2.1 Sets and subsets

Sets play an important role in almost every area of mathematics, including discrete math. Set theory is a well-developed branch of mathematics in its own right, most of which is beyond the scope of this material. We will start with some basic definitions, notations, and ideas related to sets, all of which will be used extensively in the rest of the topics covered.

A ***set*** is a collection of objects. Objects may be of various types, such as titles of books, names of bridges, or rational numbers. This material is mostly concerned with sets of mathematical objects like numbers. The objects in a set are called ***elements***. A set may contain elements of different varieties, e.g., a set whose elements are the number 2, a strawberry, and a monkey.

A set is defined by indicating which elements belong to it. If the number of elements in a set is small, the easiest way to describe the set is by listing its elements. The ***roster notation*** definition of a set is a list of the elements enclosed in curly braces with the individual elements separated by commas. The following definition of the set A uses roster notation:

$$A = \{ 2, 4, 6, 10 \}$$

The order in which the elements are listed is unimportant. So the set A can also be expressed as:

$$A = \{ 10, 6, 4, 2 \}$$

Repeating an element does not change the set. So the set A can also be expressed as:

$$A = \{ 2, 2, 4, 6, 10 \}$$

PARTICIPATION ACTIVITY | 2.1.1: Example of sets.

Animation captions:

1. The set { 2, 4, 6, 10 } has four real-number elements.
2. The set { Watermelon, Strawberry, Banana } has three fruit elements.
3. The set { 2, Strawberry, Monkey } has elements of different varieties.

PARTICIPATION ACTIVITY | 2.1.2: Sets in roster notation.

Select the statement that is false.

1) The set S = {1, 2, 3, 4}.

- ○ S = {4, 3, 2, 1}
- ○ S = {4, 4, 1, 2, 3}

- S = {1, 2, 2, 2, 3}

2) The set T = {a, A, c, d}

- T = {a, c, d, A, b}
- T = {d, A, A, c, a}
- T = {A, a, c, a, d, A}

More notation related to sets

The symbol ∈ is used to indicate that an element is in a set, as in $2 \in A$. The symbol ∉ indicates that an element is not in a set, as in $5 \notin A$. Typically, capital letters will be used as variables denoting sets, and lower case letters will be used for elements in the set. Variables can be used to indicate an unspecified member of a set. For example, if $a \in A$, then a is equal to 2, 4, 6, or 10.

The set with no elements is called the ***empty set*** and is denoted by the symbol ∅. The empty set is sometimes referred to as the ***null set*** and can also be denoted by {}. Because the empty set has no elements, for any element a, $a \notin \emptyset$ is true.

A ***finite set*** is a set that is either empty or whose elements can be numbered 1 through n for some positive integer n. An ***infinite set*** is a set that is not finite. The ***cardinality*** of a finite set A, denoted by |A|, is the number of distinct elements in A. If A = { 2, 4, 6, 10 }, then |A| = 4. The cardinality of the empty set |∅| is zero.

When there are many elements in a set, it may not be practical to provide an exhaustive list. In this case, ellipses (...) are used to denote a long (possibly infinite) sequence of numbers. However, the sequence's pattern should be clear so that the reader can infer which elements are missing. Below are some examples of the use of ellipses to describe sets:

$$B = \{ 1, 3, 5, \ldots , 99 \}$$

$$C = \{ 3, 6, 9, 12, \ldots \}$$

In the first example, the set B contains all odd integers between 0 and 100. In the second case, the set C is an infinite set containing all positive integer multiples of 3.

Two sets are equal if they have exactly the same elements. Define:

$$D = \{ 3, 4, 5 \}$$

$$E = \{ 5, 3, 4 \}$$

$$F = \{ 5, 3, 4, 6 \}$$

D = E because for any a, $a \in D$ if and only if $a \in E$. Remember that the order in which the elements are listed in the definitions of D and E is unimportant. $F \neq E$ because $6 \in F$ but $6 \notin E$.

PARTICIPATION ACTIVITY | 2.1.3: Set basics.

Animation captions:

1. In roster notation, the order in which the elements are listed does not matter.
2. |A| is the cardinality of A, which is the number of distinct elements in A. If the cardinality of A is finite, then A is a finite set.
3. $2 \in A$ indicates that 2 is an element of set A. $5 \notin A$ indicates that 5 is not an element of set A.
4. The empty set has no elements and is denoted $\varnothing$.

PARTICIPATION ACTIVITY | 2.1.4: Set membership and cardinality.

Consider the following sets:

A = { 4, 6, 3 }
B = { 2, 4, 6, ..., 20 }
C = { 2, 4, 6, ... }
D = { 3, 4, 6 }

1) A = D

- True
- False

2) $5 \in A$

- True
- False

3) $5 \in \varnothing$

- True
- False

4) C is finite.

- True
- False

5) |B| = 10

True

False

6) |A| = |D|

True

False

Example 2.1.1: Sets in applications.

Many pieces of software need to maintain sets of items. For example, a database is a large set of pieces of information. A university maintains a set of all the students enrolled. An airline maintains a set of all past and future flights. One of the most basic types of operations a programmer might perform on a set of data is to ask whether a particular item is an element of that set. Discrete math is primarily concerned with mathematical operations on sets, but software developers need to think about how to actually represent the set in order to satisfy queries to the set of items.

Some sets of numbers are used so frequently in mathematics that they have their own symbols. Here are some of the most common examples:

Table 2.1.1: Common mathematical sets.

Set	Symbol	Examples of elements
N is the set of ***natural numbers***, which includes all integers greater than or equal to 0.	**N**	0, 1, 2, 3, ...
Z is the set of all integers.	**Z**	..., -2, -1, 0, 1, 2, ...
Q is the set of ***rational numbers***, which includes all real numbers that can be expressed as a/b, where a and b are integers and b ≠ 0.	**Q**	0, 1/2, 5.23, -5/3
R is the set of real numbers.	**R**	0, 1/2, 5.23, -5/3, π, $\sqrt{2}$

The superscript + is used to indicate the positive elements of a particular set. For example, the set $\mathbf{R}^+$ is the set of all positive real numbers, and $\mathbf{Z}^+$ is the set of all positive integers. A number x is

positive if x > 0. The superscript - is used to indicate the negative elements of a particular set. For example, the set $\mathbf{R}^-$ is the set of all negative real numbers, and $\mathbf{Z}^-$ is the set of all negative integers. A number x is ***negative*** if x < 0. The number 0 is neither positive nor negative, so $0 \notin \mathbf{Z}^+$ and $0 \notin \mathbf{Z}^-$.

A number x is ***non-negative*** if x ≥ 0. The natural numbers, as defined in the table above, is equal to the set of non-negative integers. Some authors define the natural numbers to be the set of positive integers (i.e., excluding 0).

PARTICIPATION ACTIVITY | 2.1.5: Membership in numerical sets.

1) $-3 \in \mathbf{Z}^+$
   - True
   - False

2) $0 \in \mathbf{Z}^+$
   - True
   - False

3) 0 is a non-negative integer.
   - True
   - False

4) $5 \in \mathbf{R}^+$
   - True
   - False

5) -5 is a member of the set of all non-negative integers.
   - True
   - False

6) $0 \in \mathbf{Q}$
   - True
   - False

In ***set builder notation***, a set is defined by specifying that the set includes all elements in a larger set that also satisfy certain conditions. The notation would look like:

$$A = \{ x \in S : P(x) \}$$

S is the larger set from which the elements in A are taken. P(x) is some condition for membership in A. The colon symbol ":" is read "such that". The description for A above would read: "all x in S such that P(x)". Often, the set S will be one of the standard mathematical sets from the table above. For example, the set:

$$C = \{ x \in \mathbf{Z} : 0 < x < 100 \text{ and } x \text{ is prime}\}$$

would be all prime integers between 0 and 100. The set:

$$D = \{ x \in \mathbf{R} : |x| < 1 \}$$

would be all real numbers between -1 and 1, not including 1 or -1.

PARTICIPATION ACTIVITY 2.1.6: Set equality.

