9.1 Permutations with repetitions

How many ways are there to scramble the letters in the word MISSISSIPPI? The question is concerned about counting different orderings which suggests counting permutations. However, a permutation is defined as an ordering of distinct objects. The letters in MISSISSIPPI have multiple repetitions: there are four S's, four I's, two P's, and one M. A ***permutation with repetition*** is an ordering of a set of items in which some of the items may be identical to each other. To illustrate with a smaller example, there are 3! = 6 permutations of the letters CAT because the letters in CAT are all different. However, there are only 3 different ways to scramble the letters in DAD: ADD, DAD, DDA. The animation below shows how to count the number of distinct permutations of the letters in MISSISSIPPI by repeatedly applying the formula for counting r-subsets and putting together the choices using the product rule.

PARTICIPATION ACTIVITY 9.1.1: Counting the number of permutations of the letters in MISSISSIPPI.

Animation captions:

1. How many ways exist to scramble MISSISSIPPI? 11 possible locations for the 2 P's. $\binom{11}{2}$ choices to place the P's.
2. 9 locations left for the 4 I's. $\binom{9}{4}$ choices to place the I's. 5 locations left for the 4 S's. $\binom{5}{4}$ choices to place the S's. 1 location left for the M. $\binom{1}{1}$ choices to place the M.
3. $\binom{11}{2}\binom{9}{4}\binom{5}{4}\binom{1}{1} = \frac{11!}{2!9!} \cdot \frac{9!}{4!5!} \cdot \frac{5!}{4!1!} \cdot \frac{1!}{1!0!} = \frac{11!}{2!4!4!1!}$ ways exist to scramble MISSISSIPPI.

Fact 9.1.1: Formula for counting permutations with repetition.

The number of distinct sequences with n_1 1's, n_2 2's, ..., n_k k's, where $n = n_1 + n_2 + \ldots + n_k$ is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

The formula for permutations with repetition is derived from repeated use of the formula for counting r-subsets:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2\cdots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdot\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\cdots\frac{(n-n_1-n_2\cdots}{n_k!0!}$$

$$= \frac{n!}{n_1!n_2!\cdots n_k!}$$

The last 0! in the denominator on the second line comes from the fact that $n - n_1 - n_2 - ... - n_k = 0$. Recall that 0! = 1.

PARTICIPATION ACTIVITY | 9.1.2: Permutations of letters with repetition.

1) How many ways are there to permute the letters in PEPPER? (Give a numerical answer)

Check Show answer

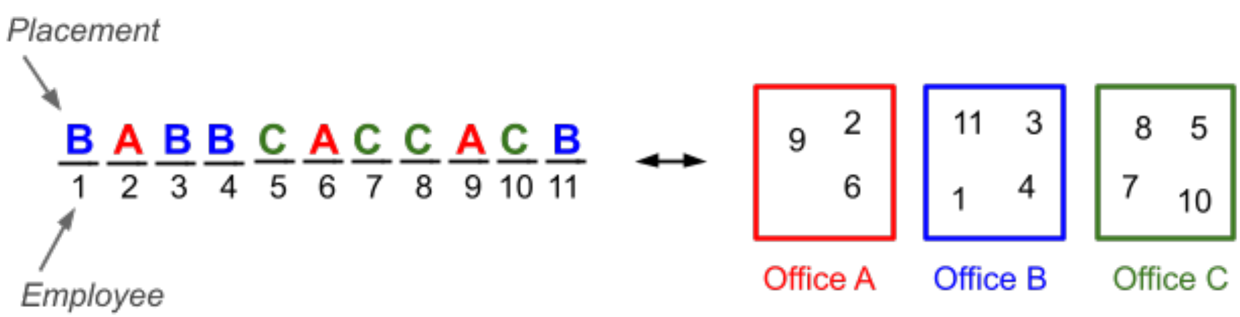
2) How many ways are there to permute the letters in HAPPY? (Give a numerical answer)

Check Show answer

Example 9.1.1: Counting assignments using permutations with repetition.

Consider a company with 11 employees that rents an office space with three offices. The first office (office A) can hold 3 people, the second office (office B) can hold 4 people and the third office (office C) can hold 4 people. How many different assignments are there for employees to offices?

Define S to be the set of permutations of the 11 characters: AAABBBBCCCC (3 A's, 4 B's, and 4 C's). There is a bijection between the set of office assignments and permutations in S. Order the employees from 1 to 11. The i^{th} character in the permutation is the office assignment for employee i:



Each permutation in S corresponds uniquely to an office assignment. Each office assignment corresponds uniquely to a permutation in S. The number of distinct assignments is therefore:

$$\frac{11!}{3!4!4!}$$

An alternative way to solve the office assignment problem is to apply the generalized product rule directly. There are $\binom{11}{3}$ ways to select the 3 people who will go into office A. After office A is filled, there are 8 people left from which to pick the 4 people for office B which results in $\binom{8}{4}$ choices. After offices A and B are filled, there are four people left from which to pick the 4 people for office C which results in $\binom{4}{4}$ choices. The choices made in each step are put together using the product rule.

$$\binom{11}{3}\binom{8}{4}\binom{4}{4} = \frac{11!}{3!4!4!}$$

The equation above can be verified by expanding the "n choose r" expressions using factorials and then cancelling.

PARTICIPATION ACTIVITY | 9.1.3: Counting assignments using permutations with repetitions.

1) Suppose that 9 desserts are handed out to 9 kids. Each kid gets one dessert. There are three ice cream sandwiches, four cupcakes and two bowls of pudding. How many ways are there to hand out the desserts to the kids?

- $\binom{9}{3}$
- $\frac{9!}{3!4!2!}$
- $9!$

2) Fifteen kids arrive at camp and are assigned a place to sleep. There are three different cabins each of which can hold five kids. How many ways are there to assign kids to cabins?

- $\frac{15!}{5!5!5!}$
- $15!$
- $\binom{15}{5}$

Additional exercises

EXERCISE | 9.1.1: Permuting letters in words.

How many ways are there to permute the letters in each of the following words?

(a) NUMBER

(b) DISCRETE

(c) SUBSETS

EXERCISE 9.1.2: Counting ternary strings.

(a) How many ternary strings (digits 0,1, or 2) are there with exactly seven 0's, five 1's and four 2's?

EXERCISE 9.1.3: Dealing cards to four players.

How many ways are there to deal hands from a standard playing deck to four players if:

(a) Each player gets exactly 13 cards.

(b) Each player gets seven cards and the rest of the cards remain in the deck?

EXERCISE 9.1.4: Distributing comic books.

20 different comic books will be distributed to five kids.

(a) How many ways are there to distribute the comic books if there are no restrictions on how many go to each kid (other than the fact that all 20 will be given out)?

(b) How many ways are there to distribute the comic books if they are divided evenly so that 4 go to each kid?

EXERCISE 9.1.5: Assigning bedrooms to daughters.

(a) A family has four daughters. Their home has three bedrooms for the girls. Two of the bedrooms are only big enough for one girl. The other bedroom will have two girls. How many ways are there to assign the girls to bedrooms?

EXERCISE 9.1.6: Assigning summer camp activities.

(a) A camp offers 4 different activities for an elective: archery, hiking, crafts and swimming. The capacity in each activity is limited so that at most 35 kids can do archery, 20 can do hiking, 25 can do crafts and 20 can do swimming. There are 100 kids in the camp. How many ways are there to assign the kids to the activities?

EXERCISE 9.1.7: Scheduling meals at a school.

A school cook plans her calendar for the month of February in which there are 20 school days. She plans exactly one meal per school day. Unfortunately, she only knows how to cook ten different meals.

(a) How many ways are there for her to plan her schedule of menus for the 20 school days if there are no restrictions on the number of times she cooks a particular type of meal?

(b) How many ways are there for her to plan her schedule of menus if she wants to cook each meal the same number of times?

