4.1 Propositions and logical operations

Logic is the study of formal reasoning. A statement in a spoken language, such as in English, is often ambiguous in its meaning. By contrast, a statement in logic always has a well defined meaning. Logic is important in mathematics for proving theorems. Logic is also used in computer science in areas such as artificial intelligence for automated reasoning and in designing digital circuits. Logic is useful in any field in which it is important to make precise statements. In law, logic can be used to define the implications of a particular law. In medicine, logic can be used to specify precisely the conditions under which a particular diagnosis would apply.

The most basic element in logic is a proposition. A ***proposition*** is a statement that is either true or false.

Table 4.1.1: Examples of propositions: Statements that are either true or false.

Proposition	Truth value
There are an infinite number of prime numbers.	True
17 is an even number.	False

Propositions are typically declarative sentences. For example, the following are *not* propositions.

Table 4.1.2: English sentences that are not propositions.

Sentence	Comment
What time is it?	A question, not a proposition. A question is neither true nor false.
Are you awake?	Even a yes/no question is neither true nor false, so is not a proposition.
Have a nice day.	A command, not a proposition. A command is neither true nor false.

A proposition's ***truth value*** is a value indicating whether the proposition is actually true or false. A proposition is still a proposition whether its truth value is known to be true, known to be false, unknown, or a matter of opinion. The following are all propositions.

Table 4.1.3: Examples of propositions and their truth values.

Proposition	Comment
Two plus two is four.	Truth value is true.
Two plus two is five.	Truth value is false.
Monday will be cloudy.	Truth value is unknown.
The movie was funny.	Truth value is a matter of opinion.

PARTICIPATION ACTIVITY | 4.1.1: Propositions.

Indicate which statements are propositions.

1) 10 is a prime number.

- Proposition
- Not a proposition

2) Shut the door.

- Proposition
- Not a proposition

3) All politicians are dishonest.

- Proposition
- Not a proposition

4) Would you like some cake?

- Proposition
- Not a proposition

5) Interest rates will rise this year.

- Proposition
- Not a proposition

The conjunction operation

Propositional variables such as p, q, and r can be used to denote arbitrary propositions, as in:

p: January has 31 days.
q: February has 33 days.

A ***compound proposition*** is created by connecting individual propositions with logical operations. A ***logical operation*** combines propositions using a particular composition rule. For example, the conjunction operation is denoted by ∧. The proposition p ∧ q is read "p and q" and is called the ***conjunction*** of p and q. p ∧ q is true if both p is true and q is true. p ∧ q is false if p is false, q is false, or both are false.

Using the definitions for p ∧ q given above, the proposition p ∧ q is expressed in English as:

p ∧ q: January has 31 days and February has 33 days.

Proposition p's truth value is true — January does have 31 days. Proposition q's truth value is false — February does not have 33 days. The compound proposition p ∧ q is therefore false, because it is not the case that both propositions are true.

A ***truth table*** shows the truth value of a compound proposition for every possible combination of truth values for the variables contained in the compound proposition. Every row in the truth table shows a particular truth value for each variable, along with the compound proposition's corresponding truth value. Below is the truth table for p ∧ q, where ***T*** represents true and ***F*** represents false.

PARTICIPATION ACTIVITY | 4.1.2: Truth table for the conjunction operation.

Animation captions:

1. p ∧ q is true only when both p and q are true.
2. p ∧ q is false for all other combinations.

Different ways to express a conjunction in English

Define the propositional variables p and h as:

p: The sauce looks disgusting.
h: The sauce tastes delicious.

There are many ways to express the proposition p ∧ h in English. The sentences below have slightly different meanings in English but correspond to the same logical meaning.

Table 4.1.4: Examples of different ways to express a conjunction in English.

p and h	The sauce looks disgusting and tastes delicious.
p, but h	The sauce looks disgusting, but tastes delicious.
Despite the fact that p, h	Despite the fact that the sauce looks disgusting, it tastes delicious.
Although p, h	Although the sauce looks disgusting, it tastes delicious.

The disjunction operation

The disjunction operation is denoted by ∨. The proposition p ∨ q is read "p or q", and is called the ***disjunction*** of p and q. p ∨ q is true if either one of p or q is true, or if both are true. The proposition p ∨ q is false only if both p and q are false. Using the same p and q from the example above, p ∨ q is the statement:

p ∨ q: January has 31 days or February has 33 days.

The proposition p ∨ q is true because January does have 31 days. The truth table for the ∨ operation is given below.

PARTICIPATION ACTIVITY | 4.1.3: Truth table for the disjunction operation.

Animation captions:

1. p ∨ q is true when either of p or q is true.
2. p ∨ q is false only when p and q are both false.

Ambiguity of "or" in English

The meaning of the word "or" in common English depends on context. Often when the word "or" is used in English, the intended meaning is that one or the other of two things is true, but not both. One would normally understand the sentence "Lucy is going to the park or the movie" to mean that Lucy is either going to the park, or is going to the movie, but not both. Such an either/or meaning corresponds to the "exclusive or" operation in logic. The ***exclusive or*** of p and q evaluates to true when p is true and q is false or when q is true and p is false. The ***inclusive or*** operation is the same as the disjunction (∨) operation and evaluates to true when one or both of the propositions are true. For example, "Lucy opens the windows or doors when warm" means she opens windows, doors, or possibly both. Since the inclusive or is most common in logic, it is just called "or" for short.

PARTICIPATION ACTIVITY | 4.1.4: Evaluating the truth value of statements using the inclusive or.

Indicate whether each statement is true or false. In each case, use the inclusive OR.

1) 4 is an even number or 4 is a prime number.

   - True
   - False

2) 5 is a prime number or 3 is a prime number.

   - True
   - False

3) 2 is a negative number or -3 is a positive number.

○ T

PARTICIPATION ACTIVITY | 4.1.5: Truth table for the exclusive or.

The exclusive or operation is usually denoted with the symbol ⊕. The proposition p ⊕ q is true if exactly one of the propositions p and q is true but not both. This question asks you to fill in the truth table for p ⊕ q.

p	q	p ⊕ q
T	T	1?
T	F	2?
F	T	3?
F	F	4?

1) What is the truth value for the square labeled 1?

○ True

○ False

2) What is the truth value for the square labeled 2?

○ True

○ False

3) What is the truth value for the square labeled 3?

○ True

○ False

4) What is the truth value for the square labeled 4?

○ True

○ False

PARTICIPATION ACTIVITY | 4.1.6: Conditions under which inclusive OR and exclusive OR are not equal.

1) Under what conditions do the logical expressions $p \oplus q$ and $p \lor q$ have different truth values?

- p = q = F
- p = T, and q = F
- p = F, and q = T
- p = q = T

The ***negation*** operation acts on just one proposition and has the effect of reversing the truth value of the proposition. The negation of proposition p is denoted ¬p and is read as "not p". Since the negation operation only acts on a single proposition, its truth table only has two rows for the proposition's two possible truth values.

PARTICIPATION ACTIVITY | 4.1.7: Truth table for the negation operation.

Animation captions:

1. The truth value of ¬p is the opposite of the truth value of p.

PARTICIPATION ACTIVITY | 4.1.8: Applying logical operations.

