9:42 AM Tue Aug 30 21%

11-1-a

For ith insertion,

the load factor

$$\alpha = \frac{i-1}{m} \leq n/m \leq \frac{1}{2}$$

$$P\{X_i \geq k\}$$

the propobility of probing an occupied slot.

$$= \frac{i-1}{m} \cdot \frac{i-2}{m-1} \cdots\cdots \frac{i-k}{m-k+1}$$

$$\leq \left(\frac{i-1}{m}\right)^k \qquad \boxed{\frac{x}{y} < \frac{x-1}{y-1} \text{ when } y>x>0}$$

$$\leq \left(\frac{1}{2}\right)^k = 2^{-k}$$

11.1-b

Make $k = 2\lg n$.

$$P\{x_i = 2\lg n\} \leq 2^{\lg \frac{1}{n^2}} = \frac{1}{n^2} = O\left(\frac{1}{n^2}\right)$$

11.1-c

$$\Pr\{X > 2\lg n\} = \Pr\{\forall_i\ X_i > 2\lg n\}$$

$$\leq \sum_i \Pr\{X_i > 2\lg n\} \leq \sum_i \frac{1}{n^2} = \frac{1}{n}$$

the maximum X

11.1-d. $E[X] = \sum X \cdot \Pr\{X\} \leq 2\lg n \cdot P\{X \leq 2\lg n\} + n \cdot \Pr\{X > 2\lg n\}$

$$\leq 2\lg n\left(1 - \frac{1}{n}\right) + n \cdot \frac{1}{n} = 2\lg n - 2\frac{\lg n}{n} + 1 = O(\lg n)$$

Ch 11

11-2-a

Event A_i: k_i is inserted into a given a particular slot.

$p(A_i) = \frac{1}{n}$ $\quad P(A_i^c) = 1 - \frac{1}{n}$

Random variable k: the number of keys inserted into a particular slot.

$$k \sim B(n, \tfrac{1}{n})$$

$$Q_k = (\tfrac{1}{n})^k (1 - \tfrac{1}{n})^{n-k} \binom{n}{k}$$

11-2-b For $i = 1, 2, \cdots n$, let Random variable X_i be the number of keys inserted into keys. Let A_i be the event that $X_i = k$.

$P_i = Pr\{\max_{i=1..n} X_i = k\} = Pr\{$There is slot i with k keys inserted while all the other slots has fewer key inserted than $k\}$

$\leq Pr\{$There is a slot with $X_i = k\} \leq P(A_1) + P(A_2) + \cdots + P(A_n) = n Q_k$

11-2-c

Stirling Approximation

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \theta\left(\frac{1}{n}\right)\right) \geq \left(\frac{n}{e}\right)^n$$

$$Q_k = \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k} \frac{n!}{k!\,(n-k)!}$$

$n-k \leq n$

$$= \left(1 - \frac{1}{n}\right)^{n-k} \cdot \frac{n \cdots (n-k)}{n^k} \cdot \frac{1}{k!}$$

$$\leq \left(1 - \frac{1}{n}\right)^{n-k} \cdot \frac{1}{k!}$$

$$\leq \frac{1}{k!} \leq \frac{e^k}{k^k}$$

11-2-d.

$$Q_{k_0} < \frac{e^{k_0}}{k_0^{k_0}} \quad \text{where } k_0 = c\lg n / \lg\lg n$$

we are tring prove

$$\frac{e^{k_0}}{k_0^{k_0}} < 1/n^3 \quad \text{or}$$

$$\frac{k_0^{k_0}}{e^{k_0}} > n^3$$

Take the logarithms on ~~both~~ sides

$$k_0 \lg k_0 - k_0 > 3\lg n$$

Substitute $k_0 = c\lg n / \lg\lg n$ into the above inequality.

$$c\cancel{\lg n} / \lg\lg n \cdot (\lg c + \lg\lg n - \lg\lg\lg n)$$

$$- c\cancel{\lg n} / \lg\lg n > 3\cancel{\lg n}$$

$$3 < c\left[\frac{\lg c - \lg e}{\lg\lg n} + 1 - \frac{\lg\lg\lg n}{\lg\lg n}\right]$$

when $n \to \infty$, $\dfrac{\lg c - \lg e}{\lg\lg n} \to 0$ $\quad \dfrac{\lg\lg\lg n}{\lg\lg n} \to 0$

$$[\cdots] \to 1$$

So there exist n_0, making $[\cdots] \geq \frac{1}{2}$

so for $n > n_0$, any $c \geq 6$ is ok.

for $3 < n < n_0$. Since n belongs to a finite field of the integer.

