复变函数论第三次作业

20234544 毛华豪

Task1:

证明 首先若假设 $f(z)=u+iv$ 我们有 $\frac{\partial f}{\partial x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x},\frac{\partial f}{\partial y}=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}$.
所以左侧式子可以表示为：

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2}&=\frac{\partial(\frac{\partial u}{\partial x})}{\partial x}+i\frac{\partial(\frac{\partial v}{\partial x})}{\partial x}\\&=\frac{\partial^2 u}{\partial x^2}+i\frac{\partial^2 v}{\partial x^2}\\\frac{\partial^2 f}{\partial y^2}&=\frac{\partial(\frac{\partial u}{\partial y})}{\partial y}+i\frac{\partial(\frac{\partial v}{\partial y})}{\partial y}\\&=\frac{\partial^2 u}{\partial y^2}+i\frac{\partial^2 v}{\partial y^2}\end{aligned}$$

对于右边我们有:

$$\begin{aligned}\frac{\partial f}{\partial z}&=\frac{1}{2}(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y})+\frac{i}{2}(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y})\\\frac{\partial f}{\partial \bar{z}}&=\frac{1}{2}(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y})+\frac{i}{2}(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})\end{aligned}$$

于是有：

$$
\begin{aligned}
\frac{\partial^2 f}{\partial z \partial \bar{z}} &= \frac{\partial \frac{\partial f}{\partial \bar{z}}}{\partial z} \\
\frac{\partial^2 f}{\partial z \partial \bar{z}} &= \frac{\partial[\frac{1}{2}(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) + \frac{i}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})]}{\partial z} \\
&= \frac{1}{2}(\frac{\partial \frac{1}{2}(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})}{\partial x} + \frac{\partial \frac{1}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})}{\partial y}) \\
&+ \frac{i}{2}(\frac{\partial \frac{1}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})}{\partial x} - \frac{\partial \frac{1}{2}(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})}{\partial y})
\end{aligned}
$$

由于 $f(z)$ 具有所需的任意性质，所以有：

$$
\begin{aligned}
\frac{\partial^2 v}{\partial x \partial y} &= \frac{\partial^2 v}{\partial y \partial x} \\
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial^2 u}{\partial y \partial x}
\end{aligned}
$$

代入上式化简得到：

$$
\begin{aligned}
&\frac{1}{2}(\frac{\partial \frac{1}{2}(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})}{\partial x} + \frac{\partial \frac{1}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})}{\partial y}) \\
&+ \frac{i}{2}(\frac{\partial \frac{1}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})}{\partial x} - \frac{\partial \frac{1}{2}(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})}{\partial y}) \\
&= \frac{1}{4}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + \frac{i}{4}(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \\
&= \frac{1}{4}(\frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2}) + \frac{1}{4}(\frac{\partial^2 u}{\partial y^2} + i\frac{\partial^2 v}{\partial y^2}) \\
&= \frac{1}{4}(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}) \to \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4\frac{\partial^2 f}{\partial z \partial \bar{z}}
\end{aligned}
$$

$\square$

Task2:

证明 $u(z) = \ln|z| \xrightarrow{z=x+iy} u(z) = \ln\left|\sqrt{x^2+y^2}\right|, (x, y \neq 0) \to u(z) =$

$u(x,y)=\frac{1}{2}\ln(x^2+y^2),(x,y\neq 0)$ 所以我们只要证明二元实函数 $u=\ln(x^2+y^2)$ 为调和函数即可。

$$\frac{\partial^2 u}{\partial x^2}=\frac{-2x^2+2y^2}{(x^2+y^2)^2}$$
$$\frac{\partial^2 u}{\partial y^2}=\frac{2x^2-2y^2}{(x^2+y^2)^2}$$
$$\to\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\equiv 0$$

所以 $u(z)=u(x,y)=\frac{1}{2}\ln(x^2+y^2),(x,y\neq 0)$ 为调和函数，除去 $(0,0)$ 点，即 $\mathbb{C}\backslash\{0\}$ 上为调和函数。 □

Task3:

证明 考虑 $Cauchy-Riemann$ 方程解题。假设 $z=x+iy,f(z)=u(x,y)+iv(x,y)$ 则 $\overline{f(\bar{z})}=u(x,-y)-iv(x,-y)\triangleq s(x,y)+it(s,y)=g(z)$。则 $g(z)$ 的定义域为所有 $f(z)$ 的定义域 $(x,y)\to(x,-y)$ 得到的。所以 D 为 $f(z)$ 定义域则 D' 为 $g(z)$ 定义域。由于 $f(z)$ 解析，所以在 D 上满足 u,v 可微。因为 u,v 可微则 s,t 一定可微，这是由于：

$$\frac{\partial s}{\partial x}(x,y)=\frac{\partial u}{\partial x}(x,-y)$$
$$\frac{\partial t}{\partial x}(x,y)=-\frac{\partial v}{\partial x}(x,-y)$$
$$\frac{\partial s}{\partial y}(x,y)=-\frac{\partial u}{\partial y}(x,-y)$$
$$\frac{\partial t}{\partial y}(x,y)=\frac{\partial v}{\partial y}(x,-y)$$

以上公式记为 $*$ 式因为 u,v 的偏导数在 D 上存在所以 s,t 的偏导数在 D'

上也存在，通过可微性条件：

$$
\begin{aligned}
&\forall(x_0,y_0)\in D',(x_0,-y_0)\in D\\
&u(x,y)=u(x_0,-y_0)+\frac{\partial u}{\partial x}(x_0,-y_0)(x-x_0)\\
&+\frac{\partial u}{\partial y}(x_0,-y_0)(y+y_0)+o(\sqrt{(x-x_0)^2+(y+y_0)^2})\\
&u(x,-y)=s(x,y)=u(x_0,y_0)+\frac{\partial u}{\partial x}(x_0,y_0)(x-x_0)\\
&-\frac{\partial u}{\partial y}(x_0,y_0)(y-y_0)+o(\sqrt{(x-x_0)^2+(y-y_0)^2})\\
&=u(x_0,y_0)+\frac{\partial s}{\partial x}(x_0,y_0)(x-x_0)\\
&+\frac{\partial s}{\partial y}(x_0,y_0)(y-y_0)+o(\sqrt{(x-x_0)^2+(y-y_0)^2})
\end{aligned}
$$

所以由 u 在 D 上满足可微性条件 s 在 D' 上满足可微性条件。同理 t 也可微。

接下来考虑 $Cauchy-Riemann$ 方程。

$$
\begin{aligned}
\frac{\partial u}{\partial x}&=\frac{\partial v}{\partial y}\\
\frac{\partial u}{\partial y}&=-\frac{\partial v}{\partial x}
\end{aligned}
$$

对于所有 $(x,y)\in D'\to(x,-y)\in D.$

$$
\begin{aligned}
\frac{\partial u}{\partial x}(x,-y)&=\frac{\partial v}{\partial y}(x,-y)\\
\frac{\partial u}{\partial y}(x,-y)&=-\frac{\partial v}{\partial x}(x,-y)
\end{aligned}
$$

将 $*$ 式代入得到:

$$
\begin{aligned}
\frac{\partial s}{\partial x}(x,y)&=\frac{\partial t}{\partial y}(x,y)\\
\frac{\partial s}{\partial y}(x,y)&=-\frac{\partial t}{\partial x}(x,y)
\end{aligned}
$$

所以满足 $Cauchy-Riemann$ 条件，$g(z)$ 在 D' 上解析。 $\square$

Task4:

证明 解析函数 f 将区域 D 一一对应映射为 Ω。我们有 $f(z)=s+it\in\Omega, z=u+iv\in D$。由 $Jacobi$ 矩阵的定义有：

$$J=\begin{pmatrix}\frac{\partial s}{\partial u} & \frac{\partial s}{\partial v}\\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v}\end{pmatrix}\to|J|=\begin{vmatrix}\frac{\partial s}{\partial u} & \frac{\partial s}{\partial v}\\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v}\end{vmatrix}=\frac{\partial s}{\partial u}\cdot\frac{\partial t}{\partial v}-\frac{\partial s}{\partial v}\cdot\frac{\partial t}{\partial u}$$

并且有:

$$f'=\frac{\partial s}{\partial u}+i\frac{\partial t}{\partial u}\to|f'|^2=(\frac{\partial s}{\partial u})^2+(\frac{\partial t}{\partial u})^2$$

