3.1 Definition of functions

Survey

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Link: *Student survey*

A function maps elements from one set X to elements of another set Y. Many functions are mathematical functions that map numbers to numbers, such as the function x^2, which maps a number to its square. Discrete mathematics is often concerned with functions that map between other kinds of sets, such as binary strings or a set of tasks. Thus, an assignment of people to teams, or of guests to hotel rooms, are also examples of functions. A function from X to Y can be viewed as a subset of X × Y: (x, y) ∈ f if f maps x to y. It is possible that X and Y are in fact the same set, in which case f is a subset of X × X.

Definition 3.1.1: Functions.

A ***function*** f that maps elements of a set X to elements of a set Y, is a subset of X × Y such that for every x ∈ X, there is *exactly one* y ∈ Y for which (x, y) ∈ f.

f: X → Y is the notation to express the fact that f is a function from X to Y. The set X is called the ***domain*** of f, and the set Y is the ***target*** of f. An alternate word for target that is sometimes used is ***co-domain***. The fact that f maps x to y (or (x, y) ∈ f) can also be denoted as f(x) = y.

If f maps an element of the domain to zero elements or more than one element of the target, then f is not ***well-defined***.

If f is a function mapping X to Y and X is a finite set, then the function f can be specified by listing the pairs (x, y) in f. Alternatively, a function with a finite domain can be expressed graphically as an arrow diagram. In an ***arrow diagram*** for a function f, the elements of the domain X are listed on the left and the elements of the target Y are listed on the right. There is an arrow from x ∈ X to y ∈ Y if and only if (x, y) ∈ f. Since f is a function, each x ∈ X has exactly one y ∈ Y such that (x, y) ∈ f, which means that in the arrow diagram for a function, there is exactly one arrow pointing out of every element in the domain.

PARTICIPATION ACTIVITY | 3.1.1: Specifying functions with finite domains.

Animation captions:

1. In an arrow diagram, domain elements are on the left, target elements on the right. An arrow for each domain element points to a target element.

For function f: X → Y, an element y is in the ***range*** of f if and only if there is an x ∈ X such that (x, y) ∈ f. Expressed in set notation:

Range of f = { y: (x, y) ∈ f, for some x ∈ X }

The range of f is a subset of the target but the range is not necessarily equal to the target. In an arrow diagram, the range is the set of elements in the target that have arrows coming into them.

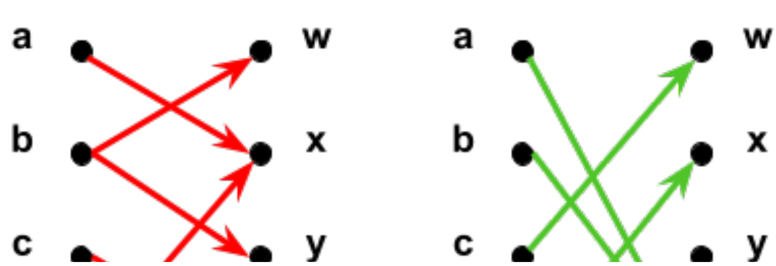
PARTICIPATION ACTIVITY | 3.1.2: Recognizing well-defined functions.

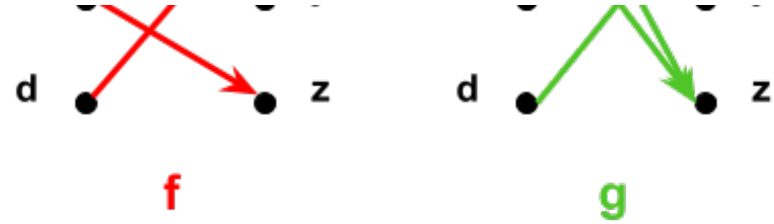
Animation captions:

1. In an arrow diagram for f, each element in the domain has exactly one arrow leaving it.
2. The range of f is the set {a, c, d} because a, c, and d are the target elements with at least one arrow coming in.
3. If the arrow from x to a is removed, f is no longer a function because f is not defined on input x.
4. If an arrow from y to b is added, f is no longer a function because f maps y to both d and b.

PARTICIPATION ACTIVITY | 3.1.3: Recognizing well-defined functions from arrow diagrams.

Below are two arrow diagrams:





1) Is f a function?

- No. There is an unmapped element in the domain.
- No. An element in the domain is mapped to two different elements in the target.
- Yes.

2) Is g a function?

- No. There is an unmapped element in the domain.
- No. An element in the domain is mapped to two different elements.
- Yes.

PARTICIPATION ACTIVITY 3.1.4: Function basics.

Sets A and X are defined as:

$$A = \{ a, b, c, d \}$$

$$X = \{ 1, 2, 3, 4 \}$$

A function f: A → X is defined to be

$$f = \{ (a, 3), (b, 1), (c, 4), (d, 1) \}$$

1) What is the target of function f?

- {a, b, c, d}
- {1, 2, 3, 4}
- {1, 3, 4}

2) What is the range of function f?

- {a, b, c, d}
- {1, 2, 3, 4}
- {1, 3, 4}

3) What is f(c)?

- c
- 3
- 4

4) Which of the following sets could be a correct well-defined function definition for g: X → A?

- { (a, 1), (b, 4), (c, 2), (d, 3) }
- { (1, a), (2, d), (2, b), (4, c) }
- { (1, a), (3, b), (4, c) }
- { (1, a), (2, a), (3, b), (4, b) }

5) Suppose that for function g: X → A, g(2) = a and g(3) = b. Which of the following sets could be the correct function definition for g?

- { (1, a), (2, a), (3, d), (4, d) }
- { (1, a), (2, d), (3, b), (4, c) }
- { (1, d), (2, a), (3, b), (4, c) }

A mathematical function f is often defined by describing how f acts on an input x, as in:

$$f(x) = x^2 - 2.$$

However, the definition is not complete until the domain and target of f are specified. For example:

$$g: \mathbf{R} \to \mathbf{R}, \text{ where } g(x) = |x|.$$

Note that g maps every real number to a real number. However, g does not map any number to a negative number.

PARTICIPATION ACTIVITY 3.1.5: Recognizing well-defined algebraic functions.

Are the expressions below well-defined functions from **R** to **R**?

1) $f(x) = 1/(x - 1)$

- Well-defined

Not well-defined

2) $g(x) = \sqrt{x^2 + 2}$

Well-defined

Not well-defined

3) $h(x) = \pm\sqrt{x^2 + 2}$

Well-defined

Not well-defined

The domain and target set for a function can also be a set of strings. For example, the function f: $\{0, 1\}^3 \rightarrow \{0, 1\}^4$ takes as input a 3-bit string and outputs a 4-bit string. Suppose that f is defined so that for any $x \in \{0, 1\}^3$, f(x) = x0. Then for any 3-bit string x, the output of f on input x is obtained by adding a 0 to the end of x. For example f(011) = 0110.

PARTICIPATION ACTIVITY 3.1.6: Functions on sets of strings.

Define $f:\{0,1\}^2 \rightarrow \{0, 1\}^3$ such that for $x \in \{0,1\}^2$, f(x) = 1x.

1) What is f(01)?

101

011

01

2) What is the range of f?

$\{0, 1\}^2$

{ 100, 101, 110, 111 }

$\{0, 1\}^3$

Function equality

Two functions, f and g, are ***equal*** if f and g have the same domain and target, and f(x) = g(x) for every element x in the domain. The notation f = g is used to denote the fact that functions f and g are equal.

PARTICIPATION ACTIVITY 3.1.7: Function equality.

According to each definition of functions f and g, is it true that f = g?

1) f: **R** → **R**, f(x) = $x^2 + 1$

g: **R** → **R**$^+$, g(x) = $(x + 1)^2 - 2x$

- Equal
- Not equal

2) f: **R** → **R**, f(x) = |x|

g: **R** → **R**, $g(x) = \sqrt{x^2}$

- Equal
- Not equal

3) X = {0, 1, 2}

f: X → X, f(x) = $(x - 1)^2$

g: X → X, g(0) = 1, g(1) = 0, g(2) = 1

- Equal
- Not equal

CHALLENGE ACTIVITY | 3.1.1: Definition of functions.

