① $e_j e_k^T = E_{jk}$, where E_{jk} is the matrix with 1 in position j, k and 0 elsewhere.

② lemma: $(I + xy^T)\left(I - \frac{xy^T}{1+y^Tx}\right)$

$$= I + xy^T - \frac{xy^T}{1+y^Tx} - \frac{(xy^T)(xy^T)}{1+y^Tx}$$

$$= I + xy^T - \frac{xy^T}{1+y^Tx} - \frac{x(y^Tx)y^T}{1+y^Tx}$$

$$= I + \frac{xy^T(1+y^Tx)}{1+y^Tx} - \frac{xy^T(1+y^Tx)}{1+y^Tx}$$

$$= I \Rightarrow (I+xy^T)^{-1} = \left(I + \frac{xy^T}{1+y^Tx}\right) \quad (\text{iff } y^Tx \neq -1)$$

verify: $(A')^{-1} = (A + uv^T)^{-1} = (I + A^{-1}uv^T)^{-1}A^{-1}$

$$= \left(I - \frac{(A^{-1}u)v^T}{1+v^T(A^{-1}u)}\right)A^{-1} \quad \text{by lemma}$$

$$= A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} \quad \checkmark \text{ verified}$$

(3) $A' = A - a_{jk}E_{jk} + a'_{jk}E_{jk}$

$$= A + (a'_{jk} - a_{jk})E_{jk}$$

$$= A + (a'_{jk} - a_{jk})e_j e_k^T$$

$$= A + uv^T$$

(4) given L, U, b:

(I) Copy matrix A $O(n^2)$ and update element a_{jk} $O(1)$

(II) compute difference $A' - A$ $O(n^2)$, which returns the matrix $(a'_{jk} - a_{jk})E_{jk} = uv^T$ above in problem #3

so, take $u = (a'_{jk} - a_{jk})e_j$ and $v = e_k$ $O(1)$

(III) compute inverse of factorized A:

$AX = LUX = I \Rightarrow X = (LU)^{-1}$

first, let $UX = D$ and solve $LD = I$ - requires only forward substitutions $O(n^2)$

then solve $UX = D$ with only backward substitutions $O(n^2)$

for $X = (LU)^{-1} = A^{-1}$

(IV) now we have A^{-1}, b, u, v, and (uv^T) stored, so it's just a matter of matrix-matrix multiplications $O(n^2)$ and matrix-vector multiplications $O(n^2)$ and ~~vector~~ vector dot products $O(n)$

compute $x = \underset{\uparrow \atop O(n^2)}{A^{-1}b} - \dfrac{A^{-1}uv^TA^{-1}b}{1 + v^TA^{-1}b}$ $\begin{array}{l} \leftarrow O(n^2+n^2+n^2) = O(n^2) \\ \leftarrow O(n^2+n+1) = O(n^2) \end{array}$

A sloppy ~~add~~ addendum showing how ~~one~~ to invert $A=LU$ and why it's $O(n^2)$ ... this was my proof-of-concept but I've included it since my answer to problem 4, step III seemed a bit ~~[illegible]~~ hand-wavey.

$LUX = I \quad , n=3$

let $UX = D = \left[\vec{d}_1 \ \vec{d}_2 \ \vec{d}_3\right]^T$

$$LD = I \rightarrow L_{11}\vec{d}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \rightarrow \vec{d}_1 = \frac{\vec{e}_1}{L_{11}}$$

$$L_{21}\vec{d}_1 + L_{22}\vec{d}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \rightarrow \vec{d}_2 = \frac{\vec{e}_2 - \frac{L_{21}}{L_{11}}\vec{e}_1}{L_{22}}$$

$$L_{31}\vec{d}_1 + L_{32}\vec{d}_2 + L_{33}\vec{d}_3 = \vec{e}_3$$

$$\rightarrow \vec{d}_3 = \frac{\vec{e}_3 - L_{32}\vec{d}_2 - L_{31}\vec{d}_1}{L_{33}}$$

now solve $UX = D$

$$U_{33}\vec{x}_3 = \vec{d}_3 \rightarrow \vec{x}_3 = \frac{\vec{d}_3}{U_{33}}$$

$$U_{22}\vec{x}_2 + U_{23}\vec{x}_3 = \vec{d}_2 \rightarrow \vec{x}_2 = \text{something}$$

$$U_{11}\vec{x}_1 + U_{12}\vec{x}_2 + U_{13}\vec{x}_3 = \vec{d}_3 \rightarrow \vec{x}_1 = \text{something}$$