12.1 Introduction to binary relations

A binary relation is a way of expressing a relationship between two sets. Mathematically, a ***binary relation*** between two sets A and B is a subset R of $A \times B$. The term *binary* refers to the fact that the relation is a subset of the Cartesian product of *two* sets. The two sets A and B may or may not be equal. For $a \in A$ and $b \in B$, the fact that $(a, b) \in R$ is denoted by aRb.

Suppose, for example, that S is the set of students at a university and C is the set of classes offered by the university. The relation E between $s \in S$ and $c \in C$ indicates whether a student is enrolled in a given class. Thus, sEc if student s is enrolled in class c. A student can be enrolled in more than one class and a class can have more than one student enrolled in it. If a student s is not currently taking any classes, then there is no c such that sEc. Similarly, if a course c is not currently offered by the university, then there is no s such that sEc.

Relations can also be defined on infinite sets. For example, we can define the relation C between $\mathbb{R}$ and $\mathbb{Z}$ to be:

$$xCy \text{ if } |x - y| \leq 1$$

That is, xCy if the distance between real number x and integer y is at most 1.

If two sets, A and B, are finite, then a binary relation R between A and B can be represented by a list of ordered pairs. A relation can also be specified in a more graphical way. In an ***arrow diagram*** of a relation R on sets A and B, the elements of A are listed on the left, the elements of B are listed on the right, and there is an arrow from $a \in A$ to $b \in B$ if aRb.

PARTICIPATION ACTIVITY | 12.1.1: Arrow diagram for a relation.

Animation captions:

1. A relation A is defined on a set of people and a set of files. Person p is related to file f under A (pAf) if person p has access to file f.
2. An arrow diagram for A lists the people in a column on the left and the files in a column on the right.
3. Each pair (p, f) in the relation is represented by an arrow pointing from p to f.

A ***matrix representation*** of relation R between A and B is a rectangular array of numbers with |A| rows and |B| columns. Each row corresponds to an element of A and each column corresponds to an element of B. For a ∈ A and b ∈ B, there is a 1 in row a, column b, if aRb. Otherwise, there is a 0.

PARTICIPATION ACTIVITY | 12.1.2: Matrix representation for a relation.

Animation captions:

1. A matrix representation for A has a row for each person and a column for each file. (Sue, File B) is represented by a 1 in the row for Sue and the column for File B.
2. Similarly, there is a 1 in the matrix for every pair in the relation.
3. All other entries in the matrix are 0.

PARTICIPATION ACTIVITY | 12.1.3: Binary relations between two sets.

Let A = { 2, 3 }. Define the relation R between the set A and $\mathbf{Z}^+$, the set of positive integers:

aRx if x is an integer multiple of a.

1) 3R8
- True
- False

2) 4R8
- True
- False

3) There is an a ∈ A, such that aR7.
- True
- False

4) There is an x ∈ $\mathbf{Z}^+$, such that 2Rx and 3Rx.
- True
- False

5) 2R2
- True
- False

Example 12.1.1: Binary relations to represent resource requirements.

Consider a manufacturing plant that must satisfy an incoming stream of orders. Different orders require different machines in the plant. The requirements for orders can be represented by a relation between the set of all orders and the set of machines in the plant. Order o is related to machine m if o requires m. Representing the requirements as a relation would be a first step in finding an efficient schedule for the orders so that machines are not needed for more than one order at the same time.

It is possible to have a relation between two sets A and B in which A = B. A ***binary relation on a set*** A is a subset of A x A. The set A is called the ***domain*** of the binary relation.

PARTICIPATION ACTIVITY 12.1.4: Binary relations on a set.

Define the relation R on $\mathbf{Z}^+$, the set of positive integers, as follows:

aRb if a and b are relatively prime (that is, the only positive integer that evenly divides both a and b is 1).

1) 8R12
   - True
   - False

2) 21R20.
   - True
   - False

3) There is a number such that x ≠ 1 and xRx.
   - True
   - False

4) There is a number x such that xR7.
   - True
   - False

An arrow diagram for a relation R on a finite set A requires only one copy of the elements of A.

There is an arrow from a to b if aRb. An element can have arrows heading into it and arrows that head out of it. An element that is related to itself is indicated by an arrow called a ***self-loop***. A self-loop leaves the element and then turns around to point to itself again.

PARTICIPATION ACTIVITY 12.1.5: Arrow diagram for a relation on a finite set.

Animation content:

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Animation captions:

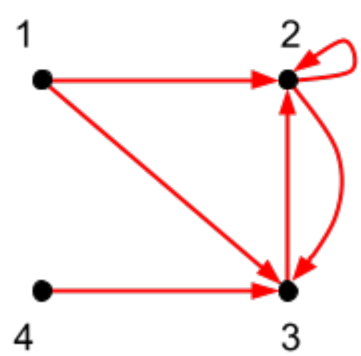
1. A pair (a,b) in the relation is represented by an arrow from a to b.
2. Every pair in the relation is represented by an arrow. The pair (d,d) is represented by a self-loop at d.

PARTICIPATION ACTIVITY 12.1.6: Representations of binary relations on a set.

Let A = {1, 2, 3, 4}. Define a relation R on A:

$$R = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 4), (4, 3)\}$$

An arrow diagram and matrix representation of the relation are supposed to be represented below, *but there are some mistakes*. Rows of the matrix are numbered 1 through 4 from top to bottom and columns are numbered 1 through 4 from left to right.



$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

1) Which pair is in the arrow diagram but not the original R? Give your answer in the form (a, b).

Check Show answer

2) Which pair is in the original R but not represented in the arrow diagram? Give your answer in the form (a, b).

Check Show answer

3) Which entry in the matrix is a 0 but should be a 1? Give your answer in the form (row-number, column-number). Note that the correct relation is given by the set R.

Check Show answer

4) Which entry in the matrix is a 1 but should be a 0? Give your answer in the form (row-number, column-number). Note that the correct relation is given by the set R.

Check Show answer

Additional exercises

EXERCISE 12.1.1: Matrices to arrow diagrams and sets.

Each matrix below represents a relation. The rows and columns are numbered 1 through 3 or 4.

Give the arrow diagram for each matrix, then express each relation as a set of ordered pairs.

(a) $$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(c) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

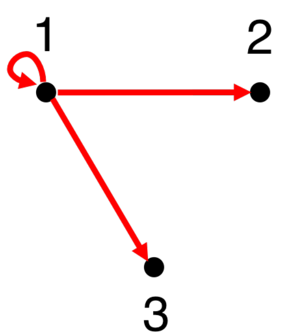
(e) $$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

(f) $$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

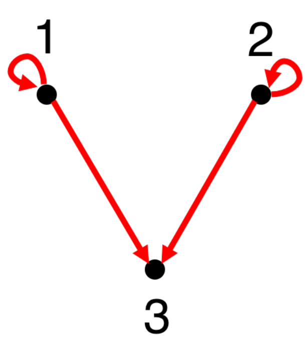
EXERCISE | 12.1.2: Arrow diagrams to matrices and sets.

Give the matrix representation for the relation depicted in each arrow diagram. Then express the relation as a set of ordered pairs.

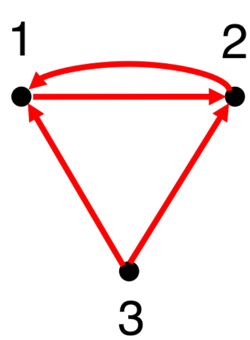
(a)



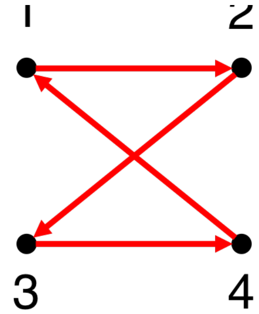
(b)



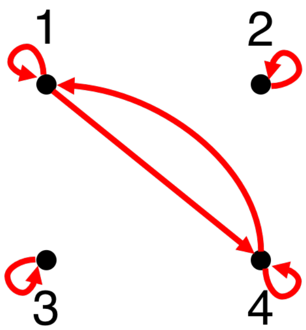
(c)



(d)



(e)



(f)

EXERCISE | 12.1.3: Matrices and arrow diagrams for relations expressed as sets of pairs.

Draw the arrow diagram and the matrix representation for each relation.

(a) Define the set A = {r, o, t, p, c} and B = {proposition, math, proof, discrete}. Define the relation R ⊆ A × B such that (letter, word) is in the relation if that letter occurs somewhere in the word.

(b) The domain for relation R is {1, 2, 3, 4}. R = { (1, 2), (3, 4), (2, 3), (3, 2), (2, 1), (3, 1), (4, 3) }.

(c) The domain for relation R is {1, 2, 3, 4}. R = { (1, 2), (2, 1), (3, 3) }.

(d) The domain for relation R is {1, 2, 3}. R = ∅.

