

Q7 Partial Differential Equations

(Dr. Quaife ; Summer 2017)

Part a

Start with 2nd degree Taylor Expansions:

$$f(x+h) = f(x) + f'(x) * h + \left(\frac{1}{2!}\right)(f''(x) * h^2) + \left(\frac{1}{3!}\right)(f'''(\xi_1) * h^3) \quad \text{Equation 1}$$

$$f(x-h) = f(x) - f'(x) * h + \left(\frac{1}{2!}\right)(f''(x) * h^2) - \left(\frac{1}{3!}\right)(f'''(\xi_2) * h^3) \quad \text{Equation 2}$$

Subtract equation 2 from equation 1:

$$f(x+h) - f(x-h) = 2f'(x) * h + \left[\left(\frac{1}{3!}\right)f'''(\xi_1) * h^3 + \left(\frac{1}{3!}\right)f'''(\xi_2) * h^3\right]$$

$$f(x+h) - f(x-h) = 2f'(x) * h + \left(\frac{1}{3!}\right)f'''(\xi) * h^3$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{f'''(\xi) * h^3}{3! * 2h}$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{f'''(\xi) * h^2}{12}, \quad \text{where } \frac{f'''(\xi) * h^2}{12} = \text{truncation error.}$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - O(h^2), \quad \text{where } O(h^2) = \text{truncation error with 2nd order accuracy.}$$

Now put in terms of function U(x,y) which is a function with variables x and y:

$$U_x = \frac{\partial u(x_i, y_i)}{\partial x} = \frac{U(x+h, y) - U(x-h, y)}{2h} - O_x(h^2)$$

Now take partial derivative with respect to x and y:

$$U_{xy} = \left(\frac{1}{2h}\right) \left[\left(\frac{U(x+h, y+h) - U(x-h, y+h)}{2h}\right) - \left(\frac{U(x+h, y-h) - U(x-h, y-h)}{2h}\right) \right] - \left(\frac{1}{3!}\right)f'''(\xi_{xy})h^3$$

$$U_{xy} = \left(\frac{U(x+h, y+h) - U(x-h, y+h)}{2h * 2h}\right) - \left(\frac{U(x+h, y-h) - U(x-h, y-h)}{2h * 2h}\right) - \left(\frac{1}{3!}\right)f'''(\xi_{xy}) * h^3 \left(\frac{1}{2h}\right)$$

$$U_{xy} = \left(\frac{U(x+h, y+h) - U(x-h, y+h)}{4h^2}\right) - \left(\frac{U(x+h, y-h) - U(x-h, y-h)}{4h^2}\right) - \left(\frac{1}{12}\right)f'''(\xi_{xy}) * h^2$$

$$U_{xy} = \frac{U(x+h, y+h) + U(x-h, y-h) - U(x+h, y-h) - U(x-h, y+h)}{4h^2} - O(h^2)$$

where $O(h^2)$ = truncation error with 2nd order accuracy.

$$U_{xy} \approx \frac{U(x+h, y+h) + U(x-h, y-h) - U(x+h, y-h) - U(x-h, y+h)}{4h^2}$$

Part b

Standard 2nd Order Centered Difference Approximation for U_{xx} :

$$U_{xx} = \frac{\partial^2 u(x_i, y_j)}{\partial x^2} = \frac{U(x+h, y) - 2U(x, y) + U(x-h, y)}{h^2} - O(h^2)$$

$$U_{xx} \approx \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2}$$

From part a :

$$U_{xy} \approx \frac{U(x+h, y+h) + U(x-h, y-h) - U(x+h, y-h) - U(x-h, y+h)}{4h^2}$$

$$U_{xy} \approx \frac{u(x_{i+1}, y_{j+1}) + u(x_{i-1}, y_{j-1}) - u(x_{i+1}, y_{j-1}) - u(x_{i-1}, y_{j+1})}{4h^2}$$

Two-Dimensional PDE:

$$U_{xx} + U_{xy} = f(x, y)$$

Plugging in the formulas for U_{xx} and U_{xy} :

$$\frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} + \frac{u(x_{i+1}, y_{j+1}) + u(x_{i-1}, y_{j-1}) - u(x_{i+1}, y_{j-1}) - u(x_{i-1}, y_{j+1})}{4h^2} = f(x_i, y_j)$$