又因为 f 是解析函数，所有满足 $Cauchy-Riemann$ 方程：

$$\frac{\partial s}{\partial u}=\frac{\partial t}{\partial v}$$
$$\frac{\partial s}{\partial v}=-\frac{\partial t}{\partial u}$$

所以有:

$$\begin{aligned}|J|&=\frac{\partial s}{\partial u}\cdot\frac{\partial t}{\partial v}-\frac{\partial s}{\partial v}\cdot\frac{\partial t}{\partial u}\\&=(\frac{\partial s}{\partial u})^2+(\frac{\partial t}{\partial u})^2=|f'|^2\end{aligned}$$

□

Task5:

解 已知 $z_0=x_0=iy_0$，所以有：$z_0^2=(x_0+iy_0)^2=x_0^2-y_0^2+2ix_0y_0$，故：

$$|e^{z_0^2}|=\left|e^{x_0^2-y_0^2+2ix_0y_0}\right|=\left|e^{x_0^2-y_0^2}\cdot e^{2ix_0y_0}\right|=\left|e^{x_0^2-y_0^2}\right|\cdot 1=\left|e^{x_0^2-y_0^2}\right|=e^{x_0^2-y_0^2}$$

又因为:

$$\frac{1}{z_0}=\frac{1}{x_0+iy_0}=\frac{x_0-iy_0}{(x_0+iy_0)\cdot(x_0-iy_0)}=\frac{x_0}{x_0^2+y_0^2}-i\frac{y_0}{x_0^2+y_0^2}$$

所以有：

$$
\begin{aligned}
e^{\frac{1}{z_0}} &= e^{\frac{x_0}{x_0^2+y_0^2}-i\frac{y_0}{x_0^2+y_0^2}} \\
&= e^{\frac{x_0}{x_0^2+y_0^2}}[\cos(\frac{-y_0}{x_0^2+y_0^2})+i\sin(\frac{-y_0}{x_0^2+y_0^2})] \\
&= e^{\frac{x_0}{x_0^2+y_0^2}}[\cos(\frac{y_0}{x_0^2+y_0^2})-i\sin(\frac{y_0}{x_0^2+y_0^2})] \\
&= e^{\frac{x_0}{x_0^2+y_0^2}}\cdot\cos(\frac{y_0}{x_0^2+y_0^2})-ie^{\frac{x_0}{x_0^2+y_0^2}}\cdot\sin(\frac{y_0}{x_0^2+y_0^2})
\end{aligned}
$$

所以有 $Re(\frac{1}{z_0})=e^{\frac{x_0}{x_0^2+y_0^2}}\cdot\cos(\frac{y_0}{x_0^2+y_0^2})$

Task6:

证明 可微一定连续，对于任意阶导数存在的函数 f 来说，它的任意阶导数都是连续的。假设 $f=u+iv$

$$
\begin{aligned}
f'(z) &= \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} \\
\Rightarrow f''(z) &= \frac{\partial}{\partial x}(\frac{\partial u}{\partial x})+i\frac{\partial}{\partial x}\frac{\partial v}{\partial x} \\
&= \frac{\partial^2 u}{\partial x^2}+i\frac{\partial^2 v}{\partial x^2}
\end{aligned}
$$

所以 $\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 v}{\partial x^2}$ 是连续的。
同理可得：

$$
\begin{aligned}
f'(z) &= \frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y} \\
\Rightarrow f''(z) &= \frac{\partial}{\partial y}(-\frac{\partial u}{\partial y})+i\frac{\partial}{\partial y}\frac{\partial v}{\partial y} \\
&= -\frac{\partial^2 u}{\partial y^2}+i\frac{\partial^2 v}{\partial y^2}
\end{aligned}
$$

所以 $\frac{\partial^2 u}{\partial y^2},\frac{\partial^2 v}{\partial y^2}$ 是连续的。

将上述两个过程结合:

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ \Rightarrow f''(z) &= \frac{\partial}{\partial y}(\frac{\partial v}{\partial x}) - i\frac{\partial}{\partial y}\frac{\partial u}{\partial x} \\ &= \frac{\partial^2 v}{\partial y \partial x} - i\frac{\partial^2 u}{\partial y \partial x} \end{aligned}$$

