Potentials and Fields

in the context of electromagnetism

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Repository & references

The slides and attached resources can be found at
https://github.com/FTurci/potential_and_fields

Additional context and information can be found in

- Chapter 23, *Physics for Scientists and Engineers*, Tipler & Mosca, 6th edition, (Freeman, 2007)
- Chapter 23, *University Physics with Modern Physics*, Young & Freedman, 15th edition, (Pearson, 2019)

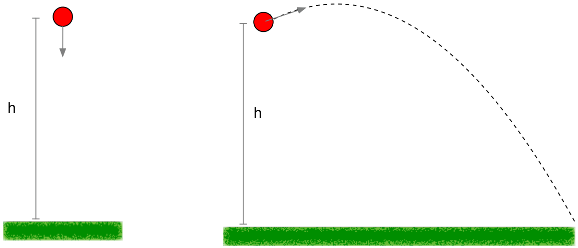
Goals

- Recall the ideas behind conservative forces and potential energy.
- Introduce the concept of potential in electrostatics.
- Contrast the electric and magnetic cases.

Conservative forces and path independence

The **work** W done by a **conservative** force only depends on the **initial** and **final** positions. It defines the **potential energy difference** ΔU:

$$\Delta U = U_f - U_i = -W = -\int_i^f \vec{F} \cdot d\vec{\ell} \tag{1}$$



Example: gravity

$$\Delta U = mgh \tag{2}$$

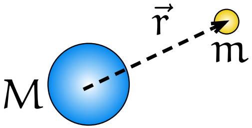
The potential difference does not depend on the paths taken by the particle but only on the initial and final heights.

Gravitational potential energy and potential

Gravity is **conservative**. For masses m and M at distance $\vec{\mathbf{r}}$ the force and the potential energy are

$$\vec{F}_g(\vec{\mathbf{r}}) = -G\frac{mM}{r^2}\hat{\mathbf{r}} \tag{3}$$

$$U(r) = -G\frac{mM}{r} \tag{4}$$



Gravitational potential energy and potential

We can also define the **gravitational vector field** probed by a point particle m at any $\vec{r}$

$$\vec{g}(\vec{r}) = \frac{\vec{F}_g}{m} = -G\frac{M}{r^2}\hat{r} \tag{5}$$

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Path independence means that for any path $\gamma(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$ there exists a **scalar function** ϕ such that

$$\phi(\vec{\mathbf{r}}_2) - \phi(\vec{\mathbf{r}}_1) = \frac{U(\vec{\mathbf{r}}_2)}{m} - \frac{U(\vec{\mathbf{r}}_1)}{m} = -\int_\gamma \vec{\mathbf{g}} \cdot d\vec{\boldsymbol{\ell}} \tag{6}$$

It is the **gravitational potential** ϕ.

Path independence, scalar potential and gradient

In one dimension, path independence is equivalent to

$$\phi(x_2) - \phi(x_1) = -\int_{x_1}^{x_2} g(x)dx \to g(x) = -\frac{d\phi}{dx} \tag{7}$$

In higher dimensions, it generalises to the **gradient theorem**:

$$\phi(\vec{\mathbf{r}}_2) - \phi(\vec{\mathbf{r}}_1) = -\int_{\gamma} \vec{\mathbf{g}} \cdot d\vec{\boldsymbol{\ell}} \to g(\vec{\mathbf{r}}) = -\vec{\boldsymbol{\nabla}}\phi(\vec{\mathbf{r}}) \tag{8}$$

The **vector field $\vec{\mathbf{g}}$** is the **gradient of the potential**.

$$\vec{\boldsymbol{\nabla}}\phi(x, y, z) = \frac{\partial\phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{k}} \tag{9}$$

Conservative fields

Gravity

Force:

$$\vec{\mathbf{F}}_g(\vec{\mathbf{r}}) = -G\frac{Mm}{r^2}\hat{\mathbf{r}}$$

Vector field:

$$\vec{\mathbf{g}}(\vec{\mathbf{r}}) = \frac{\vec{\mathbf{F}}_g(\vec{\mathbf{r}})}{m} = -G\frac{M}{r^2}\hat{\mathbf{r}}$$

Potential (scalar field):

$$\phi(\vec{\mathbf{r}}) = -G\frac{M}{r}$$

Verify that

$$\vec{\mathbf{g}} = -\vec{\boldsymbol{\nabla}}\phi$$

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Electrostatics

Force:

$$\vec{\mathbf{F}}_e(\vec{\mathbf{r}}) = \frac{kqQ}{r^2}\hat{\mathbf{r}}$$

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$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{\vec{\mathbf{F}}_e(\vec{\mathbf{r}})}{q} = \frac{kQ}{r^2}\hat{\mathbf{r}}$$

Conservative fields

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Vector field:

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Verify that

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Vector field:

$$\vec{E}(\vec{r}) = \frac{\vec{F}_e(\vec{r})}{q} = \frac{kQ}{r^2}\hat{r}$$

Potential (scalar field):

$$V(\vec{r}) = \frac{kQ}{r}$$

Again

$$\vec{E} = -\vec{\nabla}V$$

Electric potential

Two relationships between the electric field and potential

$$V(\vec{\mathbf{r}}) - V_0 = -\int_{\gamma} \vec{\mathbf{E}} \cdot d\vec{\ell} \tag{10}$$

$$\vec{\mathbf{E}} = -\vec{\nabla} V \tag{11}$$

- Only ΔV is physically significant, i.e. one must define a reference configuration $\vec{\mathbf{r}}_0$ with zero potential $V_0 = 0$.
- The potential is measured in **Volt**, $1V = 1J/1C$
- In directions **orthogonal** to the field, the potential does not vary → **equipotential contours**.

