11.1 An introduction to algorithms

Suppose you were given a list of five numbers and asked to find the smallest one. You would probably only require a quick glance at the list of numbers to find the smallest one. Suppose, however, you were given a list of temperatures recorded in a city every hour for the past ten years, and you were asked to find the lowest temperature. Finding the smallest value in such a long list of numbers would require a more systematic method. Computers are especially useful for processing large amounts of data. Using a computer to solve a problem requires a carefully-specified sequence of instructions in the form of a computer program.

An ***algorithm*** is a step-by-step method for solving a problem. A description of an algorithm specifies the input to the problem, the output to the problem, and the sequence of steps to be followed to obtain the output from the input. A recipe is an example of an algorithm in which the ingredients are the input and the final dish is the output. A good recipe gives a clear sequence of steps indicating how to produce the dish from the ingredients. A description of an algorithm usually includes:

- A name for the algorithm
- A brief description of the task performed by the algorithm
- A description of the input
- A description of the output
- A sequence of steps to follow

Algorithms are often described in ***pseudocode***, which is a language in between written English and a computer language. Steps are formatted carefully as indented lists so as to convey the structure of the approach. The steps themselves are expressed in brief English phrases or mathematical symbols.

An important type of step in an algorithm is an ***assignment***, in which a variable is given a value. An assignment operator is denoted by:

```
x := y
```

The assignment statement above would change the value of x to be the current value of the variable y. It is also possible to assign a specific value to a variable as in x := 5, after which the value of x would be 5.

The output of an algorithm is specified by a ***return*** statement. For example, the statement Return(x) would return the value of x as the output of the algorithm. A return statement in the middle of an algorithm causes the execution of the algorithm to stop at that point and return the indicated value. The following animation gives an algorithm written in pseudocode to find the sum of three numbers:

PARTICIPATION ACTIVITY 11.1.1: Algorithm to compute the sum of three numbers.

Animation content:

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Animation captions:

1. A description of an algorithm starts with the name of the algorithm.
2. The name is followed by a description of what the algorithm computes. The algorithm named "Sum of three" finds the sum of three numbers.
3. The next two lines describe the input and output of the algorithm, followed by the steps of the algorithm.
4. In a trial run of the algorithm on input a = 7, b = 8, c = 9. The variable "sum" is assigned the value a + b + c, which is 7 + 8 + 9 = 24.
5. The return statement returns the value of sum as the output of the algorithm, which is 24 for inputs a = 7, b = 8, and c = 9.

PARTICIPATION ACTIVITY 11.1.2: Assignment operations.

Give your answers to the following question in numerical form (e.g., "4" and not "four").

```
x := 5
y := 6
y := x
```

1) What is the value of variable x after the lines of code are executed?

Check Show answer

2) What is the value of variable y after the lines of code are executed?

Check Show answer

The if-statement and the if-else-statement

An ***if-statement*** tests a condition, and executes one or more instructions if the condition evaluates to true. In a single line if-statement, a condition and instruction appear on the same line. In the example below, if x currently has the value of 5, then y is assigned the value 7.

```
If (x = 5), y := 7
```


An if-statement can also be followed by a sequence of indented operations with a final end-if statement. In the block of code below, all n operations would be executed if the condition evaluates to true. If the condition evaluates to false, then the algorithm proceeds with the next instruction after the end-if.

```
If ( condition )
    Step 1
    Step 2
    . . .
    Step n
End-if
```

An ***if-else-statement*** tests a condition, executes one or more instructions if the condition evaluates to true, and executes a different set of instructions if the condition evaluates to false. The steps to be executed if the condition evaluates to true are indented below the if-statement. The steps to be executed if the condition evaluates for false are indented below the else statement, followed by a final end-if statement.

```
If ( condition )
    One or more steps
Else
    One or more steps
End-if
```

PARTICIPATION ACTIVITY 11.1.3: If-else-statement.

Animation content:

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Animation captions:

1. The condition of the if-else statement is x < y. Since x = 3 and y = 6, the condition evaluates to true. The lines under "If" are executed.
2. The variable "min" is set to x and the variable "max" is set to y. The lines under "Else" are skipped.
3. In the next block of code, x = 7 and y = 2. The condition of the if-else statement is x < y

which evaluates to false. Lines under "If" are skipped. Lines under "Else" are executed.
4. The variable "min" is set to y and max is set to x.

PARTICIPATION ACTIVITY 11.1.4: Algorithm to compute the smallest of three numbers.

Animation content:

undefined

Animation captions:

1. A description of an algorithm starts with the name of the algorithm.
2. The name is followed by a description of what the algorithm computes. The algorithm named "Smallest of three" finds the smallest of three numbers.
3. The next two lines describe the input and output of the algorithm, followed by the steps of the algorithm.
4. In a trial run of the algorithm on input a = 7, b = 5, c = 9, the first line sets min to the value of a, which is 7.
5. The next line is an if-statement that tests whether (b < min), which evaluates to true.
6. Since (b < min) is true, min is set to the value of b, which is 5.
7. The next line is an if-statement that tests whether (c < min), which evaluates to false.
8. Since (c < min) is false, the algorithm proceeds to the next line which returns the current value of min which is 5.

PARTICIPATION ACTIVITY 11.1.5: If-else-statement.

Give your answers to the following question in numerical form (e.g., "4" and not "four").

1) What is the value of abs after the following lines of code are run?


```
x := 2
If ( x > 0 )
    abs := x
Else
    abs := -x
End-if
```

Check Show answer

2) What is the value of abs after the following lines of code are run?

```
x := -2
If ( x > 0 )
    abs := x
Else
    abs := -x
End-if
```


Check Show answer

The for-loop

To solve a problem on a set of data, it is often necessary to perform an operation involving each piece of data in turn. Looping structures provide a means of specifying a sequence of steps that are to be repeated. The discussion below introduces two common kinds of loops: for-loops and while-loops.

In a ***for-loop***, a block of instructions is executed a fixed number of times as specified in the first line of the for-loop, which defines an ***index***, a starting value for the index, and a final value for the index. Each repetition of the block of instructions inside the for-loop is called an ***iteration***. The index is initially assigned a value equal to the starting value, and is incremented just before the next iteration begins. The final iteration of the for-loop happens when the index reaches the final value. In the example below, i is the index, s is the initial value, and t is the final value of the index.

```
For i = s to t
    Step 1
    Step 2
    . . .
    Step n
End-for
```

In the first iteration, i has a value of s. In the next iteration i = s + 1, and so on. In the final iteration, i = t. If t ≥ s, then the for-loop iterates t - s + 1 times. If t < s, then the for-loop is skipped entirely and the algorithm proceeds with the line after the end-for.

The animation below illustrates how the for-loop is used in an algorithm that finds the smallest number in a sequence of n numbers. The animation also shows how the algorithm works on a sample input sequence.

PARTICIPATION ACTIVITY 11.1.6: Algorithm to compute the smallest in a sequence of numbers.

Animation content:

undefined

Animation captions:

1. The algorithm finds the smallest in a sequence of numbers, a_1 through a_n. The first line sets min to the value of a_1, which is 5.
2. The for-loop has an index i which starts at 2 and goes to n. Inside the loop, an if-statement evaluates whether (a_i < 5), which is true because a_2 = 3.
3. Since the condition is true, min gets the value of a_2, which is 3. In the next iteration of the for-loop, i = 3. Again, (a_i < min) is true because a_3 = -1.
4. Min gets the value of a_3, which is -1. In the next iteration of the for-loop, i = 4 and the condition (a_i < min) is false.
5. The for-loop ends when i = n. Since n = 4 and i = 4, the algorithm proceeds to the line after the for-loop which returns current value of min = -1.

PARTICIPATION ACTIVITY 11.1.7: For-loops.

Consider the following pseudocode fragment:

```
sum := 0
For i = 2 to 5
    sum := sum + i
End-for
```

1) What is the value of "sum" after the second iteration?

Check Show answer

2) How many iterations will the for-loop execute?

Check Show answer

3) What is the final value for "sum" after executing the for-loop?

