Chapter 1

The real and complex number systems

In problems 1-19 are the questions in chapter 1 of Principles of mathematical analysis

Problem 1.1. If r is rational(r≠0)and x is irrational ,prove that r+x and rx are irrational.

Problem 1.2. Prove that there is no rational number whose square is 12.

Problem 1.3. Prove that:
(a)If $x \neq 0$ and xy=xz then y=z.
(b)If $x \neq 0$ and xy=x then y=1.
(c)If $x \neq 0$ and xy=1 then y=1/x.
(d)If $x \neq 0$ then1/(1/x)=x.

Problem 1.4. Let E be a nonempty subset of an ordered set;suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Problem 1.5. Let A be a nonempty set of real numbers which is bounded below. Let -A be the set of all numbers -x ,where x∈A.Prove that

$$inf A = -sup(-A)$$

Problem 1.6. Fix b > 1
(a)If m,n,p,q are integers, n>0 ,q>0, and r=m/n=p/q,prove that

$$(b^m)^{1/n} = (b^p)^{1/q}$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.
(b)Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
(c)If x is real ,define B(x) to be the set of all numbers b^t,where t is rational and t≤x.Prove that

$$b^r = supB(r)$$

when r is retional. Hence it makes sense to define

$$b^x = supB(x)$$

for every real x
(d)Prove that $b^{x+y} = b^x b^y$for all real x and y.

Problem 1.7. Fix b>1,y>0,and prove that there is a unique real x such that $b^x = Y$,by completing the following outline.
(a)For any positive inetger n ,$b^n - 1 \leq n(b-1)$.
(b)Hence b-1 $\leq n(b^{1/n} - 1)$.
(c)If t>1 and n >(b-1)/(t-1),then $b^{1/n} < t$.
(d)If w is such that $b^w < y$,then $b^{w+1/n} > y$ for sufficiently large n;to see this,apply part (c) with $t = yB^{-w}$.
(e)If $b^w > y$,then $b^{w-1/n} < y$ for dufficiently large n.
(f)Let A be the set of all w such that $b^w < y$,and show that x=sup A satisfies $b^x = y$.
(g)Prove that this x is unique.

Problem 1.8. Prove that no order can be defined in the complex field that turns it into an ordered field. Hint:-1 is a square.

Problem 1.9. Suppose z=a+bi,w=c+di.Define z<w if a<c,and also if a=c but b<d.Prove that this turns the set of all complex numbers in to an ordered set.(This type of order relation is called a dictionary order,or lexicographic order,for obvious reasons.)Does this ordered set have the least-upper-bound property?

Problem 1.10. Suppose z=a+bi,w=u+vi,and

$$a = (\frac{|w| + u}{2})^{1/2}, b = (\frac{|w| - u}{2})^{1/2}$$

Prove that $z^2 = w$ if v≥0 and that $(\overline{z})^2 = w$ if v≤0.Conclude that every complex number(with one exception!)has two complex square roots.

Problem 1.11. If z is a complex number, prove that there exists an r≥ 0 and a complex number w with |w|=1 such that z =rw. Are w and r always uniquely determined by z?

Problem 1.12. If $z_1,...,z_n$ are complex, prove that

$$|z_1+z_2+...+z_n| \le |z_1|+|z_2|+...+|z_n|.$$

Problem 1.13. If x,y are complex,prove that

$$||x|-|y|| \le |x-y|.$$

Problem 1.14. If z is a complex number such that |z|=1,that is ,such that $z\bar{z}=1$,compute

$$|1+z|^2+|1-z|^2.$$

Problem 1.15. Under what conditions does equality hold in the Schwarz inequality?

Problem 1.16. Suppose k≥3,x,y∈ R^k, |x-y|=d>0,and r>0.Prove:
(a)If 2r>d,there are infinitely many z∈ R^ksuch that

$$|z-x|=|z-y|=r.$$

(b)If 2r=d,there is exactly one such z.
(c)If 2r<d, there is no such z.
How must these statements be modified if k is 2 or 1?

Problem 1.17. Prove that

$$|x+y|^2+|x-y|^2=2|x|^2+2|y|^2$$

if $x \in R^k$ and $y \in R^k$.Interpret this geometrically,as a statement about parallelograms.