Assume propositions p, q, and r have the following truth values:

p is true
q is true
r is false

What are the truth values for the following compound propositions?

1) p ∧ q

- True
- False

2) ¬r

- True
- False

3) $p \wedge r$

- True
- False

4) $p \vee r$

- True
- False

5) $p \vee q$

- True
- False

Example 4.1.1: Searching the web.

The language of logic is useful in database searches, such as searching the web. Suppose one is interested in finding web pages related to higher education. A search on the term "college" could potentially miss many pages related to universities. A search on "college OR university" would yield results on both topics. A search on "dogs AND fleas" would yield pages that pertain to both dogs and fleas. A typical web search engine, though, implicitly uses an AND operation for multiple words in queries like "dogs fleas".

CHALLENGE ACTIVITY | 4.1.1: Propositions and logical operations.

422102.2723990.qx3zqy7

Start

The propositional variables p, q, r, and w are defined as follows.

- p: Felicia was tired.
- q: Felicia went to the game.
- r: Felicia went to a movie.
- w: Felicia had a headache.

Proposition in words: Felicia was tired but she went to the movie.

Proposition in symbols: [None] ¬ p

1	2	3

Check Next

Additional exercises

EXERCISE 4.1.1: Identifying propositions.

Determine whether each of the following sentences is a proposition. If the sentence is a proposition, then write its negation.

(a) Have a nice day.

(b) The soup is cold.

(c) The patient has diabetes.

(d) The light is on.

(e) It's raining today.

(f) Do you like my new shoes?

(g) The sky is purple.

(h) $2 + 3 = 6$

(i) Every prime number is even.

(j) There is a number that is larger than 17.

EXERCISE | 4.1.2: Expressing English sentences using logical notation.

Express each English statement using logical operations ∨, ∧, ¬ and the propositional variables t, n, and m defined below. The use of the word "or" means inclusive or.

t: The patient took the medication.
n: The patient had nausea.
m: The patient had migraines.

(a) The patient had nausea and migraines.

(b) The patient took the medication, but still had migraines.

(c) The patient had nausea or migraines.

(d) The patient did not have migraines.

(e) Despite the fact that the patient took the medication, the patient had nausea.

(f) There is no way that the patient took the medication.

EXERCISE 4.1.3: Applying logical operations.

Assume the propositions p, q, r, and s have the following truth values:

p is false
q is true
r is false
s is true

What are the truth values for the following compound propositions?

(a) $\neg p$

(b) $p \vee r$

(c) $q \wedge s$

(d) $q \vee s$

(e) $q \oplus s$

(f) $q \oplus r$

EXERCISE 4.1.4: Truth values for statements with inclusive and exclusive or.

Indicate whether each statement is true or false, assuming that the "or" in the sentence means the inclusive or. Then indicate whether the statement is true or false if the "or" means the exclusive or.

(a) February has 31 days or the number 5 is an integer.

(b) The number π is an integer or the sun revolves around the earth.

(c) 20 nickels are worth one dollar or whales are mammals.

(d) There are eight days in a week or there are seven days in a week.

(e) January has exactly 31 days or April has exactly 30 days.

4.2 Evaluating compound propositions

A compound proposition can be created by using more than one operation. For example, the proposition p ∨ ¬q evaluates to true if p is true or the negation of q is true.

The order in which the operations are applied in a compound proposition such as p ∨ ¬q ∧ r may affect the truth value of the proposition. In the absence of parentheses, the rule is that negation is applied first, then conjunction, then disjunction:

Figure 4.2.1: Order of operations in absence of parentheses.

1. ¬ (not)
2. ∧ (and)
3. ∨ (or)

For example, the proposition p ∨ q ∧ r should be read as p ∨ (q ∧ r), instead of (p ∨ q) ∧ r. However, good practice is to use parentheses to specify the order in which the operations are to be performed, as in p ∨ (q ∧ r). One exception to using parentheses in compound propositions is with the negation operation. Parentheses around ¬p are usually omitted to make compound propositions more readable. Because the negation operation is always applied first, the proposition ¬p ∨ q is evaluated as (¬p) ∨ q instead of ¬(p ∨ q). Also, when there are multiple ∨ operations or multiple ∧ operations, such as in the compound proposition p ∨ q ∨ r or the compound proposition p ∧ q ∧ r, parentheses are usually omitted because the order in which the operations are applied does not affect the final truth value. The exclusive or (denoted by ⊕) is not included in the table above because this material does not use the exclusive or operations in compound logical expressions.

PARTICIPATION ACTIVITY | 4.2.1: Evaluating compound propositions.

Animation captions:

1. The compound proposition $p \land \neg(q \lor r)$ is evaluated by filling in the given truth values for variables p, q, and r,
2. and then evaluating the operations in the required order.
3. The compound proposition $p \land \neg(q \lor r)$ evaluates to false.

PARTICIPATION ACTIVITY | 4.2.2: Evaluating complex compound propositions.

Assume the propositions p, q, r have the following truth values:

p is true
q is true
r is false

What are the truth values for the following compound propositions?

1) $p \lor \neg q$
- True
- False

2) $\neg r \land (p \lor \neg q)$
- True
- False

3) $\neg(p \land \neg r)$
- True
- False

4) $(p \lor r) \land \neg p$
- True
- False

CHALLENGE ACTIVITY | 4.2.1: Write proposition using symbols.

422102.2723990.qx3zqy7

Start

Define the proposition in symbols using:

- p: The weather is bad.
- q: The trip is cancelled.
- r: The trip is delayed.

Proposition in words: The trip is delayed and the weather is bad.

Proposition in symbols:

1	2	3	4	5

Check Next

Example 4.2.1: Searching the web -- continued.

Compound propositions can be created with logical operations to conduct refined web searches. Suppose one is interested in studying jaguars (the animal from the cat family). Try searching (e.g., at google.com) for the term "jaguar" — the results may include numerous hits related to the car "Jaguar". To avoid results involving cars, try a second search using the query "jaguar -car" — the "-" symbol indicates negation. Notice that results are then mostly about the animal.

Filling in the rows of a truth table

A truth table for a compound proposition has a row for every possible combination of truth assignments for the statement's variables. If a compound proposition has n variables, there are 2^n rows. The truth table for compound proposition $(p \lor r) \land \neg q$ has $2^3 = 8$ rows. The last square in each row shows the truth value for the compound proposition when the variables are set according to the truth values in that row.

Each column is labeled. The columns corresponding to single variables are to the left and the column with the compound propositions is to the right. The truth values in the variable columns are chosen methodically to create all possible combinations.

To fill in the variable columns, each column is filled in from top to bottom, beginning with T. Start with the right-most variable column and fill in the squares with an alternating T and F pattern. The next column to the left is filled in by an alternating TT and FF pattern. The next column to the left is filled in by an alternating TTTT and FFFF pattern. For each new column, the number of T's and F's in the pattern is doubled.

PARTICIPATION ACTIVITY 4.2.3: How to fill in a truth table.

Animation content:

```
undefined
```

Animation captions:

1. A column for each variable and a column for the compound proposition $(p \vee r) \wedge \neg q$. There are three variables, so there are $2^3 = 8$ rows.
2. Rightmost variable column is filled in with alternating T's and F's. Next column alternates by twos and the third column alternates by fours.
3. The column with the compound proposition shows the truth value for each truth assignment to the variables.

