5.1 Predicates and quantifiers

Many mathematical statements contain variables. The statement "x is an odd number" is not a proposition because its truth value depends on the value of variable x. If x = 5, the statement is true. If x = 4, the statement is false. The truth value of the statement can be expressed as a function P of the variable x, as in P(x). The expression P(x) is read "P of x". A logical statement whose truth value is a function of one or more variables is called a ***predicate***. If P(x) is defined to be the statement "x is an odd number", then P(5) corresponds to the statement "5 is an odd number". P(5) is a proposition because it has a well defined truth value.

A predicate can depend on more than one variable. Define the predicates Q and R as:

$$Q(x, y) : x^2 = y$$

$$R(x, y, z) : x + y = z$$

The proposition Q(5, 25) is true because $5^2 = 25$. The proposition R(2, 3, 6) is false because $2 + 3 \neq 6$.

The ***domain*** of a variable in a predicate is the set of all possible values for the variable. For example, a natural domain for the variable x in the predicate "x is an odd number" would be the set of all integers. If the domain of a variable in a predicate is not clear from context, the domain should be given as part of the definition of the predicate.

PARTICIPATION ACTIVITY | 5.1.1: Truth values for predicates on specific inputs.

The domain for all the variables in the following predicates is the set of positive integers:

P(x): x is a prime number
L(x, y): x < y
S(x, y, z): $x^2 + y^2 = z^2$

1) Is P(7) true or false?

- True
- False

2) Is L(6, 6) true or false?

- True
- False

3) Is S(3, 4, 5) true or false?

- True
- False

Statements outside the realm of mathematics can also be predicates. For example, consider the statement: "The city has a population over 1,000,000." The "city" is the variable and the domain is defined to be the set of all cities in the United States. When the city is New York, the statement becomes: "New York has a population over 1,000,000" and the statement is true. When the city is Toledo, the statement becomes: "Toledo has a population over 1,000,000" and the statement is false.

Note that it may happen that a statement P(x) is true for all values in the domain. However, if the statement contains a variable, the statement is still considered to be a predicate and not a proposition. For example, if P(x) is the statement "x + 1 > 1" and the domain is all positive integers, the statement is true for each value in the domain. However, P(x) is considered to be a predicate and not a proposition because it contains a variable.

PARTICIPATION ACTIVITY 5.1.2: Predicates and propositions.

Which sentences are propositions and which are predicates? The domain is the set of all positive integers.

1) x is odd.
- Proposition
- Predicate

2) 23 is a prime number.
- Proposition
- Predicate

3) $\frac{1}{1+x} < 1$
- Proposition
- Predicate

4) 16 = x^2
- Proposition
- Predicate

Universal quantifier

If all the variables in a predicate are assigned specific values from their domains, then the predicate becomes a proposition with a well defined truth value. Another way to turn a predicate into a proposition is to use a quantifier. The logical statement ∀x P(x) is read "for all x, P(x)" or "for every x, P(x)". The statement ∀x P(x) asserts that P(x) is true for every possible value for x in its domain. The symbol ∀ is a ***universal quantifier*** and the statement ∀x P(x) is called a ***universally quantified statement***. ∀x P(x) is a proposition because it is either true or false. ∀x P(x) is true if and only if P(n) is true for every n in the domain of variable x.

If the domain is a finite set of elements $\{a_1, a_2, ..., a_k\}$, then:

$$\forall x P(x) \equiv P(a_1) \wedge P(a_2) \wedge \cdots \wedge P(a_k)$$

The equivalence symbol means that the two expressions always have the same truth value, regardless of the truth values for $P(a_1)$, ...,$P(a_n)$. If the domain is the set of students in a class and the predicate A(x) means that student x completed the assignment, then the proposition ∀x A(x) means: "Every student completed the assignment." Establishing that ∀x A(x) is true requires showing that each and every student in the class did in fact complete the assignment.

Some universally quantified statements can be shown to be true by showing that the predicate holds for an arbitrary element from the domain. An "arbitrary element" means nothing is assumed about the element other than the fact that it is in the domain. In the following example, the domain is the set of all positive integers:

$$\forall x (\frac{1}{x+1} < 1)$$

The statement is true because when x is assigned any arbitrary value from the set of all positive integers, the inequality $\frac{1}{1+x} < 1$ holds.

PARTICIPATION ACTIVITY | 5.1.3: Proving ∀x (1/(x+1) < 1) is true for an arbitrary positive integer x.

Animation captions:

1. The proof starts with the fact that $0 < x$ for all positive integers and adds 1 to both sides to get $1 < 1 + x$.
2. Dividing both sides by $(x + 1)$ gives that $\frac{1}{(x+1)} < 1$. The inequality is true for all positive integers x.

A ***counterexample*** for a universally quantified statement is an element in the domain for which the predicate is false. A single counterexample is sufficient to show that a universally quantified statement is false. For example, consider the statement ∀x (x^2 > x), in which the domain is the set of positive integers. When x = 1, then x^2 = x and the statement x^2 > x is not true. Therefore x = 1 is a counterexample that shows the statement "∀x (x^2 > x)" is false.

PARTICIPATION ACTIVITY | 5.1.4: Truth values for universally quantified statements.

In the following questions, the domain is the set of all positive integers. Indicate whether the universally quantified statement is true or false.

1) $\forall x\ (x^2 > 0)$.

- True
- False

2) $\forall x\ (x - 1 \geq 0)$.

- True
- False

3) $\forall x\ (x - 1 > 0)$.

- True
- False

Existential quantifier

The logical statement $\exists x\ P(x)$ is read "There exists an x, such that P(x)". The statement $\exists x\ P(x)$ asserts that P(x) is true for at least one possible value for x in its domain. The symbol $\exists$ is an ***existential quantifier*** and the statement $\exists x\ P(x)$ is called a ***existentially quantified statement***. $\exists x\ P(x)$ is a proposition because it is either true or false. $\exists x\ P(x)$ is true if and only if P(n) is true for at least one value n in the domain of variable x.

If the domain is a finite set of elements $\{a_1, a_2, ..., a_k\}$, then:

$$\exists x\ P(x) \equiv P(a_1) \vee P(a_2) \vee ... \vee P(a_k)$$

If the domain is the set of students in a class and the predicate A(x) means that student x completed the assignment, then $\exists x\ A(x)$ is the statement: "There is a student who completed the assignment." Establishing that $\exists x\ A(x)$ is true only requires finding one particular student who completed the assignment. An ***example*** for an existentially quantified statement is an element in the domain for which the predicate is true. A single example is sufficient to show that an existentially quantified statement is true. However, showing that $\exists x\ A(x)$ is false requires showing that every student in the class did not complete the assignment.

Some existentially quantified statements can be shown to be false by showing that the predicate is false for an arbitrary element from the domain. For example, consider the existentially quantified statement in which the domain of x is the set of all positive integers:

$$\exists x\ (x + 1 < x)$$

The statement is false because no positive integer satisfies the expression x + 1 < x.

PARTICIPATION ACTIVITY 5.1.5: Showing ∃x (x + 1 < x) is false.

Animation captions:

1. The proof starts with the inequality $x + 1 < x$ and subtracts x from both sides to get $1 < 0$.
2. Since $1 < 0$ is false, the inequality $x + 1 < x$ is false for every x and therefore $\exists x(x + 1 < x)$ is false.

PARTICIPATION ACTIVITY 5.1.6: Truth values for existentially quantified statements.

In the following questions, the domain is the set of all positive integers. Indicate whether the existentially quantified statement is true or false.

1) $\exists x(x^2 < 0)$.

- True
- False

2) $\exists x(x - 1 > 0)$.

- True
- False

3) $\exists x(x^2 = x)$.

- True
- False

PARTICIPATION ACTIVITY 5.1.7: Proving universal and existentially quantified statements.

Match what needs to be done to show that an existential or universally quantified statement is true or false.

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

Show the following statement is true: ∃x P(x).

Show the following statement is false: ∀x P(x).

Show the following statement is false: ∃x P(x).

Show the following statement is true: ∀x P(x).

Give an example: a particular element n in the domain for which P(n) is true.

Show that for every element n in the domain, P(n) is false.

Show that for every element n in the domain, P(n) is true.

Give a counterexample: a particular element n in the domain for which P(n) is false.

Reset

CHALLENGE ACTIVITY | 5.1.1: Predicates and quantifiers.

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Start

The predicate T is defined as follows.

$$T(x, y): y = 2x - 5$$

The domain for variables x and y is the set of all positive integers.

Is $T(8, 4)$ a proposition? If yes, then give its truth value.

1	2	3

Check Next

Additional exercises

EXERCISE 5.1.1: Which expressions with predicates are propositions?

Predicates P, T and E are defined below. The domain is the set of all positive integers.

$P(x)$: x is even
$T(x, y) : 2^x = y$
$E(x, y, z) : x^y = z$

Indicate whether each logical expression is a proposition. If the expression is a proposition, then give its truth value.

(a) P(3)

(b) ¬P(3)

(c) T(5, 32)

(d) T(5, x)

(e) E(6, 2, 36)

(f) E(2, y, 8)

(g) P(3) ∨ T(5, 32)

(h) T(5, 16) → E(6, 3, 36)

EXERCISE 5.1.2: Truth values for quantified statements about integers.

In this problem, the domain is the set of all integers. Which statements are true? If an existential statement is true, give an example. If a universal statement is false, give a counterexample.

(a) $\exists x\ (x + x = 1)$

(b) $\exists x\ (x + 2 = 1)$

(c) $\forall x\ (x^2 - x \neq 1)$

(d) $\forall x\ (x^2 - x \neq 0)$

(e) $\forall x\ (x^2 > 0)$

(f) $\exists x\ (x^2 > 0)$

EXERCISE 5.1.3: Translating mathematical statements in English into logical expressions.

Consider the following statements in English. Write a logical expression with the same meaning. The domain is the set of all real numbers.

(a) There is a number whose cube is equal to 2.

(b) The square of every number is at least 0.

(c) There is a number that is equal to its square.

(d) Every number is less than or equal to its square plus 1.

EXERCISE 5.1.4: Truth values for quantified statements for a given set of predicates.

The domain for this problem is a set $\{a, b, c, d\}$. The table below shows the value of three predicates for each of the elements in the domain. For example, $Q(b)$ is false because the truth value in row b, column Q is F.

	P	Q	R
a	T	T	F
b	T	F	F
c	T	F	F
d	T	F	F

Which statements are true? Justify your answer.

(a) ∀x P(x)

(b) ∃x P(x)

(c) ∀x Q(x)

(d) ∃x Q(x)

(e) ∀x R(x)

(f) ∃x R(x)

EXERCISE 5.1.5: Converting a quantified expression to an equivalent logical expression.

P(x) is a predicate and the domain for the variable x is {1, 2, 3, 4}. For each of the logical expressions given, give an equivalent logical expression that does not use quantifiers.

(a) ∀x P(x)

(b) ∃x P(x)

5.2 Quantified statements

Universally and existentially quantified statements can also be constructed from logical operations. Consider an example in which the domain is the set of positive integers and define the following predicates:

P(x): x is prime

O(x): x is odd

The proposition $\exists x\ (P(x) \land \neg O(x))$ states that there exists a positive number that is prime and not odd. This proposition is true because of the number x = 2.

The proposition $\forall x\ (P(x) \rightarrow O(x))$ says that for every positive integer x, if x is prime then x is odd. This proposition is false, because of the counterexample x = 2. Since 2 is prime and not odd, the conditional statement $P(2) \rightarrow O(2)$ is false.

The universal and existential quantifiers are generically called ***quantifiers***. A logical statement that includes a universal or existential quantifier is called a ***quantified statement***. The quantifiers ∀ and ∃ are applied before the logical operations (∧, ∨, →, and ↔) used for propositions. This means that the statement $\forall x\ P(x) \land Q(x)$ is equivalent to $(\forall x\ P(x)) \land Q(x)$ as opposed to $\forall x\ (P(x) \land Q(x))$.

PARTICIPATION ACTIVITY | 5.2.1: Evaluating quantified statements.

For the following questions, the domain for the variable x is the set of all positive integers. The first three questions use predicates O and M which are defined as follows:

O(x): x is odd
M(x): x is an integer multiple of 4 (e.g., 4, 8, 12,...)

