Student Id 5894 3710 0798 9034

Summary 2017 Prelim Q5

Part (A)

$$-u''(x) = f(x) \qquad , \text{ for } x \in (0,1) \quad , u(0) = 1 \; , \; u(1) = 1 \; .$$

2nd Order Centered Difference Formula:

$$u''(x_i) = \frac{u(x_{i-1}) - 2\; u(x_i) + \; u(x_{i+1})}{h^2}$$

$$u(x_{i-1}) - 2\; u(x_i) + \; u(x_{i+1}) = -h^2 * f(x_i)$$

The interval of x which is [0,1] is divided in to equidistance nodes spaced at a distance of h.

The nodes are labeled $x_0, \; x_1, \ldots, x_{N+1}$. **This yields N interior points.**

This equation is used to generate a system of equations for $i = 1, 2, \ldots, N$.

The step size $h = (N + 1)^{-1}$.

$$u(x_{i-1}) - 2\,u(x_i) + \; u(x_{i+1}) = -h^2 * f(x)$$

System of equations with $i = 1, 2, \ldots, 4$:

$$
\begin{array}{llr}
i = 1 & u(x_0) - 2\,u(x_1) + u(x_2) & = -h^2 * f(x_1) \\
i = 2 & u(x_1) - 2\,u(x_2) + u(x_3) & = -h^2 * f(x_2) \\
i = 3 & u(x_2) - 2\,u(x_3) + u(x_4) & = -h^2 * f(x_3) \\
. & . \quad . \quad . & . \quad . \\
. & . \quad . \quad . & . \quad . \\
i = N-1 & u(x_{N-2}) - 2\,u(x_{N-1}) + u(x_N) & = -h^2 * f(x_{N-1}) \\
i = N & u(x_{N-1}) - 2\,u(x_N) + u(x_{N+1}) & = -h^2 * f(x_N)
\end{array}
$$

For $x_0 = 0$ and $x_{N+1} = 1$ the Boundary Conditions are satisfied by $u(x_0) = 1$ and $u(x_{N+1}) = 1$.

$$
\begin{array}{lll}
i = 1 & 1 - 2\,u(x_1) + u(x_2) & = -h^2 * f(x_1) \\
i = 2 & u(x_1) - 2\,u(x_2) + u(x_3) & = -h^2 * f(x_2) \\
i = 3 & u(x_2) - 2\,u(x_3) + u(x_4) & = -h^2 * f(x_3) \\
. & . \quad . \quad . & . \quad . \\
. & . \quad . \quad . & . \quad . \\
i = N-1 & u(x_{N-2}) - 2\,u(x_{N-1}) + u(x_N) & = -h^2 * f(x_{N-1}) \\
i = N & u(x_{N-1}) - 2\,u(x_N) + 1 & = -h^2 * f(x_N)
\end{array}
$$

Rearranging constants to the right-hand side of the equation:

$$
\begin{array}{lll}
i = 1 & -2\,u(x_1) + u(x_2) & = -h^2 * f(x_1) - 1 \\
i = 2 & u(x_1) - 2\,u(x_2) + u(x_3) & = -h^2 * f(x_2) \\
i = 3 & u(x_2) - 2\,u(x_3) + u(x_4) & = -h^2 * f(x_3) \\
. & . \quad . \quad . & . \quad . \\
. & . \quad . \quad . & . \quad . \\
i = N-1 & u(x_{N-2}) - 2\,u(x_{N-1}) + u(x_N) & = -h^2 * f(x_{N-1}) \\
i = N & u(x_{N-1}) - 2\,u(x_N) & = -h^2 * f(x_N) - 1
\end{array}
$$

Putting this into matrix form:

$$
\begin{bmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & . & . & . \\
0 & 1 & -2 & . & . & . & . \\
0 & 0 & 1 & . & . & . & . \\
0 & 0 & 0 & . & . & . & 0 \\
0 & 0 & 0 & . & -2 & 1 & 0 \\
0 & 0 & 0 & . & 1 & -2 & 1 \\
0 & 0 & 0 & . & 0 & 1 & -2
\end{bmatrix}
\begin{bmatrix}
u(x_1) \\ u(x_2) \\ u(x_3) \\ u(x_4) \\ . \\ . \\ u(x_{N-1}) \\ u(x_N)
\end{bmatrix}
=
\begin{bmatrix}
-h^2 * f(x_1) - 1 \\ -h^2 * f(x_2) \\ -h^2 * f(x_3) \\ -h^2 * f(x_4) \\ . \\ . \\ -h^2 * f(x_{N-1}) \\ -h^2 * f(x_N) - 1
\end{bmatrix}
$$