9.2 Counting by complement

Suppose we want to count the number of people in a room with red hair. We know that there are 20 people in the room and exactly 12 of them do not have red hair. Then we can deduce that the number of people in the room with red hair is 20 - 12 = 8. ***Counting by complement*** is a technique for counting the number of elements in a set S that have a property by counting the total number of elements in S and subtracting the number of elements in S that do not have the property. The principle of counting by complement can be written using set notation where P is the subset of elements in S that have the property.

$$|P| = |S| - |\overline{P}|$$

PARTICIPATION ACTIVITY 9.2.1: Counting by complement: The number of 8-bit strings with at least one 0.

Animation captions:

1. How many 8-bit strings have at least one 0? S = the set of 8-bit strings. $|S| = 2^8 = 256$.
2. P = the set of 8-bit strings with at least one 0. $\overline{P}$ = the set of 8-bit strings with no 0's.
3. Only one 8-bit string has no 0's: 11111111. $|\overline{P}| = 1.$ $|P| = |S| - |\overline{P}| = 256 - 1 = 255.$ 255 8-bit strings have at least one 0.

PARTICIPATION ACTIVITY 9.2.2: Counting by complement.

Give numerical answers for the questions below.

1) There are 10 kids on the math team. Two kids will be selected from the team to compete in the state competition. How many ways are there to select the 2 competitors?

Check Show answer

2) The math team has 6 girls and 4 boys. How many ways are there to select the two competitors if they are both girls?

Check Show answer

3) The math team has 6 girls and 4 boys. How many ways are there to select the two competitors so that at least one boy is chosen?

Check Show answer

4) Four people (John, Paul, George, and Ringo) are seated in a row on a bench. The number of ways to order the four people so that John is next to Paul is 12. How many ways are there to order the four people on the bench so that John is not next to Paul?

Check Show answer

CHALLENGE ACTIVITY | 9.2.1: Counting possibilities by complement.

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Start

An auto dealer has 4 different cars and 6 different trucks.
How many ways are there to select two vehicles?

Ex: 26 * C(36, 12) Write combination as: C(n, k)

1 2 3 4

Check Next

Additional exercises

EXERCISE | 9.2.1: Counting passwords.

How many different passwords are there that contain only digits and lower-case letters and satisfy the given restrictions?

(a) Length is 6 and the password must contain at least one digit.

(b) Length is 6 and the password must contain at least one digit and at least one letter.

EXERCISE | 9.2.2: Counting 5-card hands from a deck of standard playing cards.

A 5-card hand is drawn from a deck of standard playing cards.

(a) How many 5-card hands have at least one club?

(b) How many 5-card hands have at least two cards with the same rank?

EXERCISE 9.2.3: Counting bit strings.

(a) How many 8-bit strings have at least two consecutive 0's or two consecutive 1's?

(b) How many 8-bit strings do not begin with 000?

EXERCISE 9.2.4: Lining up club members for a photo.

Ten members of a club are lining up in a row for a photograph. The club has one president and one VP.

(a) How many ways are there for the club members to line up in which the president is not next to the VP?

(b) How many ways are there for the club members to line up if the VP is not in the leftmost position?

(c) How many ways are there for the club members to line up if the VP is not at one end (i.e. in the leftmost or rightmost positions)?

9.3 Inclusion-exclusion principle

A high school gives free admission to football games to any student who is either a senior or on the honor roll. The school is trying to determine the number of students who will be admitted for free. They know the number of seniors (denoted by the variable s) as well as the number of students on the honor roll (denoted by the variable h). Is this enough information to determine the number of students eligible to attend for free? The sum (s + h) results in over counting because seniors on the honor roll are counted twice, once for being a senior and once for being on the honor roll. The school must also know the number of students who are both seniors and on the honor roll in order to be able to determine the number of people in either group.

Define the set S to be the set of all seniors. Define H to be the set of students on the honor roll. The school would like to count the number of students in the set $S \cup H$. The ***principle of inclusion-exclusion*** is a technique for determining the cardinality of the union of sets that uses the cardinality of each individual set as well as the cardinality of their intersections. The animation below illustrates the idea with two sets:

PARTICIPATION ACTIVITY 9.3.1: Inclusion-exclusion principle illustrated with two sets.

Animation captions:

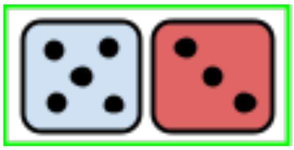
1. The number in each area of the Venn diagram is the number of times the people in that area have been counted. |S|: everyone in S is counted once.
2. Add |H|. Now everyone in (S - H) and (H - S) is counted once and everyone in (S ∩ H) is counted twice.
3. Subtract |S ∩ H|. Everyone in (S ∩ H) is counted (2 - 1 = 1) time.
4. Now, everyone in (S ∪ H) has been counted once. $|S \cup H| = |S| + |H| - |S \cap H|$.

Principle 9.3.1: The inclusion-exclusion principle with two sets.

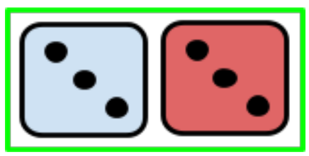
Let A and B be two finite sets, then |A ∪ B| = |A| + |B| - |A ∩ B|

Example 9.3.1: Example of the inclusion-exclusion principle: a roll of two dice.

Suppose that two six-sided dice are thrown. One is colored red and the other is colored blue. An outcome of a roll of the two dice is determined by the number that shows up on the blue die and the number that shows up on the red die. The picture below shows two different outcomes of a roll of the dice:



How many outcomes are there in which at least one of the dice comes up 3? Define B to be the set of outcomes in which the blue die comes up 3. |B| = 6 because after the blue die is determined to be 3 there are still six possibilities for the outcome of the red die. Define R to be the set of outcomes in which the red die comes up 3. Similarly, |R| = 6. There is only one element in B ∩ R, the outcomes in which both the blue and red dice are 3:



Applying the principle of inclusion-exclusion gives that the number outcomes in which at least one of the die comes up 3 is:

$$|B \cup R| = |B| + |R| - |B \cap R| = 6 + 6 - 1 = 11$$

PARTICIPATION ACTIVITY | 9.3.2: Applying the inclusion-exclusion with two sets.

1) In a group, there are 10 women, 8 blondes and 3 blonde women. How many people are either blonde or a woman?

Check Show answer

2) Erica goes swimming three out

of the seven days of the week. How many possibilities are there for her swim schedule if she goes swimming on Monday or Tuesday or both? (Define M to be the set of schedules in which Erica goes swimming on Monday. Let T be the set of schedules in which Erica goes swimming on Tuesday. Determine the numerical value of $|M \cup T|$.)

Check Show answer

3) How many positive integers less than 100 have at least one digit that is a 9? (Let T be the set of positive integers less than 100 with a 9 in the ten's place. Let O be the set of positive integers less than 100 with a 9 in the one's place. Now determine the numerical value of $|T \cup O|$).

Check Show answer

The inclusion-exclusion principle with three sets

The inclusion-exclusion principle can be applied to count the number of elements in a union of more than two sets. As the number of sets grows, the expression becomes more complex. The animation below illustrates how to compute the cardinality of the union of three sets:

PARTICIPATION ACTIVITY 9.3.3: Inclusion-exclusion principle illustrated with three sets.

Animation captions:

1. The number in each area of the Venn diagram is the number of times the people in that

area have been counted. $|A|$: everyone in A is counted once.

2. Add $|B|$. Now everyone in $(A - B)$, and $(B - A)$ is counted once and everyone in $(A \cap B)$ is counted twice.
3. Add $|C|$. Now everyone in $(A \cap B - C)$, $(A \cap C - B)$, and $(B \cap C - A)$ is counted twice and everyone in $(A \cap B \cap C)$ is counted three times.
4. Subtract $|A \cap B|$. Everyone in $(A \cap B - C)$ is counted once. Everyone in $(A \cap B \cap C)$ is counted twice.
5. Subtract $|B \cap C|$. Everyone in $(B \cap C - A)$ is counted once. Everyone in $(A \cap B \cap C)$ is counted once.
6. Subtract $|A \cap C|$. Everyone in $(A \cap C - B)$ is counted once. Everyone in $(A \cap B \cap C)$ is counted zero times.
7. Add $|A \cap B \cap C|$. Everyone in $(A \cup B \cup C)$ is counted once. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$.

