Matroid

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January 24, 2019

1 Matriod

Matriod $M = (S, l)$ is an ordered pair such that:

1. S is non-empty and finite
2. $l \subseteq 2^S$ (l is a set containing subsets of S), and $\emptyset \in l$
3. **Downward Closure**: If $B \in l$, $A \in B$, then $A \in l$. Call B or any its subset an **"independent set"**.
4. **Augmentation**: If $A \in l, B \in l$, and $|A| < |B|$, then $\exists x \in (B \backslash A)$ such that $A \cup x \in l$

2 Matriod in Graph

For an undirected graph $G = (V, E)$, define Matroid $M_G = (S_G, l_G)$ as the following:

1. $S_G = E$, or the edge set of the graph
2. If A is a subset of E, $A \in l_C$ if and only if A doesn't contain circles. Or equivalently, a collection of edges A is an independent iff $G(V, A)$ creates a **forest**.

Now we prove M_G satisfies all conditions of a matroid:

1. S_G is obviously non-empty and finite
2. **Downward Closure**: It's trivial that removing edges from a forest will also create a forest
3. **Augmentation**: It's easy to prove that **a forest with k edges have $|V| - k$ trees (counting any isolated vertex also as a tree)**. Next, if $G(V, A), G(V, B)$ are two forests of G, and $|A| < |B|$. Then $G(V, A)$ has $|V| - |A|$ trees, more than the trees in $G(V, B)$ which is $|V| - |B|$. Since $G(V, B)$ has less trees, there must be an edge $e \in B \backslash A$ that connects 2 trees (can also be vertices) of A. Since adding an edge between two trees will not create cycles, $A \cup \{e\} \in l_G$.

Definition 1. If there exists an element x not in an independent set A, and $A \cup x \in l$, we call A an **extendable** independent subset, and x an **extension** to A.

Definition 2. If A doesn't have an extension, we call A a **maximal** independent subset.

Notice that because of the augmentation property, we can derive the following theorem:

Theorem A. ll maximal independet subsets have the same size.

The proof is trivial: or else the larger independent subsets can give extension elements to smaller independet subsets.

3 Weighted Matroid

A matroid $M(S, l)$ is weighted if it's associated with a weight function $w(\cdot) : S \to \mathbb{R}^+$. The independent subset with the maximum weight is called the ***optimal*** independent subset of the matroid.

Since all the weights are positive, the ***optimal*** independent subset is also a ***maximal*** independent subset.

3.1 Matroid Greedy Algorithm

Greedy(M, w):

1. $A = \emptyset$
2. sort $M.S$ into monotonically decreasing order by weight w
3. for each $x \in M.S$, taken in monotonically decreasing order by weight $w(x)$
4. if $A \cup \{x\} \in M.\mathcal{I}$
5. $A = A \cup \{x\}$
6. return A.

Weighted matroid has the property that Matroids exhibit the greedy-choice property.

Lemma 1. Suppose that $M = (S, l)$ is a weighted matroid with weight function w and that S is sorted into monotonically decreasing order by weight. Let x be the first element of S such that $\{x\}$ is independent, if any such x exists. If x exists, then there exists an optimal subset A of S that contains x.

Proof. If such element doesn't exist, then l obviously just contains the empty set.

If such element x exist, we prove by contradiction, supposing B is the optimal independent subset. Construct independent subset A from $\{x\}$ and keep adding elements from B using augmentation to make them the same size. Therefore $A = B - y + x$ and $w(y) < w(x)$. Therefore A is obviously more weighted than B. □

Lemma 2. Let M be any matroid. If x is an element of S that is an extension of some independent subset A of S, then x is also an extension of $\emptyset$.

Proof. By downward closure or heridity, $\{x\}$ is a subset of A. Therefore $\{x\}$ is a valid independent subset of the matroid. □

Lemma 3. Let M be any matroid. If $\{x\}$ is not independent, then x is not and extension to any independent subset.

Proof. Lemma 2's contrapositive. □

Now we can prove that:

Lemma 4. The greedy algorithm produces an optimal independent subset.

Proof. Let x be the first element of S chosen by GREEDY for the weighted matroid $M = (S, l)$. The remaining problem of finding a maximum-weight indepen- dent subset containing x reduces to finding a maximum-weight independent subset of the weighted matroid $M' = (S', l')$, where

$$S' = \{y \in S : \{x, y\} \in l\}$$
$$l' = \{B \subseteq S - \{x\} : B \cup \{x\} \in l\}$$

and the weight function for M' is the weight function for M, restricted to S'. (We call M' the **contraction** of M by the element x .)

In other words, there's a bijection between optimal independet sets of M containing x and the optimal independent sets of M'. □

References

[1] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein *Introduction to Algorithms, Third Edition.* The MIT Press Cambridge, Massachusetts London, England. 2009