Population Dynamics Toolbox

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Abstract

The *population dynamics toolbox* (PDToolbox) contains a set of functions to implement evolutionary dynamics with multiple populations. We consider both small and large populations. For finite populations, we implement some revision protocols to model random interactions between agents. On the other hand, the evolution of a society with large populations is approximated by dynamical equations.

This toolbox is designed to facilitate the implementation of any game with different evolutionary dynamics or revision protocols. In particular, our attempt is to make an efficient implementation of the algorithms to compute the dynamical evolution of the society. Also, the toolbox counts with some functions to plot the state of the system and the evolution of each strategy.

In Section 1 we start by introducing the notation used along the paper and we present some ideas of population games which lead to the *mean dynamics* equation. In Section 2 we introduce some well known revision protocols and evolutionary dynamics. In Section 3 we introduce some details of the implementation of games with the toolbox and show an example of the implementation of the rock-paper-scissors game. Sections 4 and 5 contain examples of the implementation of *combined dynamics* and *multi-population games*.

Contents

1 Introduction

Let us define a society of $\mathcal{P} = \{1, \ldots, P\}$ composed by $P \geq 1$ populations. Each population consist of a large number of agents, which conform a mass $m^p > 0$, with $p \in \mathcal{P}$. Let $S^p = \{1, \ldots, n^p\}$ be the set of actions (or pure strategies) available for each agent of the p^{th} population. Each agent selects a pure strategy and the resulting state of the population is the usage proportion of each strategy. The set of population states is defined as $X^p = \{x^p \in \mathbb{R}_+^{n^p} : \sum_{i \in S^p} x_i^p = m^p\}$, where the i^{th} component of the state, denoted by $x_i^p \in \mathbb{R}_+$, is the mass of players that select the i^{th} strategy of the population p.

Population games (or large games) capture some properties of the interactions of many economic agents, e.g.,

1. large number of agents.
2. Continuity: The actions of an agent has small impact on the payoff of other agents.
3. Anonymity: means that the utility of each agent only depends on the aggregated actions of the other agents.

Game theory is useful to model decision making of agents that are rational. In game theory rationality is the ability to adopt the best actions to achieve some particular goals. This implies that agents use all the information available to make decisions. Evolutionary games relax the rationality assumption by considering myopic behavior. Thus, we assume that agents choose that actions that seem to improve their fitness, however, these actions might not be optimal (as would be the case for rational agents). Thus, evolutionary games can be useful to analyze the behavior of agents in repeated games, where rationality assumptions cannot be made.

In particular, an economic agent decides whether to modify or not its strategy according to the available information. In this respect, we assume that the agent's behavior satisfies both inertia and myopia properties. On the one hand, inertia is the tendency to remain at the status-quo, unless there exist motives to do that. Also, this implies that the strategy adjustment events are rare events. On the other hand, myopia means that the information used to make decisions is limited, e.g., each user makes decisions based on the current state of the population and do not estimate future actions. These two properties are based on the population games theoretical framework [8] and behavioral economics [4].

To accomplish the inertia property, the time between two successive updates of one agent's strategy is modeled with an exponential distribution (this distribution is used to model the occurrence of rare events). Thus, strategy actualization events could be characterized by means of stochastic alarm clocks. Particularly, a rate R_i Poisson alarm clock produces time among rings described by a rate R_i exponential distribution. The whole actualization events in the population can be considered as a rate $R = \sum_{j \in \mathcal{V}} R_j$ Poisson alarm clock. Therefore, the average number of events in a given time interval is R and the probability of selecting the i^{th} agent in a given time instant is $\frac{R_i}{R}$ [8].

At each update opportunity (revision opportunity), the i^{th} agent might compare the average profit of its strategy with the average profit of other strategies. Particularly, an agent might change its strategy with rate ρ_{ij}.

The rate of change ρ_{ij} is determined by a revision protocol, which defines the procedure used by each user to decide whether to change or not its strategy. The scalar $\rho_{ij}(\pi^p, x^p)$ is the *conditional switch rate* from strategy i to strategy j in function of a given payoff vector π and a population state x^p.

Using the law of large numbers we can approximate the evolution of the society's state to a dynamical equation defined by

$$\dot{x}_i^p = \sum_{j \in S^p} x_j^p \rho_{ji}(\pi^p, x^p) - x_i^p \sum_{j \in S^p} \rho_{ij}(\pi^p, x^p).$$

The previous equation is known as the *mean dynamic*, which is used to define some of the dynamics in the next section.

2 Revision Protocols and Evolutionary Dynamics

In this section we introduce four revision protocols, that lead to the evolutionary dynamics *logit dynamics* (Logit), *replicator dynamics* (RD), *Brown-von Neumann-Nash dynamics* (BNN), and *Smith dynamics* (Smith). These dynamics belong to the families of *perturbed optimization*, *imitative dynamics*, *excess payoff dynamics*, and *pairwise comparison dynamics*, respectively [5,8].

2.1 Pairwise Proportional Imitation (Replicator Dynamics)

With a revision opportunity the i^{th} agent observes an opponent j at random. Then it might change its strategy if its opponent has a greater fitness. The rate change is

$$\rho_{ij}^p(\pi^p, x^p) = \frac{1}{m^p}[\pi_j^p - \pi_i^p]_+,$$

where the $[\cdot]_+ : \Re \leftarrow \Re_{\geq 0}$ represents the positive part, defined as $[x]_+ \equiv \max\{0, x\}$. This protocol leads to the *replicator dynamics* defined as

$$\dot{x}_i^p = x_i^p \, \hat{F}_i^p \left(x \right),$$

where $\hat{F}_i^p$ is the excess payoff to strategy $i \in S^p$, which is defined as

$$\hat{F}_i^p(x) = F_i^p(x) - \bar{F}^p(x),$$

and $\bar{F}^p(x)$ is the average payoff the population p, i.e.,

$$\bar{F}^p(x) = \frac{1}{m^p} \sum_{j \in S^p} x_j^p F_j^p(x).$$

Algorithm

input : Society's state x
output: State update $\dot{x}$

for $p \leftarrow 1$ **to** P **do**
  $F^p \leftarrow fitness(x, p)$;
  $\bar{F}^p \leftarrow \frac{1}{m^p}(F^p)^\top x^p$;
  $\hat{F}^p \leftarrow F^p - \mathbf{1}\bar{F}^p$;
  $\dot{x}^p \leftarrow \hat{F}^p \odot x^p \frac{1}{m^p}$;
end

The running time of the algorithm is $T_{rd}(n, P) = O(P(T_f(n, P) + n))$, where $T_f(n, P)$ is the time required to calculate the fitness vector of a population.

2.2 Comparison to the Average Payoff (Brown-von Neumann-Nash Dynamics (BNN))

With a revision opportunity the i^{th} agent selects a strategy at random and might switch to it if that strategy has a payoff above the average. The agent switch strategy with probability proportional to the excess payoff

$$\rho_{ij}^p(\pi^p, x^p) = \left[\pi_j^p - \frac{1}{m^p} \sum_{k \in S^p} x_k^p \pi_k^p \right]_+,$$

This protocol leads to *Brown-von Neumann-Nash dynamics*, defined as

$$\dot{x}_i^p = \left[\hat{F}_i^p \left(\boldsymbol{x} \right) \right]_+ - x_i^p \sum_{j \in S^p} \left[\hat{F}_j^p \left(\boldsymbol{x} \right) \right]_+.$$

Algorithm

input : Society's state x
output: State update $\dot{x}$

for $p \leftarrow 1$ **to** P **do**
  $F^p \leftarrow fitness(x, p)$;
  $\bar{F}^p \leftarrow \frac{1}{m^p}(F^p)^\top x^p$;
  $\hat{F}^p \leftarrow \max\{F^p - \mathbf{1}\bar{F}^p, \mathbf{0}\}$;
  $\dot{x}^p \leftarrow \hat{F}^p - (\mathbf{1}^\top \hat{F}^p) \odot x^p \frac{1}{m^p}$;
end

The running time is $T_{BNN}(n, P) = O(P(T_f(n, P) + n))$.

