Linear Algebra Answer: Spring Semester 2016

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Background: The matrix eigenvalue problem considers the equation

$$Ax = \lambda x$$

where A is a given n by n matrix, while λ and x are an unknown scalar and vector respectively. To avoid the trivial solution $x = 0$, we must impose some additional condition on x to guarantee that it is not completely zero. Here, we choose to require $||x||_2 = 1$. There is always at least one, and there may be as many as n distinct solutions to this problem. Even if the given matrix A is real, it may be the case that some of the solutions require complex arithmetic.

A Newton-method approach: The matrix eigenvalue problem is equivalent to the following multidimensional nonlinear equation problem: Find $v = (x, \lambda)$ such that

$$f(v) = \begin{pmatrix} Ax - \lambda x \\ x'x - 1 \end{pmatrix} = 0$$

Because we can regard this as a multidimensional nonlinear equation, we can apply the multidimensional version of Newton's method.

A: In order to carry out the Newton method, we need to determine the form of the Jacobian matrix $J_{i,j} = \frac{\partial f_i}{\partial v_j}$. For the eigenvalue problem, what is J? (You should be able to express your answer as a 2x2 block matrix. Be precise in giving the values and shapes of the entries! Assume that x is a column vector, and x' is a row vector.)

Answer A:

$$J = \begin{pmatrix} A - \lambda I & -x \\ 2 * x' & 0 \end{pmatrix}$$

B: Suppose B_n is the tridiagonal matrix with entries -1, 2, -1. Suppose we are interested in solving the eigenvalue problem for this matrix. What facts can we immediately state, without computation, about the eigenvalues and eigenvectors of this matrix?

Answer B: *We note that matrix B_n is tridiagonal, and symmetric. Special algorithms exist for computing eigenvalues and eigenvectors of a tridiagonal* *matrix, but that's not the most important issue. Symmetry is very important. It means that there is a complete set of n eigenvectors, that the eigenvectors can be chosen to be orthogonal, and that all the eigenvalues are real. We will see that the existence of real eigenvalues matters when we compare questions D and E.*

C: Write a program that takes as input a 2x2 matrix A, initial guesses λ_0 and x_0, and applies m steps of the Newton iteration, seeking a solution to the eigenvalue problem. At each step, print out $||f(v^k)||_2$, the l_2 norm of the residual. Print a copy of your program and submit it with your work.

Answer C: *Here is a MATLAB function that does the job:*

```
function [ lambda, x ] = eigenvalue_newton ( m, n, a, lambda, x )

  j = zeros ( n + 1, n + 1 );
  f = zeros ( n + 1, 1 );

  for i = 1 : m

    j(1:n,1:n) = a - lambda * eye ( n );
    j(1:n,n+1) = - x(1:n);
    j(n+1,1:n) = 2.0 * x(1:n)';

    f(1:n) = a * x - lambda * x;
    f(n+1) = x' * x - 1.0;

    fn = norm ( f );
    fprintf ( 1, '  I = %d  ||F|| = %g\n', i, fn );

    v = j \ f;
    x = x - v(1:n);
    lambda = lambda - v(n+1);

  end

  return
end
```

D: Use the matrix B_2, which is simply

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

an initial eigenvalue estimate $\lambda_0 = 1.5$, an initial eigenvector estimate $x_0 = (1, 2)$. After taking $m = 10$ steps, list λ, x, $||f||_2$.

Answer D: *After 10 steps, you should have* $\lambda = 1$, $x = (0.7071, 0.7071)$, *and* $||f||_2 \approx 1.0E - 16$.

E: Repeat question D, but now use the matrix

$$C_2 = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$$

Answer E: *After 10 steps, you should have* $\lambda = 0.4560$*,* $x = (0.4735, 0.9786)$*, and* $||f||_2 \approx 2.28$ *or* $||f||_2 \approx 0.905$*, depending on whether you print the norm before or after the Newton step.*

F: Presumably, your Newton iteration in question E does not seem to converge. Give at least three reasons why any Newton iteration might not seem to be converging after a fixed number of steps. Then state a convincing reason why this Newton iteration, with the given initial data, will *never* converge.

Answer F: *Possible reasons why Newton convergence might not be observed:*

- *If the initial estimate is too far away, then convergence is not guaranteed;*
- *A Newton method can occasionally get into a cycle;*
- *The jacobian might become singular;*
- *In particular, if there are multiple roots (multiple eigenvalues) then* $A - \lambda I$ *will become singular as we approach a solution, making the jacobian singular;*
- *If convergence is slow, then we may simply not have waited long enough;*
- *There might be no solution;*
- *The solution might be complex;*

In this case, however, we can verify that the eigenvalues of C_2 *are the complex pair* $\lambda = 1 \pm 2i$*. Our Newton method uses real data, and cannot compute a complex result. Therefore, no convergence is possible.*