Seth Boren

Q7 Partial Differential Equations

(Dr. Quaife ; Summer 2017)

Part a

Start with 2nd degree Taylor Expansions:

$$f(x+h) = f(x) + f'(x) * h + (\tfrac{1}{2!})(f''(\mathrm{x}) * h^2) + (\tfrac{1}{3!})(f'''(\xi_1) * h^3) \quad \text{Equation 1}$$

$$f(x-h) = f(x) - f'(x) * h + (\tfrac{1}{2!})(f''(\mathrm{x}) * h^2) - (\tfrac{1}{3!})(f'''(\xi_2) * h^3) \quad \text{Equation 2}$$

Subtract equation 2 from equation 1:

$$f(x+h) - f(x-h) = 2 f'(x) * h + [(\tfrac{1}{3!}) f'''(\xi_1) * h^3 + (\tfrac{1}{3!}) f'''(\xi_2) * h^3]$$

$$f(x+h) - f(x-h) = 2 f'(x) * h + (\tfrac{1}{3!}) f'''(\xi) * h^3$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{f'''(\xi) * h^3}{3! * 2h}$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{f'''(\xi) * h^2}{12} \quad , \quad \text{where} \quad \frac{f'''(\xi) * h^2}{12} = \text{truncation error.}$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - O(h^2) \quad , \quad \text{where } O(h^2) = \text{truncation error with 2nd order accuracy.}$$

Now put in terms of function U(x,y) which is a function with variables x and y:

$$U_x = \frac{\boldsymbol{\partial\, u(x_i\, ,\, y_i\,)}}{\boldsymbol{\partial x}} = \frac{U(x+h\, ,y) - U(x-h\, ,y)}{2h} - O_x(h^2)$$

Now take partial derivative with respect to x and y:

$$U_{xy} = \left(\frac{1}{2h}\right) [(\frac{U(x+h\, ,\boldsymbol{y+h}) - U(x-h\, ,\boldsymbol{y+h})}{2h}) - (\frac{U(x+h\, ,\boldsymbol{y-h}) - U(x-h\, ,\boldsymbol{y-h})}{2h})] - \left(\frac{1}{3!}\right) f''' \left(\xi_{\,xy}\right) h^3$$

$$U_{xy} = (\frac{U(x+h\, ,\boldsymbol{y+h}) - U(x-h\, ,\boldsymbol{y+h})}{2h * 2h}) - (\frac{U(x+h\, ,\boldsymbol{y-h}) - U(x-h\, ,\boldsymbol{y-h})}{2h * 2h}) - (\tfrac{1}{3!}) f'''(\xi_{\,xy}) * h^3 \left(\frac{1}{2h}\right)$$

$$U_{xy} = (\frac{U(x+h\, ,\boldsymbol{y+h}) - U(x-h\, ,\boldsymbol{y+h})}{4h^2}) - (\frac{U(x+h\, ,\boldsymbol{y-h}) - U(x-h\, ,\boldsymbol{y-h})}{4h^2})] - (\tfrac{1}{12}) f'''(\xi_{\,xy}) * h^2$$

$$\boldsymbol{U_{xy} = \frac{U(x+h\, ,y+h) + U(x-h\, ,y-h) - U(x+h\, ,y-h) - U(x-h\, ,y+h)}{4h^2} - O(h^2)}$$

where $O(h^2) =$ truncation error with 2nd order accuracy.

$$\boldsymbol{U_{xy} \approx \frac{U(x+h\, ,y+h) + U(x-h\, ,y-h) - U(x+h\, ,y-h) - U(x-h\, ,y+h)}{4h^2}}$$

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Part b

Standard 2nd Order Centered Difference Approximation for $\boldsymbol{U_{xx}}$:

$$U_{xx} = \frac{\partial^2 u(x_i , y_j)}{\partial x^2} = \frac{U(x+h ,y) - 2U(x ,y) + U(x-h ,y)}{h^2} - O(h^2)$$

$$\boldsymbol{U_{xx} \approx \frac{u(x_{i+1} , y_j) - 2u(x_i , y_j) + u(x_{i-1} , y_j)}{h^2}}$$

From part a :

$$U_{xy} \approx \frac{U(x+h ,y+h) + U(x-h ,y-h) - U(x+h ,y-h) - U(x-h ,y+h)}{4h^2}$$

$$\boldsymbol{U_{xy} \approx \frac{u(x_{i+1} ,y_{j+1}) + u(x_{i-1} ,y_{j-1}) - u(x_{i+1} ,y_{j-1}) - u(x_{i-1} ,y_{j+1})}{4h^2}}$$

