Problem 4 (Spring 2015)

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a. **(i).** Explicit stage-3 Runge-Kutta method:

General s-stage explicit RK method

$$
\begin{aligned}
k_1 &= \Delta t f(t_n, Y^n) \\
k_2 &= \Delta t f(t_n + c_2 \Delta t, Y^n + a_{21} k_1) \\
k_3 &= \Delta t f(t_n + c_3 \Delta t, Y^n + a_{31} k_1 + a_{32} k_2) \\
&\vdots \\
k_s &= \Delta t f\left(t_n + c_s \Delta t, Y^n + a_{s1} k_1 + a_{s2} k_2 + \cdots + a_{ss-1} k_{s-1}\right) \\
Y^{n+1} &= Y^n + \textstyle\sum_{j=1}^{s} b_j k_j
\end{aligned}
$$

$$
\begin{array}{c|cccc}
0 & & & & \\
c_2 & a_{21} & & & \\
c_3 & a_{31} & a_{32} & & \\
\vdots & \vdots & \vdots & \ddots & \\
c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\
\hline
 & b_1 & b_2 & \cdots & b_s
\end{array}
$$

Kutta's third-order method

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 \\
1 & -1 & 2 & 0 \\
\hline
 & 1/6 & 2/3 & 1/6
\end{array}
$$

Thus

$$k_1 = \Delta t f(t_n, Y^n)$$

$$k_2 = \Delta t f(t_n + \frac{1}{2}\Delta t, Y^n + \frac{1}{2} k_1)$$

$$k_3 = \Delta t f(t_n + \Delta t, Y^n - k_1 + 2k_2)$$

$$Y^{n+1} = Y^n + \frac{1}{6} k_1 + \frac{2}{3} k_2 + \frac{1}{6} k_3$$

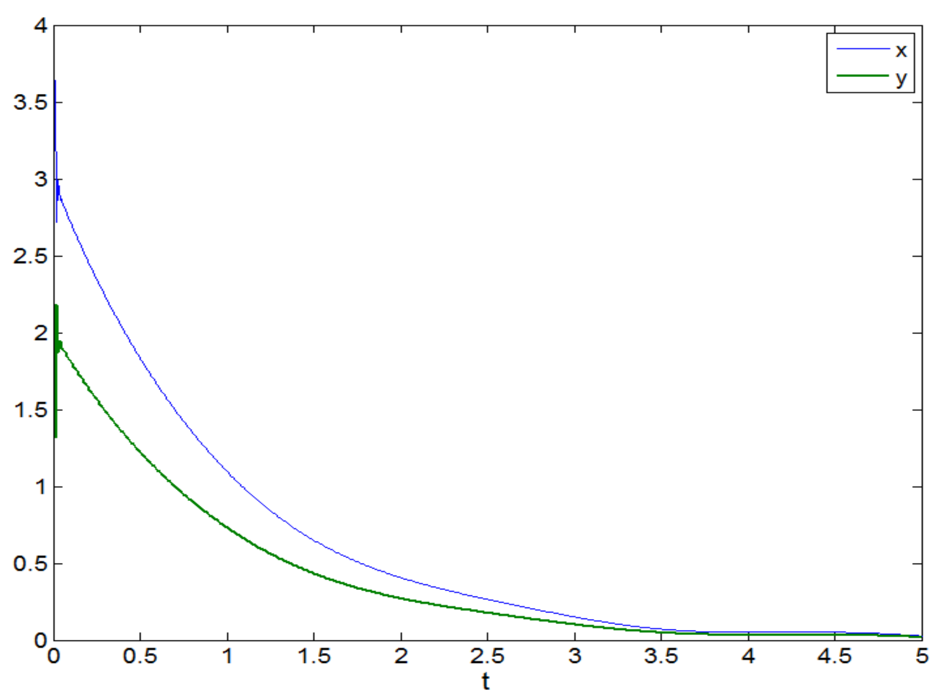
Systems:

