Math, Physics, and Engineering Practice Problems

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A collection of math, physics, and engineering problems that I like. A lot of them are from homework, exams, or classroom examples. I do not expect many of these to be solved without referencing notes, a textbook, or the internet.
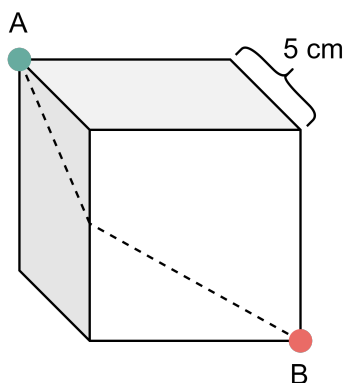
1 Minimum Surface Distance

Figure 1: A cube with points A and B that need to be connected with a line.

Consider the cube in fig. (1) that has a side length of 5 cm. On opposite corners are points A and B, and connecting them is a dashed line. What is the shortest length of the dashed line?

Solution to 1

This problem is secretly a 2D problem made to look like a 3D problem. It does not require any complex calculations. To solve for the length of the dashed line, we can first imagine disassembling the cube into six squares, as shown in fig. (2). The line that has the shortest distance
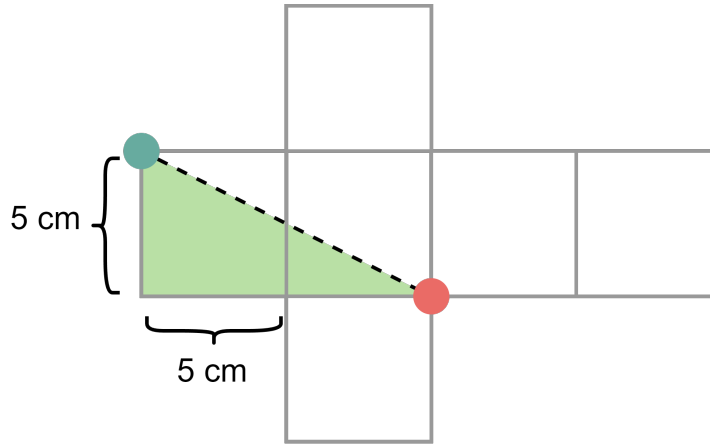
Figure 2: The opened cubed.

is the one that goes straight from point A to point B, which forms the triangle shaded in green.

The triangle has a base of 10 cm and a height of 5 cm. Using the Pythagorean Theorem, we can easily find the length

$$\text{Length} = \sqrt{(5 \text{ cm})^2 + (10 \text{ cm})^2} = \boxed{\sqrt{125} \text{ cm} \approx 11.18 \text{ cm}} \qquad (1.1)$$

2 Pillar Shadows

Two pillars are placed next to a wall in the sun. You know the green pillar is 2 meters tall, and it produces a 3 meter-long shadow. You forgot to measure the height of the red pillar, but you know the lengths of its shadow: the portion extending from the red pillar's base to the bottom of the wall is 6 meters, and the portion from the bottom of the wall to the tip of the shadow is 4 meters. Find the height of the red pillar.

Solution to 2

There are many (simpler) ways to solve for the height of the red pillar, but this is just the way that I came up with when I first solved it. We can begin by considering that the shadows of the red and green pillars are formed by the same light, so they must form similar triangles. We can glean information from the green pillar and use it to solve for the height of the red pillar.
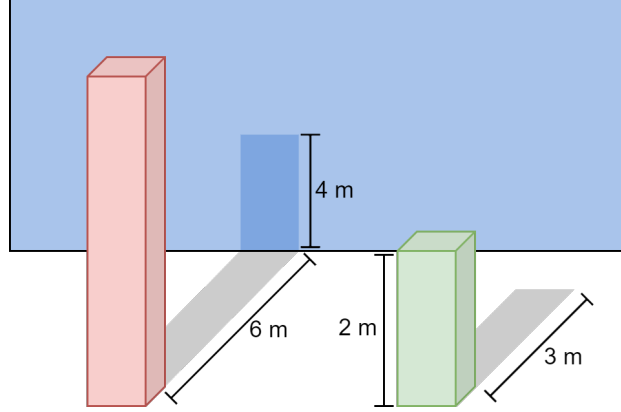
Figure 3: The red and green pillars form shadows.

We know the height of the green pillar is 2 meters and the shadow is 3 meters long. We can find the angle of the triangle θ using these

$$\theta = \arctan\left(\frac{2 \text{ meters}}{3 \text{ meters}}\right) \tag{2.1}$$

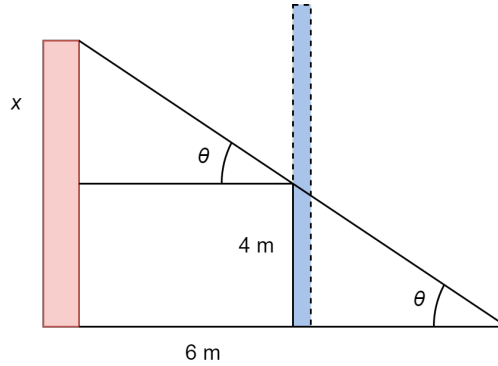We can now examine the red pillar and its shadows

Figure 4: Sideways view of the red pillar.

We notice that if the wall were not there, the shadow would continue to form a triangle with the same angle θ as the green pillar. The wall cuts the shadow such that the tip is 4 meters up the wall; using this, we can form another triangle that starts where the tip of the shadow is. We can solve for the height x of this smaller triangle using tangent

$$x = (6 \text{ meters}) \tan\left[\arctan\left(\frac{2}{3}\right)\right] = \frac{2(6 \text{ meters})}{3} = 4 \text{ meters} \tag{2.2}$$

3

Since we already accounted for the bottom 4 meters, the total height is $\boxed{4 + 4 = 8 \text{ meters}}$.

3   Simple Olympiad

Math Olympiads are competitions where people, usually high school or college students, try to solve tricky math problems, which often require a level of ingenuity. Here is one such, rather *simple*, Math Olympiad question: find the expression for y,

$$\sqrt{x + \sqrt{x + \cdots + \sqrt{x + \sqrt{x}}}} = y \tag{3.1}$$

Solution to 3

As the description of the problem says, the solution will require some ingenuity. The ellipses on the left side indicate that this pattern recurs an "infinite" number of times, before eventually ending ("infinite" is in quotes because something that is infinite cannot end). This recursion is the key to the solution – if we were to remove one of the recursions, the equation will be the same since there are an infinite number of recursions; the same would be true if we were to add another recursion. So, the following is true:

$$\sqrt{x + y} = y \tag{3.2}$$

If y is equal to that long series of recursions, then if we add another recursion on top of y and it's still equal, then we can set the y's equal to each other. Now, solving for y using the Quadratic Formula yields the solution

$$\sqrt{x + y} = y$$
$$x + y = y^2$$
$$y^2 - y - x = 0$$
$$\boxed{y = \frac{1 \pm \sqrt{1 + 4x}}{2}} \tag{3.3}$$

4   Ice Skating

Ice skating has been a fun activity for ages, yet how it actually works has been a topic of debate – people don't really know how the ice skate actually moves across the ice so smoothly. Some people have proposed that, since ice is an expanded version of liquid water in a sense, the skate applies a large enough pressure to the ice and forces it to melt, and this layer of water helps the skate glide across the ice. Using the Clapeyron equation, calculate the weight of a skater required to melt ice that is held at -10°C. Is this plausible? The enthalpy of fusion of water is $\Delta H_{fus} = 6.01$ kJ/mol, the molar volume of water is $\bar{V}_{water} = 1.81 \times 10^{-5}$ m^3/mol, and the molar volume of ice is $\bar{V}_{ice} = 1.96 \times 10^{-5}$ m^3/mol. The average ice skate is 10 inches long and 4 mm wide. Assume the skater is standing on one leg.

Solution to 4

The Clapeyron equation is

$$\frac{dP}{dT} = \frac{\Delta H_{fus}}{T \Delta \bar{V}} \tag{4.1}$$

and integrating yields

$$P_2 = P_1 + \frac{\Delta H_{fus}}{\Delta \bar{V}} \ln\left(\frac{T_2}{T_1}\right) \tag{4.2}$$

We can substitute $P_2 = F_{weight}/A_{skate}$

$$F_{weight} = A_{skate}\left[P_1 + \frac{\Delta H_{fus}}{\Delta \bar{V}} \ln\left(\frac{T_2}{T_1}\right)\right] \tag{4.3}$$

Taking $A_{skate} = 1.016 \times 10^{-3}$ m^2, $P_1 = 1$ atm $= 101,325$ Pa, $\Delta \bar{V} = 1.5 \times 10^{-6}$ m^3/mol, $T_1 = 263$K and $T_2 = 273$K, we can solve for F_{weight}

$$F_{weight} = (1.016 \times 10^{-3} \text{ m}^2)\left[101,325 \text{ Pa} + \frac{6,010 \text{ J/mol}}{1.5 \times 10^{-6} \text{ m}^3/\text{mol}} \ln\left(\frac{273\text{K}}{263\text{K}}\right)\right] \tag{4.4}$$

$$= \boxed{152,015 \text{ N} = 34,174 \text{ lbs}} \tag{4.5}$$

From this calculation, it is not plausible for a person's weight alone to cause the ice to melt under an ice skate. For those wondering, for a

person weighing 180 lbs, the ice temperature would only increase by around 0.05°C.

5 Quadratic Formula

Using the equation for a general quadratic function

$$ax^2 + bx + c = 0 \tag{5.1}$$

derive the famous Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{5.2}$$

Solution to 5

To derive the Quadratic Formula, we need to complete the square. First, we need to divide both sides by a. Then, add to both sides a term equal to half of b/a all squared

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \tag{5.3}$$

Factor the left side of the equation

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \tag{5.4}$$

Solve for x and rearrange for the Quadratic Formula

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \tag{5.5}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \tag{5.6}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \tag{5.7}$$

6   Euler's Identity

Using the Taylor series for $\sin(x)$, $\cos(x)$, and e^x, derive Euler's famous identity

$$e^{i\pi} + 1 = 0 \tag{6.1}$$

Solution to 6

We begin this solution by inspecting the Taylor series for $\sin(x)$ and $\cos(x)$. If forgotten, they can be rederived by using the Taylor series formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{6.2}$$

where it is centered around $a = 0$ (or in other words, a Maclaurin series). The Taylor/Maclaurin series of $\sin(x)$ is

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \tag{6.3}$$

and for $\cos(x)$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \tag{6.4}$$

and finally for e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ... \tag{6.5}$$

From inspection, the Taylor/Maclaurin series for e^x looks to be a combination of $\sin(x)$ and $\cos(x)$. To show that this is indeed true, we can include the imaginary number i into the argument for e^x to get e^{ix}

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - ... \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\right) \\ &= \cos(x) + i\sin(x) \end{aligned} \tag{6.6}$$

We can then let $x = \pi$ to yield the final expression for Euler's identity

$$e^{i\pi} = \cos(\pi) + i\sin(\pi)$$

$$\boxed{e^{i\pi} + 1 = 0} \qquad (6.7)$$

7 Sideways Cylinder

A sideways cylindrical tank has a 10 foot diameter and is 20 feet long. The tank is filled with water up to the 7-foot line. Approximately what percentage of the tank is filled?