Two-Dimensional PDE:

$$\boldsymbol{U_{xx} + U_{xy} = f(x, y)}$$

Plugging in the formulas for U_{xx} and U_{xy} :

$$\boldsymbol{\frac{u(x_{i+1} ,y_j) - 2u(x_i ,y_j) + u(x_{i-1} ,y_j)}{h^2} + \frac{u(x_{i+1} ,y_{j+1}) + u(x_{i-1} ,y_{j-1}) - u(x_{i+1} ,y_{j-1}) - u(x_{i-1} ,y_{j+1})}{4h^2} = f(x_i , y_j)}$$

Multiplying both sides by $4h^2$ **:**

$$4u(x_{i+1} , y_j) - 8\, u(x_i , y_j) + 4u(x_{i-1} , y_j) + u(x_{i+1} , y_{j+1}) + u(x_{i-1} , y_{j-1}) - u(x_{i+1} , y_{j-1}) - u(x_{i-1} , y_{j+1}) = 4h^2 f(x_i, y_j)$$

Re-arranging terms by rows 1st then by columns:

$$\boldsymbol{u(x_{i-1} , y_{j-1}) - u(x_{i+1} , y_{j-1}) + 4u(x_{i-1} , y_j) - 8u(x_i , y_j) + 4u(x_{i+1} , y_j) - u(x_{i-1} , y_{j+1}) + u(x_{i+1} , y_{j+1}) = 4h^2 f(x_i, y_j)}$$

The interval of x and y which are (0,1) are divided in to equidistance nodes spaced at a distance of $\boldsymbol{h}$.

To have 16 interior nodes, use $i = 1, 2, \ldots., 4$.

The interior nodes are labeled $x_{i,j}$.

For Dirichlet Boundary Conditions, the values of $u(x_i , y_j)$ are constant for any $i = 0 \text{ } or \text{ } 5 \text{ } or \text{ } j = 0 \text{ } or \text{ } 5$.

This new equation is used to generate a system of equations for $\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{4}$ and $\boldsymbol{j} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{4}$.

$$u(x_{i-1} , y_{j-1}) - u(x_{i+1} , y_{j-1}) + 4u(x_{i-1} , y_j) - 8u(x_i , y_j) + 4u(x_{i+1} , y_j) - u(x_{i-1} , y_{j+1}) + u(x_{i+1} , y_{j+1}) = 4h^2 f(x_i, y_j)$$

System of equations with $\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ and $\boldsymbol{j} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$:

$i = 1, j = 1 \quad u(x_0, y_0) - u(x_2, y_0) + 4u(x_0, y_1) - 8u(x_1 , y_1) + 4u(x_2 , y_1) - u(x_0 , y_2) + u(x_2 , y_2) = 4h^2 f(x_1, y_1)$

$i = 2, j = 1 \quad u(x_1, y_0) - u(x_3, y_0) + 4u(x_1, y_1) - 8u(x_2 , y_1) + 4u(x_3 , y_1) - u(x_1 , y_2) + u(x_3 , y_2) = 4h^2 f(x_2, y_1)$

$i = 3, j = 1 \quad u(x_2, y_0) - u(x_4, y_0) + 4u(x_2, y_1) - 8u(x_3 , y_1) + 4u(x_4 , y_1) - u(x_2 , y_2) + u(x_4 , y_2) = 4h^2 f(x_3, y_1)$

$i = 4, j = 1 \quad u(x_3, y_0) - u(x_5, y_0) + 4u(x_3, y_1) - 8u(x_4 , y_1) + 4u(x_5 , y_1) - u(x_3 , y_2) + u(x_5 , y_2) = 4h^2 f(x_4, y_1)$

$i = 1, j = 2 \quad u(x_0, y_1) - u(x_2, y_1) + 4u(x_0, y_2) - 8u(x_1 , y_2) + 4u(x_2 , y_2) - u(x_0 , y_3) + u(x_2 , y_3) = 4h^2 f(x_1, y_2)$

$i = 2, j = 2 \quad u(x_1, y_1) - u(x_3, y_1) + 4u(x_1, y_2) - 8u(x_2 , y_2) + 4u(x_3 , y_2) - u(x_1 , y_3) + u(x_3 , y_3) = 4h^2 f(x_2, y_2)$

$i = 3, j = 2 \quad u(x_2, y_1) - u(x_4, y_1) + 4u(x_2, y_2) - 8u(x_3 , y_2) + 4u(x_4 , y_2) - u(x_2 , y_3) + u(x_4 , y_3) = 4h^2 f(x_3, y_2)$

