06 - Surfaces

Acknowledgements: Olga Sorkine-Hornung

Reminder – Curves

Turning Number Theorem

Continuous world

$$\int_\gamma \kappa \, dt = 2\pi k$$

Discrete world

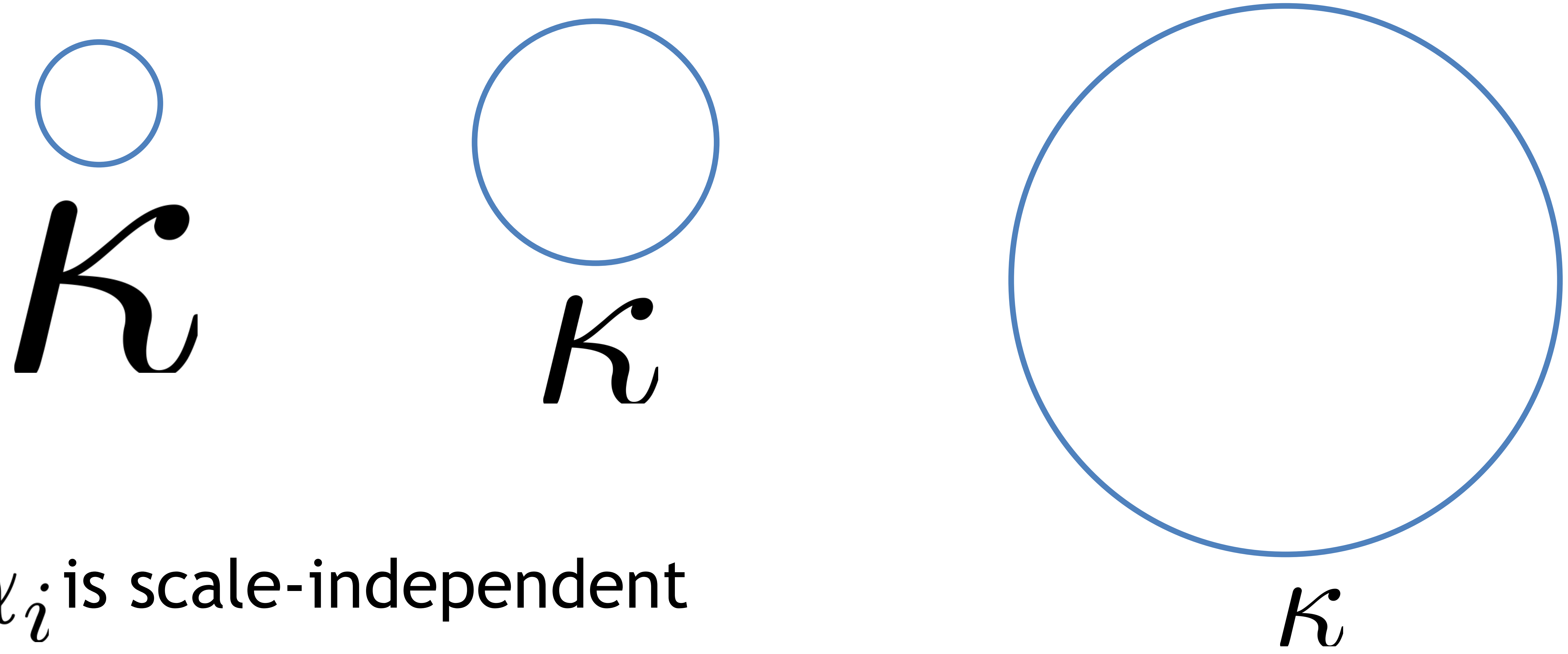
$$\sum_{i=1}^{n} \alpha_i = 2\pi k$$

k:

$$\kappa = \alpha_i \quad ??$$

Curvature is scale dependent

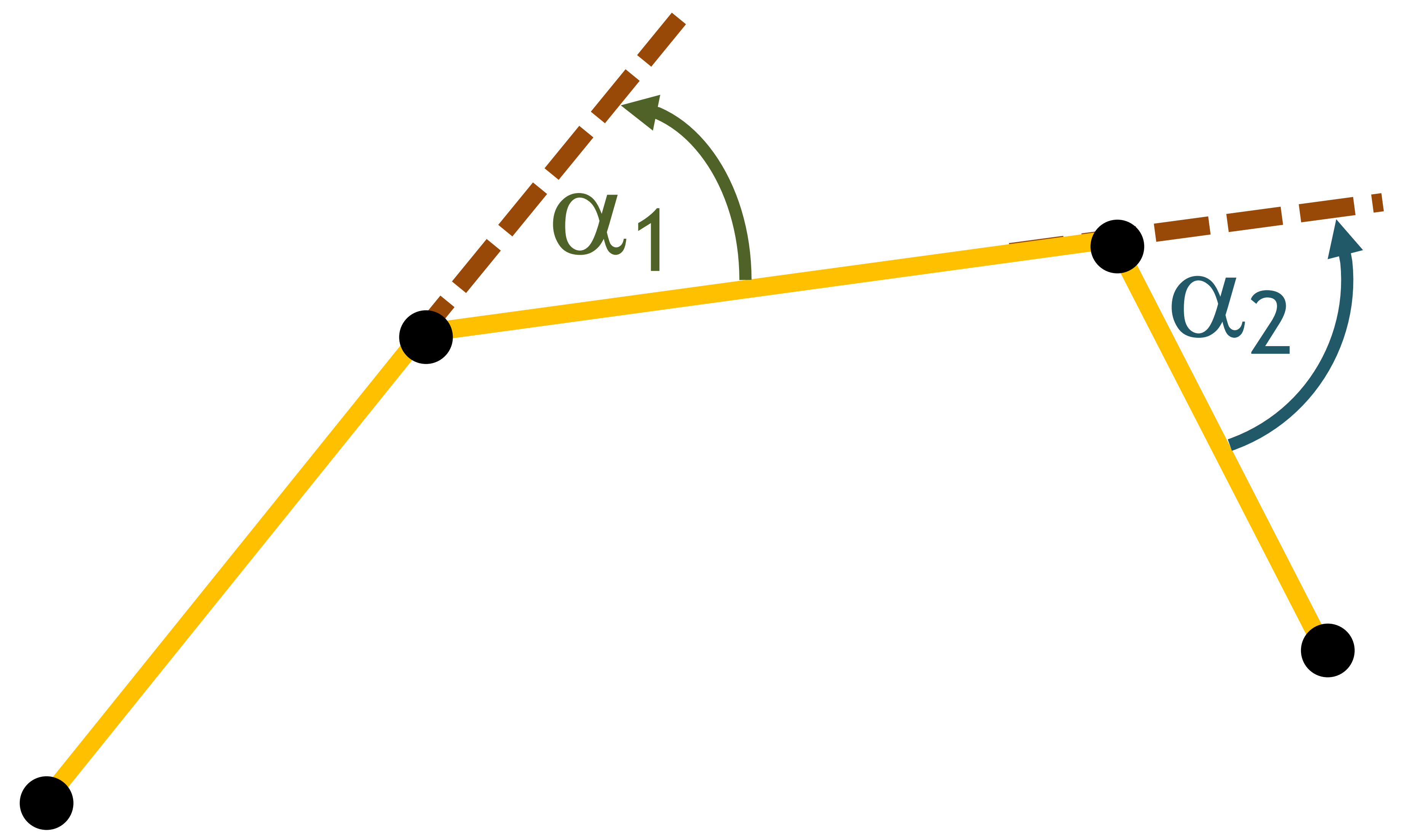
$$\kappa = \frac{1}{r}$$



α_i is scale-independent

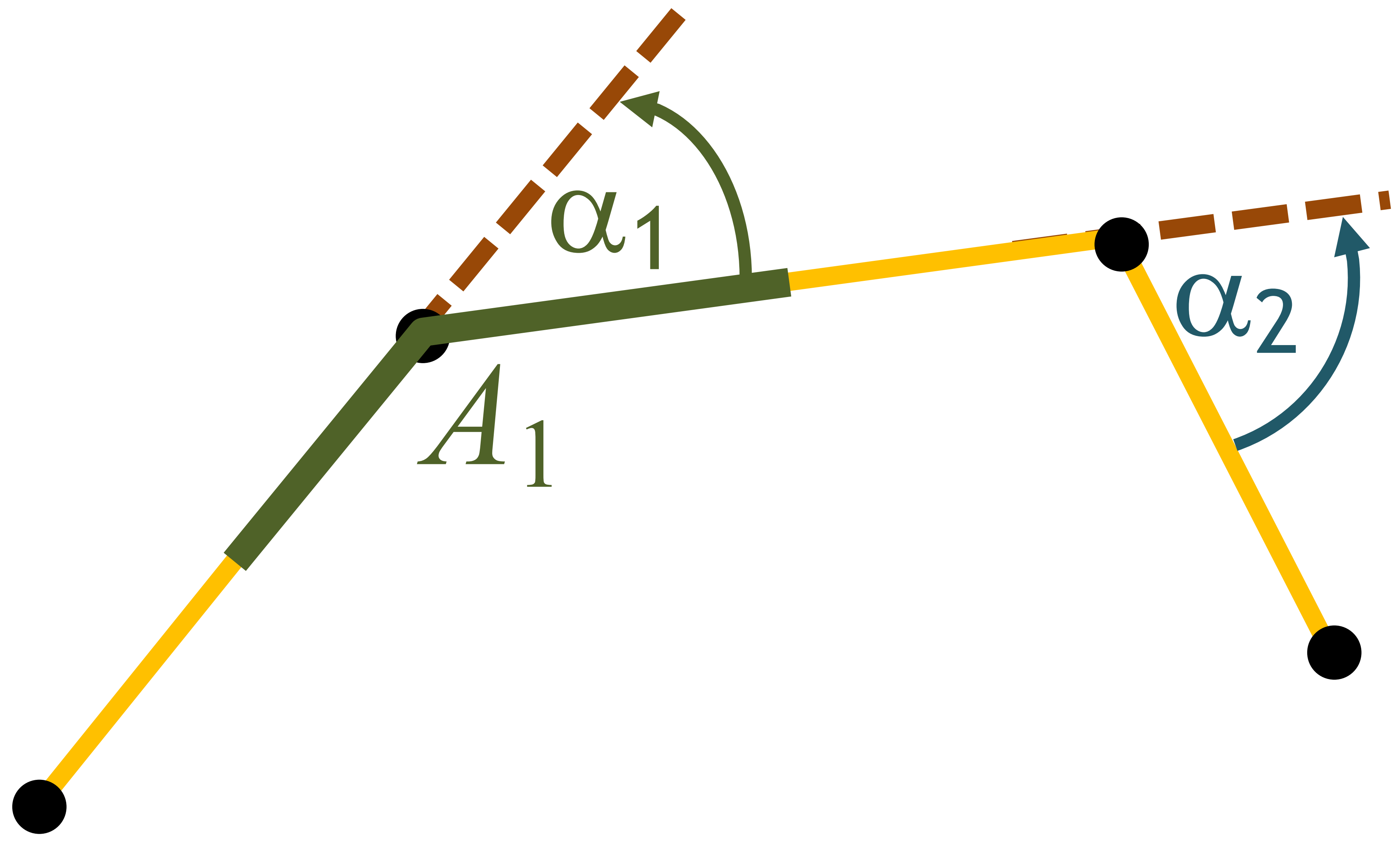
Discrete Curvature – Integrated Quantity!

- Cannot view α_i as pointwise curvature
- It is *integrated curvature* over a local area associated with vertex i



Discrete Curvature – Integrated Quantity!

- Integrated over a local area associated with vertex i



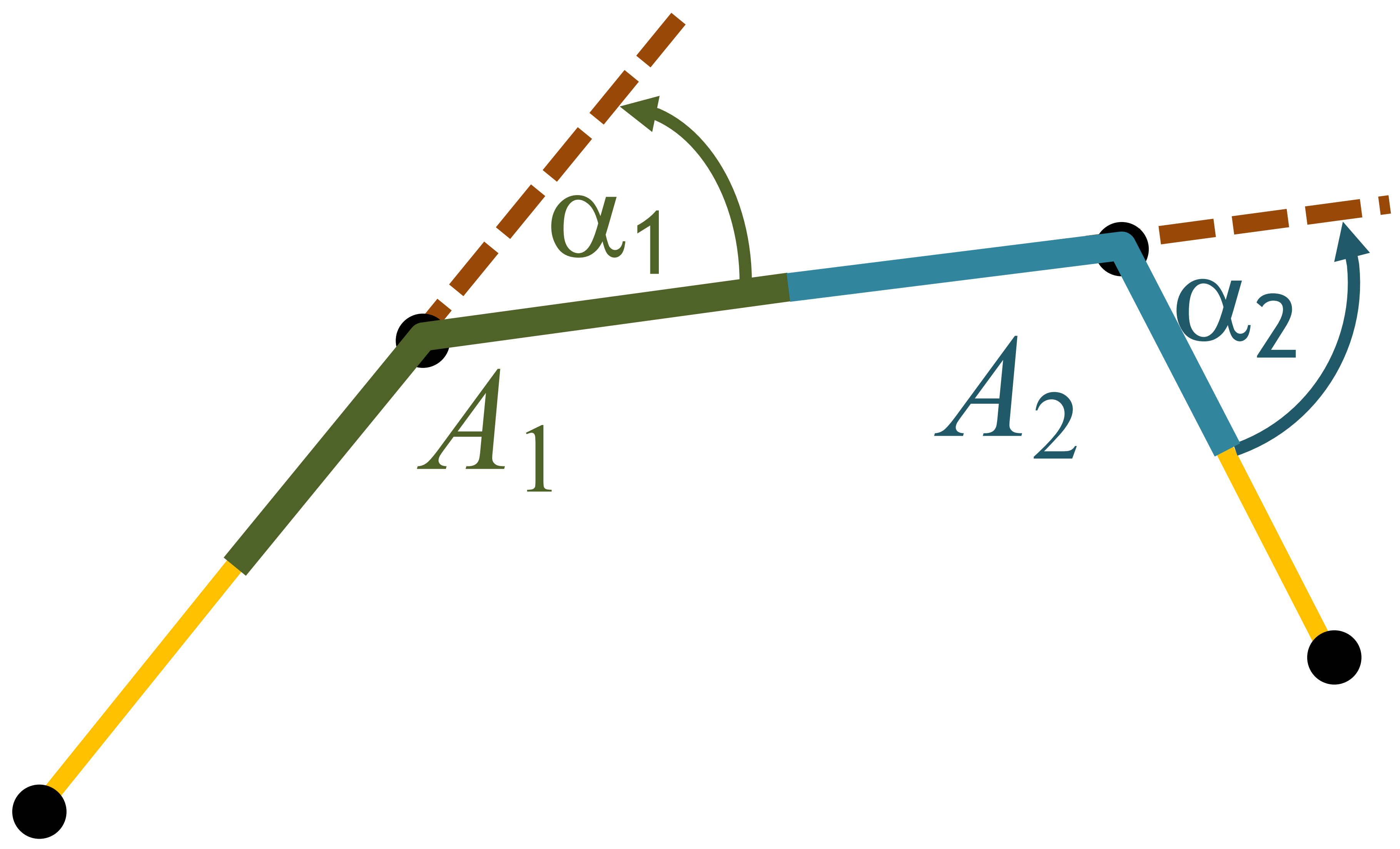
$$\alpha_1 = A_1 \cdot \kappa_1$$

Discrete Curvature – Integrated Quantity!

- Integrated over a local area associated with vertex i

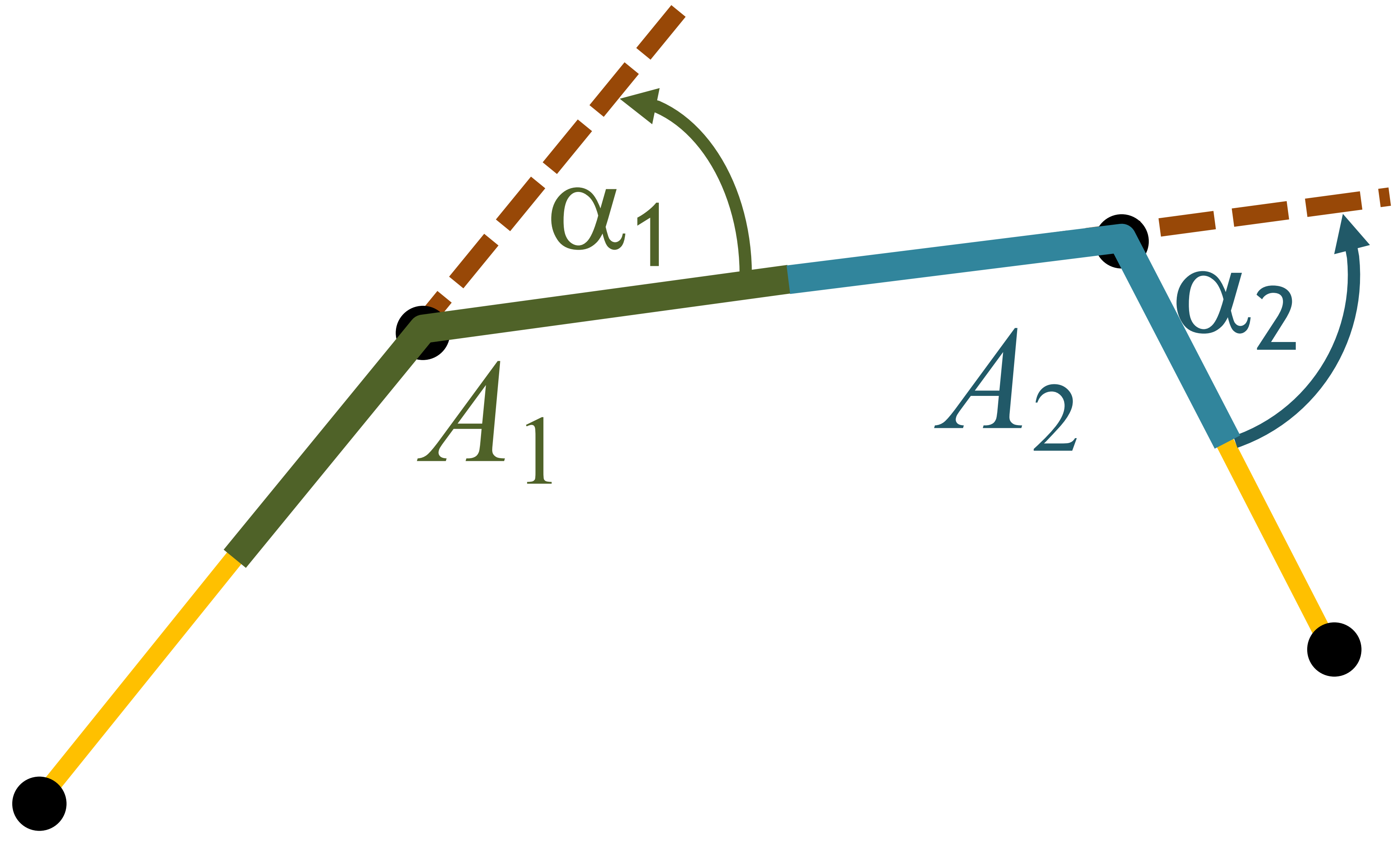
$$\alpha_1 = A_1 \cdot \kappa_1$$

$$\alpha_2 = A_2 \cdot \kappa_2$$



Discrete Curvature – Integrated Quantity!