Check Show answer

The while-loop

The while-loop is similar to the for-loop in that it provides a way to specify that a sequence of steps should be repeated. A for-loop specifies the number of iterations in advance, via the beginning and ending index values. A ***while-loop*** iterates an unknown number of times, ending when a certain condition becomes false. A while-loop is written as follows:

```
While ( condition )
    Step 1
    Step 2
    . . .
    Step n
End-while
```

The condition is an expression that evaluates to true or false. If the expression evaluates to true, then steps 1 through n are performed and the algorithm goes back to the first statement where the condition is re-evaluated. The process continues until the condition evaluates to false, at which point the algorithm proceeds with the next step after the while-loop.

The following animation illustrates the while-loop in an algorithm that searches for a particular number in a sequence of numbers:

PARTICIPATION ACTIVITY 11.1.8: Algorithm to search for an item in a sequence.

Animation content:

undefined

Animation captions:

1. The algorithm searches for a number x in a sequence of numbers a_1 through a_n. The first line sets the value of i to 1.
2. The condition of the while-loop is that $a_i \neq x$ and i < n. Since input value x = 1 and a_1 = 7, $a_i \neq x$. Also n, the number of numbers in the list is 4 and i = 1 , so i < n.
3. The line inside the while-loop, increments i by 1. The value of i is now 2 and the input value $a_i = a_2$ is 6.
4. The condition of the while-loop, $a_i \neq x$ and i < n, is true because 6 $\neq$ 1 and 2 < 4.

5. The line inside the loop increments i by 1. The value of i is now 3 and the input value $a_i = a_3$ is 1.
6. The condition of the while-loop is now false because a_i = x.
7. The next line after the while-loop is an if-statement that returns i if (a_i = x). Since i = 3 and a_3 = x, the algorithm returns 3.
8. Now the sequence is 7, 6, 1. n = 3 and x = 2. Since x is not present in the list, the while loop goes through the whole list. The condition fails when i = n.
9. In the next step, the algorithm tests whether a_3 = x. Since a_3 = 1 and x = 2, the condition is false. The final step returns -1 because x is not in the list.

PARTICIPATION ACTIVITY 11.1.9: While-loops.

Consider the following pseudocode fragment:

```
product := 1
count := 5
While ( count > 0 )
    product := product*count
    count := count - 2
End-while
```

1) What is the value of product after the second iteration?

Check Show answer

2) How many iterations will the while-loop execute?

Check Show answer

3) What is the final value for "product"?

Check Show answer

Nested loops

A ***nested loop*** is a loop that appears within another loop. The nested loop, known as the *inner loop*, is treated as a sequence of steps inside the outer loop. A nested loop is illustrated below in an algorithm that counts the number of duplicate pairs in a sequence of numbers.

PARTICIPATION ACTIVITY 11.1.10: Algorithm to count duplicate pairs in a sequence.

Animation content:

undefined

Animation captions:

1. The algorithm sets count to 0. The first iteration of the nested loops has indices i = 1 and j = 2 and tests whether (a_i = a_j): false because a_1 = 1 and a_2 = 5.
2. In the next iteration of the inside loop, j = 3. (a_i = a_j) is false because a_1 = 1 and a_3 = 6.
3. The last iteration of the inside loop is when j = 4. (a_i = a_j) is false because a_1 = 1 and a_4 = 5.
4. In the next iteration of the outside loop, i is now 2 and j is set to i+1 = 3. (a_i = a_j) is false because a_2 = 5 and a_3 = 6.
5. In the next iteration of the inside loop, j = 4. (a_i = a_j) is true because a_2 = 5 and a_4 = 5. The condition of the if-statement is true, so count is incremented by 1.
6. In the next iteration of the outside loop, i is now 3 and j is set to i+1 = 4. (a_i = a_j) is false because a_3 = 6 and a_4 = 5.
7. Both for-loops are finished and the algorithm returns the current value of "count" which is 1. There is one pair of duplicates in the list (a_2 = a_4).

PARTICIPATION ACTIVITY 11.1.11: Nested loops - example 1.

Consider the following pseudocode fragment:

1) How many times is the variable "count" increased?

Check Show answer

```
count := 0
For i = 1 to 3
    For j = 1 to 4
        count := count + i * j
    End-for
End-for
```

2) What is the final value of "count"?

Check Show answer

PARTICIPATION ACTIVITY 11.1.12: Nested loops - example 2.

Consider the following pseudocode fragment:

```
count := 0
For i = 1 to 3
    For j = i+1 to 4
        count := count + i * j
    End-for
End-for
```

1) For the iteration of the outer loop where i = 2, how many times does the inner loop iterate?

Check Show answer

2) How many times is the variable "count" increased?

Check Show answer

3) What is the final value of "count"?

Check Show answer

Additional exercises

EXERCISE 11.1.1: Writing algorithms in pseudocode.

Write an algorithm in pseudocode for each description of the input and output.

(a) Input: $a_1, a_2, \ldots, a_n$, a sequence of numbers, where $n \geq 1$
n, the length of the sequence.
Output: "True" if the sequence is non-decreasing and "False" otherwise.
A sequence of numbers is non-decreasing if each number is at least as large as the one before.

(b) Input: $a_1, a_2, \ldots, a_n$, a sequence of numbers, where $n \geq 1$
n, the length of the sequence.
Output: "True" if there are two consecutive numbers in the sequence that are the same and "False" otherwise.

(c) Input: $a_1, a_2, \ldots, a_n$, a sequence of numbers, where $n \geq 1$
n, the length of the sequence.
Output: "True" if there are any two numbers in the sequence whose sum is 0 and "False" otherwise.

(d) Input: $a_1, a_2, \ldots, a_n$, a sequence of numbers, where $n \geq 1$
n, the length of the sequence.
Output: "True" if there are any three numbers in the sequence that form a Pythagorean triple.
The numbers x, y, and z are a Pythagorean triple if $x^2 + y^2 = z^2$.

(e) Input: $a_1, a_2, \ldots, a_n$, a sequence of distinct numbers, where $n \geq 2$
n, the length of the sequence.
Output: The second smallest number in the sequence.

11.2 Asymptotic growth of functions

Let f be a function that maps positive integers to non-negative real numbers(f: $\mathbf{Z^+ \rightarrow R^{\geq}}$). The symbol $\mathbf{R^{\geq}}$ denotes the set of non-negative real numbers: $\mathbf{R^{\geq}} = \mathbf{R^+} \cup \{0\}$. The ***asymptotic growth*** of the function f is a measure of how fast the output f(n) grows as the input n grows. The classification of functions using Oh, Ω, and Θ notation (called ***asymptotic notation***) provides a way

to concisely characterize the asymptotic growth of a function. Asymptotic notation is a useful tool for evaluating the efficiency of algorithms.

The notation f = O(g) is read "f is ***Oh of*** g". f = O(g) essentially means that the function f(n) is less than or equal to g(n), if constant factors are omitted and small values for n are ignored. In the expressions $7n^3$ and $5n^2$, the 7 and the 5 are called ***constant factors*** because the values of 7 and 5 do not depend on the variable n. If f is O(g), then there is a positive number c such that when f(n) and c·g(n) are graphed, the graph of c·g(n) will eventually cross f(n) and remain higher than f(n) as n gets large.

Consider the function f(n) = $2n^3 + 3n^2 + 7$. There are many functions g(n) such that f = O(g). For example, f(n) is O($4n^3 + 2n + 1$). However, the idea is to select a function g that characterizes the asymptotic growth of f as simply as possible. Therefore, g is selected so as to eliminate all unnecessary constants and additional terms. Instead of saying that f = O($4n^3 + 2n + 1$), one would typically say that f = O(n^3).

Sometimes the functions in this material will evaluate to a negative number, such as n/2 - 10, when n < 20. In order to ensure that every function maps to a non-negative number for every possible input from $\mathbf{Z^+}$, the output value can be replaced by 0 if the output value is less than or equal to 0. Thus, the function f(n) really means max { f(n), 0 }.

