SO(2) : The exponential map

$[1\times]$

$\dot{\mathbf{R}} = \mathrm{d}\mathbf{R}/\mathrm{d}t$

$\mathbf{R} = \exp([\theta\times])$

$[\theta\times](t) = [\omega\times]t$

$\mathbf{R}(t)$

$[\theta\times] = \log(\mathbf{R})$

SO(2)

θ

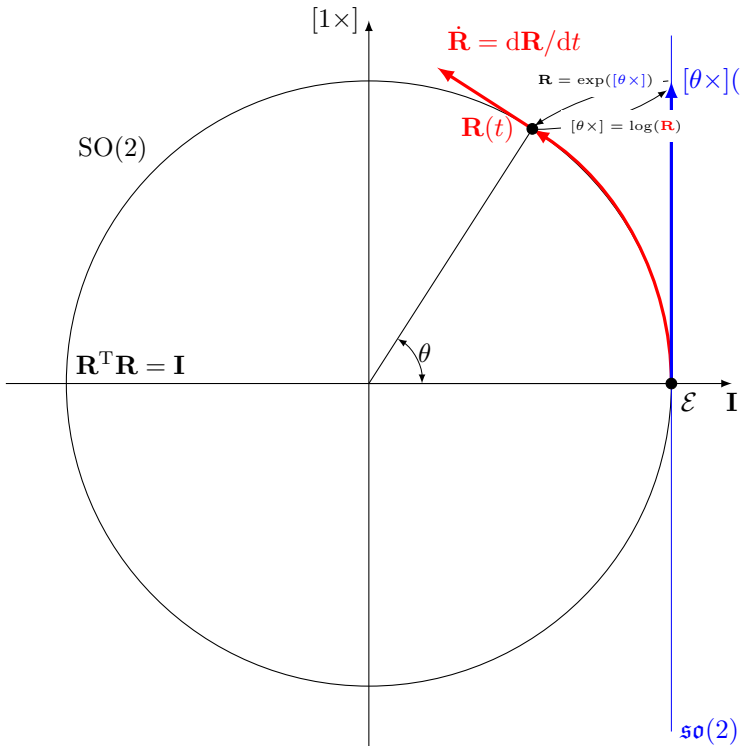$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$

$\mathcal{E}$ $\mathbf{I}$

$\mathfrak{so}(2)$

∗ Write ODE and integrate :
$$\mathbf{R}^{\mathrm{T}}\dot{\mathbf{R}} = [\omega\times] \Rightarrow \dot{\mathbf{R}} = \mathbf{R}\cdot[\omega\times]$$
$$\mathbf{R}(t) = \mathbf{R}_0 \cdot \exp([\omega\times]\,t)$$
∗ If $\mathbf{R}_0 = \mathbf{R}(0) = \mathbf{I}$ and $[\omega\times]\,t = [\theta\times] = [1\times]\cdot\theta$
$$\mathbf{R}(t) = \exp([\omega\times]\,t) = \exp([\theta\times])$$
∗ Taylor expansion, with $[1\times]^0 = \mathbf{I}$ and $[1\times]^3 = -[1\times]$
$$\exp([\theta\times]) = \mathbf{I} + [1\times]\cdot\theta + ([1\times]\cdot\theta)^2/2! + ([1\times]\cdot\theta)^3/3! + \cdots$$
$$= \mathbf{I}\cdot\left(1 - \theta^3/3! + \cdots\right) + [1\times]\cdot\left(\theta - \theta^2/2! + \cdots\right)$$
$$= \mathbf{I}\cdot\cos(\theta) + [1\times]\cdot\sin(\theta)$$
$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$