Problem 1.18. If $k \ge 2$ and $x \in R^k$,prove that there exists $y \in R^k$ such that $y \ne 0$ but xy=0.Is this also true if k=1?

Problem 1.19. Suppose $a \in R^k, b \in R^k$.Find $c \in R^k$ and r>0 such that

$$|x-a|=2|x-b|$$

if and only if |x-c|=r.

Chapter 2

Basic Topology

In problems 1-30 are the questions in chapter 2 of Principles of mathematical analysis

Problem 2.1. Prove that the empty set is a subset of every set

Problem 2.2. A complex number z is said to be algebraic if there are integers $a_0, ..., a_n$ not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. Hint: For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + ... + |a_n| = N.$$

Problem 2.3. prove that there exist real numbers which are not algebraic.

Problem 2.4. Is the set of all irratiobal real numbers countable?

Problem 2.5. Construct a bounded set of real numbers with exactly three limit points.

Problem 2.6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points.(Recall that $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points?

Problem 2.7. Let $A_1, A_2, A_3, ...$ be subsets of a metric space.
(a)If $B_n = \cup_{i=1}^{n} A_i$,prove that $\overline{B_n} = \cup_{i=1}^{n} \overline{A_i}$,for n=1,2,3,...
(b)If $B_n = \cup_{i=1}^{n} A_i$,prove that $\overline{B} \supset \cup_{i=1}^{\infty} \overline{A_i}$.
Show, by an example, that this inclusion can be proper.

Problem 2.8. Is every point of every open set E$\subset R^2$ a limit point of E? Answer the same question for closed sets in R^2.

Problem 2.9. Let E^o denote the set of all interior points of a set E?(E^o is called the inetrior of E)
(a)Prove that E^o is always open.
(b)Prove that E is open if and only if $E^o = E$.
(c)If G$\subset$E and G is open, Prove that G $\subset E^o$
(d)Prove that the complement of E^o is the closure of the complement of E.
(e)Do E and $\overline{E}$ always have the same interiors?
(f)Do E and E^o always have the same closures?

Problem 2.10. Let X be and infinite set. For P$\in$X and q $\in$X,define

$$d(p,q) = \begin{cases} 1 & (if p \neq q) \\ 0 & (if p = q) \end{cases}$$

prove that this is a metric.Which subsets of the resulting metric space are open?Which are closed?which are compact?

Problem 2.11. For x $\in R^1$ and y $\in R^1$,define

$$d_1(x,y) = (x-y)^2$$

$$d_2(x,y) = \sqrt{|x-y|}$$

$$d_3(x,y) = |x^2 - y^2|$$

$$d_4(x,y) = |x - 2y|$$

$$d_5(x,y) = \frac{|x-y|}{1+|x-y|}.$$

Determine, for each of these,whether it is a metric or not.

Problem 2.12. Let K$\subset R^1$consist if 0 and the numbers 1/n,for n = 1,2,3,... Prove that K is compact directly from the definition(without using the Heine-Borel theorem).

Problem 2.13. Construct a compact set of real numbers whose limit points form a countalble set.

Problem 2.14. Give an example of an open cover of the segment(0,1) which has no finite subcover.

Problem 2.15. Show that Theorem 2.36 and its Corollary become false(in R^1,for example)if the word "compact" is replaced by "closed" or by "bounded."
Theorem2.36:If K_α is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of k_α is nonempty, then $\cap K_\alpha$ is nonempty.
Corollary If K_n is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}(n = 1,2,3,...)$,then$\cap_1^\infty K_n$ is not empty

Problem 2.16. Regard Q,the set of all rational numbers,as a metric space,with d(p,q)=|p-q|.Let E be the set of all P∈Q such that 2<P^2<3.Show that E is closed and bounded in Q,but that E is not compact.Is E open in Q?

Problem 2.17. Let E be the set of all x ∈ [0, 1] whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in [0, 1]? Is E compact? Is E perfect?

Problem 2.18. Is there a nonempty perfect set in R^1 which contains no rational number?