2.3 Pairwise Comparison (Smith Dynamics)

With a revision opportunity the i^{th} agent selects a strategy at random. If the opponent has a higher fitness, the the agent switch strategy with probability proportional to

$$\rho_{ij}(\pi, x) = [\pi_j - \pi_i]_+$$

This protocol leads to *Smith dynamics* that are defined as

$$\dot{x}_i^p = \sum_{\gamma \in S^p} x_\gamma^p \left[F_i^p(\boldsymbol{x}) - F_\gamma^p(\boldsymbol{x})\right]_+ - x_i^p \sum_{\gamma \in S^p} \left[F_\gamma^p(\boldsymbol{x}) - F_i^p(\boldsymbol{x})\right]_+ .$$

Algorithm

Here we present two algorithms. The first one has time complexity $O(P(T_f(n, P) + n^2))$. This algorithm is implemented as 'smith.m'. A characteristic of this implementation is that might be faster under some conditions, because Matlab is optimized to operate with matrices (see more in Section 8).

input : Society's state x
output: State update $\dot{x}$

for $p \leftarrow 1$ **to** P **do**
  $F^p \leftarrow fitness(x, p)$;
  $A \leftarrow \mathbf{1}{F^p}^\top$;
  $M \leftarrow \max(\mathbf{0}_{n\times n}, A - A^\top)$;
  $F^p_{sum} \leftarrow M\mathbf{1}$;
  $F^p_{avg} \leftarrow \frac{1}{m^p} x^\top M$;
  $\dot{x}^p \leftarrow F^p_{avg} - F^p_{sum} \odot x^p \frac{1}{m^p}$;
end

Below we present an alternative algorithm that might be faster for large number of strategies. In this case we order the strategies in increasing order of fitness and then calculate the strategy's fitness difference (only the ones that are positive). This allow us to reduce the number of operations. The running time of this algorithm is $T_{smith}(n, p) = O(P(T_f(n, P) + n\log(n)))$. This algorithm is implemented as 'smith_b.m'.

input : Society's state x
output: State update $\dot{x}$

for $p \leftarrow 1$ **to** P **do**
  $F^p \leftarrow fitness(x, p)$;
  $A \leftarrow$ Fitness functions ordered in ascending order;
  $B \leftarrow$ Strategies ordered in ascending order by their fitness ;
  $A_{sum} \leftarrow \mathbf{1}^\top A$;
  $A_{avg} \leftarrow 0$;
  $x_{ord} \leftarrow x(B)\frac{1}{m^p}$;
  $x_{cum} \leftarrow 0$;
  for $i \leftarrow 1$ **to** n^p **do**
    $k \leftarrow B(i)$;
    $A_{sum} \leftarrow A_{sum} - A(i)$
    $\Gamma_a^p[k] \leftarrow A(i)x_{cum} - A_{avg}$;
    $\Gamma_b^p[k] \leftarrow A_{sum} - A(i)(n-i)$;
    $A_{avg} \leftarrow A_{avg} + A(i)x_{ord}(i)$;
    $x_{cum} \leftarrow x_{cum} + x_{ord}(i)$;
  end
  $\dot{x}^p \leftarrow \Gamma_a^p - \Gamma_b^p \odot x^p \frac{1}{m^p}$;
end

2.4 Logit Choice

With a revision opportunity the i^{th} agent selects a strategy at random and change its strategy with a probability proportional to

$$\rho_{ij}(\pi) = \frac{\exp(\pi_j \eta^{-1})}{\sum_{k \in S} \exp(\pi_k \eta^{-1})}$$

This protocol belong to target dynamics and with a large population results in the following dynamics

$$\dot{x}_i^p = \frac{\exp\left(\eta^{-1} F_i^p(\boldsymbol{x})\right)}{\sum_{\gamma \in S^p} \exp\left(\eta^{-1} F_\gamma^p(\boldsymbol{x})\right)},\ \eta > 0,$$

known ad *Logit dynamics*.

Algorithm

input : Society's state x
output: State update $\dot{x}$

for $p \leftarrow 1$ **to** P **do**
  $F^p \leftarrow fitness(x, p)$;
  $\bar{F}^p \leftarrow \frac{1}{m^p}(F^p)^\top x^p$;
  $\tilde{F}^p \leftarrow \exp(F^p \eta^{-1})$;
  $\Gamma \leftarrow \mathbf{1}^\top \tilde{F}^p$;
  $\dot{x}^p \leftarrow \frac{\tilde{F}^p}{\Gamma} - x^p$;
end

The running time is $T_{logit}(n, P) = O(P(T_f(n, P) + n))$.

3 Implementation

3.1 Parameters of the Implementation

The toolbox uses a structure that contains all the parameters required to run the simulations. The parameters of a population game are defined in Table 1. On the other hand, the parameters to run

Field	Description	Default value
$P \in \mathbb{Z}$	Number of populations	1
$n \in \mathbb{Z}$	Maximum number of pure strategies per population.	-
$S \in \mathbb{Z}^P$	Vector of pure strategies in each population, such that $1 < S(i) \leq n$.	ones(G.P, 1) * G.n
$m \in \mathbb{R}^P$	Vector with the mass of each population.	ones(G.P, 1)
$x0 \in \mathbb{R}^{P \times n}$	Initial state of the society (normalized).	random
f	Function that returns a vector with the fitness of each strategy of a population.	-

Table 1: Parameters of the game.

Field	Description	Default value
ode	ODE solver for the evolutionary dynamics	‘ode45’
dynamics	Evolutionary dynamic. Current version support combinations of ‘rd’, ‘maynard_rd’, ‘bnn’, ‘smith’, ‘logit’	‘rd’
gamma	Defines the weight given to each dynamic when using combined dynamics	$\sum \gamma(i) = 1$
step	Simulations are made using the time spam: step:step:time+step	0.01

Table 2: Parameters of the dynamical implementation.

simulations with large or small number of agents are specified in Tables 2 and 3, respectively. In this case the behavior of a population with large and small number of agents is simulated using differential equations and revision protocols, respectively. The following is an example to define a game with one population and three strategies per population:

```
G = struct('n', 3, 'f', @fitness1, 'dynamics', {rd},  'ode', 'ode113', 'x0'
    , [0.2 .7 0.1]', 'time', 60);
```

n defines the number of strategies per population, f is a function handler that calculates the fitness of the strategies in each population, and *dynamics* defines the name of the evolutionary dynamics that we want to use. The simulations are run using the ordinal differential equation (ODE) solver called $ode113$, with initial condition $x0 = [0.2, 0.7, 0.1]^\top$ during 60 time units. Note that the number of populations and the mass of each population are defined by default to one. The simulation can be started by executing

```
G.run()
```

On the other hand, the following structure is used to define a population game with small number of agents per population:

Field	Description	Default value
$N \in \mathbb{Z}$	Number of agents	100
$R \in \mathbb{R}$	Rate of the Poisson clock	1
time	Run time of the simulation (number of iterations in the discrete case).	30
revision_protocol	Revision protocol. The current version support one of the following: ‘comparison2average’, ‘pairwise_comparison’, ‘logit_choice’, ‘proportional_imitation’.	‘proportional_imitation’

Table 3: Parameters of the revision protocol.