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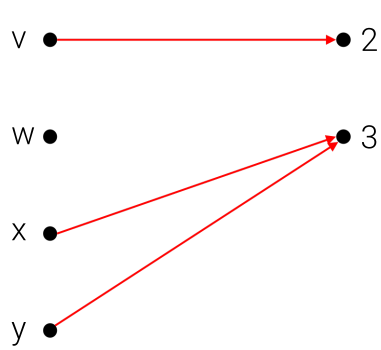
Start

Is f shown in the arrow diagram below a well-defined function?



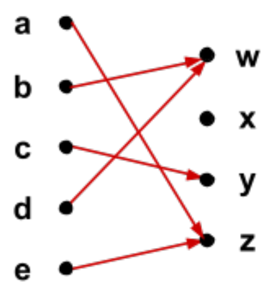
1	2	3	4

Check Next

Additional exercises

EXERCISE 3.1.1: Function basics.

The drawing below shows the arrow diagram for a function f.



Give your answers to the questions below using roster notation.

(a) What is the domain of f?

(b) What is the target of f?

(c) What is the range of f?

EXERCISE 3.1.2: Drawing arrow diagrams.

Draw an arrow diagram for each of the following functions. Give the range of the function using roster notation.

(a) $X = \{1, 2, 3, 4, 5\}$ and $Y = \{0, 1, 2, 3, 4\}$. $f: X \to Y$. $f(x) = |x - 2|$.

(b) $f: \{0, 1\}^2 \to \{0, 1\}^3$. For each $x \in \{0, 1\}^2$, $f(x) = x0$.

(c) $f: \{0, 1\}^2 \to \{0, 1\}^2$. For each $x \in \{0, 1\}^2$, $f(x)$ is obtained by swapping the two bits in x. For example, $f(01) = 10$.

EXERCISE 3.1.3: Recognizing well-defined algebraic functions and their ranges.

Which of the following are functions from **R** to **R**? If f is a function, give its range.

(a) $f(x) = \sqrt{x}$

(b) $f(x) = 1/(x^2 - 4)$

(c) $f(x) = \sqrt{x^2}$

EXERCISE 3.1.4: Function of an ordered pair.

Let B = {0, 1}. f: B × B → B × B. f(x, y) = (1 - y, 1 - x).

(a) Give the domain of the function f using roster notation. Use ordered pair notation for the Cartesian product.

(b) Draw an arrow diagram for the function *f*.

(c) Give the range of the function f using roster notation. Use ordered pair notation for the Cartesian product.

EXERCISE 3.1.5: Range of a function.

Express the range of each function using roster notation.

(a) Let A = {2, 3, 4, 5}.
f: A → **Z** such that f(x) = 2x - 1.

(b) Let A = {2, 3, 4, 5}.
f: A → **Z** such that f(x) = x^2.

(c) f: $\{0,1\}^5$ → **Z**. For x ∈ $\{0,1\}^5$, f(x) is the number of times "01" occurs in the string. For example f(01101) = 2 because the string "01" occurs twice in "01101". The first occurrence starts at the first bit. The second occurrence starts in at the fourth bit.

(d) f: $\{0,1\}^5$ → **Z**. For x ∈ $\{0,1\}^5$, f(x) is the number of 1's that occur in x. For example f(01101) = 3, because there are three 1's in the string "01101".

(e) f: $\{0,1\}^3$ → $\{0,1\}^3$. For x ∈ $\{0,1\}^3$, f(x) is obtained by replacing the last bit with 1. For example f(000) = 001.

(f) Let A = {2, 3, 4, 5}.
f: A × A → **Z**, where f(x,y) = x+y.

(g) Let A = {1, 2, 3}.
f: A × A → **Z**, where f(x,y) = x^y.

(h) Let A = {1, 2, 3}.
f: A × A → **Z**×**Z**, where f(x,y) = (y, x).

(i) Let A = {1, 2, 3}.
f: A × A → **Z**×**Z**, where f(x,y) = (x,y+1).

(j) Let A = {1, 2, 3}.
f: P(A) → **Z**. For X ⊆ A, f(X) = |X|.

(k) Let A = {1, 2, 3}.
f: P(A) → P(A). For X ⊆ A, f(X) = X ∪ {1}.

(l) Let A = {1, 2, 3}.
f: P(A) → P(A). For X ⊆ A, f(X) = X - {1}.

EXERCISE 3.1.6: Determining whether two functions are equal.

Indicate whether the two functions are equal. If the two functions are not equal, then give an element of the domain on which the two functions have different values.

(a) f: **Z** → **Z**, where $f(x) = x^2$.
g: **Z** → **Z**, where $g(x) = |x|^2$.

(b) f: **Z** → **Z**, where $f(x) = x^3$.
g: **Z** → **Z**, where $g(x) = |x|^3$.

(c) f: **Z** × **Z** → **Z**, where $f(x,y) = |x + y|$.
g: **Z** × **Z** → **Z**, where $g(x,y) = |x| +| y|$.

(d) f: **R** × **R** → **R**, where $f(x,y) = x^y$.
g: **R** × **R** → **R**, where $g(x,y) = y^x$.

(e) B = {0, 1}
f: B × B → B × B, where f(x,y) = (1-y, 1-x)
g: B × B → B × B, where g(x,y) = (x,y)

3.2 Floor and ceiling functions

The floor function and the ceiling function map real numbers onto integers. The floor and ceiling functions round real numbers to a nearby integer in different ways.

The floor function.

The ***floor function*** maps a real number to the nearest integer in the downward direction.

floor: **R** → **Z**, where floor(x) = the largest integer y such that y ≤ x.

The floor function comes up so often in mathematics that it has its own notation:

$$\text{floor}(x) = \lfloor x \rfloor$$

For example, $\lfloor 4.32 \rfloor = 4$ and $\lfloor 4 \rfloor = 4$. The floor function also rounds negative numbers to the nearest integer in the downward direction: $\lfloor -4.32 \rfloor = -5$ and $\lfloor -4 \rfloor = -4$.

The ceiling function.

The ***ceiling function*** rounds a real number to the nearest integer in the upward direction.

ceiling: **R** → **Z**, where ceiling(x) = the smallest integer y such that x ≤ y.

Like the floor function, the ceiling function also has its own notation:

$$\text{ceiling}(x) = \lceil x \rceil$$

For example, $\lceil 4.32 \rceil = 5$ and $\lceil 4 \rceil = 4$. The ceiling function also rounds negative numbers to the nearest integer in the upward direction: $\lceil -4.32 \rceil = -4$ and $\lceil -4 \rceil = -4$.

PARTICIPATION ACTIVITY 3.2.1: Computing the floor and ceiling functions.

Animation captions:

1. The floor function is computed by sliding down to the nearest integer. The floor of 1.8 is 1.
2. The floor of -1.3 is -2.
3. The ceiling function is computed by sliding up to the nearest integer. The ceiling of .3 is 1.
4. The ceiling of -.8 is 0.
5. The ceiling and the floor of an integer are the same. The floor of -1 is -1. The ceiling of -1 is -1.

PARTICIPATION ACTIVITY 3.2.2: Floor and ceiling functions.

Give the integer value for each expression below. Note: Pay attention to the brackets; the small segments at the bottom or top indicate floor or ceiling, respectively.

1) $\lfloor 5.6 \rfloor$

Check Show answer

2) $\lceil 5.6 \rceil$

Check Show answer

3) $\lceil -5.6 \rceil$

Check Show answer

4) $\lfloor -5 \rfloor$

Check Show answer

CHALLENGE ACTIVITY | 3.2.1: Computing floor and ceiling functions.

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Start

Fill in the integer.