- Integrated over a local area associated with vertex i



$$\alpha_1 = A_1 \cdot \kappa_1$$

$$\alpha_2 = A_2 \cdot \kappa_2$$

$$\sum A_i = \mathrm{len}(p)$$

The vertex areas A_i form a covering of the curve.
They are pairwise disjoint (except endpoints).

Recap

Structure-preservation

For an arbitrary (even coarse) discrete curve, the discrete measure of curvature obeys the discrete turning number theorem.

Convergence

In the limit of a refinement sequence, discrete measures of length and curvature agree with continuous measures.

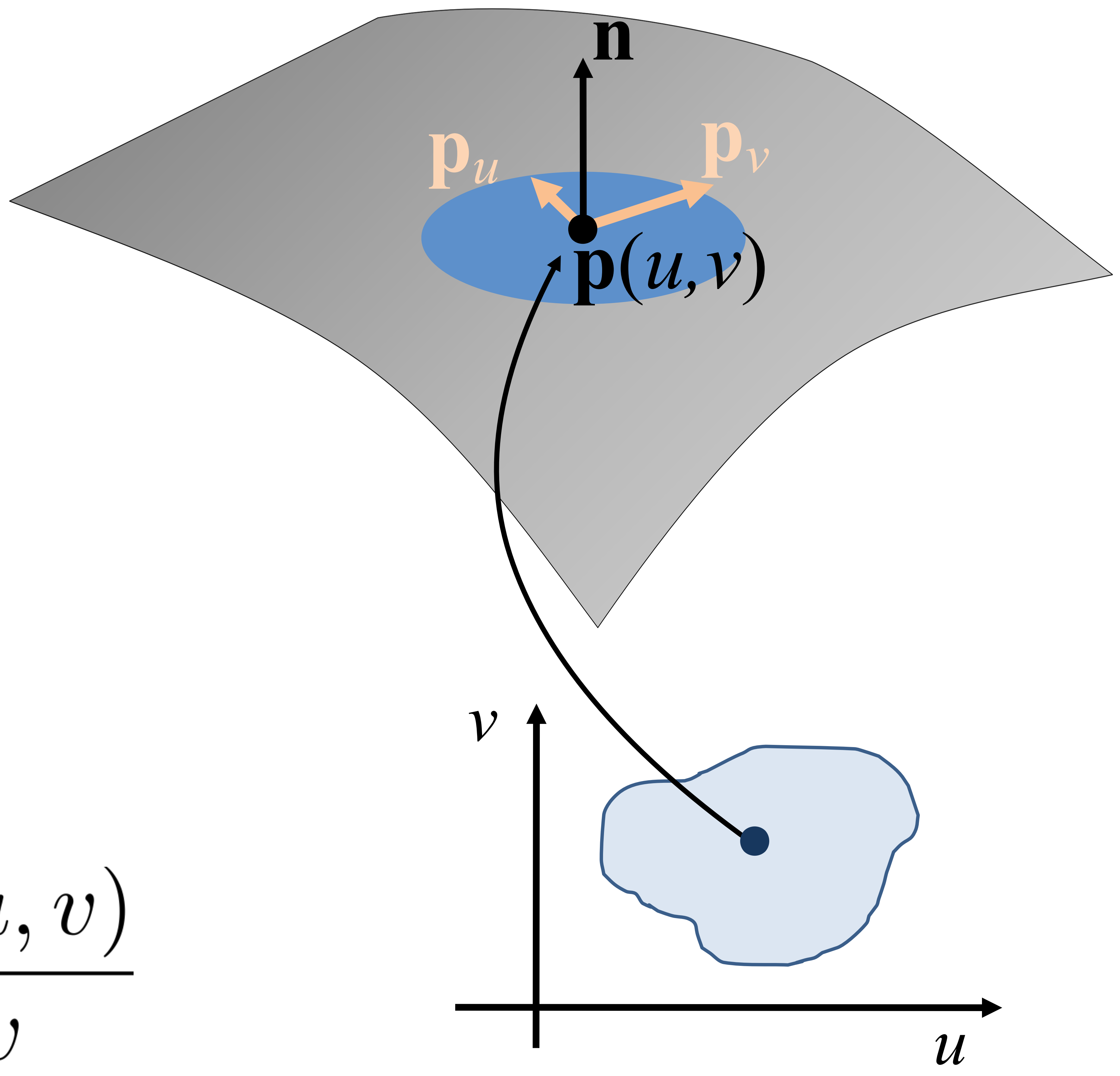
Surfaces

Surfaces, Parametric Form

- Continuous surface

$$\mathbf{p}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \quad (u,v) \in \mathbb{R}^2$$

- Tangent plane at point $\mathbf{p}(u,v)$ is spanned by



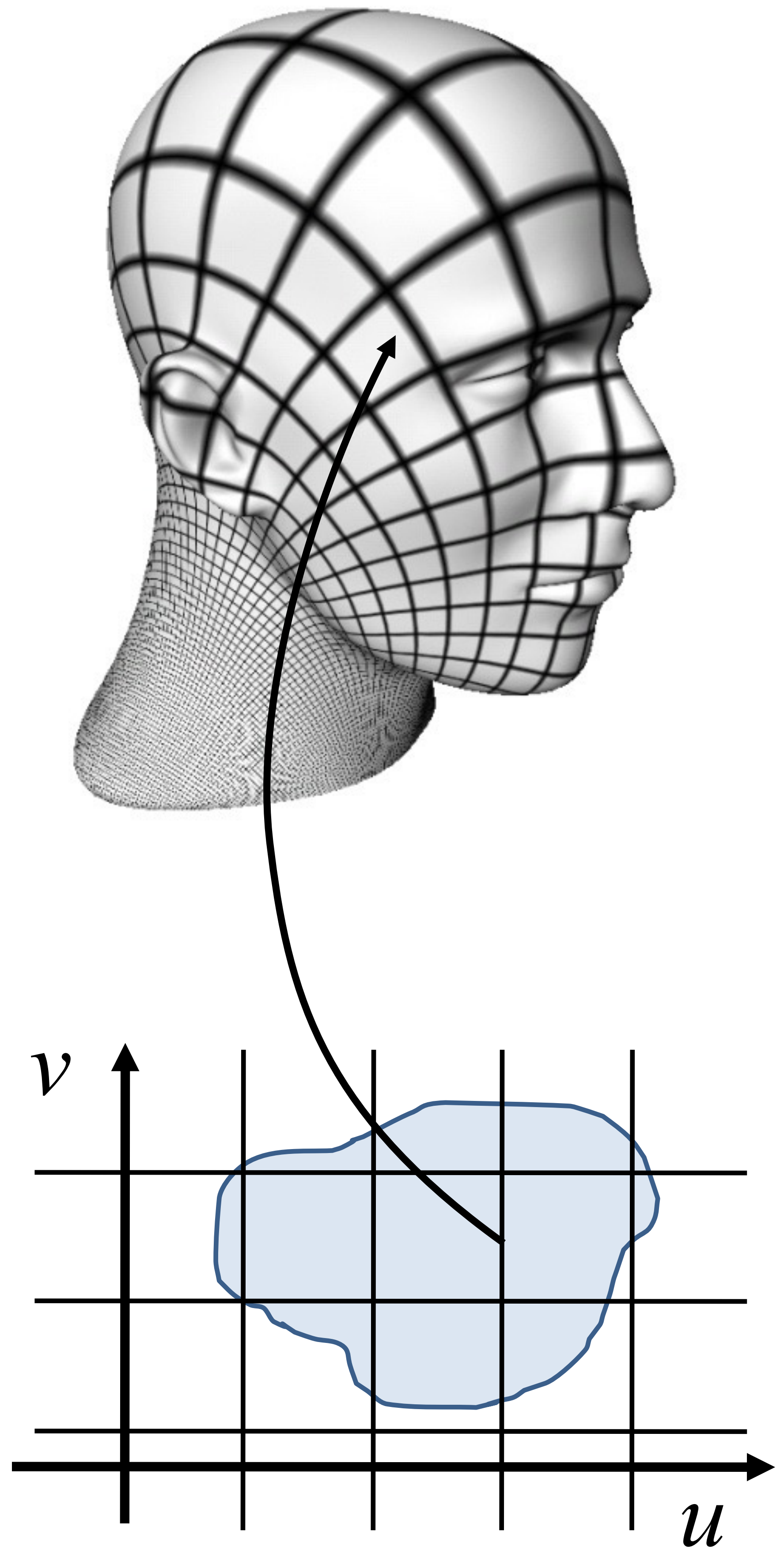
$$\mathbf{p}_u = \frac{\partial \mathbf{p}(u,v)}{\partial u}, \quad \mathbf{p}_v = \frac{\partial \mathbf{p}(u,v)}{\partial v}$$

These vectors don't have to be orthogonal

Isoparametric Lines

- Lines on the surface when keeping one parameter fixed

$$\gamma_{u_0}(v) = \mathbf{p}(u_0, v)$$
$$\gamma_{u_0}(u) = \mathbf{p}(u, v_0)$$



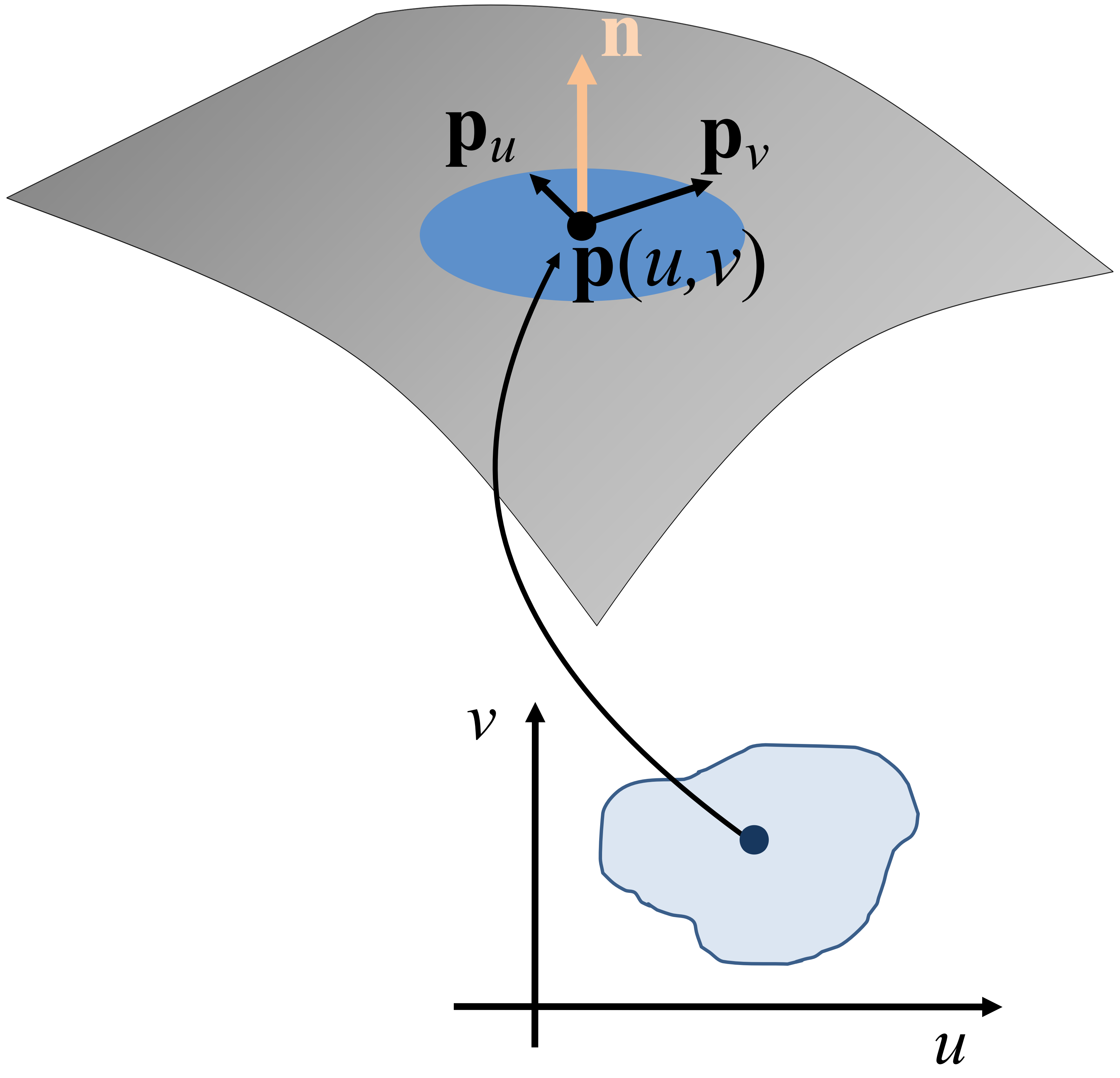
Surface Normals

- Surface normal:

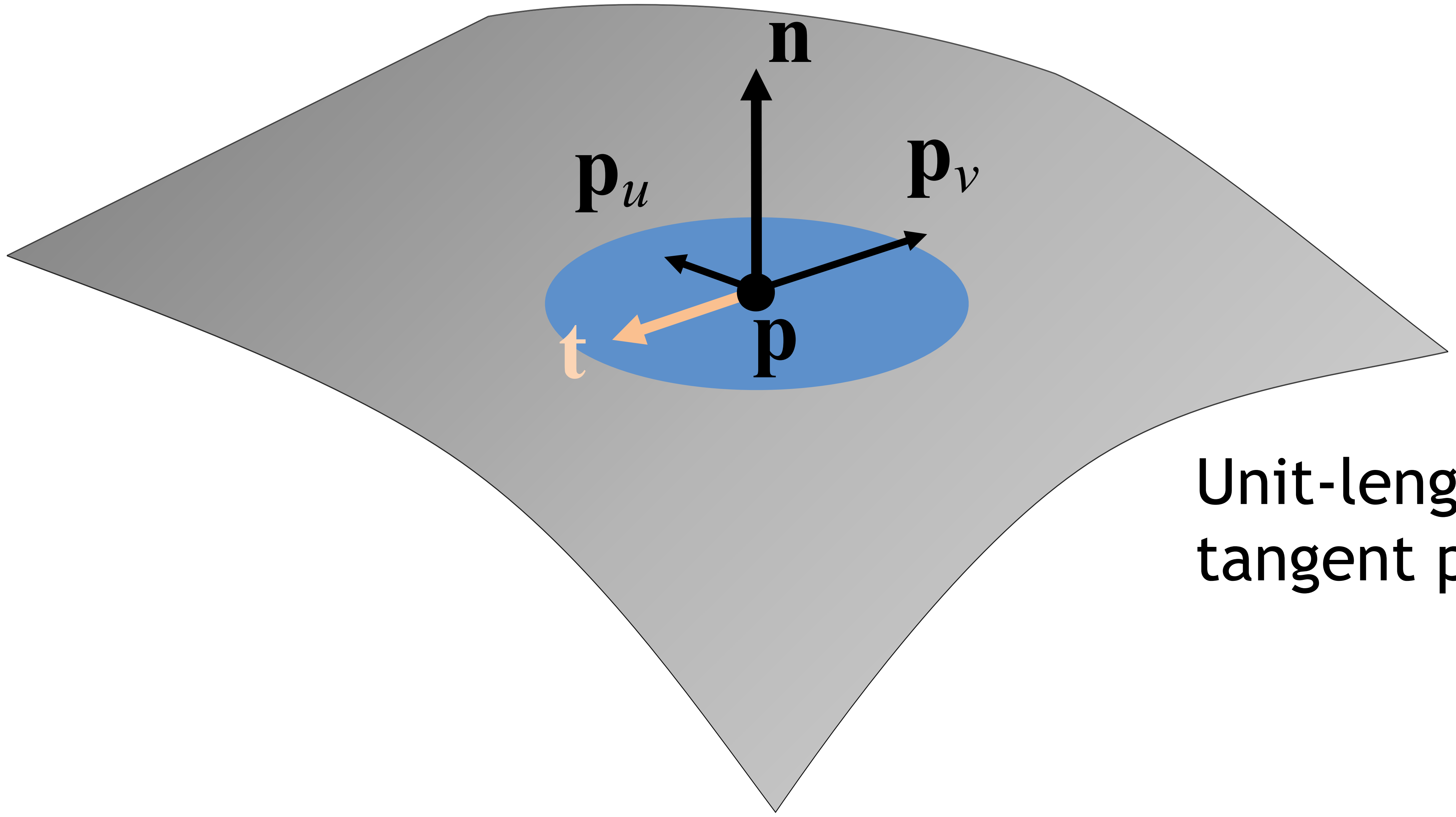
$$\mathbf{n}(u,v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\|\mathbf{p}_u \times \mathbf{p}_v\|}$$