Coulomb potential

Consider the field emanating from a charge q. Taking the integral definition of V and a reference point at $r_0 = +\infty$ with $V(\infty) = 0$ we have

$$V(\vec{r}) - V_\infty = -\int_\infty^{\vec{r}} \vec{E} \cdot d\vec{\ell} = -\int_\infty^{\vec{r}} \frac{kq}{r'^2}\hat{r'} \cdot d\vec{\ell} \tag{12}$$

$$= -\int_\infty^{r} \frac{kq}{r'^2} dr' = -kq\left(-r^{-1} - 0\right) \tag{13}$$

$$= \frac{kq}{r} \tag{14}$$

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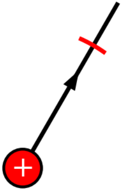
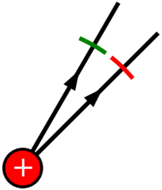
$$V(r) = \frac{kq}{r} \quad \text{(central potential)} \tag{15}$$

Coulomb potential



Coulomb potential



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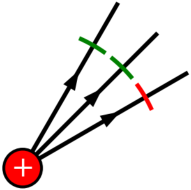

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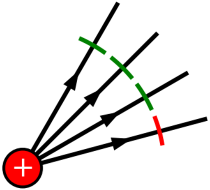

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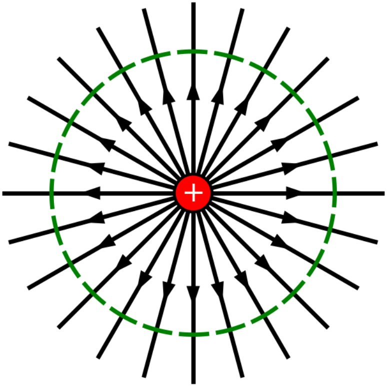

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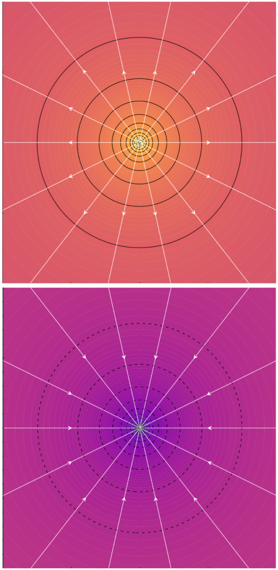

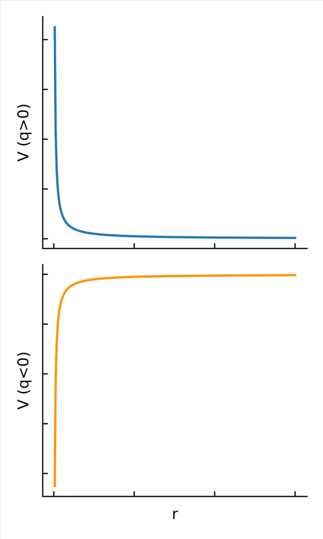

Multiple charges

For a system composed of multiple charges, the superposition principle holds for the potential as well:

$$V_{\mathrm{total}} = \sum_{k=1}^{N} V_k \tag{16}$$

The potential is a scalar function that complements the picture provided by the field lines:
https://electric-charges.herokuapp.com/

Circulation

For a closed path C, path independence implies

$$C = \oint_{\mathsf{C}} \vec{\mathsf{E}} \cdot \mathrm{d}\vec{\ell} = 0 \tag{17}$$

For **any** closed path, the circulation of a conservative field is **zero**.

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Conservative forces/fields include:

- the gravitational force and field
- the electrostatic force and field
- the elastic force

Q: Can you think of some **counter-examples**?

Lorentz force and magnetic field

The Lorentz force depends on both the magnetic field $\vec{B}$ and the velocity $\vec{v}$ of a charge q

$$\vec{F}_m = q\vec{v} \times \vec{B}, \tag{18}$$

It is perpendicular to both $\vec{v}$ and $\vec{B}$ and the $\vec{v}$ dependence makes it **non-conservative**.

The work performed by $\vec{F}_m$ along any path is identically **zero**

$$\int_i^f \vec{F}_m \cdot d\vec{\ell} = q \int_i^f (\vec{v} \times \vec{B}) \cdot d\vec{v} dt = 0, \tag{19}$$

(the scalar product of orthogonal vectors is zero).

Magnetic vs electric field

For the electric field, we can write the force on a point charge as $\vec{F}_e = q\vec{E}$.

For the magnetic field $\vec{B}$, the force is not simply proportional to the field, and there are **no magnetic charges**.

The construction of the scalar potential V from $\vec{E}$ **cannot be repeated** for $\vec{B}$.

Magnetic vs electric field

In fact, the difference between $\vec{E}$ and $\vec{B}$ can be mathematically cast in terms of **circulation**

$$\oint_C \vec{E} \cdot d\vec{\ell} = 0 \tag{20}$$

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I_C \neq 0 \tag{21}$$

$$\tag{22}$$

where I_C is the current through the closed loop C.

More on this in the next sessions.

Take-home messages

1. For conservative forces, the work done between two points in space is path-independent.
2. The electrostatic field $\vec{E}$ is a **conservative vector field** with potential V:
   - integral form $V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{\ell}$
   - local form $-\text{grad}V = \vec{E}$
3. There is no scalar potential for the magnetic field $\vec{B}$.

Remember: this is valid for **static** fields.