\(\lfloor 6.96 Ex: 1 =\)

1	2	3	4	5	6	7	8

Check Next

CHALLENGE ACTIVITY | 3.2.2: Computing nested floor and ceiling functions.

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Start

Fill in the integer

$$\lfloor -4.992 + \lfloor 8.473 \rfloor \rfloor =$$

1	2	3	4

Check Next

PARTICIPATION ACTIVITY | 3.2.3: Floor and ceiling functions applied.

1) A soccer team has x players. A pack of water contains six bottles of water. How many packs of water are needed to provide each player with one bottle of water?

- $\lfloor \frac{x}{6} \rfloor$

- $\left\lfloor \frac{x}{6} \right\rfloor$
  $\lceil \frac{x}{6} \rceil$

- $\left\lceil \frac{x}{6} \right\rceil$
  $6x$

  $6x$

2) A farmer's hens lay x eggs. How many cartons of eggs can the farmer sell if each carton must contain a dozen eggs?

- $\lfloor \frac{x}{12} \rfloor$

- $\left\lfloor \frac{x}{12} \right\rfloor$
  $\lceil \frac{x}{12} \rceil$

  $\left\lceil \frac{x}{12} \right\rceil$

Additional exercises

EXERCISE 3.2.1: Mystery functions.

Use the definition of the function f below to answer the questions:

$$f(x) = \left\lfloor x + \frac{1}{2} \right\rfloor$$

(a) Compute f(2.2), f(2.9), f(2.5), f(2), and f(3).

(b) Describe in words the relationship between x and f(x).

(c) Now define the function g as:

$$g(x) = \left\lceil x - \frac{1}{2} \right\rceil$$

$$g(x) = \left\lceil x - \frac{1}{2} \right\rceil$$

How do the functions f and g differ?

EXERCISE 3.2.2: Applying the floor and ceiling functions.

Use the ceiling and floor functions to give a mathematical expression for the following values:

(a) There are x children in the first grade at Lee Elementary school. Each child will be given five crayons to do an art project. Crayons come in boxes of 24. How many boxes need to be purchased for the art project?

(b) A baker is packaging cookies for sale in boxes of 8. He has y cookies to put into boxes. How many boxes can he sell?

EXERCISE | 3.2.3: Floor and ceiling functions.

Compute the value of each expression.

(a) $\lfloor -3.7 \rfloor$

(b) $\lceil -4.2 \rceil$

(c) $\lceil 5 \rceil$

(d) $\lfloor \lfloor 3.5 \rfloor - 4.3 \rfloor$

(e) $\left\lfloor \frac{3}{2} + \left\lceil \frac{1}{3} \right\rceil \right\rfloor$

EXERCISE | 3.2.4: Proving facts about the floor and ceiling functions.

Prove or disprove each statement

(a) If n is an even integer, then $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$

(b) If n is an odd integer, then $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$

(c) For any real number x, $\lfloor 2x \rfloor = 2\lfloor x \rfloor$.

(d) For any real number x, $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$.

(e) For any real numbers x and y, $\lceil x \rceil \lceil y \rceil = \lceil xy \rceil$.

3.3 Properties of functions

A function f: X → Y is ***one-to-one*** or ***injective*** if $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$. That is, f maps different elements in X to different elements in Y.

A function f: X → Y is ***onto*** or ***surjective*** if the range of f is equal to the target Y. That is, for every y ∈

Y, there is an $x \in X$ such that $f(x) = y$.

The properties of being one-to-one or onto are important in many situations. For example, consider a function that maps employees to offices. If the function is one-to-one, then no one has to share an office. If the function is onto, then there are no empty offices and the company's space is well utilized. Functions are used to define an assignment of processes to computers in a distributed network. If the function is one-to-one, then no computer is time-sharing between different tasks. If the function is onto, then all the resources of the network are being utilized.

A function is ***bijective*** if it is both one-to-one and onto. A bijective function is called a ***bijection***. A bijection is also called a ***one-to-one correspondence***. Here are some examples that illustrate one-to-one and onto functions:

PARTICIPATION ACTIVITY 3.3.1: One-to-one and onto functions.

Animation captions:

1. f is not one-to-one because f(w) = f(z) = c.
2. f is not onto because no elements in X map to d or e.
3. Now f is one-to-one but not onto.
4. Now f is one-to-one and onto. f is a bijection.

PARTICIPATION ACTIVITY 3.3.2: Identifying one-to-one and onto functions.

Indicate whether the functions defined below are one-to-one, onto, neither, or both:

1) h: **Z** → **Z**. h(x) = x - 4.

- Neither one-to-one nor onto
- One-to-one but not onto
- Onto but not one-to-one
- Both one-to-one and onto

2) t: **Z** → **Z**. t(x) = 2x.

- Neither one-to-one nor onto
- One-to-one but not onto
- Onto but not one-to-one
- Both one-to-one and onto

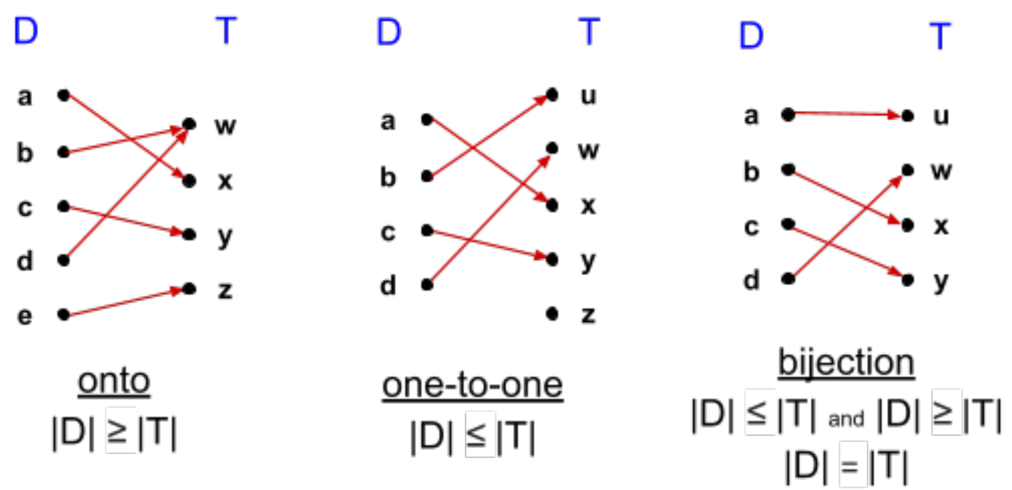
3) f: **Z** → **Z**. $f(x) = \lfloor (x + 1)/2 \rfloor$.

- Neither one-to-one nor onto
- One-to-one but not onto
- Onto but not one-to-one
- Both one-to-one and onto

When the domain and target are finite sets, it is possible to infer information about their relative sizes based on whether the function is one-to-one or onto.

- If f: D → T is onto, then for every element in the target, there is at least one element in the domain: |D| ≥ |T|.
- If f: D → T is one-to-one, then every element in the domain maps to a unique element in the target: |D| ≤ |T|.
- If f: D → T is a bijection, then f is one-to-one and onto: |D| ≤ |T| and |D| ≥ |T|, which implies that |D| = |T|.

Figure 3.3.1: Relative sizes of the domain and target from function properties.



The fact that the domain and target of a bijection have the same size may seem simple but this fact turns out to be extremely powerful. One way to count the elements in a set is to define a bijection between that set and another set whose size is already known. Counting the elements in a set is a fundamental part of discrete probability, an important tool in many areas of science.

PARTICIPATION ACTIVITY 3.3.3: Properties of functions and the relative sizes of the domain and target.

Let f be a function whose domain is $\{0, 1\}^3$ and whose target is $\{0, 1\}^2$.

1) Is it possible that f is one-to-one?

- Yes
- No

2) Is it possible that f is onto?

- Yes
- No

3) Is it possible that f is a bijection?

- Yes
- No

CHALLENGE ACTIVITY 3.3.1: Draw an arrow diagram and identify function properties.