Definition 11.2.1: Oh notation.

Let f and g be functions from $\mathbf{Z^+}$ to $\mathbf{R^{\geq}}$. Then f = O(g) if there are positive real numbers c and n_0 such that for any n $\in \mathbf{Z^+}$ such that n $\geq n_0$,

$$f(n) \leq c \cdot g(n).$$

PARTICIPATION ACTIVITY 11.2.1: Oh notation example.

Animation content:

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Animation captions:

1. $f(n) = 2n^3 + 3n^2 + 7$. $g(n) = n^3$. To prove that f is O(g), pick c = 3, n_0 = 4.
2. 3g(n) is larger than f(n) for n $\geq n_0$.

The animation shown above provides intuition for what it means for a function f to be O(g). However, a mathematical proof is required to formally establish that a function f(n) is Oh of another

function g(n).

Proof 11.2.1: Proof that f is O(g).

$f(n) = 3n^3 + 5n^2 - 7$
$g(n) = n^3$

Claim: f = O(g).

Proof.
Select c = 8 and $n_0 = 1$. We will show that for any $n \geq 1$, $f(n) \leq 8 \cdot g(n)$.

$$f(n) = 3n^3 + 5n^2 - 7 \leq 3n^3 + 5n^2$$

For $n \geq 1$, $n^2 \leq n^3$, so

$$3n^3 + 5n^2 \leq 3n^3 + 5n^3$$

Finally, $3n^3 + 5n^3 = 8n^3 = 8 \cdot g(n)$. Putting the inequalities together, we get that for any $n \geq 1$,

$$f(n) = 3n^3 + 5n^2 - 7 \leq 3n^3 + 5n^2 \leq 3n^3 + 5n^3 = 8n^3 = 8 \cdot g(n)$$

and therefore, $f(n) \leq 8 \cdot g(n)$ which means that f = O(g) . ■

Video explaining the steps in the proof.

The constants c and n_0 in the definition of Oh-notation are said to be a ***witness*** to the fact that f = O(g). In a proof that f = O(g), there are many different choices for the witness c and n_0 that will suffice. The proof given above that $3n^3 + 5n^2 - 7$ is $O(n^3)$ used witness c = 8 and $n_0 = 1$. The combination c = 4 and $n_0 = 5$ would have also worked, although the algebra in the proof would have been more involved. In showing that a polynomial function f(n) of degree k is $O(n^k)$, the following combination will work as a witness:

- $n_0 = 1$
- c = the sum of the positive coefficients in f (including the constant term)

For example, if $f(n) = 3n^5 + 7n^4 - 3n^3 + 2$, a proof that f is $O(n^5)$ could use $c = 3 + 7 + 2 = 12$ and $n_0 = 1$.

PARTICIPATION ACTIVITY 11.2.2: Proving f = O(g), where f and g are polynomials.

Define the functions

$f(n) = 5n^6 - 4n^4 + 2n^2 + 4$

$g(n) = 4n^6 + 3n^5 + 2n^2 + 1$

1) Indicate whether the following statement is true or false:
For any n ≥ 1, f(n) ≤ 11·n^6.

- True
- False

2) Indicate whether the following statement is true or false:
For any n ≥ 1, g(n) ≤ 9·n^6.

- True
- False

3) Is it true that f(n) = O(n^7)?

- True
- False

Proving that f is not O(g)

By definition, if f = O(g), then there must be a witness, c and n_0, showing that f = O(g). Showing that f is not O(g) requires showing that every possible combination of values for c and n_0 fails to be a witness. A proof that f is not O(g) must establish that for every n_0 and c, there is a value of n such that n ≥ n_0 and f(n) > c · g(n).

PARTICIPATION ACTIVITY 11.2.3: A proof that f is not O(g).

Animation content:

undefined

Animation captions:

1. $f(n) = \frac{n^3}{2} - 2n^2 + 3$. $g(n) = n^2$.Prove that f is not O(g) by contradiction. Suppose that for constants c and n_0, and all $n \geq n_0$, $\frac{n^3}{2} - 2n^2 + 3 \leq c \cdot n^2$.
2. If $c \cdot n^2$ is bigger than $\frac{n^3}{2} - 2n^2 + 3$, then $c \cdot n^2$ is also bigger than $\frac{n^3}{2} - 2n^2$. Therefore, $\frac{n^3}{2} - 2n^2 \leq c \cdot n^2$.
3. Each side of the last inequality is divided by n^2.
4. The resulting inequality is n/2 - 2 ≤ c. Now 2 is added to both sides.

5. The resulting inequality is $n/2 \leq c + 2$. Both sides are multiplied by 2 to get $n \leq 2c + 4$.
6. When n is the maximum of n_0 and $2c + 5$, the last inequality is false. All steps can be reversed, so $f(n) > c \cdot g(n)$ for some $n > n_0$, a contradiction.

PARTICIPATION ACTIVITY 11.2.4: Proofs that for functions f and g, f is not O(g).

1) A proof that f is not O(g), must show that it is **not** the case that:

   There are positive constants c and n_0 such that for every $n \geq n_0$, $f(n) \leq c \cdot g(n)$.

   Which statement is equivalent to the statement that f is not O(g)?

   - For every choice of positive values for c and n_0, there is an $n \geq n_0$ such that $f(n) \leq c \cdot g(n)$.
   - For every choice of positive values for c and n_0, there is an $n \geq n_0$ such that $f(n) > c \cdot g(n)$.
   - There are constants c and n_0 such that for every $n \geq n_0$, $f(n) > c \cdot g(n)$.

2) Is $n^2 = O(n)$?

   - Yes
   - No

Ω notation

The O notation serves as a rough upper bound for functions (disregarding constants and small input values). The Ω notation is similar, except that it provides a lower bound on the growth of a function:

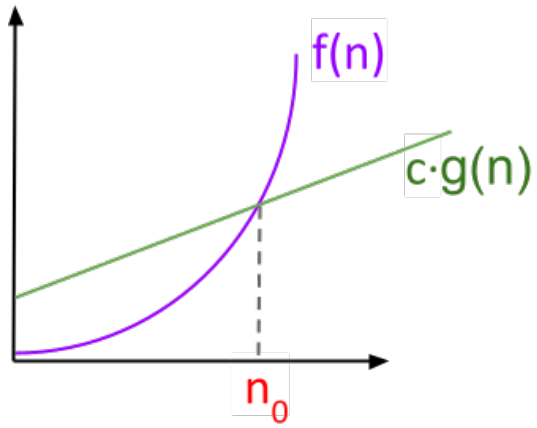
Definition 11.2.2: Ω Notation.

Let f and g be functions from $\mathbf{Z^+}$ to $\mathbf{R^{\geq}}$. Then f = Ω(g) if there are positive real numbers c and n_0 such that for any n ∈ $\mathbf{Z^+}$ such that for n ≥ n_0,

$$f(n) \geq c \cdot g(n).$$

The notation f = Ω(g) is read "f is ***Omega*** of g".

Figure 11.2.1: f is Ω(g).



There is a natural relationship between Oh-notation and Ω-notation, stated in the following theorem:

Theorem 11.2.1: Relationship of Oh-notation and Ω-notation.

Let f and g be functions from $\mathbf{Z^+}$ to $\mathbf{R^{\geq}}$. Then f = Ω(g) if and only if g = O(f).

In a proof that f is Ω(g), there are many different choices for the constants c and n_0 that will suffice as a witness. If f(n) is a polynomial of degree k, then f = Ω(n^k) only if the coefficient of the n^k term in f (call it a_k) is positive. Here are combinations for c and n_0 that suffice as a witness to show that f = Ω(n^k).

- If f has no negative coefficients, then c = a_k and n_0 = 1 suffice.
- If f has negative coefficients (but a_k > 0), then let A be the sum of the absolute values of the

negative coefficients in f(n). The choices $c = a_k/2$ and $n_0 = \max\{1, 2A/(a_k)\}$ are sufficient.