Principle 9.3.2: Inclusion-exclusion with three sets.

Let A, B and C be three finite sets, then

|A ∪ B ∪ C| = |A| + |B| + |C| - |A ∩ B| - |B ∩ C| - |A ∩ C| + |A ∩ B ∩ C|

Example 9.3.2: The inclusion-exclusion with three sets: course enrollments.

A computer science department offers three lower division classes in a given quarter: discrete math, digital logic, and introductory programming. The department would like to know the number of students enrolled in any of the three courses. Let M be the set of students enrolled in discrete math, L be the set of students enrolled in digital logic, and P the set of students enrolled in introductory programming. Here are numbers for enrollments in the courses:

- |M| = 112 students enrolled in discrete math.
- |L| = 138 students enrolled in digital logic.
- |P| = 142 students enrolled in introductory programming.
- |M ∩ L| = 25 students enrolled in both discrete math and digital logic.
- |L ∩ P| = 17 students enrolled in both digital logic and introductory programming.
- |M ∩ P| = 32 students enrolled in both introductory programming and discrete math.
- |M ∩ L ∩ P| =7 students are enrolled in all three classes.

The total number of students enrolled in any of the three courses is:

$$|M \cup L \cup P| = |M| + |L| + |P| - |M \cap L| - |L \cap P| - |M \cap P| + |M \cap L \cap P|$$

$$= 112 + 138 + 142 - 25 - 17 - 32 + 7 = 325$$

PARTICIPATION ACTIVITY | 9.3.4: Inclusion-exclusion principle example.

Animation captions:

1. How many integers from 1 to 30 are divisible by 2, 3, or 5? A_i = the set of integers between 1 and 30 divisible by i. Find $|A_2 \cup A_3 \cup A_5|$.
2. A_2 is the set {2, 4, ... , 30}. $|A_2| = 15$.
3. Add $|A_3|$. There are 30 DIV 3 = 10 numbers between 1 and 30 divisible by 3, so $|A_3| = 10$.
4. Add $|A_5|$. There are 30 DIV 5 = 6 numbers between 1 and 30 divisible by 5, so $|A_5| = 6$.
5. Subtract $|A_2 \cap A_3|$. There are 30 DIV 6 = 5 numbers between 1 and 30 divisible by 2 and 3, so $|A_2 \cap A_3| = 5$.
6. Subtract $|A_5 \cap A_3|$. There are 30 DIV 15 = 2 numbers between 1 and 30 divisible by 5 and 3, so $|A_5 \cap A_3| = 2$.
7. Subtract $|A_5 \cap A_2|$. There are 30 DIV 10 = 3 numbers between 1 and 30 divisible by 5 and 2, so $|A_5 \cap A_2| = 3$.

8. Add $|A_2 \cap A_3 \cap A_5|$. There are 30 DIV 30 = 1 number between 1 and 30 divisible by 2, 3 and 5, so $|A_2 \cap A_3 \cap A_5| = 1.$
9. $|A_2 \cup A_3 \cup A_5| = |A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_5 \cap A_3|$ $-|A_5 \cap A_2| + |A_2 \cap A_3 \cap A_5| = 15 + 10 + 6 - 5 - 2 - 3 + 1 = 22$.

PARTICIPATION ACTIVITY 9.3.5: Applying the inclusion-exclusion principle with three sets.

1) Three employees of a company, Anna, Fred and Jose, have each worked on 12 projects. Each pair of people have worked on 4 together, including one project that all three have worked on as a team. What is the total number of projects?

Check Show answer

2) How many numbers in the range from 1 through 42 are divisible by 2, 3, or 7?

Check Show answer

The general inclusion-exclusion principle

The inclusion-exclusion principle can be generalized to work for any number of sets. The pattern is the same as with two or three subsets. First, add in the size of each subset individually. Then consider each pair of sets and subtract the size of the intersection of each pair. Then consider every set of three sets and add in the three-way intersection of the triplets of sets. Continue with the pattern until the final term which is the intersection of all the sets. If the number of sets is even, the last term is subtracted. If the number of sets is odd, the last term is added. The general inclusion-principle is stated mathematically below.

Principle 9.3.3: Inclusion-exclusion with an arbitrary number of sets.

Let $A_1, A_2, \ldots, A_n$ be a set of n finite sets.

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{j=1}^{n} |A_j|$$

$$- \sum_{1 \leq j < k \leq n} |A_j \cap A_k|$$

$$+ \sum_{1 \leq j < k < l \leq n} |A_j \cap A_k \cap A_l|$$

$$\cdots$$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Figure 9.3.1: The general inclusion-exclusion principle applied to four sets.

$$
\begin{aligned}
&|A \cup B \cup C \cup D| \\
&= |A| + |B| + |C| + |D| && \text{(add the sizes} \\
&- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| && \text{(minus pairwi} \\
&+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| && \text{(plus 3-way ir} \\
&- |A \cap B \cap C \cap D| && \text{(minus 4-way}
\end{aligned}
$$

PARTICIPATION ACTIVITY | 9.3.6: Applying the general inclusion-exclusion principle.

1) Suppose you are using the inclusion-exclusion principle to compute the number of elements in the union of four sets. Each set has 15 elements. The pair-wise intersections have

5 elements each. The three-way intersections have 2 elements each. There is only one element in the intersection of all four sets. What is the size of the union?

Check Show answer

CHALLENGE ACTIVITY 9.3.1: The general inclusion-exclusion principle.

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Start

Given two sets: A and B.
A has 9 elements.
B has 8.
A and B share 4 elements.
How many elements are there in total?

Ex: 60

1 2 3

Check Next

The inclusion-exclusion principle and the sum rule

A collection of sets is ***mutually disjoint*** if the intersection of every pair of sets in the collection is empty. If we apply the principle of inclusion-exclusion to determine the union of a collection of mutually disjoint sets, then all the terms with the intersections are zero. Thus, for a collection of mutually disjoint sets, the cardinality of the union of the sets is just equal to the sum of the cardinality of each of the individual sets:

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$$

The equation above is a restatement of the sum rule which only applies when the sets are mutually disjoint.

PARTICIPATION ACTIVITY 9.3.7: The Inclusion-exclusion principle with mutually disjoint sets.

How many 5-bit strings contain the string "100" as a consecutive substring?

Define the following sets:

- A = set of 5-bit strings of the form 100**
- B = set of 5-bit strings of the form *100*
- C = set of 5-bit strings of the form **100

The *'s can be either 0 or 1.

1) Sets A, B, and C all have the same cardinality. What is the cardinality of each set?

Check Show answer

2) What is |A ∩ B|?

Check Show answer

3) What is |A ∩ C|?

Check Show answer

4) What is |B ∩ C|?

Check Show answer

5) How many 5-bit strings have "100" consecutively as a substring?

Check Show answer

Determining the cardinality of a union by complement

While the inclusion-exclusion principle is a useful tool in many situations, there are also cases in which it is not the most efficient way to determine the cardinality of a union of sets. Consider a situation in which a person has forgotten his 4-digit PIN to access his bank account. A PIN can be any string of 4 digits, e.g. 0032 or 3801. If U is the set of all possible PINs, then |U| = 10^4 because each digit can be any one of 10 choices from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. He does remember, however, that at least one of the digits is an 8. How much does the information that at least one of the digits is an 8 help him narrow down his search?

Let P_i be the set of PINs with an 8 in the i^{th} digit. For example, 0872 would be an element of P_2 because the second digit is an 8. The PIN 0882 would be in P_2 and P_3. The goal is to find $|P_1 \cup P_2 \cup P_3 \cup P_4|$. Using inclusion-exclusion would result in a mathematical expression with many terms.

Instead, counting by complement can be used to express the size of the union as:

$$|U| - |\overline{P_1 \cup P_2 \cup \cdots \cup P_n}| = |P_1 \cup P_2 \cup \cdots \cup P_n|$$

The animation below illustrates:

PARTICIPATION ACTIVITY | 9.3.8: Finding the cardinality of a union by complement.