PARTICIPATION ACTIVITY 4.2.4: Filling in the variable columns of a truth table.

The column labels for a truth table for compound proposition (p ∧ q) ∨ (¬r ∧ ¬w) is shown below.

p	q	r	w	(p ∧ q) ∨ (¬r ∧ ¬w)

1) What is the pattern for the truth values in the column labeled w?

- TFTF...
- FTFT...
- TTFF...
- FFTT...

2) What is the pattern for the truth values in the column labeled p?

- Alternating pattern of 3 T's and 3 F's.
- Alternating pattern of 4 T's and 4 F's.
- Alternating pattern of 8 T's and 8 F's.
- Alternating pattern of 16 T's and 16 F's.

PARTICIPATION ACTIVITY 4.2.5: Number of entries in a truth table.

1) How many rows are in a truth

table for a compound proposition with propositional variables p, q, and r?

Check Show answer

2) How many rows are in a truth table for the proposition (p ∧ q) ∨ (¬r ∧ ¬q) ∨ ¬(p ∧ t) ?

Check Show answer

When filling out a truth table for a complicated compound proposition, completing intermediate columns for smaller parts of the full compound proposition can be helpful.

PARTICIPATION ACTIVITY 4.2.6: Truth table with intermediate columns.

Animation captions:

1. The truth table for $\neg q \wedge (p \vee r)$ can be computed by first filling in a column for $\neg q$,
2. then filling in a column for $(p \vee r)$,
3. and finally filling in the column for $\neg q \wedge (p \vee r)$ using the intermediate columns.

PARTICIPATION ACTIVITY 4.2.7: Filling in a truth table.

Indicate how the missing items in the truth table below should be filled in:

p	q	r	¬q	p ∧ ¬q	(p ∧ ¬q) ∨ r
T	T	T	F	F	*(A)*
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	*(B)*	T
F	T	T	F	F	T

F	T	F	F	F	*(C)*
F	F	T	T	F	T
F	F	*(D)*	T	F	F

1) What is the correct value for A?

- T
- F

2) What is the correct value for B?

- T
- F

3) What is the correct value for C?

- T
- F

4) What is the correct value for D?

- T
- F

CHALLENGE ACTIVITY 4.2.2: Truth tables for compound propositions.

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Start

Using the pattern above, fill in all combinations of p and q.

p	q
T	
T	F
	T

1	2	3	4	5	6	7	8

Check Next

Example 4.2.2: Logic in electronic devices.

Most of the devices and appliances used in everyday life are controlled by electronic circuitry that works according to the laws of logic. A designer of an electronically-controlled device must first express the desired behavior in logic, and then test that the device behaves as desired under every set of circumstances.

An electronic fan automatically turns on and off depending on the humidity in a room. Since the humidity detector is not very accurate, the fan will stay on for 20 minutes once triggered in order to ensure that the room is cleared of moisture. There is also a manual off switch that can be used to override the automatic functioning of the control.

Define the following propositions:

M: The fan has been on for twenty minutes.
H: The humidity level in the room is low.
O: The manual "off" button has been pushed.

When the fan is on, the fan will turn off if the following proposition evaluates to true:

$(M \land H) \lor O$

A truth table can be useful in testing the device to make sure it works as intended under every set of circumstances. The following table might be used by a technician testing the electronic fan.

M	H	O	Should be off? (T: yes)
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

Additional exercises

EXERCISE | 4.2.1: Truth values for compound English sentences.

Determine whether the following propositions are true or false:

(a) 5 is an odd number and 3 is a negative number.

(b) 5 is an odd number or 3 is a negative number.

(c) 8 is an odd number or 4 is not an odd number.

(d) 6 is an even number and 7 is odd or negative.

(e) It is not true that either 7 is an odd number or 8 is an even number (or both).

EXERCISE | 4.2.2: Translating English statements into logic.

Express each statement in logic using the variables:

p: It is windy.
q: It is cold.
r: It is raining.

(a) It is windy and cold.

(b) It is windy but not cold.

(c) It is not true that it is windy or cold.

(d) It is raining and it is windy or cold.

(e) It is raining and windy or it is cold.

(f) It is raining and windy but it is not cold.

EXERCISE 4.2.3: Truth values for compound propositions.

The propositional variables, p, q, and s have the following truth assignments: p = T, q = T, s = F. Give the truth value for each proposition.

(a) $p \lor \neg q$

(b) $(p \land q) \lor s$

(c) $p \land (q \lor s)$

(d) $p \land \neg(q \lor s)$

(e) $\neg(q \land p \land \neg s)$

(f) $\neg(p \land \neg(q \land s))$

EXERCISE 4.2.4: Writing truth tables.

Write a truth table for each expression.

(a) $\neg p \oplus q$

(b) $\neg(p \lor q)$

(c) $r \lor (p \land \neg q)$

(d) $(r \lor p) \land (\neg r \lor \neg q)$

EXERCISE 4.2.5: Translating compound propositions into English sentences.

Express the following compound propositions in English using the following definitions:

p: I am going to a movie tonight.
q: I am going to the party tonight.

(a) $\neg p$

(b) $p \land q$

(c) $p \land \neg q$

(d) $\neg p \lor \neg q$

(e) $\neg(p \land q)$

EXERCISE 4.2.6: Multiple disjunction or conjunction operations.

Suppose that p, q, r, s, and t are all propositional variables.

(a) Describe in words when the expression $p \lor q \lor r \lor s \lor t$ is true and when it is false.

(b) Describe in words when the expression $p \land q \land r \land s \land t$ is true and when it is false.

EXERCISE 4.2.7: Expressing a set of conditions using logical operations.

Consider the following pieces of identification a person might have in order to apply for a credit card:

B: Applicant presents a birth certificate.
D: Applicant presents a driver's license.
M: Applicant presents a marriage license.

Write a logical expression for the requirements under the following conditions:

(a) The applicant must present either a birth certificate, a driver's license or a marriage license.

(b) The applicant must present at least two of the following forms of identification: birth certificate, driver's license, marriage license.

(c) Applicant must present either a birth certificate or both a driver's license and a marriage license.

EXERCISE 4.2.8: Finding truth values to make two logical expressions evaluate to different values.

Give truth values for the propositional variables that cause the two expressions to have different truth values.

For example, given $p \lor q$ and $p \oplus q$, the correct answer would be p = q = T, because when p and q are both true, $p \lor q$ is true but $p \oplus q$ is false. Note that there may be more than one correct answer.

(a) $r \land (p \lor q)$
$(r \land p) \lor q$

(b) $\neg p \land q$
$\neg(p \land q)$

(c) $p \lor q$
$(\neg p \land q) \lor (p \land \neg q)$

EXERCISE 4.2.9: Boolean expression to express a condition on the input variables.

(a) Give a logical expression with variables p, q, and r that is true if p and q are false and r is true and is otherwise false.

4.3 Conditional statements

The ***conditional operation*** is denoted with the symbol →. The proposition p → q is read "if p then q". The proposition p → q is false if p is true and q is false; otherwise, p → q is true.