For each set, find the matching set.

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

{ 0, 2, 4, 6, ... } **{ 2, 3, 5, 7, 11, 13, 17, 19 }** **{ 0, 2, 4, 6, ..., 18 }** **{ -1, 0 ,1 }**

$\{ x \in \mathbf{Z} : 0 < x < 20 \text{ and } x \text{ is prime} \}$

$\{ x \in \mathbf{R} : |x| = x^2 \}$

$\{ x \in \mathbf{N} : x \text{ is even} \}$

$\{ x \in \mathbf{N} : x \text{ is even and } x < 20 \}$

Reset

CHALLENGE ACTIVITY 2.1.1: Cardinality and equality of sets.

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John Farrell
COLOSTATECS220SeaboltFall2022

Start

Fill in the missing value such that $A = B$.

$$A = \{7, 3, 6\}$$

Ex: 5

$B = \{6, 7, \quad\quad \}$

1	2	3	4

Check Next

The ***universal set***, usually denoted by the variable *U*, is a set that contains all elements mentioned in a particular context. For example, in a discussion about certain types of real numbers, it would be understood that any element in the discussion is a real number. In this case, the universal set would be the set of real numbers. In a discussion regarding the academic standing of certain students at a school, the universal set would be the set of all students enrolled in the school.

Sets are often represented pictorially with ***Venn diagrams***. A rectangle is used to denote the universal set *U*, and oval shapes are used to denote sets within *U*. Sometimes, Venn diagrams can indicate which specific elements are inside and outside the set. An element is drawn inside the oval if it is in the set represented by the oval. The following animation illustrates the use of Venn diagrams.

PARTICIPATION ACTIVITY | 2.1.7: Venn diagrams.

Animation captions:

1. A Venn diagram depicts set A = {1, 2, 3} as an oval (or circle) that contains 1, 2, and 3. The element 4 is outside the circle for A because $4 \notin A$.
2. If the set B = {2, 3, 4} is added to the Venn diagram, then the circle for B contains 2, 3, and 4 but does not contain 1.

If every element in A is also an element of B, then A is a ***subset*** of B, denoted as $A \subseteq B$. If there is an element of A that is not an element of B, then A is not a subset of B, denoted as $A \nsubseteq B$. If the universal set is *U*, then for every set A:

$$\varnothing \subseteq A \subseteq U$$

Two sets are equal if and only if each is a subset of the other:

$$A = B \text{ if and only if } A \subseteq B \text{ and } B \subseteq A$$

If $A \subseteq B$ and there is an element of B that is not an element of A (i.e., $A \neq B$), then A is a ***proper subset*** of B, denoted as $A \subset B$. Venn diagrams are particularly useful for visualizing subset relationships between sets.

PARTICIPATION ACTIVITY | 2.1.8: Venn diagrams with subsets.

Animation captions:

1. B = {2, 4} is a subset of A = {1, 2, 3, 4}. In a Venn diagram, the oval for B is inside the circle for A. Since $3 \in A$ and $3 \notin B$, B is a proper subset of A ($B \subset A$).
2. C = {2, 4, 5} is not a subset of A ($C \not\subseteq A$) because $5 \in C$ and $5 \notin A$. In the Venn diagram, the circle for C is not contained in the circle for A.
3. If D = {1, 2, 3, 4}, $D \subseteq A$ and $A \subseteq D$, which implies that A = D. D is not a proper subset of A ($D \not\subset A$).

PARTICIPATION ACTIVITY 2.1.9: Sets and subsets.

Consider the following sets:

A = { 3, 4, 5 }
B = { 4, 5, 3 }
C = { x $\in$ **Z**: x is odd }
D = { 3, 5, 7, 9 }

1) $A \subset B$?
- Yes
- No

2) $A \subseteq B$?
- Yes
- No

3) $C \subset \mathbf{Z}$?
- Yes
- No

4) $B \subseteq D$?
- Yes
- No

5) $D \subseteq C$?
- Yes
- No

6) $\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R}$?

- Yes
- No

7) Is the following statement true?
For any two sets, X and Y, if $X \subset Y$, then $X \subseteq Y$.

- Yes
- No

8) Is the following statement true?
For any two sets, X and Y, if $X \subseteq Y$, then $X \subset Y$.

- Yes
- No

Additional exercises

EXERCISE | 2.1.1: Set membership and subsets - true or false.

Use the definitions for the sets given below to determine whether each statement is true or false:

$A = \{ x \in \mathbf{Z}: x \text{ is an integer multiple of } 3 \}$
$B = \{ x \in \mathbf{Z}: x \text{ is a perfect square} \}$
$C = \{ 4, 5, 9, 10 \}$
$D = \{ 2, 4, 11, 14 \}$
$E = \{ 3, 6, 9 \}$
$F = \{ 4, 6, 16 \}$

An integer x is a perfect square if there is an integer y such that $x = y^2$.

(a) $27 \in A$

(b) $27 \in B$

(c) $100 \in B$.

(d) $E \subseteq C$ or $C \subseteq E$.

(e) $E \subseteq A$

(f) $A \subset E$

(g) $E \in A$

EXERCISE | 2.1.2: Set membership and subsets - true or false, cont.

Use the definitions for the sets given below to determine whether each statement is true or false:

$A = \{ x \in \mathbf{Z}: x \text{ is an integer multiple of } 3 \}$
$B = \{ x \in \mathbf{Z}: x \text{ is a perfect square} \}$
$C = \{ 4, 5, 9, 10 \}$
$D = \{ 2, 4, 11, 14 \}$
$E = \{ 3, 6, 9 \}$
$F = \{ 4, 6, 16 \}$

An integer x is a perfect square if there is an integer y such that $x = y^2$.

(a) $15 \subset A$

(b) $\{15\} \subset A$

(c) $\varnothing \subset C$

(d) $D \subseteq D$

(e) $\varnothing \in B$

(f) A is an infinite set.

(g) B is a finite set.

(h) $|E| = 3$

(i) $|E| = |F|$

EXERCISE | 2.1.3: Subset relationships between common numerical sets.

Indicate whether the statement is true or false.

(a) $\mathbf{Z} \subset \mathbf{R}$

(b) $\mathbf{Z} \subseteq \mathbf{R}$

(c) $\mathbf{Z} \subseteq \mathbf{R}^+$

(d) $\mathbf{N} \subset \mathbf{R}$

(e) $\mathbf{Z}^+ \subset \mathbf{N}$

EXERCISE | 2.1.4: Sets, subsets, and set equality.

Define sets A, B and C as follows:

$A = \{2, 4, 6, 8\}$
$B = \{x \in \mathbf{Z}\text{: } x \text{ is even and } 0 < x < 10\}$
$C = \{x \in \mathbf{Z}\text{: } x \text{ is even and } 0 < x \leq 10\}$

Indicate whether each statement about the sets A, B and C is true or false.

(a) $A \subseteq B$

(b) $A \subset B$

(c) $A \subseteq C$

(d) $A \subset C$

(e) $C \subseteq B$

(f) $A = C$

(g) $A = B$

EXERCISE | 2.1.5: Expressing sets in set builder notation.

Express each set using set builder notation. Then if the set is finite, give its cardinality. Otherwise, indicate that the set is infinite.

(a) {-2, -1, 0, 1, 2}

(b) { 3, 6, 9, 12, ... }

(c) { -3, -1, 1, 3, 5, 7, 9 }

(d) { 0, 10, 20, 30, ..., 1000 }

EXERCISE | 2.1.6: Which statements are true for all sets?

Determine whether each statement is true or false for any two sets A and B. If the statement is false, explain why.

(a) If $A \subseteq B$, then $A \subset B$.

(b) If $A \subset B$, then $A \subseteq B$.

(c) If $A = B$, then $A \subseteq B$.

(d) If $A = B$, then $A \subset B$.

(e) If $A \subset B$, then $A \neq B$.