Indicate whether each quantified statement is true or false:

1) $\exists x\ (O(x) \land M(x))$
   - True
   - False

2) $\exists x\ (\neg O(x) \land \neg M(x))$
   - True
   - False

3) $\forall x\ (M(x) \rightarrow \neg O(x))$
   - True
   - False

4) $\forall x\ ((x = 1) \lor (x^2 \neq x))$

True

False

A variable x in the predicate P(x) is called a ***free variable*** because the variable is free to take on any value in the domain. The variable x in the statement ∀x P(x) is a ***bound variable*** because the variable is bound to a quantifier. A statement with no free variables is a proposition because the statement's truth value can be determined.

In the statement (∀x P(x)) ∧ Q(x), the variable x in P(x) is bound by the universal quantifier, but the variable x in Q(x) is not bound by the universal quantifier. Therefore the statement (∀x P(x)) ∧ Q(x) is not a proposition. In contrast, the universal quantifier in the statement ∀x (P(x) ∧ Q(x)) binds both occurrences of the variable x. Therefore ∀x (P(x) ∧ Q(x)) is a proposition.

PARTICIPATION ACTIVITY | 5.2.2: Free and bound variables in quantified statements.

1) The expression ∃x P(x) is a proposition.

True

False

2) The expression (∃x S(x)) ∨ R(x) is a proposition.

True

False

3) The expression ∃x (S(x) ∨ R(x)) has a free variable.

True

False

4) The expression ∀x P(x) ∨ ∃x Q(x) is a proposition.

True

False

Logical equivalence with quantified statements

Two quantified statements (whether they are expressed in English or the language of logic) have the same logical meaning if they have the same truth value regardless of value of the predicates for the elements in the domain. Consider as an example a domain consisting of a set of people invited to a party. Define the predicates:

P(x): x came to the party

S(x): x was sick

The statement "Everyone was not sick" is logically equivalent to "$\forall x\ \neg S(x)$" because the two statements have the same truth value regardless of who was invited to the party and whether they were sick.

The table below gives an example of a set of people who could have been invited to the party and the value of the predicate S(x) and P(x) for each person. For example, Gertrude came to the party (i.e., P(Gertrude) = T) because the truth value in the row labeled Gertrude and column labeled P(x) is true.

Table 5.2.1: Values for predicates S(x) and P(x) for a particular group.

Name	S(x)	P(x)
Joe	F	T
Theodore	T	F
Gertrude	F	T
Samuel	F	F

For the group of people in the domain, the statement "Someone was sick and came to the party" is false because there is no individual for whom S(x) and P(x) are true. However, $\exists x\ (S(x) \lor P(x))$ is true because, for example, $S(Joe) \lor P(Joe)$ is true. Therefore the two statements "Someone was sick and came to the party" and "$\exists x\ (S(x) \lor P(x))$" are not logically equivalent.

PARTICIPATION ACTIVITY 5.2.3: Quantified statements in logic and English.

For the following questions, the domain for the variable x is a group of employees working on a project. The predicate N(x) says that x is a new employee. The predicate D(x) says that x met his deadline. Consider the group defined in the table below:

Name	N(x)	D(x)
Happy	F	F

Sleepy	F	T
Grumpy	T	T
Bashful	T	T

1) Is the statement "Every new employee met his deadline" true or false for the group defined in the table?

- True
- False

2) Is the statement "$\forall x\ (N(x) \land D(x))$" true or false for the group defined in the table?

- True
- False

3) Select the logical expression that is equivalent to the statement: "Every new employee met his deadline".

- $\forall x\ (N(x) \land D(x))$
- $\forall x\ (N(x) \rightarrow D(x))$

PARTICIPATION ACTIVITY 5.2.4: Quantified statements in logic and English.

For the following questions, the domain for the variable x is a group of employees working on a project. The predicate N(x) says that x is a new employee. The predicate D(x) says that x met his deadline. Consider the group defined in the table below:

Name	N(x)	D(x)
Happy	F	F
Sleepy	F	T
Grumpy	T	F
Bashful	T	F

1) Is the statement "There is a new

employee who met his deadline" true or false for the group defined in the table?

- True
- False

2) Is the statement "$\exists x\ (N(x) \rightarrow D(x))$" true or false for the group defined in the table?

- True
- False

3) Select the logical expression that is equivalent to the statement: "There is a new employee who met his deadline".

- $\exists x\ (N(x) \land D(x))$
- $\exists x\ (N(x) \rightarrow D(x))$

Example 5.2.1: Translating quantified statements from English to logic.

The domain in this problem are the members of a board of directors for a company who are considering a proposal for that company. Define the following predicates:

R(x): person x read the proposal.
V(x): person x voted in favor of the proposal.

Translate each of the following sentences into an equivalent logical expression:

1. Everyone who read the proposal voted in favor of it.

   Solution: $\forall x\ (R(x) \rightarrow V(x))$. Video explanation of the solution (2:01)

2. Someone who did not read the proposal, voted in favor of it.

   Solution: $\exists x\ (\neg R(x) \land V(x))$. Video explanation of the solution (1:09)

3. Someone did not read the proposal and someone voted in favor of it.

   Solution: $\exists x\ \neg R(x) \land \exists x\ V(x)$. Video explanation of the solution (2:57)

PARTICIPATION ACTIVITY 5.2.5: Quantified statements expressed in English.

In the following question, the domain is the set of fourth graders at Lee Elementary School. The predicates P and Q are defined as follows:

P(x): x took the math test
Q(x): x is present today

Match the English sentence with the corresponding logical proposition.

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

$\forall x\,(Q(x) \rightarrow P(x))$ **$\exists x\,(Q(x) \wedge \neg P(x))$** **$\exists x\,\neg P(x)$** **$\forall x\,(Q(x) \wedge P(x))$**

Every student was present and took the math test.

There is a student who did not take the math test.

Every student who is present took the math test.

There is a student who is present and did not take the math test.

Reset

CHALLENGE ACTIVITY | 5.2.1: Give an example or counterexample of quantified statement.

Mark the statement as true or false.

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Start

$$\exists x\,(A(x) \vee B(x))$$

True | False

Names	A(x)	B(x)	C(x)	D(x)

Ann	F	F	T	T
Bob	T	T	T	F
Joe	F	F	F	T

1 2 3 4 5 6 7 8

Check Next

Additional exercises

EXERCISE 5.2.1: Determining whether a quantified statement about the integers is true.

Predicates P and Q are defined below. The domain is the set of all positive integers.

P(x): x is prime
Q(x): x is a perfect square (i.e., $x = y^2$, for some integer y)

Indicate whether each logical expression is a proposition. If the expression is a proposition, then give its truth value.

(a) $\exists x\ Q(x)$

(b) $\forall x\ Q(x) \land \neg P(x)$

(c) $\forall x\ Q(x) \lor P(3)$

(d) $\exists x\ (Q(x) \land P(x))$

(e) $\forall x\ (\neg Q(x) \lor P(x))$

EXERCISE 5.2.2: Translating quantified statements from English to logic, part 1.

In the following question, the domain is a set of students at a university. Define the following predicates:

E(x): x is enrolled in the class
T(x): x took the test

Translate the following English statements into a logical expression with the same meaning.

(a) Someone who is enrolled in the class took the test.

(b) All students enrolled in the class took the test.

(c) Everyone who took the test is enrolled in the class.

(d) At least one student who is enrolled in the class did not take the test.

EXERCISE 5.2.3: Translating quantified statements in English into logic, part 2.

In the following question, the domain is the set of employees at a company. One of the employees is named Sam. Define the following predicates:

T(x): x is a member of the executive team
B(x): x received a large bonus

Translate the following English statements into a logical expression with the same meaning.

(a) Someone did not get a large bonus.

(b) Everyone got a large bonus.

(c) Sam did not get a large bonus even though he is a member of the executive team.

(d) Someone who is not on the executive team received a large bonus.

(e) Every executive team member got a large bonus.

EXERCISE 5.2.4: Translating quantified statements from English to logic, part 3.

In the following question, the domain is a set of employees who work at a company. Ingrid is one of the employees at the company. Define the following predicates:

S(x): x was sick yesterday
W(x): x went to work yesterday
V(x): x was on vacation yesterday

Translate the following English statements into a logical expression with the same meaning.

(a) At least one person was sick yesterday.

(b) Everyone was well and went to work yesterday.

(c) Everyone who was sick yesterday did not go to work.

(d) Yesterday someone was sick and went to work.

(e) Everyone who did not go to work yesterday was sick.

(f) Everyone who missed work was sick or on vacation (or both).

(g) Someone who missed work was neither sick nor on vacation.

(h) Each person missed work only if they were sick or on vacation (or both).

(i) Ingrid was sick yesterday but she went to work anyway.

(j) Someone other than Ingrid was sick yesterday. (Note for this question, you will need the expression (x ≠ Ingrid).)

(k) Everyone besides Ingrid was sick yesterday. (Note that the statement does not indicate whether or not Ingrid herself was sick yesterday. Also, for this question, you will need the expression (x ≠ Ingrid).)

EXERCISE | 5.2.5: Translating quantified statements from English to logic, part 4.

A student club holds a meeting. The predicate M(x) denotes whether person x came to the meeting on time. The predicate O(x) refers to whether person x is an officer of the club. The predicate D(x) indicates whether person x has paid his or her club dues. The domain is the set of all members of the club. Give a logical expression that is equivalent to each English statement.

(a) Someone is not an officer.

(b) All the officers came on time to the meeting.

(c) Everyone was on time for the meeting.

(d) Everyone paid their dues or came on time to the meeting.

(e) At least one person paid their dues and came on time to the meeting.

(f) There is an officer who did not come on time for the meeting.

EXERCISE | 5.2.6: Translating quantified statements from logic to English.

In the following question, the domain is the set of employees of a company. Define the following predicates:

A(x): x is on the board of directors. (Note: members of the board of directors are also employees.)
E(x): x earns more than $100,000
W(x): x works more than 60 hours per week

Translate the following logical expressions into English:

(a) $\forall x\, (A(x) \rightarrow E(x))$

(b) $\exists x\, (E(x) \land \neg W(x))$

(c) $\forall x\, (W(x) \rightarrow E(x))$

(d) $\exists x\, (\neg A(x) \land E(x))$

(e) $\forall x\, (E(x) \rightarrow (A(x) \lor W(x)))$

(f) $\exists x\, (A(x) \land \neg E(x) \land W(x))$

EXERCISE 5.2.7: Determining whether a quantified logical statement is true and translating into English, part 1.

The domain is a group working on a project at a company. Define the following predicates.

- D(x): x missed the deadline.
- N(x): x is a new employee.

Consider a situation in which there are five people in the group. The following table gives values for the predicates D and N for each member of the group. For example, Bert did not miss the deadline because the truth value in the row labeled Bert and the column labeled D(x) is F.

Using these values, determine whether each quantified expression evaluates to true or false. Then translate the statement into English.

Name	D(x)	N(x)
Sam	T	F
Beth	T	T
Melanie	F	T
Al	T	T
Bert	F	T

(a) $\forall x\ (D(x) \lor N(x))$

(b) $\forall x\ ((x \neq \text{Sam}) \rightarrow N(x))$

(c) $\exists x\ (\neg D(x) \land N(x))$

(d) $\forall x\ (\neg D(x) \rightarrow \neg N(x))$

(e) $N(\text{Bert}) \rightarrow D(\text{Bert})$

(f) $\forall x\ (\neg N(x) \rightarrow D(x))$

(g) $\forall x\ ((x \neq \text{Bert}) \rightarrow D(x))$

(h) $\exists x\ (\neg D(x) \land \neg N(x))$

(i) $\forall x\ (D(x) \leftrightarrow N(x))$

(j) $D(\text{Sam}) \land N(\text{Sam})$

(j) D(Sam) ∧ N(Sam)

EXERCISE 5.2.8: Determining whether a quantified logical statement is true and translating into English, part 2.

In the following question, the domain is a set of male patients in a clinical study. Define the following predicates:

P(x): x was given the placebo
D(x): x was given the medication
A(x): x had fainting spells
M(x): x had migraines

Suppose that there are five patients who participated in the study. The table below shows the names of the patients and the truth value for each patient and each predicate:

Name	P(x)	D(x)	A(x)	M(x)
Frodo	T	F	F	T
Gandalf	F	T	F	F
Gimli	F	T	T	F
Aragorn	T	F	T	T
Bilbo	T	T	F	F

For each of the following quantified statements, indicate whether the statement is a proposition. If the statement is a proposition, give its truth value and translate the expression into English.