Animation content:

`undefined`

Animation captions:

1. The Universe set U is the set of all 4-digit PINs. P_i = 4-digit PINs with an 8 in place i. For example 0982 $\in P_3$.
2. $P_1 \cup P_2 \cup P_3 \cup P_4$ = PINs with an 8 in any location.
3. $\overline{P_1 \cup P_2 \cup P_3 \cup P_4}$ = PINs with no 8 in any location.
4. $|P_1 \cup P_2 \cup P_3 \cup P_4| = |U| - |\overline{P_1 \cup P_2 \cup P_3 \cup P_4}| = 10^4 - 9^4$. Thus, 9^4 is the number of length 4 strings from {0, 1, 2, 3, 4, 5, 6, 7, 9} (no 8).

PARTICIPATION ACTIVITY | 9.3.9: Calculating the cardinality of a union by complement.

1) How many 4-bit strings have a 1 in at least one of the first two places?

Check Show answer

2) Erica goes swimming three out of the seven days of the week. How many possibilities are there for her swim schedule if she goes swimming on Monday or Tuesday or both?

Check Show answer

CHALLENGE ACTIVITY 9.3.2: The inclusion-exclusion principle.

422102.2723990.qx3zqy7

Start

Bob plays hockey 2 out of the 5 weekdays. How many possible schedules are th
Friday or Monday or both?

Ex: 5

1	2	3	4	5

Check Next

Additional exercises

EXERCISE 9.3.1: Counting strings over {a, b, c}.

Count the number of strings of length 9 over the alphabet {a, b, c} subject to each of the following restrictions.

(a) The first or the last character is a.

(b) The string contains at least 8 consecutive a's.

(c) The string contains at least 8 consecutive identical characters.

(d) The first character is the same as the last character, or the last character is a, or the first character is a.

(e) The string contains at least seven consecutive a's.

(f) The characters in the string "abababa" appear consecutively somewhere in the 9-character string. (So "ccabababa" would be such a 9-character string, but "cababcaba" would not.)

(g) The string has exactly 2 a's or exactly 3 b's.

(h) The string has exactly 2 a's or exactly 2 b's or exactly 2 c's

EXERCISE 9.3.2: Counting binary strings.

Count the number of binary strings of length 10 subject to each of the following restrictions.

(a) The string has at least one 1.

(b) The string has at least one 1 and at least one 0.

(c) The string contains exactly five 1's or it begins with a 0.

EXERCISE 9.3.3: Counting calculus students.

A university offers 3 calculus classes: Math 2A, 2B and 2C. In both parts, you are given data about a group of students who have all taken at least one of the three classes.

(a) Group A contains 157 students. Of these, 51 students in Group A have taken Math 2A, 80 have taken Math 2B, and 70 have taken Math 2C. 15 have taken both Math 2A and 2B, 20 have taken both Math 2A and 2C, and 13 have taken both Math 2B and 2C. How many students in Group A have taken all three classes?

(b) You are given the following data about Group B. 28 students in Group B have taken Math 2A, 28 have taken Math 2B, and 25 have taken Math 2C. 11 have taken both Math 2A and 2B, 9 have taken both Math 2A and 2C, and 10 have taken both Math 2B and 2C. 3 have taken all three classes. How many students are in Group B?

EXERCISE 9.3.4: Counting integer multiples.

(a) How many integers in the range 1 through 120 are integer multiples of 2, 3, or 5?

(b) How many integers in the range 1 through 140 are integer multiples of 2, 5, or 7?

EXERCISE 9.3.5: Counting ways to line up for a family photo.

A family lines up for a photograph. In each of the following situations, how many ways are there for the family to line up so that the mother is next to at least one of her daughters?

(a) The family consists of two parents, two daughters and two sons.

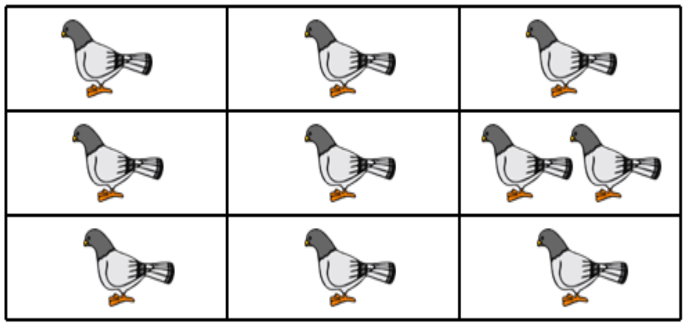
(b) The family consists of two parents, three daughters and four sons.

EXERCISE 9.3.6: Counting permutations of 100 numbers.

(a) S = {1, 2, ..., 100}. How many permutations are there of S in which the number 1 is next to at least one even number?

9.4 The pigeonhole principle

The pigeonhole principle is a mathematical tool used to establish that repetitions are guaranteed to occur in certain sets and sequences. The ***pigeonhole principle*** says that if n+1 pigeons are placed in n boxes, then there must be at least one box with more than one pigeon. The image below shows 10 pigeons placed in 9 boxes.



Although the principle itself appears to be obvious, it can be applied in clever ways to prove facts that are quite surprising. Below is a more mathematical version of the pigeonhole principle stated in the language of functions:

Theorem 9.4.1: The pigeonhole principle.

If a function f has a domain of size at least n+1 and a target of size at most n, where n is a positive integer, then there are two elements in the domain that map to the same element in the target (i.e., the function is not one-to-one).

Proof 9.4.1: The pigeonhole principle.

Proof.
Proof by contradictions. Assume that the hypothesis is true and that the negation of the conclusion is also true. Then no two elements in the domain map to the same element in the target and thus each element in the domain maps to a different element in the target. Since there are n+1 different elements in the domain they map to n+1 different elements in the target, which contradicts the hypothesis that there are n element in the target. Therefore, there must be at least two elements in the domain that f maps to the same element in the target. ■

In the mathematical statement of the pigeonhole principle, the elements of the target represent boxes and the elements of the domain represent pigeons. The function f maps the pigeons to the boxes.

Consider a person who has a drawer filled with socks. Each sock is either blue, gray, or red. How many socks does the person have to get from their drawer before he knows that he has a pair of the same color? In this case, the domain is the set of socks picked from the drawer. The target is the set of sock colors. The function f maps a sock to its color. The pigeonhole principle says that if the socks come in three colors and four socks are selected, the person is guaranteed to have a pair of the same color. The animation below illustrates:

PARTICIPATION ACTIVITY | 9.4.1: Pigeonhole principle: matching socks.

Animation captions:

1. After three socks are selected, there might not be a pair because the socks are all different colors. After the next sock is selected, it will make a pair.

Here is another application of the pigeonhole principle: among any set of 13 people, there are at least two people who have a birthday in the same month. The function f maps the set of people to the set of months such that each person is mapped to the month in which his or her birthday falls. Since the domain has size 13 and the target has size 12, the function can not be one-to-one which means that two people must be mapped to the same month and must therefore have the same birthday month. If there are 12 people or fewer, then the pigeonhole principle can not be applied. More information is required in order to determine whether or not there are two people with the same birthday month.

Now consider a fast food chain that is selecting store locations in a 2 mile by 2 mile square. The pigeonhole principle can be used to show that if 5 stores are placed in the square, then there will be two stores within $\sqrt{2}$ miles of each other. The animation below illustrates:

PARTICIPATION ACTIVITY | 9.4.2: Pigeonhole principle: locations in a square area.

Animation captions:

1. Divide the 2 mile by 2 mile area into a grid of four 1 mile by 1 mile squares.
2. If there are 5 points placed in the 4 squares, one square must have at least 2 points.
3. The farthest distance between two points in a square is $\sqrt{2}$ miles.

PARTICIPATION ACTIVITY | 9.4.3: Pigeonhole principle.

Indicate whether the pigeonhole principle can be used to prove the following statements.

1) A class of 11 students take a quiz that is worth 10 points. No partial credit is given, so every student received a score that is an integer. There must be at least two students who received the same score on the quiz.

- Yes
- No

2) Among a group of 400 people, there are at least two who have the same birthday.

- Yes
- No

3) Among a group of 350 people, there are at least two who have the same birthday.