2.2 Set of sets

It is possible that the elements of a set are themselves sets. For example, consider the set A:

$$A = \{ \{ 1, 2 \}, \varnothing, \{ 1, 2, 3 \}, \{ 1 \} \}$$

The set A has four elements: $\{ 1, 2 \}$, $\varnothing$, $\{ 1, 2, 3 \}$, and $\{ 1 \}$. For example, $\{ 1, 2 \} \in A$. Note that 1 is not an element of A, so $1 \notin A$, although $\{ 1 \} \in A$. Furthermore, $\{ 1 \} \not\subseteq A$ since $1 \notin A$.

The empty set $\varnothing$ is not the same as $\{ \varnothing \}$. The cardinality of $\{ \varnothing \}$ is one since it contains exactly one element, which is the empty set. A set can contain a combination of numbers and sets of numbers as in:

$$B = \{ 2 , \varnothing, \{ 1, 2, 3 \}, \{ 1 \} \}$$

Then 2 ∈ B, but 1 ∉ B. Also, { 2 } ⊆ B, but { 1 } ⊈ B.

PARTICIPATION ACTIVITY | 2.2.1: Sets of sets.

Animation captions:

1. The cardinality of set A = { {1, 2}, 1, 2, {1, 2, 3} } is 4. The elements are {1, 2}, 1, 2, and {1, 2, 3}.
2. The set { {1, 2}, 1 } is a proper subset of A. {1, 2} and 1 are both elements of A.

PARTICIPATION ACTIVITY | 2.2.2: Sets of sets - true or false.

Consider the following set:

$$A = \{3, 4, \{3, 4\}, \{1, 2, 3\}, 5\}$$

1) The cardinality of A is 5.

- True
- False

2) $\{3\} \subseteq A$

- True
- False

3) $\{1\} \subseteq A$

- True
- False

4) $\{1, 2, 3\} \subseteq A$

- True
- False

The ***power set*** of a set A, denoted P(A), is the set of all subsets of A. For example, if A = { 1, 2, 3 }, then:

P(A) = { ∅, { 1 }, { 2 }, { 3 }, { 1, 2 }, { 1, 3 }, { 2, 3 }, { 1, 2, 3 } }

PARTICIPATION ACTIVITY | 2.2.3: Power sets.

Animation captions:

1. The set A = { ◯, □, △} has one subset of size 0, which is ∅.
2. The subsets of A of size 1 are { ◯ }, { □ }, and { △ }.
3. The subsets of A of size 2 are { ◯, □ }, { ◯, △ }, and { □, △ }.
4. A has one subset of size 3, which is { ◯, □, △ }.
5. The power set of A (denoted P(A)) is the set whose elements are all the subsets of A.

Since |A| = 3, the cardinality of the power set of A is $2^3 = 8$. Here is the general rule for the cardinality of a power set:

Theorem 2.2.1: Cardinality of a power set.

Let A be a finite set of cardinality n. Then the cardinality of the power set of A is 2^n, or $|P(A)|=2^n$.

Example 2.2.1: Power sets - packing a care package.

Suppose that Carmen would like to pack a care package for her brother who has gone away to college. She has four items in her kitchen that she might include: a bag of caramel corn, a can of juice, a small package of fudge, and an orange. She has a cardboard box that she can use for shipping, and she would like to fill the box as much as possible without exceeding the capacity of the box. She must consider all possible subsets of the four items to see whether they fit in the box and how much of the box each selection fills. In other words, she must consider the power set of the set of four snacks. There are 16 = 2^4 options for her to consider:

{ caramel corn, juice, fudge, orange }
{ caramel corn, juice, fudge }
{ caramel corn, juice, orange }
{ caramel corn, fudge, orange }
{ juice, fudge, orange }
{ caramel corn, juice }
{ caramel corn, fudge }
{ caramel corn, orange }
{ juice, fudge }
{ juice, orange }
{ fudge, orange }
{ caramel corn }
{ juice }
{ fudge }
{ orange }
{ }

PARTICIPATION ACTIVITY | 2.2.4: Power sets - true or false.

Consider the following set:

A = { 2, 3, 5, 7, 14 }

1) 2 ∈ P(A).

- True
- False

2) { 2 } ∈ P(A).

- True
- False

3) { 2, 3 } ∈ A.
- True
- False

4) { 2, 3 } ∈ P(A).
- True
- False

5) |P(A)| = 32.
- True
- False

CHALLENGE ACTIVITY | 2.2.1: Sets of sets and power sets.

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Start

$$A = \{8, \{3\}\}$$

$|A| =$ Ex: 5

1 2 3 4

Check Next

Additional exercises

EXERCISE 2.2.1: Sets of sets - true or false.

Let $X = \{1, \{1\}, \{1, 2\}, 2, \{3\}, 4\}$. Which statements are true?

(a) $2 \in X$

(b) $\{2\} \subseteq X$

(c) $\{2\} \in X$

(d) $3 \in X$

(e) $\{1, 2\} \in X$

(f) $\{1, 2\} \subseteq X$

(g) $\{2, 4\} \subseteq X$

(h) $\{2, 4\} \in X$

(i) $\{2, 3\} \subseteq X$

(j) $\{2, 3\} \in X$

(k) $|X| = 7$

EXERCISE 2.2.2: Write out a power set in roster notation.

Write the power set of each set in roster notation.

(a) $\{a\}$

(b) $\{1, 2\}$

EXERCISE 2.2.3: The cardinality of a power set.

(a) What is the cardinality of $P(\{1, 2, 3, 4, 5, 6\})$?

EXERCISE 2.2.4: A subset of a power set.

(a) Let X = {a, b, c, d}. What is { A: A ∈ P(X) and |A| = 2 }?

(b) Let A = {1, 2, 3}. What is {X ∈ P(A): 2 ∈ X}?

EXERCISE 2.2.5: Facts about power sets - true or false.

Let X be a finite set. For each statement, indicate whether the statement is true or false, or whether more information about X is required to determine if the statement is true.

(a) ∅ ∈ P(X)

(b) ∅ ⊆ P(X)

(c) ∅ ⊂ P(X)

(d) {∅} ⊂ P(X)

(e) |P(X)| = 17

(f) |P(X)| = 0

EXERCISE 2.2.6: Power sets of power sets.

Express each set using roster notation. Then give the cardinality of the set.

(a) P(∅)

(b) P(P(∅))

(c) P(P(P(∅)))

2.3 Union and intersection

Sets can be combined in different ways to define new sets. This section introduces some standard set operations that allow new sets to be created from sets that have already been defined.

The set intersection operation

Let A and B be sets. The ***intersection*** of A and B, denoted A ∩ B and read "A intersect B", is the set of all elements that are elements of both A and B. The following animation illustrates the use of the intersection operation:

PARTICIPATION ACTIVITY | 2.3.1: The set intersection operation.

Animation captions:

1. The set A is {a, b, c, e, f}.
2. The set B is {d, e, f, g}.
3. The two elements in both A and B are e and f. Therefore A ∩ B = {e, f}.

The intersection operation can also apply to infinite sets:

$$A = \{ x \in \mathbf{Z}\text{: } x \text{ is an integer multiple of } 2 \}$$

$$B = \{ x \in \mathbf{Z}\text{: } x \text{ is an integer multiple of } 3 \}$$

$$A \cap B = \{ x \in \mathbf{Z}\text{: } x \text{ is an integer multiple of } 6 \}$$

In the above example the set A ∩ B is the set of all integers that are both an integer multiple of 2 and an integer multiple of 3, which is exactly the set of all integers that are multiples of 6.

The set union operation

The ***union*** of two sets, A and B, denoted A ∪ B and read "A union B", is the set of all elements that are elements of A or B. The definition for union uses the inclusive or, meaning that if an element is an element of both A and B, then it is also an element of A ∪ B. The following animation illustrates the union operation:

PARTICIPATION ACTIVITY | 2.3.2: The set union operation.

Animation captions:

1. The set A is {a, b, c, e, f}. The set B is {d, e, f, g}.
2. A ∪ B consists of those elements that are in A or B or both. Therefore, A ∪ B = {a, b, c, d, e, f, g}.

