Exercise 1 - Part 1

prove $$u_z(r) = -\frac{\partial P}{\partial z}\left(\frac{R_2^2 - r^2}{4\mu}\right) + U + \frac{dP}{dz}\left(\frac{R_2^2 - R_1^2}{4\mu}\right)\frac{\ln R_2/r}{\ln \frac{R_2}{R_1}}$$

starting point

the following differential equation.

$$\frac{\partial}{\partial t}(u_z \rho) = -\frac{\partial P}{\partial z} - \frac{\partial(\rho u_z^2)}{\partial z} - \frac{1}{r}\frac{\partial(r \cdot \tau_{rz})}{\partial r} \quad (1)$$

we assume that it is in steady state , $\frac{\partial(u_z \rho)}{\partial t} = 0$

$$\frac{\partial(\rho u_z^2)}{\partial z} = 0$$

u_z stays the same
ρ stays the same

then (1) transforms to :

$$-\frac{\partial P}{\partial z} - \frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r} = 0$$

$$-\frac{\partial P}{\partial z} = \frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r}$$

integration

$$-\int \frac{\partial P}{\partial z} r \partial r = \int \partial(r\tau_{rz})$$

$$-\frac{\partial P}{\partial z}\frac{r^2}{2} = r\tau_{rz} + C_1$$

using Newtons law
$\tau_{rz} = -\mu \frac{du_z}{dr}$

$$\frac{\partial P}{\partial z}\frac{r^2}{2} = \mu \frac{du_z}{\partial r} \cdot r - C_1$$

devide all terms with r
and integrate again

$$\int \frac{\partial P}{\partial z}\frac{r}{2} dr = \int \mu \frac{du_z}{dr} dr - \int \frac{1}{r} C_1 dr$$

$$\frac{\partial P}{\partial z}\frac{r^2}{4} = \mu u_z - C_1 \ln r + C_2 \quad (2)$$

now we have to apply the boundary conditions on (2)

for $r = R_1 \Rightarrow u = U$ (A)
for $r = R_2 \Rightarrow u = 0$ (B)

(2) because of (B) $\Rightarrow$

$$\frac{\partial P}{\partial z}\frac{R_2^2}{4} + C_1 \ln R_2 - C_2 = 0$$

$$C_2 = \frac{\partial P}{\partial z}\frac{R_2^2}{4} + C_1 \ln R_2$$

(2) because of (A)

$$\frac{\partial P}{\partial z}\frac{R_1^2}{4} - \mu U + C_1 \ln R_1 - C_2 = 0 \quad (3)$$

put C_2 to eq. (3)

$$\frac{\partial P}{\partial z}\frac{R_1^2}{4} - \mu U + C_1 \ln R_1 - \frac{\partial P}{\partial z}\frac{R_2^2}{4} - C_1 \ln R_2 = 0$$

$$\frac{\partial P}{\partial z}\left(\frac{R_1^2 - R_2^2}{4}\right) - \mu U + C_1 \ln\frac{R_1}{R_2} = 0$$

$$C_1 = \left[-\mu U + \frac{\partial P}{\partial z}\frac{R_1^2 - R_2^2}{4}\right]\frac{1}{\ln\frac{R_2}{R_1}}$$

$$C_1 = \frac{1}{\ln\frac{R_2}{R_1}}\left[\frac{\partial P}{\partial z}\left(\frac{R_1^2 - R_2^2}{4}\right) - \mu U\right]$$

$$C_2 = \frac{\partial P}{\partial z}\frac{R_2^2}{4} + C_1 \ln R_2$$

use both in eq(2)

$$\underbrace{\frac{\partial P}{\partial z}\frac{r^2}{4}} = \mu u_z - \frac{\ln r}{\ln\frac{R_2}{R_1}}\left[\frac{\partial P}{\partial z}\left(\frac{R_1^2 - R_2^2}{4}\right) - \mu U\right] + \underbrace{\frac{\partial P}{\partial z}\frac{R_2^2}{4}}$$

$$+ \frac{\ln R_2}{\ln\frac{R_2}{R_1}}\underbrace{\left[\frac{\partial P}{\partial z}\left(\frac{R_1^2 - R_2^2}{4}\right) - \mu U\right]}_{\text{1 minus bracket}}$$

$$u_z = -\frac{\partial P}{\partial z}\frac{R_2^2 - r^2}{4\mu} + \left[\frac{\partial P}{\partial z}\left(\frac{R_2^2 - R_1^2}{4\mu}\right) + U\right]\frac{\ln\frac{R_2}{r}}{\ln\frac{R_2}{R_1}}$$ ← 1 minus for changing the

Derive $\tau_{rz}(r) = -\frac{dP}{dz}\frac{r}{2} + \left(U + \frac{dP}{dz}\left(\frac{R_2^2 - R_1^2}{4\mu}\right)\right)\frac{\mu}{\ln\left(\frac{R_2}{R_1}\right)}\frac{1}{r}$.

we know $\tau_{rz} = -\mu \frac{du_z}{dr}$

and we have derived: $u_z(r) = -\frac{dP}{dz}\left(\frac{R_2^2 - r^2}{4\mu}\right) + \left(U + \frac{dP}{dz}\frac{(R_2^2 - R_1^2)}{4\mu}\right)\frac{\ln\left(\frac{R_2}{r}\right)}{\ln\left(\frac{R_2}{R_1}\right)}$

So we have to take the derivative of velocity with respect to the radius r.

first term $\frac{d}{dr}\left(\frac{dP}{dz}\left(\frac{R_2^2 - r^2}{4\mu}\right)\right)$

$$= -\frac{dP}{dz}\left(\frac{d}{dr}\left(\frac{R_2^2}{4\mu} - \frac{r^2}{4\mu}\right)\right) = +\frac{dP}{dz}\frac{r}{2\mu} \quad ①$$

Second term:

$$\frac{d}{dr}\left(U + \frac{dP}{dz}\left(\frac{R_2^2 - R_1^2}{4\mu}\right)\right)\frac{\ln\left(\frac{R_2}{r}\right)}{\ln\left(\frac{R_2}{R_1}\right)} =$$

$$U + \frac{dP}{dz}\left(\frac{R_2^2 - R_1^2}{4\mu}\right)\frac{1}{\ln\left(\frac{R_2}{R_1}\right)}\frac{d}{dr}\ln\left(\frac{R_2}{r}\right)$$

$$U + \frac{dP}{dz}\left(\frac{R_2^2 - R_1^2}{4\mu}\right)\frac{1}{\ln\left(\frac{R_2}{R_1}\right)}\left\{\frac{d}{dr}\left(\ln R_2 - \ln r\right)\right\}$$

$$U + \frac{dP}{dz}\left(\frac{R_2^2 - R_1^2}{4\mu}\right)\frac{1}{\ln\left(\frac{R_2}{R_1}\right)}\left(-\frac{1}{r}\right) \quad ②$$

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combining 1, 2 and multiplying by $-\mu$

$$\tau_{rz}(r) = -\frac{dP}{dz}\left(\frac{r}{2}\right) + \left(U + \frac{dP}{dz}\left(\frac{R_2^2 - R_1^2}{4\mu}\right)\frac{\mu}{\ln\left(\frac{R_2}{R_1}\right)}\right)\frac{1}{r}$$