For example, if $f(n) = (1/7)\cdot n^6 + 2n^5 + 3$, then the combination $c = 1/7$ and $n_0 = 1$ will suffice as a witness to show that f is $\Omega(n^6)$. If $f(n) = 7n^6 - 2n^5 - 3$, then the combination $c = 7/2$ and $n_0 = 2\cdot(|-2| + |-3|)/7 = 10/7$ will suffice to show that f is $\Omega(n^6)$. If $f(n) = -7n^6 + 2n^5 + 3$, then f is not $\Omega(n^6)$.

The following videos provide examples that illustrate why the values given above suffice to show that a a polynomial of degree k with a positive coefficient for x^k is $\Omega(n^k)$. The first example is a polynomial whose coefficients are all non-negative. The second example is a polynomial with some negative coefficients. Examples of formal proofs are given below.

Proof 11.2.2: Proof that f is Ω(g) for f and g with all non-negative coefficients.

$f(n) = (1/2)n^2 + 7n + 3$
$g(n) = n^2$

Claim: f = Ω(g).

Proof.
Select $c = 1/2$ and $n_0 = 1$. We will show that for any $n \geq 1$, $f(n) \geq (1/2)\cdot g(n)$.

Since $n \geq 1$, $7n \geq 0$. Adding the inequalities $7n \geq 0$ and $3 \geq 0$ gives that

$$7n + 3 \geq 0$$

Add $(1/2)n^2$ to both sides to get

$$(1/2)n^2 + 7n + 3 \geq (1/2)n^2$$

Therefore, for $n \geq 1$, $f(n) \geq (1/2)\cdot g(n)$ which means that f = Ω(g). ■

Proof 11.2.3: Proof that f is Ω(g) for f and g with some negative coefficients.

$f(n) = n^2 - 7n - 3$
$g(n) = n^2$

Claim: f = Ω(g).

Proof.
Select c = 1/2 and $n_0 = 20$. We will show that for any n ≥ 20, 2·f(n) ≥ g(n), which implies that f(n) ≥ (1/2)·g(n). Plugging in the definitions for f(n) and g(n), into 2·f(n) ≥ g(n), the goal is to show that:

$$2n^2 - 14n - 6 \geq n^2$$

Since n ≥ 20, it is also true that n ≥ 1. Start with the inequality

$$n \geq 1$$

Multiply both sides by -6 to get

$$-6n \leq -6.$$

Add $(2n^2 - 14n)$ to both sides to get:

$$2n^2 - 14n - 6 \geq 2n^2 - 14n - 6n$$

Using basic algebra:

$$2n^2 - 14n - 6n = n(2n - 14 - 6) = n(2n - 20).$$

We need to show that (2n - 20) ≥ n in order to show that $n(2n - 20) \geq n^2$. Take the inequality n ≥ 20, add n and subtract 20 from each side to get that (2n - 20) ≥ n. Multiply both sides by n to get:

$$n(2n - 20) \geq n^2.$$

Putting together all the inequalities, we get that for n ≥ 20,

$$2 \cdot f(n) = 2n^2 - 14n - 6 \geq 2n^2 - 14n - 6n = n(2n - 20) \geq n^2 = g(n)$$

and therefore, for n ≥ 20, f(n) ≥ (1/2)·g(n) which means that f = Ω(g). ■

PARTICIPATION ACTIVITY 11.2.5: Showing f is Ω(g).

Indicate whether each statement is true or false.

1) The values c = 1 and n_0 = 1 are sufficient to show that $f(n) = n^3 + n$ is $\Omega(n^3)$.
   - True
   - False

2) The values c = 1/2 and n_0 = 1 are sufficient to show that $f(n) = n^3 - (3/4)n$ is $\Omega(n^3)$.
   - True
   - False

3) The values c = 1/2 and n_0 = 2 are sufficient to show that $f(n) = n^3 - n$ is $\Omega(n^3)$.
   - True
   - False

Θ notation and polynomials

The Θ notation indicates that two functions have the same rate of growth.

Definition 11.2.3: Θ Notation.

Let f and g be functions from $\mathbf{Z^+}$ to $\mathbf{R^{\geq}}$.

$f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$.

The notation $f = \Theta(g)$ is read "f is ***Theta of*** g(n)".

If $f = \Theta(g)$, then f is said to be ***order of*** g. For example $f(n) = 4n^3 + 7n + 16$ is order of n^3. The terms 7n and 16 are called the ***lower order*** terms of the function $f(n) = 4n^3 + 7n + 16$ because removing those terms from f does not change the order of f.

The examples given so far are similar in that they are all polynomials. The following theorem establishes a general rule for bounding the growth of polynomials. The proof of the theorem is left as an exercise.

Theorem 11.2.2: Asymptotic growth of polynomials.

Let p(n) be a degree-k polynomial of the form

$$p(n) = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0$$

in which $a_k > 0$. Then p(n) is $\Theta(n^k)$.

PARTICIPATION ACTIVITY 11.2.6: Growth rates of polynomials.

$f(n) = n^5 + 4n - 3$
$g(n) = 1001 \cdot n - 100$
$h(n) = 4 \cdot n^2 - n + 3$

1) Is $f = \Omega(n^4)$?
   - Yes
   - No

2) Is $f = O(n^4)$?
   - Yes
   - No

3) Is $g = \Theta(n)$?
   - Yes
   - No

4) Is $h = \Omega(n^2)$?
   - Yes
   - No

Asymptotic growth of logarithm functions with different bases

Let a and b be two real numbers greater than 1, then

$$\log_a n = \Theta(\log_b n)$$

For example, $\log_5 n = \Theta(\log_{17} n)$. The fact that $\log_a n$ and $\log_b n$ have the same asymptotic growth rate (as long as a and b are both strictly greater than 1) follows from two facts about the logarithm function:

- If a and b are constants greater than 1, then $\log_a b$ is a positive constant.
- $\log_a n = \log_a b \cdot \log_b n$

When a function is said to be Oh of or Ω of a logarithmic function, the base is often omitted because it is understood that as long as the constant in the base is greater than 1, the value of the base does not affect the asymptotic growth of the function. For example the function $17(\log_4 n) + 4 = \Theta(\log n)$.

PARTICIPATION ACTIVITY 11.2.7: Asymptotic growth of logs.

1) Which function is not Θ(log n)?
   - $\log_{17} n$
   - $\log_{(\log n)} n$
   - $\log_{1.001} n$

The growth rate of common functions in analysis of algorithms

A function that does not depend on n at all is called a ***constant function***. f(n) = 17 is an example of a constant function. Any constant function is Θ(1).

Certain types of functions are encountered frequently in analyzing the complexity of algorithms. These functions are common enough that they have names that describe their rate of growth. For example, if a function f(n) is Θ(n), we say that f(n) is a "linear" function. The table below gives a list of common functions and their names. Except for polynomials which include linear, quadratic and cubic functions, the functions are ordered according to the rate of growth, so each function is Oh of the functions below it in the table but not Ω-of the functions lower in the table. For example, n^2 is $O(n^3)$, but n^2 is not $\Omega(n^3)$. Also n! is Ω(log n) but (log n) is not Ω(n!).

A word of warning: while the definitions allow for dropping constant factors in determining the growth rate of functions, the constant in the base of an exponential function is important. For example, it is not the case that 3^n is $O(2^n)$. Although, since $2 \leq 3$, it is true that 2^n is $O(3^n)$.

A function f(n) is said to be ***polynomial*** if f(n) is $\Theta(n^k)$ for some positive real number k. The class of polynomial functions includes the classes of linear, quadratic, and cubic functions. Every polynomial function is Oh of every exponential function, which also means that every exponential function is Ω of every polynomial function. There is no polynomial function that is Ω of any exponential function, which also means that there is no exponential function which is Oh of any polynomial function.

Table 11.2.1: Common functions in algorithmic complexity.

Function	Name
$\Theta(1)$	Constant
$\Theta(\log \log n)$	Log log
$\Theta(\log n)$	Logarithmic
$\Theta(n)$	Linear
$\Theta(n \log n)$	n log n
$\Theta(n^2)$	Quadratic
$\Theta(n^3)$	Cubic
$\Theta(n^m)$ for a positive integer m	Power
$\Theta(c^n)$, $c > 1$	Exponential
$\Theta(n!)$	Factorial

PARTICIPATION ACTIVITY 11.2.8: Asymptotic growth of common functions.

