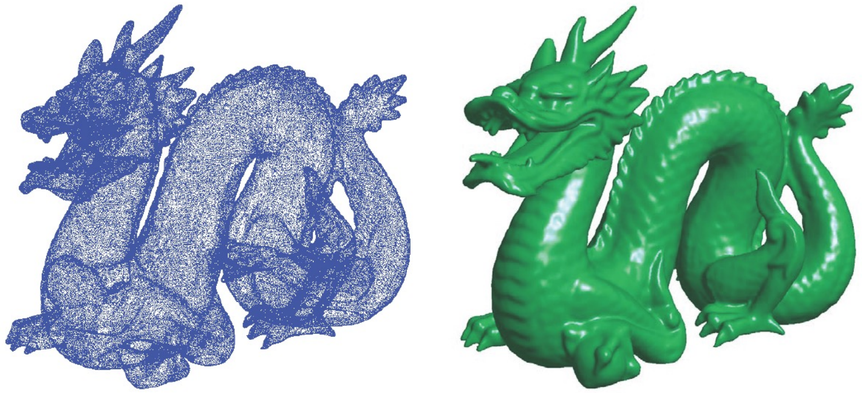
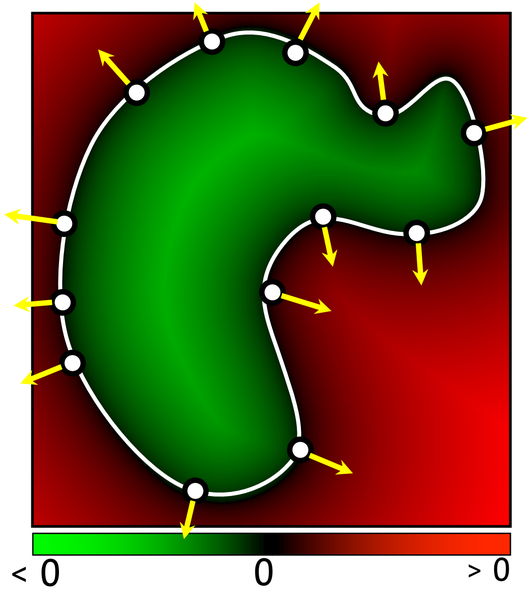
05 - Normal Estimation

Acknowledgements: Daniele Panozzo

Implicit Surface Reconstruction

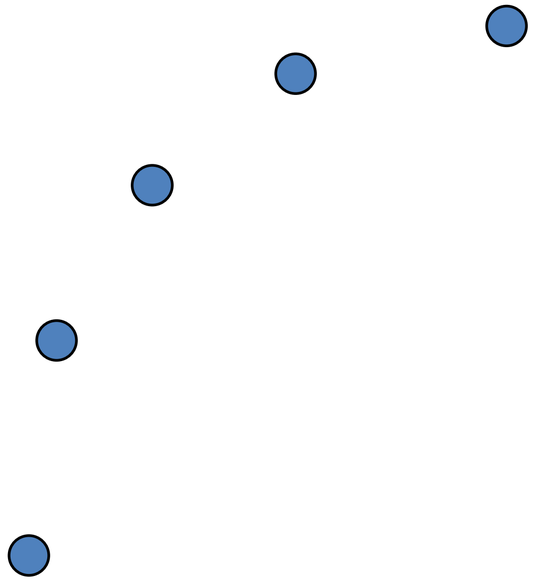
- Implicit function from point clouds
- Need consistently oriented normals





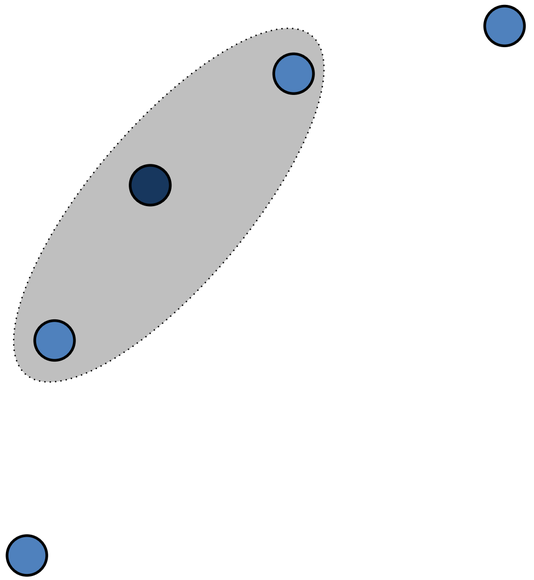
Normal Estimation

- Assign a normal vector $\mathbf{n}$ at each point cloud point $\mathbf{x}$
  - Estimate the direction by fitting a local plane
  - Find consistent global orientation by propagation (spanning tree)



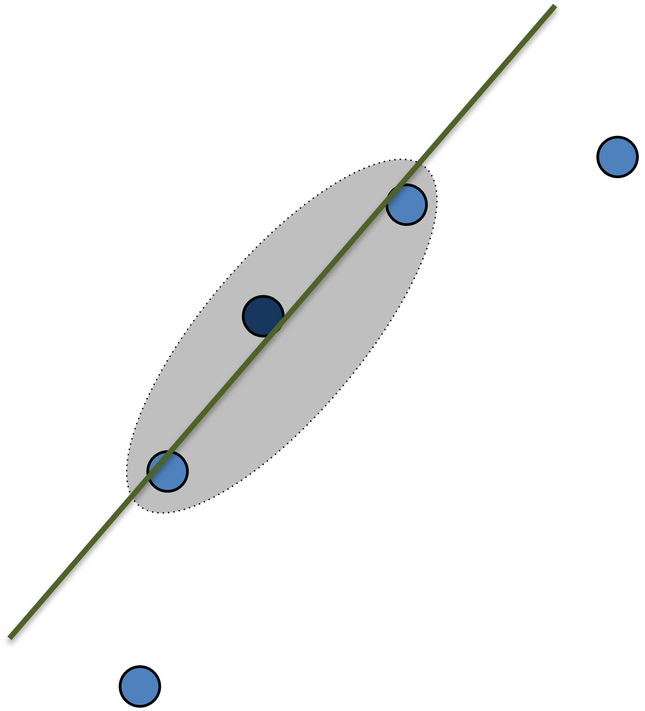
Normal Estimation

- Assign a normal vector $\mathbf{n}$ at each point cloud point $\mathbf{x}$
  - Estimate the direction by fitting a local plane
  - Find consistent global orientation by propagation (spanning tree)



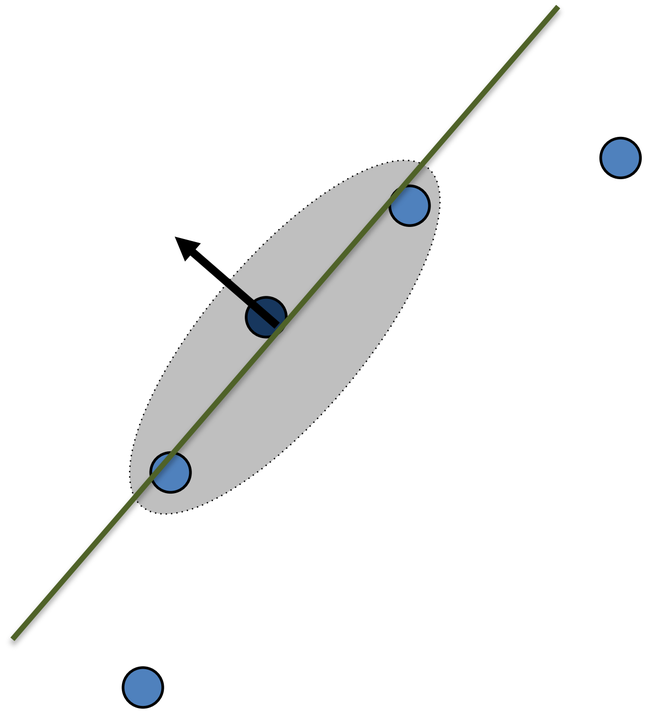
Normal Estimation

- Assign a normal vector $\mathbf{n}$ at each point cloud point $\mathbf{x}$
  - Estimate the direction by fitting a local plane
  - Find consistent global orientation by propagation (spanning tree)



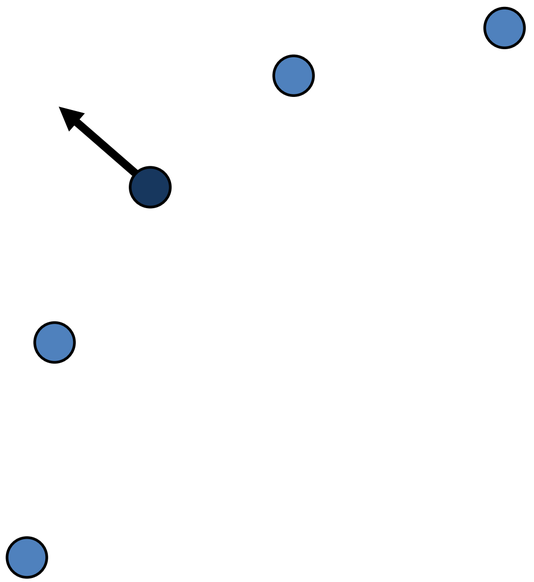
Normal Estimation

- Assign a normal vector $\mathbf{n}$ at each point cloud point $\mathbf{x}$
  - Estimate the direction by fitting a local plane
  - Find consistent global orientation by propagation (spanning tree)



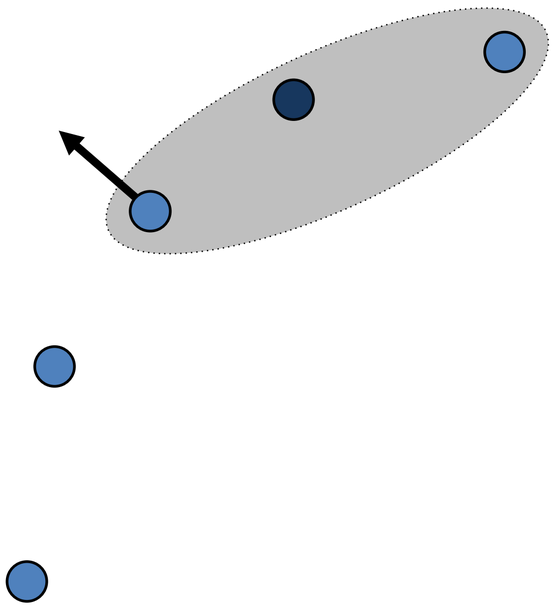
Normal Estimation

- Assign a normal vector $\mathbf{n}$ at each point cloud point $\mathbf{x}$
  - Estimate the direction by fitting a local plane
  - Find consistent global orientation by propagation (spanning tree)



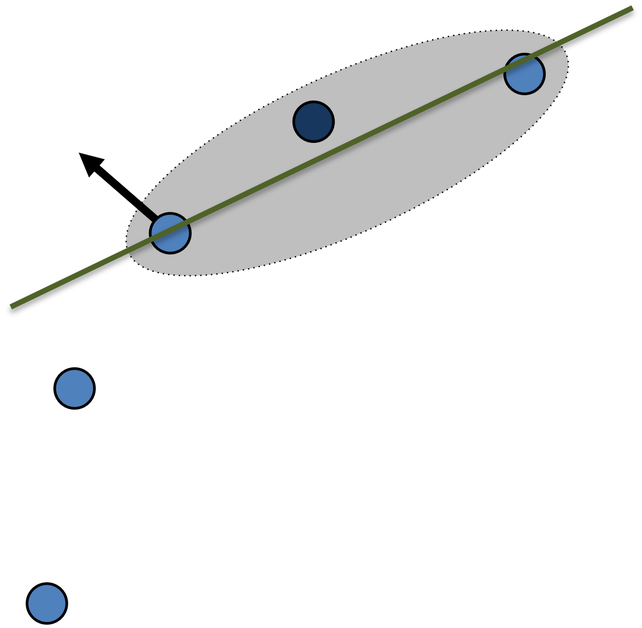
Normal Estimation

- Assign a normal vector $\mathbf{n}$ at each point cloud point $\mathbf{x}$
  - Estimate the direction by fitting a local plane
  - Find consistent global orientation by propagation (spanning tree)



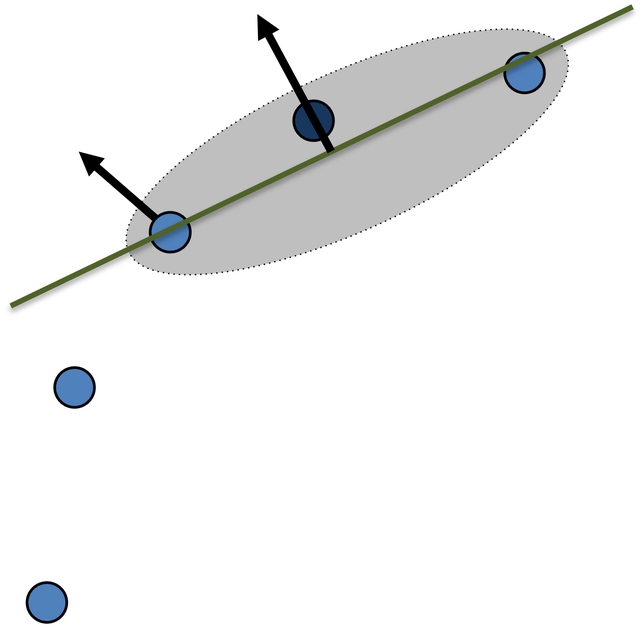
Normal Estimation

- Assign a normal vector $\mathbf{n}$ at each point cloud point $\mathbf{x}$
  - Estimate the direction by fitting a local plane
  - Find consistent global orientation by propagation (spanning tree)



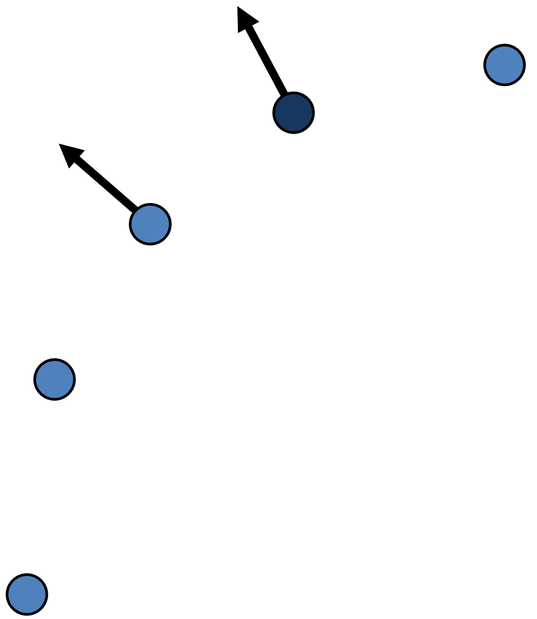
Normal Estimation

- Assign a normal vector $\mathbf{n}$ at each point cloud point $\mathbf{x}$
  - Estimate the direction by fitting a local plane
  - Find consistent global orientation by propagation (spanning tree)



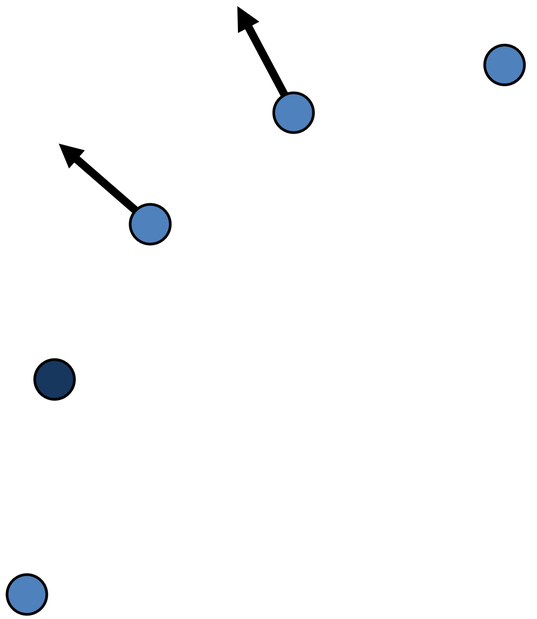
Normal Estimation

- Assign a normal vector $\mathbf{n}$ at each point cloud point $\mathbf{x}$
  - Estimate the direction by fitting a local plane
  - Find consistent global orientation by propagation (spanning tree)



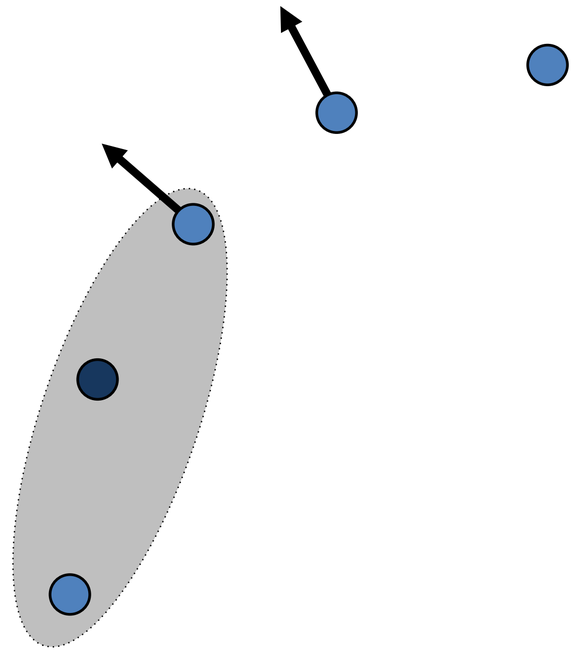
Normal Estimation

- Assign a normal vector $\mathbf{n}$ at each point cloud point $\mathbf{x}$
  - Estimate the direction by fitting a local plane
  - Find consistent global orientation by propagation (spanning tree)



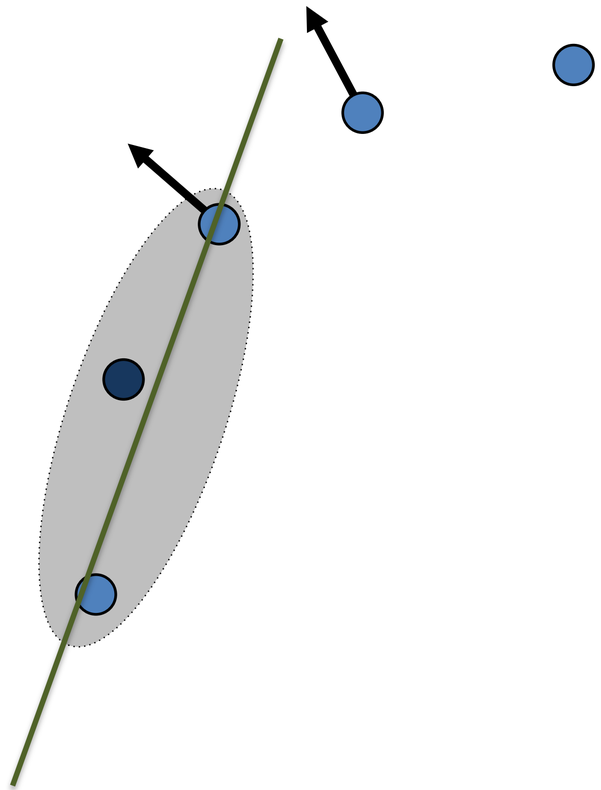
Normal Estimation

- Assign a normal vector $\mathbf{n}$ at each point cloud point $\mathbf{x}$
  - Estimate the direction by fitting a local plane
  - Find consistent global orientation by propagation (spanning tree)



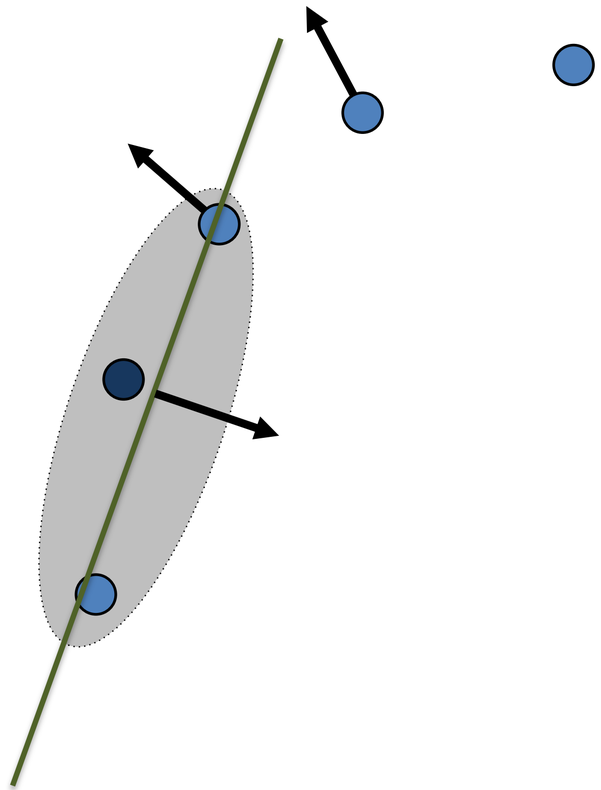
Normal Estimation

- Assign a normal vector $\mathbf{n}$ at each point cloud point $\mathbf{x}$
  - Estimate the direction by fitting a local plane
  - Find consistent global orientation by propagation (spanning tree)



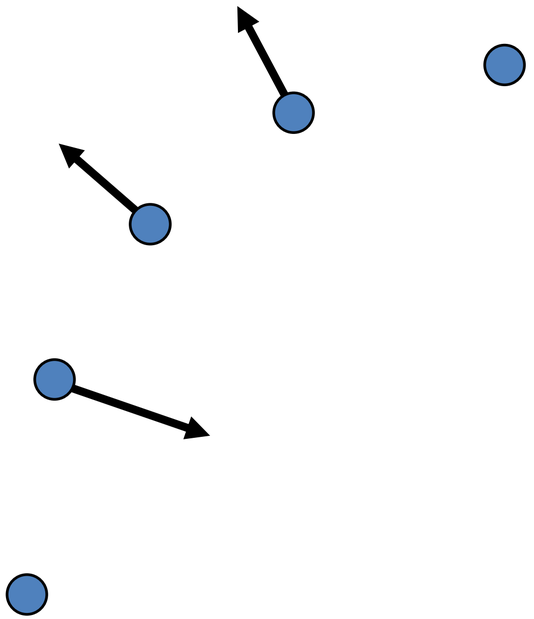
Normal Estimation

- Assign a normal vector $\mathbf{n}$ at each point cloud point $\mathbf{x}$
  - Estimate the direction by fitting a local plane
  - Find consistent global orientation by propagation (spanning tree)



