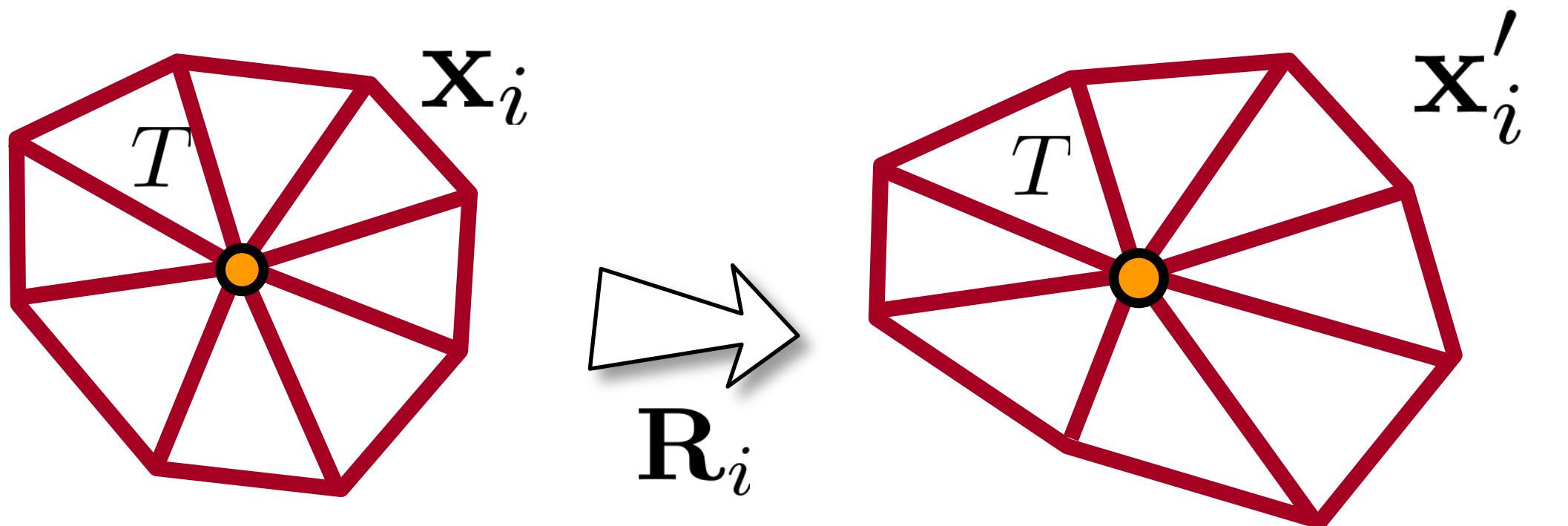


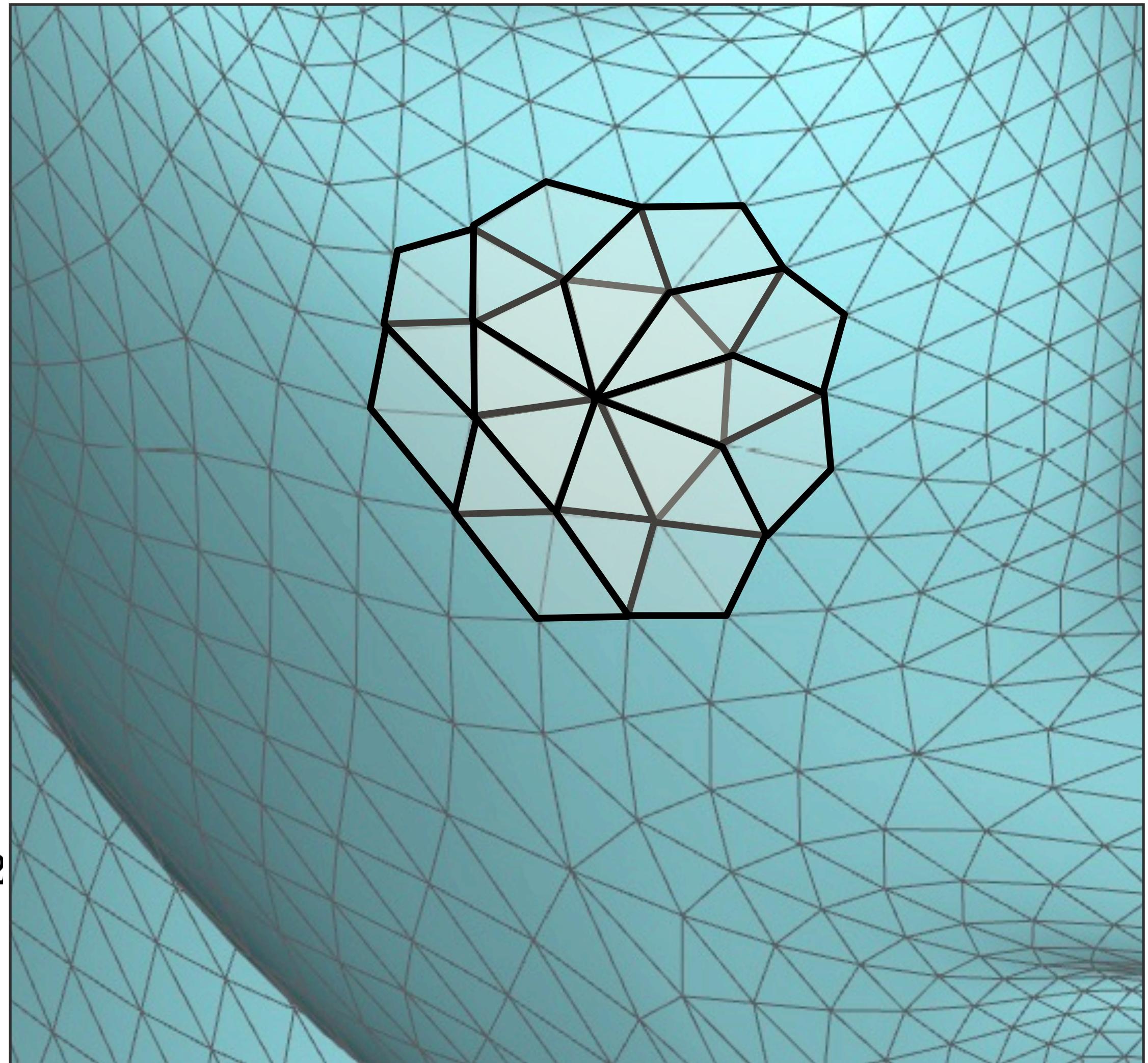
08 - As Rigid As Possible (ARAP) Deformation

As-Rigid-As-Possible Deformation

- Preserve shape of cells covering the surface
- Ask each cell i to transform **rigidly** by best-fitting rotation \mathbf{R}_i

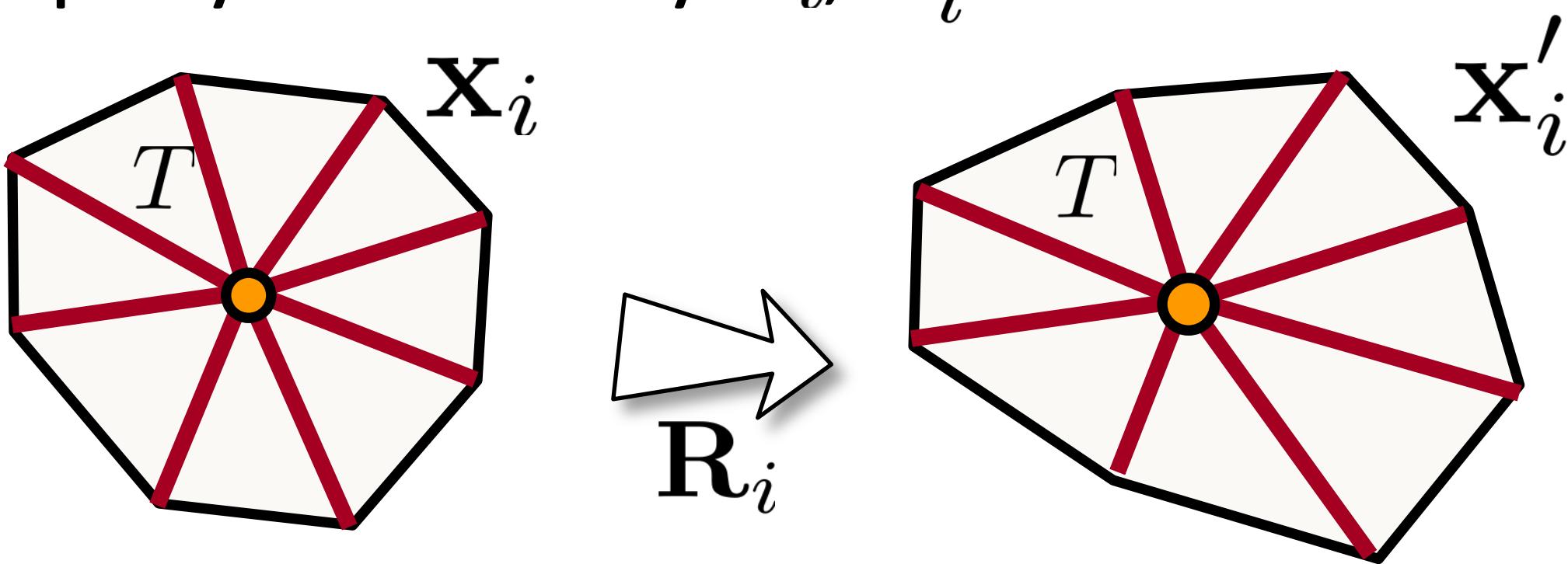


$$\min \sum_{T \in \text{Cell}_i} \sum_{(j,k) \in T} \|(\mathbf{x}'_j - \mathbf{x}'_k) - \mathbf{R}_i(\mathbf{x}_j - \mathbf{x}_k)\|^2$$



As-Rigid-As-Possible Deformation

- Optimal \mathbf{R}_i is uniquely defined by $\mathbf{x}_i, \mathbf{x}'_i$

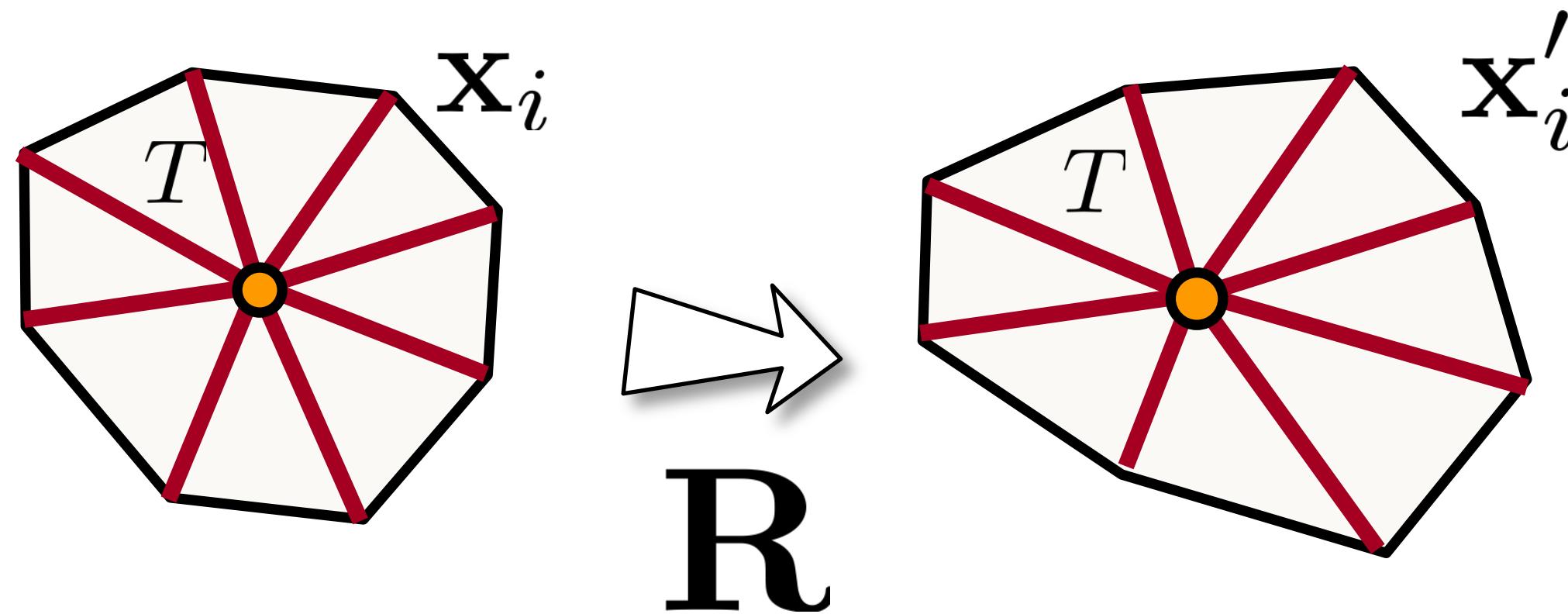


$$\min \sum_{T \in \text{Cell}_i} \sum_{(j,k) \in T} \|(\mathbf{x}'_j - \mathbf{x}'_k) - \mathbf{R}_i(\mathbf{x}_j - \mathbf{x}_k)\|^2$$

- so-called shape-matching problem,
solved by a 3x3 SVD

\mathbf{R}_i is a nonlinear
function of \mathbf{x}

Optimal Rotation

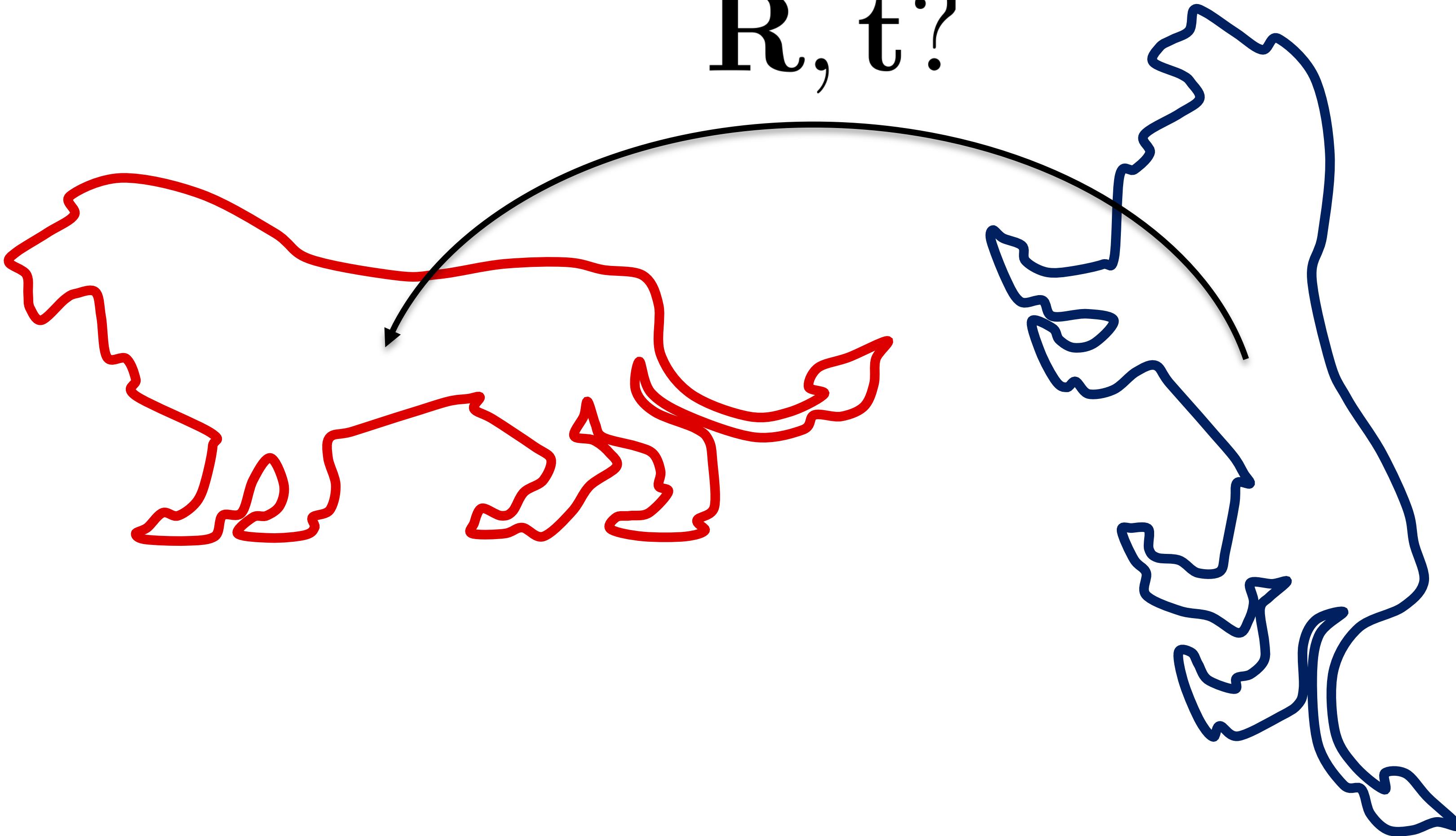


$$\min_{\mathbf{R} \in SO(3)} \sum_{T \in \text{Cell}_i} \sum_{(j,k) \in T} \|(\mathbf{x}'_j - \mathbf{x}'_k) - \mathbf{R}(\mathbf{x}_j - \mathbf{x}_k)\|^2$$

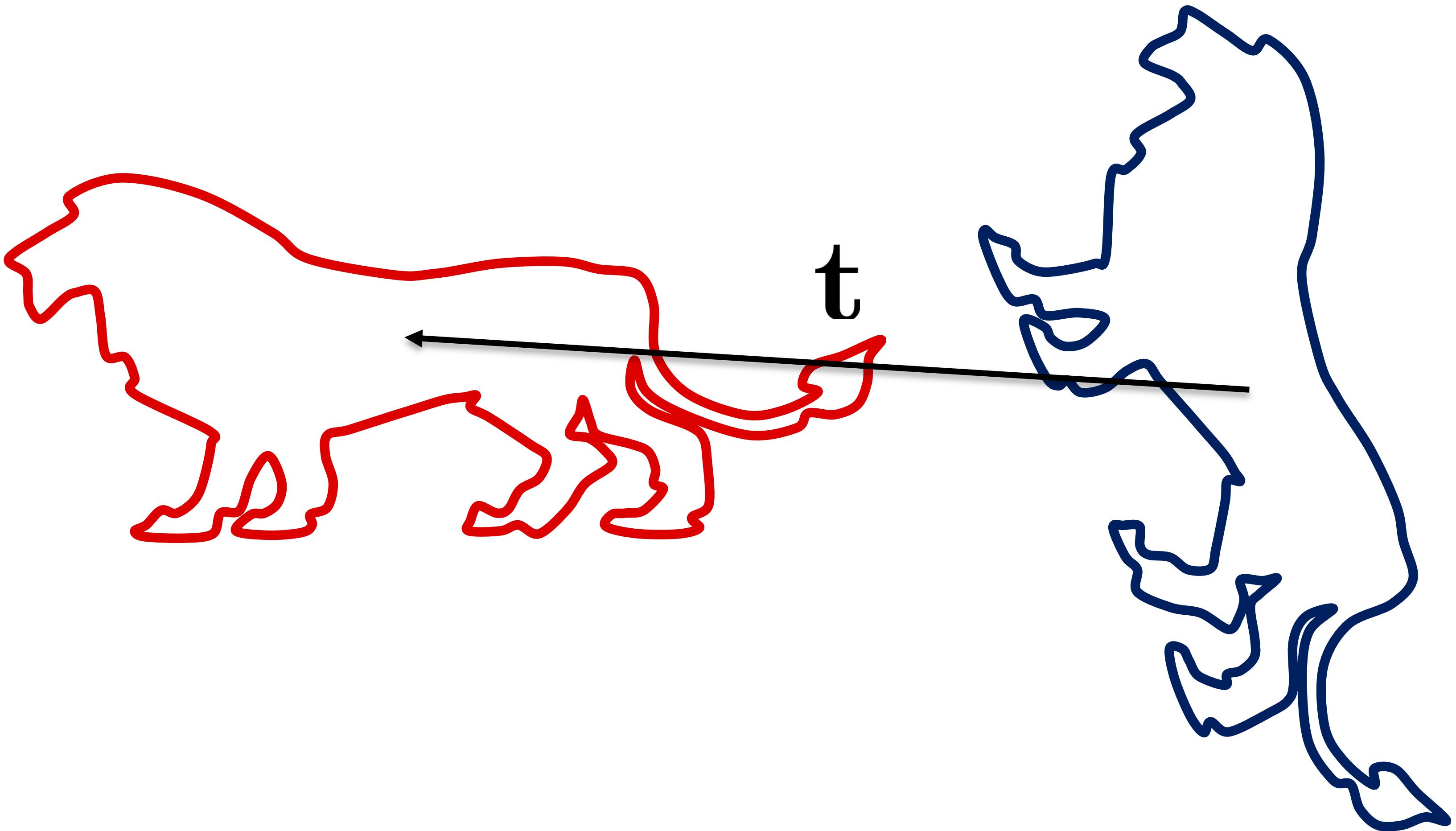
↑
Rotation group

Shape Matching Problem

$R, t?$



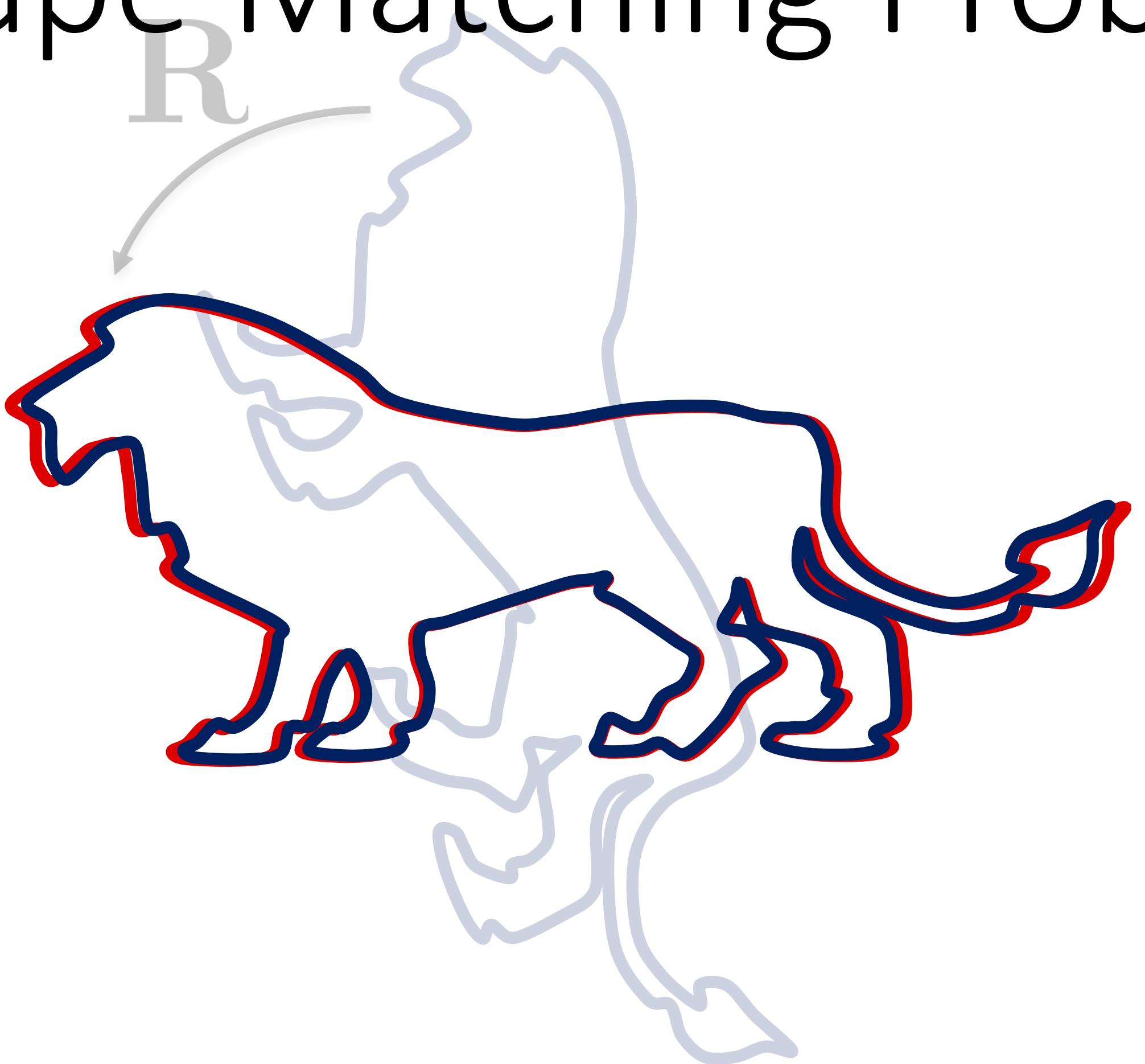
Shape Matching Problem



Shape Matching Problem



Shape Matching Problem



Shape Matching Problem

- Align two point sets

$$\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_n\} \text{ and } \mathcal{Q} = \{\mathbf{q}_1, \dots, \mathbf{q}_n\}$$

- Find a translation vector \mathbf{t} and rotation matrix \mathbf{R} so that

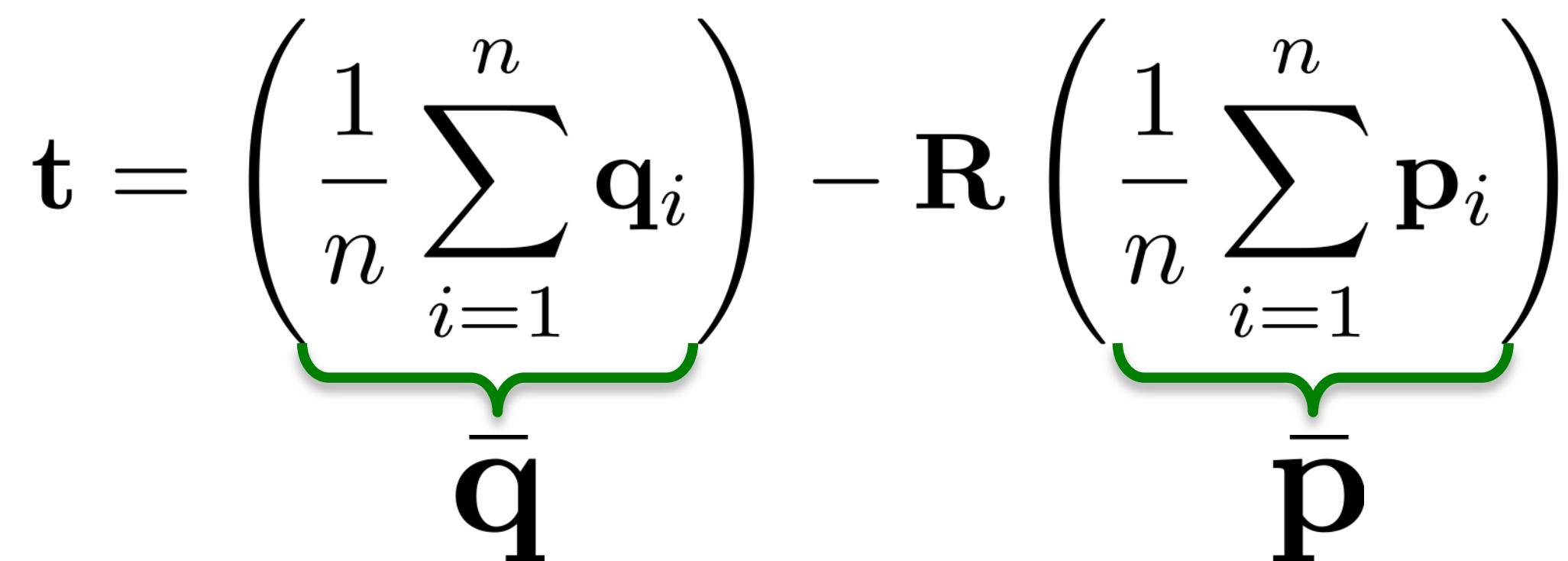
$$\sum_{i=1}^n \|(\mathbf{R}\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i\|^2 \text{ is minimized}$$

Shape Matching – Solution

- Solve for translation first (w.r.t. \mathbf{R} , \mathbf{p} , and \mathbf{q})

$$\frac{\partial}{\partial \mathbf{t}} \sum_{i=1}^n \|(\mathbf{R}\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i\|^2 = \sum_{i=1}^n 2((\mathbf{R}\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i) \stackrel{!}{=} 0$$

$$\mathbf{R} \sum_{i=1}^n \mathbf{p}_i + \sum_{i=1}^n \mathbf{t} - \sum_{i=1}^n \mathbf{q}_i = 0$$

$$\mathbf{t} = \left(\frac{1}{n} \sum_{i=1}^n \mathbf{q}_i \right) - \mathbf{R} \left(\frac{1}{n} \sum_{i=1}^n \mathbf{p}_i \right)$$


Finding the Rotation \mathbf{R}

- To find the optimal \mathbf{R} , we bring the centroids of both point sets to the origin

$$\mathbf{v}_i = \mathbf{p}_i - \bar{\mathbf{p}}, \quad \mathbf{v}'_i = \mathbf{q}_i - \bar{\mathbf{q}}$$

