$x^2 - 2x - 3 = 3^2 - 2 \cdot 3 - 3 = 9 - 6 - 3 = 0$.**

Assume x is a real number and x - 3 = 0. We shall prove that $x^2 - 2x - 3 = 0$.

Add 3 to both sides of the equation: x = 3.

Step 1.

Step 2.

Step 3.

Step 4.

Reset

A direct proof about rational numbers

The proof below proves that the sum of two rational numbers is also rational. The proof starts with the assumption that two numbers, r and s, are rational and proves that r + s is also rational. Therefore the proof is a direct proof.

The only information given about r and s is that they are rational, so the first step is to state this fact mathematically, using existential instantiation, by assigning variable names to the integers that form the ratios, e.g., $r = a/b$ and $s = c/d$, where b and d are both non-zero. Four different variable names (a, b, c, and d) are used for the four integers. Giving the four integers variable names enables the prover to express the sum r + s concretely. After plugging in the two ratios for r and s, the rest of the proof is just algebra. The goal is to express r + s as a single ratio of two integers.

Proofs about rational numbers often make use of the fact that for any two real numbers, x and y, if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.

PARTICIPATION ACTIVITY | 6.4.5: A direct proof that the sum of two rational numbers is rational.

Animation content:

undefined

Animation captions:

1. A direct proof assumes the hypotheses and then derives the conclusion.
2. The assumption is that r and s are rational numbers: r = a/b and s = c/d, for integers a, b, c, d, such that b and d are non-zero.
3. The next step is to plug the expressions for r and s into r + s and show that the result is equal to (ad +cd)/bd.
4. Since a,b, c, and d are integers, ad+cd and bd are integers. (ad + cb)/bd is the ratio of two

integers and bd is non-zero.
5. Therefore, r+s is rational.

PARTICIPATION ACTIVITY 6.4.6: Rational numbers.

Below are the steps of a direct proof of the following theorem:

Theorem: The square of any rational number is also rational.

Put the steps of the proof in the correct order so that each step follows from previous steps in the proof.

If unable to drag and drop, refresh the page.

Since r is rational, there exists integers x and y such that r = x/y and y ≠ 0.

Since x is an integer, x^2 is also an integer. Since y is an integer, y^2 is also an integer. Since y

Let r be a rational number. We will show that r^2 is also rational.

$r^2 = (x/y)^2 = x^2/y^2$.

Since $r^2 = x^2/y^2$, x^2 and y^2 are both integers, and $y^2 \neq 0$, then r^2 is a rational number.

Step 1

Step 2

Step 3

Step 4

Step 5

Reset

CHALLENGE ACTIVITY 6.4.1: Writing direct proofs.

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Start

Suppose the following theorem is proven using a direct proof.

Theorem: If x is a real number and $x < 13$, then $x^2 - 29x + 208 > 0$.

What would be assumed at the beginning of the proof?

x is a real number and $x \geq 13$.

1 | 2 | 3

Check Next

Additional exercises

EXERCISE 6.4.1: Proving statements about odd and even integers with direct proofs.

Each statement below involves odd and even integers. An odd integer is an integer that can be expressed as $2k + 1$, where k is an integer. An even integer is an integer that can be expressed as $2k$, where k is an integer.

Prove each of the following statements using a direct proof.

(a) The sum of an odd and an even integer is odd.

(b) The sum of two odd integers is an even integer.

(c) The square of an odd integer is an odd integer.

(d) The product of two odd integers is an odd integer.

(e) If x is an even integer and y is an odd integer, then $x^2 + y^2$ is odd.

(f) If x is an even integer and y is an odd integer, then $3x + 2y$ is even.

(g) If x is an even integer and y is an odd integer, then $2x + 3y$ is odd.

(h) The negative of an odd integer is also odd.

(i) If x is an even integer then $(-1)^x = 1$.

(j) If x is an odd integer then $(-1)^x = -1$.

EXERCISE 6.4.2: Proving statements about rational numbers with direct proofs.

Prove each of the following statements using a direct proof.

(a) The product of two rational numbers is a rational number.

(b) The quotient of a rational number and a non-zero rational number is a rational number.

(c) If x and y are rational numbers then $3x + 2y$ is also a rational number.

(d) If x and y are rational numbers then $3x^2 + 2y$ is also a rational number.

(e) If x and y are rational numbers, where y ≠ 0 and y ≠ -1, then $\dfrac{x}{1 + \frac{1}{y}}$ is also rational.

(f) The average of two rational numbers is also rational.

EXERCISE 6.4.3: Proving algebraic statements with direct proofs.

Prove each of the following statements using a direct proof.

(a) For any positive real numbers, x and y, $(x + y)^2 \geq xy$.

(b) If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

(c) If n is a real number and $n > 1$, then $n^2 > n$.

(d) If x is a real number such that $0 < x < 1$, then $\frac{1}{x(1-x)} \geq 4$.

EXERCISE | 6.4.4: Showing a statement is true or false by direct proof or counterexample.

Determine whether the statement is true or false. If the statement is true, give a proof. If the statement is false, give a counterexample.

(a) If x and y are even integers, then $x+y$ is an even integer.

(b) If $x+y$ is an even integer, then x and y are both even integers.

(c) If $x^2 = y^2$, then $x = y$.

(d) If x and y are real numbers and $x < y$, then $x^2 < y^2$.

(e) If x and y are positive real numbers and $x < y$, then $x^2 < y^2$.

(f) The average of two odd numbers is an odd integer.

(g) The average of two even numbers is even.

(h) The average of two odd integers is an integer.

(i) If x and y are integers such that xy is a perfect square, then x and y are also perfect squares.

(j) If x and y are perfect squares then xy is a perfect square.

(k) If x, y, and z are integers and xy|z, then x|z and y|z.

(l) If x, y, and z are integers and x|yz, then x|y or x|z.

(m) If x, y, and z are integers and x|(y+z), then x|y or x|z.

(n) If x, y, and z are integers such that x|(y+z) and x|y, then x|z.

(o) If x and y are integers and x|y^2, then x|y.

6.5 Proof by contrapositive

A ***proof by contrapositive*** proves a conditional theorem of the form p → c by showing that the contrapositive ¬c → ¬p is true. In other words, ¬c is assumed to be true and ¬p is proven as a result of ¬c.

