# Potentials and Fields

## in the context of electromagnetism

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# Repository & references

All the resources can be found on the dedicated repository:
Additional context and information can be found in

- Chapter 23, *Physics for Scientists and Engineers*, Tipler and Mosca

# Goals

- Recall the ideas behind conservative forces and potential energy.
- Introduce the concept of potential in electrostatics.
- Contrast the electric and magnetic cases.

# Conservative forces

Under the action of conservative forces, the **mechanical energy is conserved**.

$$E^i = K^i + U^i = K^f + U^f = E^f. \tag{1}$$

Hence

$$\Delta U = U_f - U_i = -(K_f - K_i) \tag{2}$$

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$$W = K_f - K_i, \tag{3}$$

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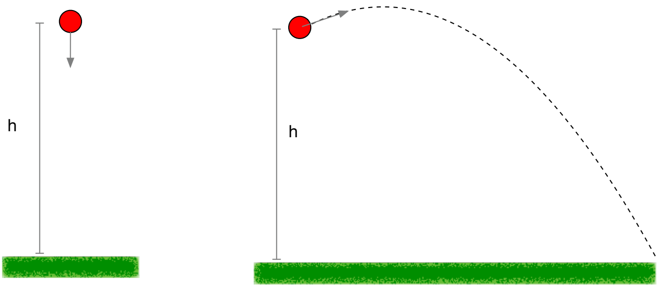
The work done by a conservative force only depends on the **initial** and **final** positions. It is a function of the spatial **configuration**.

# Potential energy and path independence

## Potential Energy Difference

$$\Delta U = U_f - U_i = -\int_i^f \vec{F} \cdot d\vec{\ell} \tag{5}$$




Same initial and final positions, same potential energy difference.

# Potential energy and path independence

For a closed path $\mathsf{C}$, the initial position is also the final position, hence

**Circulation**

$$\mathcal{C} = \oint_{\mathsf{C}} \vec{\mathsf{F}} \cdot \mathrm{d}\vec{\ell} = 0 \tag{6}$$

For **any** closed path, the circulation of a conservative force is **zero**.

# Potential energy and path independence

For a closed path C, the initial position is also the final position, hence

**Circulation**

$$C = \oint_C \vec{F} \cdot d\vec{\ell} = 0 \tag{6}$$

For **any** closed path, the circulation of a conservative force is **zero**.

Conservative forces include:

- the gravitational force
- the elastic force
- the electrostatic force

**Q:** Can you think of some **counter-examples**?

# Electric potential difference

We can define the electrostatic field from the electrostatic force using the test charge $q$:

$$\vec{E} = \frac{\vec{F}_e}{q} \tag{7}$$

For an infinitesimal displacement, the definition of $\Delta U$ (eq.5) implies

$$dU = -\vec{F}_e \cdot d\vec{\ell} = -q\vec{E} \cdot d\vec{\ell}. \tag{8}$$

This suggests defining a new quantity

**Potential Difference**

$$dV = -\vec{E} \cdot d\vec{\ell}. \tag{9}$$

and naturally $dU = q\,dV$.

# Electric potential

Let us consider $dV = -\vec{E} \cdot d\vec{\ell}$ and different possible displacments $d\vec{\ell}$.

- If $d\vec{\ell} \perp \vec{E} \to dV = 0$.
- If $d\vec{\ell} \parallel \vec{E} \to dV = dV_{\mathrm{max}}$

The projection of $\vec{E}$ along the displacement is

$$\vec{E} \cdot d\vec{\ell} = E\cos\theta d\ell \stackrel{\mathrm{def}}{=} E_t d\ell \to E_t = -\frac{dV}{d\ell} \tag{10}$$

Suppose we know $dV$ and want to retrieve $\vec{E}$. We now know that

- we need to move by $d\vec{\ell}$ in the direction of greatest change in $V$
- the magnitude of $\vec{E}$ in that direction is the derivative along $d\vec{\ell}$

These are the defining properties of the **gradient** of $V$.

# Electric potential

## Field as gradient of potential

$$-\mathrm{grad} V(\vec{\mathbf{r}}) = -\vec{\nabla} V(\vec{\mathbf{r}}) = \vec{\mathbf{E}}(\vec{\mathbf{r}}) \tag{11}$$

Naturally

$$\Delta V = V(\vec{\mathbf{r}}) - V(\vec{\mathbf{r}}_0) = \int_{\gamma(\vec{\mathbf{r}}_0, \vec{\mathbf{r}})} \vec{\nabla} V(\vec{\mathbf{r}}) d\vec{\mathbf{r}} = -\int_{\gamma(\vec{\mathbf{r}}_0, \vec{\mathbf{r}})} \vec{\mathbf{E}}(\vec{\mathbf{r}}) d\vec{\mathbf{r}}, \tag{12}$$

on any path $\gamma$ between $\vec{\mathbf{r}}_0$ and $\vec{\mathbf{r}}$.