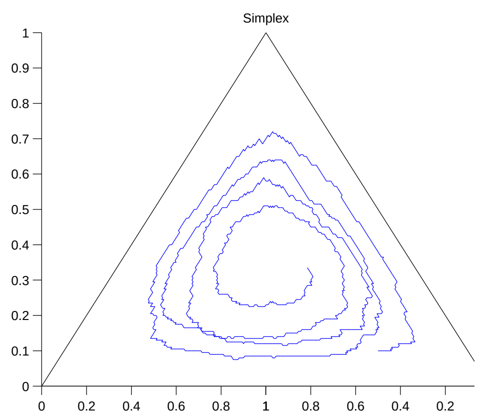
```
G = struct('N', 200, 'n', 3, 'f', @fitness1, 'x0', [0.2 .7 0.1]', 'ode', '
    ode113', 'time', 10000, 'eta', 0.02, 'revision_protocol',
    @proportional_imitation);
```

The finite population case uses the same parameters than the dynamical implementation, except for the dynamical model. However, it is necessary to define the revision protocol and the number of agents N per population. Table 3 contains the list of parameters required to run the revision protocol. The simulation of the revision protocol can be started by executing

```
G.run_finite()
```

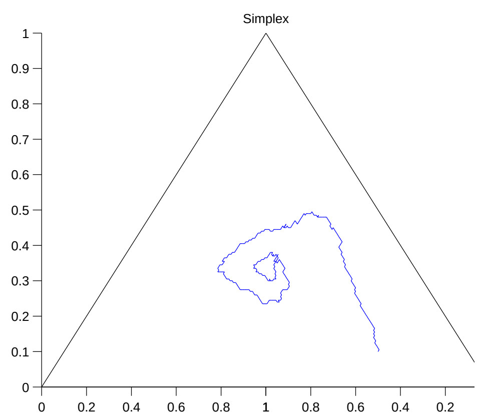
The functions `G.graph()` and `G.graph_evolution()` can be used to graph the simplex and the state evolution of the society for both cases.

3.2 Example: Rock-Paper-Scissors Game



(a) Small population. (b) Large popultion.

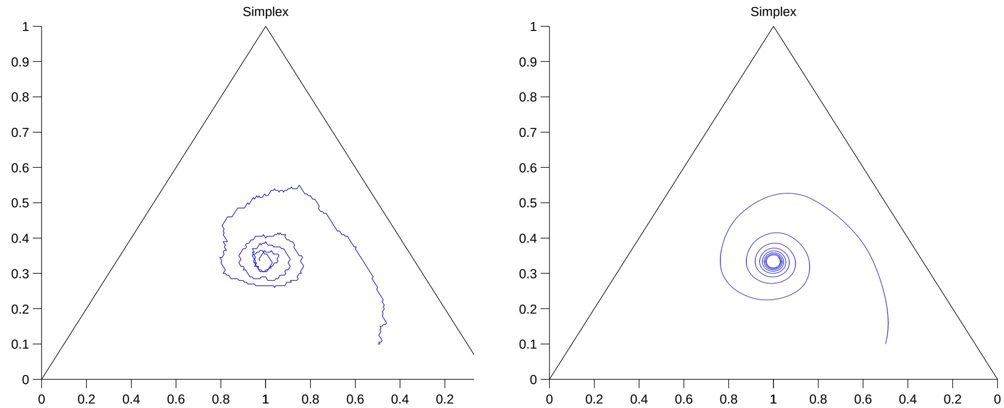
Figure 1: Rock-paper-scissors game with a) proportional imitation revision protocol and b) replicator dynamics.



(a) Small population. (b) Large popultion.

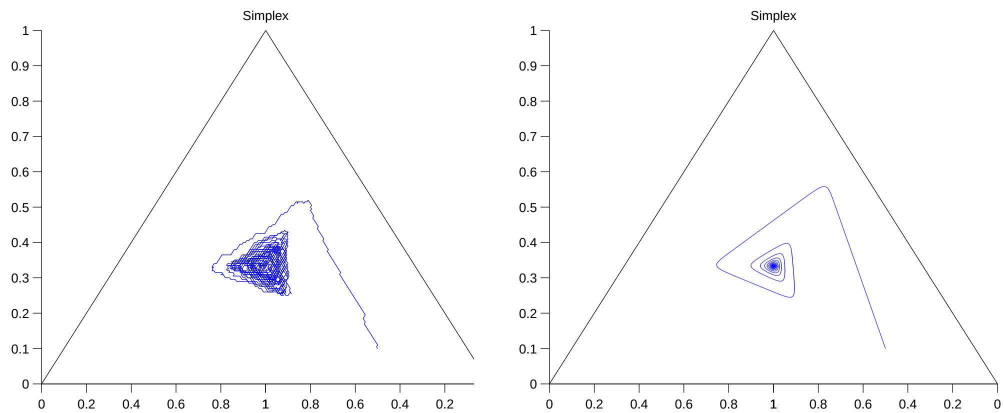
Figure 2: Rock-paper-scissors game with a) comparison to average revision protocol and b) BNN dynamics.

In this section, we implement rock-paper-scissors game with both revision protocols and evolutionary dynamics presented above, to observe the behavioral differences between a society with small number of agents and its approximation to a dynamical system. The game has only one population with three



(a) Small population. (b) Large popultion.

Figure 3: Rock-paper-scissors game with a) pairwise comparison revision protocol and b) Smith dynamics.



(a) Small population. (b) Large popultion.

Figure 4: Rock-paper-scissors game with a) logit choice revision protocol and b) Logit dynamics with $\eta = 0.02$.

strategies, denoted $x = [x_1,\ x_2,\ x_3]^\top$. The fitness function is defined as $F(x) = Ax$, where A is equal to

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$

Note that we modify the payoff matrix proposed in the literature to ensure positive payoffs. Fig. 1 to 4 show the evolution of the society with each revision protocol and its approximation to differential equations. We set the initial condition $x_0 = [0.2,\ 0.7,\ 0.1]^\top$. The small population cases are made with 200 agents and 10000 iterations. The dynamical cases are run during 30 time units.

4 Combined Dynamics

It is possible to define a set of dynamics to run a combination of the dynamics. The resulting dynamic is defined as

$$\dot{x} = \sum_{d \in \mathcal{D}} \gamma_d V_d(x),$$

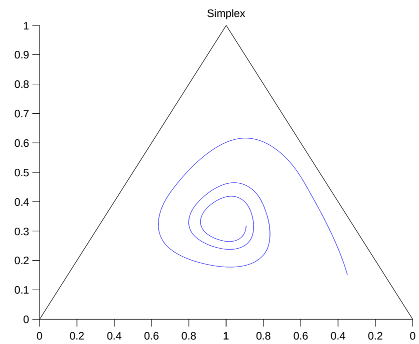
where $\mathcal{D} = \{Logit, RD, Smith, BNN\}$ denotes the set of available dynamics, $V_d()$ is the differential equation of the d^{th} dynamic and γ_d is the weight assigned to it. The dynamics should be defined in a cell array, e.g.,

```
dynamics = {'bnn', 'rd'};
```

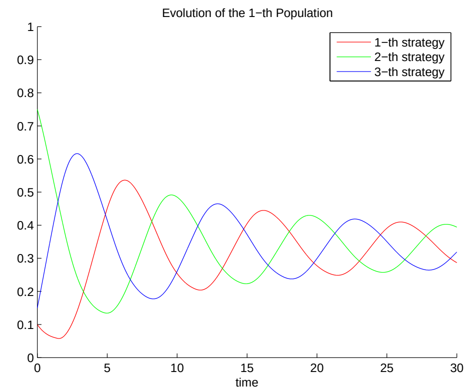
The combination is made making a linear combination between each dynamic listed in the cell array. The weight assigned to each dynamic is defined in the vector `gamma`. In this case we assign

```
gamma = [.25, .75];
```

Fig. 5 shows an example of the combined dynamics for the rock-paper-scissors game. Note that the evolution of the system is not confined to a limit cycle, as happened with the replicator dynamics in Fig. 1.



