Spring 2015 Question 2

Part a

We have to show that A is positive definite by showing $\vec{u}^T A\vec{u} > 0$.

$$\vec{u}^T A = \begin{bmatrix} u_1 & u_2 & \dots & u_{m-1} & u_m \end{bmatrix} \begin{bmatrix} \frac{2}{h^2} & \frac{-1}{h^2} & 0 \\ \frac{-1}{h^2} & \frac{2}{h^2} & \ddots \\ 0 & \ddots & \ddots \end{bmatrix} \tag{1}$$

$$\vec{u}^T A = \frac{1}{h^2}\begin{bmatrix} 2u_1 - u_2 & [-u_{i-1} + 2u_i - u_{i+1}]_{i=2}^{m-1} & 2u_m - u_{m-1} \end{bmatrix} \tag{2}$$

$$\vec{u}^T A\vec{u} = \frac{1}{h^2}\begin{bmatrix} 2u_1 - u_2 & [-u_{i-1} + 2u_i - u_{i+1}]_{i=2}^{m-1} & 2u_m - u_{m-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m-1} \\ u_m \end{bmatrix} \tag{3}$$

$$\begin{aligned}
\vec{u}^T A\vec{u} &= \frac{1}{h^2}\left(2u_1^2 - u_1u_2 + \sum_{i=2}^{m-1}\left[-u_{i-1}u_i + 2u_i^2 - u_iu_{i+1}\right] + 2u_m^2 - u_{m-1}u_m\right) \\
&= \frac{1}{h^2}\left(u_1^2 + u_1^2 - u_1u_2 + \sum_{i=2}^{m-1}\left[-u_{i-1}u_i + u_i^2 + u_i^2 - u_iu_{i+1}\right] + u_m^2 + u_m^2 - u_{m-1}u_m\right) \\
&= \frac{1}{h^2}\left(u_1^2 + \sum_{i=1}^{m-1}\left[u_i^2 - u_iu_{i-1} - u_{i-1}u_i + u_{i+1}^2\right] + u_m^2\right) \\
&= \frac{1}{h^2}\left(u_1^2 + \sum_{i=2}^{m-1}(u_i - u_{i-1})^2 + u_m^2\right)
\end{aligned} \tag{4}$$

All the squared terms are positive so $\vec{u}^T A\vec{u}$ is positive.

Part b

i

Assume the iterative method stated in the problem doesn't converge such that

$$\lim_{k\to\infty} ||\vec{x}^k - \vec{x}|| = \lim_{k\to\infty} e^k \neq 0 \tag{5}$$

Since $x^{k+1} = Px^k + c$,

$$\begin{aligned}
e^k &= ||\vec{x}^k - \vec{x}|| \\
&= ||\vec{x}^{k-1} + c - (P\vec{x} + c)|| \\
&= ||P(\vec{x}^{k-1} - \vec{x})|| \\
&= ||Pe^{k-1}|| \\
&= ||P^2 e^{k-2}|| \\
&\vdots \\
&= ||P^k e^0|| \\
&= ||P^k||
\end{aligned} \tag{6}$$

Since we assumed $\lim_{k\to\infty} e^k \neq 0$, $\lim_{k\to\infty} ||P^k|| \neq 0$. But we are given that $\lim_{k\to\infty} ||P^k||_\alpha = 0_{n\times n}$ in the question. So we have a contradiction. Therefore the method converges.

ii

We have the general problem $B\vec{x} = \vec{f}$. Using the splitting method $B = M - N$, we show

$$\begin{aligned}
B\vec{x} &= \vec{f} \\
M\vec{x} - N\vec{x} &= \vec{f} \\
\vec{x} &= M^{-1}\vec{f} + M^{-1}N\vec{x}
\end{aligned} \tag{7}$$

which is the Jacobi method $\vec{x}^{k+1} = M^{-1}N\vec{x}^k + M^{-1}\vec{f}$ with M being a diagonal matrix and N being all the off-diagonal elements (lower and upper). When we substitute $P = M^{-1}N$ and $c = M^{-1}\vec{f}$ we get the iterative method from our problem $\vec{x}^{k+1} = P\vec{x}^k + \vec{c}$. For our tridiagonal matrix A we have $P_A = 0_{n\times n}$.