Problem 2.19. (a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.
(b) Prove the same for disjoint open sets.
(c)Fix p∈ $X,\delta > 0$,define A to be the set of all q∈X for which d(p,q)<δ,define B similarly,with > in place of <.Prove that A and B are separated. (d)Prove that every connected metric space with at least two points is uncountable. Hint: Use (c).

Problem 2.20. Are closures and interiors of connected sets always connected? (Look at subsets of R^2 .)

Problem 2.21. Let A and B be separated subsets of some R^k,suppose a ∈ $A, b \in B$,and define

$$p(t) = (1-t)a + tb$$

for $t \in R^1$.Put $A_0 = p^{-1}(A), B_0 = p^{-1}(B)$.$[Thus t \in A_0 if and only if p(t) \in A.$]
(a)Prove that A_0 and B_0 are separated subsets of R^1.
(b)Prove that there exists $t_0 \in (0,1)$such that $p(t_0) \notin A \cup B$.
(c) Prove that every convex subset of R^k is connected.

Problem 2.22. A metric space is called separable if it contains a countable dense subset. Show that R^k is separable. Hint: Consider the set of points which have only rational coordinates.

Problem 2.23. A collection V_α of open subsets of X is said to be a base for X if the following is true:For every x∈X and every open set G⊂ X such that x ∈ G, we have $x \in V_\alpha \subset G$ for some α.In other words, every open set in X is the union of a subcollection ofV_α.

Prove that every separable metric space has a countable base. Hint: Take all neighborhoods with rational radius and center in some countable dense subset of X.

Problem 2.24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. Hint: Fix δ>0,and pick $x_1 \in X$.Having chosen $x_1,...,x_j \in X$,choose $x_{j+1} \in X$,if possible ,so that d(x_j,x_{j+1}) $\geq \delta$ for i=1,...,j. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ.Take $\delta = 1/n(n = 1,2,3,...)$,and consider the centers of the corresponding neighborhoods.

Problem 2.25. Prove that every compact metric space K has a countable base, and that K is therefore separable. Hint: For every positive integer n, there are finitely many neighborhoods of radius 1/n whose union covers K.

Problem 2.26. Let X be a metric space in which every infinite subset has a limit point. Prove that Xis compact. Hint: By Exercises 23 and 24, X has a countable base. It follows that every open cover of X has a countable subcover G_n,n=1,2,3,... If no finite subcollection of G_n covers X,then the complement F_n of $G_1 \cup ... \cup G_n$ is nonempty for each n, but $\cap F_n$ is empty. If E is a set which contains a point from each F_n,consider a limit point of E, and obtain a contradiction.

Problem 2.27. Define a point p in a metric space X to be a condensation point of a set E ⊂X if every neighborhood of p contains uncountably many

points of E.
Suppose E$\subset R^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. Hint: Let V_n be a countable base of R^k,let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

Problem 2.28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (Corollary: Every countable closed set in R^k has isolated points.) Hint: Use Exercise 27.

Problem 2.29. Prove that every open set in R 1 is the union of an at most countable collection of disjoint segments. Hint: Use Exercise 22.

Problem 2.30. Prove that:
If $R^k = \cup_1^\infty F_n$,where each F_n is a closed subset of R^k,then at least one F_n has a nonempty interior.
Equivalent statement: If G_n is a dense open subset of R^k, for n=1,2,3,..., then $\cap_1^\infty G_n$ is not empty(in fact, it is dense in R^k).
Hint:Imitate the proof of Theorem 2.43.

Chapter 3

Numerical sequences and series

In problems 1-25 are the questions in chapter 3 of Principles of mathematical analysis

Problem 3.1. Prove that convergence of S_n implies convergence of $|S_n|$. Is the converse true?

Problem 3.2. Calculate $\lim_{n\to\infty}(\sqrt{n^2+n}-n)$.

Problem 3.3. If $S_1=\sqrt{2}$,and

$$S_{n+1}=\sqrt{2+\sqrt{S_n}} \quad (n=1,2,3,...)$$

prove that S_n converges and that $S_n<2$ for n =1,2,3,... .

