1) iii

$$\cancel{\frac{\partial \rho v_z}{\partial z}}_{0} = \cancel{\frac{-\partial \rho v_z v_z}{\partial z}}_{0} - \frac{\partial P}{\partial z} - \frac{1}{r}\frac{\partial(\tau_{rz} r)}{\partial r} \qquad eq\ 1$$

Steady state term reduction

$$\frac{\partial P}{\partial z} = -\frac{1}{r}\frac{\partial(\tau_{rz} r)}{\partial r}$$

rearrange :

$$\int \frac{\partial P}{\partial z} r\, \partial r = \int \partial(\tau_{rz} r)$$

integrate:

$$\frac{\partial P}{\partial z}\frac{r^{\cancel{2}}}{2} = \frac{\tau_{rz} r - C_1}{r}$$

divide:

$$\frac{\partial P}{\partial z}\frac{r}{2} = \tau_{rz} - \frac{C_1}{r}$$

Substitute: $\tau = -\mu \frac{dv_z}{dr}$

$$\frac{dP}{dz}\frac{r}{2} = \mu\frac{dv_z}{dr} - \frac{C_1}{r}$$

multiply out dr:

$$\int \frac{dP}{dz}\frac{r}{2}\, dr = \mu\, dv_z - \frac{C_1}{r}\, dr$$

integrate again:

$$\frac{dP}{dz}\frac{r^2}{4} = \mu V_z - C_1 \ln(r) + C_2 \qquad eq\ 2$$

r is actually 2 different radii: R is outer, κR is inner, $V_z|_R = 0$, $V_z|_{\kappa R} = V$

$$\therefore \frac{dP}{dz}\frac{R^2}{4} = \mu \cancelto{0}{V} - C_1 \ln(R) + C_2 \rightarrow \underbrace{C_2}_{} = \frac{dP}{dz}\frac{R^2}{4} + C_1 \ln(R)$$

and

$$\frac{dP}{dz}\frac{\mathcal{R}^2}{4} = \mu V - C_1 \ln(\mathcal{R}) + C_2$$

$$\frac{dP}{dz}\frac{\mathcal{R}^2}{4} = \mu V - C_1 \ln(\mathcal{R}) + \frac{dP}{dz}\frac{R^2}{4} + C_1 \ln(R)$$

$$\frac{dP}{dz}\left(\frac{\mathcal{R}^2 - R^2}{4}\right) = \mu V + C_1\left(\ln(\mathcal{R}) - \ln(R)\right)$$

$$\frac{dP}{dz}\left(\frac{\mathcal{R}^2 - R^2}{4}\right) = \mu V + C_1 \ln\left(\frac{\mathcal{R}}{R}\right) \rightarrow \underline{\frac{dP}{dz}\left(\frac{\mathcal{R}^2 - R^2}{4}\right) - \mu V \Big/ \ln\left(\frac{\mathcal{R}}{R}\right) = C_1}$$

so

$$C_2 = \frac{dP}{dz}\frac{R^2}{4} + \left(\frac{dP}{dz}\left(\frac{\mathcal{R}^2 - R^2}{4}\right) - \mu V \Big/ \ln\left(\frac{\mathcal{R}}{R}\right)\right) \ln(R)$$

making eq2 =

$$\frac{dP}{dz}\frac{r^2}{4} = \mu V_z - \left(\frac{dP}{dz}\left(\frac{\mathcal{R}^2 - R^2}{4}\right) \Big/ \ln\left(\frac{\mathcal{R}}{R}\right)\right) \ln(r) + \frac{dP}{dz}\frac{R^2}{4} + \left(\frac{dP}{dz}\left(\frac{\mathcal{R}^2 - R^2}{4}\right) - \mu v \Big/ \ln\left(\frac{\mathcal{R}}{R}\right)\right) \ln(R)$$

$$\frac{dP}{dz}\frac{r^2}{4} = \mu V_z + \ln\left(\frac{R}{r}\right)\left(\frac{dP}{dz}\left(\frac{\mathcal{R}^2 - R^2}{4}\right) \Big/ \ln\left(\frac{\mathcal{R}}{R}\right)\right) + \frac{dP}{dz}\frac{R^2}{4}$$