A compound proposition that uses a conditional operation is called a ***conditional proposition***. A conditional proposition expressed in English is sometimes referred to as a ***conditional statement***, as in "If there is a traffic jam today, then I will be late for work."

In p → q, the proposition p is called the ***hypothesis***, and the proposition q is called the ***conclusion***. The truth table for p → q is given below.

Table 4.3.1: Truth table for the conditional operation.

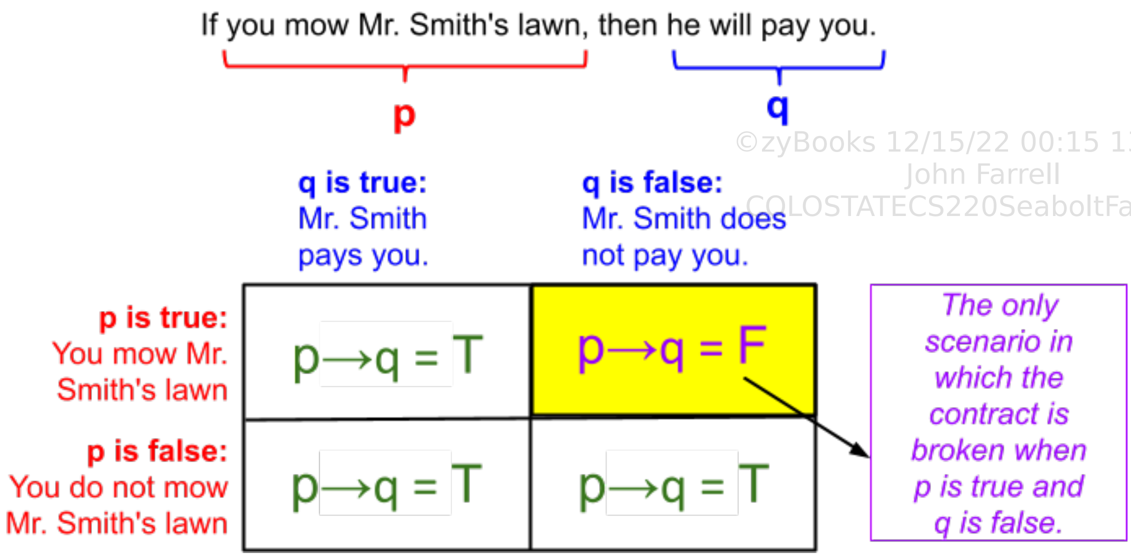
p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

A conditional proposition can be thought of like a contract between two parties, as in:

If you mow Mr. Smith's lawn, then he will pay you.

The only way for the contract between you and Mr. Smith to be broken, is for you to mow Mr. Smith's lawn and for him not to pay you. If you do not mow his lawn, then he can either pay you or not, and the contract is not broken. In the words of logic, the only way for a conditional statement to be false is if the hypothesis is true and the conclusion is false. If the hypothesis is false, then the conditional statement is true regardless of the truth value of the conclusion.

Figure 4.3.1: A conditional statement illustrated.



PARTICIPATION ACTIVITY | 4.3.1: Understanding conditional statements.

Each question has a proposition p that is a conditional statement. Truth values are also given for the individual propositions contained in that conditional statement. Indicate whether the conditional statement p is true or false.

1) p: If it rains today, I will have my umbrella.
It is raining today.
I do not have my umbrella.
- True
- False

2) p: If Sally took too long getting ready, she missed the bus.
Sally did not take too long getting ready.
Sally missed the bus.
- True
- False

3) p: If it is sunny out, I ride my bike.
It is not sunny out.

I am not riding my bike.

- True
- False

There are many ways to express the conditional statement p → q in English:

Table 4.3.2: English expressions of the conditional operation.

Consider the propositions:

p: You mow Mr. Smith's lawn.
q: Mr. Smith will pay you.

If p, then q.	If you mow Mr. Smith's lawn, then he will pay you.
If p, q.	If you mow Mr. Smith's lawn, he will pay you.
q if p	Mr. Smith will pay you if you mow his lawn.
p implies q.	Mowing Mr. Smith's lawn implies that he will pay you.
p only if q.	You will mow Mr. Smith's lawn only if he pays you.
p is sufficient for q.	Mowing Mr. Smith's lawn is sufficient for him to pay you.
q is necessary for p.	Mr. Smith's paying you is necessary for you to mow his lawn.

There is sometimes some confusion about the fact that the statement "p only if q" is the same as the proposition p → q. Both statements mean that the only way for p to be true is if q is also true.

PARTICIPATION ACTIVITY 4.3.2: Conditional proposition from English sentences.

This question uses the following propositions:

p: I will share my cookie with you.
q: You will share your soda with me.

Select the conditional statement that has the same logical meaning as the English sentence given.

1) If you share your soda with me, then I will share my cookie with you.

- $q \rightarrow p$
- $p \rightarrow q$

2) Me sharing my cookie with you is sufficient for you to share your soda with me.

- $q \rightarrow p$
- $p \rightarrow q$

3) I will share my cookie with you only if you share your soda with me.

- $q \rightarrow p$
- $p \rightarrow q$

The converse, contrapositive, and inverse

Three conditional statements related to proposition $p \rightarrow q$ are so common that they have special names. The ***converse*** of $p \rightarrow q$ is $q \rightarrow p$. The ***contrapositive*** of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.The ***inverse*** of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.

Table 4.3.3: The converse, contrapositive, and inverse.

Proposition:	$p \rightarrow q$	Ex: If it is raining today, the game will be cancelled.
Converse:	$q \rightarrow p$	If the game is cancelled, it is raining today.
Contrapositive:	$\neg q \rightarrow \neg p$	If the game is not cancelled, then it is not raining today.
Inverse:	$\neg p \rightarrow \neg q$	If it is not raining today, the game will not be cancelled.

PARTICIPATION ACTIVITY 4.3.3: Converse, contrapositive, and inverse of a conditional proposition.

Consider the conditional statement below:

If he studied for the test, then he passed the test.

Match each statement below to the term describing how it is related to the statement above.

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Inverse **Contrapositive** **Converse**

If he did not pass the test, then he did not study for the test.

If he passed the test, then he studied for the test.

If he did not study for the test, then he did not pass the test.

Reset

The biconditional operation

If p and q are propositions, the proposition "p if and only if q" is expressed with the ***biconditional operation*** and is denoted p ↔ q. The proposition p ↔ q is true when p and q have the same truth value and is false when p and q have different truth values.

Alternative ways of expressing p ↔ q in English include "p is necessary and sufficient for q" or "if p then q, and conversely". The term ***iff*** is an abbreviation of the expression "if and only if", as in "p iff q". The truth table for p ↔ q is given below:

Table 4.3.4: Truth table for the biconditional operation.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

CHALLENGE ACTIVITY 4.3.1: Convert proposition from words to symbols.

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Start

Define the proposition in symbols using:

- p: The weather is bad.
- q: The trip is cancelled.
- r: The trip is delayed.

Proposition in words: If the weather is good, then the trip will not be cancelled.

