CSC418 Assignment 1 – PartA

Yifei Yang **999590526**

Question 1.

$$\frac{dx(t)}{dt} = \frac{d\left(4\cos(2\pi t) + \frac{1}{16}\cos(32\pi t)\right)}{dt} = -8\pi\sin(2\pi t) - 2\pi\sin(32\pi t)$$

$$\frac{dy(t)}{dt} = \frac{d\left(2\sin(2\pi t) + \frac{1}{16}\sin(32\pi t)\right)}{dt} = 4\pi\cos(2\pi t) + 2\cos(32\pi t)$$

$=>$ tangent vector $= \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) = (-8\pi\sin(2\pi t) - 2\pi\sin(32\pi t), 4\pi\cos(2\pi t) + 2\cos(32\pi t))$

$\because$ tangent vector $\cdot$ normal vector $= 0$

$\therefore$ normal vector $= (4\pi\cos(2\pi t) + 2\cos(32\pi t), 8\pi\sin(2\pi t) + 2\pi\sin(32\pi t))$

$\because$ function $f(\alpha) = \cos(\alpha)$ *is even, function* $f(\alpha) = \sin(\alpha)$ *is odd*

$\therefore x(t) = 4\cos(2\pi t) + \frac{1}{16}\cos(32\pi t) = 4\cos(-2\pi t) + \frac{1}{16}\cos(-32\pi t) = x(-t)$ *for any* $t \in \mathbb{R}$

while $y(t) = 2\sin(2\pi t) + \frac{1}{16}\sin(32\pi t) = -2\sin(-2\pi t) - \frac{1}{16}\sin(-32\pi t) = -y(-t)$

$=>$ The curve is symmetric around the X – axis.

Let $t_1 = \frac{1}{32}, t_2 = \frac{1}{2} - t_1 = \frac{15}{32}$

Then $y(t_1) = \left(2\sin(2\pi t_1) + \frac{1}{16}\sin(32\pi t_1)\right) = 0.39018064403225 = \left(2\sin(2\pi t_1) + \frac{1}{16}\sin(32\pi t_1)\right)$

$= y(t_2)$

while $x(t_1) = 4\cos(2\pi t_1) + \frac{1}{16}\cos(32\pi t_1) = 3.8606411216129217$,

$-x(t_2) = 4\cos(2\pi t_2) + \frac{1}{16}\cos(32\pi t_2) = 3.9856411216129217 \neq x(t_1)$

$=>$ The curve is not symmetric around the Y – axis.

Formula to compute the curve's perimeter is $s = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}dt$

Since the cycle time of $\cos(2\pi t)$ *and* $\sin(2\pi t)$ *are* 1, cycle time of $\cos(32\pi t)$ *and* $\sin(32\pi t)$ *are* $\frac{1}{16}$.

=> The perimeter could be piecewise to 16 small curves with similar arc lengths.

Length of each small curve = $\int_0^{\frac{1}{16}} \sqrt{x'(t)^2 + y'(t)^2}dt$

Then we can approximate this perimeter as $16(\int_0^{\frac{1}{16}} \sqrt{x'(t)^2 + y'(t)^2}dt)$.

Question 2.

area of the donut $= \pi r_2^2 - \pi r_1^2$

Intersections number between the line and the boundary could be 0, 1, 2, 3 or 4

Let $\mathbf{p_0} = (p_{0x}, p_{0y}), \mathbf{d} = (d_x, d_y)$. Let the center of circles $\mathbf{p_1} = (p_{1x}, p_{1y})$
Define circle with radius r_1 *to be* C_1, *circle with radius* r_2 *to be* $C_2, r_1 < r_2$

Then function of the line is: $y = (\frac{d_y}{d_x})x + (p_{0y} + \frac{p_{0x}}{d_x})$

$$=> d_y x - d_x y + d_x p_{0y} + p_{0x} = 0$$

Let $A = d_y, B = -d_x, C = d_x p_{0y} + p_{0x}$
Then line of function is: $Ax + By + C = 0$

$\therefore$ (distance from $\mathbf{p_1}$ *to the line*) $= \dfrac{Ap_{1x} + Bp_{1y} + C}{\sqrt{A^2 + B^2}}$

- If this distance is smaller than r_1, then the line has two intersections with circle C_1.
  Since C_1 *is in* C_2, *the line has four intersections with the donut.*
- Else if this distance is equal to r_1, then the line has one intersection with circle C_1.
  Since C_1 *is in* C_2, *the line has three intersections with the donut.*
- Otherwise, this distance is larger than r_1, then the line has no intersection with circle C_1.
  The line could have at most two intersections with the whole donut. ①

Similarly, we can compare the distance from circle center to the line with r_2. And know the pecific intersection number between them. ②

By ① and ②, We can calculate the total intersection number between the donut and the line.

