Generalized Linear Recursive Function for Complex Numbers

1 Definition of the Problem

We can define a generalized linear recursive function as one that follows:

$$f(n+1) = \sum_{j=0}^{m} a_j \, f(n-j) \tag{1}$$

where a_j are the coefficients of the function.
We will see how can we generalize this function to complex numbers as:

$$f(n) = \sum_{i=0}^{r-1} \sum_{k=0}^{d_i-1} c_{i,k} \, \lambda_i^{n-k} \prod_{j=0}^{k-1} (n-j) \tag{2}$$

where:

- $c_{i,k}$ are the coefficients of the function.
- λ_i are the eigenvalues of the matrix A.
- d_i is the multiplicity of the eigenvalue λ_i.
- r is the number of distinct eigenvalues of the matrix A.
- $n \in \mathbb{C}$.

This function is exactly the same as:

$$f(n) = \sum_{i=0}^{r-1} \lambda_i^n \sum_{k=0}^{d_i-1} c'_{i,k} \, n^k \tag{3}$$

Now the coefficients $c'_{i,k}$ have a different meaning than $c_{i,k}$ in previous formula

2 Summary of the Proof

We can rewrite the recursive function as:

$$\begin{pmatrix} f(n+1) \\ f(n) \\ f(n-1) \\ \vdots \\ f(n-m+3) \\ f(n-m+2) \\ f(n-m+1) \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-2} & a_{m-1} & a_m \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f(n) \\ f(n-1) \\ f(n-2) \\ \vdots \\ f(n-m+2) \\ f(n-m+1) \\ f(n-m) \end{pmatrix} \tag{4}$$

If we have an initial conditions vector like,

$$F_0 = \begin{pmatrix} f(m) \\ f(m-1) \\ \vdots \\ f(1) \\ f(0) \end{pmatrix} \tag{5}$$

Therefore we can obtain F_n by just applying the operator A, n times to this vector.

$$F_n = A^n\, F_0 = \begin{pmatrix} f(n+m) \\ f(n+m-1) \\ \vdots \\ f(n+1) \\ f(n) \end{pmatrix} \tag{6}$$

We must obtain the eigenvalues and eigenvectors associated with matrix A.

$$P(\lambda) = \det(A - \lambda I) = 0 \tag{7}$$

which satisfy the following equation:

$$\lambda_i^{m+1} = \sum_{j=0}^{m} a_j \lambda_i^{m-j} \tag{8}$$

Each λ_i has only one associated eigenvector $V_{i,0}$,

$$V_{i,0} = \begin{pmatrix} \lambda_i^m \\ \lambda_i^{m-1} \\ \vdots \\ \lambda_i \\ 1 \end{pmatrix} \tag{9}$$

If the number of distinct eigenvalues, $r = m+1$. The problem is solved, we can express F_0 as a linear combination of the eigenvectors $V_{i,0}$, because,

$$B = (V_{0,0}, V_{1,0}, \cdots, V_{m,0}) \text{ is a basis of } R^{m+1} \tag{10}$$

We proceed as follows:

- We express F_0 as a linear combination of eigenvectors $V_{i,0}$ that solve the corresponding system of equations, solving for the coefficients c_i.

$$F_0 = \sum_{i=0}^{m} c_i V_{i,0} \tag{11}$$

- We apply n times matrix A to F_0 to obtain F_n.

$$F_n = A^n \, F_0 = \sum_{i=0}^{m} c_i \, A^n \, V_{i,0} \tag{12}$$

- Since $A^n \, V_{i,0} = \lambda_i^n \, V_{i,0}$

$$F_n = \sum_{i=0}^{m} c_i \, \lambda_i^n \, V_{i,0} \tag{13}$$

- Our interest is on f_n which is the last term of the F_n vector, since all last terms of the vectors, $V_{i,0}\big|_m = 1$, we can write:

$$f(n) = \sum_{i=0}^{m} c_i \, \lambda_i^n \qquad n \in \mathbb{C}. \tag{14}$$

Now imagine that some of our eigenvalues have degeneracy, because of this we cannot fill the space R^{m+1} with just the eigenvectors. For all eigenvalues with $d_i > 1$ we will need to build $d_i - 1$ additional vectors.

$$V_{i,k} = \frac{d^k}{d\lambda^k} V_{i,0}(\lambda)\bigg|_{\lambda=\lambda_i} \qquad k = 0, 1, \cdots, d_i - 1. \tag{15}$$

Only $V_{i,0}$ is an eigenvector.
An interesting property of these vectors is that when matrix A is applied to them, they behave as follows.

$$A \, V_{i,k} = k \, V_{i,k-1} + \lambda_i \, V_{i,k} \tag{16}$$

One shows by induction that

$$A^n \, V_{i,k} = \sum_{j=0}^{\min(n,k)} j! \binom{n}{j} \binom{k}{j} \lambda_i^{n-j} \, V_{i,k-j} \tag{17}$$

Since we know how A^n behaves when being applied to $V_{i,k}$ we just need to follow the previous steps to get our generalized formula:

3 Interesting Properties

During the demonstration, the following identities appeared:

1. **Vandermonde's identity:**

$$\binom{n+m}{k} = \sum_{j=0}^{k} \binom{n}{j} \binom{m}{k-j}.$$

2. **Shifted polynomial identity:**

$$P_n(x) = \sum_{j=0}^{n} \binom{n}{j} (-1)^{j+1} P_n(x-j).$$

3. **Nested summations:** (creo que no la tengo demostrada)

$$\sum_{i_{n-1}=0}^{i_n-1} \sum_{i_{n-2}=0}^{i_{n-1}-1} \cdots \sum_{i_1=0}^{i_2-1} \sum_{i_0=0}^{i_1-1} 1 = \binom{i_n}{n}.$$

4 Calculation of the Eigenvalues

To calculate the eigenvalues,

$$det(A - \lambda I) = det(B) = 0 \tag{18}$$

We can write the elements B_{ij} like this,

$$B_{ij} = -\lambda \delta_{ij} + \delta_{ij+1} + a_j \delta_{i0} \tag{19}$$

We start by multiplying every column by λ^{m-j}

$$B'_{ij} = B_{ij}\, \lambda^{m-j} = -\lambda^{m-j+1}\, \delta_{ij} + \lambda^{m-j}\, \delta_{ij+1} + \lambda^{m-j}\, a_j \delta_{i0} \tag{20}$$

$$det(B') = det(B)\, \lambda^{\frac{(m+1)m}{2}} \tag{21}$$

Then,

$$B''_{ij} = \sum_{k=j}^{m} B'_{ik} = -\lambda^{m-j+1}\, \delta_{ij} + \sum_{k=j}^{m} a_k \lambda^{m-k}\, \delta_{i0} \tag{22}$$

$$det(B'') = det(B') = det(B)\, \lambda^{\frac{(m+1)m}{2}} \tag{23}$$

Then,

$$B'''_{ij} = B''_{ij}\, \lambda^{j-m} = -\lambda\, \delta_{ij} + \sum_{k=j}^{m} a_k \lambda^{j-k}\, \delta_{i0} \tag{24}$$

$$det(B''') = det(B'')\,\lambda^{\frac{-(m+1)m}{2}} = det(B')\,\lambda^{\frac{-(m+1)m}{2}} = det(B) \tag{25}$$

We can observe that B''' is triangular, because of that we can write the determinant like this,

$$det(B''') = det(B) = \prod_{i=0}^{m} B_{ii} = (-\lambda + \sum_{k=0}^{m} a_k \lambda^{-k})\,\lambda^{m} \tag{26}$$

This is the equation that gives us the eigenvalues

$$\sum_{k=0}^{m} a_k \lambda^{m-k} = \lambda^{m+1} \tag{27}$$