EXERCISE 12.1.4: Arrow diagrams for relations on small finite sets.

Draw the arrow diagram for each relation.

(a) The domain of relation C is $\{0, 1\}^3$. For x, y $\in \{0, 1\}^3$, xCy if y can be obtained from x by changing only one 0 to a 1.

(b) The domain of relation P is {2, 4, 8, 16, 32, 64}. For x, y in the domain, xPy if there is a positive integer n such that $x^n = y$.

(c) The domain of relation D is {2, 3, 12, 16, 27, 48}. For x, y in the domain, xDy if y is an integer multiple of x.

(d) The domain for the relation A is the set {2, 5, 7, 8, 11}. For x, y in the domain, xAy if |x-y| is less than 2

(e) The domain for the relation P is the set {2, 4, 8, 10, 16, 64}. For x, y in the domain, xPy if there is a positive integer n such that $x^n = y$

(f) The domain of relation H is a group of four friends. For x, y in the domain, xHy if y is at least as tall as x. The table below shows each person in the domain and her height.

Name	Height
Angie	5'0"
Bernice	5'3"
Carmen	5'3"
Deirdre	5'5"

(g) The domain of relation H is a group of four friends. For x, y in the domain, xHy if y is taller than x. The table below shows each person in the domain and her height.

Name	Height
Angie	5'0"
Bernice	5'3"
Carmen	5'3"
Deirdre	5'5"

12.2 Properties of binary relations

Reflexive and anti-reflexive binary relations

Suppose that R is a binary relation on set A. R is ***reflexive*** if and only if for every x ∈ A, xRx.

Notice that the definition of reflexive is a universal statement. In order for a binary relation to be reflexive, every element in the set must be related to itself. In order to show that a relation is not reflexive, it is only necessary to show that there is a particular x ∈ A such that xRx is not true.

Suppose that the domain for a relation is a set of people. Person x is related to person y if x has the same biological mother as person y. This relation is reflexive because every person must have the same biological mother as himself or herself.

R is ***anti-reflexive*** if and only if for every x in the domain of R, it is not true that xRx. Irreflexive is an alternative term for anti-reflexive.

The definition of anti-reflexive is also a universal statement. In order for a binary relation to be anti-reflexive every element in the set must not be related to itself. In order to show that a relation is not anti-reflexive, it is only necessary to show that there is a particular x ∈ A such that xRx is true.

Suppose that the domain for a relation is a set of people. Person x is related to person y if x is taller than y. This relation is anti-reflexive because no person can be taller than himself or herself.

If some of the elements in the domain of R are related to themselves and some of the elements are not related to themselves, then R is neither reflexive nor anti-reflexive.

PARTICIPATION ACTIVITY | 12.2.1: Reflexive and anti-reflexive binary relations.

Animation content:

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Animation captions:

1. A relation R on set A is reflexive if every element in A has a self-loop in the arrow diagram for A.
2. A relation R on set A is anti-reflexive if no element in A has a self-loop in the arrow diagram for A.
3. R is not anti-reflexive because aRa and bRb. R is not reflexive because it is not true that cRc, dRd, or eRe.

PARTICIPATION ACTIVITY | 12.2.2: Identifying reflexive and anti-reflexive binary relations.

The domain for each relation described below is the set of all positive real numbers. Select the correct description of the relations.

1) x is related to y if y = x+1.

- Reflexive
- Anti-reflexive
- Neither

2) x is related to y if y = 1/x.

- Reflexive
- Anti-reflexive
- Neither

3) x is related to y if $\lfloor x \rfloor \leq \lfloor y \rfloor$.

- Reflexive
- Anti-reflexive
- Neither

Symmetric binary relations

Suppose that R is a relation on set A. R is ***symmetric*** if and only if for every pair, x and y ∈ A, xRy if and only if yRx.

A relation is symmetric if for every pair of elements x and y in the domain, one of the following situations holds:

- xRy and yRx are both true
- Neither xRy nor yRx is true

The situation that is not allowed in a symmetric relation is for there to be a pair, x and y, such that x is related to y but y is not related to x.

Notice that the definition of symmetric is a universal statement. The criteria is that for every pair, the two elements in the pair are both related to each other or both not related to each other. If there is any pair of elements, x and y, where x is related to y and y is not related to x, then the relation is not symmetric.

Suppose that the domain for a relation is a set of people. Person x is related to person y if x has the same biological mother as person y. This relation is symmetric because x has the same biological mother as y if and only if y has the same biological mother as x. The situation that never happens is that x has the same biological mother as y but y does not have the same biological mother as x.

PARTICIPATION ACTIVITY | 12.2.3: Symmetric binary relations.

Animation content:

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Animation captions:

1. A relation R on set A is symmetric if an arrow from x to y implies that there is an arrow from y to x.
2. The allowed patterns for a pair $a \neq b$ in a symmetric relation is to have aRb and bRa or to have neither aRb nor bRa.
3. Having aRb but not bRa (or vice versa) is not allowed in a symmetric relation.
4. This relation is not symmetric because e is related to a but a is not related to e.

PARTICIPATION ACTIVITY | 12.2.4: Identifying symmetric relations.

The domain for each relation described below is the set of all positive real numbers.
Select the correct description of the relations.

1) x is related to y if x/y = 2.

- Symmetric
- Not symmetric

2) x is related to y if y = 1/x.

- Symmetric
- Not symmetric

Anti-symmetric binary relations

Suppose that R is a relation on set A. R is ***anti-symmetric*** if and only if for every pair, x and y ∈ A, if $x \neq y$ then it can not be the case that xRy and yRx are both true.

A relation is anti-symmetric if for every pair of distinct elements in the domain one of the following situations holds:

- xRy, but it is not true that yRx
- yRx, but it is not true that xRy
- Neither xRy nor yRx is true

The situation that is not allowed in an anti-symmetric relation is for there to be a pair, x and y, such that x ≠ y and xRy and yRx are both true. Notice that the definition of anti-symmetric is a universal statement. If any pair, x and y, in the domain have the forbidden pattern of x ≠ y, xRy and yRx, then the relation is not anti-symmetric. One way to show that a relation is anti-symmetric is to take an arbitrary pair of elements in the domain, x and y, and show that the assumptions xRy and yRx necessarily imply that x = y.

Suppose that the domain for a relation is a set of people. Person x is related to person y if x is taller than person y. This relation is anti-symmetric because it is impossible for there to be two different people, x and y, where x is taller than y and y is taller than x.

A relation is neither symmetric nor anti-symmetric if there is a pair of distinct elements that are related to each other and another pair of elements, x and y, where x is related to y but y is not related to x.

PARTICIPATION ACTIVITY 12.2.5: Anti-symmetric binary relations.

Animation content:

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Animation captions:

1. A relation R on set A is anti-symmetric if an arrow from x to y implies that there is no arrow from y to x.
2. The allowed patterns for a pair, a and b, is for neither aRb nor bRa to be true, or for aRb to be true but not bRa, or for bRa to be true but not aRb.
3. The disallowed pattern is for there to be a pair, a and b, where $a \neq b$ and aRb and bRa are both true.
4. This relation is not anti-symmetric because bRc and cRb are both true.

PARTICIPATION ACTIVITY 12.2.6: Identifying anti-symmetric relations.

The domain for each relation described below is the set of all positive real numbers. Select the correct description of the relations.

1) x is related to y if $\lfloor x \rfloor \leq \lfloor y \rfloor$.

- Symmetric
- Anti-symmetric
- Neither

2) x is related to y if x+y = 1

- Symmetric
- Anti-symmetric
- Neither

3) x is related to y if y = x+1.

- Symmetric
- Anti-symmetric
- Neither

Transitive binary relations

Suppose that R is a relation on set A. R is ***transitive*** if and only if for every three elements, x, y, z ∈ A, if xRy and yRz, then it must also be the case that xRz. Note that in the definition of a transitive relation, the elements, x, y, and z do not necessarily have to be distinct.

The situation that is not allowed in a transitive relation is for there to be an x, y, and z , such that xRy and yRz are true but xRz is not true. Notice that the definition of transitive is a universal statement. If any x, y, and z in the domain have the forbidden pattern of xRy and yRz but not xRz, then the relation is not transitive. If there is no triple x, y, and z that has the forbidden pattern, then the relation is transitive.

Suppose that the domain for a relation is a set of people. Person x is related to person y if x is taller than person y. This relation is transitive because if x is taller than y and y is taller than z, then it must also be the case that x is taller than z.

PARTICIPATION ACTIVITY | 12.2.7: Transitive binary relations.

Animation content:

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Animation captions:

1. A relation R on set A is transitive if an arrow from e to a and an arrow a to b implies that there is an arrow from e to b.
2. The disallowed pattern in a transitive relation is to have elements, a, b, and c, where aRb

and bRc are true, but aRc is not true.
3. This relation is not transitive because aRb and bRc are true, but aRc is not true.

