200256677_5_Quaife

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[66]:
```python
import numpy as np
import cmath
import matplotlib.pyplot as plt
```

0.1 Part 1 - Verify Equation #1

Given $f(z) = z^2 + 1$ and $z_0 = 0$,

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta} - z_0} ie^{i\theta} d\theta$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{2i\theta} + 1}{e^{i\theta}} ie^{i\theta} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{2i\theta} + 1 d\theta$$

$$= \frac{1}{2\pi}\left[\frac{e^{2i\theta}}{2i} + \theta \right|_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(\frac{1}{2i} + 2\pi - \frac{1}{2i} - 0\right)$$

$$= 1 = f(z_0)$$

0.2 Part 2 - Trapezoidal Rule

[84]:
```python
# num quadrature points
N = 100
# function to integrate
f = lambda z : z**2 + 1
# z0 (point in C, must be within unit circle)
z0 = complex(0,0)
```

[89]:
```python
def complex_trapezoidal (f, N, z0) :
    # declare constant
```

```
    i = complex(0,1)

    # discretize theta uniformly on N quadrature points
    theta = np.linspace(0, 2*np.pi,N)
    # helper function - g(theta) is the integrand in equation (1) but i pulled␣
↪out the i/i
    g = lambda theta : 1 / (2 * np.pi) * f(cmath.exp(i*theta)) / (cmath.
↪exp(i*theta) - z0) * cmath.exp(i*theta)

    # sum over trapezoids
    trap = np.sum([g(theta[j-1]) + g(theta[j]) for j in range(1,N)])
    # scale by grid size
    trap = trap * (2 * np.pi - 0) / (2 * N)

    return trap
```

[90]:
```
# testing
print(f(z0))
complex_trapezoidal(f, N, z0)
```

```
(1+0j)
```

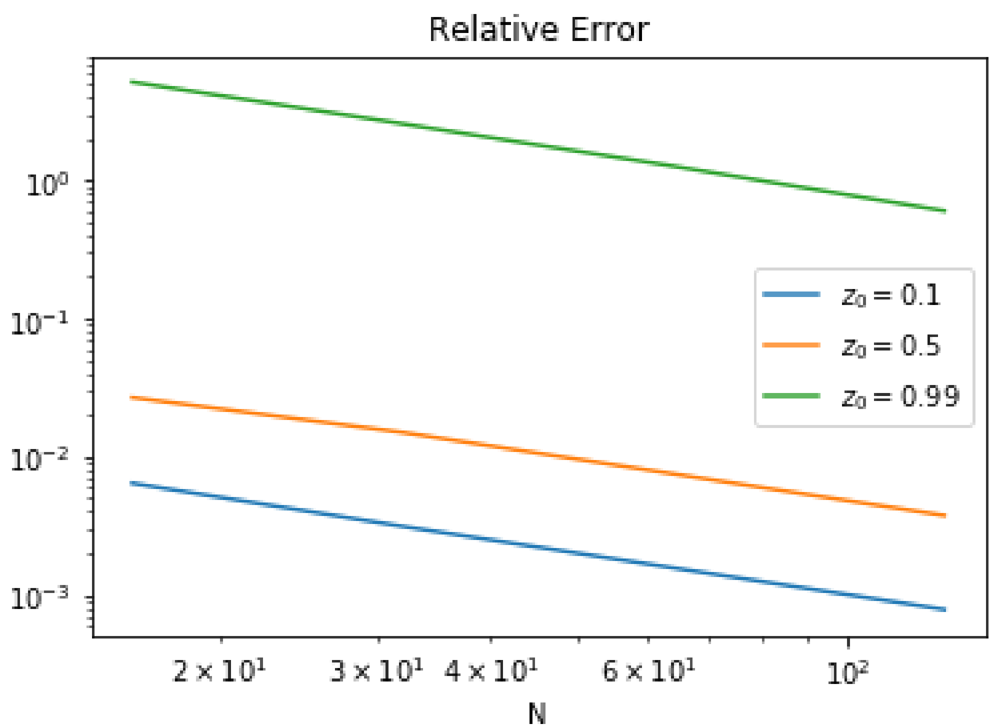
[90]:
```
(0.99+1.7875326052294235e-17j)
```