所以 $\frac{\partial^2 v}{\partial y \partial x}, \frac{\partial^2 u}{\partial y \partial x}$ 是连续的。
同理可得:

$$\begin{aligned} f'(z) &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \\ \Rightarrow f''(z) &= \frac{\partial}{\partial x}(\frac{\partial v}{\partial y}) - i\frac{\partial}{\partial x}\frac{\partial u}{\partial y} \\ &= \frac{\partial^2 v}{\partial x \partial y} - i\frac{\partial^2 u}{\partial x \partial y} \end{aligned}$$

所以 $\frac{\partial^2 v}{\partial x \partial y}, \frac{\partial^2 u}{\partial x \partial y}$ 是连续的。
综上可得:f 的实部和虚部都有二阶连续偏导数。 □

Task7:

证明 $f(z) = \frac{1}{z^2+1}$. 假设 $z = x + iy$

$$\begin{aligned} f(z) &= \frac{1}{1+z^2} = \frac{1}{(z+i)\cdot(z-i)} \\ &= \frac{1}{2i}(\frac{1}{z+i} - \frac{1}{z-i}) \\ \Rightarrow f^{(n)}(z) &= \frac{1}{2i}(\frac{d^n}{dz^n}(\frac{1}{z+i}) - \frac{d^n}{dz^n}(\frac{1}{z-i})) \end{aligned}$$

$$\because \frac{d^n}{dz^n}(\frac{1}{z-a}) = (-1)^n \frac{n!}{(z-a)^{n+1}}$$

$$\therefore f^{(n)}(z) = \frac{(-1)^n n!}{2i}\left(\frac{1}{(z+i)^{n+1}} - \frac{1}{(z-i)^{n+1}}\right)$$

$$\Rightarrow f^{(4n+3)}(z) = \frac{-(4n+3)!}{2i}\left(\frac{1}{(z+i)^{4n+4}} - \frac{1}{(z-i)^{4n+4}}\right)$$

$$\Rightarrow f^{(4n+3)}(1) = \frac{-(4n+3)!}{2i}\left(\frac{1}{(1+i)^{4n+4}} - \frac{1}{(1-i)^{4n+4}}\right)$$

$$\because (1+i)^{4n+4} = \left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^{4n+4} = \sqrt{2}e^{ik\pi}, k = n+1 \in \mathbb{Z}$$

$$(1-i)^{4n+4} = \left(\sqrt{2}e^{\frac{-i\pi}{4}}\right)^{4n+4} = \sqrt{2}e^{ik\pi} =, k = -n-1 \in \mathbb{Z}$$

$$\therefore (1+i)^{4n+4} = (1-i)^{4n+4}, (n+1 \equiv n-1 \mod 2)$$

$$\Rightarrow f^{(4n+3)}(1) = \frac{-(4n+3)!}{2i}\left(\frac{1}{(1+i)^{4n+4}} - \frac{1}{(1-i)^{4n+4}}\right) = 0$$

□

Task8:

证明 单叶函数的定义为 $\forall z_1, z_2 \in D$, 如果 $z_1 \neq z_2 \rightarrow f(z_1) \neq f(z_2)$
对于指数函数 $f(z) = e^{az}$ 来说，$\forall z_1, z_2 \in D$, 有 $-\frac{\pi}{2} < Im(z_1), Im(z_2) < \frac{\pi}{2}$。
不妨设 $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$，则 $-\frac{\pi}{2} < y_1 < \frac{\pi}{2}, -\frac{\pi}{2} < y_2 < \frac{\pi}{2}$.

$$e^{(a_1+ia_2)\cdot(x_1+iy_1)} = e^{(a_1+ia_2)\cdot(x_2+iy_2)}$$

$$\Rightarrow e^{(a_1x_1-a_2y_1)+i(a_1y_1+a_2x_1)} e^{(a_1x_2-a_2y_2)+i(a_1y_2+a_2x_2)}$$

$$\begin{cases} a_1x_1 - a_2y_1 = a_1x_2 - a_2y_2 \\ a_1y_1 + a_2x_1 = a_1y_2 + a_2x_2 + 2k\pi \end{cases} \rightarrow \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases}$$

$$\therefore \begin{cases} a_1(x_1 - x_2) = a_2(y_1 - y_2) \\ |a_1y_1 + a_2x_1 - a_1y_2 + a_2x_2| < 2\pi \end{cases} \rightarrow \left|(a_1 + \frac{a_2^2}{a_1})(y_1 - y_2)\right| < 2\pi.$$