  - Assuming *regular* parameterization, i.e.,

$$\mathbf{p}_u \times \mathbf{p}_v \neq 0$$



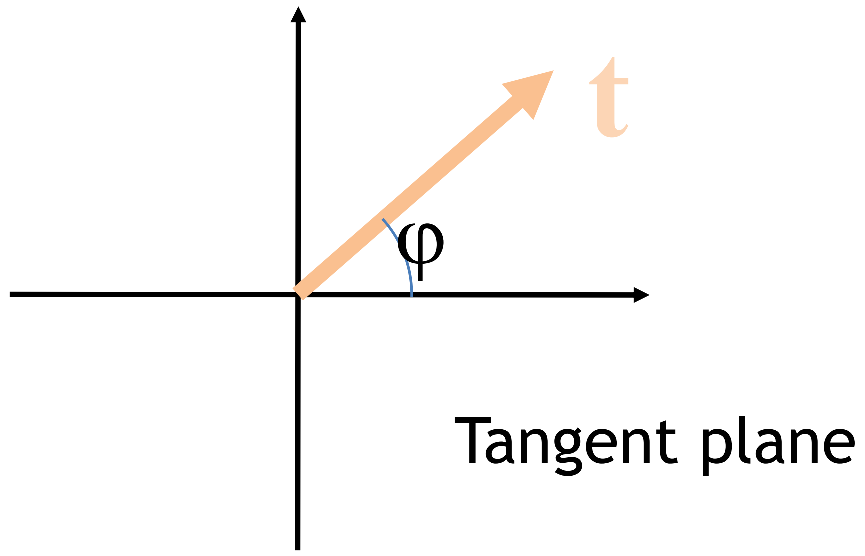
Normal Curvature



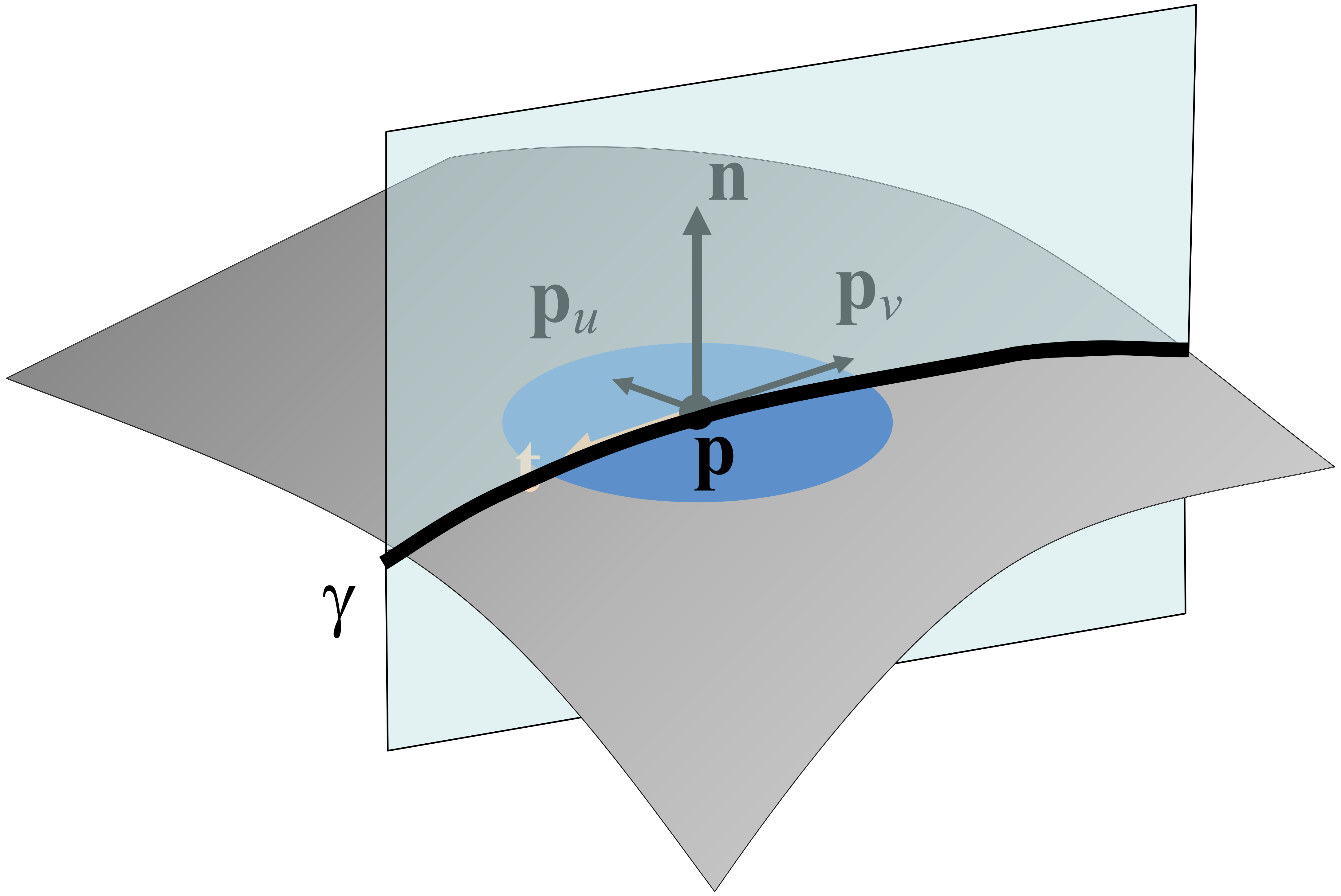
$$\mathbf{n}(u,v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\|\mathbf{p}_u \times \mathbf{p}_v\|}$$

Unit-length direction $\mathbf{t}$ in the tangent plane (if $\mathbf{p}_u$ and $\mathbf{p}_v$ are orthogonal):

$$\mathbf{t} = \cos\varphi \frac{\mathbf{p}_u}{\|\mathbf{p}_u\|} + \sin\varphi \frac{\mathbf{p}_v}{\|\mathbf{p}_v\|}$$



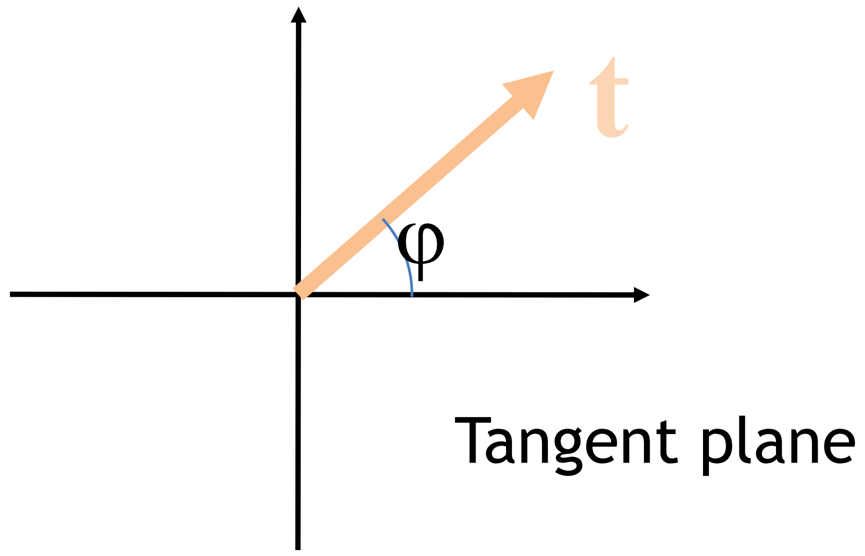
Normal Curvature



The curve γ is the intersection of the surface with the plane through $\mathbf{n}$ and $\mathbf{t}$.

Normal curvature:

$$\kappa_n(\varphi) = \kappa(\gamma(\mathbf{p}))$$



Surface Curvatures

- Principal curvatures
  - Minimal curvature
  - Maximal curvature

$$\kappa_1 = \kappa_{\min} = \min_{\varphi} \kappa_n(\varphi)$$

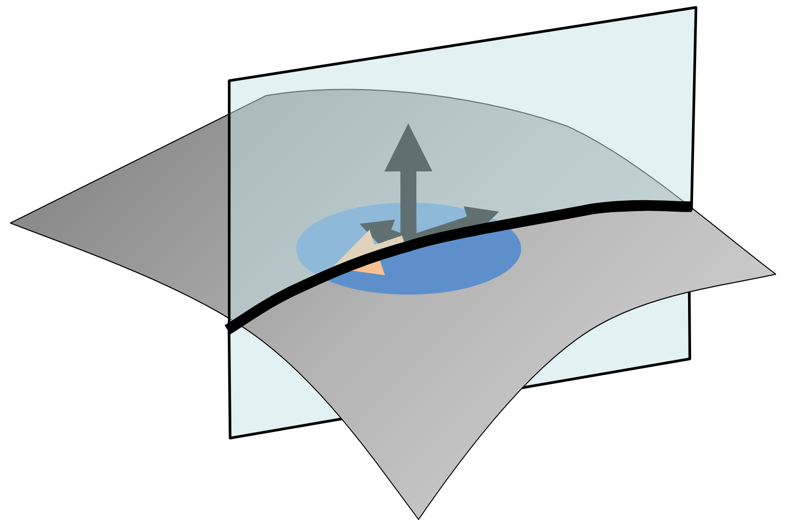
$$\kappa_2 = \kappa_{\max} = \max_{\varphi} \kappa_n(\varphi)$$

- Mean curvature

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\varphi) d\varphi$$

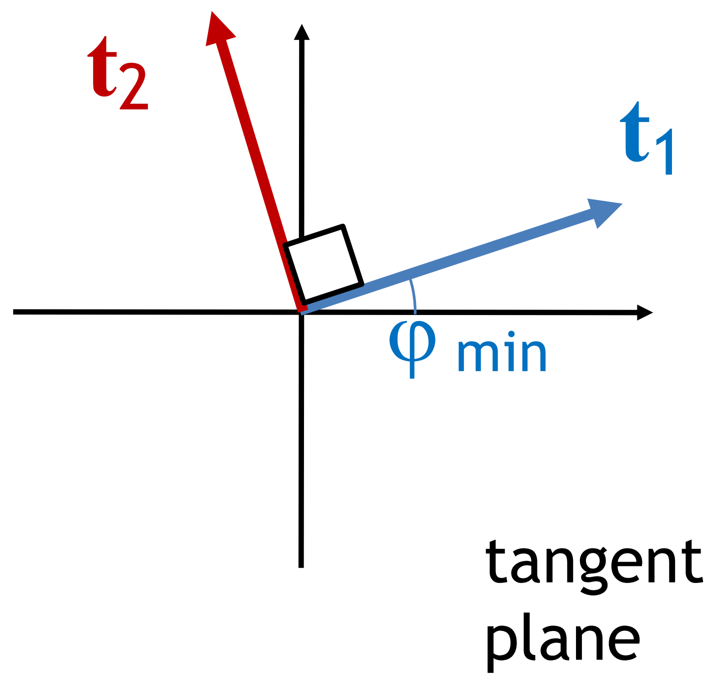
- Gaussian curvature

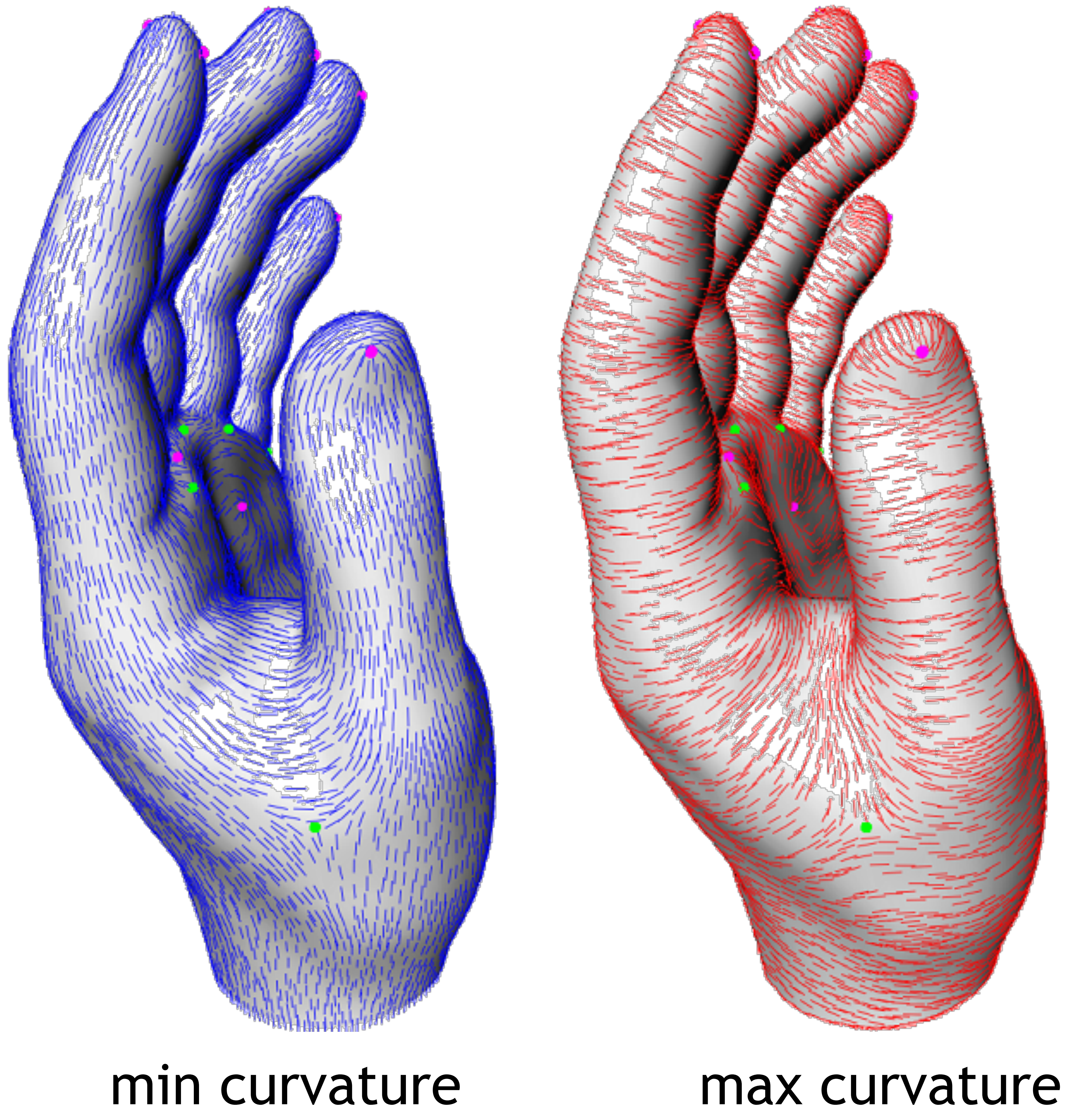
$$K = \kappa_1 \cdot \kappa_2$$



Principal Directions

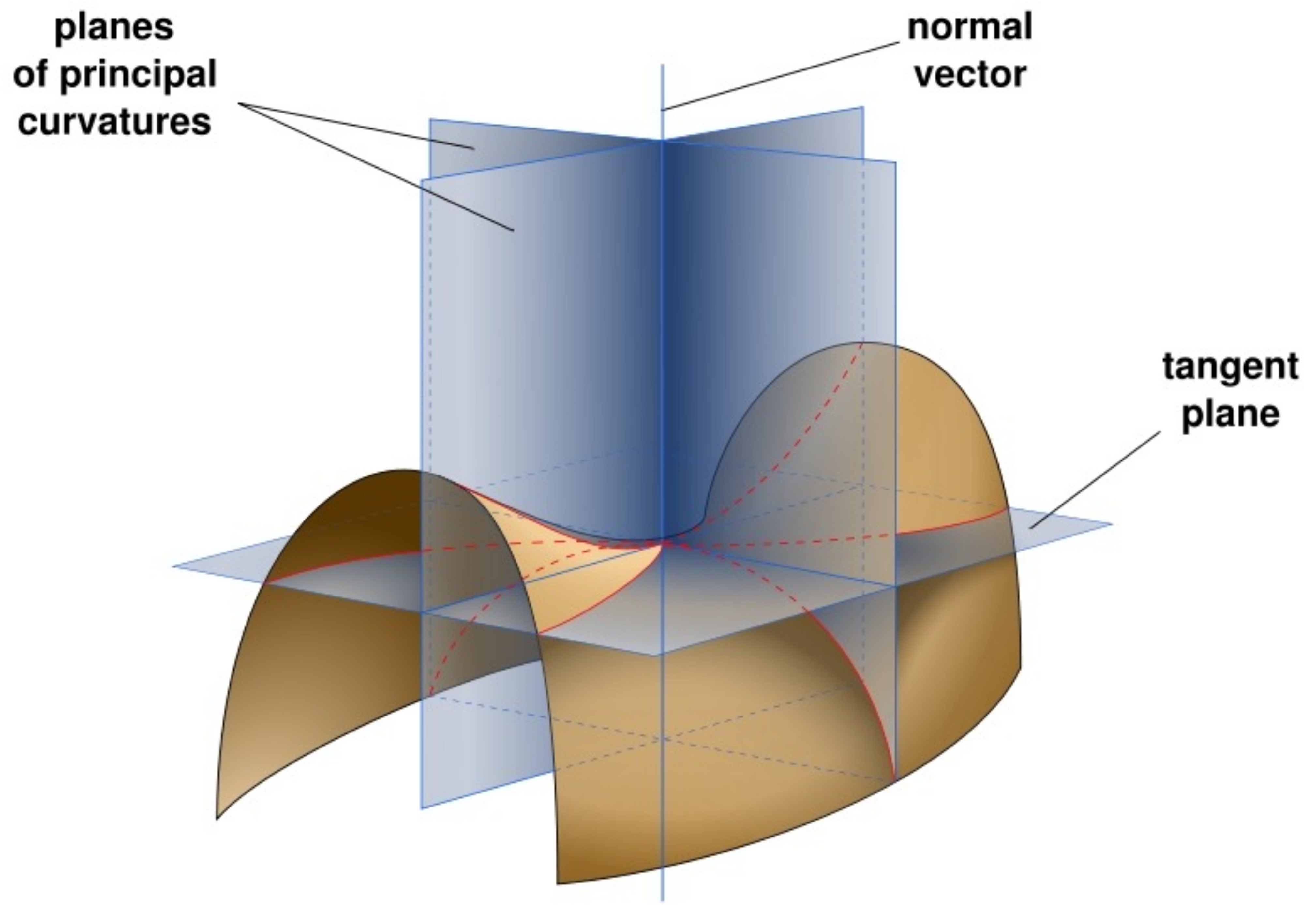
- Principal directions: tangent vectors corresponding to φ_{max} and φ_{min}





Pierre Alliez, David Cohen-Steiner, Olivier Devillers, Bruno Lévy, and Mathieu Desbrun. 2003. Anisotropic polygonal remeshing. *ACM Trans. Graph.* 22, 3 (July 2003), 485-493. DOI: http://dx.doi.org/10.1145/882262.882296

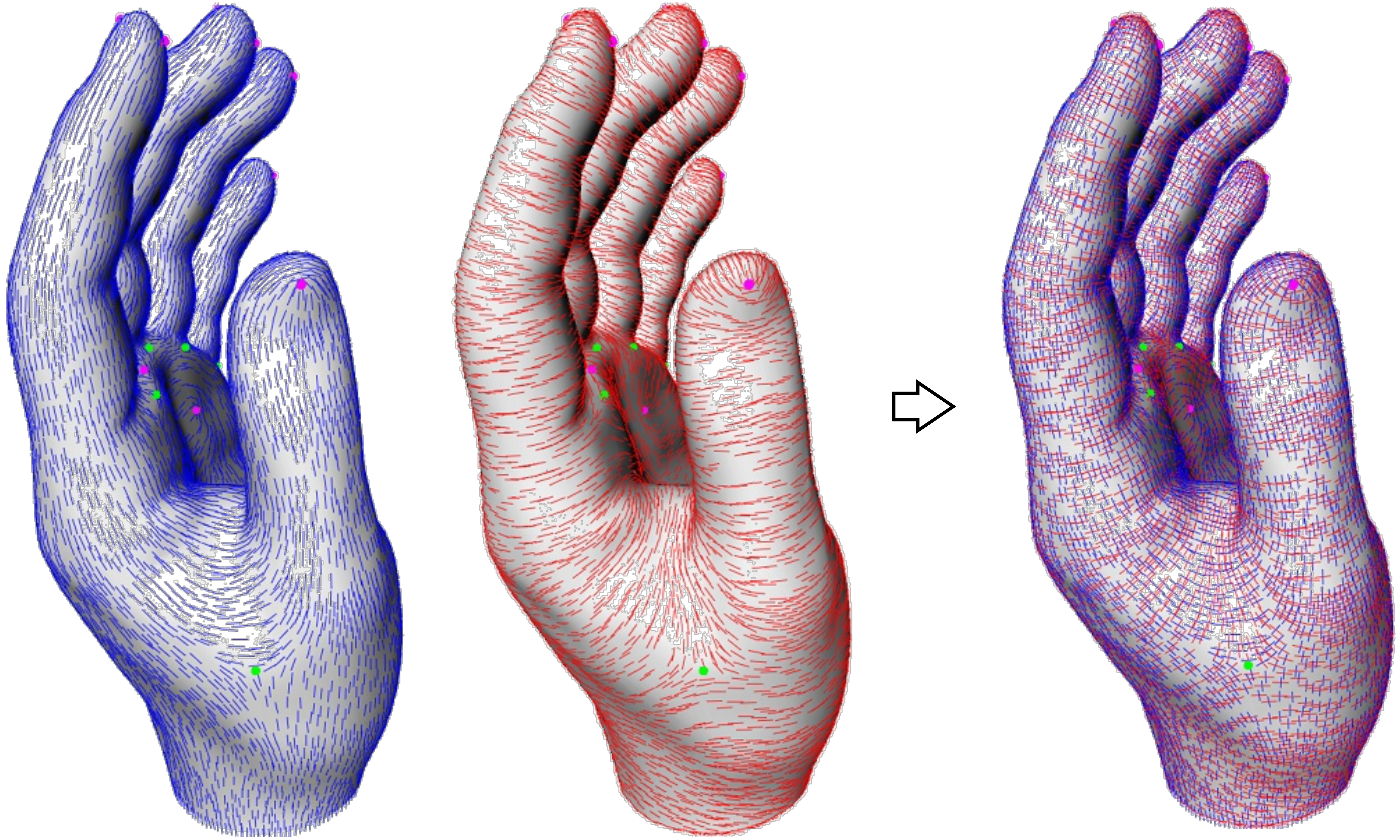
Principal Directions



Euler's Theorem: Planes of principal curvature are **orthogonal** and independent of parameterization.