When the line and Circle C_1 has intersection, then:
Since the direction of line is $\mathbf{d} = (d_1, d_2)$, then the perpendicular line from C_1 to the line is has direction $(-d_2, d_1)$

By ①, (distance from $\mathbf{q_1}$ to the line) = $\left\|(q_{1x}, q_{1y}) + u(-d_2, d_1)\right\|$ *for some* $u \in \mathbb{R}$

Solve this equation to get the number of n, then:

- If C_1 and the line has only one intersection,
  Then the location of intersection Is $(q_{1x}, q_{1y}) + n(-d_2, d_1)$.
- Otherwise, the midpoint of the line's part in C_1 has location $(q_{1x}, q_{1y}) + n(-d_2, d_1)$.

  Then, we can get the ($\frac{1}{2}$ length of the line's part in C_1) $= \sqrt{(r_1)^2 - (\text{distance from } \mathbf{q_1} \text{ to the line})^2}$

  At the same time, this ($\frac{1}{2}$ length of the line's part in C_1) $= \left\|m \cdot (d_1, d_2)\right\|$ *for some* $m \in \mathbb{R}$.

  Solve this equation to get the number of m, then the location of intersections are:

$$\left(\mathbf{q_{1x}}, \mathbf{q_{1y}}\right) + \boldsymbol{n}(-\boldsymbol{d_2}, \boldsymbol{d_1}) + \boldsymbol{m}(\boldsymbol{d_1}, \boldsymbol{d_2}) \textit{ and } \left(\mathbf{q_{1x}}, \mathbf{q_{1y}}\right) + \boldsymbol{n}(-\boldsymbol{d_2}, \boldsymbol{d_1}) - \boldsymbol{m}(\boldsymbol{d_1}, \boldsymbol{d_2})$$

Then we can apply above algorithm using r_2 to get the pacific location of other intersections with C_2.

When the line and donut are both transformed by a scale(s_x, s_y) around origin point:
=> For any point (x, y) on the line or on any circle of donut, it would be transformed to $(s_x x, s_y y)$.

Suppose$(\mathrm{x_i}, \mathrm{y_i}), i = 1, 2, 3 \ or\ 4$ is an intersection between the line and a circle of the donut.
After the scale applied:

- The number of intersections would not change.
  (i.e. for any point on either line and circles, it would still on the transformed line and circles after scale applied.)
- The location of any intersection $(\mathrm{x_i}, \mathrm{y_i})$ would becomes $(\mathrm{s_x x_i}, \mathrm{s_y y_i})$.

When the scale $(\mathrm{s_x}, \mathrm{s_y})$ is only applied on the donut, the number and locations of intersections would all change.
Since scale $(\mathrm{s_x}, \mathrm{s_y})$ is non-uniform, two circles of donut would become two ellipses and we cannot directly use the algorithm above to solve new intersections.
Suppose$(\mathrm{s_x x_i}, \mathrm{s_y y_i}), i \in [1, 4], i \in \mathbb{N}$ is an intersection between the line and scaled ellipse.
We can get another scale $\mathrm{s}' = \left(\frac{1}{\mathrm{s_x}}, \frac{1}{\mathrm{s_y}}\right)$ around origin point. Then for any point(x, y)on original donut, after applied both scale s and s', its location would still be (x, y).
=> Scale s' can scale the ellipse back to the original shape of the donut and
Therefore, we can apply s' on both the line and the ellipse donut. Then we would get a new scaled line and original donut. Then new line's function is $\mathrm{q} = \mathrm{s}'(p_0 + \lambda d) where\ q\ is\ a\ point\ on\ the\ new\ line.$
=> We can apply our algorithm on our new line and original donut. Then we may get some intersections $(\mathrm{x_i}, \mathrm{y_i}), i = 1, 2, 3 \ or\ 4.$
Therefore the new intersections between old line and ellipse donut is $(\mathrm{s_x x_i}, \mathrm{s_y y_i}), \mathrm{i} = 1, 2, 3 \text{ or } 4.$

Question 3.

(a)Let $\mathrm{f_1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ be the translate matrix and $\mathrm{f_2} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be the scaling matrix, then:

$$\mathrm{f_1} f_2 = \begin{bmatrix} 10 & 0 & 1 \\ 0 & 10 & 1 \\ 0 & 0 & 1 \end{bmatrix}, f_2 f_1 = \begin{bmatrix} 10 & 0 & 10 \\ 0 & 10 & 10 \\ 0 & 0 & 1 \end{bmatrix} \quad => \quad f_1 f_2 \neq f_2 f_1$$

$\therefore$ translation and uniform scaling is not commute

(b)Let $\mathrm{f_1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ be the translate matrix and $\mathrm{f_2} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be the scaling matrix, then:

$$\mathrm{f_1} f_2 = \begin{bmatrix} 10 & 0 & 1 \\ 0 & 20 & 1 \\ 0 & 0 & 1 \end{bmatrix}, f_2 f_1 = \begin{bmatrix} 10 & 0 & 10 \\ 0 & 20 & 20 \\ 0 & 0 & 1 \end{bmatrix} \quad => \quad f_1 f_2 \neq f_2 f_1$$