$$\begin{aligned}
\mathbf{k}_1 &= \Delta t \mathbf{F}\big(t_n, \mathbf{W}^n\big)\\
\mathbf{k}_2 &= \Delta t \mathbf{F}\big(t_n + c_2\Delta t, \mathbf{W}^n + a_{21}\mathbf{k}_1\big)\\
\mathbf{k}_3 &= \Delta t \mathbf{F}\big(t_n + c_3\Delta t, \mathbf{W}^n + a_{31}\mathbf{k}_1 + a_{32}\mathbf{k}_2\big)\\
&\vdots\\
\mathbf{k}_s &= \Delta t \mathbf{F}\big(t_n + c_s\Delta t, \mathbf{W}^n + a_{s1}\mathbf{k}_1 + a_{s2}\mathbf{k}_2 + \cdots + a_{ss-1}\mathbf{k}_{s-1}\big)\\
\mathbf{W}^{n+1} &= \mathbf{W}^n + \sum_{j=1}^{s} b_j \mathbf{k}_j\,.
\end{aligned}$$

Thus

$$\begin{aligned}
&k_1 = \Delta t f(t_n, W^n)\\
&k_2 = \Delta t f(t_n + \frac{1}{2}\Delta t, W^n + \frac{1}{2}k_1)\\
&k_3 = \Delta t f(t_n + \Delta t, W^n - k_1 + 2k_2)\\
&W^{n+1} = W^n + \frac{1}{6}k_1 + \frac{2}{3}k_2 + \frac{1}{6}k_3
\end{aligned}$$

(ii) The plot is shown below. (dt=0.01)



(iii)

From Eoin Devane, 2013. Stability theory for systems of differential equations with applications to power control in wireless networks.

2.2.3 Nonautonomous systems

In his 1892 work [7], Lyapunov also developed what has come to be known as *Lyapunov's Direct Method*, or *Lyapunov's Second Method*. This method allows us to extend our consideration to the more general nonautonomous system, where the right-hand side is allowed to depend explicitly on t. The exact form to be considered is that stated in (1)

$$\dot{x} = f(t, x),$$

with $f : [0, \infty) \times D \to \mathbb{R}^n$ piecewise continuous in t and locally Lipschitz in x, where D is a domain containing the origin.

b. **(i)** The IVP is

$$\frac{dx(t)}{dt} = -e^{-t} + 4\times 10^4 e^{-10^4 t}$$

with initial condition $x(0) = e - 3$.

(ii) dt=0.0001.

```
Using the Forward Euler method to get x:
DeltaT      Value       Error       Rate
0.00100000   39.08766124   36.00149997   0.00000000
0.00050000   19.22167632   16.13551505   1.15781739
0.00025000   9.98033715   6.89417588   1.22678962
0.00012500   6.09387735   3.00771608   1.19670972
0.00006250   4.46550995   1.37934868   1.12468120
0.00003125   3.74365062   0.65748935   1.06894777
```

```
Using the Midpoint method to get x:
DeltaT      Value       Error       Rate
0.00100000   -0.64430859   3.73046986   0.00000000
0.00050000   0.73899797   2.34716329   0.66843914
0.00025000   2.20741756   0.87874371   1.41740385
0.00012500   2.83714254   0.24901873   1.81918820
0.00006250   3.02179129   0.06436998   1.95179429
0.00003125   3.06993147   0.01622980   1.98774271
```