$$\kappa_n(\varphi) = \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi, \quad \varphi = \text{angle with } \mathbf{t}_1$$

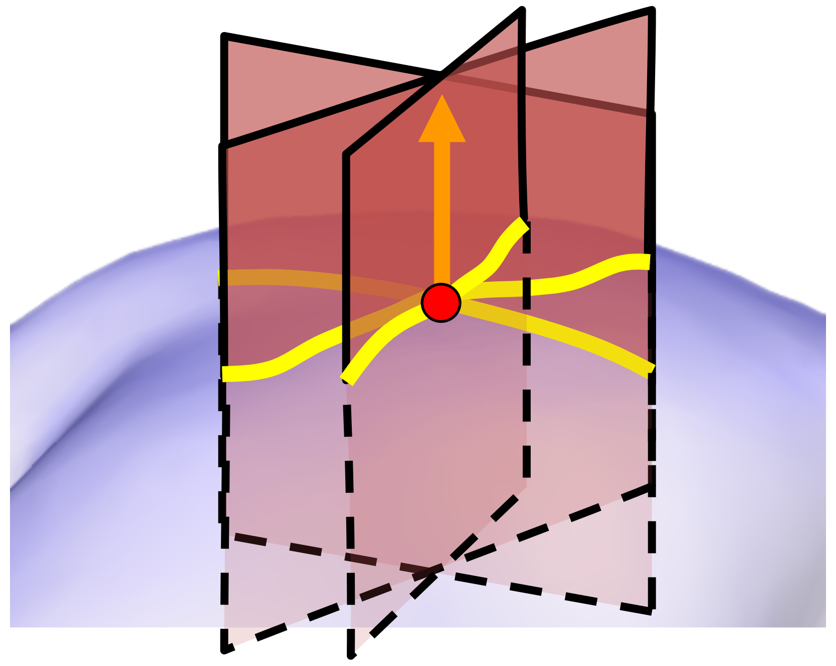
Principal Directions

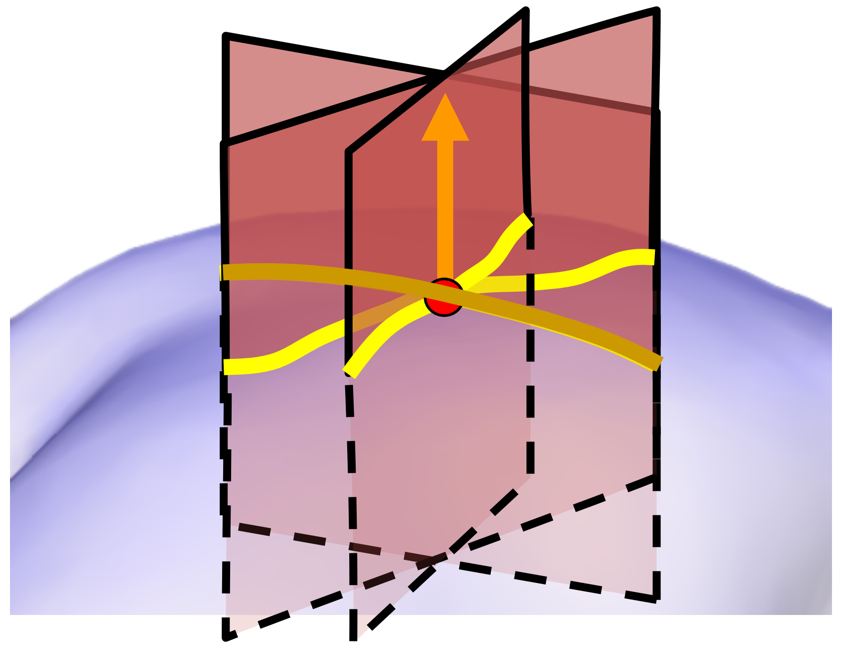


Mean Curvature

- Intuition for mean curvature

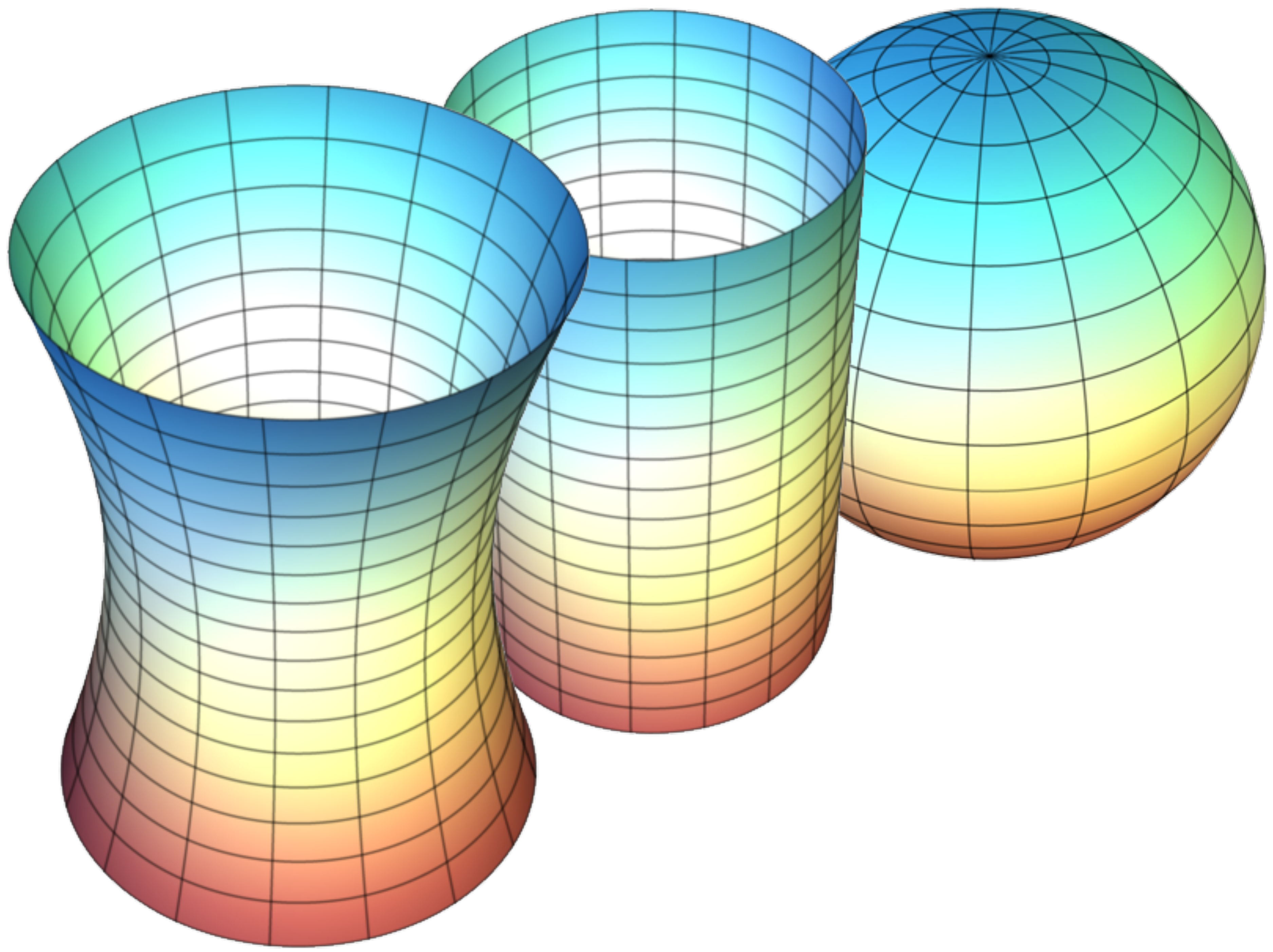
$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\varphi)d\varphi$$





Classification

- A point $\mathbf{p}$ on the surface is called
  - Elliptic, if $K > 0$
  - Parabolic, if $K = 0$
  - Hyperbolic, if $K < 0$



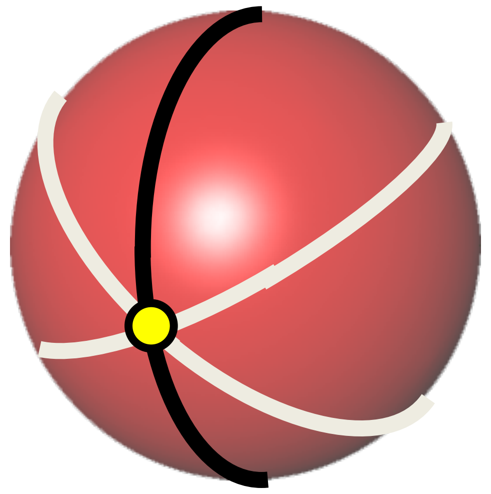
- Developable surface
  iff $K = 0$

Local Surface Shape By Curvatures

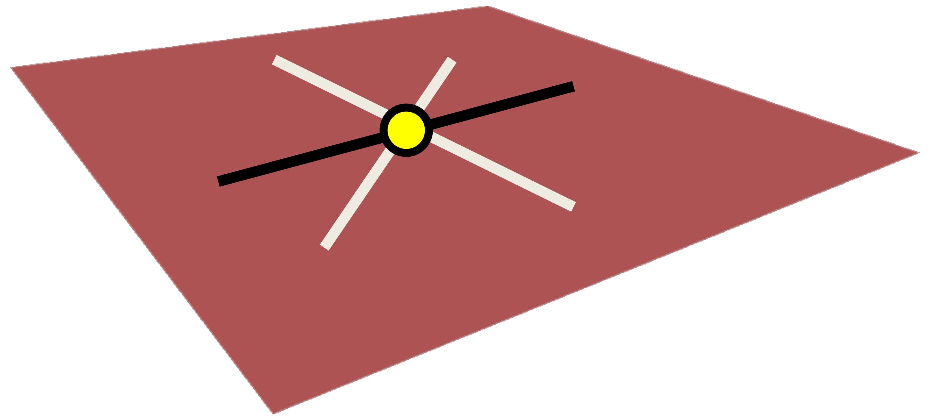
$K > 0,\ \kappa_1 = \kappa_2$

$K = 0$

Isotropic:
all directions are
principal directions



spherical (umbilical)



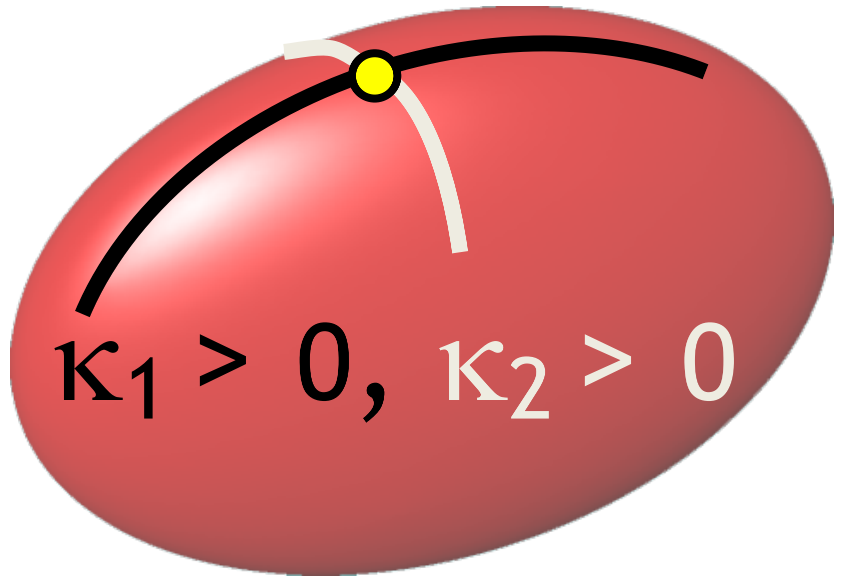
planar

$K > 0$

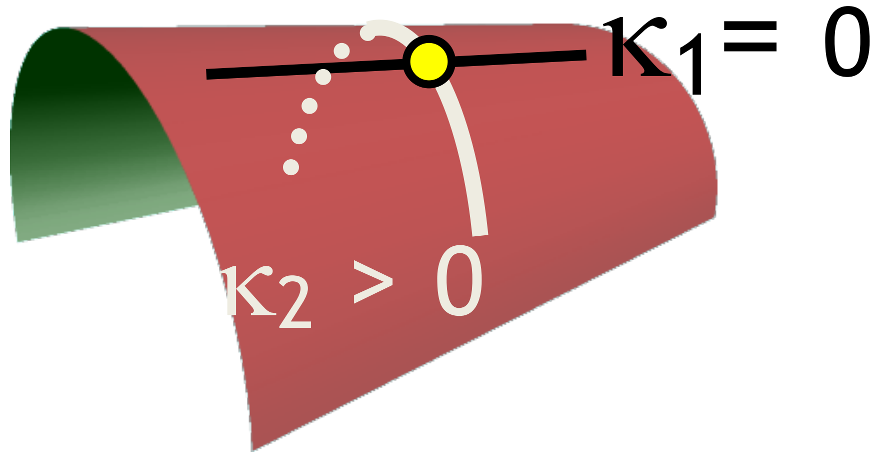
$K = 0$

$K < 0$

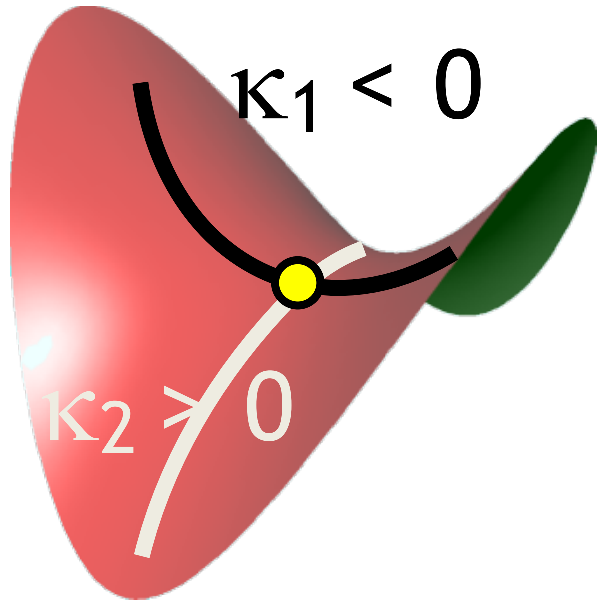
Anisotropic:
2 distinct
principal
directions



elliptic



parabolic



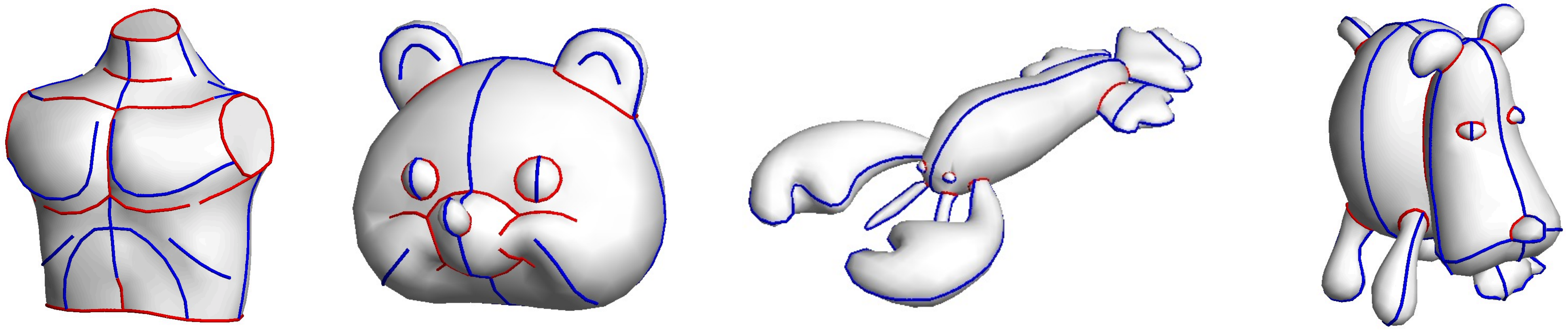
hyperbolic

Usage example: define fair surfaces

- FiberMesh

$$\min_{\mathcal{M}} \int_{\mathcal{M}} \left(\frac{d\kappa_n}{d\mathbf{t}_1}\right)^2 + \left(\frac{d\kappa_n}{d\mathbf{t}_2}\right)^2 dA$$

s.t. M interpolates the curves

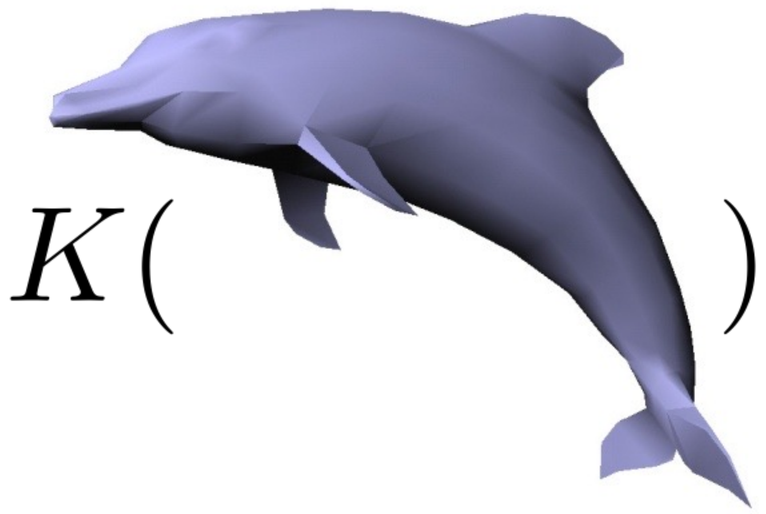
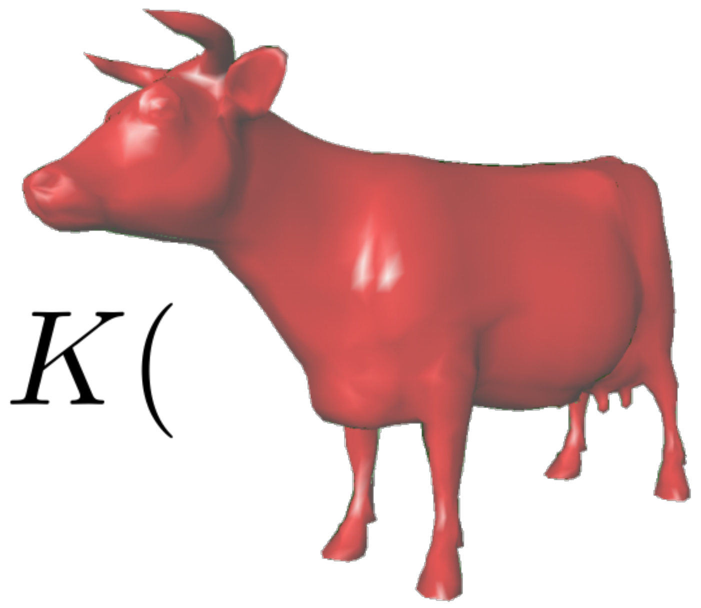
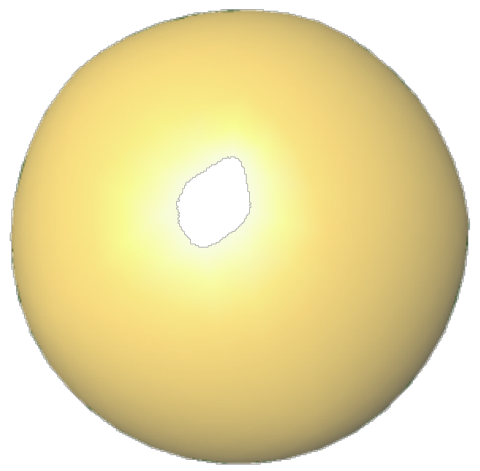