Note: If x is a real number, then |x| is the absolute value of x. If A is a finite set, then |A| refers to the cardinality of the set A. Both uses of the |*| symbols are standard mathematical notation.

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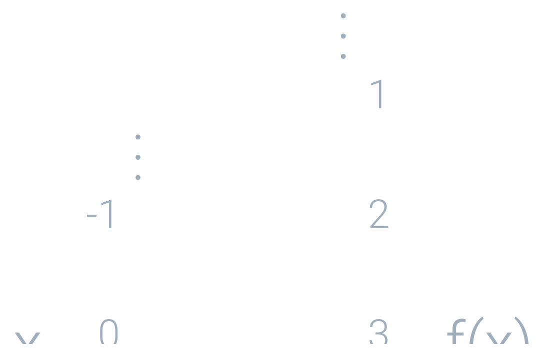
Start

$$\mathbf{Z} \rightarrow \mathbf{Z}\text{: } f(x) = x + 2$$

A. Select the element in f(x) that corresponds to the circled x.



Additional exercises

EXERCISE 3.3.1: Properties of functions that map ordered pairs of integers to integers.

Consider the following functions from **Z** × **Z** to **Z**. Which ones are onto? Justify your answer.

(a) $f(x, y) = 2x - 4y$

(b) $f(x, y) = |x| - |y|$

(c) $f(x, y) = x + y - 2$.

(d) $f(x, y) = x^{|y|}$

(e) $f(x, y) = x^{|2y|}$

EXERCISE | 3.3.2: Properties of algebraic functions.

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

(a) f: $\mathbf{R} \rightarrow \mathbf{R}$. $f(x) = x^2$

(b) g: $\mathbf{R} \rightarrow \mathbf{R}$. $g(x) = x^3$

(c) h: $\mathbf{Z} \rightarrow \mathbf{Z}$. $h(x) = x^3$

(d) f: $\mathbf{Z} \rightarrow \mathbf{Z}$, $f(x) = \left\lceil \frac{x}{5} \right\rceil - 4$

(e) f: $\mathbf{Z} \rightarrow \mathbf{Z}$, $f(x) = 5x - 4$

(f) f: $\mathbf{Z} \rightarrow \mathbf{Z}$, $f(x) = x - 4$

(g) f: $\mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z} \times \mathbf{Z}$, $f(x, y) = (x+1, 2y)$

(h) f: $\mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z} \times \mathbf{Z}$, $f(x, y) = \left(\left\lceil \frac{x}{5} \right\rceil, y - 2 \right)$

(i) f: $\mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z} \times \mathbf{Z}$, $f(x, y) = \left(\left\lceil \frac{x}{5} \right\rceil, 5y - 2 \right)$

(j) f: $\mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z} \times \mathbf{Z}$, $f(x, y) = (1-y, 1-x)$

(k) f: $\mathbf{Z}^+ \times \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$, $f(x, y) = 2^x + y$.

EXERCISE 3.3.3: Function properties and the relative sizes of the domain and target.

(a) If a function f is onto, then the domain of f is at least as large as its target. Show that the converse of the previous statement is not necessarily true by showing a function f: $\{0,1\}^3 \rightarrow \{0, 1\}^2$ that is not onto.

(b) If a function f is one-to-one, then the target of f is at least as large as its domain. Show that the converse of the previous statement is not necessarily true by showing a function f: $\{0,1\}^2 \rightarrow \{0, 1\}^3$ that is not one-to-one.

(c) If a function f is a bijection, then the domain of f is the same size as its target. Show that the converse of the previous statement is not necessarily true by showing a function f: $\{0,1\}^2 \rightarrow \{0, 1\}^2$ that is neither one-to-one nor onto.

EXERCISE 3.3.4: Properties of functions on strings and power sets.

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

(a) f: $\{0, 1\}^4 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and dropping the first bit. For example f(1011) = 011.

(b) f: $\{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, f(001) = 101 and f(110) = 110.

(c) f: $\{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example f(011) = 110.

(d) f: $\{0, 1\}^3 \rightarrow \{0, 1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string. For example, f(100) = 1001.

(e) Let A be defined to be the set {1, 2, 3, 4, 5, 6, 7, 8}. f: P(A) → {0, 1, 2, 3, 4, 5, 6, 7, 8}. For $X \subseteq A$, f(X) = |X|. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.

(f) Let A be defined to be the set {1, 2, 3, 4, 5, 6, 7, 8}. f: P(A) → P(A). For $X \subseteq A$, f(X) = A-X. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.

(g) Let A be defined to be the set {1, 2, 3, 4, 5, 6, 7, 8} and let B = {1}. f: P(A) → P(A). For $X \subseteq A$, f(X) = X - B. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.

(h) A = {a, b, c}, h: P(A) → P(A). For $X \subseteq A$, $h(X) = X \oplus \{a\}$.

(i) A = {a, b, c}, h: P(A) → P(A). For $X \subseteq A$, $h(X) = X \cup \{a\}$.

EXERCISE | 3.3.5: Functions on integers with particular properties.

Find a function whose domain is the set of all integers and whose target is the set of all positive integers that satisfies each set of properties.

(a) Neither one-to-one, nor onto.

(b) One-to-one, but not onto.

(c) Onto, but not one-to-one.

(d) One-to-one and onto.

EXERCISE | 3.3.6: Proving statements about bijections.

Let $f: \mathrm{R} \to \mathrm{R}$ and $g: \mathrm{R} \to \mathrm{R}$. Determine whether each statement is true. If the statement is true, provide a proof. If the statement is false, provide a counterexample.

(a) If f and g are both bijections, then $f + g$, defined as $(f + g)(x) = f(x) + g(x)$, is also a bijection.

(b) For any real number c, we will define cf as $(cf)(x) = c \cdot f(x)$.
Prove that if f is a bijection and $c \neq 0$, then cf is also a bijection.

3.4 The inverse of a function

If a function f: X → Y is a bijection, then the ***inverse*** of f is obtained by exchanging the first and second entries in each pair in f. The inverse of f is denoted by f^{-1}:

$$f^{-1} = \{ (y, x) : (x, y) \in f \}.$$

Reversing each pair in a function f does not always result in a well-defined function. Therefore, some functions do not have an inverse. A function f: X → Y has an inverse if and only if reversing each pair in f results in a well-defined function from Y to X. f^{-1} is a well-defined function if every element in Y is mapped to exactly one element in X.

PARTICIPATION ACTIVITY | 3.4.1: Arrow diagram for the inverse of a function.

Animation captions:

1. If arrows for f are reversed, the result is not a function because 8 has no outgoing arrow, and also 9 has two outgoing arrows. Thus, f does not have an inverse.
2. If the arrows for g are reversed, the result is a function because each left element has exactly one outgoing arrow. Thus, g has an inverse.

The finite examples in the animation above show that f^{-1} is obtained by reversing the arrows in the arrow diagram for f. The resulting f^{-1} is a function, if and only if every element in Y has exactly one outgoing arrow after the arrows are reversed which in turn holds if and only if f is a bijection.

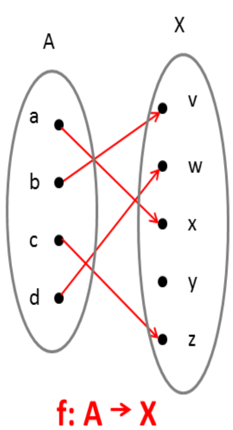
The reasoning above also applies to functions with infinite domains, and can be summed up in the following statement:

A function f has an inverse if and only if f is a bijection.

Recall that a bijection is a function that is one-to-one and onto.

PARTICIPATION ACTIVITY 3.4.2: Does a function f have an inverse?

1) Consider the function specified by the following arrow diagram. Does f have an inverse?



- Yes
- No

2) Let X = { u, v, w, x }. Define a function g: X → X to be: g = { (u, v), (v, w), (w, w), (x, u) }. Does g have an inverse?