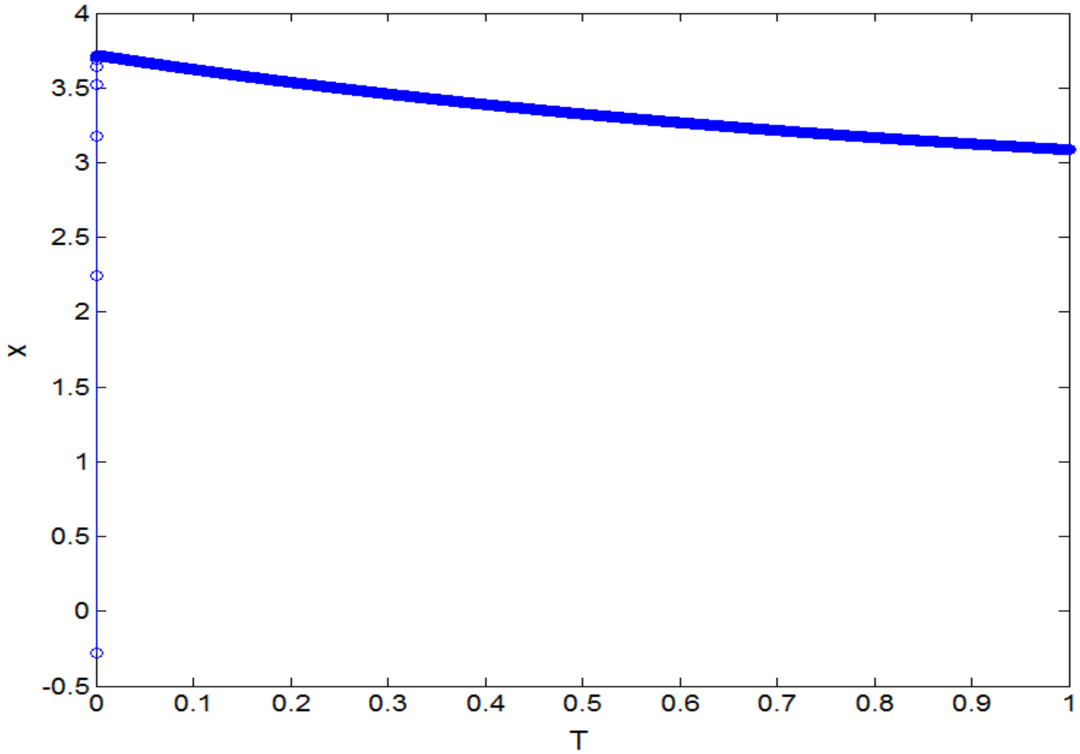
```
Using the Heun's method to get x:
DeltaT      Value       Error      Rate
0.00100000   9.12479604   6.03863477   0.00000000
0.00050000   4.65881945   1.57265818   1.94101729
0.00025000   3.35296922   0.26680795   2.55933159
0.00012500   3.12226869   0.03610742   2.88543448
0.00006250   3.09072705   0.00456578   2.98336089
0.00003125   3.08673075   0.00056948   3.00313809
```

```
Using the Stage-3 RK method to get x:
DeltaT      Value       Error      Rate
0.00100000   5.93312004   2.84695877   0.00000000
0.00050000   3.56661010   0.48044883   2.56696690
0.00025000   3.13175043   0.04558916   3.39762015
0.00012500   3.08940065   0.00323938   3.81490249
0.00006250   3.08637076   0.00020949   3.95077116
0.00003125   3.08617448   0.00001321   3.98749007
```

```
Using the Stage-4 RK method to get x:
DeltaT      Value       Error      Rate
0.00100000   4.64083986   1.55467859   0.00000000
0.00050000   3.31562602   0.22946475   2.76027191
0.00025000   3.10683098   0.02066971   3.47268260
0.00012500   3.08760838   0.00144711   3.83626872
0.00006250   3.08625450   0.00009323   3.95629792
0.00003125   3.08616714   0.00000587   3.98888374
```

```
Using the Stage-6 RK method to get x:
DeltaT      Value       Error      Rate
0.00100000   4.32681286   1.24065159   0.00000000
0.00050000   3.21175152   0.12559025   3.30430169
0.00025000   3.09257654   0.00641527   4.29107140
0.00012500   3.08639635   0.00023508   4.77025679
0.00006250   3.08616898   0.00000771   4.93083278
0.00003125   3.08616151   0.00000024   4.97813944
```

(iii) The solution is smooth due to the exact solution is smooth. The reason for so small time step is that the stability region is very small. We could use implicit Runge-Rutta method to avoid the very small time step since implicit method has a large stability region generally.



x
T
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1