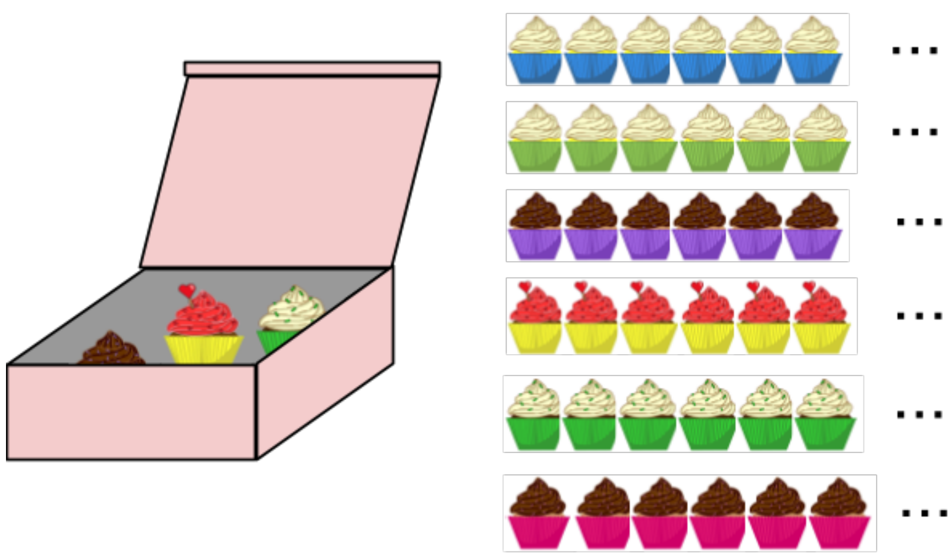
8.1 Sum and product rules

Introduction to counting

Counting, as simple as it may seem initially, is a central topic in discrete mathematics. Most children begin their education in mathematics by learning to count — 1, then 2, and so forth. In discrete mathematics the goal is to count the number of elements in (or the *cardinality* of) a finite set given a description of the set. Determining a set's cardinality often requires exploiting some mathematical structure of the set.

Figure 8.1.1: Example: Counting the number of cupcake selections.

A bakery sells 6 different varieties of cupcakes (chocolate, vanilla, red velvet, etc.). How many ways are there to fill a box with 24 cupcakes from the 6 varieties? The order in which the cupcakes are selected is unimportant; all that matters is the number of each variety in the box after they are chosen. The answer to the cupcake counting problem is 118755 — too many to count by hand.



The answer to the cupcake counting question is 118755. This material will cover a systematic technique to answer questions like the cupcake counting question. While counting cupcake selections may not seem like a compelling application, the same techniques can be used to count the number of ways 24 identical tasks can be assigned to a network of 6 processors which in turn can be used to calculate the probability that a random assignment distributes the workload evenly among the 6 processors.

Counting is an important mathematical tool to analyze many problems that arise in computer science. Counting is useful, for example, to understand the amount of a particular resource used in a computer system or computation. Counting is also used to determine the number of valid passwords for a security system or addresses in a network to ensure that there are enough unique choices to meet the demand. Counting is also at the heart of discrete probability, which is central to many areas of science.

The product rule

The two most basic rules of counting are the sum rule and the product rule. These two rules applied in different combinations can be used to handle a wide range of counting problems. The ***product rule*** provides a way to count sequences. While sequences may not seem like a particularly common type of object to count, many sets can be expressed as sets of sequences.

Theorem 8.1.1: The product rule.

Let $A_1, A_2,\ldots,A_n$ be finite sets.
Then,

$$|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$$

Consider a restaurant that has a breakfast special that includes a drink, a main course, and a side. The set of choices for each category are:

$$D = \{\text{coffee, orange juice}\}$$
$$M = \{\text{pancakes, eggs}\}$$
$$S = \{\text{bacon, sausage, hash browns}\}$$

Any particular breakfast selection can be described by a triplet indicating the choice of drink, main course, and side. For example, (coffee, pancakes, bacon) is one particular breakfast combination. The set of all possible choices is the same as the set of a triples where the first entry is a drink, the second entry is a main course, and the third entry is a side. The number of possible breakfast combinations is therefore:

$$|D \times M \times S| = |D| \cdot |M| \cdot |S| = 2 \cdot 2 \cdot 3 = 12$$

PARTICIPATION ACTIVITY | 8.1.1: An example of the product rule: counting breakfast selections.

Animation captions:

1. A breakfast special includes a choice of drink, main course, and side.
2. There are two choices for the drink: coffee or orange juice. There are two options so far for the breakfast special.
3. For each choice of drink, there are two choices for the main course: pancakes or eggs. There are $2 \cdot 2 = 4$ choices so far.
4. For each choice of drink and main course, there are three choices for the side: bacon, sausage, or hash browns. There are $4 \cdot 3 = 12$ choices total.

PARTICIPATION ACTIVITY 8.1.2: Applying the product rule: Counting burrito selections.

A burrito stand sells burritos with different choices of stuffing. The set of choices for each category are:

- Filling choices = {chicken, beef, pork}
- Bean choices = {black, pinto}
- Salsa choices = {mild, medium, hot}

1) If every burrito has a filling, beans, and salsa, then how many possible burrito combinations are there?

Check Show answer

2) Suppose that the customer can also now select grilled veggies as a filling. Now how many selections are there?

Check Show answer

3) Now suppose that the burrito stand introduces a choice between plain flour or whole wheat tortillas. The additional

option to select veggies as a filling is still available as well. Now how many selections are there?

Check Show answer

Counting strings

If Σ is a set of characters (called an ***alphabet***) then Σ^n is the set of all strings of length n whose characters come from the set Σ. For example, if Σ = {0, 1}, then Σ^6 is the set of all binary strings with 6 bits. The string 011101 is an example of an element in Σ^6. The strings xxyzx and zyyzy are examples of strings in the set $\{x, y, z\}^5$. The product rule can be applied directly to determine the number of strings of a given length over a finite alphabet:

$$|\Sigma^n| = |\underbrace{\Sigma \times \Sigma \times \cdots \times \Sigma}_{\text{n times}}| = \underbrace{|\Sigma| \cdot |\Sigma| \cdots |\Sigma|}_{\text{n times}} = |\Sigma|^n$$

For example, the number of binary strings of length n is 2^n since the size of the alphabet is 2 (e.g., |{0, 1}| = 2).

The product rule can also be used to determine the number of strings in a set when one or more of the characters are restricted. Define S to be the set of binary strings of length 5 that start and end with 0. A string is in the set S if it has the form 0***0, where each * could be a 0 or a 1.

$$|S| = |\{0\} \times \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \{0\}| = 1 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 8$$

PARTICIPATION ACTIVITY | 8.1.3: Using the product rule to count sets of strings.

1) How many six bit binary strings are there?

Check Show answer

2) How many six bit binary strings are there that begin with "01"?

Check Show answer

3) How many strings of length 4 are there over the alphabet {a, b, c}?

Check Show answer

4) How many strings of length 4 are there over the alphabet {a, b, c} that end with the character c?

Check Show answer

The sum rule

In the breakfast example, the product rule is applied because the customer selects a drink and a main course and a side. In contrast, the ***sum rule*** is applied when there are multiple choices but only one selection is made. For example, suppose a customer just orders a drink. The customer selects a hot drink or a cold drink. The hot drink selections are {coffee, hot cocoa, tea}. The cold drink selections are {milk, orange juice}. The total number of choices is 5, namely 3 hot drink choices plus 2 cold drink choices. Here is a formal statement of the sum rule, expressed in terms of sets:

Theorem 8.1.2: The sum rule.