Problem 3.4. Find the upper and lower limits of the sequence S_n defined by

$$S_1=0; \quad S_{2m}=\frac{S_{2m-1}}{2}; \quad S_{2m+1}=\frac{1}{2}+S_{2m}$$

.

Problem 3.5. For any two real sequences a_n, b_n, prove that

$$\lim_{n\to\infty}\sup(a_n+b_n)\le\lim_{n\to\infty}\sup a_n+\lim_{n\to\infty}\sup b_n$$

, provided the sum on the right is not of the form $\infty-\infty$.

Problem 3.6. Investigate the behavior (convergence or divergence) of $\sum a_n$ if
(a)$a_n = \sqrt{n+1} - \sqrt{n}$;
(b)$a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$;
(c)$a_n = (\sqrt[n]{n} - 1)^n$;
(d)$a_n = \frac{1}{1+z^n}$, for complex values of z.

Problem 3.7. Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n}$$

, if $a_n \geq 0$.

Problem 3.8. If $\sum a_n$ converges, and if b_n is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Problem 3.9. Find the 1ůadius of convergence of each of the following power series: (a)$\sum n^3 z^n$ (b)$\sum \frac{2^n}{n!} z^n$ (c)$\sum \frac{2^n}{n^2} z^n$ (d)$\sum \frac{n^3}{3^n} z^n$

Problem 3.10. Suppose that the coefficients of the power series $\sum a_n z^n$ zn are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

Problem 3.11. Suppose $a_n > 0, S_n = a_1 + ... + a_n$ and $\sum a_n$ diverges.
(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges. (b) Prove that

$$\frac{a_{N+1}}{S_{N+1}} + ... + \frac{a_{N+k}}{S_{N+k}} \geq 1 - \frac{S_N}{S_{N+k}}$$

and deduce that $\sum \frac{a_n}{S_n}$ diverges.
(c)Prove that

$$\frac{a_n}{S_n^2} \leq \frac{1}{S_{n-1}} - \frac{1}{S_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.
(d) What can be said sbout

$$\sum \frac{a_n}{1+na_n} \quad and \quad \sum \frac{a_n}{1+n^2 a_n}?$$

Problem 3.12. Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{\infty}^{m=n} a_m.$$

(a)Prove that

$$\frac{a_m}{r_m} + ... + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if m<n , and deduce that $\sum \frac{a_n}{r_n}$ diverges.
(b)Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Problem 3.13. Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Problem 3.14. If S_n is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{S_0 + ... + S_n}{n+1} \quad (n = 0, 1, 2, ...).$$

(a)If lim $S_n = S$, prove that $\lim \sigma_n = S$.
(b)Construct a sequence S_n which does not converge, although $\lim \sigma_n = 0$.
(c)Can it happen that $S_n > 0$ for all n and that $\limsup S_n = \infty$, although $\lim \sigma_n = 0$?
(d)Put $a_n = S_n - S_{n-1}$, for n≥1 . Show that

$$S_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k.$$

Assume that $\lim(na_n) = 0$ and that σ_n converges. Prove that S_n converges.[This gives a converse of (a), but under the additional assumption that $na_n \to 0$.]
(e)Derive the last conclusion from a weaker hypothesis: Assume<∞,$|na_n| \leq M$for all n,and $\lim \sigma_n = \sigma$. Prove that $\lim S_n = \sigma$,by completing the following outline: If m<n,then

$$S_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{i=m+1}^{n} (S_n - S_i).$$

For these i,

$$|S_n - S_t| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

Fix $\epsilon > 0$ and associate with each n integer m that satisfies

$$m \leq \frac{n-\epsilon}{1+\epsilon} < m+1.$$

Then (m+1)/(n-m)$\leq 1/\epsilon$ and $|S_n - S_i| \leq M\epsilon$.Hence

$$\lim_{n\to\infty} \sup |S_n - \sigma| / leM\epsilon.$$

Since ϵ wa arbitrary,$\lim S_n = \sigma$.

Problem 3.15. Definition 3.21 can be extended to the case in which the a_n lie in some fixed R^k. Absolute convergence is defined as convergence of $\sum |a_n|$Show that Theorems 3.22, 3.23, 3.25(a), 3.33, 3.34, 3.42, 3.45, 3.47, and 3.55 are true in this more general setting. (Only slight modifications are required in any of the proofs.)

