2.6 Cartesian products

An ***ordered pair*** of items is written (x, y). The first ***entry*** of the ordered pair (x, y) is x and the second entry is y. The use of parentheses ( ) for an ordered pair indicates that the order of entries is significant, unlike sets which use curly braces { }, indicating that the order in which the elements are listed does not matter. For example, (x, y) ≠ (y, x) unless x = y. By contrast, {x, y} is equal to {y, x}, with both denoting the set consisting of elements x and y. Two ordered pairs (x, y) and (u, w) are equal if and only if x = u and y = w.

For two sets, A and B, the ***Cartesian product*** of A and B, denoted A x B, is the set of all ordered pairs in which the first entry is in A and the second entry is in B. That is:

$$A \times B = \{ (a, b) : a \in A \text{ and } b \in B \}$$

Since the order of the elements in a pair is significant, A x B will not be the same as B x A, unless A = B, or either A or B is empty. If A and B are finite sets, then |A x B| = |A|·|B|.

PARTICIPATION ACTIVITY | 2.6.1: Cartesian products: Finite examples.

1 2 3 4 5 6 2x speed

B = { a, b, c }

A = { 1, 2 }

	a	b	c
1	(1 , a)	(1 , b)	(1 , c)
2	(2 , a)	(2 , b)	(2 , c)

A x B =
{ (1 , a) , (1 , b) , (1 , c) ,
(2 , a) , (2 , b) , (2 , c) }

A = { 1, 2 }

B = { a, b, c }

	1	2
a	(a , 1)	(a , 2)
b	(b , 1)	(b , 2)
c	(c , 1)	(c , 2)

B x A
{(a , 1) , (a , 2) ,
(b , 1) , (b , 2) ,
(c , 1) , (c , 2) }

B × A = { (a,1), (a,2), (b,1), (b,2), (c,1), (c,2) }.

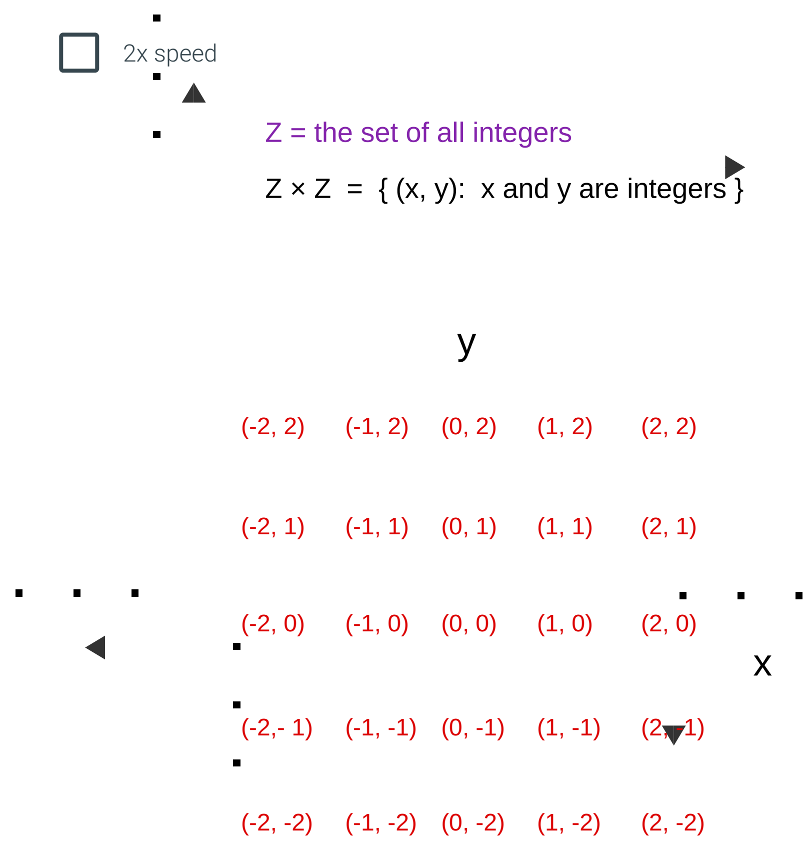
Captions

1. A = {1, 2} and B = {a, b, c}. To find A × B, make 6 pairs. Three pairs will have the form (1, *) and three pairs will have the form (2, *).
2. Then fill in the second entry of one (1,*) pair with a and one (2,*) pair with a. Do the same with b and c.
3. A × B = { (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) }.
4. To find B × A, make 6 pairs. Two pairs will have the form (a, *), two will be (b,*), and two will be (c,*).
5. Then fill in the second entry of one (a,*) pair, one (b,*) pair and one (c,*) pair with 1. Do the same with 2.
6. B × A = { (a,1), (a,2), (b,1), (b,2), (c,1), (c,2) }.