- Yes
- No

3) Define a function h: **Z** → **Z** to be h(x) =

$-x$. Does h have an inverse?

- Yes
- No

The inverse of a bijection f can also be expressed in function notation. If f is a bijection from X to Y, then for every $x \in X$ and $y \in Y$,

$$f(x) = y \quad \text{if and only if} \quad f^{-1}(y) = x.$$

Therefore the value of $f^{-1}(y)$ is the unique element $x \in X$ such that $f(x) = y$. If f^{-1} is the inverse of function f, then for every element $x \in X$, $f^{-1}(f(x)) = x$.

PARTICIPATION ACTIVITY | 3.4.3: Finding the inverse of a function with finite domains.

1) Let X = { u, v, w, x }. Define a function g: X → X to be: g = { (u, v), (v, w), (w, x), (x, u) }. What is $g^{-1}(x)$?

Check Show answer

2) Let h: $\{0, 1\}^3 \rightarrow \{0, 1\}^3$. For $x \in \{0, 1\}^3$, h(x) is obtained by moving the middle bit of x to the beginning of x. For example, h(101) = 011. What is $h^{-1}(100)$?

Check Show answer

When a function is defined on an infinite domain, it is sometimes possible to solve for the function's inverse analytically, as illustrated in the following animation:

PARTICIPATION ACTIVITY | 3.4.4: Solving for the inverse of a function analytically.

Animation captions:

1. To check that f is one-to-one, check that $x \neq x'$, implies $3x - 2 \neq 3x' - 2$. Equivalently, show

the contrapositive: $3x - 2 = 3x' - 2$, implies $x = x'$.
2. f is one-to-one because if $3x - 2 = 3x' - 2$, then $x = x'$.
3. f is onto because for every y, there is an x such that $3x - 2 = y$.
4. The inverse of f can be found by solving for an expression that gives the value of x in terms of y.
5. The inverse of f on input y is $\frac{(y+2)}{3}$.

The function f: **R** → **R** defined by $f(x) = x^2$ is not one-to-one because f(x) = f(-x) for any real number x. However, if the domain is restricted to positive reals, $\mathbf{R}^+$, then:

$$f: \mathbf{R}^+ \to \mathbf{R}^+, f(x) = x^2$$

is a bijection. To solve for f^{-1}, express $y = x^2$ and solve for x in terms of y: $x = \sqrt{y}$. Therefore: $f^{-1}(y) = \sqrt{y}$.

The use of the variable y instead of x is not important. The function $f^{-1}(y) = \sqrt{y}$ is the same function as $f^{-1}(x) = \sqrt{x}$.

PARTICIPATION ACTIVITY 3.4.5: Computing the inverse of a function analytically.

1) f: R → R, where f(x) = -x + 3. What is f^{-1}?

- $f^{-1}(x) = -x + 3$
- $f^{-1}(x) = x - 3$
- $f^{-1}(x) = -x - 3$

Example 3.4.1: Encrypting messages.

The process of encrypting messages can be expressed through the mathematical language of functions. Consider encrypting 16-digit credit card numbers. Let N be the set of all possible 16-digit numbers. An encryption scheme for N is a function e: N → N. In order to send a credit card number n ∈ N over an insecure communication channel, the sender would encrypt n by computing e(n). The encryption function e is chosen to be a bijection. The sender then sends e(n) over the channel. The receiver receives e(n) and wishes to decrypt the message to obtain the original number n. To accomplish this, the receiver applies the inverse function e^{-1} to e(n) in order to get the credit card number n: $e^{-1}(e\ (n)) = n$. The sender and receiver need to communicate ahead of time so that the receiver knows what function e^{-1} to apply to the number that is received. A secure encryption scheme would have the property that without foreknowledge of the function e^{-1}, it would be difficult to determine n from e(n).

CHALLENGE ACTIVITY 3.4.1: The inverse of a function.

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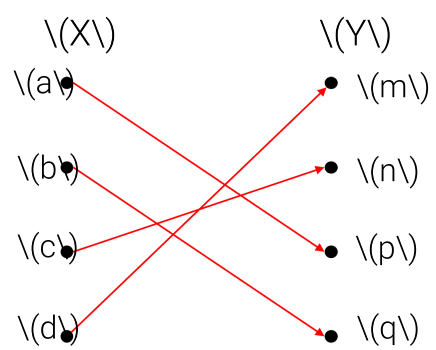
Start

A function $f : X \rightarrow Y$ is given in the arrow diagram below.



$f^{-1}(m)$ = Ex: t or undefined Enter "undefined" if the function does not have a well-defined inverse.

1 2 3

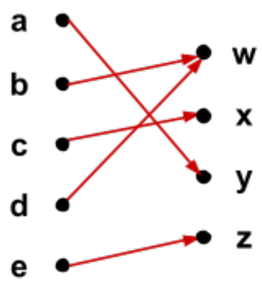
Check Next

Additional exercises

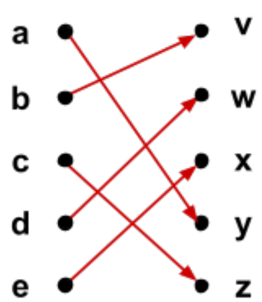
EXERCISE 3.4.1: Arrow diagrams for inverses of functions.

Each of the arrow diagrams below define a function f. For each arrow diagram, indicate whether f^{-1} is well-defined. If f^{-1} is not well-defined, indicate why. If f^{-1} is well-defined, give an arrow diagram showing f^{-1}.

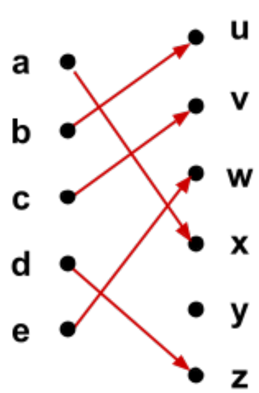
(a)



(b)



(c)



EXERCISE 3.4.2: Finding inverses of functions.

For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

(a) $f: \mathbb{Z} \to \mathbb{Z}.\ f(x) = x + 3$

(b) $f: \mathbb{Z} \to \mathbb{Z}.\ f(x) = 2x + 34$

(c) $f: \mathbb{R} \to \mathbb{R}.\ f(x) = 2x + 3$

(d) Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$
$f: P(A) \to \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$
For $X \subseteq A$, $f(X) = |X|$. Recall that for a finite set A, $P(A)$ denotes the power set of A which is the set of all subsets of A.

(e) Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. $f: P(A) \to P(A)$. For $X \subseteq A$, $f(X) = A - X$. Recall that for a finite set A, $P(A)$ denotes the power set of A which is the set of all subsets of A.

(f) $f: \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.

(g) $f: \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example, $f(011) = 110$.

(h) $f: \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string x, removing the first bit of x, and adding the bit to the end of x. For example, $f(011) = 110$.

(i) $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, f(x, y) = (x + 5, y - 2)$

(j) $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, f(x, y) = (1 - y, x)$

EXERCISE | 3.4.3: Finding the inverse of the cube of a bijective function.

For a function $f: \mathrm{R} \to \mathrm{R}$, we will define f-cube as $f\text{-cube}(x) = (f(x))^3$.

(a) Prove that if f is a bijection, then f-cube is also a bijection.
You can use the fact that for any two real numbers x and y, if $x = y$, then $x^{1/3} = y^{1/3}$.
Also for any real number x, $x^{1/3}$ is a well-defined real number.

(b) For a bijection f, what is the inverse of f-cube? Justify your answer.

3.5 Composition of functions

Let f be a function that assigns employees to offices in a company. Let g be the function that maps each office to the telephone number for the phone in that office. An employee, Rajiv, is assigned the office f(Rajiv). Rajiv's work phone number is g(f(Rajiv)). The process of applying a function to the result of another function is called ***composition***.