Problem 3.16. Fix a positive number α.Choose $x_1 > \sqrt{\alpha}$,and define $x_2, x_3, ...,$,by the recursion formula

$$x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n}).$$

(a)Prove that x_n decreases monotonically and that $\lim x_n = \sqrt{\alpha}$
(b)Put $\epsilon_n = x_n - \sqrt{\alpha}$,and Show that

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\epsilon_{n+1} < \beta(\frac{\epsilon_1}{\beta})^{2^n} \quad (n = 1, 2, 3, ...).$$

(c)This is a good algorithm for computing square roots. since the recursion formula is simple and the convergence is extremely rapid. For example,if $\alpha = 3$ and $x_1 = 2$,show that $\epsilon_1/\beta < 0.1$ and that therefore

$$\epsilon_5 < 4 * 10^{-16}, \quad \epsilon_6 < 4 * 10^{-32}.$$

Problem 3.17. Fix $\alpha > 1$.Take $x_1 > \sqrt{\alpha}$,and define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}$$

(a)Prove that $x_1 > x_3 > x_5 > \ldots.$
(b)Prove that $x_2 < x_4 < x_6 < \ldots.$
(c)Prove that $\lim x_n = \sqrt{\alpha}$.
(d)Compare the rapidity of convergence of this process with the one described in Exercise 16.

Problem 3.18. Replace the recursion formula of Exercise 16 by

$$x_{n+1} = \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1}$$

where p is a fixed positive integer, and describe the behavior of the resulting sequences x_n.

Problem 3.19. Associate to each sequence a= α_n, in which α_n is 0 or 2, the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove that the set of all x(a) is precisely the Cantor set described in Sec. 2.

Problem 3.20. Suppose P_n is a Cauchy sequence in a metric space X, and some subsequence p_{ni} converges to a point p∈X. Prove that the full sequence p_n converges to p.

Problem 3.21. Prove the following analogue of Theorem 3.l0(b): If E_n is a sequence of closed nonempty and bounded sets in a complete metric space X,if $E_n \supset E_{n+1}$,and if

$$\lim_{n\to\infty} diam E_n = 0,$$

then $\cap_1^\infty E_n$ consists of exactly one point.

Theorem 3.lO(b):If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1} (n = 1,2,3,\ldots)$ and if

$$\lim_{n\to\infty} diam K_n = 0,$$

then $\cap_1^\infty K_n$ consists of exactly one point.

Problem 3.22. Suppose Xis a nonempty complete metric space, and G_n is a sequence of dense open subsets of X. Prove Baire's theorem,namely, that $\cap_1^\infty G_n$ is not empty.(In fact, it is dense in X.)

Problem 3.23. Suppose p_n and q_n are Cauchy sequences in a metric space X. Show that the sequence d(P_n, q_n) converges.

Problem 3.24. Let X be a metric space.
(a)Call two Cauchy sequences p_n, q_n in X equivalent if

$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.
(b)Let X^* be the set of all equivalence classes so obtained. If $P \in X^*$,$Q \in X^*$,$p_n \in P, q_n \in Q$,define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n):$$

by Exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if p_n and q_n are replaced by equivalent sequences, and hence that Δ is a distance function in X^*.
(c)Prove that the resulting metric space X^* is complete.
(d)For each p∈X, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Pove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all p,q∈ X. In other words, the mapping ϕ defined by $\phi(p) = P_p$ is an isometry (i.e., a distance-preserving mapping) of X into X^*.
(e)Prove that $\phi(X)$ is dense in X^*,and that $\phi(X) = X^*$ if X is complete .By(d),we may identify X and $\phi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the completion of X.

Problem 3.25. Let X be the metric space whose points are the rational numbers, with the metric d(x, y) = | x - y | , What is the completion of this space?

Chapter 4

Countunity

In problems 1-26 are the questions in chapter 4 of Principles of mathematical analysis

Problem 4.1. Suppose f is a real function defined on R^2 which satisfies

$$\lim_{h \to 0}[f(x+h)-f(x-h)]=0$$

for every$x \in R^1$,Does this imply that f is continuous?