Proposition in symbols:

1	2	3	4	5

Check Next

Compound propositions with conditional and biconditional operations

The conditional and biconditional operations can be combined with other logical operations, as in $(p \to q) \land r$. If parentheses are not used to explicitly indicate the order in which the operations should be applied, then $\land$, $\lor$, and $\neg$ should be applied before $\to$ or $\leftrightarrow$. Thus, the proposition $p \to q \land r$ should be evaluated as $p \to (q \land r)$. Good practice, however, is to use parentheses so that the order of operations is clear.

PARTICIPATION ACTIVITY | 4.3.4: Example of evaluating a compound proposition with a biconditional.

Animation content:

undefined

Animation captions:

1. The compound proposition, $p \lor \neg(q \leftrightarrow r)$ is evaluated by first filling in the given truth values for p, q and r.
2. The biconditional operation is evaluated first,
3. then the negation and then the disjunction, yielding a truth value for the entire compound proposition.

PARTICIPATION ACTIVITY | 4.3.5: Evaluating compound propositions with conditional and biconditional operations.

Assume the propositions p, q, r, and s have the following truth values:

p is true
q is true
r is false
s is false

What are the truth values for the following compound propositions?

1) $s \rightarrow q$

- True
- False

2) $(r \leftrightarrow s) \land q$

- True
- False

3) $q \rightarrow \neg r$

- True
- False

4) $(q \land s) \rightarrow p$

- True
- False

5) $(p \leftrightarrow r) \land (\neg r \land \neg s)$

- True
- False

6) $q \rightarrow \neg(r \lor q)$

- True
- False

CHALLENGE ACTIVITY 4.3.2: Truth tables for conditional propositions.

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Start

Fill in $p \rightarrow \neg q$.

p	q	$p \rightarrow \neg q$
T	T	
T	F	
F	T	
F	F	

1 2 3 4 5

Check Next

Example 4.3.1: Automatic degree requirements check.

Large universities with thousands of students usually have an automated system for checking whether a student has satisfied the requirements for a particular degree before graduation. Degree requirements can be expressed in the language of logic so that they can be checked by a computer program. For example, let X be the proposition that the student has taken course X. For a degree in Computer Science, a student must take one of three project courses, P1, P2, or P3. The student must also take one of two theory courses, T1 or T2. Furthermore, if the student is an honors student, he or she must take the honors seminar S. Let H be the proposition indicating whether the student is an honors student. We can express these requirements with the following proposition:

$$(P1 \lor P2 \lor P3) \land (T1 \lor T2) \land (H \to S)$$

Additional exercises

EXERCISE 4.3.1: Truth values for conditional statements in English.

Which of the following conditional statements are true and why?

(a) If February has 30 days, then 7 is an odd number.

(b) If January has 31 days, then 7 is an even number.

(c) If 7 is an odd number, then February does not have 30 days.

(d) If 7 is an even number, then January has exactly 28 days.

EXERCISE 4.3.2: The inverse, converse, and contrapositive of conditional sentences in English.

Give the inverse, contrapositive, and converse for each of the following statements:

(a) If she finished her homework, then she went to the party.

(b) If he trained for the race, then he finished the race.

(c) If the patient took the medicine, then she had side effects.

(d) If it was sunny, then the game was held.

(e) If it snowed last night, then school will be cancelled.

EXERCISE 4.3.3: Truth values for the inverse, contrapositive, and converse of a conditional statement.

State the inverse, contrapositive, and converse of each conditional statement. Then indicate whether the inverse, contrapositive, and converse are true.

(a) If 3 is a prime number then 5 is an even number.

(b) If $7 < 5$, then $5 < 3$.

(c) If 5 is a negative number, then 3 is a positive number.

EXERCISE 4.3.4: Truth tables for logical expressions with conditional operations.

Give a truth table for each expression.

(a) $(\neg p \land q) \to p$

(b) $(p \to q) \to (q \to p)$

(c) $(p \lor q) \leftrightarrow (q \to \neg p)$

(d) $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$

(e) $(p \lor q) \leftrightarrow (q \land p)$

EXERCISE 4.3.5: Expressing conditional statements in English using logic.

Define the following propositions:

c: I will return to college.
j: I will get a job.

Translate the following English sentences into logical expressions using the definitions above:

(a) Not getting a job is a sufficient condition for me to return to college.

(b) If I return to college, then I won't get a job.

(c) I am not getting a job, but I am still not returning to college.

(d) I will return to college only if I won't get a job.

(e) There's no way I am returning to college.

(f) I will get a job and return to college.

EXERCISE 4.3.6: Expressing English sentences in if-then form.

Give an English sentence in the form "If...then...." that is equivalent to each sentence.

(a) Maintaining a B average is sufficient for Joe to be eligible for the honors program.

(b) Maintaining a B average is necessary for Joe to be eligible for the honors program.

(c) Rajiv can go on the roller coaster only if he is at least four feet tall.

(d) Rajiv can go on the roller coaster if he is at least four feet tall.

EXERCISE | 4.3.7: Expressing conditional statements in English using logic.

Define the following propositions:

s: a person is a senior
y: a person is at least 17 years of age
p: a person is allowed to park in the school parking lot

Express each of the following English sentences with a logical expression:

(a) A person is allowed to park in the school parking lot only if they are a senior and at least seventeen years of age.

(b) A person can park in the school parking lot if they are a senior or at least seventeen years of age.

(c) Being 17 years of age is a necessary condition for being able to park in the school parking lot.

(d) A person can park in the school parking lot if and only if the person is a senior and at least 17 years of age.

(e) Being able to park in the school parking lot implies that the person is either a senior or at least 17 years old.

EXERCISE 4.3.8: Translating logical expressions into English.

Define the following propositions:

w: the roads were wet
a: there was an accident
h: traffic was heavy

Express each of the logical expressions as an English sentence:

(a) $w \to h$

(b) $w \land a$

(c) $\neg(a \land h)$

(d) $h \to (a \lor w)$

(e) $w \land \neg h$

EXERCISE 4.3.9: Translating English propositions into logical expressions.

Use the definitions of the variables below to translate each English statement into an equivalent logical expression.

- y: the applicant is at least eighteen years old
- p: the applicant has parental permission
- c: the applicant can enroll in the course

(a) The applicant is not eighteen years old but does have parental permission.

(b) If the applicant is at least eighteen years old or has parental permission, then the applicant can enroll in the course.

(c) The applicant can enroll in the course only if the applicant has parental permission.

(d) Having parental permission is a necessary condition for enrolling in the course.

EXERCISE 4.3.10: Determining if a truth value of a compound expression is known given a partial truth assignment.

The variable p is true, q is false, and the truth value for variable r is unknown. Indicate whether the truth value of each logical expression is true, false, or unknown.

(a) $p \to (q \land r)$

(b) $(p \lor r) \to r$

(c) $(p \lor r) \leftrightarrow (q \land r)$

(d) $(p \land r) \leftrightarrow (q \land r)$

(e) $p \to (r \lor q)$

(f) $(p \land q) \to r$

EXERCISE 4.3.11: Finding logical expressions to match a truth table.

For each table, give a logical expression whose truth table is the same as the one given.