Consider a situation in which students in a class are allowed to skip the final exam if they received an A on the first or second midterm. Define sets A and B to be:

A = { x: student x received an A on midterm 1 }

B = { x: student x received an A on midterm 2 }

Then

A ∪ B = { x: student x received an A on midterm 1 or midterm 2 }

= { x: student x is eligible to skip the final exam }

PARTICIPATION ACTIVITY 2.3.3: Set intersection and union applied to 2 sets.

Use the definitions below to match equivalent sets.

A = { 1, 2, 4, 11, 12 }
B = { 2, 4, 7, 8, 11 }
C = { x ∈ **Z**: x is even }
D = { x ∈ **Z**: x is odd }

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

A ∩ C   **A ∪ B**   **C ∪ D**   **A ∩ B**   **B ∩ D**

{ 2, 4, 11 }

{ 2, 4, 12 }

Z

{ 1, 2, 4, 7, 8, 11, 12 }

{ 7, 11 }

Reset

Set operations can be combined to define even more sets. For example, the set A ∪ (B ∩ C) is the union of the set A and the set B ∩ C. The use of parentheses is important since the set (A ∪ B) ∩ C is different from the set A ∪ (B ∩ C).

Here are two animations that utilize Venn diagrams to illustrate the use of more than one set operation in defining a set:

PARTICIPATION ACTIVITY | 2.3.4: Multiple set operations.

Animation captions:

1. The set A is {1, 2, 3, 4}.
2. The set B is {3, 4, 5, 6}. The set C is {2, 3, 5, 7}.
3. To determine A ∩ (B ∪ C), first determine B ∪ C, which is {2, 3, 4, 5, 6, 7}.
4. Then take the intersection of A and B ∪ C.
5. The elements that are in A and B ∪ C are 2, 3, 4. Therefore, A ∩ (B ∪ C) = {2, 3, 4}.

PARTICIPATION ACTIVITY | 2.3.5: Multiple set operations.

Animation captions:

1. The set A is {1, 2, 3, 4}. The set B is {3, 4, 5, 6}. The set C is {2, 3, 5, 7}.
2. To determine A ∪ (B ∩ C), first determine B ∩ C, which is {3, 5}.
3. Then take the union of A and B ∩ C.
4. The elements that are in A or B ∩ C or both are 1, 2, 3, 4, and 5. Therefore, A ∪ (B ∩ C) = {1, 2, 3, 4, 5}.

The expression A ∩ B ∩ C ∩ D is well-defined because the order in which intersection operations are applied does not matter. The set defined by the expression consists of the elements that are elements of all four sets: A, B, C, and D. Similarly, the expression A ∪ B ∪ C ∪ D is also well-defined and defines the set consisting of those elements that are elements of at least one of the four sets: A, B, C, and D.

PARTICIPATION ACTIVITY | 2.3.6: Multiple intersection and union operations.

Use the definitions below to match equivalent sets.

A = { 1, 2, 4, 11, 12 }
B = { 2, 4, 7, 8, 11 }
C = { x ∈ **Z**: x is even }
D = { x ∈ **Z**: x is odd }

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

(C ∪ D) ∩ B   **A ∩ B ∩ C**   **(A ∩ D) ∪ B**   **(A ∩ D) ∪ (B ∩ C)**

{ 2, 4, 7, 8, 11 }

{ 1, 2, 4, 7, 8, 11 }

{ 1, 2, 4, 8, 11 }

{ 2, 4 }

Reset

A special notation allows for a compact representation of the intersection of a long sequence of sets A_1, A_2, ..., A_n. In the notation below, the expression "i = 1" below the intersection sign and the "n" above the intersection sign indicates that the intersection operation will be applied to all sets with integer indices i ranging from 1 through n.

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n = \{x : x \in A_i \text{ for all } i \text{ such that } 1 \leq i \leq n\}$$

Similarly, it is possible to express the union of a sequence of sets A_1, A_2, ..., A_n:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = \{x : x \in A_i \text{ for some } i \text{ such that } 1 \leq i \leq n\}$$

PARTICIPATION ACTIVITY | 2.3.7: Unions and intersections of sequences of sets.

For this question, the universal set is the set of words in the Oxford English Dictionary (OED). For each positive integer j, define A_j to be the set of all words with j letters in the OED. For example, the word "discrete" is an element of A_8 because the word "discrete" has 8 letters. The longest word in the OED is "pneumonoultramicroscopicsilicovolcanoconiosis" which has 45 letters. Match the definition of each set to its description below.

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

$\bigcap_{j=1}^{45} A_j$ $\bigcup_{j=1}^{10} A_j$ $\bigcup_{j=1}^{45} A_j$

∅

The set of all words in the OED.

The set of all words with 10 or fewer letters in the OED.

Reset

CHALLENGE ACTIVITY 2.3.1: Union and intersection operations with 3 sets.

422102.2723990.qx3zqy7

Start

Select the regions corresponding to the set denoted by the given expression.

1	2	3	4	5

Check Next

Additional exercises

EXERCISE 2.3.1: Unions and intersections of sets.

Define the sets A, B, C, and D as follows:

$A = \{-3, 0, 1, 4, 17\}$
$B = \{-12, -5, 1, 4, 6\}$
$C = \{x \in \mathbf{Z}: x \text{ is odd}\}$
$D = \{x \in \mathbf{Z}: x \text{ is positive}\}$

For each of the following set expressions, if the corresponding set is finite, express the set using roster notation. Otherwise, indicate that the set is infinite.

(a) $A \cup B$

(b) $A \cap B$

(c) $A \cap C$

(d) $A \cup (B \cap C)$

(e) $A \cap B \cap C$

(f) $A \cup C$

(g) $(A \cup B) \cap C$

(h) $A \cup (C \cap D)$

EXERCISE 2.3.2: Unions and intersections of sequences of sets.

Use the definition for A_i to answer the questions.

For i $\in \mathbf{Z}^+$, A_i is the set of all positive integer multiples of i.

(a) Describe the following set using set builder notation:

$$\bigcap_{i=1}^{5} A_i$$

(b) Describe the following set using roster notation:

$$\left(\bigcup_{i=2}^{5} A_i\right) \cap \{x \in \mathbf{Z} : 1 \leq x \leq 20\}$$

EXERCISE 2.3.3: Unions and intersections of sequences of sets, part 2.

Use the following definitions to express each union or intersection given. You can use roster or set builder notation in your responses, but no set operations. For each definition, i $\in \mathbf{Z}^+$.

- $A_i = \{i^0, i^1, i^2\}$ (Recall that for any number x, $x^0 = 1$.)
- $B_i = \{x \in \mathbf{R} : -i \leq x \leq 1/i\}$
- $C_i = \{x \in \mathbf{R} : -1/i \leq x \leq 1/i\}$

(a) $\bigcap_{i=2}^{5} A_i$

(b) $\bigcup_{i=2}^{5} A_i$

(c) $\bigcap_{i=1}^{100} B_i$

(d) $\bigcup_{i=1}^{100} B_i$

(e) $\bigcap_{i=1}^{100} C_i$

(f) $\bigcup_{i=1}^{100} C_i$

EXERCISE | 2.3.4: Power sets and set operations.

Use the set definitions A = {a, b} and B = {b, c} to express each set below. Use roster notation in your solutions.

(a) P(A ∩ B)

(b) P(A ∪ B)

(c) P(A) ∩ P(B)

(d) P(A) ∪ P(B)

2.4 More set operations

Set difference and symmetric difference

Several more operations are commonly used to define sets. The ***difference*** between two sets A and B, denoted A - B, is the set of elements that are in A but not in B. The difference operation is illustrated in the animation below.

PARTICIPATION ACTIVITY | 2.4.1: The set difference operation.

Animation captions:

1. The set A is {a, b, c, e, f} and the set B is {d, e, f, g}.
2. To determine A - B, find the elements that are in both A and B (e and f) and remove those elements from A. A - B = {a, b, c}.