$i = 4, j = 2 \quad u(x_3, y_1) - u(x_5, y_1) + 4u(x_3, y_2) - 8u(x_4 , y_2) + 4u(x_5 , y_2) - u(x_3 , y_3) + u(x_5 , y_3) = 4h^2 f(x_4, y_2)$

$i = 1, j = 3 \quad u(x_0, y_2) - u(x_2, y_2) + 4u(x_0, y_3) - 8u(x_1 , y_3) + 4u(x_2 , y_3) - u(x_0 , y_4) + u(x_2 , y_4) = 4h^2 f(x_1, y_3)$

$i = 2, j = 3 \quad u(x_1, y_2) - u(x_3, y_2) + 4u(x_1, y_3) - 8u(x_2 , y_3) + 4u(x_3 , y_3) - u(x_1 , y_4) + u(x_3 , y_4) = 4h^2 f(x_2, y_3)$

$i = 3, j = 3 \quad u(x_2, y_2) - u(x_4, y_2) + 4u(x_2, y_3) - 8u(x_3 , y_3) + 4u(x_4 , y_3) - u(x_2 , y_4) + u(x_4 , y_4) = 4h^2 f(x_3, y_3)$

$i = 4, j = 3 \quad u(x_3, y_2) - u(x_5, y_2) + 4u(x_3, y_3) - 8u(x_4 , y_3) + 4u(x_5 , y_3) - u(x_3 , y_4) + u(x_5 , y_4) = 4h^2 f(x_4, y_3)$

$i = 1, j = 4 \quad u(x_0, y_3) - u(x_2, y_3) + 4u(x_0, y_4) - 8u(x_1 , y_4) + 4u(x_2 , y_4) - u(x_0 , y_5) + u(x_2 , y_5) = 4h^2 f(x_1, y_3)$

$i = 2, j = 4 \quad u(x_1, y_3) - u(x_3, y_3) + 4u(x_1, y_4) - 8\, u(x_2 , y_4) + 4u(x_3 , y_4) - u(x_1 , y_5) + u(x_3 , y_5) = 4h^2 f(x_2, y_3)$

$i = 3, j = 4 \quad u(x_2, y_3) - u(x_4, y_3) + 4u(x_2, y_4) - 8u(x_3 , y_4) + 4u(x_4 , y_4) - u(x_2 , y_5) + u(x_4 , y_5) = 4h^2 f(x_3, y_3)$

$i = 4, j = 4 \quad u(x_3, y_3) - u(x_5, y_3) + 4u(x_3, y_4) - 8u(x_4 , y_4) + 4u(x_5 , y_4) - u(x_3 , y_5) + u(x_5 , y_5) = 4h^2 f(x_4, y_3)$

For boundary values $u(x_{0,j})$, $u(x_{5,j})$ $u(x_{i,0})$, and $u(x_{i,5})$ which are constants, **this set of equations becomes**:

$$-8u(x_1,y_1)+4u(x_2,y_1)+u(x_2,y_2)=4h^2f(x_1,y_1)-u(x_0,y_0)+u(x_2,y_0)-4u(x_0,y_1)+u(x_0,y_2)$$

$$4u(x_1,y_1)-8u(x_2,y_1)+4u(x_3,y_1)-u(x_1,y_2)+u(x_3,y_2)=4h^2f(x_2,y_1)-u(x_1,y_0)+u(x_3,y_0)$$

$$4u(x_2,y_1)-8u(x_3,y_1)+4u(x_4,y_1)-u(x_2,y_2)+u(x_4,y_2)=4h^2f(x_3,y_1)-u(x_2,y_0)+u(x_4,y_0)$$

$$4u(x_3,y_1)-8u(x_4,y_1)-u(x_3,y_2)=4h^2f(x_4,y_1)-u(x_3,y_0)+u(x_5,y_0)-4u(x_5,y_1)-u(x_5,y_2)$$

$$-u(x_2,y_1)-8u(x_1,y_2)+4u(x_2,y_2)+u(x_2,y_3)=4h^2f(x_1,y_2)-u(x_0,y_1)-4u(x_0,y_2)+u(x_0,y_3)$$

$$u(x_1,y_1)-u(x_3,y_1)+4u(x_1,y_2)-8u(x_2,y_2)+4u(x_3,y_2)-u(x_1,y_3)+u(x_3,y_3)=4h^2f(x_2,y_2)$$