Solution to 7

This problem can be solved in multiple ways, but I will show the calculus version. Instead of finding the percentage of the tank volume that is filled, we can use the area of the cross-sections instead. Consider a semi-circle with a 5-foot radius, given by the equation

$$y = \sqrt{25 - x^2} \qquad (7.1)$$

The total area of this semi-circle is $A_{tot} = 25\pi/2 \approx 39.27$ ft^2. The partial area of the semi-circle A_{part}, from $x = -5$ to $x = 2$ to designate the liquid level being up to the 7-foot line, can be found by integrating eq. (7.1)

$$A_{part} = \int_{-5}^{2} \sqrt{25 - x^2}\,dx \qquad (7.2)$$

We can solve this integral using trigonometric substitution, setting $x = 5\sin(\theta)$ and substituting into eq. (7.2) and changing the integration

bounds

$$A_{part} = \int_{\arcsin(-1)}^{\arcsin(2/5)} 5\cos(\theta)\sqrt{25 - 25\sin^2(\theta)}\,d\theta \tag{7.3}$$

$$= 25\int_{\arcsin(-1)}^{\arcsin(2/5)} \cos^2(\theta)\,d\theta \tag{7.4}$$

$$= 25\int_{\arcsin(-1)}^{\arcsin(2/5)} \frac{1+\cos(2\theta)}{2}\,d\theta \tag{7.5}$$

$$= \frac{25}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_{\arcsin(-1)}^{\arcsin(2/5)} \tag{7.6}$$

$$\approx 29.36 \text{ ft}^2 \tag{7.7}$$

The percentage of the tank that is filled is $A_{part}/A_{tot} \approx \boxed{74.76\%}$

8 Logarithmic Triangle

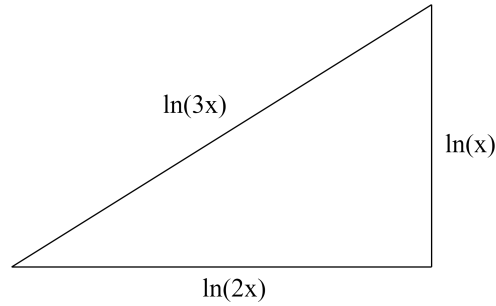Solve for the value of x in the given triangle.

Figure 5: Triangle with logarithmic side lengths.

Solution to 8

To solve for x, we need to set up the side lengths using the Pythagorean Theorem

$$\ln^2(x) + \ln^2(2x) = \ln^2(3x) \tag{8.1}$$

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Using the sum-of-logs property we can expand eq. (8.1)

$$\ln^2(x) + [\ln(2) + \ln(x)]^2 = [\ln(3) + \ln(x)]^2$$

$$\ln^2(x) + 2[\ln(2) - \ln(3)]\ln(x) + [\ln^2(2) - \ln^2(3)] = 0$$

We can now make the substitution $y = \ln(x)$

$$y^2 + 2[\ln(2) - \ln(3)]y + [\ln^2(2) - \ln^2(3)] = 0 \qquad (8.2)$$

and solve for y using the Quadratic Formula

$$y = \frac{2[\ln(3) - \ln(2)] \pm \sqrt{4[\ln(2) - \ln(3)]^2 - 4[\ln^2(2) - \ln^2(3)]}}{2} \qquad (8.3)$$

Backsubstituting $x = e^y$, we have $\boxed{x \approx 0.5837 \text{ and } 3.8549}$
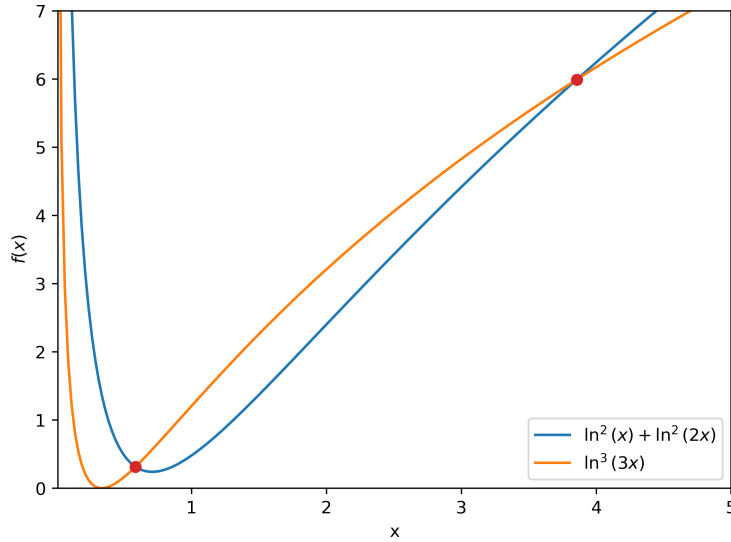

Figure 6: Solutions to the Logarithmic Triangle.

9   Gaussian Integral

The Gaussian Integral is a very famous expression that shows up in numerous fields, including statistics, heat and mass transfer, process safety, and physics. In its most simple form, the integral is

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx \tag{9.1}$$

Evaluate the integral analytically over the given domain.

Solution to 9

The Gaussian Integral is special in that there are no known ways to solve it using an elementary function or any typical integration techniques (such as integration by parts, u-substitution, trigonometric substitution, etc.). We can bypass this by considering the square of the integral, I^2. This can be represented as

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right) \tag{9.2}$$

We can rearrange the integral to yield

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)} dx dy \tag{9.3}$$

and, recognizing that $x^2 + y^2 = r^2$ for polar coordinates, we can convert from Cartesian coordinates to polar coordinates. We must also multiply the integrand by the Jacobian for polar coordinates, r.

$$I^2 = \int_0^{2\pi} \int_0^{\infty} r e^{-r^2} dr d\theta \tag{9.4}$$

The integral is now much easier to solve using u-substitution, where $u = r^2$ and $du = 2r$

$$I^2 = \frac{1}{2} \int_0^{2\pi} \int_0^{\infty} e^{-u} du d\theta \tag{9.5}$$

$$= \frac{1}{2} \left(e^0 - e^{-\infty} \right) \int_0^{2\pi} d\theta \tag{9.6}$$

$$= \frac{2\pi - 0}{2} \tag{9.7}$$

$$= \pi \tag{9.8}$$

Now we can take the square root of the integral for the answer

$$\boxed{I = \sqrt{\pi}} \qquad (9.9)$$

10 Distance Between Earth and the Moon

You pilot a rocket ship to a point between the Earth and the Moon such that the gravitational pull from either body is equal, and you can float there indefinitely. How far from the Earth is this distance? The mass of the Earth is $m_E = 5.97 \times 10^{24}$ kg, the mass of the Moon is $m_M = 7.35 \times 10^{22}$ kg, the distance between the Earth and the Moon is $D = 3.84 \times 10^8$ m, and the gravitational constant is $G = 6.67 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}.

Solution to 10

We can begin this solution by considering Newton's law of gravitation given by the equation

$$F_g = G\frac{m_1 m_2}{d^2} \qquad (10.1)$$

where G is the gravitational constant, m_1 and m_2 are the masses of both bodies, and d is the distance between them. We need to find the value of d that makes the force of gravity exerted on you from the Earth and the Moon are equal. The force of gravity between you and the Earth is

$$F_{g,E} = G\frac{m_E m}{d^2} \qquad (10.2)$$

and the force of gravity between you and the Moon is

$$F_{g,M} = G\frac{m_M m}{(D-d)^2} \qquad (10.3)$$

where D is the distance between the Earth and Moon and m is your

mass. We can equate eq. (10.2) and eq. (10.3) and solve for d

$$\cancel{G}\frac{m_E \cancel{m}}{d^2} = \cancel{G}\frac{m_M \cancel{m}}{(D-d)^2} \tag{10.4}$$

$$(D-d)^2 m_E = d^2 m_m \tag{10.5}$$

$$(D^2 - 2Dd + d^2)m_E = d^2 m_m \tag{10.6}$$

$$(m_E - m_m)d^2 - 2Dm_E d + D^2 m_E = 0 \tag{10.7}$$

We can use the Quadratic Formula to solve for d

$$d = \frac{2Dm_E \pm \sqrt{4D^2 m_E^2 - 4D^2 m_E(m_E - m_m)}}{2(m_E - m_m)} \tag{10.8}$$

Plugging in the values and taking the root less than D gives
$\boxed{d = 3.46 \times 10^8 \text{ meters}}$ from the Earth.

11 Roll of Toilet Paper

Consider a full roll of toilet paper hanging on the wall. Someone begins pulling the toilet paper from the roll at a constant linear velocity $\vec{v} = 0.5 \frac{\text{m}}{\text{s}}$, causing the roll to spin at an angular velocity $\vec{\omega}$. As more toilet paper is pulled from the roll, the radius of the roll begins to decrease until all of the toilet paper is removed, and all that is left is the cardboard tube. Derive an expression for $\vec{\omega}$ as a function of time. What is the roll's angular velocity when half of the toilet paper has been removed? Relevant quantities are listed below. The toilet paper roll experiences no friction and it spins perfectly around its moment of inertia.

Parameter	Value
Paper thickness, l	0.3 mm
Radius of Full Roll, R_o	50 mm
Radius of Cardboard Tube, R_i	6 mm
Width of Roll, w	100 mm

Solution to 11

We begin this solution by considering the equation for angular velocity, $\vec{\omega} = \vec{v}/r$. We know that $\vec{v}$ is constant and we are trying to find the

expression for $\vec{\omega}(t)$, which means that r needs to be changing with time. To find $r(t)$, we consider the volume of just the toilet paper on the roll

$$
\begin{aligned}
V_{paper} &= \pi w \left(R_o^2 - R_i^2 \right) \tag{11.1} \\
&= \pi (100 \text{ mm}) \left[(50 \text{ mm})^2 - (6 \text{ mm})^2 \right] \\
&\approx 7.74 \times 10^5 \text{ mm}^3
\end{aligned}
$$

This is the volume of paper that can be pulled off the roll. The volume of paper pulled off the roll after a time dt is equal to the decrease in volume during that same dt

$$
-\frac{dV_{paper}}{dt} = \frac{dV_{pull}}{dt} \tag{11.2}
$$

The decrease in volume can be found by taking the time derivative of eq. (11.1) after replacing R_o with $r(t)$

$$
\begin{aligned}
\frac{dV_{paper}}{dt} &= \frac{d}{dt} \left[\pi w \left(r(t)^2 - R_i^2 \right) \right] \\
&= 2\pi w r(t) \frac{dr}{dt} \tag{11.3}
\end{aligned}
$$

We can now work on finding the expression for the volume of paper pulled off the roll. First, we represent the volume of an arbitrarily small piece of pulled paper as $dV_{pull} = dx \cdot lw$, where dx is an arbitrarily small length of paper. Taking the time derivative yields

$$
\begin{aligned}
\frac{dV_{pull}}{dt} &= \frac{dx}{dt} lw \\
&= \vec{v} lw \tag{11.4}
\end{aligned}
$$

Equation (11.2) now becomes

$$
-2\pi w r(t) \frac{dr}{dt} = \vec{v} lw \tag{11.5}
$$

Solving for $r(t)$

$$
r(t) = \sqrt{R_i^2 + \frac{\vec{v} l}{\pi} (\tau - t)} \tag{11.6}
$$

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where τ is the time at which all the paper has been pulled off the roll. To find τ, one could integrate eq. (11.4) and set $V_{pull} = V_{paper} = 7.74 \times 10^5$ mm³. Doing this gives $\tau = 51.6$ s. Substituting eq. (11.6) into the angular velocity equation gives us the final expression.

$$\vec{\omega}(t) = \frac{\vec{v}}{\sqrt{R_i^2 + \frac{\vec{v}l}{\pi}(\tau - t)}} \qquad (11.7)$$

To find the angular velocity when half of the paper has been removed, we could again integrate eq. (11.4), but this time setting $V_{pull} = V_{paper}/2 = 3.87 \times 10^5$ mm³. Doing this gives a time of $t = 25.8$ s, and thus an angular velocity of $\boxed{\vec{\omega} = 14.04 \text{ s}^{-1}}$.
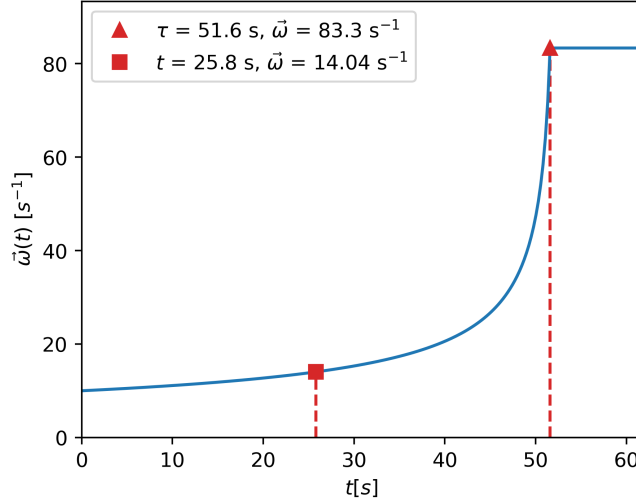

Figure 7: The change in angular velocity for a spinning toilet paper roll.