(a)

p	q	?
T	T	F
T	F	T
F	T	F
F	F	F

(b)

p	q	?
T	T	F
T	F	T
F	T	T
F	F	F

4.4 Logical equivalence

A compound proposition is a ***tautology*** if the proposition is always true, regardless of the truth value of the individual propositions that occur in it. A compound proposition is a ***contradiction*** if the proposition is always false, regardless of the truth value of the individual propositions that occur in it. The proposition p ∨ ¬p is a simple example of a tautology since the proposition is always true whether p is true or false. The fact that p ∨ ¬p is a tautology can be verified in a truth table, which shows that every truth value in the rightmost column is true.

Table 4.4.1: Truth table for tautology p ∨ ¬p.

p	¬p	p ∨ ¬p
T	F	T
F	T	T

Similarly, the proposition p ∧ ¬p is an example of a simple contradiction, because the proposition is false regardless of whether p is true or false. The truth table below shows that p ∧ ¬p is a contradiction because every truth value in the rightmost column is false.

Table 4.4.2: Truth table for contradiction p ∧ ¬p.

p	¬p	p ∧ ¬p
T	F	F
F	T	F

Showing that a compound proposition is not a tautology only requires showing a particular set of truth values for its individual propositions that cause the compound proposition to evaluate to false. For example, the proposition (p ∧ q) → r is not a tautology because when p = q = T and r = F, then (p ∧ q) → r is false. Showing that a compound proposition is not a contradiction only requires showing a particular set of truth values for its individual propositions that cause the compound proposition to evaluate to true. For example, the proposition ¬(p ∨ q) is not a contradiction because when p = q = F, then ¬(p ∨ q) is true.

PARTICIPATION ACTIVITY 4.4.1: Identifying tautologies and contradictions.

Determine whether the following compound propositions are tautologies, contradictions, or neither.

1) p ↔ ¬p

- Tautology
- Contradiction
- Neither a tautology or a contradiction

2) p → ¬p

- Tautology
- Contradiction
- Neither a tautology or a contradiction

3) (p ∧ q) → p

- Tautology
- Contradiction.
- Neither a tautology or a contradiction.

Showing logical equivalence using truth tables

Two compound propositions are said to be ***logically equivalent*** if they have the same truth value regardless of the truth values of their individual propositions. If s and r are two compound propositions, the notation s ≡ r is used to indicate that r and s are logically equivalent. For example, p and ¬¬p have the same truth value regardless of whether p is true or false, so p ≡ ¬¬p. Propositions s and r are logically equivalent if and only if the proposition s ↔ r is a tautology. Note that s ≡ r if and only if r ≡ s.

A truth table can be used to show that two compound propositions are logically equivalent.

PARTICIPATION ACTIVITY 4.4.2: Showing p → ¬p ≡ ¬p with a truth table.

Animation captions:

1. $p \rightarrow \neg p$ can be shown to be equivalent to $\neg p$ by filling in a column for $\neg p$,
2. then filling in a column for $p \rightarrow \neg p$, and verifying that the two columns are the same.

Table 4.4.3: Truth table to show: ¬p ∨ ¬q ≡ ¬(p ∧ q).

p	q	¬p	¬q	p ∧ q	¬(p ∧ q)	¬p ∨ ¬q
T	T	F	F	T	**F**	**F**
T	F	F	T	F	**T**	**T**
F	T	T	F	F	**T**	**T**
F	F	T	T	F	**T**	**T**

Showing that two propositions are not logically equivalent only requires showing a particular set of truth values for their individual propositions that cause the two compound proposition to have different truth values. For example, p ↔ r and p → r are not logically equivalent because when p = F and r = T, then p ↔ r is false but p → r is true.

PARTICIPATION ACTIVITY 4.4.3: Logical equivalence by truth table.

The table below shows the truth table for three compound propositions:

- (p ∨ ¬q) → r
- (p ↔ q) → r
- ¬r → (¬p ∧ q)

p	q	r	(p ∨ ¬q) → r	(p ↔ q) → r	¬r → (¬p ∧ q)
T	T	T	T	T	T
T	T	F	F	F	F
T	F	T	T	T	T

T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	F	F	F

1) Use the truth table to determine which logical equivalence is true.

- $(p \lor \neg q) \to r \equiv (p \leftrightarrow q) \to r$
- $(p \leftrightarrow q) \to r \equiv \neg r \to (\neg p \land q)$
- $(p \lor \neg q) \to r \equiv \neg r \to (\neg p \land q)$

De Morgan's laws

De Morgan's laws are logical equivalences that show how to correctly distribute a negation operation inside a parenthesized expression. Both versions of De Morgan's laws are particularly useful in logical reasoning. The first De Morgan's law is:

$$\neg(p \lor q) \equiv (\neg p \land \neg q)$$

When the negation operation is distributed inside the parentheses, the disjunction operation changes to a conjunction operation. Consider an English example with the following propositions for p and q.

p: The patient has migraines
q: The patient has high blood pressure

The use of the English word "or" throughout the example is assumed to be disjunction (i.e., the inclusive or). De Morgan's law says that the following two English statements are logically equivalent:

It is not true that the patient has migraines or high blood pressure.
The patient does not have migraines and does not have high blood pressure.

The logical equivalence $\neg(p \lor q) \equiv (\neg p \land \neg q)$ can be verified using a truth table. Alternatively, reasoning about which truth assignments cause the expressions $\neg(p \lor q)$ and $(\neg p \land \neg q)$ to evaluate to true provides intuition about why the two expressions are logically equivalent.

PARTICIPATION ACTIVITY | 4.4.4: Reasoning about De Morgan's laws.

1) There is only one truth assignment for p and q that makes the expression ¬(p ∨ q) evaluate to true. Which one is it?

- p = q = T
- p = T and q = F
- p = q = F

2) There is only one truth assignment for p and q that makes the expression ¬p ∧ ¬q evaluate to true. Which one is it?

- p = q = T
- p = T and q = F
- p = q = F

The second version of De Morgan's law swaps the role of the disjunction and conjunction:

$$\neg(p \land q) \equiv (\neg p \lor \neg q)$$

Continuing with the same example, the following two statements are logically equivalent:

It is not true that the patient has migraines and high blood pressure.
The patient does not have migraines or does not have high blood pressure.

The logical equivalence ¬(p ∧ q) ≡ (¬p ∨ ¬q) can be verified using a truth table. Alternatively, reasoning about which truth assignments cause the expressions ¬(p ∧ q) and (¬p ∨ ¬q) to evaluate to false provides intuition about why the two expressions are logically equivalent.

PARTICIPATION ACTIVITY | 4.4.5: Reasoning about De Morgan's laws.

1) There is only one truth assignment for p and q that makes the expression ¬(p ∧ q) evaluate to false. Which one is it?

p = q = T
p = T and q = F
p = q = F

2) There is only one truth assignment for p and q that makes the expression ¬p ∨ ¬q evaluate to false. Which one is it?

- p = q = T
- p = T and q = F
- p = q = F

PARTICIPATION ACTIVITY | 4.4.6: Matching equivalent English expressions using De Morgan's laws.

Select the English sentence that is logically equivalent to the given sentence.

1) It is not true that the child is at least 8 years old and at least 57 inches tall.

- The child is at least 8 years old and at least 57 inches tall.
- The child is less than 8 years old or shorter than 57 inches.
- The child is less than 8 years old and shorter than 57 inches.