- Yes
- No

4) There are four people whose weight ranges from 100 lbs to 130 lbs. There must be at least two people in the group whose weight differs by at

most 10 pounds.

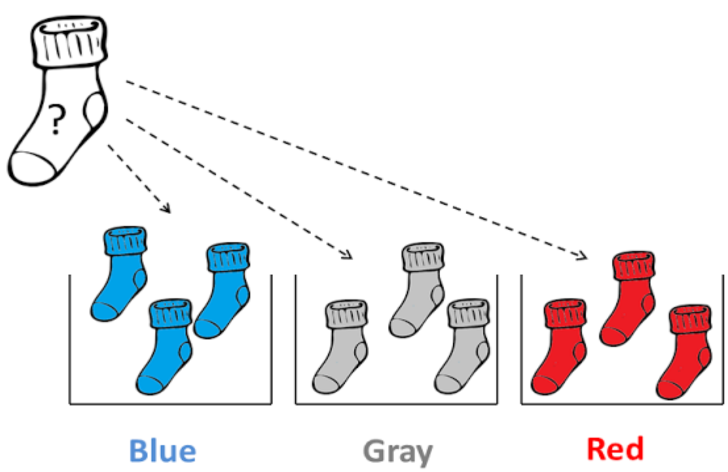
- Yes
- No

5) 37 points are placed in a 6 mile by 6 mile square area. There must be 2 points that are within $\sqrt{2}$ miles of each other.

- Yes
- No

Generalized pigeonhole principle

The pigeonhole principle can be used to show that there are at least two of a given type of item. For example, in selecting socks from a drawer, the goal was to select enough socks to have a pair. What if it is important to have more than two of a given type of item? Consider the (admittedly ridiculous) situation in which a pair of twins are picking socks from a drawer. They want to have two pairs of matching socks. Not only do the socks in each pair have to match but the twins also want to wear socks of the same color. How many socks must be drawn from the drawer to guarantee they have four of the same color? After they pick nine socks, it's possible that they have three blue, three grey and three red socks as pictured below.



The 10^{th} sock will make a set of at least four socks which are all of the same color. The general rule is stated as follows:

Theorem 9.4.2: The generalized pigeonhole principle.

Consider a function whose domain has at least n elements and whose target has k elements, for n and k positive integers. Then there is an element y in the target such that f maps at least $\lceil n/k \rceil$ elements in the domain to y.

Proof 9.4.2: The generalized pigeonhole principle.

Proof.
First we prove that there is an element in the target y such that the number of elements in the domain which f maps to y is greater than or equal to n/k. Proof by contradiction. Suppose that f maps less than n/k elements of the domain to each element in the target. Since there are k elements in the target, the total number of elements in the domain must be less than (n/k)k = n, which contradicts the fact that there are n elements in the domain. Therefore, we can conclude that there is an element in the target to which f maps at least n/k elements.

The number of elements that map to any element in the target must be an integer. $\lceil n/k \rceil$ is defined to be the smallest integer that is at least n/k, so it follows that there must be an element in the target to which f maps at least $\lceil n/k \rceil$ elements. ■

In the case of the socks, n = 10 is the number of socks and k = 3 is the number of colors. After selecting 10 socks, the twins are guaranteed that they have at least $\lceil 10/3 \rceil = 4$ socks which are all of the same color.

PARTICIPATION ACTIVITY | 9.4.4: Generalized pigeonhole principle.

Can the pigeonhole principle be used to demonstrate that each fact is true?

1) Among a group of 50 people, there are at least five whose birthday falls in the same month.

- Yes
- No

2) Alice bought 14 cups of coffee in a week. Then there was a day in the

week in which she bought at least three cups of coffee.

- Yes
- No

3) A group of 42 employees are assigned to 7 different printers. There must be a printer to which 7 employees are assigned.

- Yes
- No

In many situations one would like to know how many items are required to guarantee that there are a certain number of a given type. Consider a situation in which a school would like to select a group of students to form a committee that will meet regularly in order to plan an event. Because students from different grades have different schedules, the group must all be in the same grade. The school has students in grades 9 through 12 and the goal is to have at least four students on the committee. The school puts out a sign-up sheet for students to indicate their interest in participating in the committee. How many students need to sign up in order to guarantee that there will be at least four in the same grade?

In approaching problems like the student committee problem, it is useful to think about how one could avoid having four students from the same class sign up for the committee. The worst case for the school is the scenario when all the students who sign up are distributed evenly among the grades. If there are as many as 12 students who sign up, it is possible that the school falls short of the goal by having three students in each grade. However, after 12 students sign up, it just takes one more student before the school is guaranteed to have four in the same grade. The idea is illustrated in the animation below:

PARTICIPATION ACTIVITY | 9.4.5: Pigeonhole principle: student committee.

Animation captions:

1. The 4 grades form the 4 categories. Each student is placed in the category corresponding to their grade.
2. In the worst case the 12 students are evenly distributed among the 4 grades. The next student will result in 4 in a grade, regardless of their grade.

To generalize the approach, we prove the converse of the generalized pigeonhole principle.

Theorem 9.4.3: Converse of the generalized pigeonhole principle.

Suppose that a function f maps a set of n elements to a target set with k elements, where n and k are positive integers. In order to guarantee that there is an element y in the target to which f maps at least b elements from the domain, then n must be at least k(b - 1) + 1.

The converse shows that it is necessary to have at least k(b - 1) + 1 elements in the domain in order to guarantee that for any function f, at least b elements in the domain map to some element in the target. The pigeonhole principle establishes that having at least k(b - 1) + 1 elements in the domain is sufficient to guarantee that there is at least one element in the range to which f maps at least b elements. If n = k(b -1)+1, then

$$\left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{k(b-1)+1}{k} \right\rceil = \left\lceil (b-1) + \frac{1}{k} \right\rceil = b$$

Proof 9.4.3: Converse of the generalized pigeonhole principle.

Proof.
If n ≤ k(b -1), then it is possible that f maps b -1 element to each y in the target set, which means that there is not necessarily an element y in the target to which f maps at least b items. ■

In the example with the student committee, there are 4 grades (k = 4) and the goal was to have at least 4 students from one grade (b = 4). The number of students required to guarantee at least four students from the same grade is k(b - 1) + 1 = 4(4 - 1) + 1 = 13.

PARTICIPATION ACTIVITY | 9.4.6: Generalized pigeonhole principle - converse.

Give your answer to the following questions in numerical form (e.g., 13 instead of "thirteen").

1) A group of students is selected from a class. Every student in the class is either in the 3^{rd} grade or the 4^{th} grade, and no student is in both the 3rd and the 4th grades. How many

students must be selected in order to guarantee that at least five 3rd graders or at least five 4th graders are selected?

Check Show answer

2) A research firm is tracking the number of text messages sent by 1000 7th graders in a single day. How many texts must be sent by the group in order to guarantee that at least one of the 7th graders has sent at least 20 text messages?

Check Show answer

More uses of the pigeonhole principle

Example 9.4.1: Sums of subsets.

Below is a list of 25 six-digit numbers. The pigeonhole principle can be used to show that there are two distinct subsets of the numbers listed below that add up to the same value.

365909	937803	966435	883401	975558
595286	569510	550559	602547	705332
807399	204523	309826	910940	605665
193386	658348	439520	785793	255004
231782	717329	127235	294456	262073

- **Function:** The function maps each subset to the sum of its elements. For example the subset consisting of the first two numbers in the first column {365909, 595286} would be mapped to 961195 because 365909 + 595286 = 961195.
- **Target:** The target set is the set of all possible values for the sum of a subset of the numbers. The value of each number is at most 999999. Since there are 25 numbers, the sum of any subset of the numbers is at most 25 x 999999 = 24999975. Therefore the possible values for the sum of a subset falls in the range from 0 to 24999975. Therefore, the size of the target is at most 24999976.
- **Domain:** The domain of the function is the set of all possible subsets of the numbers. Since there are 25 numbers, there are 2^{25} possible subsets. The size of the domain is 2^{25}.