$$\frac{dP}{dz}\frac{r^2}{4} - \ln\left(\frac{R}{r}\right)\left(\frac{dP}{dz}\left(\frac{\mathcal{R}^2 - R^2}{4}\right) \Big/ \ln\left(\frac{\mathcal{R}}{R}\right)\right) - \frac{dP}{dz}\frac{R^2}{4} = \mu V_z$$

reorg'd

$$V_z = \frac{dP}{dz}\frac{r^2}{4\mu} - \ln\left(\frac{R}{r}\right)\left(\frac{dP}{dz}\left(\frac{\mathcal{R}^2 - R^2}{4\mu}\right) \Big/ \ln\left(\frac{\mathcal{R}}{R}\right)\right) - \frac{dP}{dz}\frac{R^2}{4\mu}$$

divided by μ

$$V_z = \frac{dP}{dz}\frac{r^2}{4\mu} - \frac{\ln\left(\frac{R}{r}\right)}{\ln\left(\frac{\mathcal{R}}{R}\right)}\left(\frac{dP}{dz}\left(\frac{\mathcal{R}^2 - R^2}{4\mu}\right)\right) - \frac{dP}{dz}\frac{R^2}{4\mu}$$

reorg'd ln terms

$$V_z = \frac{dP}{dz}\left(\frac{r^2 - R^2}{4\mu}\right) - \frac{\ln\left(\frac{R}{r}\right)}{\ln\left(\frac{\mathcal{R}}{R}\right)}\left(\frac{dP}{dz}\left(\frac{\mathcal{R}^2 - R^2}{4\mu}\right)\right)$$

group $\frac{dP}{dz}$ terms

$$\boxed{V_z = -\frac{dP}{dz}\left(\frac{R^2 - r^2}{4\mu}\right) + \frac{\ln\left(\frac{R}{r}\right)}{\ln\left(\frac{\mathcal{R}}{R}\right)}\left(\frac{dP}{dz}\left(\frac{R^2 - \mathcal{R}^2}{4\mu}\right)\right)}$$

flipped $r^2 - R^2$ and distributed the negative into the 2nd term

Knowing v_z, $\tau_{zr} = -\mu \frac{dv}{dr}$ for this case

$$\mu v_z = -\frac{dP}{dz}\left(\frac{R^2 - r^2}{4}\right) + \frac{\ln\left(\frac{R}{r}\right)}{\ln\left(\frac{\kappa R}{R}\right)}\left(\frac{dP}{dz}\left(\frac{R^2 - \kappa R^2}{4}\right)\right)$$

multiplied by μ

$$\tau_{zr} = \frac{d}{dr}\left[-\mu \frac{dP}{dz}\left(\frac{R^2 - r^2}{4}\right) + \frac{\mu \ln\left(\frac{R}{r}\right)}{\ln\left(\frac{\kappa R}{R}\right)}\left(\frac{dP}{dz}\left(\frac{R^2 - \kappa R^2}{4}\right)\right)\right]$$

"Derived"

$$\boxed{\tau_{zr} = -\frac{dP}{dz}\left(\frac{R^2}{4} - \frac{r}{2}\right) + \frac{\mu}{\ln\left(\frac{\kappa R}{R}\right)} \frac{1}{r}\left(\frac{dP}{dz}\left(\frac{R^2 - \kappa R^2}{4}\right)\right)}$$

actually do it...

viii) We're looking for v_θ and $\tau_{r\theta}$ so I'll start with eq 3.7-41:

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r\frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_r v_\theta}{r}} + \cancel{v_z\frac{\partial v_\theta}{\partial z}}\right)$$

$$=$$

$$\cancel{-\frac{1}{r}\frac{\partial P}{\partial \theta}} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}\right] + \cancel{\rho g_\theta}$$

$v_r = 0$, steady state, no outside forces, θ symmetry, doesn't depend on z

so only $0 = \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right)$ matters.

$$\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r} = c_1$$

focusing on the inside for now...

$$\int \partial (r v_\theta) = c_1 \int r\,\partial r$$

rearranging the inner bit for integration

$$r v_\theta = c_1\frac{r^2}{2} + c_2$$

Integrating

There are actually 2 r's, one for the outter radius R_2 and one for the inner R_1. I cannot say much about the conditions, other than ω_2 likely isn't ω_1, so $v_\theta \begin{cases} = \omega_1 R_1 \\ = \omega_2 R_2 \end{cases}$

$$\begin{cases} R_1^2\omega_1 = c_1\frac{R_1^2}{2} + c_2 \\ R_2^2\omega_2 = c_1\frac{R_2^2}{2} + c_2 \end{cases}$$

