1 Central Differnece

First lets take the Taylor series of these two terms

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2!} + \frac{f'''(x)h^3}{3!} + \frac{f^{(4)}(x)h^4}{4!} + \frac{f^{(5)}(x)h^5}{5!} + \dots$$

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)h^2}{2!} - \frac{f'''(x)h^3}{3!} + \frac{f^{(4)}(x)h^4}{4!} - \frac{f^{(5)}(x)h^5}{5!} + \dots$$

and subtract the results

$$D_1 \equiv f(x+h) - f(x-h) = 2f'(x)h + \frac{2f'''(x)h^3}{3!} + \frac{2f^{(5)}(x)h^5}{5!} + \dots$$

After some rearranging we find

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{f'''(x)h^2}{3!} - \frac{f^{(5)}(x)h^4}{5!} + \dots$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \mathcal{O}(h^2)$$

Following a similar procedure from above we find

$$D_2 \equiv f(x+2h) - f(x-2h) = 4f'(x)h + \frac{16f'''(x)h^3}{3!} + \frac{64f^{(5)}(x)h^5}{5!} + \dots$$

The h and the $2h$ results can be compined to form

$$8D_1 - D_2 = f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h) = 12h\left(f'(x) - \frac{f^{(5)}(x)h^4}{30} + \dots\right)$$

Notice that the $\mathcal{O}(h^2)$ term cancels

$$f' = \frac{f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)}{12h} + \mathcal{O}(h^4)$$

2 Numerical Results

See python code

3 Richardson Extrapolation

Lets rewrite our central difference formula as such

$$f'_h(x) = \mathcal{N}(h) - \frac{f'''(x)h^2}{6} - \frac{f^{(5)}(x)h^4}{120} + \dots$$

$$\mathcal{N}(h) \equiv \frac{f(x+h) - f(x-h)}{2h}$$

If we replace h with $h/2$ we find

$$f'_{h/2}(x) = \mathcal{N}(h/2) - \frac{f'''(x)h^2}{24} - \frac{f^{(5)}(x)h^4}{1920} + \dots$$

The h and the $h/2$ results can be compined to form

$$4f'_{h/2}(x) - f'_h(x) = 3f'(x) = 4\mathcal{N}(h/2) - \mathcal{N}(h) + \frac{f^{(5)}(x)h^4}{160} + \dots$$

Notice that the $\mathcal{O}(h^2)$ term cancels just as before but the truncation error converges much faster. Finally we are left with

$$f'(x) = \mathcal{N}(h/2) + \frac{\mathcal{N}(h/2) - \mathcal{N}(h)}{3} + \mathcal{O}(h^4)$$

For further reading I suggest `https://www.math.washington.edu/~greenbau/Math_498/lecture04_richardson.pdf`

4 Lanczos Formula

$$g'(x) = \lim_{h \to 0} \frac{3}{2h^3} \int_{-h}^{h} tg(x+t)dt$$

Newton-Cotes integration rule,

$$\int_{-h}^{h} f(x)dx = \frac{h}{4}\left(f(-h) + 3f\left(-\frac{h}{3}\right) + 3f\left(\frac{h}{3}\right) + f(h)\right) + \mathcal{O}(h^5)$$

Using NC and the definition of g' show

$$g'(x) = \frac{A}{4}\left(-g(x-h) - g\left(x - \frac{h}{3}\right) + g\left(x + \frac{h}{3}\right) + g(x+h)\right) + \mathcal{O}(h^p)$$

find A and p

Lets plug the NC formula into our definition of g'

$$g'(x) = \frac{3}{2h^3}\left[\frac{h}{4}\left(-hg(x-h) + 3\left(-\frac{h}{3}\right)g\left(x - \frac{h}{3}\right) + 3\left(\frac{h}{3}\right)g\left(x + \frac{h}{3}\right) + hg(x+h)\right) + \mathcal{O}(h^5)\right]$$

and pull out a h

$$g'(x) = \frac{3}{2h^3}\left[\frac{h^2}{4}\left(-g(x-h) - g\left(x - \frac{h}{3}\right) + g\left(x + \frac{h}{3}\right) + g(x+h)\right) + \mathcal{O}(h^5)\right]$$

We are then left with

$$g'(x) = \frac{3}{2h}\frac{1}{4}\left(-g(x-h) - g\left(x - \frac{h}{3}\right) + g\left(x + \frac{h}{3}\right) + g(x+h)\right) + \mathcal{O}(h^2)$$

$$A = \frac{3}{2h}$$
$$p = 2$$