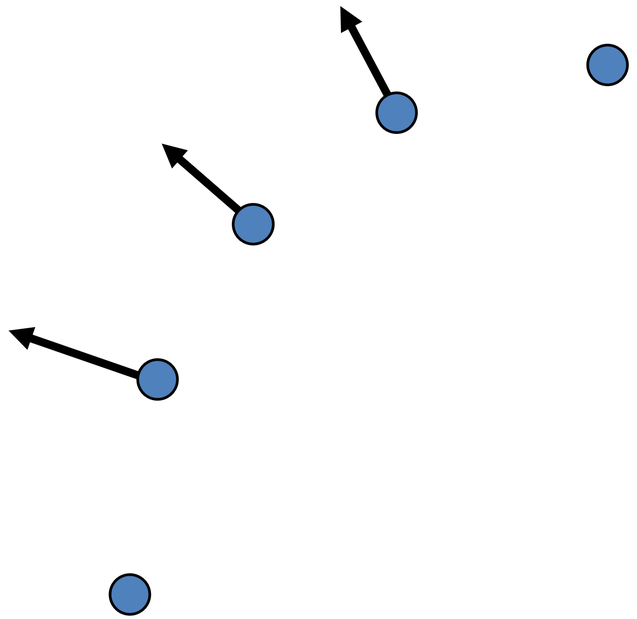
Normal Estimation

- Assign a normal vector $\mathbf{n}$ at each point cloud point $\mathbf{x}$
  - Estimate the direction by fitting a local plane
  - Find consistent global orientation by propagation (spanning tree)



Normal Estimation

- Assign a normal vector $\mathbf{n}$ at each point cloud point $\mathbf{x}$
  - Estimate the direction by fitting a local plane
  - Find consistent global orientation by propagation (spanning tree)



Local Plane Fitting

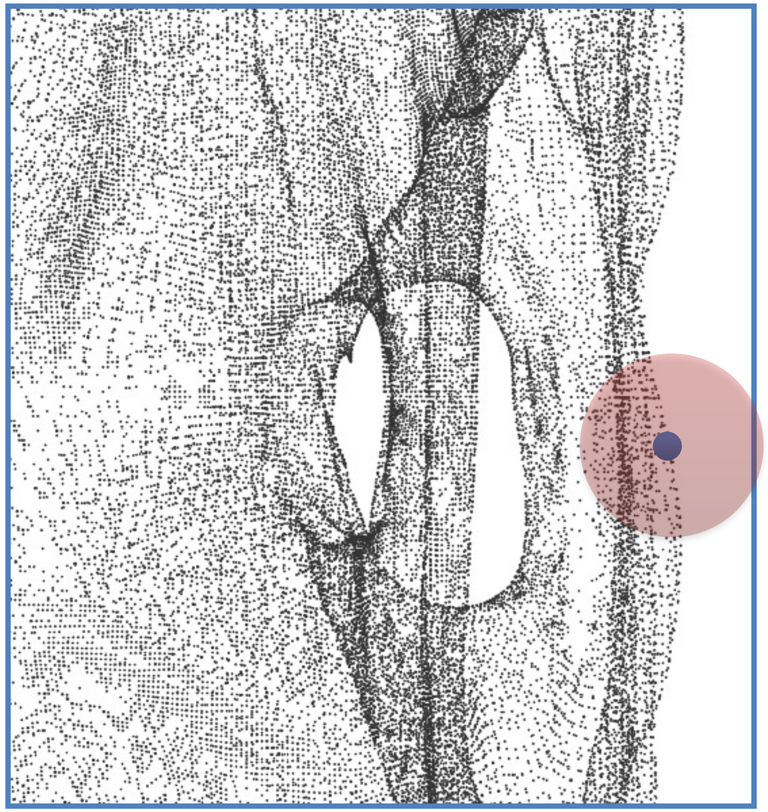
- For each point $\mathbf{x}$ in the cloud, pick k nearest neighbors or all points in r-ball:

$$\{\mathbf{x}_i \mid \|\mathbf{x}_i - \mathbf{x}\| < r\}$$

$$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$$

- Find a plane Π that minimizes the sum of square distances:

$$\min \sum_{i=1}^{n} \mathrm{dist}(\mathbf{x}_i, \Pi)^2$$



Local Plane Fitting

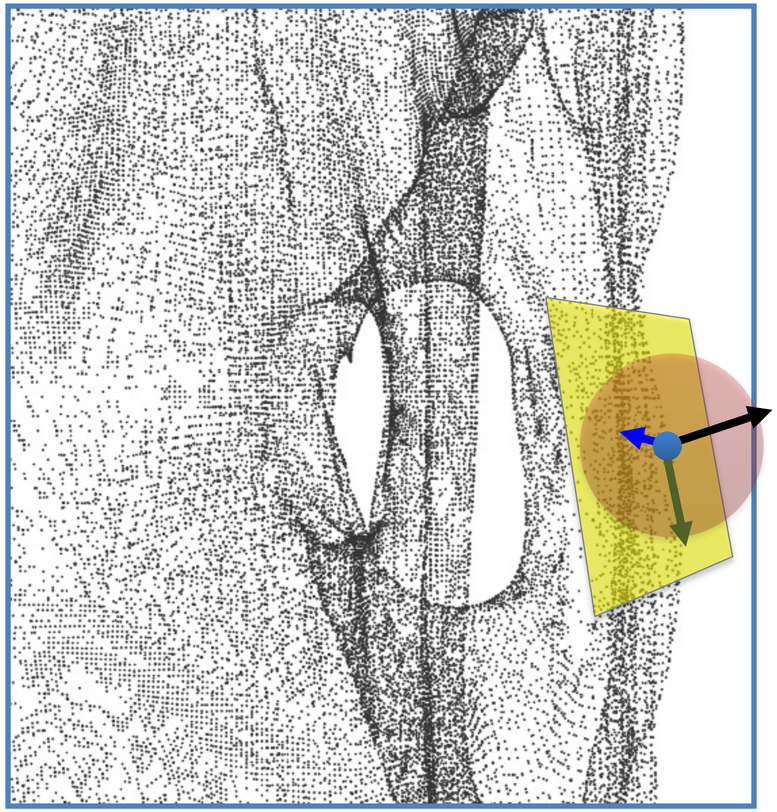
- For each point $\mathbf{x}$ in the cloud, pick k nearest neighbors or all points in r-ball:

$$\{\mathbf{x}_i \mid \|\mathbf{x}_i - \mathbf{x}\| < r\}$$

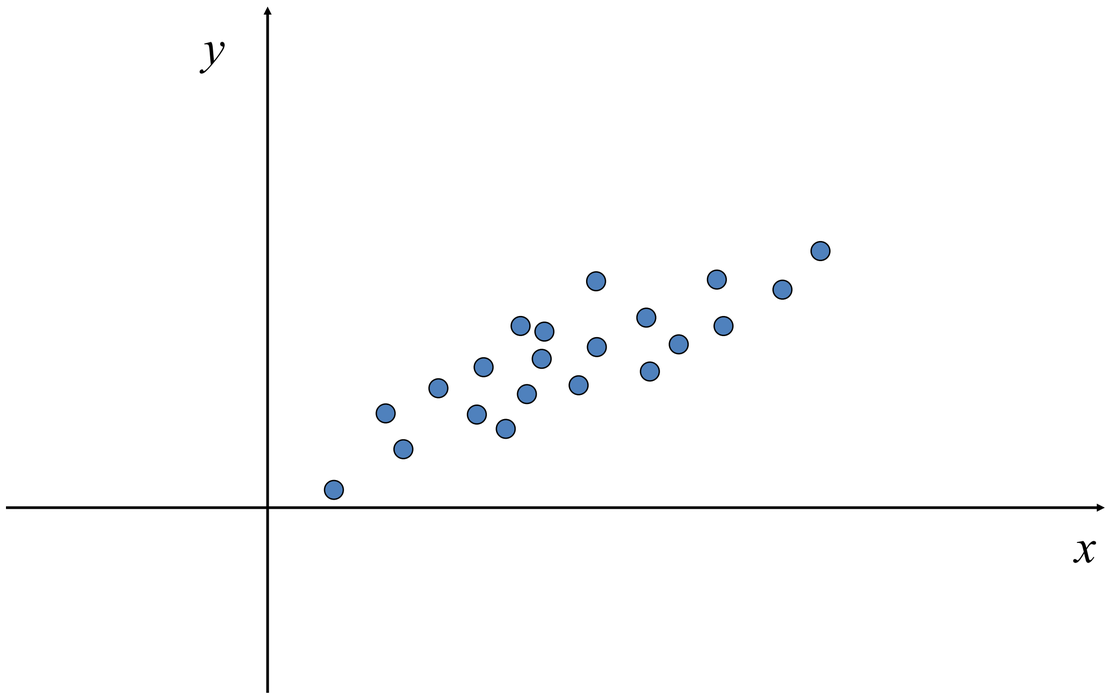
$$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$$

- Find a plane Π that minimizes the sum of square distances:

$$\min \sum_{i=1}^{n} \mathrm{dist}(\mathbf{x}_i, \Pi)^2$$

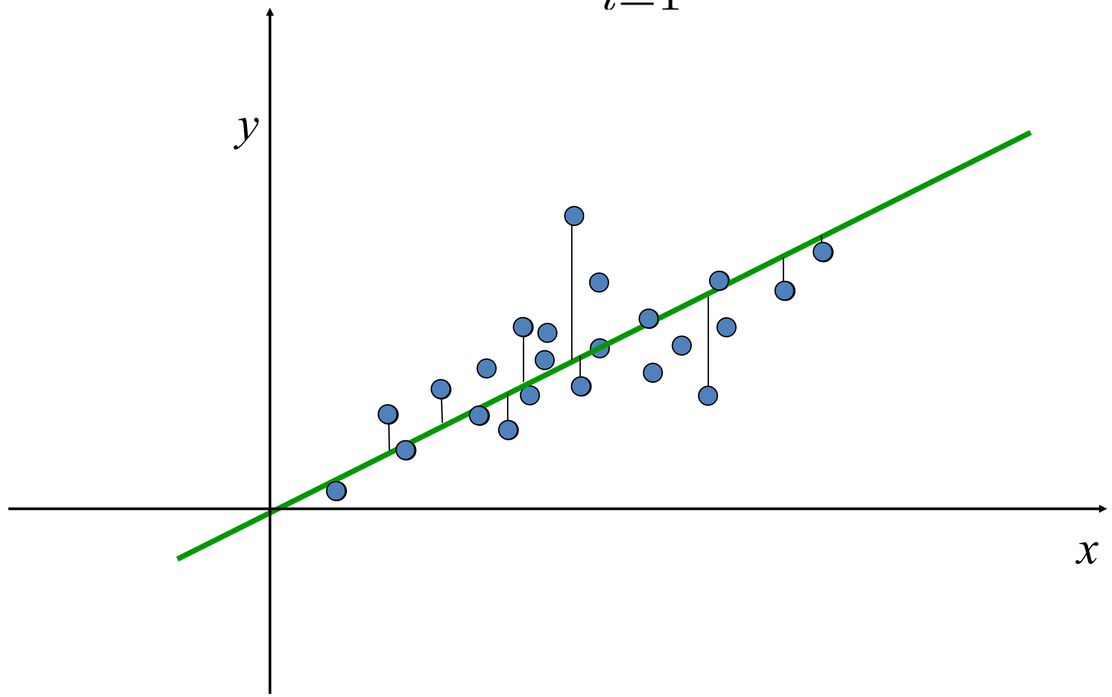


Linear Least Squares?



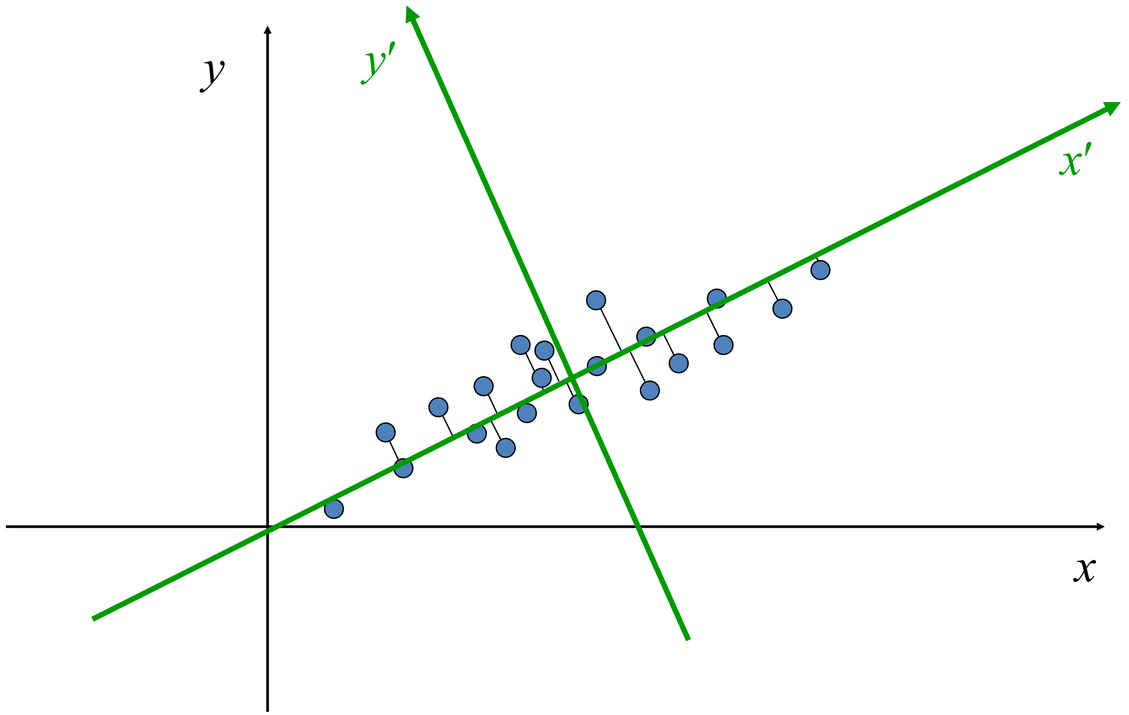
Linear Least Squares?

- Find a line $y = ax+b$ s.t. $\min \sum_{i=1}^{n} (y_i - (ax_i + b))^2$



- But we would like true orthogonal distances!

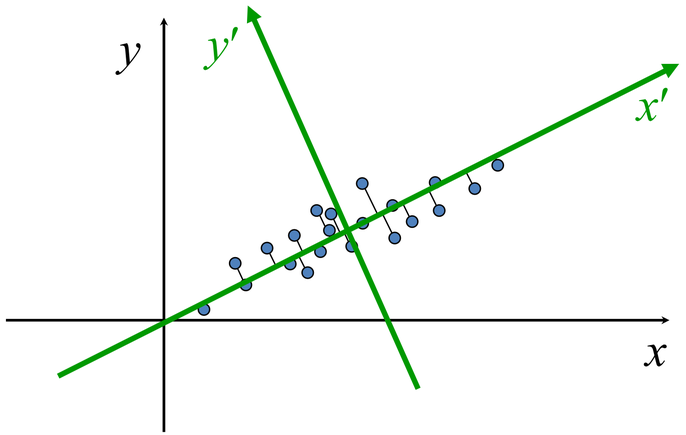
Best Fit with SSD

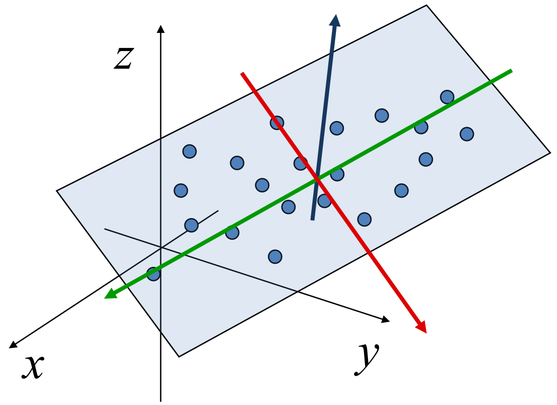


SSD = sum of squared distances (or differences)

Principle Component Analysis (PCA)

- PCA finds an orthogonal basis that best represents a given data set





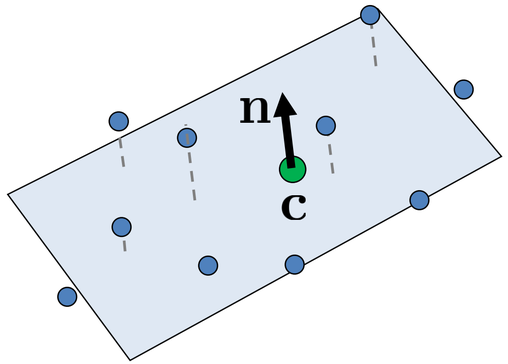
- PCA finds the best approximating line/plane/orientation… (in terms of $\Sigma distances^2$)

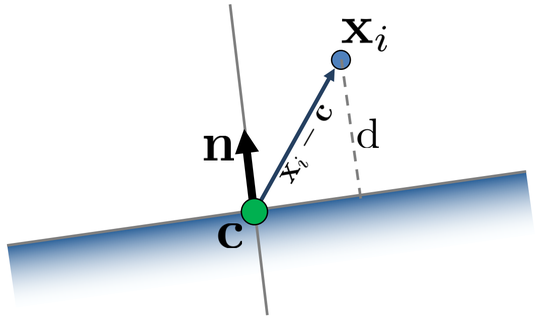
Notations

- Input points: $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^d$

- Looking for a (hyper) plane passing through $\mathbf{c}$ with normal $\mathbf{n}$ s.t.

$$\min_{\mathbf{c},\mathbf{n},\|\mathbf{n}\|=1} \sum_{i=1}^{n} \left((\mathbf{x}_i - \mathbf{c})^T \mathbf{n} \right)^2$$





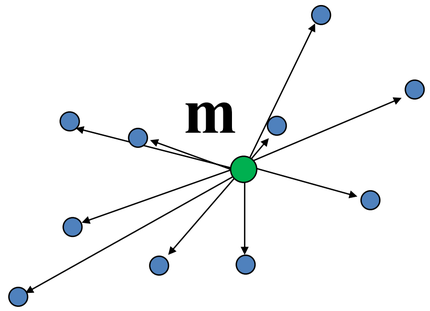
Notations

- Input points:

$$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^d$$

- Centroid:

$$\mathbf{m} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i$$



- Vectors from the centroid:

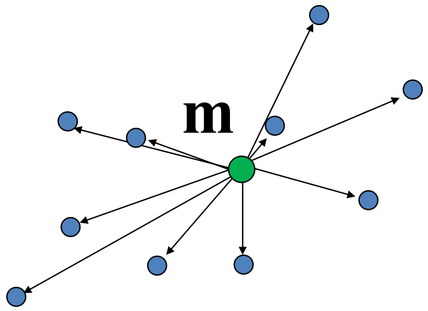
$$\mathbf{y}_i = \mathbf{x}_i - \mathbf{m}$$

Centroid: 0-dim Approximation

- It can be shown that:

$$\mathbf{m} = \underset{\mathbf{c}}{\operatorname{argmin}} \sum_{i=1}^{n} \left((\mathbf{x}_i - \mathbf{c})^T \mathbf{n} \right)^2$$

$$\mathbf{m} = \underset{\mathbf{c}}{\operatorname{argmin}} \sum_{i=1}^{n} \|\mathbf{x}_i - \mathbf{c}\|^2$$



$$\mathbf{m} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i$$

- $\mathbf{m}$ minimizes SSD
- $\mathbf{m}$ will be the origin of the (hyper)-plane
- Our problem becomes:

$$\min_{\|\mathbf{n}\|=1} \sum_{i=1}^{n} \left(\mathbf{y}_i^T \mathbf{n} \right)^2$$

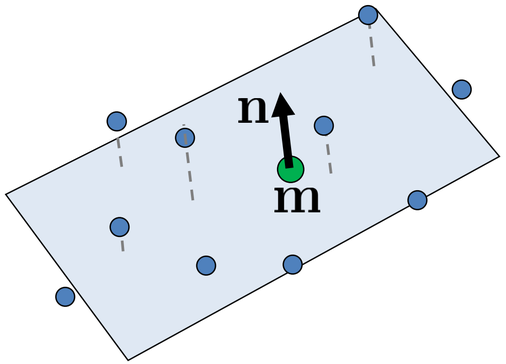
Hyperplane Normal

- Minimize!

$$\min_{\mathbf{n}^T\mathbf{n}=1} \sum_{i=1}^{n} \left(\mathbf{y}_i^T\mathbf{n}\right)^2 = \min_{\mathbf{n}^T\mathbf{n}=1} \sum_{i=1}^{n} \mathbf{n}^T\mathbf{y}_i\mathbf{y}_i^T\mathbf{n} =$$

$$\min_{\mathbf{n}^T\mathbf{n}=1} \mathbf{n}^T \left(\sum_{i=1}^{n} \mathbf{y}_i\mathbf{y}_i^T\right)\mathbf{n} = \min_{\mathbf{n}^T\mathbf{n}=1} \mathbf{n}^T \left(\mathbf{Y}\mathbf{Y}^T\right)\mathbf{n}$$

$$\mathbf{Y} = \begin{pmatrix} | & | & & | \\ \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_n \\ | & | & & | \end{pmatrix}$$



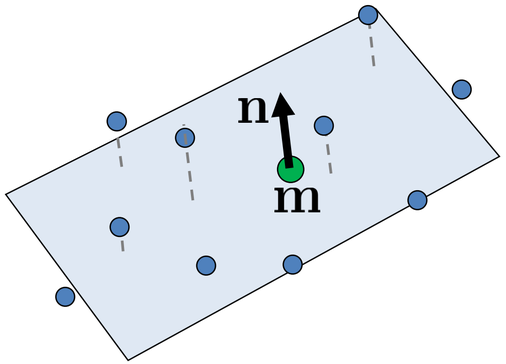
Hyperplane Normal

- Minimize!

$$\min_{\mathbf{n}^T\mathbf{n}=1} \sum_{i=1}^{n} \left(\mathbf{y}_i^T\mathbf{n}\right)^2 = \min_{\mathbf{n}^T\mathbf{n}=1} \sum_{i=1}^{n} \mathbf{n}^T\mathbf{y}_i\mathbf{y}_i^T\mathbf{n} =$$

$$\min_{\mathbf{n}^T\mathbf{n}=1} \mathbf{n}^T \left(\sum_{i=1}^{n} \mathbf{y}_i\mathbf{y}_i^T\right)\mathbf{n} = \min_{\mathbf{n}^T\mathbf{n}=1} \mathbf{n}^T \left(\mathbf{Y}\mathbf{Y}^T\right)\mathbf{n}$$

$$\mathbf{Y} = \begin{pmatrix} | & | & & | \\ \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_n \\ | & | & & | \end{pmatrix}$$