Many theorems are conditional statements that also have a universal quantifier such as:

For every integer n, if n^2 is odd then n is odd. The domain of variable n is the set of all integers. If D(n) is the predicate that says that n is odd, then the statement is equivalent to the logical expression: $\forall n\,(D(n^2) \rightarrow D(n))$. A proof by contrapositive of the theorem starts with n, an arbitrary integer, assumes that D(n) is false, and then proves that $D(n^2)$ is false.

The animation below gives a proof by contrapositive that for every integer n, if 3n + 7 is odd then n is even. The theorem is a conditional statement in which the hypothesis is that 3n + 7 is odd and the conclusion is that n is even. A contrapositive proof assumes the negation of the conclusion (n is odd) and uses the assumption to prove the negation of the hypothesis (3n + 7 is even). Why use a contrapositive proof for this proof? One reason, is that the negation of the assumption (n is odd) is a little simpler and therefore easier to work with than the hypothesis (3n + 7 is odd).

The proof implicitly uses the fact that every integer is even or odd, so if an integer is not even, then the integer is odd. This fact is proven elsewhere in the material.

PARTICIPATION ACTIVITY | 6.5.1: Proof by contrapositive example.

Animation content:

undefined

Animation captions:

1. A proof by contrapositive starts with the negation of the conclusion and derives the negation of the hypothesis.
2. The negation of the conclusion is that n is an odd integer. Therefore, $n = 2k + 1$ for some integer k.
3. The next step is to plug the expression for n into $3n + 7$ and show that the result is equal to $2(3k + 5)$, which is 2 times an integer.
4. Therefore, $3n + 7$ is even, which is the negation of the hypothesis.

PARTICIPATION ACTIVITY | 6.5.2: Proof by contrapositive - fill in the blank.

Below is a statement of a theorem and a proof by contrapositive with some parts of the argument replaced by capital letters in red font.

Theorem: For every positive integer x, if x^3 is even, then x is even.

Proof.
Let x be a positive integer. Assume A. We will prove that B.

If x is odd, then it can be written as C for some integer k. Plug in the expression for x into

x^3 to get D. The expression for x^3 can be written as E. Since $(4k^3 + 6k^2 + 3k)$ is an integer, we can conclude that x^3 is odd. ■

1) What is the correct expression for A?
   - x^3 is even
   - x^3 is odd
   - x is even
   - x is odd

2) What is the correct expression for B?
   - x^3 is even
   - x^3 is odd
   - x is even
   - x is odd

3) What is the correct expression for C?
   - 2k
   - 2k+1
   - k^3

4) What is the correct expression for D?
   - $8k^3+12k^2+6k+1$
   - $8k^3$

5) What is the best expression for E? Both choices are equal to x^3, but one does a better job showing that x^3 is odd.
   - $2(4k^3 + 6k^2 + 3k) + 1$
   - $8k^3+12k^2+6k+1$

PARTICIPATION ACTIVITY | 6.5.3: Proof by contrapositive - order the steps.

Below are the steps of a proof by contrapositive of the following theorem:

Theorem: For every real number x, if $x^3 + 2x + 1 \leq 0$, then $x \leq 0$.

Put the steps of the proof in the correct order so that each step follows from previous steps in the proof.

If unable to drag and drop, refresh the page.

Since x^3 , 2x, and 1 are all greater than 0, their sum is also greater than zero.

Assume x > 0, where x is a real number.

$x^3 + 2x + 1 > 0$. Therefore the theorem is true.

Since x > 0, then 2x > 0 and $x^3 > 0$.

Step 1.

Step 2.

Step 3.

Step 4.

Reset

When to use a direct proof vs. a proof by contrapositive

Deciding whether to prove a conditional statement using a direct proof or a proof by contrapositive often involves some trial and error. The decision should be based on whether the hypothesis or the negation of the conclusion provides a more useful assumption to work with. Consider the statement:

For every integer x, if x^2 is even, then x is even.

A direct proof assumes that x^2 is even, which in mathematical terms means that $x^2 = 2k$, for some integer k. Deriving an expression for x requires taking the square root of both sides, and it is not clear how to reason that $\sqrt{2k}$ is an even integer. Alternatively, a proof by contrapositive assumes that x is odd, which in mathematical terms means that x = 2k + 1, for some integer k. The expression for x can then be plugged into x^2 resulting in an expression that is much easier to reason about.

Proof 6.5.1: A proof by contrapositive of a theorem about even integers.

Theorem: For every integer x, if x^2 is even, then x is even.

Proof.
Let x be an integer. We assume that x is odd and prove that x^2 is odd. If x is odd, it can be expressed as 2k + 1, for some integer k. Plug in x = 2k + 1 into x^2 to get

$$x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Since k is an integer, $2k^2 + 2k$ is also an integer. Therefore x^2 can be expressed as 2j + 1, where $j = 2k^2 + 2k$ is an integer. We can conclude that x^2 is odd. ■

PARTICIPATION ACTIVITY | 6.5.4: Direct proofs and proofs by contrapositive.

Consider the following statement:

For every real number x, if $0 \le x \le 3$, then $15 - 8x + x^2 \ge 0$

1) What would be the starting assumption in a direct proof of the statement above?

- $0 \le x \le 3$
- $15-8x+x^2 \ge 0$
- $15-8x+x^2 \le 0$

2) What would be proven in a direct proof of the statement, given the assumption?

- $0 \le x \le 3$
- $15-8x+x^2 \ge 0$
- $15-8x+x^2 \le 0$

3) What would be the starting assumption in a proof by contrapositive of the statement above?

- $0 \le x \le 3$
- $15-8x+x^2 \ge 0$
- $15-8x+x^2 < 0$

4) What would be proven in a proof by contrapositive of the statement, given the assumption?

- $0 \le x \le 3$
- $x < 0$ or $x > 3$.
- $x < 0$ and $x > 3$.

PARTICIPATION ACTIVITY 6.5.5: De Morgan's laws and the contrapositive.

De Morgan's laws are

(A and B) is false if and only if A is false or B is false.

(A or B) is false if and only if A is false and B is false.

Use De Morgan's laws to answer the following statements about proofs by contrapositive.

1) What would be the starting assumption in a proof by contrapositive of the following statement?

If $x < 0$ and $xy > 0$, then $y < 0$.

- $y \ge 0$
- $y > 0$
- $y < 0$

2) What would be proven in a proof by contrapositive of the following statement?

If $x < 0$ and $xy > 0$, then $y < 0$.