Consider n sets, $A_1, A_2, \ldots, A_n$. If the sets are pairwise disjoint (which means that $A_i \cap A_j = \varnothing$ for $i \neq j$), then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$$

In the example with the customer selecting a drink, n = 2 since there are two categories of drinks: hot drinks and cold drinks. Let C be the set of cold drinks and H the set of hot drinks. The fact that H and C are disjoint ($C \cap H = \varnothing$) means that none of the drinks is categorized as both a hot drink and a cold drink. Applying the sum rule yields that the number of possible drinks is:

$$|C \cup H| = |C| + |H| = 3 + 2 = 5$$

Example 8.1.1: Product and sum rule in combination: counting passwords.

Consider a system in which a password must be a string of length between 6 and 8. The characters can be any lower case letter or digit.

Let L be the set of all lower case letters and D be the set of digits. $|L| = 26$ and $|D| = 10$. The set of all allowed characters is $C = L \cup D$. Since $D \cap L = \varnothing$, the sum rule can be applied to find the cardinality of C: $|C| = 26 + 10 = 36$.

Let A_j denote the strings of length j over the alphabet C. By the product rule, $|A_j| = 36^j$. Notice that for $j \neq k$, A_j and A_k are disjoint because a string can not have length j and length k at the same time. If the user must select a password of length 6 or 7 or 8, then the sum rule applies:

$$|A_6 \cup A_7 \cup A_8| = |A_6| + |A_7| + |A_8| = 36^6 + 36^7 + 36^8$$

In the next example, a customer purchasing a laptop can select three different sizes of screens and has a choice between two different processor speeds. For storage, the customer can select a hard disk drive or a solid state drive. The hard disk drive option comes in two sizes and the solid state option has three different sizes. The manufacturer would like to know how many different configurations are possible. The number of choices is worked out in the animation below:

PARTICIPATION ACTIVITY | 8.1.4: An example of the sum and product rule: counting laptop selections.

Animation captions:

1. Customizing a laptop includes the choice of screen size, processor speed, and storage. There are 3 screen sizes and 2 possible processor speeds.
2. The number of different laptops is $3 \cdot 2 \cdot \textbf{number of choices for storage}$. Storage can be SSD (3 different sizes) or HDD (2 different sizes).
3. The number of choices for SSD and HDD are combined by the sum rule. The total number of choices for the laptop is $3 \cdot 2 \cdot (3 + 2)$.

PARTICIPATION ACTIVITY | 8.1.5: Applying the sum and product rule in combination: Counting sets of bit strings.

A bit string consists of 0s and 1s. For example, 0101 is a bit string with four bits.

1) How many six bit strings are

there that begin and end with a 1, or start with 00?

Check Show answer

2) How many bit strings of length five or six start with a 1?

Check Show answer

CHALLENGE ACTIVITY | 8.1.1: Counting password possibilities.

422102.2723990.qx3zqy7

Start

Each character in a password is either a digit [0-9] or lowercase letter [a-z]. How many valid passwords are there with the given restriction(s)?

Length is 15.

Ex: 26 * 36^21 Write a^b as: a^b

1	2	3	4	5

Check Next

Additional exercises

EXERCISE 8.1.1: License plate combinations.

In a particular state, the license plates have 7 characters. Each character can be a capital letter or a digit except for 0. (The set of possible digits is {1; 2; 3; 4; 5; 6; 7; 8; 9}.) A person witnesses a crime and remembers some information about the license plate of the getaway car. The authorities would like to figure out how many license plates need to be checked in each case. For each constraint given below, indicate the number of license plates that satisfy that constraint.

Note: you do not need to calculate the number. You may keep the multiplications and powers in your answers

(a) No constraints

(b) The license plate starts with a digit

(c) First three are letters

(d) First three are letters and last four are numbers

EXERCISE 8.1.2: Counting passwords made up of letters, digits, and special characters.

Consider the following definitions for sets of characters:

- Digits = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 }
- Letters = { a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z }
- Special characters = { *, &, $, # }

Compute the number of passwords that satisfy the given constraints.

(a) Strings of length 6. Characters can be special characters, digits, or letters.

(b) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters.

(c) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters. The first character cannot be a letter.

EXERCISE 8.1.3: Selecting lunch specials for the week.

(a) A Chinese restaurant offers 10 different lunch specials. Each weekday for one week, Fiona goes to the restaurant and selects a lunch special. How many different ways are there for her to select her lunches for the week? Note that which lunch she orders on which day matters, so the following two selections are considered different.

One possible selection:

- Mon: Kung pao chicken
- Tues: Beef with broccoli
- Wed: Kung pao chicken
- Thurs: Moo shu pork
- Fri: Beef with broccoli

A different selection:

- Mon: Beef with broccoli
- Tues: Kung pao chicken
- Wed: Kung pao chicken
- Thurs: Moo shu pork
- Fri: Beef with broccoli

(b) Now suppose that out of the 10 dishes that the restaurant offers, only 3 of them are vegetarian. If Fiona must select a vegetarian option on Friday, how many ways are there for her to select her lunches?

(c) Again, suppose that out of the 10 dishes that the restaurant offers, only 3 of them are vegetarian. If Fiona must go with a vegetarian option on both Monday and Friday, how many ways are there for her to select her lunches?

(d) Now suppose that Fiona can select a vegetarian or a non-vegetarian lunch on any day of the week. However, in addition to selecting her main course, she must also selects between water or tea for her drink. How many ways are there for her to select her lunches?

EXERCISE 8.1.4: Dividing up a print job.

A 100-page document is being printed by four printers. Each page will be printed exactly once.

(a) Suppose that there are no restrictions on how many pages a printer can print. How many ways are there for the 100 pages to be assigned to the four printers?
One possible combination: printer A prints out pages 2-50, printer B prints out pages 1 and 51-60; printer C prints out 61-80 and 86-90; printer D prints out pages 81-85 and 91-100.

(b) Suppose the first and the last page of the document must be printed in color, and only two printers are able to print in color. The two color printers can also print black-and-white. How many ways are there for the 100 pages to be assigned to the four printers?

(c) Suppose that all the pages are black-and-white, but each group of 25 consecutive pages (1-25, 26-50, 51-75, 76-100) must be assigned to the same printer. Each printer can be assigned 0, 25, 50, 75, or 100 pages to print.
How many ways are there for the 100 pages to be assigned to the four printers?

8.2 The generalized product rule

Consider a race with 20 runners. There is a first place, a second place and a third place trophy. An outcome of the race is defined to be who wins each of the three trophies. How many outcomes are possible?

All 20 of the runners are eligible to win the first place trophy. Once the first place runner is determined, there are 19 possibilities left for the second place trophy (since no one can place both first and second). Once the top two runners are determined, there are 18 possibilities for the third place trophy. The number of possibilities for the outcome of the race is $20 \cdot 19 \cdot 18 = 6840$.

The race example illustrates that a useful way to think about counting is to imagine selecting an element from the set to be counted. The selection process is carried out in a sequence of steps. In each step, one more decision is made about the item that will be selected. At the end of the process the item to be selected is fully specified. The ***generalized product rule*** says that in selecting an item from a set, if the number of choices at each step does not depend on previous choices made, then the number of items in the set is the product of the number of choices in each step.

Definition 8.2.1: Generalized product rule.

Consider a set S of sequences of k items. Suppose there are:

- n_1 choices for the first item.
- For every possible choice for the first item, there are n_2 choices for the second item.
- For every possible choice for the first and second items, there are n_3 choices for the third item.

$$\vdots$$

- For every possible choice for the first k-1 items, there are n_k choices for the k^{th} item.

Then $|S| = n_1 \cdot n_2 \cdots n_k$.

The next animation illustrates the following example: A family of four (2 parents and 2 kids) goes on a hiking trip. The trail is narrow and they must walk single file. How many ways can they walk with a parent in the front and a parent in the rear?

