ISC5907 - Prelim Preparation Class
Spring 2016 Prelim Exam Question 1

August 5, 2016

1 Question 1: Linear Algebra LU Decomposition

1.1 Question 1a: Equations for General LU Decomposition

$$\mathbf{A} = \mathbf{LU} = \begin{pmatrix} I & 0 & 0 \\ L_2 & I & 0 \\ 0 & L_3 & I \end{pmatrix} \begin{pmatrix} U_1 & D_1 & 0 \\ 0 & U_2 & D_2 \\ 0 & 0 & U_3 \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} U_1 & D_1 & 0 \\ L_2 U_1 & L_2 D_1 + U_2 & D_2 \\ 0 & L_3 D_1 + U_2 & L_3 D_2 + U_3 \end{pmatrix} \tag{2}$$

$$= \begin{pmatrix} U_1 & B_1 & 0 \\ L_2 U_1 & L_2 B_1 + U_2 & B_2 \\ 0 & L_3 B_1 + U_2 & L_3 B_2 + U_3 \end{pmatrix} \tag{3}$$

$$= \begin{pmatrix} A_1 & B_1 & 0 \\ C_2 & A_2 & B_2 \\ 0 & C_n & A_n \end{pmatrix} \tag{4}$$

$$U_j = A_j - L_j B_j \tag{5}$$

$$D_j = B_j \tag{6}$$

$$L_i = (C_i - U_{i-1}) B_{i-2}^{-1} \tag{7}$$

In this calculation, each U_i and L_i need not necessarily be triangular. There is nothing in the definitions and equations above requiring them to be.

1.2 Question 1b: Pseudo-Code for General LU Decomposition

Algorithm 1 Block LU Decomposition

procedure COMPUTE
    $U_1 \leftarrow A_1$
    $L_2 \leftarrow C_2 U_1^{-1}$
    for `(i in 2,...,n)` **do**:
        $U_{i-1}^T = L_\star U_\star \leftarrow U_{i-1}^T L_i^T = C_i^T$
        for `(k = 1,...,p)` **do**:
            $z_k = U_\star l_k$ `#` l_k `is kth row-transpose of` L_i^T
            $L_\star z_k = C_k$ `# Forward solve for` z_k
            $U_\star l_k = z_k$ `# Back Solve`
        $D_{i-1} \leftarrow B_{i-1}$
        $U_{i-1} \leftarrow A_{i-1} - L_i B_{i-1}$

1.3 Question 1c: Algorithm for Solving $A\vec{x} = \vec{b}$

Algorithm 2 Block LU Solve

procedure COMPUTE FORWARD SOLVE
    $A \leftarrow LU$
    $y \leftarrow L^{-1} b$
procedure COMPUTE BACKWARD SOLVE
    $x \leftarrow U^{-1} y$

1.4 Question 1d: Operation Count for LU Decomposition

Using the algorithm above as a means to compute the operation count, we have that the total number of operations for LU if A and B are tridiagonal and C is diagonal, $O(2np^3)$ and if not diagonal, $O(\frac{2}{3}np^3)$.

1.5 Question 1e: Example of Discretized DE System

An example of where you'd run into a system like this, is in discretization of a Partial Differential Equation such as the Poisson Equation for Finite Differences using a 5-point stencil.

$$\Delta f = -\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 0 \tag{8}$$

$$-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i,j+1} = h^2 f(i,j) \tag{9}$$

$$\mathbf{A} = \begin{pmatrix} 4 & -1 & \dots & -1 & 0 & ... \\ -1 & 4 & -1 & \dots & -1 & ... \\ \vdots & 0 & -1 & 4 & -1 & ... \\ -1 & \ddots & 0 & \ddots & \ddots & \ddots \end{pmatrix}$$

Source: https://en.wikipedia.org/wiki/Discrete_Poisson_equation