Definition 3.5.1: Composition of functions.

f and g are two functions, where f: X → Y and g: Y → Z. The composition of g with f, denoted g o f, is the function (g o f): X → Z, such that for all x ∈ X, (g o f)(x) = g(f(x)).

PARTICIPATION ACTIVITY | 3.5.1: Composition of functions.

Animation captions:

1. Function f has domain X and target Y. g has domain Y and target Z. Therefore, the composition of g with f is a function with domain X and target Z.
2. f maps 1 to a and g maps a to 10. Therefore, g composed with f maps 1 to 10.
3. Since f(2) = a and g(a) = 10, then (g ∘ f)(2) = 10. Also, since f(3) = b and g(b) = 10, then (g ∘ f)(3) = 10.

Generally, the order in which the functions are applied is important, so f o g is not the same as g o f. Define:

$$f: \mathbf{R}^+ \to \mathbf{R}^+,\ f(x) = x^3$$

$$g: \mathbf{R}^+ \to \mathbf{R}^+, g(x) = x + 2$$

Then

$$(f \circ g)(x) = f(g(x)) = (x + 2)^3$$

$$(g \circ f)(x) = g(f(x)) = x^3 + 2$$

It is possible to compose more than two functions. Composition is associative, so the order in which one composes the functions does not matter:

$$f \circ g \circ h = (f \circ g) \circ h = f \circ (g \circ h) = f(g(h(x)))$$

The ***identity function*** always maps a set onto itself and maps every element onto itself.

The identity function on A, denoted I_A: A → A, is defined as $I_A(a) = a$, for all a ∈ A.

If a function f from A to B has an inverse, then f composed with its inverse is the identity function. If f(a) = b, then $f^{-1}(b) = a$, and $(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$.

Let f: A → B be a bijection. Then $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$.

PARTICIPATION ACTIVITY | 3.5.2: Composition of a function and its inverse.

Animation captions:

1. f has domain X and target Y. The inverse of f has domain Y and target X. Thus, the inverse of f composed with f has domain X and target X and maps 1 to 1.
2. f maps 1 to s. The inverse of f maps s to 1. Therefore, the composition of the inverse of f with f maps 1 to 1.
3. f maps 2 to t. The inverse of f maps t to 2. Therefore, the composition of the inverse of f with f maps 2 to 2.
4. The composition of the inverse of f with f maps each element to itself and is therefore the identity function on X.
5. The composition of f with the inverse of f has domain Y and target Y and maps each element to itself and is therefore the identity function on Y.

PARTICIPATION ACTIVITY | 3.5.3: Composition of functions.

Define functions f, g, h, all of which have **R** as their domain and **R** as their target.

$f(x) = 3x + 1$
$g(x) = x^2$
$h(x) = 2^x$

1) What is (f o g)(2)?

Check Show answer

2) What is (g o h)(3)?

Check Show answer

3) What is (f o g o h)(0)?

Check Show answer

4) What is $(f \circ f^{-1})(17)$?

Check Show answer

CHALLENGE ACTIVITY | 3.5.1: Composition of functions.

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Start

Given:

\(f(x) = 3^{x}\) \(g(x) = x^{2}\)

\((f \circ g)(0) = Ex: 15

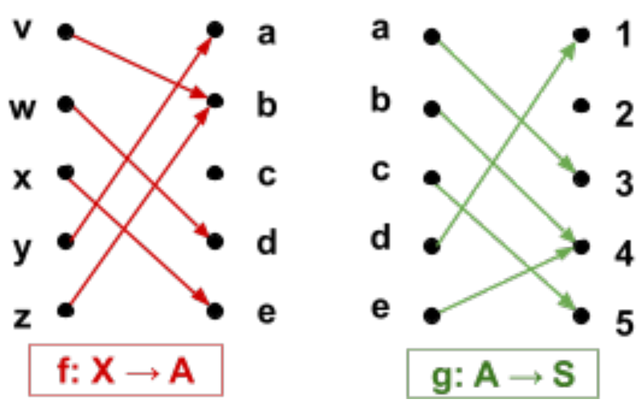
1 2 3 4 5

Check Next

Additional exercises

EXERCISE | 3.5.1: Function composition from arrow diagrams.

The drawing below gives arrow diagrams for two functions, f and g.



(a) What is the domain of g o f?

(b) What is the target of g o f?

(c) Give the arrow diagram for g o f.

(d) What is the range of g o f?

EXERCISE 3.5.2: Composition of functions on integers.

Consider three functions f, g, and h, whose domain and target are **Z**.
Let

$$f(x) = x^2 \qquad g(x) = 2^x \qquad h(x) = \left\lceil \frac{x}{5} \right\rceil$$

(a) Evaluate (f o g)(0)

(b) Evaluate (f o h)(52)

(c) Evaluate (g o h o f)(4)

(d) Give a mathematical expression for h o f.

(e) Give a mathematical expression for f o g.

EXERCISE 3.5.3: Function composition and the identity function.

Define f to be a function whose domain is X and whose target is Y such that X ∩ Y = ∅.
For each of the following functions, indicate whether the function is well-defined. If your answer is "well-defined", indicate how the function relates to f.

(a) f o I_X

(b) f o I_Y

(c) I_X o f

(d) I_Y o f

EXERCISE 3.5.4: Composition of onto and one-to-one functions.

Let f: X → Y and g: Y → Z be two functions.

(a) Is it possible that f is not onto and g o f is onto? Justify your answer. If the answer is "yes", give a specific example for f and g.

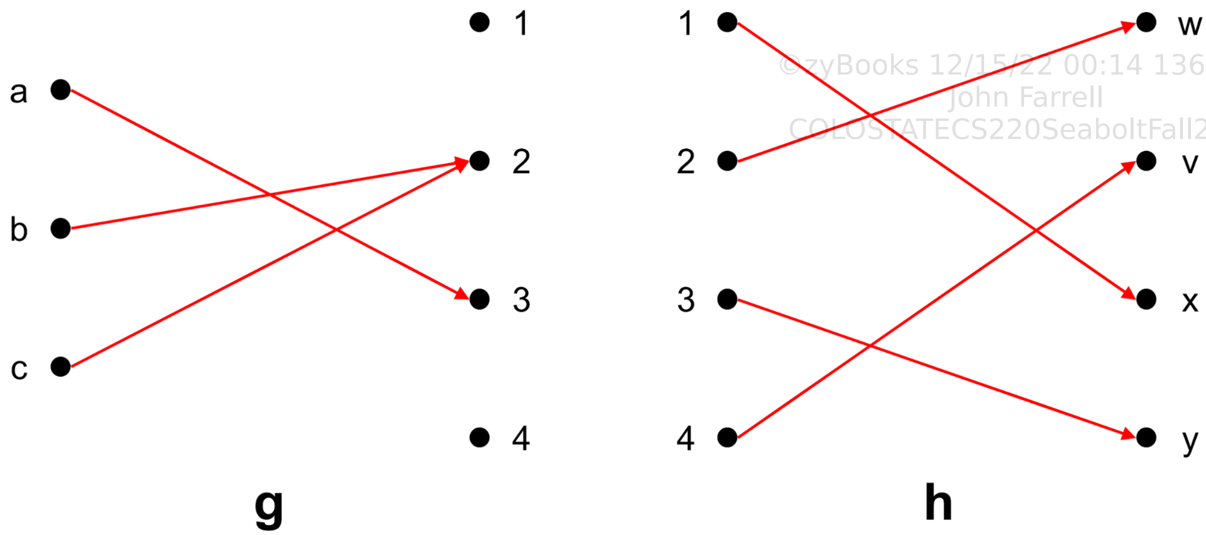
(b) Is it possible that g is not onto and g o f is onto? Justify your answer. If the answer is "yes", give a specific example for f and g.

(c) Is it possible that f is not one-to-one and g o f is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

(d) Is it possible that g is not one-to-one and g o f is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

EXERCISE | 3.5.5: Composition of functions defined by arrow diagrams.