- We want to find \mathbf{R} that minimizes

$$\sum_{i=1}^n \|\mathbf{R}\mathbf{v}_i - \mathbf{v}'_i\|^2$$

Finding the Rotation \mathbf{R}

$$\sum_{i=1}^n \|\mathbf{R}\mathbf{v}_i - \mathbf{v}'_i\|^2 = \sum_{i=1}^n (\mathbf{R}\mathbf{v}_i - \mathbf{v}'_i)^T (\mathbf{R}\mathbf{v}_i - \mathbf{v}'_i) =$$

$$= \sum_{i=1}^n \left(\underbrace{\mathbf{v}_i^T \mathbf{R}^T \mathbf{R} \mathbf{v}_i}_{\mathbf{I}} - \mathbf{v}'_i^T \mathbf{R} \mathbf{v}_i - \mathbf{v}_i^T \mathbf{R}^T \mathbf{v}'_i + \underbrace{\mathbf{v}'_i^T \mathbf{v}'_i}_{\text{constant}} \right)$$

These terms do not depend on \mathbf{R} ,
so we can ignore them in the minimization

Finding the Rotation \mathbf{R}

$$\operatorname{argmin}_{\mathbf{R} \in SO(3)} \sum_{i=1}^n \left(-\mathbf{v}'_i{}^T \mathbf{R} \mathbf{v}_i - \mathbf{v}_i^T \mathbf{R}^T \mathbf{v}'_i \right) = \operatorname{argmax}_{\mathbf{R} \in SO(3)} \sum_{i=1}^n \left(\mathbf{v}'_i{}^T \mathbf{R} \mathbf{v}_i + \underbrace{\mathbf{v}_i^T \mathbf{R}^T \mathbf{v}'_i}_{\text{green arrow}} \right) =$$

$$= \boxed{\operatorname{argmax}_{\mathbf{R} \in SO(3)} \sum_{i=1}^n \mathbf{v}'_i{}^T \mathbf{R} \mathbf{v}_i}$$

$$\mathbf{v}_i^T \mathbf{R}^T \mathbf{v}'_i = (\mathbf{v}_i^T \mathbf{R}^T \mathbf{v}'_i)^T = \mathbf{v}'_i{}^T \mathbf{R} \mathbf{v}_i$$

Finding the Rotation \mathbf{R}

$$\sum_{i=1}^n \mathbf{v}'_i{}^T \mathbf{R} \mathbf{v}_i = \text{tr} \left(\mathbf{v}'{}^T \mathbf{R} \mathbf{V} \right)$$

$$\begin{matrix} \mathbf{v}'_1{}^T \\ \mathbf{v}'_2{}^T \\ \vdots \\ \mathbf{v}'_n{}^T \end{matrix} \begin{matrix} \mathbf{R} \\ \mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n \end{matrix} \mathbf{V} = \begin{matrix} \mathbf{v}'_1{}^T \\ \mathbf{v}'_2{}^T \\ \vdots \\ \mathbf{v}'_n{}^T \end{matrix} \begin{matrix} \mathbf{R} \mathbf{v}_1 \ \mathbf{R} \mathbf{v}_2 \ \cdots \ \mathbf{R} \mathbf{v}_n \end{matrix}$$
$$\mathbf{v}'{}^T$$

Finding the Rotation \mathbf{R}

$$\sum_{i=1}^n \mathbf{v}'_i{}^T \mathbf{R} \mathbf{v}_i = \text{tr} \left(\mathbf{v}'{}^T \mathbf{R} \mathbf{V} \right)$$

$$\begin{matrix} \mathbf{v}'_1{}^T \\ \mathbf{v}'_2{}^T \\ \vdots \\ \mathbf{v}'_n{}^T \end{matrix} \begin{matrix} \mathbf{R} \mathbf{v}_1 & \mathbf{R} \mathbf{v}_2 & \cdots & \mathbf{R} \mathbf{v}_n \end{matrix} = \begin{matrix} \mathbf{v}'_1{}^T \mathbf{R} \mathbf{v}_1 \\ \mathbf{v}'_2{}^T \mathbf{R} \mathbf{v}_2 \\ \ddots \\ \vdots \\ \mathbf{v}'_n{}^T \mathbf{R} \mathbf{v}_n \end{matrix}$$

Finding the Rotation \mathbf{R}

- Find \mathbf{R} that maximizes

$$\text{tr} \left(\mathbf{V}'^T \mathbf{R} \mathbf{V} \right) = \text{tr} \left(\mathbf{R} \mathbf{V} \mathbf{V}'^T \right)$$

- SVD:

$$\mathbf{V} \mathbf{V}'^T = \mathbf{U} \boldsymbol{\Sigma} \tilde{\mathbf{U}}^T$$

Take a look at the
Matrix Cookbook!

$$\text{tr} \left(\mathbf{R} \mathbf{V} \mathbf{V}'^T \right) = \text{tr} \left(\underbrace{\mathbf{R}}_{\text{orthonormal matrix}} \underbrace{\mathbf{U} \boldsymbol{\Sigma} \tilde{\mathbf{U}}^T}_{\text{orthonormal matrix}} \right) = \text{tr} \left(\boldsymbol{\Sigma} \underbrace{\tilde{\mathbf{U}}^T \mathbf{R} \mathbf{U}}_{\text{orthonormal matrix}} \right)$$

Finding the Rotation \mathbf{R}

- We want to maximize

$$\text{tr} (\Sigma \mathbf{M})$$

\mathbf{M} : orthonormal matrix
all entries ≤ 1

$$\begin{matrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{matrix} \quad \begin{matrix} m_{11} & \dots & \\ \vdots & m_{22} & \vdots \\ & \dots & m_{33} \end{matrix}$$

$$\text{tr} (\Sigma \mathbf{M}) = \sum_{i=1}^3 \sigma_i m_{ii} \leq \sum_{i=1}^3 \sigma_i$$

Finding the Rotation \mathbf{R}

$$\text{tr} (\Sigma \mathbf{M}) = \sum_{i=1}^3 \sigma_i m_{ii} \leq \sum_{i=1}^3 \sigma_i$$

- Our best shot is $m_{ii} = 1$, i.e. to make $\mathbf{M} = \mathbf{I}$

$$\mathbf{M} = \tilde{\mathbf{U}}^T \mathbf{R} \mathbf{U} \stackrel{!}{=} \mathbf{I}$$

$$\mathbf{R} \mathbf{U} = \tilde{\mathbf{U}}$$

$$\boxed{\mathbf{R} = \tilde{\mathbf{U}} \mathbf{U}^T}$$

Summary of Rigid Alignment

- Translate the input points to the centroids

$$\mathbf{v}_i = \mathbf{p}_i - \bar{\mathbf{p}}, \quad \mathbf{v}'_i = \mathbf{q}_i - \bar{\mathbf{q}}$$

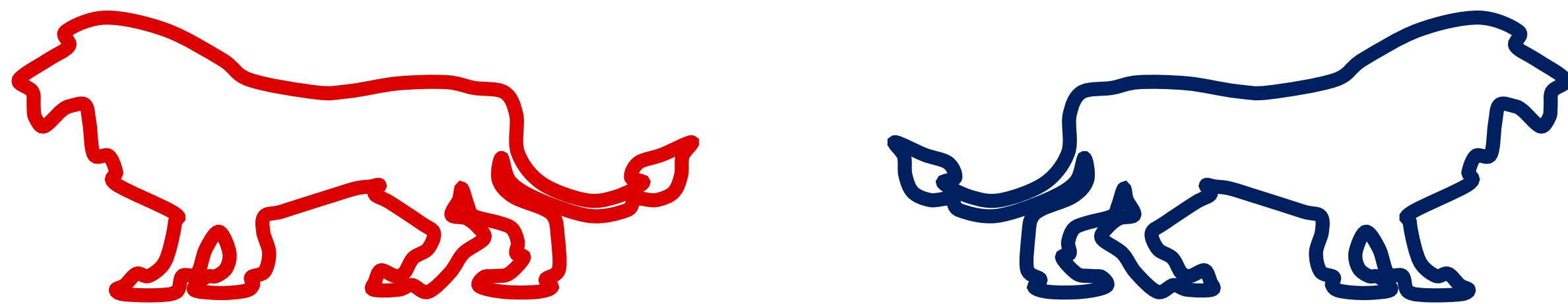
- Compute the “covariance matrix” $\mathbf{V}\mathbf{V}'^T$

- Compute its SVD: $\mathbf{V}\mathbf{V}'^T = \mathbf{U}\boldsymbol{\Sigma}\tilde{\mathbf{U}}^T$

- The optimal orthonormal \mathbf{R} is $\mathbf{R} = \tilde{\mathbf{U}}\mathbf{U}^T$

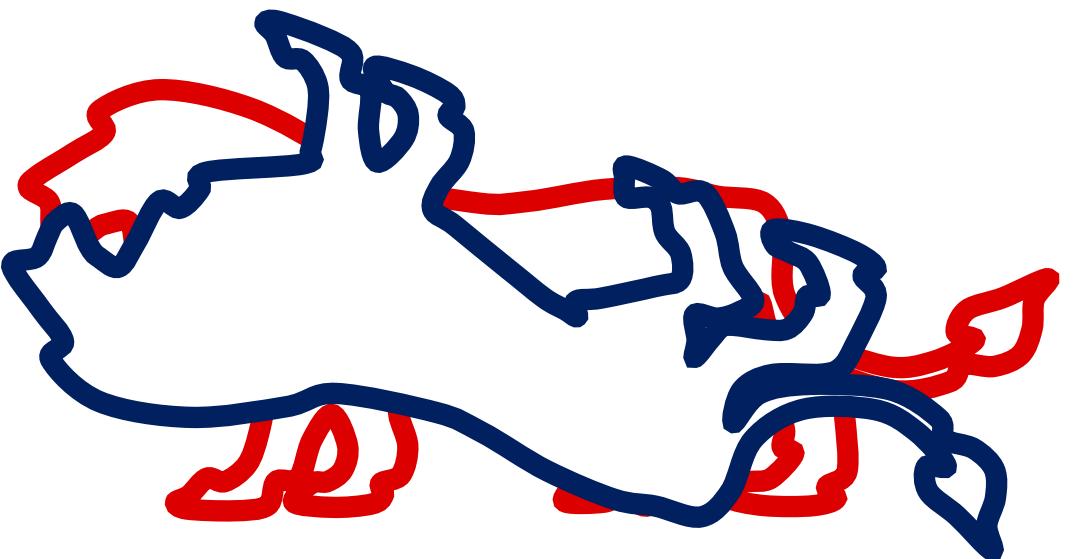
Sign Correction

- It is possible that $\det(\tilde{\mathbf{U}}\mathbf{U}^T) = -1$: sometimes reflection is the best orthonormal transform



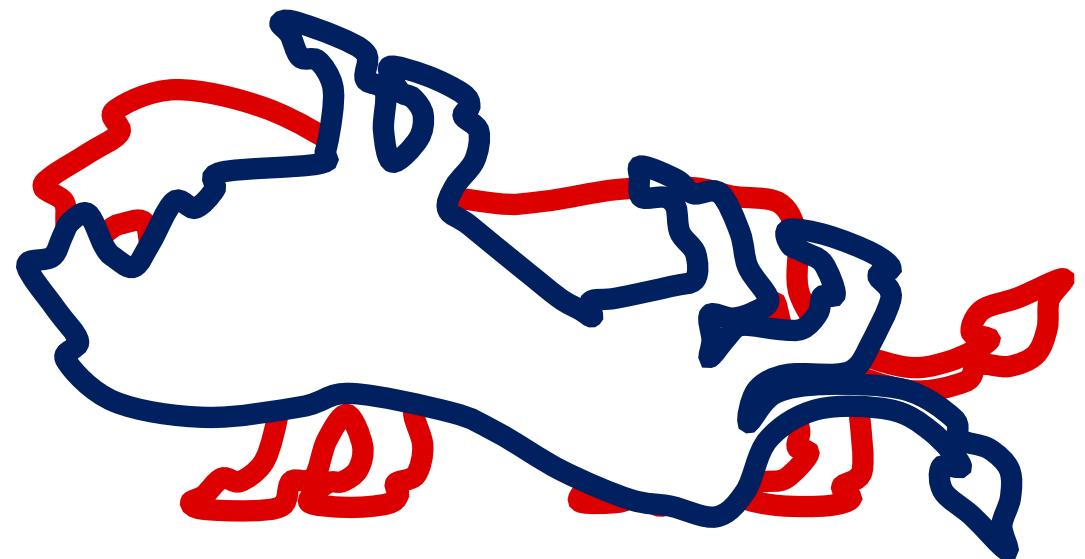
Sign Correction

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Sign Correction

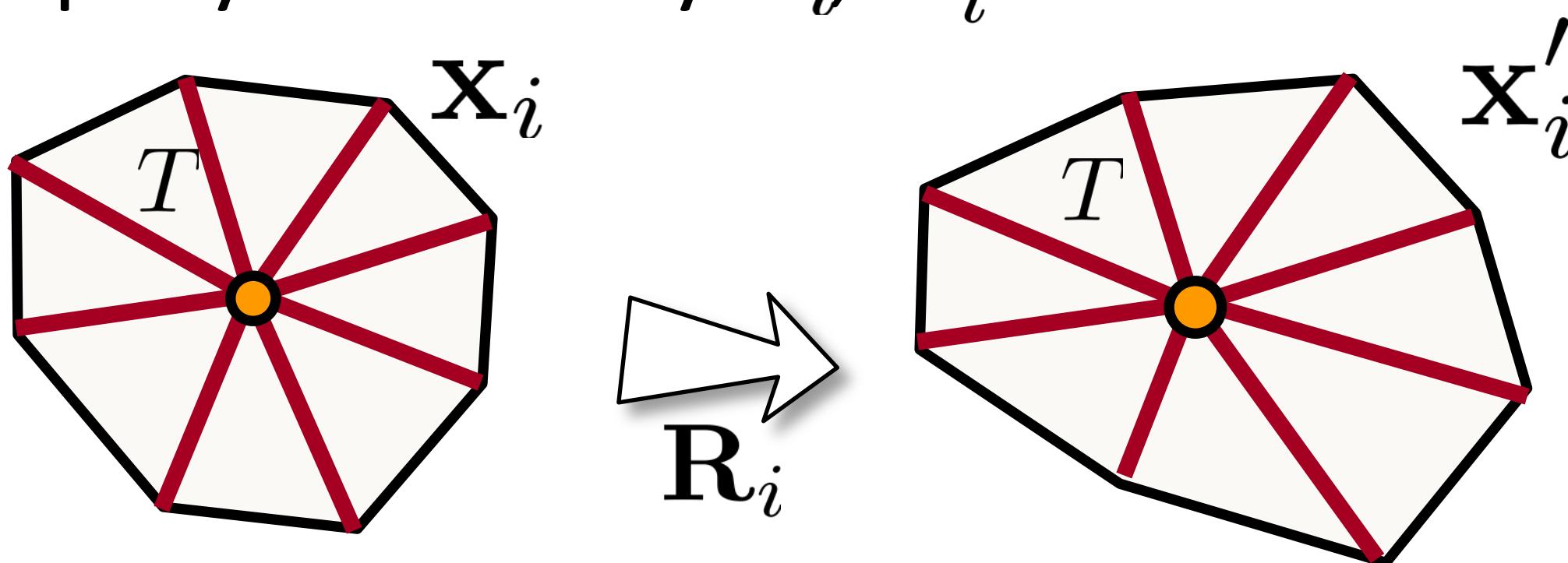
- To restrict ourselves to rotations only:
take the last column of \mathbf{U} (corresponding to the smallest singular value) and invert its sign.



- Why? See http://igl.ethz.ch/projects/ARAP/svd_rot.pdf

As-Rigid-As-Possible Deformation

- Optimal \mathbf{R}_i is uniquely defined by $\mathbf{x}_i, \mathbf{x}'_i$



$$\min \sum_{T \in \text{Cell}_i} \sum_{(j,k) \in T} \|(\mathbf{x}'_j - \mathbf{x}'_k) - \mathbf{R}_i(\mathbf{x}_j - \mathbf{x}_k)\|^2$$

- so-called shape-matching problem,
solved by a 3x3 SVD

\mathbf{R}_i is a nonlinear
function of \mathbf{x}

As-Rigid-As-Possible Deformation

- Total ARAP energy: sum up for all the cells i

$$\sum_i \sum_{T \in \text{Cell}_i} \sum_{(j,k) \in T} \|(\mathbf{x}'_j - \mathbf{x}'_k) - \mathbf{R}_i(\mathbf{x}_j - \mathbf{x}_k)\|^2$$

- Treat \mathbf{x} and \mathbf{R} as separate sets of variables
- Simple **local-global** iterative optimization process
 - Decreases the energy at each step

As-Rigid-As-Possible Deformation

- Total ARAP energy: sum up for all the cells i

$$\sum_i \sum_{T \in \text{Cell}_i} \sum_{(j,k) \in T} \|(\mathbf{x}'_j - \mathbf{x}'_k) - \mathbf{R}_i(\mathbf{x}_j - \mathbf{x}_k)\|^2$$

- Local step: keep \mathbf{x}' fixed, find optimal \mathbf{R}_i per cell i
- Global step: keep \mathbf{R}_i fixed, solve for \mathbf{x}' – quadratic minimization problem

$$\rightarrow \mathbf{L}\mathbf{x}' = \mathbf{b}$$

As-Rigid-As-Possible Deformation

- Total ARAP energy: sum up for all the cells i

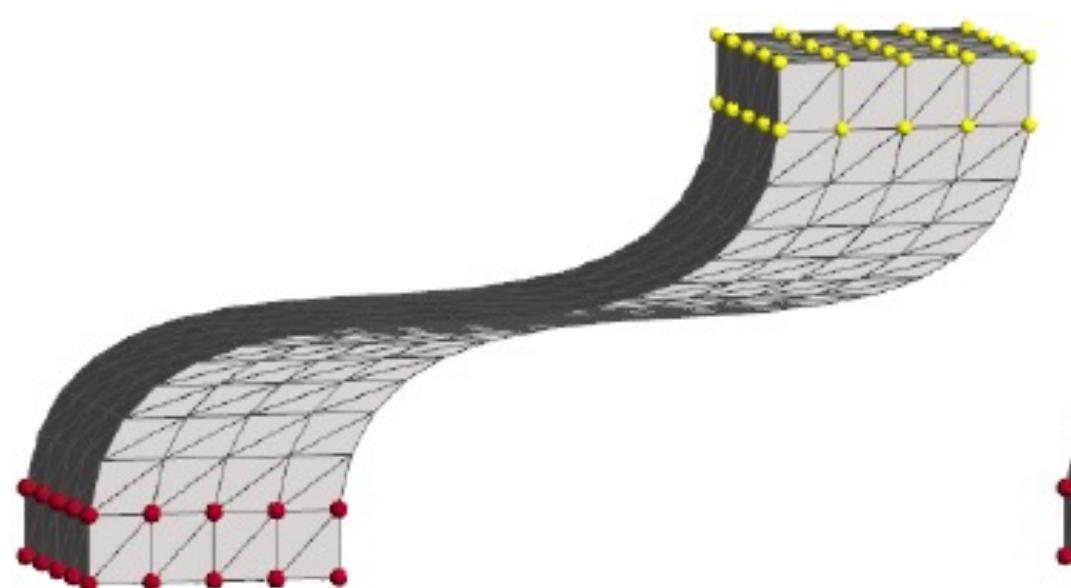
$$\sum_i \sum_{T \in \text{Cell}_i} \sum_{(j,k) \in T} \|(\mathbf{x}'_j - \mathbf{x}'_k) - \mathbf{R}_i(\mathbf{x}_j - \mathbf{x}_k)\|^2$$

- Local step: keep \mathbf{x}' fixed, find optimal \mathbf{R}_i per cell i
- Global step: keep \mathbf{R}_i fixed, solve for \mathbf{x}' – quadratic minimization problem
 - The matrix \mathbf{L} stays fixed, can pre-factorize

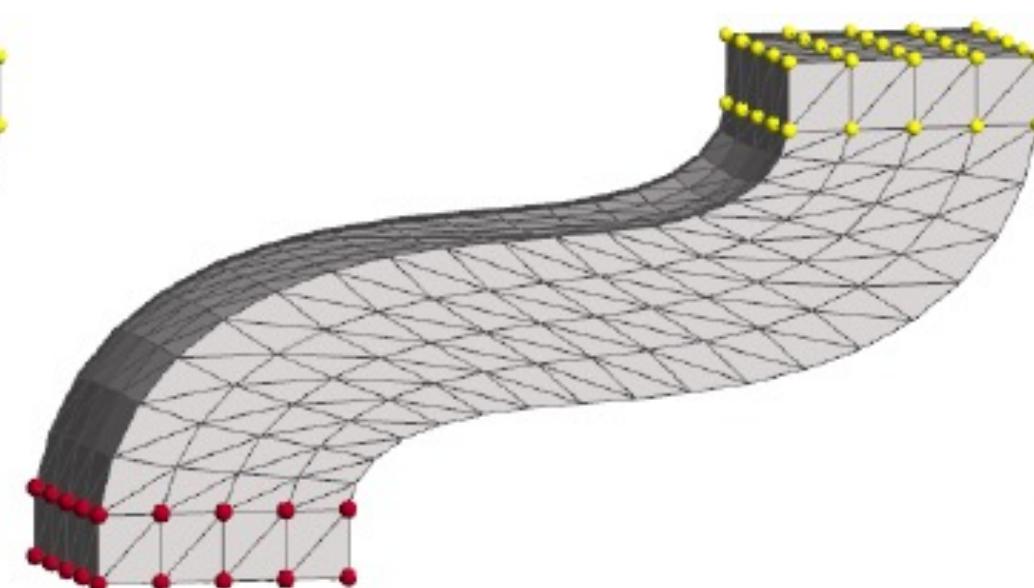
$$\rightarrow \mathbf{L}\mathbf{x}' = \mathbf{b}$$