Multiplying both sides by $4h^2$:

$$4u(x_{i+1}, y_j) - 8u(x_i, y_j) + 4u(x_{i-1}, y_j) + u(x_{i+1}, y_{j+1}) + u(x_{i-1}, y_{j-1}) - u(x_{i+1}, y_{j-1}) - u(x_{i-1}, y_{j+1}) = 4h^2 f(x_i, y_j)$$

Re-arranging terms by rows 1st then by columns:

$$u(x_{i-1}, y_{j-1}) - u(x_{i+1}, y_{j-1}) + 4u(x_{i-1}, y_j) - 8u(x_i, y_j) + 4u(x_{i+1}, y_j) - u(x_{i-1}, y_{j+1}) + u(x_{i+1}, y_{j+1}) = 4h^2 f(x_i, y_j)$$

The interval of x and y which are $(0,1)$ are divided into equidistant nodes spaced at a distance of h .

To have 16 interior nodes, use $i = 1, 2, \dots, 4$.

The interior nodes are labeled $x_{i,j}$.

For Dirichlet Boundary Conditions, the values of $u(x_i, y_j)$ are constant for any $i = 0$ or 5 or $j = 0$ or 5 .

This new equation is used to generate a system of equations for $i = 1, 2, \dots, 4$ and $j = 1, 2, \dots, 4$.

$$u(x_{i-1}, y_{j-1}) - u(x_{i+1}, y_{j-1}) + 4u(x_{i-1}, y_j) - 8u(x_i, y_j) + 4u(x_{i+1}, y_j) - u(x_{i-1}, y_{j+1}) + u(x_{i+1}, y_{j+1}) = 4h^2 f(x_i, y_j)$$

System of equations with $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$:

$$i = 1, j = 1 \quad u(x_0, y_0) - u(x_2, y_0) + 4u(x_0, y_1) - 8u(x_1, y_1) + 4u(x_2, y_1) - u(x_0, y_2) + u(x_2, y_2) = 4h^2 f(x_1, y_1)$$

$$i = 2, j = 1 \quad u(x_1, y_0) - u(x_3, y_0) + 4u(x_1, y_1) - 8u(x_2, y_1) + 4u(x_3, y_1) - u(x_1, y_2) + u(x_3, y_2) = 4h^2 f(x_2, y_1)$$

$$i = 3, j = 1 \quad u(x_2, y_0) - u(x_4, y_0) + 4u(x_2, y_1) - 8u(x_3, y_1) + 4u(x_4, y_1) - u(x_2, y_2) + u(x_4, y_2) = 4h^2 f(x_3, y_1)$$

$$i = 4, j = 1 \quad u(x_3, y_0) - u(x_5, y_0) + 4u(x_3, y_1) - 8u(x_4, y_1) + 4u(x_5, y_1) - u(x_3, y_2) + u(x_5, y_2) = 4h^2 f(x_4, y_1)$$

$$i = 1, j = 2 \quad u(x_0, y_1) - u(x_2, y_1) + 4u(x_0, y_2) - 8u(x_1, y_2) + 4u(x_2, y_2) - u(x_0, y_3) + u(x_2, y_3) = 4h^2 f(x_1, y_2)$$

$$i = 2, j = 2 \quad u(x_1, y_1) - u(x_3, y_1) + 4u(x_1, y_2) - 8u(x_2, y_2) + 4u(x_3, y_2) - u(x_1, y_3) + u(x_3, y_3) = 4h^2 f(x_2, y_2)$$

$$i = 3, j = 2 \quad u(x_2, y_1) - u(x_4, y_1) + 4u(x_2, y_2) - 8u(x_3, y_2) + 4u(x_4, y_2) - u(x_2, y_3) + u(x_4, y_3) = 4h^2 f(x_3, y_2)$$

$$i = 4, j = 2 \quad u(x_3, y_1) - u(x_5, y_1) + 4u(x_3, y_2) - 8u(x_4, y_2) + 4u(x_5, y_2) - u(x_3, y_3) + u(x_5, y_3) = 4h^2 f(x_4, y_2)$$

$$i = 1, j = 3 \quad u(x_0, y_2) - u(x_2, y_2) + 4u(x_0, y_3) - 8u(x_1, y_3) + 4u(x_2, y_3) - u(x_0, y_4) + u(x_2, y_4) = 4h^2 f(x_1, y_3)$$

$$i = 2, j = 3 \quad u(x_1, y_2) - u(x_3, y_2) + 4u(x_1, y_3) - 8u(x_2, y_3) + 4u(x_3, y_3) - u(x_1, y_4) + u(x_3, y_4) = 4h^2 f(x_2, y_3)$$