2) It is not true that the child is at least 8 years old or at least 57 inches tall.

- The child is at least 8 years old or at least 57 inches tall.
- The child is less than 8 years old or shorter than 57 inches.
- The child is less than 8 years old and shorter than 57 inches.

CHALLENGE ACTIVITY | 4.4.1: Tautologies, contradictions, and logical equivalence.

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Start

Fill out the truth table for the compound proposition shown in the table.

p	q	$p \rightarrow (p \vee q)$
T	T	
T	F	
F	T	
F	F	

1	2	3	4	5

Check Next

Additional exercises

EXERCISE 4.4.1: Proving tautologies and contradictions.

Show whether each logical expression is a tautology, contradiction or neither.

(a) $(p \vee q) \vee (q \rightarrow p)$

(b) $(p \rightarrow q) \leftrightarrow (p \wedge \neg q)$

(c) $(p \rightarrow q) \leftrightarrow p$

(d) $(p \rightarrow q) \vee p$

(e) $(\neg p \vee q) \leftrightarrow (p \wedge \neg q)$

(f) $(\neg p \vee q) \leftrightarrow (\neg p \wedge q)$

EXERCISE 4.4.2: Truth tables to prove logical equivalence.

Use truth tables to show that the following pairs of expressions are logically equivalent.

(a) $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$

(b) $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$

(c) $\neg p \rightarrow q$ and $p \vee q$

EXERCISE 4.4.3: Proving two logical expressions are not logically equivalent.

Prove that the following pairs of expressions are not logically equivalent.

(a) $p \rightarrow q$ and $q \rightarrow p$

(b) $\neg p \rightarrow q$ and $\neg p \vee q$

(c) $(p \rightarrow q) \wedge (r \rightarrow q)$ and $(p \wedge r) \rightarrow q$

(d) $p \wedge (p \rightarrow q)$ and $p \vee q$

EXERCISE 4.4.4: Proving whether two logical expressions are equivalent.

Determine whether the following pairs of expressions are logically equivalent. Prove your answer. If the pair is logically equivalent, then use a truth table to prove your answer.

(a) $\neg(p \vee \neg q)$ and $\neg p \wedge q$

(b) $\neg(p \vee \neg q)$ and $\neg p \wedge \neg q$

(c) $p \wedge (p \rightarrow q)$ and $p \rightarrow q$

(d) $p \wedge (p \rightarrow q)$ and $p \wedge q$

EXERCISE 4.4.5: Logical equivalence of two English statements.

Define the following propositions:

- j: Sally got the job.
- l: Sally was late for her interview
- r: Sally updated her resume.

Express each pair of sentences using logical expressions. Then prove whether the two expressions are logically equivalent.

(a) If Sally did not get the job, then she was late for her interview or did not update her resume.
If Sally updated her resume and did not get the job, then she was late for her interview.

(b) If Sally did not get the job, then she was late for her interview or did not update her resume.
If Sally updated her resume and was not late for her interview, then she got the job.

(c) If Sally got the job then she was not late for her interview.
If Sally did not get the job, then she was late for her interview.

(d) If Sally updated her resume or she was not late for her interview, then she got the job.
If Sally got the job, then she updated her resume and was not late for her interview.

EXERCISE 4.4.6: Applying De Morgan's laws.

Translate each English sentence into a logical expression using the propositional variables defined below. Then negate the entire logical expression using parentheses and the negation operation. Apply De Morgan's law to the resulting expression and translate the final logical expression back into English.

- p: the applicant has written permission from his parents
- e: the applicant is at least 18 years old
- s: the applicant is at least 16 years old

(a) The applicant has written permission from his parents and is at least 16 years old.

(b) The applicant has written permission from his parents or is at least 18 years old.

4.5 Laws of propositional logic

If two propositions are logically equivalent, then one can be substituted for the other within a more complex proposition. The compound proposition after the substitution is logically equivalent to the compound proposition before the substitution.

For example $p \to q \equiv \neg p \lor q$. Therefore,

$$(p \lor r) \land (\mathbf{\neg p \lor q}) \equiv (p \lor r) \land (\mathbf{p \to q})$$

In the next example, the logical equivalence $p \to q \equiv \neg p \lor q$ is applied where the variables p and q represent compound propositions:

$$(\neg t \land r) \to (\neg s \lor t) \equiv \neg(\neg t \land r) \lor (\neg s \lor t)$$

PARTICIPATION ACTIVITY | 4.5.1: Substituting logically equivalent propositions.

Use the logical equivalence $\neg(p \lor q) \equiv \neg p \land \neg q$ to match logically equivalent propositions below:

If unable to drag and drop, refresh the page.

$(s \land t) \lor \neg(t \lor r)$ **$(\neg s \land \neg t) \lor (t \lor r)$** **$\neg((s \land t) \lor (t \lor r))$**

$(s \land t) \lor (\neg t \land \neg r)$

$\neg(s \land t) \land \neg(t \lor r)$

$\neg(s \lor t) \lor (t \lor r)$

Reset

Using the laws of propositional logic to show logical equivalence

Substitution gives an alternate way of showing that two propositions are logically equivalent. If one proposition can be obtained from another by a series of substitutions using equivalent expressions, then the two propositions are logically equivalent. The table below shows several laws of propositional logic that are particularly useful for establishing the logical equivalence of compound propositions:

Table 4.5.1: Laws of propositional logic.

Idempotent laws:	p ∨ p ≡ p	p ∧ p ≡ p
Associative laws:	(p ∨ q) ∨ r ≡ p ∨ (q ∨ r)	(p ∧ q) ∧ r ≡ p ∧ (q ∧ r)
Commutative laws:	p ∨ q ≡ q ∨ p	p ∧ q ≡ q ∧ p
Distributive laws:	p ∨ (q ∧ r) ≡ (p ∨ q) ∧ (p ∨ r)	p ∧ (q ∨ r) ≡ (p ∧ q) ∨ (p ∧ r)
Identity laws:	p ∨ F ≡ p	p ∧ T ≡ p
Domination laws:	p ∧ F ≡ F	p ∨ T ≡ T
Double negation law:	¬¬p ≡ p	
Complement laws:	p ∧ ¬p ≡ F ¬T ≡ F	p ∨ ¬p ≡ T ¬F ≡ T
De Morgan's laws:	¬(p ∨ q) ≡ ¬p ∧ ¬q	¬(p ∧ q) ≡ ¬p ∨ ¬q
Absorption laws:	p ∨ (p ∧ q) ≡ p	p ∧ (p ∨ q) ≡ p
Conditional identities:	p → q ≡ ¬p ∨ q	p ↔ q ≡ (p → q) ∧ (q → p)

PARTICIPATION ACTIVITY 4.5.2: The laws of propositional logic can be used to show logical equivalence.

Animation content:

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Animation captions:

1. The proof that p → q is equivalent to¬q→ ¬p starts with p → q and applies the Conditional identity to get ¬p ∨ q.
2. The Commutative law is applied to get q ∨ ¬p.
3. The Double negation law is applied to get ¬¬q ∨ ¬p.
4. Finally, the Conditional identity is applied again to get ¬q → ¬p.

PARTICIPATION ACTIVITY | 4.5.3: The laws of propositional logic can be used to simplify compound propositions.