Initial Guess

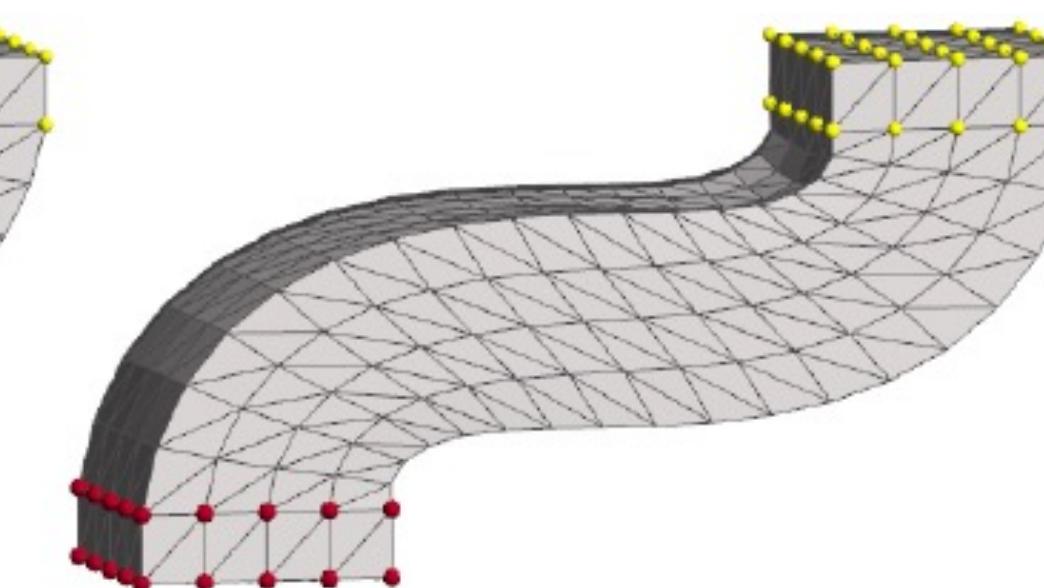
- Can use naïve Laplacian editing



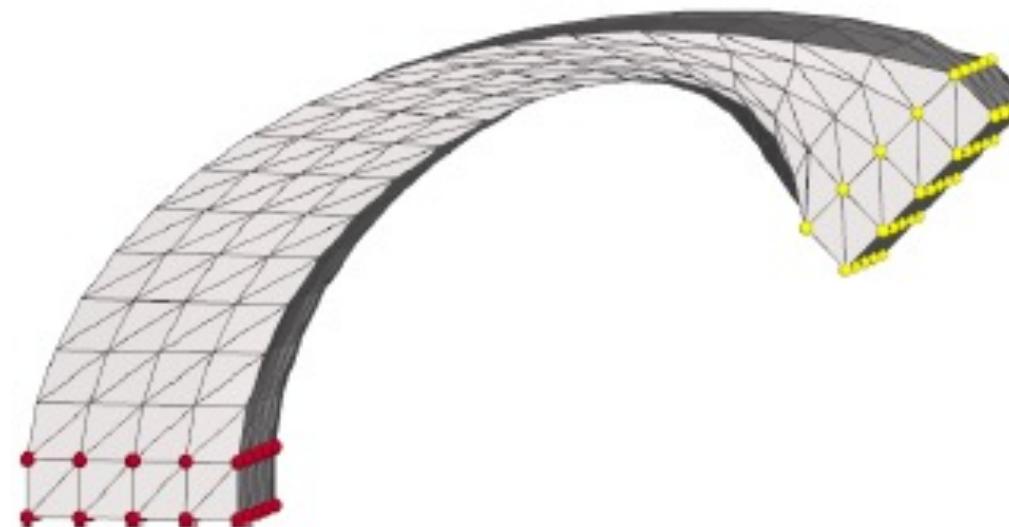
initial guess



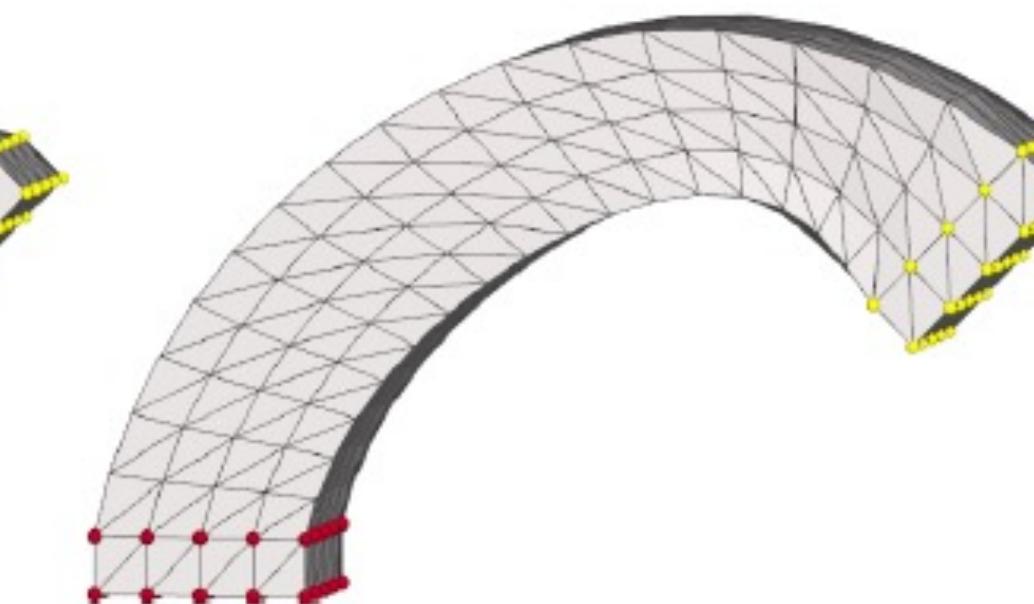
1 iterations



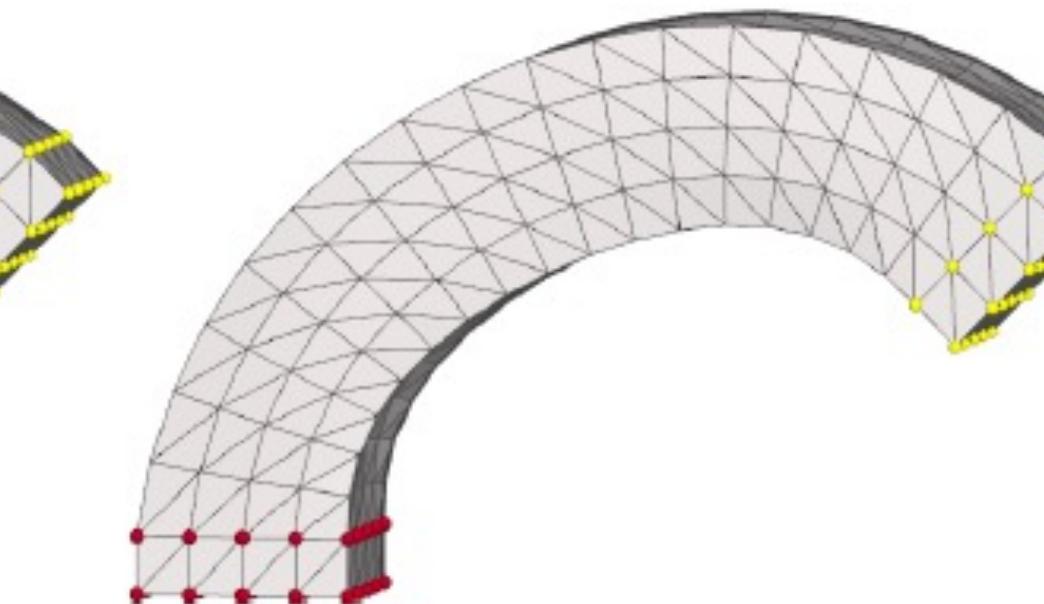
2 iterations



initial guess



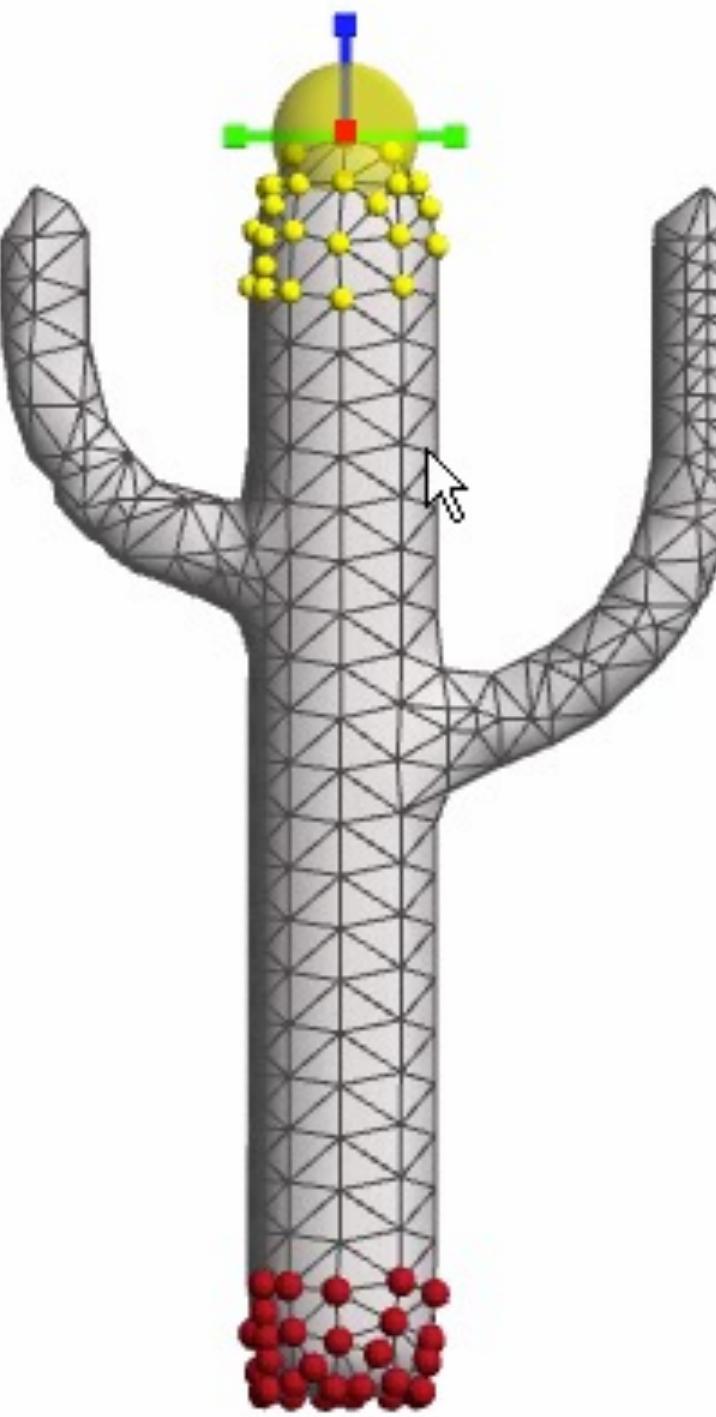
1 iterations



4 iterations

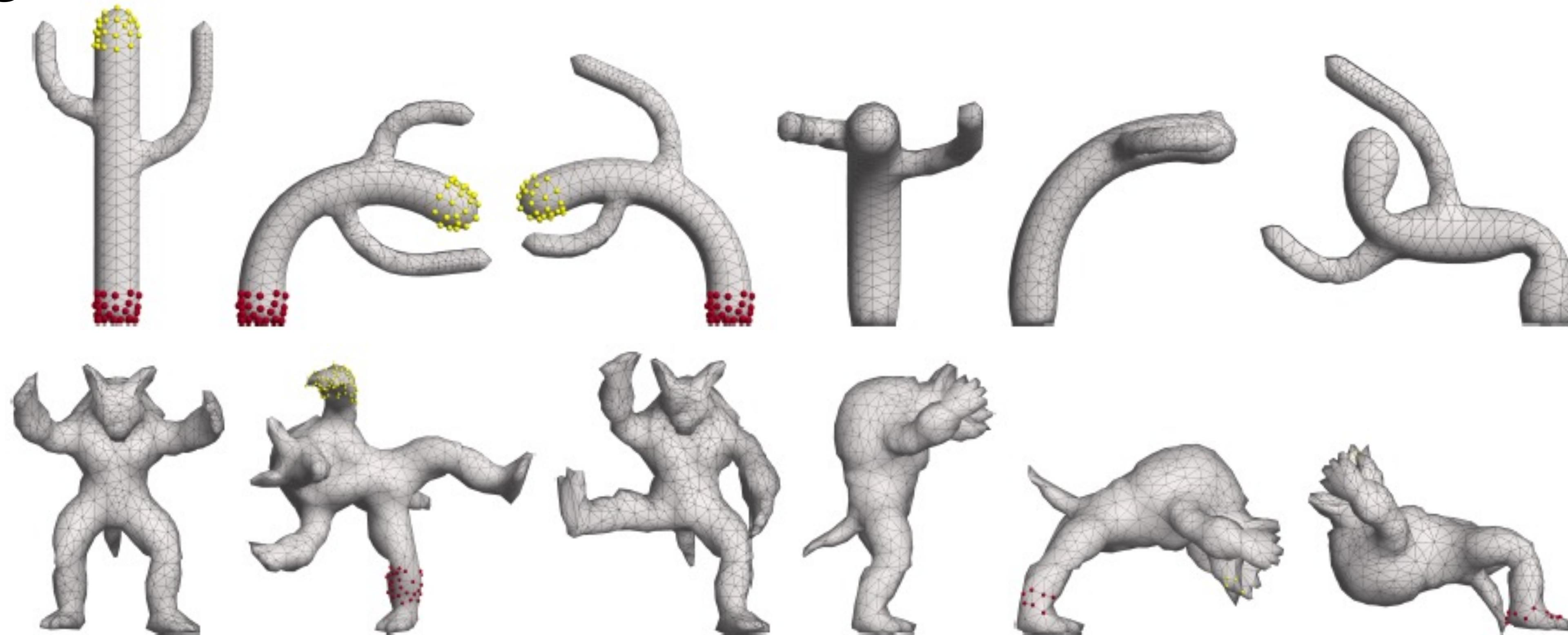
Initial Guess

- Can also use the previous frame
- Replace all handle vertex position by the currently prescribed ones
- Fast convergence

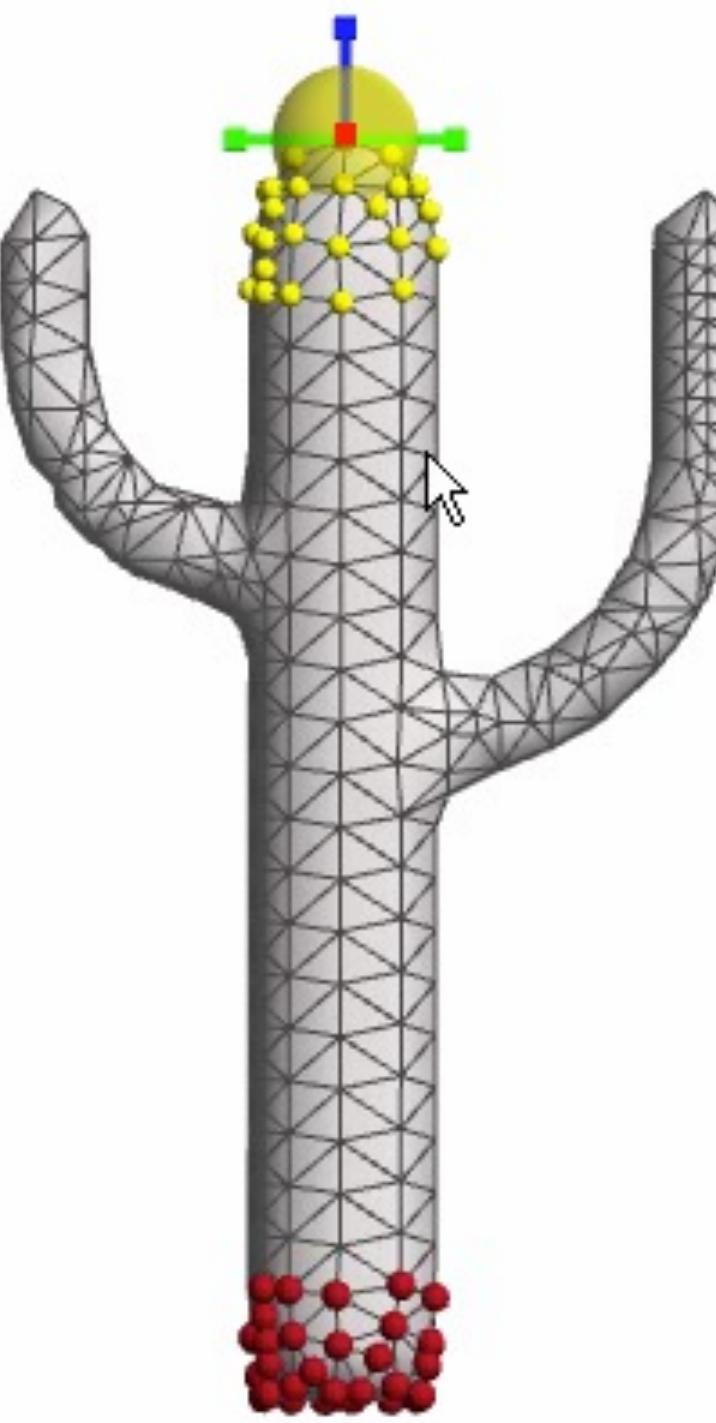


Large Rotations

- Use previous frame as the initial guess



Examples



Discussion

- Nonlinear deformation that models a kind of elastic behavior
- Very simple to implement, no parameters to tune except number of iterations
- Each step is guaranteed to not increase the energy
 - Compare with Gauss-Newton...
- Each iteration is relatively cheap, no matrix re-factorization necessary

Discussion

- Works fine on small meshes
- On larger meshes: slow convergence
 - Each iteration is more expensive
 - Need more iterations because the conditioning of the system becomes worse as the matrix grows
- Material stiffness depends on the cell size
 - lots of wrinkles for fine meshes when using 1-rings as cells

Acceleration using Subspace Techniques

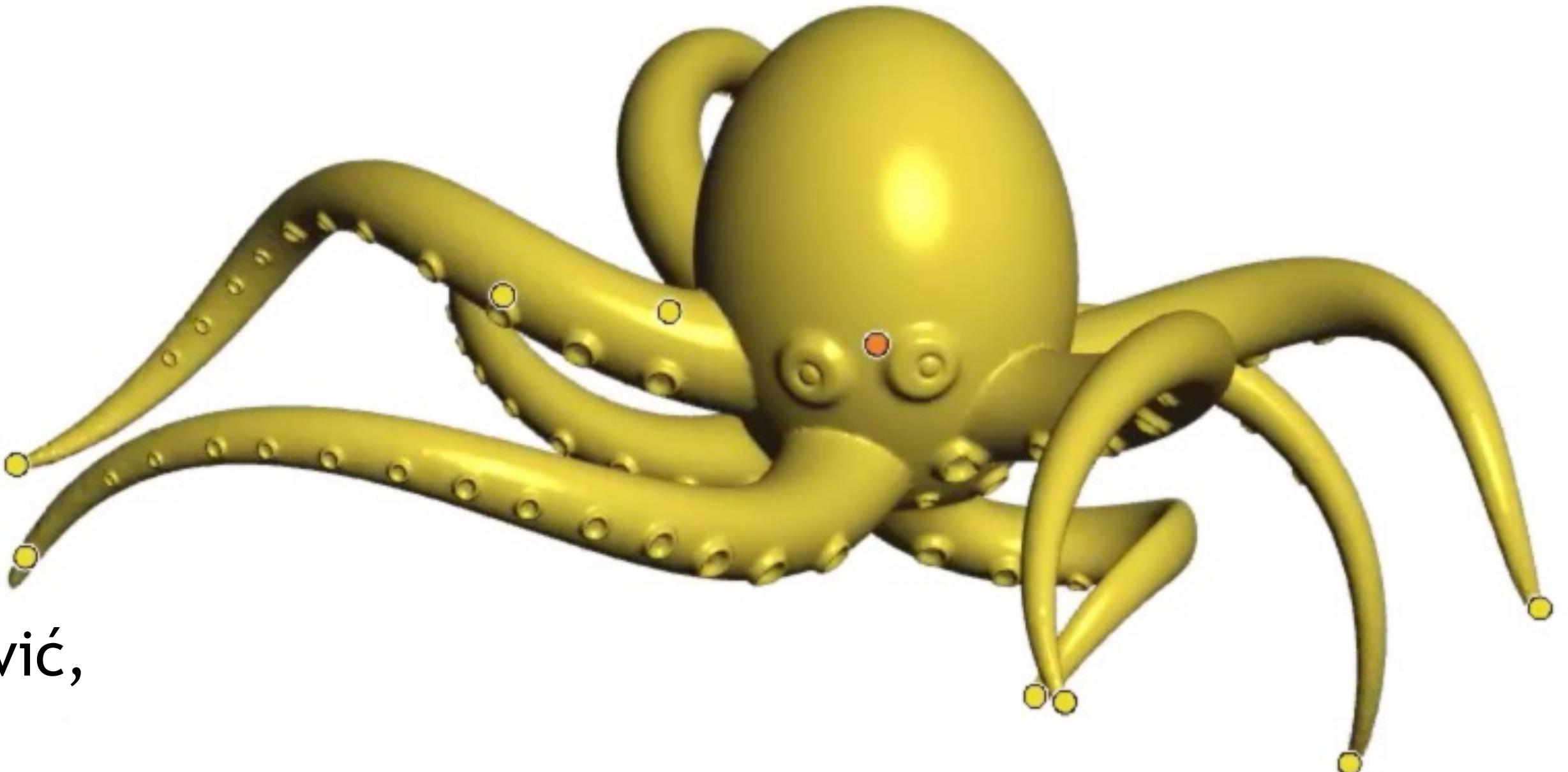
- Subspace created by int
- Drastically reduces the optimization



Alec Jacobson, Ilya Baran, Ladislav Kavan, Jovan Popović, and Olga Sorkine. [“Fast Automatic Skinning Transformations,” 2012.](#)

Acceleration using Subspace Techniques

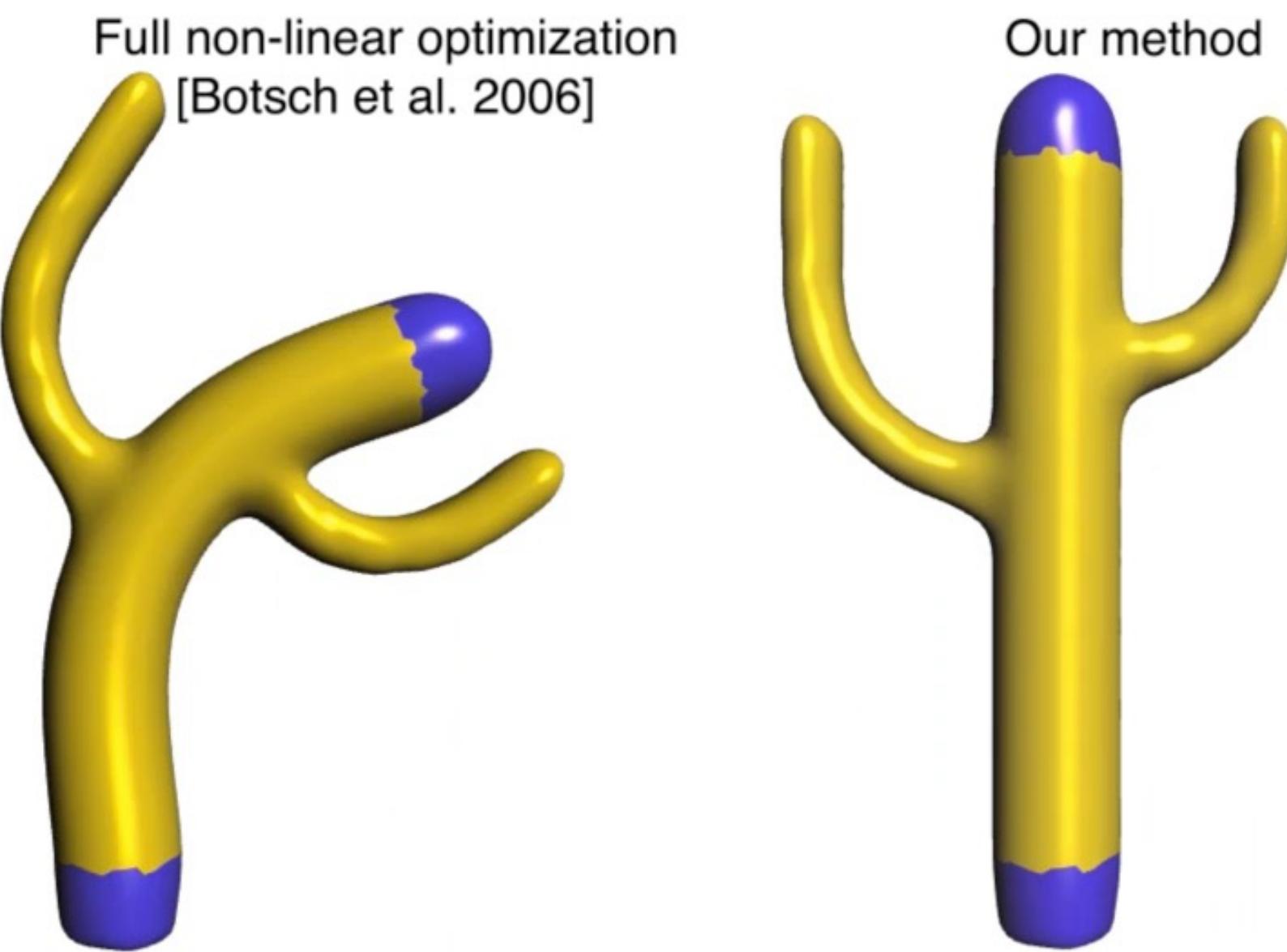
- Subspace created by influence weight functions for each handle
- Drastically reduces the number of degrees of freedom in the optimization



Alec Jacobson, Ilya Baran, Ladislav Kavan, Jovan Popović, and Olga Sorkine. [“Fast Automatic Skinning Transformations,” 2012.](#)

Acceleration using Subspace Techniques

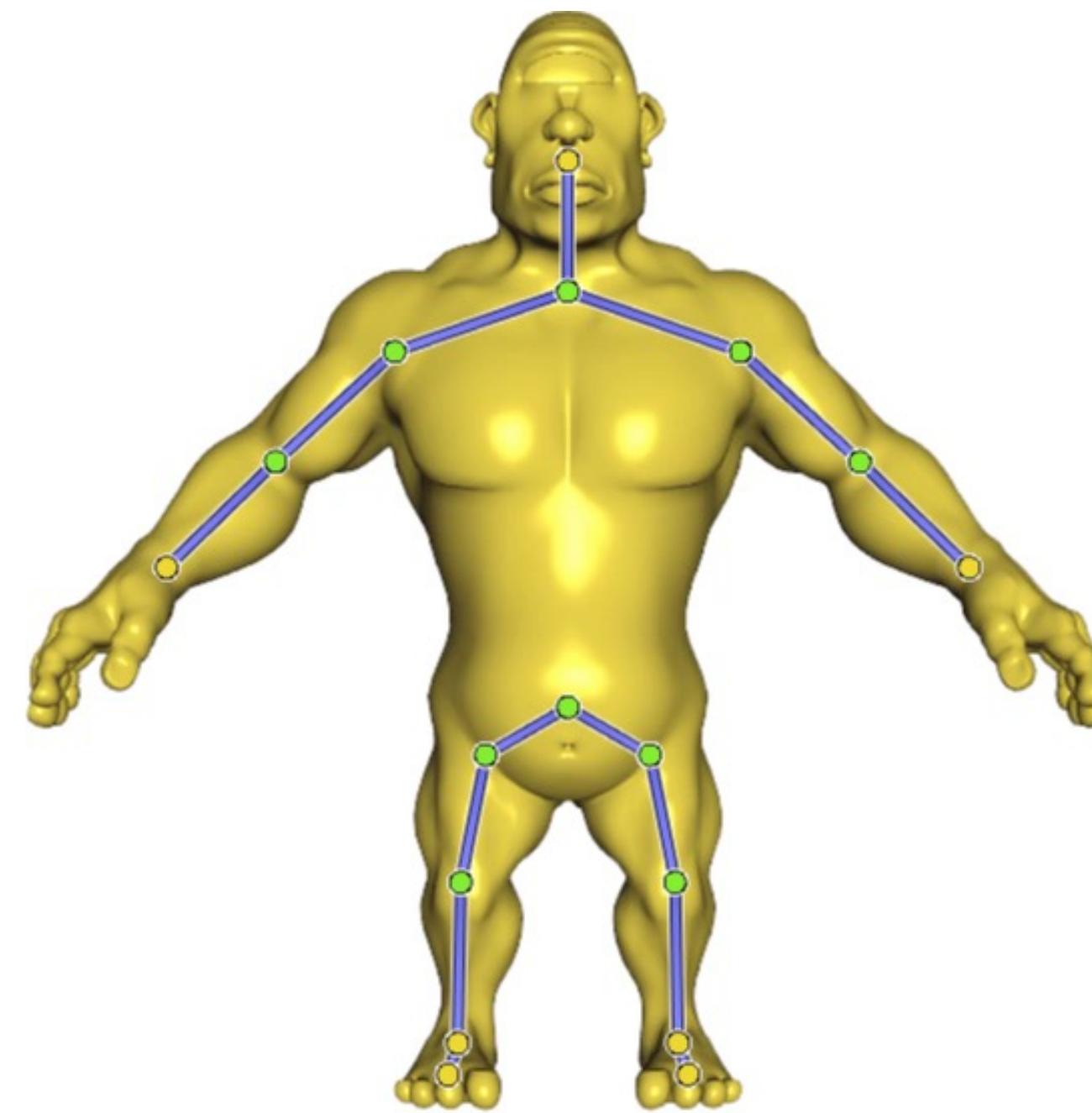
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08 - Linear Blend Skinning

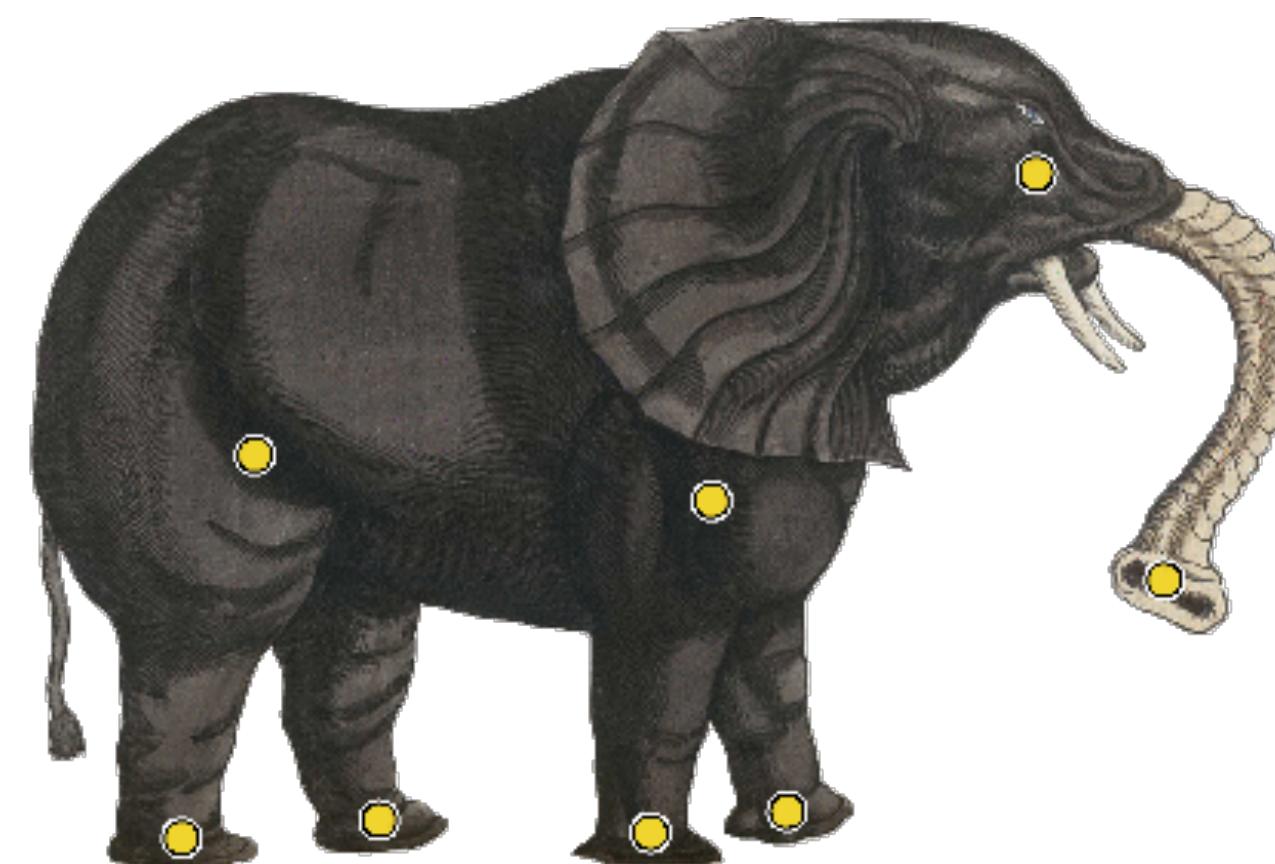
LBS generalizes to different handle types