- $x < 0$ and $xy > 0$
- $x \ge 0$ and $xy \le 0$
- $x \ge 0$ or $xy \le 0$

3) What would be the starting assumption in a proof by contrapositive of the following statement?

If x and y are integers such that xy is even, then x is even or y is even.

- x is even and y is even
- x is odd and y is odd
- x is odd or y is odd

4) What would be proven in a proof by contrapositive of the following statement?

If x and y are integers such that xy is even, then x is even or y is even.

- xy is even
- xy is odd

Irrational numbers and proofs by contrapositive

An ***irrational number*** is a real number that is not rational. Note that the definition implies that every real number is either rational or irrational but not both. Therefore if x is a real number that is not irrational, then x is rational.

The theorem proven below states that for every positive real number r, if r is irrational, then $\sqrt{r}$ is also irrational. A direct proof would assume that r is irrational which means that r cannot be expressed as a ratio of two integers. The assumption that r cannot be expressed in a certain way does not provide a concrete expression for r to work with. Alternatively, a proof by contrapositive uses the assumption that $\sqrt{r}$ is rational which provides a useful expression for r (the ratio of two integers) that can be plugged into other expressions.

PARTICIPATION ACTIVITY | 6.5.6: A proof by contrapositive about irrational numbers.

Below are the steps of a proof by contrapositive of the following theorem:

Theorem: For every positive real number r, if r is irrational, then $\sqrt{r}$ is also irrational.

Put the steps of the proof in the correct order so that each step follows from previous steps in the proof.

If unable to drag and drop, refresh the page.

Let r be a positive real number such that $\sqrt{r}$ is not irrational. We will prove that r is not irratio

Therefore $\sqrt{r} = x/y$, for some two integers x and y, where y ≠ 0.

Since r is a positive real number, $\sqrt{r}$ is a positive real number.
Since $\sqrt{r}$ is not irrational and is real, then $\sqrt{r}$ must be rational.

Since x and y are integers, x^2 and y^2 are both integers. Since y ≠ 0, then $y^2 \neq 0$.

r is equal to the ratio of two integers with a non-zero denominator, so r is a rational number. Therefore r is not irrational.

Squaring both sides of the equation gives $r = (\sqrt{r})^2 = (x/y)^2 = x^2/y^2$.

Step 1

Step 2

Step 3

Step 4

Step 5

Step 6

Reset

Proofs by contrapositive of conditional statements with multiple hypotheses

Some conditional statements have more than one hypothesis. In a proof by contrapositive it is only necessary to show that one of the hypotheses is false, assuming that the rest of the hypotheses are true and the conclusion is false. For example, consider the conditional statement:

If H_1 and H_2 are both true then C is true. $[(H_1 \land H_2) \rightarrow C]$

The contrapositive of this conditional statement is:

If C is false, then it cannot be the case that H_1 and H_2 are both true. [$\neg C \rightarrow \neg(H_1 \land H_2)$]

By De Morgan's law, the statement is equivalent to:

If C is false, then H_1 is false or H_2 is false. [$\neg C \rightarrow (\neg H_1 \lor \neg H_2)$]

which is in turn equivalent to:

If C is false and H_1 is true, then H_2 is false. [$(\neg C \land H_1) \rightarrow \neg H_2$]

Note that the expression $\neg C \rightarrow (\neg H_1 \lor \neg H_2)$ is logically equivalent to $(\neg C \land H_1) \rightarrow \neg H_2$. This fact can be verified by writing out a truth table. Alternatively, remember that a conditional statement is false if and only if the hypothesis is true and the conclusion is false. The only truth assignment in which both conclusions are false and both hypotheses are true is the assignment: C = F, H_1 = T, and H_1 = T. Therefore the two expressions are logically equivalent. Proving one statement is the same as proving the other.

Sometimes using this form of the statement can make it easier to write the proof. A proof by contrapositive in this form would start as follows:

Assume that C is false and H_1 is true. We shall show that H_2 is false.

It is also valid to swap the roles of H_1 and H_2 and start the proof with:

Assume that C is false and H_2 is true. We shall show that H_1 is false.

When the form of a proof by contrapositive is more complex as in this example, it is good practice to state explicitly at the beginning of the proof what will be assumed and what will be proven in the proof.

The hypothesis that will be shown to be false is selected to facilitate the arguments in the proof. The animation below gives a specific example.

PARTICIPATION ACTIVITY | 6.5.7: Contrapositive proofs with multiple hypotheses.

Animation content:

Static figure: A partial proof is shown. The theorem is at the top, followed by an assume statement, a prove statement, and then the start of the proof.

Step 1: If x is a rational number (hypothesis 1) and y is an irrational number (hypothesis 2), then $x+y$ is irrational (conclusion).

Proof statement: Theorem: If x is a rational number and y is an irrational number, then $x+y$ is irrational. x is a rational number is labeled hypothesis 1, y is an irrational number is labeled hypothesis 2, and $x+y$ is irrational is labeled conclusion.

Step 2: A proof can assume hypothesis 1 and the negation of the conclusion and prove the negation of hypothesis 2.

Proof statement: Assume: ¬ conclusion and hypothesis 1. Prove: ¬ hypothesis 2,

Step 3: A proof can assume that x is rational and that $x+y$ is not irrational and prove that y is not irrational.

The two lines from step 2 are modified to be Assume: $\neg x+y$ is irrational and x is rational. Prove: $\neg y$ is irrational.

Step 4: Start to the proof: "Let x and y be two real numbers such that $x+y$ is rational and x is rational. We shall prove that y is rational."

Proof statement: Start of proof: Let x and y be two real numbers such that $x+y$ is rational and x is rational. We shall prove that y is rational.

Step 5: All assumptions and facts to be proven assert that a number is rational, so the definition of a rational number can be applied.

Step 6: Note this is not a complete proof yet. The text above illustrates a way to begin a correct proof.

Animation captions:

1. If x is a rational number (hypothesis 1) and y is an irrational number (hypothesis 2), then x+y is irrational (conclusion).
2. A proof can assume hypothesis 1 and the negation of the conclusion and prove the negation of hypothesis 2.
3. A proof can assume that x is rational and that x+y is not irrational and prove that y is not irrational.
4. Start to the proof: "Let x and y be two real numbers such that x+y is rational and x is rational. We shall prove that y is rational."
5. All assumptions and facts to be proven assert that a number is rational, so the definition of a rational number can be applied.
6. Note this is not a complete proof yet. The text above illustrates a way to begin a correct proof.