Since the size of the domain is larger than the size of the target,

$$2^{25} = 33554432 > 24999976 \geq \text{size of the target set}$$

there are two distinct subsets that sum to the same number.

Note that while the pigeonhole principle establishes that there exists two subsets that sum to the same value, it does not say anything about how to actually find the two subsets. Finding two such equivalent subsets is believed to be computationally difficult. Hard problems like the equivalent subset problem described here are sometimes used as the basis of a cryptographic scheme in which decrypting an encrypted message is shown to be as hard as solving the computationally difficult problem.

Example 9.4.2: Continued fractions.

The pigeonhole principle can be used to show that the ratio of any two integers is either a finite decimal or a repeating decimal (in contrast to numbers like π whose decimal representation is infinite and non-repeating). The decimal expression of a ratio is determined by long division. If the division process terminates, then the decimal representation of the ratio is finite. Consider the other case, in which the long division continues infinitely. The animation below illustrates the process of dividing 5 by 28. In each iteration of the long division, the remainder is an integer mod 28. The only numbers that can result from a *mod 28* operation are {0, 1, ..., 27}. Therefore, some time before the 29^{th} iteration, there will be a repeated remainder which results in a repeated decimal. Fortunately, in dividing 5 by 28, the repetition occurs only after the 8^{th} repetition.

In general, when finding the decimal expansion of x/y, where x and y are positive integers, each iteration of long division will yield a remainder which is the result of a *mod y* operation. By the pigeonhole principle, there are only y possible remainders and by the end of the $y+1^{st}$ iteration, there will be a repeated remainder, resulting in a repeated decimal.

PARTICIPATION ACTIVITY | 9.4.7: Long division example.

Animation content:

`undefined`

Animation captions:

1. Divide 5.0 by 28. 50 DIV 28 = 1, so the first digit in the quotient after the decimal is 1. The remainder is 50 MOD 28 = 22.
2. Add a 0 to 22 to get 220. The next digit of the quotient is 220 DIV 28 = 7. The remainder is 220 MOD 28 = 24.
3. Add a 0 to 24 to get 240. The next digit of the quotient is 240 DIV 28 = 8. The remainder is 240 MOD 28 = 16.
4. Add a 0 to 16 to get 160. The next digit of the quotient is 160 DIV 28 = 5. The remainder is 160 MOD 28 = 20.
5. Add a 0 to 20 to get 200. The next digit of the quotient is 200 DIV 28 = 7. The remainder is 200 MOD 28 = 4.
6. Add a 0 to 4 to get 40. The next digit of the quotient is 40 DIV 28 = 1. The remainder is 40 MOD 28 = 12.
7. Add a 0 to 12 to get 120. The next digit of the quotient is 120 DIV 28 = 4. The remainder is

120 MOD 28 = 8.

8. Add a 0 to 8 to get 80. The next digit of the quotient is 80 DIV 28 = 2. The remainder is 80 MOD 28 = 24.
9. The remainder 24 occurred 6 steps earlier. Therefore the last 6 digits are repeated in the quotient: $\frac{5}{28} = .17\overline{857142}$

PARTICIPATION ACTIVITY 9.4.8: Pigeonhole principle and subset equality.

1) The largest 1 or 2-digit number is 99. How many 1-digit or 2-digit numbers must be in a set in order to apply the pigeonhole principle to conclude that there are two distinct subsets of the numbers whose elements sum to the same value? You may need a calculator for this.

Check Show answer

CHALLENGE ACTIVITY 9.4.1: Pigeonhole principle contrapositive.

422102.2723990.qx3zqy7

Start

In a standard deck of 52 cards and 4 suits, how many cards have to be picked to be sure that at least 6 are from the same suit?

Cards to pick: Ex: 20

1 2 3

Check Next

Additional exercises

EXERCISE 9.4.1: Applying the pigeonhole principle - heights and times.

Apply the pigeonhole principle to answer the following questions. If the pigeonhole principle can not be applied, give a specific counterexample.

(a) A team of three high jumpers all have a personal record that is at least 6 feet and less than 7 feet. Is it necessarily true that two of the team members must have personal records that are within four inches of each other? What if there are four jumpers? Heights are measured to within a precision of 1/4 inch.

(b) Seven members of the track team run the mile. Their mile times are all faster than 7 minutes but not faster than 6 minutes. Can you conclude that there are two runners whose times are less than nine seconds apart? What if there are eight runners? Times are measured to a precision of 0.01 second.

EXERCISE 9.4.2: Generalized pigeonhole principle.

(a) There are 121.4 million people in the United States who earn an annual income that is at least $10,000$ and less than $1,000,000$ dollars. Annual income is rounded to the nearest dollar. Show that there are 123 people who earn the same annual income in dollars.

(b) Show that among a group of 621 people, there are at least 21 who are born on the same day of the month (e.g., the 21^{st} or the 12^{th}, etc.). Is the same fact true if there are only 620 people?

EXERCISE 9.4.3: Pigeonhole principle converse.

(a) A team wishes to purchase 10 shirts of the same color. A store sells shirts in 3 different colors. What must the inventory of the store be in order to conclude that there are at least 10 shirts in one of the three colors?

(b) How many people must be selected to make sure that there are at least 20 who are born in the same month?

EXERCISE | 9.4.4: Applying the pigeonhole principle to sums of integers.

(a) Suppose a set of 8 numbers are selected from the set {1, 2, ..., 13, 14}. Show that two of the selected numbers must sum to 15.

(b) There are 14 3-digit numbers in a list. Can you conclude that there are two distinct subsets of the 14 numbers that have the same sum? Justify your answer.

EXERCISE | 9.4.5: Pigeonhole principle - a round-robin tournament.

(a) A round-robin tournament is one where each player plays each of the other players exactly once. Prove that if no person loses all their games, then there must be two players with the same number of wins.

Source: ADUni, modified by Sandy Irani.

9.5 Counting problem examples

Counting problems may require multiple counting techniques in combination. For example, Cheryl goes to the bakery to buy a dozen donuts. The bakery sells 5 varieties of donuts: plain, chocolate, jelly, maple, and custard. The bakery has a large selection of plain, chocolate, and custard donuts. However, the bakery only has 3 jelly and 5 maple donuts left. How many different selections are possible for the dozen donuts?

PARTICIPATION ACTIVITY | 9.5.1: Counting donut selections with limits on two varieties.

Animation content:

undefined

Animation captions:

1. The number of ways to select 12 donuts from 5 varieties with at most 3 jelly AND at most 5 maple = # selections with no restrictions - # selections $\neg(\leq 3$ jelly AND ≤ 5 maple).
2. # selections with no restrictions - # selections ($\neg(\leq 3$ jelly) OR $\neg(\leq 5$ maple)). By De Morgan's Law.
3. # selections with no restrictions - # selections (≥ 4 jelly OR ≥ 6 maple). Not selecting 3 or fewer jelly = selecting 4 or more jelly.

4. # selections with no restrictions - [# selections ($\geq$ 4 jelly) + # selections ($\geq$ 6 maple) - # selections ($\geq$ 4 jelly AND $\geq$ 6 maple)]. Inclusion/exclusion.
5. $\binom{12+5-1}{5-1} - \left[\binom{8+5-1}{5-1} + \binom{6+5-1}{5-1} - \binom{2+5-1}{5-1}\right]$. In all terms $m = 5$. In the first term $n = 12$. In the second term $n = 12 - 4 = 8$. In the third term $n = 12 - 6 = 6$. In the last term $n = 12 - 4 - 6 = 2$.

Example 9.5.1: Solution to the license plate counting problem.

A license plate number consists of 7 characters that are either digits or capital letters. A witness of a crime briefly sees the license plate of the getaway car. She remembers that the license plate number starts with a digit and has at least two Q's. How many license plate numbers satisfy her description?

Let D be the set of license plate numbers that start with a digit, with no other restrictions on the remaining characters. Apply the principle of counting by complement within the set D by subtracting the number of license plates that have zero or one Q. Define:

- D_0 is the set of license plate numbers that start with a digit and have zero Q's.
- D_1 is the set of license plate numbers that start with a digit and have one Q.