Andrew Nealen, Takeo Igarashi, Olga Sorkine and Marc Alexa, "FiberMesh: Designing Freeform Surfaces with 3D Curves", ACM Transactions on Computer Graphics, ACM SIGGRAPH 2007, San Diego, USA, 2007

Gauss-Bonnet Theorem

- For a closed surface M:

$$\int_{\mathcal{M}} K \, dA = 2\pi \, \chi(\mathcal{M})$$

$\int K($  $) = \int K($  $) = \int K($  $) = 4\pi$

Gauss-Bonnet Theorem

- For a closed surface M:

$$\int_{\mathcal{M}} K\, dA = 2\pi\, \chi(\mathcal{M})$$

- Compare with planar curves:

$$\int_{\gamma} \kappa\, ds = 2\pi\, k$$

Fundamental Forms

- First fundamental form

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \mathbf{p}_u^T \mathbf{p}_u & \mathbf{p}_u^T \mathbf{p}_v \\ \mathbf{p}_u^T \mathbf{p}_v & \mathbf{p}_v^T \mathbf{p}_v \end{pmatrix}$$

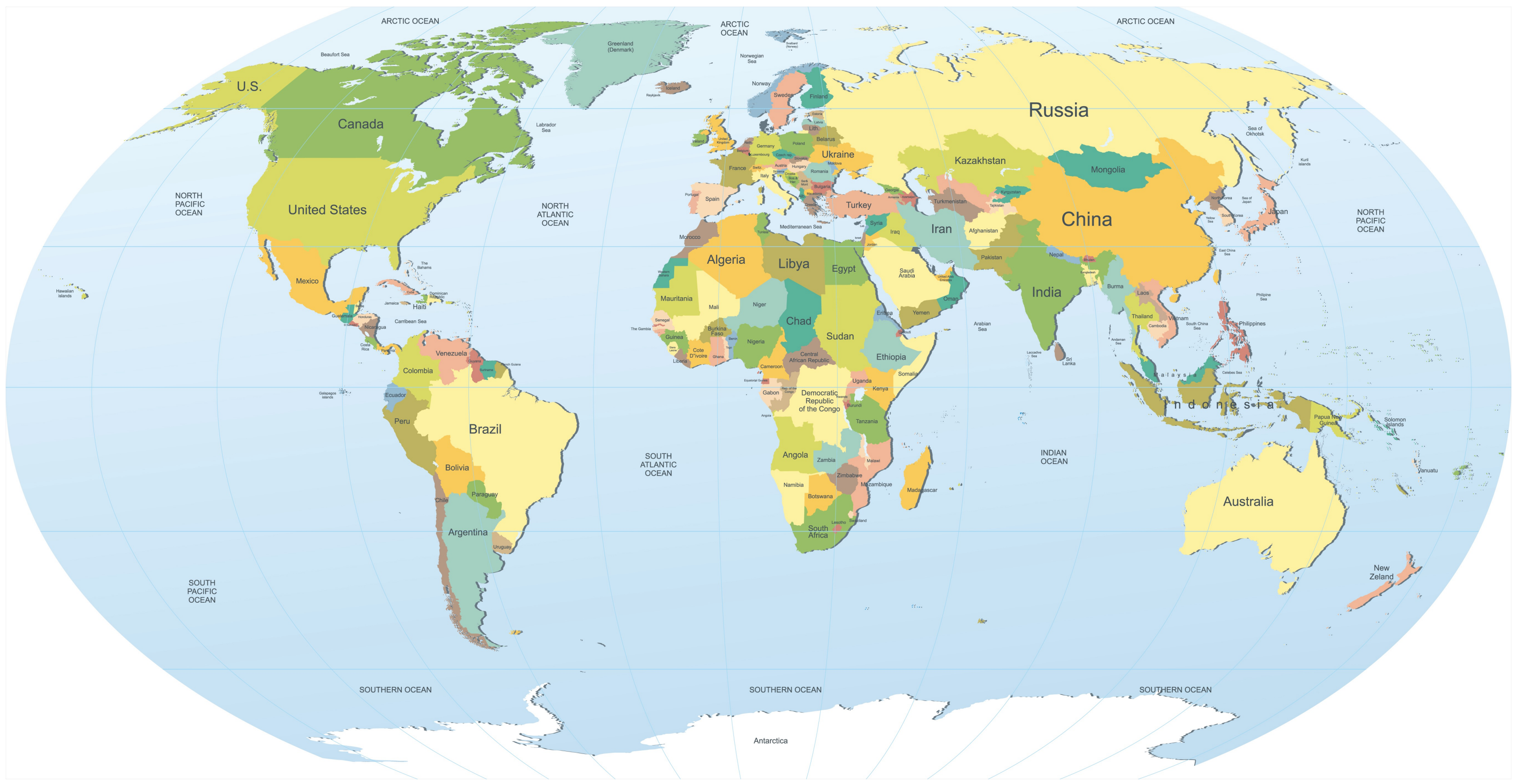
- Second fundamental form

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \mathbf{p}_{uu}^T \mathbf{n} & \mathbf{p}_{uv}^T \mathbf{n} \\ \mathbf{p}_{uv}^T \mathbf{n} & \mathbf{p}_{vv}^T \mathbf{n} \end{pmatrix}$$

- Together, they define a surface (if some compatibility conditions hold)

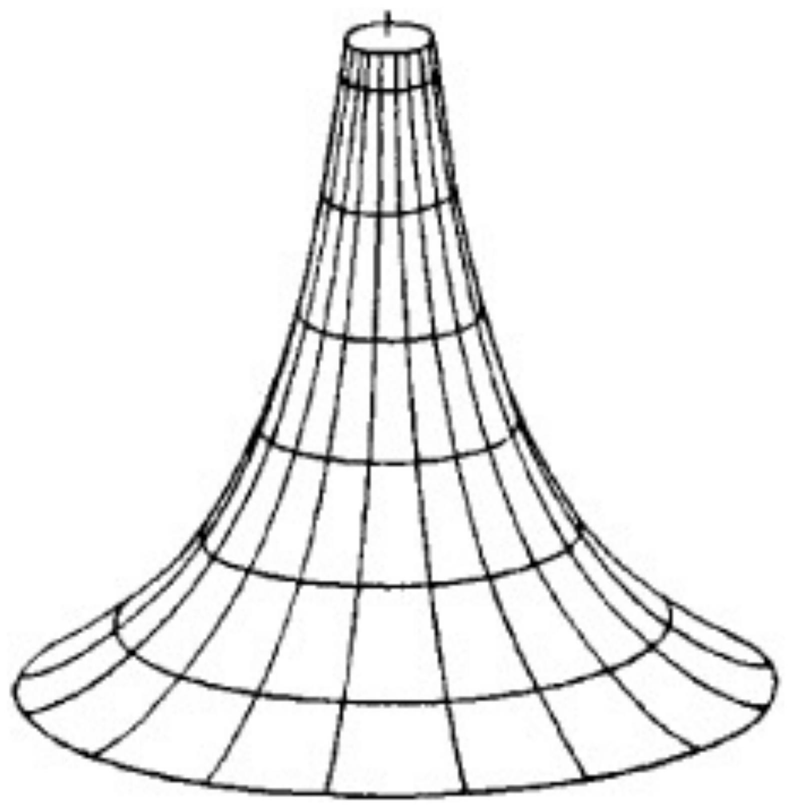
Intrinsic Geometry

- Properties of the surface that only depend on the first fundamental form
  - length
  - angles
  - Gaussian curvature

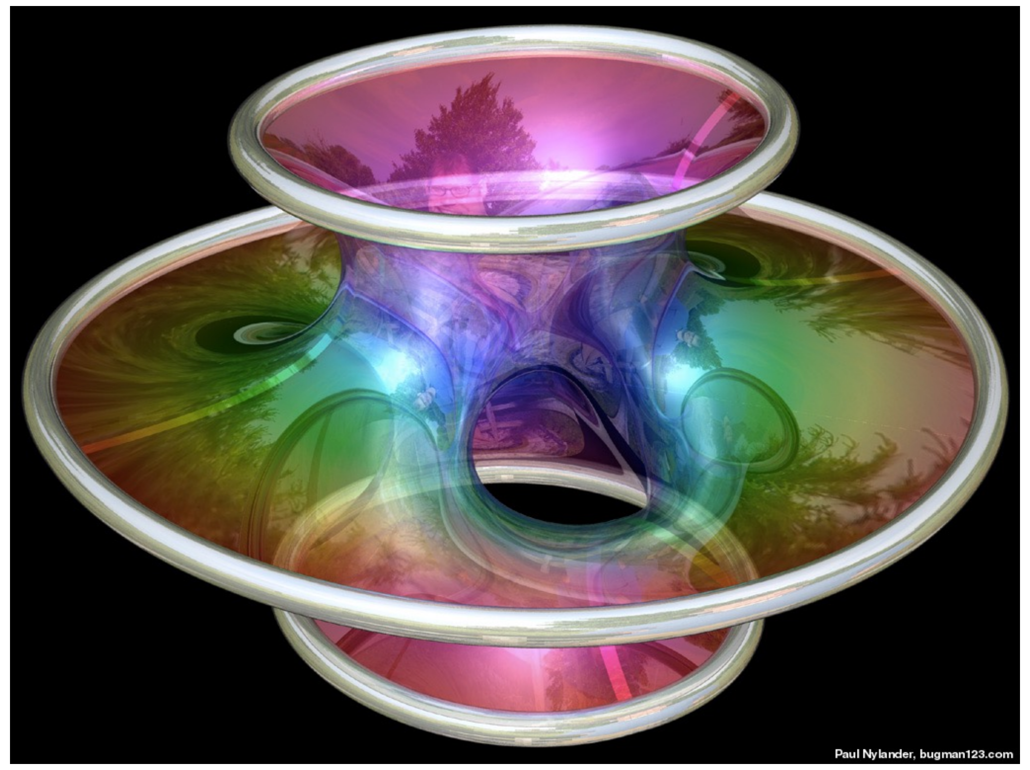


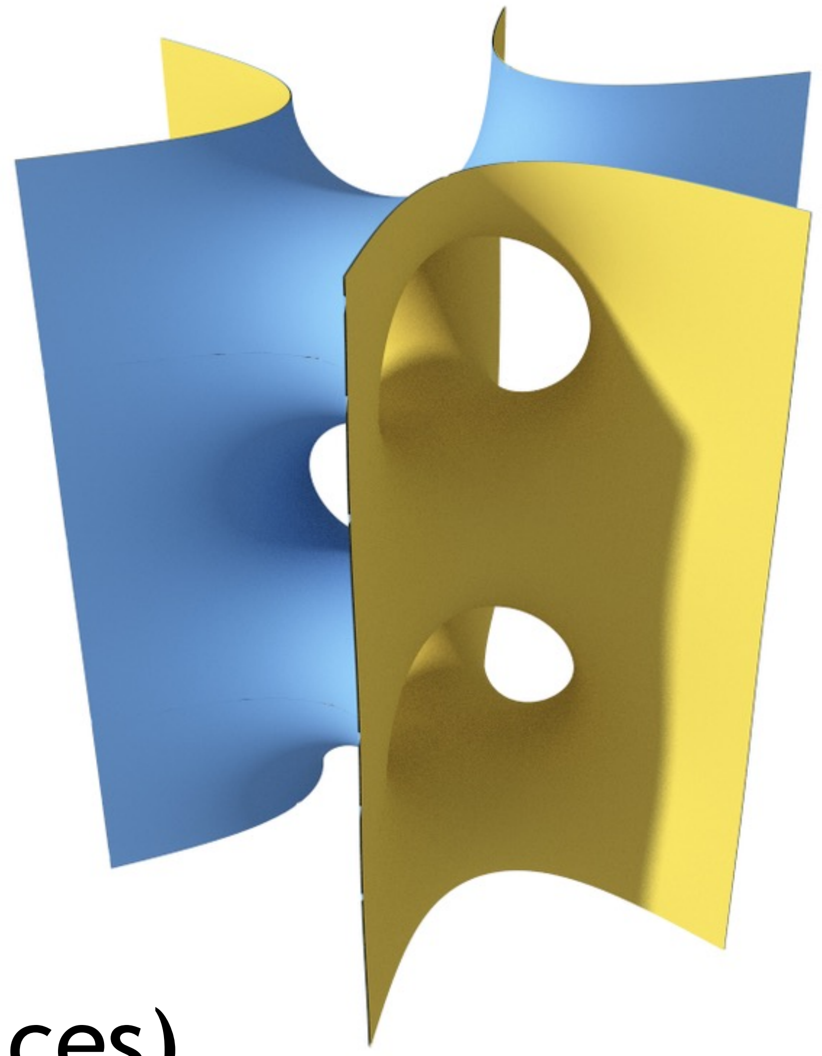
Interesting examples

- Hyperbolic geometry – spaces with constant ***negative*** Gauss curvature
- Constant mean curvature surfaces
- Minimal surfaces (H = 0)



Lobachevsky plane





Soap surfaces (minimal surfaces)

Laplace Operator

Acknowledgements: Olga Sorkine-Hornung

Laplace Operator

$$f : \mathbb{R}^3 \to \mathbb{R} \qquad \Delta f : \mathbb{R}^3 \to \mathbb{R}$$

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \mathrm{div} \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \dots$$

function in Euclidean space

divergence operator

Cartesian coordinates

$$\mathrm{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \qquad \mathrm{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Intuitive Explanation

The Laplacian $\Delta f(p)$ of a function f at a point p, up to a constant depending on the dimension, is the rate at which the average value of f over spheres centered at p deviates from $f(p)$ as the radius of the sphere grows.

Laplace-Beltrami Operator

- Extension of Laplace to functions on manifolds

$$f : \mathcal{M} \to \mathbb{R} \qquad \Delta f : \mathcal{M} \to \mathbb{R}$$

gradient operator

Laplace-Beltrami

$$\Delta_{\mathcal{M}} f = \mathrm{div}_{\mathcal{M}} \nabla_{\mathcal{M}} f$$

function on surface M

divergence operator

Laplace-Beltrami Operator

$$f(x, y, z) = x$$

- For coordinate functions:

$$\mathbf{p} = (x, y, z)$$

Laplace-
Beltrami

gradient
operator

mean
curvature

$$\Delta_{\mathcal{M}} \mathbf{p} = \mathrm{div}_{\mathcal{M}} \nabla_{\mathcal{M}} \mathbf{p} = -2H\mathbf{n} \ \in \mathbb{R}^3$$

function on
surface M

divergence
operator

unit
surface
normal

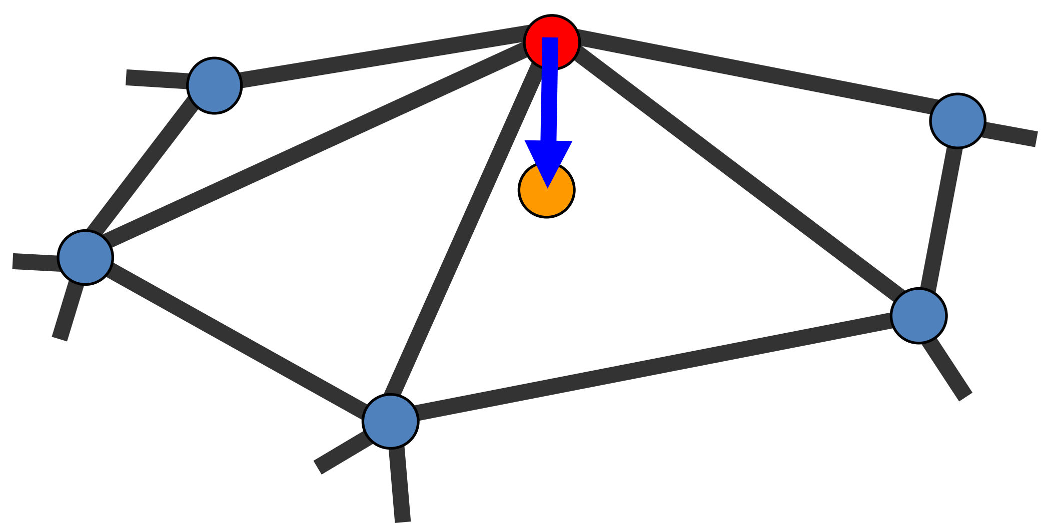
Differential Geometry on Meshes

- Assumption: meshes are piecewise linear approximations of smooth surfaces

- Can try fitting a smooth surface locally (say, a polynomial) and find differential quantities analytically
- But: it is often too slow for interactive setting and error prone

Discrete Differential Operators

- Approach: approximate differential properties at point $\mathbf{v}$ as spatial average over local mesh neighborhood $N(\mathbf{v})$ where typically
  - $\mathbf{v}$ = mesh vertex
  - $N_k(\mathbf{v})$ = k-ring neighborhood