1) 2^n is $O(n^3)$

- True
- False

2) 2^n is $\Omega(\log n)$

- True
- False

3) n^{20} is $O((1.1)^n)$

- True
- False

4) n^{20} is $\Omega((1.1)^n)$

- True
- False

Rules about asymptotic growth

There are a few rules that are useful in determining the growth rate of functions that are sums of or constant multiples of the standard functions given in the table.

Figure 11.2.2: Rules for the asymptotic growth of functions.

Let f, g, and h be functions from $\mathbf{Z}^+$ to $\mathbf{R}^{\geq}$:

- If f = O(h) and g = O(h), then f+g = O(h).
- If f = Ω(h) or g = Ω(h), then f+g = Ω(h).
- If f = O(g) and c is a positive real number, then c·f = O(g).
- If f = Ω(g) and c is a positive real number, then c·f = Ω(g) .
- If f = O(g) and g = O(h), then f = O(h).
- If f = Ω(g) and g = Ω(h), then f = Ω(h).

PARTICIPATION ACTIVITY 11.2.9: Asymptotic growth of sums of functions.

Animation content:

undefined

Animation captions:

1. f(n) = $5n^3$ + 16(n log n) + 5·2^n. $5n^3$ is O(n^3) and n^3 is O(2^n). Therefore $5n^3$ is O(2^n).
2. 16(n log n) is O(n log n) and (n log n) is O(2^n). Therefore 16(n log n) is O(2^n). Also, 5·2^n is O(2^n).
3. $5n^3$, 16(n log n), and 5·2^n are all O(2^n). Therefore, f(n) = $5n^3$ + 16(n log n) + 5·2^n is also O(2^n).
4. Since 5·2^n is $\Omega(2^n)$, f(n) = $5n^3$ + 16(n log n) + 5·2^n is also $\Omega(2^n)$. Since f(n) is O(2^n) and $\Omega(2^n)$, f(n) is $\Theta(2^n)$.

PARTICIPATION ACTIVITY 11.2.10: Growth rates of functions.

$f(n) = 5 \cdot 2^n + n + 3$
$g(n) = 10(\log n) + n! + n^2$
$h(n) = 7(\log \log n) + 17(n \log n)$

1) Is $f = \Omega(n^{1000})$?

- Yes
- No

2) Is $h = \Theta(n^2)$?

- Yes
- No

3) Is $f = O(n!)$?

- Yes
- No

4) Is $g = \Omega(n^2)$?

- Yes
- No

5) Is $h = O(1)$?

- Yes
- No

CHALLENGE ACTIVITY 11.2.1: Rate of growth.

422102.2723990.qx3zqy7

Start

Order the functions by growth rate.

$a(x) = 65x!$ $b(x) = 7890^8 \cdot 10$

$c(x) = 3x$ $d(x) = 7x^2$

1 2 3 4

Check Next

Additional exercises

EXERCISE 11.2.1: Characterizing the growth rate of functions.

Characterize the rate of growth of each function f below by giving a function g such that f = Θ(g). The function g should be one of the functions in the table of common functions.

(a) $f(n) = n^8 + 3n - 4$

(b) $f(n) = 2 \cdot 3^n$

(c) $f(n) = 2^n + 3^n$

(d) $f(n) = 7(\log \log n) + 3(\log n) + 12n$

(e) $f(n) = 9(n \log n) + 5(\log \log n) + 5$

(f) $f(n) = n \cdot \log_{37} n$

(g) $f(n) = n^{21} + (1.1)^n$

(h) $f(n) = 23n + n^3 - 2$

EXERCISE 11.2.2: Proving the growth rate for polynomials.

Give complete proofs for the growth rates of the polynomials below. You should provide specific values for c and n_0 and prove algebraically that the functions satisfy the definitions for O and Ω.

(a) $f(n) = (1/2)n^5 - 100n^3 + 3n - 1$. Prove that $f = \Theta(n^5)$.

(b) $f(n) = n^3 + 3n^2 + 4$. Prove that $f = \Theta(n^3)$.

EXERCISE 11.2.3: Proving negative results on the growth of functions.

(a) f(n) = n/100. Prove that it is not true that f = $O(\sqrt{n})$.

(b) f(n) = $n^{(3/2)}$. Prove that it is not true that f = $\Omega(n^2)$.

EXERCISE 11.2.4: Relationships between bounds on the growth rate of functions.

(a) Let f be a function whose domain is $\mathbf{Z}^+$ and whose target is $\mathbf{R}^{\geq}$. Show that if f = O(n), then f = $O(n^2)$.

(b) Let f be a function whose domain is $\mathbf{Z}^+$ and whose target is $\mathbf{R}^{\geq}$. Show that if f = O(n), then it is not true that f = $\Omega(n^2)$.

11.3 Analysis of algorithms

An algorithm describes the underlying method for how a computational problem is solved. The choice of algorithm can have a dramatic effect on how efficiently the solution is obtained. The amount of resources used by an algorithm is referred to as the algorithm's ***computational complexity***. The primary resources to optimize are the time the algorithm requires to run (time complexity) and the amount of memory used (***space complexity***). This material focuses on the time complexity of algorithms.

Naturally, a program will take more time on larger inputs. For example, a program that sorts a sequence of numbers will take a longer time to sort a long sequence than a short sequence. Therefore, time and space efficiency are measured in terms of the input size. For the problem of sorting numbers, a natural measure of the input size is the number of numbers to be sorted. For a different problem, the size of the input may depend on a different feature of the input. For example, a program that processes a digital image would likely be measured as a function of the number of pixels in the image. The input size for a problem is usually denoted by the variable n.

Atomic operations (assignment, arithmetic operations, comparison, return statements etc.) form the basic building blocks for the pseudocode. Once a set of atomic operations is identified, it is possible to define the time complexity of an algorithm as a function:

Definition 11.3.1: Time complexity as a function.

The ***time complexity*** of an algorithm is defined by a function f: $\mathbf{Z^+} \rightarrow \mathbf{Z^+}$ such that f(n) is the maximum number of atomic operations performed by the algorithm on any input of size n. $\mathbf{Z^+}$ is the set of positive integers.

Although the computational complexity of an algorithm is defined by functions that map positive integers to positive integers, sometimes algorithmic complexity is described in terms of functions that are real valued or negative for small input values (e.g., $\sqrt{n}$ or n^2 - 10). A function f(n) in the context of computational complexity means $\max\{\lceil f(n) \rceil, 1\}$. Also, some functions are not well-defined for small values of n, such as log(log(1)). In those cases, the value of the function is assumed to be 1.

PARTICIPATION ACTIVITY 11.3.1: Counting atomic operations for an algorithm ComputeSum.

Animation content:

undefined

Animation captions:

1. In the "ComputeSum" algorithm, the first line is an assignment which counts as 1 operation. The next line is a for-loop which iterates n times.
2. Each iteration of the for-loop does 4 operations: 2 to test and increment the counter i, and an addition and assignment inside the loop.
3. The line after the for-loop is a return statement which counts as 1 operation.
4. f(n) is the number of operations executed on a sequence of length n. f(n) = 1 + n(2 + 2) + 1 = 4n + 2.

PARTICIPATION ACTIVITY 11.3.2: Number of atomic operations performed by ComputeSum.

1) How many atomic operations are performed by the algorithm ComputeSum on a sequence of 100 numbers?

Check Show answer

Asymptotic complexity

A programmer writing a computer program that implements an algorithm may reduce execution time by optimizing the code to favor instructions that execute in less time than others, or by reducing the number of atomic operations inside a looping structure. Such optimizations may improve the function f(n) from $3n^2$ to $2n^2$. On the other hand, finding an entirely new approach to the problem could result in an algorithm that requires only 3n operations on an input of size n, which would have a more dramatic effect on the running time. Another consideration in evaluating algorithms is that the efficiency of the algorithm is most critical for large input sizes. Small inputs are likely to result in fast running times, so efficiency is less of a concern.