In order to make the coefficient matrix positive definite, both sides of the matrix equation are multiplied by the scalar -1 as shown below. Also bringing the h^2 term outside of the column vector.

$$
\begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & . & . & . \\
0 & -1 & 2 & . & . & . & . \\
0 & 0 & -1 & . & . & . & . \\
0 & 0 & 0 & . & . & . & 0 \\
0 & 0 & 0 & . & 2 & -1 & 0 \\
0 & 0 & 0 & . & -1 & 2 & -1 \\
0 & 0 & 0 & . & 0 & -1 & 2
\end{bmatrix}
\begin{bmatrix}
u(x_1) \\ u(x_2) \\ u(x_3) \\ u(x_4) \\ . \\ . \\ u(x_{N-1}) \\ u(x_N)
\end{bmatrix}
= h^2
\begin{bmatrix}
f(x_1) + 1/h^2 \\ f(x_2) \\ f(x_3) \\ f(x_4) \\ . \\ . \\ f(x_{N-1}) \\ f(x_N) + 1/h^2
\end{bmatrix}
$$

Resulting in the matrix equation $\boldsymbol{Au} = h^2\mathbf{f}$, where h^2 is a scalar constant**.**

For this boundary value problem, two “Dirichlet” boundary conditions were given.
Therefore, a unique solution exists.

Part (B)

For N = 8:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u(x_1) \\ u(x_2) \\ u(x_3) \\ u(x_4) \\ u(x_4) \\ u(x_4) \\ u(x_7) \\ u(x_8) \end{bmatrix} = \begin{bmatrix} h^2 * f(x_1) + 1 \\ h^2 * f(x_2) \\ h^2 * f(x_3) \\ h^2 * f(x_4) \\ h^2 * f(x_5) \\ h^2 * f(x_6) \\ h^2 * f(x_7) \\ h^2 * f(x_8) + 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Using the attached MATLAB program Q5_Part_B_C.m to find the inverse of Matrix A.

$$B = A^{-1} = \begin{bmatrix} 8/9 & 7/9 & 2/3 & 5/9 & 4/9 & 1/3 & 2/9 & 1/9 \\ 7/9 & 14/9 & 4/3 & 10/9 & 8/9 & 2/3 & 4/9 & 2/9 \\ 2/3 & 4/3 & 2 & 5/3 & 4/3 & 1 & 2/3 & 1/3 \\ 5/9 & 10/9 & 5/3 & 20/9 & 16/9 & 4/3 & 8/9 & 4/9 \\ 4/9 & 8/9 & 4/3 & 16/9 & 20/9 & 5/3 & 10/9 & 5/9 \\ 1/3 & 2/3 & 1 & 4/3 & 5/3 & 2 & 4/3 & 2/3 \\ 2/9 & 4/9 & 2/3 & 8/9 & 10/9 & 4/3 & 14/9 & 7/9 \\ 1/9 & 2/9 & 1/3 & 4/9 & 5/9 & 2/3 & 7/9 & 8/9 \end{bmatrix}$$

Next, we use the attached MATLAB program Q5_Part_B_C.m to multiply all terms of matrix B by 9.

$$B = A^{-1} = (1/9) \begin{bmatrix} 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 7 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ 6 & 12 & 18 & 15 & 12 & 9 & 6 & 3 \\ 5 & 10 & 15 & 20 & 16 & 12 & 8 & 4 \\ 4 & 8 & 12 & 16 & 20 & 15 & 10 & 5 \\ 3 & 6 & 9 & 12 & 15 & 18 & 12 & 6 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix}$$

Part (C)

$$B = A^{-1} = (1/9)\begin{bmatrix} 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 7 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ 6 & 12 & 18 & 15 & 12 & 9 & 6 & 3 \\ 5 & 10 & 15 & 20 & 16 & 12 & 8 & 4 \\ 4 & 8 & 12 & 16 & 20 & 15 & 10 & 5 \\ 3 & 6 & 9 & 12 & 15 & 18 & 12 & 6 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix}$$

$$\boldsymbol{B}_{11}^{(2)} = \begin{bmatrix} 8 & 7 \\ 7 & 14 \end{bmatrix} \qquad \boldsymbol{B}_{12}^{(2)} = \begin{bmatrix} 6 & 5 \\ 12 & 10 \end{bmatrix}$$

$$\boldsymbol{B}_{21}^{(2)} = \begin{bmatrix} 6 & 12 \\ 5 & 10 \end{bmatrix} \qquad \boldsymbol{B}_{22}^{(2)} = \begin{bmatrix} 18 & 15 \\ 15 & 20 \end{bmatrix}$$

