Spring 2013 Question 6

Part a

We have to satisfy the equation $Ax = \lambda x \iff (A - \lambda I)x = 0 \iff det(A - \lambda I)$. We know that if a matrix M is triangular, then $det(M)$ is the product of the entries on the main diagonal of M.

$$\begin{bmatrix} 0 & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ & \ddots & a_{2,3} & \dots & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & a_{n-1,n} \\ & & & & 0 \end{bmatrix} - \begin{bmatrix} \lambda & & & & \\ & \lambda & & 0 & \\ & & \lambda & & \\ & 0 & & \lambda & \\ & & & & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ & -\lambda & a_{2,3} & \dots & \vdots \\ & & -\lambda & \ddots & \vdots \\ & & & -\lambda & a_{n-1,n} \\ & & & & -\lambda \end{bmatrix} \tag{1}$$

To satisfy $det(A - \lambda I) = (-\lambda)(-\lambda)...(-\lambda) = 0$, we must have $\lambda = 0$.

Part b

Again, to satisfy $Ax = \lambda x \iff Ax = 0$

$$x = \begin{bmatrix} a \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

only when A is strictly upper triangular.

Part c

C can be diagonalized. $C = U\Lambda U^{-1}$ An efficient way to compute C^100 is to use this diagonalization such that

$$\begin{aligned} C^2 &= U\Lambda U^{-1}U\Lambda U^{-1} \\ &= U\Lambda^2 U^{-1} \\ C^{100} &= U\Lambda^{100}U^{-1} \end{aligned} \tag{2}$$

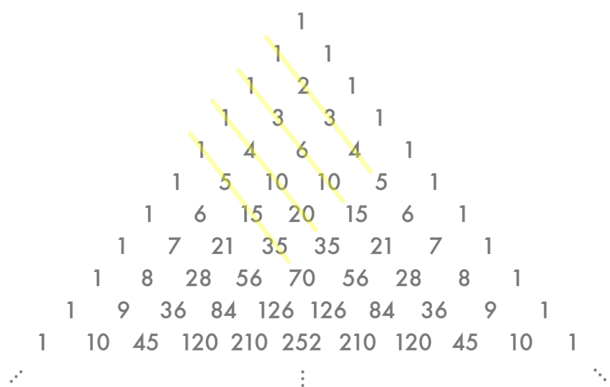
Since Λ is a diagonal matrix, each of the elements along the diagonal is simply raised to the power of the matrix, resulting in fewer computations.

Part d

Defective: An $n \times n$ matrix A is a defective matrix iff it does not have n linearly independent eigenvectors.

$$D^2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} D^3 = \begin{bmatrix} 1 & 3 & 6 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} D^4 = \begin{bmatrix} 1 & 4 & 10 & 20 \\ 0 & 1 & 4 & 10 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} D^4 = \begin{bmatrix} 1 & 5 & 15 & 35 \\ 0 & 1 & 5 & 15 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

The top row of matrix D^n corresponds to the nth diagonal in Pascal's triangle. If the top row of Pascal's triangle is 1, the element in the first row and last column of D^{100} can be found in Pascal's triangle in the 103rd row and 4th column.



Part e

Let's take a 4 by 4 matrix to see if we can find a pattern using back substitution.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \tag{4}$$

First we let $x_4 = b_4$. Then $x_3 + x_4 = b_3$ is equivalent to $x_3 + b_4 = b_3$ and so $x_3 = b_3 - b_4$.
$x_2 + x_3 + x_4 = b_2 \iff x_2 + (b_3 - b_4) + b_4) = b_2 \iff x_2 + b_3 = b_2 \iff x_2 = b_2 - b_3$
Similarly, $x_1 + b_2 = b_1$.

An algorithm would be
$x_n = b_n$
For $i = n - 1$ to 1:
$x_i = b_i - b_{i+1}$