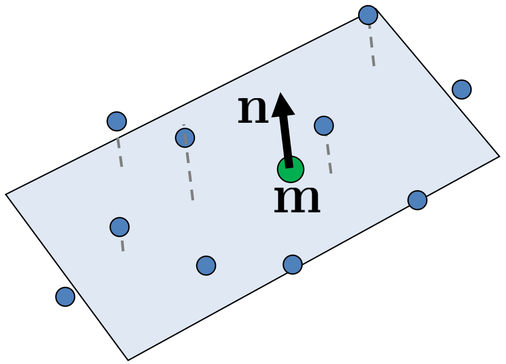
$$f(\mathbf{n}) = \mathbf{n}^T\mathbf{S}\mathbf{n} \qquad (\mathbf{S} = \mathbf{Y}\mathbf{Y}^T)$$

$$\min f(\mathbf{n}) \quad s.t.\ \mathbf{n}^T\mathbf{n} = 1$$

Hyperplane Normal

- Constrained minimization – Lagrange multipliers

$$f(\mathbf{n}) = \mathbf{n}^T\mathbf{S}\mathbf{n} \qquad (\mathbf{S} = \mathbf{Y}\mathbf{Y}^T)$$
$$\min f(\mathbf{n}) \quad s.t.\ \mathbf{n}^T\mathbf{n} = 1$$



$$\mathcal{L}(\mathbf{n}, \lambda) = f(\mathbf{n}) - \lambda(\mathbf{n}^T\mathbf{n} - 1)$$
$$\nabla\mathcal{L} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\mathbf{n}} = \frac{\partial}{\partial\mathbf{n}} f(\mathbf{n}) - \lambda\frac{\partial}{\partial\mathbf{n}}(\mathbf{n}^T\mathbf{n} - 1)$$
$$\frac{\partial\mathcal{L}}{\partial\lambda} = \mathbf{n}^T\mathbf{n} - 1$$

Matrix Cookbook!
https://archive.org/details/imm3274

$$\frac{\partial}{\partial\mathbf{n}} f(\mathbf{n}) - \lambda\frac{\partial}{\partial\mathbf{n}}(\mathbf{n}^T\mathbf{n} - 1) = (\mathbf{S} + \mathbf{S}^T)\mathbf{n} - \lambda(\mathbf{I} + \mathbf{I}^T)\mathbf{n} = 2\mathbf{S}\mathbf{n} - 2\lambda\mathbf{n}$$

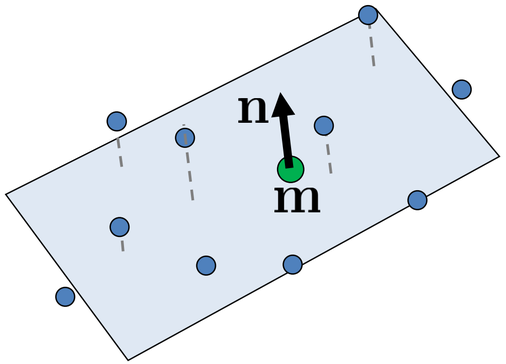
Hyperplane Normal

- Constrained minimization – Lagrange multipliers

$$f(\mathbf{n}) = \mathbf{n}^T \mathbf{S} \mathbf{n} \qquad (\mathbf{S} = \mathbf{Y}\mathbf{Y}^T)$$
$$\min f(\mathbf{n}) \quad s.t. \ \mathbf{n}^T \mathbf{n} = 1$$

$$\mathcal{L}(\mathbf{n}, \lambda) = f(\mathbf{n}) - \lambda(\mathbf{n}^T \mathbf{n} - 1)$$
$$\nabla \mathcal{L} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{n}} = 0 \iff \mathbf{S}\mathbf{n} = \lambda \mathbf{n}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \iff \mathbf{n}^T \mathbf{n} = 1$$



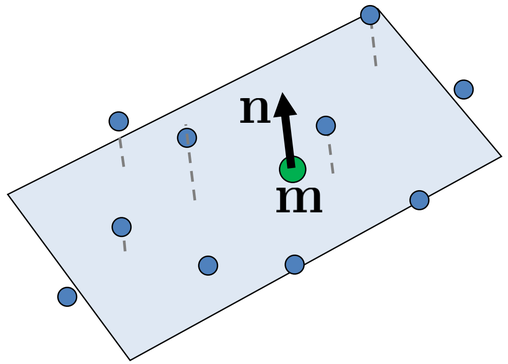
Hyperplane Normal

- Constrained minimization – Lagrange multipliers

$$f(\mathbf{n}) = \mathbf{n}^T\mathbf{S}\mathbf{n} \qquad (\mathbf{S} = \mathbf{Y}\mathbf{Y}^T)$$
$$\min f(\mathbf{n}) \quad s.t.\ \mathbf{n}^T\mathbf{n} = 1$$

$$\mathcal{L}(\mathbf{n}, \lambda) = f(\mathbf{n}) - \lambda(\mathbf{n}^T\mathbf{n} - 1)$$
$$\nabla\mathcal{L} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\mathbf{n}} = 0 \iff \mathbf{S}\mathbf{n} = \lambda\mathbf{n}$$
$$\frac{\partial\mathcal{L}}{\partial\lambda} = 0 \iff \mathbf{n}^T\mathbf{n} = 1$$

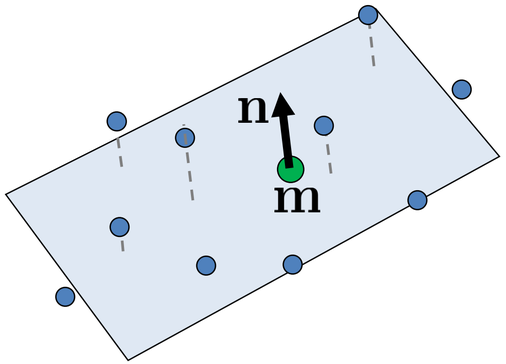


What can be said about $\mathbf{n}$??

Hyperplane Normal

- Constrained minimization – Lagrange multipliers

$$f(\mathbf{n}) = \mathbf{n}^T\mathbf{S}\mathbf{n} \qquad (\mathbf{S} = \mathbf{Y}\mathbf{Y}^T)$$
$$\min f(\mathbf{n}) \quad s.t.\ \mathbf{n}^T\mathbf{n} = 1$$



$$\mathcal{L}(\mathbf{n}, \lambda) = f(\mathbf{n}) - \lambda(\mathbf{n}^T\mathbf{n} - 1)$$
$$\nabla\mathcal{L} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\mathbf{n}} = 0 \iff \mathbf{S}\mathbf{n} = \lambda\mathbf{n}$$
$$\frac{\partial\mathcal{L}}{\partial\lambda} = 0 \iff \mathbf{n}^T\mathbf{n} = 1$$

$\mathbf{n}$ is the eigenvector of $\mathbf{S}$ with the smallest eigenvalue

Summary – Best Fitting Plane Recipe

$$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^d$$

- Input:
- Compute centroid = plane origin
- Compute scatter matrix

$$\mathbf{m} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i$$

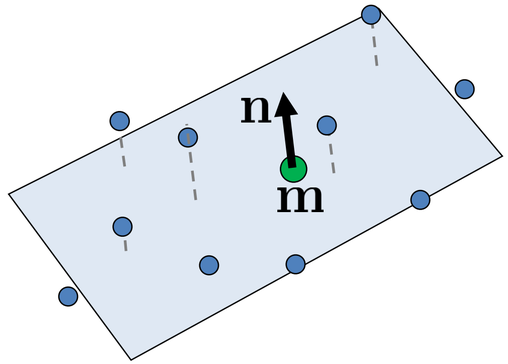
$$\mathbf{S} = \mathbf{Y}\mathbf{Y}^T$$

$$\mathbf{Y} = (\mathbf{y}_1\ \mathbf{y}_2\ \ldots\ \mathbf{y}_n)$$

$$\mathbf{y}_i = \mathbf{x}_i - \mathbf{m}$$

- The plane normal $\mathbf{n}$ is the eigenvector of $\mathbf{S}$ with the smallest eigenvalue

$$\mathbf{S} = \mathbf{V} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix} \mathbf{V}^T$$

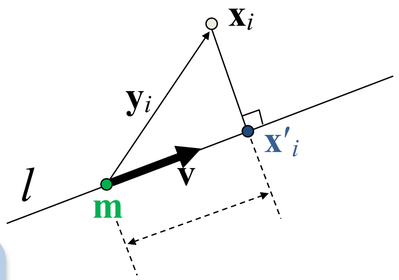


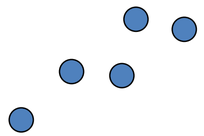
What does the Scatter Matrix do?

- Let's look at a line l through the center of mass $\mathbf{m}$ with direction vector $\mathbf{v}$, and project our points $\mathbf{x}_i$ onto it. The variance of the projected points $\mathbf{x'}_i$ is:

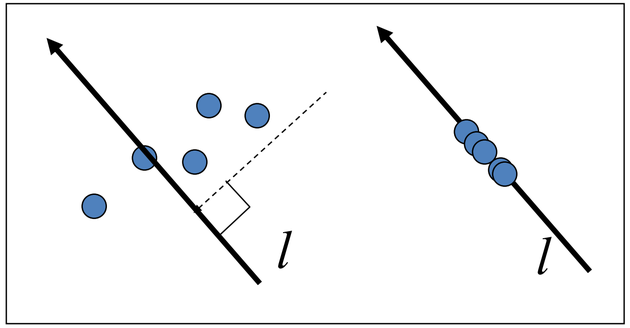
$$\mathrm{var}(\mathbf{x}_1, \ldots \mathbf{x}_n; \mathbf{v}) = \frac{1}{n}\sum_{i=1}^{n} \|\mathbf{x}'_i - \mathbf{m}\|^2 =$$

$$= \frac{1}{n}\sum_{i=1}^{n} \|(\mathbf{m} + \mathbf{v}^T\mathbf{y}_i) - \mathbf{m}\|^2 = \frac{1}{n}\sum_{i=1}^{n} (\mathbf{y}_i^T\mathbf{v})^2 = \boxed{\frac{1}{n}\mathbf{v}^T\mathbf{S}\mathbf{v}}$$

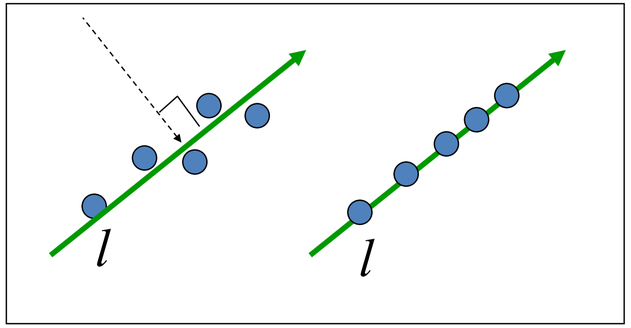




Original set



Small variance



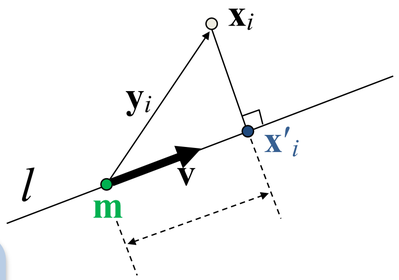
Large variance

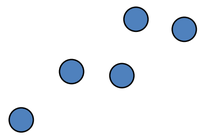
What does the Scatter Matrix do?

- The scatter matrix measures the variance of our data points along the direction $\mathbf{v}$

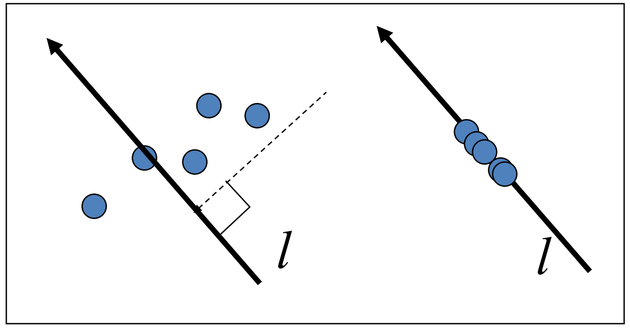
$$\mathrm{var}(\mathbf{x}_1, \ldots \mathbf{x}_n; \mathbf{v}) = \frac{1}{n}\sum_{i=1}^{n} \|\mathbf{x}'_i - \mathbf{m}\|^2 =$$

$$= \frac{1}{n}\sum_{i=1}^{n} \|(\mathbf{m} + \mathbf{v}^T\mathbf{y}_i) - \mathbf{m}\|^2 = \frac{1}{n}\sum_{i=1}^{n} (\mathbf{y}_i^T\mathbf{v})^2 = \frac{1}{n}\mathbf{v}^T\mathbf{S}\mathbf{v}$$

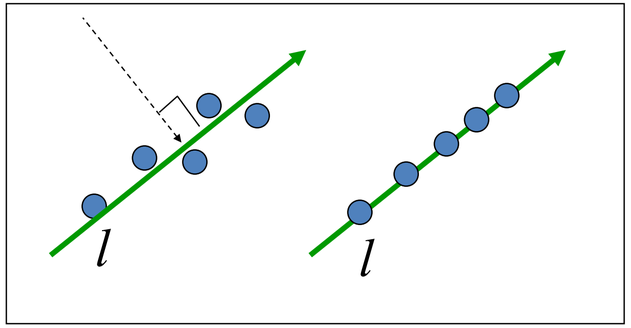




Original set



Small variance



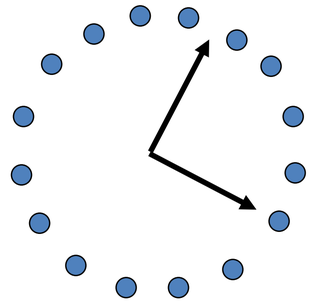
Large variance

Principal Components

- Eigenvectors of $\mathbf{S}$ that correspond to big eigenvalues are the directions in which the data has strong components (= large variance).
- If the eigenvalues are more or less the same – there is no preferable direction.

$$\mathbf{S} = \mathbf{V} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix} \mathbf{V}^T$$

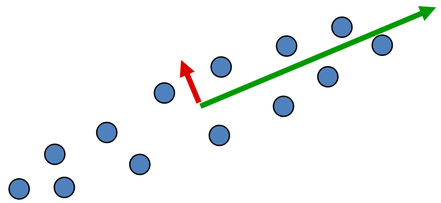
Principal Components



- There's no preferable direction
- $\mathbb{S}$ looks like this:

$$\mathbf{S} = \mathbf{V} \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \mathbf{V}^T$$

- Any vector is an eigenvector



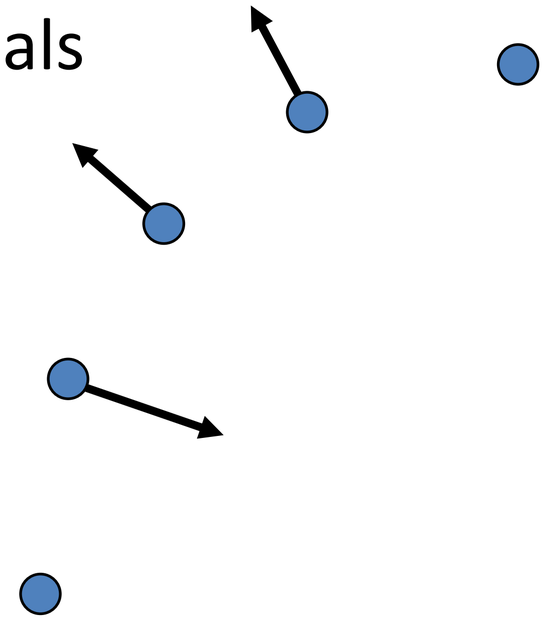
- There's a clear preferable direction
- $\mathbb{S}$ looks like this:

$$\mathbf{S} = \mathbf{V} \begin{pmatrix} \lambda & \\ & \mu \end{pmatrix} \mathbf{V}^T$$

μ is close to zero, much smaller than λ

Normal Orientation

- PCA may return arbitrarily oriented eigenvectors
- Wish to orient consistently
- Neighboring points should have similar normals



Normal Orientation

- Build graph connecting neighboring points
  - Edge (i,j) exists if $\mathbf{x}_i \in \mathrm{kNN}(\mathbf{x}_j)$ or $\mathbf{x}_j \in \mathrm{kNN}(\mathbf{x}_i)$
- Propagate normal orientation through graph
  - For neighbors $\mathbf{x}_i$, $\mathbf{x}_j$: Flip $\mathbf{n}_j$ if $\mathbf{n}_i^{\mathrm{T}}\mathbf{n}_j < 0$
  - Fails at sharp edges/corners
- Propagate along “safe” paths (parallel tangent planes)
  - Minimum spanning tree with angle-based edge weights

$$w_{ij} = 1 - \left|\mathbf{n}_i^T \mathbf{n}_j\right|$$

“Surface reconstruction from unorganized points”, Hoppe et al., SIGGRAPH 1992
http://research.microsoft.com/en-us/um/people/hoppe/recon.pdf

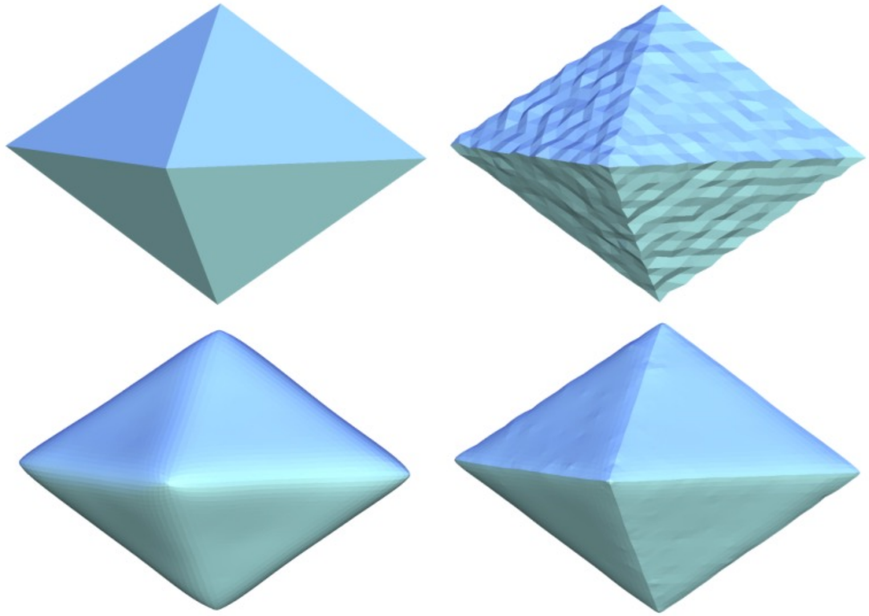
Curves

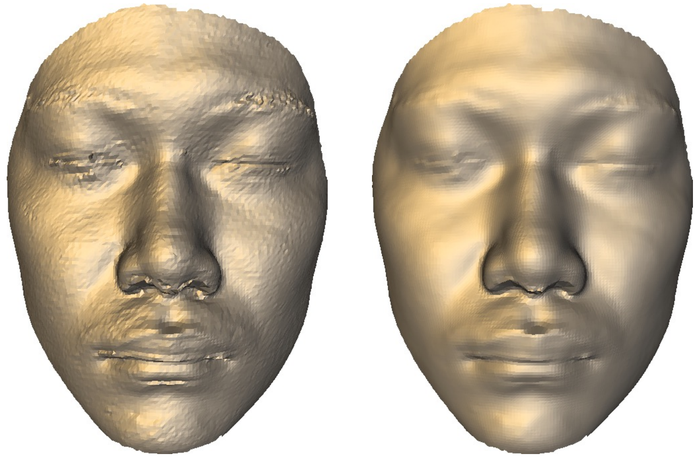
Acknowledgements: Daniele Panozzo

Elementary Differential Geometry

Differential Geometry – Motivation

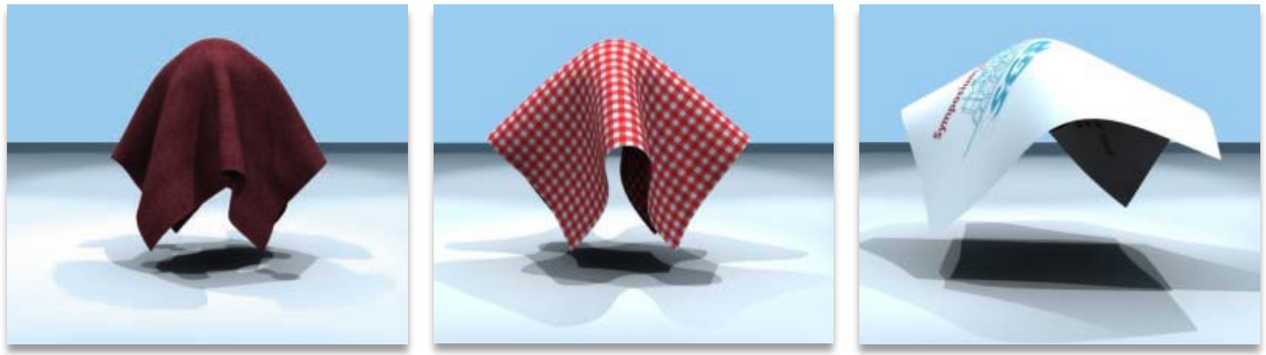
- Describe and analyze geometric characteristics of shapes
  - e.g. how smooth?