Feedback?

PARTICIPATION ACTIVITY 2.6.2: Cartesian products: An infinite example.

1 2 2x speed



The set forms an infinite grid of points when plotted on the x-y plane.

Captions

1. is the set of all integers. The set is the set of all pairs (x, y) where x and y are both integers.
2. The set forms an infinite grid of points when plotted on the x-y plane.

Feedback?

PARTICIPATION ACTIVITY | 2.6.3: Cartesian products of two sets.

Consider the following sets:

A = { 1, 2, 3 }
B = { x, y }

1) (1, y) ∈ A x B

- True
- False

Correct

A x B is the set of all ordered pairs where the first entry in the pair is in A and the second entry is in B. 1 ∈ A and y ∈ B, so (1, y) is an element of A x B.

2) (1, y) ∈ B x A

- True
- False

Correct

If a pair is in B x A, then the first entry in the pair must be in B and the second entry must be in A. However, 1 ∉ B and y ∉ A, so (1, y) cannot be an element of B x A.

3) A ⊆ A x B

- True
- False

Correct

The elements in A are single numbers such as 1. The set A x B contains only ordered pairs such as (1, x). Therefore A and A x B have no elements in common.

4) (2, 3) ∈ **Z** x **Z**

- True
- False

Correct

Z x **Z** is the set of all ordered pairs (x, y) such that x and y are integers. 2 and 3 are both integers, so (2, 3) is an element of **Z** x **Z**.

5) |A x B| = 5

- True
- False

Correct

|A x B| = |A| · |B| = 3 · 2 = 6

Feedback?

An ordered list of three items is called an ***ordered triple*** and is denoted (x, y, z). For n ≥ 4, an ordered list of n items is called an ***ordered n-tuple*** (or just ***n-tuple*** for short). For example, (w, x, y, z) is an ordered 4-tuple and (u, w, x, y, z) is an ordered 5-tuple.

The Cartesian product of three sets contains ordered triples, and for n ≥ 4, the Cartesian product of n sets contains n-tuples. The Cartesian product of n sets, A_1, A_2, ..., A_n is

$$A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots , a_n) : a_i \in A_i \text{ for all } i \text{ such that } 1 \leq i \leq n \}$$

For example, define A = {a, b}, B = {1, 2}, C = {x, y}, and D = {α, β}. Then the 4-tuples (a, 1, y, β) and (b, 1, x, α) are both examples of elements in the set A × B × C × D.

PARTICIPATION ACTIVITY | 2.6.4: Cartesian products of many sets.

Consider the following sets:

A = { 1, 2, 3 }
B = { x, y }
C = { u, v, w }
D = { +, * }

1) (w, y, 2) ∈ C x B x A

- True
- False

Correct

C x B x A contains all ordered triples in which the first entry in the triple is in C, the second entry is in B, and the third entry is in A. w ∈ C, y ∈ B, and 2 ∈ A, so (w, y, 2) is an element of C x B x A.

2) A x B x C ⊆ A x B x C x D

- True
- False

Correct

The set A x B x C contains triples. The set A x B x C x D contains 4-tuples. The two sets do not have any elements in common.

3) (1, x, u, +) ∈ B x A x C x D

- True
- False

Correct

If a 4-tuple in the set B x A x C x D, then the first entry in the tuple must be in B. However, 1 ∉ B, so (1, x, u, +) cannot be an element of B x A x C x D.

4) (1, 2, +) ∈ A x A x D

- True
- False

Correct

The set A x A x D contains all ordered triples in which the first and second entries are in A, and the third entry is in D. 1 ∈ A, 2 ∈ A, and + ∈ D, so (1, 2, +) is an element of A x A x D.

Feedback?

