Students:
Section 3.4 is a part of 1 assignment:
Reading Assignment 3

Requirements: PA
No due date

3.4 The inverse of a function

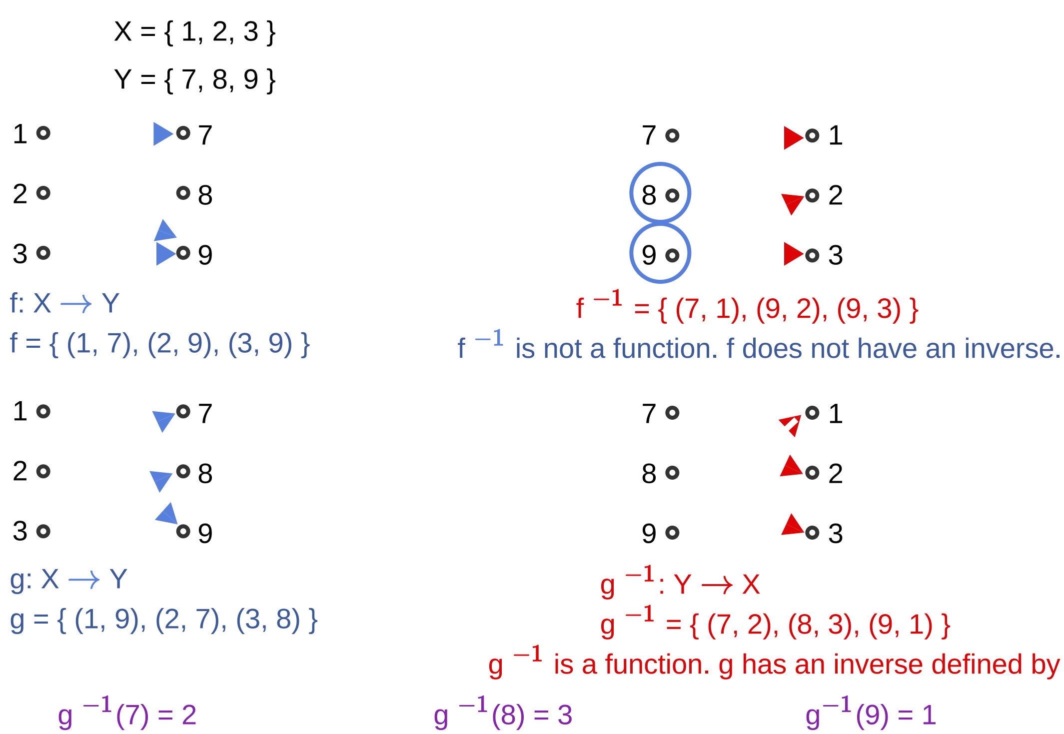
If a function f: X → Y is a bijection, then the ***inverse*** of f is obtained by exchanging the first and second entries in each pair in f. The inverse of f is denoted by f^{-1}:

$$f^{-1} = \{ (y, x) : (x, y) \in f \}.$$

Reversing each pair in a function f does not always result in a well-defined function. Therefore, some functions do not have an inverse. A function f: X → Y has an inverse if and only if reversing each pair in f results in a well-defined function from Y to X. f^{-1} is a well-defined function if every element in Y is mapped to exactly one element in X.

PARTICIPATION ACTIVITY 3.4.1: Arrow diagram for the inverse of a function.

1 2 2x speed



If the arrows for g are reversed, the result is a function because each left element has exactly one outgoing arrow. Thus, g has an inverse.

Captions

Captions

1. If arrows for f are reversed, the result is not a function because 8 has no outgoing arrow, and also 9 has two outgoing arrows. Thus, f does not have an inverse.
2. If the arrows for g are reversed, the result is a function because each left element has exactly one outgoing arrow. Thus, g has an inverse.

Feedback?

The finite examples in the animation above show that f^{-1} is obtained by reversing the arrows in the arrow diagram for f. The resulting f^{-1} is a function, if and only if every element in Y has exactly one outgoing arrow after the arrows are reversed which in turn holds if and only if f is a bijection.