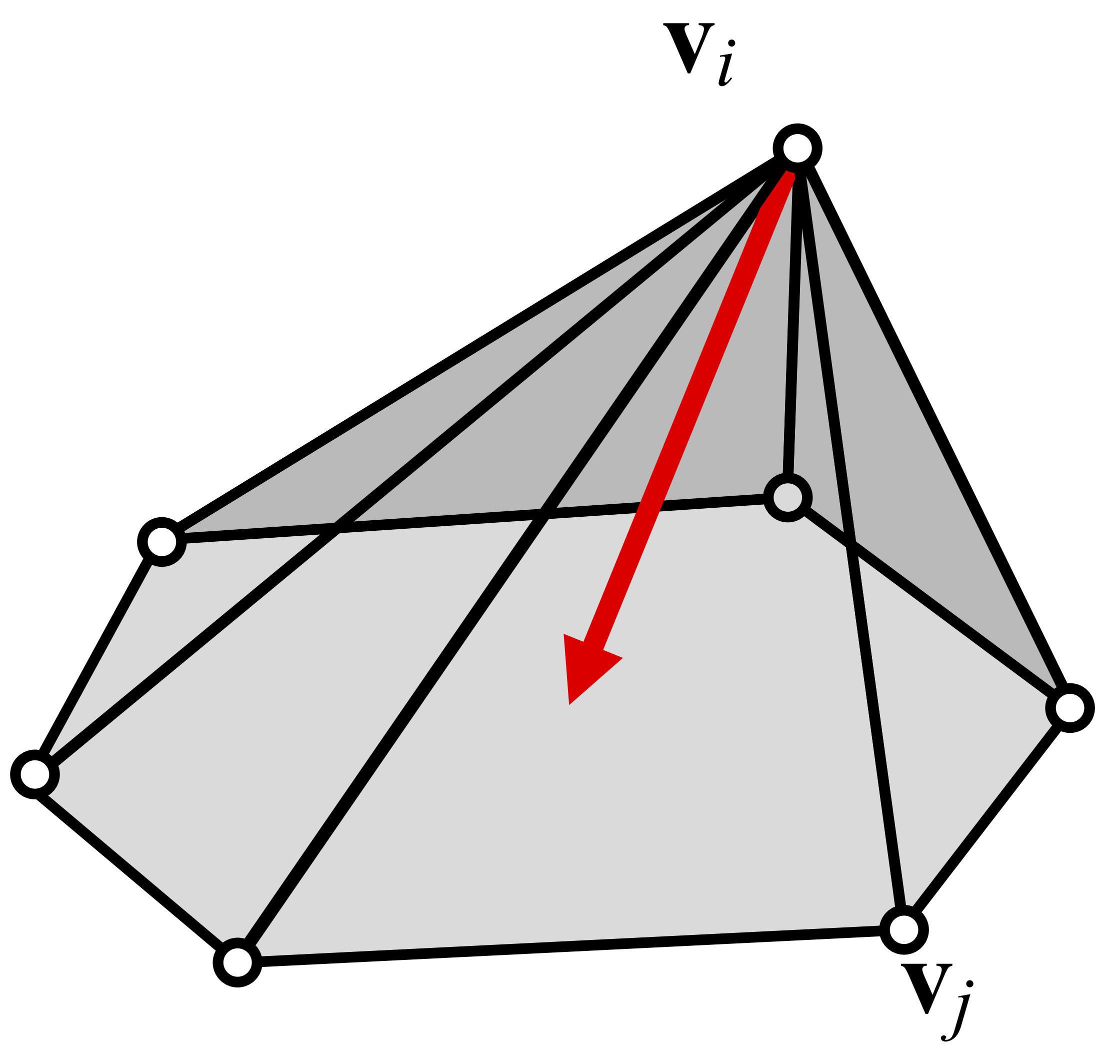


Discrete Laplace-Beltrami

$$\boxed{\Delta_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n}}$$

- Uniform discretization:

$$L_u(\mathbf{v}_i) = \frac{1}{|\mathcal{N}(i)|}\sum_{j\in\mathcal{N}(i)}(\mathbf{v}_j - \mathbf{v}_i) = \left(\frac{1}{|\mathcal{N}(i)|}\sum_{j\in\mathcal{N}(i)}\mathbf{v}_j\right) - \mathbf{v}_i$$



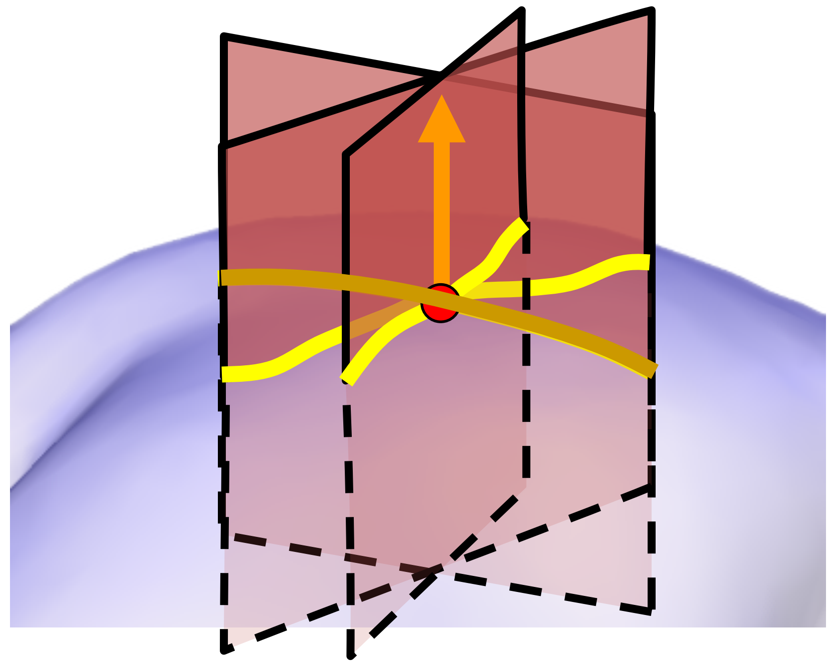
- Depends only on connectivity = simple and efficient

- Bad approximation for irregular triangulations

Discrete Laplace-Beltrami

$$\boxed{\Delta_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n}}$$

- Intuition for uniform discretization



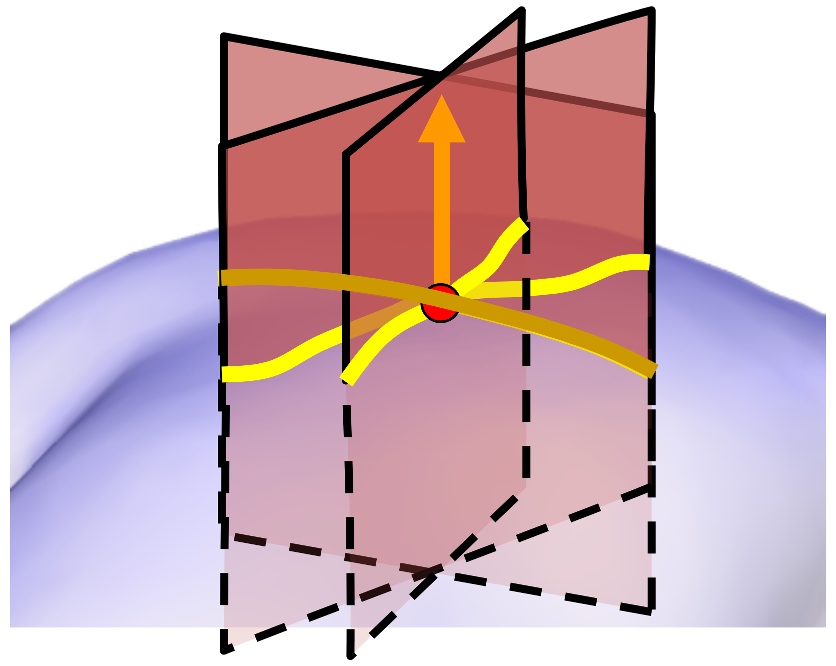
$$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa(\varphi)d\varphi \qquad\qquad \boldsymbol{\kappa}\mathbf{n} = \gamma''$$

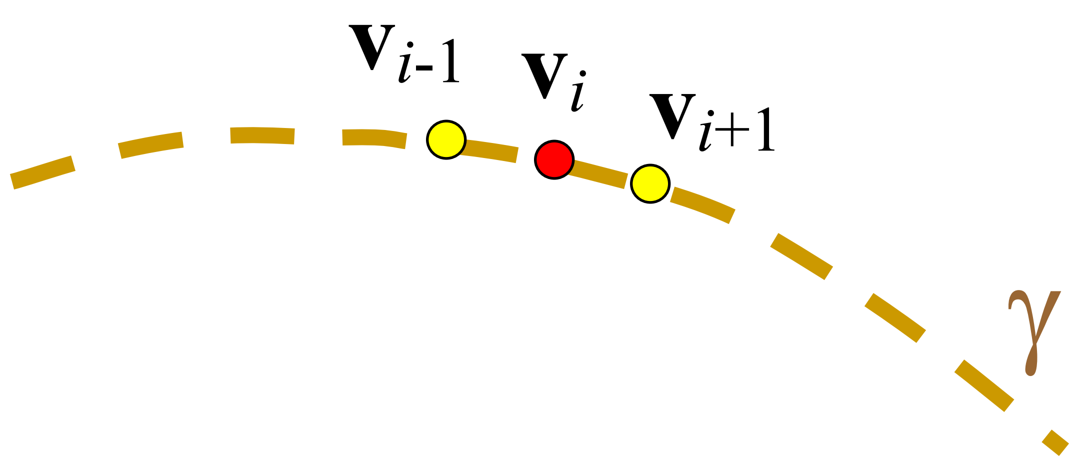
$$-2H\mathbf{n} = -2\left(\frac{1}{2\pi}\int_0^{2\pi} \kappa(\varphi)\,d\varphi\right)\mathbf{n} = -\frac{1}{\pi}\int_0^{2\pi} \kappa(\varphi)\mathbf{n}\,d\varphi = -\frac{1}{\pi}\int_0^{2\pi} \gamma''\,d\varphi$$

Discrete Laplace-Beltrami

$$\boxed{\Delta_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n}}$$

- Intuition for uniform discretization





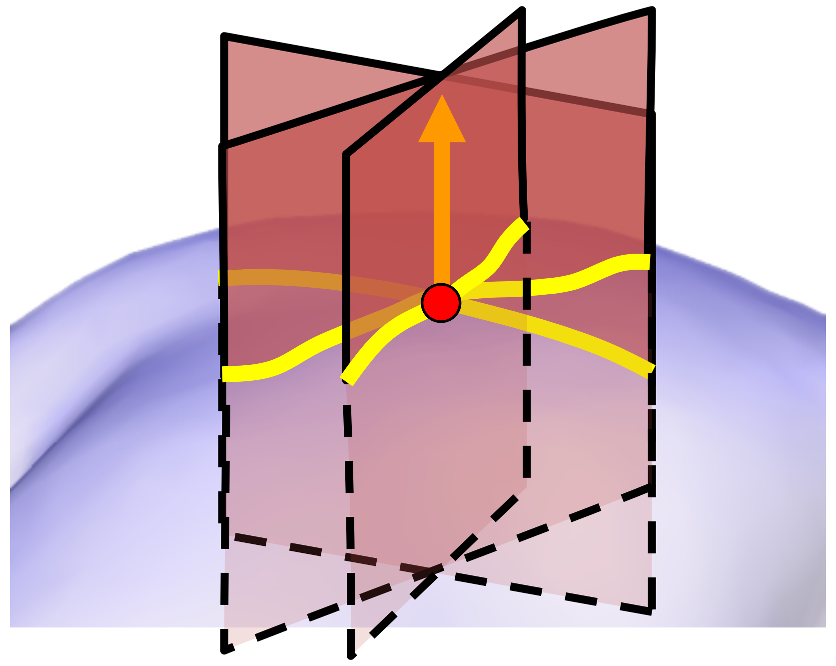
$$H = \frac{1}{2\pi}\int_0^{2\pi}\kappa(\varphi)d\varphi \qquad \kappa\mathbf{n} = \gamma''$$

$$\gamma'' \approx \frac{1}{h}\left(\frac{\mathbf{v}_{i+1}-\mathbf{v}_i}{h} - \frac{\mathbf{v}_i - \mathbf{v}_{i-1}}{h}\right) = -\frac{2}{h^2}\left(\frac{1}{2}(\mathbf{v}_{i-1}+\mathbf{v}_{i+1}) - \mathbf{v}_i\right)$$

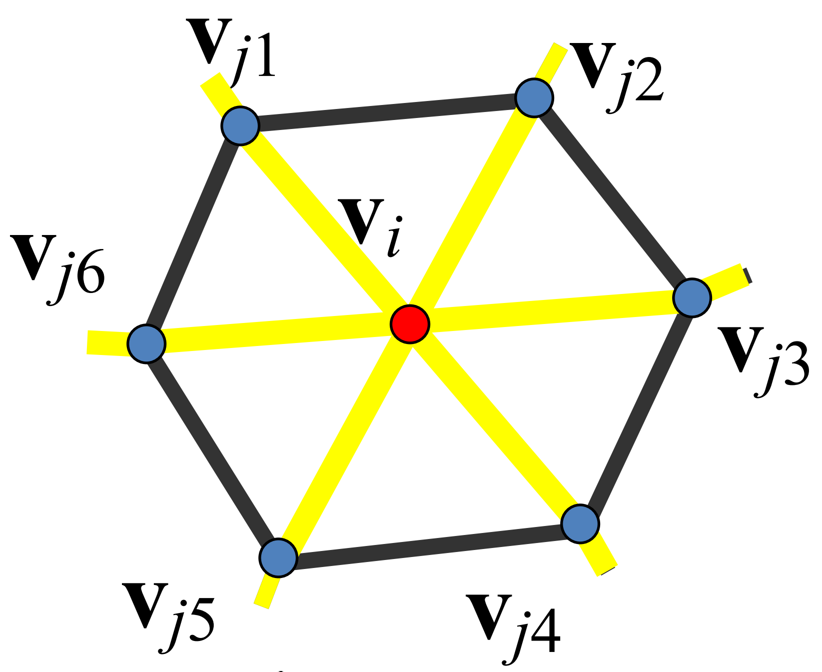
Discrete Laplace-Beltrami

$$\boxed{\Delta_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n}}$$

$$\gamma'' \approx \frac{1}{h}\left(\frac{\mathbf{v}_{i+1}-\mathbf{v}_i}{h} - \frac{\mathbf{v}_i - \mathbf{v}_{i-1}}{h}\right) = -\frac{2}{h^2}\left(\frac{1}{2}(\mathbf{v}_{i-1}+\mathbf{v}_{i+1}) - \mathbf{v}_i\right)$$



$$H = \frac{1}{2\pi}\int_0^{2\pi}\kappa(\varphi)d\varphi$$



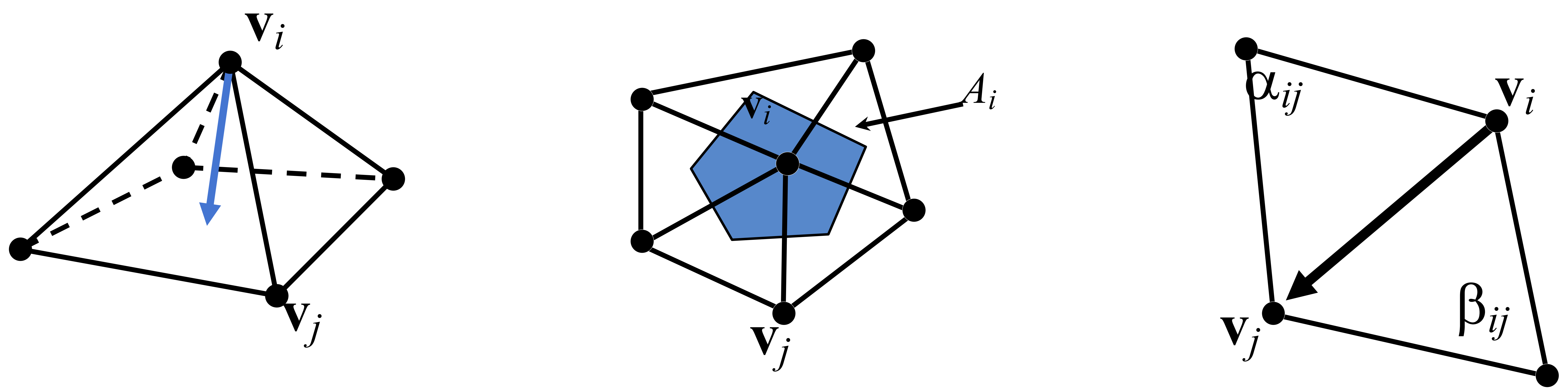
$$\frac{1}{2}(\mathbf{v}_{j_1}+\mathbf{v}_{j_4}) - \mathbf{v}_i \quad +$$
$$\frac{1}{2}(\mathbf{v}_{j_2}+\mathbf{v}_{j_5}) - \mathbf{v}_i \quad +$$
$$\frac{1}{2}(\mathbf{v}_{j_3}+\mathbf{v}_{j_6}) - \mathbf{v}_i = \frac{1}{2}\sum_{j\in\mathcal{N}(i)}\mathbf{v}_j - 3\mathbf{v}_i = 3\left(\frac{1}{6}\sum_{j\in\mathcal{N}(i)}\mathbf{v}_j - \mathbf{v}_i\right)$$

$$\boxed{L_u(\mathbf{v}_i)}$$

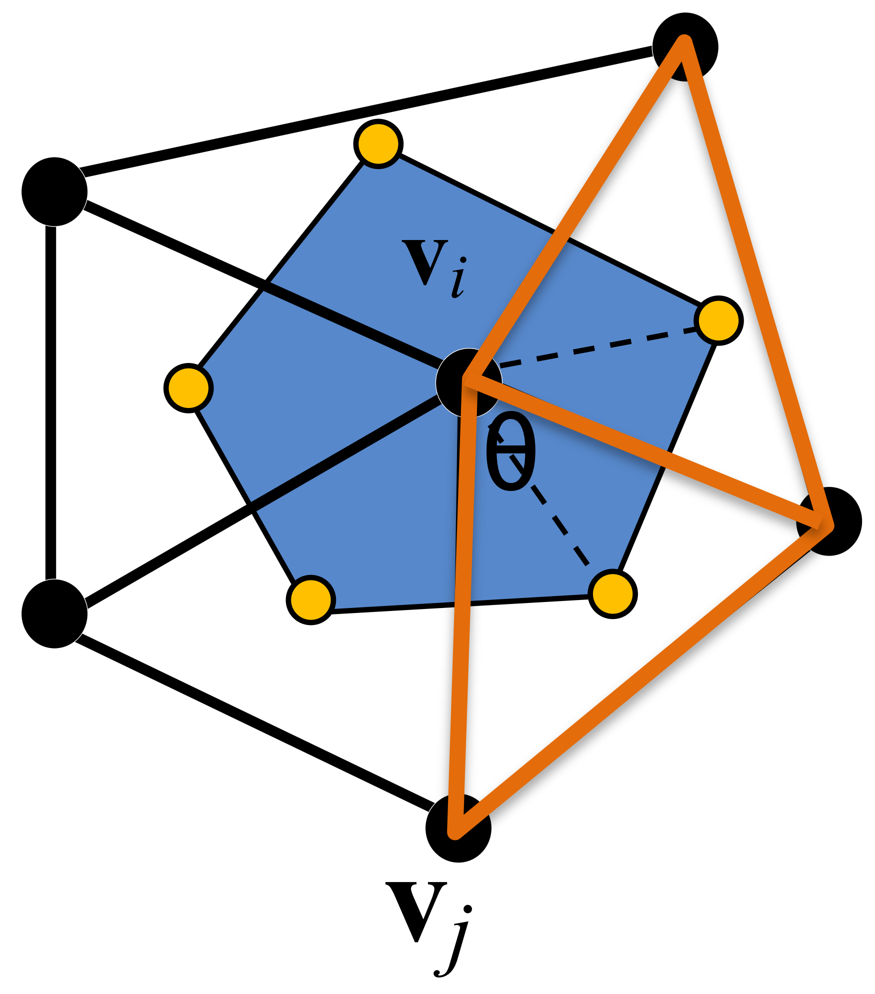
Discrete Laplace-Beltrami

- Cotangent formula

$$L_c(\mathbf{v}_i) = \frac{1}{A_i} \sum_{j \in \mathcal{N}(i)} \frac{1}{2}(\cot \alpha_{ij} + \cot \beta_{ij})(\mathbf{v}_j - \mathbf{v}_i)$$