The Cartesian product of a set A with itself can be denoted as A × A or A^2. More generally:

For example, if A = {0, 1}, then A^n is the set of all ordered n-tuples whose entries are bits (0 or 1). For n = 3:

$$\{0, 1\}^3 = \{ (0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1) \}$$

Another common example is $\mathbf{R}^n$, which is the set of all ordered n-tuples of real numbers. When n = 2, $\mathbf{R}^2$ is the set of all pairs (x, y) such that x and y are real numbers.

PARTICIPATION ACTIVITY 2.6.5: Cartesian products of the same set.

Define the set A = { a, b, c }

1) $(a, a, a, a, a) \in A^4$

- True
- False

Correct

(a, a, a, a, a) is a 5-tuple and A^4 contains only 4-tuples.

2) $(a, b, b, a) \in A^4$

- True
- False

Correct

(a, b, b, a) is a 4-tuple whose entries are all elements of A.

3) $(2, 3, 2) \in \mathbf{R}^3$

- True
- False

Correct

(2, 3, 2) is an ordered triple in which all the entries are in **R**.

Feedback?

Strings

If A is a set of symbols or characters, the elements in A^n can be written without the usual punctuation (parentheses and commas) used for ordered n-tuples. For example, if A = {x, y}, the set A^2 would be { xx, xy, yx, yy }. A sequence of characters is called a ***string***. The set of characters used in a set of strings is called the ***alphabet*** for the set of strings. The ***length*** of a string is the number of characters in the string. For example, the length of the string xxyxyx is 6.

A ***binary string*** is a string whose alphabet is {0, 1}. A ***bit*** is a character in a binary string. A string of length n is also called an ***n-bit string***. The set of binary strings of length n is denoted as $\{0,1\}^n$. An example of a binary string of length 7 (or 7-bit string) is: 0010110. Binary strings are fundamental objects in computer science: the input and output of every computer program is described by a binary string. In fact every piece of information including programs themselves are stored in computers as binary strings.

The ***empty string*** is the unique string whose length is 0 and is usually denoted by the symbol λ. Since $\{0, 1\}^0$ is the set of all binary strings of length 0, $\{0, 1\}^0 = \{\lambda\}$.

If s and t are two strings, then the ***concatenation*** of s and t (denoted st) is the string obtained by putting s and t together. If s = 010 and t = 11, then st = 01011. It is also possible to concatenate a string and a single symbol: t0 = 110. Concatenating any string x with the empty string gives back x: xλ = x.

Strings are used to specify passwords for computers or online accounts. Security systems vary with respect to the alphabet of characters allowed or required in a valid password. Strings also play an important role in discrete mathematics as a mathematical tool to help count the cardinality of sets.

PARTICIPATION ACTIVITY | 2.6.6: String basics.

1) What is the length of string zzyzx?

5

Check Show answer

Correct

5

The string zzyzx has 5 characters.

2) Let X = {x, y, z}. Is the string zzyzx an element in X^4?

no

Check Show answer

Correct

No

The strings in X^4 have length 4 and zzyzx has length 5.

3) Define the strings s = 1101 and t = 001. What is st?

1101001

Check Show answer

Correct

1101001

s = 1101 and t = 001, so concatenating the two strings results in the string st = 1101001.

4) Define the string t = 001. What is t1?

Correct

0011

Check Show answer

0011

t = 001, so putting the string t together with the bit 1 results in 0011.

5) Define the string t = 001. What is tλ?

001

Check Show answer

Correct

001

Concatenating λ to a string does not add or remove any bits to the string, so tλ = t.

Feedback?

CHALLENGE ACTIVITY | 2.6.1: Cartesian products.

422102.2723990.qx3zqy7

Jump to level 1

1
2
3
4

Select all the strings that are elements of the following set:

$\{0, 1\} \cup \{0, 1\}^2 \cup \{0, 1\}^3$

- [ ] 011
- [ ] 0111
- [ ] 0110
- [ ] 01111
- [ ] 00011
- [ ] 110
- [ ] 101
- [ ] 1110

1	2	3	**4**

Check Next

Done. Click any level to practice more. Completion is preserved.

Expected: 011, 110, 101

The strings in the set are strings that have length 1, 2, or 3 and have characters from {0, 1}.The strings 011, 110, and 101 are all in the set.

Feedback?

Feedback?

Additional exercises

EXERCISE 2.6.1: Cartesian product of three small sets.