(a) $\exists x\ (M(x) \land D(x))$

(b) $\exists x\ M(x) \land \exists x\ D(x)$

(c) $\exists x\ M(x) \land D(x)$

(d) $\forall x\ (A(x) \lor M(x))$

(e) $\forall x\ (M(x) \leftrightarrow A(x))$

(f) $\forall x\ ((M(x) \land A(x)) \to \neg D(x))$

(g) $\exists x\ (D(x) \land \neg A(x) \land \neg M(x))$

(h) $\forall x\ (D(x) \to (A(x) \lor M(x)))$

EXERCISE 5.2.9: Determining whether a quantified logical statement is true, part 1.

The domain for this question is the set {a, b, c, d, e}. The following table gives the value of predicates P, Q, and R for each element in the domain. For example, Q(c) = T because the truth value in the row labeled c and the column Q is T. Using these values, determine whether each quantified expression evaluates to true or false.

	P(x)	Q(x)	R(x)
a	T	T	F
b	T	F	F
c	F	T	F
d	T	T	F
e	T	T	T

(a) $\forall x\, (R(x) \vee Q(x) \rightarrow P(x))$

(b) $\exists x((x \neq b) \wedge \neg Q(x))$

(c) $\exists x((x = c) \rightarrow P(x))$

(d) $\exists x(Q(x) \wedge R(x))$

(e) $Q(a) \wedge P(d)$

(f) $\forall x\, ((x \neq b) \rightarrow Q(x))$

(g) $\forall x\, (P(x) \vee R(x))$

(h) $\forall x\, (R(x) \rightarrow P(x))$

(i) $\exists x(Q(x) \vee R(x))$

EXERCISE 5.2.10: Determining whether a quantified logical statement is true, part 2.

A student club holds a meeting. The predicate M(x) denotes whether person x came to the meeting on time. The predicate O(x) refers to whether person x is an officer of the club. The predicate D(x) indicates whether person x has paid his or her club dues. The domain is the set of all members of the club. The names of the members and their truth values for each of the predicates is given in the following table. Indicate whether each expression is true or false. If a universal statement is not true, give a counterexample. If an existential statement is true, give an example.

Name	M(x)	O(x)	D(x)
Hillary	T	F	T
Bernie	F	T	F
Donald	F	T	F
Jeb	F	T	T
Carly	F	T	F

(a) $\forall x \neg(O(x) \leftrightarrow D(x))$

(b) $\forall x ((x \neq \text{Jeb}) \rightarrow \neg(O(x) \leftrightarrow D(x)))$

(c) $\forall x (\neg O(x) \rightarrow D(x))$

(d) $\exists x (M(x) \land D(x))$

(e) $\forall x (M(x) \lor O(x) \lor D(x))$

(f) $\forall x \neg D(x)$

(g) $M(\text{Jeb}) \land D(\text{Hillary})$

(h) $D(\text{Bernie}) \land O(\text{Bernie})$

(i) $\exists x (O(x) \rightarrow M(x))$

(j) $\exists x (M(x) \land O(x) \land D(x))$

5.3 De Morgan's law for quantified statements

De Morgan's law for quantified statements

The negation operation can be applied to a quantified statement, such as ¬∀x F(x). If the domain for the variable x is the set of all birds and the predicate F(x) is "x can fly", then the statement ¬∀x F(x) is equivalent to:

"Not every bird can fly."

which is logically equivalent to the statement:

"There exists a bird that cannot fly."

The equivalence of the previous two statements is an example of De Morgan's law for quantified statements, which is formally stated as ¬∀x F(x) ≡ ∃x ¬F(x). The diagram below illustrates that for a finite domain, De Morgan's law for universally quantified statements is the same as De Morgan's law for propositions:

Figure 5.3.1: De Morgan's law for universally quantified statements.

Domain of discourse = $\{a_1, a_2, \ldots, a_n\}$

$$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$$

$$\equiv \qquad\qquad \equiv$$

$$\neg(P(a_1) \land P(a_2) \land \ldots \land P(a_n)) \equiv \neg P(a_1) \lor \neg P(a_2) \lor \ldots \lor \neg P(a_n)$$

Similarly, consider the statement ¬∃x A(x) in which the domain is the set of children enrolled in a class and A(x) is the predicate "x is absent today". The statement is expressed in English as:

"It is not true that there is a child in the class who is absent today."

which is logically equivalent to:

"Every child in the class is not absent today."

The logical equivalence of the last two statements is an example of the second of De Morgan's laws for quantified statements: ¬∃x P(x) ≡ ∀x ¬P(x). The diagram below illustrates that for a finite domain, De Morgan's law for existentially quantified statements is the same as De Morgan's law for propositions:

Figure 5.3.2: De Morgan's law for existentially quantified statements.

Domain of discourse = $\{a_1, a_2, \ldots, a_n\}$

$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

$$\neg(P(a_1) \vee P(a_2) \vee \ldots \vee P(a_n)) \equiv \neg P(a_1) \wedge \neg P(a_2) \wedge \ldots \wedge \neg P(a_n)$$

(Each top expression is $\equiv$ to the expression below it.)

Table 5.3.1: Summary of De Morgan's laws for quantified statements.

$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$
$\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$

PARTICIPATION ACTIVITY 5.3.1: De Morgan's law can be used to simplify an existentially quantified statement.

Animation captions:

1. Start with $\neg\exists x(P(x) \rightarrow \neg Q(x))$ and apply De Morgan's law to get $\forall x \neg(P(x) \rightarrow \neg Q(x))$.
2. Then apply the Conditional Identity to get $\forall x \neg(\neg P(x) \vee \neg Q(x))$.
3. Then apply De Morgan's law to get $\forall x(\neg\neg P(x) \wedge \neg\neg Q(x))$.
4. Finally, apply the Double Negation law to get $\forall x(P(x) \wedge Q(x))$.

PARTICIPATION ACTIVITY 5.3.2: De Morgan's law can be used to simplify a universally quantified statement.

Animation captions:

1. Start with $\neg\forall x(P(x) \wedge \neg Q(x))$ and apply De Morgan's law to get $\exists x \neg(P(x) \wedge \neg Q(x))$.
2. Then apply De Morgan's law to get $\exists x(\neg P(x) \vee \neg\neg Q(x))$.
3. Finally, apply the Double Negation law to get $\exists x(\neg P(x) \vee Q(x))$.

PARTICIPATION ACTIVITY | 5.3.3: Applying De Morgan's laws for quantified statements.

Match the logically equivalent propositions.

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

¬∀x (P(x) ∧ ¬Q(x)) **¬∃x (P(x) ∨ Q(x))** **¬∀x ¬P(x)** **¬∃x (¬P(x) ∧ Q(x))**

∃x P(x)

∃x (¬P(x) ∨ Q(x))

∀x (¬P(x) ∧ ¬Q(x))

∀x (P(x) ∨ ¬Q(x))

Reset

CHALLENGE ACTIVITY | 5.3.1: De Morgan's law for quantified statements.

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Start

Select the proposition that is logically equivalent to $\neg\exists x\, \neg P(x)$.

¬∀x P(x)

1 2 3

Check Next

Additional exercises

EXERCISE | 5.3.1: Applying De Morgan's law for quantified statements to logical expressions.

Apply De Morgan's law to each expression to obtain an equivalent expression in which each negation sign applies directly to a predicate. For example, $\exists x\ (\neg P(x) \lor \neg Q(x))$ is an acceptable final answer, but not $\neg\exists x\ P(x)$ or $\exists x\ \neg(P(x) \land Q(x))$.

(a) $\neg\exists x\ P(x)$

(b) $\neg\exists x\ (P(x) \lor Q(x))$

(c) $\neg\forall x\ (P(x) \land Q(x))$

(d) $\neg\forall x\ (P(x) \land (Q(x) \lor R(x)))$

EXERCISE | 5.3.2: Applying De Morgan's law for quantified statements to English statements.

In the following question, the domain is a set of male patients in a clinical study. Define the following predicates:

P(x): x was given the placebo
D(x): x was given the medication
M(x): x had migraines

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until each negation operation applies directly to a predicate and then translate the logical expression back into English.

Sample question: Some patient was given the placebo and the medication.

- $\exists x\ (P(x) \land D(x))$
- Negation: $\neg\exists x\ (P(x) \land D(x))$
- Applying De Morgan's law: $\forall x\ (\neg P(x) \lor \neg D(x))$
- English: Every patient was either not given the placebo or not given the medication (or both).

(a) Every patient was given the medication.

(b) Every patient was given the medication or the placebo or both.

(c) There is a patient who took the medication and had migraines.

(d) Every patient who took the placebo had migraines. (Hint: you will need to apply the conditional identity, $p \rightarrow q \equiv \neg p \lor q$.)

(e) There is a patient who had migraines and was given the placebo.

EXERCISE 5.3.3: Applying De Morgan's law for quantified statements to English statements.

In the following question, the domain is a set of students who show up for a test. Define the following predicates:

P(x): x showed up with a pencil
C(x): x showed up with a calculator

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until each negation operation applies directly to a predicate and then translate the logical expression back into English.

Sample question: Every student showed up with a calculator.

- $\forall x\, C(x)$
- Negation: $\neg\forall x\, C(x)$
- Applying De Morgan's law: $\exists x\, \neg C(x)$
- English: Some student showed up without a calculator.

(a) At least one of the students showed up with a pencil.

(b) Every student showed up with a pencil or a calculator (or both).

(c) Every student who showed up with a calculator also had a pencil.

(d) There is a student who showed up with both a pencil and a calculator.

(e) Some student showed up with a pencil or a calculator.

(f) Every student showed up with a pencil and a calculator.

EXERCISE 5.3.4: Using De Morgan's law for quantified statements to prove logical equivalence.

Use De Morgan's law for quantified statements and the laws of propositional logic to show the following equivalences:

(a) $\neg\forall x\, (P(x) \land \neg Q(x)) \equiv \exists x\, (\neg P(x) \lor Q(x))$

(b) $\neg\forall x\, (\neg P(x) \rightarrow Q(x)) \equiv \exists x\, (\neg P(x) \land \neg Q(x))$

(c) $\neg\exists x\, (\neg P(x) \lor (Q(x) \land \neg R(x))) \equiv \forall x\, (P(x) \land (\neg Q(x) \lor R(x)))$

5.4 Nested quantifiers

If a predicate has more than one variable, each variable must be bound by a separate quantifier. A logical expression with more than one quantifier that bind different variables in the same predicate is said to have ***nested quantifiers***. The examples below show several logical expressions and which variables are bound in each. The logical expression is a proposition if all the variables are bound.

Figure 5.4.1: Quantifiers and bound variables.

∀x ∃y P(x,y)	x and y are both bound.
∀x P(x,y)	x is bound and y is free.
∃y ∃z T(x,y,z)	y and z are bound. x is free.

PARTICIPATION ACTIVITY | 5.4.1: Free and bound variables with nested quantifiers.

1) Is the variable y bound in the expression ∀x Q(x, y)?
   - Yes
   - No

2) Is the following logical expression a proposition: ∀z ∃y Q(x, y, z)?
   - Yes
   - No

Nested quantifiers of the same type

Consider a scenario where the domain is a group of people who are all working on a joint project. Define the predicate M to be:

M(x, y): x sent an email to y

and consider the proposition: ∀x ∀y M(x, y). The proposition can be expressed in English as:

$$\forall x\, \forall y\, M(x, y) \;\leftrightarrow\; \text{"Everyone sent an email to everyone."}$$

The statement $\forall x\, \forall y\, M(x, y)$ is true if for every pair, x and y, $M(x, y)$ is true. The universal quantifiers include the case that $x = y$, so if $\forall x\, \forall y\, M(x, y)$ is true, then everyone sent an email to everyone else and everyone sent an email to himself or herself. The statement $\forall x\, \forall y\, M(x, y)$ is false if there is any pair, x and y, that causes $M(x, y)$ to be false. In particular, $\forall x\, \forall y\, M(x, y)$ is false even if there is a single individual who did not send himself or herself an email.

Now consider the proposition: $\exists x\, \exists y\, M(x, y)$. The proposition can be expressed in English as:

$$\exists x\, \exists y\, M(x, y) \;\leftrightarrow\; \text{"There is a person who sent an email to someone."}$$

The statement $\exists x\, \exists y\, M(x, y)$ is true if there is a pair, x and y, in the domain that causes $M(x, y)$ to evaluate to true. In particular, $\exists x\, \exists y\, M(x, y)$ is true even in the situation that there is a single individual who sent an email to himself or herself. The statement $\exists x\, \exists y\, M(x, y)$ is false if all pairs, x and y, cause $M(x, y)$ to evaluate to false.

PARTICIPATION ACTIVITY | 5.4.2: Nested quantifiers of the same type.

In the following question, the domain is the set of all non-negative integers. xy means x times y.