In evaluating algorithms, the focus is on how the function f grows with n, ignoring small input sizes and constant factors that depend on the specifics of the implementation and have less impact on the execution time.

The ***asymptotic time complexity*** of an algorithm is the rate of asymptotic growth of the algorithm's time complexity function f(n).

The figure below illustrates how dramatically the running times of algorithms with different asymptotic time complexity can vary. The figure shows how long it takes to perform f(n) instructions for different functions f and different values of n. For large n, the difference in computation time varies greatly with the rate of growth of the function f.

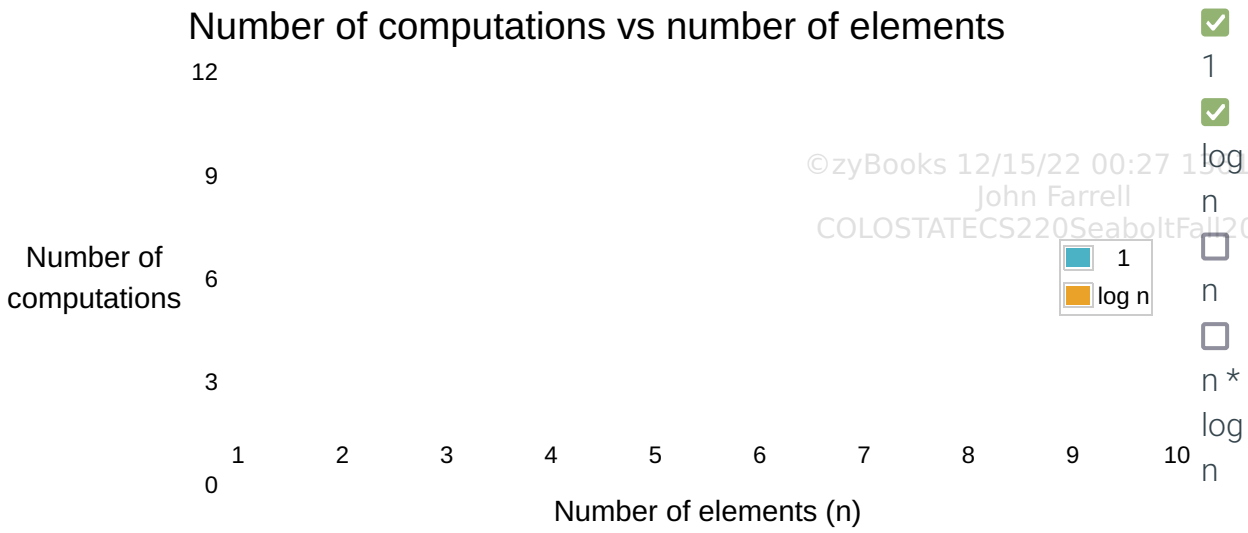
Table 11.3.1: Growth rates for different input sizes.

f(n)	n=10	n=50	n=100	n=1000	n=10000	n=100000
$\log_2 n$	3.3 µs	5.6 µs	6.6 µs	10.0 µs	13.3 µs	16.6 µs
n	10 µs	50 µs	100 µs	1000 µs	10 ms	.1 s
$n \log_2 n$	.03ms	.28 ms	.66 ms	10.0 ms	.133 s	1.67 s
n^2	.1 ms	2.5 ms	10 ms	1 s	100 s	2.8 hours
n^3	1 ms	.125 s	1 s	16.7 min	11.6 days	31.7 years
2^n	1.0 ms	35.7 years	4.0×10^{16} years	3.4×10^{287} years	6.3×10^{2996} years	*

This data assumes that a single instruction takes 1 µs to execute. A µs is a microsecond and is equal to 10^{-6} seconds. An ms is a millisecond and is equal to 10^{-3} seconds.

The interactive tool below illustrates graphically the growth rate of commonly encountered functions. For example, the growth rate of 2^n is much larger than n^2 and the growth rate of n! is much larger than 2^n.

PARTICIPATION ACTIVITY 11.3.3: Computational complexity graphing tool.



- ☐ n^2
- ☐ 2^n
- ☐

PARTICIPATION ACTIVITY 11.3.4: Running time for different growth rates.

Use the table above to answer the following questions. Assume that each atomic operation requires 1 μs to execute.

1) On an input of size n = 100000, how long would it take an algorithm that runs in time $f(n) = n^2$?
   - ○ 1.67 s
   - ○ 2.8 hours
   - ○ 31.7 years

2) On an input of size n = 100000, how long would it take an algorithm that runs in time $f(n) = n^3$?
   - ○ 1.67 s
   - ○ 2.8 hours
   - ○ 31.7 years

It may seem simplistic that all atomic operations are counted as +1 in determining the time complexity of an algorithm because in reality, some operations do take longer than others. In addition, some arbitrary decisions were made in counting atomic operations. For example, in the line "For i = 1 to n" we counted the cost of incrementing the counter i as 1 operation. However, computing "i := i+1" is technically an addition and an assignment operation, which should count as two operations. Fortunately, the distinction is not important if the goal is to understand the asymptotic complexity of the algorithm because the functions 5n + 2 and 4n + 2 are both Θ(n). In fact, it is sufficient just to determine that the number of operations per loop is some constant c and that the number of operations before and after the loop is another constant d. Since the loop is executed exactly n times on an input of size n, the number of atomic operations that will be performed is $f(n) = c \cdot n + d = \Theta(n)$.

The animation below shows the asymptotic analysis for the algorithm to find the smallest of n numbers. The time complexity of the algorithm is Θ(n). Inside the loop, the number of atomic operations performed will actually depend upon the values of the input sequence. If the condition

of the if-statement inside the loop evaluates to true, then an additional assignment is performed. If the condition evaluates to false, then the assignment is skipped. However, since the number of atomic operations performed inside the loop is at least 3 and at most 4, for all input values, the variation does not affect the asymptotic running time of the algorithm.

PARTICIPATION ACTIVITY 11.3.5: Analysis of the algorithm to find the smallest in a sequence of numbers.

Animation content:

undefined

Animation captions:

1. In the "FindSmallest" algorithm, the first line is an assignment which counts as 1 operation. The next line is a for-loop which iterates n-1 times.
2. Each iteration of the for-loop does 2 ops to test and increment the counter. 1 or 2 operations are performed inside the loop. A constant c number of operations are performed per iteration.
3. The line after the for-loop is a return statement, which counts as 1 operation. There are a constant d number of operations performed before and after the for-loop.
4. f(n) is the number of operations executed on a sequence of length n. f(n) = (n-1)c + d = nc - c + d. f(n) is Θ(n).

PARTICIPATION ACTIVITY 11.3.6: Asymptotic analysis of algorithms on sequences.

1) What is the asymptotic time complexity of the following algorithm:

SequenceProduct
This algorithm finds the product of all the elements in a sequence of n numbers.

Input: $a_1, a_2, \ldots, a_n$
n, the length of the sequence.
Output: the product of all numbers in the sequence

prod:= 1
For i = 1 to n
prod:= prod·a_i

- $\Theta(1)$
- $\Theta(n)$
- $\Theta(n^2)$

2) What is the asymptotic time complexity of the following algorithm:

AverageOfEnds
This algorithm finds the average of the first and last elements in a sequence.

Input: $a_1, a_2, \ldots, a_n$
n, the length of the sequence.
Output: the average of the first and last elements in the sequence

sum := $a_1 + a_n$
avg := sum / 2

Return(avg)

- $\Theta(1)$
- $\Theta(n)$
- $\Theta(n^2)$

Worst-case complexity

The number of operations performed by the previous algorithm FindSmallest may depend on the actual numbers in the input sequence, not just the size of the input. However, the variation does not affect the asymptotic running time of the algorithm which is $\Theta(n)$ for any input sequence. The next example is an algorithm that searches for a particular item x in a sequence of n numbers. The number of operations performed on different inputs of the same size can vary more dramatically.