PARTICIPATION ACTIVITY | 6.5.8: Conditional statements with multiple hypotheses.

1) Which selection corresponds to a valid start to a proof of the theorem below?

Theorem: If x is an even integer and y^2 is an odd integer, then x+y is an odd integer.

- Let x and y be integers such that x+y is even and x is even. We will show that y^2 is odd.
- Let x and y be integers such that x+y is even and x is odd. We will show that y^2 is even.
- Let x and y be integers such that x+y is even and x is even. We will show that y^2 is even.

CHALLENGE ACTIVITY | 6.5.1: Proof by contrapositive.

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Start

Suppose the following theorem is proven by contrapositive.

Theorem: For every real number x, if $x^2 + 20x < 0$, then $x < 0$.

Let x be a real number. What would be assumed to be true at the beginning of the proof?

$x < 0$

1 2 3

Check Next

Additional exercises

EXERCISE 6.5.1: Proof by contrapositive of statements about odd and even integers.

Prove each statement by contrapositive

(a) For every integer n, if n^2 is odd, then n is odd.

(b) For every integer n, if n^3 is even, then n is even.

(c) For every integer n, if $5n + 3$ is even, then n is odd.

(d) For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

(e) For every integer n, if n^2 is not divisible by 4, then n is odd.

(f) For every pair of integers x and y, if xy is even, then x is even or y is even.

(g) For every pair of integers x and y, if $x - y$ is odd, then x is odd or y is odd.

(h) For every integer n ≥ 3, if 2^n-1 is prime, then n is odd.

EXERCISE 6.5.2: Proof by contrapositive of statements about integer division.

Prove each statement by contrapositive

(a) If x and y are integers such that $3\nmid xy$ then $3\nmid x$.

(b) For any integers x, y, and z, if $x|y$ and $x\nmid z$, then $x\nmid(y+z)$.

(c) If x and y are positive integers such that x > 1, and x divides y, then $x\nmid(y+1)$.

EXERCISE 6.5.3: Proof by contrapositive of statements about rational numbers.

Prove each statement by contrapositive

(a) For every real number x, if x is irrational, then $-x$ is also irrational.

(b) For every pair of real numbers x and y, if $x + y$ is irrational, then x is irrational or y is irrational.

(c) For every pair of real numbers x and y, if x is rational and xy is irrational, then y is irrational.

EXERCISE 6.5.4: Proof by contrapositive of algebraic statements.

Prove each statement by contrapositive

(a) For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

(b) For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

(c) For every pair of positive real numbers x and y, if $xy > 400$, then $x > 20$ or $y > 20$.

EXERCISE | 6.5.5: Proving statements using a direct proof or by contrapositive.

Prove each statement using a direct proof or proof by contrapositive. One method may be much easier than the other.

(a) The product of any integer and an even integer is even.

(b) If p > 2 and p is a prime number, then p is odd.

(c) For every non-zero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational.

(d) If x is a real number such that $x^3 + 2x < 0$, then $x < 0$.

(e) If n and m are integers such that n^2+m^2 is odd, then m is odd or n is odd.

(f) If x, y, and z are integers and x+z and y+z are both even, then x+y is also even.

(g) The difference of two rational numbers is a rational number.

(h) If $x + 2 = 0$, then $x^2 + 6x + 8 = 0$.

(i) If x is an odd integer then $x^2 + x$ is even.

6.6 Proof by contradiction

A ***proof by contradiction*** starts by assuming that the theorem is false and then shows that some logical inconsistency arises as a result of the assumption. The reasoning behind proof by contradiction is that if the assumption that the theorem is false leads to a conclusion which cannot be true, then the theorem must be true. A proof by contradiction is sometimes called an ***indirect proof***.

If t is the statement of the theorem, the proof begins with the assumption ¬t and leads to a conclusion r ∧ ¬r, for some proposition r. A proof by contradiction often starts with: "Suppose ¬t", where ¬t is a statement that is equivalent to the negation of the theorem.

The animation below gives a proof that for every pair of positive real numbers, a and b,

$$\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}.$$

In this case, the negation of the theorem is equivalent to saying that there exists a pair positive integers, a and b such that

$$\sqrt{a} + \sqrt{b} = \sqrt{a+b}.$$

The equivalence can be restated in the notation of logic as:

$$\neg(\forall a\ \forall b\ \sqrt{a} + \sqrt{b} \neq \sqrt{a+b}) \ \equiv\ \exists a\ \exists b\ (\sqrt{a} + \sqrt{b} = \sqrt{a+b}),$$

where the domain for variables a and b is the set of all positive real numbers.

The algebra in the proof is guided by the goal of simplifying the equation. Square roots are complicated to handle, so each side of the equation is squared to help eliminate (or reduce) terms with square roots. The next steps are to distill the equation into its simplest form which will hopefully enable the prover to see a contradiction.

PARTICIPATION ACTIVITY | 6.6.1: Proof by contradiction example.

Animation content:

undefined

Animation captions:

1. A proof by contradiction shows that assuming that the theorem is false leads to an inconsistency.
2. The negation of the theorem is the same as saying that there are positive real numbers a and b such that $\sqrt{a} + \sqrt{b} = \sqrt{(a+b)}$.
3. The next step is to square both sides of the equation and derive that ab = 0 which implies that either a = 0 or b = 0.
4. The fact that a = 0 or b = 0 contradicts the assumption that a and b are both positive, which is an inconsistency.

PARTICIPATION ACTIVITY | 6.6.2: Expressing the negation of theorems.

Select the statement that is equivalent to the negation of each theorem.

1) **Theorem:** Among any pair of consecutive integers, at least one of

the integers must be even.

- There exists a pair of consecutive integers that are both odd.
- There exists a pair of consecutive integers such that at least one of the two numbers is odd.
- For every pair of consecutive integers, at least one of the integers must be odd.

2) **Theorem:** The negative of any irrational number is irrational.

- There is an irrational number s, such that -s is irrational.
- There is an irrational number s, such that -s is rational.
- The negative of any irrational number is rational.

3) **Theorem:** There exists a pair of integers, x and y, such that 6x - 21y = 1.

- There exists a pair of integers, x and y, such that 6x - 21y ≠ 1.
- For every pair of integers, x and y, 6x - 21y = 1.
- For every pair of integers, x and y, 6x - 21y ≠ 1.