1) $\forall x\, \forall y\, (xy = 1)$
   - ○ True
   - ○ False

2) $\exists x\, \exists y\, (xy = 1)$
   - ○ True
   - ○ False

3) $\exists x\, \exists y\, ((x+y = x) \land (y \neq 0))$.
   - ○ True
   - ○ False

4) $\forall x\, \forall y\, ((x+y \neq x) \lor (y = 0))$.
   - ○ True
   - ○ False

Alternating nested quantifiers

A quantified expression can contain both types of quantifiers as in: $\exists x\, \forall y\, M(x, y)$. The quantifiers are applied from left to right, so the statement $\exists x \, \forall y \, M(x, y)$ translates into English as:

$$\exists x \, \forall y \, M(x, y) \quad \leftrightarrow \quad \text{"There is a person who sent an email to everyone."}$$

Switching the quantifiers changes the meaning of the proposition:

$$\forall x \, \exists y \, M(x, y) \quad \leftrightarrow \quad \text{"Every person sent an email to someone."}$$

In reasoning whether a quantified statement is true or false, it is useful to think of the statement as a ***two player game*** in which two players compete to set the statement's truth value. One of the players is the "existential player" and the other player is the "universal player". The variables are set from left to right in the expression. The table below summarizes which variables are set by which player and the goal of each player:

Table 5.4.1: Nested quantifiers as a two-person game.

Player	Action	Goal
Existential player	Selects values for existentially bound variables	Tries to make the expression true
Universal player	Selects values for universally bound variables	Tries to make the expression false

If the predicate is true after all the variables are set, then the quantified statement is true. If the predicate is false after all the variables are set, then the quantified statement is false. Consider as an example the following quantified statement in which the domain is the set of all integers:

$$\forall x \, \exists y \, (x + y = 0)$$

The universal player first selects the value of x. Regardless of which value the universal player selects for x, the existential player can select y to be -x, which will cause the sum x + y to be 0. Because the existential player can always succeed in causing the predicate to be true, the statement $\forall x \, \exists y \, (x + y = 0)$ is true.

Switching the order of the quantifiers gives the following statement:

$$\exists x \, \forall y \, (x + y = 0)$$

Now, the existential player goes first and selects a value for x. Regardless of the value chosen for x, the universal player can select some value for y that causes the predicate to be false. For example, if x is an integer, then y = -x + 1 is also an integer and x + y = 1 $\neq$ 0. Thus, the universal player can always win and the statement $\exists x \, \forall y \, (x + y = 0)$ is false.

PARTICIPATION | 5.4.3: Examples showing reasoning about nested quantifiers using two

ACTIVITY | player games.

Animation captions:

1. Is $\forall x \exists y (y^2 = x)$ true? The domain is the set of integers. If the universal player selects $x = 13$, then the existential player cannot find an integer y such that $x = y^2$.
2. The universal player wins, so $\forall x \exists y (y^2 = x)$ is false.
3. Is $\exists x \forall y (x + y = y)$ true? If the existential player selects $x = 0$, then for any y that the universal player selects, $0 + y = y$.
4. The existential player wins, so $\exists x \forall y (x + y = y)$ is true.

PARTICIPATION ACTIVITY | 5.4.4: Truth values for statements with nested quantifiers.

In the following question, the domain is the set of all real numbers. xy means x times y.

1) ∀x ∃y (xy = 1)
   - True
   - False

2) ∃x ∀y (xy = 1)
   - True
   - False

3) ∃x ∀y (xy = y)
   - True
   - False

4) ∀x ∃y (x^2 = y)
   - True
   - False

De Morgan's law with nested quantifiers

De Morgan's law can be applied to logical statements with more than one quantifier. Each time the negation sign moves past a quantifier, the quantifier changes type from universal to existential or from existential to universal:

Table 5.4.2: De Morgan's laws for nested quantified statements.

¬∀x ∀y P(x, y) ≡ ∃x ∃y ¬P(x, y)
¬∀x ∃y P(x, y) ≡ ∃x ∀y ¬P(x, y)
¬∃x ∀y P(x, y) ≡ ∀x ∃y ¬P(x, y)
¬∃x ∃y P(x, y) ≡ ∀x ∀y ¬P(x, y)

Consider a scenario in which the domain is the set of all students in a school. The predicate L(x, y) indicates that x likes y. The statement ∃x ∀y L(x, y) is read as:

∃x ∀y L(x, y) ↔ There is a student who likes everyone in the school.

The negation of the statement is:

¬∃x ∀y L(x, y) ↔ There is no student who likes everyone in the school.

Applying De Morgan's laws yields: ¬∃x ∀y L(x, y) ≡ ∀x ∃y ¬L(x, y) which is translated into:

∀x ∃y ¬L(x, y) ↔ Every student in the school has someone that they do not like.

PARTICIPATION ACTIVITY 5.4.5: De Morgan's law for nested quantifiers.

Animation content:

`undefined`

Animation captions:

1. If De Morgan's law is applied once to $\neg\forall x\forall y P(x,y)$, the result is $\exists x\neg\forall y P(x,y)$.
2. Applying De Morgan's law again gives $\exists x\exists y\neg P(x,y)$.
3. Applying De Morgan's law once to $\neg\forall x\exists y P(x,y)$ gives $\exists x\neg\exists y P(x,y)$. The second application gives $\exists x\forall y\neg P(x,y)$.
4. Applying De Morgan's law once to $\neg\exists x\forall y P(x,y)$ gives $\forall x\neg\forall y P(x,y)$. The second application gives $\forall x\exists y\neg P(x,y)$.
5. Applying De Morgan's law once to $\neg\exists x\exists y P(x,y)$ gives $\forall x\neg\exists y P(x,y)$. The second

application gives $\forall x \forall y \neg P(x, y)$.

PARTICIPATION ACTIVITY | 5.4.6: De Morgan's law for nested quantifiers.

Match logically equivalent propositions.

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

¬∃x ∃y (¬P(x) ∧ Q(y)) **¬∀x ∃y (P(x) ∨ Q(y))** **¬∀x ∀y (P(x) ∧ Q(y))**

¬∃x ∀y (P(x) ∨ ¬Q(y))

∃x ∃y (¬P(x) ∨ ¬Q(y))

∃x ∀y (¬P(x) ∧ ¬Q(y))

∀x ∃y (¬P(x) ∧ Q(y))

∀x ∀y (P(x) ∨ ¬Q(y))

Reset

CHALLENGE ACTIVITY | 5.4.1: Nested quantifiers.

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Start

The table below shows the values of predicates \(P(x, y)\) for every possible combination
variables \(x\) and \(y\). The row number indicates the value for \(x\) and the column num
the value for \(y\). The domain for \(x\) and \(y\) is \(\{1, 2, 3, 4\}\).

Ex: \(P(3, 1)\) is false because the entry in row \(3\), column \(1\) is F.

\(y\)

\(P\)	\(1\)	\(2\)	\(3\)	\(4\)
\(1\)	F	F	T	T

\(2\)	F	F	T	T
\(3\)	F	F	F	T
\(4\)	F	T	F	T

1	2	3

Check Next

Additional exercises

EXERCISE 5.4.1: Which logical expressions with nested quantifiers are propositions?

The table below shows the value of a predicate M(x, y) for every possible combination of values of the variables x and y. The domain for x and y is {1, 2, 3}. The row number indicates the value for x and the column number indicates the value for y. For example M(1, 2) = F because the value in row 1, column 2, is F.

M		y		
		1	2	3
x	1	T	F	T
	2	T	F	T
	3	T	T	F

Indicate whether each of the logical expressions is a proposition. If so, indicate whether the proposition is true or false.

(a) $M(1, 1)$

(b) $\forall y\, M(x, y)$

(c) $\exists x\, M(x, 3)$

(d) $\exists x\, \exists y\, M(x, y)$

(e) $\exists x\, \forall y\, M(x, y)$

(f) $M(x, 2)$

(g) $\exists y\, \forall x\, M(x, y)$

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EXERCISE 5.4.2: Truth values for statements with nested quantifiers - small finite domain.

The tables below show the values of predicates P(x, y), Q(x, y), and S(x, y) for every possible combination of values of the variables x and y. The row number indicates the value for x and the column number indicates the value for y. The domain for x and y is {1, 2, 3}.

P	1	2	3
1	T	F	T
2	T	F	T
3	T	T	F

Q	1	2	3
1	F	F	F
2	T	T	T
3	T	F	F

S	1	2	3
1	F	F	F
2	F	F	F
3	F	F	F

Indicate whether each of the quantified statements is true or false.

(a) $\exists x \forall y P(x, y)$

(b) $\exists x \forall y Q(x, y)$

(c) $\exists y \forall x P(x, y)$

(d) $\exists x \exists y S(x, y)$

(e) $\forall x \exists y Q(x, y)$

(f) $\forall x \exists y P(x, y)$

(g) $\forall x \forall y P(x, y)$

(h) $\exists x \exists y Q(x, y)$

(i) $\forall x \forall y \neg S(x, y)$

EXERCISE 5.4.3: Truth values for mathematical expressions with nested quantifiers.

Determine the truth value of each expression below. The domain is the set of all real numbers.

(a) $\forall x \exists y\ (xy > 0)$

(b) $\exists x \forall y\ (xy = 0)$

(c) $\forall x \forall y \exists z\ (z = (x - y)/3)$

(d) $\forall x \exists y \forall z\ (z = (x - y)/3)$

(e) $\forall x \forall y\ (xy = yx)$

(f) $\exists x \exists y \exists z\ (x^2 + y^2 = z^2)$

(g) $\forall x \exists y\ (y^2 = x)$

(h) $\forall x \exists y\ (x < 0 \lor y^2 = x)$

(i) $\exists x\ \exists y\ (x^2 = y^2 \land x \neq y)$

(j) $\exists x\ \exists y\ (x^2 = y^2 \land |x| \neq |y|)$

(k) $\forall x\ \forall y\ (x^2 \neq y^2 \lor |x| = |y|)$

EXERCISE 5.4.4: De Morgan's law and nested quantifiers.

Write the negation of each of the following logical expressions so that all negations immediately precede predicates. In some cases, it may be necessary to apply one or more laws of propositional logic.

(a) $\forall x\ \exists y\ \exists z\ P(y, x, z)$

(b) $\forall x\ \exists y\ (P(x, y) \land Q(x, y))$

(c) $\exists x\ \forall y\ (P(x, y) \rightarrow Q(x, y))$

(d) $\exists x\ \forall y\ (P(x, y) \leftrightarrow P(y, x))$

(e) $\exists x\ \exists y\ P(x, y) \land \forall x\ \forall y\ Q(x, y)$

EXERCISE 5.4.5: Applying De Morgan's law to English statements with nested quantifiers.

The domain for variables x and y is a group of people. The predicate F(x, y) is true if and only if x is a friend of y. For the purposes of this problem, assume that for any person x and person y, either x is a friend of y or x is an enemy of y. Therefore, ¬F(x, y) means that x is an enemy of y.

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until the negation operation applies directly to the predicate and then translate the logical expression back into English.

(a) Everyone is a friend of everyone.

(b) Someone is a friend of someone.

(c) Someone is a friend of everyone.

(d) Everyone is a friend of someone.

5.5 More nested quantified statements

Using logic to express "everyone else"

Consider a scenario where the domain is a group of people who are all working on a joint project. Define the predicate M(x, y) that indicates whether x sent an email to y. The statement ∀x ∀y M(x, y) asserts that every person sent an email to every other person and every person sent an email to himself or herself. How could we use logic to express that everyone sent an email to everyone else without including the case that everyone sent an email to himself or herself? The idea is to use the conditional operation: $(x \neq y) \rightarrow M(x, y)$.

The table below shows a group of four people and the truth value of M(x, y) for each pair. For example, Agnes sent an email to Fred (i.e., M(Agnes, Fred) = T) because the truth value in the row labeled Agnes and the column labeled Fred is T.

Figure 5.5.1: Email predicate truth values.

x \ y	Agnes	Fred	Sue	Marge
Agnes	F	T	T	T
Fred	T	F	T	T
Sue	T	T	F	T
Marge	T	T	T	F

The statement ∀x ∀y M(x, y) is false because M(Fred, Fred) and M(Marge, Marge) are both false. However, the statement

$$\forall x\ \forall y\ ((x \neq y) \rightarrow M(x, y))$$

is true. The statement says that for every pair, x and y, if x and y are different people then x sent an email to y. That is, everyone sent an email to everyone else. The statement is true for the table above because for every pair that is not on the diagonal of the table (i.e., for every pair such that x ≠ y), M(x, y) is true.