PARTICIPATION ACTIVITY 12.2.8: Identifying transitive relations.

The domain for each relation described below is the set of all positive real numbers. Select the correct description of the relations.

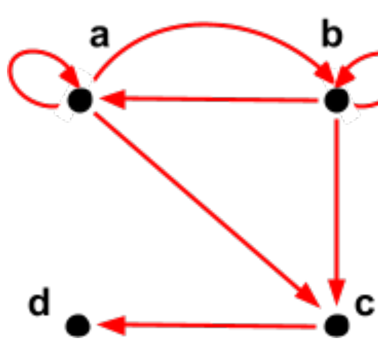
1) x is related to y if y = x+1
- Transitive
- Not transitive

2) x is related to y if y ≥ x+1
- Transitive
- Not transitive

3) x is related to y if $\lfloor x \rfloor \leq \lfloor y \rfloor$.
- Transitive
- Not transitive

PARTICIPATION ACTIVITY 12.2.9: Recognizing the properties of a relation from an arrow diagram.

The relation R on the set {a, b, c, d} is defined by the arrow diagram below.



1) Is the relation R reflexive, anti-reflexive or neither?
- Reflexive
- Anti-reflexive
- Neither

2) Is the relation R symmetric, anti-symmetric or neither?

- Symmetric
- Anti-symmetric
- Neither

3) Is the relation R transitive?

- Transitive
- Not transitive

PARTICIPATION ACTIVITY | 12.2.10: Recognizing the properties of a relation from an arrow diagram.

The arrow diagram below defines a relation Q on the set {x, y}.

1) Is the relation Q transitive?

- Transitive
- Not transitive

Proving and disproving properties of binary relations

Each of the properties of a binary relation is stated as a universal condition. Therefore in order to establish that relation has a property, the condition must be shown to be true for all the elements in the domain. Establishing that a relation does not have a property only requires showing one counter-example: specific elements in the domain which do not satisfy the condition.

For example, consider a relation whose domain is the set of all 4-bit strings. Two strings are related if they have the same first bit or the same last bit or both. For example, 1001 is related to 0101 because the two strings have the same last bit. 1001 is not related to 0100 because neither the first bits nor the last bits of the two strings are the same.

PARTICIPATION ACTIVITY | 12.2.11: Justifying that a relation does or does not have a property.

Animation content:

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Animation captions:

1. Relation R on 4-binary strings. Two strings are related if they have the same first or last bit. The relation R is reflexive: every binary string has the same first bit as itself.
2. R is symmetric because if x has the same first or last bit as y, then y has the same first or last bit as x.
3. R is not transitive. The justification is a counter-example. 1001 is related to 1000 and 1000 is related to 0000. However, 1001 is not related to 0000.

PARTICIPATION ACTIVITY | 12.2.12: Recognizing the properties of a relation: Integer multiples.

The domain of relation D is the set of positive integers. For x, y $\in \mathbf{Z}^+$, xDy if x evenly divides y. Positive integer x evenly divides positive integer y if there is another positive integer n such that y = xn.

1) Is the relation D reflexive, anti-reflexive or neither?

- Reflexive
- Anti-reflexive
- Neither

2) Is the relation D symmetric, anti-symmetric or neither?

- Symmetric
- Anti-symmetric
- Neither

3) Is the relation D transitive?

- Transitive
- Not transitive

PARTICIPATION ACTIVITY | 12.2.13: Recognizing the properties of a relation: Relatively prime positive integers.

The domain of relation P is the set of positive integers. For x, y $\in \mathbf{Z}^+$, xPy if x and y are relatively prime. Two positive integers are relatively prime if the only integer that evenly divides both numbers is 1.

For example, 4 is not relatively prime to 10 because 2 evenly divides 4 and 10. 49 is relatively prime to 15 because 1 is the only positive integer that evenly divides 49 and 15.

1) Select the pair of numbers that xPy is true.

- x = 7 and y = 7
- x = 8 and y = 27
- x = 45 and y = 18

2) Is the relation P reflexive, anti-reflexive or neither?

- Reflexive
- Anti-reflexive
- Neither

3) Is the relation P symmetric, anti-symmetric or neither?

- Symmetric
- Anti-symmetric
- Neither

4) Select the triple of numbers that show that P is not transitive.

- 7, 14, 21
- 7, 11, 15
- 4, 7, 6

Additional exercises

EXERCISE | 12.2.1: Identifying properties of relations.

For each relation, indicate whether the relation is:

- reflexive, anti-reflexive, or neither
- symmetric, anti-symmetric, or neither
- transitive or not transitive

Justify your answer.

(a) The domain of the relation L is the set of all real numbers. For x, y $\in$ **R**, xLy if x < y.

(b) The domain of the relation E is the set of all real numbers. For x, y $\in$ **R**, xEy if x $\leq$ y.

(c) The domain of relation P is the set of all positive integers. For x, y $\in$ $\mathbf{Z}^+$, xPy if there is a positive integer n such that $x^n = y$.

(d) The domain for the relation D is the set of all integers. For any two integers, x and y, xDy if x evenly divides y. An integer x evenly divides y if there is another integer n such that y = xn. (Note that the domain is the set of all integers, not just positive integers.)

(e) The domain for the relation A is the set of all real numbers. xAy if |x - y| $\leq$ 2.

(f) The domain for relation R is the set of all real numbers. xRy if x - y is rational. A real number r is rational if there are two integers a and b, such that b $\neq$ 0 and r = a/b. You can use the fact that the sum of two rational numbers is also rational.

(g) The domain for the relation is **Z**×**Z**. (a, b) is related to (c, d) if a $\leq$ c and b $\leq$ d.

(h) The domain for the relation is **Z**×**Z**. (a, b) is related to (c, d) if a $\leq$ c or b $\leq$ d (inclusive or).

(i) The domain for relation T is the set of real numbers. xTy if x + y = 0.

(j) The domain for relation Z is the set of real numbers. xZy if y = 2x.

(k) The domain for relation T is a group of people. xTy if person y is taller than person x. There are at least two people in the group who are not the same height.

(l) The domain for relation C is a group of people. xCy if person x is the first cousin of person y (i.e., a parent of person x is a sibling of a parent of person y). You can assume that there at least two people x and y such that x is the first cousin of y. You can also assume that no one has two parents who are siblings of each other.

EXERCISE 12.2.2: Properties of relations - dependence on the domain.

The players on a football team are all weighed on a scale. The scale rounds the weight of every player to the nearest pound. The number of pounds read off the scale for each player is called his measured weight. The domain for each of the following relations below is the set of players on the team. For each relation, indicate whether the relation is:

- reflexive, anti-reflexive, or neither
- symmetric, anti-symmetric, or neither
- transitive or not transitive

Justify your answer.

(a) Player x is related to player y if the measured weight of player x is at least the measured weight of player y. No two players on the team have the same measured weight.

(b) Player x is related to player y if the measured weight of player x is at least the measured weight of player y. There is at least one pair of players on the team who have the same measured weight. There is also at least one pair of players on the team who have different measured weights.

(c) Player x is related to player y if the measured weight of player x is at least the measured weight of player y. All the players on the team have exactly the same measured weight.

EXERCISE 12.2.3: Relations that are both reflexive and anti-reflexive or both symmetric and anti-symmetric.

(a) Is it possible to have a relation on the set {a, b, c} that is both reflexive and anti-reflexive? If so, give an example.

(b) Is it possible to have a relation on the set {a, b, c} that is both symmetric and anti-symmetric? If so, give an example.

(c) Is it possible to have a relation on the set {a, b, c} that is neither symmetric nor anti-symmetric? If so, give an example.

(d) Is it possible to have a relation on the set {a, b, c} that is both symmetric and transitive but not reflexive? If so, give an example.

EXERCISE | 12.2.4: Identifying properties of relations - cont.

For each relation R, indicate if the relation is

- Reflexive, anti-reflexive, or neither
- Symmetric, anti-symmetric, or neither
- Transitive or not transitive

(a) The domain of the relation R is {a}. R = {(a, a)}

(b) The domain of the relation R is {a, b, c}. R = {(a, b), (b, c), (a, c)}

(c) The domain of the relation R is {a, b, c, d}. R = {(a, b), (b, a), (c, d), (d, c)}

(d) The domain of the relation R is {a, b}. R = {(a, b), (b, a), (a, a), (b, b)}

(e) The domain of the relation R is {a, b, c, d, e, f}. R = {(a, a), (a, b), (b, c), (c, d), (d, e), (e, f), (f, a)}

EXERCISE | 12.2.5: Identifying properties of relations on a power set.

X = {a, b, c, d, e}, and P(X) is the power set of X. The domain of all of the relations defined below is P(X).