Proof by contradiction and proof by contrapositive

A proof by contrapositive is a special case of a proof by contradiction. In proving a theorem of the form p → q by contrapositive, we assume ¬q and prove ¬p. A proof by contrapositive can be recast as a proof by contradiction as follows:

- Assume ¬(p → q), which is equivalent to p ∧ ¬q
- Use the assumption that ¬q is true to derive ¬p, following the reasoning in the proof by contrapositive.

- Arrive at the contradiction p (from the assumption) and ¬p.

Proofs by contradiction are more general than proofs by contrapositive because a proof by contradiction of the theorem $p \to q$ could start with the assumption $p \land \neg q$ and conclude with a contradiction other than $p \land \neg p$. Also, proof by contrapositive is a proof technique that is specific to proofs of conditional statements, whereas a proof by contradiction can be used to prove theorems that are not of the form $p \to q$.

PARTICIPATION ACTIVITY 6.6.3: Matching proof types to logical arguments.

Match the type of the proof to its logical format. In each case the theorem being proved is $p \to q$.

If unable to drag and drop, refresh the page.

Proof by contradiction **Direct proof** **Proof by contrapositive**

Assume p. Follow a series of steps to conclude q.

Assume ¬q. Follow a series of steps to conclude ¬p.

Assume $p \land \neg q$. Follow a series of logical steps to conclude $r \land \neg r$ for some proposition r.

Reset

PARTICIPATION ACTIVITY 6.6.4: Proof by contradiction of conditional statements.

Select the statement that is a valid start to a proof by contradiction of the theorem.

1) **Theorem:** For every integer n, if n^2 is odd, then n is odd.

- Suppose that there exists an integer n such that n^2 and n are both even.
- Suppose that there exists an integer n such that n^2 is odd and n is even.
- Suppose that there exists an integer n such that n^2 is even and n is odd.

2) **Theorem:** The product of a non-zero rational number and a non-zero irrational number is irrational.

- Suppose that there are two non-zero numbers, r and s, such that r is rational, s is irrational, and rs is irrational.
- Suppose that there are two non-zero numbers, r and s, such that r and s are both irrational, and rs is rational.
- Suppose that there are two non-zero numbers, r and s, such that r is rational, s is irrational, and rs is rational.

A classic proof by contradiction

Below is a proof of the fact that $\sqrt{2}$ is irrational. The proof starts out with the assumption that $\sqrt{2}$ is rational and, by a series of logical steps, ends up with a statement known to be false. The proof makes use of two facts that are proven elsewhere:

If x is an integer and x^2 is even, then x must also be even.

This fact can be proven by contrapositive by showing that if x is an odd integer, then x^2 is also odd.

If r is rational, then r can be expressed as a ratio of two integers with no common factors.

A common factor of x and y is an integer greater than 1 that divides both x and y. The definition of rational, states that if r is rational, then r = x/y, where x and y are integers and y ≠ 0. Repeatedly dividing both x and y by any common factors, will eventually result in an expression where r is equal to the ratio of two integers with no common factors.

Theorem 6.6.1: Theorem to be proven by contradiction.

$\sqrt{2}$ is an irrational number.

Proof 6.6.1: Theorem to be proven by contradiction.

Proof.
We assume that $\sqrt{2}$ is a rational number and therefore can be expressed as the ratio of two integers n/d, where d ≠ 0 and n and d do not have any common factors. Therefore, we can assume that there is no integer greater than 1 that divides both n and d.

Squaring both sides of the equation $\sqrt{2} = n/d$ gives $2 = n^2/d^2$. Multiplying both sides by d^2 gives

$$2d^2 = n^2.$$

Since n^2 is an integer multiple of 2, n^2 is even. If the square of an integer is even, then the integer itself must be even. Therefore, n is even which means that n can be expressed as 2k for some integer k. Plugging in the expression 2k for n into n^2:

$$n^2 = (2k)^2 = 4k^2.$$

Putting the two equations above together yields that $2d^2 = 4k^2$. Dividing both sides by 2 results in the equation $d^2 = 2k^2$. Therefore d^2 is even. Now we use the fact again that if the square of an integer is even, then that integer must also be even. Therefore d is even.

We have proven that n and d are both even. This means that n and d are both divisible by 2, which contradicts the assumption that the ratio n/d is expressed in its lowest terms. Thus, we have established a contradiction and we must conclude that the assumption that $\sqrt{2}$ is a rational number is a false assumption. ■

The proof above may be difficult to follow for the reader who is not used to reading proofs. The next question gives an outline of the steps of the proof and asks you to put the parts of the outline in the correct order. When you are done with the question, see if you can prove each line in the outline on your own.

PARTICIPATION ACTIVITY | 6.6.5: Outline of the proof that $\sqrt{2}$ is irrational.

Below is an outline of the steps of the proof that $\sqrt{2}$ is irrational. Put the steps in the outline in the correct order.

If unable to drag and drop, refresh the page.

Since n is even, and $n^2 = 2d^2$, then $4k^2 = 2d^2$, for some integer k.

n^2 is even and therefore n is even.

This contradicts the assumption that there is no integer greater than 1 that divides both n an

$n^2 = 2d^2$.

Assume $\sqrt{2}$ is rational.

Since n and d are both even, 2 divides n and d.

Since $4k^2 = 2d^2$ for some integer k, d^2 is also even.

Express $\sqrt{2}$ as n/d, where n and d are integers, d ≠ 0, and there is no integer greater than 1 t

Since d^2 is even, d is even.

Step 1.

Step 2.

Step 3.

Step 4.

Step 5.

Step 6.

Step 7.

Step 8.

Step 9.

Reset

More examples of proof by contradiction

The theorem below states:

> Among any group of 25 people, there must be at least three who are all born in the same month.

In order to find a statement that is equivalent to the negation of the theorem, it will be convenient to express the theorem in a way that is closer to the language of logic:

> For every group of 25 people, there exists a subset of three from the group, such that the three people all have birthdays in the same month.

Now we apply De Morgan's law to get the negation of the theorem.

> There exists a group of 25 people, such that there does not exist a subset of three from the group that all have birthdays in the same month.

The theorem starts with an equivalent statement that is more natural sounding in English:

> Suppose that there is a group of 25 people such that no three people in the group have their birthdays in the same month.

Proof 6.6.2: Proof by contradiction - birthday months.

Theorem: Among any group of 25 people, there must be at least three who are all born in the same month.