- $V(\vec{\mathbf{r}})$ is a **scalar** function , $\mathbb{R}^3 \rightarrow \mathbb{R}$
- Only $\Delta V$ are physically significant, i.e. one must define a reference configuration $\vec{\mathbf{r}}_0$ with zero potential $V = 0$.
- The potential is measured in **Volt**, $1V = 1J/1C$

## Example 1: single point charge

The electric field at a distance $r$ from a charge $q$

$$\vec{E} = \frac{kq}{r^2}\hat{r}$$

Taking the integral definition of $V$ and a reference point at $r_0 = +\infty$ with $V(\infty) = 0$ we have

$$V(\vec{r}) - V_\infty = -\int_\infty^{\vec{r}} \vec{E} \cdot d\vec{\ell} = -\int_\infty^{\vec{r}} \frac{kq}{r'^2}\hat{r'} \cdot d\vec{\ell} = -\int_\infty^{r} \frac{kq}{r'^2} dr'$$

**Coulomb potential**

$$V(r) = \frac{kq}{r} \quad \text{(central potential)} \tag{13}$$

The work required to move a test particle $q_0$ at rest at $r = \infty$ to distance $r$ from $q$ is $W = kqq_0/r$.

# Example 2: equipotential lines and field lines

For a system composed of multiple charges, the superposition principle holds for the potential as well:

$$V_{\text{total}} = \sum_{k=1}^{N} V_k \tag{14}$$

The potential is a scalar function that complements the picture provided by the field lines:
https://electric-charges.herokuapp.com/

# Example 3: $\vec{E}$ from V

Let the potential $V(x, y, z)$ be a known function that only depends on $x$: $V(x) = 25\text{V} + 12\frac{\text{V}}{\text{m}^2}x^2$. What is the corresponding electric field?

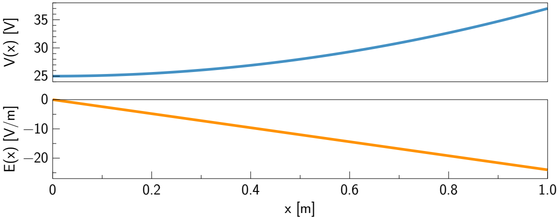
# Example 3: $\vec{E}$ from V

Let the potential $V(x, y, z)$ be a known function that only depends on $x$: $V(x) = 25\text{V} + 12\dfrac{\text{V}}{\text{m}^2}x^2$. What is the corresponding electric field?

We use $-\text{grad}V = \vec{E}$. Component-wise:

$$E_x = -\frac{\partial V}{\partial x} = -24x\frac{\text{V}}{\text{m}^2}, \quad E_y = -\frac{\partial V}{\partial y} = 0, \quad E_z = -\frac{\partial V}{\partial z} = 0$$

So $\vec{E} = -24x\dfrac{\text{V}}{\text{m}^2}\hat{\imath}$, i.e. the field points towards regions of low potential.



# Lorentz force and magnetic field

The Lorentz force depends on both the magnetic field $\vec{\mathbf{B}}$ and the velocity $v$ of a charge $q$

$$\vec{F}_m = q\vec{v} \times \vec{\mathbf{B}}, \tag{15}$$

It is perpendicular to both $\vec{v}$ and $\vec{\mathbf{B}}$ and the $\vec{v}$ dependence makes it **non-conservative**.

The work performed by $\vec{F}_m$ along any path is identically **zero**

$$\int_i^f \vec{F}_m \cdot d\vec{\ell} = q \int_i^f (\vec{v} \times \vec{\mathbf{B}}) \cdot d\vec{v} dt = 0, \tag{16}$$

(the scalar product of orthogonal vectors is zero).

# Magnetic vs electric field

For the electric field, we can write the force on a point charge as $\vec{F}_e = q\vec{E}$.

For the magnetic field $\vec{B}$, the force is not simply proportional to the field, and there are **no magnetic charges**.
The construction of the scalar potential $V$ from $\vec{E}$ **cannot be repeated** for $\vec{B}$.

# Magnetic vs electric field

In fact, the difference between $\vec{\mathsf{E}}$ and $\vec{\mathsf{B}}$ can be mathematically cast in terms of **circulation**

$$\oint_C \vec{\mathsf{E}} \cdot d\vec{\ell} = 0 \tag{17}$$

$$\oint_C \vec{\mathsf{B}} \cdot d\vec{\ell} = \mu_0 I_C \neq 0 \tag{18}$$

$$\tag{19}$$

where $I_C$ is the current through the closed loop $C$.
More on this in the next sessions.

# Take-home messages

1. For conservative forces, the work done between two points in space is path-independent.
2. The electrostatic field $\vec{E}$ is a **conservative vector field** with potential $V$:
   - integral form $V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{\ell}$
   - local form $-\mathrm{grad} V = \vec{E}$
3. There is no scalar potential for the magnetic field $\vec{\mathbf{B}}$.

**Remember:** al this is valid for **static** (= time independent) electric (and magnetic) fields.