$$i = 3, j = 3 \quad u(x_2, y_2) - u(x_4, y_2) + 4u(x_2, y_3) - 8u(x_3, y_3) + 4u(x_4, y_3) - u(x_2, y_4) + u(x_4, y_4) = 4h^2 f(x_3, y_3)$$

$$i = 4, j = 3 \quad u(x_3, y_2) - u(x_5, y_2) + 4u(x_3, y_3) - 8u(x_4, y_3) + 4u(x_5, y_3) - u(x_3, y_4) + u(x_5, y_4) = 4h^2 f(x_4, y_3)$$

$$i = 1, j = 4 \quad u(x_0, y_3) - u(x_2, y_3) + 4u(x_0, y_4) - 8u(x_1, y_4) + 4u(x_2, y_4) - u(x_0, y_5) + u(x_2, y_5) = 4h^2 f(x_1, y_4)$$

$$i = 2, j = 4 \quad u(x_1, y_3) - u(x_3, y_3) + 4u(x_1, y_4) - 8u(x_2, y_4) + 4u(x_3, y_4) - u(x_1, y_5) + u(x_3, y_5) = 4h^2 f(x_2, y_4)$$

$$i = 3, j = 4 \quad u(x_2, y_3) - u(x_4, y_3) + 4u(x_2, y_4) - 8u(x_3, y_4) + 4u(x_4, y_4) - u(x_2, y_5) + u(x_4, y_5) = 4h^2 f(x_3, y_4)$$

$$i = 4, j = 4 \quad u(x_3, y_3) - u(x_5, y_3) + 4u(x_3, y_4) - 8u(x_4, y_4) + 4u(x_5, y_4) - u(x_3, y_5) + u(x_5, y_5) = 4h^2 f(x_4, y_4)$$

For boundary values $u(x_{0,j})$, $u(x_{5,j})$, $u(x_{i,0})$, and $u(x_{i,5})$ which are constants, this set of equations becomes:

$$\begin{aligned}
 -8u(x_1, y_1) + 4u(x_2, y_1) + u(x_2, y_2) &= 4h^2 f(x_1, y_1) - u(x_0, y_0) + u(x_2, y_0) - 4u(x_0, y_1) + u(x_0, y_2) \\
 4u(x_1, y_1) - 8u(x_2, y_1) + 4u(x_3, y_1) - u(x_1, y_2) + u(x_3, y_2) &= 4h^2 f(x_2, y_1) - u(x_1, y_0) + u(x_3, y_0) \\
 4u(x_2, y_1) - 8u(x_3, y_1) + 4u(x_4, y_1) - u(x_2, y_2) + u(x_4, y_2) &= 4h^2 f(x_3, y_1) - u(x_2, y_0) + u(x_4, y_0) \\
 4u(x_3, y_1) - 8u(x_4, y_1) - u(x_3, y_2) &= 4h^2 f(x_4, y_1) - u(x_3, y_0) + u(x_5, y_0) - 4u(x_5, y_1) - u(x_5, y_2) \\
 -u(x_2, y_1) - 8u(x_1, y_2) + 4u(x_2, y_2) + u(x_2, y_3) &= 4h^2 f(x_1, y_2) - u(x_0, y_1) - 4u(x_0, y_2) + u(x_0, y_3) \\
 u(x_1, y_1) - u(x_3, y_1) + 4u(x_1, y_2) - 8u(x_2, y_2) + 4u(x_3, y_2) - u(x_1, y_3) + u(x_3, y_3) &= 4h^2 f(x_2, y_2) \\
 u(x_2, y_1) - u(x_4, y_1) + 4u(x_2, y_2) - 8u(x_3, y_2) + 4u(x_4, y_2) - u(x_2, y_3) + u(x_4, y_3) &= 4h^2 f(x_3, y_2) \\
 u(x_3, y_1) + 4u(x_3, y_2) - 8u(x_4, y_2) - u(x_3, y_3) &= 4h^2 f(x_4, y_2) + u(x_5, y_1) - 4u(x_5, y_2) - u(x_5, y_3) \\
 -u(x_2, y_2) - 8u(x_1, y_3) + 4u(x_2, y_3) + u(x_2, y_4) &= 4h^2 f(x_1, y_3) - u(x_0, y_2) - 4u(x_0, y_3) + u(x_0, y_4) \\
 u(x_1, y_2) - u(x_3, y_2) + 4u(x_1, y_3) - 8u(x_2, y_3) + 4u(x_3, y_3) - u(x_1, y_4) + u(x_3, y_4) &= 4h^2 f(x_2, y_3) \\
 u(x_2, y_2) - u(x_4, y_2) + 4u(x_2, y_3) - 8u(x_3, y_3) + 4u(x_4, y_3) - u(x_2, y_4) + u(x_4, y_4) &= 4h^2 f(x_3, y_3) \\
 u(x_3, y_2) + 4u(x_3, y_3) - 8u(x_4, y_3) - u(x_3, y_4) &= 4h^2 f(x_4, y_3) + u(x_5, y_2) - 4u(x_5, y_3) - u(x_5, y_4) \\
 -u(x_2, y_3) - 8u(x_1, y_4) + 4u(x_2, y_4) &= 4h^2 f(x_1, y_3) - u(x_0, y_3) - 4u(x_0, y_4) + u(x_0, y_5) - u(x_2, y_5) \\
 u(x_1, y_3) - u(x_3, y_3) + 4u(x_1, y_4) - 8u(x_2, y_4) + 4u(x_3, y_4) &= 4h^2 f(x_2, y_3) + u(x_1, y_5) - u(x_3, y_5) \\
 u(x_2, y_3) - u(x_4, y_3) + 4u(x_2, y_4) - 8u(x_3, y_4) + 4u(x_4, y_4) &= 4h^2 f(x_3, y_3) + u(x_2, y_5) - u(x_4, y_5) \\
 u(x_3, y_3) + 4u(x_3, y_4) - 8u(x_4, y_4) &= 4h^2 f(x_4, y_3) + u(x_5, y_3) - 4u(x_5, y_4) + u(x_3, y_5) - u(x_5, y_5)
 \end{aligned}$$