For each relation, indicate if the relation is

- Reflexive, anti-reflexive, or neither
- Symmetric, anti-symmetric, or neither
- Transitive or not transitive

(a) A is related to B if $|A - B| = 1$

(b) A is related to B if $A \cap B = \emptyset$

(c) A is related to B if $A \subset B$

(d) a is an element of X. A is related to B if $B = A \cup \{a\}$

12.3 Directed graphs, paths, and cycles

A ***directed graph*** (or ***digraph***, for short) consists of a pair (V, E). V is a set of ***vertices***, and E, a set of ***directed edges***, is a subset of V × V. An individual element of V is called a ***vertex***. A vertex is typically pictured as a dot or a circle labeled with the name of the vertex. An edge (u, v) ∈ E, is pictured as an arrow going from the vertex labeled u to the vertex labeled v. The vertex u is the ***tail*** of the edge (u, v) and vertex v is the ***head***. If the head and the tail of an edge are the same vertex, the edge is called a ***self-loop***. The graph in the animation below has a self-loop at vertex d.

The ***in-degree*** of a vertex is the number of edges pointing into it. The ***out-degree*** of a vertex is the number of edges pointing out of it.

$$\text{in-degree}(u) = |\{ v \mid (v, u) \in E \}|$$

$$\text{out-degree}(u) = |\{ v \mid (u, v) \in E \}|$$

PARTICIPATION ACTIVITY | 12.3.1: Directed graphs.

Animation captions:

1. Each edge in a directed graph is drawn as an arrow. Edge (a,b) is drawn as an arrow from vertex a to vertex b. Edge (d,d) is a self-loop at d.
2. The tail of edge (a,b) is a and the head is b.
3. The in-degree of c is 2 because there are two edges pointing into c.
4. The out-degree of b is 1 because there is one edge pointing out of b.
5. The only edge pointing into d is the self-loop (d,d), so both the in-degree and out-degree of d are 1.

The definition of a digraph should sound familiar because a digraph is the same mathematically as a relation on the set V. The set of edges E is a subset of V × V and is therefore a relation on the set V. Using the notation of relations, uEv if and only if there is a directed edge from vertex u to vertex v. A picture of a digraph is the same as the arrow diagram for the relation E.

Directed graphs are used in many different applications in computer science. They can be used, for example, to represent communication links in a network, dependencies between tasks in a large computation, or the relationships between users in a social network.

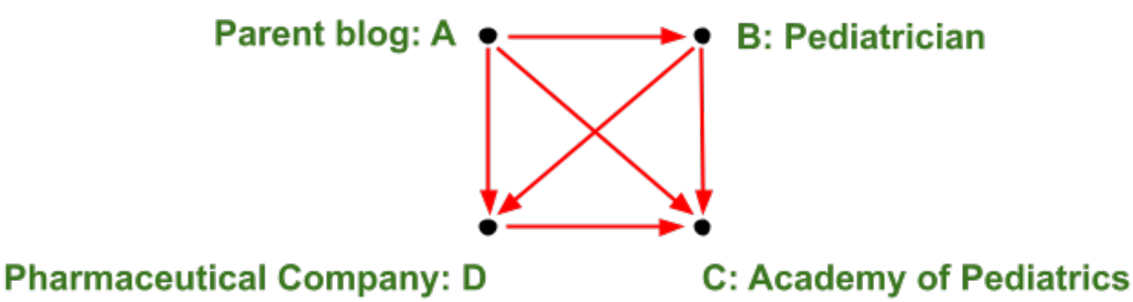
Example 12.3.1: The internet as a directed graph.

The internet can be viewed as a very large directed graph G = (V, E). The vertex set is the set of all urls. The hyperlink relationships can be expressed as directed edges: (u, v) ∈ E if there is a hyperlink from the page u to v. The structure of this graph is used to assist in a search. For example, a web page with a large in-degree is likely to be a more authoritative source and therefore should appear earlier in a list of hits returned by a search engine.

Consider for example a search on the keywords "children", "allergy", and "pollen". There are likely thousands of pages that include these three words. Which pages are likely to be of greatest interest to the user? Below is a sample of a few hits that might arise:

- A: A blog entry by a parent whose child has an allergy.
- B: An article written by a local pediatrician on children's allergies.
- C: An article on the web pages of the American Academy of Pediatrics.
- D: An article on the web pages of a pharmaceutical company selling allergy medication for children.

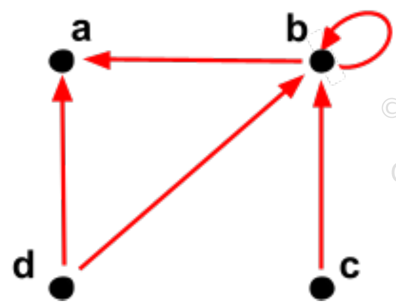
How can a computer rank these sources? In the case that there are thousands of hits, ranking is even more important because only a few will be seen by the users. Define a directed graph whose vertex set is the set of all hits. (X, Y) is an edge in the graph if page X has a hyperlink to page Y. Suppose the graph looks like:



The real algorithms used by search companies are beyond the scope of this material, but hyperlinks are a critical component of the process. Search engines like Google assign a numerical score to each web page and rank the page according to their score. The score depends on the in-degree of a page as well as the importance of the pages who have directed edges into that page. One might guess from the structure above that a helpful ranking of the pages would look like:

1. C: An article on the web pages of the American Academy of Pediatrics
2. D: An article on the web pages of a pharmaceutical company selling allergy medication for children.
3. B: An article written by a local pediatrician on children's allergies
4. A: A blog entry by a parent whose child has an allergy.

PARTICIPATION ACTIVITY | 12.3.2: In-degree and out-degree.



1) Select the ordered pair which is an edge in the graph.
   - (b, c)
   - (c, b)
   - (c, c)

2) In the graph above, which vertex has the largest in-degree?
   - a
   - b
   - c
   - d

3) In the graph above, which vertex has the smallest out-degree?
   - a
   - b
   - c
   - d

Walks in directed graphs

Consider a directed graph corresponding to airline flights. The vertex set is the set of airports used by the airline and a directed edge from airport u to airport v represents a direct flight from u to v. A person wishing to travel from Tallahassee to Peoria would naturally prefer to take a direct flight. However, if the airline does not have a direct flight, he or she may be able to make the trip from Tallahassee to Peoria by a series of direct flights. The flight graph can be used to determine when such a trip is possible.

In the language of graphs, we are asking whether it is possible to travel from one vertex to another by a series of hops along the directed edges of the graph.

Definition 12.3.1: A walk in a directed graph.

A ***walk*** from v_0 to v_l in a directed graph G is a sequence of alternating vertices and edges that starts and ends with a vertex.

$$\langle v_0, (v_0, v_1), v_1, (v_1, v_2), v_2, \ldots, v_{l-1}, (v_{l-1}, v_l), v_l \rangle$$

Each edge in the sequence appears after its tail and before its head:
$\ldots, v_{i-1}, (v_{i-1}, v_i), v_i, \ldots$

Since the edges in a walk are completely determined by the vertices, a walk can also be denoted by the sequence of vertices:

$$\langle v_0, v_1, \ldots, v_l \rangle.$$

The sequence of vertices is a walk only if $(v_{i-1}, v_i) \in E$ for each i = 1, 2,...,l. Two consecutive vertices $\langle \ldots, v_{i-1}, v_i, \ldots \rangle$ in a walk represent an occurrence of the edge (v_{i-1}, v_i) in the walk.

The ***length of a walk*** is l, the number of edges in the walk.

An ***open walk*** is a walk in which the first and last vertices are not the same. A ***closed walk*** is a walk in which the first and last vertices are the same.

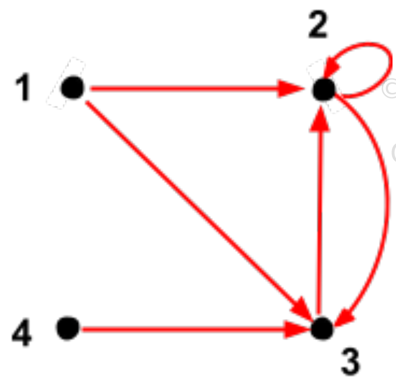
PARTICIPATION ACTIVITY | 12.3.3: Walks in directed graphs.

Animation captions:

1. ⟨a, b, c, b, d⟩ is a walk in the graph because (a, b), (b, c), (c, b), and (b, d) are all edges in the graph.
2. ⟨a, b, c, b, d⟩ is an open walk because the first and last vertices are not the same.
3. The walk ⟨a, b, c, b, d⟩ has length 4 because there are four edges in the walk.
4. ⟨b, d, c, b, c, b⟩ is a walk in the graph because (b, d), (d, c), (c, b) and (b, c) are all edges in the graph. The walk is closed because the first and last vertices are the same.
5. The length of walk ⟨b, d, c, b, c, b⟩ is 5 because there are 5 edges in the walk, including repetitions.
6. ⟨d, d⟩ is a closed walk of length 1.
7. ⟨a⟩ is a closed walk of length 0.