(a) Simplex.



(b) Evolution of the strategies in time.

Figure 5: Evolution of the combination of replicator dynamics and BNN dynamics.

5 Multi-population Games

5.1 Matching pennies

We implement a matching pennies game defining a society $\mathcal{P} = \{p_1, p_2\}$ with two populations and two strategies per population, namely *heads* and *tails*. First, note that the payoff of the game in normal form is

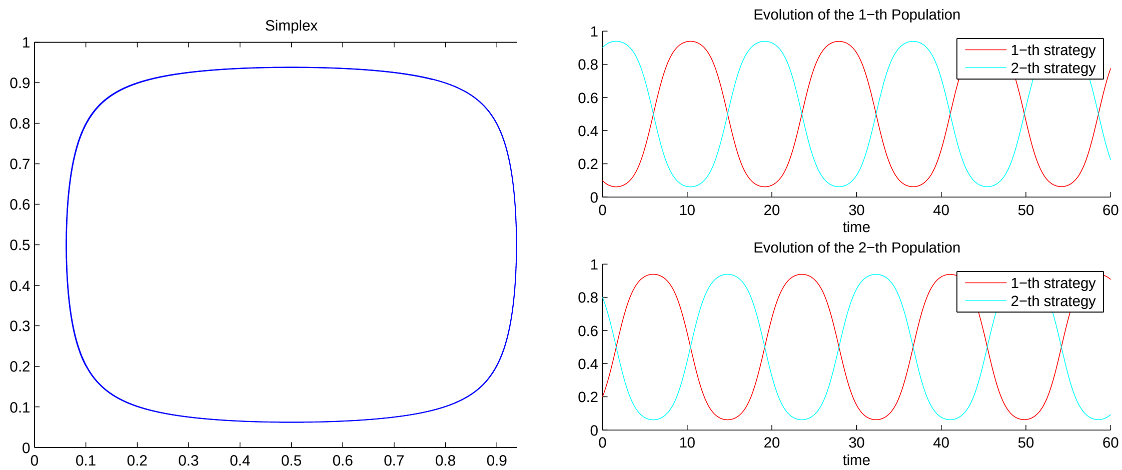
2, 1	1, 2
1, 2	2, 1

Now, the fitness vector of the population p_j can be expressed as $F^{p_j}(x^{p_k}) = A^{p_j} x^{p_k}$, for $p_j, p_k \in \mathcal{P}$ and $p_j \neq p_k$. That is, the payoff of a population is affected only by the state of the opponent population. The payoff matrices are defines as follows

$$A^1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

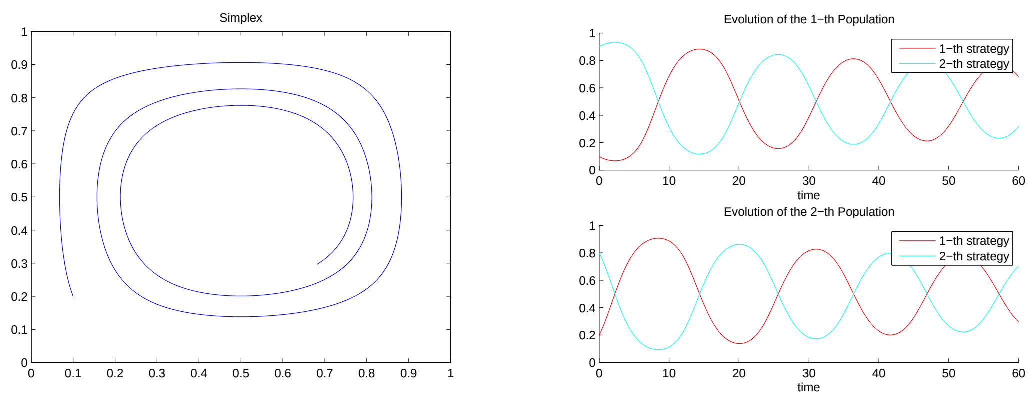
Fig. 6 to 10 show the evolution of the social state with the evolutionary dynamics presented in Section 2.



(a) Simplex.

(b) Evolution of the strategies in time.

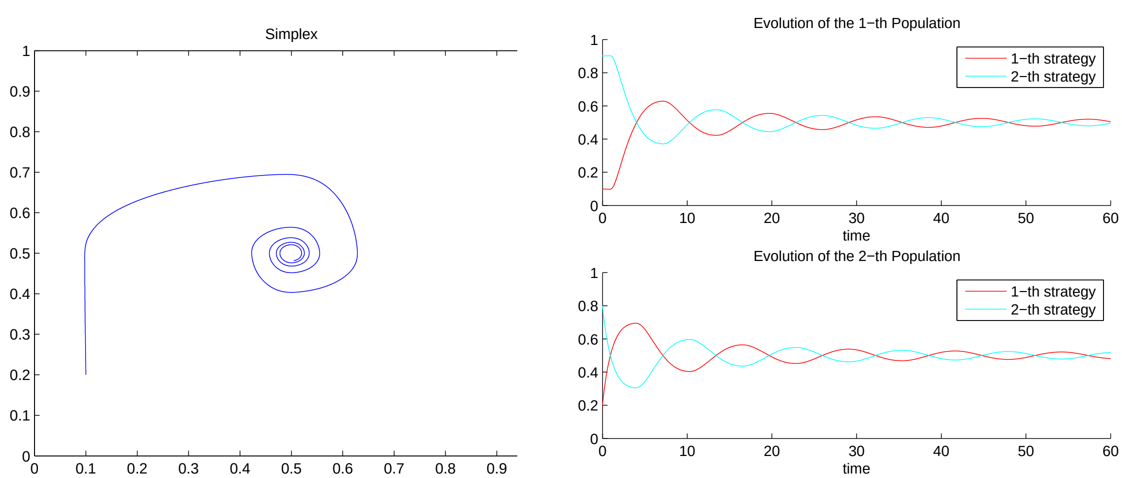
Figure 6: Matching pennies game with replicator dynamics.



(a) Simplex.

(b) Evolution of the strategies in time.

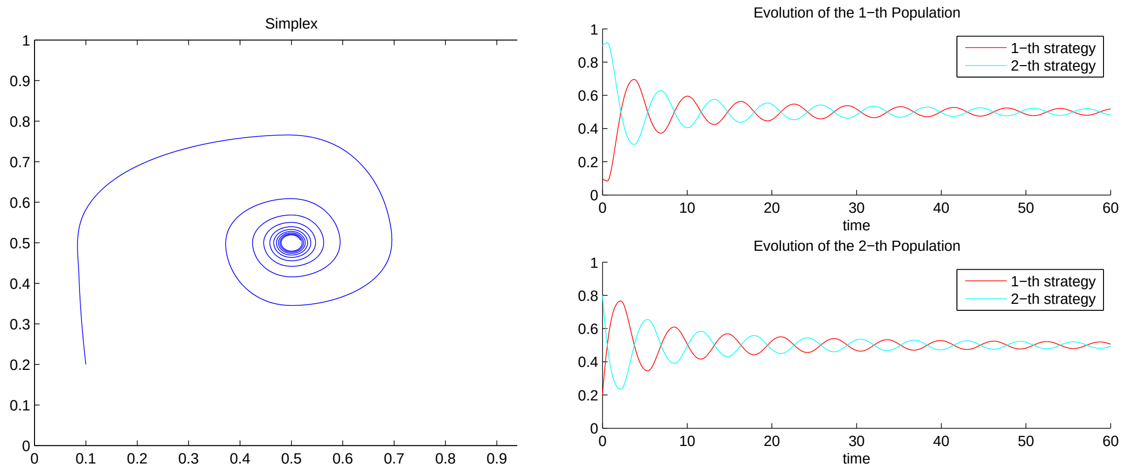
Figure 7: Matching pennies game with Maynard replicator dynamics.



(a) Simplex.

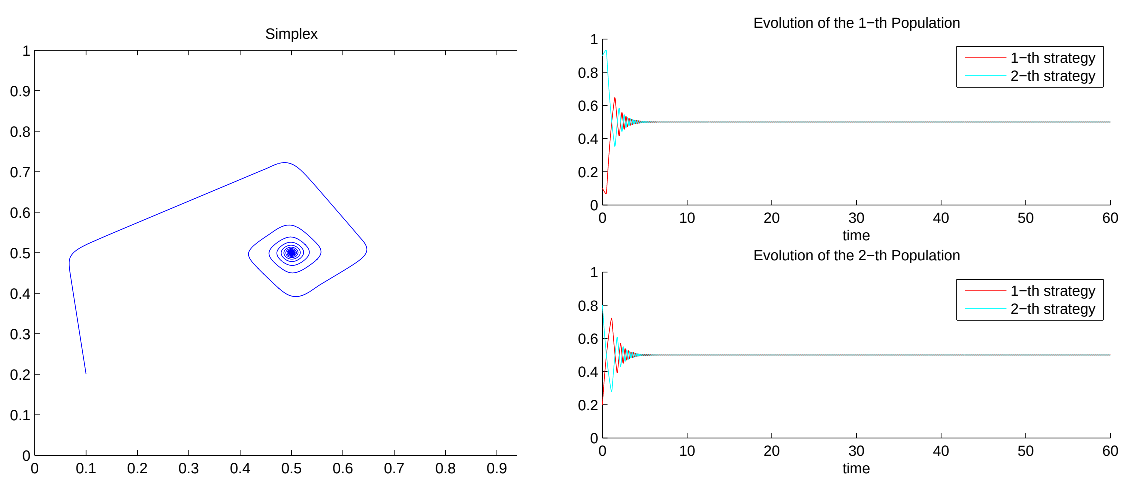
(b) Evolution of the strategies in time.

Figure 8: Matching pennies game with BNN dynamics.



(a) Simplex. (b) Evolution of the strategies in time.

Figure 9: Matching pennies game with Smith dynamics.



(a) Simplex. (b) Evolution of the strategies in time.

Figure 10: Matching pennies game with Logit dynamics with $\eta = 0.02$.

6 Maximization problems

The evolutionary dynamics can be used to solve convex optimization problems. We can use the properties of population games to design games that maximize some function $f(\boldsymbol{z})$, where $\boldsymbol{z} \in \mathbb{R}^n$ is a vector of n variables, i.e., $\boldsymbol{z} = [z_1, \ldots, z_k, \ldots, z_n]$. Below we show two alternatives to solve this optimization problem using either a single population or n populations.

6.1 Single Population Case

First, let us consider a population where each agent can choose one of the $n+1$ strategies. In this case, the first n strategies correspond one variable of the objective function and the $n+1^{th}$ strategy can be seen as a slack variable. Thus, x_k is the proportion of agents that use the k^{th} strategy, and it corresponds to the k^{th} variable, i.e., $x_k = z_k$. We define the fitness function of the k^{th} strategy F_k as the derivative of the objective function with respect to the k^{th} variable, thus, $F_k(\boldsymbol{x}) \equiv \frac{\partial}{\partial x_k} f(\boldsymbol{x})$.

Note that if $f(\boldsymbol{x})$ is a concave function, then its gradient is a decreasing function. Recall that users attempt to increase their fitness by adopting the most profitable strategy in the population, say the k^{th} strategy. This lead to an increase of x_k, which in turns decrease the fitness $F_k(\boldsymbol{x})$.

Furthermore, the equilibrium is reached when all agents that belong to the same population have the same fitness. Thus, at the equilibrium $F_i(\boldsymbol{x}) = F_j(\boldsymbol{x})$, where $i, j \in \{1, \ldots, n\}$. If we define $F_{n+1}(\boldsymbol{x}) = 0$, then at the equilibrium we have $F_i(\boldsymbol{x}) = 0$ for every strategy $i \in \{1, \ldots, n\}$. Since the fitness function decreases with the action of users, we can conclude that the strategy of the population evolves to make the gradient of the objective function equal to zero (or as close as possible). This resembles a gradient method to solve optimization problems.

Recall that the he evolution of the strategies lies in the simplex, that is, $\sum_{i \in S^p} z_i = m$. Hence, this implementation solves the following optimization problem:

$$\begin{aligned} \underset{\boldsymbol{z}}{\text{maximize}} \quad & f(\boldsymbol{z}) \\ \text{subject to} \quad & \sum_{i=1}^{n} z_i \leq m, \end{aligned} \tag{1}$$

where m is the total mass of the population. Figure 11 shows an example of the setting described above for the function

$$f(\boldsymbol{z}) = -(z_1 - 5)^2 - (z_2 - 5)^2. \tag{2}$$

The simulation is executed during 0.6 time units.

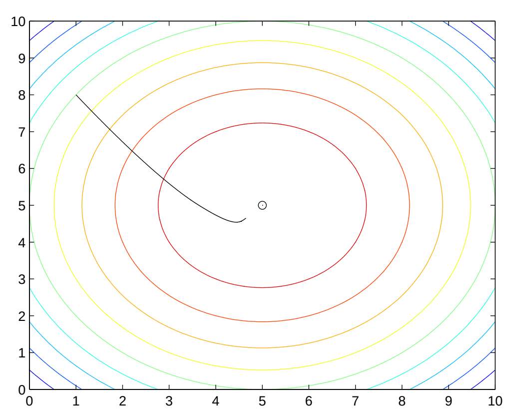


Figure 11: Evolution of the maximization setting using only one population.

6.2 Multi-population Case

Let us consider n populations where each agent can choose one of two strategies. We define a population per each variable of the maximization problem and also n additional strategies that resemble slack variables. Thus, x_i^p is the proportion of agents that use the i^{th} strategy in the p^{th} population. In this case x_1^k corresponds to the k^{th} variable, that is, $x_1^k = z_k$, while x_2^k is a slack variable. The fitness function F_1^k of the k^{th} population is defined as the derivative of the objective function with respect to the k^{th} variable, that is, $F_1^k(\boldsymbol{x}) \equiv \frac{\partial}{\partial x_1^k} f(\boldsymbol{x})$. On the other hand, $F_2^k(\boldsymbol{x}) = 0$. This implementation solves the following optimization problem:

$$\begin{aligned} & \underset{\boldsymbol{z}}{\text{maximize}} && f(\boldsymbol{z}) \\ & \text{subject to} && z_i \leq m^i, i = \{1, \ldots, n\}. \end{aligned} \tag{3}$$

Figure 12 shows an example of the setting described above for the function in Eq. (2). The simulation is executed during 0.6 time units. Note that the implementation using multiple populations reach the optimal value faster than the single population implementation.

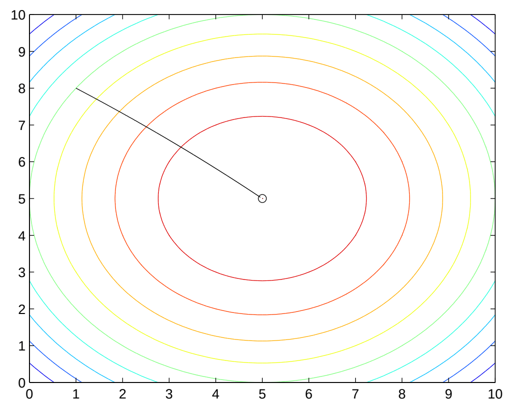


Figure 12: Evolution of the the maximization setting using n populations.

The speed of convergence to the optimum depends on the dynamics and their parameters. For instance, we observed that the equilibrium of the BNN dynamics might be closer to the optimal solution $\boldsymbol{z}^*$ if the mass of the population m^p is close to $\sum_{i=1}^{N} z_i$. Note that close to the optimum $\hat{F}^p$ is small, and if m^p is too large, then the slack variable, such as x_2^k, might be too large, making x_1^k small. These conditions might hinder the convergence to the optimum because updates in the strategies are too small.

7 Example: Demand response programs

This is an example of multiple populations used to implement demand response programs in smart grids [1, 2]. In this case, we assume that each user must decide how to distribute its electricity usage along a day. Particularly, agents might have conflicting interests because they might impose externalities on the society through the price signals, i.e., the aggregated demand might affect the profit of agents. This conflict can be seen as a game between agents, in which each agent is selfish and endeavors to maximize independently its own welfare.

In this problem we model the daily electricity allocation problem as a multi-population game with nonlinear fitness functions. Particularly, each agent can implement an evolutionary dynamic to find the best distribution of resources. Note that when implemented locally by each user, the evolutionary dynamics lead to the global efficient equilibrium (In this case the fitness is equal to the marginal utility of each agent).

A particular feature of this problem is that the Nash equilibrium of the system is inefficient. Hence, we introduce an incentives scheme (indirect revelation mechanism) to maximize the aggregated surplus of the population. The main feature of this mechanism is that it does not require private information from users, and employs a one dimensional message space to coordinate the demand profile of agents. These properties facilitate the distributed implementation of the mechanism. The mechanism entrusts the computation tasks among users, who should maximize its own utility function based the aggregated demand (that is calculated and broadcasted by a central agent). Thus, users avoid revelation of private information (e.g., preferences), but are required to report the aggregated consumption of their appliances during some time periods.

7.0.1 Problem Formulation

We consider a population composed by N consumers defined as $\mathcal{V} = 1, \ldots. N$. Also, let us divide a period of 24 hours in a set of T time intervals denoted $\tau = \{\tau_1, \ldots, \tau_T\}$. Formally, we define the set τ as a partition of $[0, 24)$, where $\cup_{t\in\{1,\ldots,T\}}\tau_t = \tau$ and $\cap_{t\in\{1,\ldots,T\}}\tau_t = \varnothing$. Let q_i^t be the electricity consumption of the i^{th} user in the t^{th} time interval. The daily electricity consumption of the i^{th} user is represented by the vector $\boldsymbol{q}_i = [q_i^1, \ldots, q_i^T]^\top \in \Re^T_{\geq 0}$. The population consumption at a given time t is defined by the vector $\boldsymbol{q}^t = [q_1^t, , q_2^t \ldots, q_N^t]^\top \in \Re^N_{\geq 0}$. On the other hand, the joint electricity consumption of the whole population is denoted by $\boldsymbol{q} = [\boldsymbol{q}_1^\top, \ldots, \boldsymbol{q}_N^\top]^\top$. Without loss of generality, we assume that the electricity consumption of the i^{th} user satisfies $q_i^t \geq 0$, in each time instant t. A *valuation function* $v_i^t(q_i^t)$ models the *valuation* that the i^{th} user gives to an electricity consumption of q_i^t units in the t^{th} time interval. Finally, let $p(\cdot) : \Re \to \Re$ be the price of electricity charged to consumers. The aggregated consumption at a given time t is defined as $||\boldsymbol{q}^t||_1 = \sum_{j=1}^N q_j^t$. Moreover, a daily valuation is $v_i(\boldsymbol{q}_i) = \sum_{t=1}^T v_i^t(q_i^t)$, where $t \in \{1, \ldots, T\}$.

Now, assuming that the electricity generation cost is the same for all t, we can express the profit function of each individual as

$$U_i(\boldsymbol{q}) = v_i(\boldsymbol{q}_i) - \sum_{t=1}^{T} q_i^t p\Big(\left\|\boldsymbol{q}^t\right\|_1 \Big),$$

where $p : \Re_+ \to \Re_+$ is the unitary price function. The consumers welfare function is maximized by solving [6]

$$\begin{aligned} \underset{\boldsymbol{q}}{\text{maximize}} \quad & \sum_{i=1}^{N} U_i(\boldsymbol{q}) = \sum_{i=1}^{N} \left(v_i(\boldsymbol{q}_i) - \sum_{t=1}^{T} q_i^t p\left(\left\|\boldsymbol{q}^t\right\|_1\right) \right) \\ \text{subject to} \quad & q_i^t \geq 0, i = \{1, \ldots, N\}, t = \{1, \ldots, T\}. \end{aligned} \tag{4}$$

7.0.2 Incentives

The solution of the optimization problem in Eq. (4) is inefficient in a strategic environment, i.e., when individuals are rational and selfish [1, 6]. In such cases, the analysis of strategic interactions among rational agents is made using game theory [3]. In particular, the Nash equilibrium (a solution concept in game theory) is sub-optimal, however, we can show that if we consider an added incentive to the individual cost function of each player, the Nash equilibrium of the game with incentives can be made efficient in the sense of Pareto [1, 2].

In particular, our DR scheme with incentives models the case when all agents keep their valuation of electricity to themselves, and have autonomous control their consumption. However, in order to incentive the agents to modify their behavior for the good of the population, the central entity sends them an incentive (e.g., a price signal or reward) to indirectly control their load.

Consider the new cost function for the i^{th} agent:

$$W_i(q_i, \boldsymbol{q}_{-i}) = v_i(q_i) - q_i p\left(\left\|\boldsymbol{q}^t\right\|_1\right) + I_i(\boldsymbol{q}).$$

where incentives are of the form:

$$I_i(\boldsymbol{q}) = \left(\left\|\boldsymbol{q}_{-i}^t\right\|_1\right)\left(h_i(\left\|\boldsymbol{q}_{-i}\right\|) - p\left(\left\|\boldsymbol{q}^t\right\|_1\right)\right).$$

The form of this incentive is inspired in the Vickrey-Clarke-Groves mechanism and the Clarke pivot rule [7]. We assign incentives according to the contribution made by an agent to the society. In particular, the function $h_i : \Re \to \Re$ is a design parameter that estimates the externalities introduced by each individual. It can be shown that these incentives can lead to an optimal equilibrium in a strategic environment. In this DR approach we consider that the utility sends a two dimensional signal to each customer, namely (q, I_i) and each customer responds with some consumption q_i. Note that the incentives modify the price paid by each user according to their relative consumption. However, two different users receive different incentives as long as their consumption are different.

7.0.3 Simulations

In this section, we illustrate some ideas of efficiency and the decentralized implementation of the incentives mechanism. We select some functions used previously in the literature. On the one hand, we define the family of valuation functions as

$$v(\boldsymbol{q}^k, \alpha_i^k) = v_i^k(q_i^k) = \alpha_i^k \log(1 + q_i^k)$$

where $\alpha_i^k > 0$ is the parameter that characterizes the valuation of the i^{th} agent at the k^{th} time instant. On the other hand, the generation cost function is defined as

$$C(\|\boldsymbol{q}\|_1) = \beta(\|\boldsymbol{q}\|_1)^2 + b\|\boldsymbol{q}\|_1,$$

and the unitary price function is

$$p(\|\boldsymbol{q}\|_1) = \frac{C(\|\boldsymbol{q}\|_1)}{\|\boldsymbol{q}\|_1} = \beta\|\boldsymbol{q}\|_1 + b.$$

Note that the generation cost only depends on the aggregated consumption, not on the time of the day. Furthermore, the fitness function of the system with incentives is

$$F_i^k(\boldsymbol{q}^k) = \frac{\alpha_i^k}{1 + q_i^k} - 2\beta \left(\sum_{j=1}^{N} q_j^k \right).$$

The evolution of utility, demand, and incentives for different dynamics is shown in Figs. 13 and 14. Note that despite using the same initial condition, the evolution of the system is different with each dynamical model. In particular, BNN and Smith dynamics converge faster to the optimum, in contrast with the Logit and replicator dynamics. This is achieved by means of a fast decrease in the power consumption.

Incentives in Fig. 14 show that, in the long run, all dynamics converge to the same level of incentives. Particularly, Smith dynamics requires more incentives during all time, except for logit dynamics, which has a sudden increase in the incentives close to the equilibrium point.

In Fig. 14 it is not clear which dynamical model moves the state of the system to the optimal equilibrium using less resources. To answer this question, we simulate the total amount of incentives used by each model. Thus, let us define the aggregated incentives in a society in a particular time t as

$$I_d(t) = \sum_{i \in \mathcal{P}} \frac{1}{|S|} \sum_{k \in S} I_i\left(\boldsymbol{q}^k(t)\right).$$

Now, the total accumulated incentives from t_0 to t is defined as

$$\Phi_d(t) = \int_{t_0}^{t} I_d(\tau) d\tau.$$

Thus, $\Phi_d(t)$ gives a measurement of the total amount subsidies required by the system with dynamic d, in the time interval $[t_0, t]$. In this case we do not have a reference to compare the subsidies requirements of each evolutionary dynamic. Hence, we compare the subsidies requirements with the average requirements of all the dynamics implemented. In order to see which dynamic requires more resources, we plot the cumulative resources required by each dynamic relative to the average. Hence, we define the cumulative incentives as

$$CI_d = \frac{\Phi_d(t)}{\sum_{d \in \mathcal{D}} \Phi_d(t)}.$$

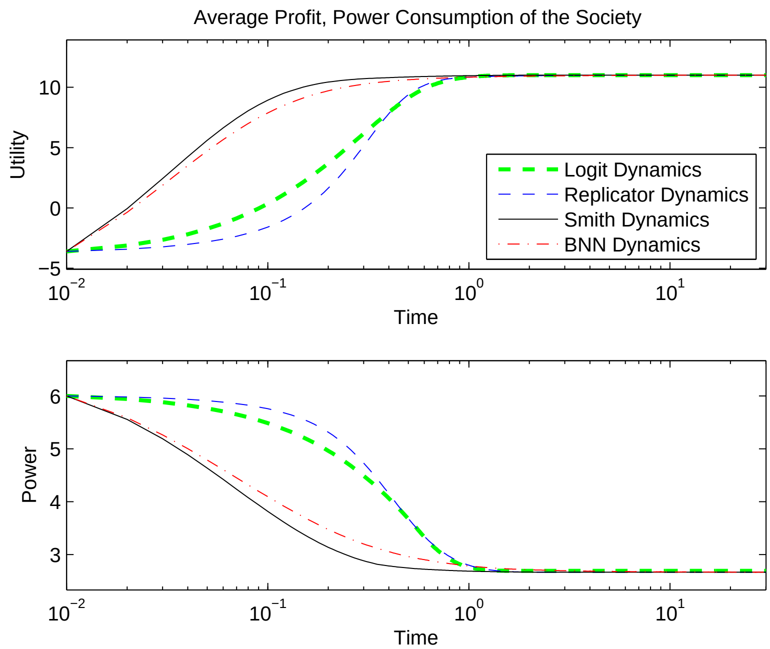


Figure 13: Evolution of profit and costs for four different dynamics.

Fig. 15 shows the results of the simulation of the relative subsidies required by each model of evolutionary dynamics.

Smith dynamics requires much more resources during all the time stamp, but is particularly high during the first stages, while logit has the lower incentives requirements. However, BNN has the lower incentives in long run.

Fig. 16 shows the final demand profile of each agent. Note that the final state corresponds to the state of each population at the equilibrium.

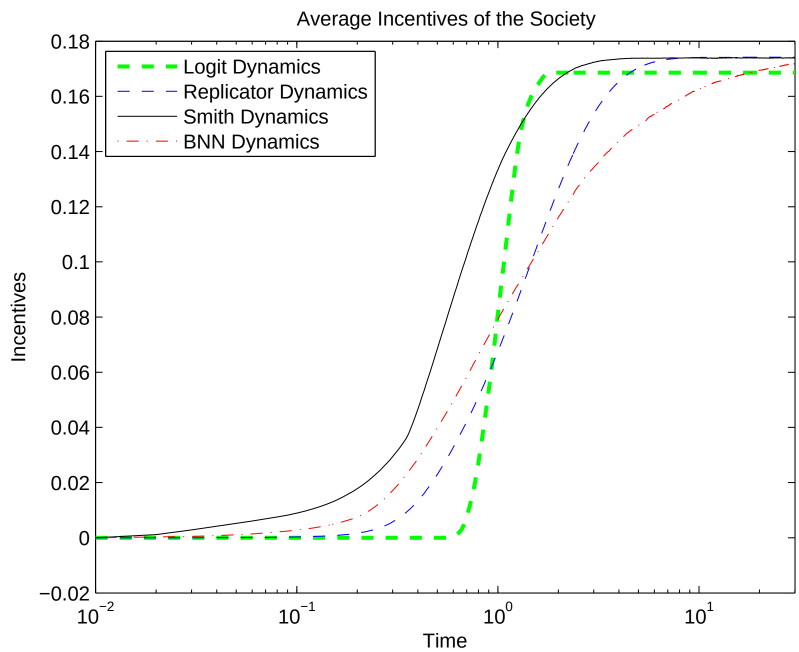


Figure 14: Evolution of the incentives with four different dynamics.

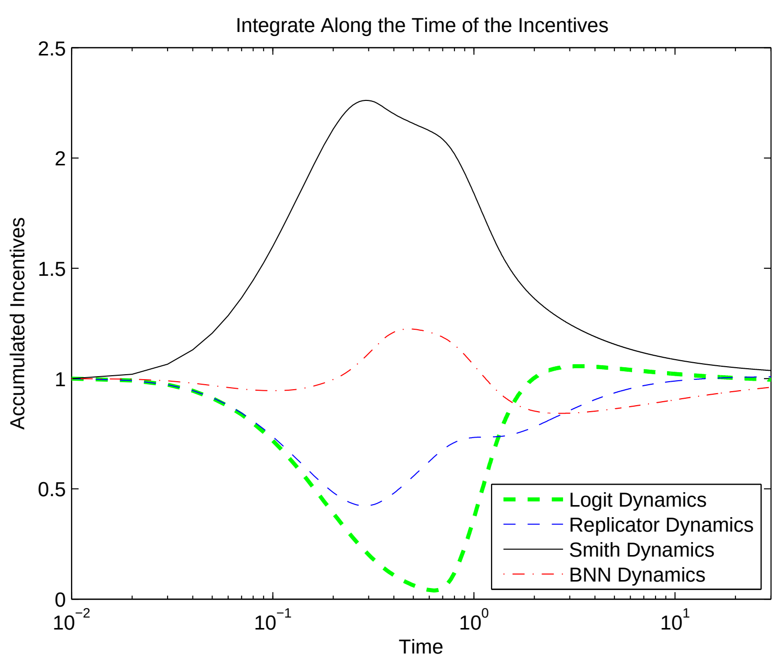


Figure 15: Accumulated incentives during the evolution of the algorithm.

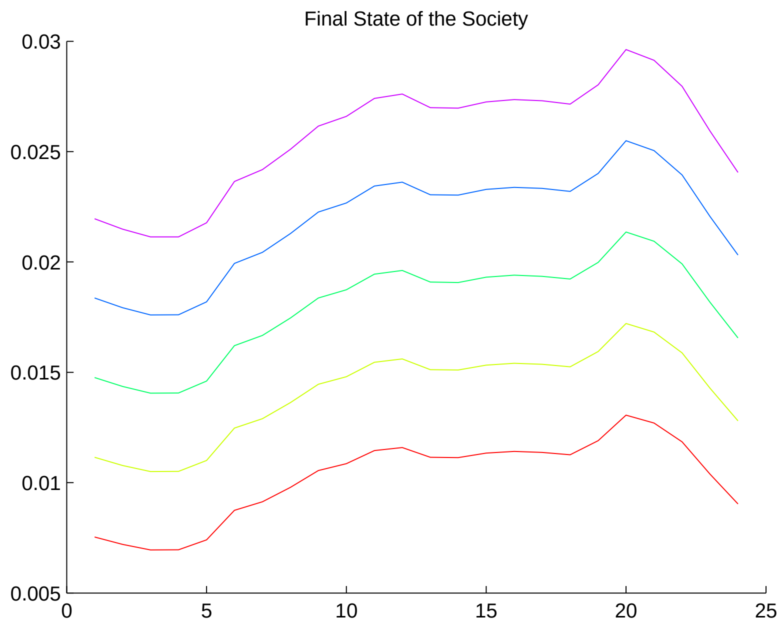


Figure 16: Final demand profile of each agent.

8 Running Time Analysis

In this section we investigate the running time of the evolutionary dynamics as a function of either the number of populations or the number of strategies per population. We use the demand response example from Section 7 to make the experiments. Our interest is to observe the time that takes to simulate the evolution of the dynamics during 10 seconds.

8.1 Running Time as a Function of the Number of Populations

Note that the time complexity of the algorithms, with respect to number of populations P, is $O(P \cdot T_f(n,P))$. The fitness function in the example satisfies $T_f(n,P) = O(nP)$. Hence, the complexity of the dynamics with respect to P is $O(P^2)$.

Although the dynamics have the same time complexity, the running time of the dynamics has substantial differences. Fig. 17a shows the running time of the game for multiple number of populations. In this case each population has 24 strategies and all populations have the same set of fitness functions. From the simulations we observe that the fastest simulations are made with replicator dynamics. Moreover, there is a large difference between the running time of Replicator and the other dynamics. One reason is that the ODE solver run less iterations with Replicator dynamics, and consequently, it makes less calls to the dynamic's algorithm.

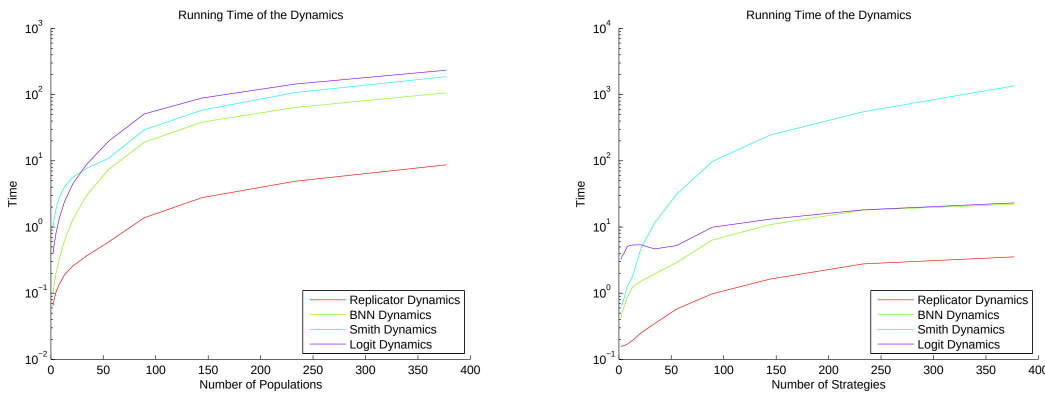
8.2 Running Time as a Function of the Number of Strategies

Fig. 17b shows the running time of the game for different number of strategies per population. In this case we define 25 populations. Note that the time complexity of the dynamics with respect to the number of strategies is $O(n)$ (or $O(n \log n)$ in the case of the Smith dynamics). In this case Replicator has the lowest running time, while smith have the largest running time. The large difference between Smith and the other dynamics is that the ODE solver makes much more iterations.

Note that, except for Smith dynamics, the running time is lower when we implement games with large number of strategies. Hence, the maximization problem in Section 6 might be solved faster using the single population case.

8.3 Running Time of Smith Dynamics

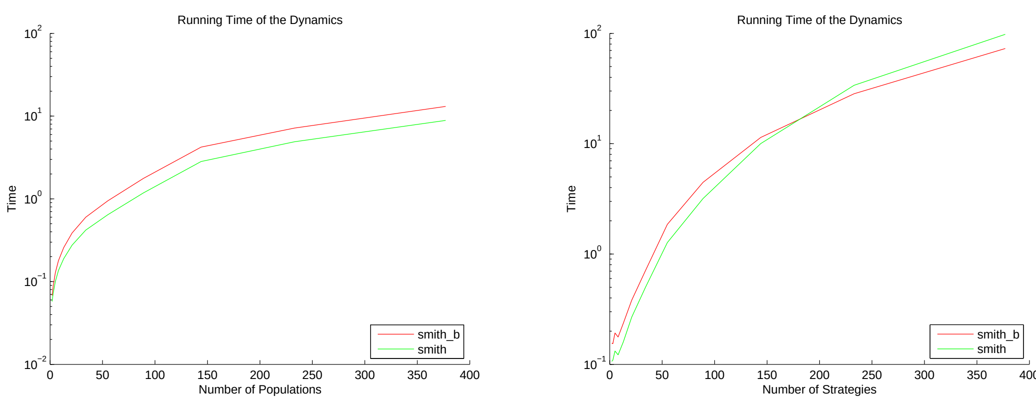
In the toolbox we include two algorithms for the Smith dynamics. One of them relies on matrix multiplications and has running time $O(P(T_f(n,P) + n^2))$ ('smith.m'), while the other has time complexity $O(P(T_f(n,P) + n \log n))$ ('smith_b.m'). Even though the time complexity of the first algorithm is higher, its simulation are faster under certain conditions. The reason is that Matlab is optimized to work with matrices. From the experiments we see that 'smith_b.m' is faster only for large number strategies (see Fig. 18).



(a) Run time for different population numbers and constant number of strategies.

(b) Run time for different number of strategies and constant number of populations.

Figure 17: Run time of four evolutionary dynamics: ‘rd’, ‘bnn’, ‘smith’, and ‘logit’.



(a) Run time for different population numbers and constant number of strategies.

(b) Run time for different number of strategies and constant number of populations.

Figure 18: Run time of two implementations of the Smith dynamics, namely ‘smith’ and ‘smith_b’.

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