Animation content:

undefined

Animation captions:

1. The proof that ¬(p ∨ q) ∨ (¬p ∧ q) is equivalent to ¬p starts with ¬(p ∨ q) ∨ (¬p ∧ q) and applies De Morgan's law to get (¬p ∧ ¬q) ∨ (¬p ∧ q).
2. The Distributive law is applied to get ¬p ∧ (¬q ∨ q) .
3. The Commutative law is applied to get ¬p ∧ (q ∨ ¬q).
4. The Complement law is applied to get ¬p ∧ T.
5. Finally, the Identity law is applied again to get ¬p.

PARTICIPATION ACTIVITY | 4.5.4: Using the laws of propositional logic to show logical equivalence.

Put the steps in the correct order to show that ¬(p → q) ≡ p ∧ ¬q. Each step should follow from the previous step using the given law.

If unable to drag and drop, refresh the page.

¬(¬p ∨ q) Conditional identity **p ∧ ¬q Double negation law**

¬¬p ∧ ¬q De Morgan's Law **¬(p → q)**

1

2

3

4

Reset

CHALLENGE ACTIVITY | 4.5.1: Reduce the proposition using laws.

Need help with this tool?

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CHALLENGE ACTIVITY | 4.5.2: Reduce the proposition using laws, including de Morgan's and conditional.

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CHALLENGE ACTIVITY | 4.5.3: Expand then reduce the proposition.

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Additional exercises

EXERCISE 4.5.1: Label the steps in a proof of logical equivalence.

Below are several proofs showing that two logical expressions are logically equivalent. Label the steps in each proof with the law used to obtain each proposition from the previous proposition. The first line in the proof does not have a label.

(a)

$(p \rightarrow q) \land (q \lor p)$
$(\neg p \lor q) \land (q \lor p)$
$(q \lor \neg p) \land (q \lor p)$
$q \lor (\neg p \land p)$
$q \lor (p \land \neg p)$
$q \lor F$
q

(b)

$(\neg p \lor q) \rightarrow (p \land q)$
$\neg(\neg p \lor q) \lor (p \land q)$
$(\neg\neg p \land \neg q) \lor (p \land q)$
$(p \land \neg q) \lor (p \land q)$
$p \land (\neg q \lor q)$
$p \land T$
p

(c)

$r \lor (\neg r \rightarrow p)$
$r \lor (\neg\neg r \lor p)$
$r \lor (r \lor p)$
$(r \lor r) \lor p$
$r \lor p$

EXERCISE 4.5.2: Using the laws of logic to prove logical equivalence.

Use the laws of propositional logic to prove the following:

(a) $\neg p \rightarrow \neg q \equiv q \rightarrow p$

(b) $p \wedge (\neg p \rightarrow q) \equiv p$

(c) $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

(d) $\neg p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \vee r)$

(e) $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

(f) $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

(g) $(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \equiv p \wedge \neg r$

(h) $p \leftrightarrow (p \wedge r) \equiv \neg p \vee r$

(i) $(p \wedge q) \rightarrow r \equiv (p \wedge \neg r) \rightarrow \neg q$

EXERCISE 4.5.3: Using the laws of logic to prove tautologies.

Use the laws of propositional logic to prove that each statement is a tautology.

(a) $(p \wedge q) \rightarrow (p \vee r)$

(b) $p \rightarrow (r \rightarrow p)$

(c) $\neg r \vee (\neg r \rightarrow p)$

(d) $\neg(p \rightarrow q) \rightarrow \neg q$

(e) $\neg p \rightarrow (p \rightarrow q)$

EXERCISE | 4.5.4: Logical relationships between the inverse, converse, and contrapositive.

Use the laws of propositional logic to prove each of the following assertions. Start by defining a generic conditional statement $p \to q$, and then restate the assertion as the equivalence or non-equivalence of two propositions using p and q. Finally prove that the two propositions are equivalent or non-equivalent.

For example, the statement: "A conditional statement is not logically equivalent to its converse" is proven by showing that that $p \to q$ is not logically equivalent to $q \to p$.

(a) A conditional statement is not logically equivalent to its converse.

(b) A conditional statement is not logically equivalent to its inverse.

(c) A conditional statement is logically equivalent to its contrapositive.

(d) The converse and inverse of a conditional statement are logically equivalent.

EXERCISE | 4.5.5: Logical equivalence of two mathematical statements.

(a) Show that the two sentences below are logically equivalent. Express each pair of sentences using a logical expression. Then prove whether the two expressions are logically equivalent. Note: you can assume that x and y are real numbers, and real numbers are either rational or irrational. So, if x is not irrational, then x is rational, and if x is not rational, then x is an irrational number.
- If x is a rational number and y is an irrational number then x-y is an irrational number.
- If x is a rational number and x-y is a rational number then y is a rational number.

4.6 Boolean functions and truth-table inputs

For this assignment you will implement a number of Boolean functions, such as implies, nand, etc.

In this assignment you will implement several Boolean functions that will later enable you to compute their truth tables. In addition, you will implement a function that creates all combinations of Boolean values for n variables.

a) **Boolean functions**

You are provided with a set of Boolean functions to implement. You will create the bodies for these

2 parameter functions, such that they return the appropriate Boolean value (True or False). The formal parameters are always p and q (in that order). The code provides the specification of the Boolean functions that need to be implemented.

b) **Truth-table inputs**

The function `make_tt_ins(n)` will create all combination of Boolean values for n variables, which is the input that needs to be provided for a truth table with n variables.

For example `make_tt_ins(2)` needs to return

```
[[False, False], [False, True], [True, False], [True, True]]
```

Note that we do not care about the order, so

```
[[True, True], [False, False], [False, True], [True, False]]
```

is also valid output. Since the input for a truth table has 2^n entries, the output for this function is a list with 2^n elements, each of which is a list of n Boolean values.

We suggest to solve this problem recursively: use the list generated by make*tt*ins(n-1) and update it accordingly by appending True/False to each row. For the base case (n==1) you should return `[[False], [True]]`

Two functions are provided: `run(f)` and `main`, so that you can test your code.

```
  python3 PA2.py <fName> tests your Boolean function <fName>
  python3 PA2.py tt <n> tests your function make_tt_ins(<n>)
```

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LAB ACTIVITY | 4.6.1: Boolean functions and truth-table inputs 0 / 100

Downloadable files

`PA2.py` **Download**

PA2.py

Load default template...

```
'''
    PA2: Boolean functions and truth-table inputs
    -_-_-_-_-_-_-_-_-_-_-_-_-_-_-_-_-_-_-_-_-_-_-

For PA2 you will implememnt a number of Boolean functions,
such as implies, nand, etc.

a) Boolean functions
You are provided with a set of Boolean functions to implement.
```

Develop mode **Submit mode**

Run your program as often as you'd like, before submitting for grading. Below, type any needed input values in the first box, then click **Run program** and observe the program's output in the second box.

Enter program input (optional)

If your code requires input values, provide them here.

Run command

`python3 PA2.py` Additional arguments

Run program

Input (from above) → **PA2.py** (Your program) → 0

Program output displayed here

Coding trail of your work What is this?

```
History of your effort will appear here once you begin
working on this zyLab.
```