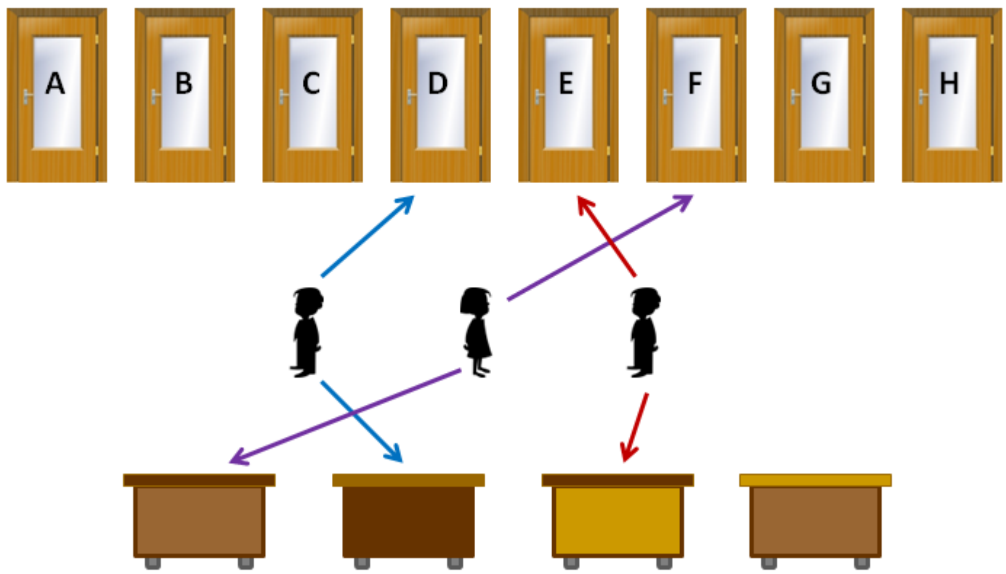
PARTICIPATION ACTIVITY | 8.2.1: An example of the generalized product rule.

Animation captions:

1. The desired sequence is (Parent, Child, Child, Parent). The first person in the sequence can be Mom or Dad. (2 choices.)
2. For each choice for the first person, there are two choices for the second person, Sister or Brother. There are $2 \cdot 2$ choices so far.
3. If the second person is Sister, then the third person must be Brother. If the second person is Brother, the third person must be Sister. Only 1 choice exists for the third person.
4. The last person must be the parent who was not chosen to be the first person. Only 1 choice exists for the fourth person. The total number of choices is $2 \cdot 2 \cdot 1 \cdot 1 = 4$.

Example 8.2.1: Counting office and desk selections by the generalized product rule.

Suppose that there are three employees in a start-up. They rent an office space with 8 offices, anticipating growth. The office space comes with four desks. Each person can select an office and a desk. The selection is done in the order that the participants joined the company with the founder going first. How many ways are there for the selection to be done?



First the founder makes his selections. He has a choice of 8 different offices and four different desks. Using the product rule, this gives 8·4 different choices. Next the second employee selects. She has a choice of 7 different offices and 3 different desks (because she can not select either the office or the desk chosen by the founder). She has 7·3 choices for her office/desk combination. Finally the number three employee picks. He has a choice of 6 offices and two desks, for a total of 6·2 choices. Overall the number of possible selections is:

$$(8 \cdot 4) \cdot (7 \cdot 3) \cdot (6 \cdot 2) = 8064$$

PARTICIPATION ACTIVITY | 8.2.2: Counting selections of officers by the generalized product rule.

In the following question, a club with 10 students elects a president, vice president, secretary and treasurer. No student can hold more than one position. Express your answer to each question as a number.

1) How many ways are there to

select the class officers?

Check Show answer

2) Now suppose that there are five ninth graders and five tenth graders in the club. How many ways are there to elect the officers if the president is a tenth grader?

Check Show answer

3) Again suppose that there are five ninth graders and five tenth graders in the club. How many ways are there to elect the officers if the president is a tenth grader and the VP is a ninth grader?

Check Show answer

Additional exercises

EXERCISE 8.2.1: Counting passwords without repeating characters.

Consider the following definitions for sets of characters:

- Digits = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 }
- Letters = { a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z }
- Special characters = { *, &, $, # }

Compute the number of passwords that satisfy the given constraints.

(a) Strings of length 6. Characters can be special characters, digits, or letters, with no repeated characters.

(b) Strings of length 6. Characters can be special characters, digits, or letters, with no repeated characters. The first character can not be a special character.

EXERCISE 8.2.2: Strings with no repetitions.

(a) How many strings are there over the set {a, b, c} that have length 10 in which no two consecutive characters are the same? For example, the string "abcbcbabcb" would count and the strings "abbbcbabcb" and "aacbcbabcb" would not count.

EXERCISE 8.2.3: Counting license plate numbers.

License plate numbers in a certain state consists of seven characters. The first character is a digit (0 through 9). The next four characters are capital letters (A through Z) and the last two characters are digits. Therefore, a license plate number in this state can be any string of the form:

Digit-Letter-Letter-Letter-Letter-Digit-Digit

(a) How many different license plate numbers are possible?

(b) How many license plate numbers are possible if no digit appears more than once?

(c) How many license plate numbers are possible if no digit or letter appears more than once?

EXERCISE 8.2.4: Selecting coders for 3 different projects.

(a) A manager must select three coders from her group to write three different software projects. There are 7 junior and 3 senior coders in her group. The first project can be written by any of the coders. The second project must be written by a senior person and the third project must be written by a junior person. How many ways are there for her to assign the three coders to the projects if no person can be assigned to more than one project?

8.3 The bijection rule

Some sets are easier to count than others. One way to approach a difficult counting problem is to show that the cardinality of the set to be counted is equal to the cardinality of a set that is easy to count. The ***bijection rule*** says that if there is a bijection from one set to another then the two sets have the same cardinality.

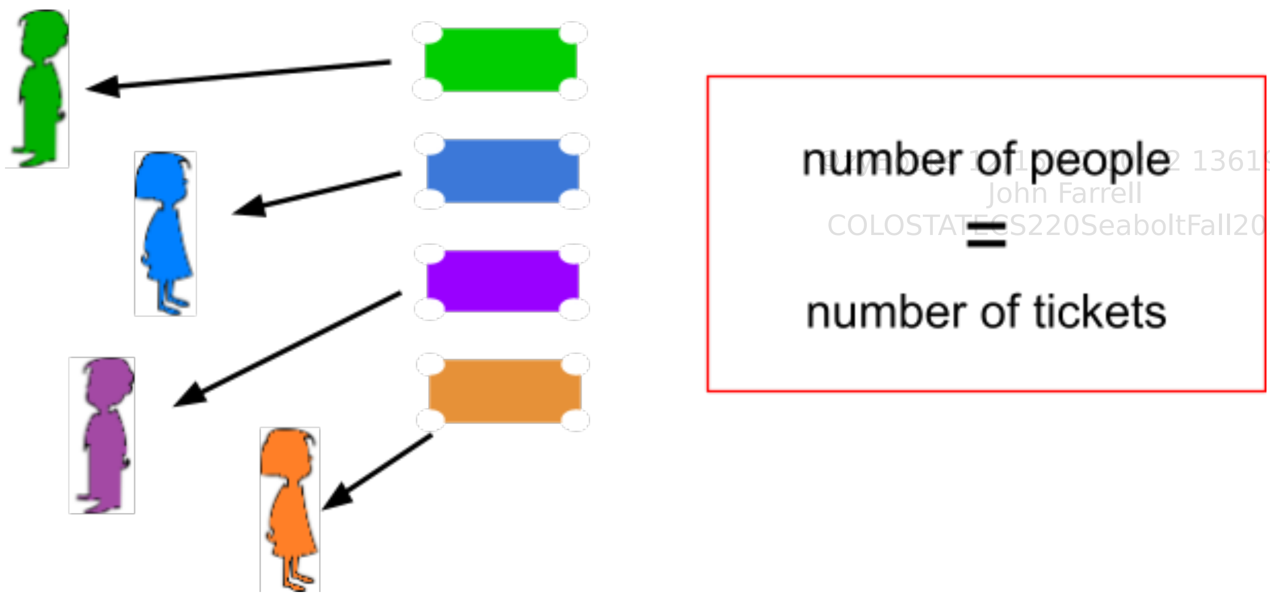
A function f from a set S to a set T is called a ***bijection*** if and only if f has a well defined inverse. The ***inverse*** of a function f that maps set S to set T is a function g that maps T to S such that for every s ∈ S and every t ∈ T, f(s) = t, if and only if g(t) = s. If a function f has an inverse, it is denoted by f^{-1}.

