$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]+\rho g_\theta$$

Because the flow is swirl in the tube so there should only be v_θ in the system.

Thus we can cross out terms that containing v_r, which means we should cross out: $v_r\frac{\partial v_\theta}{\partial r}$, $+\frac{v_r v_\theta}{r}$, $+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}$

Next, the angular velocity should remain a constant at certain point no matter how long this system is running, so we cross out: $\frac{\partial v_\theta}{\partial t}$.

Then the angular velocity doesn't change with θ since its axial symmetry, so we cross out $\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}$, $\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}$, $-\frac{1}{r}\frac{\partial P}{\partial \theta}$

Then the angular velocity doesn't change with z either, so we cross out: $v_z\frac{\partial v_\theta}{\partial z}$, $\frac{\partial^2 v_\theta}{\partial z^2}$

Last we crossed out ρg_θ since no gravity act on θ direction

$\Downarrow$

which left us with:

$$0=\mu\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r v_\theta}{\partial r}$$

the boundary conditions we are going to use is:

① $r=R_2$, $v_\theta=\omega_2 R_2$

② $r=R_1$, $v_\theta=\omega_1 R_1$

so: $\frac{1}{r}\cdot\frac{\partial(r v_\theta)}{\partial r}=\text{constant}=C_1$

$$\int \partial(r v_\theta) = \int c_1 r\, dr$$

$$r v_\theta = \frac{c_1 r^2}{2} + c_2.$$

with B.C. we get:

$$R_2^2 w_2 = \frac{c_1 R_2^2}{2} + c_2$$

$$R_1^2 w_1 = \frac{c_1 R_1^2}{2} + c_2$$

$$c_2 = R_2^2 w_2 - \frac{c_1 R_2^2}{2}$$

$$R_1^2 w_1 = \frac{c_1 R_1^2}{2} + R_2^2 w_2 - \frac{c_1 R_2^2}{2}$$

$$R_1^2 w_1 - R_2^2 w_2 = \frac{c_1 (R_1^2 - R_2^2)}{2} \Rightarrow c_1 = \frac{2(R_1^2 w_1 - R_2^2 w_2)}{R_1^2 - R_2^2}$$

$$c_2 = R_2^2 w_2 - \frac{(R_1^2 w_1 - R_2^2 w_2) \cdot R_2^2}{R_1^2 - R_2^2}$$

thus:

$$r v_\theta = \frac{\cancel{2}(R_1^2 w_1 - R_2^2 w_2) r^2}{R_1^2 - R_2^2} + R_2^2 w_2 - \frac{(R_1^2 w_1 - R_2^2 w_2) \cdot R_2^2}{R_1^2 - R_2^2}$$

$$v_\theta = \frac{(R_1^2 w_1 - R_2^2 w_2) r}{R_1^2 - R_2^2} + R_2^2 w_2 \cdot \frac{1}{r} - \frac{\frac{R_2^2}{r} R_1^2 w_1 - \frac{R_2^2}{r} R_2^2 w_2}{R_1^2 - R_2^2}$$

$$= \frac{1}{R_1^2 - R_2^2} \left(R_1^2 w_1 r - R_2^2 w_2 r - \frac{R_2^2}{r} R_1^2 w_1 + \frac{R_2^2}{r} R_2^2 w_2 \right) + \frac{R_2^2 w_2}{r}$$

$$= \frac{R_2^2}{R_1^2 - R_2^2}\left(\frac{R_1^2}{R_2^2} w_1 \cdot r - w_2 \cdot r - \frac{1}{r} R_1^2 w_1 + \frac{R_2^2}{r} w_2 \right) + \frac{R_2^2 w_2}{r}$$

$$= \frac{R_2^2}{R_1^2 - R_2^2}\left(\left(\frac{R_1^2 w_1}{R_2^2} - w_2\right) r - \frac{R_1^2 w_1 - R_2^2 w_2}{r} + \frac{R_1^2 w_2 - R_2^2 w_2}{r} \right)$$

$$= \frac{R_2^2}{R_1^2 - R_2^2}\left(\left(\frac{R_1^2 w_1}{R_2^2} - w_2\right) r + \frac{R_1^2 (w_2 - w_1)}{r} \right)$$

$$\tau_{r\theta}(r) = -\mu\left[r \frac{\partial (v_\theta / r)}{\partial r} + \underbrace{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}_{=0} \right], \quad A = \frac{R_2^2}{R_1^2 - R_2^2}$$

$$\frac{v_\theta}{r} = A\left(\frac{R_1^2 w_1}{R_2^2} - w_2 + \frac{R_1^2 (w_2 - w_1)}{r^2} \right)$$

$$\frac{\partial (v_\theta / r)}{\partial r} = A R_1^2 (w_2 - w_1) \frac{\partial}{\partial r} \frac{1}{r^2}$$

$$= A R_1^2 (w_2 - w_1) \left(-2 \cdot \frac{1}{r^3} \right)$$

$$\tau_{r\theta}(r) = -\frac{2\mu A R_1^2 (w_2 - w_1)}{r^2}$$

$$= -\frac{2\mu R_2^2 R_1^2 (w_2 - w_1)}{R_1^2 - R_2^2} \frac{1}{r^2}$$