Chapter 1

The real and complex number systems

In problems 1-19 are the questions in chapter 1 of Principles of mathematical analysis

Problem 1.1. If r is rational(r≠0)and x is irrational ,prove that r+x and rx are irrational.

Problem 1.2. Prove that there is no rational number whose square is 12.

Problem 1.3. Prove that:
(a)If $x \neq 0$ and xy=xz then y=z.
(b)If $x \neq 0$ and xy=x then y=1.
(c)If $x \neq 0$ and xy=1 then y=1/x.
(d)If $x \neq 0$ then1/(1/x)=x.

Problem 1.4. Let E be a nonempty subset of an ordered set;suppose α is a lowe bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Problem 1.5. Let A be a nonempty set of real numbers which is bounded below. Let -A be the set of all numbers -x ,where x∈A.Prove that

$$inf A = -sup(-A)$$

Problem 1.6. Fix b > 1
(a)If m,n,p,q are integers, n>0 ,q>0, and r=m/n=p/q,prove that

$$(b^m)^{1/n} = (b^p)^{1/q}$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.
(b)Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
(c)If x is real ,define B(x) to be the set of all numbers b^t,where t is rational and t≤x.Prove that

$$b^r = supB(r)$$

when r is retional. Hence it makes sense to define

$$b^x = supB(x)$$

for every real x
(d)Prove that $b^{x+y} = b^x b^y$for all real x and y.

Problem 1.7. Fix b>1,y>0,and prove that there is a unique real x such that $b^x = Y$,by completing the following outline.
(a)For any positive inetger n ,$b^n - 1 \leq n(b-1)$.
(b)Hence b-1 $\leq n(b^{1/n} - 1)$.
(c)If t>1 and n >(b-1)/(t-1),then $b^{1/n} < t$.
(d)If w is such that $b^w < y$,then $b^{w+1/n} > y$ for sufficiently large n;to see this,apply part (c) with $t = yB^{-w}$.
(e)If $b^w > y$,then $b^{w-1/n} < y$ for dufficiently large n.
(f)Let A be the set of all w such that $b^w < y$,and show that x=sup A satisfies $b^x = y$.
(g)Prove that this x is unique.

Problem 1.8. Prove that no order can be defined in the complex field that turns it into an ordered field. Hint:-1 is a square.

Problem 1.9. Suppose z=a+bi,w=c+di.Define z<w if a<c,and also if a=c but b<d.Prove that this turns the set of all complex numbers in to an ordered set.(This type of order relation is called a dictionary order,or lexicographic order,for obvious reasons.)Does this ordered set have the least-upper-bound property?

Problem 1.10. Suppose z=a+bi,w=u+vi,and

$$a = (\frac{|w| + u}{2})^{1/2}, b = (\frac{|w| - u}{2})^{1/2}$$

Prove that $z^2 = w$ if v≥0 and that $(\overline{z})^2 = w$ if v≤0.Conclude that every complex number(with one exception!)has two complex square roots.

Problem 1.11. If z is a complex number, prove that there exists an r≥ 0 and a complex number w with |w|=1 such that z =rw. Are w and r always uniquely determined by z?

Problem 1.12. If $z_1,...,z_n$ are complex, prove that

$$|z_1+z_2+...+z_n| \leq |z_1|+|z_2|+...+|z_n|.$$

Problem 1.13. If x,y are complex,prove that

$$||x|-|y|| \leq |x-y|.$$

Problem 1.14. If z is a complex number such that |z|=1,that is ,such that $z\overline{z}=1$,compute

$$|1+z|^2+|1-z|^2.$$

Problem 1.15. Under what conditions does equality hold in the Schwarz inequality?

Problem 1.16. Suppose k≥3,x,y∈ R^k, |x-y|=d>0,and r>0.Prove:
(a)If 2r>d,there are infinitely many z∈ R^ksuch that

$$|z-x|=|z-y|=r.$$

(b)If 2r=d,there is exactly one such z.
(c)If 2r<d, there is no such z.
How must these sattements be modified if k is 2 or 1?