12 Liquid Level in a Tank

Two 60-meter tall cylindrical tanks with a diameter of 5 meters are filled with water to a level of 40 meters. One of them has a constant-speed pump fitted at the bottom outlet, and the other has a valve with linear resistance fitted at the bottom outlet. Right at time $t = 0$, pipes begin pumping water into each of the tanks at a rate of 30 kg/s, and at the same time the pump is turned on and the valve is opened. If the pump is rated for a flowrate of 50 L/min and the valve has a resistance of 600 m · s/kg, will both tanks overflow? If so,

when? Will only one overflow? If so, which one and when? The density of water can be approximated as $\rho = 1,000 \text{ kg/m}^3$.

Solution to 12

We begin this solution by doing a mass balance on water for each tank, which can be modified to instead calculate the volume of water of the system

$$A\frac{dh}{dt} = F_{in} - F_{out} \tag{12.1}$$

where A is the cross-sectional area of the tank, F_{in} is the volumetric flowrate of water into the tank, and F_{out} is the volumetric flowrate of water out of the tank. Since Tank 1 has a constant-speed pump at the outlet and Tank 2 has a linear-resistance valve at the outlet, the only place where each tank's balance would be different is F_{out}. The volumetric flowrate out of Tank 1 is constant, so F_{out} can just be substituted with the rated flowrate of the pump. We can then integrate eq. (12.1) to get

$$h_1(t) = \frac{F_{in} - F_{out}}{A}t + h_o \tag{12.2}$$

Using the density of water, we can calculate the volumetric flowrate of water added to the tank as $F_{in} = 3 \times 10^{-2} \text{ } \frac{m^3}{s}$. Considering that the volumetric flowrate of water pumped out of the tank is $F_{out} = 50 \text{ } \frac{L}{min} = 8.3 \times 10^{-4} \text{ } \frac{m^3}{s}$, the tank will eventually overflow. The time to overflow can be found by using eq. (12.2), $A = \pi (5 \text{ m})^2 /4 = 19.63 \text{ m}^3$, and $h_o = 40 \text{ m}$, which yields $\boxed{t_{overflow,\,1} = 13,459 \text{ s}}$, or around 3.7 hours.

Tank 2 requires a substitution for F_{out}. Since we have a linear-resistance valve, we can treat the flow of water as a circuit, where the flowrate is equal to a driving force divided by a resistance. In this case, the driving force is the height of the water (which causes a pressure gradient downwards), and the resistance is the valve's resistance, so: $F_{out} = h(t)/R_v$. Equation (12.1) then becomes

$$A\frac{dh_2}{dt} = F_{in} - \frac{h_2(t)}{R_v} \tag{12.3}$$

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One could solve this differential equation several ways. This solution will be using Laplace Transforms, because it is fun. Using deviation variables, the Laplace Transform of eq. (12.3) is

$$h_2'(s) = \frac{K}{\tau s + 1} \tag{12.4}$$

where $K = F_{in}R_v$ and $\tau = AR_v$. Taking the inverse Laplace Transform and substituting back the dynamic variables then gives us

$$h_2(t) = K\left(1 - e^{-t/\tau}\right) + h_o \tag{12.5}$$

We can use the Final Value Theorem on eq. (12.4) to find the water level when $t \to \infty$

$$h_2(t \to \infty) = \lim_{s \to 0}\left(\frac{K}{\tau s + 1}\right) + h_o = 58 \text{ m}$$

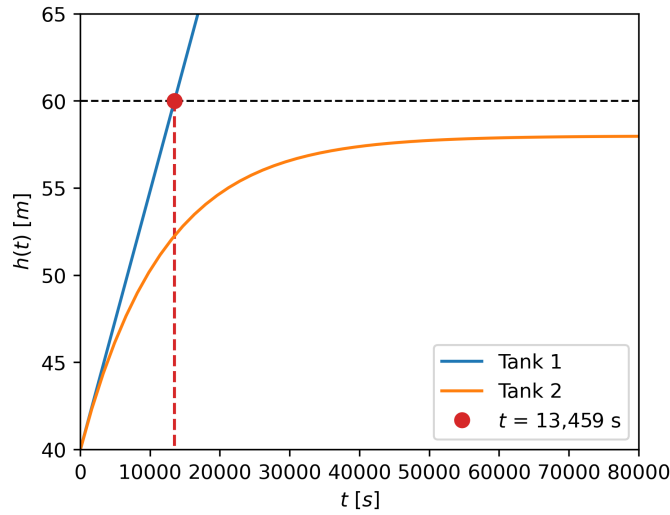From this, we can see that the water level will never reach the top of the tank, and thus Tank 2 will not overflow.

Figure 8: Liquid level within both tanks.

13   Liquid Level Disturbance Attenuation

A chemical plant contains a cylindrical surge tank that acts as a buffer between an upstream pump and a downstream reactor. The reactor carries out a reaction that is highly concentration-sensitive, and requires that the inlet flow rate is 10 ± 1 ft^3/min. One day the pump experiences a sinusoidal electrical surge with a period of 5 minutes, causing the pump to overshoot and undershoot the 10 ft^3/min setpoint by ± 8 ft^3/min. Find the cross-sectional area of the surge tank that attenuates the flow rate disturbance from the pump from ± 8 ft^3/min to the required ± 1 ft^3/min. The surge tank outlet has a linear valve with a resistance $R_v = 1$ min/ft^3.

Figure 9: The surge tank's cross-sectional area needs to be large enough to attenuate flow rate disturbances.

Solution to 13

We can begin the solution by first writing out our knowns. We know that the pump flow rate has a setpoint of 10 ft^3/min but surges to ± 8 ft^3/min every 5 minutes. In functional form, this is

$$F_0(t) = A \sin(\omega t) + F_{sp} \tag{13.1}$$

where $A = 8$ ft^3/min, $\omega = 2\pi/5$ min^{-1}, and $F_{sp} = 10$ ft^3/min. Based on the pump variance and the required reactor inlet variance, the amplitude ratio AR is

$$AR = \frac{|y(t)|_{max}}{|u(t)|_{max}} = \frac{1 \ \cancel{\text{ft}^3/\text{min}}}{8 \ \cancel{\text{ft}^3/\text{min}}} = \frac{1}{8} \tag{13.2}$$

With this, we can now do a mass balance and assume the liquid density is constant

$$\frac{dm}{dt} = \dot{m}_0 - \dot{m} \tag{13.3}$$

$$\frac{d(\rho V)}{dt} = \rho F_0 - \rho F \tag{13.4}$$

$$\frac{dV}{dt} = F_0 - F \tag{13.5}$$

and substituting $V = A_c h$ and $F = h(t)/R_v$, eq. (13.5) now becomes

$$\tau \frac{dh}{dt} + h(t) = K F_0 \tag{13.6}$$

where $\tau = A_c R_v$ and $K = R_v$. With the equation in this form, we can apply deviation variables and easily take the Laplace Transform of both sides to yield

$$\mathcal{L}\left\{\tau \frac{dh'}{dt} + h'(t)\right\} = \mathcal{L}\{K F_0'\} \tag{13.7}$$

$$\tau \left[sh'(s) - \cancel{h'(0)}\right] + h'(s) = K F_0'(s) \tag{13.8}$$

$$h'(s) \left[\tau s + 1\right] = K F_0'(s) \tag{13.9}$$

$$h'(s) = \left[\frac{K}{\tau s + 1}\right] F_0'(s) \tag{13.10}$$

We also need to find the form of $F_0'(s)$ by taking the Laplace Transform of $F_0'(t) = A\sin(\omega t)$

$$h'(s) = \left[\frac{K}{\tau s + 1}\right] \frac{A\omega}{s^2 + \omega^2} \tag{13.11}$$

We can now rearrange eq. (13.11) into a form that we can easily apply the inverse Laplace Transform to yield $h'(t)$; this can be done using partial fraction decomposition

$$\frac{\alpha}{\tau s + 1} + \frac{\beta}{s^2 + \omega^2} + \frac{\gamma s}{s^2 + \omega^2} = \left[\frac{K}{\tau s + 1}\right] \frac{A\omega}{s^2 + \omega^2} \tag{13.12}$$

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Solving this gives

$$\alpha = \frac{KA\omega\tau^2}{1+\tau^2\omega^2}, \qquad \beta = \frac{KA\omega}{1+\tau^2\omega^2}, \qquad \gamma = -\frac{KA\omega\tau}{1+\tau^2\omega^2}$$

and thus $h'(s)$ becomes

$$h'(s) = \frac{KA\omega}{1+\tau^2\omega^2}\left[\frac{1}{s^2+\omega^2} - \frac{\tau}{s^2+\omega^2} + \frac{\tau^2}{s^2+\omega^2}\right] \qquad (13.13)$$

It is now more apparent that we can take the inverse Laplace Transform of $h'(s)$ using transforms that we know

$$\mathcal{L}^{-1}\left\{h'(s)\right\} = \mathcal{L}^{-1}\left\{\frac{KA\omega}{1+\tau^2\omega^2}\left[\frac{1}{s^2+\omega^2} - \frac{\tau}{s^2+\omega^2} + \frac{\tau^2}{s^2+\omega^2}\right]\right\}$$
$$(13.14)$$

$$= \frac{KA\omega}{1+\tau^2\omega^2}\left[\frac{\sin(\omega t)}{\omega} - \tau\cos(\omega t) + \tau\exp\left(-\frac{t}{\tau}\right)\right] \quad (13.15)$$

The long time response, which dictates whether the inlet flow to the reactor will eventually be within the required variance, is

$$h'_l(t) = \frac{KA\omega}{1+\tau^2\omega^2}\left[\frac{\sin(\omega t)}{\omega} - \tau\cos(\omega t)\right] \qquad (13.16)$$

$$= \frac{KA}{\sqrt{1+\tau^2\omega^2}}\sin\left[\omega t - \arctan(\omega\tau)\right] \qquad (13.17)$$

The maximum value of $h'(t)$ is when $\sin\left[\omega t - \arctan(\omega\tau)\right] = 1$, so

$$h'_{l,max} = \frac{KA}{\sqrt{1+\tau^2\omega^2}} \qquad (13.18)$$

Using this, we now need to compare $h'_{l,max}$ and $F'_{0,max}$, also recognizing that the ratio of the two (and dividing $h'_{l,max}$ by R_v to make the units match) is the amplitude ratio AR

$$\frac{h'_{l,max}/R_v}{F'_{0,max}} = \frac{1}{8} = \frac{KA}{R_v(A+8)\sqrt{1+\tau^2\omega^2}} \qquad (13.19)$$

where solving for τ yields

$$\tau = \frac{1}{\omega}\sqrt{\left(\frac{8KA}{R_v(A+8)}\right)^2 - 1} \tag{13.20}$$

and, using the relationship $\tau = A_c R_v$, the cross-sectional area of the surge tank is given by

$$A_c = \frac{1}{R_v\omega}\sqrt{\left(\frac{8KA}{R_v(A+8)}\right)^2 - 1} \tag{13.21}$$

Plugging in all the values gives $\boxed{A_c = 3.08 \text{ ft}^2}$

14 Concentration Profile in a Gas-Filled Tower

A cylindrical tower of height H is filled with species A, a gas. The top of the tower is removed so that A is free to escape. A constant gust of wind flows over the top of the tower, instantly carrying away any particles of A that may have escaped. You want to know the concentration profile of A within the tower after one day. Derive an expression for the concentration profile of species A within the tower as a function of height and time. Relevant quantities are listed below. The gust of wind does not cause any convection within the tower. There is no reaction occurring.