skeletons



regions

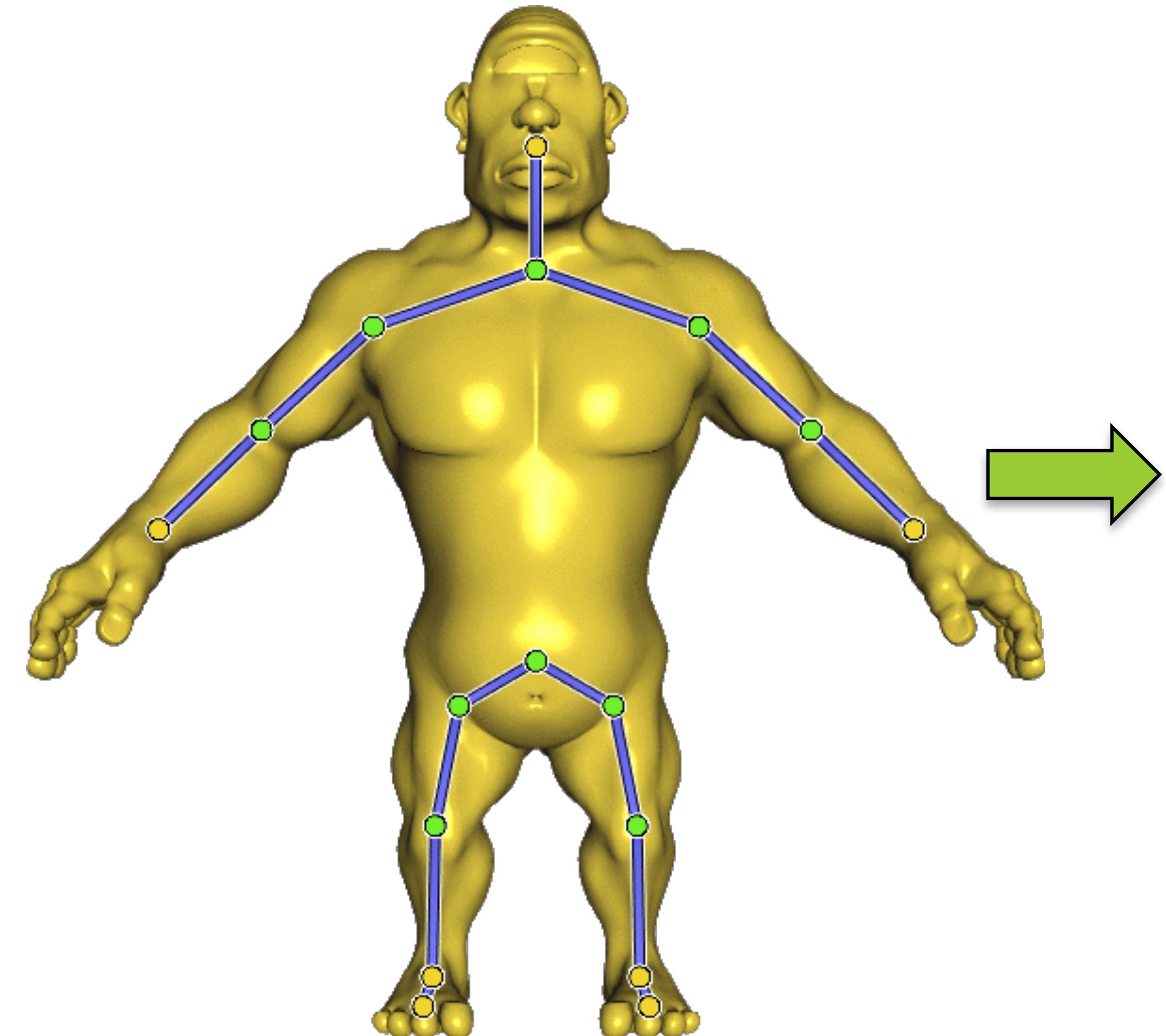


points



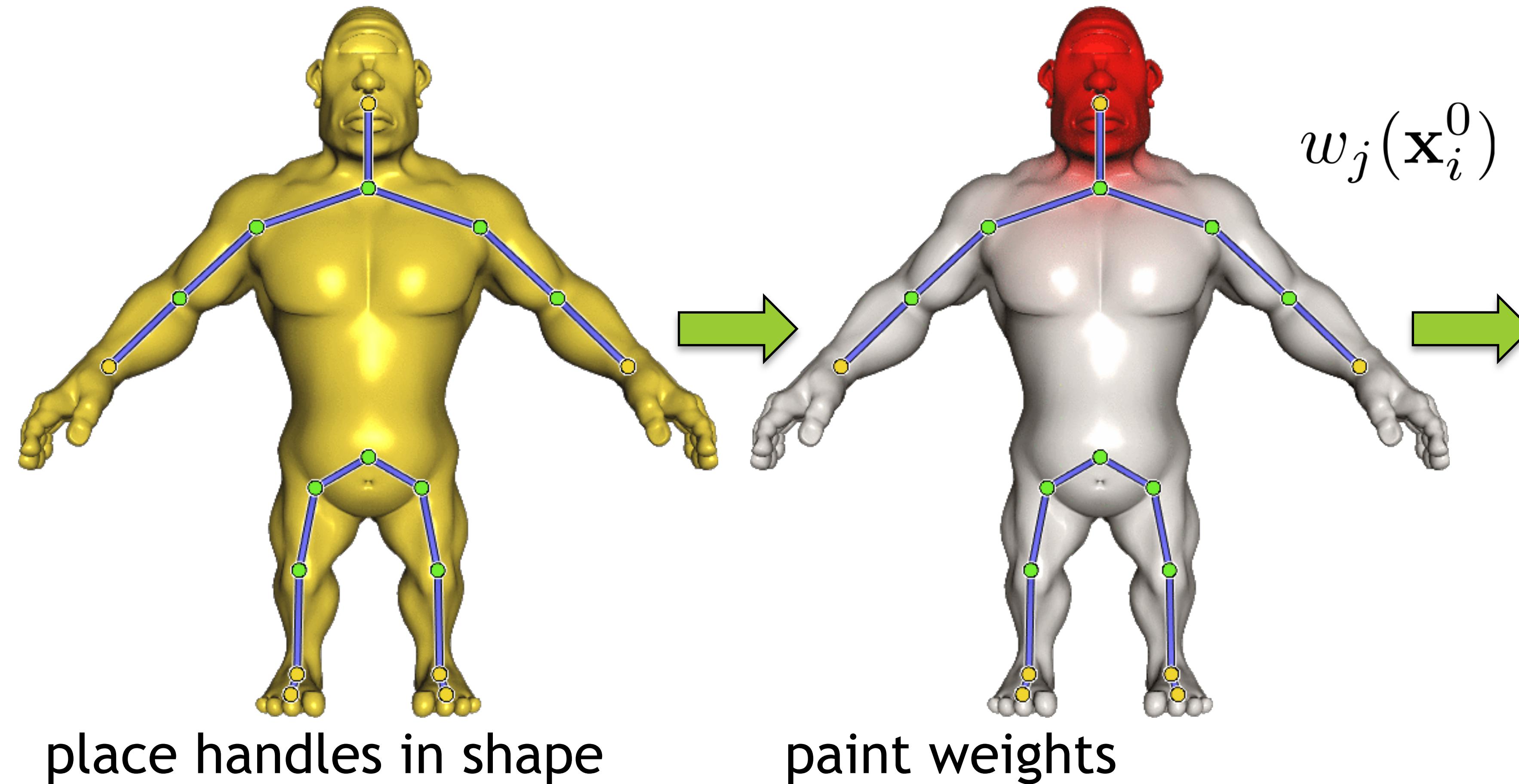
cages

Linear Blend Skinning rigging preferred for its real-time performance

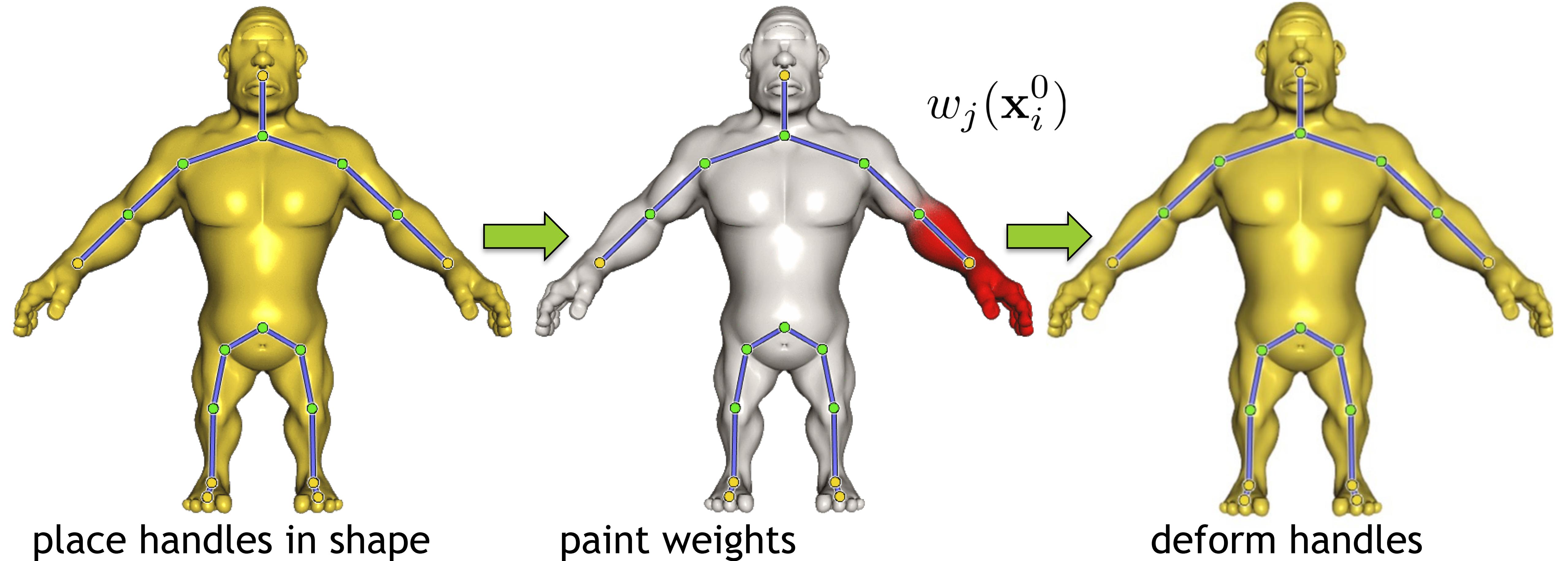


place handles in shape

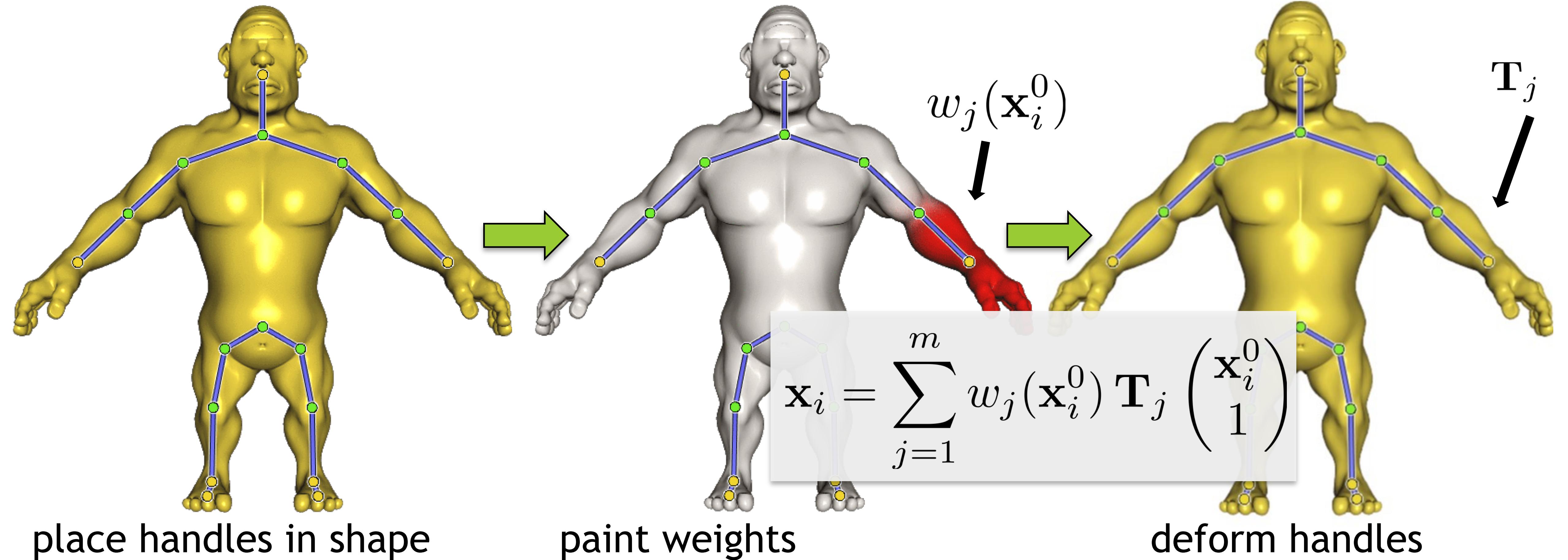
Linear Blend Skinning rigging preferred for its real-time performance



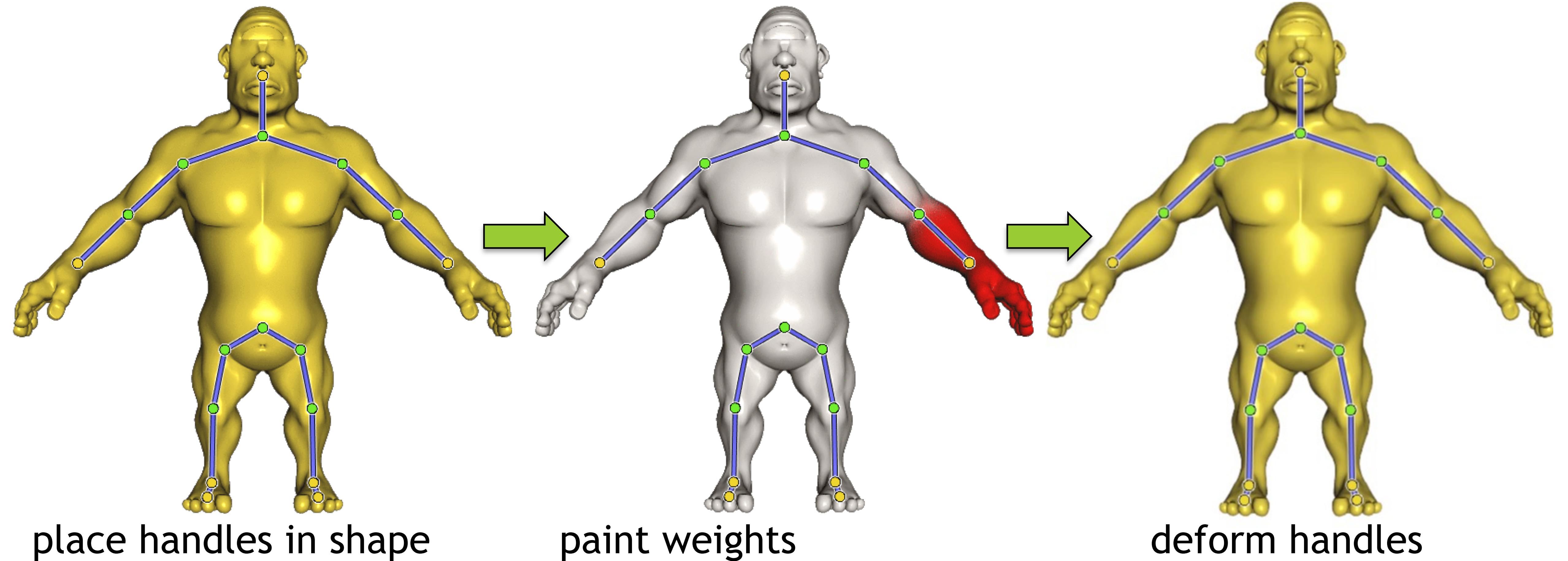
Linear Blend Skinning rigging preferred for its real-time performance



Linear Blend Skinning rigging preferred for its real-time performance



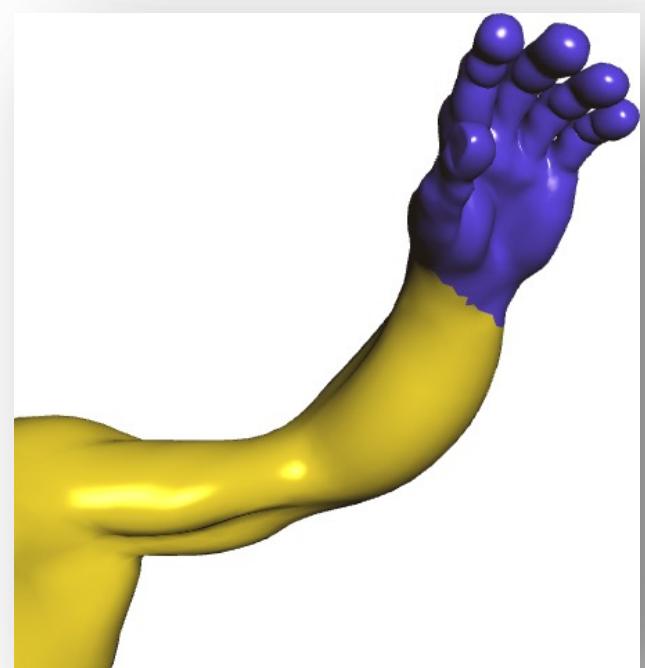
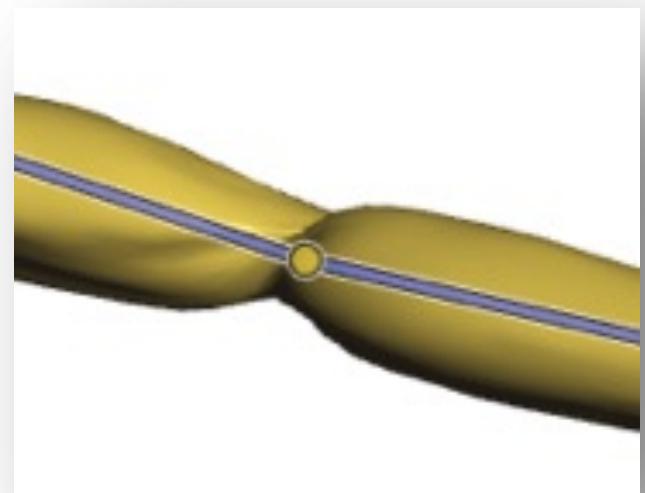
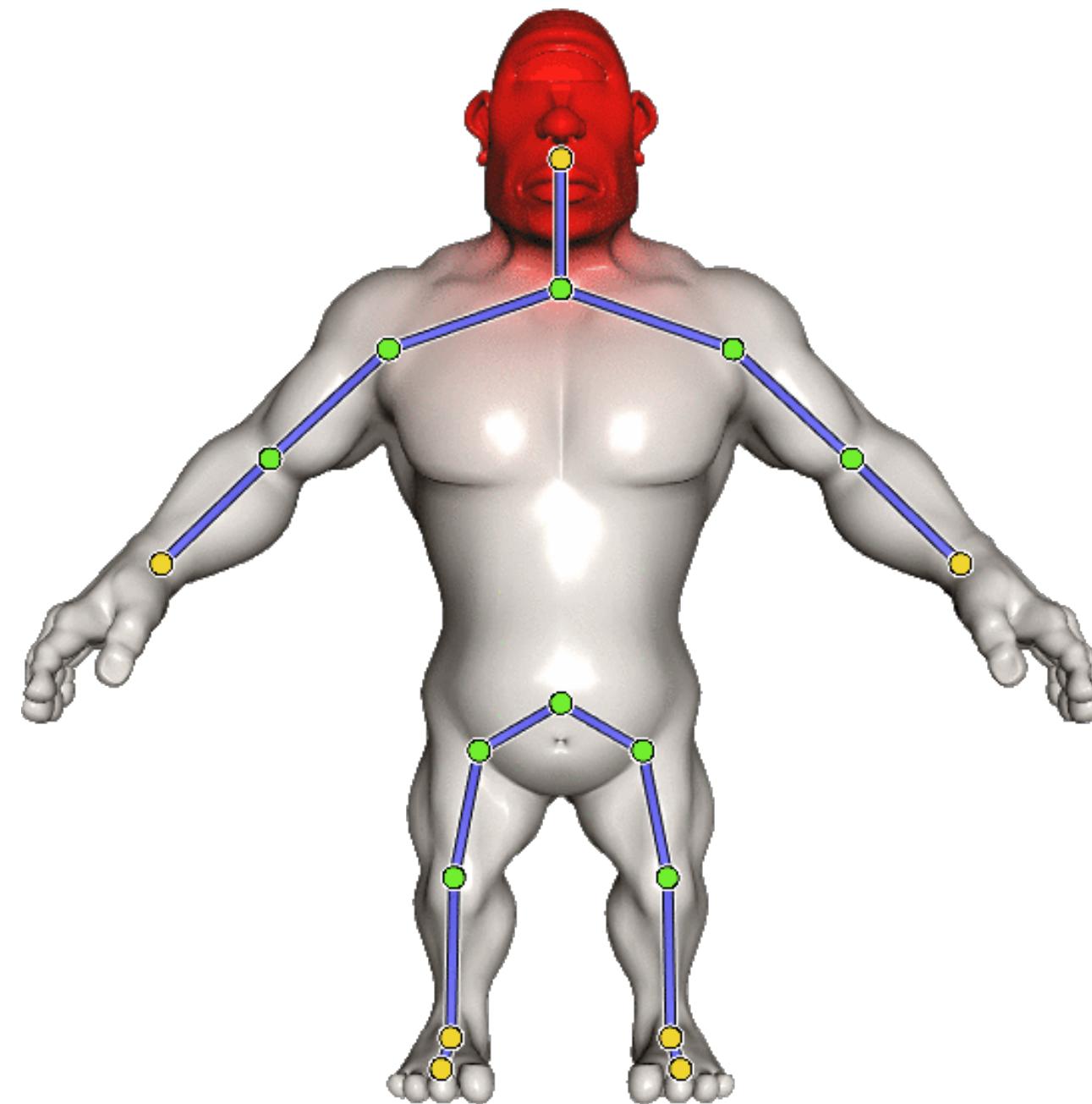
Linear Blend Skinning rigging preferred for its real-time performance



Challenges with LBS

- Weight functions w_j
 - Need intuitive, general and automatic weights
- Degrees of freedom \mathbf{T}_j
 - Let the energy decide!
- Richness of achievable deformations
 - Want to avoid common LBS pitfalls – candy wrapper, collapses

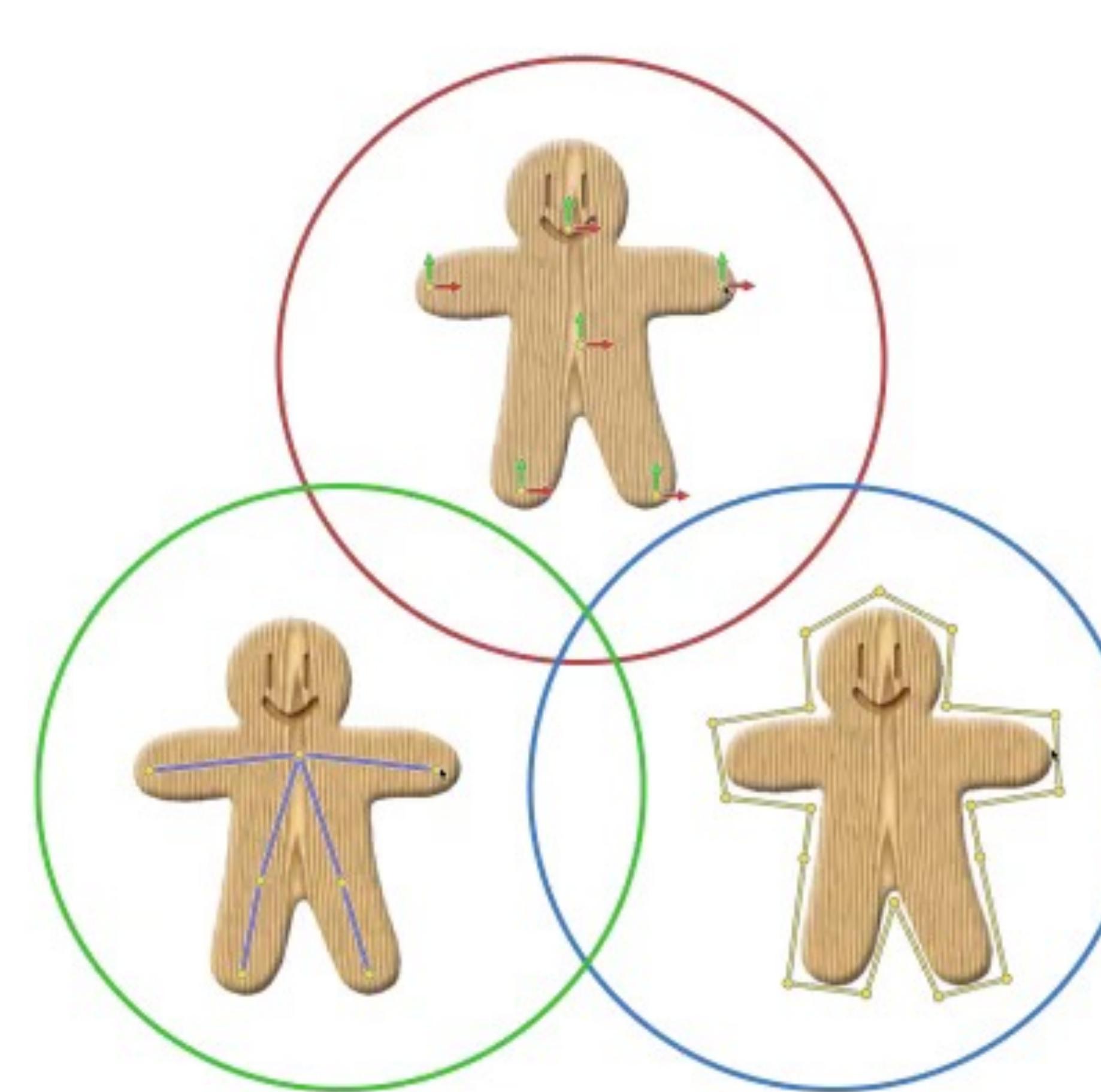
$$\mathbf{x}_i = \sum_{j=1}^m w_j(\mathbf{x}_i^0) \mathbf{T}_j \begin{pmatrix} \mathbf{x}_i^0 \\ 1 \end{pmatrix}$$



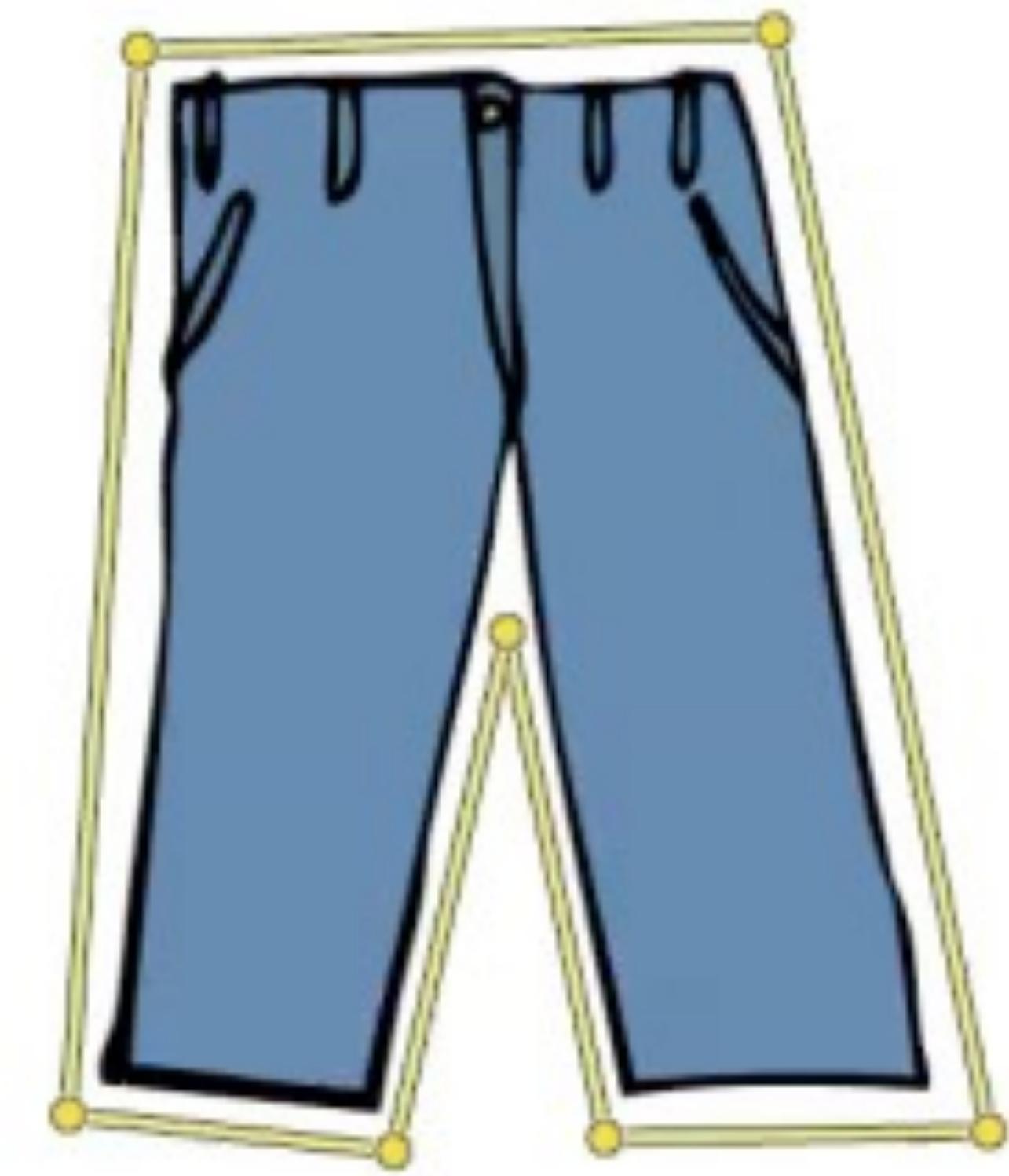
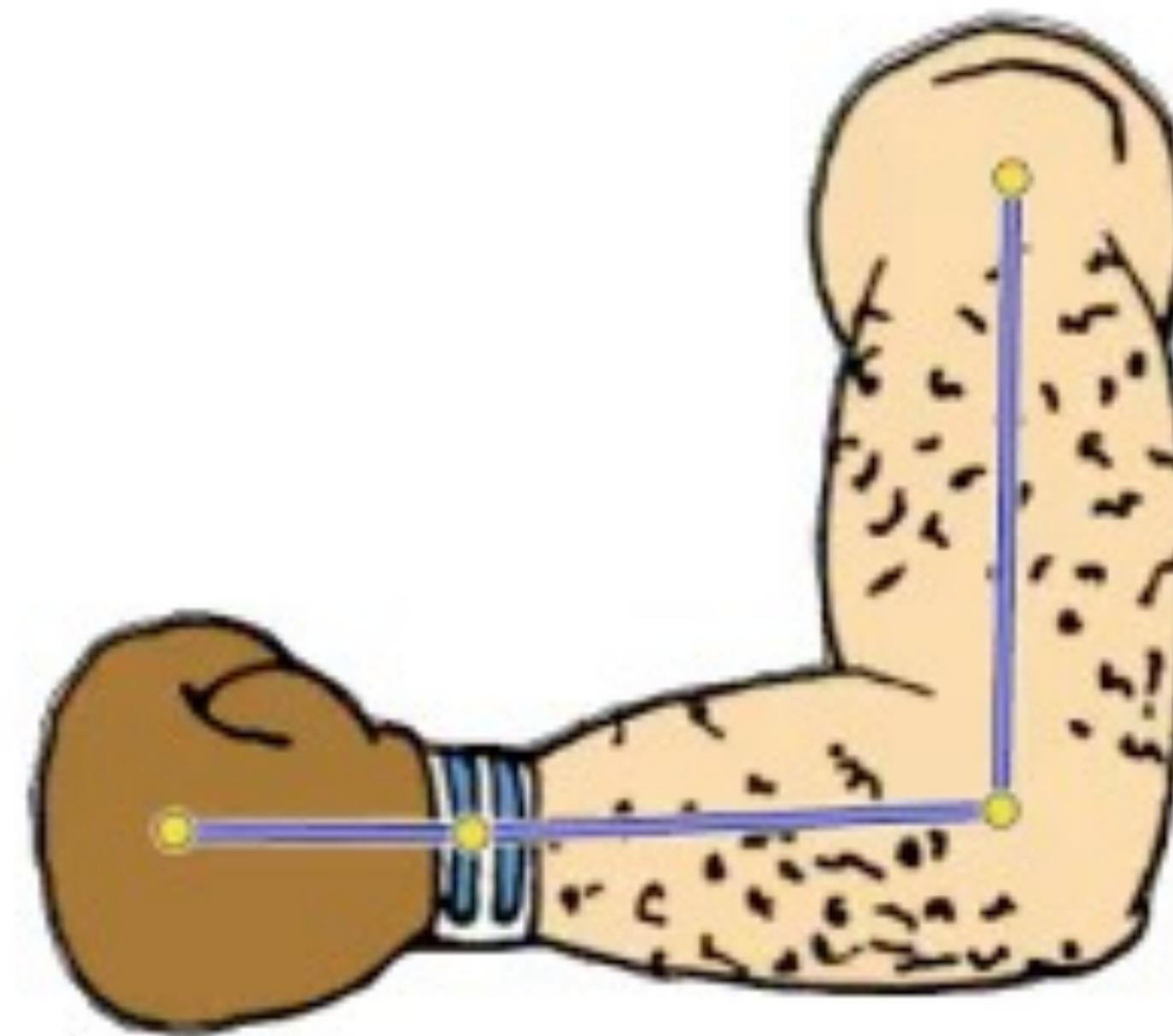
Bounded Biharmonic Weights