Proof.

Proof by contradiction: assume that there is a group of 25 people such that no three people in the group have their birthdays in the same month. We introduce twelve variables: $x_1,...,x_{12}$. The variable x_i is the number of people in the group whose birthday falls in the i^{th} month. x_1 is the number of people in the group with January birthdays, x_2 is the number of people with February birthdays, etc.

Since everyone is born in some month, the number of people in the group must be $x_1 + x_2 + x_3 + \cdots x_{12}$. By our assumption, there are no three people born in the same month, so each x_i is at most 2. Therefore the number of people in the group is at most 24:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12}$$

$$\leq\ 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2$$

$$=\ 2 * 12 = 24.$$

The fact that the number of people is at most 24 contradicts the fact that there are 25 people in the group. Therefore, there must be at least three people in the group who are all born in the same month. ■

If you had trouble following the equations in this proof, here is a video explanation.

PARTICIPATION ACTIVITY | 6.6.6: Proof by contradiction from geometry - fill in the blank.

Below is a statement of a theorem and a proof by contradiction with some parts of the argument replaced by capital letters in red font.

Theorem: Every triangle has at least one acute angle.

Proof.

Assume that the theorem is false which means that A. If an angle is not acute, it is right or obtuse, meaning it has degree 90 or greater. Let the degree of the three angles be x, y, and z. We will show that if B, then x + y + z > 180.

Since B, x + y + z ≥ C. Since C > 180, then

$$x + y + z \geq C > 180$$

The conclusion that x + y + z > 180 is a contradiction because it violates the known theorem that the sum of the degrees of the angles of a triangle equals 180. Thus, the assumption that there is a triangle which does not have an acute angle is false. ■

1) What is the correct expression for A?

- Every triangle has at least one angle that is not acute.
- There is a triangle with at least one angle that is not acute.
- There is a triangle which does not have an acute angle.

2) What is the correct expression for B?

- x ≥ 90, y ≥ 90, and z ≥ 90.
- x ≥ 90, y ≥ 90, or z ≥ 90.
- x < 90, y < 90, or z < 90.

3) What is the correct expression for C?

- 180
- 360
- 270

PARTICIPATION ACTIVITY 6.6.7: Proof by contradiction - no smallest positive real number.

Below are the steps for a proof by contradiction of the following theorem:

Theorem: There is no smallest positive real number.

Put the steps of the proof in the correct order so that each step follows from previous steps in the proof.

If unable to drag and drop, refresh the page.

Since r is positive, r/2 is also positive. Since r is positive, r > r/2.

Assume there is a smallest positive real number called r.

This contradicts the assumption that r is the smallest positive real number.

r/2 is a positive real number that is smaller than r.

Step 1.

Step 2.

Step 3.

Step 4.

Reset

Additional exercises

EXERCISE | 6.6.1: Rational and irrational numbers.

You can use the fact that $\sqrt{2}$ is irrational to answer the questions below. You can also use other facts proven within this exercise.

(a) Prove that $\sqrt{2}/2$ is irrational.

(b) Prove that $2 - \sqrt{2}$ is irrational.

(c) Is it true that the sum of two positive irrational numbers is also irrational? Prove your answer.

(d) Is it true that the product of two irrational numbers is also irrational? Prove your answer.

(e) Is the following statement true? Prove your answer.
If x is a non-zero rational number and y is an irrational number, then y/x is irrational.

EXERCISE 6.6.2: Adapting a proof about irrational numbers, Part 1.

(a) Prove that if n is an integer such that n^3 is even, then n is even.

(b) $\sqrt[3]{2}$ is irrational.
You can use the fact that if n is an integer such that n^3 is even, then n is even.
Your proof will be a close adaptation of the proof that $\sqrt{2}$ is irrational.

EXERCISE 6.6.3: Adapting a proof about irrational numbers, Part 2.

For this problem, you will need to use the following fact which is proven elsewhere in this material.

For every integer n, exactly one of the following three facts is true:

- n = 3k, for some integer k.
- n = 3k+1, for some integer k.
- n = 3k+2, for some integer k.

(a) Prove that if n is an integer such that $3|n^2$, then $3|n$.

(b) $\sqrt{3}$ is irrational.
You can use the fact that if n is an integer such that $3|n^2$, then $3|n$.
Your proof will be a close adaptation of the proof that $\sqrt{2}$ is irrational.

EXERCISE 6.6.4: Negating statements and proofs by contradiction.

(a) Give an English sentence that is equivalent to the negation of the following statement:
Among any set of 16 days chosen from the calendar, at least three of the chosen days fall on the same day of the week.

(b) Give a proof by contradiction of the following statement:
Among any set of 16 days chosen from the calendar, at least three of the chosen days fall on the same day of the week.

(c) Give an English sentence that is equivalent to the negation of the following statement:
Among any group of 1000 people, at least three of the people have the same birthday.

(d) Give a proof by contradiction of the following statement:
Among any group of 1000 people, at least three of the people have the same birthday.

EXERCISE 6.6.5: Proof by contrapositive vs. proof by contradiction.

For each statement, write what would be assumed and what would be proven in a proof by contrapositive of the statement. Then write what would be assumed and what would be proven in a proof by contradiction of the statement.

(a) If x and y are a pair of consecutive integers, then x and y have opposite parity.

(b) For all integers n, if n^2 is odd, then n is also odd.

EXERCISE | 6.6.6: Proofs by contradiction.

Give a proof for each statement.

(a) If a group of 9 kids have won a total of 100 trophies, then at least one of the 9 kids has won at least 12 trophies.

(b) If a person buys at least 400 cups of coffee in a year, then there is at least one day in which the person has bought at least two cups of coffee.

(c) The average of three real numbers is greater than or equal to at least one of the numbers.

(d) There is no smallest integer.

(e) There is no largest odd integer.

(f) There is no largest rational negative number.
(Note that there is a well-defined ordering of all negative rational numbers. For example $-1/2$ is larger than -5, because $-1/2 > -5$.)

(g) For all even integers n, n^2 is a multiple of 4.

(h) For all integers x and y, $x^2 - 4y \neq 2$.
You can use the following fact in your proof:
If n^2 is an even integer, then n is also an even integer.

(i) If the product of two positive real numbers is larger than 400, then at least one of the two numbers is larger than 20.

(j) If the sum of two positive real numbers is larger than 400, then at least one of the two numbers is larger than 200.