Problem 1.17. Prove that

$$|x+y|^2+|x-y|^2=2|x|^2+2|y|^2$$

if $x \in R^k$ and $y \in R^k$.Interpret this geometrically,as a statement about parallelograms.

Problem 1.18. If $k \geq 2$ and $x \in R^k$,prove that there exists $y \in R^k$ such that $y \neq 0$ but xy=0.Is this also true if k=1?

Problem 1.19. Suppose $a \in R^k, b \in R^k$.Find $c \in R^k$ and r>0 such that

$$|x-a|=2|x-b|$$

if and only if |x-c|=r.

Chapter 2

Basic Topology

In problems 1-30 are the questions in chapter 2 of Principles of mathematical analysis

Problem 2.1. Prove that the empty set is a subset of every set

Problem 2.2. A complex number z is said to be algebraic if there are integers $a_0,...,a_n$ not all zero, such that

$$a_0z^n + a_1z^{n-1} + ... + a_{n-1}z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. Hint: For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + ... + |a_n| = N.$$

Problem 2.3. prove that there exist real numbers which are not algebraic.

Problem 2.4. Is the set of all irratiobal real numbers countable?

Problem 2.5. Construct a bounded set of real numbers with exactly three limit points.

Problem 2.6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points.(Recall that $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points?

Problem 2.7. Let $A_1, A_2, A_3, ...$ be subsets of a metric space.
(a)If $B_n = \cup_{i=1}^{n} A_i$,prove that $\overline{B_n} = \cup_{i=1}^{n} \overline{A_i}$,for n=1,2,3,...
(b)If $B_n = \cup_{i=1}^{n} A_i$,prove that $\overline{B} \supset \cup_{i=1}^{\infty} \overline{A_i}$.
Show, by an example, that this inclusion can be proper.

Problem 2.8. Is every point of every open set E$\subset R^2$ a limit point of E? Answer the same question for closed sets in R^2.

Problem 2.9. Let E^o denote the set of all interior points of a set E?(E^o is called the inetrior of E)
(a)Prove that E^o is always open.
(b)Prove that E is open if and only if $E^o = E$.
(c)If G$\subset$E and G is open, Prove that G $\subset E^o$
(d)Prove that the complement of E^o is the closure of the complement of E.
(e)Do E and $\overline{E}$ always have the same interiors?
(f)Do E and E^o always have the same closures?

Problem 2.10. Let X be and infinite set. For P$\in$X and q $\in$X,define

$$d(p,q) = \begin{cases} 1 & (if p \neq q) \\ 0 & (if p = q) \end{cases}$$

prove that this is a metric.Which subsets of the resulting metric space are open?Which are closed?which are compact?

Problem 2.11. For x $\in R^1$ and y $\in R^1$,define

$$d_1(x,y) = (x-y)^2$$

$$d_2(x,y) = \sqrt{|x-y|}$$

$$d_3(x,y) = |x^2 - y^2|$$

$$d_4(x,y) = |x-2y|$$

$$d_5(x,y) = \frac{|x-y|}{1+|x-y|}.$$

Determine, for each of these,whether it is a metric or not.

Problem 2.12. Let K$\subset R^1$consist if 0 and the numbers 1/n,for n = 1,2,3,... Prove that K is compact directly from the definition(without using the Heine-Borel theorem).

Problem 2.13. Construct a compact set of real numbers whose limit points form a countalble set.

Problem 2.14. Give an example of an open cover of the segment(0,1) which has no finite subcover.

Problem 2.15. Show that Theorem 2.36 and its Corollary become false(in R^1,for example)if the word "compact" is replaced by "closed" or by "bounded."
Theorem2.36:If K_α is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of k_α is nonempty, then $\cap K_\alpha$ is nonempty.
Corollary If K_n is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}(n = 1,2,3,...)$,then$\cap_1^\infty K_n$ is not empty

Problem 2.16. Regard Q,the set of all rational numbers,as a metric space,with d(p,q)=|p-q|.Let E be the set of all P∈Q such that 2<P^2<3.Show that E is closed and bounded in Q,but that E is not compact.Is E open in Q?

