Students:
Section 3.3 is a part of 1 assignment:
Reading Assignment 3

Requirements: PA CA
No due date

3.3 Properties of functions

A function f: X → Y is ***one-to-one*** or ***injective*** if $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$. That is, f maps different elements in X to different elements in Y.

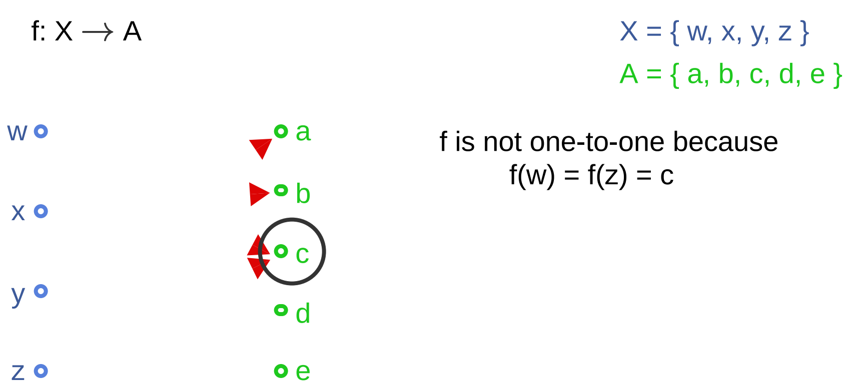
A function f: X → Y is ***onto*** or ***surjective*** if the range of f is equal to the target Y. That is, for every y ∈ Y, there is an x ∈ X such that f(x) = y.

The properties of being one-to-one or onto are important in many situations. For example, consider a function that maps employees to offices. If the function is one-to-one, then no one has to share an office. If the function is onto, then there are no empty offices and the company's space is well utilized. Functions are used to define an assignment of processes to computers in a distributed network. If the function is one-to-one, then no computer is time-sharing between different tasks. If the function is onto, then all the resources of the network are being utilized.

A function is ***bijective*** if it is both one-to-one and onto. A bijective function is called a ***bijection***. A bijection is also called a ***one-to-one correspondence***. Here are some examples that illustrate one-to-one and onto functions:

PARTICIPATION ACTIVITY 3.3.1: One-to-one and onto functions.

1 2 3 4 2x speed



f is not one-to-one because f(w) = f(z) = c.

Captions

1. f is not one-to-one because f(w) = f(z) = c.
2. f is not onto because no elements in X map to d or e.
3. Now f is one-to-one but not onto.
4. Now f is one-to-one and onto. f is a bijection.

Feedback?

PARTICIPATION ACTIVITY | 3.3.2: Identifying one-to-one and onto functions.

Indicate whether the functions defined below are one-to-one, onto, neither, or both:

1) h: **Z** → **Z**. h(x) = x - 4.

- Neither one-to-one nor onto
- One-to-one but not onto
- Onto but not one-to-one
- Both one-to-one and onto

Correct

If x ≠ y, then x - 4 ≠ y - 4, so h is one-to-one. For every y, there is an integer x such that x - 4 = y, so h is onto.

2) t: **Z** → **Z**. t(x) = 2x.

- Neither one-to-one nor onto
- One-to-one but not onto
- Onto but not one-to-one
- Both one-to-one and onto

Correct

If x ≠ y, then 2x ≠ 2y. Therefore t is one-to-one. For any odd integer y, there is no x such that 2x = y. For example, there is no integer x such that 2x = 5. Therefore t is not onto.

3) f: **Z** → **Z**.