Figure 11.3.1: Searching a sequence.

SearchSequence

This algorithm returns the index of the first occurrence of a particular number in a sequence of numbers.
If the number does not occur in the sequence, the algorithm returns -1.

Input: A value x to search for.
A sequence $a_1, a_2,\ldots,a_n$.
n, the length of the sequence
Output: the index of the first occurrence of x in the sequence, or -1 if x is not in the sequence.

```
i := 1
While ( a_i ≠ x and i < n )
    i := i+1
End-while

If ( a_i = x ), Return( i )
Return( -1 )
```

PARTICIPATION ACTIVITY 11.3.7: The number of iterations in the SearchSequence algorithm.

For each input (value for x and sequence $a_1,\ldots,a_n$) given, indicate the number of times the instruction inside the while loop (i := i + 1) is executed.

1) x = 5. The sequence is: 3, 2, 5, 7,
1

Check Show answer

2) x = 5. The sequence is: 5, 2, 3, 7,
1

Check Show answer

3) x = 5. The sequence is: 6, 2, 3, 7, 1

Check Show answer

The algorithm starts at the beginning of the list and compares each item a_i to x, starting with a_1. If x occurs early in the list (for example, if a_1 = x), then the algorithm returns right away after only performing O(1) operations. However, if the first occurrence of x is last in the list or if x does not occur in the list at all, the algorithm must check every item in the list which takes Ω(n) time. Thus, the time complexity of the algorithm depends greatly on the input. What is the correct complexity of the algorithm?

The most common way to analyze the complexity of algorithms whose run time varies with different features of the input is to count the number of operations performed in the worst (most time-consuming) case. The ***worst-case analysis*** of an algorithm evaluates the time complexity of the algorithm on the input of a particular size that takes the longest time.

The ***worst-case*** time complexity function f(n) is defined to be the maximum number of atomic operations the algorithm requires, where the maximum is taken over all inputs of size n.

The input of a given size that maximizes f(n) is the worst-case input for the algorithm. For the searching algorithm given above, the worst-case input is when the item x does not occur in the list at all. Determining an upper bound and a lower bound for the time complexity of an algorithm require two different tasks:

- When proving an ***upper bound*** on the worst-case complexity of an algorithm (using Oh-notation), the upper bound must apply for every input of size n.
- When proving a ***lower bound*** for the worst-case complexity of an algorithm (using Ω notation), the lower bound need only apply for at least one possible input of size n.

Worst-case analysis can lead to overly pessimistic results in some cases if the worst-case input for an algorithm is highly unusual and the algorithm takes much less time on typical inputs. Average-case analysis is an alternative to worst-case analysis that tries to address this shortcoming. ***Average-case analysis*** evaluates the time complexity of an algorithm by determining the average running time of the algorithm on a random input. Average case analysis uses probability theory to formally define a "random" input and is not covered in this material.

PARTICIPATION ACTIVITY 11.3.8: Analysis of the algorithm SearchSequence.

Animation content:

undefined

Animation captions:

1. In the "SearchSequence" algorithm, there is 1 operation before the while-loop and 2 after. The while-loop executes 4 operations per iteration.
2. The while-loop iterates at most n-1 times. If x is not present in the list, then the while-loop executes exactly n-1 times.
3. f(n) is the number of operations on a sequence of length n. $f(n) \leq 1 + (n-1) \cdot 4 + 2 = 4n - 1 = O(n)$.
4. In the worst case, $f(n) \geq 1 + 4(n-1) + 2 = 4n - 1 = \Omega(n)$.

PARTICIPATION ACTIVITY 11.3.9: Determining the worst case input for SearchSequence.

1) Each of the choices is a possible input to the algorithm SearchSequence. The size of the input is the same for all three choices. Which input will result in the worst case running time for the algorithm?
   - x = 9. The sequence is 4, 1, 6, 8, 3, 5, 2.
   - x = 9. The sequence is 4, 9, 6, 8, 3, 5, 2.
   - x = 9. The sequence is 4, 1, 6, 8, 9, 5, 2.

Analyzing an algorithm with nested loops

Figure 11.3.2: Algorithm to count number of duplicate pairs in a sequence.

CountDuplicatePairs

This algorithm counts the number of duplicate pairs in a sequence.

Input: $a_1, a_2,...,a_n$, where n is the length of the sequence.
Output: count = the number of duplicate pairs.

```
count := 0

For i = 1 to n-1
    For j = i+1 to n
        If ( a_i = a_j ), count := count +1
    End-for
End-for

Return( count )
```

The outer loop iterates n times on a sequence of length n. The number of iterations of the inner loop depends on the value of i, the index for the outer loop. In the first iteration of the outer loop, i = 1 and the inner loop iterates n - 1 times. In the next iteration of the outer loop, i = 2 and the inner loop iterates n - 2 times. In general, in the i^{th} iteration of the outer loop, the inner loop iterates n - i times. The process continues until i = n - 1 and the inner loop iterates only one time. The total number of iterations of the inner loop is:

$$(n-1) + (n-2) +... + 2 + 1$$

The material on summations gives a method to analyze sums like the one in the expression above. For now, we will just use the fact that:

$$(n-1) + (n-2) + \cdots + 2 + 1 = \frac{n(n-1)}{2} = \Theta(n^2)$$

Note that for the CountDuplicatePairs algorithm, the number of times the loops iterate are the same, regardless of the input. The number of operations inside the inner loop may be 1 or 2, because the variable count is incremented only when $a_i = a_j$. However, the difference between 1 and 2 only affects the constant factor which does not appear in the $\Theta(n^2)$. Therefore, it is sufficient to say that the number of instructions per iteration is some constant c. There are also a constant

number of operations for each iteration of the outer loop (incrementing and checking the index i) and a constant number of operations before and after the nested loop (initializing and returning the variable count). These numbers end up as constant factors and can be given arbitrary names such as "c" or "d" because constants factors do not appear in the final asymptotic notation.

PARTICIPATION ACTIVITY 11.3.10: Analysis of the algorithm CountDuplicatePairs.

Animation captions:

1. In the "CountDuplicates" algorithm, the inner for-loop executes c operations per iteration.
2. When the index of the outer for-loop is i, the inner for-loop executes (n-i) times.
3. The number of operations to update the index and check the termination condition for the for-loop is a constant b.
4. The total number of operations in the iteration of the outer for-loop with index i is $c(n-i)+b$. The index i goes from 1 to n-1.
5. d operations are executed after the nested for-loops.
6. The total number of operations executed is $c[(n-1)+(n-2)+\cdots+1]+b\cdot(n-1)+d$.
7. $c[(n-1)+(n-2)+\cdots+1]+b\cdot(n-1)+d$ is $\Theta(n^2)$.

The formal proof that the time complexity of CountDuplicatePairs is $\Theta(n^2)$ is given below. The lower bound can be simplified by omitting the constant number of operations before and after the nested loop because the additive constants do not affect the asymptotic bound. In proving a lower bound on the time complexity, it is not necessary to count every instruction because the goal is to show that the number of operations is at least a certain number. In particular with nested loops, the number of times the innermost loop executes often dominates the running time of the algorithm, so it is often sufficient to count instructions executed in the innermost loop. The upper bound can also be simplified by using n (instead of n - i) as an upper bound on the number of iterations of the inner loop. Even with these simplifications, the asymptotic growth rates of the upper and lower bounds match.

Proof 11.3.1: Proof that the growth rate of the worst-case time complexity of CountDuplicatePairs is quadratic.

Proof.

Lower bound: For inputs of size n, the number of times the inner loop iterates with outer index i is (n-i). The outer index ranges from 1 through (n-1). Therefore the total number of times that the inner loop iterates in the entire nested loop is (n-1) + (n-2) + ... + 1. The value of the sum is $\Omega(n^2)$, so the worst-case complexity of CountDuplicatePairs is $\Omega(n^2)$.