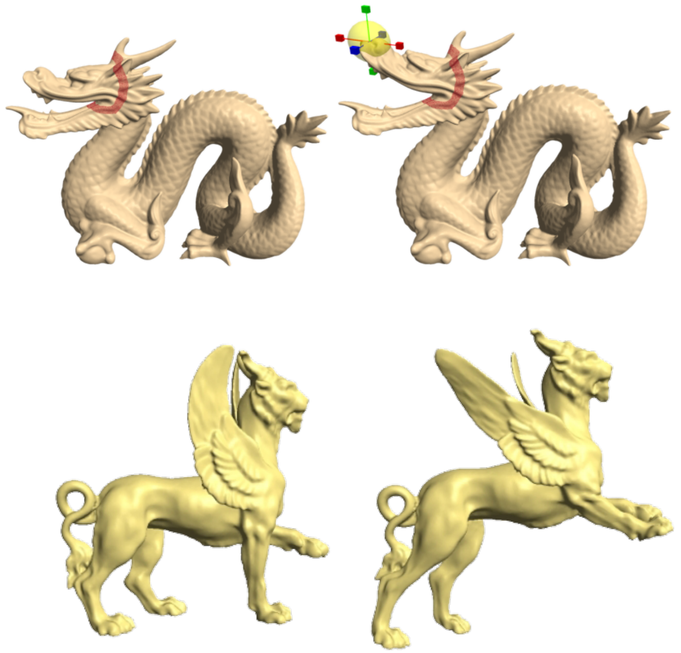




Differential Geometry – Motivation

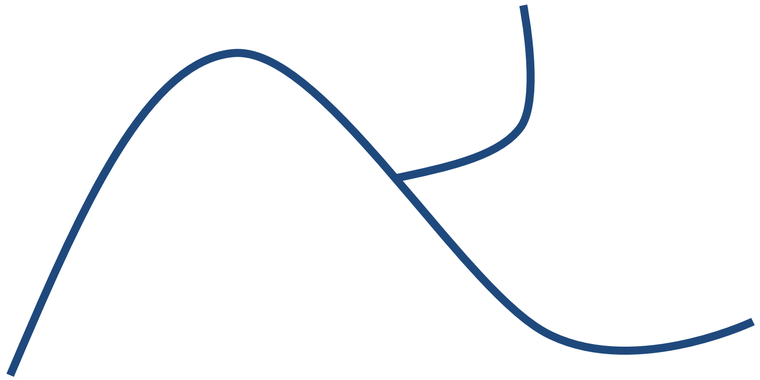
- Describe and analyze geometric characteristics of shapes
  - e.g. how smooth?
  - how shapes deform





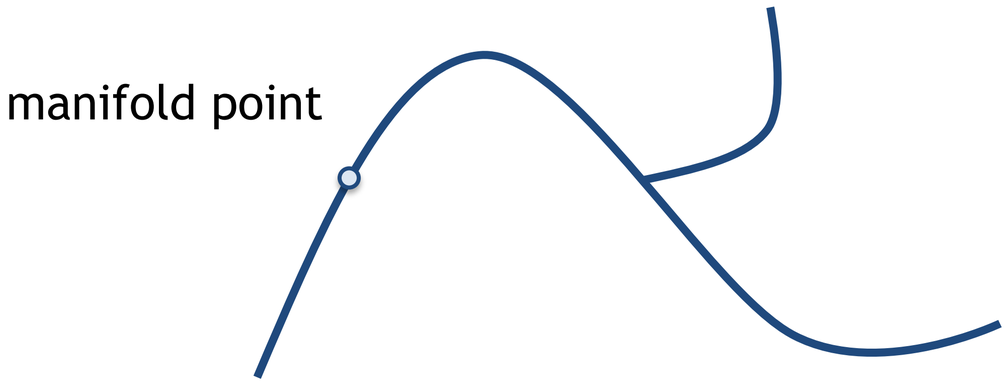
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



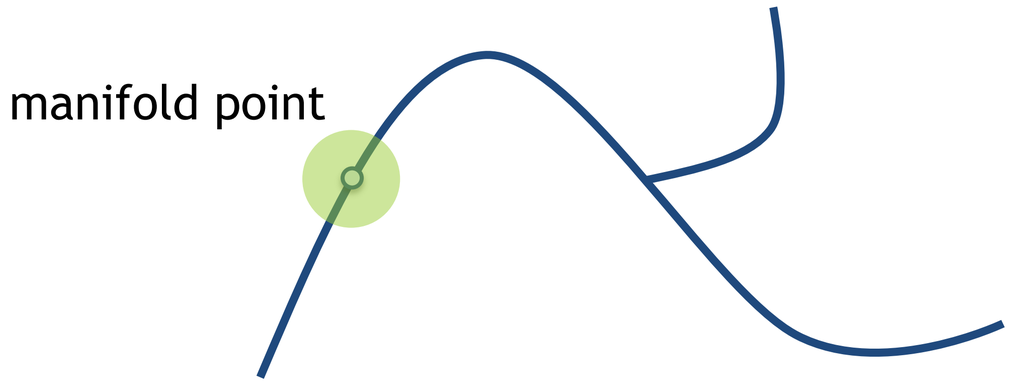
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



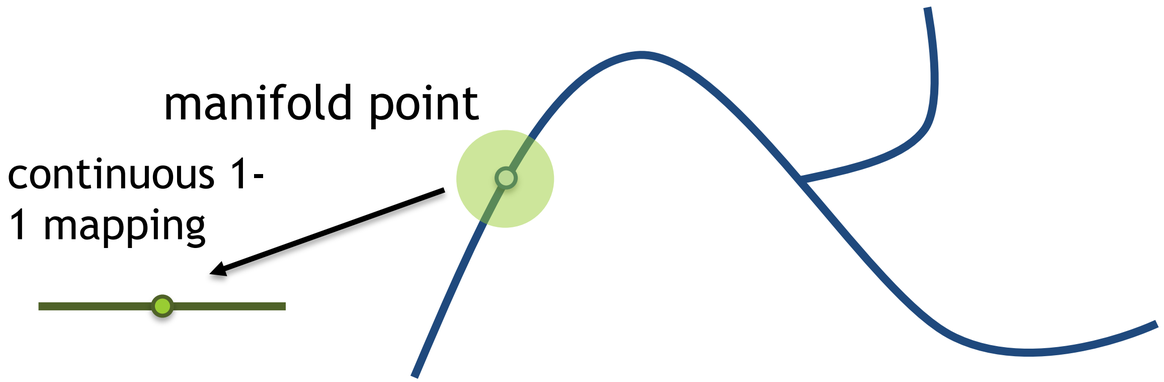
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



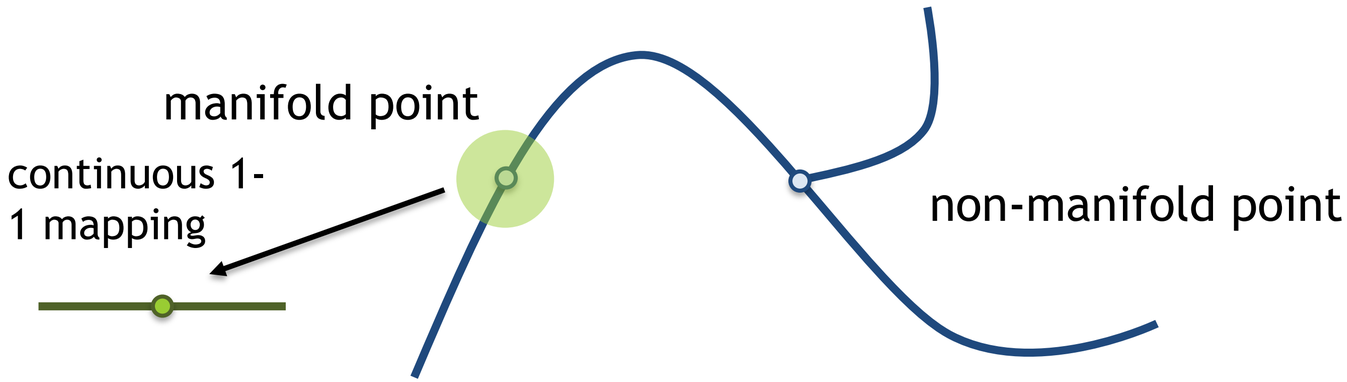
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



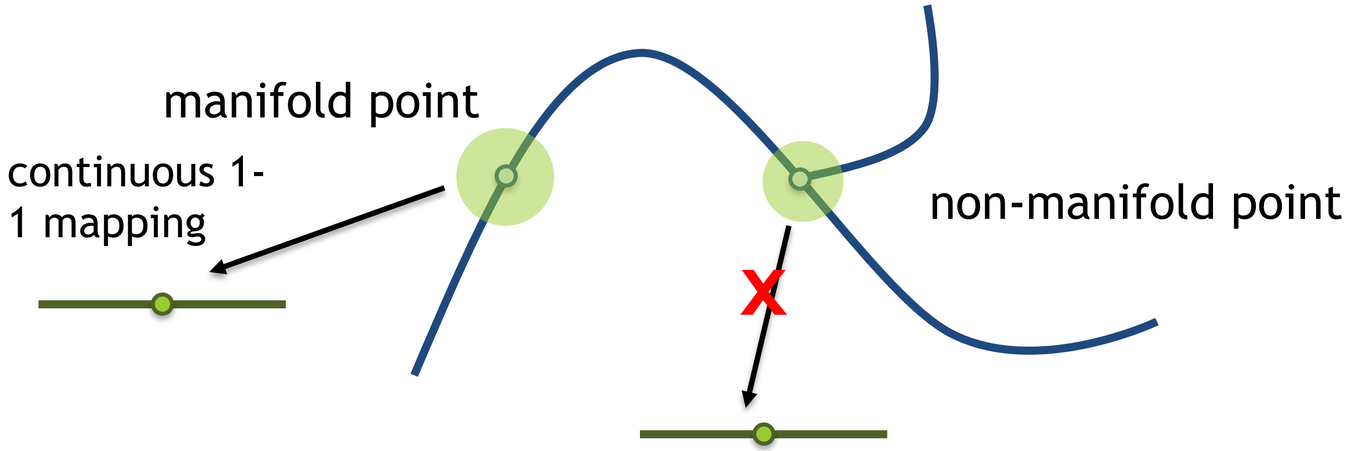
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



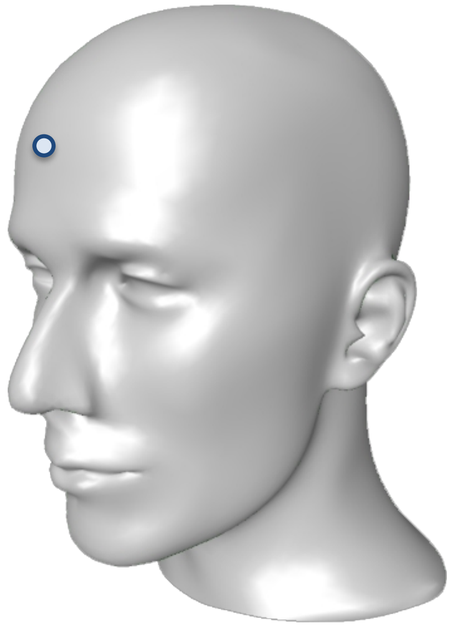
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



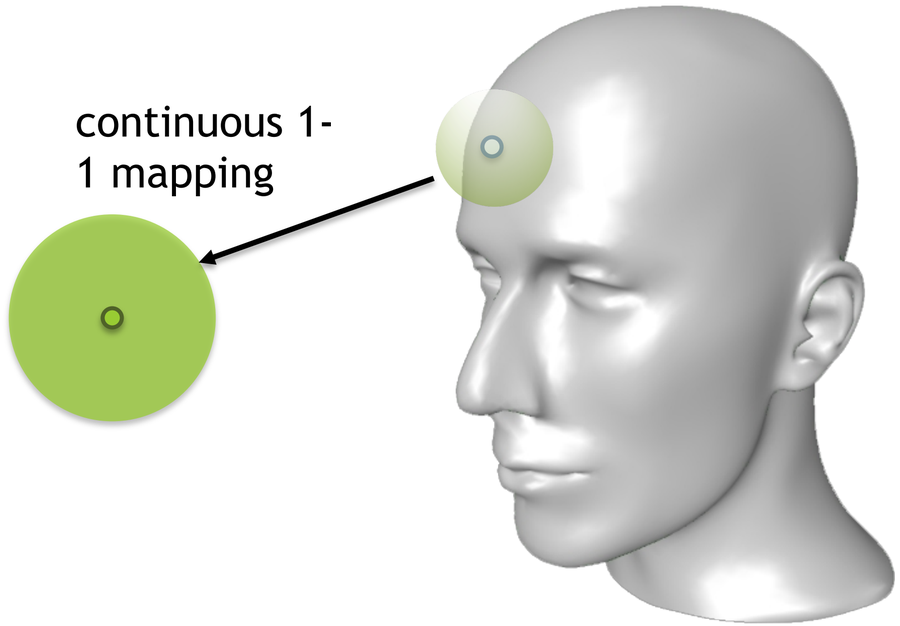
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



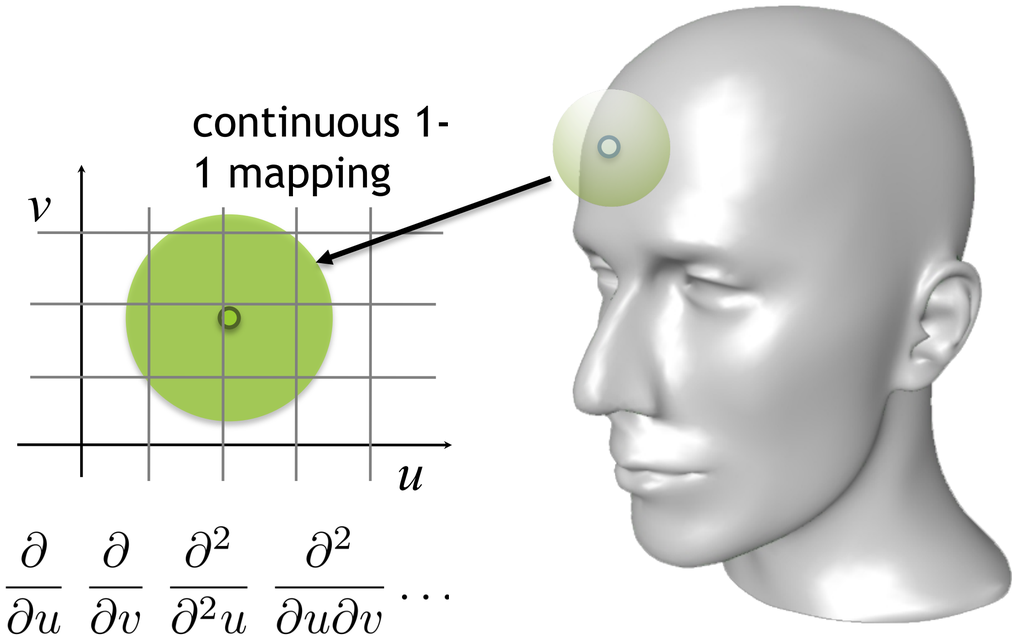
Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



Differential Geometry Basics

- Geometry of manifolds
- Things that can be discovered by local observation: point + neighborhood



If a sufficiently smooth mapping can be constructed, we can look at its first and second derivatives

Tangents, normals, curvatures, curve angles

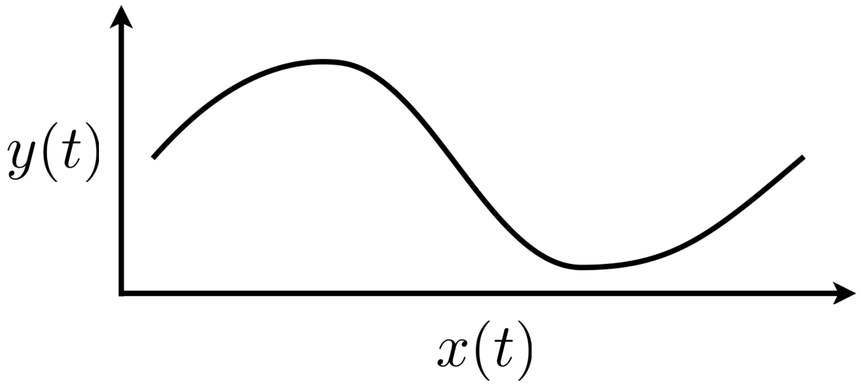
Distances, topology

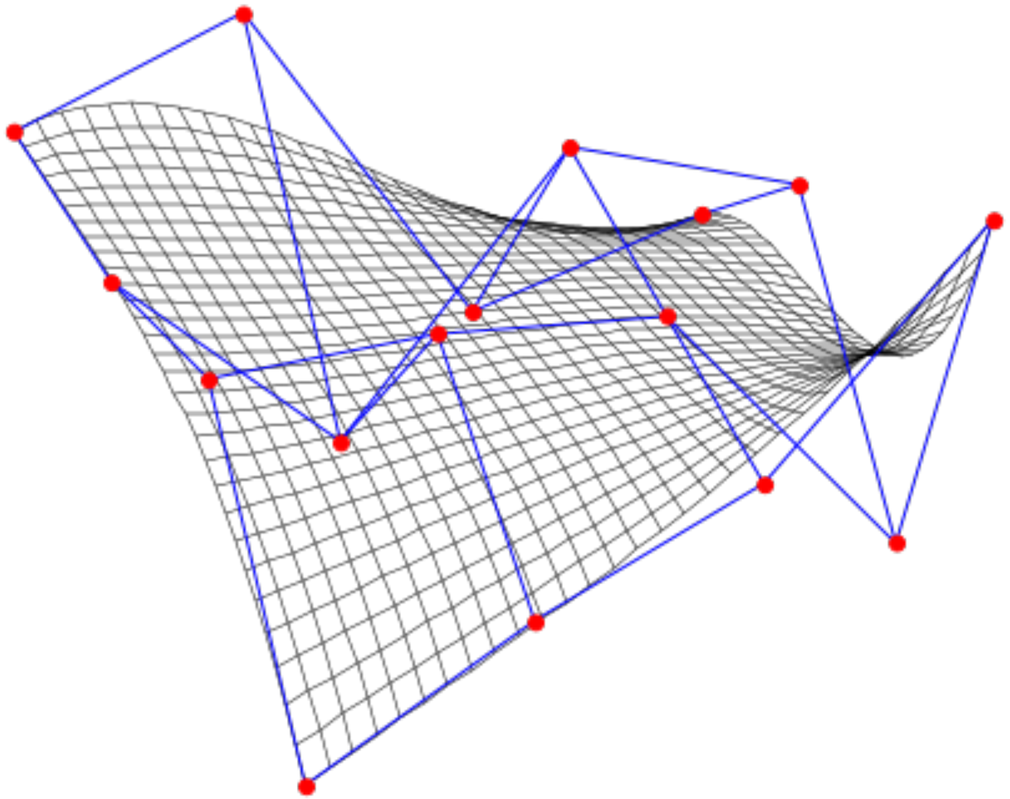
Curves

How do we model shapes?

Delcam Plc. [CC BY-SA 3.0 (http://creativecommons.org/licenses/by-sa/3.0) or GFDL (http://www.gnu.org/copyleft/fdl.html)], via Wikimedia Commons

Building blocks: curves and surfaces





By Wojciech mula at Polish Wikipedia, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=9853555

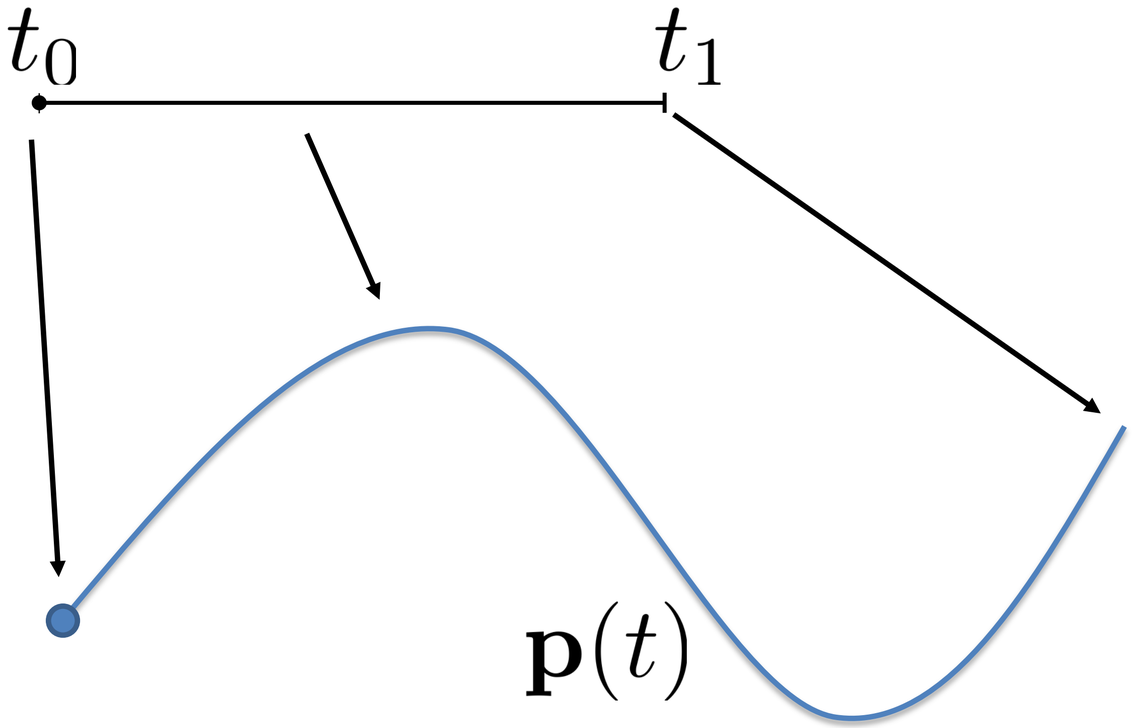
Modeling curves

- We need **mathematical concepts** to characterize the desired curve properties
- Notions from **curve geometry** help with designing user interfaces for curve creation and editing

2D parametric curve

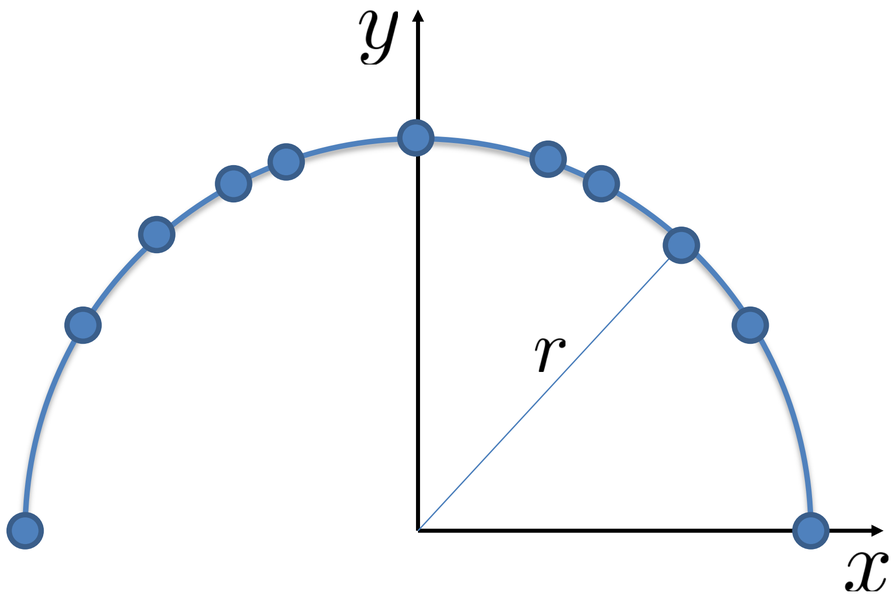
- must be continuous

$$\mathbf{p}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \; t \in [t_0, t_1]$$

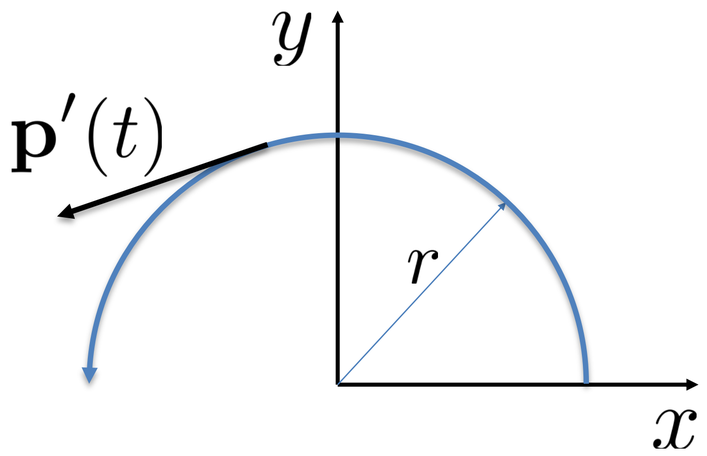


A curve can be parameterized in many different ways

$$\begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}, t \in [0, \pi] \qquad \begin{pmatrix} -rt \\ r\sqrt{1-t^2} \end{pmatrix}, t \in [-1, 1]$$



Tangent vector



$$\mathbf{p}(t) = \begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}$$

$$\mathbf{p}'(t) = \begin{pmatrix} -r\sin t \\ r\cos t \end{pmatrix}$$

$$\|\mathbf{p}'(t)\| = \text{speed}$$

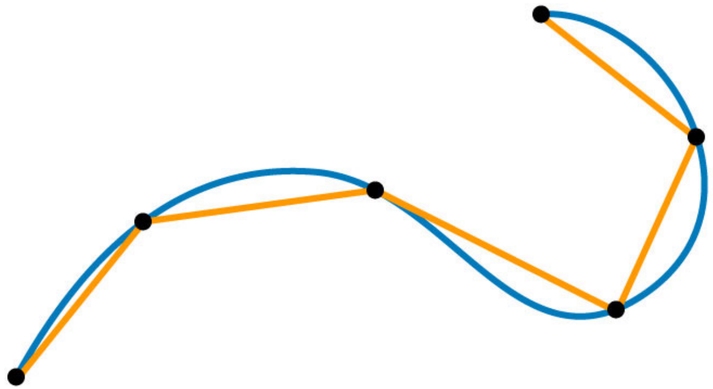
$$\frac{\mathbf{p}'(t)}{\|\mathbf{p}'(t)\|} = \mathbf{T}(t) = \text{unit tangent}$$

Parametrization-independent!