$$P_A = \begin{bmatrix} \frac{h^2}{2} & 0 & 0 \\ 0 & \frac{h^2}{2} & \ddots \\ 0 & \ddots & \ddots \end{bmatrix} \begin{bmatrix} 0 & \frac{-1}{h^2} & 0 \\ \frac{-1}{h^2} & 0 & \ddots \\ 0 & \ddots & \ddots \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots \\ 0 & 0 & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix} \tag{8}$$

Part c

i

Our matrix A (in part a) is symmetric positive definite so the splitting $A = M - N$ has M containing the diagonal elements of A, and $-N$ containing all the subdiagonal elements of A. So $M = M^T$ which means $M^T - N$ is also s.p.d. (as $M - N$ is also s.p.d. which can be shown by part a above). By the Lemma given, the spectral radius $\rho(M^{-1}N) < 1$. This iterative method converges iff for matrix P, $\rho(P) < 1$. In part b.ii we showed P_A is a matrix of 0s so $\rho(P) = 0$.

ii

To find which matrix will converge faster we can find the spectral radius of each matrix. The matrix with the smaller spectral radius converges faster; in this case, $\rho(B_1) = 6$ and $\rho(B_1) = 10$, so B_1 will converge faster using the Jacobi method.

Part d

We can use the power method to find the spectral radius. Using the file

`sp2015_2d.py`

we get the number of iterations for convergence of the power method, the dominant eigenvector, and the corresponding eigenvalue.

```
After 8178 iterations.
--------------------------
Dominant eigenvector:
[[-0.03077979]
 [ 0.06153076]
 [-0.09222411]
 [ 0.12283109]
 [-0.15332301]
 [ 0.18367131]
 [-0.21384751]
 [ 0.2438233 ]
 [-0.27357053]
 [ 0.30306126]
 [-0.33226774]
 [ 0.36116247]
 [-0.38971823]
 [ 0.41790806]
 [-0.44570533]
 [ 0.47308375]
 [-0.50001737]
 [ 0.52648063]
 [-0.55244836]
 [ 0.57789585]
 [-0.60279879]
 [ 0.62713339]
 [-0.65087631]
 [ 0.67400475]
 [-0.69649644]
 [ 0.71832967]
 [-0.73948329]
 [ 0.75993677]
 [-0.77967019]
 [ 0.79866427]
 [-0.81690037]
```

```
[ 0.83436056]
[-0.85102758]
[ 0.86688489]
[-0.88191668]
[ 0.89610789]
[-0.90944422]
[ 0.92191215]
[-0.93349896]
[ 0.94419275]
[-0.95398243]
[ 0.96285774]
[-0.9708093 ]
[ 0.97782857]
[-0.98390791]
[ 0.98904052]
[-0.99322055]
[ 0.99644303]
[-0.99870389]
[ 1.        ]
[-1.00032916]
[ 0.99969009]
[-0.99808245]
[ 0.99550686]
[-0.99196487]
[ 0.98745898]
[-0.98199263]
[ 0.97557024]
[-0.96819713]
[ 0.95987962]
[-0.95062493]
[ 0.94044124]
[-0.92933767]
[ 0.91732425]
[-0.90441195]
[ 0.89061265]
[-0.87593912]
[ 0.86040504]
[-0.84402496]
[ 0.82681433]
[-0.80878943]
[ 0.78996739]
[-0.77036616]
[ 0.75000453]
[-0.72890204]
[ 0.70707906]
[-0.68455666]
[ 0.66135669]
```

```
 [-0.6375017 ]
 [ 0.61301492]
 [-0.58792026]
 [ 0.56224228]
 [-0.53600616]
 [ 0.50923767]
 [-0.48196314]
 [ 0.45420946]
 [-0.42600402]
 [ 0.39737471]
 [-0.36834985]
 [ 0.3389582 ]
 [-0.30922893]
 [ 0.27919155]
 [-0.24887591]
 [ 0.21831216]
 [-0.18753074]
 [ 0.15656229]
 [-0.12543768]
 [ 0.09418793]
 [-0.06284421]
 [ 0.03143779]]
Spectral radius:
[[ 40794.13036297]]
```