Parameter	Value
Tower Height, H	50 m
Diffusivity of A, D_A	1×10^{-5} $\frac{\text{m}^2}{\text{s}}$
Concentration, C_{A0}	1 mol/m^3

Solution to 14

We begin this solution by considering the rate of diffusion of A up the column. This can be estimated by doing a scaling analysis

$$\tau \gg \frac{H^2}{D_A} = \frac{50 \text{ m}}{1 \times 10^{-5} \text{ m}^2/\text{s}} = 5 \times 10^6 \text{ s} \approx 58 \text{ days} \tag{14.1}$$

Since you are only interested in the concentration profile after one day, this inequality allows us to approximate the tower as being infinitely

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tall and can simplify our solution. We can now set up the governing equation for mass transfer of species A in cylindrical coordinates

$$\frac{\partial C_A}{\partial t} + \vec{v} \cdot \vec{\nabla} C_A = D_A \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial C_A}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 C_A}{\partial \theta^2} + \frac{\partial^2 C_A}{\partial z^2} \right] + R_A \quad (14.2)$$

where C_A is the concentration of A, $\vec{v}$ is the velocity of gas in each coordinate, r is the radius of the tower, and R_A is the reaction rate of A. Since there is no reaction, $R_A = 0$. This leaves us with the equation

$$\frac{\partial C_A}{\partial t} + \vec{v} \cdot \vec{\nabla} C_A = D_A \frac{\partial^2 C_A}{\partial z^2} \quad (14.3)$$

We must now consider the velocity term. Assuming that there is no r- or θ-dependence, we are left with velocity in the z-direction

$$\frac{\partial C_A}{\partial t} + \frac{\partial}{\partial z} \left(v_z^* C_A \right) = D_A \frac{\partial^2 C_A}{\partial z^2} \quad (14.4)$$

where v_z^* is the mole-averaged velocity in the z-direction. The term $v_z^* C_A$ is known as the total molar flux of A in the z-direction, denoted as N_{A_z}. In this scenario, A is carried off by the wind and causes an upward flux; this means that a second gas, species B, takes its place. B has a downward flux that is equal to the negative of A's upward flux ($N_{A_z} = -N_{B_z}$), however it will eventually reach the bottom of the tower and cannot go any further. This means that at the bottom, $N_{B_z} = 0$. Consequently, if $N_{B_z} = 0$ anywhere in the system, then $N_{B_z} = 0$ everywhere in the system to maintain the continuity of total concentration. Using this relation, we can neglect the second term in eq. (14.4), leaving us with the equation

$$\frac{\partial C_A}{\partial t} = D_A \frac{\partial^2 C_A}{\partial z^2} \quad (14.5)$$

This is a second-order PDE that needs one initial condition and two boundary conditions. In our situation, those conditions are: $C_A(t = 0, z) = C_{A0}$; $C_A(t, z = 0) = 0$; and $C_A(t, z \to \infty) = C_{A0}$ (setting the top of the tower as $z = 0$ and the bottom as $z = H$).

We can now employ a similarity variable, $\eta = z/l(t)$. We can transform the C_A differentials to depend on η instead of t and z by using the following expressions

$$\frac{\partial C_A}{\partial z} = \frac{dC_A}{d\eta}\frac{d\eta}{dz} = \frac{1}{l(t)}\frac{\partial C_A}{\partial \eta} \tag{14.6}$$

$$\frac{\partial^2 C_A}{\partial z^2} = \frac{1}{l(t)}\left[\frac{d^2 C_A}{d\eta^2}\frac{\partial \eta}{\partial z}\right] = \frac{1}{l(t)^2}\frac{d^2 C_A}{d\eta^2} \tag{14.7}$$

$$\frac{\partial C_A}{\partial t} = \frac{dC_A}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{\eta}{l(t)}\frac{dl}{dt}\frac{dC_A}{d\eta} \tag{14.8}$$

Take the results from eq. (14.7) and eq.(14.8) and plug them into eq. (14.5) to yield

$$-\eta l\frac{dl}{dt}\frac{dC_A}{d\eta} = D_A\frac{d^2 C_A}{d\eta^2} \tag{14.9}$$

$$\frac{d^2 C_A}{d\eta^2} + \frac{\eta l(t)}{D_A}\frac{dl}{dt}\frac{dC_A}{d\eta} = 0 \tag{14.10}$$

The C_A differentials now depend only on η, however there is still time-dependence with the coefficient of the second C_A differential. To get rid of the time-dependence, we can set the coefficient to a constant β

$$\frac{l(t)}{D_A}\frac{dl}{dt} = \beta \tag{14.11}$$

$$l = \sqrt{2\beta D_A t} \tag{14.12}$$

Our equation now becomes

$$\frac{d^2 C_A}{d\eta^2} + \beta\eta\frac{dC_A}{d\eta} = 0 \tag{14.13}$$

This is now a second-order ODE. We need to change the boundary

conditions and initial condition to fit our system

$$\text{BC1:} \quad C_A(t, z = 0) \quad = 0 \quad \Rightarrow \quad \eta = 0, \quad C_A = 0 \tag{14.14}$$

$$\text{BC2:} \quad C_A(t, z \to \infty) = C_{A0} \quad \Rightarrow \quad \eta \to \infty, C_A = C_{A0} \tag{14.15}$$

$$\text{IC:} \quad C_A(t = 0, z) \quad = C_{A0} \quad \Rightarrow \quad \eta \to \infty, C_A = C_{A0} \tag{14.16}$$

We can see that BC2 and the IC have identical conditions after they are mapped to η; this allows us to collapse both into one condition. We can now begin solving the second-order ODE using integrating factors $\mu(\eta)$

$$\mu(\eta)\frac{d^2 C_A}{d\eta^2} + \beta\eta\mu(\eta)\frac{dC_A}{d\eta} = 0 \tag{14.17}$$

$$\frac{d}{d\eta}\left[\exp\left(\frac{\beta}{2}\eta^2\right)\frac{dC_A}{d\eta}\right] = 0 \tag{14.18}$$

$$\exp\left(\frac{\beta}{2}\eta^2\right)\frac{dC_A}{d\eta} = \text{A} \tag{14.19}$$

$$\frac{dC_A}{d\eta} = \text{A}\exp\left(-\frac{\beta}{2}\eta^2\right) \tag{14.20}$$

We can integrate both sides to obtain an expression for $C_A(\eta)$

$$\int_0^{C_A} dC_A = \text{A}\int_0^\eta \exp\left(-\frac{\beta}{2}\eta^2\right)d\eta \tag{14.21}$$

$$C_A(\eta) = \text{A}\int_0^\eta \exp\left(-\frac{\beta}{2}\eta^2\right)d\eta + \text{B} \tag{14.22}$$

We need to find the values of A and B. From BC1 we get

$$C_A(\eta = 0) = 0 \quad \Rightarrow \quad \text{A}\cancel{\int_0^0 \exp\left(-\frac{\beta}{2}\eta^2\right)d\eta} + \text{B} = 0 \tag{14.23}$$

$$\text{B} = 0$$

and from the collapsed condition we get

$$C_A(\eta \to \infty) = C_{A0} \quad \Rightarrow \quad \text{A}\int_0^\infty \exp\left(-\frac{\beta}{2}\eta^2\right)d\eta = C_{A0} \tag{14.24}$$

$$\text{A} = \frac{C_{A0}}{\int_0^\infty \exp\left(-\frac{\beta}{2}\eta^2\right)d\eta}$$

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Plugging in our values for A and B into eq. (14.22) we get

$$C_A(\eta) = C_{A0} \frac{\int_0^\eta \exp\left(-\frac{\beta}{2}\eta^2\right) d\eta}{\int_0^\infty \exp\left(-\frac{\beta}{2}\eta^2\right) d\eta} \tag{14.25}$$

We can make the substitution $\eta = \sqrt{2/\beta}u \implies u = \eta\sqrt{\beta/2}$ and rewrite eq. (14.25)

$$C_A(\eta) = C_{A0} \frac{\int_0^\eta \exp\left(-u^2\right) du}{\int_0^\infty \exp\left(-u^2\right) du} \tag{14.26}$$

$$= C_{A0} \frac{\text{erf}\left(u\right)}{\text{erf}(\infty) - \text{erf}(0)} \tag{14.27}$$

$$= C_{A0}\text{erf}\left(u\right) \tag{14.28}$$

We can now back-substitute to get from u- to η-dependence

$$C_A(\eta) = C_{A0}\text{erf}\left(\sqrt{\frac{\beta}{2}}\eta\right) \tag{14.29}$$

We can now back-substitute our similarity variable η with its definition, and eq. (14.12)

$$C_A(z,t) = C_{A0}\text{erf}\left(\frac{z\sqrt{\beta/2}}{l(t)}\right) \tag{14.30}$$

$$= C_{A0}\text{erf}\left(\frac{z\sqrt{\beta/2}}{\sqrt{2\beta D_A t}}\right) \tag{14.31}$$

where, after reorganizing, the final solution is

$$C_A(z,t) = C_{A0}\text{erf}\left(\frac{z}{2\sqrt{D_A t}}\right) \tag{14.32}$$

This function yields the following graph

From the figure, it appears that even after 24 hours not much of A has diffused out of the tower. This makes sense because, from eq. (14.1), we estimated the time for diffusion to take place as around 58 days. This solution went more deeply into the derivation of the argument for
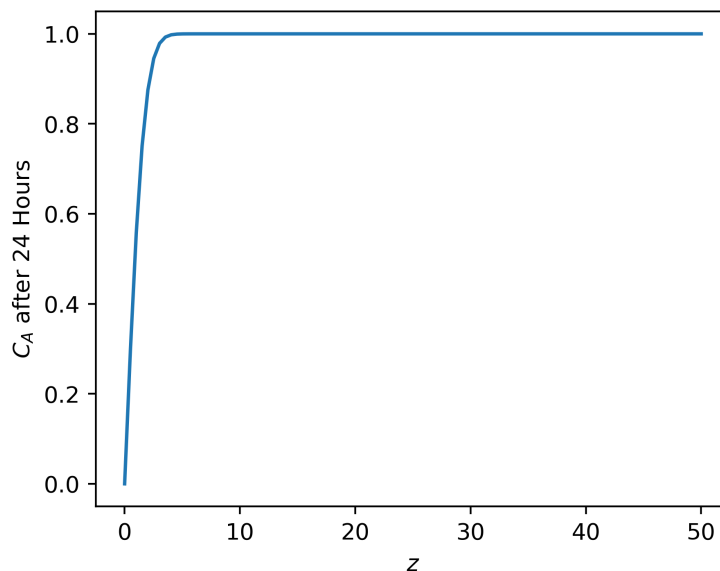
Figure 10: Concentration of A within the tower.

the error function, however many people instead opt to set $\beta = 2$, and therefore

$$\frac{l(t)}{D_A}\frac{dl}{dt} = 2 \tag{14.33}$$

which leads to a straighter path for the solution. Keep in mind that we are able to follow through with this solution because we assumed the tower to be semi-infinite with respect to the diffusion front. If it had clear boundaries (i.e., the tower was 5 meters tall or so), then we would need to solve the diffusion equation as a partial differential equation, as problem 16 does.