Arc length

- How long is the curve between t_0 and t? How far does the particle travel?
- Speed is $\|\mathbf{p}'(t)\|$, so:

$$s(t) = \int_{t_0}^{t} \|\mathbf{p}'(t)\| dt$$

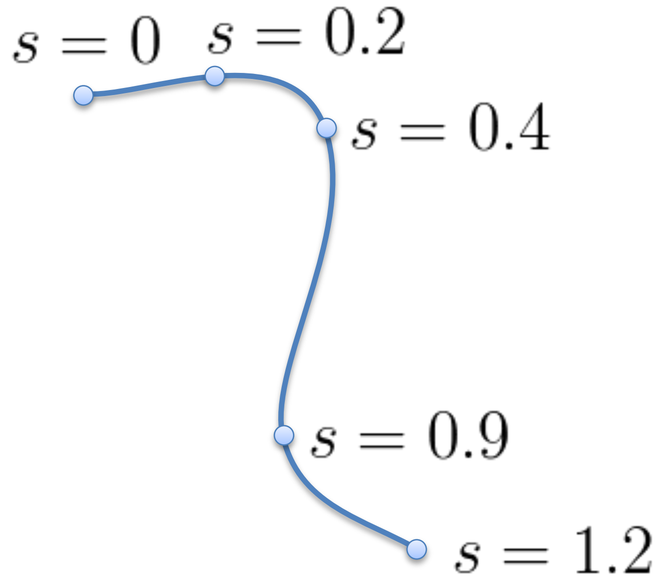


- Speed is nonnegative, so $s(t)$ is non-decreasing

Arc length parameterization

- Every curve has a natural parameterization:

$$\mathbf{p}(s), \text{ such that } \|\mathbf{p}'(s)\| = 1$$



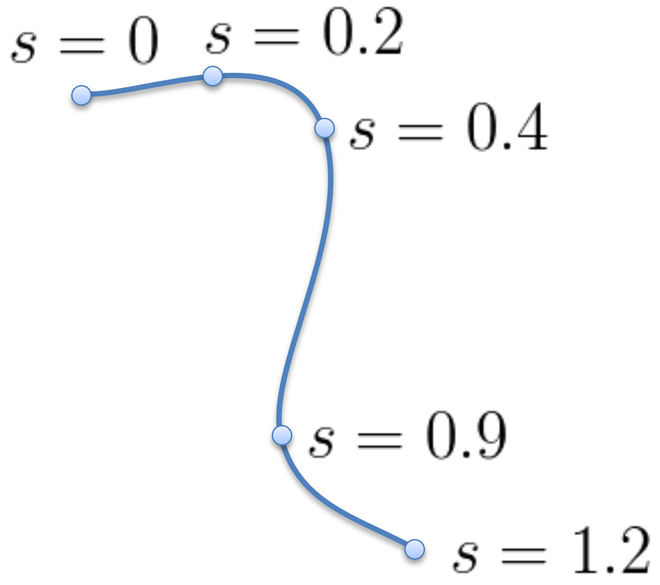
Arc length parameterization

- Every curve has a natural parameterization:

$$\mathbf{p}(s), \text{ such that } \|\mathbf{p}'(s)\| = 1$$

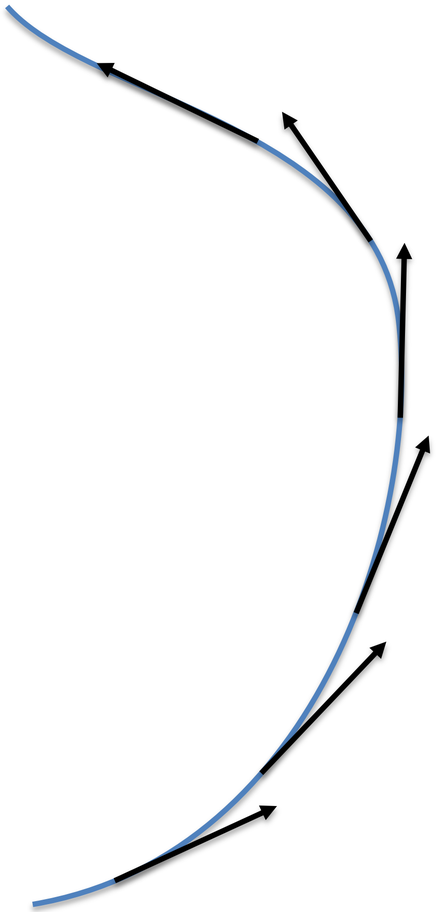
- Isometry between parameter domain and curve
- Tangent vector is unit-length:

$$\mathbf{p}'(s) = \mathbf{T}(s)$$

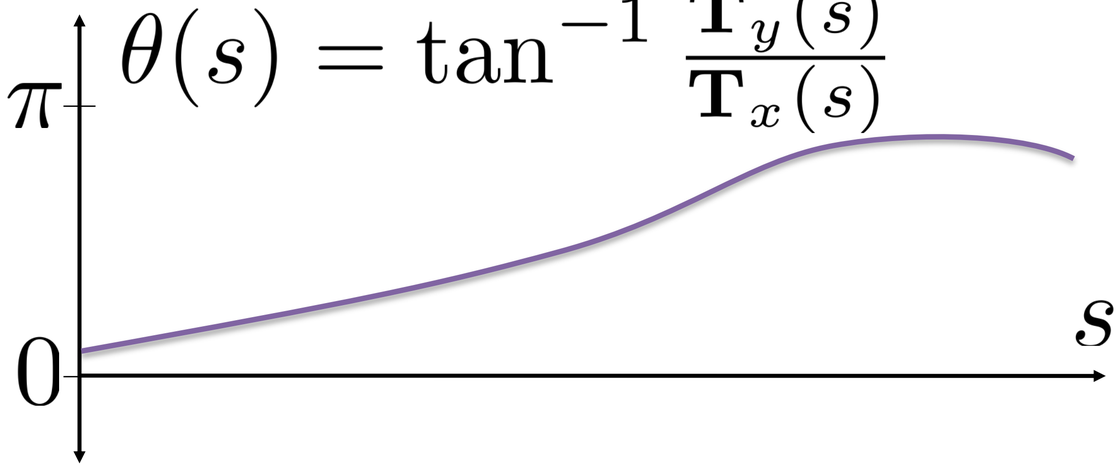


Curvature

- How much does the curve turn per unit s?

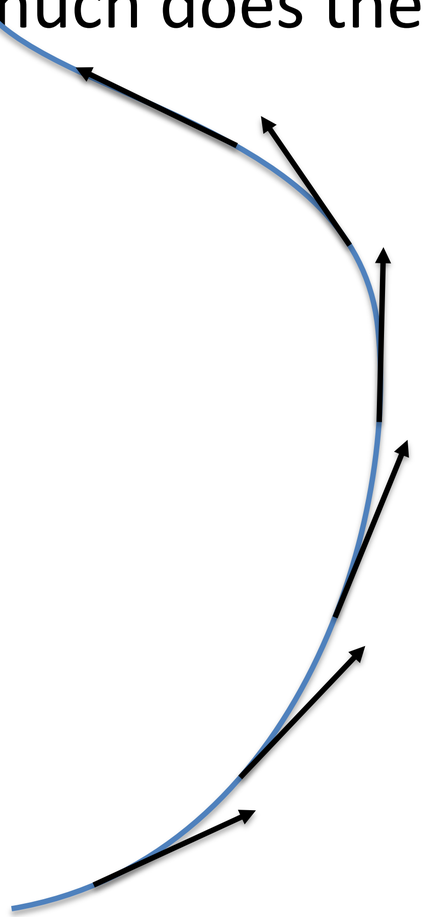


$$\theta(s) = \tan^{-1} \frac{\mathbf{T}_y(s)}{\mathbf{T}_x(s)}$$

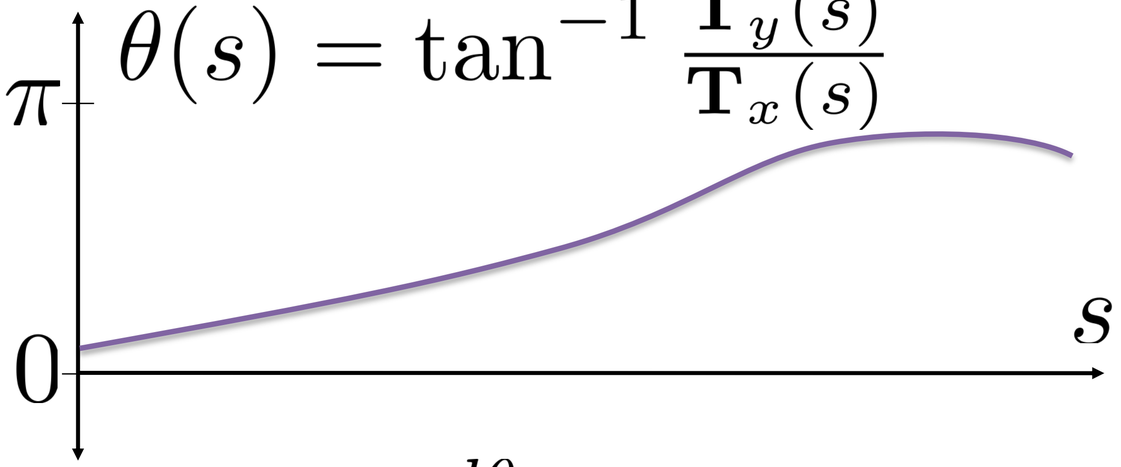


Curvature

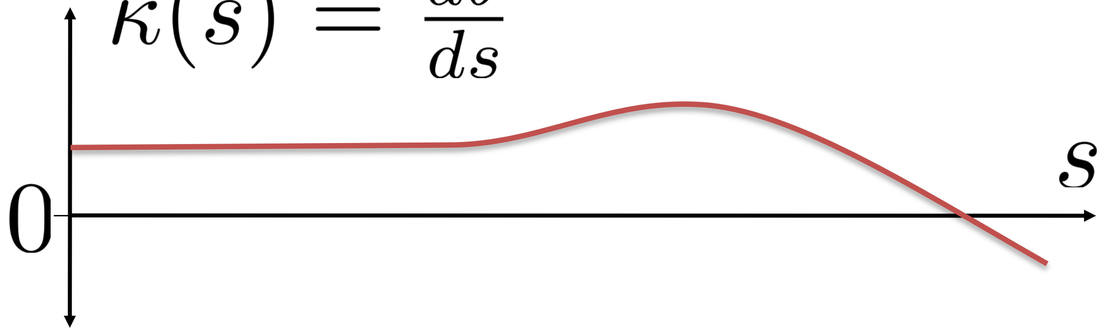
- How much does the curve turn per unit s ?



$$\theta(s) = \tan^{-1} \frac{\mathbf{T}_y(s)}{\mathbf{T}_x(s)}$$



$$\kappa(s) = \frac{d\theta}{ds}$$



Curvature of a circle

- Curvature of a circle:

$$\mathbf{p}(s) = \begin{pmatrix} r\cos(s/r) \\ r\sin(s/r) \end{pmatrix}$$

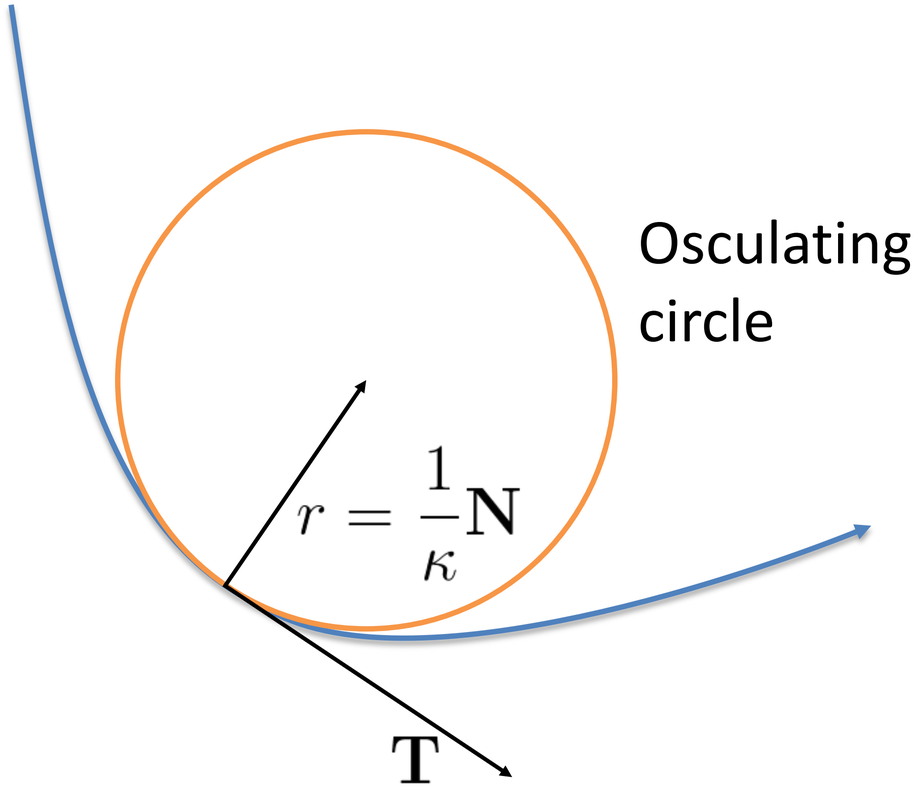
$$\mathbf{p}'(s) = \begin{pmatrix} -\sin(s/r) \\ \cos(s/r) \end{pmatrix}$$

$$\theta(s) = \tan^{-1}\frac{\cos(s/r)}{-\sin(s/r)}$$

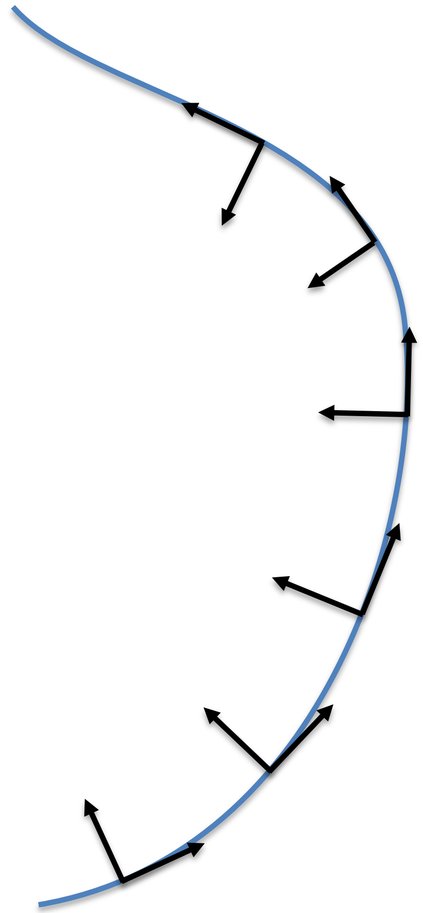
$$= s/r - \pi/2$$

$$\kappa(s) = 1/r$$

$$\mathbf{N}(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{T}(t)$$ = Unit Normal



Frenet Frame



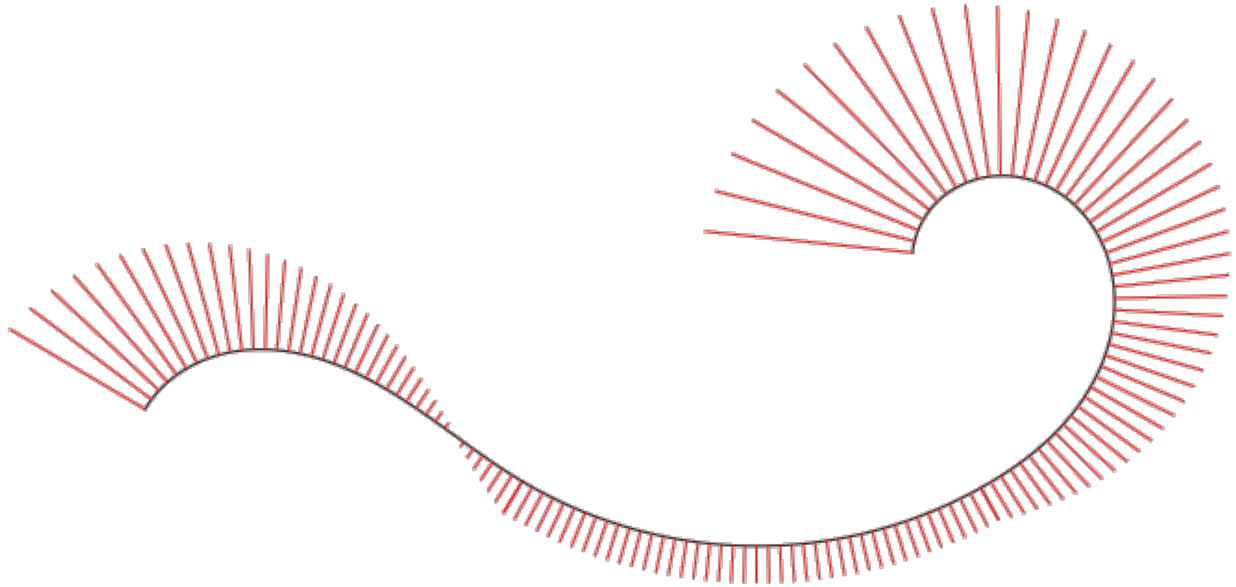
$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$$
$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s)$$

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}$$

Curvature normal

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$$

- Points inward
- $-\kappa(s)\mathbf{N}(s)$ useful for evaluating curve quality



Smoothness

Two kinds, parametric and geometric:

$$C^1 : \mathbf{p}(t) \text{ is continuously differentiable}$$
$$G^1 : \mathbf{p}(s) \text{ is continuously differentiable}$$