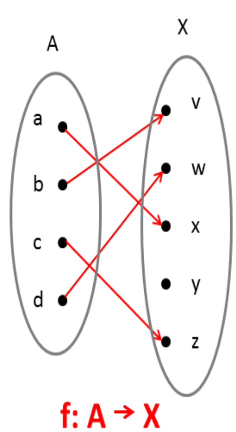
The reasoning above also applies to functions with infinite domains, and can be summed up in the following statement:

A function f has an inverse if and only if f is a bijection.

Recall that a bijection is a function that is one-to-one and onto.

PARTICIPATION ACTIVITY | 3.4.2: Does a function f have an inverse?

1) Consider the function specified by the following arrow diagram. Does f have an inverse?



- Yes
- No

Correct

f is not onto. There is no element in A that maps to y ∈ X.

2) Let X = { u, v, w, x }. Define a function g: X → X to be: g = { (u, v)

Correct

g is not one-to-one because g(v) = g(w) = w

→ X to be: g = { (u, v), (v, w), (w, w), (x, u) }. Does g have an inverse?

- Yes
- No

g is not one-to-one because g(v) = g(w) = w.

3) Define a function h: **Z** → **Z** to be h(x) = -x. Does h have an inverse?

- Yes
- No

Correct

h is one-to-one and onto.

Feedback?

The inverse of a bijection f can also be expressed in function notation. If f is a bijection from X to Y, then for every x ∈ X and y ∈ Y,

$$f(x) = y \quad \text{if and only if} \quad f^{-1}(y) = x.$$

Therefore the value of $f^{-1}(y)$ is the unique element x ∈ X such that f(x) = y. If f^{-1} is the inverse of function f, then for every element x ∈ X, $f^{-1}(f(x)) = x$.

PARTICIPATION ACTIVITY 3.4.3: Finding the inverse of a function with finite domains.

1) Let X = { u, v, w, x }. Define a function g: X → X to be: g = { (u, v), (v, w), (w, x), (x, u) }. What is $g^{-1}(x)$?

w

Check **Show answer**

Correct

w

g(w) = x. Therefore, $g^{-1}(x)$ = w.

2) Let h: $\{0, 1\}^3 \to \{0, 1\}^3$. For x ∈ $\{0, 1\}^3$, h(x) is obtained by moving the middle bit of x to the beginning of x. For example, h(101) = 011. What is $h^{-1}(100)$?

010

Correct

010

h(010) = 100. Therefore, $h^{-1}(100)$ = 010.

Check Show answer

Feedback?

When a function is defined on an infinite domain, it is sometimes possible to solve for the function's inverse analytically, as illustrated in the following animation:

PARTICIPATION ACTIVITY | 3.4.4: Solving for the inverse of a function analytically.

1 2 3 4 5 2x speed

f: R R f(x) = 3x - 2

Check that f is one to one: x x' 3x - 2 3x' - 2 f is one to one

Show the contrapositive: 3x - 2 = 3x' - 2 x = x'

Check that f is onto: for every real number y
there is an x s.t.
f(x) = 3x - 2 = y

Solve for x in terms of y: $x = \frac{y + 2}{3}$ This is f

f : R R

f (y) = $\frac{y + 2}{3}$

The inverse of f on input y is .

Captions

1. To check that f is one-to-one, check that , implies . Equivalently, show the contrapositive: , implies .
2. f is one-to-one because if , then .
3. f is onto because for every y, there is an x such that .
4. The inverse of f can be found by solving for an expression that gives the value of x in terms of y.
5. The inverse of f on input y is .

Feedback?

The function f: **R** → **R** defined by f(x) = x^2 is not one-to-one because f(x) = f(-x) for any real number x. However, if the domain is restricted to positive reals, $\mathbf{R}^+$, then:

number x. However, if the domain is restricted to positive reals, $\mathbf{R}^+$, then.

$$f: \mathbf{R}^+ \rightarrow \mathbf{R}^+, f(x) = x^2$$

is a bijection. To solve for f^{-1}, express $y = x^2$ and solve for x in terms of y: x = . Therefore: $f^{-1}(y)$ = .

The use of the variable y instead of x is not important. The function $f^{-1}(y)$ = is the same function as $f^{-1}(x)$ = .

PARTICIPATION ACTIVITY | 3.4.5: Computing the inverse of a function analytically.