The difference operation is not commutative since it is not necessarily the case that A - B = B - A. By contrast the symmetric difference is commutative. The ***symmetric difference*** between two sets, A and B, denoted A ⊕ B, is the set of elements that are a member of exactly one of A and B, but not both. An alternative definition of the symmetric difference operation is:

$$A \oplus B = (A - B) \cup (B - A)$$

The symmetric difference operation is illustrated in the animation below:

PARTICIPATION ACTIVITY | 2.4.2: The symmetric difference operation.

Animation captions:

1. The set A is {a, b, c, e, f} and the set B is {d, e, f, g}.
2. To determine A $\oplus$ B, remove the elements that are in both A and B (e and f) and take the remaining elements that are in A or B. A $\oplus$ B = {a, b, c, d, g}

The set complement operation

The set operations defined so far have not required a formal definition of the universal set *U*. For example, A ∩ B is well-defined even if *U* is unknown, as long as A and B are well-defined. By contrast, determining the complement of a set requires that the universal set be well-defined. The ***complement*** of a set A, denoted $\overline{A}$, is the set of all elements in *U* that are not elements of A. An alternative definition of $\overline{A}$ is *U* - A. For example, let *U* = **Z**, and define:

$$A = \{ x \in \mathbf{Z} : x \text{ is odd} \}$$

The complement of A is the set of all even integers. The animation below illustrates the complement of a finite set.

PARTICIPATION ACTIVITY | 2.4.3: The complement operation.

Animation captions:

1. The universal set U is {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. The set A is {4, 5, 6, 7}.
2. The complement of A is found by removing the elements of A from U. Therefore, the complement of A is {1, 2, 3, 8, 9, 10}.

The table below summarizes the set operations defined in this material.

Table 2.4.1: Summary of set operations.

Operation	Notation	Description
Intersection	A ∩ B	{ x : x ∈ A and x ∈ B }
Union	A ∪ B	{ x : x ∈ A or x ∈ B or both }
Difference	A - B	{ x : x ∈ A and x ∉ B }
Symmetric difference	A ⊕ B	{ x : x ∈ A - B or x ∈ B - A }
Complement	$\overline{A}$	{ x : x ∉ A }

Set operations can be combined in different ways to define new sets. Here is an animation that uses Venn diagrams to illustrate how set operations can be combined:

PARTICIPATION ACTIVITY 2.4.4: Combining set operations to create new sets.

Animation captions:

1. To determine $\overline{(A \oplus B)} \cap C$ using a Venn diagram, first shade in the areas for $A \oplus B$.
2. Then use a second color to shade in the areas not shaded in with the first color. The second color corresponds to $\overline{(A \oplus B)}$, the complement of $A \oplus B$.
3. Then use a third color to shade in the areas in the intersection of the complement of $A \oplus B$ and C. The third color corresponds to $\overline{(A \oplus B)} \cap C$.

CHALLENGE ACTIVITY 2.4.1: Set operations with 2 sets.

422102.2723990.qx3zqy7

Start

Select the regions corresponding to the set denoted by the given expression.

1 2 3 4 5

CHALLENGE ACTIVITY | 2.4.2: Set operations with 3 sets.

422102.2723990.qx3zqy7

Start

Select the regions corresponding to the set denoted by the given expression.

1	2	3	4	5

Check Next

PARTICIPATION ACTIVITY | 2.4.5: Combinations of set operations.

Use the definitions below to match equivalent sets. The universal set is **Z**.

A = { 1, 2, 4, 11, 12 }
B = { 2, 4, 7, 8, 11 }

C = { x ∈ **Z**: x is even }
D = { x ∈ **Z**: x is odd }

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

C ⊕ D **D - B** **A - (B ∩ C)** **($\overline{\text{B ∪ C}}$) ∩ A** **(A - B) ∩ C**

{ 12 }

The set of odd numbers except 7 and 11

Z

{ 1 }

{ 1, 11, 12 }

Reset

CHALLENGE ACTIVITY 2.4.3: List all elements of given set expression.

422102.2723990.qx3zqy7

Start

$$A = \{1, 4, 5, 8\}$$
$$B = \{1, 3, 4\}$$

$B \cap A = \{$ Ex: 1, 2, 3 $\}$

1 2 3 4 5 6 7

Check Next

Additional exercises

EXERCISE 2.4.1: Set subtraction is not associative.

(a) Give an example showing that subtraction is not associative: A - (B - C) ≠ (A - B) - C.

EXERCISE 2.4.2: Symmetric difference applied to many sets.

(a) Calculate $A \oplus B \oplus C$ for A = {1, 2, 3, 5}, B = {1, 2, 4, 6}, C = {1, 3, 4, 7}.
Note that the symmetric difference operation is associative: $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

(b) Let A, B, and C be any finite sets. Give a concise description for which elements from A, B, and C are in $A \oplus B \oplus C$.

(c) Let $A_1, A_2, \ldots, A_n$ be any finite sets. Give a concise description for which elements from $A_1, A_2, \ldots, A_n$ are in $A_1 \oplus A_2 \oplus \ldots \oplus A_n$.

EXERCISE 2.4.3: Set operations.

Define the following sets.

- A = {x ∈ **Z**: x is a multiple of 3}
- B = {3, 5, 7, 9}
- C = {2, 3, 4, 5}

Indicate whether each statement is true or false.

(a) $|B| = |C|$

(b) $|A \cap B| = |A \cap C|$

(c) $A \cap C \subseteq A \cap B$

(d) $C - B \subseteq \overline{A}$

(e) $B \cup C = \{3, 5\}$

(f) $2 \in A \cup C$

(g) $\{2, 3\} \in C$

(h) $\{3\} \in P(C)$

(i) $\overline{A} \cap B \cap C = \varnothing$

(j) $\varnothing \in A$

EXERCISE 2.4.4: Set operations, part 2.

Sets A through H are defined as follows.

- $A = \{1, 2, 3, 4\}$
- $B = \{-1, -2, -3\}$
- $C = \{-1, 0, 1, 2, 3\}$
- $D = \{2, 3, 4, 5, 6, 7\}$
- $E = \{x \in \mathbf{Z}: x \text{ is odd}\}$
- $F = \{x \in \mathbf{Z}^+: x \leq 7\}$
- $G = \{x \in \mathbf{Z}^+: x < 7\}$
- $H = \{x \in \mathbf{Z}^+: x \leq 6\}$

Indicate whether each statement is true or false.

(a) $|A \cap B| = 1$

(b) $\{1, 2\} \subset P(A)$

(c) $G \subseteq H$

(d) $|C - F| = 1$

(e) $A \cup B = A \oplus B$

(f) $1 \in A \cap B \cap C$

(g) $\varnothing \in C$

(h) $\{\{0\}\} \subseteq P(C)$

(i) $C \cap F = C \cap G$

(j) $E \cup F \subseteq \mathbf{R}$

(k) $\varnothing \in P(B)$

EXERCISE 2.4.5: Reasoning about the symmetric difference operation.

Select from the following options to describe each set given below.

- $\varnothing$
- A
- B
- Not enough information given.

(a) $A \oplus \varnothing$

(b) $A \oplus A$

(c) $A \oplus B$

(d) $A \oplus A \oplus A$

(e) $A \oplus (A \oplus B)$

2.5 Set identities

The set operations intersection, union and complement are defined in terms of logical operations. All the elements are assumed to be contained in a universal set *U*.

$$x \in A \cap B \leftrightarrow (x \in A) \land (x \in B)$$

$$x \in A \cup B \leftrightarrow (x \in A) \lor (x \in B)$$

$$x \in \overline{A} \leftrightarrow \neg(x \in A)$$

The sets *U* and $\varnothing$ correspond to the constants true (T) and false (F):

$$x \in \varnothing \leftrightarrow F$$

$$x \in U \leftrightarrow T$$

The laws of propositional logic can be used to derive corresponding set identities. A ***set identity*** is an equation involving sets that is true regardless of the contents of the sets in the expression. The idea is similar to an equivalence in logic which holds regardless of the truth values of the individual variable in the expressions. The animation below shows the derivation of De Morgan's set identity using De Morgan's law for propositional logic.