PARTICIPATION ACTIVITY | 5.5.1: Expressing 'someone else' in logic.

The diagram below shows a scenario for the predicate "x sent an email to y" for a particular group of people.

x \ y	Agnes	Fred	Sue	Marge
Agnes	F	F	T	F
Fred	F	T	F	F
Sue	F	F	F	T

Marge	F	T	F	F

1) Indicate whether the following statement is true or false for the group in the table:
"Everyone sent an email to someone else"

- True
- False

2) Is the statement "∀x ∃y M(x, y)" true or false for the group in the table?

- True.
- False

3) Consider the statement ∀x ∃y ((x ≠ y) ∧ M(x, y)). In the two player game for the group in the table, if the universal player selects x = Sue, who will the existential player select for y?

- Fred
- Sue
- Marge

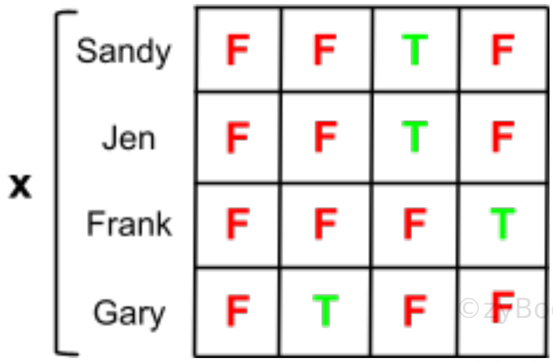
4) Is the statement "∀x ∃y ((x ≠ y) ∧ M(x, y))" true or false for the group in the table?

- True
- False

PARTICIPATION ACTIVITY | 5.5.2: Expressing 'someone else' in logic, cont.

The diagrams below shows a scenario for the predicate "x sent an email to y" for a particular group of people.

	y			
	Sandy	Jen	Frank	Gary



x				
Sandy	F	F	T	F
Jen	F	F	T	F
Frank	F	F	F	T
Gary	F	T	F	F

1) Indicate whether the following statement is true or false for the group in the table:
"Everyone sent an email to someone else"

- True
- False

2) Is the statement "∀x ∃y ((x ≠ y) ∧ M(x, y))" true or false for the group in the table?

- True
- False

3) Are the two statements below logically equivalent?
"Everyone sent an email to someone else"
and
∀x ∃y ((x ≠ y) ∧ M(x, y))

- Yes
- No

Expressing uniqueness in quantified statements

An existentially quantified statement evaluates to true even if there is more than one element in the domain that causes the predicate to evaluate to true. If the domain is a set of people who attend a meeting and the predicate L(x) indicates whether or not x came late to the meeting, then the statement ∃x L(x) is true if there are one, two or more people who came late.

PARTICIPATION ACTIVITY | 5.5.3: Using logic to express that exactly one person came late to the meeting.

Animation content:

undefined

Animation captions:

1. How to express: "Exactly one person was late to the meeting." L(x) means x was late to the meeting. ∃x L(x) means that someone was late to the meeting.
2. A way is needed to express that x is the only person who came late to the meeting.
3. Add that for every y, if y $\neq$ x, then y was not late to the meeting: $\exists x(L(x) \land \forall y((x \neq y) \rightarrow \neg L(y)))$.

PARTICIPATION ACTIVITY | 5.5.4: Expressing uniqueness in quantified statements.

Consider a domain consisting of a set of people attending a meeting. The predicate L(x) indicates whether the person was late to the meeting. The table below shows a sample domain and the value of the predicate L for each person attending the meeting.

Name	L(x)
Shirley	F
Rob	T
Ted	F
Mindy	T

1) Indicate whether the following statement is true or false for the group in the table:
"Exactly one person was late for the meeting."

- ○ True
- ○ False

2) Is the following statement true or false? L(Shirley) ∧ ∀y ((Shirley ≠ y) →

¬L(y))

True

False

3) Is the following statement true or false? L(Rob) ∧ ∀y ((Rob ≠ y) → ¬L(y))

True

False

PARTICIPATION ACTIVITY 5.5.5: Expressing uniqueness in quantified statements, cont.

Consider a domain consisting of a set of people attending a meeting. The predicate L(x) indicates whether the person was late to the meeting. The table below shows a sample domain and the value of the predicate L for each person attending the meeting.

Name	L(x)
Shirley	F
Rob	T
Ted	F
Mindy	F

1) Indicate whether the following statement is true or false for the group in the table: "Exactly one person was late for the meeting."

True

False

2) If x = Rob, is the following statement true: L(Rob) ∧ ∀y ((Rob ≠ y) → ¬L(y))

True

False

Moving quantifiers in logical statements

Now consider a set of people at a party as the domain. We would like to find a logical expression that is equivalent to the statement: "Every adult is married to someone at the party." There are two predicates:

M(x, y): x is married to y.
A(x): x is an adult.

Here is an equivalent statement that is closer in form to a logical expression: "For every person x, if x is an adult, then there is a person y to whom x is married." The logic is expressed as:

∀x (A(x) → ∃y M(x, y))

Since y does not appear in the predicate A(x), "∃y" can be moved to the left so that it appears just after the ∀x resulting in the following equivalent expression:

∀x ∃y(A(x) → M(x, y))

Note a quantifier can not be moved in front of another quantifier without changing the meaning of the expression. For example, ∀x ∃y(A(x) → M(x, y)) is not logically equivalent to ∃y ∀x(A(x) → M(x, y)).

PARTICIPATION ACTIVITY 5.5.6: Example of moving quantifiers in logical statements.

Animation content:

undefined

Animation captions:

1. How to express: "Every adult is married to exactly one person." A(x) means x is an adult. M(x, y) means x is married to y.
2. The first step is to express: "Every adult is married to someone." For every x, if x is an adult, then there is a y such that x is married to y: $\forall x(A(x) \to \exists y M(x, y))$.
3. The next step is to express that y is the only person x is married to: for every z, if z is not y, then x is not married to z: $\forall x(A(x) \to (\exists y M(x, y) \wedge \forall z((z \neq y) \to \neg M(x, z))))$.
4. The quantifier for y can be moved to the front just after ∀x because ∃y is not moved past of any occurrences of y or any other quantifiers.
5. The quantifier for z can be moved to the front just after ∃y because ∀z does not move past any occurrences of z or any other quantifiers.

PARTICIPATION ACTIVITY 5.5.7: Nested quantifiers expressed in English.

Match each proposition to the corresponding English sentence. The domain is the set of all students in a math class. The two predicates are defined as:

P(x, y): x knows y's phone number.
H(x): x has the homework assignment.

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

$\forall x\ \forall y\ P(x, y)$ **$\exists x\ \forall y\ P(x, y)$** **$\exists x\ \forall y\ (H(x) \land P(x, y))$** **$\forall x\ \exists y\ (P(x, y) \land H(y))$**

Every student knows every student's phone number.

Some student knows every student's phone number.

Every student knows the phone number of a student who has the homework assignment.

There is a student who has the homework assignment and knows every student's phone number.

Reset

CHALLENGE ACTIVITY 5.5.1: More nested quantified statements.

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Additional exercises

EXERCISE 5.5.1: Truth values for expressions with nested quantifiers.

The domain for this problem is a group of three people who are working on a project. To make notation easier, the people are numbered 1, 2, 3. The predicate M(x, y) indicates whether x has sent an email to y, so M(2, 3) is read "Person 2 has sent an email to person 3." The table below shows the value of the predicate M(x,y) for each (x,y) pair. The truth value in row x and column y gives the truth value for M(x,y).

M	1	2	3
1	T	T	T
2	T	F	T
3	T	T	F

Indicate whether the quantified statement is true or false. Justify your answer.

(a) $\forall x \forall y\ M(x,y)$

(b) $\forall x \forall y\ ((x \neq y) \rightarrow M(x,y))$

(c) $\exists x \exists y\ \neg M(x,y)$

(d) $\exists x \exists y\ ((x \neq y) \wedge \neg M(x,y))$

(e) $\forall x \exists y\ \neg M(x,y)$

(f) $\exists x \forall y\ M(x,y)$

EXERCISE 5.5.2: Truth values for mathematical statements with nested quantifiers.

The domain for all variables in the expressions below is the set of real numbers. Determine whether each statement is true or false. Justify your answer.

(a) $\forall x \, \exists y \, (x + y = 0)$

(b) $\exists x \, \forall y \, (x + y = 0)$

(c) $\exists x \, \forall y \, (xy = y)$

(d) $\exists x \, \exists y \, ((x^2 = y^2) \land (x \neq y))$

(e) $\forall x \, \forall y \, \exists z \, (z = (x + y)/2)$

(f) $\forall x \, \exists y \, \forall z \, (z = (x + y)/2)$

EXERCISE 5.5.3: Showing non-equivalence for expressions with nested quantifiers.

Show that the two quantified statements in each problem are not logically equivalent by filling in a table so that, for the domain {a, b, c}, the values of the predicate P you select for the table causes one of the statements to be true and the other to be false. For example, the table below shows that $\forall x \, \forall y \, P(x, y)$ and $\exists x \, \exists y \, P(x, y)$ are not logically equivalent because for the given values of the predicate P, $\forall x \, \forall y \, P(x, y)$ is false and $\exists x \, \exists y \, P(x, y)$ is true.

P	a	b	c
a	T	T	T
b	T	F	T
c	T	T	F

(a) $\forall x \, \exists y \, P(x, y)$ and $\exists x \, \forall y \, P(x, y)$

(b) $\forall x \, \exists y \, ((x \neq y) \land P(x, y))$ and $\forall x \, \exists y \, P(x, y)$

(c) $\exists x \, \exists y \, (P(x, y) \land P(y, x))$ and $\exists x \, \exists y \, P(x, y)$

EXERCISE | 5.5.4: Mathematical statements into logical statements with nested quantifiers.

Translate each of the following English statements into logical expressions. The domain is the set of all real numbers.

(a) There are two numbers whose ratio is less than 1.

(b) The reciprocal of every positive number is also positive.

(c) There are two numbers whose sum is equal to their product.

(d) The ratio of every two positive numbers is also positive.

(e) The reciprocal of every positive number less than one is greater than one.

(f) There is no smallest number.

(g) Every number other than 0 has a multiplicative inverse.

(h) Every number other than 0 has a unique multiplicative inverse.

EXERCISE | 5.5.5: Statements with nested quantifiers: English to logic, part 1.

The domain is the members of a chess club. The predicate B(x, y) means that person x has won a match against person y at some point in time. Give a logical expression equivalent to the following English statements. You can assume that it is possible for a person to win a match against himself or herself.

(a) Sam has lost a match to someone.

(b) Everyone has lost a match before.

(c) No one has ever won a match against Nancy.

(d) Everyone has won a match against at least one person.

(e) No one has won a match against both Ingrid and Dominic.

(f) Josephine has won a match against everyone else.

(g) Nancy has won a match against exactly one person.

(h) There are at least two members who have never lost a match.

EXERCISE 5.5.6: Statements with nested quantifiers: English to logic, part 2.

The domain for the variables x and y are the set of musicians in an orchestra. The predicates S, B, and P are defined as:

S(x): x plays a string instrument
B(x): x plays a brass instrument
P(x, y): x practices more than y

Give a quantified expression that is equivalent to the following English statements:

(a) There are no brass players in the orchestra.

(b) Someone in the orchestra plays a string instrument and a brass instrument.

(c) There is a brass player who practices more than all the string players.

(d) All the string players practice more than all the brass players.

(e) Exactly one person practices more than Sam.

(f) Sam practices more than anyone else in the orchestra.

EXERCISE 5.5.7: Statements with nested quantifiers: English to logic, part 3.

The domain is a group working on a project at a company. One of the members of the group is named Sam. Define the following predicates.

- P(x, y): x knows y's phone number. (A person may or may not know their own phone number.)
- D(x): x missed the deadline.
- N(x): x is a new employee.

Give a logical expression for each of the following sentences.

(a) Someone knows everyone's phone number.

(b) Everyone knows someone's phone number.

(c) There is at least one new employee who missed the deadline.

(d) Sam knows the phone number of everyone who missed the deadline.

(e) There is a new employee who knows everyone's phone number.

(f) Exactly one new employee missed the deadline.

EXERCISE | 5.5.8: Statements with nested quantifiers: English to logic, part 4.

A student club holds an election for officers. Before the voting, members can nominate each other. It is also possible for a member to nominate himself or herself. Some of the members are new members. Some of the members are currently officers. The domain is the set of members of the club. One of the members of the club is named Sam. Define the following predicates.