Definition 8.3.1: The bijection rule.

Let S and T be two finite sets. If there is a bijection from S to T, then |S| = |T|.

Suppose that every person in a theater must submit a ticket to an usher in order to enter. One way to count the number of people in the theater is to count the number of tickets submitted. In this case, the bijection is from the set of submitted tickets to the set of people in the theater. Each ticket is mapped to the person who submits the ticket to the usher. The inverse function maps each person to his or her ticket.

Figure 8.3.1: Bijection mapping tickets to people.



The next example uses the bijection rule to determine the cardinality of the power set of a finite set X. Recall that the power set of X (denoted P(X)) is the set of all subsets of X. Let |X| = n. The animation below illustrates a bijection f from P(X) to the set of binary strings of length n:

PARTICIPATION ACTIVITY 8.3.1: An example of the bijection rule.

Animation captions:

1. A function f is a bijection from the power set of {a, b, c} to the set of 3-bit strings. The set ∅ does not include a,b, or c. Therefore, f(∅) = 000.
2. The set {a} includes a, so the first bit of f({a}) is 1. The set {a} does not include b or c, so the second and third bits of f({a}) are 0. f({a}) = 100.
3. The value of f can be determined the same way for all the subsets of {a,b,c}, ending with f({a,b,c}) = 111.
4. f defines a bijection between the set $\{0,1\}^3$ and the power set of {a, b, c}. Therefore $|P(X)| = |\{0,1\}^3|$.

The bijection illustrated in the animation can be defined formally. Order the elements of X in some way: x_1, x_2, ...x_n. The order is arbitrary but it is important to pick one ordering and stick with it. For each Y ⊆ X, f(Y) is an n-bit string y whose bits are $y_1y_2...y_n$. The string y is defined by the rule: $y_i = 1$ if $x_i \in Y$, otherwise $y_i = 0$. The inverse of f maps binary strings of length n back to subsets of X. $f^{-1}(y)$ is a subset Y of X such that $x_i \in Y$ if $y_i = 1$, otherwise $x_i \notin Y$. For every Y ⊆ X and every y ∈ {0,

$1\}^n$,

$$f(Y) = y \leftrightarrow f^{-1}(y) = Y.$$

PARTICIPATION ACTIVITY 8.3.2: The bijection mapping power sets to strings.

Let X = {1, 2, 3, 4}. Define the function f from P(X) to $\{0, 1\}^4$ as defined above.

1) What is f({1,4})?

Check Show answer

2) Which element is not in $f^{-1}(1101)$?

Check Show answer

3) How many elements are in the set $f^{-1}(0000)$?

Check Show answer

The k-to-1 rule

A group of kids at a slumber party all leave their shoes in a big pile at the door. One way to count the number of kids at the party is to count the number of shoes and divide by 2. Of course, it is important to establish that each kid has exactly one pair of shoes in the pile. Counting kids by counting shoes and dividing by 2 is an example of the k-to-1 rule with k = 2. Applying the k-to-1 rule requires a well defined function from objects we can count to objects we would like to count. In the example with the shoes, the function maps each shoe to the kid who owns it. Here is a definition of the kind of function that is required:

Definition 8.3.2: k-to-1 correspondence.

Let X and Y be finite sets. The function $f:X \to Y$ is a ***k-to-1 correspondence*** if for every $y \in Y$, there are exactly k different $x \in X$ such that $f(x) = y$.

A 1-to-1 correspondence is another term for a bijection, so a bijection is a k-to-1 correspondence with k = 1. The ***k-to-1 rule*** uses a k-to-1 correspondence to count the number of elements in the range by counting the number of elements in the domain and dividing by k.

Definition 8.3.3: k-to-1 rule.

Suppose there is a k-to-1 correspondence from a finite set A to a finite set B. Then $|B| = |A|/k$.

PARTICIPATION ACTIVITY | 8.3.3: An example of the k-to-1 rule.

Animation captions:

1. 6 cans of juice per pack. f(c) = p if can c belongs to pack p. f is a 6-to-1 function.
2. (# cans of juice)/6 = (# packs of juice).

PARTICIPATION ACTIVITY | 8.3.4: K-to-1 rule.

1) A farm orders x horse shoes for its horses. The farm does not order extras and all the horses will get new horse shoes. Apply the k-to-1 rule to determine the number of horses on the farm. Give your answer as a mathematical expression in terms of x.

Check Show answer

Additional exercises

EXERCISE 8.3.1: Devising a 3-to-1 correspondence.

(a) Find a function from the set {1, 2, ..., 30} to {1, 2, ..., 10} that is a 3-to-1 correspondence. (You may find that the division, ceiling or floor operations are useful.)

EXERCISE 8.3.2: Using the bijection rule to count palindromes.

If x is a string, then x^R is the reverse of the string. For example, if x = 1011, then x^R = 1101. A string is a palindrome if the string is the same backwards and forwards (i.e., if $x = x^R$). Let B = {0, 1}. The set B^n is the set of all n-bit strings. Let P_n be the set of all strings in B^n that are palindromes.

(a) Show a bijection between P_6 and B^3.

(b) What is $|P_6|$?

(c) Determine the cardinality of P_7 by showing a bijection between P_7 and B^n for some n.

EXERCISE 8.3.3: Using the bijection rule to count binary strings with even parity.

Let B = {0, 1}. B^n is the set of binary strings with n bits. Define the set E_n to be the set of binary strings with n bits that have an even number of 1's. Note that zero is an even number, so a string with zero 1's (i.e., a string that is all 0's) has an even number of 1's.

(a) Show a bijection between B^9 and E_{10}. Explain why your function is a bijection.

(b) What is $|E_{10}|$?

EXERCISE 8.3.4: Using the bijection rule to count ternary strings whose digits sum to a multiple of 3.

Let T = {0, 1, 2}. A string $x \in T^n$ is said to be balanced if the sum of the digits is an integer multiple of 3.

(a) Show a bijection between the set of strings in T^6 that are balanced and T^5. Explain why your function is a bijection.

(b) How many strings in T^6 are balanced?

EXERCISE 8.3.5: Using the k-to-1 rule for counting ways to line up a group.

Ten kids line up for recess. The names of the kids are:

{Abe, Ben, Cam, Don, Eli, Fran, Gene, Hal, Ike, Jan}.

Let S be the set of all possible ways to line up the kids. For example, one ordering might be:

(Fran, Gene, Hal, Jan, Abe, Don, Cam, Eli, Ike, Ben)

The names are listed in order from left to right, so Fran is at the front of the line and Ben is at the end of the line.

Let T be the set of all possible ways to line up the kids in which Gene is ahead of Don in the line. Note that Gene does not have to be immediately ahead of Don. For example, the ordering shown above is an element in T.

Now define a function f whose domain is S and whose target is T. Let x be an element of S, so x is one possible way to order the kids. If Gene is ahead of Don in the ordering x, then f(x) = x. If Don is ahead of Gene in x, then f(x) is the ordering that is the same as x, except that Don and Gene have swapped places.