PARTICIPATION ACTIVITY | 2.5.1: Deriving De Morgan's set identity using logic.

Animation captions:

1. The definition of set complement says that x is an element of the complement of $A \cap B$ if and only if $\neg(x \in A \cap B)$.
2. The definition of set intersection says that $\neg(x \in A \cap B)$ if and only if $\neg(x \in A \wedge x \in B)$.
3. De Morgan's law for propositions says that $\neg(x \in A \wedge x \in B)$ is equivalent to $\neg(x \in A) \vee \neg(x \in B)$.
4. By the definition of set complement: $\neg(x \in A)$ if and only if x is in the complement of A $(\overline{A})$ and $\neg(x \in B)$ if and only if x is in the complement of B $(\overline{B})$.
5. The definition of set union says that $x \in \overline{A} \vee x \in \overline{B}$ if and only if $x \in (\overline{A} \cup \overline{B})$.
6. Therefore, De Morgan's set identity is true: the complement of $A \cap B$ is equal to the union of the complement of A and the complement of B.

All the set identities in the table below can be proven in a similar manner using the definitions for set operations and the laws of propositional logic.

Table 2.5.1: Set identities.

Name	Identities	
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	$A \cup \varnothing = A$	$A \cap U = A$
Domination laws	$A \cap \varnothing = \varnothing$	$A \cup U = U$
Double complement law	$\overline{\overline{A}} = A$	
Complement laws	$A \cap \overline{A} = \varnothing$ $\overline{U} = \varnothing$	$A \cup \overline{A} = U$ $\overline{\varnothing} = U$
De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$

PARTICIPATION ACTIVITY | 2.5.2: Using the set identities.

Match equivalent sets.

Mouse: Drag/drop. Refresh the page if unable to drag and drop.

C **B** ***U*** **∅** **B ∪ C**

$\varnothing \cap (B \cup C)$

$B \cup \varnothing$

$B \cup \overline{B}$

$C \cup (C \cap B)$

$\overline{B \cup C}$

Reset

CHALLENGE ACTIVITY | 2.5.1: Set identities.

422102.2723990.qx3zqy7

Start

Select the set that is equivalent to:

$$(B \cup A) \cup (B \cup A)$$

- [ ] A
- [ ] $\varnothing$
- [ ] $B \cup A$
- [ ] B

Select the set identity that shows the two sets are equivalent.

Complement law

1	2	3	4

Check Next

Additional exercises

EXERCISE | 2.5.1: Name the set identity.

Name the set identity that is used to justify each of the identities given below.

(a) $(B \cap C) \cup \overline{B \cap C} = U$

(b) $\overline{A \cup (A \cap B)} = \overline{A}$

(c) $A \cup (\overline{B \cap C}) = A \cup (\overline{B} \cup \overline{C})$

(d) $(B \cap \overline{\overline{C}}) = B \cap C$

(e) $(B - A) \cup (B - A) = (B - A)$

(f) $((A \oplus B) - C) \cap \varnothing = \varnothing$

EXERCISE | 2.5.2: Proving set identities.

Use the set identities given in the table to prove the following new identities. Label each step in your proof with the set identity used to establish that step.

(a) $(\overline{A} \cap C) \cup (A \cap C) = C$

(b) $(B \cup A) \cap (\overline{B} \cup A) = A$

(c) $\overline{A \cap \overline{B}} = \overline{A} \cup B$

(d) $\overline{A} \cap (A \cup B) = \overline{A} \cap B$

(e) $\overline{A} \cup (A \cap B) = \overline{A} \cup B$

(f) $A \cap (B \cap \overline{B}) = \varnothing$

(g) $A \cup (B \cup \overline{B}) = U$

EXERCISE 2.5.3: Showing set equations that are not identities.

A set equation is not an identity if there are examples for the variables denoting the sets that cause the equation to be false. For example $A \cup B = A \cap B$ is not an identity because if $A = \{1, 2\}$ and $B = \{1\}$, then $A \cup B = \{1, 2\}$ and $A \cap B = \{1\}$, which means that $A \cup B \neq A \cap B$.

Show that each set equation given below is not a set identity.

(a) $A \cup (B \cap A) = B$

(b) $A - (B \cap A) = A$

(c) $(A \cup B) - (A \cap B) = A - B$

(d) $(B - A) \cup A = A$

(e) $A \cup B = A \oplus B$

EXERCISE 2.5.4: Proving set identities with the set difference operation.

The set subtraction law states that $A - B = A \cap \overline{B}$. Use the set subtraction law as well as the other set identities given in the table to prove each of the following new identities. Label each step in your proof with the set identity used to establish that step.

(a) $A - (B \cap A) = A - B$

(b) $A \cap (B - A) = \emptyset$

(c) $A \cup (B - A) = A \cup B$

(d) $A - (B - A) = A$

(e) $(A - B) - A = \emptyset$

2.6 Cartesian products

An ***ordered pair*** of items is written (x, y). The first ***entry*** of the ordered pair (x, y) is x and the second entry is y. The use of parentheses ( ) for an ordered pair indicates that the order of entries is significant, unlike sets which use curly braces { }, indicating that the order in which the elements are listed does not matter. For example, $(x, y) \neq (y, x)$ unless $x = y$. By contrast, $\{x, y\}$ is equal to $\{y, x\}$, with both denoting the set consisting of elements x and y. Two ordered pairs (x, y) and (u, w) are equal if and only if x = u and y = w.

For two sets, A and B, the ***Cartesian product*** of A and B, denoted A x B, is the set of all ordered pairs in which the first entry is in A and the second entry is in B. That is:

$$A \times B = \{ (a, b) : a \in A \text{ and } b \in B \}$$

Since the order of the elements in a pair is significant, A x B will not be the same as B x A, unless A = B, or either A or B is empty. If A and B are finite sets, then |A x B| = |A|·|B|.

PARTICIPATION ACTIVITY | 2.6.1: Cartesian products: Finite examples.

Animation content:

undefined

Animation captions:

1. A = {1, 2} and B = {a, b, c}. To find A × B, make 6 pairs. Three pairs will have the form (1, *) and three pairs will have the form (2, *).
2. Then fill in the second entry of one (1,*) pair with a and one (2,*) pair with a. Do the same with b and c.
3. A × B = { (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) }.
4. To find B × A, make 6 pairs. Two pairs will have the form (a, *), two will be (b,*), and two will be (c,*).
5. Then fill in the second entry of one (a,*) pair, one (b,*) pair and one (c,*) pair with 1. Do the same with 2.
6. B × A = { (a,1), (a,2), (b,1), (b,2), (c,1), (c,2) }.

PARTICIPATION ACTIVITY | 2.6.2: Cartesian products: An infinite example.

Animation captions:

1. Z is the set of all integers. The set $Z \times Z$ is the set of all pairs (x, y) where x and y are both integers.
2. The set $Z \times Z$ forms an infinite grid of points when plotted on the x-y plane.

PARTICIPATION ACTIVITY | 2.6.3: Cartesian products of two sets.

Consider the following sets:

A = { 1, 2, 3 }
B = { x, y }

1) (1, y) ∈ A x B

- True
- False

2) (1, y) ∈ B x A

- True
- False

3) A ⊆ A x B

- True
- False

4) (2, 3) ∈ **Z** x **Z**

- True
- False

5) |A x B| = 5

- True
- False

An ordered list of three items is called an ***ordered triple*** and is denoted (x, y, z). For n ≥ 4, an ordered list of n items is called an ***ordered n-tuple*** (or just ***n-tuple*** for short). For example, (w, x, y, z) is an ordered 4-tuple and (u, w, x, y, z) is an ordered 5-tuple.

