9 a) $$\boxed{\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0,\ u(x,0) = u_0(x),\ x \in \mathbb{R},\ t > 0}$$

Show u constant on char. curves by showing $\frac{du}{dt} = 0$

$$\frac{d}{dt}\left[u(x(t),t)\right] = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt}$$

$$= -c(x)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\frac{dx}{dt}$$

$$= \left[\frac{dx}{dt} - c(x)\right]\frac{\partial u}{\partial x}$$

and, on the characteristic curves given, where $\frac{dx}{dt} = c(x(t))$,

$$= \left[c(x(t)) - c(x(t))\right]\frac{\partial u}{\partial x} = 0$$

b) $$\boxed{\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0,\ u(x,0) = u_0(x),\ x \in \mathbb{R},\ t > 0}$$

Find characteristics same way as before

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt}$$

$$= \left(\frac{dx}{dt} - x\right)\frac{\partial u}{\partial x}$$

$$\frac{du}{dt} = 0 \Rightarrow \frac{dx}{dt} = x$$

solving this ODE,

~~$x(t) = \xi e^t$ where ξ is a constant~~

$$x(t) = x(0)e^t$$

d) $\boxed{\dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = 0, \quad u(x,0) = u_0(x) \qquad \text{characteristics} \quad x(t) = x(0)e^t}$

given an arbitrary point (x,t), follow its characteristic curve backwards:

$$x(t) = \tilde{x}e^t \rightarrow \tilde{x} = xe^{-t}$$

and, since u constant on characteristic curve,

$$u(x,t) = u(\tilde{x},0) = u_0(\tilde{x}) = u_0(xe^{-t})$$

e) to verify, first check IC:

$$u(x,0) = u_0(xe^0) = u_0(x) \quad \checkmark$$

then check PDE holds true:

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = u_0'(xe^{-t})(-xe^{-t}) + xu_0'(xe^{-t})(e^{-t})$$

$$= 0 \quad \checkmark$$

200256677_9_Quaife

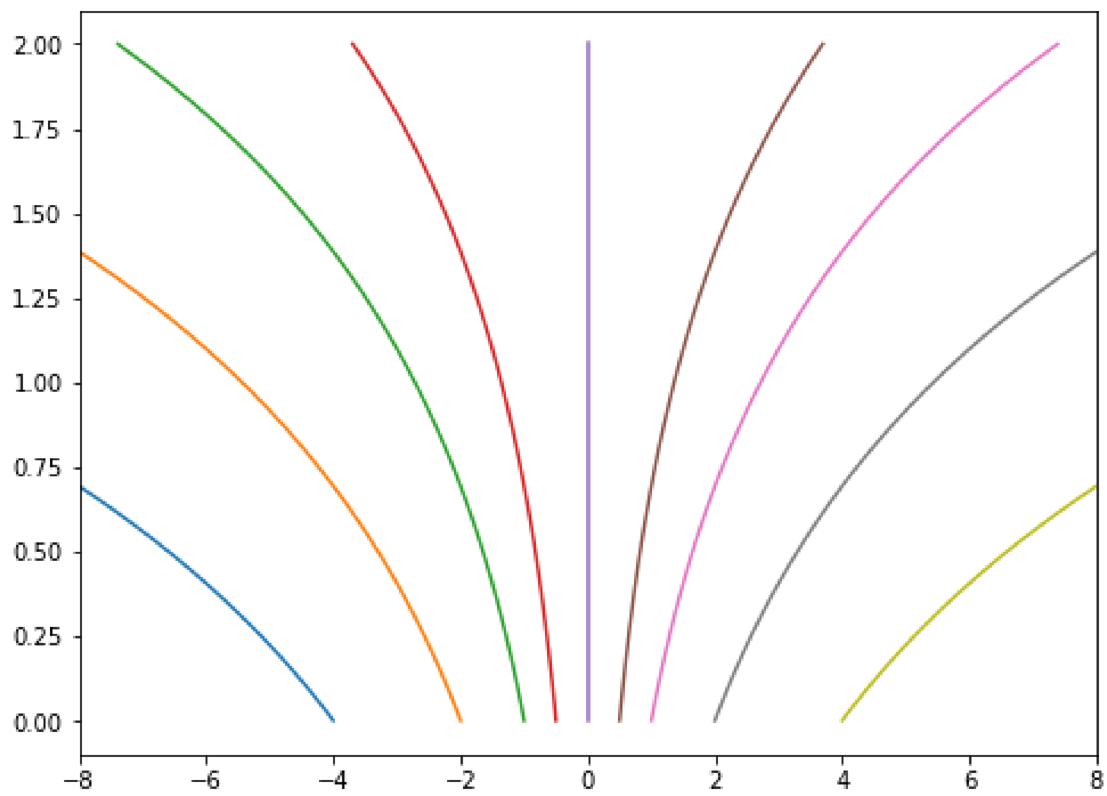
May 29, 2020

[51]:
```python
import numpy as np
import matplotlib.pyplot as plt
plt.rcParams['figure.figsize'] = (8,6)
```