$\therefore \left|(a_1 + \frac{a_2^2}{a_1})\right| |(y_1 - y_2)| = \left|(a_1 + \frac{a_2^2}{a_1})(y_1 - y_2)\right| < 2\pi. \quad \forall |y_1 - y_2| < \frac{\pi}{2} - (-\frac{\pi}{2}).$

$\therefore \left|(a_1 + \frac{a_2^2}{a_1})\right| < (\frac{2\pi}{|(y_1 - y_2)|})_{min}$

$\because (\frac{2\pi}{|(y_1 - y_2)|})_{min} \to \pi$

$\therefore \left|(a_1 + \frac{a_2^2}{a_1})\right| \leq 2$

因为 $a_1, \frac{a_2^2}{a_1}$ 一定同号，所以原式可以写成:

$$|a_1| + \left|\frac{a_2^2}{a_1}\right| \leq 2$$
$$\Leftrightarrow |a_1|^2 + |a_2|^2 \leq 2|a_1|$$

以上过程都是可逆的，所以满足充要条件。 □

Task9:

解 要满足单叶并且解析，$\forall z_1, z_2 \in D$, 如果 $z_1 \neq z_2 \to f(z_1) \neq f(z_2)$ 则单叶。

对于 $f(z_1) = f(z_2) \to z_1 = z_2$ 则说明单叶，由于原函数 $f(z) = z^2 + az + 3$ 为解析函数 $z^n, n \in \mathbb{N}$ 的复合所以可以知道原函数也解析。所以仅需考虑单叶性。

$$\begin{aligned} z_1^2 + az_2 + 3 = f(z_1) = f(z_2) &= z^2 + az_2 + 3 \\ &\to (z_1^2 - z_2^2) + a(z_1 - z_2) = 0 \\ &\to (z_1 - z_2)(z_1 + z_2 + a) = 0 \end{aligned}$$

要得出 $z_1 = z_2$，只要 $(z_1 + z_2 + a) \neq 0$ 即可，即 $a \neq -(z_1 + z_2)$，再结合我们知道的条件 $z_1 = z_2$ 有 $a \neq -2z_1, z_1 \in \{z \in \mathbb{C} : |z| < 1\}$. 所以 $|a| \notin (-2, 2) \to |a| \geq 2$

Task10:

证明 结合 $Cauchy - Riemann$ 条件，函数 $f(z) = u(z) + iv(z)$ 在区域 D 上解析，且 $u = v^2$。令 $u(z) = u(x, y), v(z) = v(x, y)$，根据解析的条件有：

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

由于 $u = v^2$ 所以条件变为:

$$\frac{\partial v^2}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial v^2}{\partial y} = -\frac{\partial v}{\partial x}$$
$$\Rightarrow$$
$$2v\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}$$
$$2v\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}$$
$$\Rightarrow$$
$$\begin{aligned} 0 = 2v\frac{\partial v}{\partial x} - 2v\frac{\partial v}{\partial x} &= \frac{\partial v}{\partial y} + 2v \cdot 2v\frac{\partial v}{\partial y} \\ &= (1 + 4v^2)\frac{\partial v}{\partial y} \rightarrow \frac{\partial v}{\partial y} = 0 \end{aligned}$$

结合前面的 $Cauchy - Riemann$ 方程，我们知道:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0$$
$$\longrightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$$

所以 f 在 D 上恒为常数。 □

Task11:

证明 假设 $f(z)=u(x,y)+iv(x,y)$ 在 D 内解析。则有

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

又因为 $|f(z)|^2=u^2+v^2$，所以有：

$$\frac{\partial^2|f(z)|^2}{\partial z\partial\bar{z}}=\frac{\partial}{\partial z}\frac{\partial|f(z)|^2}{\partial\bar{z}}=\frac{\partial}{\partial z}\frac{\partial(u^2+v^2)}{\partial\bar{z}}$$
$$=\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial u^2}{\partial x}+\frac{\partial v^2}{\partial x}+i\frac{\partial u^2}{\partial y}+i\frac{\partial v^2}{\partial y}\right)$$
$$=\frac{1}{4}\left(\left(\frac{\partial\frac{\partial u^2}{\partial x}+\frac{\partial v^2}{\partial x}}{\partial x}+\frac{\partial\frac{\partial u^2}{\partial y}+\frac{\partial v^2}{\partial y}}{\partial y}\right)+i\left(\frac{\partial\frac{\partial u^2}{\partial y}+\frac{\partial v^2}{\partial y}}{\partial x}-\frac{\partial\frac{\partial u^2}{\partial x}+\frac{\partial v^2}{\partial x}}{\partial y}\right)\right)$$
$$=\frac{1}{4}(2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x}+2u\frac{\partial^2 u}{\partial x^2}+2\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial x}+2v\frac{\partial^2 v}{\partial x^2}+$$
$$2\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial y}+2u\frac{\partial^2 u}{\partial y^2}+2\frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial y}+2v\frac{\partial^2 v}{\partial y^2})+$$
$$\frac{i}{4}(2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}+2u\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial x}+2v\frac{\partial^2 v}{\partial x\partial y}$$
$$-2\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial x}-2u\frac{\partial^2 u}{\partial y\partial x}-2\frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial x}-2v\frac{\partial^2 v}{\partial y\partial x})$$

结合柯西黎曼方程和解析函数存在二阶连续偏导数的性质可知上式等于：

$$\xrightarrow{\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\frac{\partial v}{\partial y}=-\frac{\partial}{\partial y}\frac{\partial v}{\partial x}=\frac{\partial^2 u}{\partial y^2}\cdots}=\frac{1}{4}\left(2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial y}+2\frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial y}\right)$$
$$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

又因为：

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\Rightarrow$$
$$|f'(z)|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

所以有：

$$\frac{\partial^2|f(z)|^2}{\partial z\partial\bar{z}}=|f'(z)|^2$$

接下来证明第二小问原式可以等价于证明：

$$\frac{\partial^2 |f(z)|}{\partial z\partial \bar{z}} \cdot (4|f(z|) = |f'(z)|^2 = \frac{\partial^2 |f(z)|^2}{\partial z\partial \bar{z}}$$

同上证明方法：

$$\begin{aligned}
\frac{\partial^2 |f(z)|}{\partial z\partial \bar{z}} &= \frac{\partial}{\partial z}\frac{\partial\sqrt{u^2+v^2}}{\partial \bar{z}} \\
&= \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial\sqrt{u^2+v^2}}{\partial x} + i\frac{\partial\sqrt{u^2+v^2}}{\partial y}\right) \\
&= \frac{1}{4}(\frac{\partial}{\partial x}(\frac{\partial\sqrt{u^2+v^2}}{\partial x}) + \frac{\partial}{\partial y}(\frac{\partial\sqrt{u^2+v^2}}{\partial y}) \\
&+ i\frac{\partial}{\partial x}\cdot\frac{\partial\sqrt{u^2+v^2}}{\partial y} - i\frac{\partial}{\partial y}\cdot\frac{\partial\sqrt{u^2+v^2}}{\partial x})
\end{aligned}$$

由于 $f(z)$ 解析，可以得出求偏导数之后 $\frac{\partial}{\partial x}\cdot\frac{\partial\sqrt{u^2+v^2}}{\partial y} = \frac{\partial}{\partial y}\cdot\frac{\partial\sqrt{u^2+v^2}}{\partial x}$，所以原式等于:

$$\begin{aligned}
&\frac{1}{4}(\frac{\partial}{\partial x}(\frac{\partial\sqrt{u^2+v^2}}{\partial x}) + \frac{\partial}{\partial y}(\frac{\partial\sqrt{u^2+v^2}}{\partial y})) \\
=&\frac{1}{4}(\frac{\partial}{\partial x}(\frac{u}{\sqrt{u^2+v^2}}\frac{\partial u}{\partial x} + \frac{v}{\sqrt{u^2+v^2}}\frac{\partial v}{\partial x})+ \\
&\frac{\partial}{\partial y}(\frac{u}{\sqrt{u^2+v^2}}\frac{\partial u}{\partial y} + \frac{v}{\sqrt{u^2+v^2}}\frac{\partial v}{\partial y})) \\
=&\frac{1}{4}(\frac{\partial}{\partial x}(\frac{u}{\sqrt{u^2+v^2}})\cdot\frac{\partial u}{\partial x} + \frac{u}{\sqrt{u^2+v^2}}\frac{\partial^2 u}{\partial x^2})+ \\
&\frac{1}{4}(\frac{\partial}{\partial x}(\frac{v}{\sqrt{u^2+v^2}})\cdot\frac{\partial v}{\partial x} + \frac{v}{\sqrt{u^2+v^2}}\frac{\partial^2 v}{\partial x^2})+ \\
&\frac{1}{4}(\frac{\partial}{\partial y}(\frac{u}{\sqrt{u^2+v^2}})\cdot\frac{\partial u}{\partial y} + \frac{u}{\sqrt{u^2+v^2}}\frac{\partial^2 u}{\partial y^2})+ \\
&\frac{1}{4}(\frac{\partial}{\partial y}(\frac{v}{\sqrt{u^2+v^2}})\cdot\frac{\partial v}{\partial y} + \frac{v}{\sqrt{u^2+v^2}}\frac{\partial^2 v}{\partial y^2})
\end{aligned}$$