Applying BCs

$$\begin{cases} c_2 = R_1^2\omega_1 - c_1\frac{R_1^2}{2} \end{cases}$$

isolating c_2

$$\left\{ R_2^2 \omega_2 = C_1 \frac{R_2^2}{2} + R_1^2 \omega_1 - C_1 \frac{R_1^2}{2} \right.$$

Sustituting into R_2 eq

$$R_2^2 \omega_2 = C_1 \left(\frac{R_2^2}{2} - \frac{R_1^2}{2} \right) + R_1^2 \omega_1$$

rearranging

$$C_1 = \frac{R_2^2 \omega_2 - R_1^2 \omega_1}{\left(\frac{R_2^2}{2} - \frac{R_1^2}{2} \right)} = 2 \, \frac{R_2^2 \omega_2 - R_1^2 \omega_1}{R_2^2 - R_1^2}$$

found consts by substitution

$$\therefore C_2 = R_1^2 \omega_1 - \cancel{2} \, \frac{R_2^2 \omega_2 - R_1^2 \omega_1}{R_2^2 - R_1^2} \, \frac{R_1^2}{\cancel{2}}$$

$$r v_\theta = \cancel{2} \, \frac{R_2^2 \omega_2 - R_1^2 \omega_1}{R_2^2 - R_1^2} \, \frac{r^2}{\cancel{2}} + R_1^2 \omega_1 - R_1^2 \frac{R_2^2 \omega_2 - R_1^2 \omega_1}{R_2^2 - R_1^2}$$

Applying C_1 & C_2

$$v_\theta = \frac{R_2^2 \omega_2 - R_1^2 \omega_1}{R_2^2 - R_1^2} r + \frac{R_1^2 \omega_1}{r} - \frac{R_2^2 \omega_2 - R_1^2 \omega_1}{R_2^2 - R_1^2} \cdot \frac{R_1^2}{r}$$

Divide by r

$$v_\theta = \frac{r\left(R_2^2 \omega_2 - R_1^2 \omega_1\right) - \frac{R_1^2}{r}\left(R_2^2 \omega_2 - R_1^2 \omega_1\right)}{R_2^2 - R_1^2} + \frac{R_1^2 \omega_1}{r}$$

rearrange

$$v_\theta = \frac{R_1^2}{R_2^2 - R_1^2} \left[r \left(\frac{R_2^2 \omega_2}{R_1^2} - \omega_1 \right) + \frac{R_2^2 \omega_1 - R_1^2 \omega_1}{r} - \frac{R_2^2 \omega_2 - R_1^2 \omega_1}{r} \right]$$

Plugged into CAS for sanity

$$\boxed{v_\theta = \frac{R_2^2}{R_2^2 - R_1^2}\left[r\left(\omega - \frac{\omega_1 R_1^2}{R_2^2}\right) - \frac{(\omega_2 - \omega_1) R_1^2}{r} \right]}$$

$$\tau_{r\theta}(r) = -\mu\left[r\,\frac{\partial\left(\frac{v_\theta}{r}\right)}{\partial r} + \cancel{\frac{1}{r}\,\frac{\partial v_r}{\partial \theta}} \right]$$

$$\frac{v_\theta}{r} = \frac{R_2^2}{R_2^2 - R_1^2}\left(\underbrace{\frac{R_1^2 \omega_1}{R_2^2} - \omega_2}_{\text{no } r} + \frac{R_1^2(\omega_2 - \omega_1)}{r^2} \right)$$

Divide by r

$$\frac{\partial\left(\frac{v_\theta}{r}\right)}{\partial r} = \frac{R_2^2}{R_2^2 - R_1^2}\left(0 - R_1^2(\omega_2 - \omega_1)\left(\frac{-2}{r^3}\right) \right)$$

Derive

$$\therefore\ \tau_{r\theta}(r) = -\mu\left[\cancel{r}\,\frac{R_2^2}{R_2^2 - R_1^2}\left(R_1^2(\omega_2 - \omega_1)\left(\frac{-2}{r^{\cancel{3}\,2}}\right) \right) \right]$$

$$\boxed{\tau_{r\theta}(r) = \frac{-2\mu R_2^2 R_1^2(\omega_2 - \omega_1)}{R_2^2 - R_1^2} \Big/ r^2}$$

rearranged