$$\boldsymbol{B}_{21}^{(1)} = \begin{bmatrix} 4 & 8 & 12 & 16 \\ 3 & 6 & 9 & 12 \\ 2 & 4 & 6 & 8 \\ 1 & 2 & 3 & 4 \end{bmatrix} \qquad \boldsymbol{B}_{12}^{(1)} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 8 & 6 & 4 & 2 \\ 12 & 9 & 6 & 3 \\ 16 & 12 & 8 & 4 \end{bmatrix}$$

$$\boldsymbol{B}_{33}^{(2)} = \begin{bmatrix} 20 & 15 \\ 15 & 18 \end{bmatrix} \qquad \boldsymbol{B}_{34}^{(2)} = \begin{bmatrix} 10 & 5 \\ 12 & 6 \end{bmatrix}$$

$$\boldsymbol{B}_{43}^{(2)} = \begin{bmatrix} 10 & 12 \\ 5 & 6 \end{bmatrix} \qquad \boldsymbol{B}_{44}^{(2)} = \begin{bmatrix} 14 & 7 \\ 7 & 8 \end{bmatrix}$$

The adjacent off-diagonal blocks with $\boldsymbol{i} \neq \boldsymbol{j}$ are transposes of one another.

More plainly written:

$\boldsymbol{B}_{12}^{(2)}$ **is the transpose of** $\boldsymbol{B}_{21}^{(2)}$ **and vice versa.**

$\boldsymbol{B}_{12}^{(1)}$ **is the transpose of** $\boldsymbol{B}_{21}^{(1)}$ **and vice versa.**

$\boldsymbol{B}_{34}^{(2)}$ **is the transpose of** $\boldsymbol{B}_{43}^{(2)}$ **and vice versa.**

Part (D)

$$\boldsymbol{B}_{12}^{(1)} = \begin{bmatrix} \mathbf{4} & \mathbf{3} & \mathbf{2} & \mathbf{1} \\ \mathbf{8} & \mathbf{6} & \mathbf{4} & \mathbf{2} \\ \mathbf{12} & \mathbf{9} & \mathbf{6} & \mathbf{3} \\ \mathbf{16} & \mathbf{12} & \mathbf{8} & \mathbf{4} \end{bmatrix}$$ **has a rank of 1.**

Row 3 is a multiple of row 1 and row 4 is a multiple of row 2, and row 2 is a multiple of row 1.

$$\boldsymbol{B}_{12}^{(2)} = \begin{bmatrix} \mathbf{6} & \mathbf{5} \\ \mathbf{12} & \mathbf{10} \end{bmatrix}$$ **has a rank of 1.**

Row 2 is a multiple of row 1.

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SVD for the matrix $B_{12}^{(1)} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 8 & 6 & 4 & 2 \\ 12 & 9 & 6 & 3 \\ 16 & 12 & 8 & 4 \end{bmatrix}$ $\quad$ $B_{12}^{(1)}$ **= U Σ V**

Using the attached MATLAB program Q5_Part_B_C.m to perform the SVD process:

$$U_{12}^{(1)} = \begin{pmatrix} -461/2525 & 1348/1431 & 510/1811 & 0 \\ -505/1383 & -477/3499 & 1158/5281 & -2584/2889 \\ -505/922 & 189/1346 & -499/605 & 0 \\ -1427/1954 & -319/1170 & 1067/2433 & 1292/2889 \end{pmatrix}$$

$$\Sigma_{12}^{(1)} = \begin{pmatrix} 30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V_{12}^{(1)} = \begin{pmatrix} -1427/1954 & 809/1263 & -241/1015 & 0 \\ -505/922 & -57/167 & 702/919 & 0 \\ -505/1383 & -910/1479 & -650/1211 & 1292/2889 \\ -461/2525 & -455/1479 & -325/1211 & -2584/2889 \end{pmatrix}$$

The matrix $U_{12}^{(1)}\Sigma_{12}^{(1)}\,V_{12}^{(1)}$ **where is a rank one matrix, which is a key point to the whole process.**

SVD for the matrix $B_{12}^{(2)} = \begin{bmatrix} 6 & 5 \\ 12 & 10 \end{bmatrix}$ $\quad$ $B_{12}^{(2)}$ **= U Σ V**

Using the attached MATLAB program Q5_Part_B_C.m to perform the SVD process:

$$U_{12}^{(2)} = \begin{pmatrix} \mathbf{-1292/2889} & \mathbf{-2584/2889} \\ \mathbf{-2584/2889} & \mathbf{1292/2889} \end{pmatrix}$$

$$\Sigma_{12}^{(2)} = \begin{pmatrix} \mathbf{16853/965} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$V_{12}^{(2)} = \begin{pmatrix} \mathbf{-527/686} & \mathbf{-1249/1951} \\ \mathbf{-1249/1951} & \mathbf{527/686} \end{pmatrix}$$