- N(x, y): person x nominated person y for a position.
- W(x): person x is a new member.
- O(x): person x is currently an officer.

Give a quantified expression that is logically equivalent to each of the following statements.

(a) All the new members nominated all the officers.

(b) One of the current officers did not nominate anyone.

(c) Everyone nominated someone.

(d) Someone nominated everyone.

(e) Everyone nominated someone other than themselves.

(f) Exactly one person nominated Sam.

EXERCISE 5.5.9: Statements with nested quantifiers: English to logic, part 5.

The domain is a set of animals on a farm. One of the bunnies on the farm is named Fluffy. Use the definitions of the predicates below to translate each English statement into an equivalent logical expression.

- B(x): x is a bunny
- H(x): x is a horse
- F(x): x has been fed
- W(x, y): x weighs more than y

(a) There is a horse who has not been fed.

(b) Every bunny has been fed.

(c) Exactly one horse has not been fed.

(d) Every horse weighs more than every bunny.

(e) Fluffy weighs more than every other bunny.

EXERCISE 5.5.10: Statements with nested quantifiers: variables with different domains.

The domain for the first input variable to predicate T is a set of students at a university. The domain for the second input variable to predicate T is the set of Math classes offered at that university. The predicate T(x, y) indicates that student x has taken class y. Sam is a student at the university and Math 101 is one of the courses offered at the university. Give a logical expression for each sentence.

(a) Sam has taken Math 101.

(b) Every student has taken at least one math class.

(c) Every student has taken at least one class other than Math 101.

(d) There is a student who has taken every math class other than Math 101.

(e) Everyone other than Sam has taken at least two different math classes.

(f) Sam has taken exactly two math classes.

5.6 Logical reasoning

The language of logic allows us to formally establish the truth of logical statements, assuming that a set of hypotheses is true. An ***argument*** is a sequence of propositions, called ***hypotheses***, followed by a final proposition, called the ***conclusion***. An argument is ***valid*** if the conclusion is true whenever the hypotheses are all true, otherwise the argument is ***invalid***. An argument will be denoted as:

p_1
p_2
....
p_n
$\therefore c$

$p_1 \ldots p_n$ are the hypotheses and c is the conclusion. The symbol ∴ reads "therefore". The argument is valid whenever the proposition $(p_1 \land p_2 \land \ldots \land p_n) \to c$ is a tautology. According to the commutative law, reordering the hypotheses does not change whether an argument is valid or not. Therefore two arguments are considered to be the same even if the hypotheses appear in a different order. For example, the following two arguments are considered to be the same:

p
$p \to q$
$\therefore q$

$p \to q$
p
$\therefore q$

p and $p \to q$ are the hypotheses. q is the conclusion.

PARTICIPATION ACTIVITY 5.6.1: The components of a logical argument.

Consider the following argument:

$\neg q$
$p \to q$
$\therefore \neg p$

1) The conclusion is:

- p → q
- ¬p
- ¬q

2) The proposition ¬q is:

- a hypothesis
- the conclusion
- an argument

3) The argument is valid if which proposition is a tautology:

- ((p → q) ∧ ¬q) → ¬p
- ¬p → ((p → q) ∧ ¬q)
- ¬p ↔ ((p → q) ∧ ¬q)

One way to establish the validity of an argument is to use a truth table.

PARTICIPATION ACTIVITY 5.6.2: Using a truth table to establish the validity of an argument.

Animation content:

`undefined`

Animation captions:

1. To show that the argument with hypotheses p and $p \to q$ and conclusion q is valid, fill in the truth table for $p, q,$ and $p \to q$.
2. There is only one row in which both hypotheses are true.
3. Since the conclusion is also true in that row, the argument is valid.

In order to use a truth table to establish the validity of an argument, a truth table is constructed for all the hypotheses and the conclusion. Each row in which all the hypotheses are true is examined. If the conclusion is true in each of the examined rows, then the argument is valid. If there is any row in which all the hypotheses are true but the conclusion is false, then the argument is invalid.

PARTICIPATION ACTIVITY 5.6.3: Validity of an argument from truth tables.

Consider the argument below:

$p \to q$
$p \lor q$
$\therefore q$

The truth table below shows the truth values for the two hypotheses and the conclusion for every possible truth assignment to p and q.

p	q	$p \to q$	$p \lor q$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

1) In which rows are both the hypotheses true?

- Row 1
- Rows 1 and 3
- Rows 1, 3, and 4

2) Is the conclusion true in all of the rows where both the hypotheses are true?

- Yes
- No

3) Is the argument valid?

- Yes
- No

An argument can be shown to be invalid by showing an assignment of truth values to its variables that makes all the hypotheses true and the conclusion false. For example, the argument:

is invalid, because when p = F and q = T, the hypotheses $p \to q$ and $\neg p$ are both true, but the conclusion $\neg q$ is false. In some cases, it may be necessary to build the whole truth table in order to actually find a truth assignment that shows an argument is invalid. However, the final proof of

$\neg p$
$p \rightarrow q$
$\therefore \neg q$

invalidity only requires a single truth assignment for which all the hypotheses are true and the conclusion is false.

PARTICIPATION ACTIVITY 5.6.4: Proving an argument is invalid.

Consider the argument below:

$p \lor q$
p
$\therefore \neg q$

The truth table below shows the truth values for the two hypotheses and the conclusion for every possible truth assignment to p and q.

p	q	p ∨ q	¬q
T	T	T	F
T	F	T	T
F	T	T	F
F	F	F	T

1) In which rows are both the hypotheses true?

- Row 1
- Rows 1 and 2
- Rows 1 and 3

2) Is the conclusion true in all of the rows where both the hypotheses are true?

- Yes
- No

3) Is the argument valid or invalid?

- Valid
- Invalid

4) Which truth assignment is a proof that the argument is invalid?

- ○ p = q = T
- ○ p = T and q = F
- ○ p = F and q = T

The form of an argument

The hypotheses and conclusion in a logical argument can also be expressed in English, as in:

It is raining today.
If it is raining today, I will not ride my bike to school.
∴ I will not ride my bike to school.

The ***form*** of an argument expressed in English is obtained by replacing each individual proposition with a variable. While it is common to express a logical argument in English, the validity of an argument is established by analyzing its form. Define propositional variables p and q to be:

p: It is raining today.
q: I will not ride my bike to school.

The argument's form, given below, was already shown to be valid by using a truth table:

p
$p \rightarrow q$
$\therefore q$

PARTICIPATION ACTIVITY | 5.6.5: Determining the form of an argument expressed in English.

Use the following variable definitions to match each argument to its form:

p: Sam studied for his test.
q: Sam passed his test.

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

$p \lor q$	p	$p \rightarrow q$
$\neg q$	q	$\neg q$
$\therefore p$	$\therefore p \land q$	$\therefore \neg p$

If Sam studied for his test, then
Sam passed his test.
Sam did not pass his test.
∴ Sam did not study for his test.

Sam studied for his test or Sam
passed his test.
Sam did not pass his test.
∴ Sam studied for his test.

Sam studied for his test.
Sam passed his test.
∴ Sam studied for his test and Sam
passed his test.

Reset

When arguments are expressed in English, the propositions sometimes have known truth values. In the argument below the hypotheses and conclusion are all true:

5 is not an even number.
If 5 is an even number, then 7 is an even number.
∴ 7 is not an even number.

The argument is, nonetheless, invalid because its form

$\neg p$
$p \rightarrow q$
$\therefore \neg q$

was shown to be invalid.

In a valid argument, the conclusion must follow from the hypotheses for every possible combination of truth values for the individual propositions. The invalid argument form shown above can be used with a different set of propositions to reach a false conclusion even when the hypotheses are true, as in:

5 is not an even number.
If 5 is an even number, then 6 is an even number.
∴ 6 is not an even number.

PARTICIPATION ACTIVITY | 5.6.6: Valid and invalid arguments in English.

In the examples below, all of the hypotheses and the conclusions are true. Indicate which arguments are valid. You can use the fact that the argument form A given below is valid and argument form B is invalid:

(Valid) Argument form A:

$\neg q$
$p \rightarrow q$
$\therefore \neg p$

(Invalid) Argument form B:

$\neg p$
$p \rightarrow q$
$\therefore \neg q$

1)
6 is not a prime number.
If 6 is a prime number, then 4 is a prime number.
∴ 4 is not a prime number.

- Valid
- Invalid

2)
4 is not a prime number.
If 6 is a prime number, then 4 is a prime number.
∴ 6 is not a prime number.

- Valid
- Invalid

3)

π is not a rational number.
If π is a rational number, then 2π is a rational number.
∴ 2π is not a rational number

- Valid
- Invalid

CHALLENGE ACTIVITY | 5.6.1: Logical reasoning.

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Additional exercises

EXERCISE 5.6.1: Valid and invalid arguments expressed in logical notation.

Indicate whether the argument is valid or invalid. For valid arguments, prove that the argument is valid using a truth table. For invalid arguments, give truth values for the variables showing that the argument is not valid.

(a)

$p \lor q$
p
$\therefore q$

(b)

$p \leftrightarrow q$
$p \lor q$
$\therefore p$

(c)

p
q
$\therefore p \leftrightarrow q$

(d)

$p \lor q$
$\neg q$
$\therefore p \leftrightarrow q$

(e)

$(p \land q) \rightarrow r$
$\therefore (p \lor q) \rightarrow r$

(f)

$(p \lor q) \rightarrow r$
$\therefore (p \land q) \rightarrow r$

(g)

$q \rightarrow p$
$\neg q$
$\therefore p$

(h)

$\neg(p \rightarrow q)$
$q \rightarrow p$
$\therefore \neg q$

(i)

$\neg(p \rightarrow q)$

$(p \quad q)$
p
$\therefore q \to p$

(j)
$q \to p$
p
$\therefore \neg(p \to q)$

EXERCISE 5.6.2: Converse and inverse errors.

Converse and inverse errors are typical forms of invalid arguments. Prove that each argument is invalid by giving truth values for the variables showing that the argument is invalid. You may find it easier to find the truth values by constructing a truth table.

(a) Converse error

$p \to q$
q
$\therefore p$

(b) Inverse error

$p \to q$
$\neg p$
$\therefore \neg q$

EXERCISE 5.6.3: Valid and invalid arguments in English.

Which of the following arguments are invalid and which are valid? Prove your answer by replacing each proposition with a variable to obtain the form of the argument. Then prove that the form is valid or invalid.

(a)

The patient has high blood pressure or diabetes or both.
The patient has diabetes or high cholesterol or both.
∴ The patient has high blood pressure or high cholesterol.

(b)

He studied for the test or he failed the test or both.
He passed the test.
∴ He studied for the test.

(c)

If $\sqrt{2}$ is an irrational number, then $2\sqrt{2}$ is an irrational number.
$2\sqrt{2}$ is an irrational number.
∴ $\sqrt{2}$ is an irrational number.

(d)

4 is an odd integer or 4 is a negative integer.
4 is not a negative integer.
∴ 4 is an odd integer.

EXERCISE 5.6.4: Valid arguments with false conclusions and invalid arguments with true conclusions.

Use the propositions given below to create the arguments as described.

- p: 4 is a prime number. (False)
- q: 5 is a prime number. (True)

(a) Give an example of a valid argument expressed in English sentences with a conclusion that is a false statement. Prove that the argument is valid by giving the form of the argument and showing that the argument is valid using a truth table.

(b) Give an example of an invalid argument expressed in English sentences with a conclusion that is a true statement. Prove that the argument is invalid by giving the form of the argument and showing truth values for the variables in which all the hypotheses are true but the conclusion is false.

5.7 Rules of inference with propositions

Using truth tables to establish the validity of an argument can become tedious, especially if an argument uses a large number of variables. Fortunately, some arguments can be shown to be valid by applying rules that are themselves arguments that have already been shown to be valid. The laws of propositional logic can also be used in establishing the validity of an argument.

Table 5.7.1: Rules of inference known to be valid arguments.

Rule of inference	Name
p $p \to q$ $\therefore q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	Modus tollens
p $\therefore p \lor q$	Addition
$p \land q$ $\therefore p$	Simplification
p q $\therefore p \land q$	Conjunction
$p \to q$ $q \to r$ $\therefore p \to r$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	Disjunctive syllogism
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	Resolution

The validity of an argument can be established by applying the rules of inference and laws of propositional logic in a ***logical proof***. A logical proof of an argument is a sequence of steps, each of which consists of a proposition and a justification. If the proposition in a step is a hypothesis, the justification is "Hypothesis". Otherwise, the proposition must follow from previous steps by applying one law of logic or rule of inference. The justification indicates which rule or law is used and the previous steps to which it is applied. For example, having justification "Modus ponens, 2, 3" on line 4 means that the Modus ponens rule is applied to the propositions on lines 2 and 3, resulting in the proposition on line 4. The proposition in the last step in the proof must be the conclusion of the argument being proven. For example, in the animation below, the last proposition "t" in step 6 is the same as the conclusion of the argument being proven.

