

# Integral Calculus

## 3.0 Objectives

This chapter covers the following topics related to integral calculus. After successful completion of this section, you will be able to:

- Evaluate indefinite and definite integrals.
- Use the substitution method and integration-by-parts to evaluate integrals.
- Integration of trigonometric and hyperbolic functions.
- Derive reduction formulae and use these to evaluate integrals.
- Integrate using other methods, such as the method of partial fractions and completing the square.
- Evaluate the area under the curve and between the curves.
- Find the volume of a solid of revolution.
- Determine the lengths of plane curves.
- Find the centre of mass.
- Determine the MacLaurin series expansion for some common functions.
- Define power series and use power series to evaluate integrals.

## 3.1 Indefinite Integrals

- i) The video in the link below demonstrates how to apply the indefinite integral for the following trigonometric functions.

a)  $\int \cos x dx$       b)  $\int \sin x dx$       c)  $\int \sec^2 x dx$

d)  $\int \frac{3}{7} \sin x dx$       e)  $\int \frac{4 \sec^2 x}{5} dx$       f)  $\int \frac{5}{\sec x} dx$

g)  $\int \frac{2}{7} \sin x dx$       h)  $\int \frac{3}{8 \csc x} dx$       i)  $\int \frac{-4 \cos x}{7} dx$

j)  $\int 3x - \frac{\sec^2 x}{8} dx$       k)  $\int 1 + \tan^2 x dx$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/M0WcSCxY12Y>



- ii) The video in the link below demonstrates how to apply the indefinite integral for the following exponential functions.

a)  $\int \frac{2}{7}e^x dx$

b)  $\int \frac{3}{5}e^{2x} dx$

c)  $\int \frac{4}{3e^{5x}} dx$

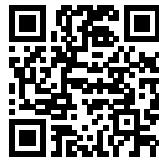
d)  $\int \frac{2}{3}e^{3x-2} dx$

e)  $\int 3e^{7x} - \frac{2}{3}e^{5x-1} dx$

f)  $\int 3e^{5-2x} - \frac{3}{2e^{4x}} dx$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/S8-nsBjcnF8>



### 3.2 Definite Integration

- i) The video in the link below explains what a definite integral is and demonstrates how to apply it to the following functions.

a)  $\int_1^3 3x^2 dx$

b)  $\int_{-1}^2 (4x^3 - 3) dx$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/6AS2qyKbbNI>



- ii) The video below further demonstrates how to find the definite integral for algebraic expressions with various indices, including trigonometric functions.

a)  $\int_2^4 x^3 dx$

b)  $\int_4^{10} 7 dx$

c)  $\int_1^2 (3x^2 - 5x + 2) dx$

d)  $\int_{-1}^3 (2x + 3)^2 dx$

e)  $\int_{\frac{1}{2}}^1 \frac{1}{x^2} dx$

f)  $\int_1^e \frac{1}{x} dx$

g)  $\int_4^9 \sqrt{x} dx$

h)  $\int_2^3 \frac{x^3 - 5x^2}{x} dx$

i)  $\int_0^{\frac{1}{3}} e^{3x} dx$

j)  $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x dx$

k)  $\int_0^{\frac{\pi}{4}} \sin(2x) dx$

l)  $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sec^2 x dx$

m)  $\int_0^1 x^2(x^3 + 5)^2 dx$

n)  $\int_1^2 4xe^{x^2} dx$

o)  $\int_0^1 xe^x dx$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/rCW0dfQ3cwQ>



### 3.3 Integration techniques

The following section provides worked examples of integration by substitution and integration by parts techniques for both definite and indefinite integrals.

i) Integration by substitution: Indefinite integrals

$$\begin{array}{lll} \text{a)} \int 4x(x^2 + 5)^3 dx & \text{b)} \int 8 \cos(4x) dx & \text{c)} \int x^3 e^{x^4} dx \\ \text{d)} \int 8\sqrt{40 - 2x^2} dx & \text{e)} \int \frac{x^3}{(2+x^4)^2} dx & \text{f)} \int \sin^4 x \cos x dx \\ \text{g)} \int \sqrt{5x+4} dx & \text{h)} \int x\sqrt{3x+2} dx & \text{i)} \int 2x\sqrt{4x-5} dx \end{array}$$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/sdYdnpYn-1o>

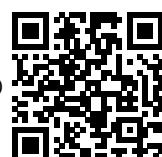


ii) Integration by substitution: Definite integrals

$$\begin{array}{lll} \text{a)} \int_0^2 2x(x^2 + 4)^2 dx & \text{b)} \int_0^4 4x\sqrt{16 - x^2} dx & \text{c)} \int_1^2 \frac{2x}{(1+x^2)^3} dx \end{array}$$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/tM4RWc9ryx0>



iii) Integration by parts: Indefinite integrals

- a)  $\int xe^x dx$       b)  $\int x \sin x dx$       c)  $\int x^2 \ln x dx$   
 d)  $\int \ln x dx$       e)  $\int x^2 \sin x dx$       f)  $\int x \cos x dx$   
 g)  $\int x^2 e^x dx$       h)  $\int (\ln x)^2 dx$       i)  $\int \ln x^7 dx$   
 j)  $\int e^x \sin x dx$       k)  $\int \frac{(\ln x)^2}{x} dx$       l)  $\int e^{3x} \cos(4x) dx$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/sWSLL03DS1I>



#### iv) Integration by parts: Definite integrals

a)  $\int_1^e x^2 \ln x dx$       b)  $\int_0^1 x^2 e^x dx$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/E6VLoKoJWXY>



### 3.4 Integration of Trigonometric and Hyperbolic Functions

The videos in this section demonstrate how to integrate trigonometric, inverse trigonometric, and hyperbolic functions using various techniques, including identities and standard formulas.

#### i) Integration using trigonometric identities

- a)  $\int \frac{3}{7} \sin x dx$       b)  $\int \frac{4 \sec^2 x}{5} dx$       c)  $\int \frac{5}{\sec x} dx$   
 d)  $\int \frac{2}{7} \sin x dx$       e)  $\int \frac{3}{8 \csc x} dx$       f)  $\int \frac{-4 \cos x}{7} dx$   
 g)  $\int \left(3x - \frac{\sec^2 x}{8}\right) dx$       h)  $\int (1 + \tan^2 x) dx$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/M0WcSCxYl2Y>



$$\begin{array}{lll} \text{a) } \int \frac{3}{4} \sin(5x - 2) dx & \text{b) } \int \frac{4 \sec^2(2 - 3x)}{5} dx & \text{c) } \int \frac{3}{\sec 2x} dx \\ \text{d) } \int 5 \cos(4x - 7) dx & \text{e) } \int \frac{2 \sin(3 - 8x)}{7} dx & \text{f) } \int \frac{3}{\cos^2(2x - 5)} dx \end{array}$$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/vRBLjaFlCMk>



$$\begin{array}{lll} \text{a) } \int (1 + \tan^2 x) dx & \text{b) } \int (1 + \tan^2(5\theta)) d\theta & \int (3 + 3 \tan^2 2x) dx \end{array}$$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/3RIxvN-7IFI>



$$\begin{array}{ll} \text{a) } \int \sin x \cos x dx & \text{b) } \int 5 \cos\left(\frac{3}{2}x\right) \sin\left(\frac{3}{2}x\right) dx \end{array}$$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/yc8Kepa1s20>



$$\begin{array}{ll} \text{a) } \int \sin^2 x dx & \text{b) } \int 3 \sin^2(5\theta) d\theta \end{array}$$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/KrWPb6JTVRc>



**ii) Integration using inverse trigonometric functions**

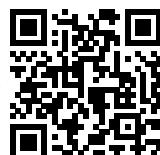
a)  $\int \frac{2}{\sqrt{16 - x^2}} dx$

b)  $\int \frac{dx}{5 + 16x^2}$

c)  $\int \frac{dx}{x\sqrt{9x^2 - 4}}$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/J6MvP8SYVfo>



**iii) Integration by substitution of hyperbolic functions**

Set 1

a)  $\int \frac{1}{\sqrt{x^2 + 4}} dx$    b)  $\int \frac{1}{\sqrt{x^2 - 9}} dx$    c)  $\int \frac{1}{25 - x^2} dx$

Set 2

a)  $\int \frac{1}{\sqrt{3x^2 + 27}} dx$    b)  $\int \frac{1}{\sqrt{x^2 - 6x}} dx$   
c)  $\int \frac{1}{\sqrt{7 - 6x - x^2}} dx$    d)  $\int \frac{1}{\sqrt{12x + 2x^2}} dx$

Set 3

a)  $\int \sqrt{1 + x^2} dx$    b)  $\int_0^6 \frac{x^3}{\sqrt{x^2 + 9}} dx$    c)  $\int \frac{1}{\sqrt{4x^2 - 12x - 7}} dx$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/j0MwIMtrUeQ>



### 3.5 Reduction Formulae

The following link demonstrates how to use the reduction formula method to evaluate the integrals listed below.

a)  $\int \cos^n x dx$

b)  $\int_0^{\frac{\pi}{2}} x^n \sin x dx$

c)  $\int \frac{x^n}{\sqrt{x+1}} dx$

d)  $\int x(\ln x)^{2n} dx$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/an8vp2Qh0NE>



### 3.6 Integration by using partial fractions and completing the square

- i) The link below demonstrates how to find the partial fractions for the following improper rational functions, but **not the integral**.

$$\text{a) } \frac{5x - 3}{x^2 - 3x - 4}$$

$$\text{b) } \frac{6x - 22}{2x^2 + 7x - 15}$$

$$\text{c) } \frac{7x - 11}{(x - 2)^2}$$

$$\text{d) } \frac{3x^2 - 24x + 53}{x^3 - 6x^2 + 9x}$$

$$\text{e) } \frac{6x^2 + 21x + 11}{(x^2 + 3)(x + 5)}$$

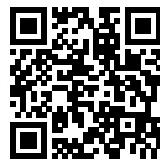
$$\text{f) } \frac{3x^2 + 5x - 4}{(x^2 - 7)(x + 1)}$$

$$\text{g) } \frac{3x^4 - 2x^3 + 6x^2 - 3x + 3}{(x^2 + 2)^2(x + 3)}$$

$$\text{h) } \frac{x^3 + 3}{x^2 - 2x - 3}$$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/2bMndF920qo>



- ii) The link below demonstrates how to evaluate integrals of the following rational functions by using partial fractions.

$$\text{a) } \int \frac{1}{x^2 - 4} dx$$

$$\text{b) } \int \frac{x - 4}{x^2 + 2x - 15} dx$$

$$\text{c) } \int \frac{x}{(x - 1)(x - 2)^2} dx$$

$$\text{d) } \int \frac{x^2 + 9}{(x^2 - 1)(x^2 + 4)} dx$$

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/6rXByMcuAyI>



- iii) The video in the link below demonstrates how to evaluate integrals using the completing the square method for the following questions.

a)  $\int \frac{dx}{x^2 - 6x + 13}$

b)  $\int \frac{x - 5}{x^2 + 8x + 22} dx$

### 3.7 Evaluate the Area Under the Curve and Between the Curves

The video in the link below demonstrates how to find the area between a curve and the X-axis, between two curves along the X-axis, between a curve and the Y-axis and between two curves along the Y-axis.

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/kgg5Rspf1Js>



Then, it demonstrates how to evaluate the area of the region bounded by:

- a) the line with equation  $y = 8 - 2x$ , the X-axis and the Y-axis.
- b) the line with equation  $y = x$  and the curve with equation  $y = x^2$
- c) the curves with equations  $y = x^2$  and  $x = y^2$
- d) the curves with equations  $x = 1 - y^2$  and  $x = y^2 - 1$
- e) the curve with equation  $y = x^2 - 4x$  and the X-axis
- f) the line with equation  $y = 6 - 3x$  and the curve with equation  $y = x^2 - 4x$
- g) the line with equation  $x = 3y - 2$  and the curve with equation  $x = 2y^2 - 4$

### 3.8 Volume of a Solid of Revolution

The video at the link below demonstrates the methodology for finding the volume of revolution along the X- and Y-axes.

It also demonstrates evaluating the volume of the solid generated by the following curves between given values:

**Video** Visit the URL below to view a video:

[https://www.youtube.com/embed/SAHSVg7Jw\\_A](https://www.youtube.com/embed/SAHSVg7Jw_A)



- a)  $y = \sqrt{x}$  rotating along the X-axis for  $360^\circ$  between  $x = 0$  and  $x = 4$

- b)  $y = \frac{1}{x}$  rotating along the X-axis for  $360^\circ$  between  $x = 1$  and  $x = 3$
- c)  $y = x^2$  rotating along the Y-axis for  $360^\circ$  between  $y = 0$  and  $y = 4$
- d)  $y = \frac{x^3}{2}$  rotating along the Y-axis for  $360^\circ$  between  $x = 0$  and  $y = 1$

### 3.9 The Length of a Curve

The video in the link below demonstrates how to evaluate the length of a curve for the following examples.

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/DNDAwWIL5FY>



- a)  $f(x) = 1 + 6x^{\frac{3}{2}}$ ,  $0 \leq x \leq 1$
- b)  $f(x) = \frac{3}{2}x^{\frac{2}{3}}$ ,  $1 \leq x \leq 8$
- c)  $x = \frac{1}{3}(y^2 + 2)^{\frac{3}{2}}$ ,  $0 \leq y \leq 4$

### 3.10 Centre of Mass

The video that is comprised in the link below demonstrates the concept of the moment of a point mass about a point and extends the definition to a system of point masses about the origin. The centre of mass is then defined, and the physical significance is explained in terms of doors and see-saws.

**Video** Visit the URL below to view a video:

[https://www.youtube.com/embed/Zx0IlCjh\\_pw](https://www.youtube.com/embed/Zx0IlCjh_pw)



- i) Moreover, the video also explains how to find the answer to the following question.

Two particles with masses 30 kg, 20 kg, and 50 kg are attached at points  $A$ ,  $B$ , and  $C$  on a light, uniform rod.

The rod is supported at point  $O(0, 0)$ , acting as a pivot.

- (a) Will the system be balanced about point  $O$ ?
- (b) If not, where should the fulcrum (pivot) be placed to achieve balance?

- ii)** The video in the link below demonstrates the method of finding the centre of mass for a non-uniform rod. It also demonstrates how to find the centre of mass of two examples that are listed below.

**Video** Visit the URL below to view a video:

<https://www.youtube.com/embed/xxXUCYzdrlM>



- (a) Find the centre of mass of a uniform rod of length  $L$  with constant density  $\rho$ .
- (b) Find the centre of mass of a non-uniform rod of length  $L$  with density  $x$ .
- iii)** The video below demonstrates how to prove that the centre of mass of a uniform rod with mass  $M$  is at its middle.

**Video** Visit the URL below to view a video:

[https://www.youtube.com/embed/HCIhzGG\\_ZVY](https://www.youtube.com/embed/HCIhzGG_ZVY)