15 Fourier Series

A *Fourier Series* is a way of representing a function as an infinite sum of sines and cosines. The Fourier Series $f(x)$ of a function $g(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \sin\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi x}{L}\right) \right] \tag{15.1}$$

where

$$a_0 = \frac{1}{L} \int_{-L}^{L} g(x) dx \tag{15.2}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} g(x) \sin \left(\frac{n\pi x}{L} \right) dx \tag{15.3}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} g(x) \cos \left(\frac{n\pi x}{L} \right) dx \tag{15.4}$$

and where L is a given interval. Find the Fourier Series of the following functions:

$$g(x) = 5 \qquad\qquad x \in [0, 5] \tag{15.5}$$

$$h(x) = \sin(x) \qquad\qquad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \tag{15.6}$$

$$i(x) = 5H(x - 3) \qquad\qquad x \in [-4, 4] \tag{15.7}$$

Solution to 15

The Fourier Series of $g(x)$ can be found by applying eq. (15.1) and eq. (15.2-15.4). a_0 is thus

$$a_0 = \frac{1}{L} \int_{-L}^{L} 5 dx \tag{15.8}$$

$$= \frac{5}{L} [x]_{-L}^{L} \tag{15.9}$$

$$= 10 \tag{15.10}$$

a_n is thus

$$a_n = \frac{1}{L} \int_{-L}^{L} 5 \sin \left(\frac{n\pi x}{L} \right) dx \tag{15.11}$$

$$= -\frac{5}{n\pi} \left[\cos \left(\frac{n\pi x}{L} \right) \right]_{-L}^{L} \tag{15.12}$$

$$= -\frac{5}{n\pi} \left[\cancel{\cos(n\pi) - \cos(-n\pi)} \right] \tag{15.13}$$

$$= 0 \tag{15.14}$$

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and b_n is thus

$$b_n = \frac{1}{L} \int_{-L}^{L} 5 \cos\left(\frac{n\pi x}{L}\right) dx \qquad (15.15)$$

$$= \frac{5}{n\pi} \left[\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} \qquad (15.16)$$

$$= \frac{5}{n\pi} \left[\sin(n\pi) - \sin(-n\pi)\right] \qquad (15.17)$$

$$= \frac{10}{n\pi} \sin(n\pi) \qquad (15.18)$$

$$= 0 \text{ for } n > 0 \qquad (15.19)$$

Substituting a_0, a_n, and b_n into eq. (15.1) yields

$$f(x) = \frac{10}{2} + \sum_{n=1}^{\infty} \left[(0)\sin\left(\frac{n\pi x}{L}\right) + (0)\cos\left(\frac{n\pi x}{L}\right)\right] \qquad (15.20)$$

$$= \boxed{5} \qquad (15.21)$$

The sines and cosines have canceled each other out, and all that is left is the y-intercept of 5. The Fourier Series of $h(x)$ can also be found by going through the general process as before. a_0 is thus

$$a_0 = \frac{1}{L} \int_{-L}^{L} \sin(x) dx \qquad (15.22)$$

$$= -\frac{1}{L} \left[\cos(x)\right]_{-L}^{L} \qquad (15.23)$$

$$= -\frac{1}{L} \left[\cos(L) - \cos(-L)\right] \qquad (15.24)$$

$$= 0 \qquad (15.25)$$

a_n is thus

$$a_n = \frac{1}{L} \int_{-L}^{L} \sin(x) \sin\left(\frac{n\pi x}{L}\right) dx \tag{15.26}$$

$$= \frac{1}{2L} \int_{-L}^{L} \cos\left[x\left(1 - \frac{n\pi}{L}\right)\right] - \cos\left[x\left(1 + \frac{n\pi}{L}\right)\right] dx \tag{15.27}$$

$$= \frac{1}{2} \left\{ \frac{1}{L - n\pi} \sin\left[x\left(1 - \frac{n\pi}{L}\right)\right] - \frac{1}{L + n\pi} \sin\left[x\left(1 + \frac{n\pi}{L}\right)\right] \right\}_{-L}^{L} \tag{15.28}$$

$$= \frac{n\pi}{(L - n\pi)(L + n\pi)} \left[\sin(L - n\pi) + \sin(L + n\pi)\right] \tag{15.29}$$

and b_n is thus

$$b_n = \frac{1}{L} \int_{-L}^{L} \sin(x) \cos\left(\frac{n\pi x}{L}\right) dx \tag{15.30}$$

$$= \frac{1}{2L} \int_{-L}^{L} \sin\left[x\left(1 - \frac{n\pi}{L}\right)\right] + \sin\left[x\left(1 + \frac{n\pi}{L}\right)\right] dx \tag{15.31}$$

$$= \frac{1}{2} \left\{ \frac{1}{L - n\pi} \cos\left[x\left(1 - \frac{n\pi}{L}\right)\right] + \frac{1}{L + n\pi} \cos\left[x\left(1 + \frac{n\pi}{L}\right)\right] \right\}_{-L}^{L} \tag{15.32}$$

$$= \frac{n\pi}{(L - n\pi)(L + n\pi)} \left[\cos(L - n\pi) - \cos(L + n\pi)\right] \tag{15.33}$$

Substituting a_0, a_n, and b_n into eq. (15.1) yields (too large to box in LaTeX)

$$f(x) = \sum_{n=1}^{\infty} \frac{n\pi}{(L - n\pi)(L + n\pi)} \left\{ \left[\sin(L - n\pi) + \sin(L + n\pi)\right] \sin\left(\frac{n\pi x}{L}\right) \right.$$
$$\left. + \left[\cos(L - n\pi) - \cos(L + n\pi)\right] \cos\left(\frac{n\pi x}{L}\right) \right\} \tag{15.34}$$

The Fourier Series of $i(x)$ can again be found by going through the general process. Remember that $H(x)$ is the Heaviside step function, where

$$H(x - a) = \begin{cases} 0 & \text{for } x < a \\ 1 & \text{for } x > a \end{cases} \tag{15.35}$$
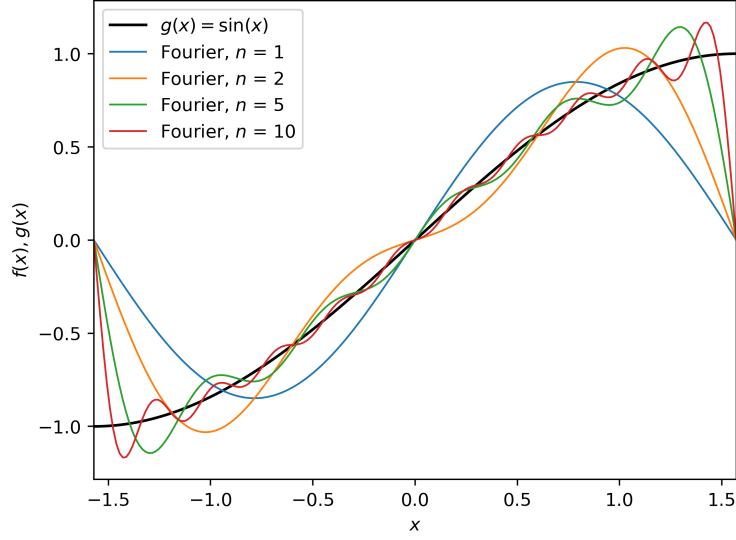
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Figure 11: Fourier Series representations of $h(x) = \sin(x)$ with increasing n.

and its integral is

$$\int_{-\infty}^{x} H(\hat{x} - a)d\hat{x} = (x - a)H(x - a) \tag{15.36}$$

Realizing that $H(x < 3) = 0$ and $H(x > 3) = 1$, a_0 becomes

$$a_0 = \frac{1}{L} \int_{-L}^{L} 5H(x - 3)dx \tag{15.37}$$

$$= \frac{5}{L} \left[(x - 3)H(x - 3) \right]_{-L}^{L} \tag{15.38}$$

$$= \frac{5}{L} \left[(L - 3)H(L - 3) - (-L - 3)\cancel{H(-L-3)} \right] \tag{15.39}$$

$$= \frac{5}{L} (L - 3)H(L - 3) \tag{15.40}$$

$$= \frac{5}{4} \tag{15.41}$$

And realizing that $H(x - 3)$ "limits" our integration bounds to $[3, L]$,

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a_n then becomes

$$a_n = \frac{1}{L} \int_{-L}^{L} 5H(x-3) \sin\left(\frac{n\pi x}{L}\right) dx \qquad (15.42)$$

$$= \frac{5}{L} \int_{3}^{L} \sin\left(\frac{n\pi x}{L}\right) dx \qquad (15.43)$$

$$= -\frac{5}{n\pi} \left[\cos\left(\frac{n\pi x}{L}\right)\right]_{3}^{L} \qquad (15.44)$$

$$= -\frac{5}{n\pi} \left[\cos(n\pi) - \cos\left(\frac{3n\pi}{L}\right)\right] \qquad (15.45)$$

$$= -\frac{5}{n\pi} \left[(-1)^n - \cos\left(\frac{3n\pi}{L}\right)\right] \qquad (15.46)$$

and b_n thus becomes

$$b_n = \frac{1}{L} \int_{-L}^{L} 5H(x-3) \cos\left(\frac{n\pi x}{L}\right) dx \qquad (15.47)$$

$$= \frac{5}{L} \int_{3}^{L} \cos\left(\frac{n\pi x}{L}\right) dx \qquad (15.48)$$

$$= \frac{5}{n\pi} \left[\sin\left(\frac{n\pi x}{L}\right)\right]_{3}^{L} \qquad (15.49)$$

$$= \frac{5}{n\pi} \left[\sin(n\pi) - \sin\left(\frac{3n\pi}{L}\right)\right] \qquad (15.50)$$

$$= -\frac{5}{n\pi} \sin\left(\frac{3n\pi}{L}\right) \qquad (15.51)$$

Substituting a_0, a_n, and b_n into eq. (15.1) yields

$$f(x) = \frac{5}{8} - \sum_{n=1}^{\infty} \frac{5}{n\pi} \left\{ \left[(-1)^n - \cos\left(\frac{3n\pi}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right) \right. \qquad (15.52)$$

$$\left. + \sin\left(\frac{3n\pi}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \right\} \qquad (15.53)$$

16 Temperature Distribution in a Finite Rod

A rod of length L, thermal conductivity K, and initial temperature T_0 is placed in an insulating bath that does not transfer any heat to or from the rod. At time $t = 0$, the rod comes into perfect contact with a surface that holds a constant temperature of T_1 at $x = 0$.
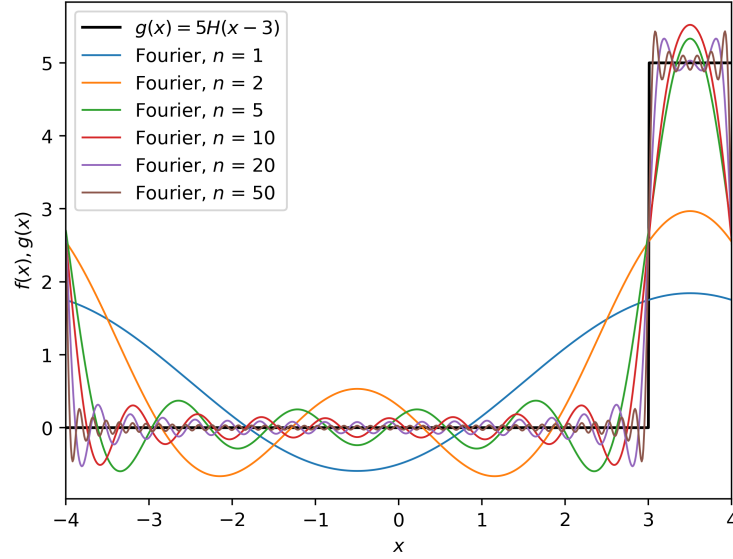
Figure 12: Fourier Series representations of $i(x) = 5H(x-3)$ with increasing n.