Problem 2.17. Let E be the set of all x ∈ [0, 1] whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in [0, 1]? Is E compact? Is E perfect?

Problem 2.18. Is there a nonempty perfect set in R^1 which contains no rational number?

Problem 2.19. (a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.
(b) Prove the same for disjoint open sets.
(c)Fix p∈ $X,\delta > 0$,define A to be the set of all q∈X for which d(p,q)<δ,define B similarly,with > in place of <.Prove that A and B are separated. (d)Prove that every connected metric space with at least two points is uncountable. Hint: Use (c).

Problem 2.20. Are closures and interiors of connected sets always connected? (Look at subsets of R^2 .)

Problem 2.21. Let A and B be separated subsets of some R^k,suppose a ∈ $A, b \in B$,and define

$$p(t) = (1-t)a + tb$$

for $t \in R^1$.Put $A_0 = p^{-1}(A), B_0 = p^{-1}(B)$.$[Thus t \in A_0 if and only if p(t) \in A.$]
(a)Prove that A_0 and B_0 are separated subsets of R^1.
(b)Prove that there exists $t_0 \in (0,1)$such that $p(t_0) \notin A \cup B$.
(c) Prove that every convex subset of R^k is connected.

Problem 2.22. A metric space is called separable if it contains a countable dense subset. Show that R^k is separable. Hint: Consider the set of points which have only rational coordinates.

Problem 2.23. A collection V_α of open subsets of X is said to be a base for X if the following is true:For every x∈X and every open set G⊂ X such that x ∈ G, we have $x \in V_\alpha \subset G$ for some α.In other words, every open set in X is the union of a subcollection ofV_α.

Prove that every separable metric space has a countable base. Hint: Take all neighborhoods with rational radius and center in some countable dense subset of X.

Problem 2.24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. Hint: Fix δ>0,and pick $x_1 \in X$.Having chosen $x_1,...,x_j \in X$,choose $x_{j+1} \in X$,if possible ,so that d(x_j,x_{j+1}) $\geq \delta$ for i=1,...,j. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ.Take $\delta = 1/n(n = 1,2,3,...)$,and consider the centers of the corresponding neighborhoods.

Problem 2.25. Prove that every compact metric space K has a countable base, and that K is therefore separable. Hint: For every positive integer n, there are finitely many neighborhoods of radius 1/n whose union covers K.

Problem 2.26. Let X be a metric space in which every infinite subset has a limit point. Prove that Xis compact. Hint: By Exercises 23 and 24, X has a countable base. It follows that every open cover of X has a countable subcover G_n,n=1,2,3,... If no finite subcollection of G_n covers X,then the complement F_n of $G_1 \cup ... \cup G_n$ is nonempty for each n, but $\cap F_n$ is empty. If E is a set which contains a point from each F_n,consider a limit point of E, and obtain a contradiction.

Problem 2.27. Define a point p in a metric space X to be a condensation point of a set E ⊂X if every neighborhood of p contains uncountably many

points of E.
Suppose E$\subset R^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. Hint: Let V_n be a countable base of R^k,let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

Problem 2.28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (Corollary: Every countable closed set in R^k has isolated points.) Hint: Use Exercise 27.

Problem 2.29. Prove that every open set in R 1 is the union of an at most countable collection of disjoint segments. Hint: Use Exercise 22.

Problem 2.30. Prove that:
If $R^k = \cup_1^\infty F_n$,where each F_n is a closed subset of R^k,then at least one F_n has a nonempty interior.
Equivalent statement: If G_n is a dense open subset of R^k, for n=1,2,3,..., then $\cap_1^\infty G_n$ is not empty(in fact, it is dense in R^k).
Hint:Imitate the proof of Theorem 2.43.

Chapter 3

Numerical sequences and series

In problems 1-25 are the questions in chapter 3 of Principles of mathematical analysis

Problem 3.1. ...