(a) What is the output of f on the following input?
(Fran, Gene, Hal, Jan, Abe, Don, Cam, Eli, Ike, Ben)

(b) What is the output of f on the following input?
(Eli, Ike, Don, Hal, Jan, Abe, Ben, Fran, Gene, Cam)

(c) Is the function f a k-to-1 correspondence for some positive integer k? If so, for what value of k? Justify your answer.

(d) There are 3628800 ways to line up the 10 kids with no restrictions on who comes before whom. That is, |S| = 3628800. Use this fact and the answer to the previous question to determine |T|.

(e) Let Q be the set of orderings in which Gene comes before Don and Jan comes before Abe (again, not necessarily immediately before). Define a k-to-1 correspondence from S to Q. Use the value of k to determine |Q|.

8.4 Counting permutations

One of the most common applications of the generalized product rule is in counting permutations. An ***r-permutation*** is a sequence of r items with no repetitions, all taken from the same set. Consider the set X = {John, Paul, George, Ringo}. The sequences (Paul, Ringo, John) and (John, George, Paul) are both examples of 3-permutations over X. In a sequence, order matters, so the sequence (Paul, Ringo, John) is different from the sequence (Ringo, Paul, John).

PARTICIPATION ACTIVITY | 8.4.1: Using the generalized product rule to count the number of 5-permutations from a set of size 8.

Animation captions:

1. Select a 5-permutation from a set with 8 elements {A, B, C, D, E, F, G, H}. 8 choices exist for the first item. If F is selected, the remaining choices are {A, B, C, D, E, G, H}.
2. There are 7 choices for the second item, 6 for the third, 5 for the fourth, and 4 for the fifth. There are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$, 5 permutations from a set of 8.

Fact 8.4.1: The number of r-permutations from a set with n elements.

Let r and n be positive integers with r ≤ n. The number of r-permutations from a set with n elements is denoted by P(n, r):

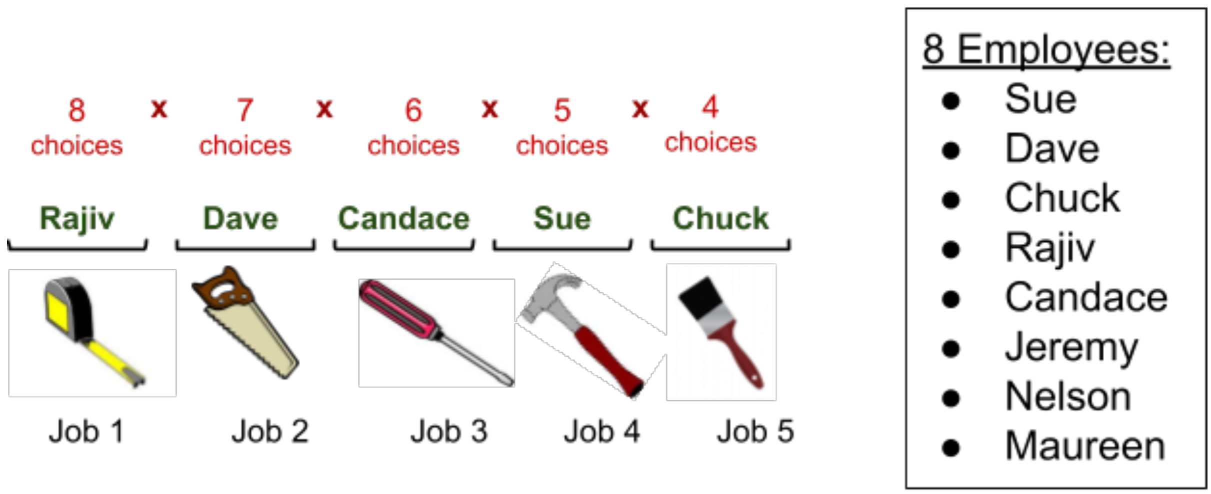
$$
\begin{aligned}
P(n,r) &= \frac{n!}{(n-r)!} \\
&= \frac{n(n-1)\cdots(n-r+1)\ \cancel{(n-r)}\ \cancel{(n-r-1)}\ \cdots \cancel{1}}{\cancel{(n-r)}\ \cancel{(n-r-1)}\ \cdots \cancel{1}} \\
&= n(n-1)\cdots(n-r+1)
\end{aligned}
$$

The closed form for P(n, r) is a consequence of the generalized product rule. There are n choices for the first item in the sequence because the set from which the items are drawn has n elements. Once the choice of the first item in the sequence is made, there are n - 1 choices for the next item because the first item in the sequence can not be repeated. In general, once the first i items in the sequence have been chosen, there are n - i remaining elements from which the next one can be chosen. The selection process continues until r items have been chosen for the sequence. Just before the last (r^{th}) item is chosen, r - 1 items have already been chosen and there are n - (r - 1) = n - r + 1 items from which to select the last item.

Example 8.4.1: Counting job assignments by counting r-permutations.

A manager has five different jobs that need to get done on a given day. She has eight employees whom she can assign to the jobs. A job only requires one person and no person can be assigned more than one job. How many possible ways can she do the assignment?

Order the jobs arbitrarily so that one job is first, one is second, etc. An assignment is a 5-permutation from the set of 8 employees. The first person gets the first job, the second person gets the second job,..., the fifth person gets the fifth job. The 5-permutation will look like:



Source: Tape measure (G.E.Sattler Creative Commons); other pictures are Public Domain.

The number of 5-permutations from a set of 8 people is $P(8, 5) = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$.

The number of assignments can also be derived using the generalized product rule directly. There are 8 choices for job 1. Once the person for job 1 has been selected, there are 7 remaining choices for job 2, then 6 choices for job 3, 5 choices for job 4, and 4 choices for job 5. Applying the product rule, the total number of assignments is $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$.

PARTICIPATION ACTIVITY 8.4.2: Counting r-permutations.

1) A red, blue, and green die are thrown. Each die has six possible outcomes. How many outcomes are possible in which

the three dice all show different numbers?

Check Show answer

2) There are 5 computers and 3 students. How many ways are there for the students to sit at the computers if no computer has more than one student and each student is seated at a computer?

Check Show answer

3) A class has ten students. A teacher will give out three prizes: One student gets a gift card, one gets a book, and one gets a movie ticket. No student can receive more than one prize. How many ways can the teacher distribute the prizes?

Check Show answer

A ***permutation*** (without the parameter r) is a sequence that contains each element of a finite set exactly once. For example, the set {a, b, c} has six permutations:

Table 8.4.1: Permutations of the set {a, b, c}.

(a, b, c)	(b, a, c)	(c, a, b)
(a, c, b)	(b, c, a)	(c, b, a)

Fact 8.4.2: The number of permutations of a finite set.

The number of permutations of a finite set with n elements is

$$P(n, n) = n \times (n-1) \times \ldots \times 2 \times 1 = n!$$

PARTICIPATION ACTIVITY 8.4.3: Counting line-ups: Permutations.

1) A wedding party consisting of a bride, a groom, two bridesmaids, and two groomsmen line up for a photo. How many ways are there for the wedding party to line up?

Check Show answer

The next example combines the product rule with counting permutations. Consider again the set {John, Paul, George, Ringo}. These four would like to sit on a bench together, but Paul and John would like to sit next to each other. How many possible seatings are there? In order to apply the generalized product rule, view the set of possibilities as a process in which a seating is specified. The first step is to determine whether Paul sits to the left or right of John. There are two possible choices: Paul is to the left of John or Paul is to the right of John. Then glue Paul and John together in the chosen order to satisfy the constraint that they sit together. Now there are three items to order: two of them are people (George and Ringo), the other is a pair that is bound together

[John+Paul]. The next step is to select a permutation of the three items. The animation illustrates the final count:

PARTICIPATION ACTIVITY | 8.4.4: Using the product rule with counting permutations.