Upper bound: For any input of size n, the number of times the inner loop iterates with outer index i is (n-i). Since n - i ≤ n, the inner loop iterates at most n times for every iteration of the outer loop. The outer loop iterates at most n times, so the total number of iterations of the inner loop is at most n^2. The total number of operations executed in the entire nested loop is at most cn^2, for some constant c. There are at most d operations performed before and after the nested loop, so the total number of operations performed is at most $cn^2 + d$ which is $O(n^2)$. ■

PARTICIPATION ACTIVITY 11.3.11: Worst-case complexity - searching for a duplicate.

The algorithm below determines whether or not there is a pair of duplicate numbers in a sequence of numbers.

```
FindDuplicate
This algorithm determines if there is a pair of duplicate entries in the sequence.

Input: a1, a2,...,an
       n, the length of the sequence.
Output: "Yes" if there is a duplicate pair and "No" if the elements in the sequence are
distinct.

For i = 1 to n - 1
    For j = i+1 to n
        If ( ai = aj ), Return("Yes")
    End-for
End-for

Return( "No" )
```

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1) Which description corresponds to the type of input that will cause FindDuplicate to execute the most number of instructions?

- A sequence of n numbers in which the first and second entries are the same.
- A sequence of n numbers that are all distinct, except the first and last numbers which are equal.
- A sequence of n numbers that are all distinct.

2) Use your answer in the previous question to identify the best possible lower bound for the asymptotic time complexity of the algorithm.

- $\Omega(1)$
- $\Omega(n)$
- $\Omega(n^2)$

3) What is the best upper bound for the asymptotic time complexity of the algorithm?

- $O(n)$
- $O(n^2)$

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Additional information: 11.3.1: Efficient algorithms.

In practice, the criteria for an algorithm to be "efficient" may depend greatly on the context. An algorithm that processes terabytes of data may be completely impractical unless its time complexity is at most linear.

An algorithm is said to run in ***polynomial time*** if its time complexity is $O(n^k)$ for some fixed constant k. As a shorthand, an algorithm is called "efficient" if the algorithm runs in polynomial time. For example, an algorithm whose time complexity is $O(n^5)$ would be considered efficient. However, an algorithm whose time complexity is $\Omega(n^{\log(n)})$ or $\Omega(2^n)$ would be considered inefficient. In reality, an algorithm that runs in time $\Omega(n^{100})$ would only be practical for very small input sizes. However, the distinction between algorithms that run in polynomial time and those that do not is a useful starting point to understanding the algorithm's efficiency in practice.

CHALLENGE ACTIVITY 11.3.1: Code complexity.

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Start

The below code has a worst-case complexity of Θ(1)

Input: $a_1, a_2, a_3, \ldots, a_n$
n, the length of the sequence.

```
index := 1
While (index < n)
    Output: index
    index := index + 1
End-while
```

Check Next

Additional exercises

EXERCISE | 11.3.1: Worst-case time complexity - counting numbers in a sequence less than a target value.

CountValuesLessThanT

Input: $a_1, a_2, \ldots, a_n$
n, the length of the sequence.
T, a target value.
Output: The number of values in the sequence that are less than T.

count := 0

For i = 1 to n
If (a_i < T), count := count + 1
End-for

Return(count)

(a) Characterize the asymptotic growth of the worst-case time complexity of the algorithm. Justify your answer.

EXERCISE 11.3.2: Worst-case time complexity - maximum subsequence sum.

The input is a sequence of numbers $a_1, a_2,..., a_n$. A subsequence is a sequence of numbers found consecutively within $a_1, a_2,..., a_n$. For example, in the sequence: -3, -1, 17, 5, 66, 22, -5, 42, both 17, 5, 66 and -1, 17 are subsequences. However, 66, -5 is not a subsequence because 66 and -5 do not appear consecutively in the sequence. A subsequence can contain only one number such as 66. It can also be empty.

In the maximum subsequence sum problem, the input is a sequence of numbers and the output is the maximum number that can be obtained by summing the numbers in a subsequence of the input sequence. If the input was the sequence -3, -1, 17, 5, 66, 22, -5, 42, then the output would be 147 because the sum of subsequence 17, 5, 66, -5, 42 is 147. Any other subsequence will sum to an equal or smaller number. The empty subsequence sums to 0, so the maximum subsequence sum will always be at least 0.

The algorithm below computes the maximum subsequence sum of an input sequence.

```
MaximumSubsequenceSum

Input: a_1, a_2,...,a_n
       n, the length of the sequence.
Output: The value of the maximum subsequence sum.

maxSum := 0

For i = 1 to n
    thisSum := 0
    For j = i to n
        thisSum := thisSum + a_j
        If ( thisSum > maxSum ), maxSum := thisSum
    End-for
End-for

Return( maxSum )
```


(a) Characterize the asymptotic growth of the worst-case time complexity of the algorithm. Justify your answer.

(b) Can you find an algorithm that solves the same problem whose worst-case time complexity is linear?

EXERCISE | 11.3.3: Worst-case time complexity - finding the maximum value of a function.

The function M takes three input values that are positive integers and outputs a positive integer. We don't know much about the function M but we are given some instructions to compute M. The line M(x, y, z) in the algorithm below computes the value of M on input values x, y, and z. You can assume that it takes O(1) operations to compute the value of M on any input. The input to the algorithm is a sequence of n numbers. We would like to find the maximum value of the function M where the three input values are numbers from the sequence. Numbers can be repeated, so $M(a_1, a_1, a_1)$ is a possibility.


```
FindMaxFunctionValue

Input: a1, a2,...,an
       n, the length of the sequence.
Output: The largest values of M on input values from the sequence.

max := M(a1, a1, a1)

For i = 1 to n
    For j = 1 to n
        For k = 1 to n
            new := M(ai, aj, ak)
            If ( new > max ), max := new
        End-for
    End-for
End-for

Return( max )
```

(a) Characterize the asymptotic growth of the worst-case time complexity of the algorithm. Justify your answer.

EXERCISE | 11.3.4: Worst-case time complexity - finding a target product of two numbers.

FindProduct

Input: $a_1, a_2, \ldots, a_n$
n, the length of the sequence.
prod, a target product.
Output: "Yes" if there are two numbers in the sequence whose product equals the input "prod". Otherwise, "No".


```
For i = 1 to n-1
    For j = i+1 to n
        If (a_i·a_j = prod), Return( "Yes" )
    End-for
End-for

Return( "No" )
```

(a) Characterize the input that will cause the "If" statement in the inner loop to execute the most number of times.

(b) Give an asymptotic lower bound for the running time of the algorithm based on your answer to the previous question.

(c) Was it important to use the worst-case input for the lower bound? That is, would any input of a given length have resulted in the same asymptotic lower bound?

(d) Give an upper bound (using O-notation) for the time complexity of the algorithm that matches your asymptotic lower bound for the algorithm.

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EXERCISE | 11.3.5: Worst-case time complexity - mystery algorithm.

The algorithm below makes some changes to an input sequence of numbers.

```
MysteryAlgorithm

Input: a1, a2,...,an
       n, the length of the sequence.
       p, a number.
Output: ??

i := 1
j := n

While (i < j)
    While (i < j and ai < p)
        i := i + 1
    End-while
    While (i < j and aj ≥ p)
        j := j - 1
    End-while
    If (i < j), swap ai and aj
End-while

Return( a1, a2,...,an )
```


(a) Describe in English how the sequence of numbers is changed by the algorithm. (Hint: try the algorithm out on a small list of positive and negative numbers with p = 0)

(b) What is the total number of times that the lines "i := i + 1" or "j := j - 1" are executed on a sequence of length n? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the two lines are executed.

(c) What is the total number of times that the swap operation is executed? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the swap is executed.

(d) Give an asymptotic lower bound for the time complexity of the algorithm. Is it important to consider the worst-case input in determining an asymptotic lower bound (using Ω) on the time complexity of the algorithm? (Hint: argue that the number of swaps is at most the number of times that i is incremented or j is

number of swaps is at most the number of times that I is incremented or J is decremented).

(e) Give a matching upper bound (using O-notation) for the time complexity of the algorithm.

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