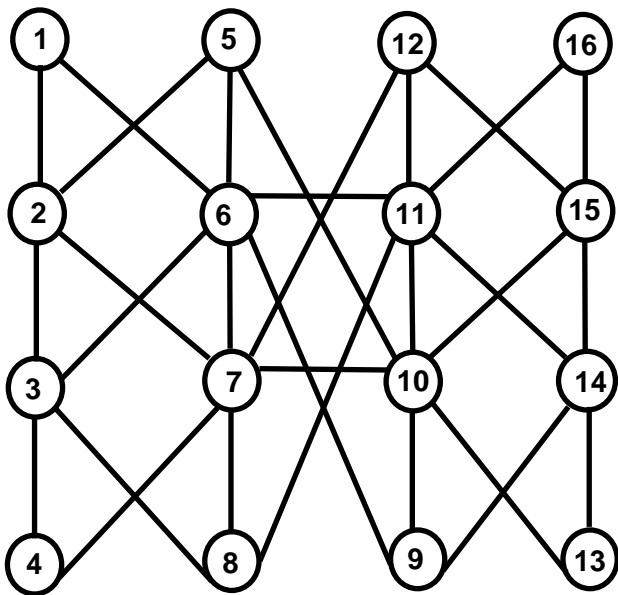
$$\begin{bmatrix}
 -8 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 4 & -8 & 4 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 4 & -8 & 4 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 4 & -8 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & -1 & 0 & 0 & -8 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & -1 & 0 & 4 & -8 & 4 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & -1 & 0 & 4 & -8 & 4 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 4 & -8 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -8 & 4 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 4 & -8 & 4 & 0 & -1 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 4 & -8 & 4 & 0 & -1 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & -8 & 0 & 0 & -1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -8 & 4 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 4 & -8 & 4 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 4 & -8 & 4 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & -8
 \end{bmatrix}
 \begin{bmatrix}
 u(x_1, y_1) \\
 u(x_2, y_1) \\
 u(x_3, y_1) \\
 u(x_4, y_1) \\
 u(x_1, y_2) \\
 u(x_2, y_2) \\
 u(x_3, y_2) \\
 u(x_4, y_2) \\
 u(x_1, y_3) \\
 u(x_2, y_3) \\
 u(x_3, y_3) \\
 u(x_4, y_3) \\
 u(x_1, y_4) \\
 u(x_2, y_4) \\
 u(x_3, y_4) \\
 u(x_4, y_4)
 \end{bmatrix}
 =
 \begin{bmatrix}
 F_1 \\
 F_2 \\
 F_3 \\
 F_4 \\
 F_5 \\
 F_6 \\
 F_7 \\
 F_8 \\
 F_9 \\
 F_{10} \\
 F_{11} \\
 F_{12} \\
 F_{13} \\
 F_{14} \\
 F_{15} \\
 F_{16}
 \end{bmatrix}$$

Note: Multiplying both sides by scalar -1 will make the coefficient matrix positive definite.