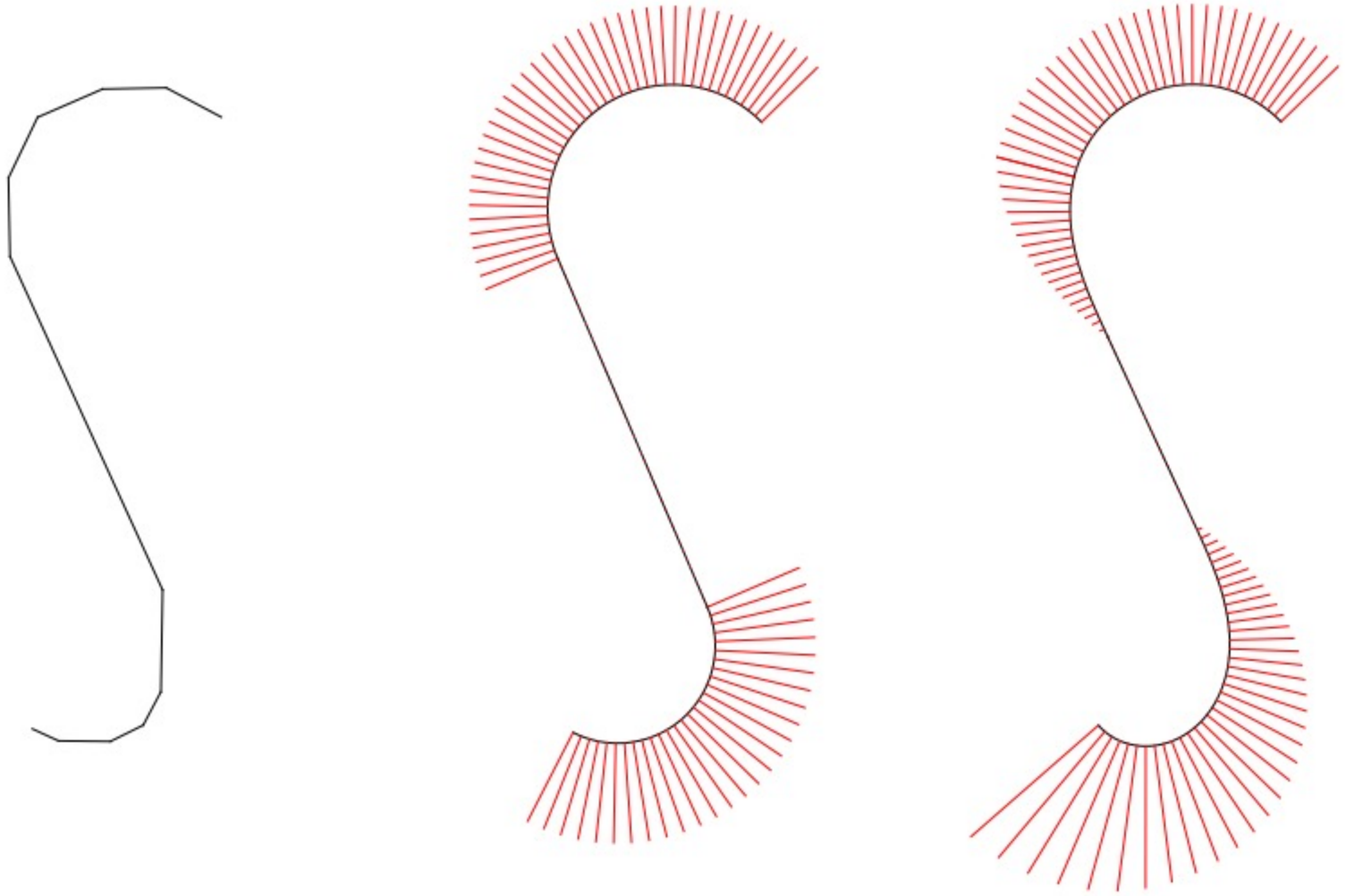
Parametrization-Independent

$$C^1 \qquad \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$G^1 \qquad \begin{pmatrix} \cos \hat{t} \\ \sin \hat{t} \end{pmatrix}, \; \hat{t} = \begin{cases} t+1 & \text{if } t < 1 \\ 2t & \text{if } t \geq 1 \end{cases}$$

Smoothness example

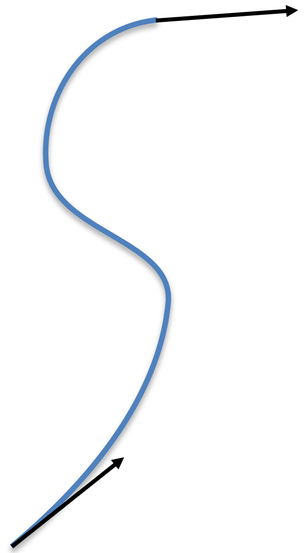
G^0 G^1 G^2



Turning

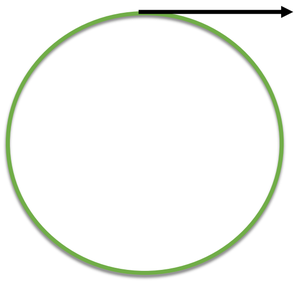
- Angle from start tangent to end tangent:

$$\int_{s_0}^{s_1} \kappa(s)ds = \int_{t_0}^{t_1} \kappa(t)\|\mathbf{p}'(t)\|dt$$



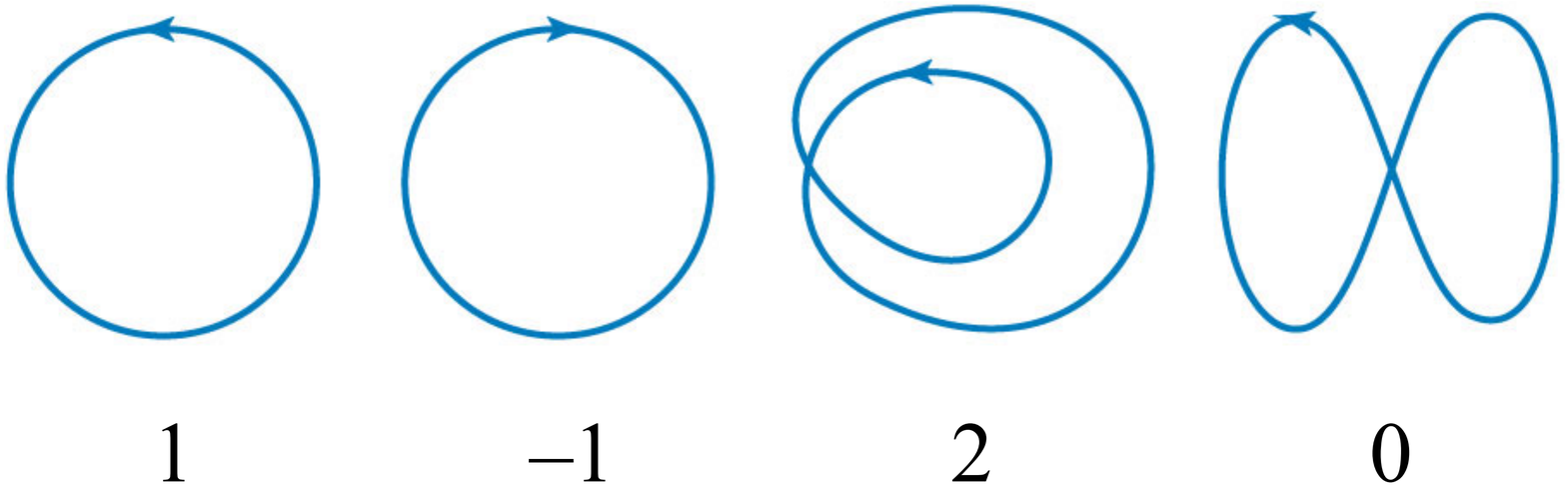
- If curve is closed, the tangent at the beginning is the same as the tangent at the end

$$\oint_{\mathbf{p}} \kappa(s)ds = 2\pi n$$



Turning numbers

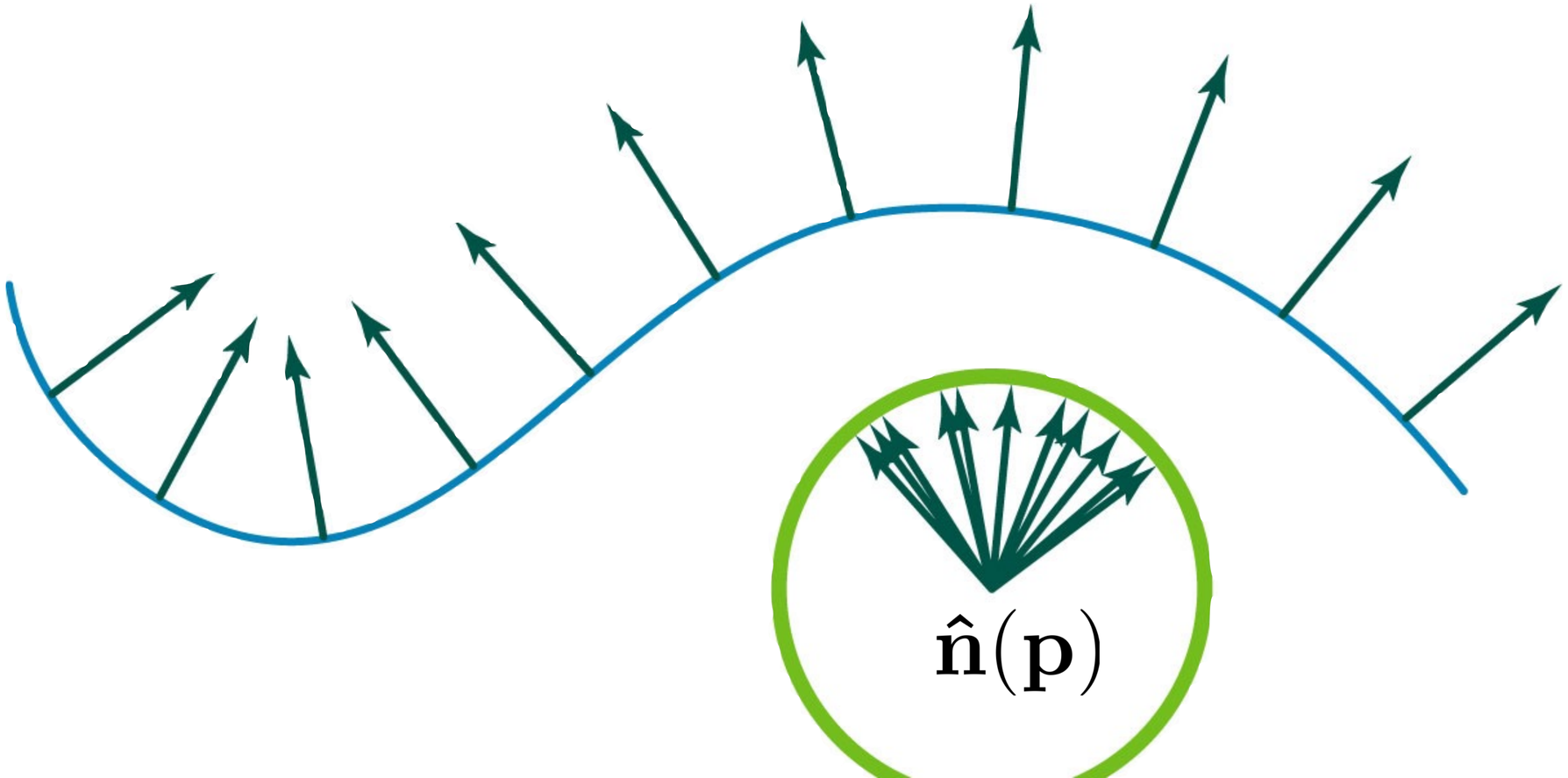
$$\oint_{\mathbf{p}} \kappa(s)ds = 2\pi n$$



- n measures how many full turns the tangent makes.

Gauss map

- Point on curve maps to point on unit circle.



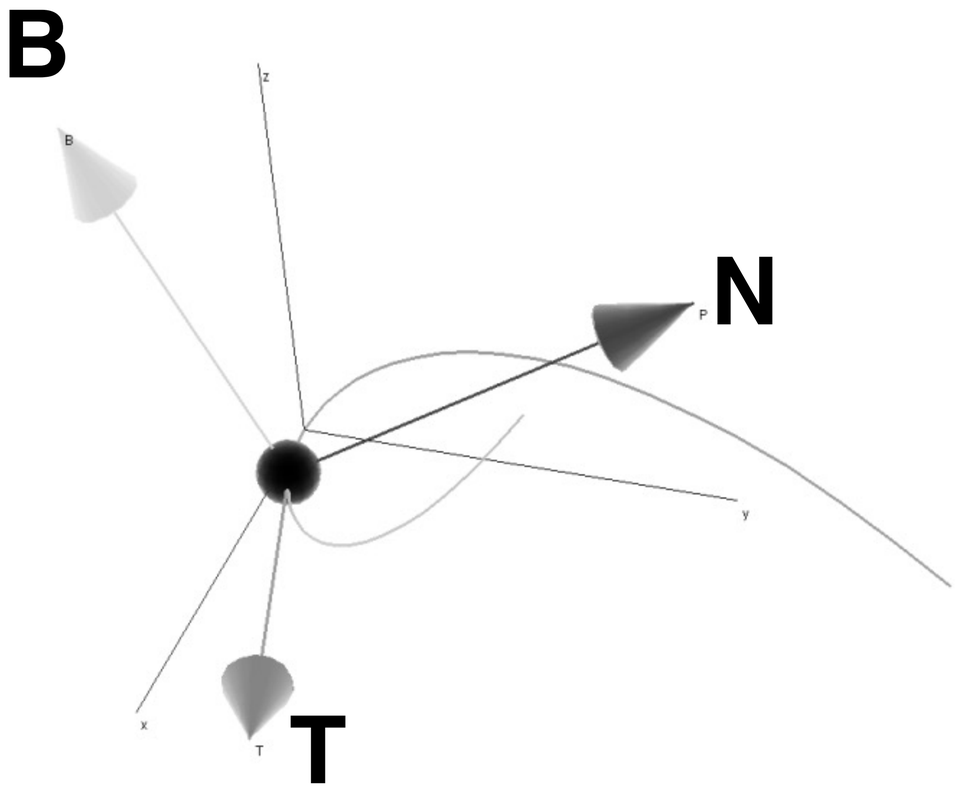
Space curves (3D)

- In 3D, many vectors are orthogonal to **T**

$$\mathbf{N}(s) := \mathbf{T}'(s)/\|\mathbf{T}'(s)\|$$
$$\mathbf{B}(s) := \mathbf{T}(s) \times \mathbf{N}(s)$$

- $\mathbf{T}, \mathbf{N}, \mathbf{B}$ are the “Frenet frame”
- τ is torsion: non-planarity

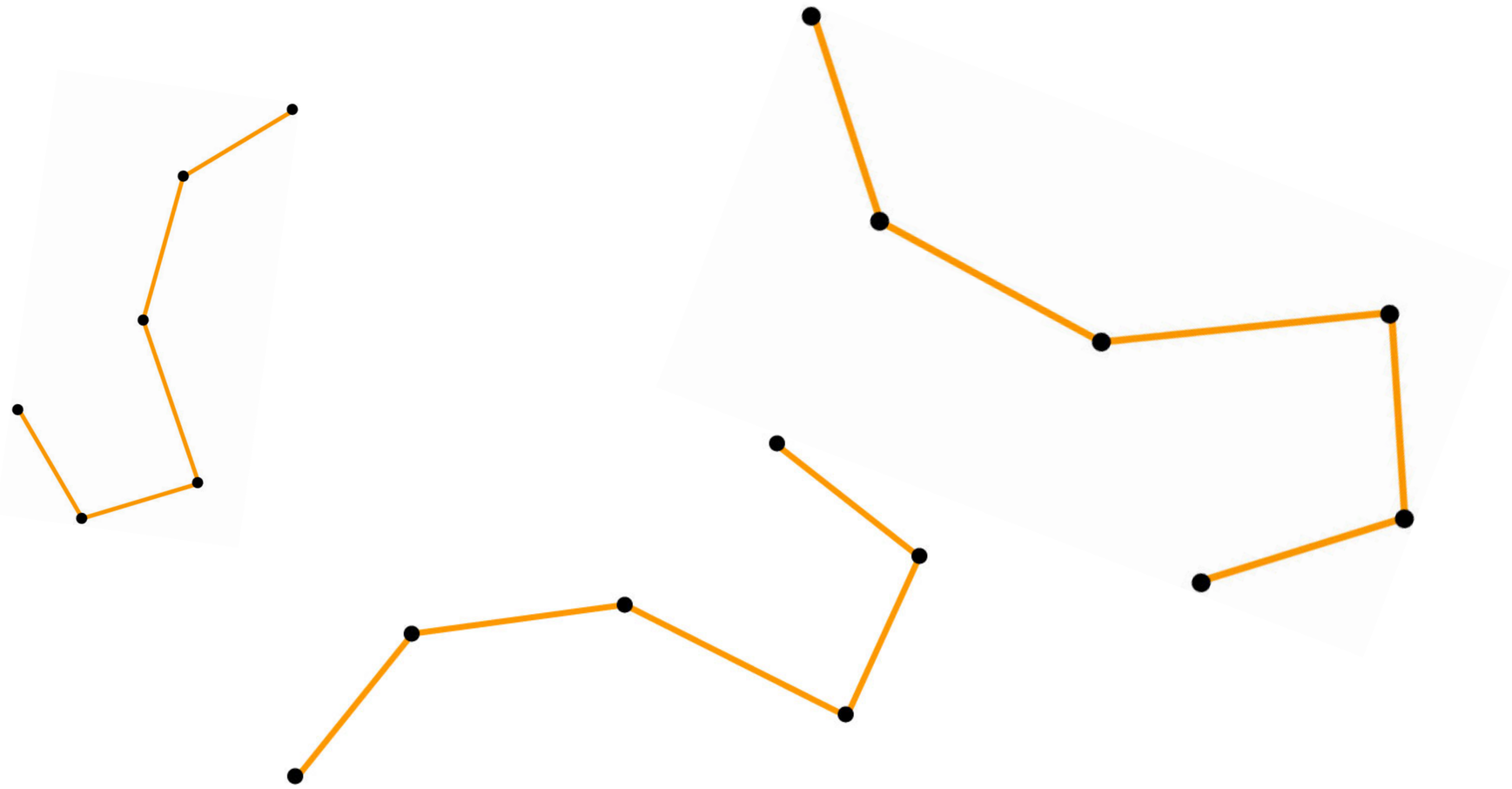
$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} & \kappa & \\ -\kappa & & \tau \\ & -\tau & \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$



Discrete Differential Geometry of Curves

Some references: see
http://ddg.cs.columbia.edu/

Discrete Planar Curves

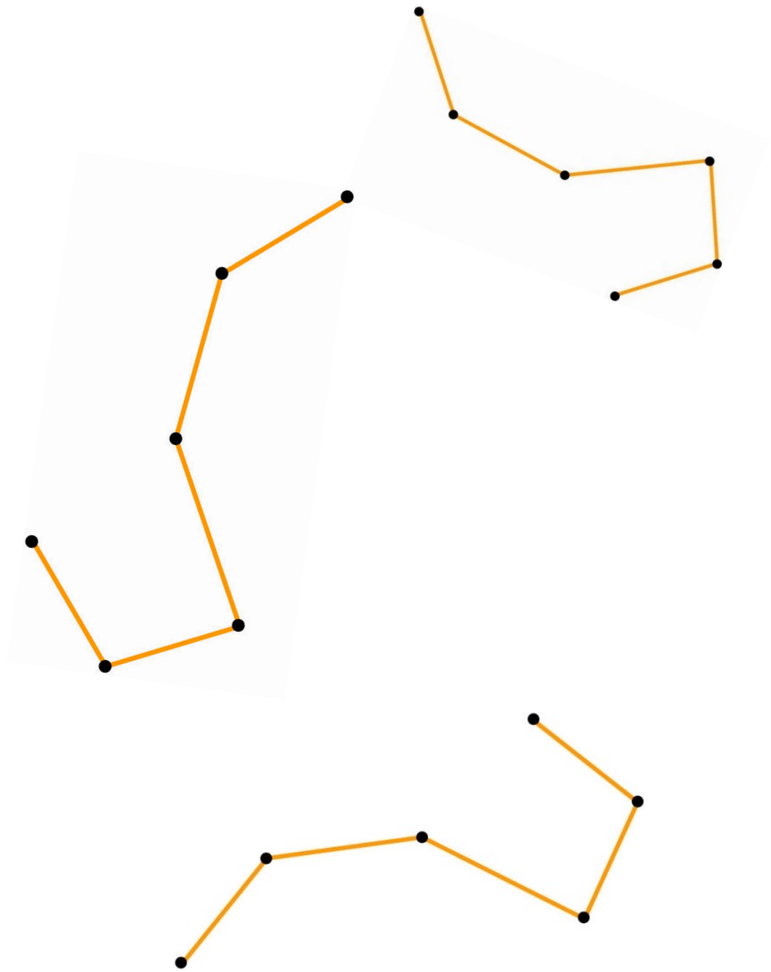


Discrete Planar Curves

- Piecewise linear curves
- Not smooth at vertices
- Can’t take derivatives

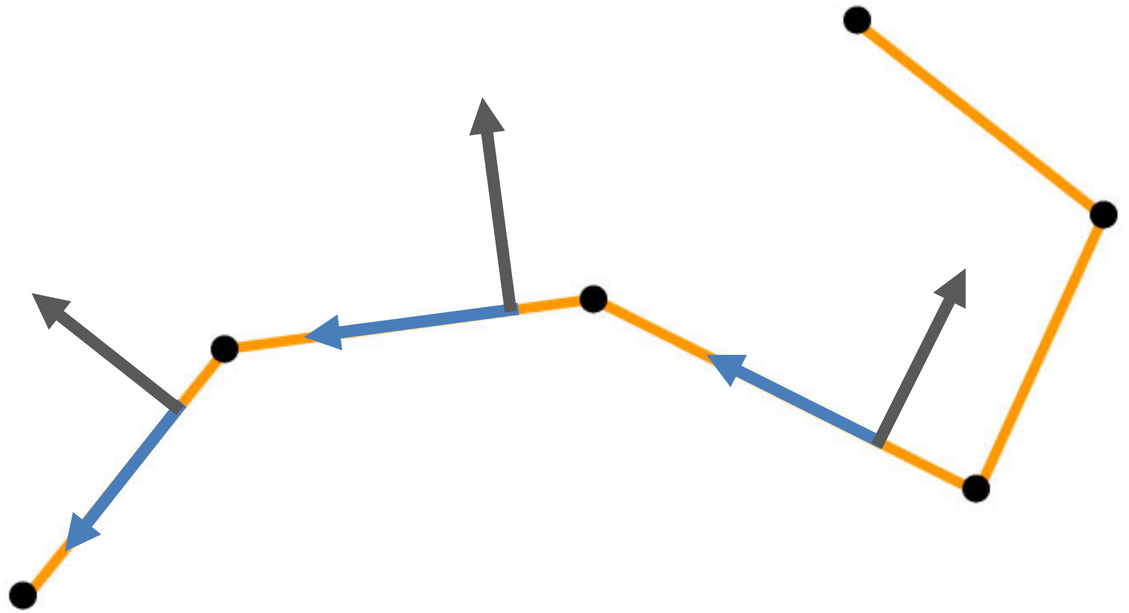
- Generalize notions from the smooth world for the discrete case!