Alec Jacobson, Ilya Baran, Jovan Popović, Olga Sorkine-Hornung

Automatic weights that
unify points, skeletons and cages



Weights should be smooth,
shape-aware, positive and *intuitive*



Weights must be smooth everywhere, *especially* at handles

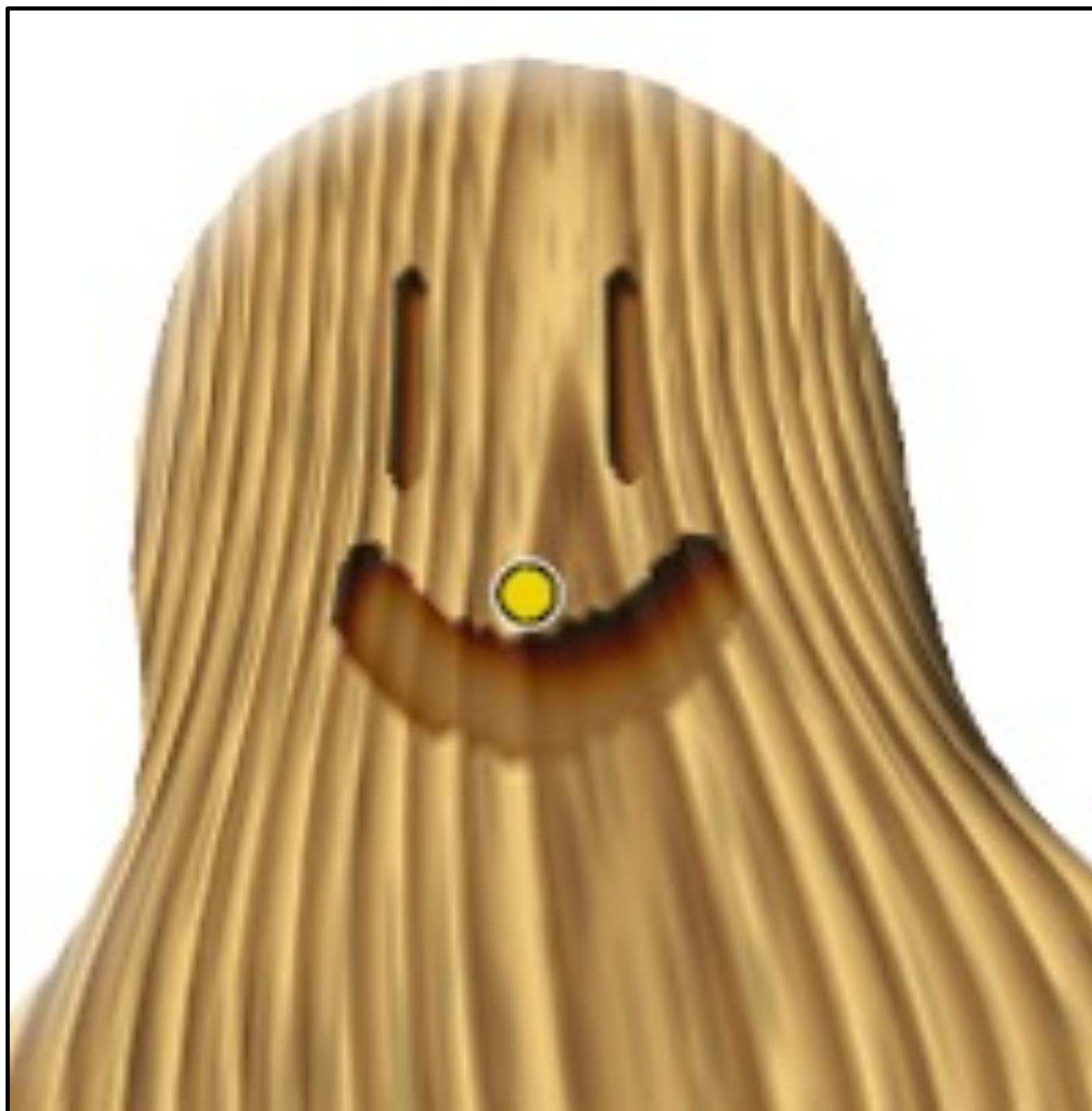


Bounded Biharmonic Weights

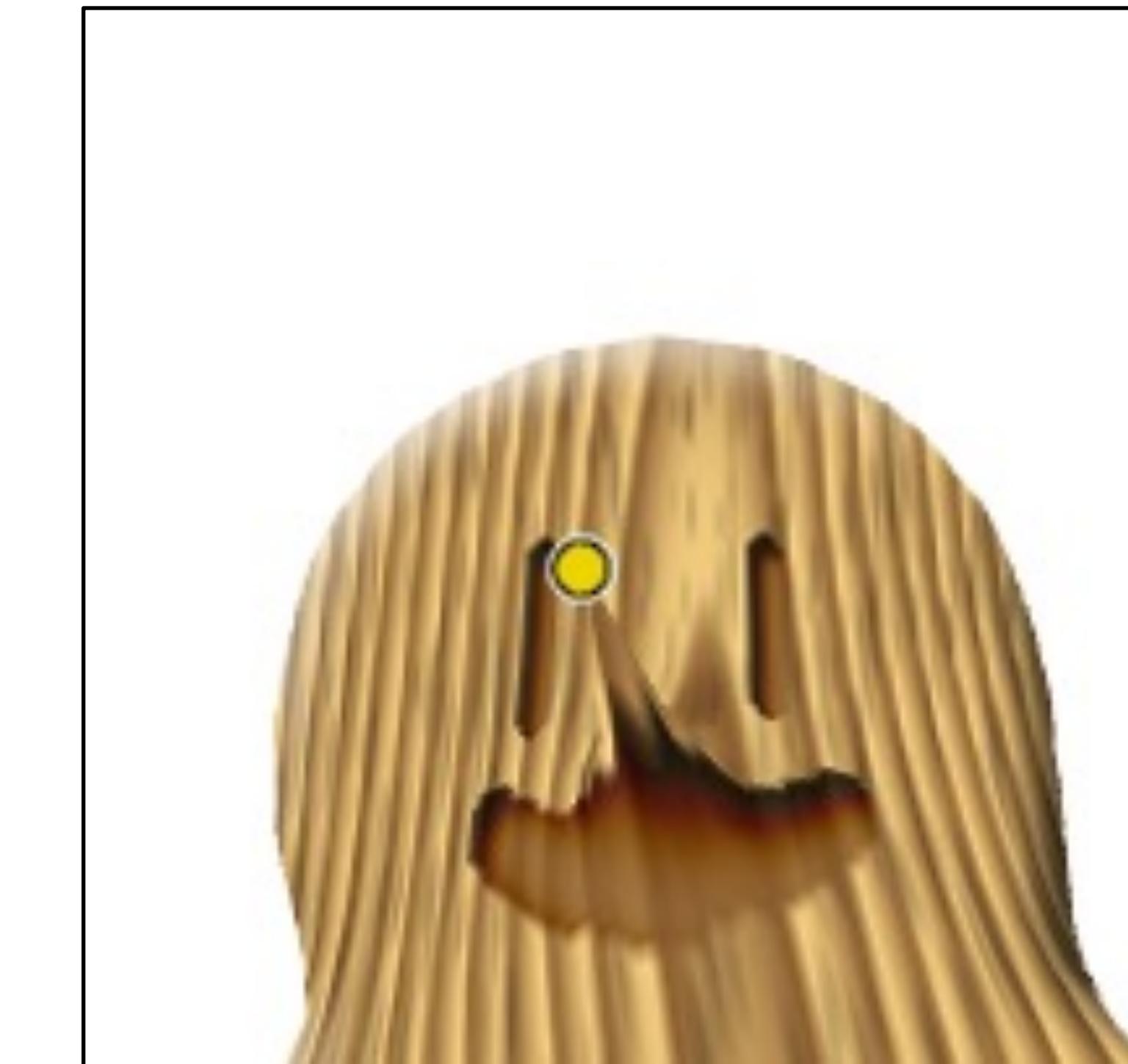


Extension of Harmonic Coordinates
[Joshi et al. 2005]

Weights must be smooth everywhere, *especially* at handles



Bounded Biharmonic Weights

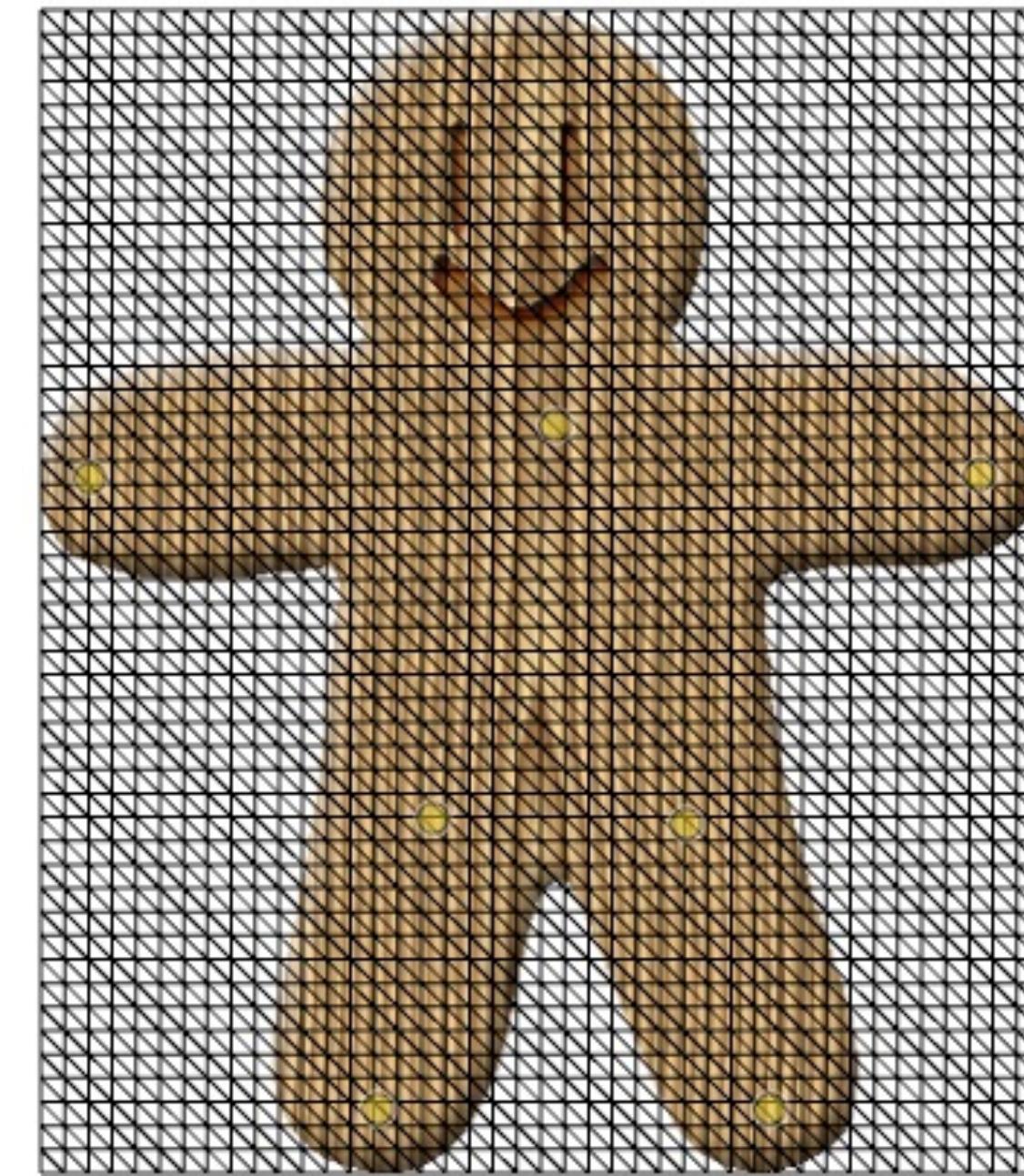


Extension of Harmonic Coordinates
[Joshi et al. 2005]

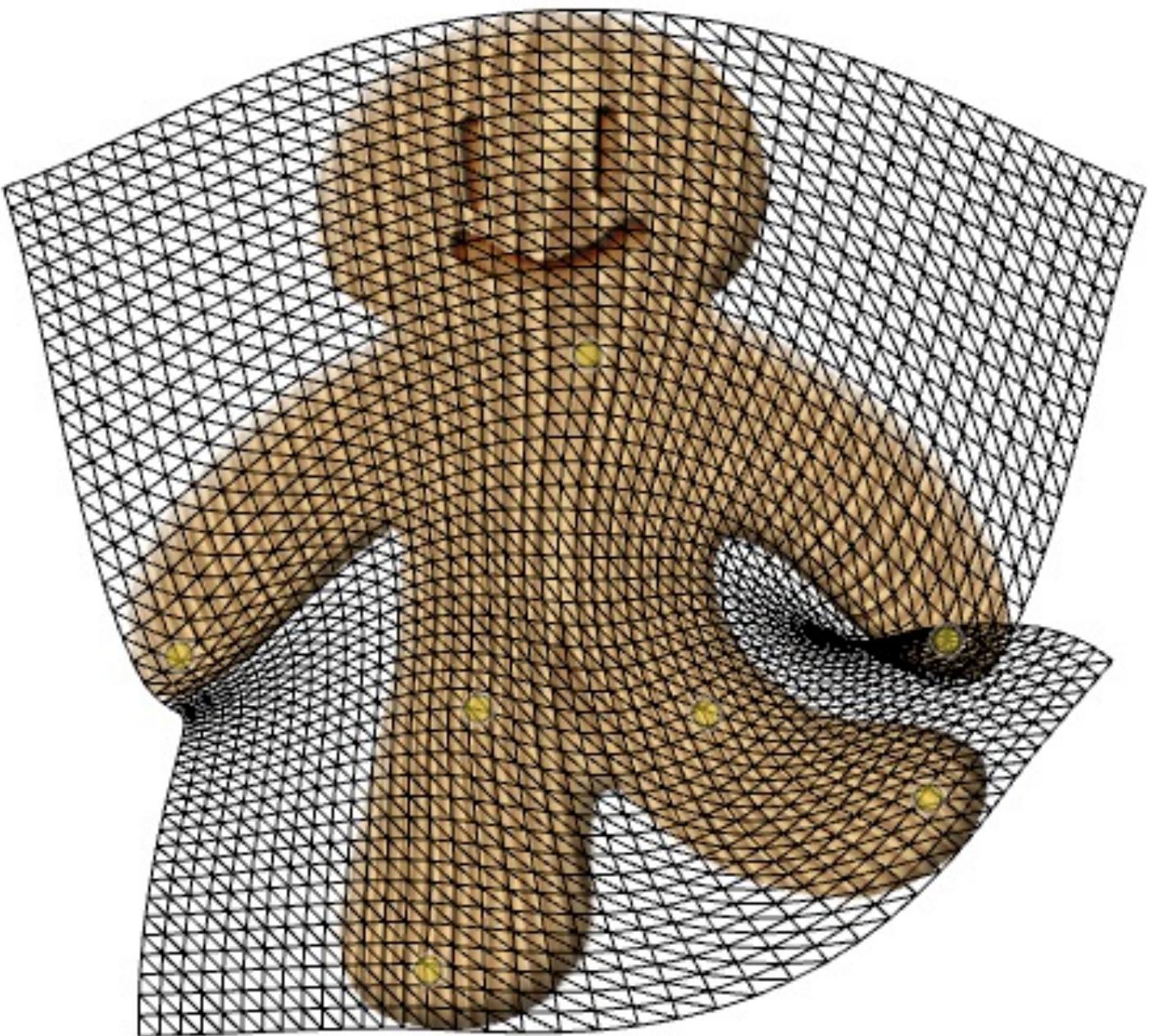
Shape-awareness ensures respect of domain's features



Bounded Biharmonic Weights

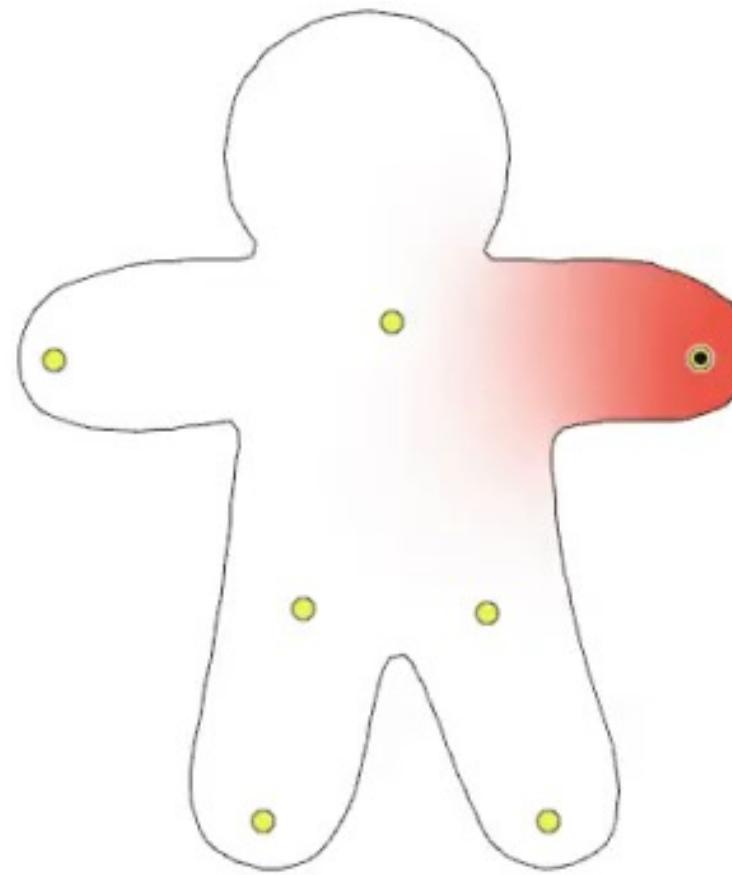


Non-shape-aware methods
e.g. [Schaefer et al. 2006]

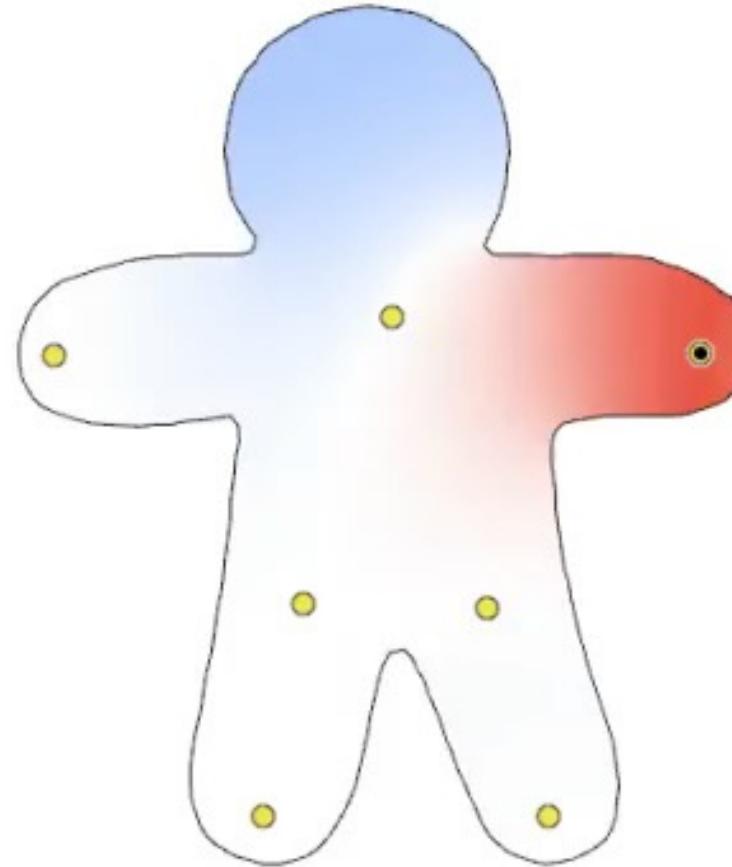


Non-negative weights are necessary for intuitive response

Bounded Biharmonic Weights



Unconstrained biharmonic
[Botsch and Kobbelt 2004]



Weights must maintain other simple, but important properties

$$\sum_{j \in H} w_j(\mathbf{x}^0) = 1$$

Partition of unity

Handle vertices

$$w_j \Big|_{H_k} = \delta_{jk}$$

w_j is linear along cage faces

Interpolation of handles

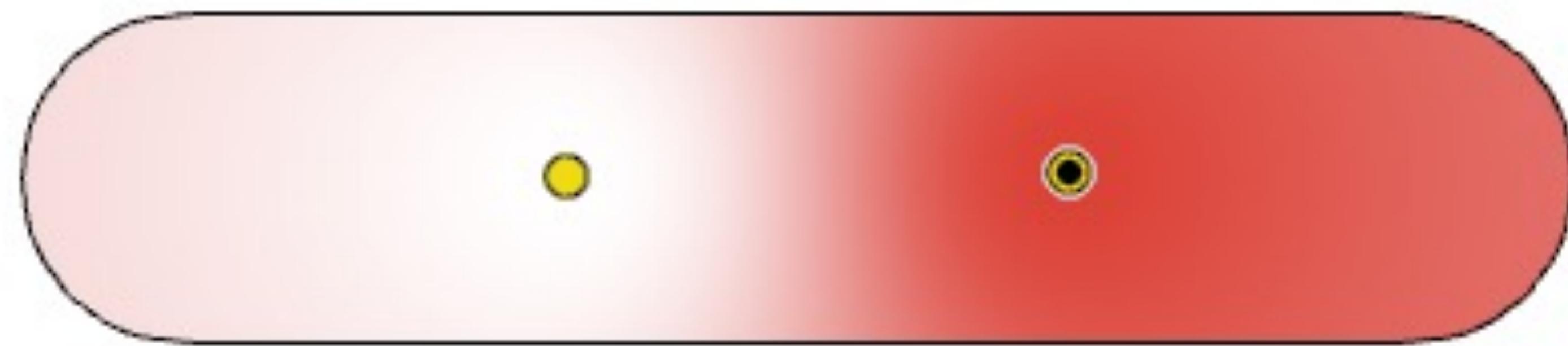
How about $w_j(\mathbf{x}^0) = d(\mathbf{x}^0, H_j)^{-1}$?



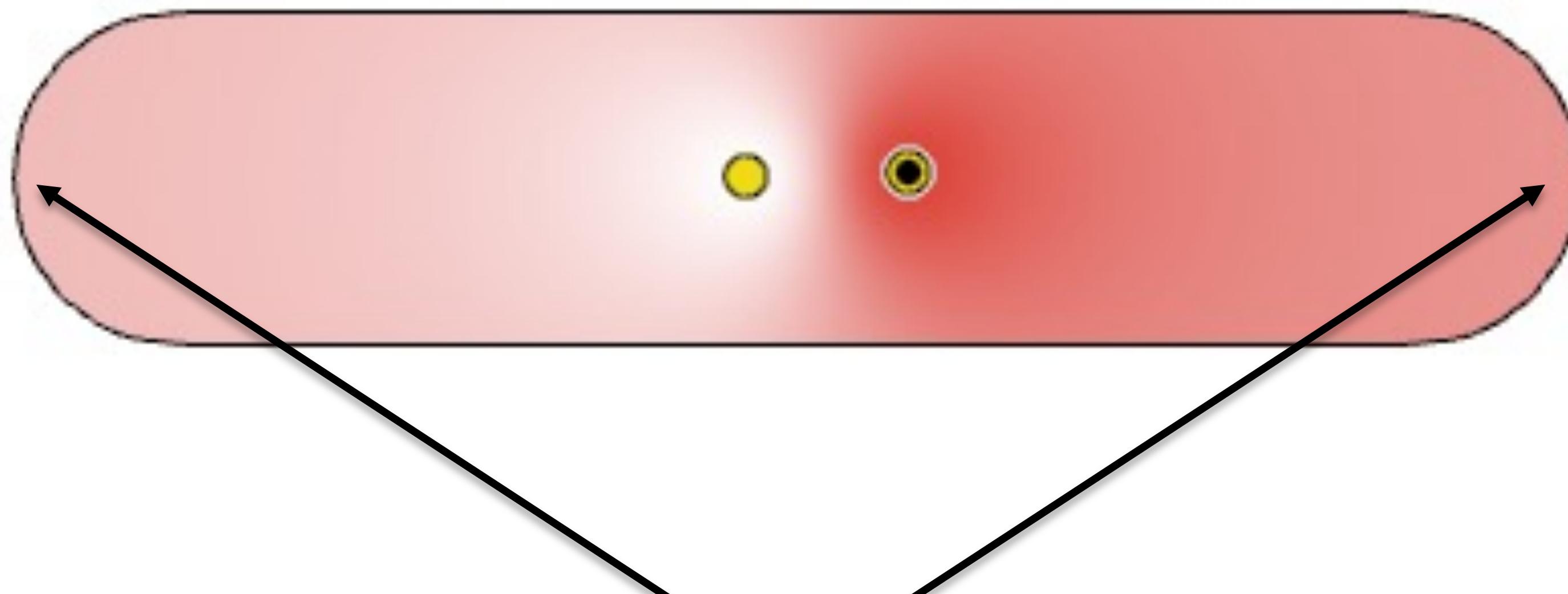
Inverse distance methods inherently suffer from *fall-off effect*



Inverse distance methods inherently suffer from *fall-off effect*



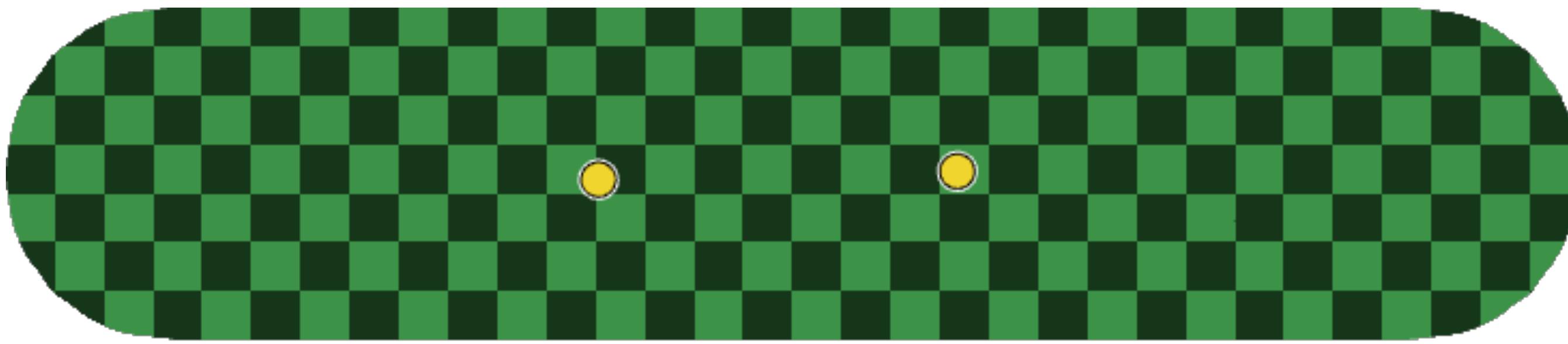
Inverse distance methods inherently suffer from *fall-off effect*



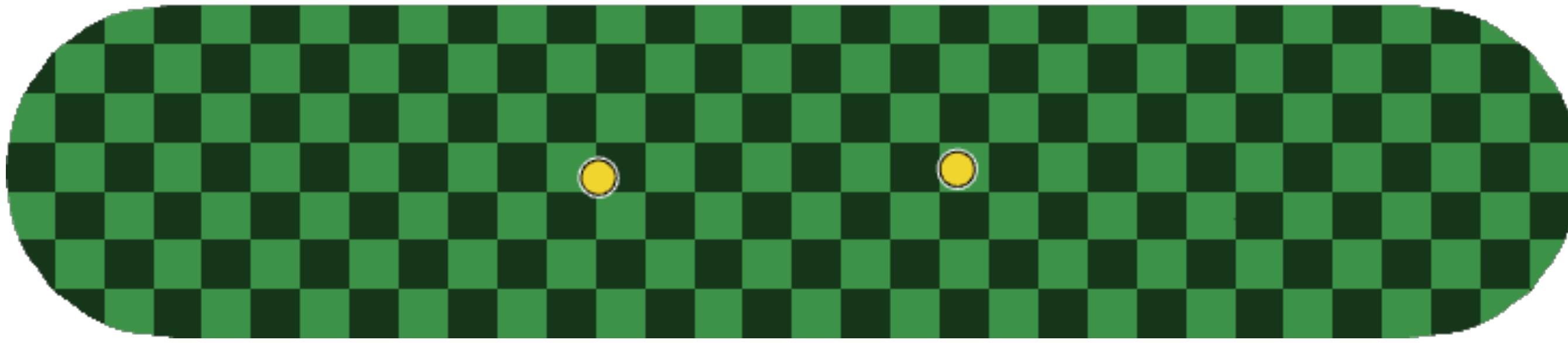
Approaching 0.5

Inverse distance methods inherently suffer from *fall-off effect*

Inverse-distance weights



BBW



Bounded biharmonic weights enforce properties
as constraints to minimization

$$\arg \min_{w_j} \frac{1}{2} \int_{\Omega} |\Delta w_j|^2 dV$$

$$w_j \Big|_{H_k} = \delta_{jk}$$

w_j is linear along cage faces

Bounded biharmonic weights enforce properties
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$$w_j \Big|_{H_k} = \delta_{jk}$$

w_j is linear along cage faces

Constant inequality constraints

$$0 \leq w_j(\mathbf{x}^0) \leq 1$$

Partition of unity

$$\sum_{j \in H} w_j(\mathbf{x}^0) = 1$$

Bounded biharmonic weights enforce properties as constraints to minimization

$$\arg \min_{w_j} \frac{1}{2} \int_{\Omega} |\Delta w_j|^2 dV$$

$$w_j \Big|_{H_k} = \delta_{jk}$$

w_j is linear along cage faces

Constant inequality constraints

$$0 \leq w_j(\mathbf{x}^0) \leq 1$$

Solve independently and normalize

$$w_j(\mathbf{x}^0) = \frac{w_j(\mathbf{x}^0)}{\sum_{i \in H} w_i(\mathbf{x}^0)}$$