$n_1 = \operatorname{argmax} [\cdots\cdots]$

For $3 < n < n_0$, any $c \geq \frac{3}{[\cdots]_{max}}$ is ok.

$$P_k \leq n Q_k \leq n \cdot \frac{1}{n^3} = \frac{1}{n^2}$$

11.-2-e.

$$E(M) = \sum_{k=1}^{n} Pr\{M=k\} \cdot M$$

$$= \sum_{k=1}^{n} P_k \cdot k$$

$$= \sum_{k=1}^{k_0} P_k \cdot k + \sum_{k=k_0+1}^{n} P_k \cdot k$$

$$\leq \sum_{k=1}^{k_0} P_k \cdot k_0 + \sum_{k=k_0+1}^{n} P_k \cdot n$$

$$= k_0 \sum_{k=1}^{k_0} P_k + n \cdot \sum_{k=k_0+1}^{n} P_k$$

$$= k_0 \, Pr\{M < k_0\} + n \cdot Pr\{M > k_0\}$$

$$Pr\{M < k_0\} \leq 1$$

$$Pr\{M > k_0\}$$

$$= \sum_{k=k_0+1}^{n} P_k$$

$$\leq \sum_{k=k_0+1}^{n} \cdot \frac{1}{n^2}$$

$$\leq n \cdot \frac{1}{n^2}$$

$$\leq \frac{1}{n}$$

$$E(M) \leq k_0 \cdot 1 + n \cdot \frac{1}{n}$$

$$\leq k_0$$

$$= O\left(\frac{c \lg n}{\lg \lg n}\right)$$

11-3-a

i	j
0	$h(k)$
1	$1+h(k)$
2	$2+1+h(k)$
3	$3+2+1+h(k)$

$$\frac{i(i+1)}{2}+h(k)$$

$$h'(k)=\frac{1}{2}i^2+\frac{1}{2}i+h(k)$$

$$C_2=\frac{1}{2}\quad C_1=+\frac{1}{2}$$

11-3-b

They are inserted into different slot.

11-4-a

For $\langle x^{(1)}, x^{(2)}\rangle$ 2 distinct keys and for any h chosen at random from $\mathcal{H}$, the sequence $\langle h(x^{(1)}), h(x^{(2)})\rangle$ is equally likely to be any of the m^2 sequence of length 2 with elements drawn from $\{0, \ldots, m-1\}$

$$P\{h(x^{(1)})=h(x^{(2)})\}=\frac{1}{m}$$

So $\mathcal{H}$ is universal.

11-4-b

when $x=\langle 0, \cdots, 0\rangle$

$h_a(x)\equiv 0 \mod p$.

For distinct keys of $x^{(1)}$ and $x^{(2)}$ $(x^{(1)}=\langle 0, \cdots, 0\rangle$

In the sequence $\langle h_a(x^{(1)}), h_a(x^{(2)})\rangle$, $h_a(x^{(1)})$ is always 0, which breaks 2-universalness

The $Pr\{h_a(x^{(1)})=h_a(x^{(2)})\}=\frac{1}{m}$, which maintains universalness.

11-4-c

$\langle h_{ab}(x^{(1)}), h_{ab}(x^{(2)}) \rangle$ can be any sequence of length 2 with elements drawn from $(0, \ldots, p-1)$.

11-4-d

$Pr\{h(m') = t'\}$

$$= Pr\{\underbrace{h(m')}_{\text{fixed}} = \underbrace{h'(m')}_{\text{Random}}\}$$

$$= Pr\{h'(m') = h(m')\}$$

$$= \frac{1}{p}$$