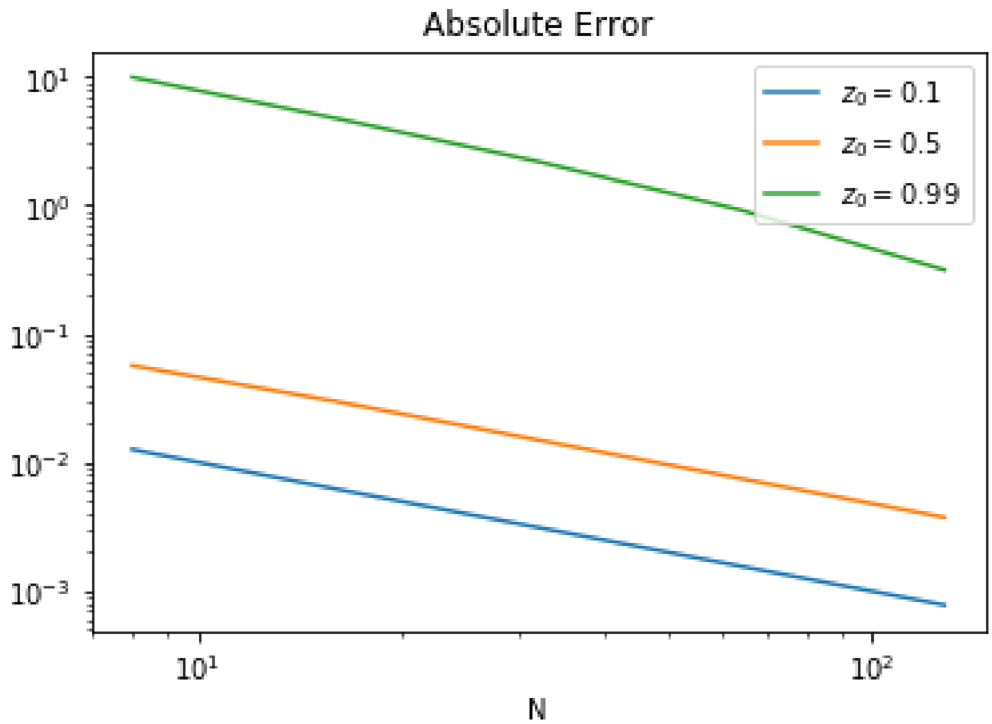
0.3 Part 3 - Convergence Study

[119]:
```
f = lambda z : cmath.sin(z)
z0s = [0.1, 0.5, 0.99]
Ns = [8, 16, 32, 64, 128]

# plot relative error over increasing N for range of z0
fig1 = plt.figure(1)
relative_plot = fig1.add_subplot()
relative_plot.title.set_text("Relative Error")
relative_plot.xaxis.label.set_text("N")
# plot absolute error over increasing N for range of z0
fig2 = plt.figure(2)
absolute_plot = fig2.add_subplot()
absolute_plot.title.set_text("Absolute Error")
absolute_plot.xaxis.label.set_text("N")

for z0 in z0s:

    relative_error = []
    absolute_error = []

    for N in Ns:
```

```
        approximation = complex_trapezoidal(f, N, z0)

        exact = f(z0)
        absolute_error.append(abs(exact - approximation))

        # why are we asked to plot relative error when absolute error is␣
↪available?
        if(N != Ns[0]):
            relative_error.append(abs(approximation - old_approximation))
        old_approximation = approximation

    absolute_plot.loglog(Ns, absolute_error, label=r"$z_0 = $"+str(z0))
    relative_plot.loglog(Ns[1:],relative_error, label=r"$z_0 = $"+str(z0))

relative_plot.legend()
absolute_plot.legend()
```