Weights optimized as precomputation at bind-time

$$\arg \min_{w_j} \frac{1}{2} \int_{\Omega} |\Delta w_j|^2 dV$$

$$w_j|_{H_k} = \delta_{jk}$$

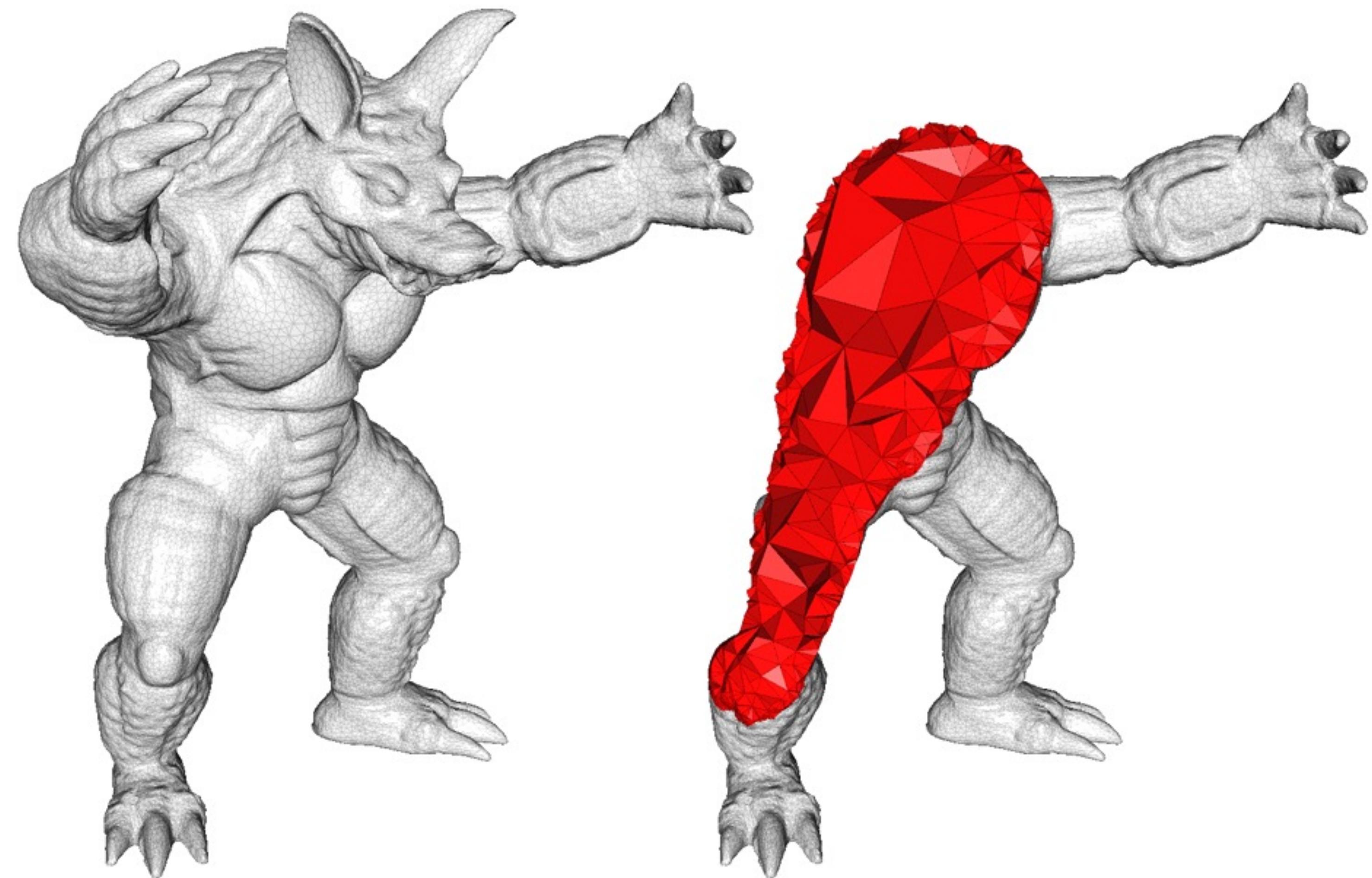
w_j is linear along cage faces

$$0 \leq w_j(\mathbf{x}^0) \leq 1$$

FEM discretization

2D → Triangle mesh

3D → Tet mesh



Weights optimized as precomputation at bind-time

$$\arg \min_{w_j} \frac{1}{2} \int_{\Omega} |\Delta w_j|^2 dV$$

$$w_j|_{H_k} = \delta_{jk}$$

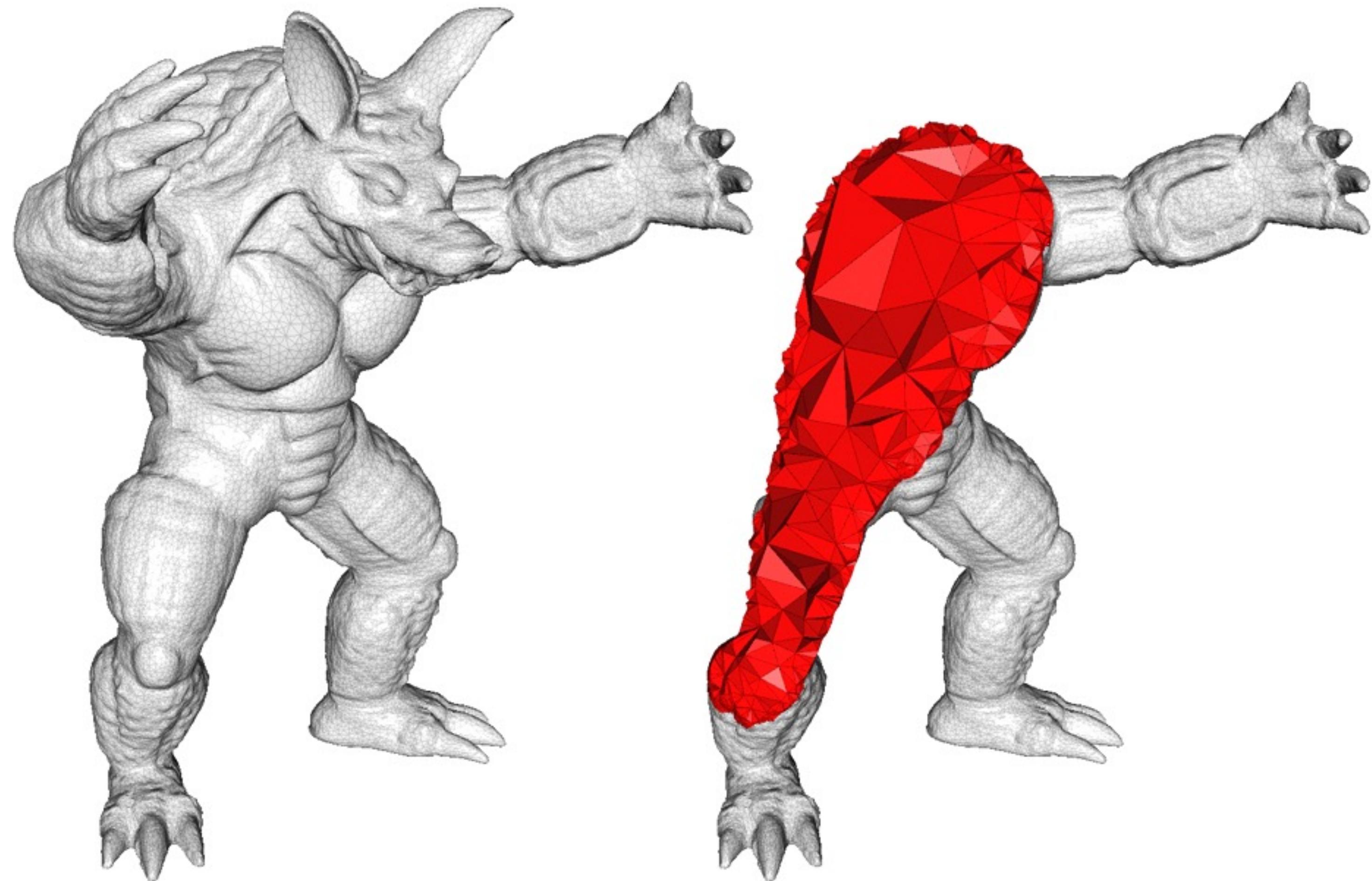
w_j is linear along cage faces

$$0 \leq w_j(\mathbf{x}^0) \leq 1$$

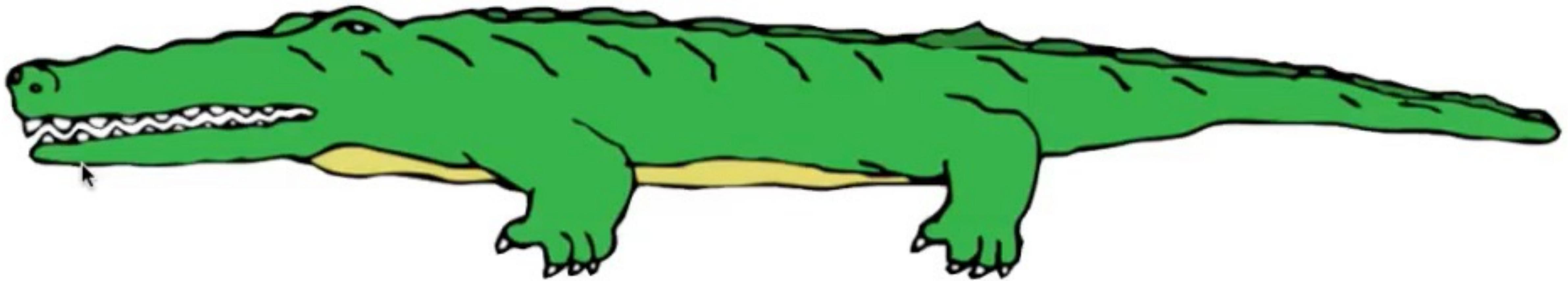
Sparse quadratic programming with
constant inequality constraints

2D → less than second per handle

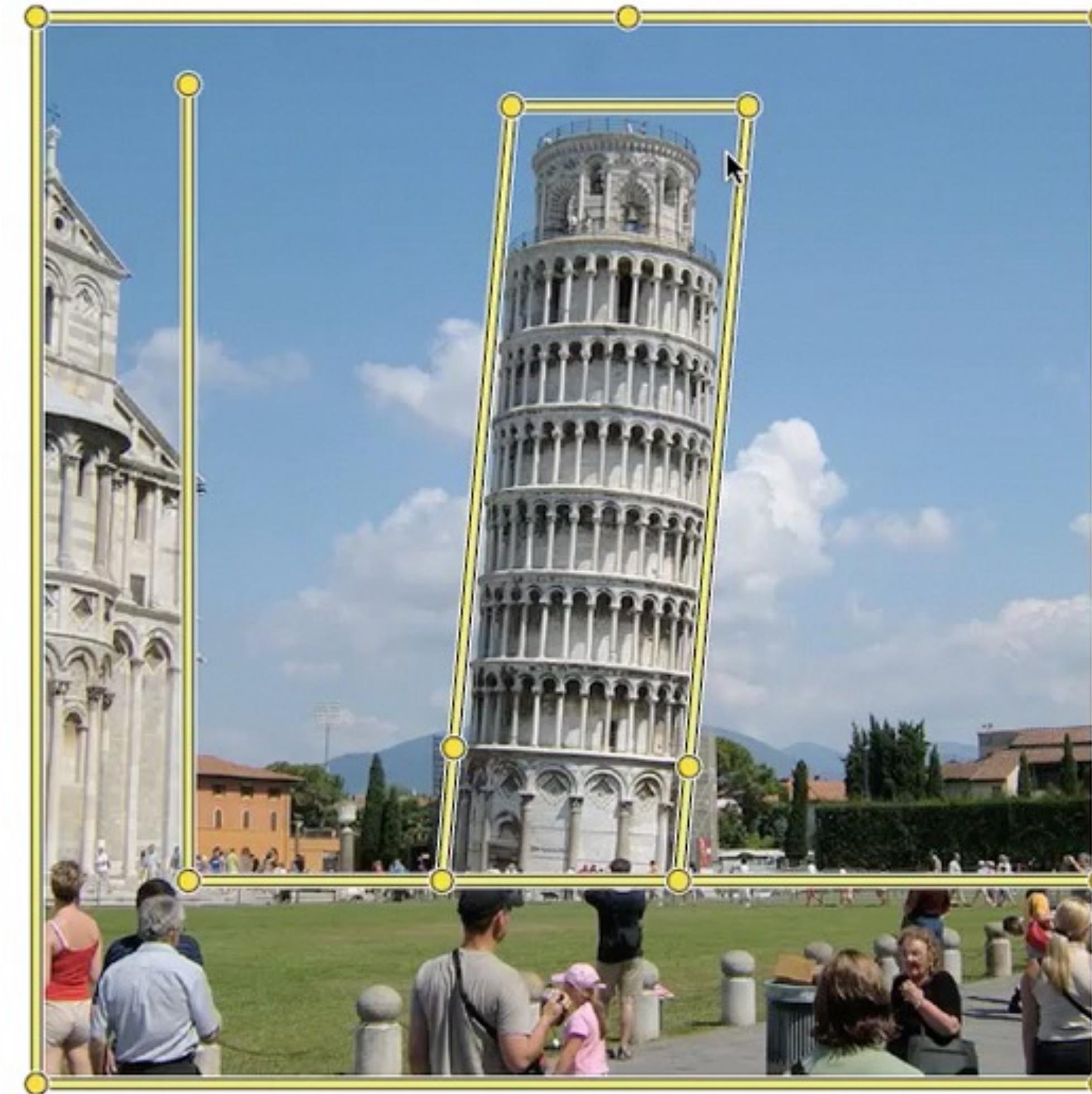
3D → tens of seconds per handle



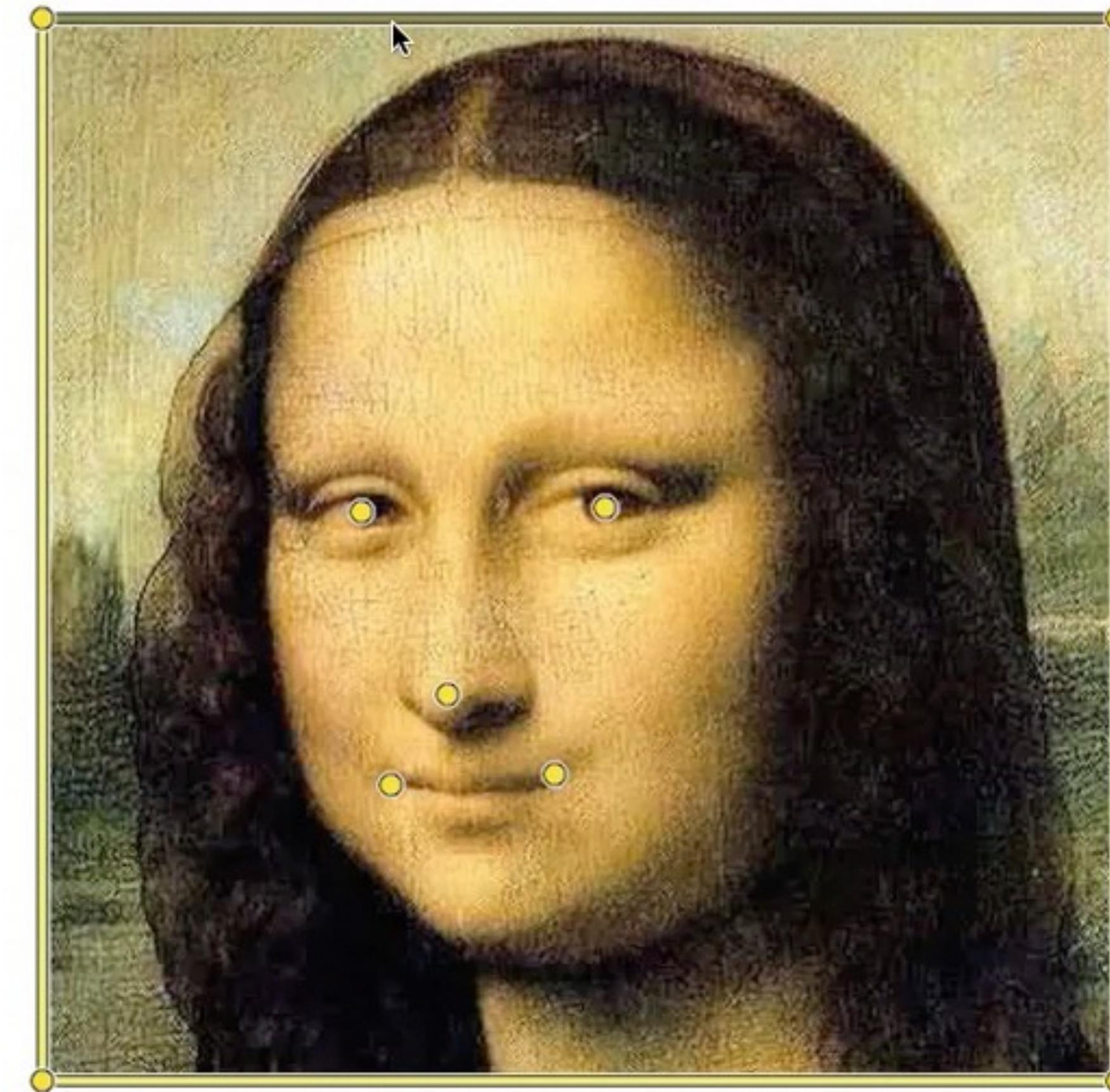
Some examples of BBW in action



Some examples of BBW in action



Some examples of BBW in action



3D Characters



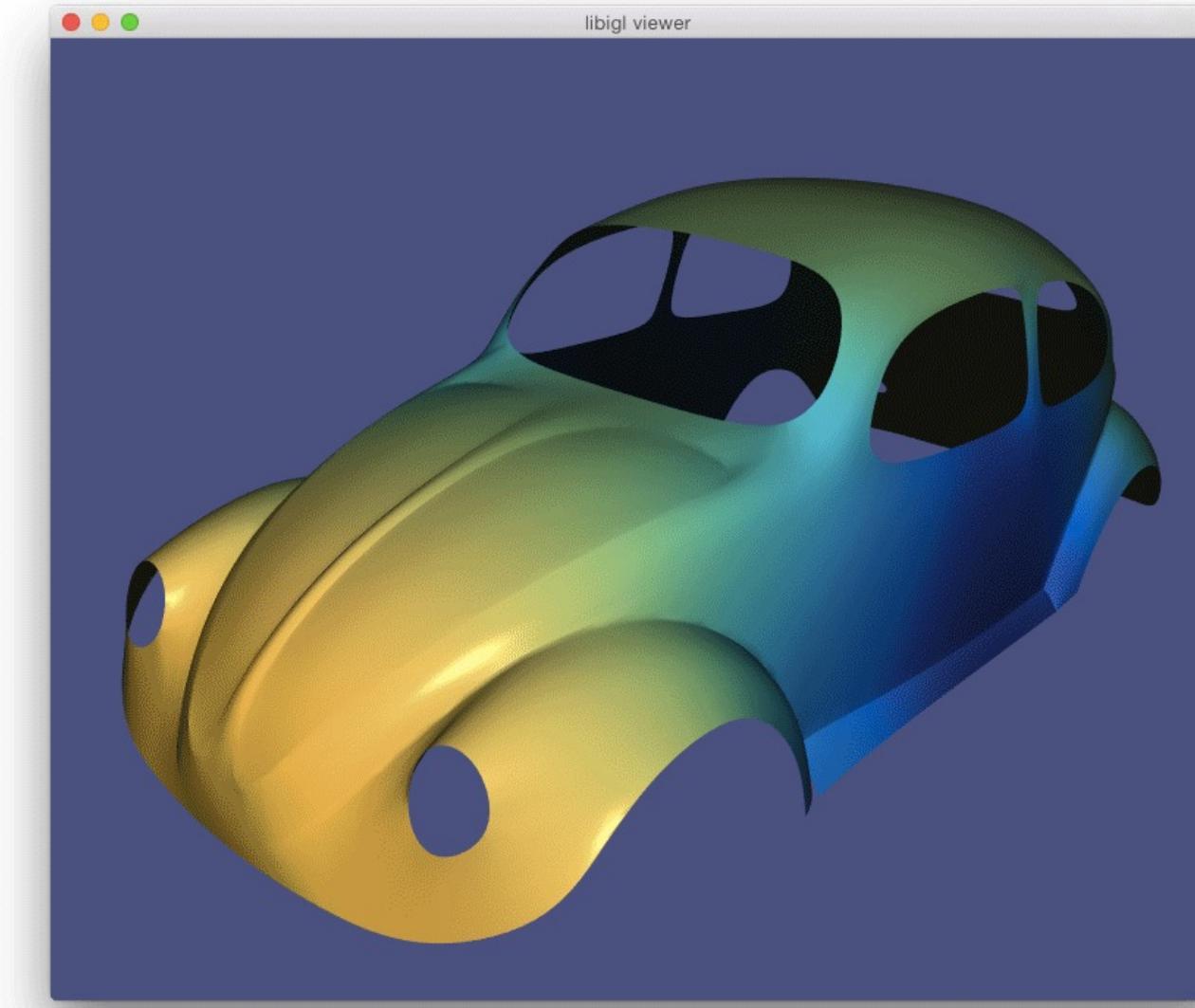
Demo

- Libigl demo 403

08 - Summary

Geometric Modeling and Processing

- The shape of an object is an important characteristic (not the only one...)
- Geometry processing: computerized modeling of 2D/3D geometry

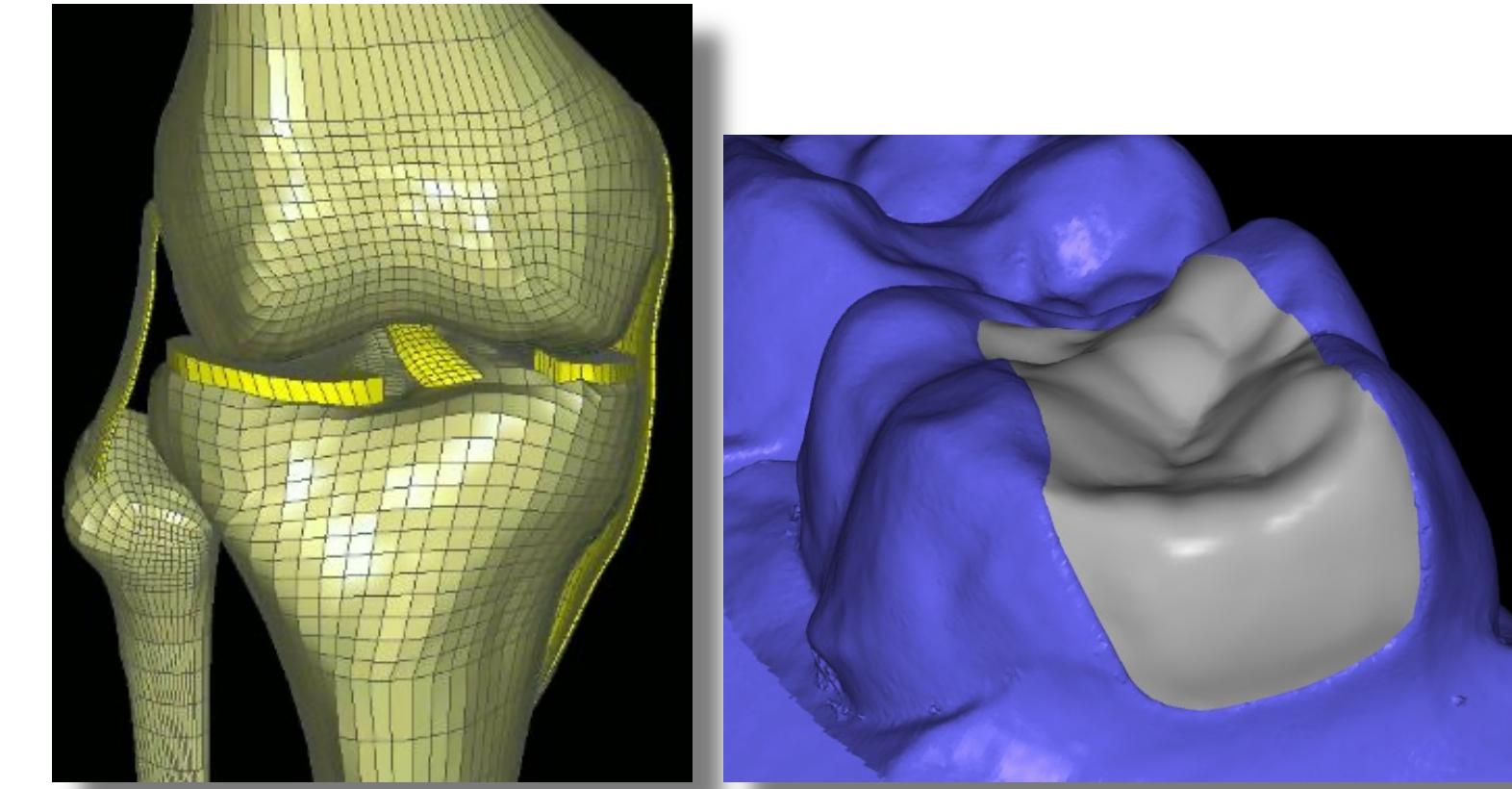


Copyright: Blender

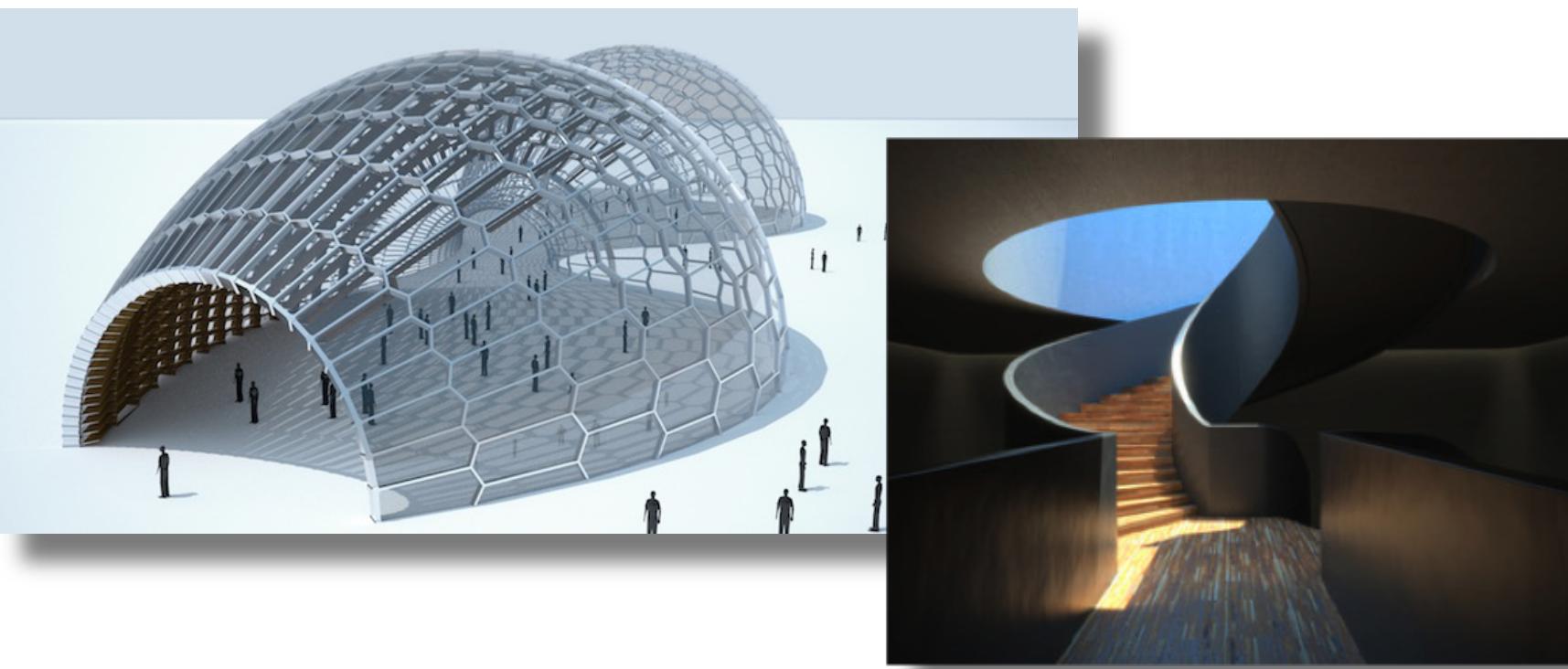
Applications



Product design and prototyping



Medicine, prosthetics



Architecture



Cultural heritage

C++

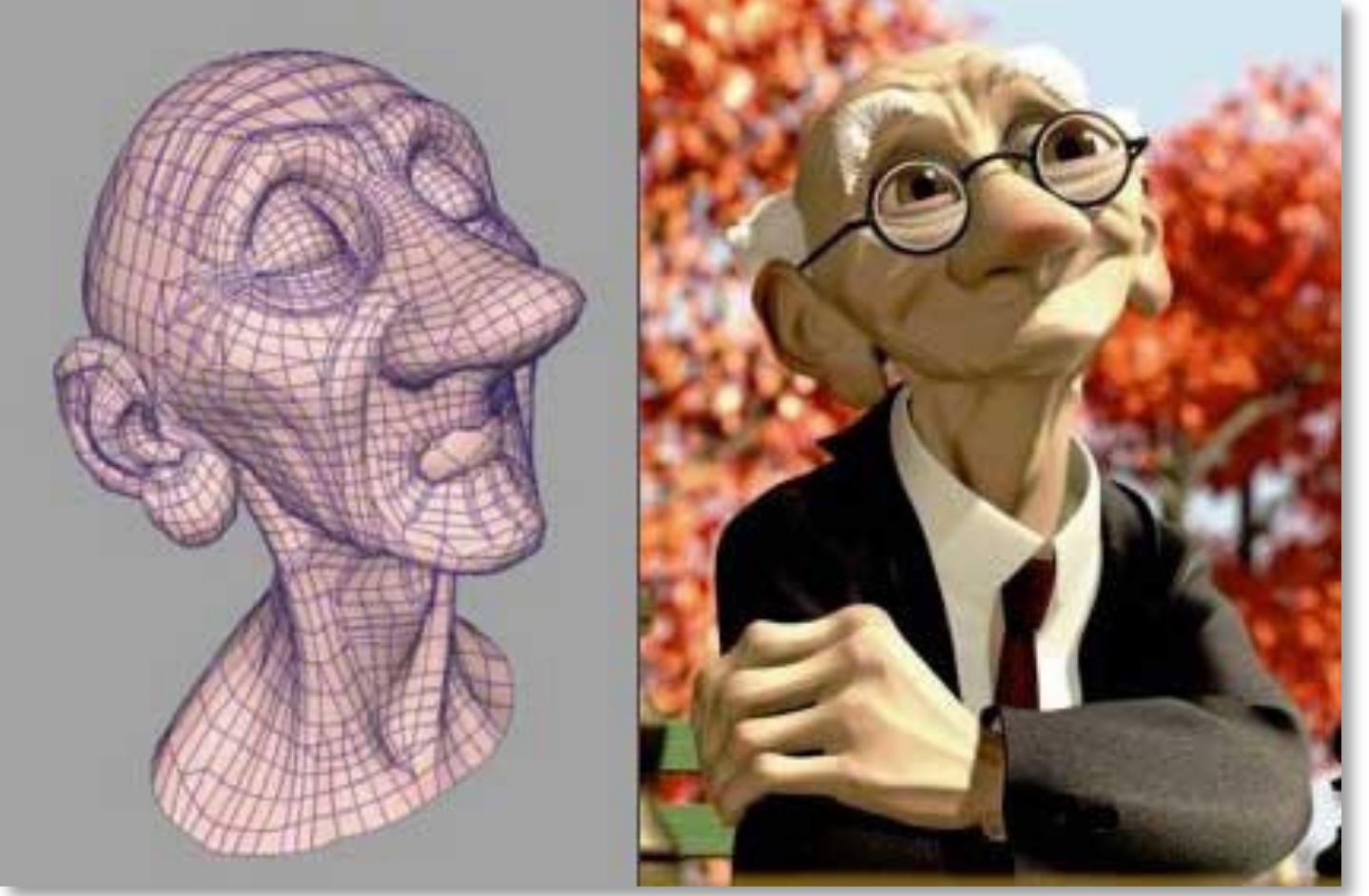
- It is flexible, and many of the features are optional:
 - it can be used as an extension of C, with no objects
 - it can be used as a fully object oriented language
 - it has many advanced features such as “templates” that are useful to write efficient code that *is also readable*