Animation captions:

1. First decide if John is to the left of Paul: John, Paul or Paul, John. There are 2 choices so far.
2. There are 3! permutations of George, Ringo, and (John+Paul).
3. Put the choices together. For each permutation of George, Ringo, and (John+Paul), there are 2 ways to order John and Paul. $(3!) \cdot 2 = 12$ choices.

PARTICIPATION ACTIVITY | 8.4.5: Counting line-ups: Permutations and the product rule.

1) A wedding party consisting of a bride, a groom, two bridesmaids, and two groomsmen line up for a photo. How many ways are there for the wedding party to line up so that the bride is next to the groom?

Check Show answer

CHALLENGE ACTIVITY | 8.4.1: Counting permutations of passwords.

422102.2723990.qx3zqy7

Start

Each character in a password is either a digit [0-9] or lowercase letter [a-z]. How many valid passwords are there with the given restriction(s)?

Length is 14.
No character repeats.

Ex: P(11, 4) * P(4, 3)

Write permutations as: P(n, k)

1	2	3	4	5	6

Check Next

Additional exercises

EXERCISE | 8.4.1: Counting functions from a set to itself.

Count the number of different functions with the given domain, target and additional properties.

(a) f: $\{0,1\}^7 \rightarrow \{0,1\}^7$.

(b) f: $\{0,1\}^7 \rightarrow \{0,1\}^7$. The function f is one-to-one.

(c) f: $\{0,1\}^5 \rightarrow \{0,1\}^7$.

(d) f: $\{0,1\}^5 \rightarrow \{0,1\}^7$. The function f is one-to-one.

EXERCISE | 8.4.2: Counting telephone numbers.

At a certain university in the U.S., all phone numbers are 7-digits long and start with either 824 or 825.

(a) How many different phone numbers are possible?

(b) How many different phone numbers are there in which the last four digits are all different?

EXERCISE | 8.4.3: Lining up club members for a photo.

Ten members of a club are lining up in a row for a photograph. The club has one president, one VP, one secretary, and one treasurer.

(a) How many ways are there to line up the ten people?

(b) How many ways are there to line up the ten people if the VP must be beside the president in the photo?

(c) How many ways are there to line up the ten people if the president must be next to the secretary and the VP must be next to the treasurer?

EXERCISE | 8.4.4: Lining up a girl scout troop.

(a) A girl scout troop with 10 girl scouts and 2 leaders goes on a hike. When the path narrows, they must walk in single file with a leader at the front and a leader at the back. How many ways are there for the entire troop (including the scouts and the leaders) to line up?

8.5 Counting subsets

Consider a class with 20 students who must elect three representatives to the student council. The teacher conducts a vote and reveals the names of the three students who received the most votes. He does not reveal how many votes each student received or which one received more votes than the other two. How many ways are there to select the three representatives?

The outcome of the election is a set of three students, not a sequence because there is no particular order imposed on the three representatives. The outcome {Joshua, Karen, Ingrid} is the same outcome as {Karen, Ingrid, Joshua}. A subset of size r is called an ***r-subset***. In counting the number of ways to elect the three representatives, we are counting the number of different 3-subsets of students from a class of size 20.

An r-subset is sometimes referred to as an ***r-combination***. The counting rules for sequences and subsets are commonly referred to as "permutations and combinations". The term "combination" in the context of counting is another word for "subset".

PARTICIPATION ACTIVITY | 8.5.1: Distinguishing between r-subsets and r-permutations.

Let S = {a, b, c}.

1) Is (b, a) a 2-permutation or a 2-subset from S?

- 2-permutation
- 2-subset

2) Is {b, a} a 2-permutation or a 2-subset from S?

- 2-permutation
- 2-subset

3) How many different 2-permutations from S are there?

- 6
- 3
- 1

4) How many different 2-subsets from S are there?

- 6
- 3
- 1

Using the k-to-1 rule to count subsets

Consider a small example in which a subset of three colors is selected from the set

Colors = {blue, green, orange, pink, red}

The number of 3-permutations from the set of five colors is P(5, 3) = 5!/2! = 60. Now define a function mapping 3-permutations to 3-subsets. The function is defined by just removing the ordering, so (orange, pink, blue) and (blue, orange, pink) both map to the set {orange, blue, pink}. The animation below shows that the function is (3!)-to-1, so by the k-to-1 rule (with k = 3!):

$$\text{Number of 3-subsets of colors} = \frac{P(5,3)}{3!} = \frac{5!}{3!2!} = 10$$

PARTICIPATION ACTIVITY 8.5.2: Mapping r-permutations to r-subsets: A small example.

Animation captions:

1. How many permutations map to the selection {orange, blue, pink}? 3! permutations because there are 3! ways to permute the 3 colors.
2. 3! 3-permutations map to one 3-subset.

To derive the general rule for counting r-subsets, define a mapping between r-permutations from a set of size n and r-subsets. The k-to-1 rule will be applied with k = r! as illustrated in the animation below:

PARTICIPATION ACTIVITY 8.5.3: Mapping r-permutations to r-subsets to subsets: A general rule.

Animation captions:

1. There are P(n,r) r-permutations from set S = {1, 2, ..., n}.
2. r! r-permutations map to each r-subset from S.
3. Each r- subset from S ({1, 2, ..., r} through {n-r+1, ..., n}) has r! r-permutations that map onto it.
4. $(\# \text{ r-subsets from S}) = \frac{(\# \text{ r-permutations from S})}{\text{r!}} = \frac{\text{P(n, r)}}{\text{r!}} = \frac{\text{n!}}{\text{(r!(n-r)!)}}$.

PARTICIPATION ACTIVITY 8.5.4: Mapping r-permutations to r-subsets.

1) Consider a function that maps 5-permutations from a set S = {1, 2, ..., 20} to 5-subsets from S. The function takes a 5-permutation and removes the ordering on the elements. How many 5-permutations map on to the subset {2, 5, 13, 14, 19}?

Check Show answer

Counting subsets comes up so frequently that the formula for counting subsets has its own notation and terminology:

Definition 8.5.1: Counting subsets: 'n choose r' notation.

The number of ways of selecting an r-subset from a set of size n is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$\binom{n}{r}$ is read "n choose r". The notation C(n, r) is sometimes used for $\binom{n}{r}$.

PARTICIPATION ACTIVITY | 8.5.5: Calculating 'n choose r'.

1) Calculate a numerical value for $\binom{7}{3}$.
(Hint: write out the factorials as products and cancel numbers before multiplying).

Check Show answer

2) Calculate a numerical value for $\binom{7}{4}$.

Check Show answer

3) Calculate a numerical value for $\binom{100}{1}$, the number of ways to select a subset of size 1 from a set of size 100.

Check Show answer

4) Calculate a numerical value for

$\binom{100}{100}$, the number of ways to select a subset of size 100 from a set of size 100. Note that 0! = 1.

Check Show answer

We can calculate an expression for $\binom{n}{n-r}$ by replacing r with n - r in the expression for $\binom{n}{r}$:

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}.$$

An equation is called an ***identity*** if the equation holds for all values for which the expressions in the equation are well defined. The equation $\binom{n}{r} = \binom{n}{n-r}$ is an identity because the equality holds for any non-negative integer n and any integer r in the range from 0 through n. The identity means that for any set S with n elements, the number of r-subsets from S is equal to the number of (n - r)-subsets from S. In fact, there is a bijection between r-subsets of S and (n - r)-subsets of S: each r-subset X of S corresponds uniquely to a subset of (n - r) elements consisting of the elements that are not in X.

PARTICIPATION ACTIVITY | 8.5.6: Examples of counting subsets.

For the following questions, express your answer as "n choose r". Ex: 8 choose 3.

1) A teacher must select four members of the math club to participate in an upcoming competition. How many ways are there for her to make her selection if the club has 12 members?

Check Show answer

2) A file will be replicated on 3 different computers in a distributed network of 15 computers. How many ways are

there to select the locations for the file?