Define two functions: g: {a, b, c} → {1, 2, 3, 4} and h: {1, 2, 3, 4} → {w, v, x, y}. The functions are shown in the arrow diagrams below.



(a) What is the range of g?

(b) What is the domain of h o g?

(c) What is $h^{-1}(y)$?

(d) What is the domain of h^{-1} o h?

(e) What is (h o g)(b)?

(f) Is g one-to-one or onto?

(g) Are either g or h a bijection?

EXERCISE 3.5.6: Composition of functions on sets of strings.

Define the following functions f, g, and h:

- f: $\{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, f(001) = 101 and f(110) = 110.
- g: $\{0, 1\}^3 \to \{0, 1\}^3$. The output of g is obtained by taking the input string and reversing the bits. For example, g(011) = 110.
- h: $\{0, 1\}^3 \to \{0, 1\}^3$. The output of h is obtained by taking the input string x, and replacing the last bit with a copy of the first bit. For example, h(011) = 010.

(a) What is (g o f)(010)?

(b) What is (g o h)(010)?

(c) What is (h o f)(010)?

(d) What is the range of h o f?

(e) What is the range of g o f?

EXERCISE 3.5.7: Composition of functions on sets of strings, part 2.

Let d, f, and g be defined as follows.

- d: $\{0, 1\}^4 \to \{0, 1\}^4$. d(x) is obtained from x by removing the second bit and placing it at the end. For example, d(1011) = 1110.
- f: $\{0, 1\}^4 \to \{0, 1\}^4$. f(x) is obtained from x by replacing the last bit with 1. For example, f(1000) = 1001.
- g: $\{0, 1\}^4 \to \{0, 1\}^3$. g(x) is obtained from x by removing the first bit. For example, g(1000) = 000.

(a) What is $d^{-1}(1001)$?

(b) Which of the following functions is not well defined, f o g or g o f?

(c) What is the range of g o f?

(d) What is (f o d)(1011)?

EXERCISE 3.5.8: Explicit formulas for compositions of functions.

The domain and target set of functions f, g, and h are **Z**. The functions are defined as:

- $f(x) = 2x + 3$
- $g(x) = 5x + 7$
- $h(x) = x^2 + 1$

Give an explicit formula for each function given below.

(a) f o g

(b) g o f

(c) f o h

(d) h o f

3.6 Logarithms and exponents

The ***exponential function*** $\exp_b$:**R** → **R⁺** is defined as:

$$\exp_b(x) = b^x$$

where b is a positive real number and b ≠ 1. The parameter b is called the ***base of the exponent*** in the expression b^x. The input x to the function b^x is called the ***exponent***.

Figure 3.6.1: Properties of exponents.

For any positive real numbers b, c, and any real numbers x, and y, the following equalities are always true:

$$b^x b^y = b^{x+y}$$

$$(b^x)^y = b^{xy}$$

$$\frac{b^x}{b^y} = b^{x-y}$$

$$(bc)^x = b^x c^x$$

PARTICIPATION ACTIVITY 3.6.1: Applying the properties of exponents.

Match mathematical expressions that have the same value.

If unable to drag and drop, refresh the page.

$\mathbf{2^{3k+3}}$ $\mathbf{4}$ $\mathbf{2^{3k}}$ $\mathbf{2^{2k}}$ $\mathbf{4 \cdot 2^k}$

$$2^{k+2}$$

$$\frac{2^{k+1}}{2^{k-1}}$$

$$\left(2^k\right)^2$$

$$(4 \cdot 2)^k$$

$$\left(2^{k+1}\right)^3$$

Reset

CHALLENGE ACTIVITY 3.6.1: Properties of exponents.

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Start

Ex: 5

\(9^6\cdot 9^5 = 9\)

1	2	3	4	5	6

Check Next

The exponential function is one-to-one and onto, and therefore has an inverse. The ***logarithm function*** is the inverse of the exponential function. For real number b > 0 and b ≠ 1, $\log_b$:**R⁺** → **R** is defined as:

$$b^x = y \quad \Leftrightarrow \quad \log_b y = x$$

The parameter b is called the ***base of the logarithm*** in the expression $\log_b$ y.

Figure 3.6.2: Sketch of the exponential and logarithm function for b greater than 1.

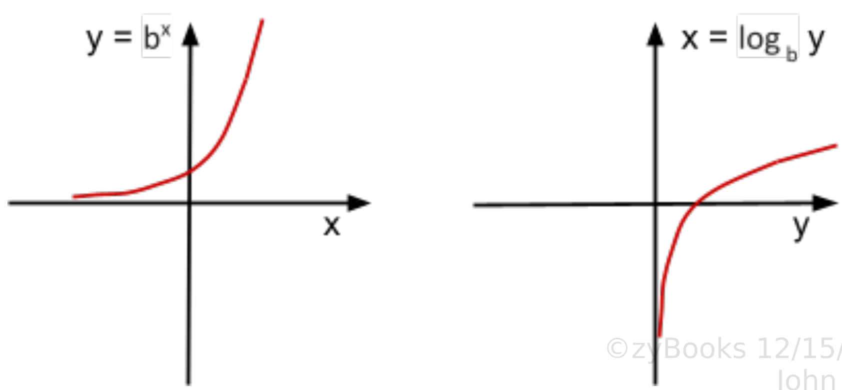

Figure 3.6.3: Properties of logarithms.

For any positive numbers b, c, x, and y, such that b ≠ 1 and c ≠ 1, the following equalities are always true:

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$\log_b(x^y) = y\log_b x$$

$$\log_c x = \frac{\log_b x}{\log_b c}$$

PARTICIPATION ACTIVITY | 3.6.2: Applying the properties of logarithms.

Match mathematical expressions that have the same value.

If unable to drag and drop, refresh the page.

$\log_{10}1000$ **$\log_3 100$** **$\log_2 10$** **$\log_3 1000$** **$\log_3 10$**

$\log_3 10 + \log_3 100$

$\log_3 1000 - \log_3 100$

$2 \cdot \log_3 10$

$\log_3 10/\log_3 2$

3

Reset

CHALLENGE ACTIVITY | 3.6.2: Properties of logarithms.

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A function f is said to be ***strictly increasing*** if whenever $x_1 < x_2$, then $f(x_1) < f(x_2)$. A function f is said to be ***strictly decreasing*** if whenever $x_1 < x_2$, then $f(x_1) > f(x_2)$. If b > 1, then the functions $f(x) = b^x$ and $f(x) = \log_b x$ are both strictly increasing. The fact that both functions are strictly increasing can help in approximating the value of the functions. For example, the value of $\log_3 200$ is not an integer and would be difficult to determine without a calculator. However, since $3^4 = 81$ and $3^5 = 243$, so by definition $\log_3 81 = 4$ and $\log_3 243 = 5$. Because the log function is strictly increasing and 81 < 200 < 243, the value of $\log_3 200$ is in between 4 and 5.

PARTICIPATION ACTIVITY 3.6.3: Using the fact that exp and log are increasing to estimate values.

Use the equalities below to answer the questions without a calculator.

$2^2 = 4$	$2^5 = 32$	$2^8 = 256$
$2^3 = 8$	$2^6 = 64$	$2^9 = 512$
$2^4 = 16$	$2^7 = 128$	$2^{10} = 1024$

1) What is the value of $\log_2 64$?

Check Show answer

2) What is the value of $\lceil \log_2 100 \rceil$?

Check Show answer

3) What is the value of $\lfloor \log_2 927 \rfloor$?

Check Show answer

4) What is the value of the smallest power of 2 that is greater than $2^{9.2}$? A power of 2 is a number that can be expressed as 2^k,

where k is a non-negative integer.

Check Show answer

Example 3.6.1: Exponential functions and population growth.