The other side of the rod, at $x = L$, is perfectly insulated. Derive an expression (from the heat equation) for the temperature distribution within the rod as a function of length and time. The rod's diameter is very small compared to its length.

Solution to 16

We begin this solution by constructing the proper heat equation. We are given that the diameter of the rod is very small compared to its length, allowing us to make the thin-fin assumption (no variation in the y- and z-directions in Cartesian coordinates, or r- and θ-directions in cylindrical coordinates). We also know that all sides except one are insulated, so there is no heat transfer occurring anywhere except for the surface that is in contact with the heater. This gives us the heat equation

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \tag{16.1}$$

where u is the temperature and K is the thermal conductivity. We now need an initial condition and two boundary conditions. The rod is initially at $u(0, x) = T_0$, which is our initial condition. When the rod comes into contact with the heater, the rod is held at T_1: $u(t, 0) = T_1$. The opposite side of the rod is insulated, so there is no heat transfer:

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$\frac{\partial u}{\partial t}\big|_{x=L} = 0$. Those are our boundary conditions.

We can make our boundary conditions equal to zero to make calculations easier by making a change of variables, $V(t, x) = u(t, x) - T_1$. This gives us a new heat equation

$$\frac{\partial V}{\partial t} = K\frac{\partial^2 V}{\partial x^2} \tag{16.2}$$

The boundary conditions now become $V(t, 0) = 0$ and $\frac{\partial V}{\partial t}\big|_{x=L} = 0$, and the initial condition becomes $V(0, x) = T_0 - T_1$.

We can make a tentative solution for $V(t, x)$ – considering that we know that it is a function of length and time, we can postulate that the actual expression is some product of a length-dependent function and a time-dependent function

$$V(t, x) = \tau(t)\chi(x) \tag{16.3}$$

We can substitute eq. (16.3) into eq. (16.1) which yields

$$\frac{\tau'}{K\tau} = \frac{\chi''}{\chi} \tag{16.4}$$

Since the left side and right side are equal yet depend on two different independent variables, they must only be equal at some eigenvalue $-\lambda$ (the negative sign will allow the solution to converge)

$$\frac{\tau'}{K\tau} = \frac{\chi''}{\chi} = -\lambda \tag{16.5}$$

We can break apart eq. (16.5) into a time part and a length part. The time part can be solved trivially

$$\frac{\tau'}{K\tau} = -\lambda \quad \Rightarrow \quad \tau(t) = \tau_0 e^{-K\lambda t} \tag{16.6}$$

The length part can be solved by using the boundary conditions, and can later help us find the value of λ. We can reorganize the expression

$$\chi'' = -\lambda\chi \tag{16.7}$$

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which has the solution

$$\chi(x) = A\sin(\alpha x) + B\cos(\alpha x) \qquad (16.8)$$

Since we know that the boundary condition at $x = 0$ is zero, we can conclude that we have the sine portion of eq. (16.8). We also know that there is no heat transfer at the other boundary at $x = L$ – since the time part of the solution has no effect on the heat transfer at the boundary, we can substitute the expression $\chi(x) = A\sin(x)$ into it with no consequences

$$\frac{d\chi}{dx} = \frac{d}{dx}\left[A\sin(\alpha x)\right] = -A\cos(\alpha x) = 0 \qquad (16.9)$$

Using eq. (16.9), we can deduce that the argument in the cosine needs to be zero when $x = L$. This yields the expression

$$\frac{d\chi}{dx} = \cos\left(\frac{n\pi x}{2L}\right) = 0 \qquad (16.10)$$

where n is an odd integer that denotes how many times the sine wave oscillates between 0 and L. From this, we see that $\alpha = \frac{n\pi}{2L}$. We can apply this to eq. (16.7) to find λ

$$\chi'' = \frac{d^2\chi}{dx^2} = -\frac{n^2\pi^2}{4L^2}\sin\left(\frac{n\pi x}{2L}\right) \qquad (16.11)$$

and by analogy, we can see that $\lambda = \alpha^2 = \frac{n^2\pi^2}{4L^2}$. This gives us the final expressions for $\tau(t)$ and $\chi(x)$

$$\tau(t) = \tau_0\exp\left(-K\frac{n^2\pi^2}{4L^2}t\right) \quad \text{and} \quad \chi(x) = \sin\left(\frac{n\pi x}{2L}\right) \qquad (16.12)$$

The tentative solution for $V(t, x)$ in eq. (16.3) now becomes

$$V(t, x) = \tau_0\exp\left(-K\frac{n^2\pi^2}{4L^2}t\right)\sin\left(\frac{n\pi x}{2L}\right) \qquad (16.13)$$

which is true for all odd integer values of n. The heat equation is a linear partial differential equation, meaning that superposition (the

addition of several solutions creates another solution) is valid. We can thus sum over all possible values of n for completeness

$$V(t,x) = \sum_{n=1,3,5,\ldots}^{\infty} \left[\tau_0 \exp\left(-K\frac{n^2\pi^2}{4L^2}t\right) \sin\left(\frac{n\pi x}{2L}\right) \right] \qquad (16.14)$$

We now need to find the value of τ_0, which needs to be true for all odd integer values of n. By evaluating the infinite sum at $t = 0$, we can set it equal to the initial condition

$$V(0,x) = T_0 - T_1 = \sum_{n=1,3,5,\ldots}^{\infty} \tau_0 \sin\left(\frac{n\pi x}{2L}\right) \qquad (16.15)$$

To extract τ_0, we can multiply both sides by $\sin\left(\frac{m\pi x}{2L}\right)$ and integrate from $x = 0$ to $x = L$. This takes advantage of the orthogonality of sine and allows us to ignore all summed terms where $n \neq m$

$$\int_0^L (T_0 - T_1) \sin\left(\frac{n\pi x}{2L}\right) dx = \int_0^L \tau_0 \sin^2\left(\frac{n\pi x}{2L}\right) dx \qquad (16.16)$$

The value of $\int_0^L \sin^2\left(\frac{n\pi x}{2L}\right) dx$ is $L/2$ for all odd values of n, so the expression for τ_0 is

$$\tau_0 = \frac{2}{L} \int_0^L (T_0 - T_1) \sin\left(\frac{n\pi x}{2L}\right) dx \qquad (16.17)$$

Plugging this expression for τ_0 into eq. (16.14) yields the expression

$$V(t,x) = \sum_{n=1,3,5,\ldots}^{\infty} \left[\frac{2}{L}(T_0 - T_1) \exp\left(-K\frac{n^2\pi^2}{4L^2}t\right) \sin\left(\frac{n\pi x}{2L}\right) \int_0^L \sin\left(\frac{n\pi \hat{x}}{2L}\right) d\hat{x} \right]$$
$$(16.18)$$

Back-substituting $u(t,x) = V(t,x) + T_1$ yields the final solution

$$\boxed{u(t,x) = \sum_{n=1,3,5,\ldots}^{\infty} \left[\frac{2}{L}(T_0 - T_1) \exp\left(-K\frac{n^2\pi^2}{4L^2}t\right) \sin\left(\frac{n\pi x}{2L}\right) \int_0^L \sin\left(\frac{n\pi \hat{x}}{2L}\right) d\hat{x} \right] + T_1}$$
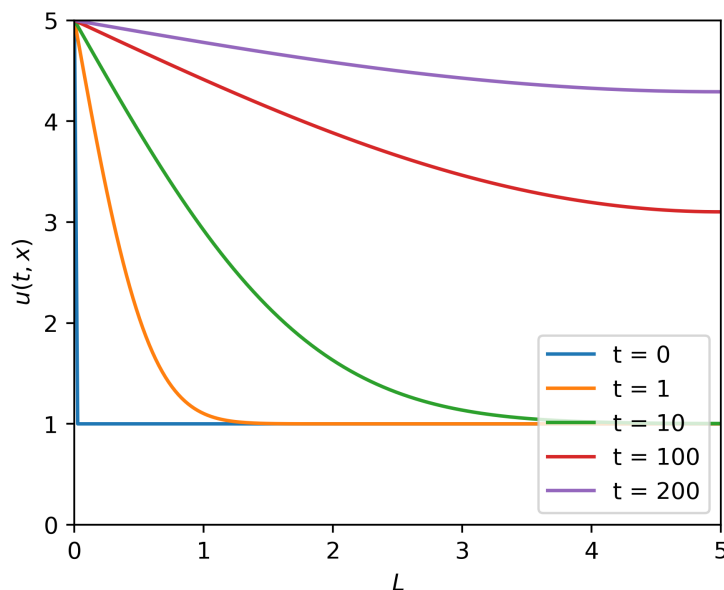$$(16.19)$$

Figure 13: Temperature distribution within the rod at different times.

In the figure it can be seen that the temperature at $x = 0$ is constant at T_1 (in this case I arbitrarily set $T_1 = 5$) and at $x = L = 5$ (arbitrary), the slope is zero, indicating there is no heat transfer out. For the figure I calculated the sum to 200 terms.

One can think about this solution as an exponentially-decaying Fourier series of the difference between two temperatures. The shape is attributed to the linear combination of sine waves (which behave according to the boundary conditions), and does not blow up to infinity with infinite terms because the decaying exponential normalizes their contributions to the sum. The greater the number of terms (large values of n), the smaller the exponential becomes, which allows for the fine-tuning of the solution. If there are few terms (small values of n), then the oscillations of the sine waves become more apparent because it hasn't been fine-tuned. Keep in mind, the sum with only a few terms is still a solution to the heat equation, however it is not physically observed.

17 Temperature Distribution in a Cylindrical Can

A can of soda of height H, radius R, and initial temperature T_h is placed in an ice bath with a constant temperature T_c, such that the can is standing on one face. The ice bath's level

is H, leaving the top of the can exposed to air (constant temperature T_a and heat transfer coefficient h). The surface the can is placed on is a perfect insulator. Assuming no forced convection and that the outer surface of the can is held at a constant temperature T_c, derive an expression (from the heat equation) for the temperature distribution within the rod as a function of height, radius, and time. The can's diameter is not much smaller than its height.