Problem 4.2. If f is a continuous mapping of a metric space X into a metric space Y,prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set E⊂X,(E denotes the closure of E.) Show ,by an example, that F($\overline{E}$) can be a proper subset of $\overline{F(E)}$.

Problem 4.3. Let f be a continuous real function on a metric space X.Let Z(f) (the zero set of f) be the set of all p∈X at which f(p)=0,Prove that Z(f) is closed.

Problem 4.4. Let f and g be continuous mapping of a metric space X into a metric space Y,and let E be a dense subset of X,Prove that f(E) is dense in f(X),If g(p)=f(p) for all p∈E, prove that g(p)=f(p) for all p∈X,(In other words, a continuous mapping is determined by its values on a dense subset of its domain.)

Problem 4.5. If f is a real continuous function defined on a closed set E⊂ R^1 ,prove that there exist continuous real functions g on R^1 such that g(x)=f(x) for all x∈E.(Such functions g are called continuous extensions of f from E to R^1.)Show that the result becomes false if the word 'closed' is omitted. Extend the result to vector-valued functions.

Hint:Let the graph of g be a straight line on each of the segments which consitute the complement of E.

Problem 4.6. If f is defined on E, the graph of f is the set of points(x,f(x)),for x∈E,In particualr,If E is a set of real numbers, and f is real-values, the graph of f is a subset of the plane.

Suppose E is compact ,and prove that f is continuous on E if and only if its graph is compact.

Problem 4.7. If E⊂X and if f is a function defined on X,the restriction of f to E is the function g whose domain of definition is E, such that g(p)=f(p) fro p ∈ E,Define f and g on R^2 by :$f(0,0) = g(0,0) = 0, f(x,y) = xy^2/(x^2 + y^4), g9x,y) = xy^2/(x^2 + y^6)$,if $(x,y) \neq (0,0)$. Prove that f is bounded on R^2,that g is unbounded in every neighborhood of (0,0),and that f is not countinuous at (0,0);neertheless,the restrictions of both f and g to every straight line in R^2 are continuous!

Problem 4.8. Let f be a real uniformly continuous function on the bounded set E in R^2.Prove that f is bounded on E.

Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

Problem 4.9. Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets:To every $\epsilon > 0$there exists a $\delta > 0$ such that diam$f(E) < \epsilon$ for all e⊂X with diam E< δ.

Problem 4.10. Complete the details of the following alternative proof of Theorem 4.19:If f is not uniformly continuous, then for some $\epsilon > 0$ there are sequencesp_n,q_n in X such that $d_x(p_n, q_n) \to 0$ but $d_y(f(p_n), f(q_n)) > \epsilon$. Use theorem 2.37 to obtain a contradiction.

Problem 4.11. Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $f(x_n)$ is a Cauchy sequence in Y for every Cauchy sequenceX_n in X. Use this result to give an alternative proof the theorem stated in Exercise 13.

Problem 4.12. A uniformly continuous function of a uniformly continuous function is uniformly continuous.

State this more precisely and prove it.

Problem 4.13. Let E be a dense subset of a metric space X, and Let f be a uniformly continuous real function defined on E.Prove that f has a continuous extension from E to X.

Hint:For each p∈ X and each positive integer n, let $V_m(p)$ be the set of all q∈E with d(p,q)<1/n.Use Exercise 9 to show that the intersection of the closures of the sets $f(V_1(p), f(V_2(p)))$,...,consist of a single point, say g(p),of R^1.prove that the function g so defined on X is the desired extension of f.

Could the range space R^1 be replaced by R^k? By any compact metric space? By any complete metric space? By any metric space?

Problem 4.14. Let I=[0,1] be the closed unit interval.Suppose f is a continuous mapping of I into I.Prove that f(x)=x for at least one x∈I.

Problem 4.15. Call a mapping of X into Y open if f(V) is an open set in Y whenever V is an open set in X.

Prove that every continuous open mapping of R^1 into R^1 is monotonic.

Problem 4.16. Let [x] denote the largest integer contained in x,that is ,[x] is the integer such that x-1<[x]≤x;and let (x)=x-[x] denote the fractional part of x,What discontinuities do the functions [x] and (x) have?

Problem 4.17. Let f be a real function defined on (a,b).Prove that the set of points at which f has a simple discontinuity is at most countable.

Hint:Let E be the set on which f(x-)<f(x+). With each point x of E,associate a triple (q,p,r) of rational numbers such that

(a)f(x-)<p<f(x+)

(b)a<q<t<x implies f(t)<p

(c)x<t<r<b implies f(t)>p

The set of all such triples is countable.Show that each triple is associated with at most one point if E. Deal similarly with the other possible types of simple discontnuities.

Problem 4.18. Every rational x can be written in the form x =m/n,where n>0,and m and n are integers without any common divisors. When x=0 ,we take n=1.Consider the function f defined on R^1 by

$$f(x) = \begin{cases} 0 \quad (x \quad irrational) \\ 1/n \qquad\qquad (x = \frac{m}{n}) \end{cases} \tag{4.1}$$

Prove that f is continuous at every irrational point,and that f has a simple discontinuity at every rational point.

Problem 4.19. Suppose f is a real function with domain R^1 which has the intermediate value property :If f(a)<c<f(b),then f(x)=c for some x between a and b.

Suppose also, for evert rational r, that the set of all x with f(x)=r is closed. Prove that f is countinuous.

Problem 4.20. If E is a nonempty subset of a metric space X, define the distance from x∈X to E by

$$\rho_E(x) = \inf_{z \in E} d(x,z)$$

(a) Prove that $\rho_E(x) = 0$ if and only if x∈E.
(b)Prove that ρ_E is a uniformly continuous function on X,by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x,y)$$

for all x∈E,y∈E.

Problem 4.21. Suppose K and F are disjoint sets in a metric space X,K is compact,F is closed. Prove that there exists$\delta > 0$ such that d(p,q)>δ if p∈K,q∈F.

Show that the conclusion may fail for two disjoint closed sets if neither is compact.

Problem 4.22. Let A and B be disjoint nonempty colsed sets in a metric space X,and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)} (p \in X)$$

Show that f is a countinuous function on X whose range lies in [0,1],that f(p)=0 precisely on A and f(p)=1 precisely on B. This establishes a converse of Exercises 3: Every closed set A $\subset$ X is Z(f) for some continuous real F on X. Setting

$$V = f^{-1}([0,1/2), \quad WF^{-1}((1/2,1])$$

Show that V and W are open and disjoint, and that a$\subset$V,B$\subset$W.

Problem 4.23. A real-valued function f defined in (a,b) is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

whenever a<x<b ,a<y<b,0<λ<1. Prove that every convex fuction is continuous . Prove that every increasing convex function of a convex function is convex.

If f is convex in (a,b) and if a<s<t<u<b, show that

$$\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}$$

Problem 4.24. Assume that f is a continuous real function defined in (a,b) such that

$$f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$$

for all x ,y $\in$(a,b). Prove that f is convex.

Problem 4.25. If A$\subset R^k$and B $\subset R^k$,define A+B to be the set of all sums x+y with x$\in$A,y$\in$B.

(a) If K is compact and C is closed in R^k,prove that k+c is closed.

(b)Let α be an irrational real number. Let C_1 be the set of all integers ,let C_2 be the set of all nα with n$\in C_1$. Shwo that C_1 and C_2 are closed subsets of R^1 whose sum $C_1 + C_2$ is not closed, by showing that $C_1 + C_2$ is a countable dense subset of R^1.

Problem 4.26. Suppose X,Y,Z are metric spaces,and Y is compact.Let f map X into Y,let g be a continuous one-to-one mapping of Y into Z,and put h(x)=g(f(x)) for x$\in$X.

Prove that f is uniformly continuous if h is uniformly continuous.Prove also that f is continuous if h is continuous.

Show that the compactness of Y cannot ve omitted from the hypotheses,even when X and Z are compact.

Chapter 5

DIFFERENTIATION

In problems 1-25 are the questions in chapter 5 of Principles of mathematical analysis