Unit 1 - Application of Derivative

Paper - 5

5 Dec 2022

(1) Review of Function.

Invertible → condition Fun must Bijective

- To have a function to set's a and b is required. let set A be domain and set B be codomain then the function f is defn as $f: A \to B$ $x \in A$ & s.t for all $\forall x \in A$ & $\forall y \in B$

. domain codomain

then $f^{-1}(y) = x$

* classification of function

(1) injective function.

- A function $f: A \to B$ is said to be injective. if $f(x) = f(y) \Rightarrow x = y$

(2) surjective function.

- A function f is said to be surjective. If $\forall y \in B$, ~~s.t~~ $\exists x \in A$ s.t $f(x) = y$

(3) Bijective function.

- A function $f: A \to B$ is said to be Bijective if it is both injective & surjective.

(4) Invertible function.

- If function is bijective. for $f: A \to B$ it's inverse if $f^{-1}: B \to A$ is defn as
if $f(x) = y$ $\forall x \in A$ & $y \in B$.
then $f^{-1}(y) = x$

Injective ↘
Bijectivity ⟷ Invertible.
Surjective ↗

* composition of function.
- If 2 functions $f : A \to B$ & $g : B \to C$ then there composition $g \circ f$ reanging from $g \circ f : A \to C$ is defn as
$(g \circ f)(x) = g(f(x))$
$\forall x \in A$.

* The Limit of a function :

Defn - for every $\varepsilon > 0$ δ
Let A be an open interval contaning a point 'a' let $f : A \to \mathbb{R}$ be a function
We say that the limit as $x \to a$ is the No. 'L'
For every $(\varepsilon) > 0$, $\exists\ \delta > 0$
s.t $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

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Q.1 Show that using $\varepsilon - \delta$ defn of lim of fn.
s.T $\lim_{x \to 3} \frac{x^3 - 4x^2 + 13x - 30}{x - 3} = 16$

$\rightarrow$ $x \to a$ $\quad f(x) = \frac{x^3 - 4x^2 + 13x - 30}{x - 3}$ $\quad L = 16$
$x \to 3$ $\quad a = 3$

for every $\varepsilon > 0$, $\exists\ \delta > 0$
s.t $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$
s.t $0 < |x - 3| < \delta$
$|f(x) - L| < \varepsilon$

Consider, $\left| \frac{x^3 - 4x^2 + 13x - 30}{x - 3} - 16 \right| < \varepsilon$

by the synthetic method
Solve the $x^3 - 4x^2 + 13x - 30$

3	1	-4	+13	-30
		3	-3	30
	1	-1	10	0

$x^2 - x + 10 = 0$

$= \left| \frac{(x-3)(x^2 - x + 10)}{(x-3)} - 16 \right|$

$= |x^2 - x + 10 - 16|$ Factors

$= |x^2 - x - 6|$
$= |(x-3)(x+2)| < \varepsilon$

$x - 3 < \frac{\varepsilon}{x+2} = \delta$

choosing $\delta = \frac{\varepsilon}{x+2}$

Again, for every $\varepsilon > 0$, choosing

$\delta = \frac{\varepsilon}{x+2}$, assuming that $0 < |x-3| < \delta = \frac{\varepsilon}{x+2}$

$|f(x) - L|$

$\Rightarrow \left| \frac{x^3 - 4x^2 + 13x - 30}{x - 3} - 16 \right|$

$\left| \frac{(x-3)(x^2 - x + 10)}{(x-3)} - 16 \right|$

$= |(x-3)(x+2)| < \varepsilon$

put the value of $(x-3)$ as $\delta = \frac{\varepsilon}{x+2}$

$< \frac{\varepsilon}{x+2} \times (x+2)$

$= < \varepsilon$

Hence $\lim_{x \to 3} \left\{ \frac{x^3 - 4x^2 + 13x - 30}{x - 3} \right. = 16.$

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Q. 2 S.T $\lim_{x \to 2} 5x - 4 = 6$ using ε, δ defn

→ $x \to a$, $a = 2$, $f(x) = 5x - 4$, $L = 6$
$x \to 2$

For every $\varepsilon > 0$, $\exists \delta > 0$
s.t $0 < |x - 2| < \delta$
$\Rightarrow |f(x) - L| < \varepsilon$

$|5x - 4 - 6| < \varepsilon$

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$|5x - 10| < \varepsilon$

$|5(x-2)| < \varepsilon$
Taking $(x-2)$ on the oneside
$|x - 2| < \frac{\varepsilon}{5}$

Choosing $\delta = \frac{\varepsilon}{5}$

Again, for every $\varepsilon > 0$, choosing $\delta = \frac{\varepsilon}{5}$, assuming at $0 < |x-2| < \delta = \frac{\varepsilon}{5}$

$|f(x) - L|$
$\Rightarrow |5x - 4 - 6|$
$\Rightarrow |5(x-2)| < 5 \frac{\varepsilon}{5}$

$\Rightarrow \varepsilon$
Hence $\lim_{x \to 2} 5x - 4 = 6$

Q. 3 S.T $\lim_{x \to 5} 4x + 3 = 2~~3~~3$ using ε-δ defn

→ $x \to a$, $a = 5$, $f(x) = 4x + 3$, $L = 2~~5~~3$
$x \to 5$

For every $\varepsilon > 0$, $\exists \delta > 0$
s.t $0 < |x - 5| < \delta$
$\Rightarrow |f(x) - L| < \varepsilon$

$|4x + 3 - 2\cancel{3}| < \varepsilon$

$|4x - \cancel{23}^{20}| < \varepsilon$

$|4(x - 5)| < \varepsilon$

$|x - 5| < \frac{\varepsilon}{4}$

choosing $\delta = \frac{\varepsilon}{4}$

Again, for every $\varepsilon > 0$, choosing $\delta = \frac{\varepsilon}{4}$ assuming at $0 < |x-2| < \delta = \frac{\varepsilon}{4}$

$|f(x) - L|$
$\Rightarrow |4x + 3 - 23|$
$\Rightarrow |4(x-5)| < 4 \frac{\varepsilon}{4}$

$\Rightarrow \varepsilon$
Hence $\lim_{x \to 5} 4x + 3 = 23$

eg:- $\lim_{x \to 2} 5x+4 = 14$ limit value (L) = function value $f(a)$
$14 = 14$
continouse

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Continuity of the Function -

Defn - Let A be an open interval & $a \in A$. a function $f : A \to \mathbb{R}$ is said to be continous at $x = a$ if for $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

$\lim_{x \to a^-} = \lim_{x \to a^+} = f(a)$

Q.1) discusse the continuity of $f(x) = \frac{x^2-4}{x-2}$ in $[0,2]$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)}$

$= \lim_{x \to 2} x+2 = 4$

$\lim_{x = 2^-} f(x) = 4$

$\lim_{x = 2^+} f(x) = 4.$

lim of $f(x)$ exist & it is continouse in $[0,2]$

Q.2 $f(x) = \frac{x^2-49}{x-7}$ in $[0,7]$

$\to \lim_{x \to 7} f(x) = \lim_{x \to 7} \frac{x^2-49}{x-7} = \lim_{x \to 7} \frac{(x-7)(x+7)}{(x-7)}$

$= \lim_{x \to 7} x+7 = 14$

$\lim_{x = 7^-} f(x) = 14$

$\lim_{x = 7^+} f(x) = 14$

lim of $f(x)$ exist & it is continouse in $[0,7]$

Q.3 Discusse the continuity of a function defined by

$f(x) = 2x \quad , x < 0$
$= 0 \quad , 0 \le x \le 1$
$= 4x \quad , x > 1$

$\to$ case (1) at $x = 0$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} 2x = 0$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} 0 = 0$

$f(x)$ at $x = 0$ exist & $f(x)$ is continouse at $x = 0$

case (2) at $x = 1$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 0 = 0$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4x = 4(1) = 4$

limit at $x = 1$ exist but $f(x)$ is discontinous at $x = 1$

$\lim_{x \to 1^-} f(x) \ne \lim_{x \to 1^+} f(x)$

H.W (1) Discuss the continuity of $f(x) = (2x-3)^{1/5}$ at $x = 3/2$.

$\to \lim_{x \to 3/2} f(x) = \lim_{x \to 3/2} (2x-3)^{1/5} = \lim_{x \to 3/2} (2x-3)^{1/5}$ $\quad 2\frac{3}{2} - 3$

$\lim_{x \to 3/2} 2x-3 = 2\,3/2 - 3 = 0$

$\lim_{x = 3/2^-} f(x) = 0 \qquad \lim_{x = 3/2^+} f(x) = 0.$

Differenable if any only if then they continuous

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* Differentiability of a function! –

Defn – let f be a function ranging from $f: A \to \mathbb{R}$ for of interval $A = [a, b]$ let, point p belows $p \in \mathbb{R}$ to $\mathbb{R}$ then the function is side to be differentiable if any $x \in A$.

$\lim_{x \to p} \frac{f(x) - f(p)}{x - p}$ exist & the above limit is known as derivative of function f at point p

denoted by $f'(p)$ or df/dp $\left(\frac{df}{dp} \Big|_{at\, x = p} \right)$

or $Df(p)$

so for differentiability $\frac{df}{dp}\Big|_{at\, x=p} = f'(p) = Df(p) =$

$= \lim_{x \to p} \frac{f(x) - f(p)}{x - p}$.

lefthand & righthand derivative of f at $x = p$ is as follows.

$\lim_{x \to p^-} \frac{f(x) - f(p)}{x - p}$ $Df(p^{-1}) = L.H.D.$, $x < p$.

$\lim_{x \to p^+} \frac{f(x) - f(p)}{x - p} = Df(p+) = R.H.D$, $x \geq p$

Note! – f is differentiable at point p if and only if $Df(p^-)$ & $Df(p^+)$ exist & they are equal

$Df(p^-) = Df(p^+)$

f is diff at $x = p$.

eg! – $f(x) = 4x + 7$, $x < 3$

$= x^2 + 3x + 1$, $x \geq 3$

→ check the differentiability at $x = 3$ $(p = 3)$.

$Df(p^-) = \lim_{x \to p^-} \frac{f(x) - f(p)}{x - p} = \lim_{x \to 3^-} \frac{4x + 7 - f(3)}{x - 3}$.

$f(3) = 4x + 7 = 12 + 7 = 19$

$= \frac{4x + 7 - 19}{x - 3} = \frac{4x - 12}{x - 3} = \frac{4(x - 3)}{x - 3} = 4$

$Df(p^-) = 4$ — (1)

$Df(p^+) = \lim_{x \to p^+} \frac{f(x) - f(p)}{x - p} = \lim_{x \to 3^+} \frac{x^3 + 3x + 1 - (9 + 9 + 1)}{x - 3}$

$= \frac{x^3 + 3x + 1 - 19}{x - 3} = \lim_{x \to 3^+} \frac{x^3 + 3x - 18}{x - 3} = \lim_{x \to 3^+} \frac{(x + 6)(x - 3)}{(x - 3)}$

$= 3 + 6 = 9$

$Df(p+) = 9$ — (2)

from (1) & (2) $Df(p^-) \neq Df(p^+)$

$\therefore$ f is not differentiable at $x = 3$

X-2 $f(x) = 4x + 1$, $x \leq 2$

$= x^2 + 5$, $x > 2$

→ $f(2) = 4(2) + 1 = 9$

$Df(p^{-1}) = \lim_{x \to p^-} \frac{f(x) - f(p)}{x - p} = \lim_{x \to 2^-} \frac{4x + 1 - 9}{x - 2}$

$= \frac{4x - 8}{x - 2} = 4 \frac{(x - 2)}{x - 2} = 4$.

$Df(p^-) = 4 \longrightarrow (1)$

$Df(p^+) = \lim_{x \to p^+} \frac{f(x) - f(p)}{x - p} = \lim_{x \to 2} \frac{x^2+5-9}{x-2}$

$= \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{(x-2)} = x+2 = 4$

$Df(p^+) = 4 \longrightarrow (2)$

from (1) & (2)

$Df(p^-) = Df(p^+)$

$\therefore f$ is diff at $x = 2$

* Derivative of rational Functions
- In form of

$h(x) = \frac{f(x)}{g(x)} = \frac{u}{v}$

$h'(x) = \frac{g(x) f'(x) - f(x) g'(x)}{[g(x)]^2}$

$= \frac{v \cdot du/dx - u \frac{dv}{dx}}{v^2}$