The Cartesian product of three sets contains ordered triples, and for n ≥ 4, the Cartesian product of n sets contains n-tuples. The Cartesian product of n sets, A_1, A_2, ..., A_n is

$$A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots , a_n) : a_i \in A_i \text{ for all } i \text{ such that } 1 \leq i \leq n \}$$

For example, define A = {a, b}, B = {1, 2}, C = {x, y}, and D = {α, β}. Then the 4-tuples (a, 1, y, β) and (b, 1, x, α) are both examples of elements in the set A × B × C × D.

PARTICIPATION ACTIVITY 2.6.4: Cartesian products of many sets.

Consider the following sets:

A = { 1, 2, 3 }
B = { x, y }
C = { u, v, w }
D = { +, * }

1) (w, y, 2) ∈ C x B x A

- True
- False

2) A x B x C ⊆ A x B x C x D

- True
- False

3) (1, x, u, +) ∈ B x A x C x D

- True
- False

4) (1, 2, +) ∈ A x A x D

- True
- False

The Cartesian product of a set A with itself can be denoted as A × A or A^2. More generally:

$$A^k = \underbrace{A \times A \times \cdots \times A}_{\text{k times}}$$

For example, if A = {0, 1}, then A^n is the set of all ordered n-tuples whose entries are bits (0 or 1). For n = 3:

$\{0, 1\}^3$ = { (0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1) }

Another common example is $\mathbf{R}^n$, which is the set of all ordered n-tuples of real numbers. When n = 2, $\mathbf{R}^2$ is the set of all pairs (x, y) such that x and y are real numbers.

PARTICIPATION ACTIVITY | 2.6.5: Cartesian products of the same set.

Define the set A = { a, b, c }

1) (a, a, a, a, a) ∈ A^4

True

2) $(a, b, b, a) \in A^4$

False

True

False

3) $(2, 3, 2) \in \mathbf{R}^3$

True

False

Strings

If A is a set of symbols or characters, the elements in A^n can be written without the usual punctuation (parentheses and commas) used for ordered n-tuples. For example, if A = {x, y}, the set A^2 would be { xx, xy, yx, yy }. A sequence of characters is called a ***string***. The set of characters used in a set of strings is called the ***alphabet*** for the set of strings. The ***length*** of a string is the number of characters in the string. For example, the length of the string xxyxyx is 6.

A ***binary string*** is a string whose alphabet is {0, 1}. A ***bit*** is a character in a binary string. A string of length n is also called an ***n-bit string***. The set of binary strings of length n is denoted as $\{0,1\}^n$. An example of a binary string of length 7 (or 7-bit string) is: 0010110. Binary strings are fundamental objects in computer science: the input and output of every computer program is described by a binary string. In fact every piece of information including programs themselves are stored in computers as binary strings.

The ***empty string*** is the unique string whose length is 0 and is usually denoted by the symbol λ. Since $\{0, 1\}^0$ is the set of all binary strings of length 0, $\{0, 1\}^0 = \{\lambda\}$.

If s and t are two strings, then the ***concatenation*** of s and t (denoted st) is the string obtained by putting s and t together. If s = 010 and t = 11, then st = 01011. It is also possible to concatenate a string and a single symbol: t0 = 110. Concatenating any string x with the empty string gives back x: xλ = x.

Strings are used to specify passwords for computers or online accounts. Security systems vary with respect to the alphabet of characters allowed or required in a valid password. Strings also play an important role in discrete mathematics as a mathematical tool to help count the cardinality of sets.

PARTICIPATION ACTIVITY | 2.6.6: String basics.

1) What is the length of string zzyzx?

Check Show answer

2) Let X = {x, y, z}. Is the string zzyzx an element in X^4?

Check Show answer

3) Define the strings s = 1101 and t = 001. What is st?

Check Show answer

4) Define the string t = 001. What is t1?

Check Show answer

5) Define the string t = 001. What is tλ?

Check Show answer

CHALLENGE ACTIVITY | 2.6.1: Cartesian products.

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Start

D = { *, +, # }
Y = { 1, 2, 3, 4 }

Select the elements of Y × D.

1 2 3 4

Check Next

Additional exercises

EXERCISE 2.6.1: Cartesian product of three small sets.

The sets A, B, and C are defined as follows:

A = {tall, grande, venti}
B = {foam, no-foam}
C = {non-fat, whole}

Use the definitions for A, B, and C to answer the questions. Express the elements using n-tuple notation, not string notation.

(a) Write an element from the set $A \times B \times C$.

(b) Write an element from the set $B \times A \times C$.

(c) Write the set $B \times C$ using roster notation.

EXERCISE 2.6.2: Cartesian product of two small sets.

Define the sets X and Y as: X = {*, +, $} and Y = {52, 67}. Use the definitions for X and Y to answer the questions.

(a) Write the set X × Y using roster notation.

(b) Give an element of X^4. Express your answer as a 4-tuple, not as a string.

(c) Give an element of X × X × Y × Y × X. Express your answer as a 5-tuple, not as a string.

EXERCISE 2.6.3: Cartesian product - true or false.

Indicate which of the following statements are true.

(a) $\mathbf{R}^2 \subseteq \mathbf{R}^3$

(b) $\mathbf{Z}^2 \subseteq \mathbf{R}^2$

(c) $\mathbf{Z}^2 \cap \mathbf{Z}^3 = \emptyset$

(d) For any two sets, A and B, if A ⊆ B, then $A^2 \subseteq B^2$.

(e) For any three sets, A, B, and C, if A ⊆ B, then A × C ⊆ B × C.

EXERCISE 2.6.4: Expressing sets defined by Cartesian products in roster notation.

Express each set in roster notation. Express the elements as strings, not n-tuples.

(a) A^2, where A = {+, -}.

(b) A^3, where A = {0, 1}.

EXERCISE | 2.6.5: Cardinality of a set defined by a Cartesian product.

(a) What is $|\{0, 1\}^7|$?

(b) What is $|\{a, b, c, d\}^3|$?

EXERCISE | 2.6.6: Roster notation for sets defined using set builder notation and the Cartesian product.

Express the following sets using the roster method. Express the elements as strings, not n-tuples.

(a) $\{0x: x \in \{0, 1\}^2\}$

(b) $\{0, 1\}^0 \cup \{0, 1\}^1 \cup \{0, 1\}^2$

(c) $\{0x: x \in B\}$, where $B = \{0, 1\}^0 \cup \{0, 1\}^1 \cup \{0, 1\}^2$.

(d) $\{xy:$ where $x \in \{0\} \cup \{0\}^2$ and $y \in \{1\} \cup \{1\}^2\}$

(e) $\{xy: x \in \{aa, ab\}$ and $y \in \{a\} \cup \{a\}^2\}$

EXERCISE 2.6.7: Cartesian products, power sets, and set operations.

Use the following set definitions to specify each set in roster notation. Except where noted, express elements of Cartesian products as strings.

- A = {a}
- B = {b, c}
- C = {a, b, d}

(a) $A \times (B \cup C)$

(b) $A \times (B \cap C)$

(c) $(A \times B) \cup (A \times C)$

(d) $(A \times B) \cap (A \times C)$

(e) $(C \times B) \cap (B \times C)$

(f) $P(A \times B)$

(g) $P(A) \times P(B)$. Use ordered pair notation for elements of the Cartesian product.

EXERCISE 2.6.8: Proving set identities with Cartesian products.

Use the following three definitions and the laws of logic to prove the two identities given below.

- Definition of Cartesian product: $(x,y) \in A \times B \leftrightarrow (x \in A) \land (y \in B)$
- Definition of intersection: $x \in A \cap B \leftrightarrow (x \in A) \land (x \in B)$
- Definition of union: $x \in A \cup B \leftrightarrow (x \in A) \lor (x \in B)$

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

EXERCISE | 2.6.9: Cartesian products and the empty set.

(a) If $A = \{1, 2, 3\}$, then what is $A \times \varnothing$?

(b) If A and B are finite sets and $A \times B = \varnothing$, then what can you conclude about the sets A and B? Justify your answer.