(k) If n is composite, then n has a factor that is greater than 1 and less than or equal to $\sqrt{n}$.

6.7 Proof by cases

Many theorems can be phrased as ∀x P(x), where the value of variable x can be any element from some domain. Sometimes proving such a theorem simultaneously for all elements in the domain is difficult, but the proof becomes more approachable if the domain is broken down into different classes where each class can be addressed separately. A ***proof by cases*** of a universal statement such as ∀x P(x) breaks the domain for the variable x into different classes and gives a different proof for each class. The proof for each class is called a ***case***. Every value in the domain must be included in at least one class. In a proof by cases, the cases are numbered, and each case begins with "Case n:", where n is the number of that case. The number is followed by a statement of the assumptions for that case.

Take as an example, the theorem:

Theorem 6.7.1: Theorem to be proven by cases.

For every integer x, x^2 - x is an even integer.

The domain is the set of all integers. This problem would be easier to approach if there were a mathematical expression to substitute for x in x^2 - x in order to determine the parity of the expression. Even though we don't know whether x is odd or even, we can prove the theorem by treating each case separately. The key point is that the set of odd integers together with the set of even integers account for all possible integers, ensuring that all integers are addressed.

PARTICIPATION ACTIVITY | 6.7.1: Proof by cases.

Animation content:

undefined

Animation captions:

1. The two cases (that x is odd and x is even) cover all possible integers.
2. If x is even, then $x = 2k$ for some integer k. The next step plugs $2k$ in for x into $x^2 - x$ and shows that the result is 2 times an integer, which is even.
3. If x is odd, then $x = 2k + 1$ for some integer k. The next step plugs $2k + 1$ in for x into $x^2 - x$ and shows that the result is 2 times an integer, which is even.

PARTICIPATION ACTIVITY | 6.7.2: Dividing a domain into separate cases.

Indicate whether the division of each domain is a valid set of cases for a proof.

1) x is an integer.

Case 1: $x \leq 5$

Case 2: $x \geq 6$

- Valid
- Invalid

2) x is an integer.

Case 1: $x \leq 0$

Case 2: $x \geq 0$

- Valid
- Invalid

3) x is a real number.

Case 1: $x < 0$

Case 2: $x > 0$

- Valid
- Invalid

4) x is a real number.

Case 1: $x \leq -1$

Case 2: $x = 0$

Case 3: $x \geq 1$

- Valid
- Invalid

5) x is a real number.

Case 1: x is rational

Case 2: x is irrational

- Valid
- Invalid

6) p is a proposition.

Case 1: p is True.

Case 2: p is False.

Valid

The next example gives a proof by cases for a theorem that is not about the properties of numbers:

Proof 6.7.1: Proof by cases - mutual friends and enemies.

Theorem: Consider a group of six people. Each pair of people are either friends or enemies with each other. Then there are three people in the group who are all mutual friends or all mutual enemies.

This video gives a pictorial overview of the proof which may make the formal proof given below easier to understand.

Proof.
Select a particular individual from the group and call that person x. Person x is either friends or enemies with each of the five other people in the group. Since five is an odd number, person x cannot have the same number of friends as enemies in the group. We consider the case in which person x has more friends than enemies and the case where person x has more enemies than friends.

Case 1: person x has more friends than enemies in the group. Then it must be the case that person x has at least three friends. Select three friends of person x. If the three selected friends of x are all mutual enemies, then the theorem holds. If they are not all mutual enemies, then at least two of them must be friends with each other. Thus, there are two people who are friends with each other who are also both friends of person x. They form a set of three people in the group who are all mutual friends.

Case 2: person x has more enemies than friends in the group. Then it must be the case that person x has at least three enemies. Select three enemies of person x. If the three selected enemies of x are all mutual friends, then the theorem holds. If they are not all mutual friends, then at least two of them must be enemies of each other. Thus, there are two people who are enemies of each other who are also both enemies of person x. They form a set of three people in the group who are all mutual enemies. ■

PARTICIPATION ACTIVITY | 6.7.3: Proof by cases - multiple of 4.

Below is a statement of a theorem and a proof by cases with some parts of the argument replaced by capital letters in red font.

Theorem: Every perfect square is either a multiple of 4 or a multiple of 4 plus 1.

Proof.

Every perfect square can be expressed as n^2 for some integer n. We consider two cases:

Case 1: n is even. If n is even, then n can be expressed as A, for some integer k. Plug in the expression for n into n^2:

$$n^2 = (A)^2 = B$$

Since k^2 is an integer, we can conclude that if n is even, then n^2 is a multiple of 4.

Case 2: n is C. If n is C, then n can be expressed as D, for some integer k. Plug in the expression for n into n^2:

$$n^2 = (D)^2 = E$$

Since (k^2+k) is an integer, we can conclude that if n is C, then n^2 is a multiple of 4 plus 1.

■

1) What is the correct expression for A?

- 2k
- 2k+1
- even

2) What is the correct expression for B?

- 2k+1
- $4k^2$
- multiple of 4

3) What is the correct expression for C?

- even
- odd
- a multiple of 4

4) What is the correct expression for D?

- 2k
- 2k+1
- odd

5) What is the best expression for E? Note that more than one of the choices may make the equality true. However, the goal is to express n^2 as a multiple of 4 or one plus a multiple of 4.

- $4k^2 + 2k + 1$
- $(2k)^2 + 4k + 1$.
- $4(k^2+k) + 1$

Proof by cases and the absolute value function

The following example uses the absolute value function. The ***absolute value*** of a real number x is defined to be |x| = -x if x < 0, and |x| = x if x ≥ 0. For example, if x = -3, then x < 0 and

$$|x| = |-3| = -(-3) = 3.$$

If x = 3, then x ≥ 0 and

$$|x| = |3| = 3.$$

In a proof by cases, it is acceptable for a situation to be included in more than one case. For example, in the proof below the condition in which x - 5 = 0 is covered in both cases.

Proof 6.7.2: Proof by cases - using the absolute value function.

Theorem: For any real number x, $|x+5| - x > 1$.