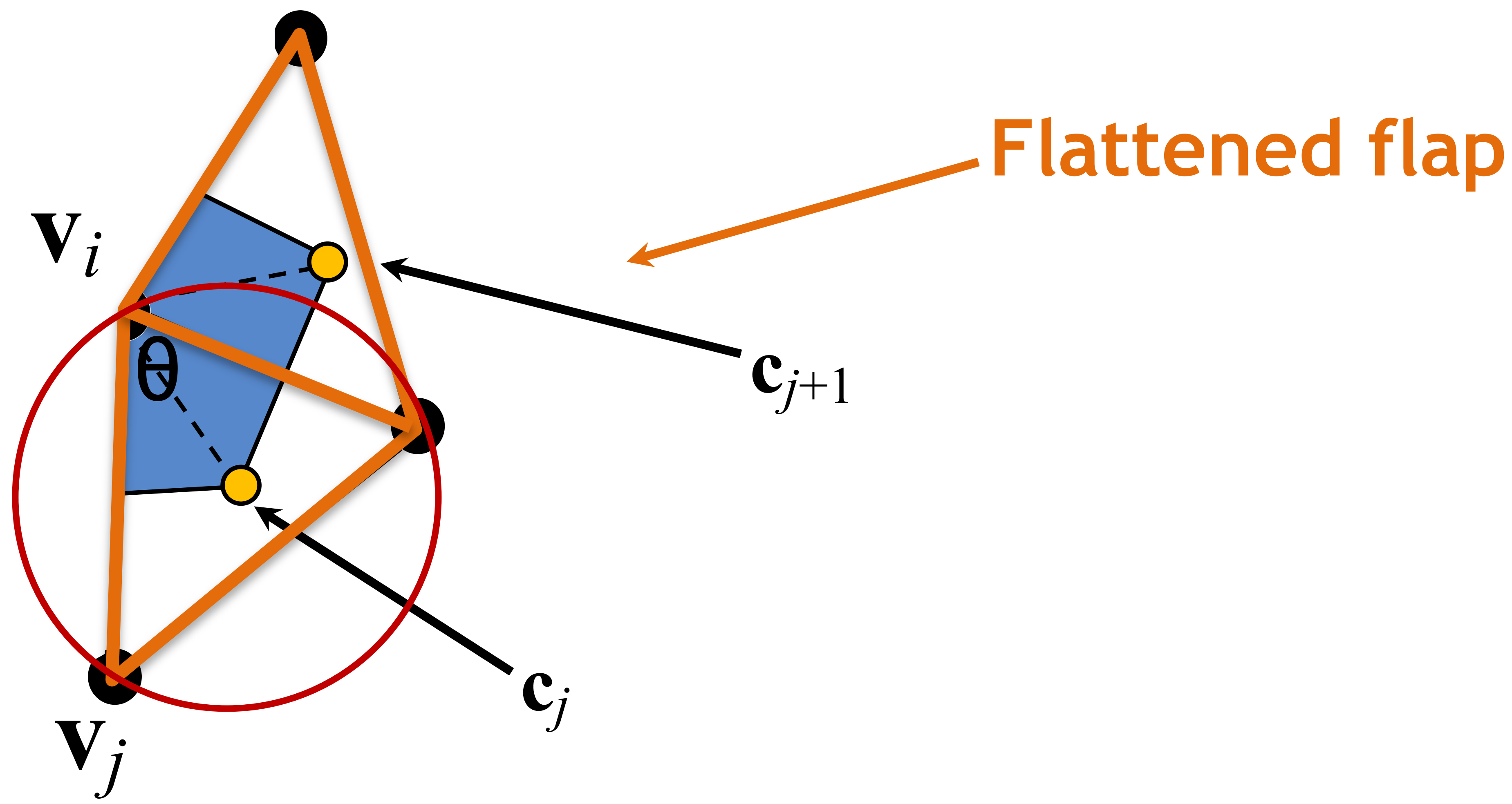


Voronoi Vertex Area



- Unfold the triangle flap onto the plane (without distortion)

Voronoi Vertex Area



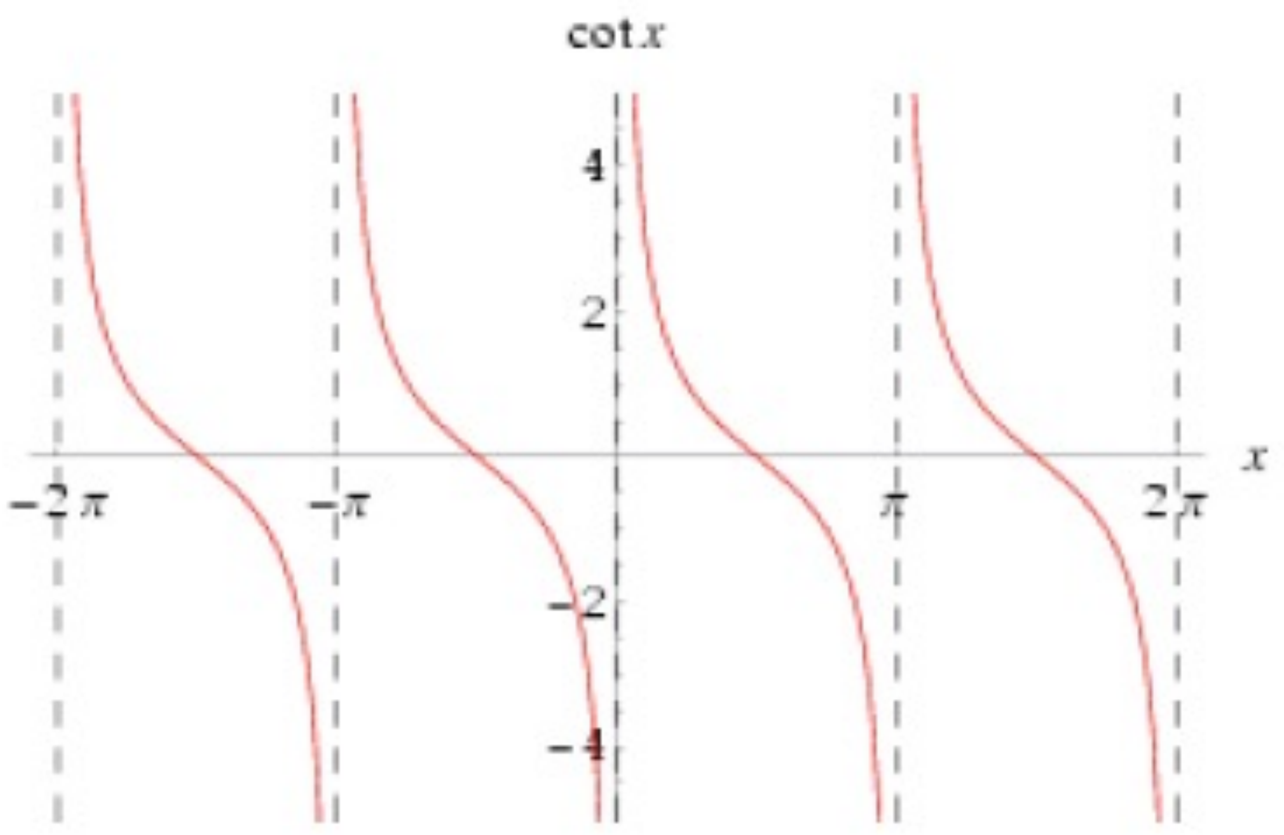
$$\mathbf{c}_j = \begin{cases} \text{circumcenter of } \triangle(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_{j+1}) & \text{if } \theta < \pi/2 \\ \text{midpoint of edge } (\mathbf{v}_j, \mathbf{v}_{j+1}) & \text{if } \theta \geq \pi/2 \end{cases}$$

$$A_i = \sum_j \text{Area}\left(\triangle(\mathbf{v}_i, \mathbf{c}_j, \mathbf{c}_{j+1})\right)$$

Discrete Laplace-Beltrami

- Cotangent formula

$$L_c(\mathbf{v}_i) = \frac{1}{A_i} \sum_{j \in \mathcal{N}(i)} \frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij})(\mathbf{v}_j - \mathbf{v}_i)$$



- Accounts for mesh geometry
- Potentially negative/ infinite weights

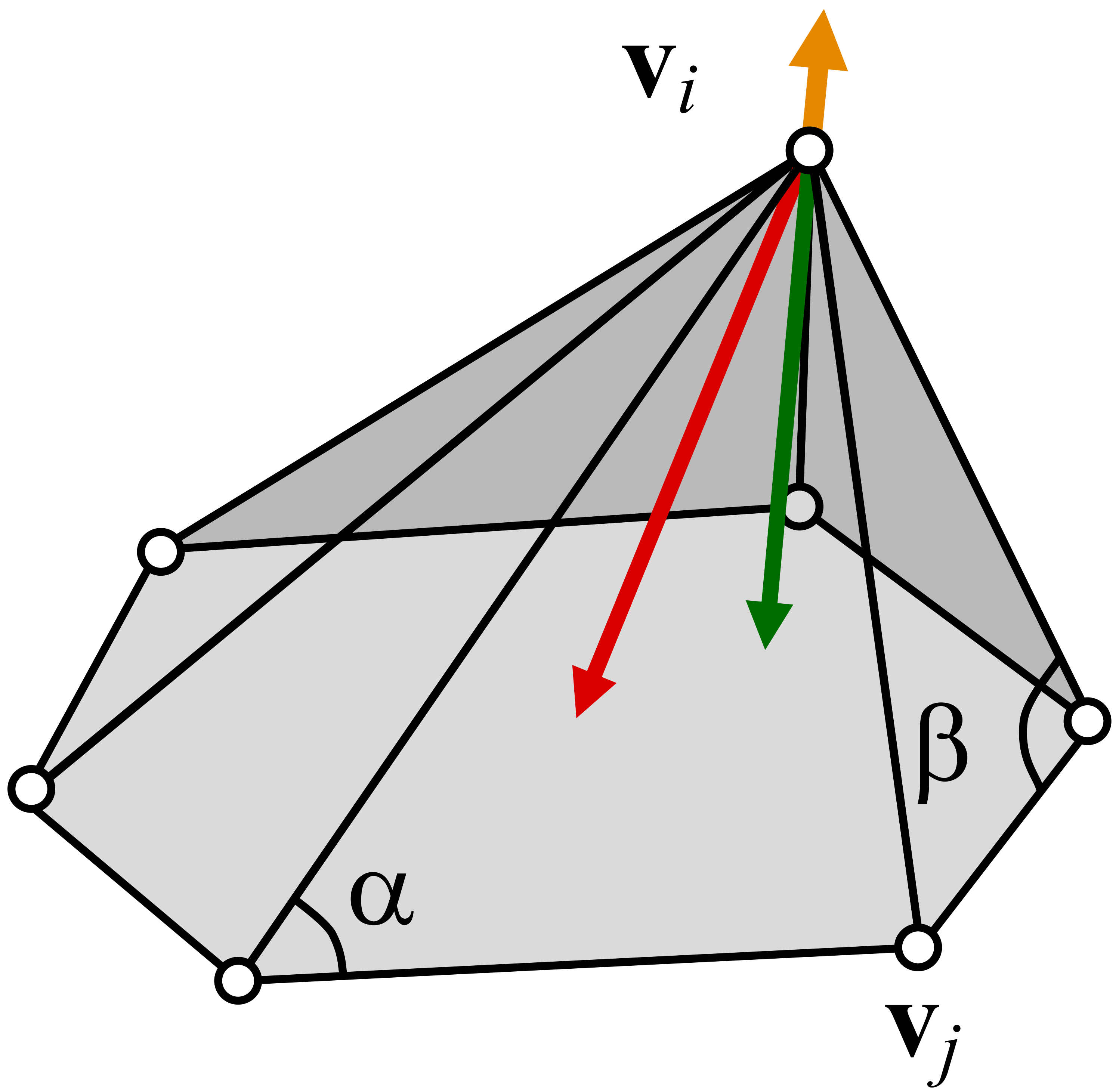
Discrete Laplace-Beltrami

- Cotangent formula

$$L_c(\mathbf{v}_i) = \frac{1}{A_i} \sum_{j \in \mathcal{N}(i)} \frac{1}{2}(\cot \alpha_{ij} + \cot \beta_{ij})(\mathbf{v}_j - \mathbf{v}_i)$$

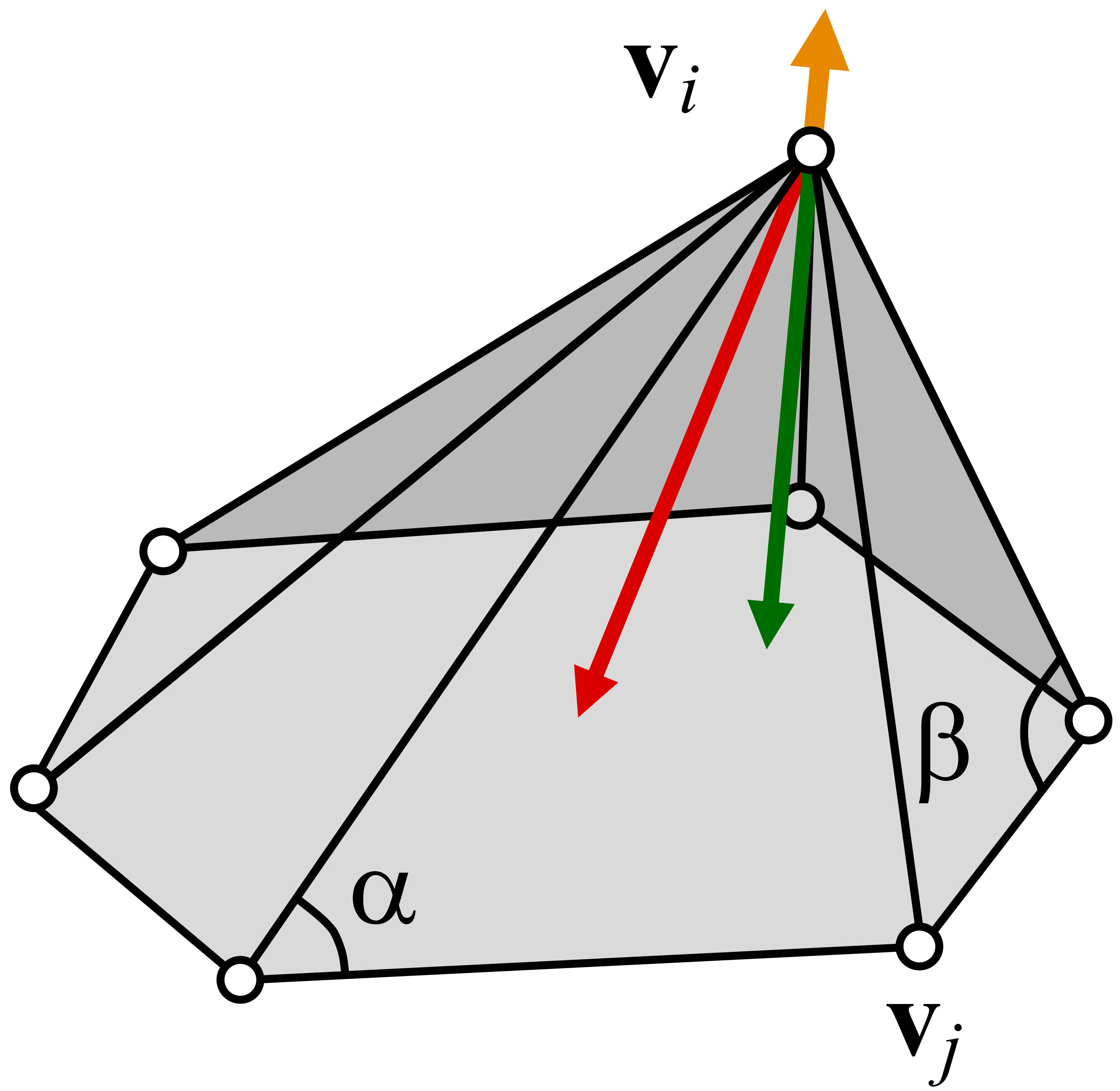
  - Can be derived using linear Finite Elements
  - Nice property: gives zero for planar 1-rings!

Discrete Laplace-Beltrami



- Uniform Laplacian $\mathbf{L}_u(\mathbf{v}_i)$
- Cotangent Laplacian $\mathbf{L}_c(\mathbf{v}_i)$
- Normal

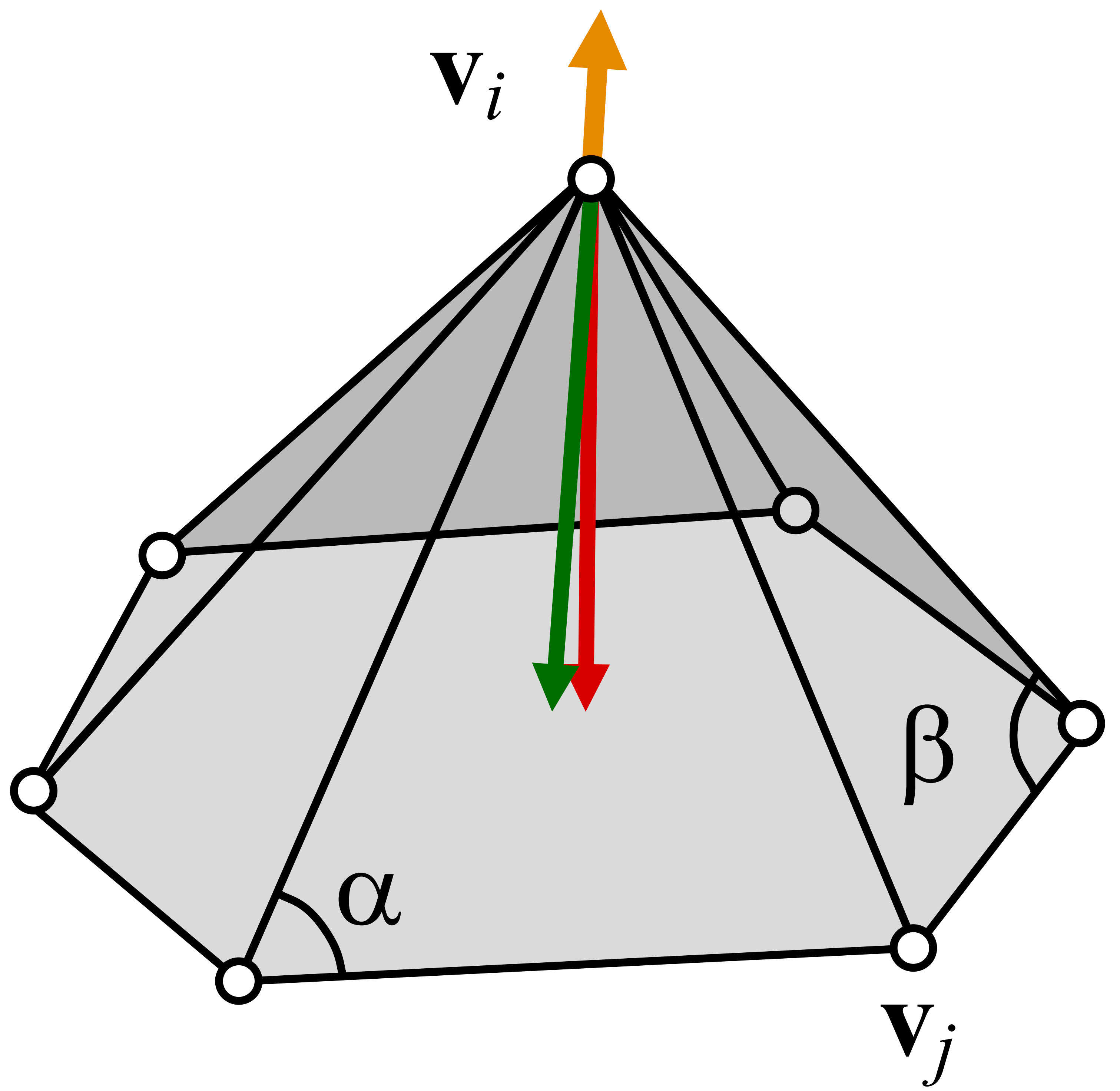
Discrete Laplace-Beltrami



- Uniform Laplacian $\mathbf{L}_u(\mathbf{v}_i)$
- Cotangent Laplacian $\mathbf{L}_c(\mathbf{v}_i)$
- Normal