Check Show answer

Counting binary strings with a fixed number of 1's

The ideas used to count subsets of a particular size can also be used to count binary strings with a particular number of 1's. The animation below shows how to count the number of 5-bit strings that have exactly two 1's by showing a bijection between the strings to be counted and the number of 2-subsets from a set of size 5.

PARTICIPATION ACTIVITY 8.5.7: Counting 5-bit strings with exactly two 1's.

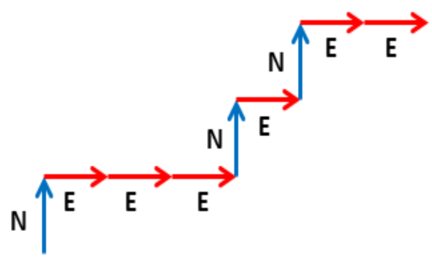
Animation captions:

1. Bijection from 5-bit strings with exactly 2 1's to 2-subsets of {1, 2, 3, 4, 5}. The bits are numbered 1 through 5 from left to right.
2. {1, 2} maps to 11000 because the two 1's are in places 1 and 2. Each 2-subset of {1, 2, 3, 4, 5} maps to a 5-bit string with 2 1's in the same way.
3. Since the mapping is a bijection, (# of 5-bit strings with exactly 2 1's) = (# of 2-subsets of {1, 2, 3, 4, 5} = $\binom{5}{2}$

Example 8.5.1: Counting paths on grid.

Consider a city whose streets are laid out as a grid. Streets run north-south or east-west. A visitor is dropped off by a cab at a certain intersection and would like to visit a museum that is 6 blocks to the east and 3 blocks to the north of his current location. The visitor always takes a direct path and never travels west or south in the course of his walk. How many distinct paths are there for the visitor to walk to the museum?

The idea in counting paths is to show a bijection between distinct paths and certain kinds of sequences. The sequences are like the directions for the visitor, telling him in which direction he should walk at each block. The visitor needs to walk a total of nine blocks. For six of the blocks, he walks east and 3 blocks he walks north, so the sequences have length 9 and there are 6 E's and 3 N's. The path below corresponds to the sequence NEEENENEE:



Counting the number of sequences with 6 E's and 3 N's is similar to the problem of counting the number of binary strings with a particular number of 1's. There are a total of 6 + 3 = 9 characters in the sequence where each character is an N or an E. Once the location of the three N's is chosen from the 9 possible places in the sequence, E's are placed in the remaining six locations, and the sequence is determined.

There are $\binom{9}{3}$ sequences with 6 E's and 3 N's. In general, the number of paths that take the visitor n blocks north and m blocks east is $\binom{m+n}{m}$.

PARTICIPATION ACTIVITY | 8.5.8: Counting strings by counting subsets.

1) Define a bijection between 5-subsets of the set S = {1, 2, 3, 4, 5, 6, 7, 8} and 8-bit strings with exactly five 1's. A subset X of S with five elements maps on to a string x so that $j \in X$ if and only if the j^{th} bit of x is 1. What

string corresponds to the set {1, 3, 4, 5, 8}?

Check Show answer

2) How many 8-bits strings have exactly five 1's?

Express your answer as 'n choose r'. Ex: 6 choose 2.

Check Show answer

3) How many strings over the alphabet {a, b} have length 20 and exactly 8 a's?

Express your answer as 'n choose r'. Ex: 6 choose 2.

Check Show answer

CHALLENGE ACTIVITY 8.5.1: Counting subsets of bit strings.

422102.2723990.qx3zqy7

Start

A bit string contains 1's and 0's. How many different bit strings can be constructed given the restriction(s)?

Length is 27.

Ex: 2^40 or 26 * C(36, 12)

Write a^b as: a^b
Write combination as: C(n, k)

1	2	3	4	5

Check Next

Additional exercises

EXERCISE 8.5.1: Mapping permutations to subsets.

Consider a function f that maps 5-permutations from the set S = {1, ..., 20} to 5-subsets from S. The function takes a 5-permutation and creates an unordered set whose elements are the five numbers included in the permutation.

(a) What is the value of f on input (12, 1, 3, 15, 9)?

(b) Is (12, 3, 12, 4, 19) a 5-permutation? Why or why not?

(c) How many permutations are mapped onto the subset {12, 3, 13, 4, 19}?

EXERCISE 8.5.2: Permutations and combinations from a set of letters.

Define the set S = {a, b, c, d, e, f, g}.

(a) Give an example of a 4-permutation from the set S.

(b) Give an example of a 4-subset from the set S.

(c) How many subsets of S have exactly four elements?

(d) How many subsets of S have either three or four elements?

EXERCISE 8.5.3: Counting bit strings.

How many 10-bit strings are there subject to each of the following restrictions?

(a) No restrictions.

(b) The string starts with 001.

(c) The string starts with 001 or 10.

(d) The first two bits are the same as the last two bits.

(e) The string has exactly six 0's.

(f) The string has exactly six 0's and the first bit is 1.

(g) There is exactly one 1 in the first half and exactly three 1's in the second half.

EXERCISE 8.5.4: Counting strings of letters.

How many different strings of length 12 containing exactly five a's can be chosen over the following alphabets?

(a) The alphabet {a, b}

(b) The alphabet {a, b, c}

EXERCISE 8.5.5: Choosing a chorus.

(a) There are 30 boys and 35 girls that try out for a chorus. The choir director will select 10 girls and 10 boys from the children trying out. How many ways are there for the choir director to make his selection?

EXERCISE 8.5.6: Counting possible computer failures.

Suppose a network has 40 computers of which 5 fail.

(a) How many possibilities are there for the five that fail?

(b) Suppose that 3 of the computers in the network have a copy of a particular file. How many sets of failures wipe out all the copies of the file? That is, how many 5-subsets contain the three computers that have the file?

EXERCISE 8.5.7: Choosing a student committee.

14 students have volunteered for a committee. Eight of them are seniors and six of them are juniors.

(a) How many ways are there to select a committee of 5 students?

(b) How many ways are there to select a committee with 3 seniors and 2 juniors?

(c) Suppose the committee must have five students (either juniors or seniors) and that one of the five must be selected as chair. How many ways are there to make the selection?

EXERCISE 8.5.8: Counting five-card poker hands.

This question refers to a standard deck of playing cards. If you are unfamiliar with playing cards, there is an explanation in "Probability of an event" section under the heading "Standard playing cards." A five-card hand is just a subset of 5 cards from a deck of 52 cards.

(a) How many different five-card hands are there from a standard deck of 52 playing cards?

(b) How many five-card hands have exactly two hearts?

(c) How many five-card hands are made entirely of hearts and diamonds?

(d) How many five-card hands have four cards of the same rank?

(e) A "full house" is a five-card hand that has two cards of the same rank and three cards of the same rank. For example, {queen of hearts, queen of spades, 8 of diamonds, 8 of spades, 8 of clubs}. How many five-card hands contain a full house?

(f) How many five-card hands do not have any two cards of the same rank?

8.6 Subset and permutation examples

Questions about counting subsets and permutations often do not explicitly use the words "subset" or "permutation". Selecting a sample object from the set is helpful to determine whether the formulas P(n, k) or $\binom{n}{k}$ apply. If the order in which the elements of the permutation/subset are selected is important, then the question is asking about permutations. If order is not important, then the question is asking about subsets.

Consider as an example a distribution process in which a teacher is distributing a set of four prizes to the ten students in his class. Each student can get at most one prize. How many ways are there to distribute the prizes if:

- The prizes are all identical.
- The prizes are all different from each other.

PARTICIPATION ACTIVITY 8.6.1: Distributing identical vs. distinct prizes.

Animation captions:

1. Distribute 4 identical prizes. A sample outcome shows the subset of 4 students who get prizes. $\binom{10}{4}$ different choices.
2. Distribute 4 different prizes. A sample outcome shows which students get 1st, 2nd, 3rd, and 4th prizes.
3. The outcome is different if two students swap prizes. The outcome is a sequence of students who get prizes. P(10,4) choices.