Part c

Grids	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	-8	4	0	0	0	1	0	0	0	0	0	0	0	0	0	0
2	4	-8	4	0	-1	0	1	0	0	0	0	0	0	0	0	0
3	0	4	-8	4	0	-1	0	1	0	0	0	0	0	0	0	0
4	0	0	4	-8	0	0	-1	0	0	0	0	0	0	0	0	0
5	0	-1	0	0	-8	4	0	0	0	1	0	0	0	0	0	0
6	1	0	-1	0	4	-8	4	0	-1	0	1	0	0	0	0	0
7	0	1	0	-1	0	4	-8	4	0	-1	0	1	0	0	0	0
8	0	0	1	0	0	0	4	-8	0	0	-1	0	0	0	0	0
9	0	0	0	0	0	-1	0	0	-8	4	0	0	0	1	0	0
10	0	0	0	0	1	0	-1	0	4	-8	4	0	-1	0	1	0
11	0	0	0	0	0	1	0	-1	0	4	-8	4	0	-1	0	1
12	0	0	0	0	0	0	1	0	0	0	4	-8	0	0	-1	0
13	0	0	0	0	0	0	0	0	0	-1	0	0	-8	4	0	0
14	0	0	0	0	0	0	0	0	1	0	-1	0	4	-8	4	0
15	0	0	0	0	0	0	0	0	0	1	0	-1	0	4	-8	4
16	0	0	0	0	0	0	0	0	0	0	1	0	0	0	4	-8

Undirected Adjacency graph for linear system from part b:

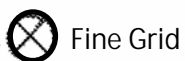


(Grid Number shown inside bubbles)