CMAKE

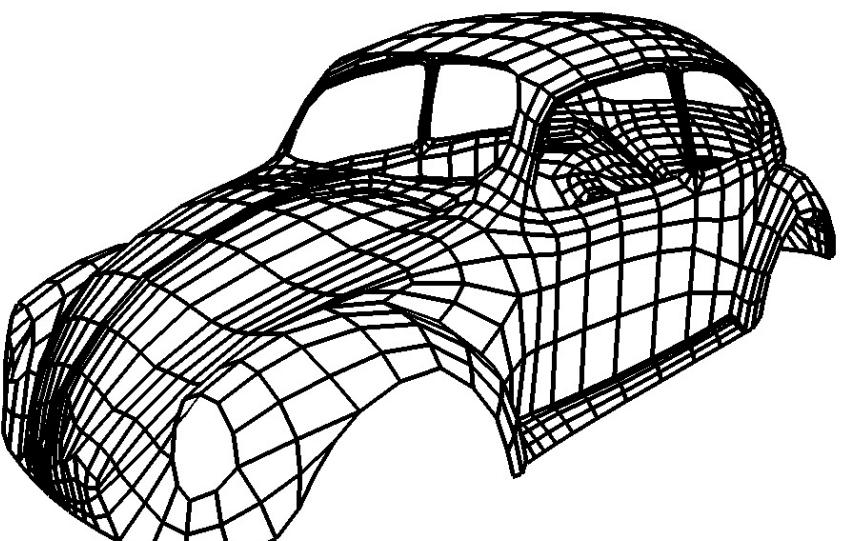
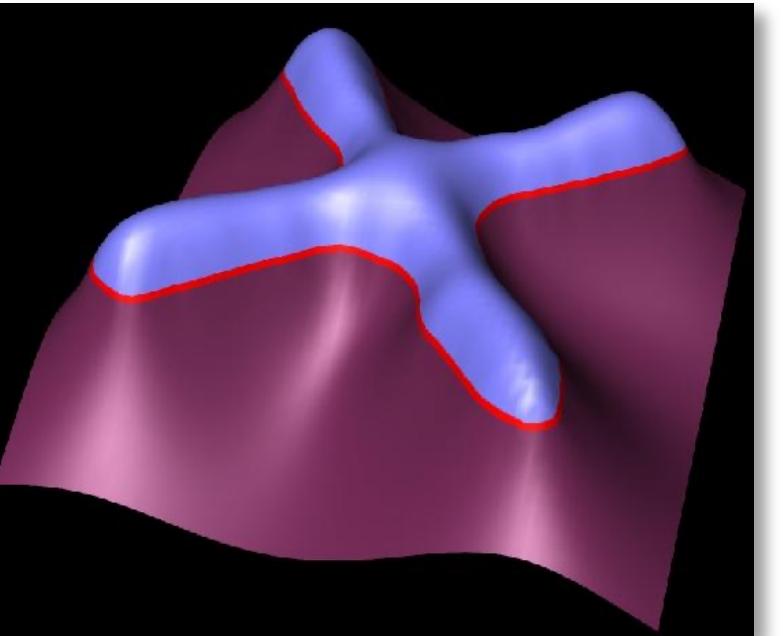
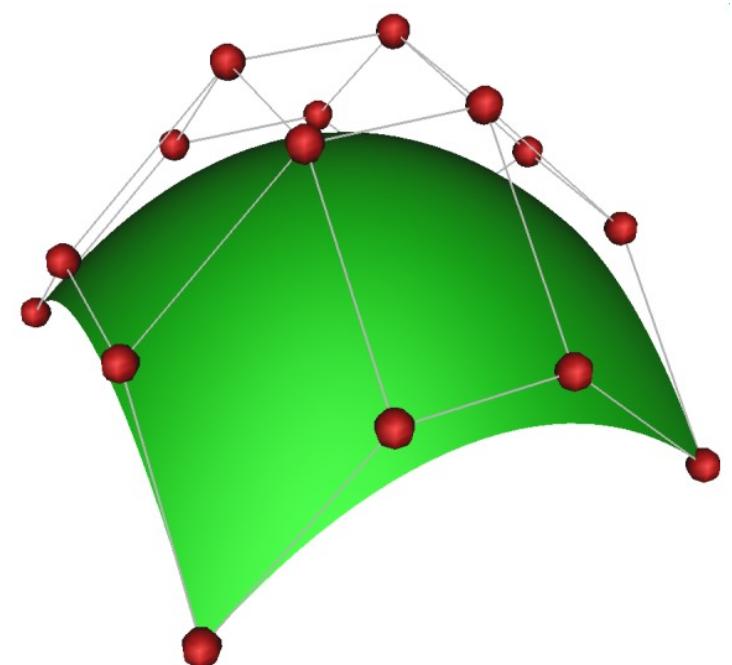
- If your project is in a folder `Assignment_1`, you need to:
 - `mkdir build; cd build`
 - `cmake ..`
- This will create the project. To compile it:
 - `make` (macosx/linux)
 - Open the project with visual studio on windows
- As an alternative, you can use Clion/Xcode/VS code that does all of this for you!

Course Topics

- Overview of shape representations
 - Parametric curves/surfaces
 - Implicit
 - Polygonal meshes

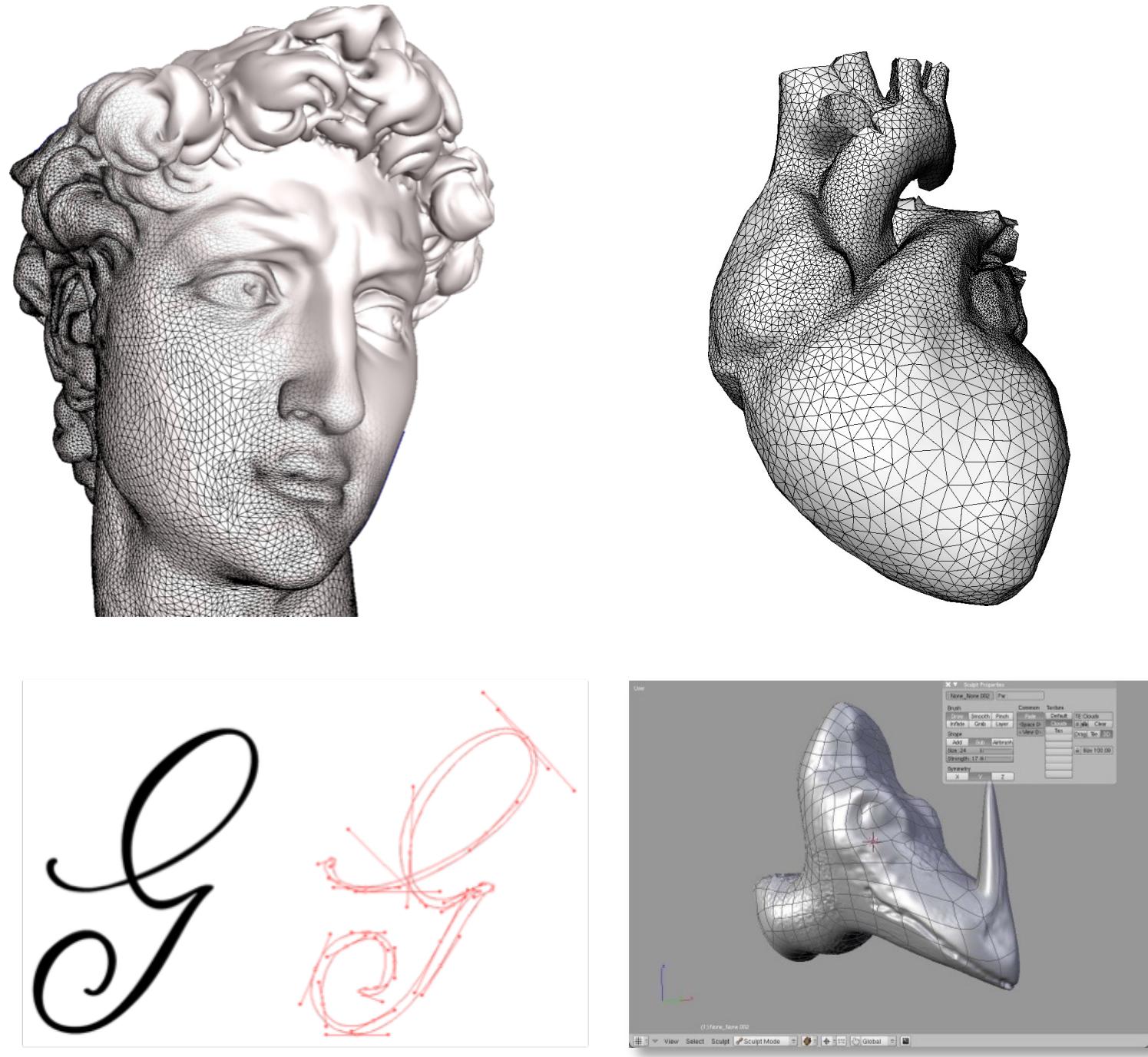


Pixar



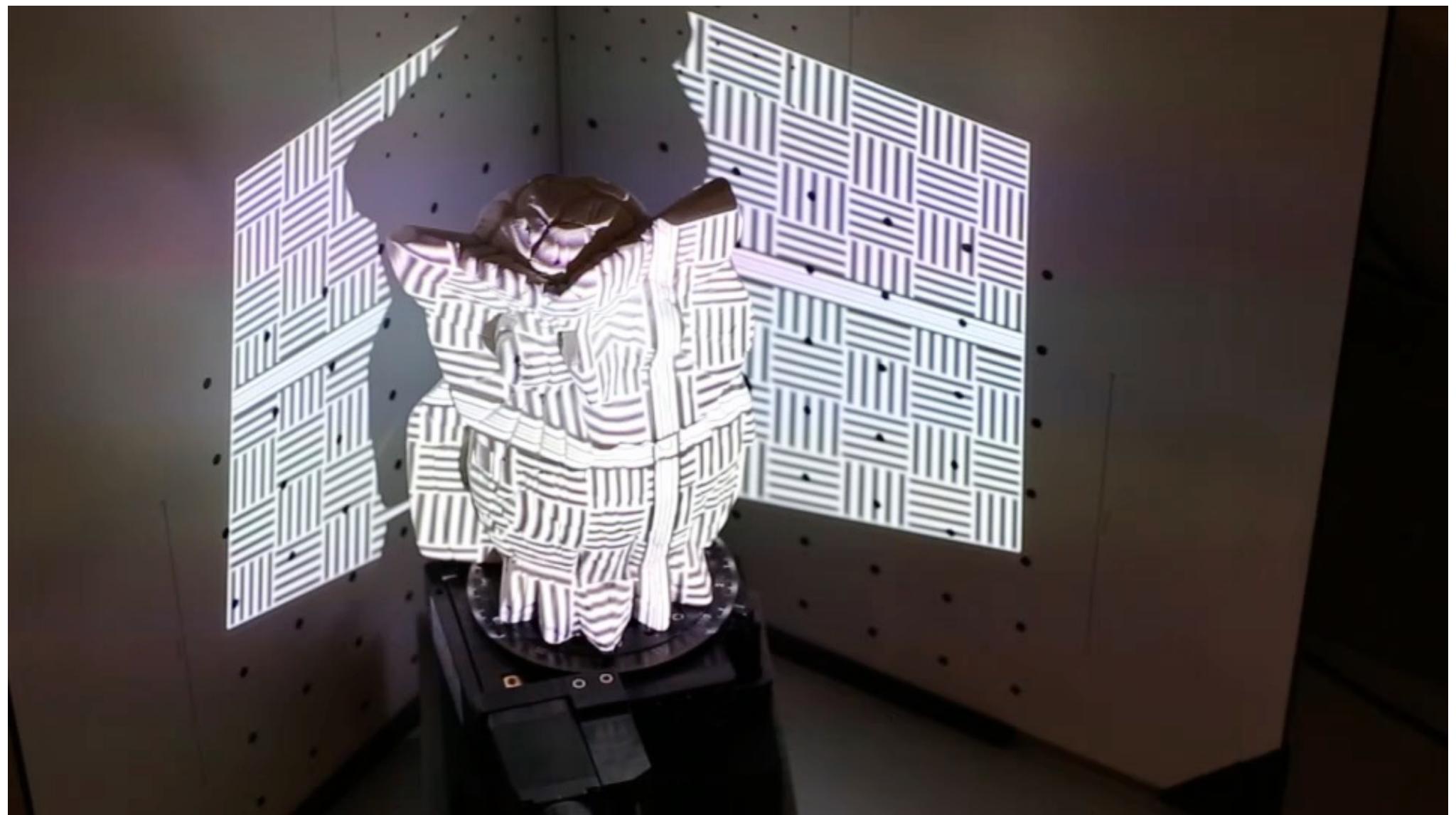
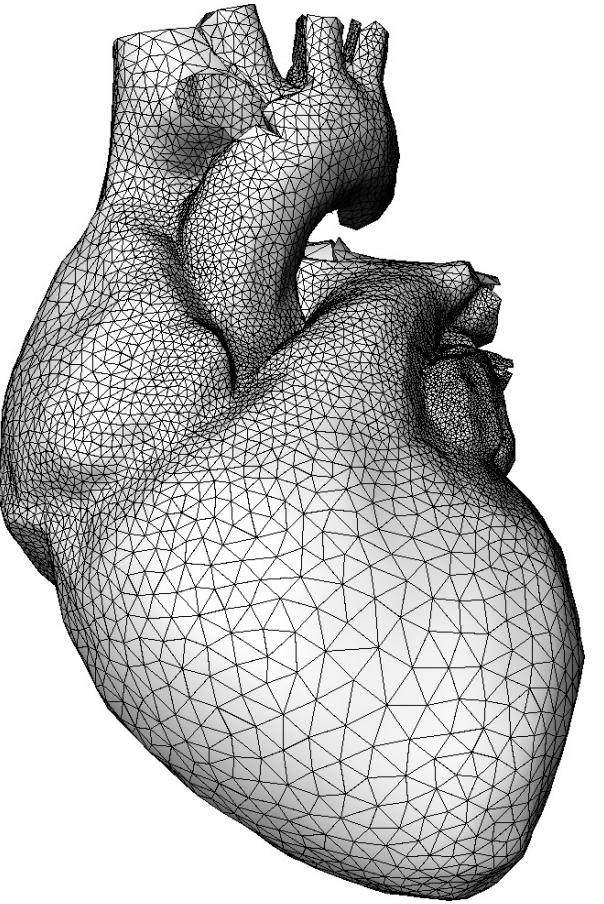
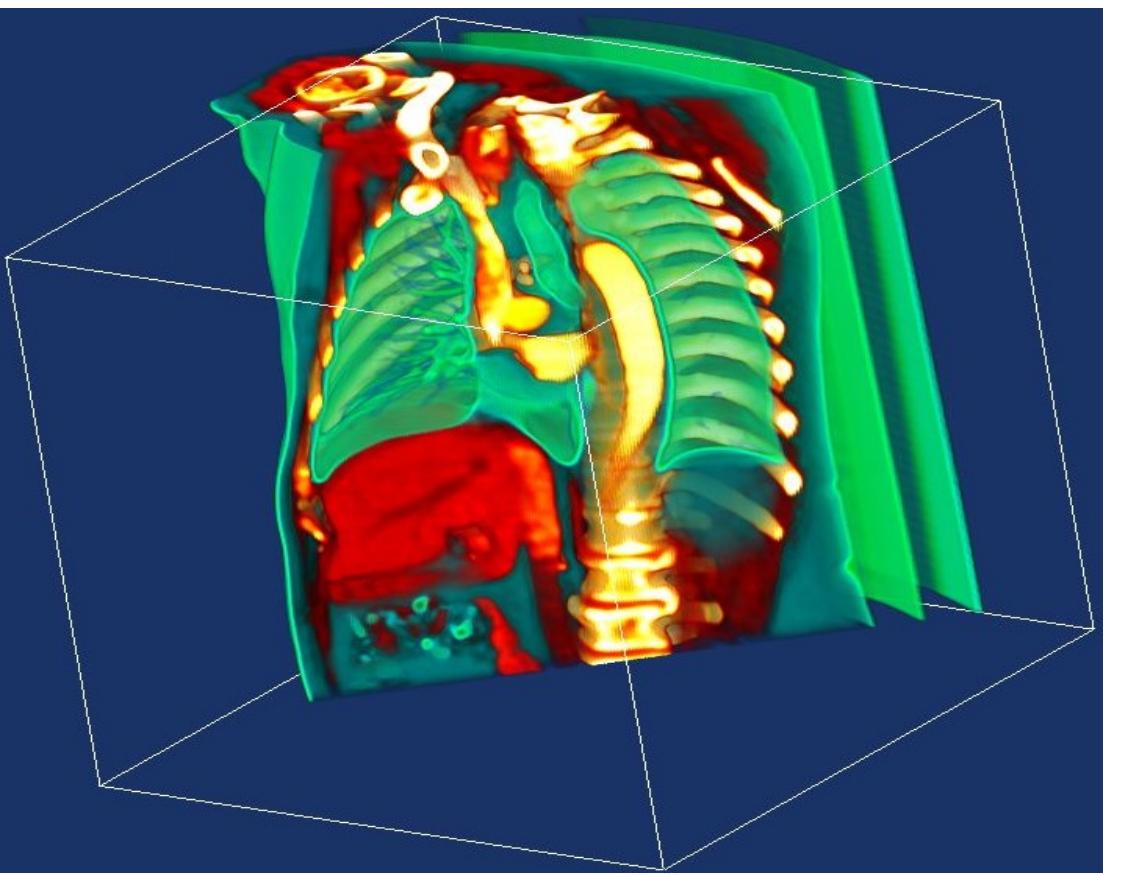
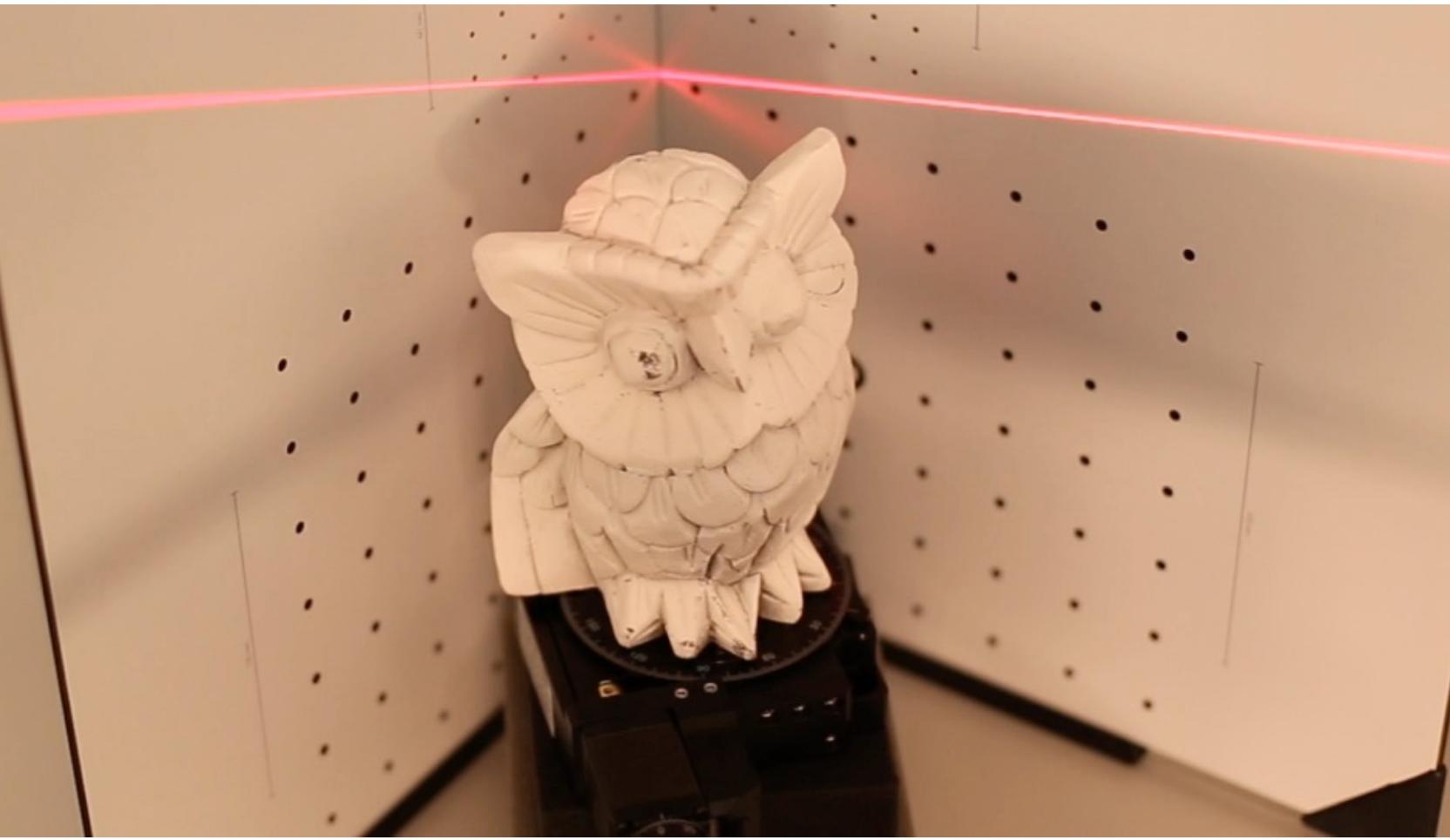
Shape Representation: Origin- and Application-Dependent

- Acquired real-world objects:
 - Discrete sampling
 - Points, meshes
 - Modeling “by hand”:
 - Higher-level representations, amendable to modification, control
 - Parametric surfaces, subdivision surfaces, implicits
 - Procedural modeling
 - Algorithms, grammars

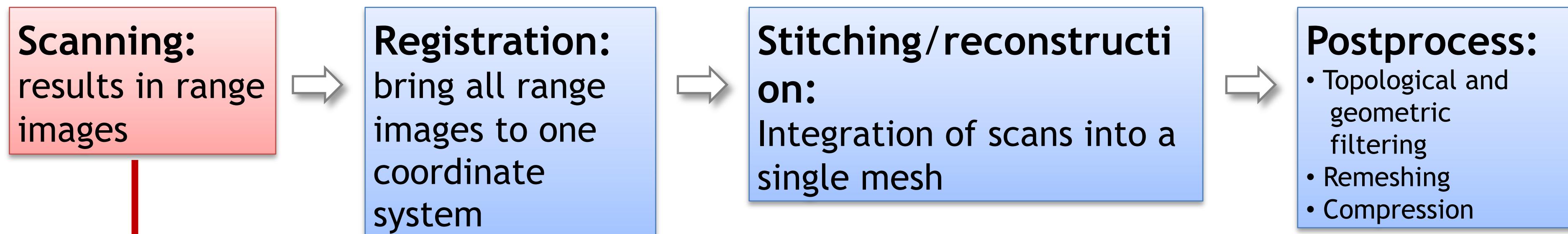


Course Topics

- Shape acquisition
 - Scanning/imaging
 - Reconstruction



Geometry Acquisition Pipeline



Euler-Poincaré Formula

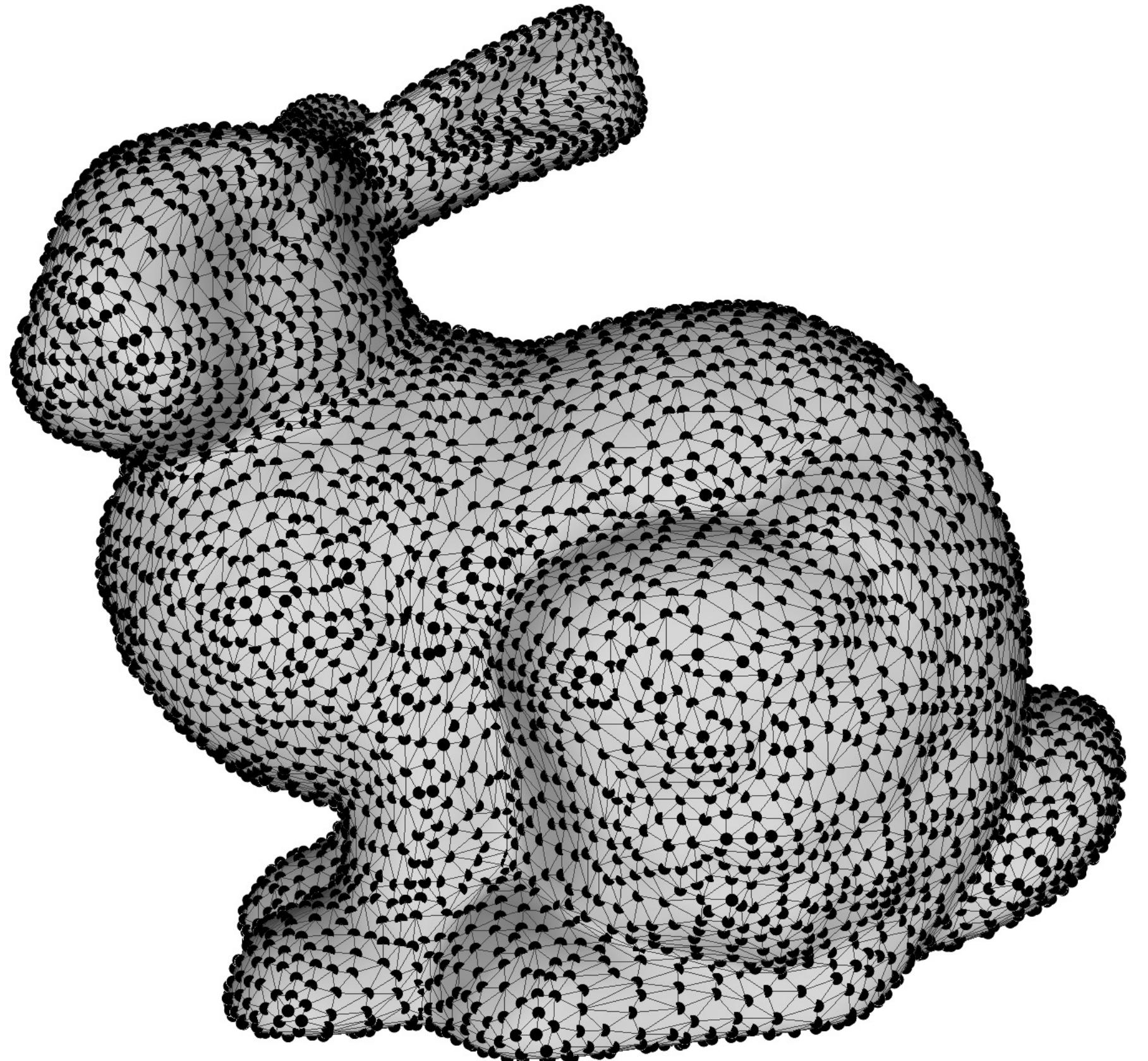
- For orientable meshes:

$$v - e + f = 2(c - g) - b = \chi(M)$$

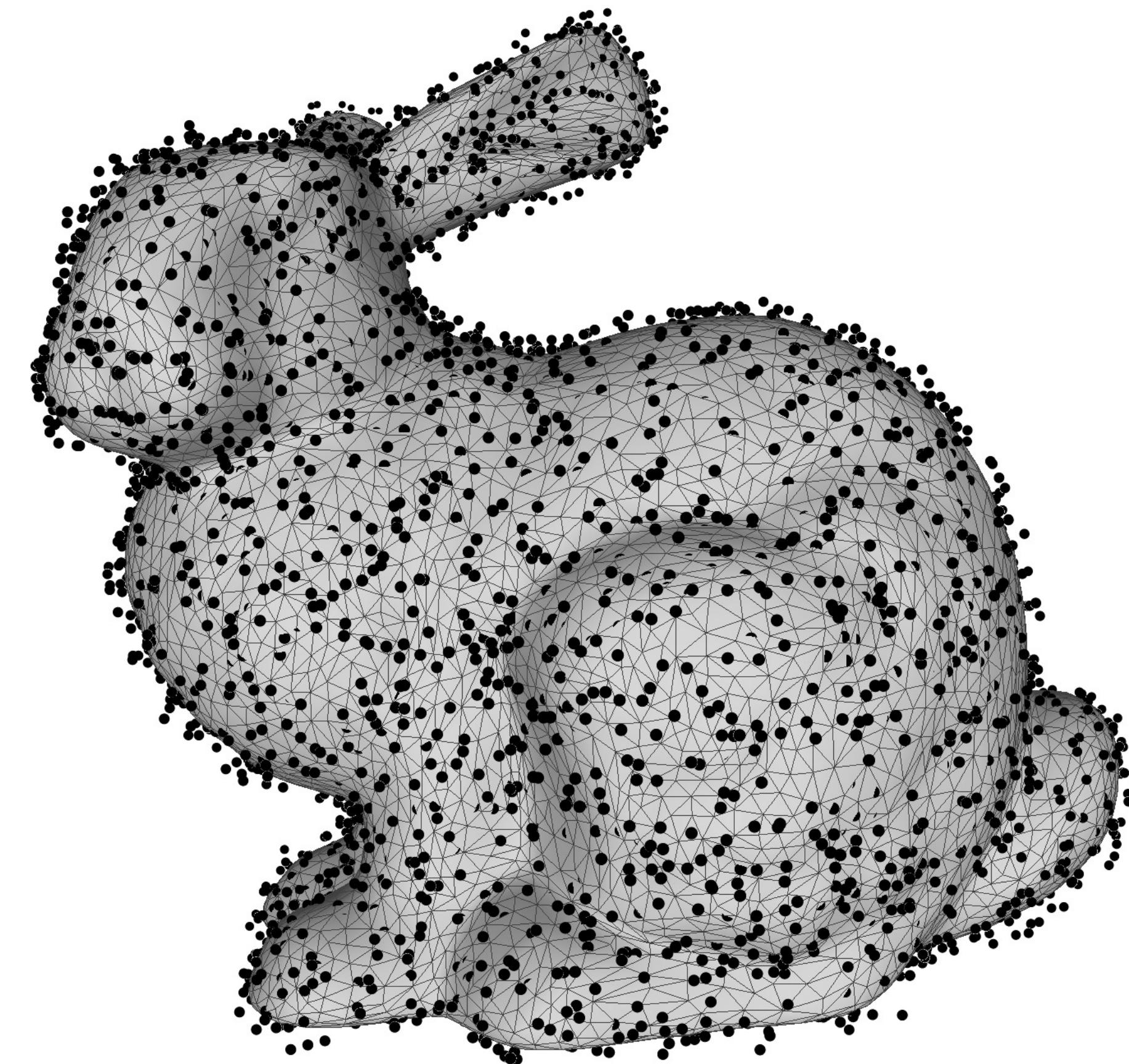
- c = number of connected components
- g = genus
- b = number of boundary loops

$$\chi(\text{Sphere}) = 2 \quad \chi(\text{Torus}) = 0$$

Problem Assumptions



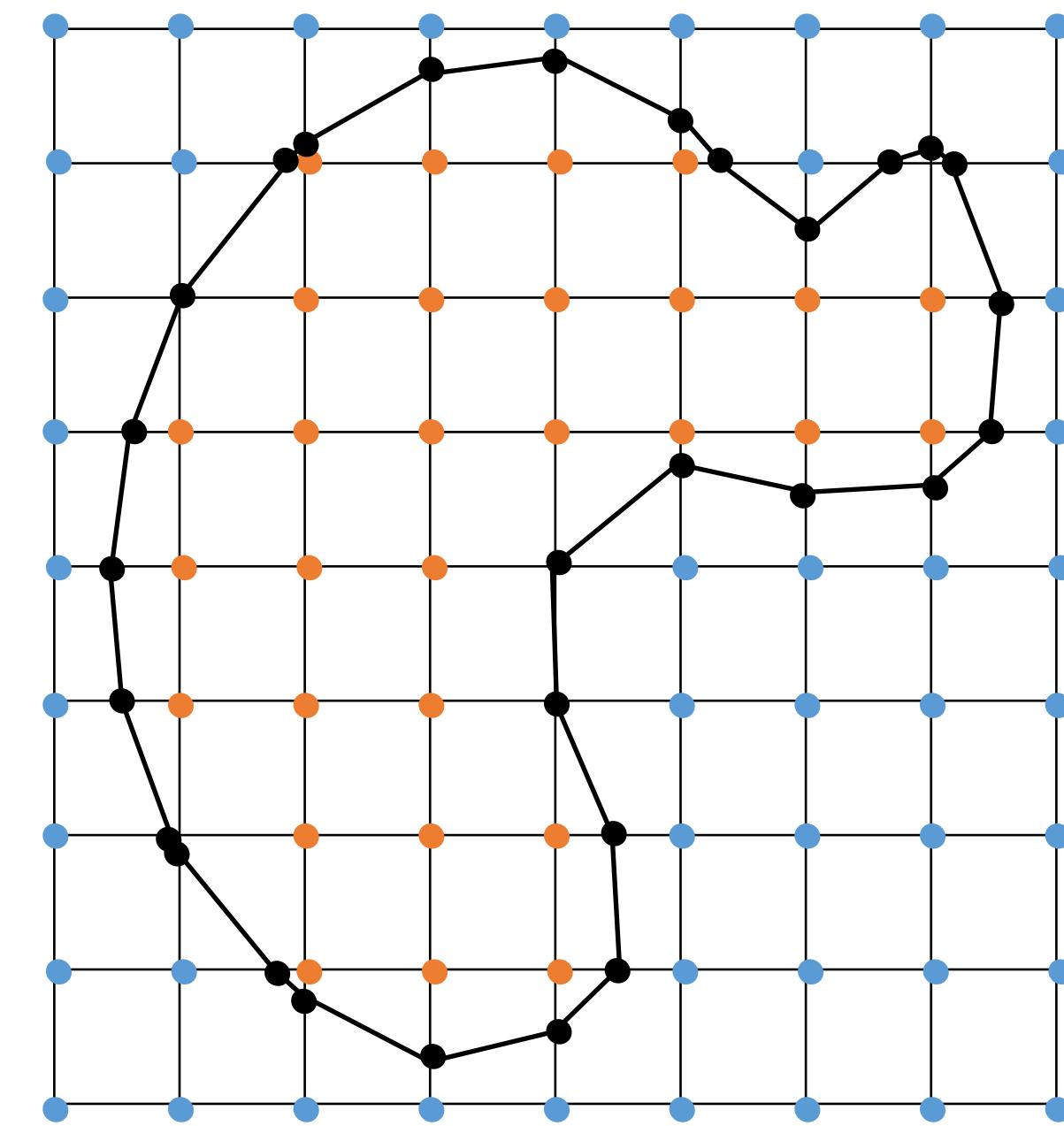
Interpolation



Approximation

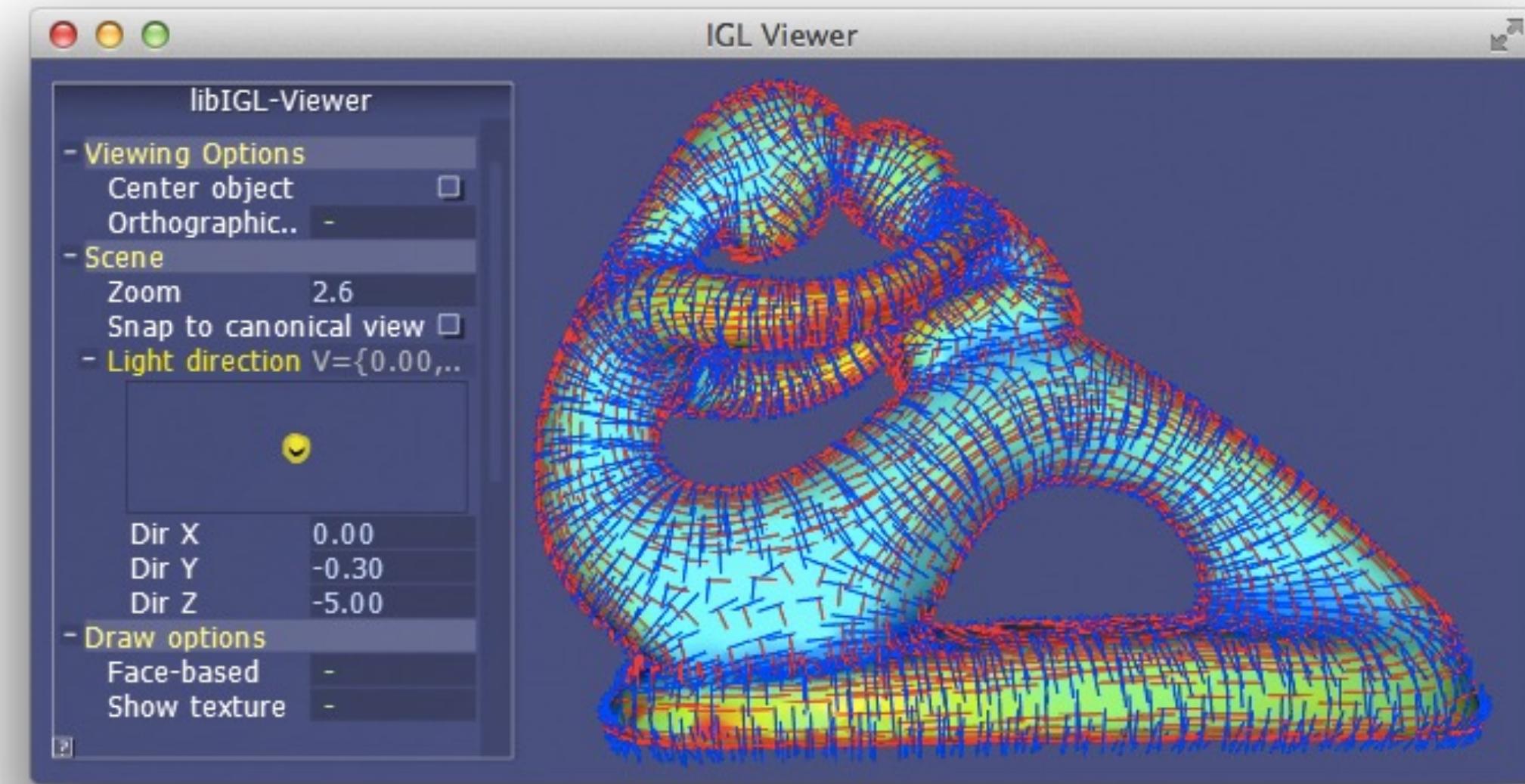
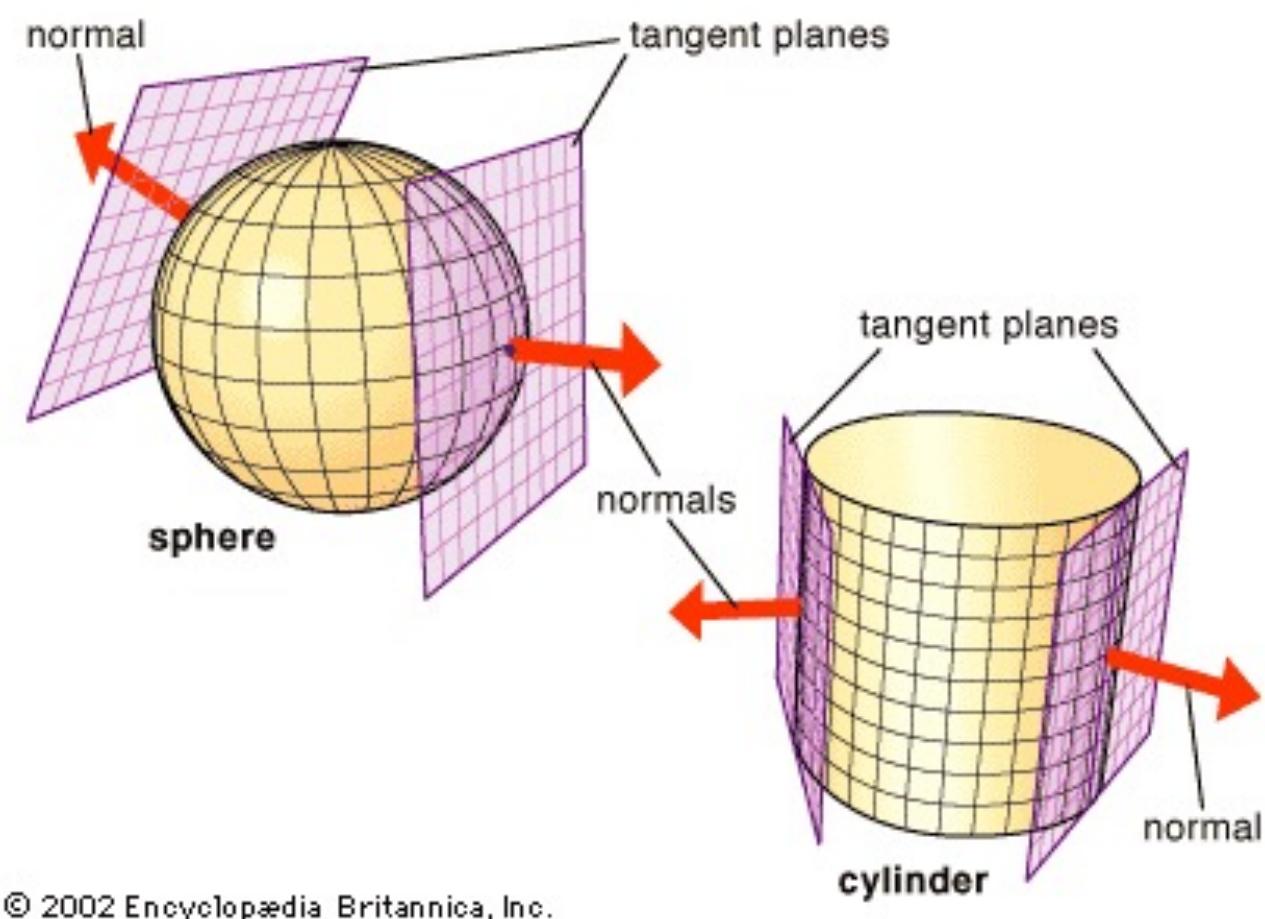
Implicit Reconstruction

1. Estimate normal (Optional)
2. Compute function $F(x)$
3. Discretize function $F(x)$
4. Extract zero Iso-surface



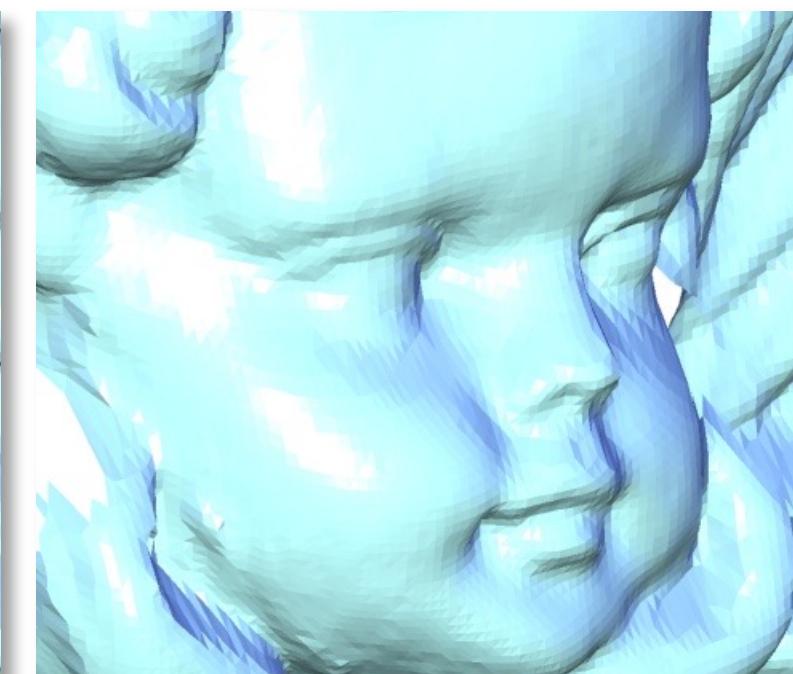
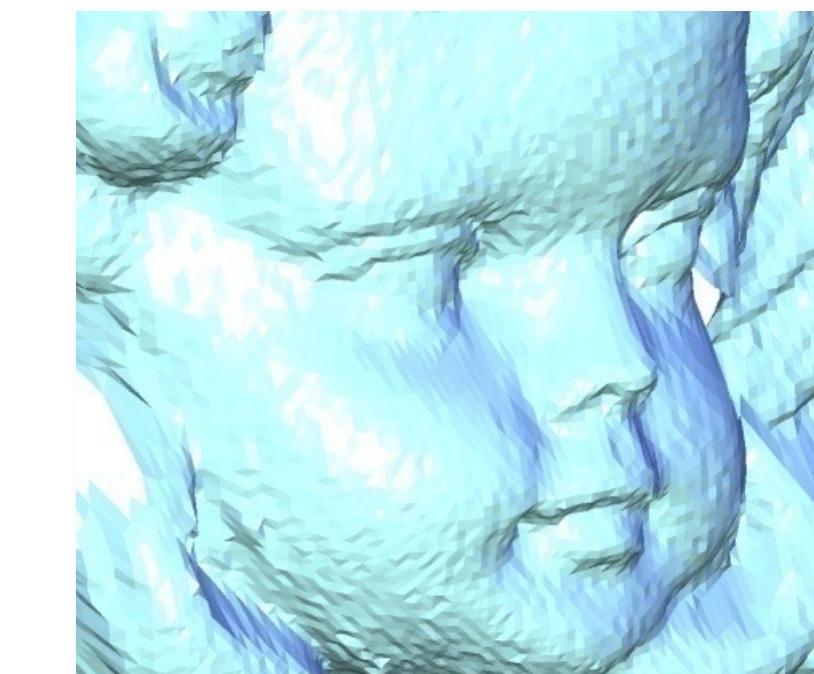
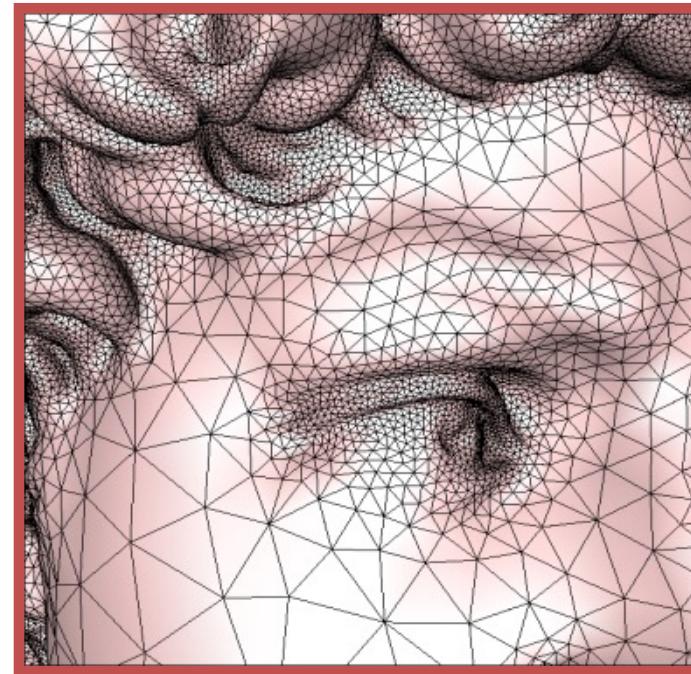
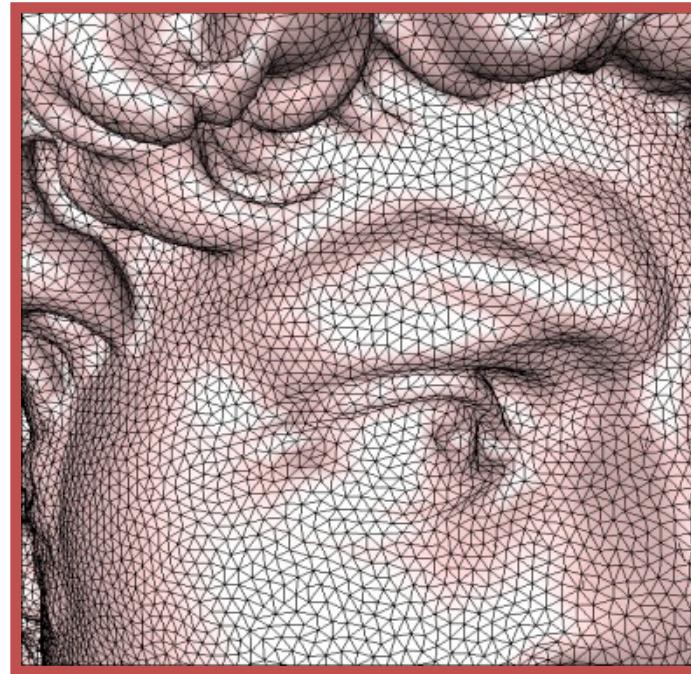
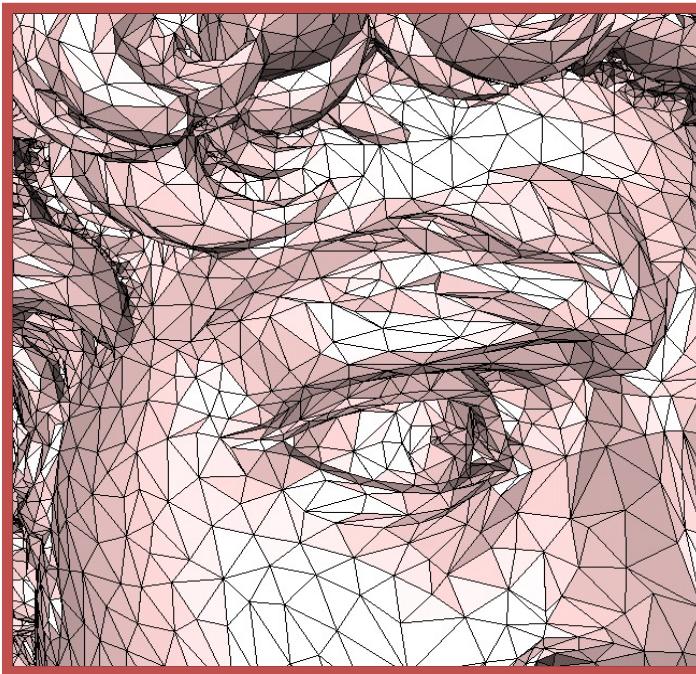
Course Topics

- Differential geometry
 - Continuous and (mostly) discrete
 - Powerful tool to analyze and model shapes



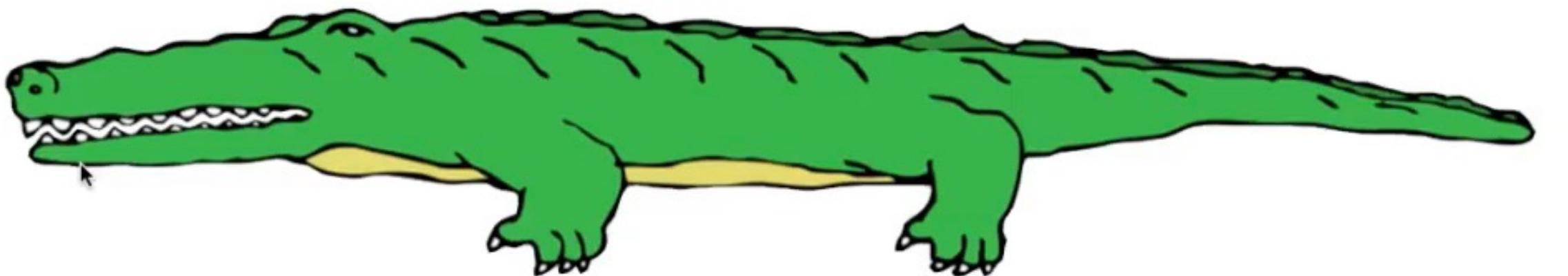
Course Topics

- Digital geometry processing
 - Denoising, smoothing, simplification, remeshing, parameterization, compression



Course Topics

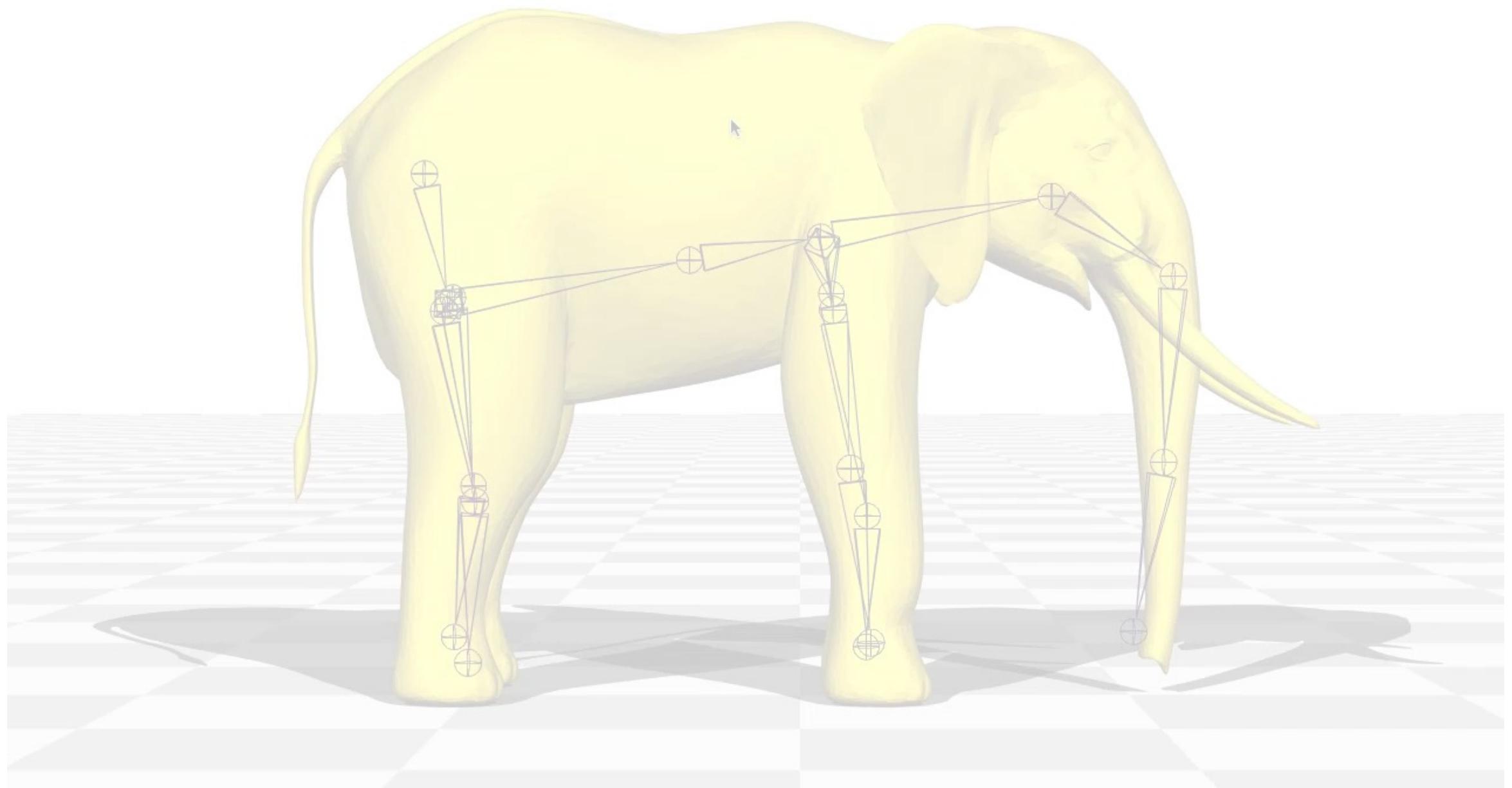
- Skinning, animation



<http://youtu.be/P9fqm8vgdB8>



<http://youtu.be/Pjg33pH9RKo>



Thank you