PARTICIPATION ACTIVITY | 12.3.4: Identifying open and closed walks.

The next four questions refer to the directed graph shown:



1) The sequence ⟨ 1, 2, 4 ⟩ is an open walk.

- True
- False

2) The sequence ⟨ 2, 3, 2, 2 ⟩ is a closed walk.

- True
- False

3) The sequence ⟨ 1, 2, 3, 2 ⟩ is an open walk.

- True
- False

4) The sequence ⟨ 2, 2, 2 ⟩ is a closed walk.

- True
- False

Trails, circuits, paths, and cycles

In many contexts, walks that do not repeat vertices or edges are preferable. For example, an airline trip that arrived at the same airport twice would be inefficient.

- A ***trail*** is an open walk in which no edge occurs more than once.

- A ***circuit*** is a closed walk in which no edge occurs more than once.
- A ***path*** is a trail in which no vertex occurs more than once.
- A ***cycle*** is a circuit of length at least 1 in which no vertex occurs more than once, except the first and last vertices which are the same.

⟨a, c, d, a⟩ is a cycle because the only repeated vertices are the first and the last, a. The circuit ⟨a, c, a, d, a⟩ is not a cycle because the vertex a appears in the middle of the circuit as well as at the beginning and the end. The circuit ⟨b, c, d, c, b⟩ is also not a cycle because the vertex c is repeated in the middle of the circuit.

PARTICIPATION ACTIVITY | 12.3.5: Trails, circuits, paths, and cycles.

Animation captions:

1. ⟨a, b, c, b, d⟩ is a walk in the graph because (a, b), (b, c), (c, b), and (b, d) are all edges in the graph.
2. No edge occurs more than once. So the open walk ⟨a, b, c, b, d⟩ is a trail.
3. Vertex b is reached twice, so this trail is not a path.
4. ⟨b, d, c, b⟩ is a walk in the graph because (b, d), (d, c), and (c, b) are all edges.
5. The closed walk ⟨b, d, c, b⟩ is a circuit because no edge occurs more than once. The circuit is a cycle because only the first and last vertices are repeated.

PARTICIPATION ACTIVITY | 12.3.6: Trails, circuits, paths, and cycles.

The sequences below are all walks in a graph. Select the correct description for each sequence.

1) ⟨ 1, 2, 3, 2, 1 ⟩

- ○ Neither a circuit nor a cycle.
- ○ A circuit but not a cycle.
- ○ A circuit and a cycle.

2) ⟨ 2, 1, 3, 4 ⟩

- ○ Neither a trail nor a path.
- ○ A trail but not a path.
- ○ A trail and a path.

3) ⟨ 2, 3, 2 ⟩

- Neither a circuit nor a cycle.
- A circuit but not a cycle.
- A circuit and a cycle.

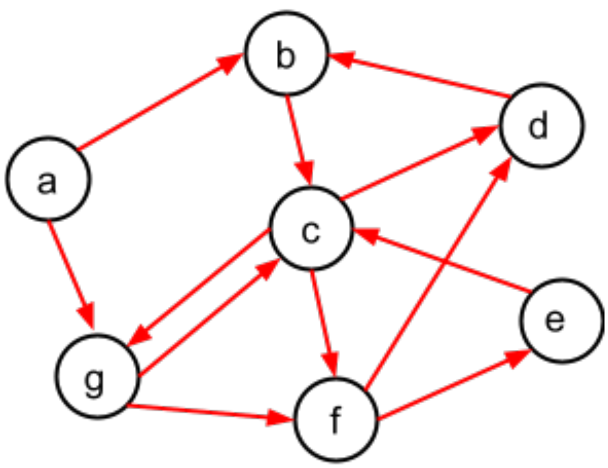
4) ⟨ 1, 4, 3, 1, 2, 5, 1 ⟩

- Neither a circuit nor a cycle.
- A circuit but not a cycle.
- A circuit and a cycle.

Additional exercises

EXERCISE | 12.3.1: Directed graph definitions.

The diagram below shows a directed graph.



(a) What is the in-degree of vertex d?

(b) What is the out-degree of vertex c?

(c) What is the head of edge (b, c)?

(d) What is the tail of edge (g, f)?

(e) List all the self-loops in the graph.

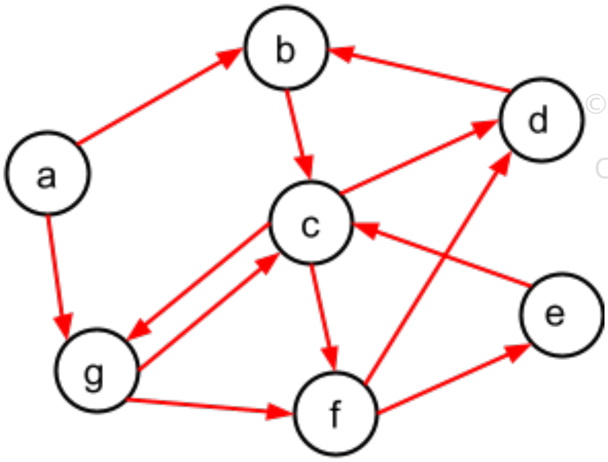
(f) Is ⟨a, g, f, c, d⟩ a walk in the graph? Is it a trail? Is it a path?

(g) Is ⟨a, g, f, d, b⟩ a walk in the graph? Is it a trail? Is it a path?

(h) Is ⟨c, g, f, e⟩ a circuit in the graph? Is it a cycle?

EXERCISE | 12.3.2: Directed graph definitions, cont.

The diagram below shows a directed graph.



(a) Is $\langle$d, b, c, g, c, f, e, c, d$\rangle$ a circuit in the graph? Is it a cycle?

(b) What is the longest cycle in the graph?

(c) Give an example of a cycle of length 4.

(d) Give an example of a path of length 5.

(e) Is there a path of length 3 from vertex d to vertex f? If so, give an example.

(f) Is there a path of length 3 from vertex a to vertex c? If so, give an example.

(g) Give an example of a trail of length 4 that is not a path.

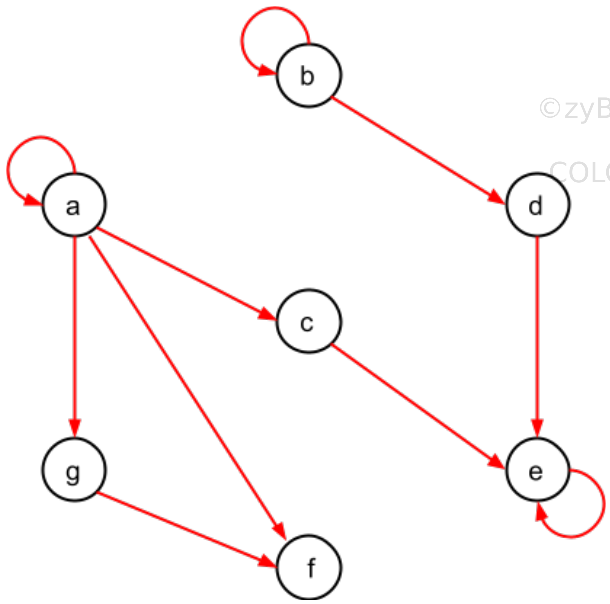
(h) Give an example of a circuit of length 5 that is not a cycle.

(i) Give an example of a circuit of length 6 that is not a cycle.

(j) Is it true that for each pair of vertices there is a path from one to the other?

EXERCISE 12.3.3: Directed graph definitions, cont.

The diagram below shows a directed graph.



(a) Which vertex has the largest in-degree? What is the in-degree for that vertex?

(b) Which vertex has the largest out-degree? What is the out-degree for that vertex?

(c) List all the self-loops in the diagram.

(d) Is the sequence $\langle b, d, e, e \rangle$ a walk in the graph? If it is, is it an open walk?

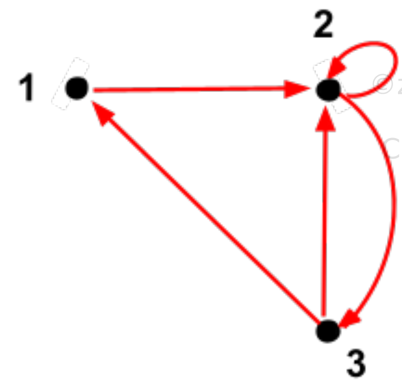
(e) Is the sequence $\langle a, c, f, g \rangle$ a walk in the graph? If it is, is it a circuit? Is it a cycle?

(f) Is the sequence $\langle a, a, c, e, e \rangle$ a walk in the graph? If it is, is it a trail? Is it a path?

(g) Give another example of a trail of length 4 that is not a path.