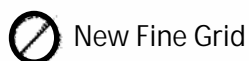
Part d



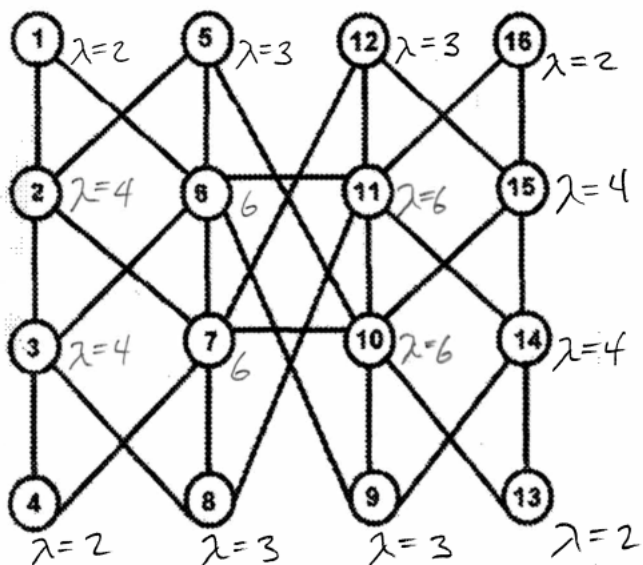
Coarse Grid



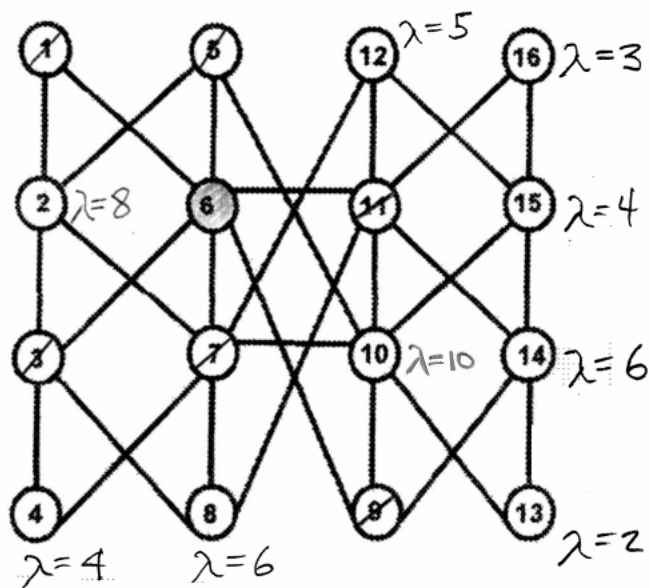
Fine Grid



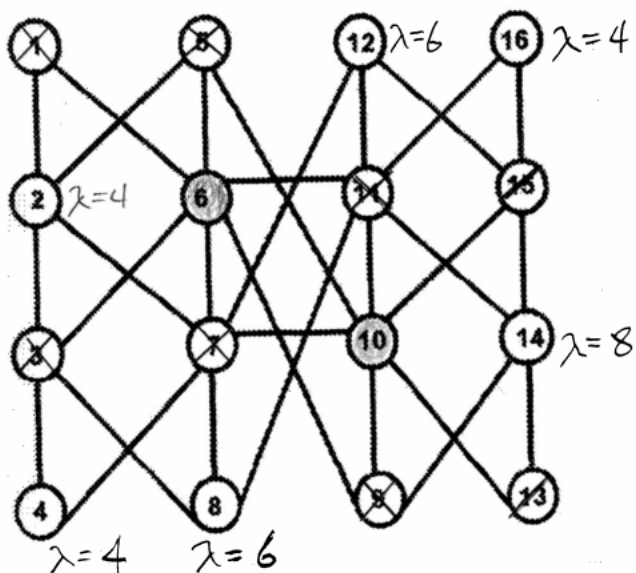
New Fine Grid



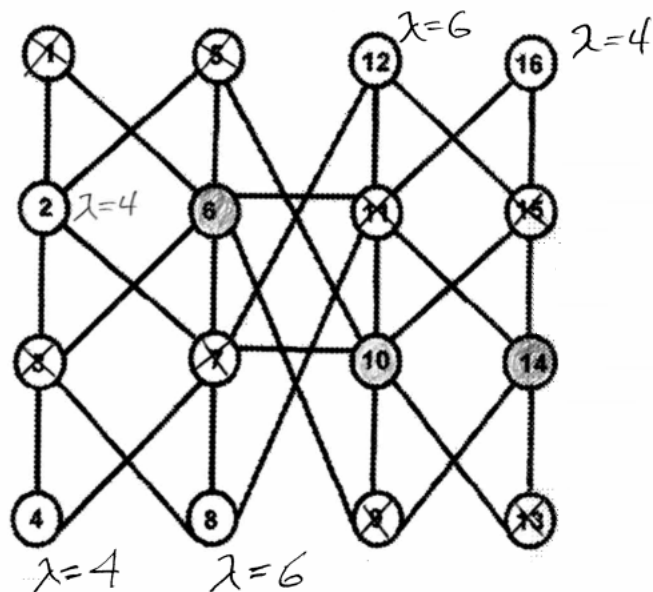
Algebraic Multigrid Coloring Scheme Step 1