Since the sets D_0 and D_1 are disjoint, the sum rule applies to count the cardinality of their union: $|D_0 \cup D_1| = |D_0| + |D_1|$. The solution to the problem is then:

$$|D| - |D_0 \cup D_1| = |D| - (|D_0| + |D_1|) = |D| - |D_0| - |D_1|$$

The cardinality of D, D_0, and D_1 is determined by:

- $|D| = 10 \cdot 36^6$. There are 10 choices for the first character, which is a digit from 0 to 9. There are 36 choices for each of the remaining six characters, which are digits or capital letters (i.e. 0 to 9 or A to Z).
- $|D_0| = 10 \cdot 35^6$. There are 10 choices for the first character, which is a digit from 0 to 9. There are 35 choices for each of the remaining six characters, which are digits or capital letters other than Q (i.e. 0 to 9, A to P, or R to Z).
- $|D_1| = 10 \cdot 6 \cdot 35^5$. There are 10 choices for the first character, which is a digit from 0 to 9. There are $\binom{6}{1} = 6$ choices for the location of the one Q among the remaining six characters, and 35 choices for each of the remaining five characters, which are digits or capital letters other than Q (i.e. 0 to 9, A to P, or R to Z).

The final solution is:

$$|D| - |D_0| - |D_1| = 10 \cdot 36^6 - 10 \cdot 35^6 - 10 \cdot 6 \cdot 35^5 = 10 \cdot (36^6 - 35^6 - 6 \cdot 35^5) = 233854610$$

PARTICIPATION ACTIVITY 9.5.2: A counting problem using multiple techniques.

A manager of a grocery store purchases 50 crates of soda to stock the store. There are 5 varieties of soda. How many ways are there for her to purchase the 50 crates of soda if she does not order more than 25 of any particular variety?

The following series of questions will lead to the final answer.

1) How many ways are there for the manager to purchase the soda if there are no restrictions on the number of each variety she chooses?

- $\binom{50}{5}$
- $\binom{54}{4}$
- $\binom{54}{49}$

2) The number of ways that the manager can purchase 50 crates of soda with at most 25 chosen from any variety is $\binom{54}{4} - X$. What is the correct description of X?

- The number of ways to purchase soda with at least 25 of some variety chosen.
- The number of ways to purchase soda with at least 26 of some variety chosen.
- The number of ways to purchase soda with at least 26 of every variety chosen.

3) Let V_i be the set of ways to choose 50 crates of soda from 5 varieties with at least 26 chosen of the i^{th} variety. What is the correct expression for X from the previous problem?

- $|V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5|$
- $|V_1 \cap V_2 \cap V_3 \cap V_4 \cap V_5|$

4) $|V_1|$ is the number of ways to choose 50 crates of soda from 5 varieties with at least 26 chosen of the first variety. What is $|V_1|$?

- $\binom{29}{4}$
- $\binom{28}{5}$
- $\binom{28}{4}$

5) What is $|V_1 \cap V_2|$?

- 0
- $\binom{28}{4}$

6) What is $|V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5|$?

- $\binom{28}{4}$
- $5 \cdot \binom{28}{4}$
- $5 \cdot \binom{28}{4} - 10$

7) How many ways are there for the manager to purchase 50 crates of soda from 5 varieties if she does not purchase more than 25 of any particular variety?

- $\binom{54}{4} - 5 \cdot \binom{28}{4}$
- $\binom{54}{5} - 5 \cdot \binom{28}{4}$
- $\binom{54}{5} - \binom{28}{4}$

Additional exercises

EXERCISE 9.5.1: Ordering soda at a grocery store.

An employee of a grocery store is placing an order for soda. There are 8 varieties of soda and they are sold in cases. Each case contains all the same variety. The store will order 50 cases total.

(a) How many ways are there to place the order?

(b) How many ways are there to place the order if she does not order more than 20 of any single variety?

EXERCISE 9.5.2: Counting solutions to integer equations.

How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 25$ in which each x_i is a non-negative integer and ...

(a) There are no other restrictions.

(b) $x_i \geq 3$ for $i = 1, 2, 3, 4, 5, 6$

(c) $3 \leq x_1 \leq 10$

(d) $3 \leq x_1 \leq 10$ and $2 \leq x_2 \leq 7$

EXERCISE 9.5.3: More on distributing coupons.

10 coupons are given to 20 shoppers in a store. Each shopper can receive at most one coupon. 5 of the shoppers are women and 15 of the shoppers are men.

(a) If the coupons are identical, how many ways are there to distribute the coupons so that at least one woman receives a coupon?

(b) If the coupons are different, how many ways are there to distribute the coupons so that at least one woman receives a coupon?

(c) If the coupons are identical, how many ways are there to distribute the coupons so that at least one woman does not receive a coupon?

(d) If the coupons are different, how many ways are there to distribute the coupons so that at least one woman does not receive a coupon?

EXERCISE | 9.5.4: Selecting a championship soccer team.

Two soccer teams in a youth league tie for first place. The "Breakaways" have 20 players and the "Cyclones" have 18 players. The league must select 12 players to form a new team that will go on to the regional championship.

(a) How many ways are there to select the 12 players from the Cyclone and Breakaway players?

(b) How many ways are there to select the 12 players so that the new team has the same number of players from the Breakaways as from the Cyclones?

(c) How many ways are there to select the 12 players so that none of the players from the Breakaways are chosen?

EXERCISE | 9.5.5: Ordering jobs in a printer queue.

There are eight different jobs in a printer queue. Each job has a distinct tag which is a string of three upper case letters. The tags for the eight jobs are:

{ LPW, QKJ, CDP, USU, BBD, PST, LSA, RHR }

(a) How many different ways are there to order the eight jobs in the queue?

(b) How many different ways are there to order the eight jobs in the queue so that job USU comes immediately before CDP?

(c) How many different ways are there to order the eight jobs in the queue so that either QKJ or LPW come last?

(d) How many different ways are there to order the eight jobs in the queue so that QKJ is either last or second-to-last?

(e) How many different ways are there to order the eight jobs in the queue so that job USU comes somewhere before CDP in the queue, although not necessarily immediately before?

(f) How many different ways are there to order the eight jobs in the queue so that job USU comes somewhere before CDP in the queue (although not necessarily immediately before) and CDP comes somewhere before BBD (again, not necessarily immediately before)?

EXERCISE | 9.5.6: Counting strings.

This question concerns strings over the alphabet { a, b, c, d, e }.

(a) How many strings have length 9 and exactly four a's?

(b) How many strings have length 9 and exactly four a's and exactly two b's?

(c) How many strings have length 9 and begin with "ab" or "cab"?

(d) How many strings have length 9 and begin with "ab" or "cab" and have exactly three d's?

(e) How many strings have length 9 and begin with "ab" or "cab" and have exactly three or four d's?

EXERCISE | 9.5.7: Selecting long-haired and short-haired cats.

A family goes to the animal shelter to select three pet cats. The shelter currently has 27 cats. At the shelter there are 10 long-haired cats and 17 short-haired cats.

(a) How many ways are there for the family to select their cats?

(b) How many ways are there for the family to make their selection if they want two long-haired cats and one short-haired cat?

(c) How many ways can they make their selection if they want at least one long-haired cat?

EXERCISE | 9.5.8: Distributing identical homework passes.

A teacher distributes 10 identical homework passes to her class of 20 students. How many ways are there for her to distribute the passes if:

(a) Each student gets at most one homework pass.

(b) There are no restrictions on the number of homework passes a student can get.

(c) There are no restrictions on the number of homework passes a student can get, except that one particular student, Sam, gets at least two homework passes.

(d) There are no restrictions on the number of homework passes a student can get, except that one particular student, Sam, gets at most two homework passes.

EXERCISE | 9.5.9: Distributing donuts to kids.

Two dozen donuts are given out to 24 kids so that each kid gets exactly one donut.

(a) How many ways are there to distribute the donuts if the donuts are all different?

(b) One of the kids is name Antonio. One of the donuts is lemon-filled. How many ways are there to distribute the donuts if the donuts are all different and Antonio must get the lemon-filled donut?