- **There is no one single way**



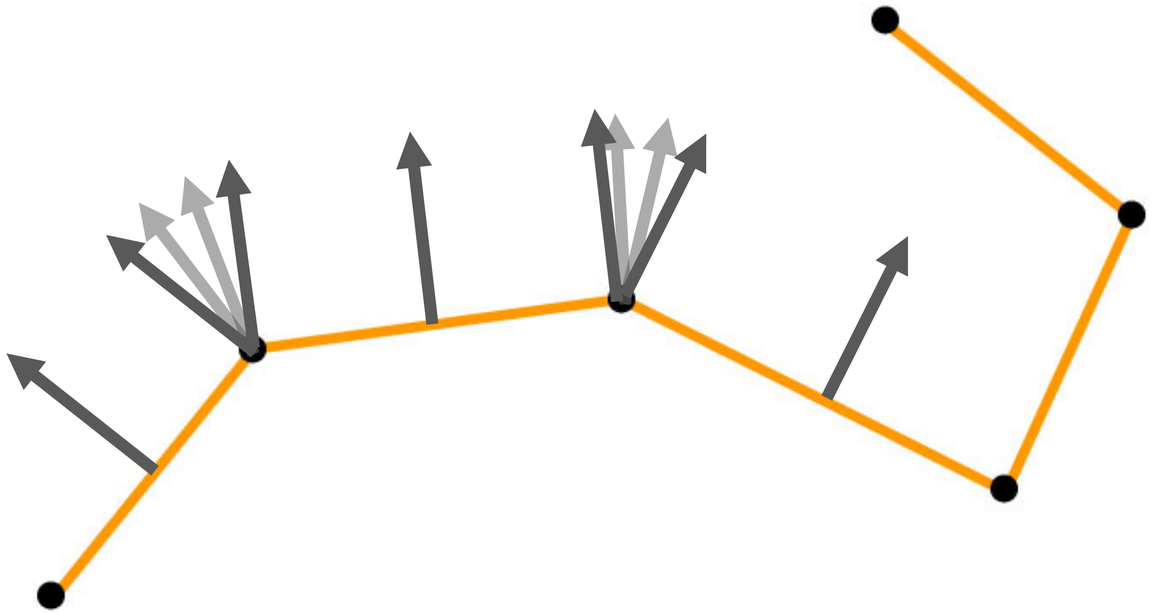
Tangents, Normals

- For any point on the edge, the tangent is simply the unit vector along the edge and the normal is the perpendicular vector



Tangents, Normals

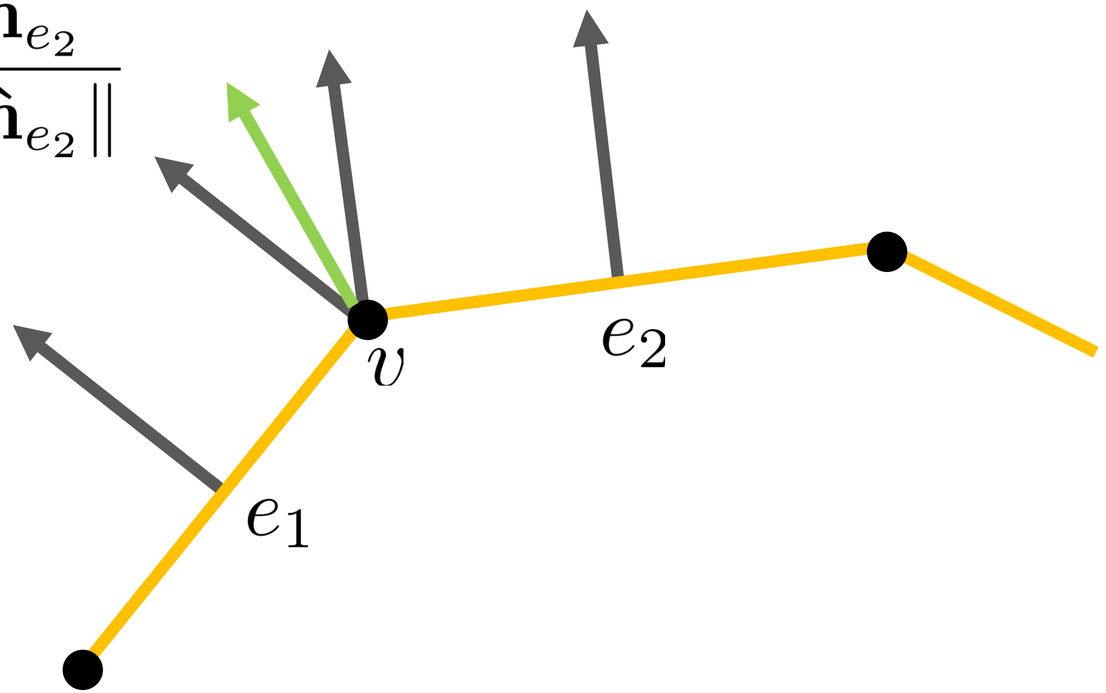
- For vertices, we have many options



Tangents, Normals

- Can choose to average the adjacent edge normals

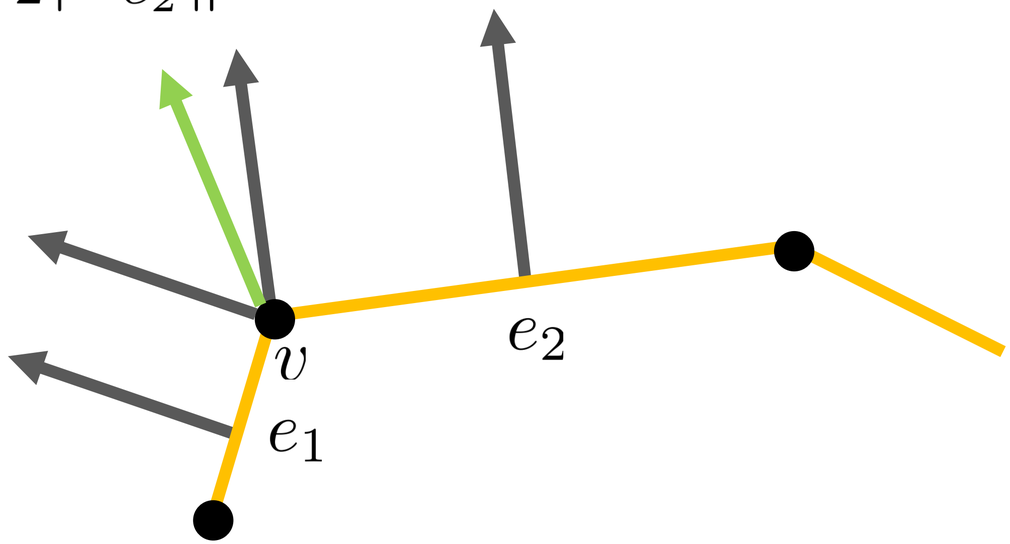
$$\hat{\mathbf{n}}_v = \frac{\hat{\mathbf{n}}_{e_1} + \hat{\mathbf{n}}_{e_2}}{\|\hat{\mathbf{n}}_{e_1} + \hat{\mathbf{n}}_{e_2}\|}$$



Tangents, Normals

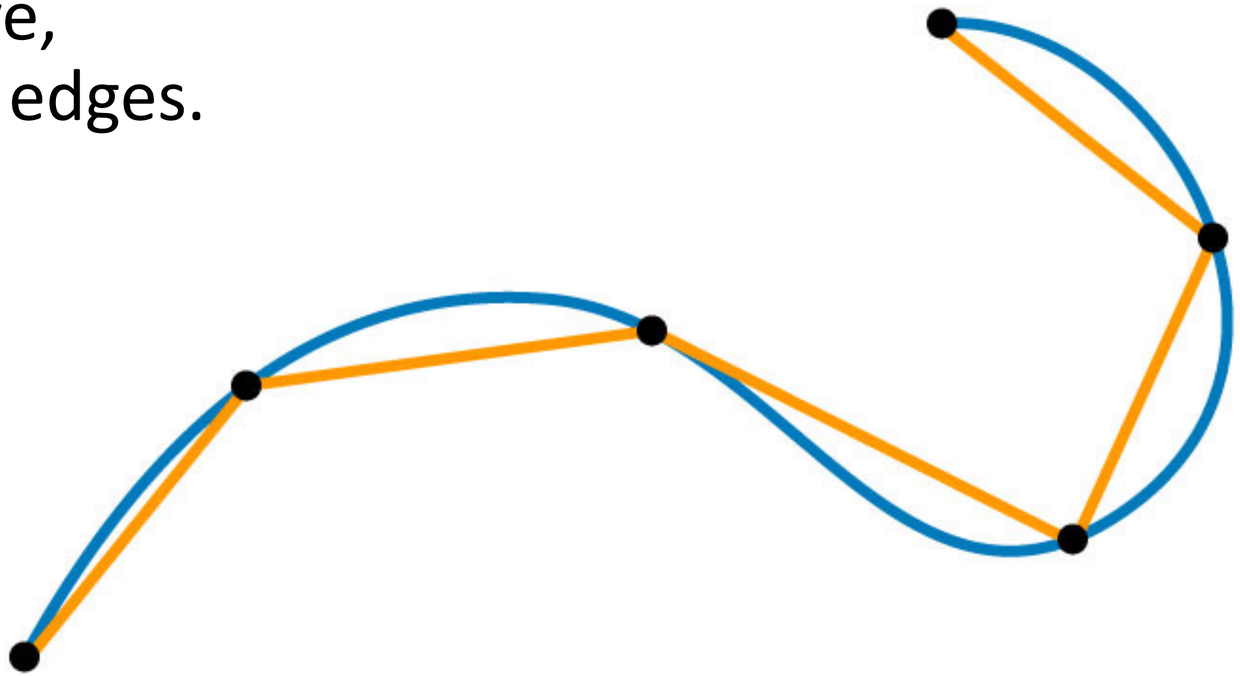
- Weight by edge lengths

$$\hat{\mathbf{n}}_v = \frac{|e_1|\hat{\mathbf{n}}_{e_1} + |e_2|\hat{\mathbf{n}}_{e_2}}{\||e_1|\hat{\mathbf{n}}_{e_1} + |e_2|\hat{\mathbf{n}}_{e_2}\|}$$



Inscribed Polygon, p

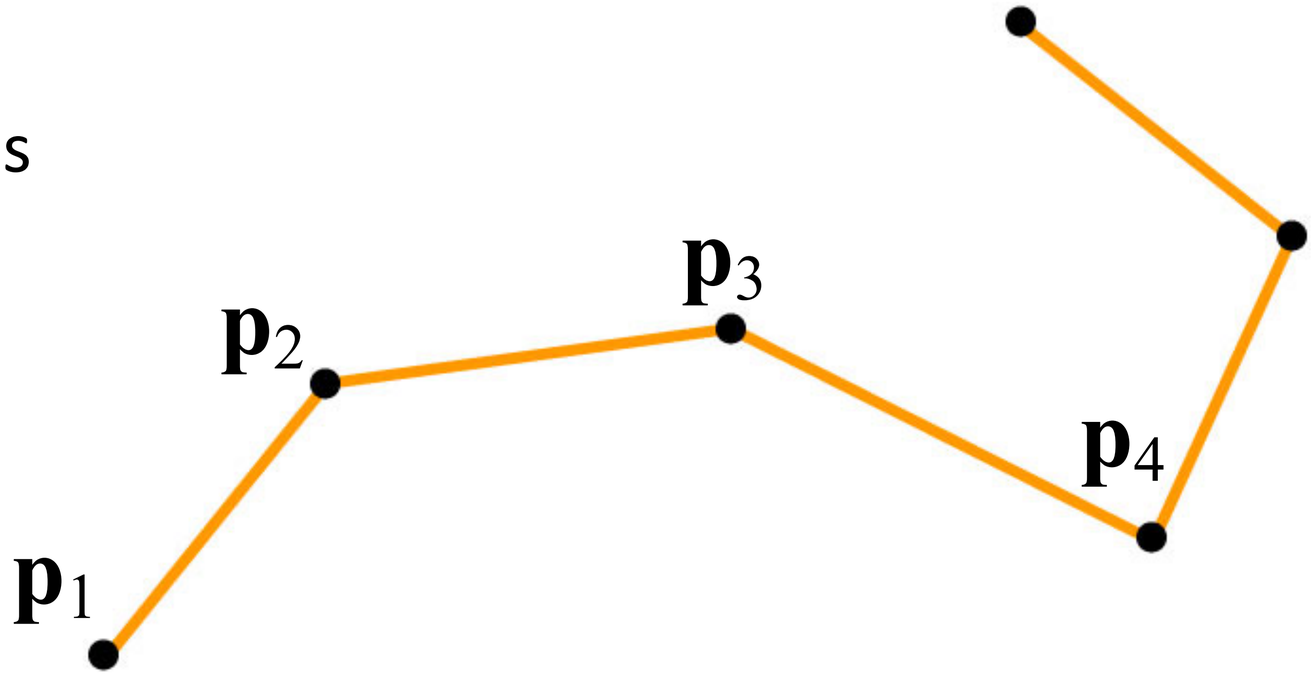
- Connection between discrete and smooth
- Finite number of vertices
  each lying on the curve,
  connected by straight edges.



The Length of a Discrete Curve

$$\mathrm{len}(p) = \sum_{i=1}^{n-1} \|\mathbf{p}_{i+1} - \mathbf{p}_i\|$$

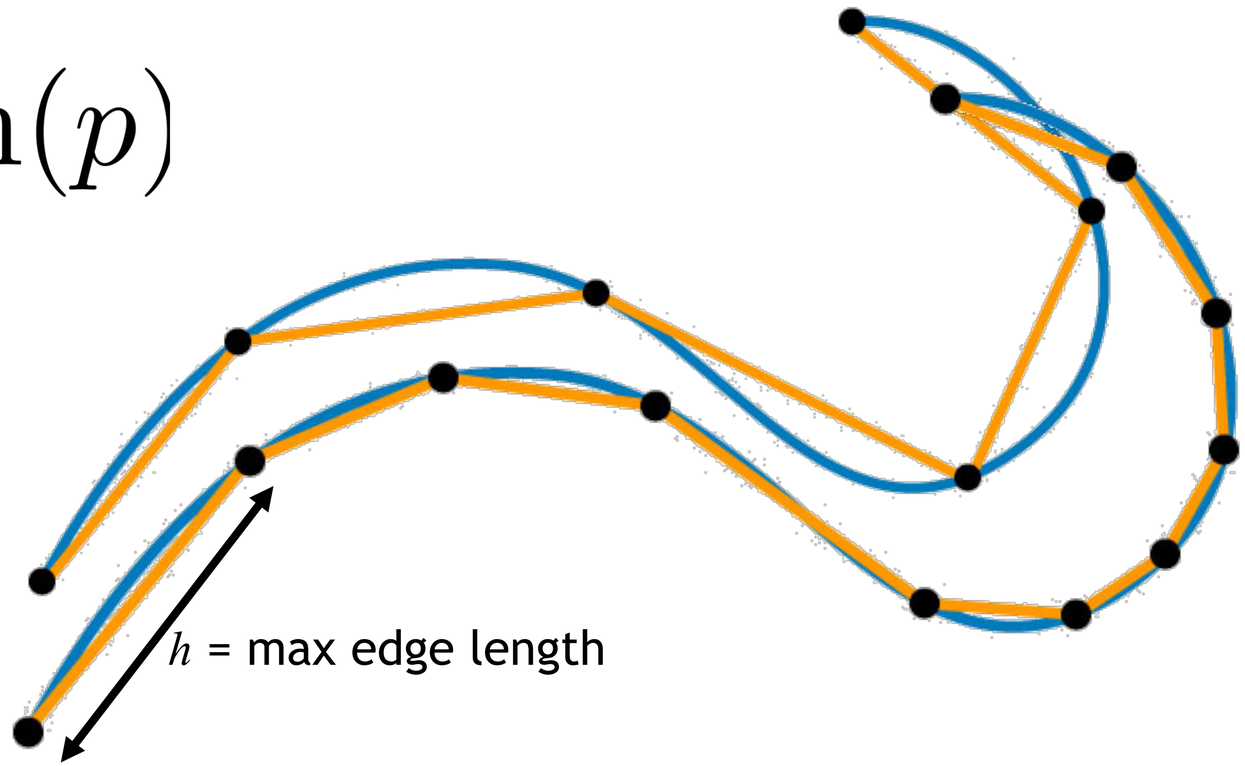
- Sum of edge lengths



The Length of a Continuous Curve

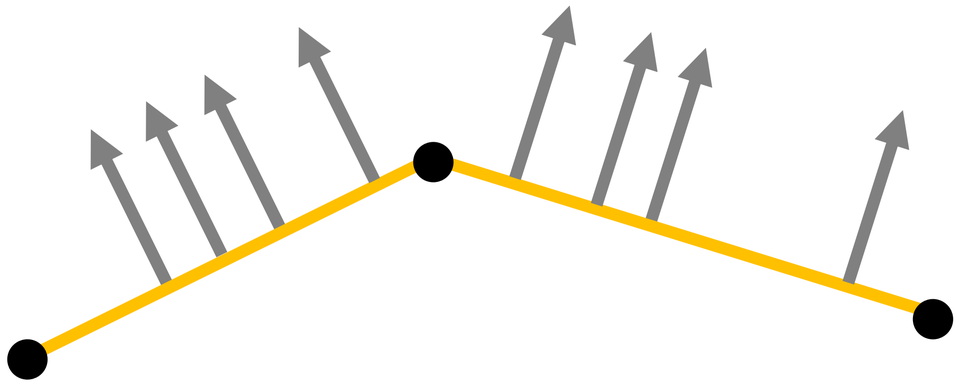
- Take limit over a refinement sequence

$$\lim_{h \to 0} \operatorname{len}(p)$$



Curvature of a Discrete Curve

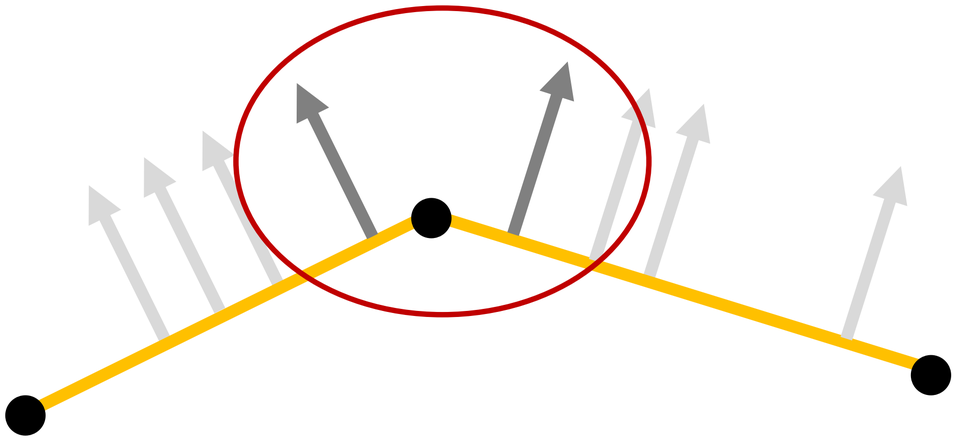
- Curvature is the change in normal direction as we travel along the curve



no change along each edge –
curvature is zero along edges

Curvature of a Discrete Curve

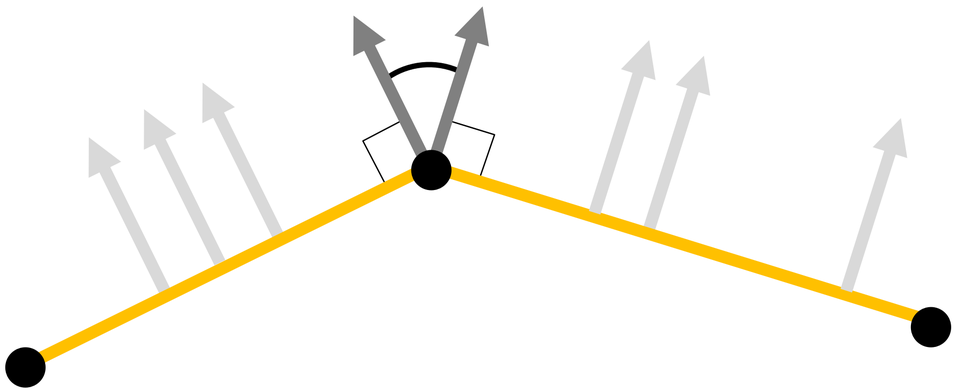
- Curvature is the change in normal direction as we travel along the curve



normal changes at vertices –
record the turning angle!

Curvature of a Discrete Curve

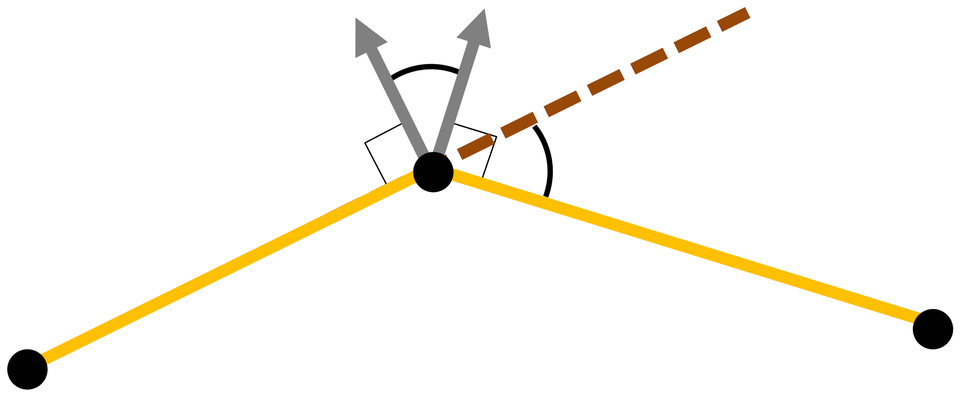
- Curvature is the change in normal direction as we travel along the curve



normal changes at vertices –
record the turning angle!