$\therefore$ translation and non $-$ uniform scaling is not commute

(c)Let both of fixed point of scalling and rotation to be $(0, 0)$, then:

Let $\mathrm{f_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be the scaling matrix ,

$$Let\ f_2 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\cos(\frac{\pi}{4})&-\sin(\frac{\pi}{4})&0\\\sin(\frac{\pi}{4})&\cos(\frac{\pi}{4})&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}\cos(\frac{\pi}{4})&-\sin(\frac{\pi}{4})&0\\\sin(\frac{\pi}{4})&\cos(\frac{\pi}{4})&0\\0&0&1\end{bmatrix}$$ be the rotation matrix.

$$f_1 f_2 = \begin{bmatrix}10\cos(\frac{\pi}{4})&-\sin(\frac{\pi}{4})&0\\\sin(\frac{\pi}{4})&20\cos(\frac{\pi}{4})&0\\0&0&1\end{bmatrix}, f_2 f_1 = \begin{bmatrix}10\cos(\frac{\pi}{4})&-20\sin(\frac{\pi}{4})&0\\10\sin(\frac{\pi}{4})&20\cos(\frac{\pi}{4})&0\\0&0&1\end{bmatrix} \quad => \quad f_1 f_2 \neq f_2 f_1$$

∴ translation and uniform scaling is not commute

(d)Letfixed point of two scallings to be $(-2,0)$ and $(-1,0)$, then:

$$\text{Let } f_1 = \begin{bmatrix}1&0&-2\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}2&0&2\\0&1&0\\0&0&1\end{bmatrix}$$ be one scaling matrix ,

$$\text{Let } f_2 = \begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}3&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}3&0&0\\0&1&0\\0&0&1\end{bmatrix}$$ be another scaling matrix.

$$f_1 f_2 = \begin{bmatrix}6&0&6\\0&1&0\\0&0&1\end{bmatrix}, f_2 f_1 = \begin{bmatrix}6&0&8\\0&1&0\\0&0&1\end{bmatrix} \quad => \quad f_1 f_2 \neq f_2 f_1$$

∴ translation and uniform scaling is not commute

(e)Let $f_1 = \begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}$ be the translate matrix and $f_2 = \begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}$ be the shear matrix, then:

$$f_1 f_2 = \begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}, f_2 f_1 = \begin{bmatrix}1&1&2\\0&1&1\\0&0&1\end{bmatrix} \quad => \quad f_1 f_2 \neq f_2 f_1$$

∴ translation and uniform scaling is not commute

Question 4.

Let $q = (q_x, q_y)$

Then define triangle to be ΔABC, let AB = $(a,b) = B - A, AC\ (c,d) = C - A$. by definition of triangle, q is in the triangle iff:

$q = uAB + vAC$, where $u \geq 0, v \geq 0, u + v \leq 1$

$$=> \begin{cases} q_x = ua + vc \\ q_y = ub + vd \end{cases}, where\ u \geq 0, v \geq 0, u + v \leq 1$$

Since $(q_x, q_y), (a,b), (c,d)$ $are\ already\ known$, by these two equations, we can solve the value of u and v.

Then if u or v violates $u \geq 0, v \geq 0, u + v \leq 1$, q is not in the triangle.

Otherwise, if u = 0, then q is on edge AC.

If v = 0, then q is on edge AB.

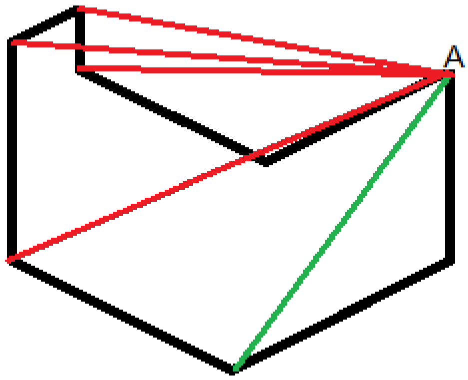
If u + v = 1, then q is on edge BC.

Otherwise, q is in the triangle.

For any quadrilateral, pick one of its vertexes A. For all the other vertexes $\mathrm{B_i}, i \in \mathbb{N}, i \in [1,3]$. If edge $\mathrm{AB_i}$ does not exist, then link them. After that this quadrilateral has already been triangulated.

For any n-sided polygon, pick one of its vertexes A. For all the other vertexes $\mathrm{B_i}, i \in \mathbb{N}, i \in [1, n-1]$. If edge $\mathrm{AB_i}$ does not exist, then link them. After that this n-sided polygon has already been triangulated.

For the follow case, this procedure will not work since red lines are located in outside of the polygon:



After the polygon triangulated, for all triangle in it, check if the point is outside this triangle:

- If the point is not in any of all the triangles, then this point is out the polygon.
- If the point is in any triangle, then this point is in the polygon.
- If the point is on an edge of some triangles, then:
  - If the edge contains this point is an edge of the polygon, then this point is on the polygon.
  - Otherwise, the point is in the polygon.