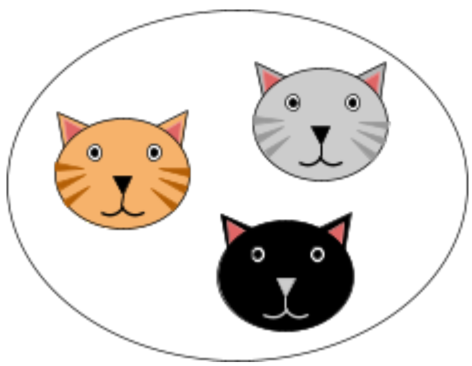
If the prizes are all different from each other, then the problem is asking about the number of 4-permutations from a set of 10 elements because the order in which prizes are distributed is important. If the prizes are all identical, then the problem is asking about the number of 4-subsets because the order in which prizes are distributed is not important.

Example 8.6.1: Two different cat selection problems: Subsets vs. permutations.

Consider two closely related counting problems:

1. A family goes to the animal shelter to adopt 3 cats. The shelter has 20 different cats from which to select. How many ways are there for the family to make their selection?
2. Three different families go to the animal shelter to adopt a cat. Each family will select one cat. How many ways are there for the families to make their selections? Note that which family gets which cat matters.

In the first problem in which one family selects three cats, the number of ways to make the selection is $\binom{20}{3}$ because the order in which the cats are selected is not important. The outcome of a selection is just the set of cats that the family selects.



In the second problem in which three families are each selecting one cat, the specific cat selected by each family is important. The order in which families make their selection is fixed but arbitrary. The first family selects their cat from the set of 20 cats at the shelter. Then, since no cat can belong to two families, the second family selects their cat from the 19 remaining cats, and finally the third family selects their cat from the 18 cats left after the first two families have chosen. Thus, there are $20 \cdot 19 \cdot 18 = P(20, 3)$ ways for the three families to make their selections. The fact that the order in which the cats are selected matters corresponds to the fact that which family gets which cat is important. The two sample selections below are considered different even though the same three cats leave the shelter that day.

PARTICIPATION ACTIVITY 8.6.2: Selecting subsets or permutations.

Provide solutions in either the form "P(n, r)" or "n choose r" (e.g. P(8, 3) or 8 choose 3).

1) Dave swims three times in the week. How many ways are there to plan his workout schedule (i.e. which days he will swim) for a given week?

Check Show answer

2) Dave will swim one day, run one day, and bike another day in a week. He does at most one activity on any particular day. How many ways are there for him to select his workout schedule (i.e. which activities he does which days)?

Check Show answer

3) The students in a class elect a president, vice president, secretary, and treasurer. There are 30 students in the class and no student can have more than one job. Note that it matters who is elected into which

position. How many different outcomes are there from the election process?

Check Show answer

4) A class of 30 students elects four students to serve on a student leadership council. The teacher tallies the votes and only reveals the names of the four students who received the most votes. How many different outcomes are there from the election process?

Check Show answer

Additional exercises

EXERCISE 8.6.1: Selecting students for jobs.

(a) A teacher selects 4 students from her class of 37 to work together on a project. How many ways are there for her to select the students?

(b) A teacher selects students from her class of 37 students to do 4 different jobs in the classroom: pick up homework, hand out permission slips, staple worksheets, and organize the classroom library. Each job is performed by exactly one student in the class and no student can get more than one job. How many ways are there for her to select students and assign them to the jobs?

EXERCISE 8.6.2: Results of a piano competition.

120 pianists compete in a piano competition.

(a) In the first round, 30 of the 120 are selected to go on to the next round. How many different outcomes are there for the first round?

(b) In the second round, the judges select the first, second, third, fourth and fifth place winners of the competition from among the 30 pianists who advanced to the second round. How many outcomes are there for the second round of the competition?

EXERCISE 8.6.3: Choosing a lineup for a traveling basketball team.

There are 20 members of a basketball team.

(a) The coach must select 12 players to travel to an away game. How many ways are there to select the players who will travel?

(b) From the 12 players who will travel, the coach must select her starting line-up. She will select a player for each of the five positions: center, power forward, small forward, shooting guard and point guard. How many ways are there for her to select the starting line-up?

(c) From the 12 players who will travel, the coach must select her starting line-up. She will select a player for each of the five positions: center, power forward, small forward, shooting guard and point guard. However, there are only three of the 12 players who can play center. Otherwise, there are no restrictions. How many ways are there for her to select the starting line-up?

EXERCISE | 8.6.4: Hiring a software engineer.

A search committee is formed to find a new software engineer.

(a) If 100 applicants apply for the job, how many ways are there to select a subset of 9 for a short list?

(b) If 6 of the 9 are selected for an interview, how many ways are there to pick the set of people who are interviewed? (You can assume that the short list is already decided).

(c) Based on the interview, the committee will rank the top three candidates and submit the list to their boss who will make the final decision. (You can assume that the interviewees are already decided.) How many ways are there to select the list from the 6 interviewees?

EXERCISE | 8.6.5: Counting orders at a restaurant.

A group of five friends go to a restaurant for dinner. The restaurant offers 20 different main dishes.

(a) Suppose that the group collectively orders five different dishes to share. The waiter just needs to place all five dishes in the center of the table. How many different possible orders are there for the group?

(b) Suppose that each individual orders a main course. The waiter must remember who ordered which dish as part of the order. It's possible for more than one person to order the same dish. How many different possible orders are there for the group?

(c) Suppose that each individual orders a main course. The waiter must remember who ordered which dish as part of the order. However the friends agree that no two people will order the same dish. How many different possible orders are there for the group?

EXERCISE 8.6.6: Selecting a committee of senators.

A country has two political parties, the Demonstrators and the Repudiators. Suppose that the national senate consists of 100 members, 44 of which are Demonstrators and 56 of which are Repudiators.

(a) How many ways are there to select a committee of 10 senate members with the same number of Demonstrators and Repudiators?

(b) Suppose that each party must select a speaker and a vice speaker. How many ways are there for the two speakers and two vice speakers to be selected?

8.7 Combinations

This fourth python programming assignment, PA4, is about combinations. You will write a function comb(A,n,k,p,lo) that prints all k out of n combinations of 0...n-1 in lexicographical order. The parameters p and lo represent the current location to be filled (p) and the first number to pick in that location (lo). The array A is used to create and store the current combination. The algorithm for enumerating combinations is discussed in lecture 17 Permutations.

```
python3 comb.py 5 3
```

produces

```
[0, 1, 2]
[0, 1, 3]
[0, 1, 4]
[0, 2, 3]
[0, 2, 4]
[0, 3, 4]
[1, 2, 3]
[1, 2, 4]
[1, 3, 4]
[2, 3, 4]
```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LAB ACTIVITY 8.7.1: Combinations 0 / 100 comb.py

Load default template...

```
import sys

def comb(A,n,k,p,lo):
  """
    n>=1, k<=n, p: position to fill, lo: first number to pick
    print all possible subsets of k out of n
  """

if __name__ == "__main__":
  d = len(sys.argv)>3
  n = int(sys.argv[1])
  k = int(sys.argv[2])
  A = []
  for i in range(k):
      A.append(0)
  if d: print("n:",n,"k:",k)
  comb(A,n,k,0,0)
```


Develop mode **Submit mode**

Run your program as often as you'd like, before submitting for grading. Below, type any needed input values in the first box, then click **Run program** and observe the program's output in the second box.

Enter program input (optional)

If your code requires input values, provide them here.

Run command

python3 comb.py Additional arguments

Run program

Input (from above) → **comb.py** (Your program) → 0

Program output displayed here

Coding trail of your work What is this?

History of your effort will appear here once you begin working on this zyLab.