- Neither one-to-one nor onto
- One-to-one but

Correct

f(3) = f(4) = 2, so f is not one-to-one. For every y, there is an x such that f(x)=y. Check that f(2y - 1) = y for every integer y, so f is onto.

not onto

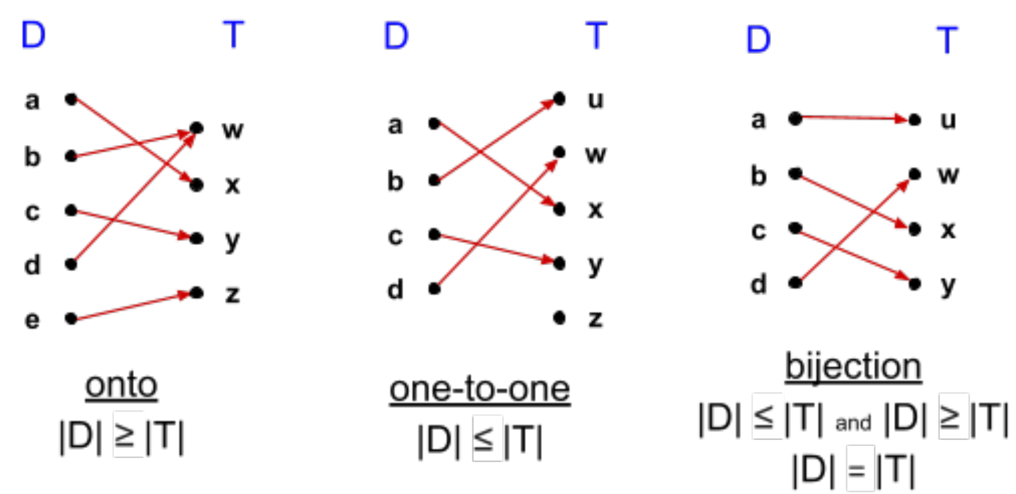
- Onto but not one-to-one
- Both one-to-one and onto

Feedback?

When the domain and target are finite sets, it is possible to infer information about their relative sizes based on whether the function is one-to-one or onto.

- If f: D → T is onto, then for every element in the target, there is at least one element in the domain: |D| ≥ |T|.
- If f: D → T is one-to-one, then every element in the domain maps to a unique element in the target: |D| ≤ |T|.
- If f: D → T is a bijection, then f is one-to-one and onto: |D| ≤ |T| and |D| ≥ |T|, which implies that |D| = |T|.

Figure 3.3.1: Relative sizes of the domain and target from function properties.



Feedback?

The fact that the domain and target of a bijection have the same size may seem simple but this fact turns out to be extremely powerful. One way to count the elements in a set is to define a bijection between that set and another set whose size is already known. Counting the elements in a set is a fundamental part of discrete probability, an important tool in many areas of science.

PARTICIPATION ACTIVITY | 3.3.3: Properties of functions and the relative sizes of the domain and target.

Let f be a function whose domain is $\{0, 1\}^3$ and whose target is $\{0, 1\}^2$.

1) Is it possible that f is one-to-one?

- Yes
- No

2) Is it possible that f is onto?

- Yes
- No

3) Is it possible that f is a bijection?

- Yes
- No

Feedback?

CHALLENGE ACTIVITY | 3.3.1: Draw an arrow diagram and identify function properties.

Note: If x is a real number, then |x| is the absolute value of x. If A is a finite set, then |A| refers to the cardinality of the set A. Both uses of the |*| symbols are standard mathematical notation.

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Start

$\mathbf{Z} \to \mathbf{Z}$: f(x) = x - 5

A. Select the element in f(x) that corresponds to the circled x.

B. Select the properties that descr

onto	not onto
one-to-one	not one-to-one

-7

0 -6

x		f(x)
1	-5	
2	-4	
3	-3	
	-2	

Undo

1 2 3 4 5

Check Next

Feedback?

Additional exercises

EXERCISE | 3.3.1: Properties of functions that map ordered pairs of integers to integers.

Consider the following functions from **Z** × **Z** to **Z**. Which ones are onto? Justify your answer.

(a) $f(x, y) = 2x - 4y$

(b) $f(x, y) = |x| - |y|$

(c) $f(x, y) = x + y - 2$.