The sets A, B, and C are defined as follows:

A = {tall, grande, venti}
B = {foam, no-foam}
C = {non-fat, whole}

Use the definitions for A, B, and C to answer the questions. Express the elements using n-tuple notation, not string notation.

(a) Write an element from the set A × B × C.

(b) Write an element from the set B × A × C.

(c) Write the set B × C using roster notation.

Feedback?

EXERCISE 2.6.2: Cartesian product of two small sets.

Define the sets X and Y as: X = {*, +, $} and Y = {52, 67}. Use the definitions for X and Y to answer the questions.

(a) Write the set X × Y using roster notation.

(b) Give an element of X^4. Express your answer as a 4-tuple, not as a string.

(c) Give an element of X × X × Y × Y × X. Express your answer as a 5-tuple, not as a string.

Feedback?

EXERCISE 2.6.3: Cartesian product - true or false.

Indicate which of the following statements are true.

(a) $\mathbf{R}^2 \subseteq \mathbf{R}^3$

(b) $\mathbf{Z}^2 \subseteq \mathbf{R}^2$

(c) $\mathbf{Z}^2 \cap \mathbf{Z}^3 = \emptyset$

(d) For any two sets, A and B, if $A \subseteq B$, then $A^2 \subseteq B^2$.

(e) For any three sets, A, B, and C, if $A \subseteq B$, then $A \times C \subseteq B \times C$.

Feedback?

EXERCISE | 2.6.4: Expressing sets defined by Cartesian products in roster notation.

Express each set in roster notation. Express the elements as strings, not n-tuples.

(a) A^2, where A = {+, -}.

(b) A^3, where A = {0, 1}.

Feedback?

EXERCISE | 2.6.5: Cardinality of a set defined by a Cartesian product.

(a) What is $|\{0, 1\}^7|$?

(b) What is $|\{a, b, c, d\}^3|$?

Feedback?

EXERCISE | 2.6.6: Roster notation for sets defined using set builder notation and the Cartesian product.

Express the following sets using the roster method. Express the elements as strings, not n-tuples.

(a) $\{0x: x \in \{0, 1\}^2\}$

(b) $\{0, 1\}^0 \cup \{0, 1\}^1 \cup \{0, 1\}^2$

(c) $\{0x: x \in B\}$, where $B = \{0, 1\}^0 \cup \{0, 1\}^1 \cup \{0, 1\}^2$.

(d) $\{xy\text{: where } x \in \{0\} \cup \{0\}^2 \text{ and } y \in \{1\} \cup \{1\}^2\}$

(e) $\{xy\text{: } x \in \{aa, ab\} \text{ and } y \in \{a\} \cup \{a\}^2\}$

Feedback?

EXERCISE 2.6.7: Cartesian products, power sets, and set operations.

Use the following set definitions to specify each set in roster notation. Except where noted, express elements of Cartesian products as strings.

- $A = \{a\}$
- $B = \{b, c\}$
- $C = \{a, b, d\}$

(a) $A \times (B \cup C)$

(b) $A \times (B \cap C)$

(c) $(A \times B) \cup (A \times C)$

(d) $(A \times B) \cap (A \times C)$

(e) $(C \times B) \cap (B \times C)$

(f) $P(A \times B)$

(g) $P(A) \times P(B)$. Use ordered pair notation for elements of the Cartesian product.

Feedback?

EXERCISE 2.6.8: Proving set identities with Cartesian products.

Use the following three definitions and the laws of logic to prove the two identities given below.

- Definition of Cartesian product: $(x,y) \in A \times B \leftrightarrow (x \in A) \wedge (y \in B)$
- Definition of intersection: $x \in A \cap B \leftrightarrow (x \in A) \wedge (x \in B)$
- Definition of union: $x \in A \cup B \leftrightarrow (x \in A) \vee (x \in B)$

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Feedback?

EXERCISE | 2.6.9: Cartesian products and the empty set.

(a) If $A = \{1, 2, 3\}$, then what is $A \times \varnothing$?

(b) If A and B are finite sets and $A \times B = \varnothing$, then what can you conclude about the sets A and B? Justify your answer.

Feedback?

How was this section? Provide feedback

Activity summary for assignment: Reading Assignment 2 100 / 105 pts

No due date 100 / 105 pts submitted to canvas

Completion details