$$u(x_2,y_1)-u(x_4,y_1)+4u(x_2,y_2)-8u(x_3,y_2)+4u(x_4,y_2)-u(x_2,y_3)+u(x_4,y_3)=4h^2f(x_3,y_2)$$

$$u(x_3,y_1)+4u(x_3,y_2)-8u(x_4,y_2)-u(x_3,y_3)=4h^2f(x_4,y_2)+u(x_5,y_1)-4u(x_5,y_2)-u(x_5,y_3)$$

$$-u(x_2,y_2)-8u(x_1,y_3)+4u(x_2,y_3)+u(x_2,y_4)=4h^2f(x_1,y_3)-u(x_0,y_2)-4u(x_0,y_3)+u(x_0,y_4)$$

$$u(x_1,y_2)-u(x_3,y_2)+4u(x_1,y_3)-8u(x_2,y_3)+4u(x_3,y_3)-u(x_1,y_4)+u(x_3,y_4)=4h^2f(x_2,y_3)$$

$$u(x_2,y_2)-u(x_4,y_2)+4u(x_2,y_3)-8u(x_3,y_3)+4u(x_4,y_3)-u(x_2,y_4)+u(x_4,y_4)=4h^2f(x_3,y_3)$$

$$u(x_3,y_2)+4u(x_3,y_3)-8u(x_4,y_3)-u(x_3,y_4)=4h^2f(x_4,y_3)+u(x_5,y_2)-4u(x_5,y_3)-u(x_5,y_4)$$

$$-u(x_2,y_3)-8u(x_1,y_4)+4u(x_2,y_4)=4h^2f(x_1,y_3)-u(x_0,y_3)-4u(x_0,y_4)+u(x_0,y_5)-u(x_2,y_5)$$

$$u(x_1,y_3)-u(x_3,y_3)+4u(x_1,y_4)-8u(x_2,y_4)+4u(x_3,y_4)=4h^2f(x_2,y_3)+u(x_1,y_5)-u(x_3,y_5)$$

$$u(x_2,y_3)-u(x_4,y_3)+4u(x_2,y_4)-8u(x_3,y_4)+4u(x_4,y_4)=4h^2f(x_3,y_3)+u(x_2,y_5)-u(x_4,y_5)$$

$$u(x_3,y_3)+4u(x_3,y_4)-8u(x_4,y_4)=4h^2f(x_4,y_3)+u(x_5,y_3)-4u(x_5,y_4)+u(x_3,y_5)-u(x_5,y_5)$$

$$\begin{bmatrix}
-8 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
4 & -8 & 4 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 4 & -8 & 4 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 4 & -8 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & -8 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & -1 & 0 & 4 & -8 & 4 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & -1 & 0 & 4 & -8 & 4 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 4 & -8 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -8 & 4 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 4 & -8 & 4 & 0 & -1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 4 & -8 & 4 & 0 & -1 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & -8 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -8 & 4 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 4 & -8 & 4 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 4 & -8 & 4\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & -8
\end{bmatrix}
\begin{bmatrix}
u(x_1,y_1)\\ u(x_2,y_1)\\ u(x_3,y_1)\\ u(x_4,y_1)\\ u(x_1,y_2)\\ u(x_2,y_2)\\ u(x_3,y_2)\\ u(x_4,y_2)\\ u(x_1,y_3)\\ u(x_2,y_3)\\ u(x_3,y_3)\\ u(x_4,y_3)\\ u(x_1,y_4)\\ u(x_2,y_4)\\ u(x_3,y_4)\\ u(x_4,y_4)
\end{bmatrix}
=
\begin{bmatrix}
F_1\\ F_2\\ F_3\\ F_4\\ F_5\\ F_6\\ F_7\\ F_8\\ F_9\\ F_{10}\\ F_{11}\\ F_{12}\\ F_{13}\\ F_{14}\\ F_{15}\\ F_{16}
\end{bmatrix}$$

Note: Multiplying both sides by scalar -1 will make the coefficient matrix positive definite.

Seth Boren

Part c

$$
\begin{array}{c}
\\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16
\end{array}\\
\begin{bmatrix}
-8 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
4 & -8 & 4 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 4 & -8 & 4 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 4 & -8 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & -8 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & -1 & 0 & 4 & -8 & 4 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & -1 & 0 & 4 & -8 & 4 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 4 & -8 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -8 & 4 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 4 & -8 & 4 & 0 & -1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 4 & -8 & 4 & 0 & -1 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & -8 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -8 & 4 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 4 & -8 & 4 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 4 & -8 & 4\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & -8
\end{bmatrix}
\end{array}
$$

Undirected Adjacency graph for linear system from part b: