Project 1 Proposal

Fangyi Guo

ODE: Driven, damped harmonic oscillator

Consider the harmonic oscillator with mass m, spring constant k, damping constant c, sinusoidal drive force $F(t) = F_0 \cos(\omega t)$. The the equation of motion is given by

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega t).$$

Let $\omega_0 = \sqrt{k/m}$ and $\beta = \frac{c}{2m}$, then the above equation can be written as

$$\ddot{x} + 2\beta\dot{x} + \omega^2 x = \frac{F_0}{m} \cos(\omega t).$$

The analytic solution has three regimes depending on the relation of β and ω_0. The homogeneous solutions are

$$x_h(t) = \begin{cases} e^{-\beta t}\left(A_1 \cos\omega_d t + A_2 \sin\omega_d t\right), & \beta < \omega_0 \\ (B_1 + B_2 t)\, e^{-\omega_0 t}, & \beta = \omega_0 \\ C_1 e^{r_+ t} + C_2 e^{r_- t}, & \beta > \omega_0 \end{cases}$$

where

$$\omega_d = \sqrt{\omega_0^2 - \beta^2}, \quad r_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2}.$$

The constants can be determined by initial conditions. The particular solution to the equation of motion is given by

$$x_p(t) = X(\omega)\cos(\omega t - \phi(\omega)),$$

where

$$X(\omega) = \frac{F_0/m}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + (2\beta\omega)^2}}, \quad \phi(\omega) = \arctan\frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

The full solution is thus given by $x(t) = x_h(t) + x_p(t)$. Given the initial conditions $x(0) = x_0, \dot{x}(0) = v_0$, we conclude the following:

Underdamped ($\beta < \omega_0$)

$$x(t) = e^{-\beta t}\left(A_1 \cos\omega_d t + A_2 \sin\omega_d t\right) + X\cos(\omega t - \phi)$$

with

$$A_1 = x_0 - X\cos\phi, \quad A_2 = \frac{v_0 + \beta A_1 - \omega X \sin\phi}{\omega_d}.$$

Critically damped $(\beta = \omega_0)$

$$x(t) = (B_1 + B_2 t)\, e^{-\omega_0 t} + X \cos(\omega t - \phi)$$

with

$$B_1 = x_0 - X\cos\phi, \quad B_2 = v_0 + \beta\,(x_0 - X\cos\phi) - \omega X \sin\phi.$$

Overdamped $(\beta > \omega_0)$

$$x(t) = C_1 e^{r_+ t} + C_2 e^{r_- t} + X\cos(\omega t - \phi)$$

with

$$C_1 = \frac{(v_0 - \omega X\sin\phi) - r_-\,(x_0 - X\cos\phi)}{r_+ - r_-}, \quad C_2 = \frac{r_+\,(x_0 - X\cos\phi) - (v_0 - \omega X \sin\phi)}{r_+ - r_-}$$

The energy is given by

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2.$$

Taking derivative and using the equation of motion, we have

$$\dot{E} = F_0\cos(\omega t)\dot{x} - c\dot{x}^2.$$

One can check numerically that this identity holds. For simplicity, one can also check the special cases for undriven damped $F_0 = 0$ where $E_(t)$ decreases monotonically, or for undriven undamped $(F_0 = 0, \beta = 0)$ where $E_(t)$ is constant.

We know from the analytic solution that the resonance appears for the underdamped case at

$$\omega_{\mathrm{res}} = \sqrt{\omega_0^2 - 2\beta^2},$$

where the amplitude of the particular solution $X(\omega)$ peaks. One can find the numerical value for this maximum amplitude.

Integral: Orbital motion in a $1/r$ central force

Consider a particle of mass m in a central potential $V(r) = -\frac{\mu m}{r}$, then we have the equation of motion

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\mathrm{d}V\,(r)}{\mathrm{d}r},$$
$$m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0,$$

where $\mu = GM$ is constant in our physical background. The second equation implies that the angular momentum, defined as

$$L = mr^2\dot{\theta},$$

is constant. Let $u\,(\theta) = 1/r$, after some algebra the equation of motion can be simplified as

$$u'' + u = \frac{\mu m^2}{L^2}.$$

The ODE has the solution

$$u(\theta) = \frac{\mu m^2}{L^2}\left(1 + e\cos(\theta - \theta_0)\right),$$

and then the trajectory of the particle is given by

$$r(\theta) = \frac{p}{1 + e\cos(\theta - \theta_0)}, \quad p = \frac{L^2}{\mu m^2}.$$

This expression defines conic curves for the trajectory, and we will focus on ellipse case where the eccentricity $e > 1$. Using the definition of the angular momentum, we can compute the period of the orbit

$$T = \int_0^T \mathrm{d}t = \frac{m}{L}\int_0^{2\pi} r(\theta)^2\,\mathrm{d}\theta = \frac{p^2 m}{L}\int_0^{2\pi} \frac{\mathrm{d}\theta}{(1 + e\cos\theta)^2}.$$

The last inequality involves a constant shift $\theta \to \theta + \theta_0$, and hence this integral is independent of θ_0. Analytically, the integral evaluates to be

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{(1 + e\cos\theta)^2} = \frac{2\pi}{(1 - e^2)^{3/2}}.$$

For the simulation purpose, one can choose the parametrization using a, e, μ, m, which then gives p, L. Using the definition of p and the length of the semi-major axis $a = p/(1-e^2)$, we derive Kepler's third law,

$$T = 2\pi\sqrt{\frac{a^3}{\mu}},$$

which can be a physical quantity to compare with the simulation result.