同理由于 $f(z)$ 解析所以满足 $Cauchy-Riemann$ 方程，所以

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y}$$
$$\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\frac{-\partial v}{\partial x}$$

，结合解析函数存在二阶连续偏导数的性质 ($Task6$)，原式变成：

$$\frac{1}{4}(\frac{\partial}{\partial x}(\frac{u}{\sqrt{u^2+v^2}})\cdot\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}(\frac{v}{\sqrt{u^2+v^2}})\cdot\frac{\partial v}{\partial x}+ \\ \frac{\partial}{\partial y}(\frac{u}{\sqrt{u^2+v^2}})\cdot\frac{\partial u}{\partial y}+\frac{\partial}{\partial y}(\frac{v}{\sqrt{u^2+v^2}})\cdot\frac{\partial v}{\partial y})$$

分组求和得到：

$$\begin{aligned}&\frac{\partial}{\partial x}\left(\frac{u}{\sqrt{u^2+v^2}}\right)+\frac{\partial}{\partial y}\left(\frac{v}{\sqrt{u^2+v^2}}\right)\\&=\frac{v^2\frac{\partial u}{\partial x}-uv\frac{\partial v}{\partial x}+u^2\frac{\partial u}{\partial x}+uv\frac{\partial v}{\partial x}}{(u^2+v^2)^{3/2}}\\&\xrightarrow{v^2\frac{\partial u}{\partial x}+u^2\frac{\partial u}{\partial x}=(u^2+v^2)\frac{\partial u}{\partial x}}\frac{\partial}{\partial x}\left(\frac{u}{\sqrt{u^2+v^2}}\right)+\frac{\partial}{\partial y}\left(\frac{v}{\sqrt{u^2+v^2}}\right)\\&=\frac{(u^2+v^2)\frac{\partial u}{\partial x}}{(u^2+v^2)^{3/2}}=\frac{\frac{\partial u}{\partial x}}{\sqrt{u^2+v^2}}\end{aligned}$$

同理有：

$$-\frac{\partial}{\partial y}\left(\frac{u}{\sqrt{u^2+v^2}}\right)+\frac{\partial}{\partial x}\left(\frac{v}{\sqrt{u^2+v^2}}\right)=\frac{\frac{\partial v}{\partial x}}{\sqrt{u^2+v^2}}$$

所以原式等于：

$$\frac{1}{4\sqrt{u^2+v^2}}\left((\frac{\partial u}{\partial x})^2+(\frac{\partial v}{\partial x})^2\right)$$

所以:

$$\frac{\partial^2|f(z)|}{\partial z\partial\bar{z}}\cdot(4|f(z|)=|f'(z)|^2=\frac{\partial^2|f(z)|^2}{\partial z\partial\bar{z}}=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

成立。 □

Task12:

证明 因为 $f(z)$ 解析的充要条件可以表述为 $\dfrac{\partial f}{\partial \bar{z}} \equiv 0.$

$$\frac{\partial f}{\partial \bar{z}} = e^{\sin(z^2+\bar{z})} \cdot \cos(z^2+\bar{z}) = e^{\sin(z^2+\bar{z})} \cdot \frac{e^{i(z^2+\bar{z})} + e^{-i(z^2+\bar{z})}}{2} \not\equiv 0$$

所以由于指数函数始终不为 0，所以原函数在 $\mathbb{C}$ 上无处解析。 $\square$