EXERCISE 12.3.4: Finding walks of a specific length.

The questions below refer to the directed graph shown:



(a) List all the numbers x, such that there is a trail in the graph of length x. Justify your answer.

(b) List all the numbers x, such that there is a path in the graph of length x. Justify your answer.

(c) List all the numbers x, such that there is a circuit in the graph of length x. Justify your answer.

(d) List all the numbers x, such that there is a cycle in the graph of length x. Justify your answer.

12.4 Composition of relations

There is a one-to-one correspondence between directed graphs and binary relations in that the arrow diagram for a binary relation is a directed graph. Similarly the edge set of a directed graph defines a binary relation on the set of vertices of that graph. If a directed graph G has a walk of length k from vertex a to vertex b, then what does that say about the binary relation corresponding to G? Before defining the counterpart of a walk in a binary relation, we need to first define the composition of two relations.

The ***composition*** of relations R and S on set A is another relation on A, denoted S o R. The pair (a, c) ∈ S o R if and only if there is a b ∈ A such that (a, b) ∈ R and (b, c) ∈ S.

The ordering of the relations R and S in the expression "S o R" may seem unusual because it is natural to read an expression from left to right. However, the ordering defined here for composition of relations is consistent with the ordering defined for composition of functions in which R is applied first and then S. The following animation illustrates the composition of two relations:

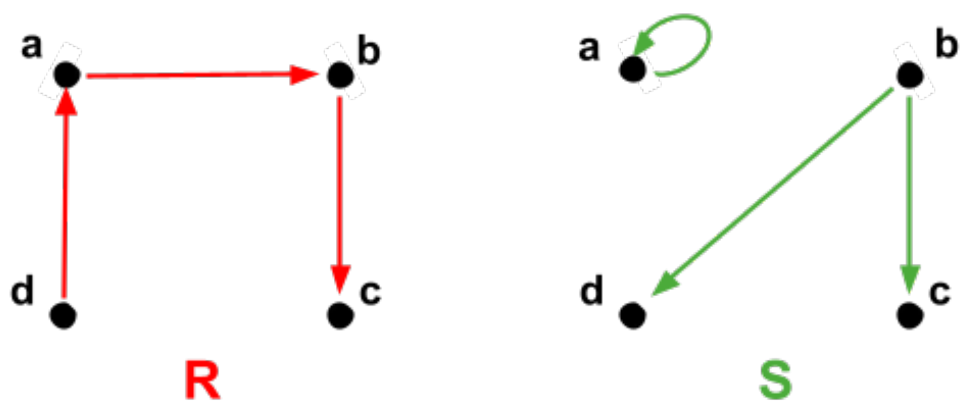
PARTICIPATION ACTIVITY | 12.4.1: Composition of relations.

Animation captions:

1. R and S are relations on set A. (a, c) ∈ S ∘ R because there is a vertex d such that (a,d) ∈ R and (d,c) ∈ S.
2. Since (c,d) ∈ R and (d,c) ∈ S, (c,c) ∈ S ∘ R.
3. Since (b,b) ∈ R and (b,d) ∈ S, (b,d) ∈ S ∘ R.
4. Since (b,b) ∈ R and (b,a) ∈ S, (b,a) ∈ S ∘ R.

PARTICIPATION ACTIVITY | 12.4.2: Composition of relations.

Consider the following two relations on the set {a, b, c, d}:



1) Is (a, d) in S o R?

- Yes
- No

2) Is (b, d) in S o R?

- Yes
- No

3) Is (b, c) in S o R?

- Yes
- No

4) Is (d, a) in S o R?

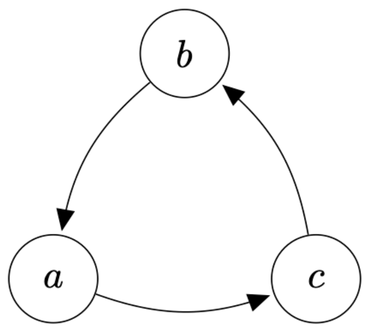
- ○ Yes
- ○ No

CHALLENGE ACTIVITY | 12.4.1: Composition of relations.

422102.2723990.qx3zqy7

Start

Given the relation U below, use ordered pair notation to express the relation U o U.



U

U o U = { Ex: (a, b), (b, c) }

1	2	3	4

Check Next

Additional exercises

EXERCISE 12.4.1: Composition of relations expressed as a set of pairs.

Here are two relations defined on the set {a, b, c, d}:

S = { (a, b), (a, c), (c, d), (c, a) }
R = { (b, c), (c, b), (a, d), (d, b) }

Write each relation as a set of ordered pairs.

(a) S o R

(b) R o S

(c) S o S

(d) R o R

EXERCISE 12.4.2: Composition of relations on the real numbers.

Here are four relations defined on **R**, the set of real numbers:

$R_1 = \{ (x, y): x \leq y \}$
$R_2 = \{ (x, y): x > y \}$
$R_3 = \{ (x, y): x < y \}$
$R_4 = \{ (x, y): x = y \}$

Describe each relation below. (Hint: each of the answers will be one of the relations R_1 through R_4 or the relation **R**x**R**.)

(a) R_1 o R_2

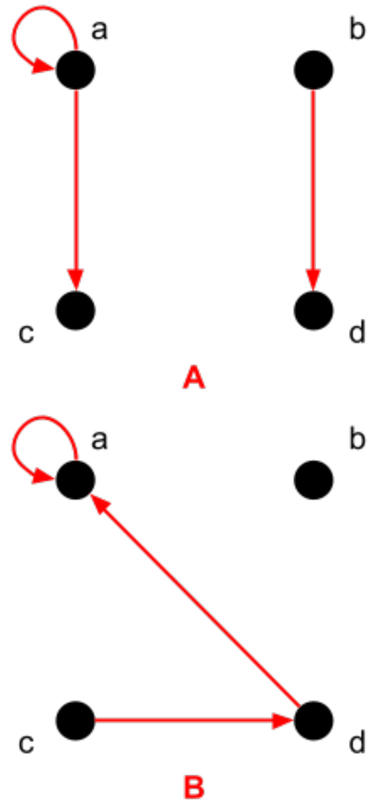
(b) R_4 o R_1

(c) R_1 o R_1

(d) R_3 o R_1

EXERCISE 12.4.3: Composition of relations and arrow diagrams.

The arrow diagrams for relations A and B are shown below. Both relations have the domain {a, b, c, d}.

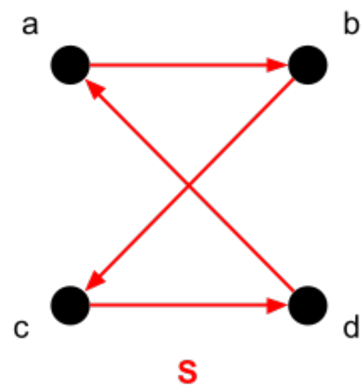


(a) Draw the arrow diagram for $B \circ A$.

(b) Draw the arrow diagram for $A \circ B$.

EXERCISE 12.4.4: Composition of relations and arrow diagrams, cont.

Below is the arrow diagram for relation S with the domain {a, b, c, d}. Define relation T to be $S \circ S$.



(a) Express relation T as a set of related pairs.

(b) Draw the arrow diagram for $S \circ T$.

EXERCISE 12.4.5: Composition and relation properties.

For the following statements, provide a proof if the statement is true or give a counterexample if the statement is false. S and R are binary relations over the same domain.

(a) If S and R are both reflexive, then $S \circ R$ is reflexive.

(b) If S is reflexive, then $S \circ S$ is reflexive.

(c) If S and R are both anti-reflexive, then $S \circ R$ is anti-reflexive.

(d) If S is anti-reflexive, then $S \circ S$ is anti-reflexive.

(e) If S and R are both anti-symmetric, then $S \circ R$ is anti-symmetric.

(f) If S is anti-symmetric, then $S \circ S$ is anti-symmetric.

(g) If S and R are both symmetric, then $S \circ R$ is symmetric.

(h) If S is symmetric, then $S \circ S$ is symmetric.

(i) If S and R are both transitive, then $S \circ R$ is transitive.

(j) If S is transitive, then $S \circ S$ is transitive.

12.5 Graph powers and the transitive closure

A relation on a set can be composed with itself. For example, consider a relation P on a set of people that expresses parent-child relationships. xPy means that x is the parent of y. x(P o P)z holds if there is a person y such that x is the parent of y and y is the parent of z. In other words, x(P o P)z means that x is a grandparent of z.

In the directed graph corresponding to the relation P o P, there is an edge from x to z if there is a vertex y such that there is an edge from x to y in P and an edge from y to z in P. Thus, the directed graph corresponding to P o P represents all walks of length 2 in P. More generally:

$R^1 = R$
$R^k = R \circ R^{k-1}$, for all $k \geq 2$

The edge set E of a directed graph G can be viewed as a relation. E^k is the relation E composed with itself k times. The graph G^k is defined to be the directed graph whose edge set is E^k and is called the k^{th} ***power of G***. G^k expresses walk relationships between vertices in the following natural way:

Theorem 12.5.1: The Graph Power Theorem.