The growth of a population is often modeled mathematically by an exponential function. Consider an example in which the population of lizards on an island at the beginning of a study is denoted by a number p. After a time period of t days, the population is described by the function liz(t) = $p \cdot b^t$ for some number b. The number of lizards on the island must be an integer, but the function liz(t) will map positive real numbers to positive real numbers. It is understood that population models are approximate and the true population at time t will be an integer that is close to liz(t). The number b indicates how fast the population of lizards is increasing. The value of b will depend on the conditions on the island and the characteristics of the lizards. Developing a good mathematical model for population growth (finding an accurate function liz(t)) would likely require significant field work and study—but, once such a model was developed, it would be possible to answer questions such as: (approximately) how many days will it take until the population of lizards reaches n?

The value of t that satisfies the equation

$$n = p \cdot b^t \Leftrightarrow \frac{n}{p} = b^t$$

is $\log_b(n/p)$. Therefore, the number of days for the population to reach n would be approximately $\log_b(n/p)$.

Divide-and-conquer is a common strategy in computer science in which a problem is solved for a large set of items by dividing the set of items into two evenly sized groups, solving the problem on each half and then combining the solutions for the two halves. For example, one approach to sorting a list of numbers is to divide the list in half, sort each half separately and then merge the two sorted lists. The logarithm function plays an important role in analyzing divide-and-conquer algorithms. Consider the following simpler scenario:

There is an ambiguity in the problem of Ingrid and her chocolates which requires clarification. If Ingrid has an odd number of chocolates, the chocolates can not be divided exactly evenly. For example, if Ingrid has 17 chocolates, she would divide them into one pile with 8 chocolates and one

Ingrid has a bag with n chocolates. n > 0, so the bag is not empty. Ingrid meets a friend and splits her chocolates into two evenly sized piles and gives her friend the chocolates in one of the piles and puts the rest in her bag. Then she meets another friend and splits her chocolates again. Every time she meets a friend, she shares half of her chocolates with the friend. How many friends can she meet and share half her chocolates before she is down to one chocolate in her bag?

pile with 9 chocolates. In general, if there are n chocolates, one pile will have $\lceil n/2 \rceil$ chocolates and the other pile will have $\lfloor n/2 \rfloor$. Does Ingrid keep the bigger or smaller pile for herself? Does the choice of which pile she chooses affect the number of friends with whom she can divide her pile?

PARTICIPATION ACTIVITY | 3.6.4: Dividing chocolates.

Animation content:

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Animation captions:

1. Dividing a pile of 8 chocolates in half results in a pile with 4 chocolates.
2. Dividing the pile of 4 chocolates in half results in a pile with 2 chocolates.
3. Dividing the pile of 2 chocolates in half results in a pile with 1 chocolate. The number of times the pile is divided is 3 which is equal to the log base 2 of 8.
4. A pile with 7 chocolates does not divide evenly. If the smaller pile is kept, the number of times the pile is divided is 2 which is the floor of the log base 2 of 7.
5. Keeping the larger pile results in a pile with 4, then 2, then 1 chocolate. The number of times the pile is divided is 3 which is the ceiling of the log base 2 of 7.

If it happens that the number of chocolates in Ingrid's bag is a power of two ($n = 2^k$ for non-negative integer k), then every time she meets a friend, the chocolates can be evenly divided into two piles. She starts with 2^k chocolates. After meeting the first friend, she is down to $2^k/2 = 2^{k-1}$ chocolates. After the next friend, she is down to $2^{k-1}/2 = 2^{k-2}$ chocolates. Each time she meets a friend, the exponent in the number of chocolates goes down by 1, until she is down to $1 = 2^0$ chocolates. She starts with 2^k and ends up with 2^0 chocolates. The number of friends she can meet is $k = \log_2 n$.

If n is not a power of 2, then $\log_2 n$ is not an integer. Certainly, the number of encounters with friends has to be an integer. Also, if n is not a power of 2, Ingrid will be forced to divide her pile unevenly at some point in the process. If Ingrid always takes the larger half, then the number of times she can divide her pile of chocolates in half is $\lceil \log_2 n \rceil$. If she always takes the smaller half, then the number of times she can divide her pile of chocolates in half is $\lfloor \log_2 n \rfloor$.

Theorem 3.6.1: Dividing piles and the logarithm function.

Let n and b be positive integers with b > 1. Consider a process in which in each step, n is replaced with $\lfloor n/b \rfloor$, until n < b. The process lasts for $\lfloor \log_b n \rfloor$ steps.

If instead in each step, n is replaced with $\lceil n/b \rceil$, until n = 1. The process lasts for $\lceil \log_b n \rceil$ steps.

PARTICIPATION ACTIVITY | 3.6.5: Dividing chocolates and the logarithm function.

1) Suppose that Ingrid starts with 54 chocolates. In each step, she divides the chocolates into two piles as evenly as possible. If the piles can not be divided perfectly evenly, then she always takes the larger pile. Then how many times can Ingrid divide her pile?

Check | Show answer

2) Suppose that Ingrid has 27 chocolates. In every step, she meets two friends and divides her pile three ways between herself and her two friends. How many steps until she is down to one chocolate?

Check | Show answer

3) Suppose that Ingrid has 63 chocolates. In every step, she meets two friends and divides

her pile three ways between herself and her two friends. She divides the chocolates as evenly as possible, but in the event that the chocolates can not be divided perfectly evenly into three piles, she takes the larger pile for herself. How many steps until she is down to one chocolate?

Check Show answer

Additional exercises

EXERCISE 3.6.1: Find an equivalent expression - exponents.

For each expression, give an equivalent expression that is a power of 6. That is your answer should have the form 6^*, where * is an expression with numbers and possibly the variable k.

(a) $\left(6^k\right)^k$

(b) $\left(6^{2k}\right)^3$

(c) $\left(6^{2^k}\right)^2$

(d) $\dfrac{6^{2k-1}}{6^{-k}}$

(e) $\dfrac{6^{2k}}{6}$

(f) $36 \cdot 6^k$

(g) $6^{k^2} \cdot 6$

EXERCISE 3.6.2: Find an equivalent expression - logarithms.

For each expression, give an equivalent expression that is of the form $\log_5(*)$, where * is an expression with numbers and possibly the variable k.

(a) $\log_5 k + \log_5 2$

(b) $2 \cdot \log_5 k$

(c) $\log_5 k - \log_5 7$

(d) $(\log_3 k)/(\log_3 5)$

(e) $(\log_3 (k^2))/(\log_3 25)$

EXERCISE 3.6.3: Find an equivalent expression - exponents and logarithms.

For each expression, give an equivalent expression that does not use the log function.

(a) $$2^{\log_2 x}$$
(Hint: Set the expression equal to y and take the log base 2 of both sides of the equation. Remember that the log function is one-to-one, so if $\log_2 x = \log_2 y$, then x = y.)

(b) $4^{\log_2 x}$

(c) $2^{\log_4 x}$

EXERCISE 3.6.4: Approximating the log function.

The table below shows integer powers of 3 up to 6:

$3^0 = 1$
$3^1 = 3$
$3^2 = 9$
$3^3 = 27$
$3^4 = 81$
$3^5 = 243$
$3^6 = 729$

Use the values in the table to calculate the values for the expressions below. You should not need a calculator!

(a) $\lceil \log_3 189 \rceil$

(b) $\lfloor \log_3 536 \rfloor$

(c) $\lfloor \log_3 2 \rfloor$

(d) $\lceil \log_3 55 \rceil$

EXERCISE | 3.6.5: Repeated division.

(a) Start with the number n = 54527. Divide n by 5 and round the result down to an integer. Keep repeating the division and rounding step until the resulting number is less than 5. How many divisions are performed? You can use a calculator for this problem, but you should not have to actually perform all of the divisions.

(b) Start with the number n = 54527. Divide n by 5 and round the result up to an integer. Keep repeating the division and rounding step until the resulting number is equal to 1. How many divisions are performed? You can use a calculator for this problem, but you should not have to actually perform all of the divisions.

(c) Suppose that you have a calculator that performs the log function but you do not know the base of the log. How can you use your calculator to solve the two problems above?