Solution to 17

We begin this solution by constructing the proper heat equation. Because we are dealing with a cylindrical can, we need to use cylindrical coordinates. The full heat equation for this situation is

$$\frac{\partial u}{\partial t} = K \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \right] \tag{17.1}$$

We know that the top of the can (which we will define as $z = 0$) is subject to natural convective cooling (Newton's law of cooling) and the bottom is perfectly insulated, so the boundary conditions in the z-direction are

$$u(t, r, \theta, 0) = T_c + \frac{k}{h} \frac{\partial u(t, r, \theta, 0)}{\partial z} \quad \text{and} \quad \frac{\partial u(t, r, \theta, H)}{\partial z} = 0 \tag{17.2}$$

The outside of the can is held at a constant T_c, and the inside is some finite value (we cannot keep the inside of the can a specific temperature because it will eventually change with time). Assuming that the can has constant properties, the boundary conditions in the r-direction are:

$$u(t, 0, \theta, z) = \text{finite} \quad \text{and} \quad u(t, R, \theta, z) = T_c \tag{17.3}$$

The value of $u(t, r, \theta, z)$ is periodic in θ, which acts as our boundary condition for θ). The initial condition is $u(0, r, \theta, z) = T_h$.

Due to the complexity of this problem, we need to use eigenfunction expansion. The main idea behind eigenfunction expansion is to break apart the PDE into functions that we know how to solve. This can be done via substitution, where we use eigenfunctions for one dimension that we know, and cram the other dimensions into an arbitrary func-

tion, which we can solve later. These substitutions can be made for every dimension.

The first substitution we will make is $V(t, r, \theta, z) = u(t, r, \theta, z) - T_c$ to make the boundary conditions zero. Equation (17.1) then becomes

$$\frac{\partial V}{\partial t} = K \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} \right] \qquad (17.4)$$

We can now use eigenfunction expansion – since there are more than two dimensions, we need multiple eigenfunction expansions. Our first expansion will take the form

$$V(t, r, \theta, z) = \sum_{m=-\infty}^{\infty} C_m(t, r) e^{im\theta} \qquad (17.5)$$

18 McCabe-Thiele Diagram for Methanol and Water

You are designing a distillation column to distill a mixture of methanol and water. The feed stream contains 75 mol% methanol and 25 mol% water; the distillate stream contains 99.99 mol% methanol and 0.01 mol% water; and the bottoms stream contains 0.002 mol% methanol and 99.998 mol% water. The reflux ratio is designed to be 1.4, and the feed conditions result in $q = 1.15$. Determine the theoretical number of stages by graphically stepping off stages of a McCabe-Thiele diagram. The equilibrium data for methanol and water is given below. All calculations and plots must be made on a computer. Use only information given here (no need to search for any thermodynamic quantities).

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	0.417	0.579	0.669	0.729	0.780	0.825	0.871	0.915	0.959	1.0

Solution to 18

The solution for this problem will first begin by performing a mass balance on the distillation column. All mass entering in the feed F must leave through either the vapor stream V or the bottoms stream B. All methanol entering in the feed stream must also either leave through the vapor stream or the bottoms stream. We can develop a total mass balance and a component mass balance

$$F = V + B \tag{18.1}$$

$$Fx_{M,F} = Vx_{M,V} + Bx_{M,B} \tag{18.2}$$

Using these equations, we can solve for V and B

$$V = F\left(1 - \frac{x_{M,F} - x_{M,V}}{x_{M,B} - x_{M,V}}\right) \quad \text{and} \quad B = F\left(\frac{x_{M,F} - x_{M,V}}{x_{M,B} - x_{M,V}}\right) \tag{18.3}$$

The rectifying line equation is

$$y_{n+1} = \frac{R}{R+1}x_n + \frac{x_D}{R+1} \tag{18.4}$$

the feed line equation is

$$y = -\frac{q}{1-q}x + \frac{x_F}{1-q} \tag{18.5}$$

Since we don't quite have the information to properly make the stripping line, we will instead create a line that connects the bottoms stream methanol mole fraction $y_{M,B} = x_{M,B}$ and the point where the rectifying line and feed line intersect.

At this point the plots can be created and the number of theoretical stages can be calculated. One thing that poses an issue however is that the data points are discretized, and we do not have an exact function that calculates the equilibrium curve in between points. To deal with this, instead of attempting to fit the data to a curve or look up thermodynamic data online, for each data point pair we can estimate the in-between values through interpolation.

A solution that I came up with is in appendix A, however there are several ways to do it (and mine may not be that well-optimized in the first place). My solution was written in Python.

The McCabe-Thiele diagram is shown in fig. (14). There are 26 theoretical stages.
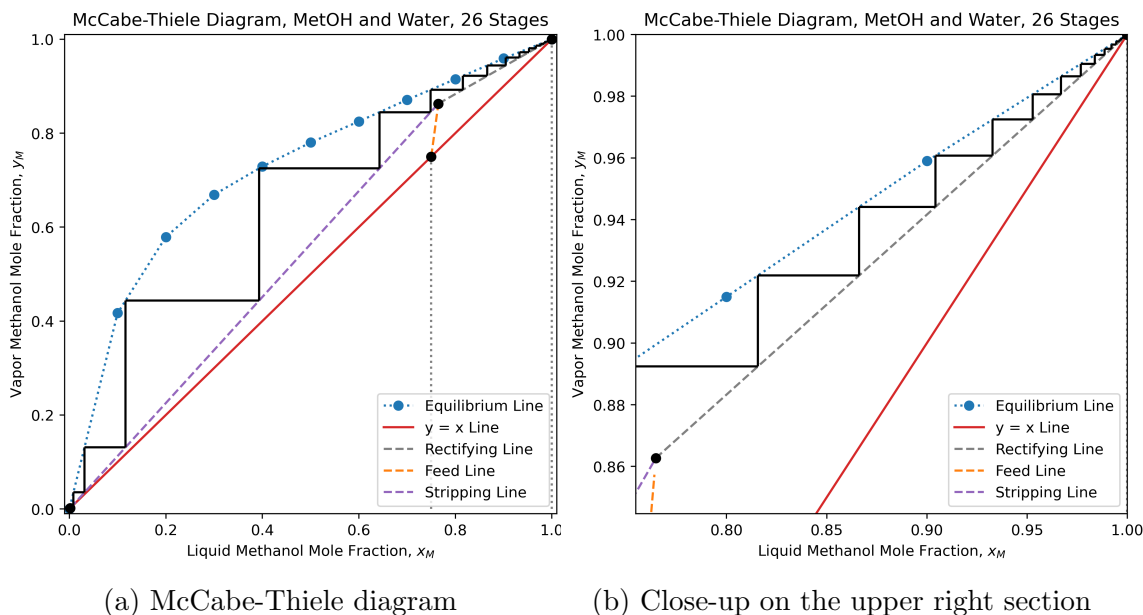
(a) McCabe-Thiele diagram   (b) Close-up on the upper right section

Figure 14: The McCabe-Thiele diagram for methanol and water.

19 Tetration Tower

Tetration is a mathematical operation that can be thought of as "the step after exponentiation." For a base number α and height n, tetration is defined as

$$\alpha \uparrow\uparrow n = \alpha^{\alpha^{\alpha^{\cdots}}} \tag{19.1}$$

where α is raised to itself n times. With this, show that

$$\sqrt{2} \uparrow\uparrow \infty = 2 \tag{19.2}$$

Solution to 19

To begin this solution, we must first consider that we are asked to evaluate an infinitely-tall tetration tower – though a calculator will show that we will get to more than 99% accuracy by brute forcing it with only 9 iterations, an analytical answer would be more elegant. Let us first notice that

$$\sqrt{2} \uparrow\uparrow \infty = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}} \tag{19.3}$$

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We can set this tetration equal to some value x, and recognize that since the tetration tower is infinitely-tall, the exponent is equivalent to the tetration itself

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}} = x \quad \Rightarrow \quad \sqrt{2}^{x} = x \tag{19.4}$$

We can now solve for x

$$\sqrt{2}^{x} = x \tag{19.5}$$

$$x \ln(\sqrt{2}) = \ln(x) \tag{19.6}$$

$$e^{\ln(x)} \ln(\sqrt{2}) = \ln(x) \tag{19.7}$$

$$\ln(\sqrt{2}) = \ln(x)e^{-\ln(x)} \tag{19.8}$$

In this situation, typical functions are not able to solve for x; instead, we need to use the 0th branch of the Lambert W function for real values, which is defined as

$$W\left(\varphi e^{\varphi}\right) = \varphi \tag{19.9}$$

Using the W function we can continue

$$W\left[-\ln(\sqrt{2})\right] = W\left[-\ln(x)e^{-\ln(x)}\right] \tag{19.10}$$

$$W\left[-\ln(\sqrt{2})\right] = \ln\left(\frac{1}{x}\right) \tag{19.11}$$

Now dealing with the left side, we can use an online calculator to evaluate the W function, but it is possible to evaluate without a calculator as such

$$W\left[-\ln(\sqrt{2})\right] = W\left[\ln\left(2^{-\frac{1}{2}}\right)\right] \tag{19.12}$$

$$= W\left[\frac{1}{2}\ln\left(\frac{1}{2}\right)\right] \tag{19.13}$$

$$= W\left[e^{\ln\left(\frac{1}{2}\right)}\ln\left(\frac{1}{2}\right)\right] \tag{19.14}$$

$$= \ln\left(\frac{1}{2}\right) \tag{19.15}$$

and thus

$$\ln\left(\frac{1}{2}\right) = \ln\left(\frac{1}{x}\right) \tag{19.16}$$

$$\boxed{x = 2} \tag{19.17}$$

20   Tricky ODE

Let the function f be differentiable everywhere, and $f(x + y) = f(x)f(y)$ for all x, y in the real numbers. Given that $f'(0) = 3$, find $f(x)$.

Solution to 20

This problem seems very tricky, but once you see the solution it is not as bad as it seems. We first need to consider what $f(x)$ is in terms of x and y, and we can do this by setting $y = 0$

$$f(x) = f(x)f(0) \tag{20.1}$$

Before we divide by $f(x)$ to solve for $f(0)$, we need to make sure that $f(x) \neq 0$. If $f(x) = 0$, then it would be a horizontal line; if it were a horizontal line, then its derivative would be zero, contradicting the fact that we were given $f'(0) = 3$. Thus, we know that dividing by $f(x)$ is allowed. This gives us $f(0) = 1$.

Let us now consider the definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \tag{20.2}$$

Applying the function given to us, this becomes

$$f'(x) = \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h}$$
$$= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h} \tag{20.3}$$

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which, when evaluating the limit, gives

$$f'(x) = f(x) \left(\frac{f(0) - 1}{0} \right)$$
$$= f(x) \left(\frac{0}{0} \right) \tag{20.4}$$

which allows us to perform L'Hospital's Rule on eq. (20.3). This yields

$$f'(x) = f(x) \lim_{h \to 0} \left(\frac{\frac{d}{dh}[f(h) - 1]}{\frac{d}{dh}[h]} \right)$$
$$= f(x) \lim_{h \to 0} \frac{f'(h)}{1}$$
$$= f(x) f'(0)$$
$$= 3f(x) \tag{20.5}$$

Keep in mind that L'Hospital's Rule is valid because it was stated that the function f is differentiable everywhere (also implying that the function exists everywhere). This gives us the ODE

$$f'(x) = 3f(x)$$

which has the solution

$$\boxed{f(x) = e^{3x}} \tag{20.6}$$

We can verify that this is the solution by substituting it into the expression we were given

$$f(x + y) = f(x)f(y)$$
$$e^{3x+3y} = e^{3x}e^{3y}$$

which, by exponent rules, is indeed true.