Algebraic Multigrid Coloring Scheme Step 2



Algebraic Multigrid Coloring Scheme Step 3



Algebraic Multigrid Coloring Scheme Step 4

Part d



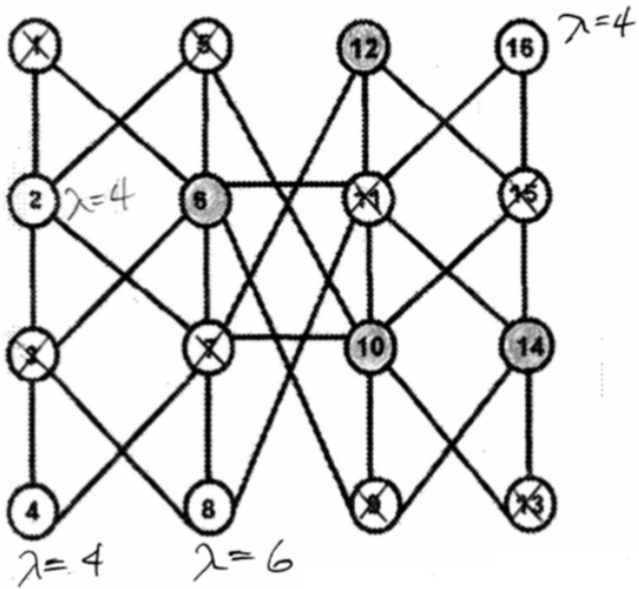
Coarse Grid



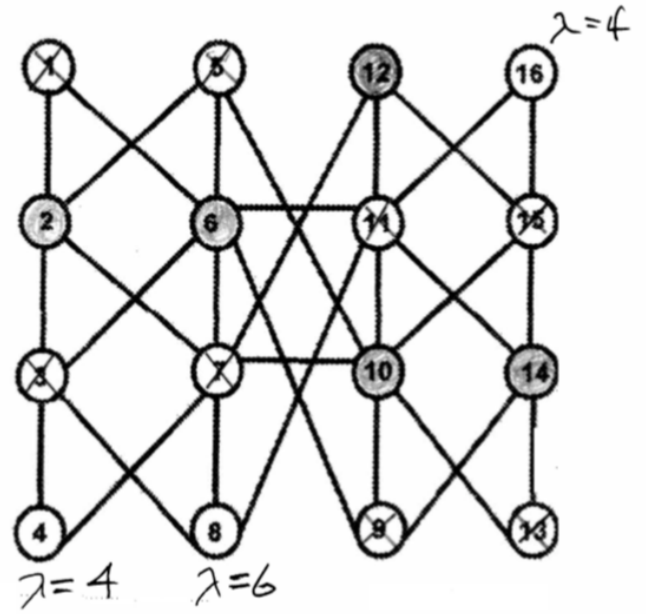
Fine Grid



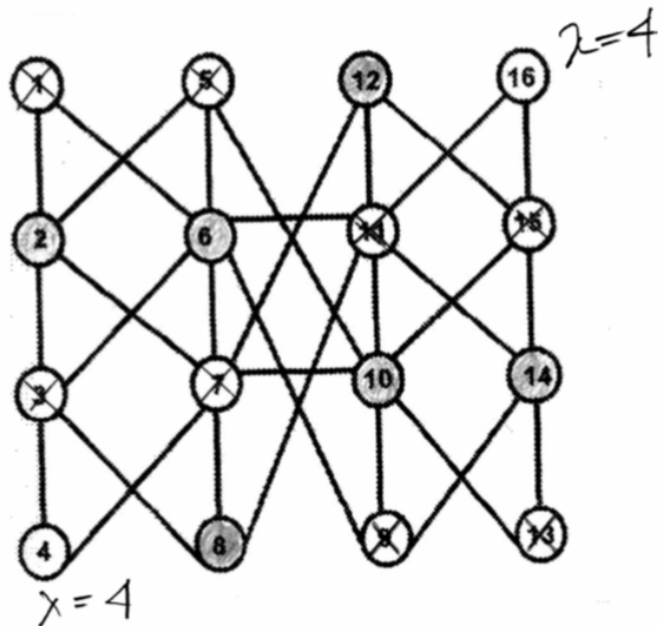
New Fine Grid



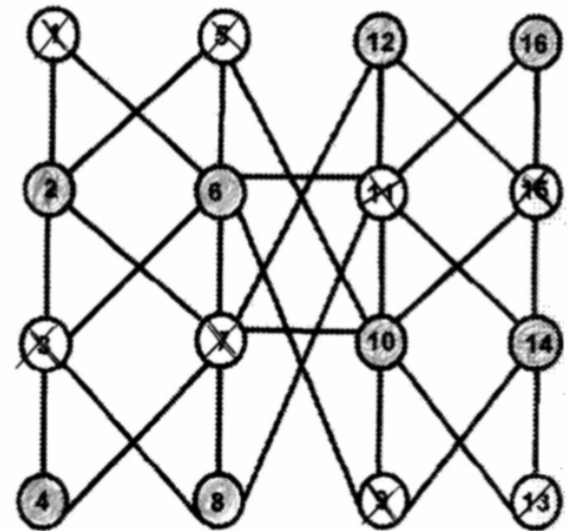
Algebraic Multigrid Coloring Scheme Step 5



Algebraic Multigrid Coloring Scheme Step 6



Algebraic Multigrid Coloring Scheme Step 7



Algebraic Multigrid Coloring Scheme Step 8, 9

Part e

The results of part d "coloring scheme method for algebraic multigrid" and the results of one step of "geometric multigrid method" seem to produce the same results.

However, there is a difference between the two methods.

The algebraic multigrid method selects coarse grids associated with a **subset** of the original values of $u(x_i, y_j)$.

The geometric multigrid method selects coarse grids that **may are not be** associated with the original values of $u(x_i, y_j)$.