The matrix $\Sigma_{12}^{(2)}$ **is a rank one matrix, which is a key point to the whole process.**

$$U_{12}^{(2)}\Sigma_{12}^{(2)}V_{12}^{(2)} = \begin{pmatrix} \mathbf{6} & \mathbf{5} \\ \mathbf{12} & \mathbf{10} \end{pmatrix}$$

Where $\Sigma_{12}^{(2)} \epsilon R^{k*k} = R^1$

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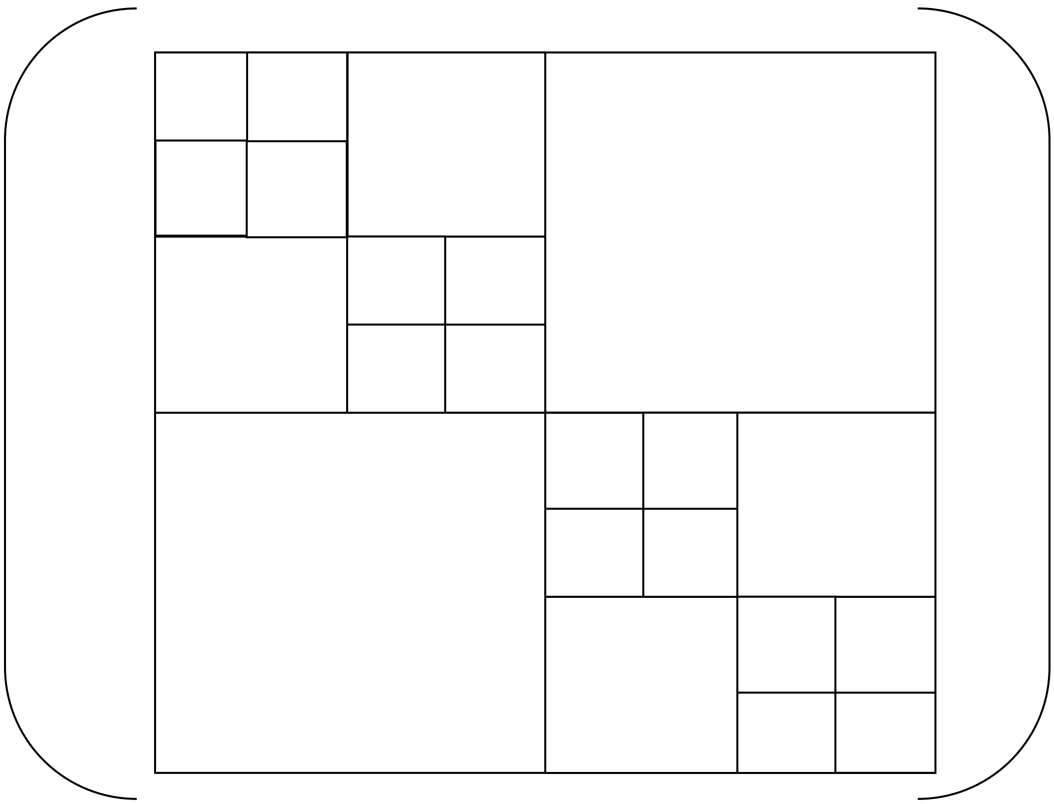
Part (E)

If **N** is a power of 2, the same symmetry advantages apply, they are just a little harder keep up with in the small blocks. The large off-diagonal blocks at the bottom left and the top right are still transposes of one another regardless of the size of **N**, if **N** is a power of 2.

As you can see in the drawing below additional new large off-diagonal blocks are created in each new quadrant with these new off-diagonal pairs being transposes of one another which is a huge computational advantage. This effect keeps happening with each increase in the size of the power of **N**. The off-diagonal blocks are all rank 1.



Part (F)

Applying the "structure" of the inverse of Matrix **A to a column vector** (multiplying the inverse of Matrix **A** by a Column Vector) can increase the speed and efficiency of the process by taking advantage of the structure as follows: For the large off-diagonal blocks, the column vector multiplication of the first row of the block can be multiplied by 2, 3, …. To obtain what would have been the result of the column vector multiplication of the second, third, …. rows of the block. This saves many calculations per block.

Dense matrix vector multiplication requires **O(N^2)** operations. If the matrix has rank **k**, the matrix vector multiplication can be done in **O(k*N)** operations. Since **k** is much smaller than **N** (**k** = 1 in this problem), there is the potential for more efficient computation.

The other advantage, which is less direct, is that the small off-diagonal blocks are transposes of the adjacent small off-diagonal blocks. The large off-diagonal blocks are also transposes of the adjacent large off-diagonal blocks.