PARTICIPATION ACTIVITY | 5.7.1: An example of a logical proof to establish the validity of an argument.

Animation captions:

1. An argument has hypotheses $(p \lor r) \to q, q \to t$, and r. The conclusion is t. The first line of the proof is r, a hypothesis.
2. Line 2 of the proof is $p \lor r$, by Addition applied to line 1.
3. Line 3 of the proof is $(p \lor r) \to q$, a hypothesis.
4. Line 4 of the proof is q, by Modus Ponens applied to lines 2 and 3.
5. Line 5 of the proof is $q \to t$, a hypothesis.
6. Line 6 of the proof is t, by Modus Ponens applied to lines 4 and 5, which is the conclusion of the argument.

Here is an argument expressed in English:

If it is raining or windy or both, the game will be cancelled.
The game will not be cancelled.
It is not windy.

The first step in proving the validity of the argument is to assign variable names to each of the individual propositions:

w: It is windy
r: It is raining
c: The game will be cancelled

Replacing English phrases with variable names results in the following argument form:

(r ∨ w) → c
¬c
∴ ¬w

The final step is to prove that the argument is valid using a logical proof.

Table 5.7.2: An example of a logical proof to establish the validity of an argument.

1.	(r ∨ w) → c	Hypothesis
2.	¬c	Hypothesis
3.	¬(r ∨ w)	Modus tollens, 1, 2
4.	¬r ∧ ¬w	De Morgan's law, 3
5.	¬w ∧ ¬r	Commutative law, 4
6.	¬w	Simplification, 5

PARTICIPATION ACTIVITY 5.7.2: Applying the laws of inference.

Establish the validity of the following argument.

¬(p ∧ ¬q)
p
∴ q ∨ r

Complete the proof that establishes the validity of the argument above by indicating the rule used to justify each step. The number after the Double negation law indicates the line number to which the rule is applied. The proof starts with the following line:

Hypothesis	1. ¬(p ∧ ¬q)

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

De Morgan's law, 1 **Addition, 6** **Double negation law, 2** **Double negation law, 4**

Disjunctive syllogism, 3, 5 **Hypothesis**

2. $\neg p \vee \neg\neg q$

3. $\neg p \vee q$

4. p

5. $\neg\neg p$

6. q

7. $q \vee r$

Reset

CHALLENGE ACTIVITY | 5.7.1: Rules of inference with propositions.

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Additional exercises

EXERCISE 5.7.1: Determining whether an argument in English is valid or invalid.

Indicate whether each argument is valid or invalid. If an argument is valid, then the argument uses exactly one of the rules of inference. Indicate which rule is used for the argument. If the argument is invalid, give truth assignments to the propositions "Sally took the medication" and "Sally had side effects" that prove the argument is invalid. For example a solution might be: Invalid. Sally took the medication = F, Sally had side effects = T.

(a)

Sally had a side effect or Sally took the medication.
Sally took the medication.
∴ Sally did not have side effects.

(b)

If Sally took the medication then she had side effects.
Sally did not take the medication.
∴ Sally did not have side effects.

(c)

Sally took the medication.
∴ Sally took the medication or had side effects.

(d)

If Sally had side effects, then she took the medication.
Sally did not take the medication.
∴ Sally did not have side effects.

EXERCISE 5.7.2: Proving arguments are valid using rules of inference.

Use the rules of inference and the laws of propositional logic to prove that each argument is valid. Number each line of your argument and label each line of your proof "Hypothesis" or with the name of the rule of inference used at that line. If a rule of inference is used, then include the numbers of the previous lines to which the rule is applied.

(a)

$p \to q$
$q \to r$
$\neg r$
$\therefore \neg p$

(b)

$p \to (q \land r)$
$\neg q$
$\therefore \neg p$

(c)

$(p \land q) \to r$
$\neg r$
q
$\therefore \neg p$

(d)

$(p \lor q) \to r$
p
$\therefore r$

(e)

$p \lor q$
$\neg p \lor r$
$\neg q$
$\therefore r$

(f)

$p \to q$
$r \to u$
$p \land r$
$\therefore q \land u$

EXERCISE 5.7.3: Proving the rules of inference using other rules.

Some of the rules of inference can be proven using the other rules of inference and the laws of propositional logic.

(a) One of the rules of inference is Modus tollens:

$p \rightarrow q$
$\neg q$
$\therefore \neg p$

Prove that Modus tollens is valid using the laws of propositional logic and any of the other rules of inference besides Modus tollens. (Hint: you will need one of the conditional identities from the laws of propositional logic).

(b) One of the rules of inference is Modus ponens:

$p \rightarrow q$
p
$\therefore q$

Prove that Modus ponens is valid using the laws of propositional logic and any of the other rules of inference besides Modus ponens. (Hint: you will need one of the conditional identities from the laws of propositional logic).

(c) One of the rules of inference is Disjunctive syllogism :

$p \vee q$
$\neg p$
$\therefore q$

Prove that Disjunctive syllogism is valid using the laws of propositional logic and any of the other rules of inference besides Disjunctive syllogism. (Hint: you will need one of the conditional identities from the laws of propositional logic).

(d) One of the rules of inference is Resolution:

$p \vee q$
$\neg p \vee r$
$\therefore q \vee r$

Prove that Resolution is valid using the laws of propositional logic and any of the other rules of inference besides Resolution. (Hint: you will need one of the conditional identities from the laws of propositional logic).

EXERCISE 5.7.4: Proving arguments in English are valid using rules of inference.

Prove that each argument is valid by replacing each proposition with a variable to obtain the form of the argument. Then use the rules of inference to prove that the form is valid.

(a)

If I drive on the freeway, I will see the fire.
I will drive on the freeway or take surface streets (or both).
I am not going to take surface streets.
∴ I will see the fire.

(b)

If it was not foggy or it didn't rain (or both), then the race was held and there was a trophy ceremony.
The trophy ceremony was not held.
∴ It rained.

(c)

If I work out hard, then I am sore.
If I am sore, I take an aspirin.
I did not take an aspirin.
∴ I did not work out hard.

EXERCISE 5.7.5: Proving arguments in English are valid or invalid.

Give the form of each argument. Then prove whether the argument is valid or invalid. For valid arguments, use the rules of inference to prove validity.

(a)

If I get a job then I will buy a new car and a new house.
I won't buy a new house.
∴ I will not get a job.

(b)

If I get a job then I will buy a new car or a new house (or both).
I won't buy a new house.
∴ I will not get a job.

(c)

I will buy a new car and a new house only if I get a job.
I am not going to get a job.
∴ I will not buy a new car.

(d)

I will buy a new car and a new house only if I get a job.
I am not going to get a job.
I will buy a new house.
∴ I will not buy a new car.

5.8 Rules of inference with quantifiers

In order to apply the rules of inference to quantified expressions, such as $\forall x\ \neg(P(x) \land Q(x))$, we need to remove the quantifier by plugging in a value from the domain to replace the variable x. A value that can be plugged in for variable x is called an ***element*** of the domain of x. For example, if the domain is the set of all integers, $\neg(P(3) \land Q(3))$ is a proposition to which De Morgan's law can be applied. Elements of the domain can be defined in a hypothesis of an argument. In the example below, the domain is a set of employees at a company. The second line defines Linda as an element of the domain.

Elements of the domain can also be introduced within a proof in which case they are given generic names such as "c" or "d". There are two types of named elements used in logical proofs. An ***arbitrary*** element of a domain has no special properties other than those shared by all the elements of the domain. A ***particular*** element of the domain may have properties that are not shared by all the elements of the domain. For example, if the domain is the set of all integers, 3 is a

Every employee who received a large bonus works hard.
Linda is an employee at the company.
Linda received a large bonus.
∴ Some employee works hard.

particular element of the domain. The number 3 is odd which is not a property that is shared by all integers. Every domain element referenced in a proof must be defined on a separate line of the proof. If the element is defined in a hypothesis, it is always a particular element and the definition of that element in the proof is labeled "Hypothesis". If an element is introduced for the first time in the proof, the definition is labeled "Element definition" and must specify whether the element is arbitrary or particular.

PARTICIPATION ACTIVITY 5.8.1: Definitions of arbitrary and particular elements of a domain.

For each definition of an element in the domain, indicate whether the element defined is particular or arbitrary.

1) The domain is the set of all integers.

   3 is an integer. Hypothesis.

   - Particular
   - Arbitrary

2) The domain is the set of all employees at a company.

   c is an arbitrary employee of the company. Element definition.

   - Particular
   - Arbitrary

3) The domain is the set of all integers.

   c is a particular integer. Element definition.

   - Particular
   - Arbitrary

4) The domain is the set of students enrolled for a class.

   Larry is enrolled in the class. Hypothesis.

○ Particular

○ Arbitrary

The rules ***existential instantiation*** and ***universal instantiation*** replace a quantified variable with an element of the domain. The rules ***existential generalization*** and ***universal generalization*** replace an element of the domain with a quantified variable. Note that the rules only apply to non-nested quantifiers. Applying the rules of inference to nested quantifiers would require more constraints on which rules could be applied in particular situations.

Table 5.8.1: Rules of inference for quantified statements.

Rule of Inference	Name	Example
c is an element (arbitrary or particular) ∀x P(x) ∴ P(c)	Universal instantiation	Sam is a student in the class. Every student in the class completed the assignment. Therefore, Sam completed his assignment.
c is an arbitrary element P(c) ∴ ∀x P(x)	Universal generalization	Let c be an arbitrary integer. $c \leq c^2$ Therefore, every integer is less than or equal to its square.
∃x P(x) ∴ (c is a particular element) ∧ P(c)	Existential instantiation*	There is an integer that is equal to its square. Therefore, $c^2 = c$, for some integer c.
c is an element (arbitrary or particular) P(c) ∴ ∃x P(x)	Existential generalization	Sam is a particular student in the class. Sam completed the assignment. Therefore, there is a student in the class who completed the assignment.

*Note: each use of Existential instantiation must define a new element with its own name (e.g., "c" or "d").

PARTICIPATION ACTIVITY | 5.8.2: A logical proof that uses the laws of inference for quantified statements.

Animation captions:

1. An argument has hypotheses ∀x (P(x) ∨ Q(x)), "3 is an integer", and ¬P(3). The conclusion is Q(3). The first two lines in the proof are the first two hypotheses.
2. Line 3 of the proof is (P(3) ∨ Q(3)), by Universal instantiation applied to lines 1 and 2.
3. Line 4 of the proof is ¬P(3), a hypothesis.
4. Line 5 of the proof is Q(3), by Disjunctive syllogism applied to lines 3 and 4.

PARTICIPATION ACTIVITY | 5.8.3: Correct and incorrect use of generalization and instantiation.

Indicate whether the proof fragment is a correct or incorrect use of the rule of inference.

1)

1.	c is an element	Hypothesis
2.	P(c)	Hypothesis
3.	∀x P(x)	Universal generalization, 1, 2

- Correct
- Incorrect

2)

1.	c is an element	Hypothesis
2.	∀x P(x)	Hypothesis
3.	P(c)	Universal instantiation, 1, 2

- Correct
- Incorrect

3)

1.	c is an element	Hypothesis
2.	P(c)	Hypothesis
3.	d is an element	Hypothesis
4.	Q(d)	Hypothesis
5.	P(c) ∧ Q(d)	Conjunction, 2, 4
6.	∃x (P(x) ∧ Q(x))	Existential generalization, 1, 3, 5

- Correct
- Incorrect

PARTICIPATION ACTIVITY 5.8.4: Completing a proof of validity with quantified statements.

The proof below establishes the validity of the following argument:

∃x P(x)
∀x Q(x)
∴ ∃x (P(x) ∧ Q(x))

Match each rule to the correct location in the proof.

1. ∃x P(x)	Hypothesis
2. (c is a particular element) ∧ P(c)	Reason A
3. P(c) ∧ (c is a particular element)	Commutative law, 2
4. ∀x Q(x)	Hypothesis
5. c is a particular element	Simplification, 2
6. Q(c)	Reason B

8. P(c) ∧ Q(c)	Reason D
9. ∃x(P(x) ∧ Q(x))	Reason E

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

Existential instantiation, 1 **Simplification, 3** **Existential generalization, 5, 8**

Conjunction, 6, 7 **Universal instantiation, 4, 5**

Reason A

Reason B

Reason C

Reason D

Reason E

Reset

Here is an argument expressed in English. The domain is the set of students enrolled in a class:

Every student who stayed up too late missed the test.
Juan is enrolled in the class.
Juan did not miss the test.
∴ Some student did not stay up too late.