(d) $f(x, y) = x^{|y|}$

(e) $f(x, y) = x^{|2y|}$

Feedback?

EXERCISE | 3.3.2: Properties of algebraic functions.

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

(a) f: **R** → **R**. $f(x) = x^2$

(b) g: **R** → **R**. $g(x) = x^3$

(c) h: **Z** → **Z**. $h(x) = x^3$

(d) f: **Z** → **Z**,

(e) f: **Z** → **Z**, $f(x) = 5x - 4$

(f) f: **Z** → **Z**, $f(x) = x - 4$

(g) f: **Z** × **Z** → **Z** × **Z**, $f(x, y) = (x+1, 2y)$

(h) f: **Z** × **Z** → **Z** × **Z**,

(i) f: **Z** × **Z** → **Z** × **Z**,

(j) f: **Z** × **Z** → **Z** × **Z**, $f(x, y) = (1-y, 1-x)$

(k) f: $\mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$, $f(x, y) = 2^x + y$.

Feedback?

EXERCISE | 3.3.3: Function properties and the relative sizes of the domain and target.

(a) If a function f is onto, then the domain of f is at least as large as its target. Show that the converse of the previous statement is not necessarily true by showing a function f: $\{0,1\}^3 \to \{0, 1\}^2$ that is not onto.

(b) If a function f is one-to-one, then the target of f is at least as large as its domain. Show that the converse of the previous statement is not necessarily true by showing a function f: $\{0,1\}^2 \to \{0, 1\}^3$ that is not one-to-one.

(c) If a function f is a bijection, then the domain of f is the same size as its target. Show that the converse of the previous statement is not necessarily true by showing a function f: $\{0,1\}^2 \to \{0, 1\}^2$ that is neither one-to-one nor onto.

Feedback?

Feedback?

EXERCISE 3.3.4: Properties of functions on strings and power sets.

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

(a) f: $\{0, 1\}^4 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and dropping the first bit. For example f(1011) = 011.

(b) f: $\{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, f(001) = 101 and f(110) = 110.

(c) f: $\{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example f(011) = 110.

(d) f: $\{0, 1\}^3 \to \{0, 1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string. For example, f(100) = 1001.

(e) Let A be defined to be the set {1, 2, 3, 4, 5, 6, 7, 8}. f: P(A) → {0, 1, 2, 3, 4, 5, 6, 7, 8}. For X ⊆ A, f(X) = |X|. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.

(f) Let A be defined to be the set {1, 2, 3, 4, 5, 6, 7, 8}. f: P(A) → P(A). For X ⊆ A, f(X) = A-X. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.

(g) Let A be defined to be the set {1, 2, 3, 4, 5, 6, 7, 8} and let B = {1}. f: P(A) → P(A). For X ⊆ A, f(X) = X - B. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.

(h) A = {a, b, c}, h: P(A) → P(A). For X ⊆ A, h(X) = X ⊕ {a}.

(i) A = {a, b, c}, h: P(A) → P(A). For X ⊆ A, h(X) = X ∪ {a}.

Feedback?

EXERCISE 3.3.5: Functions on integers with particular properties.

Find a function whose domain is the set of all integers and whose target is the set of all positive integers that satisfies each set of properties.

(a) Neither one-to-one, nor onto.

(b) One-to-one, but not onto.

(c) Onto, but not one-to-one.

(d) One-to-one and onto.

Feedback?

EXERCISE 3.3.6: Proving statements about bijections.

Let and . Determine whether each statement is true. If the statement is true, provide a proof. If the statement is false, provide a counterexample.

(a) If and are both bijections, then , defined as , is also a bijection.

(b) For any real number , we will define as . Prove that if is a bijection and , then is also a bijection.

Feedback?

How was this section? Provide feedback

Activity summary for assignment: Reading Assignment 3 40 / 90 pts

No due date 40 / 90 pts submitted to canvas

Completion details