(c) One of the kids is name Antonio and one of the kids is named Rachel. One of the donuts is lemon-filled. How many ways are there to distribute the donuts if the donuts are all different and either Antonio or Rachel must get the lemon-filled donut?

(d) How many ways are there to distribute the donuts if there are four varieties of donuts and exactly six of each variety (and there are no restrictions on who can get which variety)?

EXERCISE 9.5.10: Counting ternary strings.

A ternary string has characters from the set {0, 1, 2}. For example 122010 and 0011210 are examples of ternary strings.

(a) How many ternary strings are there whose length is in the range 6 through 8?

(b) How many ternary strings of length seven start with a 1 or a 2?

(c) How many ternary strings are there of length 8 with exactly three 1's?

EXERCISE 9.5.11: Photo line up.

Eight kids line up for a photo. One of the kids is named Felicia.

(a) How many ways are there to line up the eight kids?

(b) Felicia has one best friend named Bob. How many ways are there to line up the eight kids so that Felicia is next to Bob?

(c) Felicia has two best friends named Bob and Hubert. How many ways are there to line up the eight kids so that Felicia is next to Bob and Hubert?

(d) Felicia has three best friends named Bob, Cassandra, and Hubert. How many ways are there to line up the eight kids so that Felicia is next to exactly one of her three best friends?

EXERCISE 9.5.12: Distributing gifts to a set of shoppers.

A grocery store offers a promotion in which five customers visiting the store on a particular day are each given a gift. No customer can receive more than one gift. 250 customers enter the store on the day of the promotion.

(a) If the gifts are all identical, how many different ways are there to distribute the gifts?

(b) If the gifts are all different from each other, how many different ways are there to distribute the gifts?

EXERCISE 9.5.13: Selecting an honors choir.

There are 30 singers in a school choir. 20 of the singers are women, and the other 10 are men. The choir director must select 12 singers from the choir to be in the honors choir.

(a) How many ways are there to select the honors choir if there is no restriction on the number of men and women selected?

(b) How many ways are there to select the honors choir if there must be at least one man in the honors choir?

(c) How many ways are there to select the honors choir if there must be the same number of men and women?

EXERCISE 9.5.14: Counting strings with a fixed number of c's.

(a) How many strings over the alphabet {a, b, c, d, e} of length 13 have exactly four c's?

EXERCISE 9.5.15: Selecting actors for roles in a play.

(a) 30 girls audition for a play. There are four different female roles in the play. How many possible outcomes are there for the auditions? Note that it is important which girl gets assigned which part.

EXERCISE 9.5.16: Lining up kids - boys ahead of girls.

(a) Ten kids line up for recess. There are five boys and five girls. How many ways are there for the 10 kids to line up so that all the boys are ahead of all the girls? That is, none of the girls are ahead of any of the boys.

EXERCISE 9.5.17: Counting binary strings with a fixed number of 1's.

(a) How many binary strings of length 12 have six 1's or seven 1's?

EXERCISE 9.5.18: Counting schedules for school lunches.

(a) A chef is deciding on the schedule for the week of lunch specials for her restaurant. She makes 22 different dishes that she can put on the schedule. For each of the seven days of the week, she must select one dish for the lunch special. How many ways are there for her to select the schedule for the week if she does not select the same dish more than once? Note that it matters which dish is served on which day for the schedule of daily specials.

EXERCISE 9.5.19: Counting PINs.

A PIN is a string of four digits. Each of the four digits can be any digit from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} except that the last digit must be even and the second-to-last digit must be odd. (Note that 0 is even.)

(a) How many different choices are there for a PIN if there are no restrictions on the number of times a digit can appear in the PIN?

(b) How many different choices are there for a PIN if the four digits in the PIN must all be different?

EXERCISE 9.5.20: Counting binary strings, cont.

(a) How many binary strings of length 12 do not have exactly four 1's?

(b) How many binary strings of length 12 start with 101 or 1110?

(c) How many binary strings of length 12 start with 101 or 1110 and have exactly four 1's?

(d) How many binary strings of length 12 start with 101 or 1110 and do not have exactly four 1's?

(e) How many binary strings of length 12 start with 00 or end with 00 or both?

EXERCISE 9.5.21: Assigning tasks to processors.

A set of 30 tasks are assigned to a set of 10 processors. The processors are all distinct. Each task is assigned to exactly one processor. A processor can be assigned more than one task.

(a) How many ways are there to assign the tasks if the tasks are all different and there are no restrictions on the number of tasks that can go to any particular processor?

(b) How many ways are there to assign the tasks if the tasks are all identical and there are no restrictions on the number of tasks that can go to any particular processor?

(c) How many ways are there to assign the tasks if the tasks are all different and each processor must receive the same number of tasks?

(d) How many ways are there to assign the tasks if the tasks are all identical and each processor must receive the same number of tasks?

EXERCISE 9.5.22: Cardinality of a power set.

(a) S = {a, b, c, d, e, f, g}. What is |P(S)|?

EXERCISE 9.5.23: Purchasing candy bars.

Felicity goes to a grocery store to purchase 13 candy bars. The store sells 8 varieties of candy bars. One of the varieties is Snickers and one of the varieties is Twix.

(a) How many ways are there for Felicity to make her selection?

(b) Suppose that Felicity would like at least two Snickers bars. Then how many ways are there for Felicity to make her selection?

(c) Suppose that Felicity would like exactly two Snickers bars. Then how many ways are there for Felicity to make her selection?

(d) Suppose that Felicity would like at most three Twix bars. Then how many ways are there for Felicity to make her selection?

(e) Suppose that Felicity would like at most three Twix bars and at least two Snickers bars. Then how many ways are there for Felicity to make her selection?

EXERCISE 9.5.24: Solutions to equations involving a sum of integer valued variables.

(a) How many solutions are there to the equation
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 37$
where each x_i is a non-negative integer?

(b) How many solutions are there to the equation
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 37$
where each x_i is an integer that satisfies
$x_i \geq 2$?

(c) How many solutions are there to the equation
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 37$
where each x_i is an integer that satisfies
$x_i \geq 2$?

EXERCISE 9.5.25: Storing widgets in warehouses.

A company has different 10 warehouses for storing their inventory. They will be storing 100 identical crates of widgets in the 10 warehouses.

(a) How many ways are there for the company to distribute the crates of widgets among the warehouses?

(b) How many ways are there for the company to distribute the crates of widgets among the warehouses if at least 5 crates must be stored at each warehouse?

(c) How many ways are there for the company to distribute the crates of widgets among the warehouses if at most 50 crates can be stored at any one warehouse?

(d) How many ways are there for the company to distribute the crates of widgets among the warehouses if at most 40 crates can be stored at any one warehouse?

9.6 Counting

This third python programming assignment, PA3, is about counting. You will write two functions partitions(n,k) that counts in how many ways n distinct elements can be grouped into k (non empty) partitions, and mkCh(a,c) that counts in how many ways amount a can be paid with coins {1,5,10,25}. Both algorithms are discussed in lecture 15: counting.

Start with the skeleton code. A correct implementation of counting:

```
python3 counting.py 3 2
```

produces

```
n: 3 k: 2 partitions: 3
amount: 32 coins: [1, 5, 10, 25] ways: 18
```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LAB ACTIVITY | 9.6.1: Counting | 0 / 100

main.py

Load default template...

```
import sys

coins = [1,5,10,25]

def partitions(n,k):
    """
     pre 0<k<=n, n>0
     post return the number of ways k partitions
           can be formed out of n distinct elements
    """

    # if k==n or k==1 :
    #  there is only one way to form partitions
    # else :
    # select an element a, and
    #   either
    #     form k partitions with the rest of the elements
```

Develop mode | Submit mode

Run your program as often as you'd like, before submitting for grading. Below, type any needed input values in the first box, then click **Run program** and observe the program's output in the second box.

Enter program input (optional)

If your code requires input values, provide them here.

Run command

python3 main.py Additional arguments

Run program

Input (from above) → **main.py** (Your program) → O

Program output displayed here

Coding trail of your work What is this?

History of your effort will appear here once you begin working on this zyLab.