Curvature of a Discrete Curve

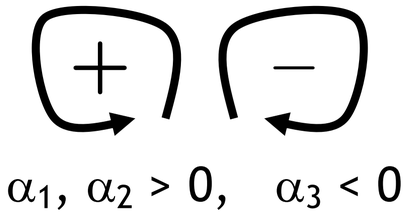
- Curvature is the change in normal direction as we travel along the curve



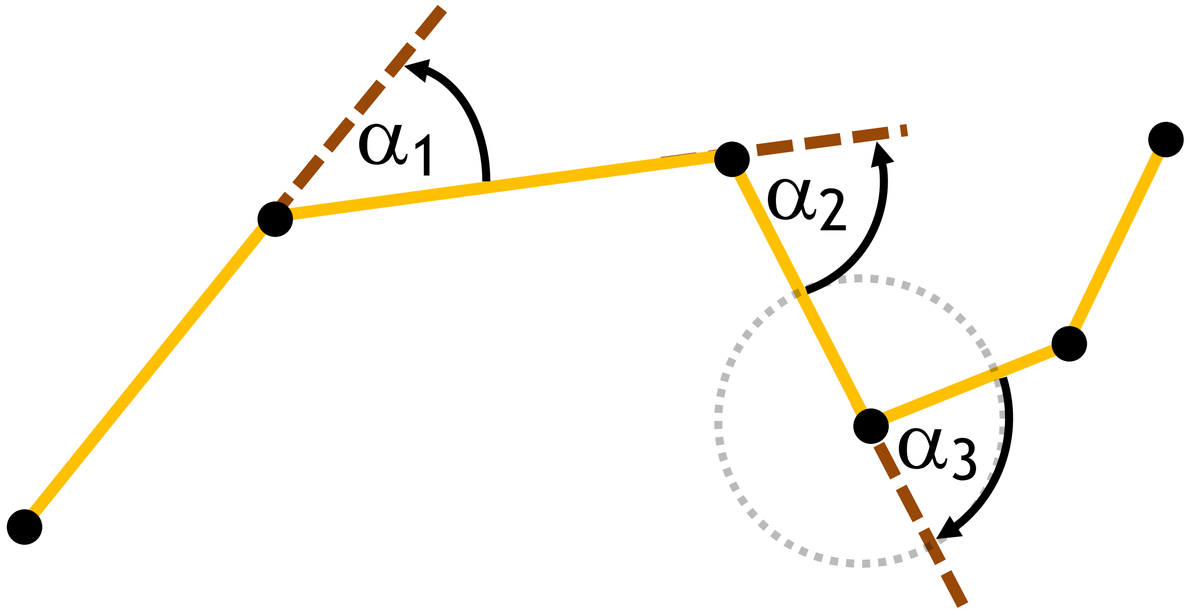
same as the turning angle
between the edges

Curvature of a Discrete Curve

- Zero along the edges
- Turning angle at the vertices
  = the change in normal direction



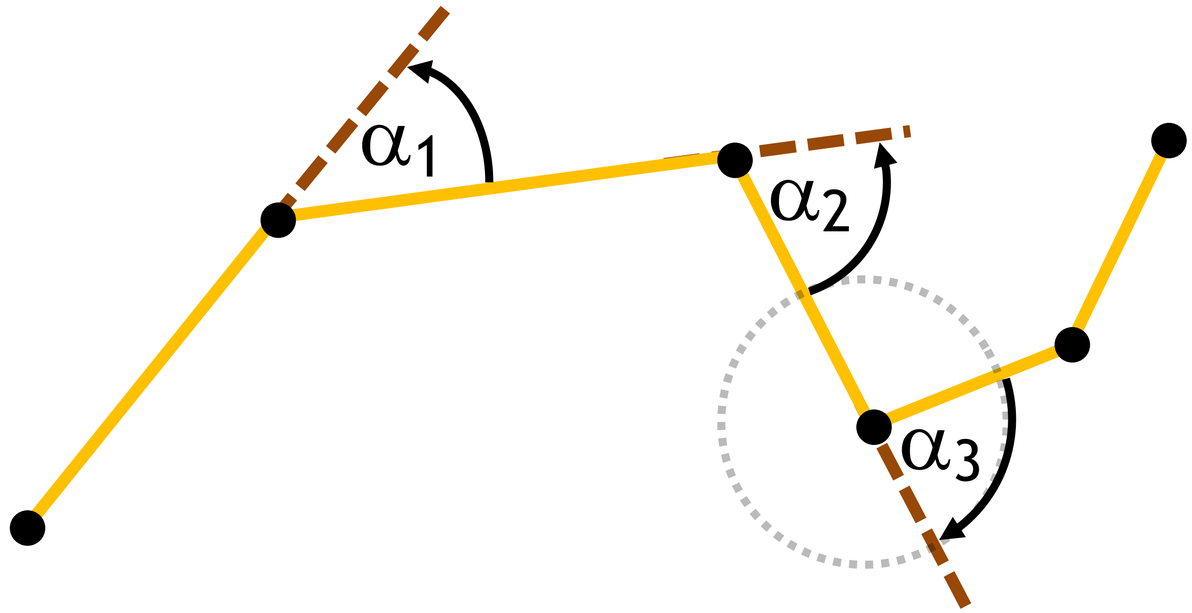
$\alpha_1, \alpha_2 > 0, \quad \alpha_3 < 0$



Total Signed Curvature

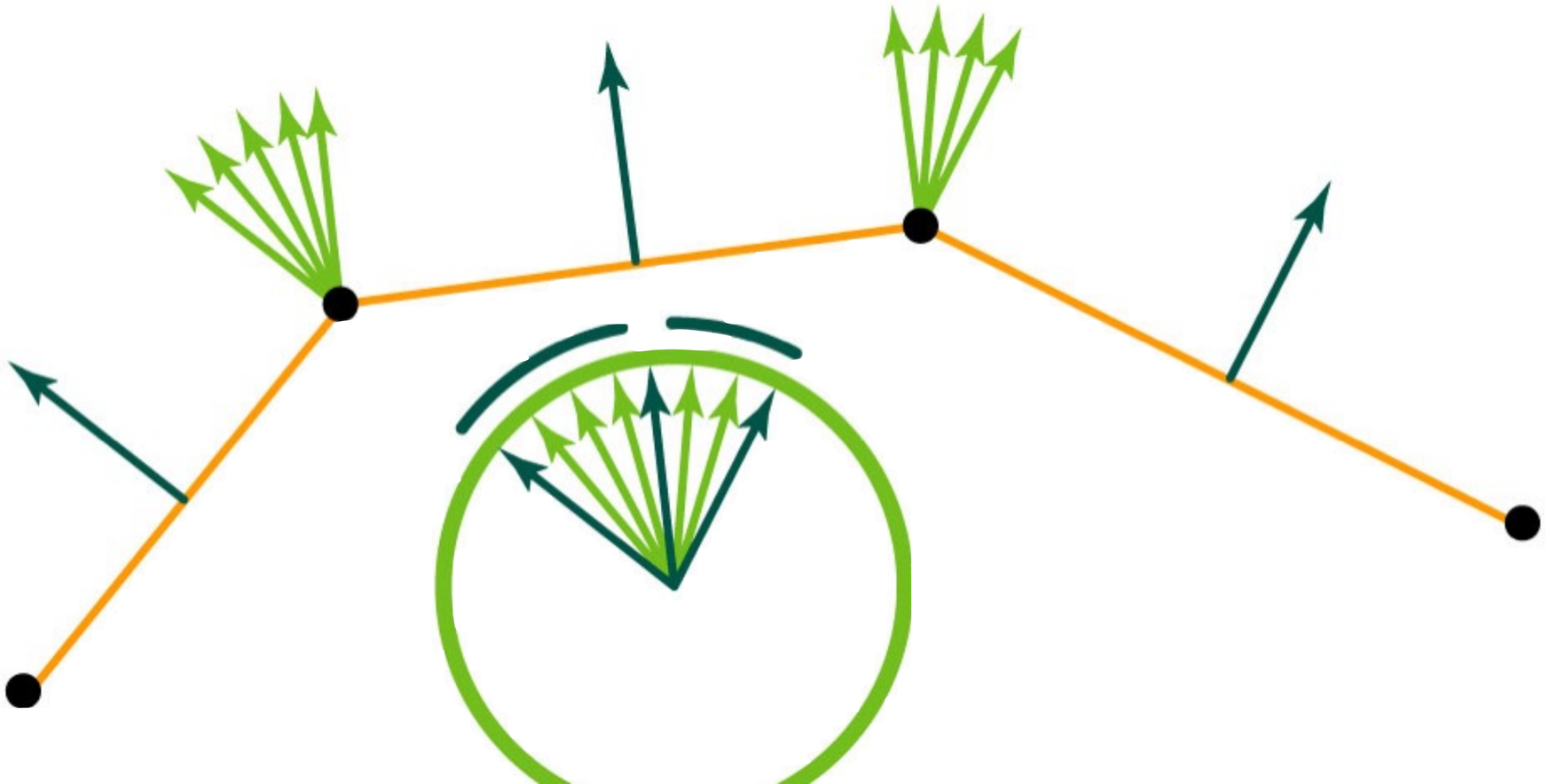
$$\mathrm{tsc}(p) = \sum_{i=1}^{n} \alpha_i$$

- Sum of turning angles



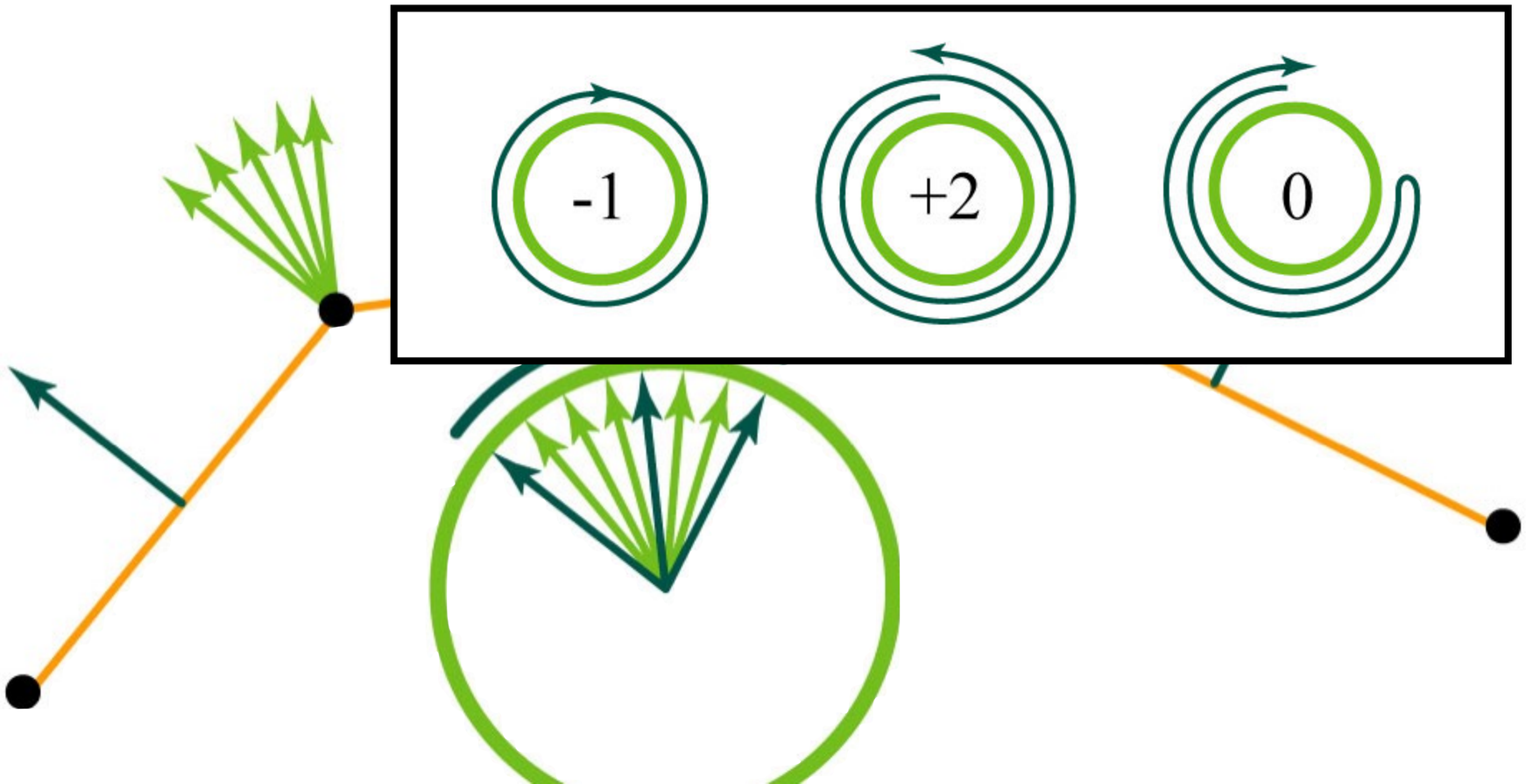
Discrete Gauss Map

- Edges map to points, vertices map to arcs.



Discrete Gauss Map

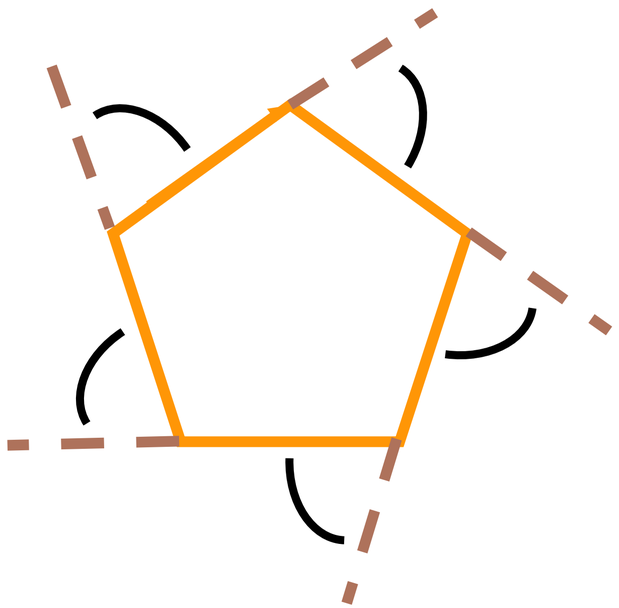
- Turning number well-defined for discrete curves.



Discrete Turning Number Theorem

$$\mathrm{tsc}(p) = \sum_{i=1}^{n} \alpha_i = 2\pi k$$

- For a closed curve,
  the total signed curvature is
  an integer multiple of 2π.
  - proof: sum of exterior angles



Turning Number Theorem

Continuous world

$$\int_\gamma \kappa \, dt = 2\pi k$$

Discrete world

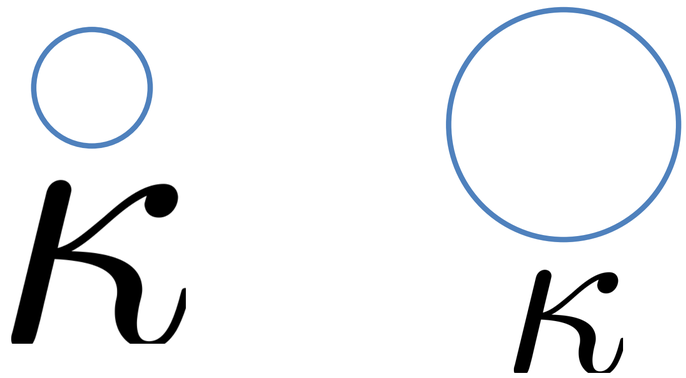
$$\sum_{i=1}^{n} \alpha_i = 2\pi k$$

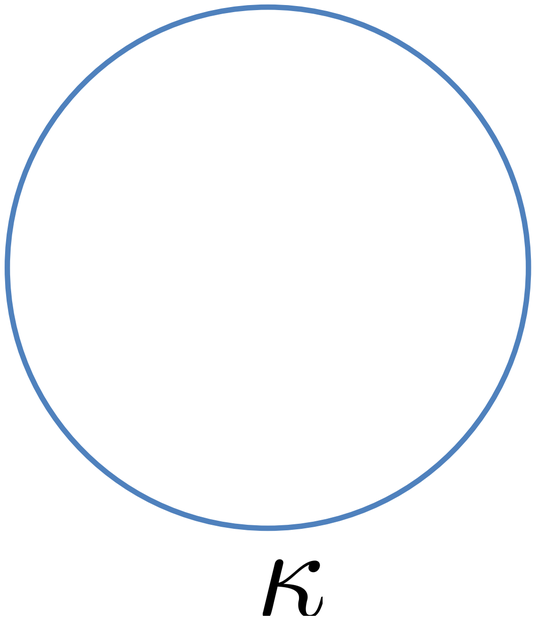
k:

$$\kappa = \alpha_i \quad ??$$

Curvature is scale dependent

$$\kappa = \frac{1}{r}$$

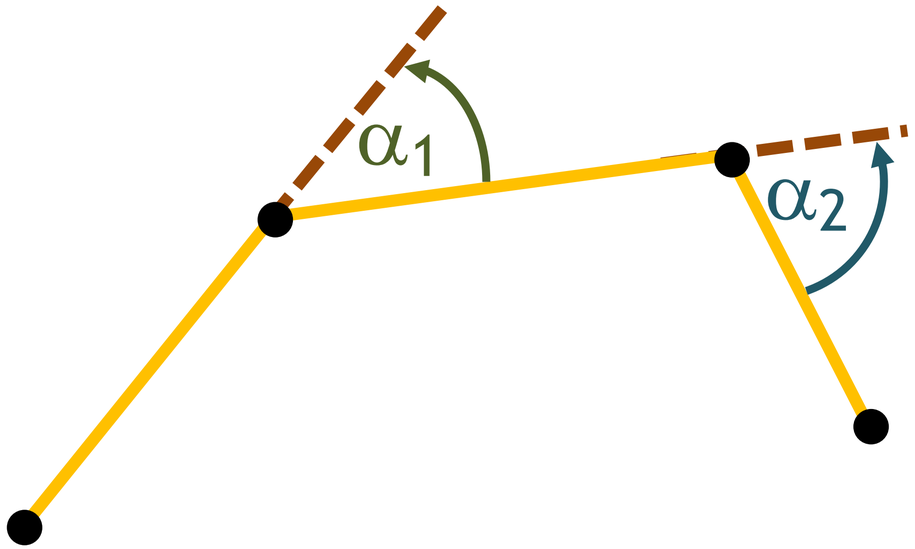




α_i is scale-independent

Discrete Curvature – Integrated Quantity!

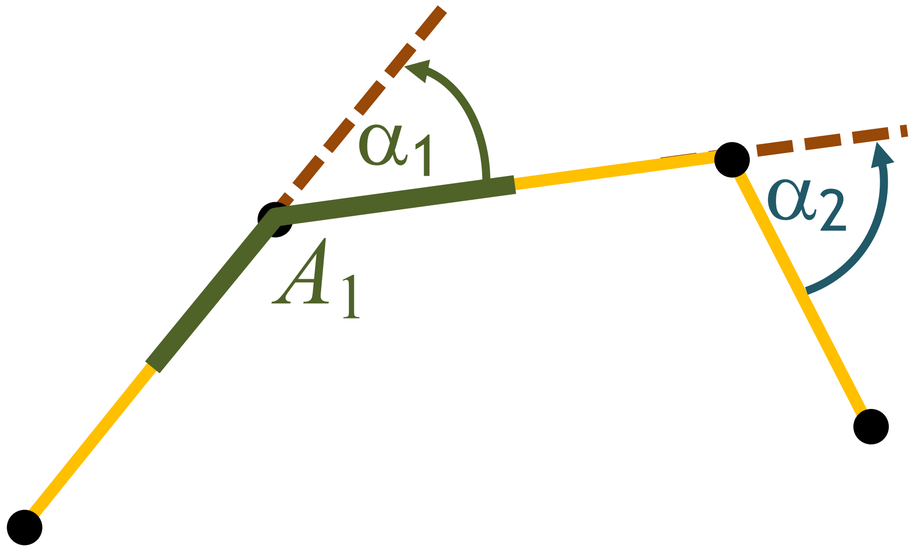
- Cannot view α_i as pointwise curvature
- It is *integrated curvature* over a local area associated with vertex i



Discrete Curvature – Integrated Quantity!

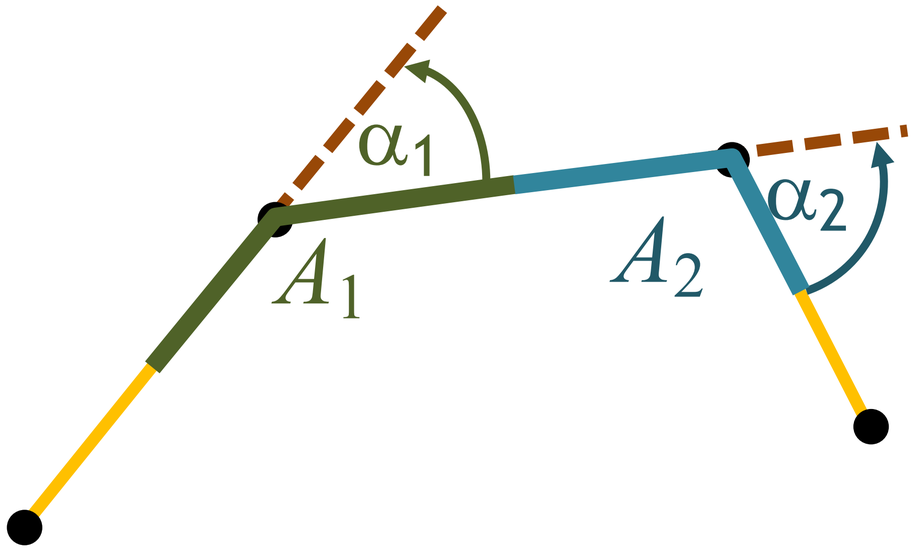
- Integrated over a local area associated with vertex i

$$\alpha_1 = A_1 \cdot \kappa_1$$



Discrete Curvature – Integrated Quantity!

- Integrated over a local area associated with vertex i



$$\alpha_1 = A_1 \cdot \kappa_1$$

$$\alpha_2 = A_2 \cdot \kappa_2$$

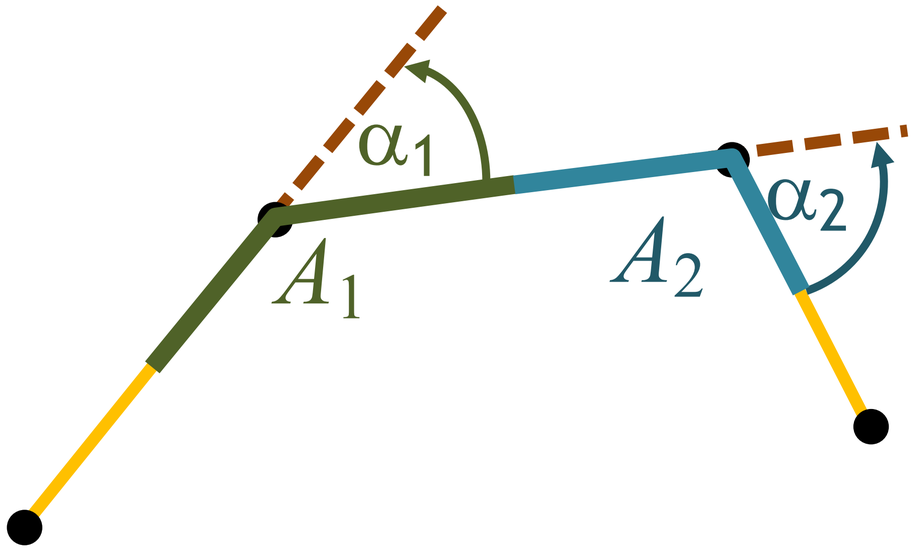
Discrete Curvature – Integrated Quantity!

- Integrated over a local area associated with vertex i

$$\alpha_1 = A_1 \cdot \kappa_1$$

$$\alpha_2 = A_2 \cdot \kappa_2$$

$$\sum A_i = \text{len}(p)$$



The vertex areas A_i form a covering of the curve.
They are pairwise disjoint (except endpoints).

Thank you