1) $f: R \rightarrow R$, where $f(x) = -x + 3$. What is f^{-1}?

- $f^{-1}(x) = -x + 3$
- $f^{-1}(x) = x - 3$
- $f^{-1}(x) = -x - 3$

Correct

The result of solving the equation $y = -x + 3$ for x is $x = -y + 3$. Therefore, $f^{-1}(y) = -y + 3$ which is the same as $f^{-1}(x) = -x + 3$.

Feedback?

Example 3.4.1: Encrypting messages.

The process of encrypting messages can be expressed through the mathematical language of functions. Consider encrypting 16-digit credit card numbers. Let N be the set of all possible 16-digit numbers. An encryption scheme for N is a function e: $N \rightarrow N$. In order to send a credit card number $n \in N$ over an insecure communication channel, the sender would encrypt n by computing e(n). The encryption function e is chosen to be a bijection. The sender then sends e(N) over the channel. The receiver receives e(N) and wishes to decrypt the message to obtain the original number n. To accomplish this, the receiver applies the inverse function e^{-1} to e(N) in order to get the credit card number n: $e^{-1}(e(n)) = n$. The sender and receiver need to communicate ahead of time so that the receiver knows what function e^{-1} to apply to the number that is received. A secure encryption scheme would have the property that without foreknowledge of the function e^{-1}, it would be difficult to determine n from e(n).

Feedback?

CHALLENGE ACTIVITY | 3.4.1: The inverse of a function.

422102.2723990.qx3zqy7

Jump to level 1

1

2

3

A function is defined by .

-6

1	2	**3**

Check

Next

Done. Click any level to practice more. Completion is preserved.

✔ Expected: -6

Starting with and solving for results in: .

Therefore . The inverse of is well-defined.

Substituting for gives:

Thus, .

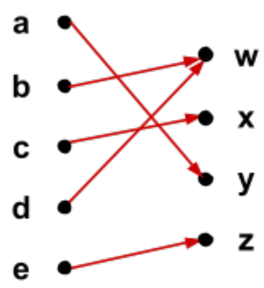
Feedback?

Additional exercises

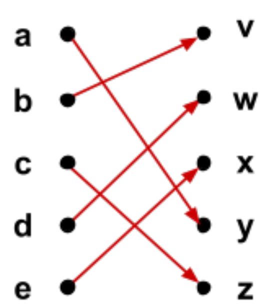
EXERCISE | 3.4.1: Arrow diagrams for inverses of functions.

Each of the arrow diagrams below define a function . For each arrow diagram, indicate whether is well-defined. If is not well-defined, indicate why. If is well-defined, give an arrow diagram showing .

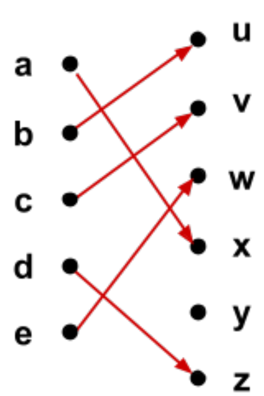
(a)



(b)



(c)



Feedback?

EXERCISE | 3.4.2: Finding inverses of functions.

For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of .

(a) .

(b) .

(c) .

(d) Let be defined to be the set

For , . Recall that for a finite set , denotes the power set of which is the set of all subsets of .

(e) Let be defined to be the set . .
For , . Recall that for a finite set , denotes the power set of which is the set of all subsets of .

(f) . The output of is obtained by taking the input string and replacing the first bit by , regardless of whether the first bit is a or . For example, and .

(g) . The output of is obtained by taking the input string and reversing the bits. For example, .

(h) . The output of is obtained by taking the input string , removing the first bit of , and adding the bit to the end of . For example, .

(i) ,

(j) ,

Feedback?

EXERCISE | 3.4.3: Finding the inverse of the cube of a bijective function.

For a function , we will define as .

(a) Prove that if is a bijection, then is also a bijection.
You can use the fact that for any two real numbers and , if , then . Also for any real number , is a well-defined real number.

(b) For a bijection , what is the inverse of ? Justify your answer.

Feedback?

How was this Provide feedback

section?

Activity summary for assignment: Reading Assignment 3 57 / 90 pts

No due date

57 / 90 pts submitted to canvas

Completion details