Finite Element Method Answers: Summer 2015

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Consider the following partial differential equation:

$$-u'' + u = -e^x \text{ for } x \in [10, 22]$$
$$u(10.0) = 70.0; u(22.0) = 145$$

Suppose the finite element method is applied, so that the interval [10,22] is broken up into three elements of equal size, each of which uses piecewise **quadratic** basis functions. Thus, there will be a total of 7 equally spaced nodes x_i. For each node x_i, we may define a corresponding basis function $\psi_i(x)$ which is continuous everywhere, is a quadratic polynomial within each element, attains the value 1 at the node x_i and is 0 at the other nodes.

We may assume that the finite element approximate solution is represented by

$$u^h(x) = \sum_{i=1}^{7} c_i \psi_i(x)$$

Because of the boundary conditions, c_1 and c_7 are known immediately. To find the remaining coefficients, we need to set up a 5x5 linear system $Ac = f$.

a) Forming the matrix A requires evaluating the stiffness matrix $K(i,j) = \int_{10}^{22} \psi_i'(x)\psi_j'(x)dx$, for i from 2 to 6. Simply because of the choice of piecewise quadratic basis functions, many entries of K are guaranteed to be zero or positive. Fill the matrix below with '0', '+' or '?' according to whether the corresponding entry is guaranteed to be zero, positive, or is not automatically known. *This answer does not require any computation.*

	2	3	4	5	6
2	+	?	0	0	0
3	?	+	?	?	0
4	0	?	+	?	0
5	0	?	?	+	?
6	0	0	0	?	+

b) Give a formula for the basis function associated with the node $x = 14.0$. Your formula must be correct for any possible value of x in [10,22]. Your formula will depend on the locations of the nodes. Use explicit values for these quantities; that is, instead of a factor like $(x - x_3)$ say $(x - 14)$.

$$\psi_3(x) = \begin{cases} (x-10)(x-12)/(14-10)/(14-12), & \text{if } 10 \le x \le 14, \\ (x-16)(x-18)/(14-16)/(14-18), & \text{if } 14 \le x \le 18, \\ 0, & \text{otherwise} \end{cases}$$

c) Suppose that the table below contains the finite element coefficients c_i for this problem. Use this information to determine the value of $u^h(15.0)$.

i	1	2	3	4	5	6	7
x_i	10.0	12.0	14.0	16.0	18.0	20.0	22.0
c_i	70.0	85.0	100.0	50.0	75.0	125.0	145.0

$$\begin{aligned} u^h(x) &= 100.0 * \psi_3(x) + 50.0 * \psi_4(x) + 75.0 * \psi_5(x) \\ \psi_3(x) &= (x-16)/(14-16) * (x-18)/(14-18) \\ \psi_4(x) &= (x-14)/(16-14) * (x-18)/(16-18) \\ \psi_5(x) &= (x-14)/(18-14) * (x-16)/(18-16) \\ u^h(15.0) &= 100.0 * 0.375 + 50.0 * 0.750 - 75.0 * 0.1250 = 65.6250 \end{aligned}$$

d) The computation described above used a number of elements $ne = 3$ and a typical element size $h = 12/ne = 4$. Suppose that the problem is solved repeatedly, reducing the value of h each time, and that the exact solution $u(x)$ is known, and that we have recorded the L^2 norm of the error $e(h) = ||u - u^h||$ for each value of h in the table below.

For finite element applications, the L^2 norm of the error typically has an asymptotic convergence rate, related to the element size h, such as $O(1)$, $O(h^1)$, $O(h^2)$, and so on.

Using the h and $e(h)$ data in the table, fill in the row labeled "norm rate" that estimates the convergence rate by comparing successive pairs of element sizes and errors. You will need to transform the data in the usual way in order to produce estimates of the convergence rate exponent (which is typically a small integer).

ne	3	8	10	20
h	4	1.5	1.2	0.6
$e(h)$	11.8496	0.126155	0.0646452	0.00808974
norm rate	XXX	4.63133	2.99625	2.99838

Does your observed norm rate agree with the expected value from the theory?
YES, AS H DECREASES, NORM RATE IS GOING TO 3.