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References/Sources

1. The framework for problem 3 was borrowed from Professor Nael El-Farra's exam problem from ECH 157

2. Problem 4 was adapted from Professor Stephanie Dungan's homework problem from ECH 143

3. The framework for problem 5 was borrowed from Professor Gregory Miller's in-class examples and homework problems from ECH 140

4. Problem 6 is heavily based on one of Professor Gregory Miller's in-class examples from ECH 140

5. Problem 7 is pretty much exactly like a homework question from Professor Matthew Ellis in ECH 158B

A   Code for Problem 7 – McCabe-Thiele Diagram

```python
# Import libraries
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import fsolve


# Equilibrium liquid and vapor mole fractions
equi_x = [1.0,
          0.9,
          0.8,
          0.7,
          0.6,
          0.5,
          0.4,
          0.3,
          0.2,
          0.1,
          0.0]


equi_y = [1.000,
          0.959,
          0.915,
          0.871,
          0.825,
          0.780,
          0.729,
          0.669,
          0.579,
          0.417,
          0.000]


# Feed stream conditions
x_mf  = 0.75
x_wf  = 1 - x_mf


# Vapor stream conditions
x_mv  = 0.9999
x_wv  = 1 - x_mv


# Bottoms stream conditions
x_mb  = 0.002
x_wb  = 1 - x_mb



# Mass balance
F = 100  # randomly chosen basis
B = F * (x_mf - x_mv) / (x_mb - x_mv)
V = F - B


# Reflux ratio
R = 1.4


# Feed line q-value (don't set q = 1)
```

```
q = 1.15

# Rectifying Line
def rectify(xn):
    yn = R / (R + 1) * xn + x_mv / (R + 1)
    return yn

# Feed line
def q_line(x):
    y = -q / (1 - q) * x + x_mf / (1 - q)
    return y

# y = x line
def y_x(x):
    return x

# Rectifying line and Feed line intersection
def intersect(x):
    return rectify(x) - q_line(x)

# Triple intersection point
intersection_point = fsolve(intersect, x_mf) # returns liquid mole fraction of methanol

# Stripping Line
def strip(x):
    m = (rectify(intersection_point) - x_mb) / (intersection_point - x_mb)
    y = m * (x - x_mb) + x_mb
    return y

# Mole fraction range, just for plotting
x_frac   = np.arange(0, 1, 0.0001)

# Make the correct q-line
if x_mf < intersection_point:
    x_q_line = np.arange(x_mf, intersection_point, 0.001)
else:
    x_q_line = np.arange(intersection_point, x_mf, 0.001)

x_r_line = np.arange(intersection_point, x_mv, 0.001)
x_s_line = np.arange(x_mb, intersection_point, 0.001)


# Initialize counter for number of stages and mole fraction for iteration
N_r = 0
N_s = 0
i   = 0
mole_fraction = x_mv
y_n_plus_1    = mole_fraction

rect_line_xcoords = []
rect_line_ycoords = []

strip_line_xcoords = []
strip_line_ycoords = []
```

```python
# Find number of stages for rectifying section
while x_mv >= mole_fraction >= intersection_point:

    if equi_y[i + 1] <= y_n_plus_1 <= equi_y[i]:

        # Calculate liquid mole fraction through interpolation
        def x_n_plus_1(x):

            # Rectifying line
            m1 = R / (R + 1)
            b1 = x_mv / (R + 1)

            # Data line
            m2 = (equi_y[i] - equi_y[i + 1]) / (equi_x[i] - equi_x[i + 1])
            b2 = m2 * (-equi_x[i + 1]) + equi_y[i + 1]

            # Calculate mole fraction at equilibrium line
            x2 = (m1 * x + b1 - b2) / m2

            return x2

        x2 = x_n_plus_1(mole_fraction)

        # Append x- and y-coords to lists
        rect_line_xcoords.append([x2, mole_fraction])
        rect_line_ycoords.append([y_n_plus_1, rectify(x2)])

        # Re-assign x- and y-values
        y_n_plus_1    = rectify(x2)
        mole_fraction = x2

        # Modify step once mole fraction is less than intersection point
        if x2 < intersection_point:
            y_n_plus_1 = strip(x2)
            rect_line_ycoords[-1][1] = y_n_plus_1

        # Add to stage counter
        N_r += 1

    else:
        # Go to next data index
        i += 1


# Find number of stages for stripping section
while intersection_point >= mole_fraction >= x_mb:

    if equi_y[i + 1] <= y_n_plus_1 <= equi_y[i]:

        # Calculate liquid mole fraction through interpolation
        def x_n_plus_1(x):

            # Stripping line
            m1 = (rectify(intersection_point) - x_mb) / (intersection_point - x_mb)
            b1 = m1 * (-x_mb) + x_mb
```

```
            # Data line
            m2 = (equi_y[i] - equi_y[i + 1]) / (equi_x[i] - equi_x[i + 1])
            b2 = m2 * (-equi_x[i + 1]) + equi_y[i + 1]

            # Calculate mole fraction at equilibrium line
            x2 = (m1 * x + b1 - b2) / m2

            return x2

        x2 = x_n_plus_1(mole_fraction)

        # Append x- and y-coords to lists
        strip_line_xcoords.append([x2, mole_fraction])
        strip_line_ycoords.append([y_n_plus_1, strip(x2)])

        # Re-assign x- and y-values
        y_n_plus_1    = strip(x2)
        mole_fraction = x2

        # Add to stage counter
        N_s += 1

    else:
        # Go to next data index
        i += 1

print("Total number of stages for rectifying section:", N_r)
print("Total number of stages for stripping section: ", N_s)
print("Total number of stages:                       ", N_r + N_s)
print("Total number of actual stages:                ", int(np.ceil((N_r + N_s + 1) / 0.8)))


# Plot results
plt.figure(figsize = (6, 6))
plt.plot(equi_x, equi_y, label = "Equilibrium Line",
         marker = "o", linestyle = ":", color = "tab:blue")
plt.plot(x_frac, y_x(x_frac), label = "y = x Line", color = "tab:red")
plt.plot(x_r_line, rectify(x_r_line), label = "Rectifying Line",
         linestyle = "--", color = "gray")

plt.plot(x_q_line, q_line(x_q_line), linestyle = "--", label = "Feed Line",
         color = "tab:orange")

plt.plot(x_s_line, strip(x_s_line), linestyle = "--", label = "Stripping Line",
         color = "tab:purple")

# Plot rectifying line stages
for i in range(len(rect_line_xcoords)):
    plt.hlines(y = rect_line_ycoords[i][0], xmin = rect_line_xcoords[i][0],
               xmax = rect_line_xcoords[i][1], color = "k")

for j in range(len(rect_line_xcoords) - 1):
    plt.vlines(x = rect_line_xcoords[j + 1][1], ymin = rect_line_ycoords[j][1],
               ymax = rect_line_ycoords[j][0], color = "k")
```

```python
# Plot stripping line stages
for k in range(len(strip_line_xcoords)):
    plt.hlines(y = strip_line_ycoords[k][0], xmin = strip_line_xcoords[k][0],
               xmax = strip_line_xcoords[k][1], color = "k")


for l in range(len(strip_line_xcoords) - 1):
    plt.vlines(x = strip_line_xcoords[l + 1][1], ymin = strip_line_ycoords[l][1],
               ymax = strip_line_ycoords[l][0], color = "k")


plt.vlines(x = rect_line_xcoords[-1][0], ymin = rect_line_ycoords[-1][1],
           ymax = rect_line_ycoords[-1][0], color = "k")


plt.vlines(x = x_mb, ymin = 0, ymax = x_mb, color = "gray", linestyle = ":")
plt.vlines(x = x_mf, ymin = 0, ymax = x_mf, color = "gray", linestyle = ":")
plt.vlines(x = x_mv, ymin = 0, ymax = x_mv, color = "gray", linestyle = ":")


plt.plot(x_mf, x_mf, marker = "o", color = "k")
plt.plot(intersection_point, rectify(intersection_point), marker = "o",
         color = "k", linestyle = "")
plt.plot(x_mb, x_mb, marker = "o", color = "k")
plt.plot(x_mv, x_mv, marker = "o", color = "k")
plt.xlabel(r"Liquid Methanol Mole Fraction, $x_M$")
plt.ylabel(r"Vapor Methanol Mole Fraction, $y_M$")
plt.title("McCabe-Thiele Diagram, MetOH and Water, {} Stages".format(N_r + N_s))
plt.xlim(xmin = -0.01, xmax = 1.01)
plt.ylim(ymin = -0.01, ymax = 1.01)
plt.legend(loc = "lower right")
plt.show()



# Plot close-up for dramatic effect o_0
plt.figure(figsize = (6, 6))
plt.plot(equi_x, equi_y, label = "Equilibrium Line", marker = "o",
         linestyle = ":", color = "tab:blue")
plt.plot(x_frac, y_x(x_frac), label = "y = x Line", color = "tab:red")
plt.plot(x_r_line, rectify(x_r_line), label = "Rectifying Line",
         linestyle = "--", color = "gray")
plt.plot(x_q_line, q_line(x_q_line), linestyle = "--", label = "Feed Line",
         color = "tab:orange")
plt.plot(x_s_line, strip(x_s_line), linestyle = "--", label = "Stripping Line",
         color = "tab:purple")

# Plot rectifying line stages
for i in range(len(rect_line_xcoords)):
    plt.hlines(y = rect_line_ycoords[i][0], xmin = rect_line_xcoords[i][0],
               xmax = rect_line_xcoords[i][1], color = "k")


for j in range(len(rect_line_xcoords) - 1):
    plt.vlines(x = rect_line_xcoords[j + 1][1], ymin = rect_line_ycoords[j][1],
               ymax = rect_line_ycoords[j][0], color = "k")


# Plot stripping line stages
for k in range(len(strip_line_xcoords)):
```

```python
        plt.hlines(y = strip_line_ycoords[k][0], xmin = strip_line_xcoords[k][0],
                   xmax = strip_line_xcoords[k][1], color = "k")


for l in range(len(strip_line_xcoords) - 1):
    plt.vlines(x = strip_line_xcoords[l + 1][1], ymin = strip_line_ycoords[l][1],
               ymax = strip_line_ycoords[l][0], color = "k")


plt.vlines(x = rect_line_xcoords[-1][0], ymin = rect_line_ycoords[-1][1],
           ymax = rect_line_ycoords[-1][0], color = "k")


plt.vlines(x = x_mb, ymin = 0, ymax = x_mb, color = "gray", linestyle = ":")
plt.vlines(x = x_mf, ymin = 0, ymax = x_mf, color = "gray", linestyle = ":")
plt.vlines(x = x_mv, ymin = 0, ymax = x_mv, color = "gray", linestyle = ":")


plt.plot(x_mf, x_mf, marker = "o", color = "k")
plt.plot(intersection_point, rectify(intersection_point), marker = "o",
         color = "k", linestyle = "")
plt.plot(x_mb, x_mb, marker = "o", color = "k")
plt.plot(x_mv, x_mv, marker = "o", color = "k")
plt.xlabel(r"Liquid Methanol Mole Fraction, $x_M$")
plt.ylabel(r"Vapor Methanol Mole Fraction, $y_M$")
plt.title("McCabe-Thiele Diagram, MetOH and Water, {} Stages".format(N_r + N_s))
plt.xlim(xmin = intersection_point - 0.01, xmax = 1)
plt.ylim(ymin = intersection_point + 0.08, ymax = 1)
plt.legend(loc = "lower right")
plt.show()
```