[119]: <matplotlib.legend.Legend at 0x7f53b3020cd0>



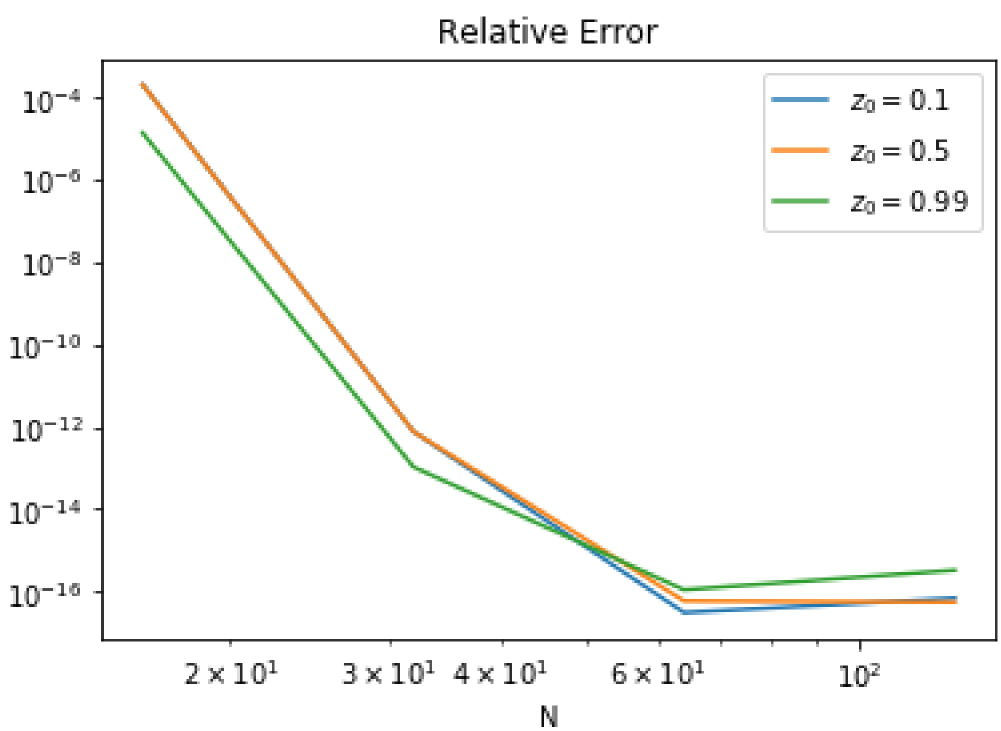


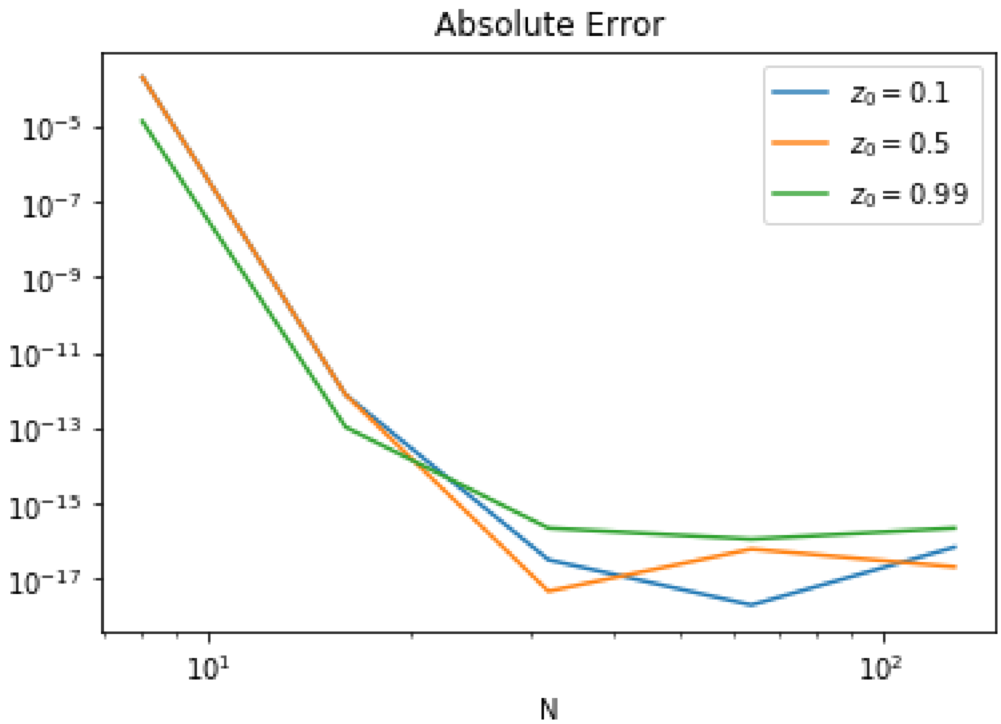
0.4 Part 4 - Residue

[124]:
```
f = lambda z : cmath.sin(z)
g = lambda z : complex(1,0)
z0s = [0.1, 0.5, 0.99]
Ns = [8, 16, 32, 64, 128]

# plot relative error over increasing N for range of z0
fig1 = plt.figure(1)
relative_plot = fig1.add_subplot()
relative_plot.title.set_text("Relative Error")
relative_plot.xaxis.label.set_text("N")
# plot absolute error over increasing N for range of z0
fig2 = plt.figure(2)
absolute_plot = fig2.add_subplot()
absolute_plot.title.set_text("Absolute Error")
absolute_plot.xaxis.label.set_text("N")

for z0 in z0s:

    relative_error = []
    absolute_error = []
```

```
    for N in Ns:
        numerator = complex_trapezoidal(f, N, z0)
        denominator = complex_trapezoidal(g, N, z0)
        approximation = numerator / denominator

        exact = f(z0)
        absolute_error.append(abs(exact - approximation))

        # why are we asked to plot relative error when absolute error is␣
↪available?
        if(N != Ns[0]):
            relative_error.append(abs(approximation - old_approximation))
        old_approximation = approximation

    absolute_plot.loglog(Ns, absolute_error, label=r"$z_0 = $"+str(z0))
    relative_plot.loglog(Ns[1:],relative_error, label=r"$z_0 = $"+str(z0))

relative_plot.legend()
absolute_plot.legend()
```

[124]: <matplotlib.legend.Legend at 0x7f53b378d110>





This time around, the rate of convergence slows rapidly as we add more and more quadrature nodes, and seems to be independent of the chosen z_0.

```
[ ]:
```