

Integral Calculus

3.0 Objectives

This chapter covers the following topics related to integral calculus. After successful completion of this section, you will be able to:

- Evaluate indefinite and definite integrals.
- Use the substitution method and integration-by-parts to evaluate integrals.
- Integration of trigonometric and hyperbolic functions.
- Derive reduction formulae and use these to evaluate integrals.
- Integrate using other methods, such as the method of partial fractions and completing the square.
- Evaluate the area under the curve and between the curves.
- Find the volume of a solid of revolution.
- Determine the lengths of plane curves.
- Find the centre of mass.
- Determine the MacLaurin series expansion for some common functions.
- Define power series and use power series to evaluate integrals.

3.1 Indefinite Integrals

- i) The video in the link below demonstrates how to apply the indefinite integral for the following trigonometric functions.

a) $\int \cos x \, dx$	b) $\int \sin x \, dx$	c) $\int \sec^2 x \, dx$
d) $\int \frac{3}{7} \sin x \, dx$	e) $\int \frac{4 \sec^2 x}{5} \, dx$	f) $\int \frac{5}{\sec x} \, dx$
g) $\int \frac{2}{7} \sin x \, dx$	h) $\int \frac{3}{8 \csc x} \, dx$	i) $\int \frac{-4 \cos x}{7} \, dx$
j) $\int 3x - \frac{\sec^2 x}{8} \, dx$	k) $\int 1 + \tan^2 x \, dx$	

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/M0WcSCxY12Y>



- ii) The video in the link below demonstrates how to apply the indefinite integral for the following exponential functions.

$$\begin{array}{lll} \text{a)} \quad \int \frac{2}{7} e^x dx & \text{b)} \quad \int \frac{3}{5} e^{2x} dx & \text{c)} \quad \int \frac{4}{3e^{5x}} dx \\ \text{d)} \quad \int \frac{2}{3} e^{3x-2} dx & \text{e)} \quad \int 3e^{7x} - \frac{2}{3} e^{5x-1} dx & \text{f)} \quad \int 3e^{5-2x} - \frac{3}{2e^{4x}} dx \end{array}$$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/S8-nsBjcnF8>



3.2 Definite Integration

- i) The video in the link below explains what a definite integral is and demonstrates how to apply it to the following functions.

$$\text{a)} \quad \int_1^3 3x^2 dx \quad \text{b)} \quad \int_{-1}^2 (4x^3 - 3) dx$$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/6AS2qyKbbNI>



- ii) The video below further demonstrates how to find the definite integral for algebraic expressions with various indices, including trigonometric functions.

$$\begin{array}{lll} \text{a)} \quad \int_2^4 x^3 dx & \text{b)} \quad \int_4^{10} 7 dx & \text{c)} \quad \int_1^2 (3x^2 - 5x + 2) dx \\ \text{d)} \quad \int_{-1}^3 (2x + 3)^2 dx & \text{e)} \quad \int_{\frac{1}{2}}^1 \frac{1}{x^2} dx & \text{f)} \quad \int_1^e \frac{1}{x} dx \\ \text{g)} \quad \int_4^9 \sqrt{x} dx & \text{h)} \quad \int_2^3 \frac{x^3 - 5x^2}{x} dx & \text{i)} \quad \int_0^{\frac{1}{3}} e^{3x} dx \\ \text{j)} \quad \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x dx & \text{k)} \quad \int_0^{\frac{\pi}{4}} \sin(2x) dx & \text{l)} \quad \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sec^2 x dx \\ \text{m)} \quad \int_0^1 x^2 (x^3 + 5)^2 dx & \text{n)} \quad \int_1^2 4xe^{x^2} dx & \text{o)} \quad \int_0^1 xe^x dx \end{array}$$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/rCW0dfQ3cwQ>



3.3 Integration techniques

The following section provides worked examples of integration by substitution and integration by parts techniques for both definite and indefinite integrals.

i) Integration by substitution: Indefinite integrals

$$\begin{array}{lll} \text{a)} & \int 4x(x^2 + 5)^3 dx & \text{b)} \quad \int 8 \cos(4x) dx \quad \text{c)} \quad \int x^3 e^{x^4} dx \\ \text{d)} & \int 8\sqrt{40 - 2x^2} dx & \text{e)} \quad \int \frac{x^3}{(2 + x^4)^2} dx \quad \text{f)} \quad \int \sin^4 x \cos x dx \\ \text{g)} & \int \sqrt{5x + 4} dx & \text{h)} \quad \int x\sqrt{3x + 2} dx \quad \text{i)} \quad \int 2x\sqrt{4x - 5} dx \end{array}$$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/sdYdnpYn-1o>

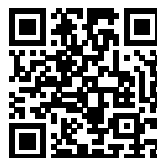


ii) Integration by substitution: Definite integrals

$$\text{a)} \quad \int_0^2 2x(x^2 + 4)^2 dx \quad \text{b)} \quad \int_0^4 4x\sqrt{16 - x^2} dx \quad \text{c)} \quad \int_1^2 \frac{2x}{(1 + x^2)^3} dx$$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/tM4RWc9ryx0>



iii) Integration by parts: Indefinite integrals

$$\begin{array}{lll}
\text{a)} \quad \int x e^x dx & \text{b)} \quad \int x \sin x dx & \text{c)} \quad \int x^2 \ln x dx \\
\text{d)} \quad \int \ln x dx & \text{e)} \quad \int x^2 \sin x dx & \text{f)} \quad \int x \cos x dx \\
\text{g)} \quad \int x^2 e^x dx & \text{h)} \quad \int (\ln x)^2 dx & \text{i)} \quad \int \ln x^7 dx \\
\text{j)} \quad \int e^x \sin x dx & \text{k)} \quad \int \frac{(\ln x)^2}{x} dx & \text{l)} \quad \int e^{3x} \cos(4x) dx
\end{array}$$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/sWSSL03DS1I>



iv) Integration by parts: Definite integrals

$$\text{a)} \quad \int_1^e x^2 \ln x dx \quad \text{b)} \quad \int_0^1 x^2 e^x dx$$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/E6VLoKoJWXY>



3.4 Integration of Trigonometric and Hyperbolic Functions

The videos in this section demonstrate how to integrate trigonometric, inverse trigonometric, and hyperbolic functions using various techniques, including identities and standard formulas.

i) Integration using trigonometric identities

$$\begin{array}{lll}
\text{a)} \quad \int \frac{3}{7} \sin x dx & \text{b)} \quad \int \frac{4 \sec^2 x}{5} dx & \text{c)} \quad \int \frac{5}{\sec x} dx \\
\text{d)} \quad \int \frac{2}{7} \sin x dx & \text{e)} \quad \int \frac{3}{8 \csc x} dx & \text{f)} \quad \int \frac{-4 \cos x}{7} dx \\
\text{g)} \quad \int \left(3x - \frac{\sec^2 x}{8} \right) dx & \text{h)} \quad \int (1 + \tan^2 x) dx &
\end{array}$$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/M0WcSCxYl2Y>



$$\begin{array}{lll} \text{a)} \quad \int \frac{3}{4} \sin(5x - 2) \, dx & \text{b)} \quad \int \frac{4 \sec^2(2 - 3x)}{5} \, dx & \text{c)} \quad \int \frac{3}{\sec 2x} \, dx \\ \text{d)} \quad \int 5 \cos(4x - 7) \, dx & \text{e)} \quad \int \frac{2 \sin(3 - 8x)}{7} \, dx & \text{f)} \quad \int \frac{3}{\cos^2(2x - 5)} \, dx \end{array}$$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/vRBLjaFlCMk>



$$\text{a)} \quad \int (1 + \tan^2 x) \, dx \quad \text{b)} \quad \int (1 + \tan^2(5\theta)) \, d\theta \quad \int (3 + 3 \tan^2 2x) \, dx$$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/3RIxvN-7IFI>



$$\text{a)} \quad \int \sin x \cos x \, dx \quad \text{b)} \quad \int 5 \cos\left(\frac{3}{2}x\right) \sin\left(\frac{3}{2}x\right) \, dx$$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/yc8Kepa1s20>



$$\text{a)} \quad \int \sin^2 x \, dx \quad \text{b)} \quad \int 3 \sin^2(5\theta) \, d\theta$$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/KrWPb6JTVRc>



ii) Integration using inverse trigonometric functions

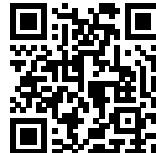
a) $\int \frac{2}{\sqrt{16-x^2}} dx$

b) $\int \frac{dx}{5+16x^2}$

c) $\int \frac{dx}{x\sqrt{9x^2-4}}$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/J6MvP8SYVfo>



iii) Integration by substitution of hyperbolic functions

Set 1

a) $\int \frac{1}{\sqrt{x^2+4}} dx$ b) $\int \frac{1}{\sqrt{x^2-9}} dx$ c) $\int \frac{1}{25-x^2} dx$

Set 2

a) $\int \frac{1}{\sqrt{3x^2+27}} dx$ b) $\int \frac{1}{\sqrt{x^2-6x}} dx$
c) $\int \frac{1}{\sqrt{7-6x-x^2}} dx$ d) $\int \frac{1}{\sqrt{12x+2x^2}} dx$

Set 3

a) $\int \sqrt{1+x^2} dx$ b) $\int_0^6 \frac{x^3}{\sqrt{x^2+9}} dx$ c) $\int \frac{1}{\sqrt{4x^2-12x-7}} dx$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/j0MwIMtrUeQ>



3.5 Reduction Formulae

The following link demonstrates how to use the reduction formula method to evaluate the integrals listed below.

a) $\int \cos^n x dx$

b) $\int_0^{\frac{\pi}{2}} x^n \sin x dx$

c) $\int \frac{x^n}{\sqrt{x+1}} dx$

d) $\int x(\ln x)^{2n} dx$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/an8vp2Qh0NE>



3.6 Integration by using partial fractions and completing the square

- i) The link below demonstrates how to find the partial fractions for the following improper rational functions, but **not the integral**.

a) $\frac{5x-3}{x^2-3x-4}$

b) $\frac{6x-22}{2x^2+7x-15}$

c) $\frac{7x-11}{(x-2)^2}$

d) $\frac{3x^2-24x+53}{x^3-6x^2+9x}$

e) $\frac{6x^2+21x+11}{(x^2+3)(x+5)}$

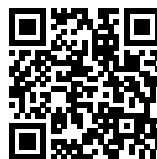
f) $\frac{3x^2+5x-4}{(x^2-7)(x+1)}$

g) $\frac{3x^4-2x^3+6x^2-3x+3}{(x^2+2)^2(x+3)}$

h) $\frac{x^3+3}{x^2-2x-3}$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/2bMndF920qo>



- ii) The link below demonstrates how to evaluate integrals of the following rational functions by using partial fractions.

a) $\int \frac{1}{x^2-4} dx$

b) $\int \frac{x-4}{x^2+2x-15} dx$

c) $\int \frac{x}{(x-1)(x-2)^2} dx$

d) $\int \frac{x^2+9}{(x^2-1)(x^2+4)} dx$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/6rXByMcuAyI>



- iii) The video in the link below demonstrates how to evaluate integrals using the completing the square method for the following questions.

a) $\int \frac{dx}{x^2 - 6x + 13}$

b) $\int \frac{x - 5}{x^2 + 8x + 22} dx$

1.2 Domain and range of a function

- 1) The following video demonstrates how to find the domain for the functions listed below

a) $f(x) = \frac{x + 3}{x - 7}$

e) $f(x) = \sqrt{x - 9}$

i) $f(x) = \frac{\sqrt{x + 5}}{x^2 - 5}$

b) $f(x) = \frac{2x + 1}{x^2 - 16}$

f) $f(x) = \sqrt{x^2 + 5x - 21}$

j) $f(x) = \frac{3x - 4}{\sqrt{x + 9}}$

c) $f(x) = \frac{x^2}{x^2 + 9}$

g) $f(x) = \sqrt[3]{-x + 5}$

k) $f(x) = \frac{\sqrt{x + 1}}{\sqrt{x^2 - 36}}$

- 2) The following video demonstrates how to find the domain and range for the following functions.

- (i) Linear functions

a) $f(x) = x + 3$

- (ii) Quadratic functions

a) $f(x) = x^2$

b) $f(x) = x^2 - 3$

c) $f(x) = 4 - x^2$

d) $f(x) = x^2 - 4x + 5$

- (iii) Cubic functions

a) $f(x) = x^3$

b) $f(x) = x^3 + 5x^2 - 8$

c) $f(x) = 4 - x^2$

- (iv) Absolute (modulus) functions

a) $f(x) = |x|$

b) $f(x) = |x| + 3$

c) $f(x) = |x - 2| - 3$

d) $f(x) = 2 - |x - 3|$

- (v) Radical functions (Functions with roots)

a) $f(x) = \sqrt{x}$

b) $f(x) = -\sqrt{x}$

c) $f(x) = \sqrt{-x}$

d) $f(x) = -\sqrt{-x}$

e) $f(x) = -\sqrt{x - 3} + 4$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/Si2vmzUWfJE>



1.3 Functions and relations

The following video demonstrates the distinction between function and relationship. It also explains the types of functions, such as one-to-one and many-to-one.

Video Visit the URL below to view a video:

https://www.youtube.com/embed/6u9Qh1Vi_Qw



1.4 Functions and their inverses

The following video demonstrates the relationship between functions and their inverses, along with the following worked examples.

(i) **Linear functions** a) $f(x) = 7x + 2$ b) $f(x) = 8x - 5$ c) $f(x) = \frac{x}{5} + 1$

ii) **Quadratic functions**

a) $f(x) = x^2 + 4$

iii) **Radical functions**

a) $f(x) = \sqrt{2x^3 + 58}$

iv) **Inverse composite functions**

a) $f(x) = x + 5$ and $g(x) = x^2 - 2$, workout $(gf)^{-1}(x)$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/1aBkyMVvlsU>



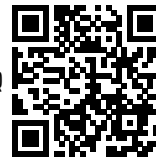
1.5 Exponential and Logarithmic Functions

The videos included in the links below demonstrate expressions and equations involving logarithms and exponents. They also define and graph logarithmic and exponential functions, as well as find their domain and range.

- i) The link below demonstrates how to find the inverse of the following logarithmic functions.
- | | |
|-------------------------------|---------------------------------------|
| a) $f(x) = \log_2(x - 3) - 5$ | b) $f(x) = 2 \log_4(x - 5) + 12$ |
| c) $f(x) = \ln(x - 3) + 4$ | d) $f(x) = 5 - \ln(6 - \frac{1}{3}x)$ |

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/hNsvGz7JPJQ>



- ii) The link below demonstrates how to find the inverse of the following exponential functions

- | | |
|-----------------------|-------------------------|
| a) $f(x) = 3^x + 5$ | b) $f(x) = 2(4^x) - 10$ |
| c) $f(x) = 4e^x - 20$ | d) $f(x) = 8 - 2e^{5x}$ |

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/ihkYZOwPetM>



1.6 Trigonometric and hyperbolic functions, their graphs, inverses, equations, and identities

- i) The following video demonstrates the relationship between the standard trigonometry functions and their inverse.

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/YXWKpgmLgHk>



- ii) The following videos demonstrate sketching the inverse trigonometric functions listed below.

a) $y = \sin^{-1}(x)$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/YNstPOESndU>



b) $y = \cos^{-1}(x)$

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/YNstPOESndU>



iii) The following videos demonstrate various aspects of hyperbolic functions, including graphing and solving identities.

a) Introduction to hyperbolic functions

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/PJRSu0Vf0r0>



b) Graphs of hyperbolic functions

Video Visit the URL below to view a video:

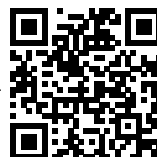
https://www.youtube.com/embed/w_UEjfADQQc



c) Graphs of inverse hyperbolic functions

Video Visit the URL below to view a video:

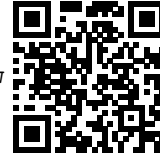
<https://www.youtube.com/embed/TaVdqXsSksA>



d) Hyperbolic identities

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/m9nwdn55Z2w>



e) Solving hyperbolic equations using identities

Video Visit the URL below to view a video:

<https://www.youtube.com/embed/gIZwP5r08Rk>