Let G be a directed graph. Let u and v be any two vertices in G. There is an edge from u to v in G^k if and only if there is a walk of length k from u to v in G.

The proof is given at the end of this section. The following animation gives some intuition about why the theorem holds using a small example. The notation ⟨ a, b, *, c ⟩ used in the animation indicates a path of length 3 in which the first vertex is a, the second vertex is b, the fourth vertex is c, and the third vertex is unknown.

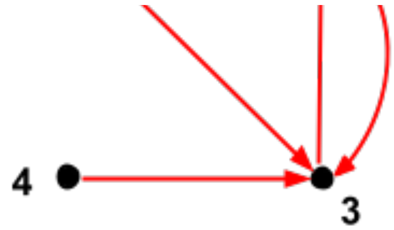
PARTICIPATION ACTIVITY | 12.5.1: Graph powers and walks.

Animation captions:

1. The walk ⟨a, d, a⟩ in G implies that (a,a) is an edge in G^2.
2. The walk ⟨b, a, d⟩ in G implies that (b,d) is an edge in G^2.
3. The walk ⟨c, b, a⟩ in G implies that (c,a) is an edge in G^2.
4. ⟨c, d, a⟩ is a walk in G, but (c,a) is already present in G^2.
5. The walk ⟨d, a, d⟩ in G implies that (d,d) is an edge in G^2.
6. G^3 represents walks of length 3 in G and is obtained by composing G with G^2.
7. The walk ⟨a,*,a⟩ in G (represented by edge (a,a) in G^2) and the edge (a,d) in G imply an edge (a,d) in G^3.
8. The walk ⟨b,*,d⟩ in G (represented by edge (b, d) in G^2) and the edge (d, a) in G imply an edge (b, a) in G^3.
9. The walk ⟨c,*,a⟩ in G (represented by edge (c, a) in G^2) and the edge (a, d) in G imply an edge (c, d) in G^3.
10. The walk ⟨d,*,d⟩ in G (represented by edge (d, d) in G^2) and the edge (d, a) in G imply an edge (d, a) in G^3.

PARTICIPATION ACTIVITY | 12.5.2: Graph powers.

The figure below shows directed graph G:



1) Which one is not an edge in G^2?

- (1, 3)
- (1, 2)
- (4, 3)

2) How many edges are in G^1?

- 0
- 6
- 4

3) Is (4, 2) an edge in G^3?

- Yes
- No

The transitive closure

The union of G^k for all $k \geq 1$ (denoted G^+) represents reachability by walks of any positive length in G. In taking the union of graphs, there is only one copy of the vertex set and the union is taken over the edge sets of the respective graphs.

$$G^+ = G^1 \cup G^2 \cup G^3 \cup G^4 \ldots.$$

(u, v) is an edge in G^+ if vertex v can be reached from vertex u in G by a walk of any length. While the expression given above for $G^+$is an infinite union, if the vertex set is finite, then only graph powers up to |V| are required. Let G be a graph on a finite vertex set with n vertices. Then

$$G^+ = G^1 \cup G^2 \cup G^3 \cup \ldots \cup G^n$$

The same definition holds for a relation R. Let R be a relation on a finite domain with n elements. Then

$$R^+ = R^1 \cup R^2 \cup R^3 \cup \ldots \cup R^n$$

The relation R^+ is called the ***transitive closure of R*** and is the smallest relation that is both transitive and includes all the pairs from R. In other words, any relation that contains all the pairs from R and is transitive must include all the pairs in R^+. If G is a directed graph, then G^+ is called the ***transitive closure of G***. The animation below shows how the graph G^+(with 4 vertices) is determined from the graphs G^1 through G^4.

PARTICIPATION ACTIVITY | 12.5.3: The transitive closure of G as a union of graph powers.

Animation captions:

1. G has 4 vertices. Therefore G^+ is the union of graphs G, G^2, G^3, and G^4. Every edge in G, G^2, G^3, and G^4 becomes an edge in G^+.

There is an alternative way to find the transitive closure of a graph or relation that does not require computing the powers directly. The process repeatedly looks for three elements, x, y, z, such that (x, y) and (y, z) are pairs in the relation but (x, z) is not in the relation. If there is such a triplet of elements, then the pair (x, z) is added to the relation. The process eventually ends when there is no such triplet of elements. The procedure outlined below will eventually terminate with R^+:

Figure 12.5.1: Procedure to find the transitive closure of a relation R on a set A.

Repeat the following step until no pair is added to R:

- If there are three elements x, y, z $\in$ A such that (x, y) $\in$ R, (y, z) $\in$ R and (x, z) $\notin$ R, then add (x, z) to R.

The following animation illustrates the process described above:

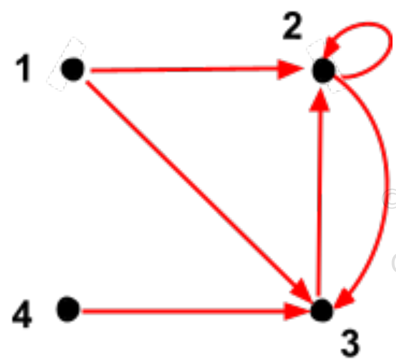
PARTICIPATION ACTIVITY | 12.5.4: Finding the transitive closure of a relation.

Animation captions:

1. To find the transitive closure of relation R, start with the pairs in R and add pairs. Since (a,c) and (c,d) are present, (a,d) is added.
2. Since (b,c) and (c,d) are present, (b,d) is added. After (a,e), (b,e), and (c,e) are added, there are no more walks $\langle x, *, y\rangle$ without the presence of edge (x,y).

PARTICIPATION ACTIVITY | 12.5.5: The transitive closure of a graph.

The figure below shows directed graph G:



1) Is (1, 3) an edge in G^+?

- Yes
- No

2) Is (3, 3) an edge in G^+?

- Yes
- No

3) What is the in-degree of vertex 4 in G^+?

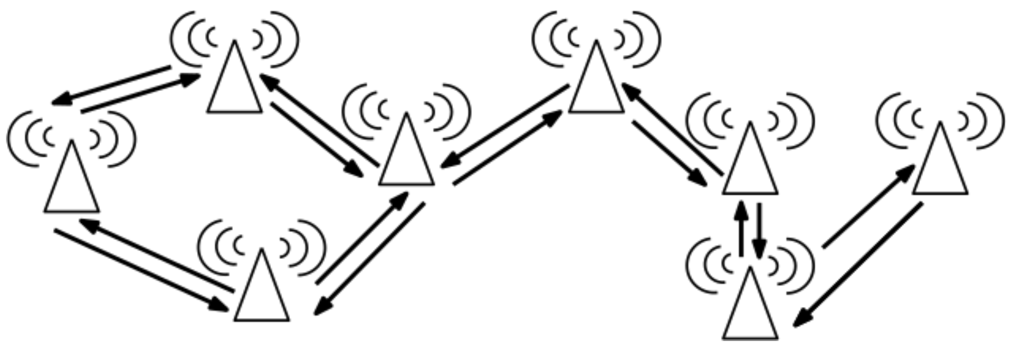
- 3
- 2
- 0

4) What is the in-degree of vertex 3 in G^+?

- 1
- 3
- 4
- 2

Example 12.5.1: Connectivity in sensor networks.

A sensor network consists of a set of sensors distributed over a geographical area. The sensors themselves are typically small, low-cost devices with a limited amount of power that fades over time with use. Sensor networks are used in many industrial and military applications to monitor processes, machinery, or people. Typically, a sensor is only able to transmit a message to other nearby sensors. The radius of a sensor's range may depend in part on its remaining power. The communication graph for the network has a vertex set corresponding to the sensors in the network with a directed edge from sensor x to sensor y if x can send a message directly to y. Messages can also be transmitted through the network along a path: x sends a message to y, y transmits the message to z, and so on. One of the goals in the design of sensor networks is to maintain connectivity as long as possible. In order to test connectivity, it is important to be able to answer questions like: Is there a path from x to y in the network along the directed edges of the communication graph? This question can be answered for all pairs of sensors by computing G^+ where G is the communication graph.



Induction overview for the proof of the Graph Power Theorem

The proof of the Graph Power Theorem requires a technique called induction which is covered in more depth elsewhere. Here is a brief introduction to induction followed by a proof of the theorem. The theorem states that for any k ≥ 1:

there is an edge (u, v) in G^k ↔ there is a walk from u to v of length k in G.