2.7 Partitions

Two sets, A and B, are said to be ***disjoint*** if their intersection is empty ($A \cap B = \varnothing$). A sequence of sets, $A_1, A_2, \ldots, A_n$, is ***pairwise disjoint*** if every pair of distinct sets in the sequence is disjoint (i.e., $A_i \cap A_j = \varnothing$ for any i and j in the range from 1 through n where $i \neq j$).

A ***partition*** of a non-empty set A is a collection of non-empty subsets of A such that each element of A is in exactly one of the subsets. $A_1, A_2, \ldots, A_n$ is a partition for a non-empty set A if all of the following conditions hold:

- For all i, $A_i \subseteq A$.
- For all i, $A_i \neq \varnothing$
- $A_1, A_2, \ldots, A_n$ are pairwise disjoint.
- $A = A_1 \cup A_2 \cup \ldots \cup A_n$

The animation below gives an example of a partition of a finite and an infinite set.

PARTICIPATION ACTIVITY | 2.7.1: Set partitions.

Animation captions:

1. $A = \{1,2,3,4,5,6\}$. $A_1 = \{1,4,5\}$, $A_2 = \{2,3\}$, and $A_3 = \{6\}$. A_1, A_2, and A_3 form a partition of A.
2. $B_1 = \{x \in R: x < 1\}$, $B_2 = \{x \in R: 1 \leq x \leq 3\}$, $B_3 = \{x \in R: x > 3\}$. B_1, B_2, and B_3 form a partition of R.

PARTICIPATION ACTIVITY | 2.7.2: Set partitions.

For each question, indicate whether the sets form a partition of the universal set.

1) The universal set is **R**.

$A_1 = \{ x \in \mathbf{R}: x \leq -4 \}$
$A_2 = \{ x \in \mathbf{R}: -4 < x < 4 \}$
$A_3 = \{ x \in \mathbf{R}: x \geq 4 \}$

- A_1, A_2, and A_3 form a partition of **R**.
- A_1, A_2, and A_3 are not pairwise disjoint.
- $A_1 \cup A_2 \cup A_3 \neq \mathbf{R}$.

2) The universal set is $\mathbf{Z}^+$.

$A_1 = \{ x \in \mathbf{Z}^+: x \text{ is prime} \}$
$A_2 = \{ x \in \mathbf{Z}^+: x \text{ is odd and } x \text{ is a composite number} \}$
$A_3 = \{ x \in \mathbf{Z}^+: x \text{ is even} \}$
$A_4 = \{ 1 \}$

- A_1, A_2, A_3, and A_4 form a partition of $\mathbf{Z}^+$.
- A_1, A_2, A_3, and A_4 are not pairwise disjoint.
- $A_1 \cup A_2 \cup A_3 \cup A_4 \neq \mathbf{Z}^+$

3) The universal set is **Z**.

$O = \{ x \in \mathbf{Z}: x \text{ is odd} \}$
$E = \{ x \in \mathbf{Z}: x \text{ is even} \}$

- O and E form a partition of **Z**.
- O and E are not disjoint.
- $O \cup E \neq \mathbf{Z}$

CHALLENGE ACTIVITY 2.7.1: Partitions.

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Start

A = {1, 2, 3, 4, 5, 6, 7, 8, 9}

Select the sets that form a partition of A

1	2	3

Check Next

Additional exercises

EXERCISE 2.7.1: Recognizing partitions - small finite sets.

Define the sets A, B, C, D, and E as follows:

- A = {1, 2, 6}
- B = {2, 3, 4}
- C = {5}
- D = $\{x \in \mathbf{Z}: 1 \leq x \leq 6\}$
- E = $\{x \in \mathbf{Z}: 1 < x < 6\}$

Use the definitions for A, B, C, D, and E to answer the questions.

(a) Do the sets A, B, and C form a partition of the set D? If not, which condition of a partition is not satisfied?

(b) Do the sets B and C form a partition of the set D? If not, which condition of a partition is not satisfied?

(c) Do the sets B and C form a partition of the set E? If not, which condition of a partition is not satisfied?

EXERCISE | 2.7.2: Finding a partition of a set.

Let sets A through F be defined as follows.

- A = {000}
- B = {111}
- C = {0x: x $\in$ $\{0, 1\}^2$}
- D = {01x: x $\in$ {0, 1}}
- E = {1x: x $\in$ $\{0, 1\}^2$}
- F = {00x: x $\in$ {0, 1}}

(a) Give two different partitions of the set $\{0, 1\}^3$ using one or more of the sets defined above. Give your answer by writing the letters corresponding to the sets in each partition. Ex: A, B, C.

EXERCISE | 2.7.3: Recognizing partitions - the real numbers.

Define the sets A, B, C, D, and E as follows:

- A = {x $\in$ **R**: x < -2}
- B = {x $\in$ **R**: x > 2}
- C = {x $\in$ **R**: |x| < 2}
- D = {x $\in$ **R**: |x| $\leq$ 2}
- E = {x $\in$ **R**: x $\leq$ -2}

Use the definitions for A, B, C, D, and E to answer the questions.

(a) Do the sets A, B, and C form a partition of **R**? If not, which condition of a partition is not satisfied?

(b) Do the sets A, B, and D form a partition of **R**? If not, which condition of a partition is not satisfied?

(c) Do the sets B, D, and E form a partition of **R**? If not, which condition of a partition is not satisfied?

EXERCISE 2.7.4: Recognizing partitions - sets of strings.

(a) Suppose that every student at a university is assigned a unique 8-digit ID number. For $i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, define the set A_i to be the set of currently enrolled students whose ID number begins with the digit i. For each digit, i, there is at least one student whose ID starts with i.
Do the sets $A_0, \ldots, A_9$ form a partition of the set of currently enrolled students?

(b) Let A be the set of words in the Oxford English Dictionary (OED). For each positive integer j, define A_j to be the set of all words with j letters in the OED. For example, the word "discrete" is an element of A_8 because the word "discrete" has 8 letters. The longest word in the OED is "pneumonoultramicroscopicsilicovolcanoconiosis" which has 45 letters. You can assume that for any integer i in the range 1 through 45, there is at least one word in the OED that has i letters. Do the sets $A_1, \ldots, A_{45}$ form a partition of the set of words in the OED?

2.8 Set Operations in Python

In this assignment you will program a few functions that allow a list to have the functionality of a set data structure (remember list can have duplicates but sets do not so your code should not return duplicate values even if the input list contain duplicates). This includes creating a new set, adding/removing elements etc. Complete details are provided in the code template below. We have also (mostly) implemented two of the functions to give you an idea of what's expected (set_new(), and set_remove()). All the functions should be just a few lines of code.

In this assignment we will represent sets using the Python list data structure. For example:

s = [1,'a',3]

represents the mathematical set object {1, 'a', 3}.

Note that since order does not matter, the same set can be represented by a list that has the elements of the set in a different order. In your implementation, the result returned by your code should not depend on the order in which the elements appear in the list.

Your job is to implement the following set operations:

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LAB ACTIVITY 2.8.1: Set Operations in Python 0 / 100

Downloadable files

sets.py Download

sets.py

Load default template...

```
def set_new() :
    """Return a new set"""
    return []

def set_remove(s, value):
    """Remove the given value from the set s"""
    # perform some type checking to see that the user
    # has provided the right kind of input:
    if type(s)!=type([]) :
        raise ValueError
    # we can simply use the "remove" method of a list:
    s.remove(value)
    # to be complete before returning we should make sure there are no duplicates - not sh
    return s

def set_union(s1, s2) :
    """Return the union of sets s1 and s2 as a list"""
    if type(s1)!=type([]) or type(s2)!=type([]):
        raise ValueError
```


Develop mode **Submit mode**

Run your program as often as you'd like, before submitting for grading. Below, type any needed input values in the first box, then click **Run program** and observe the program's output in the second box.

Enter program input (optional)

If your code requires input values, provide them here.

Run program

Input (from above) → **sets.py** (Your program) → O

Program output displayed here

Coding trail of your work What is this?

History of your effort will appear here once you begin working on this zyLab.

2.9 Copy of Set Operations in Python

This section's content is not available for print.