Proof.
Case 1: $x + 5 \geq 0$. Since $x + 5 \geq 0$, $|x+5| = x + 5$. Therefore

$$|x+5| - x = (x+5) - x = 5 > 1.$$

Case 2: $x + 5 \leq 0$. Since $x + 5 \leq 0$, $|x+5| = -(x+5)$. Therefore

$$|x+5| - x = -(x+5) - x = 2(-x) - 5.$$

Subtracting 5 from both sides of the inequality $x + 5 \leq 0$ gives $x \leq -5$. If both sides of the inequality are multiplied by -1, the result is $-x \geq 5$. Therefore

$$2(-x) - 5 \geq 2 \cdot 5 - 5 = 5 > 1.$$

We have established that $|x+5| - x = 2(-x) - 5$ and that $2(-x) - 5 > 1$. Therefore $|x+5| - x > 1$. ■

PARTICIPATION ACTIVITY 6.7.4: Proof by cases - absolute value.

Below are the arguments for a proof by cases.

Theorem: For every pair of real numbers, x and y, |x·y| = |x|·|y|.

Match the assumptions of each case with the argument that matches the case.

If unable to drag and drop, refresh the page.

x ≥ 0 and y ≥ 0. **x ≥ 0 and y ≤ 0.** **x ≤ 0 and y ≥ 0.** **x ≤ 0 and y ≤ 0.**

xy ≥ 0 and therefore |x·y| = xy.
Meanwhile |x| = -x and |y| = -y, so
|x|·|y| = (-x)(-y) = xy.

xy ≤ 0 and therefore |x·y| = -xy.
Meanwhile |x| = -x and |y| = y, so
|x|·|y| = (-x)(y) = -xy.

xy ≤ 0 and therefore |x·y| = -xy.
Meanwhile |x| = x and |y| = -y, so
|x|·|y| = (x)(-y)= -xy.

xy ≥ 0 and therefore |x·y| = xy.
Meanwhile |x| = x and |y| = y, so
|x|·|y| = (x)(y)= xy.

Reset

Without loss of generality

Sometimes the proofs for two different cases are so similar, that it is repetitive to include both cases. When this happens, the two cases can be merged into one case. The term ***without loss of generality*** (sometimes abbreviated ***WLOG*** or ***w.l.o.g.***) is used in mathematical proofs to narrow the scope of a proof to one special case in situations when the proof can be easily adapted to apply to the general case. Consider a proof of the following theorem:

Theorem: For any two integers x and y, if x is even or y is even, then xy is even.

A proof of the theorem could have two cases: one case assumes that x is even and the other case assumes that y is even. Since the assumption of the theorem is that at least one of x or y is even, the two cases cover all the possibilities for x and y. The proofs for the two cases would be identical except that the roles of x and y would be reversed. Instead a proof could address only one case and use the term "without loss of generality".

Proof 6.7.3: Use of - without loss of generality - in a proof.

Theorem: For any two integers x and y, if x is even or y is even, then xy is even.

Proof.
Without loss of generality, assume that x is even. Then x = 2k for some integer k. Plugging in the expression 2k for x in xy gives xy = 2ky = 2(ky). Since k and y are integers, ky is also an integer. Since xy is equal to two times an integer, xy is even. ■

PARTICIPATION ACTIVITY | 6.7.5: When to apply - without loss of generality.

1) Consider the following theorem and proof. Can the following two cases be merged with the use of "without loss of generality"?

If x and y have the same parity then 3x and 5y also have the same parity.

Proof.
Case 1: x and y are both even. Since x is even, x = 2k for some integer k. Since y is even, y = 2j for some integer j. 3x = 3 ·2k = 2·3k. Since 3 and k are both integers, 3k is also an integer and 3x is even. 5y = 5 ·2j = 2·5j. Since 5 and j are both integers, 5j is also an integer and 5y is even. 3x and 5y have the same parity since they are both even.

Case 2: x and y are both odd. Since x is odd, x = 2k+1 for some integer k. Since y is odd, y = 2j+1 for some integer j. 3x = 3(2k+1) = 2(3k+1) + 1. Since k, 3, and 1 are integers, 3k+1 is also an integer and 3x is odd. 5y = 5 (2j+1) = 2(5j+2)+1. Since j, 5, and 1 are integers, 5j+2 is also an integer and 5y is odd. 3x and 5y have the same parity since they are both odd.

■

- Yes
- No

2) Consider the following theorem and proof. Can the following two cases be merged with the use of "without loss of generality"?

If x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$.

Proof.
Case 1: $x \leq y$. If $x \leq y$, then $\min(x, y) = x$ and $\max(x, y) = y$. Therefore, $\max(x, y) + \min(x, y) = x + y$.

Case 2: $x \geq y$. If $x \geq y$, then $\min(x, y) = y$ and $\max(x, y) = x$. Therefore, $\max(x, y) + \min(x, y) = x + y$.
■

- Yes
- No

Additional exercises

EXERCISE | 6.7.1: Proofs by cases - statements about numbers.

Prove each statement.

(a) For every real number x, $x^2 \geq 0$.

(b) For every integer n, $n^2 \geq n$.

(c) If x is a real number such that $x^2 + 2x - 3 < 0$, then $-3 < x < 1$.

(d) If x is a real number such that $x^2 - 3x - 10 < 0$, then $-2 < x < 5$.

EXERCISE 6.7.2: Proofs by cases - even/odd integers and divisibility.

Prove each statement.

(a) If x is an integer, then $x^2 + 5x - 1$ is odd.

(b) If integers x and y have the same parity, then $x + y$ is even.
The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.

(c) If integers x and y where $x < y$ are consecutive, then they have opposite parity.

(d) For integers x and y, if xy is odd, then x is odd and y is odd.

(e) If x and y are integers such that $x^3(y + 5)$ is odd, then x is odd and y is even.

(f) Let x and y be two integers. If xy is not an integer multiple of 5, then neither x nor y is an integer multiple of 5.

(g) If x and y are two numbers such that xy and x+y are both even, then x and y are both even.

(h) If n is an odd integer then 8|(n^2-1).

EXERCISE 6.7.3: Proofs by cases - absolute value.

Prove each statement.

(a) For any real number x, $|x| \geq 0$.

(b) For any real number x, $|x| \geq x$ and $|x| \geq -x$.
You can use the fact proven in the previous problem that for any real number x, $|x| \geq 0$.

(c) For all real numbers x and y, $|x - y| = |y - x|$.

(d) For real numbers x and y, $|x + y| \leq |x| + |y|$.
You can use the fact proven in the previous problem that for any real number z, $z \leq |z|$ and $-z \leq |z|$.

(e) For any real number x, $|x - 6| + x > 3$.

(f) For all real numbers x and y, there is a number z such that $|x - z| = |y - z|$.