0.1 Part C - Space-Time Diagram

[52]:
```python
t = np.linspace(0,2)
x = [-4, -2, -1, -0.5, 0, 0.5, 1, 2, 4]

for x0 in x:
    plt.plot(x0*np.exp(t), t)

axes = plt.gca()
axes.set_xlim([-8,8])
```

[52]:
```
(-8, 8)
```

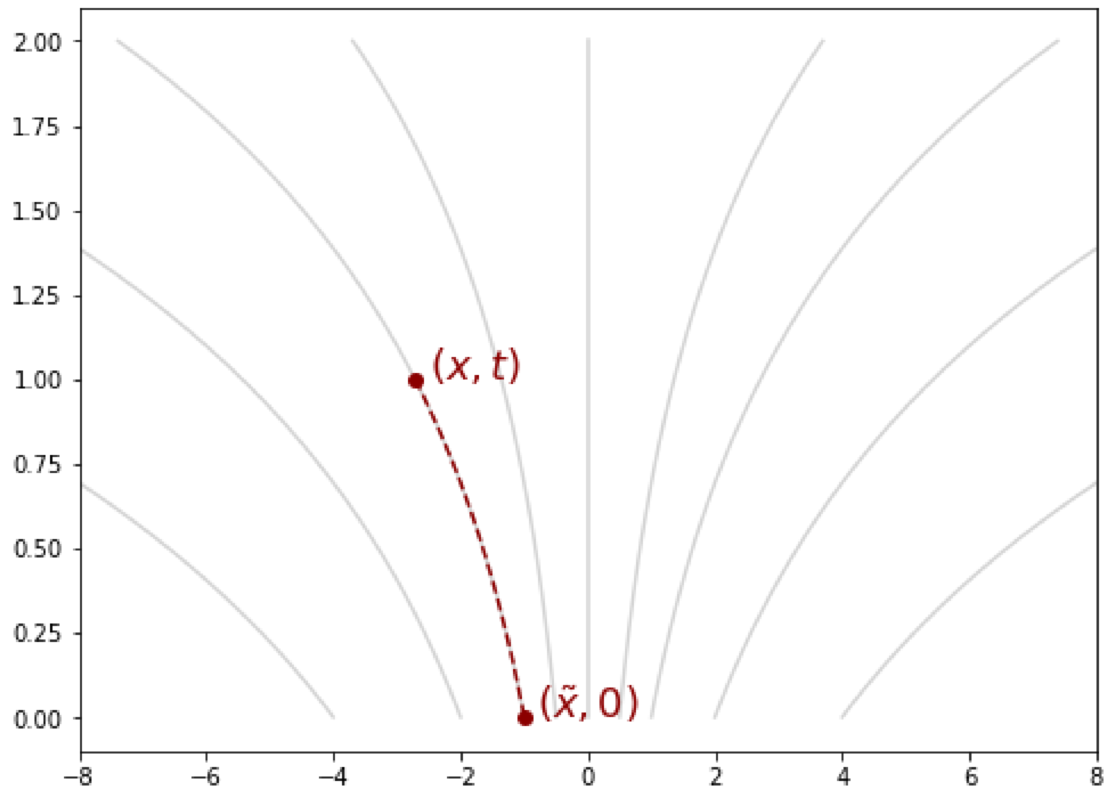


0.2 Part D - Solution

[102]:
```
t = np.linspace(0,2)
x = [-4, -2, -1, -0.5, 0, 0.5, 1, 2, 4]

for x0 in x:
    plt.plot(x0*np.exp(t), t, color="lightgrey")

# (x tilde, 0) correspon ding to (x, t)
plt.plot(-1, 0, 'o', color='darkred')
plt.annotate(r"$(\tilde{x},0)$", (-0.8, 0), color='darkred', size=18,␣
 ↪weight='bold')
# arbitrary (x, t)
plt.plot(-np.exp(1), 1, 'o', color='darkred')
plt.annotate(r"$(x,t)$", (-np.exp(1)+.2, 1), color='darkred', size=18,␣
 ↪weight='bold')
# dashed line
t = np.linspace(0,1)
plt.plot(-np.exp(t), t, '--', color='darkred')

axes = plt.gca()
```

```
axes.set_xlim([-8,8])
```

[102]: (-8, 8)



[ ]:

[ ]: