Chapter 1

The real and complex number systems

In problems 1-19 are the questions in chapter 1 of Principles of mathematical analysis

Problem 1.1. If r is rational(r≠0)and x is irrational ,prove that r+x and rx are irrational.

Problem 1.2. Prove that there is no rational number whose square is 12.

Problem 1.3. Prove that:
(a)If $x \neq 0$ and xy=xz then y=z.
(b)If $x \neq 0$ and xy=x then y=1.
(c)If $x \neq 0$ and xy=1 then y=1/x.
(d)If $x \neq 0$ then1/(1/x)=x.

Problem 1.4. Let E be a nonempty subset of an ordered set;suppose α is a lowe bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Problem 1.5. Let A be a nonempty set of real numbers which is bounded below. Let -A be the set of all numbers -x ,where x∈A.Prove that

$$infA = -sup(-A)$$

Problem 1.6. Fix b > 1
(a)If m,n,p,q are integers, n>0 ,q>0, and r=m/n=p/q,prove that

$$(b^m)^{1/n} = (b^p)^{1/q}$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.
(b)Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
(c)If x is real ,define B(x) to be the set of all numbers b^t,where t is rational and t≤x.Prove that

$$b^r = supB(r)$$

when r is retional. Hence it makes sense to define

$$b^x = supB(x)$$

for every real x
(d)Prove that $b^{x+y} = b^x b^y$for all real x and y.

Problem 1.7. Fix b>1,y>0,and prove that there is a unique real x such that $b^x = Y$,by completing the following outline.
(a)For any positive inetger n ,$b^n - 1 \leq n(b-1)$.
(b)Hence b-1 $\leq n(b^{1/n} - 1)$.
(c)If t>1 and n >(b-1)/(t-1),then $b^{1/n} < t$.
(d)If w is such that $b^w < y$,then $b^{w+1/n} > y$ for sufficiently large n;to see this,apply part (c) with $t = yB^{-w}$.
(e)If $b^w > y$,then $b^{w-1/n} < y$ for dufficiently large n.
(f)Let A be the set of all w such that $b^w < y$,and show that x=sup A satisfies $b^x = y$.
(g)Prove that this x is unique.

Problem 1.8. Prove that no order can be defined in the complex field that turns it into an ordered field. Hint:-1 is a square.

Problem 1.9. Suppose z=a+bi,w=c+di.Define z<w if a<c,and also if a=c but b<d.Prove that this turns the set of all complex numbers in to an ordered set.(This type of order relation is called a dictionary order,or lexicographic order,for obvious reasons.)Does this ordered set have the least-upper-bound property?

Problem 1.10. Suppose z=a+bi,w=u+vi,and

$$a = (\frac{|w| + u}{2})^{1/2}, b = (\frac{|w| - u}{2})^{1/2}$$

Prove that $z^2 = w$ if v≥0 and that $(\overline{z})^2 = w$ if v≤0.Conclude that every complex number(with one exception!)has two complex square roots.

Problem 1.11. If z is a complex number, prove that there exists an r≥ 0 and a complex number w with |w|=1 such that z =rw. Are w and r always uniquely determined by z?

Problem 1.12. If $z_1,...,z_n$ are complex, prove that

$$|z_1+z_2+...+z_n| \leq |z_1|+|z_2|+...+|z_n|.$$

Problem 1.13. If x,y are complex,prove that

$$||x|-|y|| \leq |x-y|.$$

Problem 1.14. If z is a complex number such that |z|=1,that is ,such that $z\bar{z}=1$,compute

$$|1+z|^2+|1-z|^2.$$

Problem 1.15. Under what conditions does equality hold in the Schwarz inequality?

Problem 1.16. Suppose k≥3,x,y∈ R^k, |x-y|=d>0,and r>0.Prove:
(a)If 2r>d,there are infinitely many z∈ R^ksuch that

$$|z-x|=|z-y|=r.$$

(b)If 2r=d,there is exactly one such z.
(c)If 2r<d, there is no such z.
How must these sattements be modified if k is 2 or 1?

Problem 1.17. Prove that

$$|x+y|^2+|x-y|^2=2|x|^2+2|y|^2$$

if $x \in R^k$ and $y \in R^k$.Interpret this geometrically,as a statement about parallelograms.

Problem 1.18. If $k \geq 2$ and $x \in R^k$,prove that there exists $y \in R^k$ such that $y \neq 0$ but xy=0.Is this also true if k=1?

Problem 1.19. Suppose $a \in R^k, b \in R^k$.Find $c \in R^k$ and r>0 such that

$$|x-a|=2|x-b|$$

if and only if |x-c|=r.