Chapter 1

The real and complex number systems

In problems 1-20 are the questions in chapter 1 of Principles of mathematical analysis

Problem 1.1. If r is rational($r \neq 0$)and x is irrational ,prove that r+x and rx are irrational.

Problem 1.2. Prove that there is no rational number whose square is 12.

Problem 1.3. Let E be a nonempty subset of an ordered set;suppose α is a lowe bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Problem 1.4. Let A be a nonempty set of real numbers which is bounded below. Let -A be the set of all numbers -x ,where $x \in A$.Prove that

$$inf A = -sup(-A)$$

Problem 1.5. Fix b > 1
(a)If m,n,p,q are integers, n>0 ,q>0, and r=m/n=p/q,prove that

$$(b^m)^{1/n} = (b^p)^{1/q}$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.
(b)Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
(c)If x is real ,define B(x) to be the set of all numbers b^t,where t is rational and $t \leq x$.Prove that

$$b^r = supB(r)$$

when r is retional. Hence it makes sense to define

$$b^x = supB(x)$$

for every real x
(d)Prove that $b^{x+y} = b^x b^y$for all real x and y.

Problem 1.6. Fix b>1,y>0,and prove that there is a unique real x such that $b^x = Y$,by completing the following outline.
(a)For any positive inetger n ,$b^n - 1 \leq n(b-1)$.
(b)Hence b-1 $\leq n(b^{1/n} - 1)$.
(c)If t>1 and n >(b-1)/(t-1),then $b^{1/n} < t$.
(d)If w is such that $b^w < y$,then $b^{w+1/n} > y$ for sufficiently large n;to see this,apply part (c) with $t = yB^{-w}$.
(e)If $b^w > y$,then $b^{w-1/n} < y$ for dufficiently large n.
(f)Let A be the set of all w such that $b^w < y$,and show that x=sup A satisfies $b^x = y$.
(g)Prove that this x is unique.