The proof must show that the theorem holds for an infinite sequence values for k: k = 1, 2, ... ***Induction*** starts by showing that a theorem is true for k = 1.Then an inductive proof shows that for any k > 1, if the theorem is true for k - 1, then the theorem also holds for k. The fact that the Graph Power Theorem is true for k = 1 follows almost immediately from the definitions.

Proof 12.5.1: Proof of the Graph Power Theorem.

Theorem: Let G be a directed graph. Let u and v be any two vertices in G. There is an edge from u to v in G^k if and only if there is a walk of length k from u to v in G.

Proof.

By induction on k.

Base case: k = 1. G^1 = G, by definition. Moreover an edge (u, v) is a walk $\langle u, v\rangle$ of length 1. Therefore, there is an edge (u, v) in G^1 if and only if there is a walk $\langle u, v\rangle$ of length 1 in G.

Inductive step: Prove that for k > 1, if it is true that:

there is an edge (u, v) in G^{k-1} $\leftrightarrow$ there is a walk from u to v of length k - 1 in G

then it is true that:

there is an edge (u, v) in G^k $\leftrightarrow$ there is a walk from u to v of length k in G

E, E^{k-1}, and E^k are the edge sets for G, G^{k-1}, and G^k. By definition, E^k is obtained by composing E and E^{k-1}: $E^k = E \circ E^{k-1}$. By definition of composition, there is an edge (u, v) in G^k if and only if there is a vertex x, such that (u, x) is an edge in G^{k-1} and (x, v) is an edge in G. By the assumption that the theorem holds for k - 1, there is an edge (u, x) in G^{k-1} if and only if there is a walk from u to x of length k - 1 in G. The edge (x, v) in G is a walk of length 1 in G: $\langle x, v\rangle$.

We have shown that there is an edge (u, v) in G^k if and only if there is a vertex x such that there is a walk of length k-1 from u to x and a walk of length 1 from x to v in G.

It remains to show that there is an x such that there is a walk of length k-1 from u to x and a walk of length 1 from x to v in G if and only if there is a walk of length k in G from u to v. Each direction of the "if and only if" is proven separately.

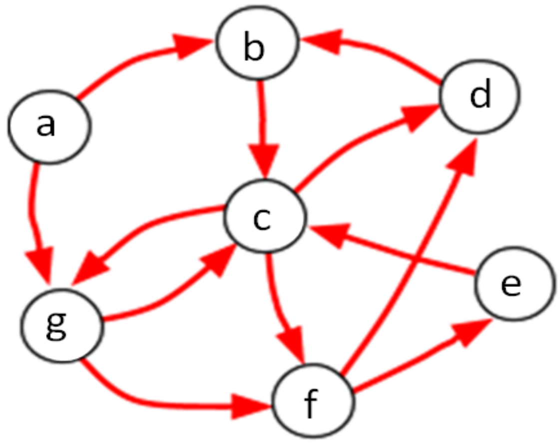
Suppose that $\langle u, \ldots, x\rangle$ is a walk of length k-1 and $\langle x, v\rangle$ is a walk of length 1 in G. The walks $\langle u, \ldots, x\rangle$ and $\langle x, v\rangle$ can be put together to form a walk $\langle u, \ldots, x, v\rangle$. The length of the walk from u to v is the sum of the lengths of walks from u to x and from x to v:
(k - 1) + 1 = k.

Now suppose that there is a walk of length k from u to v in G. Let x be the second-to-last vertex reached on the walk: $\langle u, \ldots x, v\rangle$. Then $\langle u, \ldots, x\rangle$ must be a walk of length k - 1 from u to x in G and (x, v) must be an edge in G. ■

Additional exercises

EXERCISE 12.5.1: Edges of graph powers.

The diagram below shows a directed graph G.



(a) Is (a, b) in G^2?

(b) Is (b, e) in G^3?

(c) Is (g, g) in G^3?

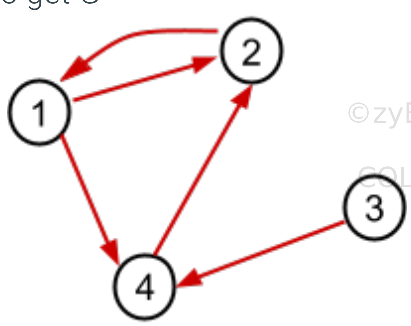
(d) Is (g, g) in G^4?

(e) Is (b, b) in G^3?

(f) Is (b, d) in G^5?

EXERCISE 12.5.2: Drawing graph powers.

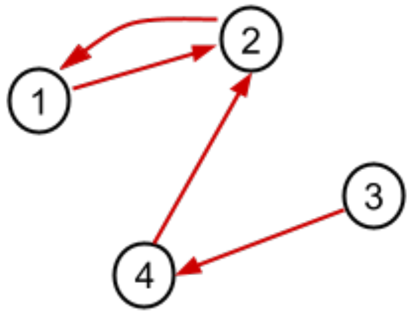
(a) The drawing below shows a graph G. Draw G^2, G^3, and G^4. Then take the union of all of the graphs (including G) to get G^+



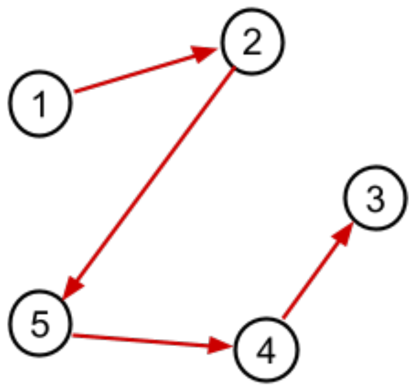
EXERCISE | 12.5.3: Finding the transitive closure of a graph.

Draw the transitive closure of each graph.

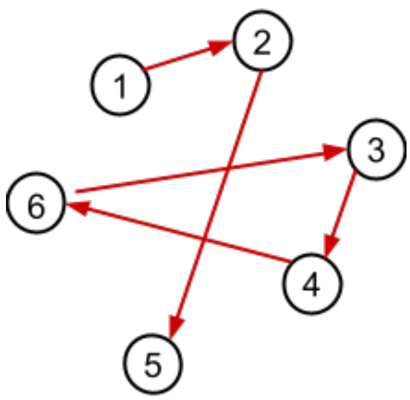
(a)



(b)

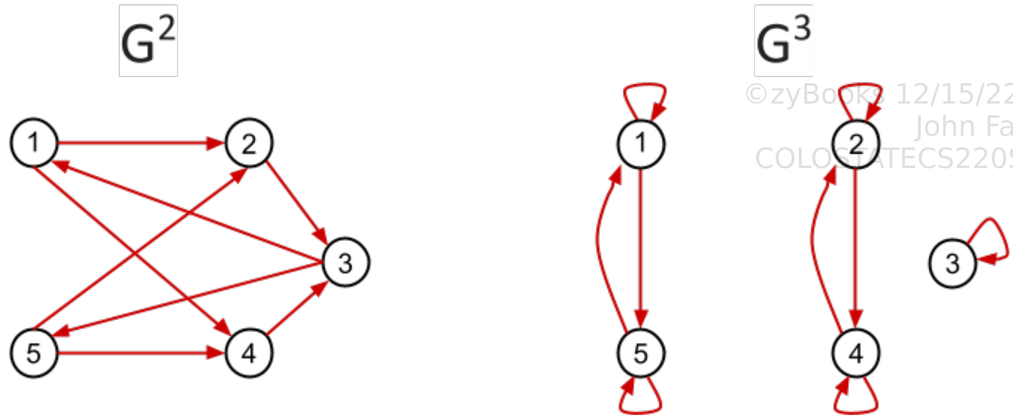


(c)



EXERCISE 12.5.4: Inferring facts about a graph from its graph powers.

The drawing below shows G^2 and G^3 for a graph G.



Use the information provided in G^2 and G^3 to answer the following questions about G.

(a) Is there a walk of length 3 from vertex 4 to vertex 5 in G?

(b) Is there any closed walk of length 2 in G?

(c) Is there any closed walk of length 3 in G?

(d) Is there a walk of length 4 from vertex 2 to vertex 1 in G?

(e) Is there a walk of length 5 from vertex 2 to vertex 3 in G?

(f) Is there a walk from vertex 2 to vertex 4 in G?

(g) Which vertices can be reached from vertex 3 in G by a walk of length 2?

(h) Which vertices have a walk of length 3 to vertex 1 in G?

(i) Is it possible to infer the out-degree of vertex 4 in G^+ from the information given?

EXERCISE 12.5.5: Properties of relations and the transitive closure.

(a) Consider a digraph G in which each vertex has an in-degree of at least one. Suppose that the relation defined by the edges of G is symmetric. Is G^+ reflexive? Why or why not?

EXERCISE 12.5.6: Composition and relation properties, cont.

(a) Show that if G is reflexive then every edge in G^k is also an edge in G^{k+1}.

(b) Show that if G is reflexive then $G^+ = G^n$.