The first step in proving that the argument is valid is to determine the form of the argument. Define the following two predicates:

S(x): x stayed up too late
M(x): x missed the test

Here is the form of the argument:

∀x (S(x) → M(x))
Juan, a student in the class
¬M(Juan)
∴ ∃x ¬S(x)

PARTICIPATION ACTIVITY 5.8.5: Completing a proof of validity with quantified statements.

Complete the proof of the argument given above by putting the rule used to justify each step next to the step. The proof starts with the following two lines:

Hypothesis	1. ∀x (S(x) → M(x))
Hypothesis	2. Juan, a student in the class

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

Existential generalization, 2, 5 **Hypothesis** **Modus tollens, 3, 4**

Universal instantiation, 1, 2

3. S(Juan) → M(Juan)

4. ¬M(Juan)

5. ¬S(Juan)

6. ∃x ¬S(x)

Reset

PARTICIPATION ACTIVITY 5.8.6: Ordering the steps of a proof of validity.

The proof below establishes the validity of the following argument:

$\forall x\,(Q(x) \lor P(x))$
$\forall x\,\neg Q(x)$
$\therefore \forall x\,P(x)$

Complete the proof of the argument above by putting each step on the correct line. The proof starts with the following line:

$\forall x\,(Q(x) \lor P(x))$	1. Hypothesis

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

c is an arbitrary element of the domain **$\forall x\,\neg Q(x)$** **$Q(c) \lor P(c)$** **$P(c)$** **$\neg Q(c)$** **$\forall x\,P(x)$**

2. Element definition.

3. Universal instantiation, 1, 2

4. Hypothesis

5. Universal instantiation, 2, 4

6. Disjunctive syllogism, 3, 5

7. Universal generalization, 2, 6

Reset

Multiple uses of existential instantiation: a common mistake

It is important to define a new particular element with a new name for each use of existential instantiation within the same logical proof in order to avoid a faulty proof that an invalid argument is valid.

The mistake in the proof below is in the assumption that the value c (introduced in Step 2), which causes P(x) to be true, is the same value c (introduced in Step 5) that causes Q(x) to be true. A correct use of existential instantiation in line 4 would first introduce a new particular element, d that is not necessarily equal to c, and assert that Q(d) is true. For example, if P(x) means that x owns a cat and Q(x) means that x owns a dog, then the two hypotheses say that there is someone who owns a cat and there is someone who owns a dog. However, the two hypotheses together do not imply that there is someone who owns a cat and a dog.

Table 5.8.2: Incorrect use of existential instantiation leading to an erroneous proof of an invalid argument.

∃x P(x)
∃x Q(x)
∴ ∃x (P(x) ∧ Q(x))

1.	∃x P(x)	Hypothesis
2.	(c is a particular element) ∧ P(c)	Existential instantiation, 1
3.	P(c) ∧ (c is a particular element)	Commutative law, 2
4.	∃x Q(x)	Hypothesis
5.	(c is a particular element) ∧ Q(c)	Existential instantiation, 4
6.	Q(c) ∧ (c is a particular element)	Commutative law, 5
7.	P(c)	Simplification, 3
8.	Q(c)	Simplification, 6
9.	P(c) ∧ Q(c)	Conjunction, 7, 8
10.	c is a particular element	Simplification, 2
11.	∃x (P(x) ∧ Q(x))	Existential generalization, 9, 10

PARTICIPATION ACTIVITY 5.8.7: Correct and incorrect use of generalization and instantiation.

Indicate whether the proof fragment is a correct or incorrect use of the rule of inference.

1)

2.	(c is a particular element) ∧ P(c)	Existential instantiation, 1
3.	∃x Q(x)	Hypothesis
4.	(c is a particular element) ∧ Q(c)	Existential instantiation, 3

- Correct
- Incorrect

Showing an argument with quantified statements is invalid

An argument with quantified statements can be shown to be invalid by defining a domain and predicates for which the hypotheses are all true but the conclusion is false. For example, consider the invalid argument:

∃x P(x)
∃x Q(x)
∴ ∃x (P(x) ∧ Q(x))

Suppose that domain is the set {c, d}. The two hypotheses, ∃x P(x) and ∃x Q(x), are both true for the values for P and Q on elements c and d given in the table. However, the conclusion ∃x (P(x) ∧ Q(x)) is false. Therefore, the argument is invalid.

	P	Q
c	T	F
d	F	T

PARTICIPATION ACTIVITY | 5.8.8: Showing an argument with quantified statements is invalid: finite domain.

1) The argument below is invalid. Suppose that the domain of x is the set {c, d}. Select the table that proves

the argument is invalid.

∀x P(x) ∨ ∀x Q(x)
∴ ∃x (P(x) ∧ Q(x))

○

	P	Q
c	T	F
d	T	F

○

	P	Q
c	T	F
d	F	T

○

	P	Q
c	T	F
d	T	T

PARTICIPATION ACTIVITY | 5.8.9: Showing an argument with quantified statements is invalid: integer domain.

The following argument is invalid:

∃x P(x)
∃x Q(x)
∴ ∃x (P(x) ∧ Q(x))

Which definitions for predicates P and Q show that the argument is invalid? In each question the domain of x is the set of positive integers.

1) Suppose that the predicates P and Q are defined as follows:

P(x): x is prime
Q(x): x is even

Do the definitions for P and Q show that the argument is invalid?

- Yes
- No

2) Suppose that the predicates P and Q are defined as follows:

P(x): x is prime
Q(x): x is multiple of 4

Do the definitions for P and Q show that the argument is invalid?

- Yes
- No

3) Suppose that the predicates P and Q are defined as follows:

P(x): x is prime
Q(x): $x^2 < x$

Do the definitions for P and Q show that the argument is invalid?

- Yes
- No

CHALLENGE ACTIVITY 5.8.1: Rules of inference with quantifiers.

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Additional exercises

$\forall x\ (P(x) \lor Q(x))$
$\exists x\ \neg P(x)$
$\therefore \forall x\ Q(x)$

EXERCISE | 5.8.1: Proving the validity of arguments with quantified statements.

Prove that the given argument is valid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. Then use the rules of inference to prove that the form is valid.

(a) The domain is the set of musicians in an orchestra.

Everyone practices hard or plays badly (or both).
Someone does not practice hard.
∴ Someone plays badly.

(b) The domain is the set of people who live in a city. Linda lives in the city.

Linda lives in the city.
Linda owns a Ferrari.
Everyone who owns a Ferrari has gotten a speeding ticket.
∴ Linda has gotten a speeding ticket.

(c) The domain is the set of all paintings.

All of the paintings by Matisse are beautiful.
The museum has a painting by Matisse.
∴ The museum has a beautiful painting.

(d) The domain is the set of students at an elementary school.

Every student who has a permission slip can go on the field trip.
Every student has a permission slip.
∴ Every student can go on the field trip.

(e) The domain is the set of students at a university.

Larry is a student at the university.
Hubert is a student at the university.
Larry and Hubert are taking Boolean Logic.
Any student who takes Boolean Logic can take Algorithms.
∴ Larry and Hubert can take Algorithms.

EXERCISE | 5.8.2: Which arguments are valid?

Which of the following arguments are valid? Explain your reasoning.

(a) I have a student in my class who is getting an A. Therefore, John, a student in my class is getting an A.

(b) Every girl scout who sells at least 50 boxes of cookies will get a prize. Suzy, a girl scout, got a prize. Therefore Suzy sold at least 50 boxes of cookies.

EXERCISE | 5.8.3: Show an argument with quantified statements is invalid.

Show that the given argument is invalid by giving values for the predicates P and Q over the domain {a, b}.

(a)

$\forall x\,(P(x) \rightarrow Q(x))$
$\exists x\,\neg P(x)$
$\therefore \exists x\,\neg Q(x)$

(b)

$\exists x\,(P(x) \lor Q(x))$
$\exists x\,\neg Q(x)$
$\therefore \exists x\,P(x)$

EXERCISE 5.8.4: Determine and prove whether an argument is valid or invalid.

Determine whether each argument is valid. If the argument is valid, give a proof using the laws of logic. If the argument is invalid, give values for the predicates P and Q over the domain {a, b} that demonstrate the argument is invalid.

(a)

$\exists x\, (P(x) \land Q(x))$
$\therefore \exists x\, Q(x) \land \exists x\, P(x)$

(b)

$\exists x\, Q(x) \land \exists x\, P(x)$
$\therefore \exists x\, (P(x) \land Q(x))$

(c)

$\forall x\, (P(x) \land Q(x))$
$\therefore \forall x\, Q(x) \land \forall x\, P(x)$

(d)

$\forall x\, (P(x) \lor Q(x))$
$\therefore \forall x\, Q(x) \lor \forall x\, P(x)$

EXERCISE 5.8.5: Determine and prove whether an argument in English is valid or invalid.

Prove whether each argument is valid or invalid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. If the argument is valid, then use the rules of inference to prove that the form is valid. If the argument is invalid, give values for the predicates you defined for a small domain that demonstrate the argument is invalid.

The domain for each problem is the set of students in a class.

(a)

Every student on the honor roll received an A.
No student who got a detention received an A.
No student who got a detention is on the honor roll.

(b)

No student who got an A missed class.
No student who got a detention received an A.
No student who got a detention missed class.

(c)

Every student who missed class got a detention.
Penelope is a student in the class.
Penelope got a detention.
Penelope missed class.

(d)

Every student who missed class got a detention.
Penelope is a student in the class.
Penelope did not miss class.
Penelope did not get a detention.

(e)

Every student who missed class or got a detention did not get an A.
Penelope is a student in the class.
Penelope got an A.
Penelope did not get a detention.

5.9 Populating Truth Tables with Boolean Functions

IMPORTANT: You will need to use the drop down above the code, below, to select "PA2.py" and paste your PA2 code in that file in order for your PA3 code to work.

For this assignment you will be provided with a function `callf2(f,p,q)` whose input is a function and its two arguments, and returns the value of the function applied to the arguments. You are also given a "main" to test your code.

Your task is to implement three functions: `eval_tt_f2(f)` and `equivalent(tt1,tt2)` and `is_tautology(tt)`.

The function `eval_tt_f2(f)` will, given a two argument Boolean function f as input (one of the functions you created in PA2.py), return the truth table for f. Use the function `make_tt_ins(n)` function you implemented in PA2.py to create a list with the inputs for the truth table, and then use `callf2` to append the truth value for f to each row. For example, eval*tt*f2(iff) should return:

[[False, False, True], [False, True, False], [True, False, False], [True, True, True]]

The function `equivalent(tt1,tt2)` should return True if tt1 and tt2 are logically equivalent and False otherwise. For example, `equivalent(eval_tt_f2(PA2.implies), eval_tt_f2(PA2.nqIMPnp))`
should return True.

The function `is_tautology(tt)` should return True if the tt parameter has a True value as the output for all possible tt inputs, otherwise it should return false.

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LAB ACTIVITY 5.9.1: Populating Truth Tables with Boolean Functions 0 / 100

Current file: **PA3.py** Load default template...

```
'''
          PA3: Truth tables and equivalence
          -_-_-_-_-_-_-_-_-_-_-_-_-_-_-_-_-_-

For PA3 you will be provided with a function callf2(f,p,q)
which will cal f(p,q), and with a main to test your code.
You will implement two functions: eval_tt_f2(f) and
equivalent(tt1,tt2).

The function eval_tt_f2(f) will, given a two argument Boolean
function f ( one of the functions you created in PA2.py), return
the truth table for f. Use your make_tt_ins(n) function from
PA2.py to create an arrayList with n inputs, then use callf2
to append the truth value for f to each row of that arrayList.
For example, eval_tt_f2(iff) will return:

[[False, False, True], [False, True, False], [True, False, False], [True, True, True]]

The function equivalent(tt1,tt2) will return True if tt1 and tt2
```

Develop mode **Submit mode**

Run your program as often as you'd like, before submitting for grading. Below, type any needed input values in the first box, then click **Run program** and observe the program's output in the second box.

Enter program input (optional)

If your code requires input values, provide them here.

Run program

Input (from above) → **PA3.py** (Your program) → O

Program output displayed here

Coding trail of your work What is this?

History of your effort will appear here once you begin working on this zyLab.