Numerical Quadrature Question: Summer Semester 2016

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Background: A numerical quadrature rule Q for a 2-dimensional region Ω is a set of n triples $(w_i, x_i, y_i), i = 1, ..., n$, where w_i is a *weight* and (usually) $(x_i, y_i) \in \Omega$. Suppose $f(x, y)$ is an integrable function defined over Ω, and symbolize its integral by $I(f) = \int_\Omega f(x, y)\, d\Omega$. The quadrature rule may be used to approximate such integrals by

$$I(f) \approx Q(f) \equiv \sum_{i=1}^{n} w_i f(x_i, y_i)$$

Let Ω be the *unit quarter disk*, that is, the set of points such that

$$\Omega = \{(x, y)|0 <= x \text{ and } 0 <= y \text{ and } x^2 + y^2 <= 1\}$$

Using the notation $\Gamma(x)$ for the standard mathematical gamma function, it can be shown that, for nonnegative integer exponents p and q, we have:

$$I(x^p y^q) = \Gamma(\frac{p+3}{2})\Gamma(\frac{q+1}{2})/\Gamma(\frac{p+q+4}{2})/2/(1+p)$$

The total degree d of a monomial $x^p y^q$ is defined as $d = p + q$. We say that a quadrature rule for a 2-dimensional region has *precision* d if $I(f) = Q(f)$ for all monomials of total degree d or less.

A: We wish to compute a quadrature rule Q for Ω which has precision 2, using $n = 3$ points.

- A1: List the monomials which Q must integrate exactly.
- A2: What is the value of m, the number of monomials which Q must integrate exactly?
- A3: What is the value of k, the number of degrees of freedom we have in specifying the rule Q?

B: The rule Q must integrate the monomial of total degree 0, the two monomials of total degree 1, and the three monomials of total degree 2 exactly.

- B1: Write these statements down as a system of nonlinear equations involving the data $(w_i, x_i, y_i), i = 1, n)$. Where convenient, you may use symbolic expressions rather than numeric values.
- B2: Write a computer function **resid()** that accepts arbitrary values for the data $(w_i, x_i, y_i), i = 1, n)$ and returns the vector r of residuals, that is, $r(1) = Q(1) - I(1), r(2) = Q(x) - I(x), ...$
- B3: Include a copy of your **resid()** function.

C: Write a program that guesses initial values for Q, and then solves the equations $r = 0$ by calling some appropriate builtin-in library function for the solution of a rectangular $(m \times k)$ system of nonlinear equations. The MATLAB functions $fsolve()$ or $lsqnonlin()$ are examples of such software.

- C1: What routine did you use to solve the system of nonlinear equations?
- C2: What initial values did you use for the points and weights?
- C3: What final values did your program return for the points and weights?
- C4: Include a copy of your program.

D: Let $f(x) = 4 + 8xy$.

- D1: What is your estimated integral Q(f)?
- D2: Assuming exact arithmetic, should $Q(f) = I(f)$ in this case?