- For nearly equal edge lengths
  Uniform ≈ Cotangent

Discrete Laplace-Beltrami



- Uniform Laplacian $\mathbf{L}_u(\mathbf{v}_i)$
- Cotangent Laplacian $\mathbf{L}_c(\mathbf{v}_i)$
- Normal

- **For nearly equal edge lengths**
  Uniform ≈ Cotangent

Cotan Laplacian allows computing discrete normal

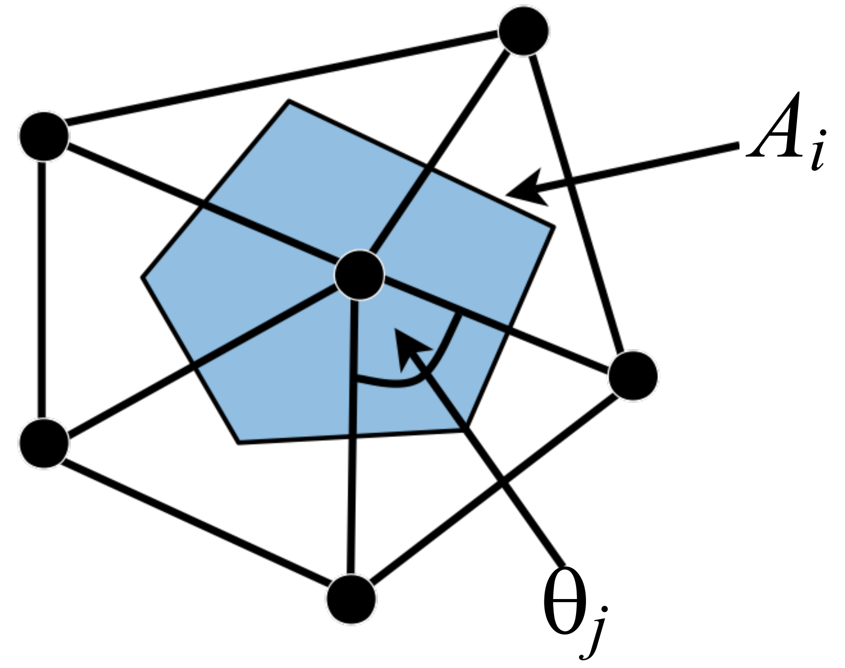
Discrete Curvatures

- Mean curvature (sign defined according to normal)

$$|H(\mathbf{v}_i)| = \|L_c(\mathbf{v}_i)\|/2$$

- Gaussian curvature

$$K(\mathbf{v}_i) = \frac{1}{A_i}(2\pi - \sum_j \theta_j)$$



- Principal curvatures

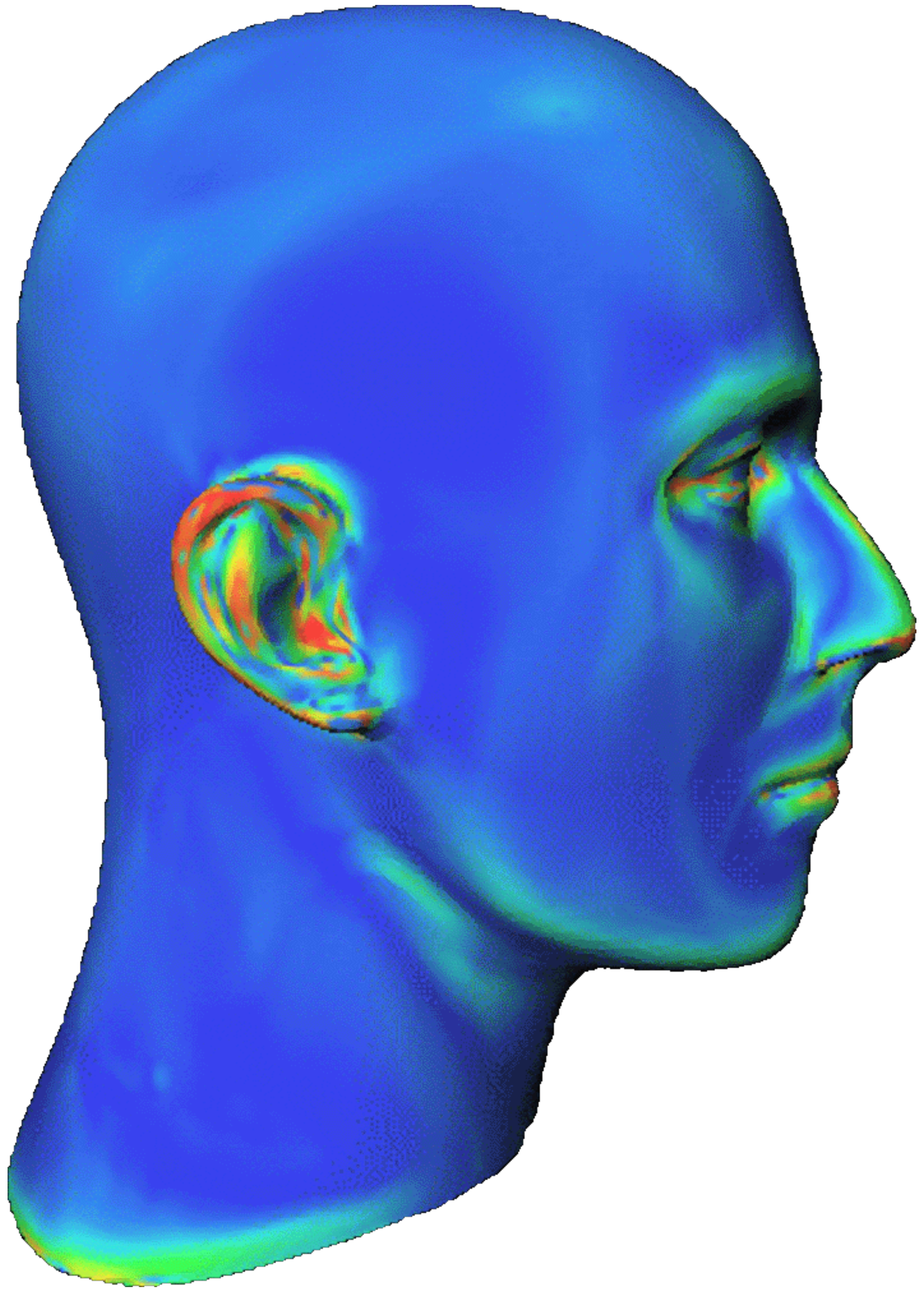
$$\kappa_1 = H - \sqrt{H^2 - K} \qquad \kappa_2 = H + \sqrt{H^2 - K}$$

Discrete Gauss-Bonnet Theorem

- Total Gaussian curvature is fixed for a given topology

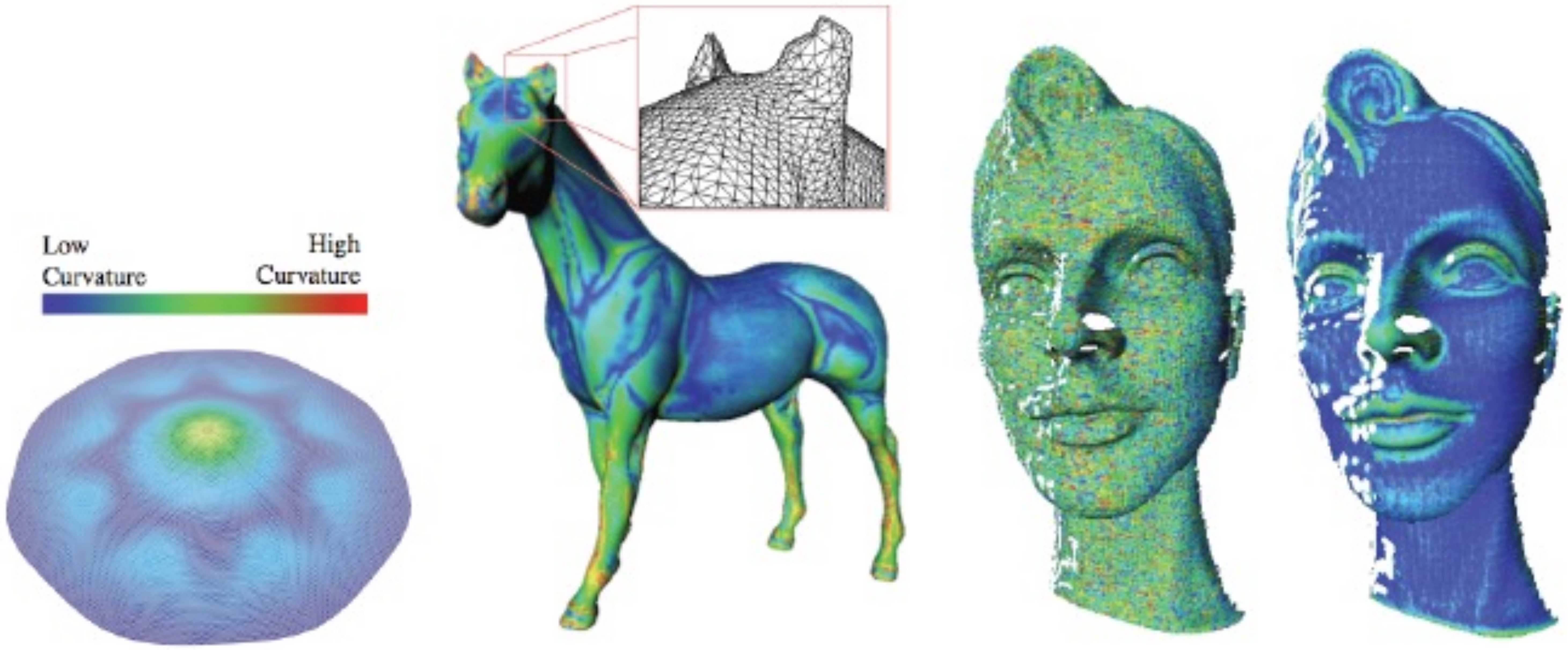
$$\int_{\mathcal{M}} K\,dA = \sum_i A_i K(\mathbf{v}_i) = \sum_i \left[2\pi - \sum_{j\in\mathcal{N}(i)} \theta_j\right] == 2\pi\chi(\mathcal{M})$$

Example: Discrete Mean Curvature



Links and Literature

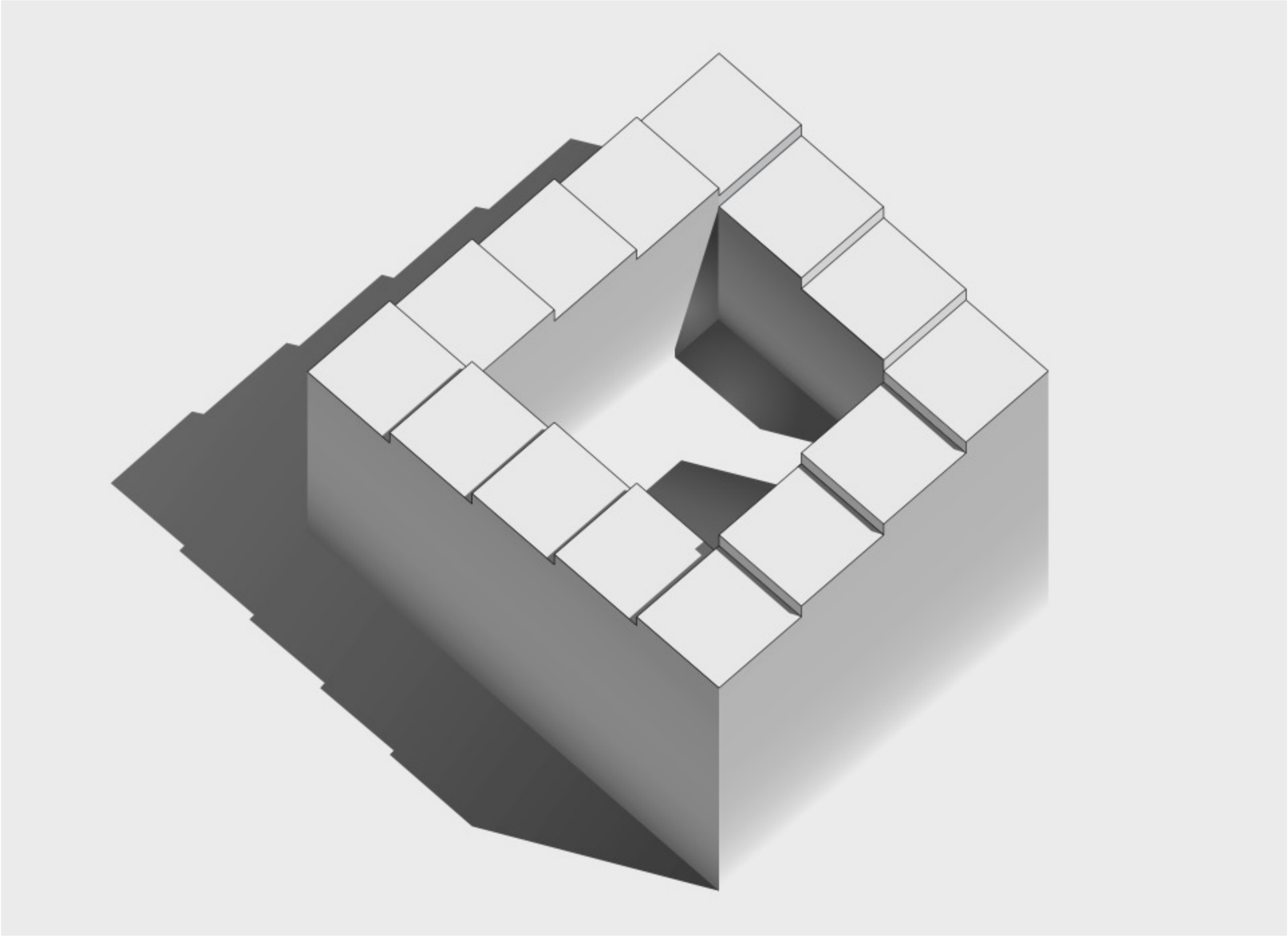
- M. Meyer, M. Desbrun, P. Schroeder, A. Barr
  Discrete Differential-Geometry Operators for Triangulated 2-Manifolds, VisMath, 2002



Links and Literature

Discrete Differential Geometry: An Applied Introduction

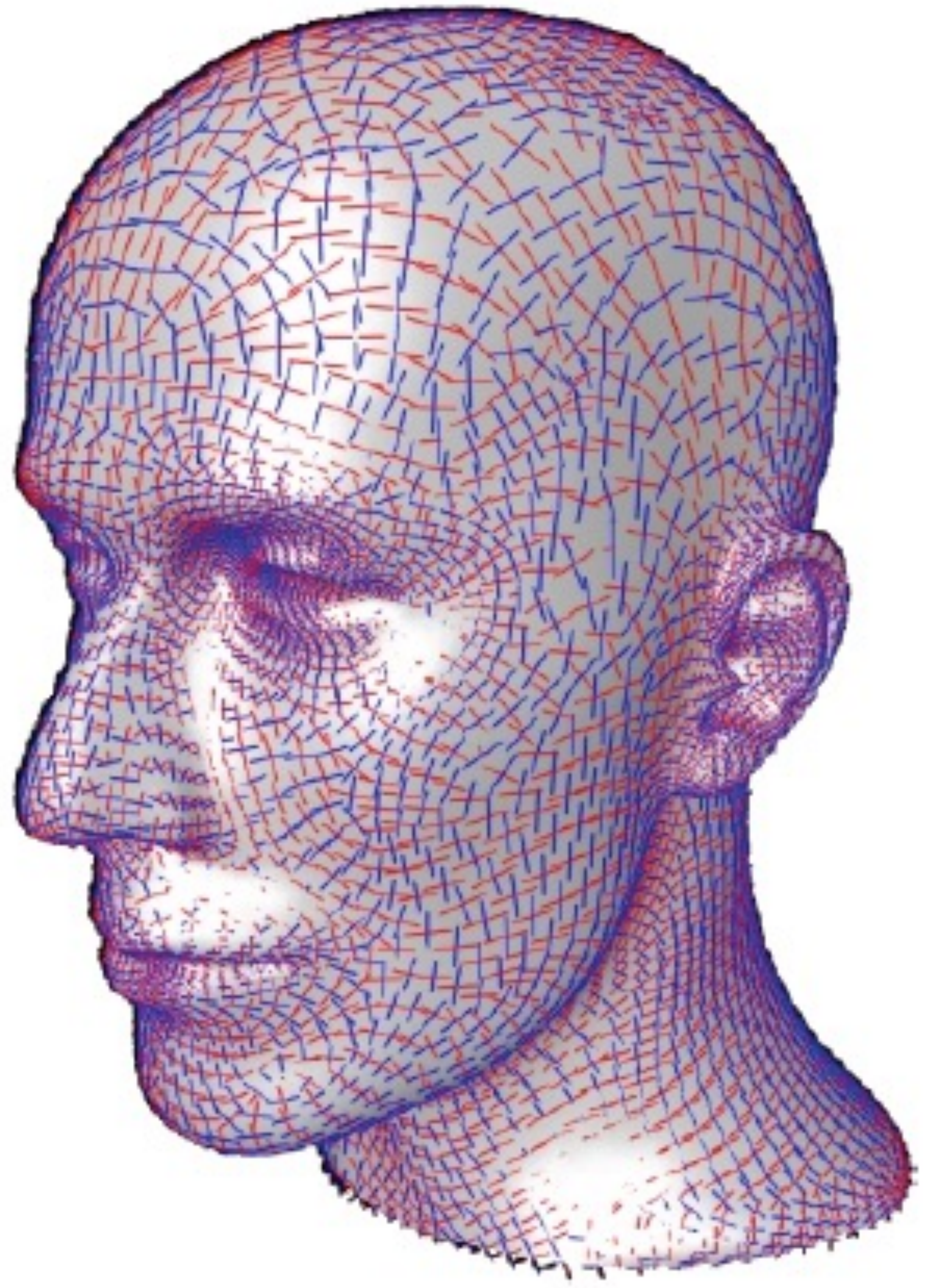
Keenan Crane



https://www.cs.cmu.edu/~kmcrane/Projects/DGPDEC/

Links and Literature

- libigl implements many discrete differential operators
- See the tutorial!
- http://libigl.github.io/libigl/tutorial/tutorial.html



Principal Directions

Thank you